BLOCKS IN DELIGNE'S CATEGORY $\underline{\mathrm{Re}}\mathrm{p}(S_t)$

by

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We give an exposition of Deligne's tensor category $\underline{\mathrm{Rep}}(S_t)$ where t is not necessarily an integer. Thereafter, we give a complete description of the blocks in $\underline{\mathrm{Rep}}(S_t)$ for arbitrary t. Finally, we use our result on blocks to decompose tensor products and classify tensor ideals in $\underline{\mathrm{Rep}}(S_t)$.

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CHAPTER I

INTRODUCTION

The subject of this dissertation lies in the theory of tensor categories. Let us begin by considering the collection of all finite dimensional complex representations of a finite (or affine algebraic) group G. These representations along with the maps among them form the category $\mathrm{Rep}(G)$. The tensor product of representations gives the category $\mathrm{Rep}(G)$ extra structure, making $\mathrm{Rep}(G)$ a basic example of a tensor category (see definition II.1.1). Many people have contributed to the theory of tensor categories, most notably Tannaka, Krein, Grothendieck, Saavedra Rivano, and Deligne. If we restrict ourselves to working over the complex numbers then one of the biggest results on tensor categories is due to Deligne, and states that any tensor category satisfying certain "mild" conditions can be realized as a category of representations of some supergroup[1] (see [8]). This result is quite remarkable since tensor categories with no clear underlying group arise in many areas of mathematics (e.g. algebraic geometry, differential Galois theory, algebraic quantum field theory). There do, however, exist tensor categories which do not satisfy the "mild" conditions in Deligne's result, and therefore cannot be realized as a category of representations of a supergroup. The purpose of this dissertation is to give a detailed description of a family of tensor categories which cannot be realized as categories of representations of any supergroup.

In this dissertation we study the tensor categories denoted $\underline{\mathrm{Rep}}(S_t)$ indexed by t which is not necessarily a nonnegative integer (for the purposes of this introduction, assume t is an arbitrary complex number; see definition III.2.9). Deligne introduced the category $\underline{\mathrm{Rep}}(S_t)$ in [9]; the notation was chosen because $\underline{\mathrm{Rep}}(S_t)$ "interpolates" representations of the symmetric group S_t when t is a nonnegative integer (see section IV.3). It is shown in [9] that $\underline{\mathrm{Rep}}(S_t)$ is semisimple if and only if t is not a nonnegative integer. Loosely speaking, this means that the structure of $\underline{\mathrm{Rep}}(S_t)$ is more complicated, and thus more interesting, when t is a nonnegative integer. The main result of

[1] A supergroup is a generalization of an affine group (see example II.1.2.3).

this dissertation is the complete description of the category $\underline{\text{Rep}}(S_t)$ in the non-semisimple cases. More precisely, we completely describe the blocks of $\underline{\text{Rep}}(S_t)$ as additive categories (see theorems VI.0.6, VII.1.4, and VII.3.1).

The category $\underline{\text{Rep}}(S_t)$ has been studied prior to Deligne's work in [9] in different guises. In the mid 1900's, Murnaghan and Littlewood worked on the problem of decomposing tensor products of symmetric group representations (see for example [23] and [17]). Their resulting formulae often involved "meaningless representations" which were discarded. These formulae are actually telling one how to decompose tensor products in $\underline{\text{Rep}}(S_t)$, and the "meaningless representations" can be explained by the precise connection between the symmetric groups and $\underline{\text{Rep}}(S_t)$ (see section IV.3). Moreover, $\underline{\text{Rep}}(S_t)$ is intimately related with the partition algebras introduced by Martin in [19] and [20] (see definition III.2.3). The partition algebras have been studied by many people including Doran and Wales (see [10]) as well as Halverson and Ram (see [14]).

This dissertation is organized as follows. In chapter II we recall the notions of tensor categories and pseudo-abelian envelopes. We give all necessary definitions and a few basic results which will be useful later in the text. In chapter III we carefully define the tensor category $\underline{\text{Rep}}(S_t)$ with emphasis on motivation. In chapter IV we classify indecomposable objects in $\underline{\text{Rep}}(S_t)$ and prove some basic properties of $\underline{\text{Rep}}(S_t)$. In chapter V we construct endomorphisms of the identity functor on $\underline{\text{Rep}}(S_t)$ which play a key role in proving our description of blocks of $\underline{\text{Rep}}(S_t)$. In chapters VI and VII we give a complete description of the blocks in $\underline{\text{Rep}}(S_t)$, the main result of this dissertation. In chapter VIII we use our results on blocks along with a classical formula due to Littlewood to decompose tensor products in $\underline{\text{Rep}}(S_t)$. Finally, in chapter IX we classify tensor ideals in $\underline{\text{Rep}}(S_t)$.

Lastly, there are other families of tensor categories which, like $\underline{\text{Rep}}(S_t)$, are generically semisimple. There exists such families which "interpolate" complex general linear and orthogonal groups (see for example [9, sections 9 and 10]). Also, Knop defined many more examples including tensor categories related to finite general linear groups in [15] and [16]. Some work has been done towards the description of blocks in these categories in the guise of studying Brauer algebras (see for example [6], [7], [5], and [22]). However, to this point in time, $\underline{\text{Rep}}(S_t)$ is the only family for which a complete description exists in the non-semisimple cases. Hopefully, the methods used in this text will be useful in furthering our understanding of these other families of tensor categories.

CHAPTER II

PRELIMINARIES

In this chapter we give a brief introduction to tensor categories, blocks of additive categories, and pseudo-abelian envelopes. We present basic definitions and results relevant to this dissertation. For a more complete treatment of tensor categories, see [1] or [8]. For more on pseudo-abelian envelopes, we refer the reader to [9, §1.7-8]. We close the chapter by fixing Young diagram conventions.

II.1 Tensor Categories

The main topic of study in this paper is the structure of certain tensor categories. In this section we will define the term tensor category and give a few examples.

Let F be a field.

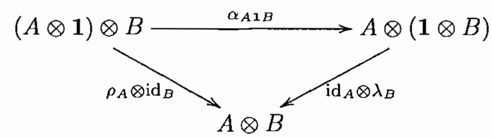
Definition II.1.1. A *tensor category* is an F-linear[1] category $\mathcal{T}$ equipped with

- an F-linear bifunctor $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$.
- (**associativity**) a functorial isomorphism $\alpha_{ABC} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$ for each $A, B, C \in \mathrm{ob}(\mathcal{T})$.
- (**commutativity**) a functorial isomorphism $\beta_{AB} : A \otimes B \xrightarrow{\sim} B \otimes A$ for each $A, B \in \mathrm{ob}(\mathcal{T})$, which satisfies $\beta_{BA}\beta_{AB} = \mathrm{id}_{A \otimes B}$.
- (**unit**) an object $\mathbf{1} \in \mathrm{ob}(\mathcal{T})$ and functorial isomorphisms $\lambda_A : \mathbf{1} \otimes A \xrightarrow{\sim} A$, $\rho_A : A \otimes \mathbf{1} \xrightarrow{\sim} A$ for each $A \in \mathrm{ob}(\mathcal{T})$.

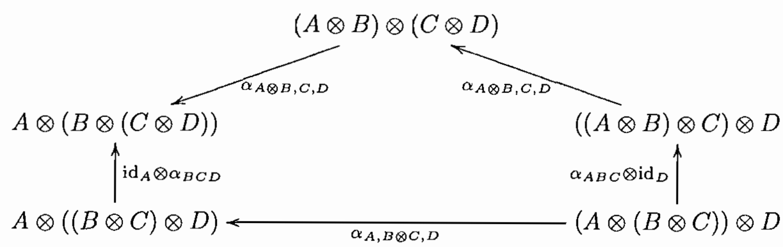
[1] An F-linear category is a category in which all Hom's are F-vector spaces and composition is F-bilinear.

satisfying:

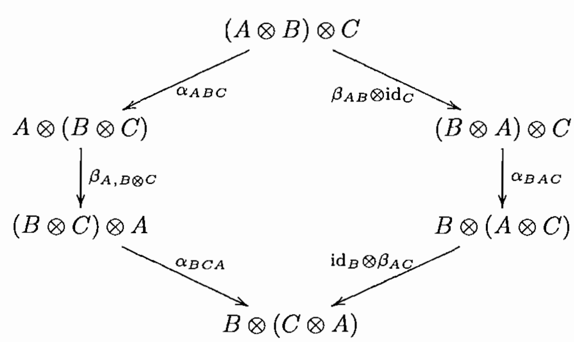
- **(triangle axiom)** For any objects A, B, the following diagram commutes.



- **(pentagon axiom)** For any objects A, B, C, D, the following diagram commutes.



- **(hexagon axiom)** For any objects A, B, C, the following diagram commutes.



Furthermore, a tensor category is assumed to be rigid. In other words, for any object A, there is a dual object $A^\vee$ and morphisms $ev_A : A^\vee \otimes A \to \mathbf{1}$, $coev_A : \mathbf{1} \to A \otimes A^\vee$ such that the compositions $A \xrightarrow{coev_A \otimes \mathrm{id}_A} A \otimes A^\vee \otimes A \xrightarrow{\mathrm{id}_A \otimes ev_A} A$ and $A^\vee \xrightarrow{\mathrm{id}_{A^\vee} \otimes coev_A} A^\vee \otimes A \otimes A^\vee \xrightarrow{ev_A \otimes \mathrm{id}_{A^\vee}} A^\vee$ are equal to identity morphisms (here I am skipping the associativity and unit isomorphisms). Finally, in a tensor category we require $\mathrm{End}(\mathbf{1}) = F$.

Example II.1.2. The following are all examples of tensor categories. In each example the tensor product as well as the associativity, commutativity, unit, and dual constraints are the usual ones.

(1) The category Vec of finite dimensional vector spaces over F.

(2) The category $\mathrm{Rep}(G)$ of finite dimensional representations of G over F, where G is an affine algebraic group.

(3) An *affine supergroup* G consists of data $(\mathcal{O}(G), \mu, \eta, \Delta, \varepsilon, S)$ where $\mathcal{O}(G) = \mathcal{O}(G)_0 \oplus \mathcal{O}(G)_1$ is a supercommutative super Hopf algebra with multiplication $\mu : \mathcal{O}(G) \otimes \mathcal{O}(G) \to \mathcal{O}(G)$, unit $\eta : F \to \mathcal{O}(G)$, comultiplication $\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$, counit $\varepsilon : \mathcal{O}(G) \to F$, and antipode $S : \mathcal{O}(G) \to \mathcal{O}(G)$. By a *representation of a supergroup* G, we mean a super comodule of $\mathcal{O}(G)$. Let $e : \mathcal{O}(G) \to F$ be any map satisfying

- the morphism $m \circ (e \otimes e) \circ \Delta : \mathcal{O}(G) \to F$ (Where $m : F \otimes F \to F$ is multiplication) is equal to the counit ε.
- the automorphism $(e \otimes \mathrm{id}_{\mathcal{O}(G)} \otimes (e \circ S)) \circ (\Delta \otimes \mathrm{id}_{\mathcal{O}(G)}) \circ \Delta : \mathcal{O}(G) \to \mathcal{O}(G)$ is equal to the map $h \mapsto (-1)^{\deg(h)} h$.

If V is a representation of G with comodule structure $\xi : V \to \mathcal{O}(G) \otimes V$, then e acts on V by the composition $V \xrightarrow{\xi} \mathcal{O}(G) \otimes V \xrightarrow{\mathrm{id}_V \otimes e} F \otimes V = V$. Let $\mathrm{Rep}(G, e)$ denote all finite dimensional representations of the supergroup G where e acts as the parity automorphism $v \mapsto (-1)^{\deg v}$. Given any supergroup G and any such e, $\mathrm{Rep}(G, e)$ is a tensor category. $\diamond$

Remark II.1.3. Deligne has shown that any tensor category satisfying certain "mild" constraints is equivalent as a tensor category to $\mathrm{Rep}(G, e)$ for some supergroup G (see [8]). In this paper we will study tensor categories which are not equivalent as tensor categories to $\mathrm{Rep}(G, e)$ for any supergroup G.

We close this section with definitions involving tensor categories relevant to this paper.

Definition II.1.4. A *tensor functor* between tensor categories $\mathcal{T}$ and $\mathcal{T}'$ is a functor $\mathcal{G} : \mathcal{T} \to \mathcal{T}'$ along with an isomorphism $1 \xrightarrow{\sim} \mathcal{G}(1)$ and functorial isomorphisms $\mathcal{G}(A) \otimes \mathcal{G}(B) \xrightarrow{\sim} \mathcal{G}(A \otimes B)$ which are compatible with the associativity, commutativity, and unit constraints.

Definition II.1.5. A *tensor ideal* $\mathcal{I}$ in a tensor category $\mathcal{T}$ is a subspace $\mathcal{I}(X, Y) \subset \mathrm{Hom}_{\mathcal{T}}(X, Y)$ for each pair of objects X, Y in $\mathcal{T}$ such that

(a) $ghk \in \mathcal{I}(X, W)$ for each $k \in \mathrm{Hom}_{\mathcal{T}}(X, Y)$, $h \in \mathcal{I}(Y, Z)$, $g \in \mathrm{Hom}_{\mathcal{T}}(Z, W)$.

(b) $g \otimes \mathrm{id}_Z \in \mathcal{I}(X \otimes Z, Y \otimes Z)$ for every object Z and every $g \in \mathcal{I}(X, Y)$.

A tensor ideal $\mathcal{I}$ is said to be *proper* if $\mathcal{I}(X, Y) \neq \mathrm{Hom}_{\mathcal{T}}(X, Y)$ for some objects X, Y. The tensor ideal consisting of all zero morphisms is called the *zero tensor ideal.*

Definition II.1.6. Given a tensor category $\mathcal{T}$ and a tensor ideal $\mathcal{I}$, the *quotient category,* $\mathcal{T}/\mathcal{I}$, is defined to be the tensor category with

- objects: $\mathrm{ob}(\mathcal{T}/\mathcal{I}) = \mathrm{ob}(\mathcal{T})$.
- morphisms: $\mathrm{Hom}_{\mathcal{T}/\mathcal{I}}(X, Y) := \mathrm{Hom}_{\mathcal{T}}(X, Y)/\mathcal{I}(X, Y)$.

Property (a) of a tensor ideal guarantees that composition in $\mathcal{T}/\mathcal{I}$ is well defined, whereas property (b) guarantees that $\mathcal{T}/\mathcal{I}$ is in fact a tensor category.

II.2 Linear Algebra in Tensor Categories

In this section we will extend the definitions of some common linear algebra terms to arbitrary tensor categories. We begin with dual maps.

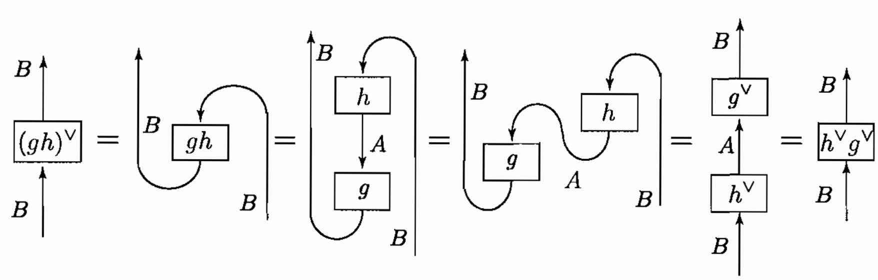
Definition II.2.1. Given a morphism $g : A \to B$ in a tensor category, define the *dual map* $g^\vee : B^\vee \to A^\vee$ to be the composition

$$B^\vee \xrightarrow{\mathrm{id}_{B^\vee} \otimes coev_A} B^\vee \otimes A \otimes A^\vee \xrightarrow{\mathrm{id}_{B^\vee} \otimes g \otimes \mathrm{id}_{A^\vee}} B^\vee \otimes B \otimes A^\vee \xrightarrow{ev_B \otimes \mathrm{id}_{A^\vee}} A^\vee.$$

The following familiar property of dual maps will be important for us in later chapters.

Proposition II.2.2. $(gh)^\vee = h^\vee g^\vee$ for any morphisms $g : A \to B$ and $h : B \to A$. In particular, if $e : A \to A$ is an idempotent then so is $e^\vee$.

Proof. We will use graphical calculus for morphisms in a tensor category (see [25] or [1]).



□

Next, we define the trace of an endomorphism as well as the dimension of an object.

Definition II.2.3. Given an endomorphism $g : A \to A$ in a tensor category $\mathcal{T}$, define its *trace* $\mathrm{tr}(g) \in \mathrm{End}_{\mathcal{T}}(\mathbf{1}) = F$ to be the composition

$$\mathbf{1} \xrightarrow{coev_A} A \otimes A^{\vee} \xrightarrow{g \otimes \mathrm{id}_{A^{\vee}}} A \otimes A^{\vee} \xrightarrow{\beta_A} A^{\vee} \otimes A \xrightarrow{ev_A} \mathbf{1}.$$

Given an object A in $\mathcal{T}$, define its *dimension* $\dim A := \mathrm{tr}(\mathrm{id}_A)$.

Just as with dual maps, there are familiar properties of trace which will be useful later. For a proof of the following proposition we refer the reader to [25, lemma 1.5.1].

Proposition II.2.4. (1) $\mathrm{tr}(gh) = \mathrm{tr}(hg)$ for any morphisms $g : A \to B$ and $h : B \to A$.

(2) $\mathrm{tr}(g \otimes h) = \mathrm{tr}(g)\mathrm{tr}(h)$ for any endomorphisms $g : A \to A$ and $h : B \to B$.

II.3 Blocks and Semisimple Categories

The main results of this dissertation concern the blocks of a specific tensor category. In this section we define the terms "block" and "semisimple" and say a few words on their connection.

Let $\mathcal{A}$ denote an arbitrary F-linear category which satisfies the Krull-Schmidt property[2].

Definition II.3.1. Consider the weakest equivalence relation on the set of isomorphism classes of indecomposable objects in $\mathcal{A}$ where two indecomposable objects are equivalent whenever there exists a nonzero morphism between them. We call the equivalence classes in this relation *blocks.* We will also use the term block to refer to a full subcategory of $\mathcal{A}$ which is $\oplus$-generated by the indecomposable objects in a single block. We will say a block is *trivial* if it contains only one indecomposable object (up to isomorphism) and its endomorphism ring is F.

It follows that $\mathcal{A}$ is equivalent to the direct sum of its blocks. Hence, to understand the structure of $\mathcal{A}$ it suffices to understand the structure of all the blocks in $\mathcal{A}$. The following definition is closely related to blocks.

Definition II.3.2. $\mathcal{A}$ is called *semisimple* if the following two conditions are satisfied.

- The only nonzero morphisms between indecomposable objects in $\mathcal{A}$ are isomorphisms.

[2] Every object can be decomposed as a direct sum of indecomposable objects. Moreover, the isomorphism types of the indecomposable summands are unique up to reordering.

- $\mathrm{End}_{\mathcal{A}}(X)$ is a division algebra which is finite dimensional over F for each indecomposable object X.

If one wishes to study the structure of an F-linear category $\mathcal{A}$, it is common to ask "is $\mathcal{A}$ semisimple?" If the answer is no, one may ask "by what amount does $\mathcal{A}$ fail to be semisimple?" One way to answer this question is to describe the blocks in $\mathcal{A}$. This is precisely what we aim to do with a specific tensor category in subsequent chapters.

II.4 Pseudo-abelian Envelopes

In this section we will define the "pseudo-abelian envelope" of an arbitrary pre-additive category. We then prove a Krull-Schmidt-type proposition concerning indecomposable objects in pseudo-abelian envelopes. With hopes of making the definitions a bit more comprehensible, there will be the running example.

For this section assume $\mathcal{A}$ is a pre-additive category (i.e. a category such that all Hom's are abelian groups and composition is bilinear).

Example II.4.1. Given a ring R, the category with one object whose endomorphism ring is R is a pre-additive category. We denote this category simply by R. ◇

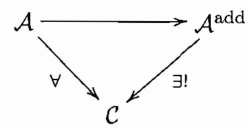
Definition II.4.2. The *additive envelope* of $\mathcal{A}$ is the category $\mathcal{A}^{\mathrm{add}}$ with

- objects: Finite tuples of objects in $\mathcal{A}$ written as $A_1 \oplus \cdots \oplus A_k$ for $A_1, \dots, A_k \in \mathrm{ob}(\mathcal{A})$, $k > 0$. We also include the empty tuple which is the zero object in $\mathcal{A}^{\mathrm{add}}$.
- morphisms: $\mathrm{Hom}_{\mathcal{A}^{\mathrm{add}}}\left(\bigoplus_{i=1}^{k} A_i, \bigoplus_{i=1}^{l} B_i\right)$ is the set of all $l \times k$-matrices whose (i,j)-entry is a morphism $A_j \to B_i$ in $\mathcal{A}$. Composition in $\mathcal{A}^{\mathrm{add}}$ is given by matrix multiplication along with the induced composition from $\mathcal{A}$.

Composition in $\mathcal{A}^{\mathrm{add}}$ is clearly associative. The identity morphism in $\mathrm{End}_{\mathcal{A}^{\mathrm{add}}}\left(\bigoplus_{i=1}^{k} A_i\right)$ is the diagonal matrix $\mathrm{diag}(\mathrm{id}_{A_1}, \dots, \mathrm{id}_{A_k})$.

Informally, $\mathcal{A}^{\mathrm{add}}$ is the smallest additive category containing $\mathcal{A}$. More precisely, it is easy to check that $\mathcal{A}^{\mathrm{add}}$ along with the functor $\mathcal{A} \to \mathcal{A}^{\mathrm{add}}$ which takes objects to 1-tuples has the following universal property.

Universal Property of $\mathcal{A}^{\mathrm{add}}$. If $\mathcal{C}$ is an additive category and $\mathcal{A} \to \mathcal{C}$ is an additive functor (i.e. a functor where the induced maps on Hom's are abelian group homomorphisms), then there exists a functor $\mathcal{A}^{\text{add}} \to \mathcal{C}$, unique up to natural equivalence, making the following diagram commute.



Example II.4.3. Let $\mathcal{A} = R$ where R is a ring. Then $\mathcal{A}^{\text{add}}$ is isomorphic as a category to the category of free right R-modules of finite rank. $\diamond$

Next, we need the notion of a "Karoubian category."

Definition II.4.4. A category $\mathcal{K}$ is called *Karoubian* if for every object A and every idempotent $e \in \text{End}_{\mathcal{K}}(A)$ there is an object B and morphisms $i : B \to A$ and $p : A \to B$ such that $p \circ i = \text{id}_B$ and $i \circ p = e$.

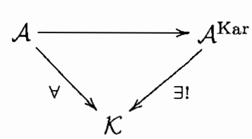
Definition II.4.5. The *Karoubian envelope* of $\mathcal{A}$ is the category $\mathcal{A}^{\text{Kar}}$ with

- objects: (A, e) for each $A \in \text{ob}(\mathcal{A})$ and idempotent $e \in \text{End}_{\mathcal{A}}(A)$.
- morphisms: $\text{Hom}_{\mathcal{A}^{\text{Kar}}}((A, e), (B, f)) := f \,\text{Hom}_{\mathcal{A}}(A, B)e$. Composition in $\mathcal{A}^{\text{Kar}}$ is induced from composition in $\mathcal{A}$.

Composition in $\mathcal{A}^{\text{Kar}}$ is clearly associative. The identity morphism in $\text{End}_{\mathcal{A}^{\text{Kar}}}((A, e))$ is the morphism e.

Just as $\mathcal{A}^{\text{add}}$ is the smallest additive category containing $\mathcal{A}$, $\mathcal{A}^{\text{Kar}}$ is the smallest Karoubian category containing $\mathcal{A}$. More precisely, it is easy to check that $\mathcal{A}^{\text{Kar}}$ along with the functor $\mathcal{A} \to \mathcal{A}^{\text{Kar}}$ which takes an object A to (A, id_A) has the following universal property.

Universal Property of $\mathcal{A}^{\text{Kar}}$. If $\mathcal{K}$ is a Karoubian category and $\mathcal{A} \to \mathcal{K}$ is a functor, then there is a functor $\mathcal{A}^{\text{Kar}} \to \mathcal{K}$, unique up to natural equivalence, making the following diagram commute.



We are now ready to define the pseudo-abelian envelope.

Definition II.4.6. The *pseudo-abelian envelope* of a pre-additive category $\mathcal{A}$ is defined to be $\mathcal{A}^{\text{ps ab}} := (\mathcal{A}^{\text{add}})^{\text{Kar}}$.

Example II.4.7. Let $\mathcal{A} = R$ where R is a ring. Then $\mathcal{A}^{\text{ps ab}}$ is equivalent to the category of finitely generated projective right R-modules. ◇

We close this section with a proposition concerning indecomposable objects in pseudo-abelian envelopes.

Proposition II.4.8. Suppose $\mathcal{A}$ is an F-linear category such that all Hom spaces are finite dimensional. An object in $\mathcal{A}^{\text{ps ab}}$ is indecomposable if and only if it is isomorphic to an object of the form (A, e) where A is an object in $\mathcal{A}$ and $e \in \text{End}_{\mathcal{A}}(A)$ is a primitive[3] idempotent.

Proof. Suppose (X, f) is an object in $\mathcal{A}^{\text{ps ab}}$. We will first show that (X, f) is indecomposable if and only if f is a primitive idempotent. To do so, suppose $f = f_1 + f_2$ where f_1 and f_2 are orthogonal idempotents. Let $p_j : (X, f_1) \oplus (X, f_2) \to (X, f_j)$ and $i_j : (X, f_j) \to (X, f_1) \oplus (X, f_2)$ for $j = 1, 2$ be the usual biproduct maps. Then $i_1 f_1 + i_2 f_2$ is an isomorphism from $(X, f) \to (X, f_1) \oplus (X, f_2)$ with inverse $f_1 p_1 + f_2 p_2$. It follows that (X, f) is indecomposable if and only if f is primitive.

Now suppose (X, f) is an indecomposable object in $\mathcal{A}^{\text{ps ab}}$ and write

$$\text{id}_X = f + f_1 + \cdots + f_r$$

where $f, f_1, \ldots, f_r$ are mutually orthogonal primitive idempotents. If $X = \bigoplus_{i=1}^{m} A_i$ and $\text{id}_{A_i} = e_{i,1} + \cdots + e_{i,r_i}$ is an orthogonal decomposition of id_{A_i} into primitive idempotents for each i, then

$$\text{id}_X = \sum_{1 \leq i \leq m} \sum_{1 \leq j \leq r_i} e_{i,j}$$

is another orthogonal decomposition of id_X into primitive idempotents. Thus, by the Krull-Schmidt theorem (see for example [2]), f is conjugate to $e_{i,j}$ for some i, j. If we let $u \in \text{End}_{\mathcal{A}^{\text{add}}}(X)$ be such that $f = u e_{i,j} u^{-1}$, then $f u e_{i,j} : (A_i, e_{i,j}) \to (X, f)$ is an isomorphism with inverse $e_{i,j} u^{-1} f$. □

Remark II.4.9. Suppose $\mathcal{A}$ is an F-linear category with all Hom spaces are finite dimensional. Then using proposition II.4.8 one can show the categories $(\mathcal{A}^{\text{add}})^{\text{Kar}}$ and $(\mathcal{A}^{\text{Kar}})^{\text{add}}$ are equivalent. This is not true for arbitrary $\mathcal{A}$. For example, consider the category $R = \mathbb{Z}/6\mathbb{Z}$ (see example

[3]An idempotent $e \in A$ is *primitive* if there do not exist nonzero idempotents $e_1, e_2 \in A$ with $e = e_1 + e_2$ and $e_1 e_2 = 0$.

II.4.3). Since $R^{\mathrm{Kar}} = R$, $(R^{\mathrm{Kar}})^{\mathrm{add}}$ is the category of free R-modules of finite rank. On the other hand, $(R^{\mathrm{add}})^{\mathrm{Kar}}$ is the category of finitely generated projective R-modules. But $\mathbb{Z}/2\mathbb{Z}$ is a finitely generated projective R-module which is not free.

II.5 Pseudo-abelian Envelopes of Tensor Categories

In this section we will show how to extend the structure of a tensor category to its pseudo-abelian envelope. We will do so by showing that the additive and Karoubian envelopes of a tensor category both inherit the structure of a tensor category.

