A GOAL PROGRAMMING MODEL FOR SCHEDULING NURSES WITH NONUNIFORM WORK SCHEDULES AND VARIABLE SHIFT LENGTHS

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To my parents and

my grandmother

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CHAPTER 1

INTRODUCTION

The environment in which decision-makers operate has become increasingly complex. To deal with the vast amount of data that needs to be assimilated and the many objectives that must be considered, managers have called upon mathematical modeling to help them make the best decisions. In many instances the use of these models is the only way that a person can hope to attain the best solution.

This paper examines an application of mathematical modeling to nurse scheduling. It is suggested in Chapter 3 that the Social Security Amendments of 1983 contain incentives that will make cost containment and efficiency a primary concern for hospital administrators and augment the demand for such models. In Chapter 2, a technique known as goal programming is shown to be superior to linear programming and is later used to formulate the scheduling model. The introduction demonstrates the need for models which consider the multiple, conflicting goals of management.

According to classical economic theory, the consumer and the businessman are perfectly rational people who always act in their own self-interest. This premise allows the economist to construct models and make predictions. Critics have argued that such an assumption is not realistic, because people may have several goals which they would like to achieve and no one is completely rational. The traditional response of economists is given by Milton Friedman: "Viewed as a body of substantive hypotheses, theory is to be judged by its predictive power for the class of phenomenon which it is intended to 'explain.'"[1] The assumption of the rational man may or may not be true; the only issue is whether this assumption makes accurate predictions, and economists find that it does.

But a theory that is a valid predictor of how people act on average does not necessarily apply to people on the individual level, because on an individual basis, one must deal with multiple goals. An individual firm is motivated by goals other than profit. A study by Martin Shubuk of the goals of twenty-five corporations found the following:[2]

Primary Goal	Number of Firms That Named it as a Primary Goal
Personnel Relations	21
Duties and responsibility to society in general	19
Consumers' Needs	19
Stockholders' Interests	16
Profit	13
Quality of Product	11
Technological Progress	9
Supplier Relations	9
Corporate Growth	8
Managerial Efficiency	7
Duties to Government	4
Distributor Relations	4
Prestige	2
Religion as an explicit guide to business	1

It is evident that managers of industrial organizations have many goals and that different decision-makers assign a different priority to these primary goals.

Nuclear power provides a lucid example of the way in which goals can come in conflict with one another. Two major goals of current Western society are finding new energy sources and preserving the environment. By pursuing a policy of nuclear power, energy is provided, but at the same time the environment is endangered because radioactive waste is created. Depending on how one values these two goals relative to one another, one either favors or opposes the development of nuclear power. The decisions one faces usually involve choosing between two social goods or two social evils.

The decision-maker is further limited by a lack of information, limited resources, and an inability to analyze the decision environment accurately. When a decision is reached, it may not be the absolute optimum--the point where all goals have been achieved. Usually, only a "satisficing" solution can be attained: not every goal has been completely achieved, but the firm has come as close as is possible. Modern decision analysis introduces a scientific approach that aids the decision-maker in achieving the best nonoptimum, satisficing value.

Decisions are limited by the many constraints that are placed upon them. There are two types of constraints which limit the options of decision-makers. System constraints are imposed by the decision environment. These include limits on time, manpower, the production capacity of equipment, government regulations, and collective bargaining agreements. Decision constraints are imposed by the organizational goal structure and can change as new policies are adopted. If these goals are ranked and weights are placed upon each one according to its importance, the decision analysis will indicate the best decision. Possible goals include: sales goals, profit goals, pollution control, labor stabilization, and goals for external growth.

A good model will take all of these factors into consideration. Chapter 2 contains explanations of two operations research techniques: linear programming and goal programming. A detailed description of these types of mathematical modeling will facilitate a better understanding of the nurse scheduling model.

CHAPTER 2

LINEAR AND GOAL PROGRAMMING

Section 2.1: An Introduction to Operations Research

Operations Research involves the application of scientific principles to decision-making. Its development as a formal discipline can be traced to World War II when these techniques were used by the British military in order to determine how best to use radar devices. The name "Operations Research" was coined because scientists were used to study operational problems. The allied forces also used operations research for strategic bombing, anti-submarine, and mining operations.

The growth of operations research since World War II has been due primarily to the development of the digital computer. Many of the techniques that were developed during the war also could be applied to industry problems after the war. Production, inventory, maintenance and scheduling techniques in particular were readily transferable. New models eventually were developed for applications in budgeting, capital, marketing and other areas. Today, Operations Research is being used in almost every field where complex decisions must be made. It is used not only by industries, but also by local and federal governments for public health, regional planning, transportation, education, meteorology and countless other areas.

Operation research is used by complex systems in order to determine the best way to allocate scarce resources. Economics, by comparison, is the analysis and description of institutional processes by which scarce resources are allocated. Operations Research determines how to make a decision whereas economics studies the decisions that are made.

The analysis of a problem involves many steps. The formulation phase is the most crucial, because if the problem is not set up correctly, the solution found may be the correct answer to the wrong problem. The system next must be studied and statistics gathered so that a model can be developed. An Operations Research model usually consists of a system of mathematical equations that contains all information that is relevant to the decision. The model, which is a microcosm of reality, can be manipulated to examine the results that would occur if the status quo were changed. Computational algorithms have been developed that will solve the system of equations and find the satisficing solution.

Section 2.2: The General Formulation of the Linear Program

The most popular version of mathematical modeling is called linear programming, short for "programming of interdependent activities in a linear structure." The term "programming" is used here to indicate planning, not the writing of instructions for a computer. In order to utilize linear programming, the equations must be linear, that is, $f(cx) = cf(x)$ where $f(x)$ is a function and c is a constant. The linear program consists of an objective function and a set of equations that "constrain" the objective function.

The purpose of the problem is usually to maximize or minimize the objective function subject to a set of constraints. The objective function consists of a linear combination of the (decision) variables. The constraints also are linear combinations of the decision variables, but they are expressed in terms of inequalities.

Let c, x and b be vectors, such that c, $x \varepsilon R^n$ and $b \varepsilon R^m$. Let A be a real mxn matrix. Then the general form of the linear program is:

$$\begin{aligned} &\max C'X \\ &\text{st } Ax \sim b \end{aligned}$$

where the symbol "'" indicates that the scalar (inner) product of the vectors C and X is being taken, and the symbol "$\sim$" is a vector of relations to be read as $\geqslant$, or =, or $\leqslant$ for each coordinate of A, x, and b. The notation "max" means maximize the following objective function, although this can be replaced by "min" when the equation is being minimized. "st" means subject to the following set of constraints.

An example will help to illustrate the form of the linear program. The following problem has three decision variables and two constraints. Suppose that:

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

and all of the inequalities are $\leqslant$. Then the problem becomes:

$$\max X_1 + 3X_2 + X_3$$

$$\begin{aligned} \text{st} \quad X_1 + 3X_2 + 4X_3 &\leqslant 1 \\ -2X_1 \qquad\quad + X_3 &\leqslant 3 \end{aligned}$$

Similar results can be derived for the case where the objective function is to be minimized.

Some examples will help to demonstrate how linear programming might be applied in various situations. Later, goal programming will be used to solve these same problems in order to compare and contrast the techniques and solutions that are derived by the two methods.

Section 2.3: Fred's Furniture Factory: An Example of Linear Programming

Fred's Furniture Factory produces two products: dressers and nightstands. He has 5 workers, each of whom works 40 hours per week. The following table contains information about the dressers and nightstands that is necessary to construct a model.

	Dressers	Nightstands
production time	10 workhours	10 workhours
profit	\$100	\$50
maximum number that can be sold	18	12

The problem is to construct a linear programming model that will tell Fred the product mix that will maximize his profits.

First, the variables are defined:

D = the number of dressers to be produced.
N = the number of nightstands to be produced.

The first constraint is the limited number of workhours available to the factory. Two hundred hours are available and it takes ten workhours to produce a good. Mathematically, this means that:

1) $$D + N \leqslant 20$$

There are other constraints because Fred does not wish to produce more dressers and nightstands than he can sell. These limits are:

2) $$D \leqslant 18$$

3) $$N \leqslant 12.$$

The objective function relates the profit level to D and N. Since the profit on these is one hundred dollars and fifty dollars, respectively:

$$100D + 50N$$

is the objective function. Now the constraints and the objective function are combined to make the following linear program:

$$\begin{array}{ll} \text{Max} & 100D + 50N \\ \text{st} & 1)\ D + N \leqslant 20 \\ & 2)\ D \leqslant 18 \\ & 3)\ N \leqslant 12 \\ & 4)\ D,\ N \geqslant 0 \end{array}$$

Linear programs are usually solved by computers using the simplex method, but a problem with two or fewer variables can be solved graphically.

An equation with an inequality represents a region rather than a line. $D + N \leqslant 20$ therefore represents all points between the origin and the line $D + N = 20$, as shown in Fig. 2.1. This line is bounded by the x and y-axes because D and N are required to be greater than zero.