Let $\mathcal{T}$ be a tensor category (see definition II.1.1). First, we give $\mathcal{T}^{\mathrm{add}}$ the structure of a tensor category.

Definition II.5.1. (the tensor structure for $\mathcal{T}^{\mathrm{add}}$)

- objects: Define the tensor product of two objects in $\mathcal{T}^{\mathrm{add}}$ by requiring that $\otimes$ distributes over $\oplus$ and using the tensor product of objects in $\mathcal{T}$.
- morphisms: The tensor product of two morphisms in $\mathcal{T}^{\mathrm{add}}$ is induced from the tensor product of morphisms in $\mathcal{T}$ along with the usual matrix tensor product.
- associativity: $\alpha_{\bigoplus_i A_i, \bigoplus_j B_j, \bigoplus_k C_k}$ is the diagonal matrix whose diagonal entries are α_{A_i,B_j,C_k}.
- commutativity: $\beta_{\bigoplus_i A_i, \bigoplus_j B_j}$ is the diagonal matrix whose diagonal entries are β_{A_i,B_j}.
- unit: The unit object in $\mathcal{T}^{\mathrm{add}}$ is the 1-tuple $\mathbf{1}$. $\lambda_{\bigoplus_i A_i}$ (resp. $\rho_{\bigoplus_i A_i}$) is the diagonal matrix whose diagonal entries are λ_{A_i} (resp. ρ_{A_i}).
- duals: Set $(\bigoplus_i A_i)^\vee := \bigoplus_i A_i^\vee$. $ev_{\bigoplus_i A_i}$ (resp. $coev_{\bigoplus_i A_i}$) is the matrix whose $A_i^\vee \otimes A_i \to \mathbf{1}$ (resp. $\mathbf{1} \to A_i \otimes A_i^\vee$) entries are ev_{A_i} (resp. $coev_{A_i}$) and all other entries are zero.

The following proposition is easy to check.

Proposition II.5.2. With the constraints listed above, $\mathcal{T}^{\mathrm{add}}$ is a tensor category.

Next we define a tensor structure on $\mathcal{T}^{\mathrm{Kar}}$.

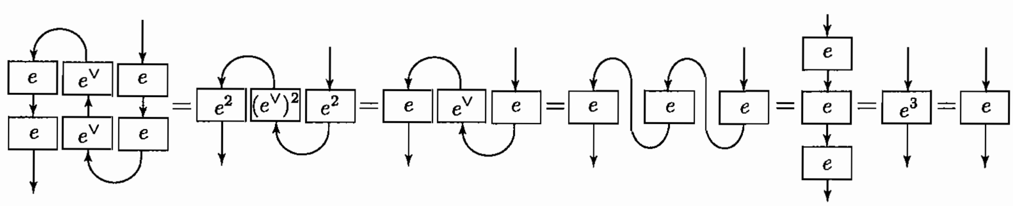
Definition II.5.3. (the tensor structure for $\mathcal{T}^{\mathrm{Kar}}$)

- objects: $(A, e) \otimes (B, f) := (A \otimes B, e \otimes f)$.

- morphisms: The tensor product of two morphisms in $\mathcal{T}^{\mathrm{Kar}}$ is defined to be the tensor product of the morphisms viewed as morphisms in $\mathcal{T}$.
- associativity: $\alpha_{(A,e),(B,f),(C,g)} := (e \otimes (f \otimes g))\alpha_{ABC}((e \otimes f) \otimes g)$.
- commutativity: $\beta_{(A,e),(B,f)} := (f \otimes e)\beta_{AB}(e \otimes f)$.
- unit: The unit object in $\mathcal{T}^{\mathrm{Kar}}$ is $(\mathbf{1}, \mathrm{id}_{\mathbf{1}})$. Set $\lambda_{(A,e)} := e\lambda_A(\mathrm{id}_{\mathbf{1}} \otimes e)$ and $\rho_{(A,e)} := e\rho_A(e \otimes \mathrm{id}_{\mathbf{1}})$.
- duals: Set $(A,e)^\vee := (A^\vee, e^\vee)$ (see definition II.2.1). Set $ev_{(A,e)} := ev_A(e^\vee \otimes e)$ and $coev_{(A,e)} := (e \otimes e^\vee)coev_A$.

Proposition II.5.4. With the constraints listed above, $\mathcal{T}^{\mathrm{Kar}}$ is a tensor category.

Proof. The only axioms of a tensor category which are not obviously satisfied are the rigidity axioms. To show $(\mathrm{id}_{(A,e)} \otimes ev_{(A,e)})(coev_{(A,e)} \otimes \mathrm{id}_{(A,e)}) = \mathrm{id}_{(A,e)}$ in $\mathcal{T}^{\mathrm{Kar}}$, we need to show that $(e \otimes (ev_A(e^\vee \otimes e)))(((e \otimes e^\vee)coev_A) \otimes e) = e$ in $\mathcal{T}$. We will do this using graphical calculus for morphisms in a $\mathcal{T}$. All the arrows in the following diagrams are labelled A.



The proof that $(ev_{(A,e)} \otimes \mathrm{id}_{(A,e)^\vee})(\mathrm{id}_{(A,e)^\vee} \otimes coev_{(A,e)}) = \mathrm{id}_{(A,e)^\vee}$ is similar. □

II.6 Notation and Conventions for Young Diagrams

Throughout this dissertation, Young diagrams will be used to index indecomposable objects in various categories. In this section we will fix our notation and drawing conventions for Young diagrams.

Definition II.6.1. A *Young diagram* $\lambda = (\lambda_1, \lambda_2, \ldots)$ is an infinite sequence of nonnegative integers with $\lambda_i \geq \lambda_{i+1}$ for all i, such that all but finitely many of the λ_i are zero. Also, the *size* of λ is set to be $|\lambda| := \sum_{i=1}^{\infty} \lambda_i$.

We will identify a Young diagram with an array of boxes as follows: Label the rows of the array $1, 2, 3, \ldots$ in increasing order from top to bottom. Place λ_i boxes in the ith row so that the number of boxes in each column decreases from left to right (gravity goes up and to the left). With this identification, $|\lambda|$ denotes the number of boxes.

Example II.6.2. The Young diagram $\lambda = (4, 4, 3, 3, 1, 0, \ldots)$ has $|\lambda| = 15$ and is pictured below.

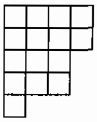


Figure 1: A Young Diagram $\diamond$

Occasionally it will be useful to give a Young diagram by its *multiplicities*. We will write $(l_1^{m_1}, l_2^{m_2}, \ldots, l_r^{m_r})$ to denote the Young diagram with m_i rows of length l_i for each $i = 1, \ldots, r$.

Example II.6.3. $(4^2, 3^2, 1)$ denotes the Young diagram in example II.6.2. $\diamond$

Next, we define a total order on the set of all Young diagrams.

Definition II.6.4. Given Young diagrams $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ write $\lambda \prec \lambda'$ if $|\lambda| < |\lambda'|$ or if $|\lambda| = |\lambda'|$ and there exists i such that $\lambda_i < \lambda'_i$ and $\lambda_j = \lambda'_j$ for all $j < i$.

Finally, let us fix the following notation concerning Young diagrams:

- Let Ψ denote the set of all Young diagrams and set $\Psi_d := \{\lambda \in \Psi \mid |\lambda| = d\}$.
- Let $\varnothing$ denote the "empty" Young diagram $(0, \ldots)$.
- Given a Young diagram λ, let L_λ denote the simple $S_{|\lambda|}$-module corresponding to λ (see e.g. [12, 4.2]).
- Given a Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots)$ and an element $t \in F$ (where F is a field of characteristic zero), set

$$\lambda(t) := (t - |\lambda|, \lambda_1, \lambda_2, \ldots).$$

CHAPTER III

THE TENSOR CATEGORY $\underline{\mathrm{Rep}}(S_t; F)$

In this chapter we carefully define Deligne's category $\underline{\mathrm{Rep}}(S_t; F)$. Our construction of Deligne's category will be motivated by the connection between symmetric groups and partition algebras introduced by Martin in [19] and [20]. We begin with an exposition of this connection.

III.1 Motivation: Representations of S_d and Partition Diagrams

Let d be a nonnegative integer, and let F be a field of characteristic zero. Let us consider the tensor category $\mathrm{Rep}(S_d; F)$ of finite dimensional representations over F of the symmetric group S_d. Note that we take S_0 to be the trivial group whose one element is the identity permutation of the empty set. Let V_d denote the natural d-dimensional representation of S_d with basis $\{v_1, \dots, v_d\}$, so that S_d acts by permuting the basis elements (V_0 is taken to be 0). Setting $V_d^{\otimes 0} = F$ for all $d \geq 0$, we have the following well known result:

Proposition III.1.1. *Any irreducible representation of S_d is a direct summand of $V_d^{\otimes n}$ for some nonnegative integer n.*

Proof. If $d = 0$ the statement is certainly true. Assume now that $d > 0$. Let χ denote the character of V_d. Consider the virtual character

$$\psi := \prod_{\substack{\sigma \in S_d \\ \sigma \neq 1}} (\chi - \chi(\sigma)).$$

Since V_d is a faithful representation, $\psi(\sigma) \neq 0$ if and only if $\sigma = 1$. Hence $d!\psi$ is a nonzero integer multiple of the character of the regular representation of S_d, which contains all irreducible representations. As ψ is an integer linear combination of characters of the form χ^n ($n \in \mathbb{Z}_{\geq 0}$), the result follows. $\square$

Remark III.1.2. The proof of proposition III.1.1 works for any faithful representation of any finite group G.

As a consequence of proposition III.1.1, one way to understand $\mathrm{Rep}(S_d; F)$ is to study objects of the form $V_d^{\otimes n}$ and morphisms between those objects. We will now use set partitions to construct some such morphisms. Our notation will be similar to that of [14].

By a *partition* π of a finite set S we mean a collection $\pi_1, \ldots, \pi_n$ of disjoint, nonempty subsets of S with $S = \bigcup_i \pi_i$. The sets π_i will be called *parts* of π. Given a partition π of $\{1, \ldots, n, 1', \ldots, m'\}$, a *partition diagram of* π is a graph with vertices labelled $\{1, \ldots, n, 1', \ldots, m'\}$ whose connected components partition the vertices into the parts of π. We will always draw partition diagrams using the following convention:

- Vertices $1, \ldots, n$ (resp. $1', \ldots, m'$) are aligned horizontally and increasing from left to right with i directly above i'.

- Edges lie entirely below the vertices labelled $1, \ldots, n$ and above the vertices labelled $1', \ldots, m'$.

We will identify a partition with its partition diagram and write π for both the partition and the partition diagram.

Example III.1.3. Figure 1 shows a partition diagram for the partition $\{\{1, 3, 2', 3'\}, \{2, 4\}, \{1'\}\}$.

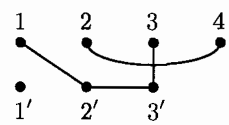


Figure 1: A Partition Diagram

Notice that a partition diagram representing this partition is not unique, but its connected components are. ◇

To each partition of $\{1, \ldots, n, 1', \ldots, m'\}$ we will associate a linear map $V_d^{\otimes n} \to V_d^{\otimes m}$. Before doing so, we introduce some notation.

- Let $P_{n,m}$ denote the set of all partitions of $\{1, \ldots, n, 1', \ldots, m'\}$, let $P_{n,0}$ denote the set of all partitions of $\{1, \ldots, n\}$, let $P_{0,m}$ denote the set of all partitions of $\{1', \ldots, m'\}$, and, by convention, let $P_{0,0} := \{\varnothing\}$ where $\varnothing$ denotes the *empty partition diagram.* Finally, let $FP_{n,m}$ denote the F-vector space with basis $P_{n,m}$.

- For nonnegative integers n and d, let $[n,d]$ denote the set of all functions from $\{j \mid 1 \le j \le n\}$ to $\{j \mid 1 \le j \le d\}$. In particular, $[0,d] = \{\varnothing\}$ for all d, and $[n,0] = \varnothing$ for all $n \neq 0$. Given $\boldsymbol{i} \in [n,d]$ and $j \in \{j \mid 1 \le j \le n\}$, write i_j for the image of j under $\boldsymbol{i}$.

- For $\boldsymbol{i} \in [n,d]$ and $\boldsymbol{i}' \in [m,d]$, the $(\boldsymbol{i},\boldsymbol{i}')$-*coloring* of a partition $\pi \in P_{n,m}$ is obtained by coloring the vertices of π labelled j (resp. j') by the integer i_j (resp. i'_j). Saying an $(\boldsymbol{i},\boldsymbol{i}')$-coloring of π is *good* means vertices are colored the same whenever they are in the same connected component of π. Saying an $(\boldsymbol{i},\boldsymbol{i}')$-coloring of π is *perfect* means vertices are colored the same if and only if they are in the same connected component of π.

- For $n \neq 0$ and $\boldsymbol{i} \in [n,d]$, set $v_{\boldsymbol{i}} := v_{i_1} \otimes \cdots \otimes v_{i_n} \in V_d^{\otimes n}$. Set $v_\varnothing := 1 \in V_d^{\otimes 0}$ where $\varnothing$ is the unique element of $[0,d]$.

We are now ready to associate partitions in $P_{n,m}$ with linear maps $V_d^{\otimes n} \to V_d^{\otimes m}$.

Definition III.1.4. For $n,m,d \in \mathbb{Z}_{\ge 0}$, define the F-linear map $f : FP_{n,m} \to \mathrm{Hom}_{S_d}(V_d^{\otimes n}, V_d^{\otimes m})$ by setting

$$f(x)(v_{\boldsymbol{i}}) = \sum_{\boldsymbol{i}' \in [m,d]} f(x)^{\boldsymbol{i}}_{\boldsymbol{i}'} v_{\boldsymbol{i}'} \qquad (x \in FP_{n,m}, \boldsymbol{i} \in [n,d])$$

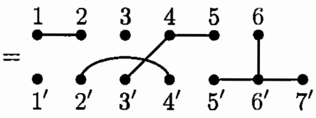
where

$$f(\pi)^{\boldsymbol{i}}_{\boldsymbol{i}'} := \begin{cases} 1, & \text{if the } (\boldsymbol{i},\boldsymbol{i}')\text{-coloring of } \pi \text{ is good,} \\ 0, & \text{otherwise.} \end{cases} \qquad (\pi \in P_{n,m})$$

Indeed, $f(x)$ commutes with the action of S_d which merely permutes the colors.

Example III.1.5. (1) $f : FP_{0,0} \to \mathrm{End}_{S_d}(F)$ sends the unique element of $P_{0,0}$ to the identity map on F.

(2) Assume $d > 0$ and let

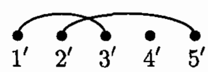
$\pi =$ 

Then $f(\pi) : V_d^{\otimes 6} \to V_d^{\otimes 7}$ is given by

$$f(\pi)(v_{\boldsymbol{i}}) = \begin{cases} \sum\limits_{1 \leq k,j \leq d} v_k \otimes v_j \otimes v_{i_4} \otimes v_j \otimes v_{i_6} \otimes v_{i_6} \otimes v_{i_6}, & \text{if } i_1 = i_2 \text{ and } i_4 = i_5, \\ 0, & \text{otherwise.} \end{cases}$$

for $\boldsymbol{i} \in [6, d]$.

(3) Assume $d > 0$ and let

$\pi =$ 

Then $f(\pi) : F \to V_d^{\otimes 5}$ is given by

$$f(\pi)(1) = \sum_{1 \leq i,j,k \leq d} v_i \otimes v_j \otimes v_i \otimes v_k \otimes v_j.$$

◇

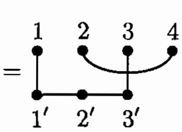
Next we wish to show that f is surjective. A proof of this fact when $n = m$ can be found in [14, theorem 3.6]. Their proof extends to the case $n \neq m$ without difficulty. For completeness we will give a modified version of their proof. Before doing so, we must introduce a bit more notation.

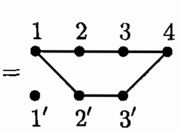
- Let $\leq$ be the partial order on $P_{n,m}$ defined by $\pi \leq \mu$ whenever the partition μ is courser than the partition π (i.e. r and s are in the same part of μ whenever they are in the same part of π for each pair $r, s \in \{1, \ldots, n, 1', \ldots, m'\}$).
- Define the basis $\{x_\pi \mid \pi \in P_{n,m}\}$ of $FP_{n,m}$ inductively by setting

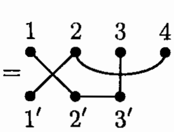
$$x_\pi := \pi - \sum_{\mu \gneqq \pi} x_\mu. \tag{III.1}$$

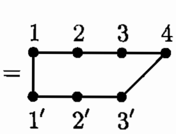
Example III.1.6. (1) $x_\pi = \pi$ when π is any partition consisting of one part.

(2) Let $\pi \in P_{4,3}$ be the partition with the partition diagram given in example III.1.3. Then $x_\pi = \pi - \mu_1 - \mu_2 - \mu_3 + 2\mu_4$ where

$\mu_1 =$ 

$\mu_2 =$ 

$\mu_3 =$ 

$\mu_4 =$ 

◇

We are now ready to prove the following:

Theorem III.1.7. For integers $n, m, d \geq 0$, the map $f : FP_{n,m} \to \mathrm{Hom}_{S_d}(V_d^{\otimes n}, V_d^{\otimes m})$ (see definition III.1.4) has the following properties:

(1) f is surjective.

(2) $\ker(f) = \mathrm{Span}_F\{x_\pi \mid \pi \text{ has more than } d \text{ parts}\}$.

In particular, f is an isomorphism of F-vector spaces whenever $d \geq n + m$.

Proof. (compare with [14, proof of theorem 3.6(a)]) If $d = 0$ then the theorem is certainly true. Now assume $d > 0$. For $g \in \mathrm{Hom}_{S_d}(V_d^{\otimes n}, V_d^{\otimes m})$ write

$$g(v_{\boldsymbol{i}}) := \sum_{\boldsymbol{i}' \in [m,d]} g^{\boldsymbol{i}}_{\boldsymbol{i}'} v_{\boldsymbol{i}'} \qquad (\boldsymbol{i} \in [n, d])$$

Since g commutes with the action of S_d, the matrix entries $g^{\boldsymbol{i}}_{\boldsymbol{i}'}$ are constant on the S_d-orbits of the matrix coordinates $\{(\boldsymbol{i}, \boldsymbol{i}')\}_{\boldsymbol{i} \in [n,d], \boldsymbol{i}' \in [m,d]}$. Each S_d-orbit of $\{(\boldsymbol{i}, \boldsymbol{i}')\}_{\boldsymbol{i} \in [n,d], \boldsymbol{i}' \in [m,d]}$ corresponds to a partition $\pi \in P_{n,m}$ as follows: $(\boldsymbol{i}, \boldsymbol{i}')$ is in the orbit corresponding to π if and only if the $(\boldsymbol{i}, \boldsymbol{i}')$-coloring of π is perfect. A straightforward induction argument shows

$$f(x_\pi)^{\boldsymbol{i}}_{\boldsymbol{i}'} = \begin{cases} 1, & \text{if the } (\boldsymbol{i}, \boldsymbol{i}')\text{-coloring of } \pi \text{ is perfect,} \\ 0, & \text{otherwise.} \end{cases} \qquad (\boldsymbol{i} \in [n, d], \boldsymbol{i}' \in [m, d]) \qquad \text{(III.2)}$$

Thus g is a F-linear combination of the $f(x_\pi)$'s. This proves part (1).

To prove part (2) notice that for $\pi \in P_{n,m}$, there exists $\boldsymbol{i} \in [n, d]$ and $\boldsymbol{i}' \in [m, d]$ such that the $(\boldsymbol{i}, \boldsymbol{i}')$-coloring of π is perfect if and only if π has at most d parts. Hence, by (III.2), $f(x_\pi)$ is the zero map if and only if π has more than d parts. Part (2) now follows since $\{x_\pi \mid \pi \in P_{n,m}\}$ is a basis for $FP_{n,m}$. □

We conclude our investigation of morphisms of the form $f(\pi) : V_d^{\otimes n} \to V_d^{\otimes m}$ by studying the composition of such morphisms. First we require the following:

Definition III.1.8. Given partition diagrams $\pi \in P_{n,m}$ and $\mu \in P_{m,l}$, construct a new diagram $\mu \star \pi$ by identifying the vertices $1', \ldots, m'$ of π with the vertices $1, \ldots, m$ of μ and renaming them $1'', \ldots, m''$ as illustrated in figure 2.

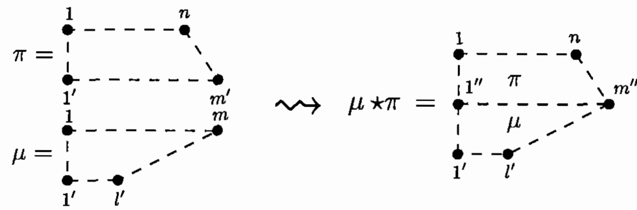


Figure 2: Composition of Partition Diagrams

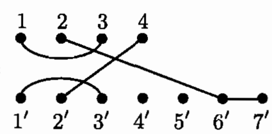
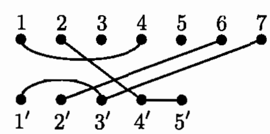
Now let $\ell(\mu,\pi)$ denote the number of connected components of $\mu \star \pi$ whose vertices are not among $1,\ldots,n,1',\ldots,l'$. Finally, let $\mu\cdot\pi \in P_{n,l}$ be the partition obtained by restricting $\mu\star\pi$ to $\{1,\ldots,n,1',\ldots,l'\}$ (i.e. r and s are in the same part of $\mu\cdot\pi$ if and only if r and s are in the same part of $\mu\star\pi$).

We are now ready to state

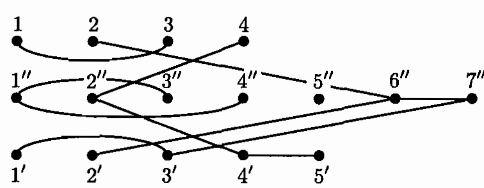
Proposition III.1.9. $f(\mu)f(\pi) = d^{\ell(\mu,\pi)}f(\mu\cdot\pi)$ for any $\pi \in P_{n,m}$, $\mu \in P_{m,l}$.

Before we give a proof of proposition III.1.9 let us consider an example.

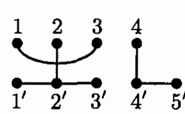
Example III.1.10. In this example we will verify proposition III.1.9 when

$\pi =$  and $\mu =$ 

On the one hand,

$\mu\star\pi =$ 

Thus $\ell(\mu,\pi) = 2$ and

$\mu\cdot\pi =$ 

Therefore we have $d^{\ell(\mu,\pi)}f(\mu\cdot\pi)(v_{i_1}\otimes v_{i_2}\otimes v_{i_3}\otimes v_{i_4}) = d^2\delta_{i_1,i_3}v_{i_2}\otimes v_{i_2}\otimes v_{i_2}\otimes v_{i_4}\otimes v_{i_4}$ where $\delta_{i,j}$ is the Kronecker delta function.

On the other hand,

$$\begin{aligned}
f(\mu)f(\pi)(v_{\boldsymbol{i}}) &= f(\mu)\left(\delta_{i_1,i_3}\sum_{1\le j,k,l\le d} v_j\otimes v_{i_4}\otimes v_j\otimes v_k\otimes v_l\otimes v_{i_2}\otimes v_{i_2}\right)\\
&= \delta_{i_1,i_3}\sum_{1\le j,k,l\le d} f(\mu)\left(v_j\otimes v_{i_4}\otimes v_j\otimes v_k\otimes v_l\otimes v_{i_2}\otimes v_{i_2}\right)\\
&= \delta_{i_1,i_3}\sum_{1\le j,k,l\le d} \delta_{j,k}\; v_{i_2}\otimes v_{i_2}\otimes v_{i_2}\otimes v_{i_4}\otimes v_{i_4}\\
&= \delta_{i_1,i_3}\sum_{1\le j,l\le d} v_{i_2}\otimes v_{i_2}\otimes v_{i_2}\otimes v_{i_4}\otimes v_{i_4}\\
&= d^2\delta_{i_1,i_3}\; v_{i_2}\otimes v_{i_2}\otimes v_{i_2}\otimes v_{i_4}\otimes v_{i_4}.
\end{aligned}$$

for any $\boldsymbol{i} \in [4, d]$, as desired. $\diamond$

Now to show proposition III.1.9 holds in general.

Proof of proposition III.1.9. Suppose $\pi \in P_{n,m}$ and $\mu \in P_{m,l}$ for some integers $n, m, l \geq 0$. By definition III.1.4 the matrix coordinates of $f(\mu)f(\pi) : V_d^{\otimes n} \to V_d^{\otimes l}$ are given by

$$(f(\mu)f(\pi))_{\boldsymbol{i}'}^{\boldsymbol{i}} = \sum_{\boldsymbol{i}''\in[m,d]} f(\mu)_{\boldsymbol{i}'}^{\boldsymbol{i}''} f(\pi)_{\boldsymbol{i}''}^{\boldsymbol{i}}. \tag{III.3}$$

Hence $(f(\mu)f(\pi))_{\boldsymbol{i}'}^{\boldsymbol{i}}$ is the number of $\boldsymbol{i}'' \in [m,d]$ such that the $(\boldsymbol{i},\boldsymbol{i}'')$-coloring of π and the $(\boldsymbol{i}'',\boldsymbol{i}')$-coloring of μ are simultaneously good. These are exactly the $\boldsymbol{i}'' \in [m,d]$ such that coloring the vertices j, j', j'' of $\mu \star \pi$ with integers i_j, i'_j, i''_j respectively, gives a good coloring of $\mu \star \pi$. Any good coloring of $\mu \star \pi$ gives rise to a good coloring of $\mu \cdot \pi$. Clearly any good coloring of $\mu \cdot \pi$ arrises in this way. Moreover, two good colorings of $\mu \star \pi$ give the same good coloring of $\mu \cdot \pi$ if and only if the two colorings of $\mu \star \pi$ differ only at connected components whose vertices are not among $1, \ldots, n, 1', \ldots, l'$. Since there are d choices of color for each component, we see the number of $\boldsymbol{i}'' \in [m,d]$ such that the $(\boldsymbol{i},\boldsymbol{i}'')$-coloring of π and the $(\boldsymbol{i}'',\boldsymbol{i}')$-coloring of μ are simultaneously good is $d^{\ell(\mu,\pi)} f(\mu\cdot\pi)_{\boldsymbol{i}'}^{\boldsymbol{i}}$. The result follows. $\square$

Remark III.1.11. From proposition III.1.9 the structure constants of the composition $f(\mu)f(\pi)$ are polynomial in the integer d. We will exploit this fact in section III.2 when we define the category $\underline{\mathrm{Rep}}(S_t; F)$ which "interpolates" the category $\mathrm{Rep}(S_t; F)$ for nonnegative integer t, but is defined for arbitrary $t \in F$.

III.2 Definition of $\underline{\mathbf{Rep}}(S_t; F)$

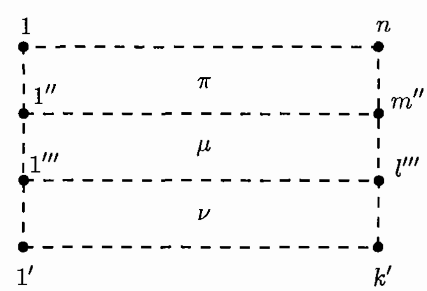
In this section we define Deligne's tensor category $\underline{\mathrm{Rep}}(S_t; F)$ for arbitrary $t \in F$ following [9, §8] (so our definition is different from the one given in [9, §2]). To construct $\underline{\mathrm{Rep}}(S_t; F)$ we will first use partitions to construct the smaller category $\underline{\mathrm{Rep}}_0(S_t; F)$. We then obtain $\underline{\mathrm{Rep}}(S_t; F)$ from $\underline{\mathrm{Rep}}_0(S_t; F)$ using pseudo-abelian envelopes.