The other two constraints place additional restrictions on the solution. The solution is now restricted to the area in which the three primary regions intersect, as shown in Figure 2.2. Fred will want to

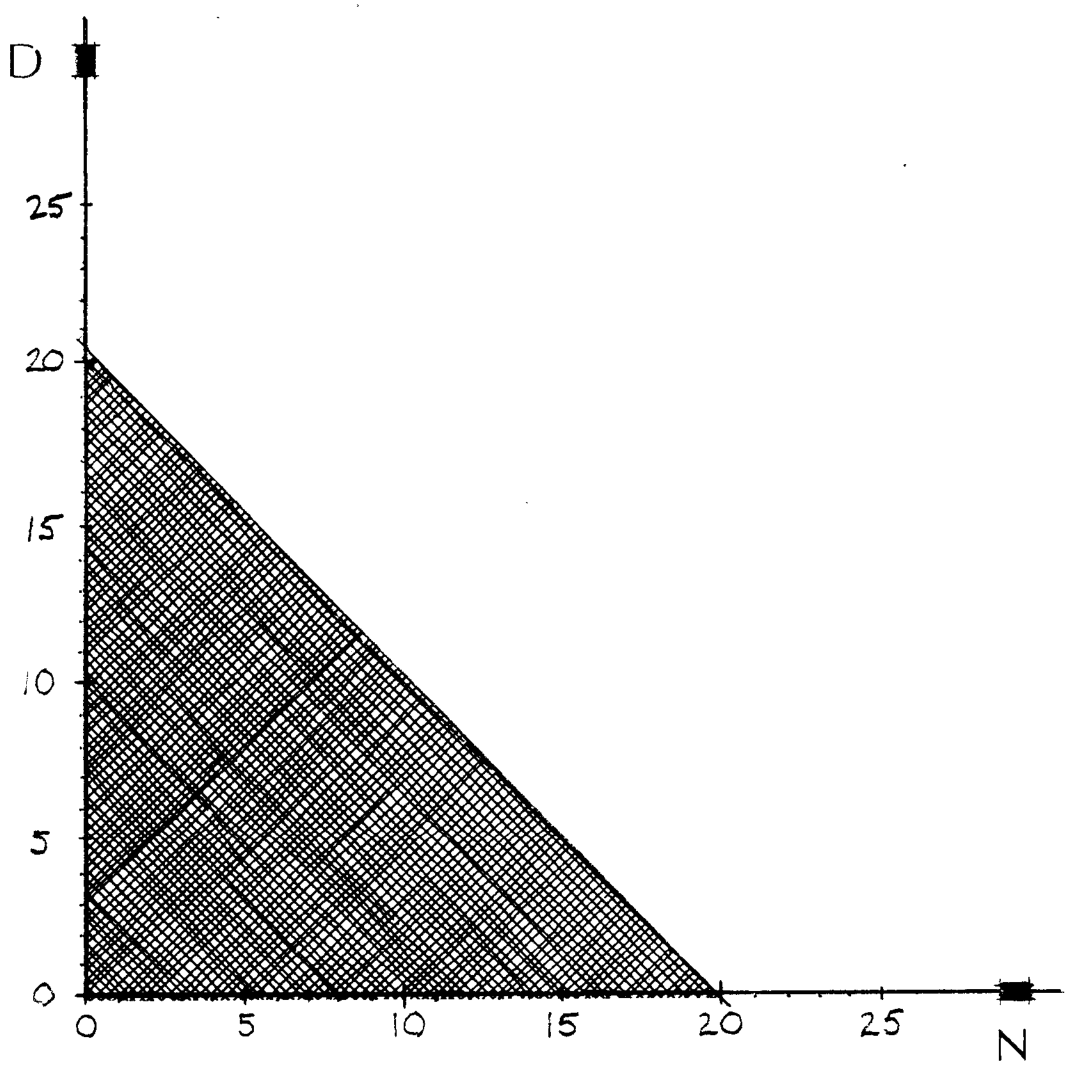

Figure 2.1

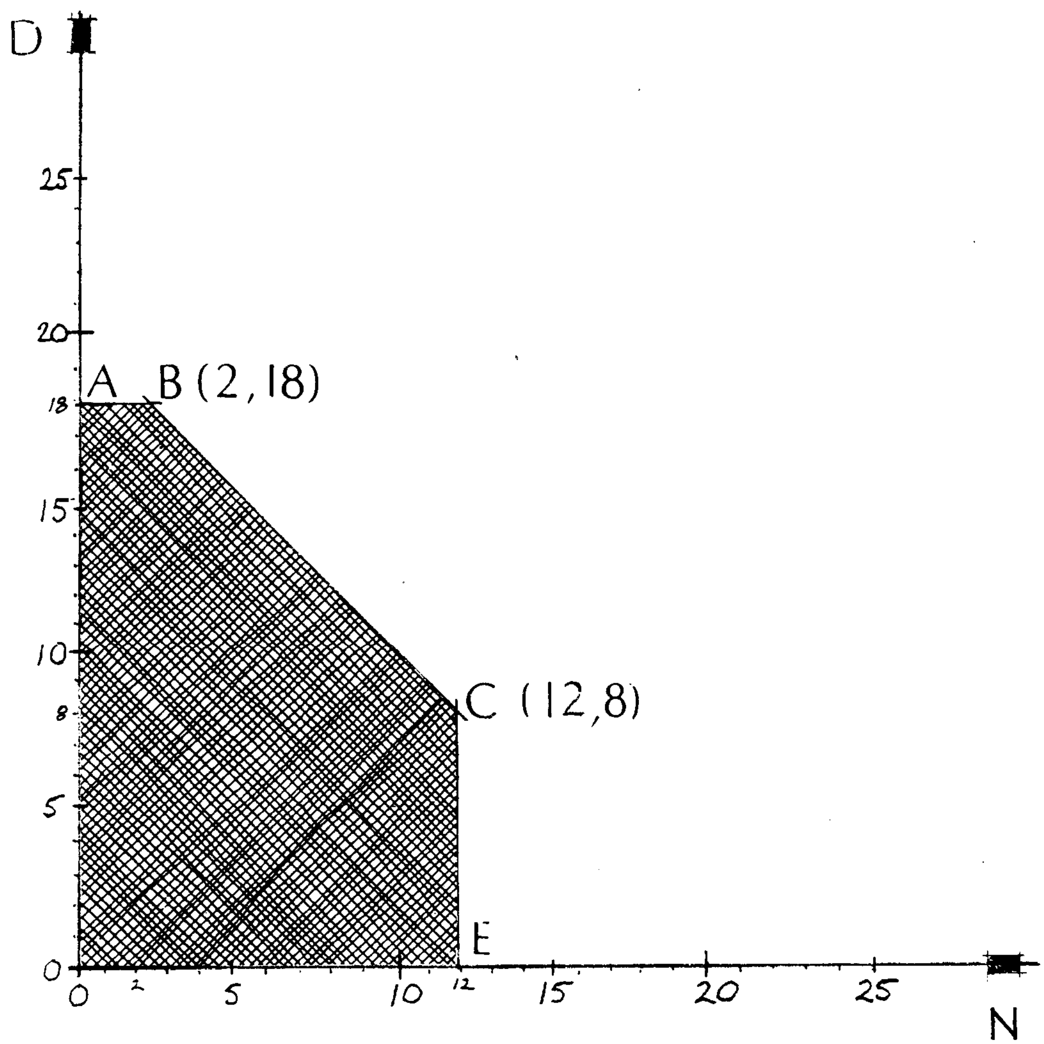

Figure 2.2

produce somewhere on the outer boundary of the region because points close to the origin minimize profit. The point that maximizes profit must be either A, B, C or E. If the coordinates at each point are substituted into the profit formula, 100D + 50N :

at point A, Profit = \$1800
at point B, Profit = \$1900
at point C, Profit = \$1400
at point E, Profit = \$ 600

Fred will therefore produce at point B, making 2 nightstands and 18 dressers. Note that if production took place at a point between B and C, the profit would be somewhere between \$1400 and \$1900, if it took place between A and B it would be between \$1800 and \$1900, and between C and E, profit would be between \$600 and \$1400.

Section 2.4: The Israeli Motor Company: An Infeasible Linear Programming Problem

Sometimes linear programming doesn't work very well, as the following example illustrates.

The Israeli Motor Company produces two models of cars: the Sabra and the Samaria. The following table contains the relevant statistics for these cars:

	Sabra	Samaria
mpg	35	20
profit	100,000 shekels	200,000 shekels
production capacity	80,000	60,000

The Israeli government has mandated that the overall gas mileage for all cars produced be 30 mpg. In order to maintain its market share, the company needs to sell 130,000 cars. How many Sabras and Samarias should be produced?

Let SAB = the number of Sabras sold and

SAM = the number of Samarias sold.