As in the previous section, assume d is a nonnegative integer and that F is a field of characteristic zero. Let $\mathrm{Rep}_0(S_d; F)$ denote the full subcategory of $\mathrm{Rep}(S_d; F)$ whose objects are of the form $V_d^{\otimes n}$ for $n \geq 0$. Clearly the objects in $\mathrm{Rep}_0(S_d; F)$ are indexed by nonnegative integers, and by theorem III.1.7 the morphisms in $\mathrm{Rep}_0(S_d; F)$ are given (albeit not uniquely) by F-linear combinations of maps $f(\pi)$ indexed by set partitions. Moreover, the structure constants of compositions of the $f(\pi)$'s are polynomials in d (see proposition III.1.9). Using this data, we now define a tensor category similar to $\mathrm{Rep}_0(S_d; F)$ replacing the integer d with an arbitrary element of F.

Let $t \in F$.

Definition III.2.1. The category $\underline{\mathrm{Rep}}_0(S_t; F)$ has
Objects: $[n]$ for each $n \in \mathbb{Z}_{\geq 0}$.
Morphisms: $\mathrm{Hom}_{\underline{\mathrm{Rep}}_0(S_t;F)}([n],[m]) := FP_{n,m}$.
Composition: $FP_{m,l} \times FP_{n,m} \to FP_{n,l}$ is defined to be the bilinear map satisfying $\mu \circ \pi = t^{\ell(\mu,\pi)} \mu \cdot \pi$ for each $\pi \in P_{n,m}$, $\mu \in P_{m,l}$.

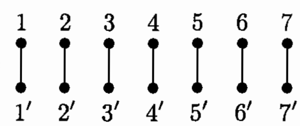
To see that composition is associative, it is enough to show $\nu \circ (\mu \circ \pi) = (\nu \circ \mu) \circ \pi$ for all $\pi \in P_{n,m}$, $\mu \in P_{m,l}$, $\nu \in P_{l,k}$. To do so, consider the partition

$\nu \star \mu \star \pi =$ 

Notice that $\nu \cdot (\mu \cdot \pi)$, $(\nu \cdot \mu) \cdot \pi \in P_{n,k}$ are both obtained by restricting the partition $\nu \star \mu \star \pi$ to the set $\{1, \ldots, n, 1', \ldots, k'\}$. Furthermore, $\ell(\mu, \pi) + \ell(\nu, \mu \cdot \pi)$ and $\ell(\nu, \mu) + \ell(\nu \cdot \mu, \pi)$ are both the number of connected components of $\nu \star \mu \star \pi$ whose vertices are not among $\{1, \ldots, n, 1', \ldots, k'\}$. Hence composition is associative.

One can easily check that the partition in $P_{n,n}$ whose parts are all of the form $\{j, j'\}$ is the identity morphism $\mathrm{id}_n : [n] \to [n]$.

Example III.2.2. The identity morphism $\mathrm{id}_7 : [7] \to [7]$ is given by

$\mathrm{id}_7 =$ 

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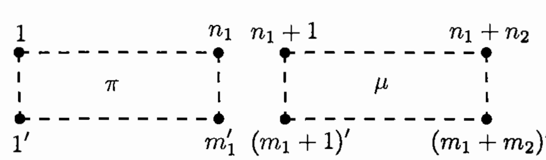
Before giving $\underline{\mathrm{Rep}}_0(S_t; F)$ the structure of a tensor category, we pause to give the following definition which will play an important role in later sections.

Definition III.2.3. (compare with [19], [20]) The *partition algebra* $FP_n(t)$ is defined to be the endomorphism algebra $\mathrm{End}_{\underline{\mathrm{Rep}}_0(S_t;F)}([n])$.

Remark III.2.4. We identify each element of the symmetric group S_n with a partition in $P_{n,n}$ as follows: $\sigma \leftrightarrow \{\{i, \sigma(i)'\} \mid 1 \leq i \leq n\}$. This identification extends linearly to an inclusion of algebras $FS_n \hookrightarrow FP_n(t)$ for each $t \in F$ (here FS_n denotes the group algebra of S_n).

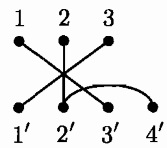
Now to define tensor products. While reading the following definitions, the reader may find it helpful to keep in mind the analogy studied in section III.1 between the objects $[n]$ (resp. morphisms π) in $\underline{\mathrm{Rep}}_0(S_t; F)$ and the objects $V_d^{\otimes n}$ (resp. morphisms $f(\pi)$) in $\mathrm{Rep}(S_d; F)$.

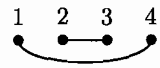
Definition III.2.5. For objects $[n], [m]$ in $\underline{\mathrm{Rep}}_0(S_t; F)$ set $[n] \otimes [m] := [n+m]$. For morphisms we let $\otimes : FP_{n_1,m_1} \times FP_{n_2,m_2} \to FP_{n_1+n_2,m_1+m_2}$ be the bilinear map such that

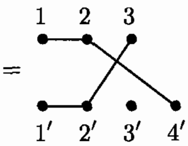
$\pi \otimes \mu :=$ 

for all $\pi \in P_{n_1,m_1}$, $\mu \in P_{n_2,m_2}$.

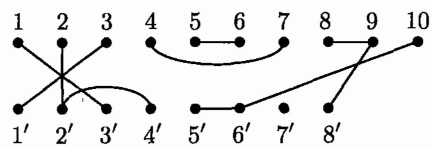
Example III.2.6. Suppose

$\pi =$ 

$\mu =$ 

$\nu =$ 

Then

$\pi \otimes \mu \otimes \nu =$ 

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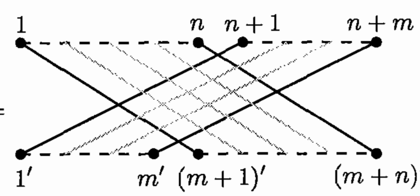
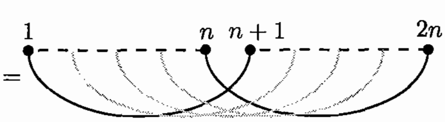
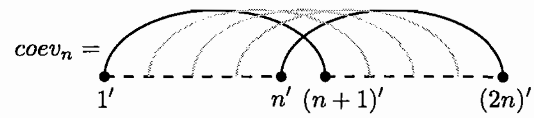
Proposition III.2.7. The following constraints make $\underline{\mathrm{Rep}}_0(S_t; F)$ a tensor category.

- (associativity) $\alpha_{n,m,l} : ([n] \otimes [m]) \otimes [l] \to [n] \otimes ([m] \otimes [l])$ is the identity morphism id_{n+m+l}.
- (commutativity) $\beta_{n,m} : [n] \otimes [m] \to [m] \otimes [n]$ is the partition in $P_{n+m,n+m}$ whose parts are of the form $\{j, (m+j)'\}$ or $\{n+j, j'\}$. That is to say

$\beta_{n,m} =$ 

- (unit) Set $\mathbf{1} := [0]$. Both unit morphisms $[0] \otimes [n] \to [n]$ and $[n] \otimes [0] \to [n]$ are the identity morphism id_n.
- (duals) Set $[n]^\vee := [n]$ with the morphism $ev_n : [n]^\vee \otimes [n] \to \mathbf{1}$ (resp. $coev_n : \mathbf{1} \to [n] \otimes [n]^\vee$) given by the partition in $P_{2n,0}$ (resp. $P_{0,2n}$) whose parts are of the form $\{j, n+j\}$ (resp. $\{j', (n+j)'\}$). That is to say

$ev_n =$ 



Proof. The triangle and pentagon axioms are easily satisfied, as all morphisms in both diagrams are identity morphisms. Figure 3 illustrates the hexagon axiom.

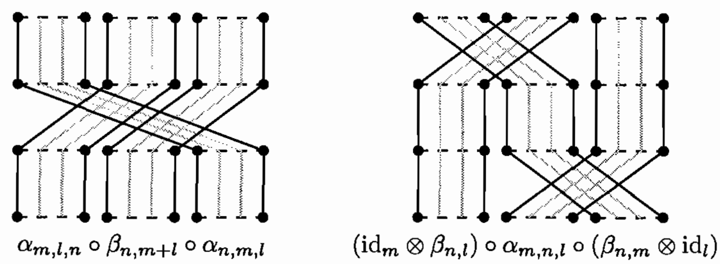


Figure 3: The Hexagon Axiom

Figure 4 shows that the choice of dual objects as well as the evaluation and coevaluation morphisms make $\underline{\mathrm{Rep}}_0(S_t; F)$ into a rigid category.

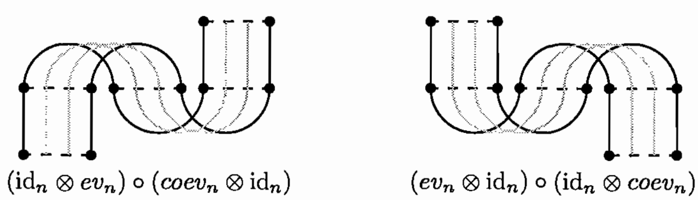


Figure 4: The Rigidity Axioms

Finally, $\mathrm{End}_{\underline{\mathrm{Rep}}_0(S_t;F)}(\mathbf{1}) = F$ as $P_{0,0}$ contains only the empty partition. Thus $\underline{\mathrm{Rep}}_0(S_t; F)$ is a tensor category. □

Remark III.2.8. For $d \in \mathbb{Z}_{\geq 0}$, the connection between $\underline{\mathrm{Rep}}(S_d; F)$ and $\mathrm{Rep}(S_d; F)$ can be made more precise as follows. The functor $\mathcal{F}_0 : \underline{\mathrm{Rep}}_0(S_d; F) \to \mathrm{Rep}_0(S_d; F)$ defined on objects by $[n] \mapsto V_d^{\otimes n}$, and on morphisms as the F-linear map $\pi \mapsto f(\pi)$ is a tensor functor. $\mathcal{F}_0$ is certainly surjective on objects. Moreover, by theorem III.1.7.1, $\mathcal{F}_0$ is surjective morphisms. However, by theorem III.1.7.2, $\mathcal{F}_0$ does not give an equivalence of categories.

Now we are ready to use pseudo-abelian envelopes studied in sections II.4 and II.5 to define Deligne's category.

Definition III.2.9. $\underline{\text{Rep}}(S_t; F) := \underline{\text{Rep}}_0(S_t; F)^{\text{ps ab}}$.

Remark III.2.10. (1) $\underline{\text{Rep}}(S_t; F)$ inherits the structure of a tensor category from $\underline{\text{Rep}}_0(S_t; F)$ (see propositions II.5.2, II.5.4, and III.2.7).

(2) Speaking informally, studying the category $\underline{\text{Rep}}_0(S_t; F)$ is a way to simultaneously study the partition algebras $FP_n(t)$ for all $n \geq 0$ (see definition III.2.3 along with example II.4.1). In this line of thinking, studying $\underline{\text{Rep}}(S_t; F)$ is a way to simultaneously study all finitely generated projective right $FP_n(t)$-modules for all $n \geq 0$ (see example VII.2).

III.3 The Trace of an Endomorphism in $\underline{\text{Rep}}_0(S_t; F)$

We close this section by examining the trace of a morphism in $\underline{\text{Rep}}_0(S_t; F)$. First, notice that tr : $\text{End}_{\underline{\text{Rep}}_0(S_t;F)}([n]) \to F$ is an F-linear map. Furthermore, if $\pi : [n] \to [n]$ is a partition diagram (not equal to id_0) then, by the definition of trace (see definition III.3), $\text{tr}(\pi) = t^\ell$ where ℓ is the number of connected components of the leftmost diagram in figure 5. Clearly ℓ is also the

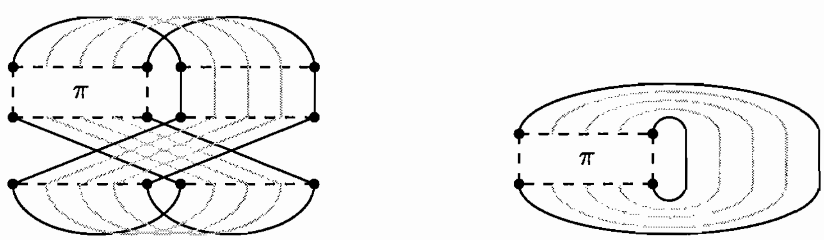


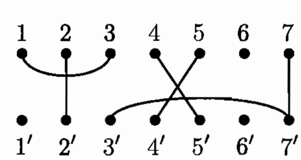
Figure 5: The Trace of π (Left) and the Trace Diagram of π (Right)

number of connected components in the *trace diagram* of π shown on the right in figure 5.

Example III.3.1. (1) The only endomorphisms in $\underline{\text{Rep}}_0(S_0; F)$ with nonzero trace are nonzero scalar multiples of id_0.

(2) In $\underline{\text{Rep}}_0(S_t; F)$, $\dim([0]) = \text{tr}(\text{id}_0) = 1$ and $\dim([n]) = \text{tr}(\text{id}_n) = t^n$ for all positive n.

(3) Consider $\pi : [7] \to [7]$ given by

$\pi =$ 

$\mathrm{tr}(\pi) = t^4$ since there are 4 connected components in the the trace diagram of π, as shown in figure 6.

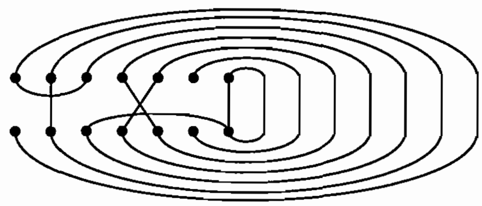


Figure 6: An Example of a Trace Diagram

◇

CHAPTER IV

INDECOMPOSABLE OBJECTS

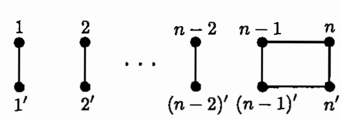
This chapter is organized as follows. In section IV.1 we give a classification of indecomposable objects in $\underline{\mathrm{Rep}}(S_t; F)$ in terms of Young diagrams. In section IV.2 we show that $\underline{\mathrm{Rep}}(S_t; F)$ is semisimple for generic t. In section IV.3 we explain the connection between $\underline{\mathrm{Rep}}(S_d; F)$ and $\mathrm{Rep}(S_d; F)$ when d is a nonnegative integer. Finally, in section IV.4 we study a hook length formula which gives the dimension of a generic indecomposable object.

IV.1 Classification of Indecomposable Objects in $\underline{\mathrm{Rep}}(S_t; F)$

In this section we will classify indecomposable objects in $\underline{\mathrm{Rep}}(S_t; F)$ for arbitrary $t \in F$. It will be convenient for us to let K denote the field of fractions of $F[[T - t]]$ where T is an indeterminate.

We start with the following

Lemma IV.1.1. For $n > 1$ let e denote the following idempotent in $FP_n(t)$

$e =$ 

Then for each $n > 1$ we have the following algebra isomorphisms:

(1) $eFP_n(t)e \cong FP_{n-1}(t)$.

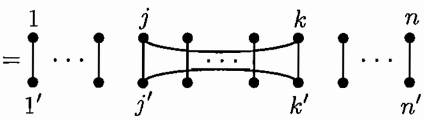
(2) $FP_n(t)/(e) \cong FS_n$.

Proof. To prove (1) notice we can embed $FP_{n-1}(t)$ into $FP_n(t)$ as the F-span of

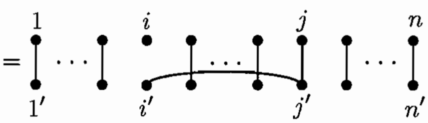
$$\{\pi \in P_{n,n} \mid n \text{ (resp. } n') \text{ is in the same part of } \pi \text{ as } n-1 \text{ (resp. } (n-1)')\}.$$

This span is exactly $eFP_n(t)e$.

To prove (2) recall (Remark III.2.4) that we can embed FS_n into $FP_n(t)$ by identifying $\sigma \in S_n$ with the partition $\{\{1,\sigma(1)'\},\ldots,\{n,\sigma(n)'\}\}$. Since $FS_n \cap (e) = 0$, it suffices to show a partition $\pi \in P_{n,n}$ has $\pi \in (e)$ whenever $\pi \notin S_n$. Notice for fixed j and k, the partition

$\pi_{j,k} =$ 

is in (e). Indeed, $\pi_{j,k} = \sigma e \sigma$ where $\sigma \in S_n \subset P_{n,n}$ is the product of transpositions $(j, n-1)(k,n)$. Now suppose $\mu \in P_{n,n} \backslash S_n$. Then either μ has a part of the form $\{i\}$ for some $i \in \{1,\ldots,n\}$ or there exist $j,k \in \{1,\ldots,n\}$ which are in the same part of μ. If the latter is true, then $\mu = \mu\pi_{j,k} \in (e)$. If the former is true, then $\mu = \mu\pi_{i,j}\nu_{i,j} \in (e)$ where $j \neq i$ and

$\nu_{i,j} =$ 

□

Remark IV.1.2. In fact the proof shows that the composition of the embedding $FS_n \subset FP_n(t)$ and the projection $FP_n(t) \to FP_n(t)/(e) \cong FS_n$ is the identity map.

In view of Lemma IV.1.1.2 we can consider any irreducible representation of S_n as an irreducible representation of $FP_n(t)$. Now suppose λ is a Young diagram. Let E_λ be an irreducible representation of $S_{|\lambda|}$ corresponding to λ (see e.g. [12, 4.2]) considered as a representation of $FP_{|\lambda|}(t)$ and let $P(E_\lambda)$ be the projective cover of E_λ. Then $P(E_\lambda)$ is isomorphic to $FP_{|\lambda|}(t)-$module of the form $FP_{|\lambda|}(t)e_\lambda$ where $e_\lambda \in FP_{|\lambda|}(t)$ is a primitive idempotent. The idempotent e_λ is not unique but it is unique up to conjugation[1]; hence the object $L(\lambda) = ([|\lambda|], e_\lambda) \in \underline{\text{Rep}}(S_t; F)$ is an indecomposable object (see proposition II.4.8) which is well defined up to isomorphism.

[1]Recall that two idempotents e, e' in a finite dimensional algebra A are conjugate if and only if the modules Ae and Ae' are isomorphic.

Theorem IV.1.3. The assignment $\lambda \mapsto L(\lambda)$ induces a bijection

$$\left\{ \begin{array}{c} \text{Young diagrams of} \\ \text{arbitrary size} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{indecomposable objects in} \\ \underline{\text{Rep}}(S_t; F) \text{ up to isomorphism} \end{array} \right\}$$

This bijection enjoys the following properties:

(1) Except for the case $t = 0, \lambda = \varnothing$, there exists an indecomposable object of the form $([n], e)$ corresponding to the Young diagram λ if and only if $n \geq |\lambda|$.

(2) (Lifting Idempotents) Suppose $e \in FP_n(t)$ is a primitive idempotent such that $([n], e) \cong L(\lambda)$ in $\underline{\text{Rep}}(S_t; F)$.. Then there is an idempotent $\varepsilon \in KP_n(T)$ with $\varepsilon|_{T=t} = e$[2]. Moreover, if $\varepsilon = \varepsilon_1 + \cdots + \varepsilon_r$ is an orthogonal decomposition of ε into primitive idempotents, then there is a unique i such that $([n], \varepsilon_i) \cong L(\lambda)$ in $\underline{\text{Rep}}(S_T; K)$. Finally, if $([n], e)$ and $([n'], e')$ are isomorphic in $\underline{\text{Rep}}(S_t; F)$ and e, e' lift to $\varepsilon, \varepsilon'$ respectively, then $([n], \varepsilon)$ and $([n'], \varepsilon')$ are isomorphic in $\underline{\text{Rep}}(S_T; K)$

The remainder of section IV.1 is devoted to the proof of theorem IV.1.3. With proposition II.4.8 in mind, we start by classifying primitive idempotents (up to conjugation) in partition algebras. We will use the following well known lemma (see e.g. [2]):

Lemma IV.1.4. Suppose A is a finite dimensional F-algebra and e is an idempotent in A. Let (e) denote the two-sided ideal of A generated by e. There are bijective correspondences

$$\left\{ \begin{array}{c} \text{primitive} \\ \text{idempotents} \\ \text{in } A \text{ up to} \\ \text{conjugation} \end{array} \right\} \overset{\text{bij.}}{\leftrightarrow} \left\{ \begin{array}{c} \text{simple} \\ A\text{-modules} \\ \text{up to} \\ \text{isomorphism} \end{array} \right\} \overset{\text{bij.}}{\leftrightarrow} \left\{ \begin{array}{c} \text{simple} \\ eAe\text{-modules} \\ \text{up to} \\ \text{isomorphism} \end{array} \right\} \sqcup \left\{ \begin{array}{c} \text{simple} \\ A/(e)\text{-modules} \\ \text{up to} \\ \text{isomorphism} \end{array} \right\}$$

satisfying the following properties.

(1) Suppose f is a primitive idempotent in A and L is a simple A-module. f corresponds to L in the leftmost bijection above if and only if $fL \neq 0$. Moreover, if f corresponds to L, then the F-linear map $L \to L$ given by $x \mapsto fx$ has trace[3] equal to 1.

[2]Evaluating $T = t$ does not give a well-defined map $KP_n(T) \to FP_n(t)$. Part of the theorem is that we can find such an $\varepsilon \in KP_n(T)$ so that $\varepsilon|_{T=t}$ makes sense.

[3]This refers to the trace in the category Vec_F, i.e. the usual trace of an F-linear map.

(2) In the rightmost bijection above, the simple A-modules which correspond to simple $A/(e)$-modules are exactly the simple A-modules for which e acts as zero.

We start with classifying the primitive idempotents of partition algebras. Part (1) of the following theorem is originally due to Martin (See [21]). However, his proof does not extend to the case $t = 0$. Our proof is similar to one found in [10].

Theorem IV.1.5. (1) When $t \neq 0$ we have the following bijection.

$$\left\{ \begin{array}{c} \text{primitive idempotents in} \\ FP_n(t) \text{ up to conjugation} \end{array} \right\} \xleftrightarrow{\text{bij.}} \left\{ \begin{array}{c} \text{Young diagrams } \lambda \\ \text{with } |\lambda| \leq n \end{array} \right\}$$

(2) When $n > 0$ we have the following bijection.

$$\left\{ \begin{array}{c} \text{primitive idempotents in} \\ FP_n(0) \text{ up to conjugation} \end{array} \right\} \xleftrightarrow{\text{bij.}} \left\{ \begin{array}{c} \text{Young diagrams } \lambda \\ \text{with } 0 < |\lambda| \leq n \end{array} \right\}$$

Proof. Part (1) is true when $n = 0$ since $FP_0(t) = F$. To show part (1) holds for $n = 1$, let

$$\pi = \begin{array}{c} 1 \\ \bullet \\ \\ \bullet \\ 1' \end{array}$$

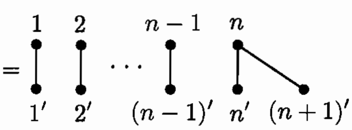
and let f denote the idempotent $\frac{1}{t}\pi$. It is easy to show that $1 = f + (1 - f)$ is a nontrivial decomposition of 1 into primitive idempotents in $FP_n(t)$ when $t \neq 0$. Thus part (1) holds when $n = 1$. Now we proceed by induction on n. For $n > 1$ let $e \in FP_n(t)$ denote the idempotent in lemma IV.1.1. Then by lemma IV.1.1 we have $FP_n(t)/(e) \cong FS_n$ and $eFP_n(t)e \cong FP_{n-1}(t)$. Since the simple FS_n-modules up to isomorphism are in bijective correspondence with all Young diagrams λ with $|\lambda| = n$, part (1) will follow by induction along with lemma IV.1.4.

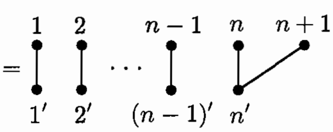
The proof of (2) is similar, except $FP_1(0) \cong F[\pi]/(\pi^2)$ has no nontrivial idempotents. □

We are now in position to prove part (1) of theorem IV.1.3.

Proof of Theorem IV.1.3 part (1). Let us first consider the case when $t \neq 0$. By proposition II.4.8 we know every indecomposable object in $\underline{\text{Rep}}(S_t; F)$ is isomorphic to one of the form $([n], e)$

for some nonzero primitive idempotent $e \in FP_n(t)$. By theorem IV.1.5 it suffices to show that for every integer $n \geq 0$ and every nonzero primitive idempotent $e \in FP_n(t)$, there exists and idempotent $f \in FP_{n+1}(t)$ such that $([n], e)$ is isomorphic to $([n+1], f)$. When $n = 0$, the only primitive idempotent is the empty partition id_0. Let μ and μ' be the only elements of $P_{0,1}$ and $P_{1,0}$ respectively, and let π be the idempotent in the proof of theorem IV.1.5. Then the morphism $\frac{1}{t}\pi\mu\mathrm{id}_0 : ([0], \mathrm{id}_0) \to ([1], \pi)$ is an isomorphism with inverse $\frac{1}{t}\mathrm{id}_0\mu'\pi$. Now suppose $n > 0$ and consider the partition diagrams

$\phi_n =$ 

$\phi'_n =$ 

Notice that the morphism $(\phi_n e \phi'_n)\phi_n e : ([n], e) \to ([n+1], \phi_n e \phi'_n)$ is an isomorphism with inverse $e\phi'_n(\phi_n e \phi'_n)$.

When $t = 0$ the argument above shows for every integer $n > 0$ and every nonzero primitive idempotent $e \in FP_n(t)$, there exists an idempotent $f \in FP_{n+1}(t)$ such that $([n], e)$ is isomorphic to $([n+1], f)$. However, there is no nonzero composition $[0] \to [n] \to [0]$ in $\underline{\mathrm{Rep}}_0(S_0)$ for any $n > 0$. Thus there is no integer $n > 0$ and idempotent $f \in FP_n(0)$ such that $([n], f)$ is isomorphic to the indecomposable object $([0], \mathrm{id}_0)$. The case when $t = 0$ now follows from theorem IV.1.5.2. □

To prove part (2) of theorem IV.1.3 we will use the following well known lemma (see for example [2, theorem 1.7.3]).

Lemma IV.1.6. Suppose A is an algebra and $N \subset A$ is a nilpotent ideal. If e is an idempotent in A/N, then there is an idempotent f in A lifting e (i.e. the quotient map $A \twoheadrightarrow A/N$ sends $f \mapsto e$). Moreover, if e and e' are conjugate idempotents in A/N lifting to idempotents f and f' in A respectively, then f and f' are conjugate.

We are now ready to prove part (2) of theorem IV.1.3.

Proof of Theorem IV.1.3 part (2). Let e and λ be as in the statement of theorem IV.1.3(2). Set $R_i := F[T]/(T-t)^i$ for each positive integer i, and $R_\infty := F[[T-t]]$. By lemma IV.1.6 we can lift idempotents from the partition algebra $R_{i-1}P_n(t) = R_iP_n(t)/(T-t)^{i-1}$ to the algebra $R_iP_n(t)$ for each $i > 0$. Set $e_1 = e$ and recursively pick $e_i \in R_iP_n(T)$ to be an idempotent which lifts e_{i-1} for $i > 1$. Finally, let $\varepsilon \in R_\infty P_n(T)$ be the unique element such that $\varepsilon \mapsto e_i$ under the quotient map $R_\infty P_n(T) \twoheadrightarrow R_i P_n(T)$ for each $i > 0$. Then ε is an idempotent in $KP_n(T)$ such that $\varepsilon|_{T=t} = e$. Suppose that $\varepsilon = \varepsilon_1 + \cdots + \varepsilon_r$ is an orthogonal decomposition of ε into primitive idempotents. We will show by induction on n that there is a unique i such that the indecomposable object $([n], \varepsilon_i)$ in $\underline{\text{Rep}}(S_T; K)$ corresponds to λ.

If $n = |\lambda|$, then by parts (2) of lemmas IV.1.4 and IV.1.1, e corresponds to the simple FS_n-module L_λ labelled by the Young diagram λ. Furthermore, e acts on L_λ via the quotient map $FP_n(t) \twoheadrightarrow FS_n$ along with the action of FS_n. Similarly, ε acts on $K \otimes_F L_\lambda$, the simple KS_n-module corresponding to λ, through the quotient map $KP_n(T) \twoheadrightarrow KS_n$. Since ε is an idempotent, the trace of the K-linear map $K \otimes_F L_\lambda \to K \otimes_F L_\lambda$ given by $x \mapsto \varepsilon \cdot x$ is a nonnegative integer. But $\varepsilon|_{T=t} = e$, and by lemma IV.1.4.1 we know the F-linear map $L_\lambda \to L_\lambda$ given by $x \mapsto e \cdot x$ has trace equal to 1. Hence the trace of the map $K \otimes_F L_\lambda \to K \otimes_F L_\lambda$ given by $x \mapsto \varepsilon \cdot x$ must also equal 1. By lemma IV.1.4.1, this is only possible if there exists a unique i with $([n], \varepsilon_i)$ corresponding to λ.