The first constraint is the average gas mileage requirement:

$$\frac{35\ SAB + 20\ SAM}{SAB + SAM} \geqslant 30$$

Cross-multiplying and subtracting from the right-hand side,

$$5\ SAB - 10\ SAM \geqslant 0$$

so

1) $$SAB - 2\ SAM \geqslant 0$$

The production capacity constraints can be simplified by expressing the right-hand side values in terms of thousands:

2) $$SAM \leqslant 60$$

3) $$SAB \leqslant 80.$$

In order to maintain the company's market share, the following constraint must be present:

4) $SAB + SAM \geq 130.$

Once again, the objective function will be maximized in order to make as much profit as possible. The complete linear program is:

$$\text{Max } 200{,}000 \text{ SAM} + 100{,}000 \text{ SAB}$$

st

$$\begin{array}{lr} 1) & SAB - 2\,SAM \geq 0 \\ 2) & SAM \leq 60 \\ 3) & SAB \leq 80 \\ 4) & SAB + SAM \geq 130 \\ 5) & SAB,\ SAM \geq 0 \end{array}$$

The graphical solution is shown in Figure 2.3. Unfortunately, the region defined by constraint 4 is not contained by constraints 1-3. The solution is said to be "infeasible," because no points in that region lie in the remaining solution space. Because all constraints have equal importance and because all conditions cannot be fulfilled, there is no way to solve the problem. This type of problem is the major reason for the development of goal programming.

Although linear programming is an excellent technique for many kinds of decision problems, it has some major weaknesses. It can be used only to solve one goal. Usually linear programming is used to determine how to maximize profits or minimize costs. However, in complex decision environments, managers usually have several conflicting goals. If goals conflict, there is no way to determine the proper course of action unless the goals are ranked. This is a second weakness of linear programming: all constraints are given equal weights when in reality, decision-makers may place different levels of importance on them. Finally, linear programming only allows for a cardinal solution.