If $n > |\lambda|$, set $e' = \phi'_{n-1} e \phi_{n-1}$ where ϕ_{n-1}, ϕ'_{n-1} are as in the proof of theorem IV.1.31. Since $([n], e)$ and $([n-1], e')$ are isomorphic, inductively we we can find an $\varepsilon' \in KP_{n-1}(T)$ so that the pair e', ε' satisfies part (2) of theorem IV.1.3. Set $\varepsilon = \phi_{n-1} \varepsilon' \phi'_{n-1}$.

Finally, assume $([n], e)$ and $([n'], e')$ are isomorphic in $\underline{\text{Rep}}(S_t; F)$ and e, e' lift to $\varepsilon, \varepsilon'$ respectively. Without loss of generality assume $n \geq n'$ and set

$$\phi := \begin{cases} \text{id}_n & \text{if } n = n' \\ \phi_{n-1}\phi_{n-2}\cdots\phi_{n'} & \text{if } n > n' \end{cases} \quad \text{and} \quad \phi' := \begin{cases} \text{id}_n & \text{if } n = n' \\ \phi'_{n'}\phi'_{n'+1}\cdots\phi'_{n-1} & \text{if } n > n' \end{cases}$$

Then e is conjugate to $\phi e' \phi'$ in $FP_n(t)$. Hence, by lemma IV.1.6, ε and $\phi \varepsilon' \phi'$ are conjugate in $KP_n(T)$. Thus $([n], \varepsilon)$ and $([n'], \varepsilon')$ are isomorphic in $\underline{\text{Rep}}(S_T; K)$. □

We close this section with an observation concerning field extensions and idempotents. The following proposition will be quite useful in subsequent sections.

Proposition IV.1.7. Suppose $F \subset E$ is a field extension and e is a primitive idempotent in $FP_n(t)$. Then e is also a primitive idempotent in $EP_n(t)$.

Proof. The assumption implies that the object $([n], e) \in \underline{\text{Rep}}(S_t; F)$ is indecomposable. Thus by Theorem IV.1.3 it is isomorphic to the object $L(\lambda) = ([|\lambda|], e_\lambda)$. The representation E_λ of $FP_{|\lambda|}(t)$ is absolutely irreducible (since the corresponding representation of $S_{|\lambda|}$ is), so the idempotent e_λ

is primitive in $EP_{|\lambda|}(t)$. Hence the object $([n], e) \cong ([|\lambda|], e_\lambda) \in \underline{\mathrm{Rep}}(S_t, E)$ is indecomposable and we are done. □

Corollary IV.1.8. Every indecomposable object in $\underline{\mathrm{Rep}}(S_T; K)$ is isomorphic to one of the form $([n], \varepsilon)$ where $\varepsilon = \sum_{\pi \in P_{n,n}} a_\pi \pi$ with $a_\pi \in F(T)$ for all π (here $F(T)$ denotes the field of fractions of the polynomial ring $F[T]$). In particular, if ε corresponds to the Young diagram λ, then for all but finitely many integers d, $\varepsilon|_{T=d}$ is a primitive idempotent in $FP_n(d)$ corresponding to λ.

Proof. Applying proposition IV.1.7 to the field extension $F(T) \subset K$ shows that every primitive idempotent in $KP_n(T)$ is conjugate to one with coefficients in $F(T)$. □

IV.2 On $\underline{\mathrm{Rep}}(S_t; F)$ for Generic t

In this section we will show that $\underline{\mathrm{Rep}}(S_t; F)$ is semisimple for "generic" values of t. Deligne showed that $\underline{\mathrm{Rep}}(S_t; F)$ is not semisimple if and only if t is a nonnegative integer (see [9]). That result will follow from our description of the blocks in $\underline{\mathrm{Rep}}(S_t; F)$ (see corollary VI.4.5). For now we will confine ourselves to prove the following, weaker theorem.

Theorem IV.2.1. $\underline{\mathrm{Rep}}(S_t; F)$ is semisimple for all but countably many values of t. Moreover, if $\underline{\mathrm{Rep}}(S_t; F)$ is not semisimple, then t is an algebraic integer.

We will use the following well known lemma in our proof of theorem IV.2.1.

Lemma IV.2.2. Suppose A is a finite dimensional F-algebra. For $a \in A$, let ϕ_a denote the F-linear map $A \to A$ given by $x \mapsto ax$. Define the *trace form* on A by $(a, b) := \mathrm{tr}(\phi_a \phi_b)$. Then A is a semisimple algebra if and only if the trace form on A is non-degenerate.

Proof. Let $S := \{a \in A \mid (a, b) = 0 \text{ for all } b \in A\}$. We will show that S is equal to $J(A)$ (the Jacobson radical of A). Suppose $a \in J(A)$. Since $J(A)$ is a nilpotent ideal, ab is nilpotent for every $b \in A$. Hence $(a, b) = 0$ for all $b \in A$, so $J(A) \subset S$. On the other hand, if $a \in S$ then $\mathrm{tr}(\phi_a^n) = 0$ for all integers $n > 0$. This implies ϕ_a, and thus a, is nilpotent. Hence S is a nilpotent two sided ideal. As $J(A)$ is the largest nilpotent two sided ideal in A, we conclude $S \subset J(A)$. □

Before proving theorem IV.2.1 we give two examples illustrating the usefulness of lemma IV.2.2. The reader may find these examples helpful when reading the proof of theorem IV.2.1.

Example IV.2.3. (1) Consider the partition algebra $FP_1(t)$. Let π be as in the proof of theorem IV.1.5 and fix the ordered basis $P_{1,1} = \{\mathrm{id}_1, \pi\}$ for $FP_1(t)$. Under this ordered basis the matrix with entries (x, y) for $x, y \in P_{1,1}$ is

$$\begin{pmatrix} 2 & t \\ t & t^2 \end{pmatrix}$$

Since the determinant of the matrix above is t^2, we conclude (by lemma IV.2.2) that $FP_1(t)$ is semisimple if and only if $t \neq 0$.

(2) Consider the partition algebra $FP_2(t)$. Using the ordered basis

$P_{2,2} =$ 

the matrix with entries (π, μ) for $\pi, \mu \in P_{2,2}$ is

$$\begin{pmatrix}
2t^4 & 2t^3 & 2t^3 & 2t^3 & 2t^3 & 2t^3 & 2t^3 & 2t^2 & 2t^2 & 2t^2 & 2t^2 & 2t^2 & 2t^2 & 2t^2 & 2t \\
2t^3 & 5t^2 & 2t^2 & 2t^2 & 2t^2 & 2t^2 & 2t^2 & 5t & 2t & 2t & 2t & 5t & 5t & 2t & 5 \\
2t^3 & 2t^2 & 5t^2 & 2t^2 & 2t^2 & 2t^2 & 2t^2 & 5t & 2t & 2t & 5t & 2t & 2t & 5t & 5 \\
2t^3 & 2t^2 & 2t^2 & 2t^2 & 2t^3 & 2t^2 & 2t^2 & 2t & 2t^2 & 2t & 2t^2 & 2t^2 & 2t & 2t & 2t \\
2t^3 & 2t^2 & 2t^2 & 2t^3 & 2t^2 & 2t^2 & 2t^2 & 2t & 2t^2 & 2t & 2t & 2t & 2t^2 & 2t^2 & 2t \\
2t^3 & 2t^2 & 2t^2 & 2t^2 & 2t^2 & 2t^2 & 5t^2 & 2t & 2t & 5t & 5t & 2t & 5t & 2t & 5 \\
2t^3 & 2t^2 & 2t^2 & 2t^2 & 2t^2 & 5t^2 & 2t^2 & 2t & 2t & 5t & 2t & 5t & 2t & 5t & 5 \\
2t^2 & 5t & 5t & 2t & 2t & 2t & 2t & 15 & 2t & 7 & 5 & 5 & 5 & 5 & 5 \\
2t^2 & 2t & 2t & 2t^2 & 2t^2 & 2t & 2t & 2t & 2t^2 & 2t & 2t & 2t & 2t & 2t & 2t \\
2t^2 & 2t & 2t & 2t & 2t & 5t & 5t & 7 & 2t & 15 & 5 & 5 & 5 & 5 & 5 \\
2t^2 & 2t & 5t & 2t^2 & 2t & 5t & 2t & 5 & 2t & 5 & 5 & 5 & 2t & 5t & 5 \\
2t^2 & 5t & 2t & 2t^2 & 2t & 2t & 5t & 5 & 2t & 5 & 5 & 5 & 5t & 2t & 5 \\
2t^2 & 5t & 2t & 2t & 2t^2 & 5t & 2t & 5 & 2t & 5 & 2t & 5t & 5 & 5 & 5 \\
2t^2 & 2t & 5t & 2t & 2t^2 & 2t & 5t & 5 & 2t & 5 & 5t & 2t & 5 & 5 & 5 \\
2t & 5 & 5 & 2t & 2t & 5 & 5 & 5 & 2t & 5 & 5 & 5 & 5 & 5 & 5
\end{pmatrix}$$

The determinant of the matrix above is $1259712t^{14}(t-1)^4(t-2)^6$. Hence, by lemma IV.2.2, $FP_2(t)$ is semisimple if and only if $t \neq 0, 1, 2$. ◇

Now to prove theorem IV.2.1.

Proof of theorem IV.2.1. Suppose L_1 and L_2 are indecomposable objects in $\underline{\mathrm{Rep}}(S_t; F)$. By theorem IV.1.3.1 there exists $n \in \mathbb{Z}_{geq0}$ and idempotents $e_1, e_2 \in FP_n(t)$ so that L_1 and L_2 are isomorphic to $([n], e_1)$ and $([n], e_2)$ respectively. Whence $\mathrm{Hom}_{\underline{\mathrm{Rep}}(S_t;F)}(L_1, L_2) = e_2FP_n(t)e_1$.

Thus, in order to show $\text{Hom}_{\underline{\text{Rep}}(S_t;F)}(L_1, L_2)$ is either zero or a finite dimensional division algebra over F, it suffices to show $FP_n(t)$ is a semisimple algebra. Therefore, for a fixed $t \in F$, $\underline{\text{Rep}}(S_t; F)$ is semisimple whenever $FP_n(t)$ are semisimple for all $n \geq 0$.

Let $M_n(t)$ denote the matrix whose rows and columns are labelled by the elements of $P_{n,n}$ (in some fixed order) with the x, y-entry equal to (x, y) (the trace form on $FP_n(t)$, see lemma IV.2.2). Then the entries of $M_n(t)$ are in $\mathbb{Z}[t]$. Hence $\det M_n(t) \in \mathbb{Z}[t]$. It follows from lemma IV.2.2 that $FP_n(t)$ is semisimple if and only if $\det M_n(t) \neq 0$. However, from theorem III.1.7, we know that $FP_n(d)$ is semisimple for integers $d \geq 2n$. Hence, $\det M_n(t)$ is a polynomial in t which is not identically zero. Thus, for each $n \geq 0$ there are only finitely may values of t for which $\det M_n(t) = 0$. Therefore there are only countably many values of t for which $FP_n(t)$ is not semisimple for all $n \geq 0$. □

Remark IV.2.4. If $t \in F$ is not an algebraic integer, then by theorem IV.2.1 we know $\underline{\text{Rep}}(S_t; F)$ is semisimple. However, given an arbitrary $t \in F$, neither theorem IV.2.1 nor its proof allow us to determine if $\underline{\text{Rep}}(S_t; F)$ is semisimple. As mentioned at the beginning of this section, we will eventually show that $\underline{\text{Rep}}(S_t; F)$ is semisimple if and only if t is not a nonnegative integer.

We close this section with one final observation.

Corollary IV.2.5. $\underline{\text{Rep}}(S_T; K)$ is semisimple.

Proof. This follows from theorem IV.2.1 as T is not an algebraic integer. □

IV.3 The Interpolation Functor $\underline{\text{Rep}}(S_d; F) \to \text{Rep}(S_d; F)$

Throughout this section we assume d is a nonnegative integer. In this section we will explain how $\underline{\text{Rep}}(S_d; F)$ "interpolates" the category $\text{Rep}(S_d; F)$. More precisely, we will show $\text{Rep}(S_d; F)$ is equivalent to the quotient of $\underline{\text{Rep}}(S_t; F)$ by the so called "negligible morphisms." To start, let us define the *interpolation functor*.

Definition IV.3.1. $\mathcal{F} : \underline{\text{Rep}}(S_d; F) \to \text{Rep}(S_d; F)$ is the functor defined on indecomposable objects by $\mathcal{F}([n], e) = f(e)(V_d^{\otimes n})$, and on morphisms $\alpha : ([n], e) \to ([n'], e')$ by $\mathcal{F}(\alpha) = f(\alpha)$. Here f and V_d are as in section III.1.

Notice that $\mathcal{F}$ is clearly a tensor functor. Moreover, from the discussion in section III.1 we have the following.

Proposition IV.3.2. $\mathcal{F}$ is surjective on objects and morphisms.

Proof. This follows from proposition III.1.1 and theorem III.1.7.1. □

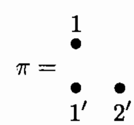
However, by theorem III.1.7.2 we know that $\mathcal{F}$ does not induce an equivalence of tensor categories. To illustrate the amount by which $\mathcal{F}$ fails to induce an equivalence of categories we need the following definition.

Definition IV.3.3. A morphism $g : X \to Y$ in a tensor category is called *negligible* if $\mathrm{tr}(gh) = 0$ for all $h : Y \to X$. Set $\mathcal{N}(X,Y) := \{g : X \to Y \mid f \text{ is negligible}\}$.

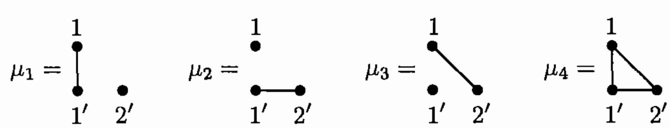
Example IV.3.4. (1) The only morphisms in $\underline{\mathrm{Rep}}_0(S_0; F)$ which are not negligible are nonzero scalar multiples of id_0 (see example III.3.1.1).

(2) Let $\pi : [1] \to [1]$ be as in the proof of theorem IV.1.5. Then $x_\pi : [1] \to [1]$, defined by equation (III.1), is given by $x_\pi = \pi - \mathrm{id}_1$. Since $\mathrm{tr}(\pi) = t = \mathrm{tr}(\mathrm{id}_1)$, we have $\mathrm{tr}(x_\pi) = 0$. Moreover, $x_\pi \pi = (t-1)\pi$ so that $\mathrm{tr}(x_\pi \pi) = t(t-1)$. We conclude that x_π is negligible if and only if $t = 0, 1$.

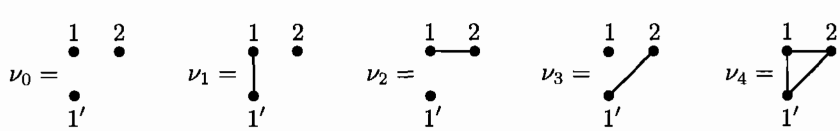
(3) Consider the morphism $\pi \in \mathrm{Hom}_{\underline{\mathrm{Rep}}_0(S_t;F)}([1],[2])$ given by



From equation (III.1) we get $x_\pi = \pi - \mu_1 - \mu_2 - \mu_3 + 2\mu_4$ where



If we set



then one can compute $\mathrm{tr}(x_\pi \nu_0) = t(t-1)(t-2)$, and $\mathrm{tr}(x_\pi \nu_i) = 0$ for $i = 1, 2, 3, 4$. Thus x_π is negligible if and only if $t = 0, 1, 2$. ◇

Remark IV.3.5. Each of the examples IV.3.4 follow from the following fact: In $\underline{\mathrm{Rep}}_0(S_t; F)$

$$\mathcal{N}([n],[m]) = \begin{cases} \mathrm{Span}_F\{x_\pi \mid \pi \in P_{n,m} \text{ has more than } t \text{ parts}\}, & \text{if } t \in \mathbb{Z}_{\geq 0} \\ 0, & \text{otherwise.} \end{cases}$$

For $t \in \mathbb{Z}_{\geq 0}$ this fact follows from theorem III.1.7.2, remark III.2.8, along with the following proposition IV.3.6.2. For $t \notin \mathbb{Z}_{\geq 0}$ we will eventually show that the larger category $\underline{\mathrm{Rep}}(S_t; F)$ is semisimple (see corollary VI.4.5). It is well known that there are no nonzero negligible morphisms in a semisimple category.

Proposition IV.3.6. The following statements hold in any tensor category.

(1) $\mathcal{N}$ is a tensor ideal.

(2) The image under a full tensor functor of a morphism g is negligible if and only if g is negligible.

Proof. Statement (1) follows from proposition II.2.4. Statement (2) follows from the fact that a tensor functor preserves the trace of a morphism. □

Since there are no nonzero negligible morphisms in $\mathrm{Rep}(S_d; F)$, by proposition IV.3.6.2 we conclude the functor $\mathcal{F} : \underline{\mathrm{Rep}}(S_d; F) \to \mathrm{Rep}(S_d; F)$ sends all negligible morphisms to zero. Thus $\mathcal{F}$ induces a functor $\overline{\mathcal{F}} : \underline{\mathrm{Rep}}(S_d; F)/\mathcal{N} \to \mathrm{Rep}(S_d; F)$.

Theorem IV.3.7. $\overline{\mathcal{F}}$ induces an equivalence of categories $\underline{\mathrm{Rep}}(S_d; F)/\mathcal{N} \cong \mathrm{Rep}(S_d; F)$.

Proof. This follows from proposition IV.3.2 and proposition IV.3.6.2. □

We finish this section with the following proposition concerning the functor $\mathcal{F}$. For proof of the proposition we refer the reader to [9, proposition 6.4].

Proposition IV.3.8. Suppose d is a nonnegative integer and $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a Young diagram. If $d - |\lambda| \geq \lambda_1$, then $\mathcal{F}(L(\lambda)) = L_{\lambda(d)}$. If $d - |\lambda| < \lambda_1$, then $\mathcal{F}(L(\lambda)) = 0$.

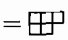
IV.4 Dimensions

In this section we study a hook length formula which gives the dimension of indecomposable objects in $\underline{\text{Rep}}(S_T; K)$. Let us start by defining the hook length formula.

Definition IV.4.1. The *hook length* of a fixed box in a Young diagram λ is the number of boxes in λ which are either directly below or directly to the right of the fixed box, counting the fixed box itself once. Given a Young diagram λ let P_λ denote the unique polynomial such that

$$P_\lambda(d) = \frac{d!}{\prod(\text{hook lengths of } \lambda(d))}$$

for every integer $d \geq 2|\lambda|$.

Example IV.4.2. Let $\lambda =$  and suppose $d \geq 10$ is an integer. In figure 1 each box of $\lambda(d)$ is labeled by its hook length.

$d-3$	$d-4$	$d-6$	$d-8$	$d-9$	$\cdots$	2	1
4	3	1					
2	1						

Figure 1: An Example of Hook Lengths

Thus $P_\lambda(d) = \frac{d!}{24(d-3)(d-4)(d-6)(d-8)!} = \frac{1}{24}d(d-1)(d-2)(d-5)(d-7)$. ◇

First, we show how P_λ is related to the indecomposable object in $\underline{\text{Rep}}(S_T; K)$ corresponding to λ.

Proposition IV.4.3. $\dim_{\underline{\text{Rep}}(S_T;K)} L(\lambda) = P_\lambda(T)$

Proof. By proposition II.4.8 there exists and idempotent $\varepsilon \in KP_n(T)$ with $([n], \varepsilon) \cong L(\lambda)$ in the category $\underline{\text{Rep}}(S_T; K)$. Since $\dim_{\underline{\text{Rep}}(S_T;K)} L(\lambda) = \text{tr}(\varepsilon)$, it follows from corollary IV.1.8 that $\dim_{\underline{\text{Rep}}(S_T;K)} L(\lambda)$ is an element of $F(T)$. Moreover, for all but finitely many integers d, evaluating $T = d$ in $\dim_{\underline{\text{Rep}}(S_T;K)} L(\lambda)$ gives $\dim_{\underline{\text{Rep}}(S_d;F)} L(\lambda)$ (again by corollary IV.1.8). Since $\mathcal{F}$ is a tensor functor, $\dim_{\underline{\text{Rep}}(S_d;F)} L(\lambda) = \dim_{\text{Rep}(S_d;F)} L_{\lambda(d)}$ whenever d is a sufficiently large integer (see proposition IV.3.8). Furthermore, it is well know that $\dim_{\text{Rep}(S_d;F)} L_{\lambda(d)} = P_\lambda(d)$ (see e.g. [12, 4.12]). Thus, the rational function $\dim_{\underline{\mathrm{Rep}}(S_T;K)} L(\lambda)$ agree with the polynomial $P_\lambda(T)$ whenever T is a sufficiently large integer. Hence they must always agree. $\square$

Next, we wish to determine the roots of P_λ. The following combinatorics will be useful towards that endeavor.

Definition IV.4.4. Given a Young diagram λ and an integer $d \geq 2|\lambda|$, create the (λ, d) *grid marking* as follows: Start with a grid of $(|\lambda|+1) \times (d - |\lambda|)$ black boxes. Place the Young diagram $\lambda(d)$ (with white boxes) atop the grid so that the upper left corner of $\lambda(d)$ is atop the upper left corner of the grid. Now place the numbers $0, 1, \ldots, d-1$ into the boxes of the grid using the following rules:

- Begin by placing the number 0 in the lower left box of the grid.
- If the number i is in a black box, place $i+1$ into the box directly above i.
- If the number i is in a white box, place $i+1$ into the box directly to the right of i.

Label the rows of the grid $0, \ldots, |\lambda|$ (from top to bottom) and the columns of the grid $1, \ldots d - |\lambda|$ (from left to right).

Example IV.4.5. Set $\lambda = (4, 3, 1, 1, 0, \ldots)$ and $d = 25$. Figure 2 shows the (λ, d) grid marking.

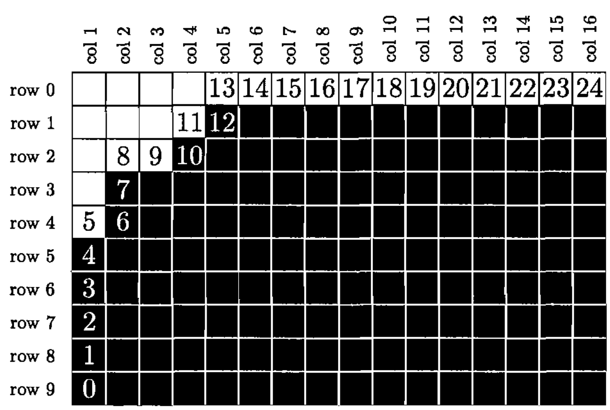


Figure 2: The (λ, d) Grid Marking $\diamond$

The following proposition records the properties of the (λ, d) grid marking which will be useful for determining the roots of P_λ.

Proposition IV.4.6. The (λ, d) grid marking has the following properties:

(1) If $\colorbox{black}{\textcolor{white}{k}}$ appears in the ith row, then $k = |\lambda| + \lambda_i - i$.

(2) If $\boxed{k}$ appears in the ith column, then $d - k$ is the hook length of the row 0, column i box in the Young diagram $\lambda(d)$.

Proof. (1) If $\colorbox{black}{\textcolor{white}{k}}$ appears in the ith row, then $\colorbox{black}{\textcolor{white}{k}}$ must be in column $\mu_i + 1$. Hence k is the number of up/right moves it takes to get from the lower left corner to the upper right corner in a $(|\lambda| - i) \times (\lambda_i + 1)$ grid. The result follows.

(2) Let c_i denote the number of boxes in the ith column of λ. Notice there are c_i boxes below, and $d - |\lambda| - i$ boxes to the right of the row 0, column i box in $\lambda(d)$. Hence the hook length of that box is $d - |\lambda| - i + c_i + 1$. On the other hand, if $\boxed{k}$ appears in the ith column, then $\boxed{k}$ must be in row c_i. Hence k is the number of up/right moves it takes to get from the lower left corner to the upper right corner in a $(|\lambda| - c_i) \times i$ grid. Thus $k = |\lambda| - c_i + i - 1$. $\square$

Using proposition IV.4.6, we can determine all the roots of the polynomial P_λ.

Proposition IV.4.7. P_λ is a degree $|\lambda|$ polynomial with $|\lambda|$ distinct, integer roots given by $|\lambda| + \lambda_i - i$ for each $i = 1, \ldots, |\lambda|$.

Proof. Suppose d is an integer with $d > 2|\lambda|$. It follows from definition IV.4.1 that the roots of P_λ are exactly the integers $0 \leq k < d$ such that $d - k$ is not a hook length of a box in the top row of $\lambda(d)$. By proposition IV.4.6.2, those are exactly the values of k for which $\colorbox{black}{\textcolor{white}{k}}$ appears in the (λ, d) grid marking. The result now follows from proposition IV.4.6.1. $\square$

CHAPTER V

ENDOMORPHISMS OF THE IDENTITY FUNCTOR

In this chapter we study endomorphisms of the identity functor on $\underline{\mathrm{Rep}}_0(S_t; F)$ constructed using certain central elements in group algebras of symmetric groups. These endomorphisms of the identity functor will play a key role in describing the blocks in $\underline{\mathrm{Rep}}(S_t; F)$. This role is analogous to the role the Casimir element plays in Lie theory.

V.1 Interpolating Sums of r-cycles

In this section we define morphisms in $\underline{\mathrm{Rep}}_0(S_t; F)$ which "interpolate" the action of the sum of all r-cycles on representations of symmetric groups. To begin, let r and d be positive integers with $r \leq d$.

Definition V.1.1. Let $\Omega_{r,d} \in FS_d$ denote the sum of all r-cycles in S_d.

Since $\Omega_{r,d}$ is in the center of FS_d, the action of $\Omega_{r,d}$ on $V_d^{\otimes n}$ gives and element of $\mathrm{End}_{S_d}(V_d^{\otimes n})$ for each integer $n \geq 0$. This, along with theorem III.1.7, shows the following definition is valid.

Definition V.1.2. For nonnegative integers r, n, and d with $r \leq d$ and $2n \leq d$, let $C_n^r(d)$ denote the unique element of $FP_n(d)$ with $f(C_n^r(d)) \in \mathrm{End}_{S_d}(V_d^{\otimes n})$ given by the action of $\Omega_{r,d}$.

The first goal of this section is to define elements of $FP_n(t)$ for arbitrary $t \in F$ which agree with $C_n^r(t)$ when t is a sufficiently large integer. We are able to do this because, as we will show, $C_n^r(d)$ depends polynomially on d. The fact that $C_n^r(d)$ depends polynomially on d boils down to the following combinatorial proposition.

Proposition V.1.3. Suppose n is a nonnegative integers and $\pi \in P_{n,n}$. Fix the following notation.

- Let a denote the number of parts of π.
- Let b denote the number of parts π_k of π such that $j, j' \in \pi_k$ for some $1 \leq j \leq n$.
- Let c denote the number of connected components in the trace diagram of π (see section III.3).
- Suppose r and d are positive integers with $d \geq r$, and $\boldsymbol{i}, \boldsymbol{i}' \in [n, d]$ are such that the $(\boldsymbol{i}, \boldsymbol{i}')$-coloring of π is perfect. Let $S(\pi, r, d)$ denote the number of r-cycles, $\sigma \in S_d$, such that $\sigma(i_j) = i'_j$ for all $1 \leq j \leq n$.