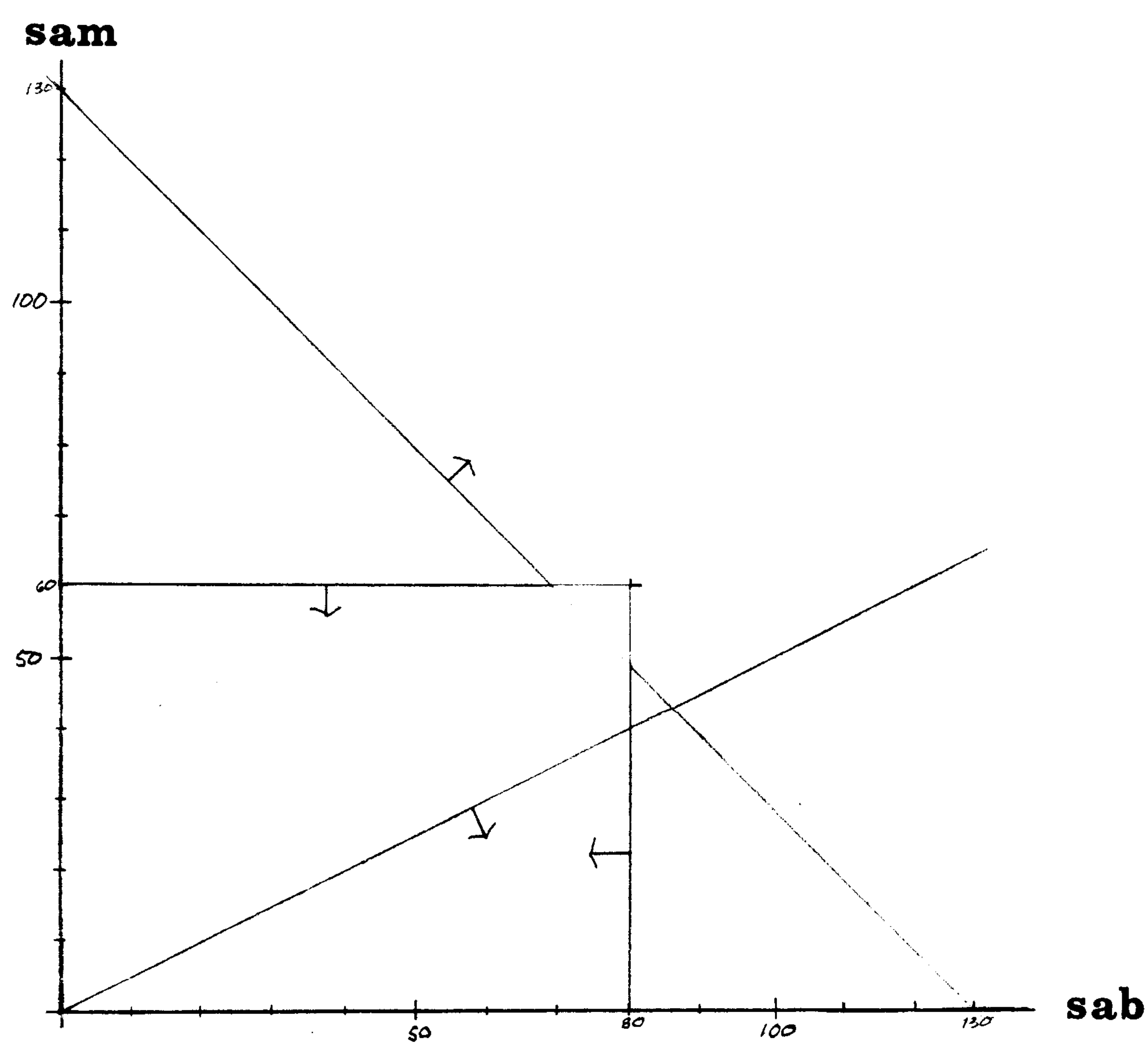

Figure 2.3

If a manager has a different goal than cost minimization or profit maximization, then an ordinal solution will be more appropriate than a cardinal one.

Section 2.5: The General Formulation of the Goal Program

These weaknesses can be overcome by goal programming, a technique that was invented by Charnes and Cooper in 1961 to solve infeasible linear programming problems. The first application used in business was published by Charnes, Cooper and Neihaus in 1968; the model was for manpower planning. Goal programming can be applied to every imaginable area. Some of these include: scheduling of employees, deciding where to locate fire stations, determining the best way to bus students in order to integrate a school district, deciding how much of a product to produce during different time periods, investment of financial portfolios, hospital management, and scheduling of airline flights. In short, there is a use for goal programming in almost any kind of complex decision-making process.

The constraints in a goal program are not absolute as in a linear program. Consequently, a more realistic model can be constructed because an organization may have goals that are of very high priority but whose nonachievement can be tolerated. The goal programming model also allows for the situation in which goals have different priorities. If goals can be enumerated and prioritized then a goal programming model can be formulated. A goal programming constraint is written in the form:

$$\left(\sum_{i=1}^{n} X_i\right) + d^- - d^+ = 5$$

The variables of the form X_i are known as decision variables and the variables d^- and d^+ are known as "deviational variables." In this instance, it is desirable that $\sum_{i=1}^{n} X_i = 5$. d^- represents underachievement of this objective and d^+ represents overachievement. When $d^+ > 0$, $d^- = 0$ and when $d^- > 0$, $d^+ = 0$. Suppose it turns out that $\sum_{i=1}^{n} X_i = 6$ is the closest that the model can come to achieving the objective. Then $d^+ = 1$ and $d^- = 0$. If, however, $\sum_{i=1}^{n} X_i = 4$, then $d^- = 1$ and $d^+ = 0$.

The goal program begins with the goal that has been given the highest priority and minimizes d^+ and d^- as much as possible. That is, the deviation between what the organization would like to achieve and what it is possible to achieve is minimized. The same is then done for the second goal given the constraints which the minimization of the first goal's deviational variables has added to the model. Usually the first two or three goals can be completely attained. But eventually lower ranked goals come into conflict with previous goals and cannot be entirely achieved.

The objective function contains constants, deviational variables and "preemptive" priority factors. These priority factors, usually denoted as P_i, indicate the relative importance that is being attached to the minimization of each deviational variable. For each P_i, $P_i >>> P_{i+1}$, meaning that there is no constant, s, for which $sP_{i+1} \geqslant P_i$.

The values a_i, represent constants that take values other than 1 if the model requires different weights at the same priority level.

The general form of the model is too mathematically complex to serve any useful purpose in this paper. The following is the general form for a goal program with two goals, two constraints, and two decision variables:

$$\begin{aligned} \text{Min } Z = {} & a_1P_1d_1^- + a_2P_2d_1^- + a_3P_1d_1^+ + a_4P_2d_1^+ + a_5P_1d_2^- \\ & + a_6P_2d_2^- + a_7P_1d_2^+ + a_8P_2d_2^+ \\ \text{st } & C_{11}X_1 + C_{12}X_2 + d_1^- - d_1^+ = b_1 \\ & C_{21}X_1 + C_{22}X_2 + d_2^- - d_2^+ = b_2. \end{aligned}$$

where all variables are real.

Suppose $a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0$, $a_1 = a_8 = 1$, $C_{11} = 1$, $C_{12} = 0$, $C_{21} = 2$, $C_{22} = -1$, $b_1 = 1$, and $b_2 = 2$.