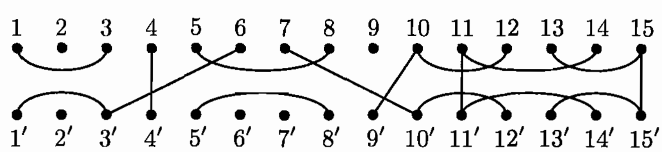
Fix a positive integer r. If $S(\pi, r, d)$ is nonzero for some integer $d \geq r$, then $S(\pi, r, d)$ is nonzero for all $d \geq r + b$. Moreover, if $S(\pi, r, d)$ is nonzero, then

$$S(\pi, r, d) = \frac{(r - a + c - 1)!}{(r - a + b)!} \prod_{k=1}^{r+b-a} (d - r - b + k).^{1} \tag{V.1}$$

The proof of proposition V.1.3 is a simple counting argument which we leave to the reader. However, a proof can be obtained by simply generalizing the following example.

Example V.1.4. Let $n = 15$ and

$\pi =$



Then $a = 14$, $b = 3$, and $c = 7$. For $d \geq 14$, let $\boldsymbol{i}, \boldsymbol{i}' \in [n, d]$ be the functions which give the perfect $(\boldsymbol{i}, \boldsymbol{i}')$-coloring of π shown in figure 1.

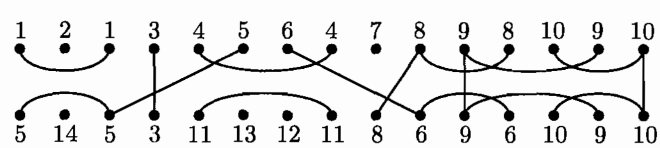


Figure 1: A Perfect $(\boldsymbol{i}, \boldsymbol{i}')$-coloring of π

[1] If $r = a - b$ then the empty product $\prod_{k=1}^{r+b-a} (d - r - b + k)$ is equal to 1, and (V.1) gives $S(\pi, r, d) = (c - b - 1)!$. If $r < a - b$ or $r \leq a - c$ then $S(\pi, r, d)$ is zero for all d, so (V.1) does not apply.

For an r-cycle $\sigma \in S_d$ to satisfy $\sigma(i_j) = i'_j$ for $j = 1, \ldots, n$, σ must fix the $3 = b$ numbers $3, 9, 10$, and map

$$1 \to 5 \to 13, \qquad 2 \to 14, \qquad 4 \to 11, \qquad 7 \to 8 \to 6 \to 12. \tag{V.2}$$

Clearly, such an r-cycle exists if and only if $r \geq 11 = a - b$ and $d \geq r + 3 = r + b$. In that case, the number of such r-cycles can be counted as follows. There are $(r-8)! = (r-a+c-1)!$ ways to arrange the $4 = c - b$ "chains" listed in (V.2) along with the $r - 11 = r - a + b$ other entries within the cycle. Moreover, there are $d - 14 = d - a$ choices for the remaining $r - 11 = r - a + b$ entries within the r-cycle after the "chains" in (V.2) have been taken into account. Hence the number of desired r-cycles is given by $(r-8)!\binom{d-14}{r-11} = (r-a+c-1)!\binom{d-a}{r-a+b}$ which agrees with (V.1). $\diamond$

With proposition V.1.3 in mind, we are now ready for the following definition.

Definition V.1.5. For $t \in F$ and integers $r > 0$, and $n \geq 0$, define $\omega_n^r(t) \in FP_n(t)$ as follows. Using the basis $\{x_\pi\}$ for $FP_n(t)$ defined in (III.1), set

$$\omega_n^r(t) = \sum_{\pi \in P_{n,n}} q_{\pi,r,t} x_\pi$$

where, using the notation set up in proposition V.1.3,

$$q_{\pi,r,t} = \begin{cases} 0, & \text{if } S(\pi, r, d) = 0 \text{ for all integers } d > 1, \\ \frac{(r-a+c-1)!}{(r-a+b)!} \prod_{k=1}^{r+b-a} (t - r - b + k), & \text{otherwise.} \end{cases}$$

Although the definition of $\omega_n^r(t)$ may seem a bit complicated, for the rest of this paper we will only be concerned with the following (less complicated) properties of $\omega_n^r(t)$.

Proposition V.1.6. (1) Fix integers $r > 0$ and $n \geq 0$. Whenever d is a sufficiently large[2] integer, $\omega_n^r(d) = C_n^r(d)$. In other words, when d is a sufficiently large integer, the map $f(\omega_n^r(d)) : V_d^{\otimes n} \to V_d^{\otimes n}$ is given by the action of $\Omega_{r,d} \in S_d$.

(2) Fix $t \in F$ and an integer $r > 0$. The morphisms $\omega_n^r(t) : [n] \to [n]$ for each nonnegative integer n form an endomorphism of the identity functor on $\underline{\mathrm{Rep}}_0(S_t; F)$. In particular, $\omega_n^r(t)$ is in the center of $FP_n(t)$ for every $t \in F$ and integer $n \geq 0$.

[2]The statement is certainly true for $d \geq 2n + r$, although this bound is not sharp.

Proof. For $\boldsymbol{i}, \boldsymbol{i}' \in [n,d]$, let $\pi(\boldsymbol{i},\boldsymbol{i}') \in P_{n,n}$ denote the unique partition which has a perfect $(\boldsymbol{i},\boldsymbol{i}')$-coloring. Then the action of $\Omega_{r,d}$ on $V_d^{\otimes n}$ maps the basis vector $v_{\boldsymbol{i}} \mapsto \sum_{\boldsymbol{i}' \in [n,d]} S(\pi(\boldsymbol{i},\boldsymbol{i}'),r,d)v_{\boldsymbol{i}'}$. On the other hand, (III.2) shows that $f(\omega_n^r(t))$ maps $v_{\boldsymbol{i}} \mapsto \sum_{\boldsymbol{i}' \in [n,d]} q_{\pi(\boldsymbol{i},\boldsymbol{i}'),r,d} v_{\boldsymbol{i}'}$. By proposition V.1.3, $S(\pi(\boldsymbol{i},\boldsymbol{i}'),r,d) = q_{\pi(\boldsymbol{i},\boldsymbol{i}'),r,d}$ for sufficiently large d. This proves part (1).

To prove part (2), choose $\mu \in P_{n,m}$. For an integer $d > r$, we know that $f(\mu) : V_d^{\otimes n} \to V_d^{\otimes m}$ commutes with the action of $\Omega_{r,d} \in S_d$. Hence, by part (1), $f(\omega_m^r(d)\mu) = f(\mu\omega_n^r(d))$ when d is a sufficiently large integer. Thus, by part (2) of theorem III.1.7, $\omega_m^r(d)\mu = \mu\omega_n^r(d)$ when d is a sufficiently large integer. If we set $\omega_m^r(t)\mu =: \sum_{\pi \in P_{n,m}} a_\pi(t)\pi$ and $\mu\omega_n^r(t) =: \sum_{\pi\in P_{n,m}} a'_\pi(t)\pi$ for each $t \in F$, then we have shown the polynomials $a_\pi(t)$ and $a'_\pi(t)$ are equal when t is a sufficiently large integer. Hence they are always equal. □

V.2 Frobenius' Formula

This section will be devoted to studying how $\omega_n^r(t)$ interacts with indecomposable objects in $\underline{\mathrm{Rep}}(S_t; F)$. We start with the following proposition.

Proposition V.2.1. Fix $t \in F$ along with integers $r > 0$ and $n \geq 0$. If e is a primitive idempotent in $FP_n(t)$, then there exists $\xi \in F$ and a positive integer m such that $(\omega_n^r(t) - \xi)^m e = 0$.

Proof. Let $\bar{F}$ denote the algebraic closure of F and write $\omega := \omega_n^r(t)$. Let $a(x)$ (resp. $a'(x)$) denote the monic polynomial of minimal degree in $F[x]$ (resp. $\bar{F}[x]$) with $a(\omega)e$ (resp. $a'(\omega)e$) equal to zero.[3] First we will show that $a(x)$ is a power of an irreducible polynomial in $F[x]$. To do so, suppose $b(x)$ and $c(x)$ are relatively prime monic polynomials in $F[x]$ with $a(x) = b(x)c(x)$. Then there exist polynomials $g(x), h(x) \in F[x]$ with $\deg g(x) < \deg c(x)$, $\deg h(x) < \deg b(x)$, and $g(x)b(x) + h(x)c(x) = 1$. Hence $g(\omega)b(\omega)e + h(\omega)c(\omega)e = e$ is a decomposition of e into orthogonal idempotents (here we are using the fact that ω is in the center of $FP_n(t)$, see proposition V.1.6.2). Since e is primitive, this implies $g(\omega)b(\omega)e = 0$ or $h(\omega)c(\omega)e = 0$. The minimality of $a(x)$ implies that either $g(x) = 0$ or $h(x) = 0$, which implies $c(x) = 1$ or $b(x) = 1$. Thus $a(x)$ is a power of an irreducible polynomial in $F[x]$. Since e is primitive in $\bar{F}P_n(t)$ (see proposition IV.1.7), the same line of reasoning shows $a'(x)$ is a power of an irreducible polynomial in $\bar{F}[x]$. Hence $a'(x) = (x - \xi)^m$ for some positive integer m and $\xi \in \bar{F}$. Since $a(x)$ is a power of an irreducible polynomial in $F[x]$ and $(x-\xi)^m$ divides $a(x)$ in $\bar{F}[x]$, we conclude $\xi \in F$. □

[3]The polynomial $a(x)$ (resp. $a'(x)$) exists since $FP_n(t)$ (resp. $\bar{F}P_n(t)$) is finite dimensional over F (resp. $\bar{F}$).

Next, we use a classical result of Frobenius to produce a formula for the scalar ξ in proposition V.2.2. The study of this formula will be the key to describing the blocks of $\underline{\mathrm{Rep}}(S_t, F)$ in section VI.4. First we state Frobenius' result on the symmetric group:

Theorem V.2.2 (Frobenius' formula[4]). Fix positive integers $d \geq r$. Given a Young diagram $\lambda = (\lambda_0, \lambda_1, \ldots)$ of size d, set $\mu_i = \lambda_i - i$ for each $i \geq 0$. Then $\Omega_{r,d}$, (definition V.1.1), acts on the simple S_d-module corresponding to λ by the scalar

$$\xi^{\lambda}_{r,k} := \frac{1}{r}\sum_{i=0}^{k} (\mu_i + k - 1)(\mu_i + k - 2)\cdots(\mu_i + k - r) \prod_{\substack{0\leq j\leq k\\ j\neq i}} \frac{\mu_i - \mu_j - r}{\mu_i - \mu_j} \tag{V.3}$$

where k is any positive integer such that $\lambda_{k+1} = 0$.

The result of theorem V.2.2 was first appeared in [11]. A modern proof of theorem V.2.2 is outlined in [12, exercise 4.17], (see also [18, example 7 in section I.7]).

To close this section we show that ξ in proposition V.2.2 is given by Frobenius' formula.

Proposition V.2.3. Fix $t \in F$ and a positive integer r. Suppose $([n], e)$ is an indecomposable object in $\underline{\mathrm{Rep}}(S_t, F)$ corresponding to the Young diagram λ. If k is a positive integer such that $\lambda_{k+1} = 0$, then $(\omega^r_n(t) - \xi^{\lambda(t)}_{r,k})^m e = 0$ for some positive integer m.

Proof. By theorem IV.1.3.1 we may assume $n = |\lambda|$. Let ξ and m be as in proposition V.2.2, so that $(\omega^r_n(t) - \xi)^m e = 0$. Applying the quotient map $\psi : FP_n(t) \twoheadrightarrow FS_n$ (lemma IV.1.1.2) to this equation yields $(\psi(\omega^r_n(t)) - \xi)^m e_\lambda = 0$ in FS_n where e_λ is a primitive idempotent in FS_n corresponding to λ. Since $\psi(\omega^r_n(t))$ is central in FS_n, this implies $\psi(\omega^r_n(t))e_\lambda = \xi e_\lambda$. Hence, by definition V.1.5, ξ depends polynomially on t.

Now, assume d is a positive integer such that $d \geq \lambda_1 + |\lambda|$. By proposition IV.3.8, applying the functor $\mathcal{F}$ to the equation $(\omega^r_n(d) - \xi)^m e = 0$ yields $(\Omega_{r,d} - \xi)^m e_{\lambda(d)} = 0$. Thus, by theorem V.2.2, $\xi = \xi^{\lambda(t)}_{r,k}$ whenever $t = d$ is a sufficiently large integer. Since ξ depends polynomially on t, $\xi^{\lambda(t)}_{r,k}$ is a rational function in t, and $\xi = \xi^{\lambda(t)}_{r,k}$ for infinitely many values of t, we conclude that $\xi = \xi^{\lambda(t)}_{r,k}$ for all $t \in F$. □

[4] We take formula (V.3) to be the definition of $\xi^{\lambda}_{r,k}$ even if $\lambda = (\lambda_0, \lambda_1, \ldots)$ is not a Young diagram.

CHAPTER VI

BLOCKS OF INDECOMPOSABLE OBJECTS

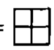
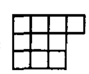
Consider the following equivalence class on the set of Young diagrams of arbitrary size.

Definition VI.0.4. For $t \in F$ and a Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots)$, set

$$\mu_\lambda(t) := (t - |\lambda|, \lambda_1 - 1, \lambda_2 - 2, \ldots).$$

For Young diagrams λ and λ' write $\mu_\lambda(t) = (\mu_0, \mu_1, \ldots)$ and $\mu_{\lambda'}(t) = (\mu'_0, \mu'_1, \ldots)$. We write $\lambda \overset{t}{\sim} \lambda'$ whenever there exists a bijection $\tau : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ with $\mu_i = \mu'_{\tau(i)}$ for all $i \geq 0$.

Example VI.0.5. Let

$\lambda =$  $\qquad \lambda' =$ 

Then $\mu_\lambda(7) = (3, 1, 0, -3, -4, -5, \ldots)$ and $\mu_{\lambda'}(7) = (-3, 3, 1, 0, -4, -5, \ldots)$. Hence $\lambda \overset{7}{\sim} \lambda'$. $\diamond$

Clearly, for each $t \in F$, $\overset{t}{\sim}$ defines an equivalence relation on the set of all Young diagrams. The main goal of chapter VI is to prove the following theorem.

Theorem VI.0.6. $L(\lambda)$ and $L(\lambda')$ are in the same block of $\underline{\mathrm{Rep}}(S_t; F)$ if and only if $\lambda \overset{t}{\sim} \lambda'$.

VI.1 What Does Frobenius' Formula Tell Us About Blocks?

In this section we use Frobenius' formula (V.3) to show $\overset{t}{\sim}$-equivalence classes correspond to unions of blocks in $\underline{\mathrm{Rep}}(S_t; F)$.

Lemma VI.1.1. Suppose λ and λ' are Young diagrams and $k \in \mathbb{Z}_{\geq 0}$ with $\lambda_{k+1} = \lambda'_{k+1} = 0$.

(1) If $L(\lambda)$ and $L(\lambda')$ are in the same block in $\underline{\mathrm{Rep}}(S_t; F)$, then $\xi_{r,k}^{\lambda(t)} = \xi_{r,k}^{\lambda'(t)}$ for every $r > 0$.

(2) If $\xi_{r,k}^{\lambda(t)} = \xi_{r,k}^{\lambda'(t)}$ for every $r > 0$, then $\lambda \overset{t}{\sim} \lambda'$.

Proof. To prove part (1), let us first fix some notation. Let n, n' be nonnegative integers and $e \in FP_n(t), e' \in FP_{n'}(t)$ be idempotents with $L(\lambda) \cong ([n], e)$ and $L(\lambda') \cong ([n'], e')$. Fix r and write $\xi := \xi_{r,k}^{\lambda(t)}$, $\xi' := \xi_{r,k}^{\lambda'(t)}$, $\omega := \omega_n^r(t)$, $\omega' := \omega_{n'}^r(t)$. Finally, let m be a positive integer with $(\omega - \xi)^m e = 0$ and $(\omega' - \xi')^m e' = 0$ (such an m exists by proposition V.2.3). Now, suppose $\xi \neq \xi'$. Then there are polynomials $p(x), q(x) \in F[x]$ with $p(x)(x-\xi)^m + q(x)(x-\xi')^m = 1$. Hence, given any morphism $\phi : ([n'], e') \to ([n], e)$ in $\underline{\mathrm{Rep}}(S_t; F)$, we have

$$\phi = p(\omega)(\omega - \xi)^m \phi + q(\omega)(\omega - \xi')^m \phi = p(\omega)(\omega - \xi)^m e\phi + q(\omega)(\omega - \xi')^m \phi e'.$$

By proposition V.1.6.2, the right hand side of the equation above is equal to

$$p(\omega)(\omega - \xi)^m e\phi + \phi q(\omega')(\omega' - \xi')^m e' = 0.$$

Thus, if there exists a nonzero morphism $([n'], e') \to ([n], e)$ in $\underline{\mathrm{Rep}}(S_t; F)$, then $\xi = \xi'$.

To prove part (2), notice $\xi_{r,k}^{\lambda(t)}$ is symmetric in $\mu_0, \ldots, \mu_k$. Thus $\xi_{r,k}^{\lambda(t)} \prod_{0 \le i < j \le k} (\mu_i - \mu_j)$ is an antisymmetric polynomial in $\mu_0, \ldots, \mu_k$. However, every antisymmetric polynomial in $\mu_0, \ldots, \mu_k$ is divisible by $\prod_{0 \le i < j \le k} (\mu_i - \mu_j)$. Thus $\xi_{r,k}^{\lambda(t)}$ is a symmetric polynomial in $\mu_0, \ldots, \mu_k$. Moreover, from equation (V.3) it is apparent that as a polynomial in $\mu_0, \ldots, \mu_k$,

$$\xi_{r,k}^{\lambda(t)} = \frac{1}{r} \sum_{i=0}^{k} \mu_i^r + (\text{terms of total degree less than } r).$$

Thus, if $\xi_{r,k}^{\lambda(t)} = \xi_{r,k}^{\lambda'(t)}$ for every $r > 0$ then $\sum_{i=0}^k \mu_i^r = \sum_{i=0}^k (\mu_i')^r$ for every $r > 0$. This is only possible if the list $\mu_0, \ldots, \mu_k$ is a permutation of $\mu_0', \ldots, \mu_k'$. □

VI.2 On the Equivalence Relation $\overset{t}{\sim}$

In this section we prove some elementary properties of the equivalence relation $\overset{t}{\sim}$ and give examples. We say a $\overset{t}{\sim}$-equivalence class is *trivial* if it contains only one Young diagram. Soon we will see that the $\overset{t}{\sim}$-equivalence classes are all trivial unless t is a nonnegative integer (see corollary VI.2.2.1). First, we prove the following easy proposition.

Proposition VI.2.1. Suppose λ is a Young diagram and write $\lambda_\mu(t) = (\mu_0, \mu_1, \ldots)$. Suppose further that $\tau : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is a bijection and set $\mu' = (\mu'_0, \mu'_1, \ldots)$ where $\mu'_i = \mu_{\tau^{-1}(i)}$. There exists a Young diagram λ' such that $\mu' = \mu_{\lambda'}(t)$ if and only if $\mu'_i \in \mathbb{Z}$ with $\mu'_i > \mu'_{i+1}$ for all $i > 0$.

Proof. Suppose λ' satisfies $\mu' = \mu_{\lambda'}(t)$. Then $\mu'_i = \lambda'_i - i > \lambda'_{i+1} - i - 1 = \mu'_{i+1}$ for all $i > 0$.

On the other hand, suppose $\mu'_i > \mu'_{i+1}$ for all $i > 0$. Set $\lambda'_i = \mu_i + i$ for all $i \geq 0$ and $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$. Since $\mu_i > \mu_{i+1}$ and $\mu'_i > \mu'_{i+1}$ for all $i > 0$, $\tau(i)$ must equal i for all $i > \max\{\tau(0), \tau^{-1}(0)\}$. Thus $\lambda'_i = \lambda_i$ for all $i > \max\{\tau(0), \tau^{-1}(0)\}$. This shows $\lambda'_i = 0$ for all but finitely many values of i. Moreover, $\mu_i > \mu_{i+1}$ for $i > 0$ implies $\lambda'_i \geq \lambda'_{i+1}$ for all $i > 0$. Thus λ' is indeed a Young diagram. Finally, choose $k > \max\{\tau(0), \tau^{-1}(0)\}$ with $\lambda_k = 0$. Then $\lambda'_k = 0$ as well. Furthermore, $t = \sum_{i=0}^k \lambda_i = \sum_{i=0}^k \mu_i + \frac{k(k+1)}{2} = \sum_{i=0}^k \mu'_i + \frac{k(k+1)}{2} = \sum_{i=0}^k \lambda'_i$, which implies $\lambda'_0 = t - |\lambda'|$. □

Corollary VI.2.2. (1) The $\overset{t}{\sim}$-equivalence classes are all trivial unless $t \in \mathbb{Z}_{\geq 0}$.

(2) Suppose $d \in \mathbb{Z}_{\geq 0}$ and λ is a Young diagram. The $\overset{d}{\sim}$-equivalence class containing λ is nontrivial if and only if the coordinates of $\mu_\lambda(d)$ are pairwise distinct.

Proof. Suppose λ is a Young diagram. It follows from proposition VI.2.1 that the $\overset{t}{\sim}$-equivalence class containing λ is nontrivial if and only if the coordinates of $\mu_\lambda(t)$ are all integers which are pairwise distinct. This proves part (2). Also, this implies the $\overset{t}{\sim}$-equivalence classes are all trivial unless $t \in \mathbb{Z}$. Finally, if t is a negative integer, then $\lambda_{|\lambda|-t} = 0$ which implies $\mu_{|\lambda|-t} = \mu_0$. This proves part (1). □

Example VI.2.3. (1) Let $\lambda = (2, 1, 0, \ldots)$. Then $\mu_\lambda(t) = (t-3, 1, -1, -3, -4, \ldots)$. Thus, by corollary VI.2.2, λ is in a nontrivial $\overset{t}{\sim}$-equivalence class if and only if $t \in \mathbb{Z}_{\geq 0}$ and $t \neq 0, 2, 4$.

(2) If we let $\varnothing$ denote the Young diagram $(0, \ldots)$, then $\mu_\varnothing(t) = (t, -1, -2, -3, \ldots)$. Thus, by corollary VI.2.2, λ is in a nontrivial $\overset{t}{\sim}$-equivalence class if and only if $t \in \mathbb{Z}_{\geq 0}$. ◇

The following proposition gives a complete description of nontrivial $\overset{t}{\sim}$-equivalence classes.

Proposition VI.2.4. Suppose d is a nonnegative integer. Each nontrivial $\overset{d}{\sim}$-equivalence class is infinite. Moreover, a Young diagram λ is the minimal element in a nontrivial $\overset{d}{\sim}$-equivalence class if and only if $\lambda(d)$ is a Young diagram of size d. In particular, the number of nontrivial $\overset{d}{\sim}$-equivalence classes is equal to the number of Young diagrams of size d. Finally, suppose $\{\lambda^{(0)} \prec \lambda^{(1)} \prec \cdots\}$

is a nontrivial $\overset{d}{\sim}$-equivalence class with $\lambda^{(0)}(d) = (l_1^{m_1}, \ldots, l_r^{m_r})$. If we set $m := \sum_{j=1}^{r} m_j$, then $\lambda^{(i)} = ((l_1+1)^{m_1}, \ldots, (l_r+1)^{m_r}, 1^{i-m})$ for all $i \geq m$.

Proof. Suppose λ is a Young diagram in a nontrivial $\overset{d}{\sim}$-equivalence class. Label the coordinates of $\mu_\lambda(d)$ by $\mu_0, \mu_1, \ldots$ so that $\mu_i > \mu_{i+1}$ for all $i \geq 0$ (such a labeling is possible by corollary VI.2.2.2). For each nonnegative integer i, set $\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots)$ where

$$\lambda_j^{(i)} := \begin{cases} \mu_{j-1} + j & \text{if } j \leq i, \\ \mu_j + j & \text{if } j > i. \end{cases} \tag{VI.1}$$

for each $j > 0$. Then $\lambda^{(i)}$ is a Young diagram with $\mu_{\lambda^{(i)}}(d) = (\mu_0^{(i)}, \mu_1^{(i)}, \ldots)$ where

$$\mu_j^{(i)} = \begin{cases} \mu_i & \text{if } j = 0, \\ \mu_{j-1} & \text{if } 0 < j \leq i, \\ \mu_j & \text{if } j > i. \end{cases}$$

Hence $\lambda^{(i)} \overset{d}{\sim} \lambda$ for all $i \geq 0$. Moreover, it follows from proposition VI.2.1 that $\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \ldots$ is a complete list of Young diagrams which are $\overset{d}{\sim}$-equivalent to λ. Furthermore,

$$\begin{aligned} |\lambda^{(i)}| &= \textstyle\sum_{j>1} \lambda_j^{(i)} \\ &= \textstyle\sum_{j=1}^{i} (\mu_{j-1} + j) + \mu_{i+1} + i + 1 + \sum_{j>i+1} (\mu_j + j) \\ &< \textstyle\sum_{j=1}^{i} (\mu_{j-1} + j) + \mu_i + i + 1 + \sum_{j>i+1} (\mu_j + j) \\ &= \textstyle\sum_{j>1} \lambda_j^{(i+1)} = |\lambda^{(i+1)}|. \end{aligned}$$

Next, given a Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots)$, $d - |\lambda| \geq \lambda_1$ if and only if $\mu_i > \mu_{i+1}$ for all $i \geq 0$ where $\mu_\lambda(d) = (\mu_0, \mu_1, \ldots)$, which occurs if and only if λ is the minimal element of a nontrivial $\overset{d}{\sim}$-equivalence class.

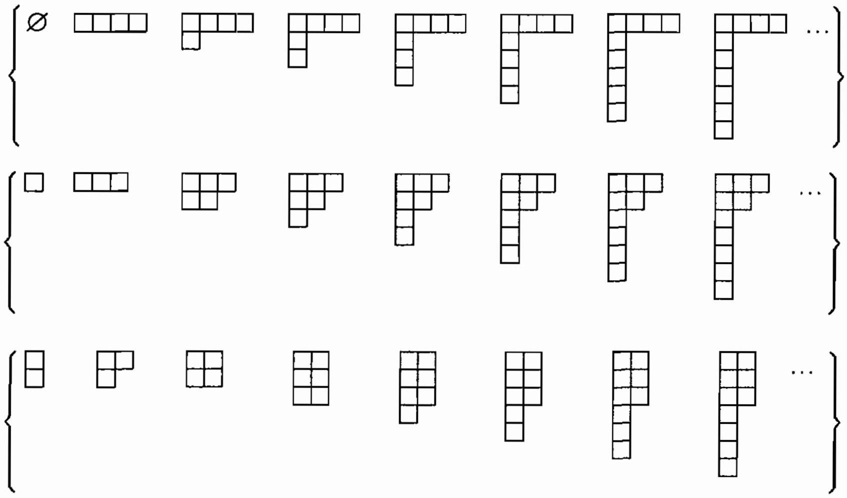
Finally, If $\lambda^{(0)}(d) = (l_1^{m_1}, \ldots, l_r^{m_r})$ and $i \geq m = \sum_{j=1}^{r} m_j$, then by (VI.1)

$$\lambda_j^{(i)} := \begin{cases} d - |\lambda^{(0)}| + 1 & \text{if } j = 1, \\ \lambda_{j-1}^{(0)} + 1 & \text{if } j \leq i, \\ 0 & \text{if } j > i. \end{cases}$$

Thus $\lambda^{(i)} = ((l_1+1)^{m_1}, \ldots, (l_r+1)^{m_r}, 1^{i-m})$ whenever $i \geq m$. □

Example VI.2.5. (1) The only nontrivial $\stackrel{0}{\sim}$-equivalence class is $\{\varnothing, (1), (1^2), (1^3), \ldots\}$.

(2) Below are the three nontrivial $\stackrel{3}{\sim}$-equivalence classes of Young diagrams.



$\diamond$

Next we show how the polynomials P_λ defined in section IV.4 can be used to determine when a Young diagram is in a trivial $\stackrel{d}{\sim}$-equivalence class.

Proposition VI.2.6. Suppose λ is a Young diagram and d is a nonnegative integer. λ is in a trivial $\stackrel{d}{\sim}$-equivalence class if and only if $P_\lambda(d) = 0$.

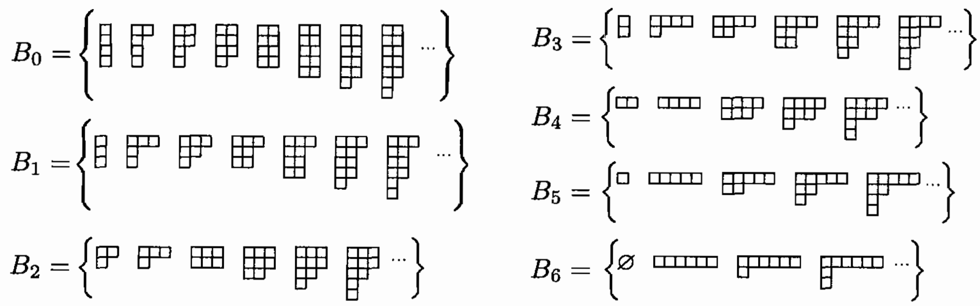
Proof. By corollary VI.2.2, λ is in a trivial $\stackrel{d}{\sim}$-equivalence class if and only if the coordinates of $\mu_\lambda(d)$ are not distinct, which occurs if and only if $d - |\lambda| = \lambda_i - i$ for some $i > 0$. However, $d - |\lambda| > \lambda_i - i$ when $i > |\lambda|$, so λ is in a trivial $\stackrel{d}{\sim}$-equivalence class if and only if $d = |\lambda| + \lambda_i - i$ for some $0 < i \leq |\lambda|$. The result now follows from proposition IV.4.7. $\square$

We conclude this section by defining a total order on the nontrivial $\stackrel{d}{\sim}$-equivalence classes. This ordering will be useful in section VI.4.

Definition VI.2.7. If B and B' are nontrivial $\stackrel{d}{\sim}$-equivalence classes with minimal diagrams λ and λ' respectively, we write $B \prec B'$ if $\lambda(d) \prec \lambda'(d)$ (see definition II.6.4).

Example VI.2.8. (1) The nontrivial $\stackrel{3}{\sim}$-equivalence classes in example VI.2.5.2 are listed in decreasing order.

(2) Below are the seven nontrivial $\overset{5}{\sim}$-equivalence classes with $B_0 \prec \cdots \prec B_6$.



$\diamond$

VI.3 The Functor $- \otimes L(\square)$

In this section we explain how to decompose the tensor product $L(\lambda) \otimes L(\square)$ where λ is an arbitrary Young diagram. The following lemma will be our main tool in our study of decomposing tensor products.

Lemma VI.3.1. Fix Young diagrams λ, μ. If $L(\lambda) \otimes L(\mu) = \bigoplus_\nu L(\nu)^{\oplus a_\nu}$ in the category $\underline{\mathrm{Rep}}(S_T, K)$, then $L_{\lambda(d)} \otimes L_{\mu(d)} = \bigoplus_\nu L_{\nu(d)}^{\oplus a_\nu}$ in the category $\mathrm{Rep}(S_d; F)$ whenever d is a sufficiently large integer[1].

Proof. Suppose ε and ε' are primitive idempotents in $KP_n(T)$ corresponding to λ and μ respectively. Then a_ν is the number of primitive idempotents corresponding to ν in an orthogonal decomposition of $\varepsilon \otimes \varepsilon'$ into primitive idempotents. By corollary IV.1.8, for all but finitely many integers d, evaluating $T = d$ in a decomposition of $\varepsilon \otimes \varepsilon'$ gives a decomposition of $e \otimes e' \in FP_{2n}(d)$ where e and e' are primitive idempotents in $FP_n(d)$ corresponding to λ and μ respectively. Hence, $L(\lambda) \otimes L(\mu) = \bigoplus_{\nu \in \Psi} L(\nu)^{\oplus a_\nu}$ in the category $\underline{\mathrm{Rep}}(S_d, F)$ whenever d is a sufficiently large integer. As long as d is chosen large enough so that $\lambda(d), \mu(d)$, as well as all $\nu(d)$ (when $a_\nu \neq 0$) are Young diagrams, applying the tensor functor $\mathcal{F}$ gives the desired result (see proposition IV.3.8). $\square$

[1]In [17], Littlewood showed for sufficiently large d, the decomposition of $L_{\lambda(d)} \otimes L_{\mu(d)}$ in $\mathrm{Rep}(S_d; F)$ depends only on λ and μ, not on d. Hence, even though $\nu(d)$ is only a Young diagram when $d \geq \nu_1 + |\nu|$, we can choose d large enough so that $a_\nu = 0$ whenever $d < \nu_1 + |\nu|$. Thus, if we set $L_{\nu(d)}^{\oplus 0} = 0$, the formula $L_{\lambda(d)} \otimes L_{\mu(d)} = \bigoplus_{\nu \in \Psi} L_{\nu(d)}^{\oplus a_\nu}$ makes sense when d is sufficiently large.

From lemma VI.3.1, along with the well known algorithm for decomposing $L_\lambda \otimes L_{(d-1,1,0,\ldots)}$ in $\mathrm{Rep}(S_d; K)$ (see e.g. [17]), we have the following algorithm for decomposing $L(\lambda) \otimes L(\square)$ in the category $\underline{\mathrm{Rep}}(S_T; K)$.

Proposition VI.3.2. $L(\lambda) \otimes L(\square) = \bigoplus_\nu L(\nu)^{\oplus a_\nu}$ where a_ν is the number of times the Young diagram ν is obtained from step 2 in the following algorithm.

Step 1: Delete zero or one box from λ wherever doing so results in a Young diagram.

Step 2: Add one box to λ wherever doing so results in a Young diagram. To every Young diagram not equal to λ obtained from step 1, add zero or one box wherever doing so results in a Young diagram.

Example VI.3.3. In this example we will apply proposition VI.3.2 to $L(\lambda) \otimes L(\square)$ in $\underline{\mathrm{Rep}}(S_T; K)$ where $\lambda = (2, 1, 0, \ldots)$. Figure 1 illustrates the algorithm in proposition VI.3.2 as follows: At the top of the figure is λ; the middle level lists all Young diagrams obtained in step 1 of the algorithm; the bottom level lists all Young diagrams obtained from step 2; the arrows indicate adding or deleting zero or one box as prescribed by the algorithm.

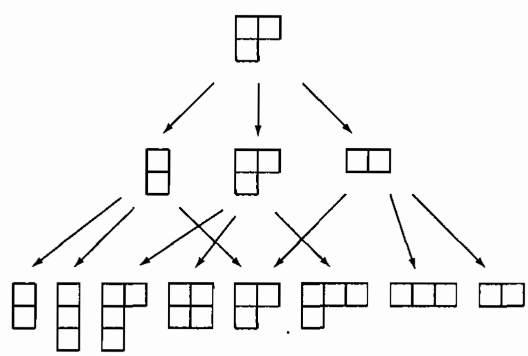


Figure 1: Decomposing $L(2, 1, 0, \ldots) \otimes L(\square)$.

Hence, the multiplicity of $L(\nu)$ in the decomposition of $L(\lambda) \otimes L(\square)$ is the number of paths from the top λ to ν (in the bottom level) in figure 1. $\diamond$

We conclude this section with a technical lemma concerning tensoring with $L(\square)$ which we will need in section VI.4.

Lemma VI.3.4. Fix a nonnegative integer d and suppose $B = \{\lambda^{(0)} \prec \lambda^{(1)} \prec \cdots\}$ is a nontrivial $\overset{d}{\sim}$-equivalence class.

(1) If B is minimal with respect to $\prec$, then for each integer $i > 0$ there exists a Young diagram ρ, in a trivial $\overset{d}{\sim}$-equivalence class, with $L(\rho) \otimes L(\Box) = \bigoplus_\nu L(\nu)^{\oplus a_\nu}$ in $\underline{\text{Rep}}(S_T; K)$ where

$$a_{\lambda^{(j)}} = \begin{cases} 1 & \text{if } j \in \{i, i-1\}, \\ 0 & \text{if } j \notin \{i, i-1\}. \end{cases}$$

(2) If B is not minimal with respect to $\prec$, then there exists a nontrivial $\overset{d}{\sim}$-equivalence class $B' = \{\rho^{(0)} \prec \rho^{(1)} \prec \cdots\}$ such that for each integer $i \geq 0$, $L(\rho^{(i)}) \otimes L(\Box) = \bigoplus_\nu L(\nu)^{\oplus a_\nu}$ and $L(\lambda^{(i)}) \otimes L(\Box) = \bigoplus_\nu L(\nu)^{\oplus b_\nu}$ in $\underline{\text{Rep}}(S_T; K)$ where

$$a_{\lambda^{(j)}} = b_{\rho^{(j)}} = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

Moreover, B' can be chosen with $B' \prec B$.

Proof. (1) Assume B is the minimal $\overset{d}{\sim}$-equivalence class. Then

$$\lambda^{(i)} = \begin{cases} (2^i, 1^{d-i-1}) & \text{if } 0 \leq i < d, \\ (2^d, 1^{i-d}) & \text{if } i \geq d. \end{cases} \tag{VI.2}$$

Now fix $i > 0$ and set

$$\rho := \begin{cases} (1^d) & \text{if } i = 1, \\ (3, 2^{i-2}, 1^{d-i}) & \text{if } 1 < i < d, \\ (2^{d-1}, 1) & \text{if } i = d, \\ (3, 2^{d-1}, 1^{i-d-1}) & \text{if } i > d. \end{cases}$$

If we set $\mu_\rho(d) = (\mu_0, \mu_1, \ldots)$, then it is easy to check that

$$\mu_0 = \begin{cases} \mu_i & \text{if } 1 \leq i \leq d, \\ \mu_{i+1} & \text{if } i > d. \end{cases}$$

Hence ρ is in a trivial $\overset{d}{\sim}$-equivalence class (see corollary VI.2.2). Finally, comparing ρ to the Young diagrams in (VI.2) and using proposition VI.3.2.2, it is easy to check that ρ satisfies part (1).

(2) Suppose B is not minimal so that $\lambda^{(0)}(d)$ is a Young diagram of size d distinct from (1^d). Let $\rho^{(0)}$ be the Young diagram with $\rho^{(0)}(d)$ obtained from $\lambda^{(0)}(d)$ by adding one box to the first row with zero boxes and removing a box from the last row containing more than one box (such a row exists since $\lambda^{(0)}(d) \neq (1^d)$); see figure 2.

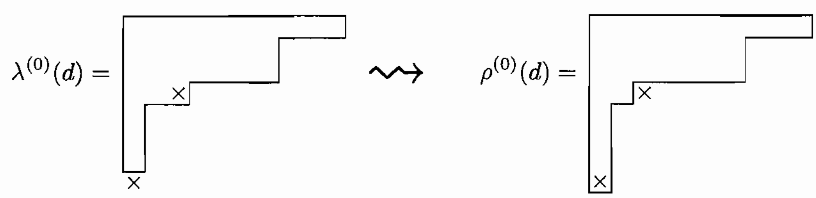


Figure 2: Constructing $\rho^{(0)}$ From $\lambda^{(0)}$

Since $\rho^{(0)}(d)$ is a Young diagram of size d with $\rho^{(0)}(d) \prec \lambda^{(0)}(d)$, $\rho^{(0)}$ is the minimal element in a nontrivial $\overset{d}{\sim}$-equivalence class $B' = \{\rho^{(0)} \prec \rho^{(1)} \prec \cdots\}$ which satisfies $B' \prec B$ (see proposition VI.2.4). Moreover, given an integer $i \geq 0$, it follows from the construction of $\rho^{(0)}$ that the coordinates of $\lambda^{(i)}(d) - \rho^{(i)}(d)$ are all zero except for one 1 and one -1. Thus, by proposition VI.3.2, B' satisfies part (2). $\square$

VI.4 Lift of Idempotents

In this section we examine the idempotents in $KP_n(T)$ lifted from $FP_n(t)$ (see theorem IV.1.3.2). Then, our results on lifted idempotents along with lemma VI.1.1 will be used to prove theorem VI.0.6. We begin with the following proposition.

Proposition VI.4.1. Suppose e is a primitive idempotent in $FP_n(t)$ corresponding to the Young diagram λ. Suppose further that e lifts to an idempotent $\varepsilon \in KP_n(T)$ and $\varepsilon = \varepsilon_1 + \cdots + \varepsilon_m$ is an orthogonal decomposition of ε into primitive idempotents with ε_i corresponding to Young diagram $\lambda^{(i)}$ for each $i = 1, \ldots, m$. Then $\lambda^{(i)} \overset{t}{\sim} \lambda$ and $|\lambda^{(i)}| \leq |\lambda|$ for all $i = 1, \ldots, m$.

Proof. By part theorem IV.1.3 we can assume $n = |\lambda|$, so that $|\lambda^{(i)}| \leq |\lambda|$ for all $i = 1, \ldots, m$. To show $\lambda^{(i)} \overset{t}{\sim} \lambda$ for all $i = 1, \ldots, m$ it suffices to show $\xi_{r,k}^{\lambda^{(i)}(t)} = \xi_{r,k}^{\lambda(t)}$ for all $r > 0$ and $i = 1, \ldots, m$ where k is an integer such that $\lambda_k^{(i)} = \lambda_k = 0$ for all i (see lemma VI.1.1.2). Fix positive integers r and k with $\lambda_k^{(i)} = \lambda_k = 0$ for all i and let $A(x) \in K[x]$ be the minimal monic polynomial with $A(\omega_n^r(T))\varepsilon = 0$. Then $A(x)$ is the product of linear terms of the form $x - \xi_{r,k}^{\lambda^{(i)}(T)}$ (see proposition

V.2.3). Let $B(x), C(x) \in K[x]$ be the unique monic polynomials with $A = BC$ such that $B(x)$ (resp. $C(x)$) is the product of linear terms of the form $x - \xi_{r,k}^{\lambda^{(i)}(T)}$ with $\xi_{r,k}^{\lambda^{(i)}(t)}$ equal to (resp. not equal to) $\xi_{r,k}^{\lambda(t)}$. Suppose for a contradiction that $C(x) \neq 1$. Since $B(x)$ and $C(x)$ are relatively prime polynomials of positive degree, there exist nonzero polynomials $G(x), H(x) \in K[x]$ with $\deg G(x) < \deg C(x)$, $\deg H(x) < \deg B(x)$ and

$$G(x)B(x) + H(x)C(x) = 1. \tag{VI.3}$$

Let N be the minimal nonnegative integer such that all coefficients of both $G'(x) := (T-t)^N G(x)$ and $H'(x) := (T-t)^N H(x)$ lie in $F[[T-t]]$. Then from equation (VI.3) we have

$$G'(x)B(x) + H'(x)C(x) = (T-t)^N. \tag{VI.4}$$

Let $b(x), c(x), g(x), h(x) \in F[x]$ be the polynomials obtained by evaluating $T = t$ in the polynomials $B(x), C(x), G'(x), H'(x)$ respectively. If $N > 0$, then equation (VI.4) implies $b(x)$ (resp. $c(x)$) divides $h(x)$ (resp. $g(x)$). On the other hand, $\deg b(x) = \deg B(x) > \deg H(x) > \deg h(x)$. Similarly $\deg c(x) > \deg g(x)$. Thus $h(x) = g(x) = 0$ which contradicts the minimality of N. Hence $N = 0$ which implies the coefficients of $H(x) = H'(x)$ and $G(x) = G'(x)$ are in $F[[T-t]]$. Thus evaluating $T = t$, $x = \omega_n^r(t)$ in equation (VI.3) and multiplying by e gives

$$g(\omega_n^r(t))b(\omega_n^r(t))e + h(\omega_n^r(t))c(\omega_n^r(t))e = e. \tag{VI.5}$$

Since $\omega_n^r(t)$ is in the center of $FP_n(t)$ (proposition V.1.6.2) and $b(\omega_n^r(t))c(\omega_n^r(t))e = 0$, the two summands on the left side of equation (VI.5) are idempotents. As e is assumed to be primitive, either $g(\omega_n^r(t))b(\omega_n^r(t))e = 0$ or $h(\omega_n^r(t))c(\omega_n^r(t))e = 0$. This implies that either $G(\omega_n^r(T))B(\omega_n^r(T))\varepsilon = 0$ or $H(\omega_n^r(T))C(\omega_n^r(T))\varepsilon = 0$ (see theorem IV.1.3.2) which contradicts the minimality of $A(x)$. This completes the proof. $\square$

Among other things, proposition VI.4.1 implies that the lifting a primitive idempotent in $FP_n(t)$ corresponding to Young diagram in a trivial $\overset{t}{\sim}$-equivalence class yields a primitive idempotent in $KP_n(T)$. The following lemma describes the result of lifting a primitive idempotent in $FP_n(t)$ which corresponds to Young diagram in a nontrivial $\overset{t}{\sim}$-equivalence class.

Lemma VI.4.2. Fix a nonnegative integer d. Suppose $B = \{\lambda^{(0)} \prec \lambda^{(1)} \prec \cdots\}$ is a nontrivial $\overset{d}{\sim}$-equivalence class, and $e \in FP_n(d)$ is a primitive idempotent corresponding to $\lambda^{(i)}$. Suppose further that e lifts to an idempotent $\varepsilon \in KP_n(T)$.

(1) If $i = 0$, then ε is a primitive idempotent in $KP_n(T)$ corresponding to $\lambda^{(i)}$.

(2) If $i > 0$, then there is an orthogonal decomposition of ε given by $\varepsilon = \varepsilon_i + \varepsilon_{i-1}$ where ε_i (resp. ε_{i-1}) is a primitive idempotent in $KP_n(T)$ corresponding to $\lambda^{(i)}$ (resp. $\lambda^{(i-1)}$).

Proof. Part (1) follows from theorem IV.1.3.2 and proposition VI.4.1. To prove part (2) we will first show that $\varepsilon = \varepsilon_i + \delta\varepsilon_{i-1}$ where $\delta \in \{0, 1\}$. To do so, it suffices to find a positive integer m and an idempotent in $\tilde{e} \in FP_m(t)$ which lifts to an idempotent in $\tilde{\varepsilon} \in KP_m(T)$ such that the number of primitive idempotents corresponding to λ in any orthogonal decomposition of $\tilde{\varepsilon}$ into primitive idempotents is one if $\lambda = \lambda^{(i)}$; at most one if $\lambda = \lambda^{(i-1)}$; and zero if $\lambda \in B \setminus \{\lambda^{(i)}, \lambda^{(i-1)}\}$. Indeed, if such an $\tilde{e}$ exists, then by proposition VI.4.1, an orthogonal decomposition of $\tilde{e}$ must contain an idempotent corresponding to $\lambda^{(i)}$ whose lift in $KP_m(T)$ (and hence ε (see theorem IV.1.3.2)) is the orthogonal sum of one idempotent corresponding to $\lambda^{(i)}$ and at most one idempotent corresponding to $\lambda^{(i-1)}$. We now proceed by induction on $\prec$.

For our base case, assume that B is the minimal $\overset{d}{\sim}$-equivalence class. Let ρ be as in lemma VI.3.4.1, $m = |\rho| + 1$, and $\tilde{e} := e' \otimes \mathrm{id}_1 \in FP_m(t)$ where $e' \in FP_{|\rho|}(t)$ is a primitive idempotent corresponding to ρ. Since ρ is in a trivial $\overset{d}{\sim}$-equivalence class, by proposition VI.4.1 along with theorem IV.1.3.2, e' lifts to a primitive idempotent $\varepsilon' \in KP_{|\rho|}(T)$ corresponding to ρ. Hence $\tilde{\varepsilon} := \varepsilon' \otimes \mathrm{id}_1 \in KP_m(T)$ is a lift of the idempotent $\tilde{e}$. By theorem IV.1.5.1, $\mathrm{id}_1 \in KP_1(T)$ is the sum of a primitive idempotent corresponding to $\square$ and a primitive idempotent corresponding to $\varnothing$. Thus, by lemma VI.3.4.1, an orthogonal decomposition of $\tilde{\varepsilon}$ into primitive idempotents will contain exactly one idempotent corresponding to each $\lambda^{(i)}$ and $\lambda^{(i-1)}$ and no idempotents corresponding to $\lambda^{(j)}$ when $j \neq i, i-1$.

Now assume B is not minimal and let $B' = \{\rho^{(0)} \prec \rho^{(1)} \prec \cdots\}$ be as in lemma VI.3.4.2. Set $m = |\rho^{(i)})| + 1$ and $\tilde{e} := e' \otimes \mathrm{id}_1 \in FP_m(t)$ where $e' \in FP_{|\rho^{(i)}|}(t)$ is a primitive idempotent corresponding to $\rho^{(i)}$. Since $B' \prec B$, by induction e' lifts to an idempotent $\varepsilon'_i + \delta\varepsilon'_{i-1} \in KP_{|\rho|}(T)$ where ε'_i, ε'_{i-1} are mutually orthogonal primitive idempotents corresponding to $\rho^{(i)}$, $\rho^{(i-1)}$ respectively, and $\delta \in \{0, 1\}$. Hence, $\tilde{e}$ lifts to the idempotent $\tilde{\varepsilon} := (\varepsilon'_i + \delta\varepsilon'_{i-1}) \otimes \mathrm{id}_1 \in KP_m(T)$. By lemma VI.3.4.2, an orthogonal decomposition of $\tilde{\varepsilon}$ into primitive idempotents will contain exactly one idempotent corresponding to $\lambda^{(i)}$, δ idempotents corresponding to $\lambda^{(i-1)}$, and zero idempotents corresponding to $\lambda^{(j)}$ when $j \neq i, i-1$.

It remains to show $\delta = 1$. First, since $\mathcal{F}$ (see definition IV.3.1) is a tensor functor, $\dim_{\underline{\mathrm{Rep}}(S_d;F)}([n], e) = \dim_{\mathrm{Rep}(S_d;F)} \mathcal{F}([n], e)$ which, by proposition IV.3.8, is equal to 0. On the other hand, $\dim_{\underline{\mathrm{Rep}}(S_d;F)}([n], e) = (\dim_{\underline{\mathrm{Rep}}(S_T;K)}([n], \varepsilon))|_{T=d}$ which, by proposition IV.4.3, is equal to $P_{\lambda^{(i)}}(d) + \delta P_{\lambda^{(i-1)}}(d)$. By corollary VI.2.6 we know $P_{\lambda^{(i)}}(d) \neq 0$, hence $\delta = 1$. □

Before proving our description of blocks in $\underline{\mathrm{Rep}}(S_t, F)$, we need one more proposition concerning idempotent lifting.

Proposition VI.4.3. If $e, e' \in FP_n(t)$ are idempotents which lift to $\varepsilon, \varepsilon' \in KP_n(T)$ respectively, then $\dim_F(e'FP_n(t)e) = \dim_K(\varepsilon' KP_n(T)\varepsilon)$.

Proof. Set $R = FP_n(t)$ and $S = KP_n(T)$. Also, let $f_1, f_2, f_3, f_4, \eta_1, \eta_2, \eta_3, \eta_4$ denote the idempotents $e, 1-e, e', 1-e', \varepsilon, 1-\varepsilon, \varepsilon', 1-\varepsilon'$ respectively. Since $\eta_i|_{T=t} = f_i$ for $1 \leq i \leq 4$, $\dim_F f_i R f_j \leq \dim_K \eta_i S \eta_j$ for each $1 \leq i, j \leq 4$. Hence

$$\dim_F R = \dim_F \bigoplus_{\substack{j=1,2\\ i=3,4}} f_i R f_j \leq \dim_K \bigoplus_{\substack{j=1,2\\ i=3,4}} \eta_i S \eta_j = \dim_K S.$$

However, $\dim_F R = |P_{n,n}| = \dim_K S$. Thus $\dim_F f_i R f_j = \dim_K \eta_i S \eta_j$ for each $1 \leq i, j \leq 4$. □

We are now ready to prove the main result section VI, which we restate now.

Theorem VI.0.6. $L(\lambda)$ and $L(\lambda')$ are in the same block of $\underline{\mathrm{Rep}}(S_t; F)$ if and only if $\lambda \overset{t}{\sim} \lambda'$.

Proof. If $L(\lambda)$ and $L(\lambda')$ are in the same block of $\underline{\mathrm{Rep}}(S_t; F)$, then lemma VI.1.1 implies $\lambda \overset{t}{\sim} \lambda'$. To prove the converse, by corollary VI.2.2.1, we may assume $t = d \in \mathbb{Z}_{\geq 0}$. Suppose $\{\lambda^{(0)} \prec \lambda^{(1)} \prec \cdots\}$ is a nontrivial $\overset{d}{\sim}$-equivalence class. It suffices to show $\mathrm{Hom}_{\underline{\mathrm{Rep}}(S_t;F)}(L(\lambda^{(i)}), L(\lambda^{(i+1)})) \neq 0$ for all $i \geq 0$. Since $\mathrm{Hom}_{\underline{\mathrm{Rep}}(S_t;F)}(L(\varnothing), L(\square)) = FP_{0,1}$, we may assume we are not in the case where $t = 0$ and $\lambda^{(i)} = \varnothing$. Now, fix $i \geq 0$ and set $n = \max\{|\lambda^{(i)}|, |\lambda^{(i+i)}|\}$. By theorem IV.1.3.1 we can find primitive idempotents $e, e' \in FP_n(t)$ with $([n], e)$ (resp. $([n], e')$) isomorphic to $L(\lambda^{(i)})$ (resp. $L(\lambda^{(i+i)})$). Hence, it suffices to show $e'FP_n(t)e$ is nonzero. Suppose $\varepsilon, \varepsilon'$ are idempotents in $KP_n(T)$ lifting e, e' respectively. By proposition VI.4.3, $\dim_F(e'FP_n(t)e) = \dim_K(\varepsilon' KP_n(T)\varepsilon)$. Moreover, it follows from lemma VI.4.2 that $\dim_K(\varepsilon' KP_n(T)\varepsilon) \neq 0$. □

We close this section by examining the dimensions of the Hom spaces between indecomposable objects in $\underline{\mathrm{Rep}}(S_t; F)$.

Proposition VI.4.4. (1) $\dim_F \mathrm{End}_{\underline{\mathrm{Rep}}(S_t;F)}(L(\lambda)) = 1$ whenever λ is in a trivial $\overset{t}{\sim}$-equivalence class. In particular, the block corresponding to a trivial $\overset{t}{\sim}$-equivalence class is trivial.

(2) Given a nontrivial block $\{\lambda^{(0)} \prec \lambda^{(1)} \prec \cdots\}$ in $\underline{\mathrm{Rep}}(S_t; F)$,

$$\dim_F \mathrm{Hom}_{\underline{\mathrm{Rep}}(S_t;F)}(L(\lambda^{(i)}), L(\lambda^{(j)})) = \begin{cases} 2 & \text{if } i = j > 0, \\ 0 & \text{if } |i-j| > 2, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. By lemma VI.4.2 and propositions VI.4.1 and VI.4.3, it suffices to prove

$$\dim_K \mathrm{Hom}_{\underline{\mathrm{Rep}}(S_T;K)}(L(\lambda), L(\lambda')) = \begin{cases} 1 & \text{if } \lambda = \lambda', \\ 0 & \text{if } \lambda \neq \lambda'. \end{cases}$$

By proposition IV.1.7 it suffices to consider the case when K is algebraically closed. As $\underline{\mathrm{Rep}}(S_T; K)$ is semisimple (corollary IV.2.5), the result follows from Schur's lemma. □

Corollary VI.4.5. $\underline{\mathrm{Rep}}(S_t; F)$ is semisimple if and only if t is not a nonnegative integer.

Proof. This follows from proposition VI.4.4.1 along with corollary VI.2.2.1 and proposition VI.2.4. □

CHAPTER VII

QUIVER DESCRIPTION OF A NON-SEMISIMPLE BLOCK

In this chapter we give a complete description of nontrivial blocks in $\underline{\text{Rep}}(S_d; F)$ for all $d \in \mathbb{Z}_{\geq 0}$. In particular, we prove that all nontrivial blocks are equivalent as additive categories. We begin by describing the nontrivial block in $\underline{\text{Rep}}(S_0; F)$. Next, we show that for fixed $d \in \mathbb{Z}_{\geq 0}$, the nontrivial blocks in $\underline{\text{Rep}}(S_d; F)$ are all equivalent. We then state a conjecture which would allow us to compare blocks in $\underline{\text{Rep}}(S_d; F)$ with those in $\underline{\text{Rep}}(S_{d-1}; F)$ using a "restriction" functor. Finally, we use Martin's results on the partition algebras to give a complete description of nontrivial blocks.