Then the problem becomes:

$$\begin{aligned} \text{Min } Z = {} & P_1d_1^- + P_2d_2^+ \\ \text{st } & X_1 + d_1^- - d_1^+ = 1 \\ & 2X_1 - X_2 + d_2^- - d_2^+ = 2 \end{aligned}$$

The top priority is to minimize d_1^- and the second priority is to minimize d_2^+. Since d_2^- and d_1^+ are not in the objective function, the minimization of these variables is not a goal. If one of the deviational variables could not be allowed to be positive, it would be omitted from the constraints.

Section 2.6: Fred's Furniture Factory: An Example of Goal Programming

The two problems solved above with linear programming also can be solved with goal programming. The first example involved Fred's Furniture Factory. Suppose instead of maximization of profits, Fred has developed the following three goals in descending order of priority:

1. Employment stability: 200 hours of work should be performed each week.
2. Sales goal: 18 dressers and 12 nightstands.
3. Minimize overtime: more than 200 hours of work can be performed, but this is undesirable.

System Constraint

1. Only 20 items can be produced in 200 hours.

The first constraint can be written as:

1) $$D + N + d_1^- - d_1^+ = 20$$

The second constraint reflects the sales goal of D = 18. Since it is impossible to sell more than 18 dressers per week, Fred does not want a model that would allow for such overproduction. By eliminating d_2^+ from the constraint, this goal is achieved.

2) $$D + d_2^- = 18.$$

Because Fred does not wish to produce more than 12 nightstands per week, d_3^+ is omitted from the third constraint:

3) $$N + d_3^- = 12$$

4) $$D, N, d_1^-, d_2^-, d_3^-, d_1^+ \geq 0.$$

Employment stability will be achieved if each worker is able to work at least a full forty hour work week. If this is the case, then at least twenty items will be manufactured. So the first goal is to minimize d_1^-. The first term of the objective function is $P_1 d_1^-$.

The sales goal will be achieved if 18 dressers and 12 nightstands are produced. Since a positive value for d_2^- or d_3^- would indicate an underachievement of this goal, these two deviational variables must be minimized. Fred is twice as concerned that the sales goal for dressers be realized because he makes twice as much profit on dressers. The second goal will be expressed in the objective function as:

$$2P_2 d_2^- + P_2 d_3^-.$$

Finally, Fred would like to minimize the overtime of his workers. When they work overtime, they produce more than 20 pieces of furniture so d_1^+ is positive. The third goal is to minimize d_1^+: $P_3 d_1^+$. The entire goal program is:

$$\text{Min } Z = P_1 d_1^- + 2P_2 d_2^- + P_2 d_3^- + P_3 d_1^+$$

st

1) $D + N + d_1^- - d_1^+ = 20$

2) $D + d_2^- = 18$

3) $N + d_3^- = 12$

4) $D, N, d_1^-, d_2^-, d_3^-, d_1^+ \geqslant 0.$

The graphical solution to this system of equations is different from the solution derived for linear programming. The constraints are graphed in Figure 2.4. The first priority is to minimize d_1^-. So the solution will not be below the line D + N = 20. The second goal is to produce eighteen dressers and twelve nightstands. This goal is completely attained, because it is not in conflict with the first goal. The third goal is to produce at or below the line D + N = 20. This goal is in direct conflict with the second goal. d_1^+ will be equal to ten, meaning that there will be one hundred hours of overtime. The satisficing production level will be eighteen dressers and twelve nightstands. $d_1^- = d_3^- = d_2^- = 0$, $d_1^+ = 10$.

Now suppose that the priorities change so that minimizing overtime is more important than the sales goal. The first goal is still to produce at or above the line D + N = 20. But now the second priority is to produce at or below the line D + N = 20 so the solution will be found on this line. The sales goal is in direct conflict with the goal of reducing overtime and will not be completely achieved. Because there is twice as much profit on dressers, Fred wants to produce as many of these as possible before manufacturing nightstands. The satisficing point is now eighteen dressers and two nightstands--this is the same solution which the linear program identified as the optimum. The