VII.1 The Nontrivial Block in $\underline{\text{Rep}}(S_0; F)$

In this section we give a complete description of the one nontrivial block in $\underline{\text{Rep}}(S_0; F)$. In this particular case the constructions of all idempotents are easy enough that we are able to fully describe the block by brute force computations. We expect this method is too computationally complicated in other cases. Throughout this section we consider the group algebra of the symmetric group FS_n as a subalgebra of the partition algebra $FP_n(0)$ (see remark III.2.4). With this in mind, we have the following idempotents:

$$s_n := \frac{1}{n!} \sum_{\sigma \in S_n} sgn(\sigma)\sigma \in FP_n(0) \qquad (n \geq 0).$$

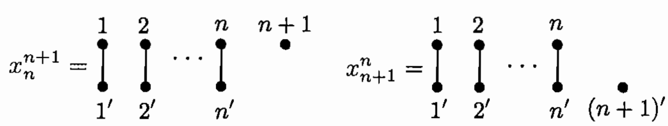
Proposition VII.1.1. In $\underline{\text{Rep}}(S_0; F)$, $([n], s_n) \cong L((1^n))$ for all $n \geq 0$.

Proof. The proposition is certainly true when $n = 0$, so we assume $n > 0$. Since the projection $FP_n(0) \twoheadrightarrow FS_n$ maps $s_n \mapsto s_n$, and in FS_n the idempotent s_n is primitive corresponding to (1^n), we know any orthogonal decomposition of s_n into primitive idempotents in $FP_n(0)$ must contain a summand corresponding to $L((1^n))$ in $\underline{\text{Rep}}(S_0; F)$. Thus, by example VI.2.5.1 and proposition VI.4.4 it suffices to show $\dim_F s_n FP_n(0) s_n = 2$.

Suppose $\mu \in P_{n,n}$ has the property that $\tau\mu = \pi$ for some transposition $\tau \in S_n \subset P_{n,n}$. Then $s_n\mu s_n = s_n\tau\mu s_n = -s_n\pi s_n$, which implies $s_n\mu s_n = 0$. Similarly, if $\mu\tau = \mu$ for some transposition τ, then $s_n\mu s_n = 0$. The only elements of $\mu \in P_{n,n}$ such that $\tau\mu \neq \mu$ and $\mu\tau \neq \mu$ for all transpositions $\tau \in S_n$ are either elements of S_n or of the form $\sigma x \sigma'$ for some $\sigma, \sigma' \in S_n$ with $x = \mathrm{id}_{n-1} \otimes \pi$ where $\pi \in P_{1,1}$ is as in the proof of theorem IV.1.5. If $\mu \in S_n$, then $s_n\mu s_n = \pm s_n$. If $\mu = \sigma x \sigma'$ for some $\sigma, \sigma' \in S_n$, then $s_n\mu s_n = \pm s_n x s_n$. Hence, s_n and $s_n x s_n$ span $s_n FP_n(0) s_n$.

□

As a consequence of example VI.2.5.1 and proposition VII.1.1, describing the nontrivial block in $\underline{\mathrm{Rep}}(S_0; F)$ amounts to describing morphisms among the objects $([n], s_n)$ for $n \geq 0$. To describe such morphisms, let x_0^1 denote the unique element of $P_{1,0}$ and let $x_n^{n+1} = \mathrm{id}_n \otimes x_0^1$ for all $n > 0$. Finally, let $x_{n+1}^n = (x_n^{n+1})^\vee$ for all $n \geq 0$ as pictured below.



The following lemma records a couple properties useful in forthcoming calculations.

Lemma VII.1.2. The following identities hold in $\underline{\mathrm{Rep}}(S_0; F)$:

(1) $s_n x_n^{n+1} s_{n+1} = x_n^{n+1} s_{n+1}$ for all $n \geq 0$.

(2) $-n s_n x_n^{n-1} x_{n-1}^n s_n = (n+1) x_n^{n+1} s_{n+1} x_{n+1}^n$ for all $n > 0$.

(3) $x_{n-1}^n x_n^{n+1} s_{n+1} = 0$ for all $n > 0$.

Proof. To prove part (1), let $s_n' := s_n \otimes \mathrm{id}_1 \in FP_{n+1}(0)$ for all $n \geq 0$. It is easy to see $s_n x_n^{n+1} s_{n+1} = x_n^{n+1} s_n' s_{n+1} = x_n^{n+1} s_{n+1}$.

To prove part (2), notice that for $\sigma \in S_{n+1}$ there are exactly $(n-1)!$ pairs (τ_1, τ_2) with $\tau_1, \tau_2 \in S_n$ such that $\tau_1 x_n^{n-1} x_{n-1}^n \tau_2 = x_n^{n+1} \sigma x_{n+1}^n$. Moreover, for such a pair $sgn(\tau_1\tau_2) = -sgn(\sigma)$. Conversely, given any $\tau_1, \tau_2 \in S_n$, either $\tau_1 x_n^{n-1} x_{n-1}^n \tau_2 = 0$ or there exists a unique $\sigma \in S_{n+1}$ with $\tau_1 x_n^{n-1} x_{n-1}^n \tau_2 = x_n^{n+1} \sigma x_{n+1}^n$. Therefore, $-(n! s_n) x_n^{n-1} x_{n-1}^n (n! s_n) = (n-1)! x_n^{n+1} (n+1)! s_{n+1} x_{n+1}^n$ which is equivalent to the identity in part (2).

To prove part (3), let $\tau \in S_{n+1}$ denote the transposition $n \leftrightarrow n+1$. Then it is apparent that $x_{n-1}^n x_n^{n+1} s_{n+1} = (x_{n-1}^n x_n^{n+1} \tau) s_{n+1} = x_{n-1}^n x_n^{n+1} (\tau s_{n+1}) = -x_{n-1}^n x_n^{n+1} s_{n+1}$. The result follows. □

Next, define the following morphisms:

$$\alpha_n := (-1)^n(n+1)! s_{n+1} x_{n+1}^n s_n, \qquad \beta_n := \frac{1}{n!} s_n x_n^{n+1} s_{n+1} \quad (n \geq 0),$$

$$\gamma_n := (-1)^n n s_n x_n^{n-1} x_{n-1}^n s_n \quad (n > 0).$$

The next lemma contains all calculations needed to describe the nontrivial block in $\underline{\mathrm{Rep}}(S_0; F)$.

Lemma VII.1.3. The following equations hold in $\underline{\mathrm{Rep}}(S_0; F)$:

(1) $\alpha_n \neq 0$ for $n \geq 0$.

(2) $\beta_n \neq 0$ for $n \geq 0$.

(3) $\gamma_n \neq 0$ for $n > 0$.

(4) $\beta_0 \alpha_0 = 0$.

(5) $\beta_n \alpha_n = \gamma_n$ for $n > 0$.

(6) $\alpha_{n-1}\beta_{n-1} = \gamma_n$ for $n > 0$.

(7) $\alpha_n \alpha_{n-1} = 0$ for $n > 0$.

(8) $\beta_{n-1}\beta_n = 0$ for $n > 0$.

Proof. Up to a nonzero scalar multiple, α_n and $\beta_n^\vee$ are equal. Hence, parts (1) and (7) will follow from parts (2) and (8) respectively.

(2) Write $\beta_n = \sum_{\pi \in P_{n+1,n}} b_\pi \pi$. Then $b_{x_n^{n+1}} = \frac{1}{n!(n+1)!}$.

(3) Write $\gamma_n = \sum_{\pi \in P_{n+1,n}} c_\pi \pi$. Then $c_{x_n^{n-1} x_{n-1}^n} = \frac{(-1)^n}{n!}$.

(4) $\beta_0 \alpha_0 = -x_0^1 x_1^0 = 0$ in $\underline{\mathrm{Rep}}(S_0; F)$.

(5) $\beta_n \alpha_n = (-1)^{n+1}(n+1) s_n x_n^{n+1} s_{n+1} x_{n+1}^n s_n = \gamma_n$, (lemma VII.1.2.2).

(6) $\alpha_{n-1}\beta_{n-1} = (-1)^n n s_n x_n^{n-1} s_{n-1} x_{n-1}^n s_n = \gamma_n$, (lemma VII.1.2.1).

(8) $\beta_{n-1}\beta_n = \frac{1}{(n-1)!n!} s_{n-1} x_{n-1}^n s_n x_n^{n+1} s_{n+1}$, which (by lemma VII.1.2.1) is equal to the expression $\frac{1}{(n-1)!n!} s_{n-1} x_{n-1}^n x_n^{n+1} s_{n+1}$, which (by lemma VII.1.2.3) is equal to zero. □

The following theorem describes the nontrivial block in $\underline{\mathrm{Rep}}(S_0; F)$.

Theorem VII.1.4. Let $L_n := L((1^n))$ for all $n \geq 0$. The nontrivial block in $\underline{\mathrm{Rep}}(S_0; F)$ has the following associated quiver

$$L_0 \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} L_1 \circlearrowleft^{\gamma_1} \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} L_2 \circlearrowleft^{\gamma_2} \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots$$

with relations: $\beta_0\alpha_0 = 0$, $\alpha_n\alpha_{n-1} = 0$, $\beta_{n-1}\beta_n = 0$, and $\beta_n\alpha_n = \alpha_{n-1}\beta_{n-1} = \gamma_n$ for all $n > 0$.

Proof. The objects in the block follow from example VI.2.5.1. The fact that the arrows (along with the identity maps) form bases of the appropriate Hom spaces follows from lemma VI.4.4 and lemma VII.1.3.1-3. The relations follow from lemma VII.1.3.4-8. □

Remark VII.1.5. The block described in theorem VII.1.4 is equivalent to the nontrivial blocks in the category of tilting modules of $U_q(sl_2)$ (when q is a root of unity).

VII.2 Comparison of Non-semisimple Blocks in $\underline{\mathrm{Rep}}(S_d; F)$

In this section we show that for fixed $d \in \mathbb{Z}_{\geq 0}$, the non-semisimple blocks in $\underline{\mathrm{Rep}}(S_d; F)$ are all equivalent as additive categories. Thereafter we conjecture that a restriction functor induces an equivalence of categories between certain non-semisimple blocks in $\underline{\mathrm{Rep}}(S_d; F)$ and $\underline{\mathrm{Rep}}(S_{d-1}; F)$. First, let us fix some notation. Given a block $\mathcal{B}$ in $\underline{\mathrm{Rep}}(S_d; F)$, let $\mathrm{Inc}_{\mathcal{B}} : \mathcal{B} \to \underline{\mathrm{Rep}}(S_d; F)$ and $\mathrm{Proj}_{\mathcal{B}} : \underline{\mathrm{Rep}}(S_d; F) \to \mathcal{B}$ denote the inclusion and projection functors respectively.

Proposition VII.2.1. Suppose $\mathcal{B}$ is a nontrivial block in $\underline{\mathrm{Rep}}(S_d; F)$. There exists a block $\mathcal{B}'$ in $\underline{\mathrm{Rep}}(S_d; F)$ with $\mathcal{B}' \prec \mathcal{B}$ such that $\mathrm{Proj}_{\mathcal{B}} \circ (- \otimes L(\Box)) \circ \mathrm{Inc}_{\mathcal{B}'} : \mathcal{B}' \to \mathcal{B}$ is an equivalence of additive categories. Hence, all nontrivial blocks in $\underline{\mathrm{Rep}}(S_d; F)$ are all equivalent as additive categories.

Proof. Let B denote the set of Young diagrams corresponding to indecomposable objects in $\mathcal{B}$. Let B' be the set of Young diagrams given by lemma VI.3.4.2, and $\mathcal{B}'$ the corresponding block. It follows from lemma VI.3.4.2 that $\mathrm{Proj}_{\mathcal{B}} \circ (- \otimes L(\Box)) \circ \mathrm{Inc}_{\mathcal{B}'}$ and $\mathrm{Proj}_{\mathcal{B}'} \circ (- \otimes L(\Box)) \circ \mathrm{Inc}_{\mathcal{B}}$ are inverse to one another on objects. Moreover, since $(- \otimes L(\Box))$ is self adjoint and $\mathrm{Proj}_{\mathcal{B}}$ is both right and left adjoint to $\mathrm{Inc}_{\mathcal{B}}$, it follows that $\mathrm{Proj}_{\mathcal{B}} \circ (- \otimes L(\Box)) \circ \mathrm{Inc}_{\mathcal{B}'}$ is adjoint to $\mathrm{Proj}_{\mathcal{B}'} \circ (- \otimes L(\Box)) \circ \mathrm{Inc}_{\mathcal{B}}$. The result follows. □

Next, we use the universal property of $\underline{\mathrm{Rep}}(S_t; F)$ (see section 8.3 in [9]) to define a restriction functor.

Definition VII.2.2. Let $\underline{\mathrm{Res}}_{S_{t-1}}^{S_t} : \underline{\mathrm{Rep}}(S_t; F) \to \underline{\mathrm{Rep}}(S_{t-1}; F)$ denote the functor given by the universal property of $\underline{\mathrm{Rep}}(S_t; F)$ which sends $([1], \mathrm{id}_1) \mapsto ([1], \mathrm{id}_1) \oplus ([0], \mathrm{id}_0)$.

Conjecture VII.2.3. For $d \in \mathbb{Z}_{\geq 0}$, let $\mathcal{B}_d$ denote the nontrivial block in $\underline{\mathrm{Rep}}(S_d; F)$ containing the object $L(\varnothing)$. Then the functor $\mathrm{Proj}_{\mathcal{B}_d} \circ \underline{\mathrm{Res}}_{S_d}^{S_{d+1}} \circ \mathrm{Inc}_{\mathcal{B}_{d+1}}$ induces an equivalence of additive categories $\mathcal{B}_{d+1} \cong \mathcal{B}_d$.

Remark VII.2.4. It is not hard to show that $\mathrm{Proj}_{\mathcal{B}_d} \circ \underline{\mathrm{Res}}_{S_d}^{S_{d+1}} \circ \mathrm{Inc}_{\mathcal{B}_{d+1}}$ is bijective on objects. Hence, to prove conjecture VII.2.3 it suffices to show $\mathrm{Proj}_{\mathcal{B}_d} \circ \underline{\mathrm{Res}}_{S_d}^{S_{d+1}} \circ \mathrm{Inc}_{\mathcal{B}_{d+1}}$ is either full or faithful.

VII.3 Description of Blocks Via Martin

In this section we give a general description of the nontrivial blocks based on the results of Martin. We start by reviewing the main result in [21].

Assume $d \neq 0$ and let $\lambda^{(0)} \prec \lambda^{(1)} \prec \cdots$ denote the Young diagrams associated to a fixed nontrivial block in $\underline{\mathrm{Rep}}(S_d; F)$. For each $m \geq |\lambda^{(n)}|$, let $E_m^{(n)}$ denote the simple $FP_m(d)$-module associated to $\lambda^{(n)}$ (see theorem IV.1.5.1), and let $P_m^{(n)}$ denote its projective cover. According to [21, proposition 9], these modules have Loewy structure

$$P_m^{(0)} = \begin{array}{c} E_m^{(0)} \\ \\ E_m^{(1)} \end{array} \qquad (m \geq |\lambda^{(1)}|),$$

$$P_m^{(n)} = \begin{array}{ccc} & E_m^{(n)} & \\ E_m^{(n-1)} & & E_m^{(n+1)} \\ & E_m^{(n)} & \end{array} \qquad (n > 0, m \geq |\lambda^{(n+1)}|).$$

In other words, for $m > 0$, $P_m^{(0)}$ has a maximal simple submodule $B_0 \cong E_m^{(1)}$ with $P_m^{(0)}/B_0 \cong E_m^{(0)}$.

Moreover, for each $m \geq |\lambda^{(n+1)}|$ there exists a chain of submodules

$$\begin{array}{ccccc} & & P_m^{(n)} & & \\ & & \cup & & \\ & & A_n & & \\ & \subset & & \supset & \\ B_n^- & & & & B_n^+ \\ & \supset & & \subset & \\ & & C_n & & \end{array}$$

with $P_m^{(n)}/A_n \cong E_m^{(n)}, A_n/B_n^{\pm} \cong E_m^{(n\mp1)}, B_n^{\pm}/C_n \cong E_m^{(n\pm1)}$, and $C_n \cong E_m^{(n)}$ for $n > 0$. Using the notation above, define the following maps[1]:

$$\begin{aligned} &\alpha_0 = \alpha_{0,m} : P_m^{(0)} \twoheadrightarrow P_m^{(0)}/B_0 \cong C_1 \hookrightarrow P_m^{(1)} \\ &\beta_0 = \beta_{0,m} : P_m^{(1)} \twoheadrightarrow P^{(1)}/B_1^+ \cong P_m^{(0)} \\ &\alpha_n = \alpha_{n,m} : P_m^{(n)} \twoheadrightarrow P_m^{(n)}/B_n^- \cong B_{n+1}^- \hookrightarrow P_m^{(n+1)} \qquad (n > 0) \\ &\beta_n = \beta_{n,m} : P_m^{(n+1)} \twoheadrightarrow P_m^{(n+1)}/B_{n+1}^+ \cong B_n^+ \hookrightarrow P_m^{(n)} \quad (n > 0) \\ &\gamma_n = \gamma_{n,m} : P_m^{(n)} \twoheadrightarrow P_m^{(n)}/A_n \cong C_n \hookrightarrow P_m^{(n)} \qquad (n > 0) \end{aligned} \tag{VII.1}$$

We are now ready to give a general description of the nontrivial blocks in $\underline{\mathrm{Rep}}(S_d; F)$.

Theorem VII.3.1. Suppose d is a nonnegative integer and $\mathcal{B}$ is a nontrivial block in $\underline{\mathrm{Rep}}(S_d; F)$. Then $\mathcal{B}$ is equivalent as an additive category to the nontrivial block described in theorem VII.1.4.

Proof. We may assume $d \neq 0$. Notice

$$\mathrm{Hom}_{\underline{\mathrm{Rep}}(S_d;F)}(L(\lambda^{(n)}), L(\lambda^{(n')})) = \mathrm{Hom}_{FP_m(d)}(P_m^{(n)}, P_m^{(n')})$$

whenever $m \geq |\lambda^{(n)}|, |\lambda^{(n')}|$. Hence, by proposition VI.4.4.2, it suffices to prove the maps defined by (VII.1) satisfy equations (1)-(7) in lemma VII.1.3. Equations (1)-(5) are clearly satisfied. The fact that equations (6) and (7) are satisfied follows from the observation that the compositions $B_{n\pm1}^{\mp} \hookrightarrow P_m^{(n\pm1)} \twoheadrightarrow P_m^{(n\pm1)}/B_{n\pm1}^{\pm} \cong B_n^{\pm}$ factor through $B_{n\pm1}^{\mp} \twoheadrightarrow B_{n\pm1}^{\mp}/C_{n\pm1} \cong C_n \hookrightarrow B_n^{\pm}$. □

Remark VII.3.2. Our proof of Theorem VII.3.1 is a bit unsatisfactory since it is based on rather deep results from [21]. On the other hand Theorem VII.3.1 follows from theorem VII.1.4,

[1]The equations in (VII.1) are only defined when m is sufficiently large.

proposition VII.2.1, along with conjecture VII.2.3. Hence a proof of conjecture VII.2.3 would complete a proof of theorem VII.3.1 which is independent from [21].

CHAPTER VIII

DECOMPOSING TENSOR PRODUCTS

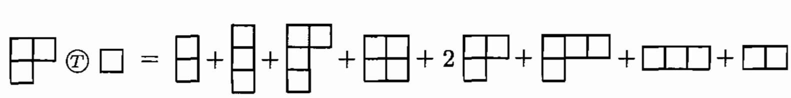
In this chapter we first show how a classical result of Littlewood can be used to decompose tensor products in $\underline{\text{Rep}}(S_T; K)$. We then give an example illustrating how lemma VI.4.2 along with knowledge of tensor decomposition in $\underline{\text{Rep}}(S_T; K)$ can be used to decompose tensor products in $\underline{\text{Rep}}(S_t; F)$. First, we introduce some notation which will simplify decomposition formulas.

Definition VIII.0.3. Set $Y := \bigoplus_{\lambda \in \Psi} \mathbb{Z}\lambda$. We now define multiple ring structures on Y.

- For $\lambda, \mu \in \Psi_d$ set $\lambda * \mu := \sum_{\nu \in \Psi_d} a^{\nu}_{\lambda\mu}\nu$ where $a^{\nu}_{\lambda\mu}$ are the nonnegative integers defined by $L_\lambda \otimes L_\mu = \bigoplus_{\nu \in \Psi_d} L_\nu^{\oplus a^{\nu}_{\lambda\mu}}$ in $\text{Rep}(S_d; F)$. If $\lambda, \mu \in \Psi$ have $|\lambda| \neq |\mu|$, set $\lambda * \mu = 0$.
- For $\lambda, \mu \in \Psi$ and $t \in F$, set $\lambda \circled{t} \mu := \sum_{\nu \in \Psi} b^{\nu}_{\lambda\mu}\nu$ where $b^{\nu}_{\lambda\mu}$ are the nonnegative integers defined by $L(\lambda) \otimes L(\mu) = \bigoplus_{\nu \in \Psi} L(\nu)^{\oplus b^{\nu}_{\lambda\mu}}$ in $\underline{\text{Rep}}(S_t; F)$.
- For $\lambda, \mu \in \Psi$, set $\lambda \cdot \mu := \sum_{\nu \in \Psi} c^{\nu}_{\lambda\mu}\nu$ where $c^{\nu}_{\lambda\mu}$ are the *Littlewood-Richardson coefficients* defined by $\text{Ind}^{S_{|\lambda|+|\mu|}}_{S_{|\lambda|} \times S_{|\mu|}}(L_\lambda \otimes L_\mu) = \bigoplus_{\nu \in \Psi} L_\nu^{\oplus c^{\nu}_{\lambda\mu}}$.

Example VIII.0.4. (1) Given Young diagrams λ and μ, if $\lambda \circled{T} \mu = \sum_{\nu \in \Psi} b^{\nu}_{\lambda,\mu}\nu$ then, by lemma VI.3.1, $\lambda(d) * \mu(d) = \sum_{\nu \in \Psi} b^{\nu}_{\lambda,\mu}\nu(d)$ whenever d is a sufficiently large integer.

(2) From example VI.3.3 we have



$\diamond$

VIII.1 The Generic Case

The following theorem gives a method for decomposing tensor products in $\underline{\mathrm{Rep}}(S_T; K)$. The result is a direct consequence of lemma VI.3.1 and a classical formula due to Littlewood (see theorem IX in [17]). For a more modern proof of Littlewood's formula, see theorem 1.1 in [24].

Theorem VIII.1.1. For Young diagrams α, β, η, and λ, let $\Gamma^{\lambda}_{\alpha\beta\eta}$ be the nonnegative integer defined by

$$\alpha \cdot \beta \cdot \eta = \sum_{\lambda \in \Psi} \Gamma^{\lambda}_{\alpha\beta\eta} \lambda.$$

Then for Young diagrams λ and μ,

$$\lambda \mathbin{\textcircled{\scriptsize T}} \mu = \sum_{\alpha,\beta,\beta',\eta,\eta' \in \Psi} \Gamma^{\lambda}_{\alpha\beta\eta} \Gamma^{\mu}_{\alpha\beta'\eta'} \eta \cdot \eta' \cdot (\beta * \beta'). \tag{VIII.1}$$

Example VIII.1.2. The *Littlewood-Richardson rule* can be used to compute the numbers $\Gamma^{\lambda}_{\alpha\beta\eta}$ (see [18, §I.9] or [12, A.1]). Table 1 lists nonzero $\Gamma^{\lambda}_{\alpha\beta\eta}$ when $\lambda = (1,1,0,\ldots), (2,1,0,\ldots), (2,2,0,\ldots)$ respectively.

$\{\alpha,\beta,\eta\}$	$\Gamma^{(1,1)}_{\alpha\beta\eta}$
$\{(1,1),\varnothing,\varnothing\}$	1
$\{\square,\square,\varnothing\}$	1

$\{\alpha,\beta,\eta\}$	$\Gamma^{(2,1)}_{\alpha\beta\eta}$
$\{(2,1),\varnothing,\varnothing\}$	1
$\{(1,1),\square,\varnothing\}$	1
$\{(2),\square,\varnothing\}$	1
$\{\square,\square,\square\}$	2

$\{\alpha,\beta,\eta\}$	$\Gamma^{(2,2)}_{\alpha\beta\eta}$
$\{(2,2),\varnothing,\varnothing\}$	1
$\{(2,1),\square,\varnothing\}$	1
$\{(1,1),(1,1),\varnothing\}$	1
$\{(1,1),\square,\square\}$	1
$\{(2),(2),\varnothing\}$	1
$\{(2),\square,\square\}$	1

Table 1: Nonzero $\Gamma^{\lambda}_{\alpha\beta\eta}$ for Various λ $\diamond$

Example VIII.1.3. In the following examples we use theorem VIII.1.1 to compute $\lambda \mathbin{\textcircled{\scriptsize T}} \mu$ for various Young diagrams λ and μ. In each example, we use example VIII.1.2 along with the Littlewood-Richardson rule to compute all nonzero terms of the right hand side of equation (VIII.1).

(1) In this example we will compute $(1^2) \textcircled{T} (2,1)$. Table 2 gives all nonzero terms of the right hand side of equation (VIII.1).

α	β	η	β'	η'	$\eta \cdot \eta' \cdot (\beta * \beta')$	$\Gamma^{(1^2)}_{\alpha\beta\eta}\Gamma^{(2,1)}_{\alpha\beta'\eta'}$
$\varnothing$	$\varnothing$	(1^2)	$\varnothing$	$(2,1)$	$(3,2)+(3,1,1)+(2,2,1)+(2,1,1,1)$	1
	(1)	(1)	(1)	(2)	$(4)+2(3,1)+(2,2)+(2,1,1)$	1
	(1)	(1)	(1)	(1^2)	$(3,1)+(2,2)+2(2,1,1)+(1^4)$	1
	(1^2)	$\varnothing$	(2)	(1)	$(2,1)+(1^3)$	1
	(1^2)	$\varnothing$	(1^2)	(1)	$(3)+(2,1)$	1
(1)	$\varnothing$	(1)	$\varnothing$	(1^2)	$(2,1)+(1^3)$	1
	$\varnothing$	(1)	$\varnothing$	(2)	$(3)+(2,1)$	1
	(1)	$\varnothing$	(1)	(1)	$(2)+(1^2)$	2
(1^2)	$\varnothing$	$\varnothing$	$\varnothing$	(1)	(1)	1

Table 2: Calculations for Computing $(1^2) \textcircled{T} (2,1)$

Hence

$$(1^2) \textcircled{T} (2,1) = (3,2)+(3,1,1)+(2,2,1)+(2,1,1,1)+(4)+3(3,1)+2(2,2)+3(2,1,1)+(1^4)+2(3)+4(2,1)+2(1^3)+2(2)+2(1^2)+(1).$$

(2) In this example we will compute $(1^2) \textcircled{T} (2^2)$. Table 3 gives all nonzero terms of the right hand side of equation (VIII.1).

α	β	η	β'	η'	$\eta \cdot \eta' \cdot (\beta * \beta')$	$\Gamma^{(1^2)}_{\alpha\beta\eta}\Gamma^{(2^2)}_{\alpha\beta'\eta'}$
$\varnothing$	$\varnothing$	(1^2)	$\varnothing$	(2^2)	$(3,3)+(3,2,1)+(2,2,1,1)$	1
	(1)	(1)	(1)	$(2,1)$	$(4,1)+2(3,2)+2(2,2,1)+(2,1,1,1)$	1
	(1^2)	$\varnothing$	(1^2)	(1^2)	$(3,1)+(2,1,1)$	1
	(1^2)	$\varnothing$	(2)	(2)	$(3,1)+(2,1,1)$	1
(1)	$\varnothing$	(1)	$\varnothing$	$(2,1)$	$(3,1)+(2,2)+(2,1,1)$	1
(1^2)	$\varnothing$	$\varnothing$	$\varnothing$	(1^2)	(1^2)	1

Table 3: Calculations for Computing $(1^2) \textcircled{T} (2^2)$

Hence $(1^2) \textcircled{T} (2^2) = (3,3)+(3,2,1)+(2,2,1,1)+(4,1)+2(3,2)+2(2,2,1)+(2,1,1,1)+3(3,1)+(2,2)+3(2,1,1)+(1^2)$.

(3) In this example we will compute $(2,1) \textcircled{T} (2,1)$. Table 4 gives all nonzero terms of the right hand side of equation (VIII.1).

α	β	η	β'	η'	$\eta \cdot \eta' \cdot (\beta * \beta')$	$\Gamma^{(2,1)}_{\alpha\beta\eta}\Gamma^{(2,1)}_{\alpha\beta'\eta'}$
$\emptyset$	$\emptyset$	$(2,1)$	$\emptyset$	$(2,1)$	$(4,2)+(4,1,1)+(3,3)+2(3,2,1)+(3,1,1,1)+(2,2,2)+(2,2,1,1)$	1
	(1)	$(1,1)$	(1)	$(1,1)$	$(3,2)+(3,1,1)+2(2,2,1)+2(2,1,1,1)+(1,1,1,1,1)$	1
	(1)	$(1,1)$	(1)	(2)	$(4,1)+(3,2)+2(3,1,1)+(2,2,1)+(2,1,1,1)$	1
	(1)	(2)	(1)	$(1,1)$	$(4,1)+(3,2)+2(3,1,1)+(2,2,1)+(2,1,1,1)$	1
	(1)	(2)	(1)	(2)	$(5)+2(4,1)+2(3,2)+(3,1,1)+(2,2,1)$	1
	$(1,1)$	(1)	$(1,1)$	(1)	$(4)+2(3,1)+(2,2)+(2,1,1)$	1
	$(1,1)$	(1)	(2)	(1)	$(3,1)+(2,2)+2(2,1,1)+(1,1,1,1)$	1
	(2)	(1)	$(1,1)$	(1)	$(3,1)+(2,2)+2(2,1,1)+(1,1,1,1)$	1
	(2)	(1)	(2)	(1)	$(4)+2(3,1)+(2,2)+(2,1,1)$	1
	$(2,1)$	$\emptyset$	$(2,1)$	$\emptyset$	$(3)+(2,1)+(1,1,1)$	1
(1)	$\emptyset$	$(1,1)$	$\emptyset$	$(1,1)$	$(2,2)+(2,1,1)+(1,1,1,1)$	1
	$\emptyset$	$(1,1)$	$\emptyset$	(2)	$(3,1)+(2,1,1)$	1
	$\emptyset$	(2)	$\emptyset$	$(1,1)$	$(3,1)+(2,1,1)$	1
	$\emptyset$	(2)	$\emptyset$	(2)	$(4)+(3,1)+(2,2)$	1
	(1)	(1)	(1)	(1)	$(3)+2(2,1)+(1,1,1)$	4
	$(1,1)$	$\emptyset$	$(1,1)$	$\emptyset$	(2)	1
	$(1,1)$	$\emptyset$	(2)	$\emptyset$	$(1,1)$	1
	(2)	$\emptyset$	$(1,1)$	$\emptyset$	$(1,1)$	1
	(2)	$\emptyset$	(2)	$\emptyset$	(2)	1
$(1,1)$	$\emptyset$	(1)	$\emptyset$	(1)	$(2)+(1,1)$	1
	(1)	$\emptyset$	(1)	$\emptyset$	(1)	1
(2)	$\emptyset$	(1)	$\emptyset$	(1)	$(2)+(1,1)$	1
	(1)	$\emptyset$	(1)	$\emptyset$	(1)	1
$(2,1)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	1

Table 4: Calculations for Computing $(2,1) \textcircled{T} (2,1)$

Hence

$$(2,1) \textcircled{T} (2,1) = (4,2)+(4,1,1)+(3,3)+2(3,2,1)+(3,1,1,1)+(2,2,2)+(2,2,1,1)+(5)+3(4,1)+5(3,2)+6(3,1,1)+5(2,2,1)$$
$$+4(2,1,1,1)+(1,1,1,1,1)+3(4)+9(3,1)+6(2,2)+9(2,1,1)+3(1,1,1,1)+5(3)+9(2,1)+5(1,1,1)+4(2)+4(1,1)+2(1)+\emptyset.$$

(4) In this example we will compute $(2,1)\ \textcircled{T}\ (2^2)$. Table 5 gives all nonzero terms of the right hand side of equation (VIII.1).

α	β	η	β'	η'	$\eta\cdot\eta'\cdot(\beta*\beta')$	$\Gamma^{(2,1)}_{\alpha\beta\eta}\Gamma^{(2^2)}_{\alpha\beta'\eta'}$
$\varnothing$	$\varnothing$	$(2,1)$	$\varnothing$	$(2,2)$	$(4,3)+(4,2,1)+(3,3,1)+(3,2,2)+(3,2,1,1)+(2,2,2,1)$	1
	(1)	$(1,1)$	(1)	$(2,1)$	$(4,2)+(4,1,1)+(3,3)+3(3,2,1)+2(3,1,1,1)+(2,2,2)+2(2,2,1,1)+(2,1,1,1,1)$	1
	(1)	(2)	(1)	$(2,1)$	$(5,1)+2(4,2)+2(4,1,1)+(3,3)+3(3,2,1)+(3,1,1,1)+(2,2,2)+(2,2,1,1)$	1
	$(1,1)$	(1)	$(1,1)$	$(1,1)$	$(4,1)+(3,2)+2(3,1,1)+(2,2,1)+(2,1,1,1)$	1
	$(1,1)$	(1)	(2)	(2)	$(4,1)+(3,2)+2(3,1,1)+(2,2,1)+(2,1,1,1)$	1
	(2)	(1)	$(1,1)$	$(1,1)$	$(4,1)+(3,2)+(3,1,1)+2(2,2,1)+2(2,1,1,1)+(1,1,1,1,1)$	1
	(2)	(1)	(2)	(2)	$(5)+2(4,1)+2(3,2)+(3,1,1)+(2,2,1)$	1
	$(2,1)$	$\varnothing$	$(2,1)$	(1)	$(4)+2(3,1)+(2,2)+2(2,1,1)+(1,1,1,1)$	1
(1)	$\varnothing$	$(1,1)$	$\varnothing$	$(2,1)$	$(3,2)+(3,1,1)+(2,2,1)+(2,1,1,1)$	1
	$\varnothing$	(2)	$\varnothing$	$(2,1)$	$(4,1)+(3,2)+(3,1,1)+(2,2,1)$	1
	(1)	(1)	(1)	$(1,1)$	$(3,1)+(2,2)+2(2,1,1)+(1,1,1,1)$	2
	(1)	(1)	(1)	(2)	$(4)+2(3,1)+(2,2)+(2,1,1)$	2
	$(1,1)$	$\varnothing$	$(1,1)$	(1)	$(3)+(2,1)$	1
	$(1,1)$	$\varnothing$	(2)	(1)	$(2,1)+(1,1,1)$	1
	(2)	$\varnothing$	$(1,1)$	(1)	$(2,1)+(1,1,1)$	1
	(2)	$\varnothing$	(2)	(1)	$(3)+(2,1)$	1
$(1,1)$	$\varnothing$	(1)	$\varnothing$	$(1,1)$	$(2,1)+(1,1,1)$	1
	(1)	$\varnothing$	(1)	(1)	$(2)+(1,1)$	1
(2)	$\varnothing$	(1)	$\varnothing$	(2)	$(3)+(2,1)$	1
	(1)	$\varnothing$	(1)	(1)	$(2)+(1,1)$	1
$(2,1)$	$\varnothing$	$\varnothing$	$\varnothing$	(1)	(1)	1

Table 5: Calculations for Computing $(2,1)\ \textcircled{T}\ (2^2)$

Hence

$$\begin{aligned}(2,1)\ \textcircled{T}\ (2,2) = {} & (4,3)+(4,2,1)+(3,3,1)+(3,2,2)+(3,2,1,1)+(2,2,2,1)+(5,1)+3(4,2)+3(4,1,1)+2(3,3)+6(3,2,1)\\ & +3(3,1,1,1)+2(2,2,2)+3(2,2,1,1)+(2,1,1,1,1)+(5)+6(4,1)+7(3,2)+8(3,1,1)+7(2,2,1)+5(2,1,1,1)+(1,1,1,1,1)+3(4)\\ & +8(3,1)+5(2,2)+8(2,1,1)+3(1,1,1,1)+3(3)+6(2,1)+3(1,1,1)+2(2)+2(1,1)+(1).\end{aligned}$$

◇

VIII.2 The Non-semisimple Case

In this section we Illustrate how to use lemma VI.4.2 along with theorem VIII.1.1 to decompose tensor products in $\underline{\mathrm{Rep}}(S_t; F)$ in non-semisimple cases. First, we introduce some notation.

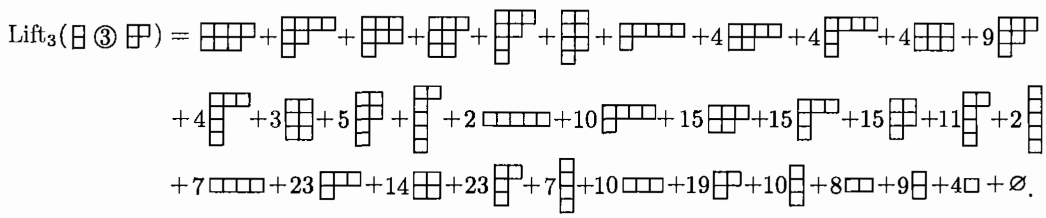
Notation. Fix $t \in F$ and suppose λ is a Young diagram and $e \in FP_n(t)$ is a primitive idempotent corresponding to λ. Suppose further that e lifts to the idempotent $\varepsilon \in KP_n(T)$ and $\varepsilon = \sum_{i=1}^{m} \varepsilon_i$ is an orthogonal decomposition of ε into primitive idempotents. Finally, suppose $\lambda^{(i)}$ is the Young diagram corresponding to ε_i for $i = 1, \ldots, m$. Let $\mathrm{Lift}_t : Y \to Y$ denote the $\mathbb{Z}$-linear map which sends $\lambda \mapsto \sum_{i=1}^{m} \lambda^{(i)}$.

By proposition VI.4.1 and lemma VI.4.2, Lift_t is a bijection for all $t \in F$. Moreover, it is easy to see

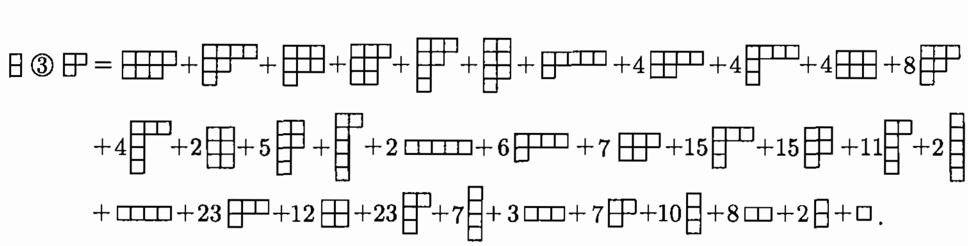
$$\mathrm{Lift}_t(\lambda) \textcircled{\scriptsize T} \mathrm{Lift}_t(\mu) = \mathrm{Lift}_t(\lambda \textcircled{\scriptsize t} \mu). \tag{VIII.2}$$

We are now ready to give an example illustrating how to decompose tensor products in $\underline{\mathrm{Rep}}(S_t; F)$ in non-semisimple cases.

Example VIII.2.1. In this example we will compute $(2,1) \textcircled{\scriptsize 3} (2,2)$. By lemma VI.4.2 and example VI.2.5.2, $\mathrm{Lift}_3((2,1)) = (1,1) + (2,1)$ and $\mathrm{Lift}_3((2,2)) = (2,1) + (2,2)$. Hence, by (VIII.2) and example VIII.1.3,



Thus, by lemma VI.4.2 and example VI.2.5.2,



$\diamond$

CHAPTER IX

TENSOR IDEALS

In this chapter we use our results on blocks along with an argument of Deligne's to classify tensor ideals in $\underline{\mathrm{Rep}}(S_t; F)$. More precisely, we will prove the following theorem.

Theorem IX.0.2. If $t \notin \mathbb{Z}_{\geq 0}$, then $\underline{\mathrm{Rep}}(S_t; F)$ has no nonzero proper tensor ideals. If $t \in \mathbb{Z}_{\geq 0}$, then the only nonzero proper tensor ideal in $\underline{\mathrm{Rep}}(S_t; F)$ is the ideal of negligible morphisms.

IX.1 Deligne's Lemma

Suppose n is a nonnegative integer and consider $x_{\mathrm{id}_n} \in FP_n(t)$ (see (III.1)). x_{id_n} is an idempotent. Indeed, if $t \in \mathbb{Z}_{\geq 0}$ then by (III.2) $f(x_{\mathrm{id}_n}) : V_t^{\otimes n} \to V_t^{\otimes n}$ maps

$$v_{\boldsymbol{i}} \mapsto \begin{cases} v_{\boldsymbol{i}} & \text{if } i_j \neq i_k \text{ for } j \neq k, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $f(x_{\mathrm{id}_n})$ is an idempotent whenever $t \in \mathbb{Z}_{\geq 0}$. Hence, by theorem III.1.7.2 and the fact that $f : FP_n(t) \to \mathrm{End}_{S_t}(V_t^{\otimes n})$ is an algebra homomorphism, $x_{\mathrm{id}_n} \in FP_n(t)$ is an idempotent whenever $t \in \mathbb{Z}_{\geq n}$. Since the condition $x_{\mathrm{id}_n}^2 = x_{\mathrm{id}_n}$ in $FP_n(t)$ is polynomial in t, it follows that $x_{\mathrm{id}_n} \in FP_n(t)$ is an idempotent for all $t \in F$.

Now suppose d is a nonnegative integer and let Δ denote the object $([d+1], x_{\mathrm{id}_{d+1}})$ in $\underline{\mathrm{Rep}}(S_d; F)$. The content of the following lemma concerning Δ is contained in a hand written letter from P. Deligne. A proof of lemma IX.1.1 will appear in [3].

Lemma IX.1.1. The endofunctor $-\otimes\Delta$ on $\underline{\mathrm{Rep}}(S_d; F)$ factors through the category $\underline{\mathrm{Rep}}(S_{-1}; F)$. In other words, there exist functors $\underline{\mathrm{Rep}}(S_d; F) \to \underline{\mathrm{Rep}}(S_{-1}; F)$ and $\underline{\mathrm{Rep}}(S_{-1}; F) \to \underline{\mathrm{Rep}}(S_d; F)$

making the following diagram commute.

$$\begin{array}{ccc} \underline{\mathrm{Rep}}(S_d;F) & \xrightarrow{\quad -\otimes\Delta \quad} & \underline{\mathrm{Rep}}(S_d;F) \\ & \searrow \qquad \nearrow & \\ & \underline{\mathrm{Rep}}(S_{-1};F) & \end{array}$$

Corollary IX.1.2. Every nonzero tensor ideal in $\underline{\mathrm{Rep}}(S_d;F)$ contains a nonzero identity endomorphism.

Proof. Suppose $\mathcal{I}$ is a nonzero tensor ideal in $\underline{\mathrm{Res}}(S_d;F)$. Since tensor ideals are closed under composition, it suffices to show that $\mathcal{I}$ contains a morphism which has a nonzero isomorphism as a direct summand. Let f be a nonzero morphism in $\mathcal{I}$. Then $f \otimes \mathrm{id}_\Delta$ is also a nonzero morphism in $\mathcal{I}$. By lemma IX.1.1 there exists a functor $\mathcal{G} : \underline{\mathrm{Rep}}(S_{-1};F) \to \underline{\mathrm{Rep}}(S_d;F)$ such that $f \otimes \mathrm{id}_\Delta = \mathcal{G}(\phi)$ for some nonzero morphism ϕ in $\underline{\mathrm{Rep}}(S_{-1};F)$. By corollary VI.4.5, $\underline{\mathrm{Rep}}(S_{-1};F)$ is semisimple. Hence ϕ (and therefore $\mathcal{G}(\phi)$) is the direct sum of isomorphisms and zero morphisms. □

IX.2 Proof of Theorem IX.0.2

The following proposition, which holds in any tensor category, will be useful in the proof theorem IX.0.2.

Proposition IX.2.1. All morphisms in a proper tensor ideal are negligible.

Proof. (compare with [13, proposition 3.1]) Suppose $\mathcal{I}$ is a tensor ideal in a tensor category $\mathcal{T}$. Suppose further that there exist objects X, Y in $\mathcal{T}$ and a morphism $f \in \mathcal{I}(X,Y)$ which is not negligible. Then $\mathrm{tr}(fg) \neq 0$ for some $g : Y \to X$. Thus $\mathrm{tr}(fg) = ev_Y \circ (\mathrm{id}_Y \otimes fg) \circ coev_Y$ is a nonzero morphism in $\mathcal{I}(\mathbf{1},\mathbf{1})$. Since all nonzero elements of $\mathrm{End}_{\mathcal{T}}(\mathbf{1}) = F$ are invertible, $\mathrm{id}_{\mathbf{1}} \in \mathcal{I}(\mathbf{1},\mathbf{1})$. Finally, any morphism $h : A \to B$ in $\mathcal{T}$ is equal to the composition $A = A \otimes \mathbf{1} \stackrel{h\otimes \mathrm{id}_1}{\longrightarrow} B \otimes \mathbf{1} = B$. Hence $\mathcal{I}$ must contain all morphisms in $\mathcal{T}$. □

Next, we introduce an equivalence relation on Young diagrams.

Definition IX.2.2. Consider the weakest equivalence relation on the set of all Young diagrams such that λ and μ are equivalent whenever $L(\lambda)$ is a direct summand of $L(\mu) \otimes ([1], \mathrm{id}_1)$ in $\underline{\mathrm{Rep}}(S_d;F)$. When λ and μ are in the same equivalence class we write $\lambda \stackrel{d}{\approx} \mu$.

The following proposition consists of all remaining information used in our upcoming proof of theorem IX.0.2.

Proposition IX.2.3. Assume d is a nonnegative integer and λ, μ are Young diagrams.

(1) $\mathrm{id}_{L(\lambda)}$ is a negligible morphism in $\underline{\mathrm{Rep}}(S_d; F)$ if and only if λ is not the minimal Young diagram in a nontrivial $\stackrel{d}{\sim}$-equivalence class.

(2) $\lambda \stackrel{d}{\approx} \mu$ whenever λ and μ are in trivial $\stackrel{d}{\sim}$-equivalence classes.

(3) $\lambda \stackrel{d}{\approx} \mu$ whenever λ is a non-minimal element of a nontrivial $\stackrel{d}{\sim}$-equivalence class and μ is in a trivial $\stackrel{d}{\sim}$-equivalence class.

(4) $\lambda \stackrel{d}{\approx} \mu$ whenever neither λ nor μ is a minimal Young diagram in a nontrivial $\stackrel{d}{\sim}$-equivalence class.

(5) If $\mathcal{I}$ is a tensor ideal in $\underline{\mathrm{Rep}}(S_d; F)$ containing $\mathrm{id}_{L(\lambda)}$ and $\lambda \stackrel{d}{\approx} \mu$, then $\mathrm{id}_{L(\mu)}$ is also in $\mathcal{I}$.

Proof. Part (1) follows from propositions IV.3.8 and VI.2.4. Part (2) follows from propositions VI.3.2 and VI.4.1 along with equation VIII.2. Part (4) follows from parts (2) and (3). Part (5) is easy to check. Hence, it suffices to prove part (3). To do so, let B denote the nontrivial $\stackrel{d}{\sim}$-equivalence class containing λ. We will proceed by induction on B with respect to $\prec$.

If B is the minimal with respect to $\prec$, then we are done by lemma VI.3.4.1 along with lemma VI.4.2.2 and equation VIII.2. Now suppose B is not minimal with respect to $\prec$. Then, by lemma VI.3.4.2 along with lemma VI.4.2.2 and equation VIII.2, $\lambda \stackrel{d}{\approx} \lambda'$ where λ' is a non-minimal Young diagram in a nontrivial $\stackrel{d}{\sim}$-equivalence class B' such that $B' \prec B$. By induction $\lambda' \stackrel{d}{\approx} \mu$ for some Young diagram μ in a trivial $\stackrel{d}{\sim}$-equivalence class. □

We are now ready to prove our classification of tensor ideals.

Proof of theorem IX.0.2. If $t \notin \mathbb{Z}_{\geq 0}$ then by corollary VI.4.5, $\underline{\mathrm{Rep}}(S_t; F)$ is semisimple. Hence $\underline{\mathrm{Rep}}(S_t; F)$ contains no nonzero negligible morphisms and we are done by proposition IX.2.1.

Now assume d is a nonnegative integer and $\mathcal{I}$ is a nonzero proper tensor ideal of $\underline{\mathrm{Rep}}(S_d; F)$. Suppose λ is a Young diagram which is not the minimal Young diagram in a nontrivial $\stackrel{d}{\sim}$-equivalence class. By propositions IX.2.1 and IX.2.3.1 it suffices to show that $\mathrm{id}_{L(\lambda)}$ is contained in $\mathcal{I}$. By corollary IX.1.2, there exists a nonzero identity morphism in $\mathcal{I}$. It follows that $\mathcal{I}$ contains

$\mathrm{id}_{L(\mu)}$ for some Young diagram μ. By proposition IX.2.1, $\mathrm{id}_{L(\mu)}$ is negligible. Hence, by proposition IX.2.3.1, μ is not the minimal Young diagram in a nontrivial $\overset{d}{\sim}$-equivalence class. Thus by proposition IX.2.3.4, $\lambda \overset{d}{\approx} \mu$. Finally, by proposition IX.2.3.5, $\mathrm{id}_{L(\lambda)}$ is contained in $\mathcal{I}$. □

APPENDIX

LIST OF SYMBOLS

$\mathcal{A}^{\mathrm{add}}$	additive envelope of the category $\mathcal{A}$	8
$\mathcal{A}^{\mathrm{Kar}}$	Karoubian envelope of the category $\mathcal{A}$	9
$\mathcal{A}^{\mathrm{ps\ ab}}$	pseudo-abelian envelope of the category $\mathcal{A}$	9
$\mathcal{B}$	block	62
$C_n^r(d)$	element of $FP_n(d)$ related to $\Omega_{r,d}$	41
$coev_A$	coevaluation morphism $\mathbf{1} \to A \otimes A^\vee$	4
d	nonnegative integer	14
Δ	the object $([d+1], x_{\mathrm{id}_{d+1}})$ in $\underline{\mathrm{Rep}}(S_d; F)$	72
ev_A	evaluation morphism $A^\vee \otimes A \to \mathbf{1}$	4
f	linear map $FP_{n,m} \to \mathrm{Hom}_{S_d}(V_d^{\otimes n}, V_d^{\otimes m})$	16
F	field (assumed to be of characteristic zero after chapter II)	3
$\mathcal{F}$	interpolation functor $\underline{\mathrm{Rep}}(S_d; F) \to \mathrm{Rep}(S_d; F)$	35
$FP_{n,m}$	F-vector space with basis $P_{n,m}$	16
$FP_n(t)$	partition algebra $\mathrm{End}_{\underline{\mathrm{Rep}}_0(S_t;F)}([n])$	22
FS_n	group algebra of the symmetric group	22
$\Gamma^\lambda_{\alpha\beta\eta}$	coefficients in Littlewood's formula	67
i	element of $[n,d]$	16
$\mathcal{I}$	tensor ideal	5

i_j	$i(j)$	16
$\mathrm{Inc}_{\mathcal{B}}$	inclusion functor $\mathcal{B} \to \underline{\mathrm{Rep}}(S_d; F)$	62
K	field of fractions of $F[[T-t]]$	27
$\ell(\mu, \pi)$	number of "middle components" in $\mu \star \pi$	19
L_λ	simple S_d-representation corresponding to λ	13
$L(\lambda)$	indecomposable object in $\underline{\mathrm{Rep}}(S_t; F)$ corresponding to λ	28
$(l_1^{m_1}, \ldots, l_r^{m_r})$	Young diagram given by multiplicities	13
λ	Young diagram	12
$\|\lambda\|$	size of λ (number of boxes)	12
$\lambda(t)$	$(t - \|\lambda\|, \lambda_1, \lambda_2, \ldots)$	13
$\mu_\lambda(t)$	$(t - \|\lambda\|, \lambda_1 - 1, \lambda_2 - 2, \ldots)$	46
$\mathcal{N}$	tensor ideal of negligible morphisms	36
$[n]$	object in $\underline{\mathrm{Rep}}_0(S_t; F)$	21
$[n, d]$	set of functions $\{j \mid 1 \le j \le n\} \to \{j \mid 1 \le j \le d\}$	16
$\omega_n^r(t)$	element of $FP_n(t)$ which interpolates action of $\Omega_{r,t}$ when $t \in \mathbb{Z}_{\geq 0}$	43
$\Omega_{r,d}$	sum of all r-cycles in S_d	41
$P_{n,m}$	set of partitions of $\{1, \ldots, n, 1', \ldots, m'\}$	15
P_λ	polynomial defined with hook lengths of $\lambda(d)$	38
$\mathrm{Proj}_{\mathcal{B}}$	projection functor $\underline{\mathrm{Rep}}(S_d; F) \to \mathcal{B}$	62
Ψ	set of all Young diagrams	13
Ψ_d	$\{\lambda \in \Psi \mid \|\lambda\| = d\}$	13
$\mathrm{Rep}(S_d; F)$	category of finite dimensional representations of S_d over F	14

$\underline{\mathrm{Rep}}(S_t;F)$	Deligne's category which interpolates $\mathrm{Rep}(S_t;F)$ when $t \in \mathbb{Z}_{\geq 0}$	25
$\mathrm{Rep}_0(S_d;F)$	full subcategory of $\mathrm{Rep}(S_d;F)$ with objects $V_d^{\otimes n}$	21
$\underline{\mathrm{Rep}}_0(S_t;F)$	category which interpolates $\mathrm{Rep}_0(S_t;F)$ when $t \in \mathbb{Z}_{\geq 0}$	21
$\underline{\mathrm{Res}}^{S_t}_{S_{t-1}}$	restriction functor $\underline{\mathrm{Rep}}(S_t;F) \to \underline{\mathrm{Rep}}(S_{t-1};F)$	63
S_d	symmetric group on $\{1,\ldots,d\}$	14
s_n	idempotent in $FP_n(0)$	59
t	element of F	21
T	indeterminate	27
$\mathcal{T}$	tensor category	3
tr	trace	25
v_i	element of fixed basis of V_d	14
$v_{\boldsymbol{i}}$	$v_{i_1} \otimes \cdots \otimes v_{i_n}$	16
V_d	natural d dimensional representation of S_d	14
x_π	alternative basis element in $FP_{n,m}$ corresponding to $\pi \in P_{n,m}$	17
x_0^1	unique element of $P_{1,0}$	60
x_n^{n+1}	$\mathrm{id}_n \otimes x_0^1$	60
x_{n+1}^n	$(x_n^{n+1})^\vee$	60
$\xi_{r,k}^\lambda$	scalar given by Forbenius' formula	45
Y	$\bigoplus_{\lambda \in \Psi} \mathbb{Z}\lambda$	66
$\mathbf{1}$	unit object	3
$\varnothing$	empty Young diagram $(0,\ldots)$	13
	empty partition diagram in $P_{0,0}$	16
$\prec$	total order on Young diagrams	13
	total order on $\overset{t}{\sim}$-equivalence classes	50

$(\ ,\ )$	trace form	33
$\overset{d}{\approx}$	equivalence of Young diagrams related to tensor products	73
$\overset{t}{\sim}$	equivalence of Young diagrams related to $\mu_\lambda(t)$	46
$\star$	"stack" partition diagrams	18
$\cdot$	concatenation of partition diagrams	19
	multiplication on Y related to Littlewood-Richardson coefficients	66
$*$	multiplication on Y related to tensor products in $\mathrm{Rep}(S_d; F)$	66
ⓣ	multiplication on Y related to tensor products in $\underline{\mathrm{Rep}}(S_t; F)$	66
$\vee$	dual object	4
	dual map	6

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