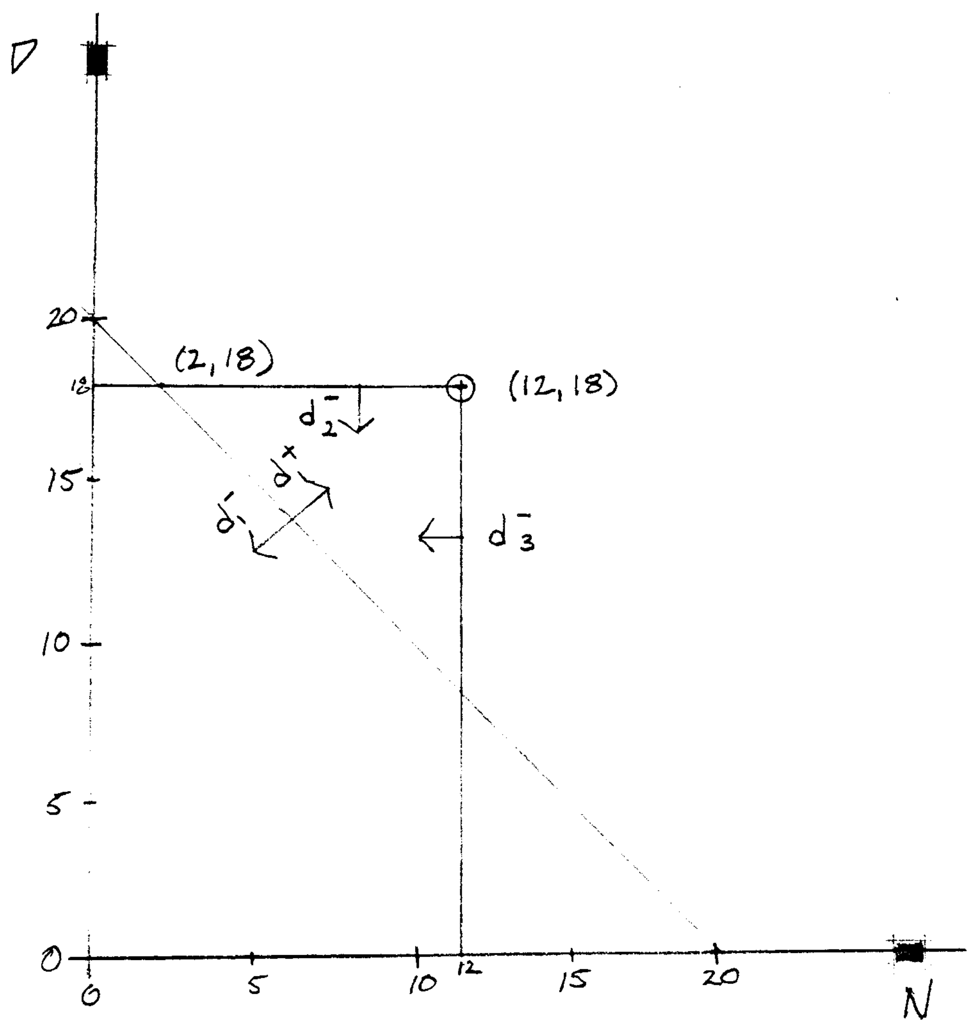

Figure 2.4

solution which is derived from the goal programming model may or may not be the linear programming solution because the way in which the decision-maker ranks the organization's goals determines the solution.

Section 2.7. The Israeli Motor Company: A Goal Programming Solution to an Infeasible Linear Programming Problem

Goal programming also will successfully solve the automobile problem that linear programming was unable to handle. The Israeli Motor Company has prioritized its management goals as follows:

1. Meet the government's average gas mileage requirement of 30 mpg.
2. Achieve the sales goal of 130,000.
3. Maximize profit.

The first constraint of the goal program is the same as the first constraint of the linear program except the inequality is replaced by an equation with deviational variables:

1) $$SAB - 2SAM + d_1^- - d_1^+ = 0$$

A system constraint caused by production limits requires that a maximum of 60,000 Samarias and 80,000 Sabras be produced. The model should allow for underproduction but not overproduction. Thus d_2^+ and d_3^+ are omitted:

2) $$SAB + d_2^- = 80$$

3) $$SAM + d_3^- = 60$$

The sales goal can be written:

4) $$SAB + SAM + d_4^- - d_4^+ = 130.$$

The profit goal does not require an additional constraint because it will be incorporated into the objective function.

The first priority level is: $P_1d_1^-$ because if d_1^- is positive then the average mileage required is not achieved. The company loses nothing if its cars average over 30 mpg so d_1^+ need not be minimized.

The second priority is to minimize the underachievement of the sales goal. Overachievement is desirable so d_4^+ need not be minimized. The second term of the objective function is: $P_2d_4^-$.

The third priority, maximizing profit, can be achieved by selling as many cars as possible. Since the profit on Samarias is twice as much as that on Sabras, the third priority level is:

$$P_3d_2^- + 2P_3d_3^-$$

The resulting goal program is:

$$\text{Min } Z = P_1d_1^- + P_2d_4^- + P_3d_2^- + 2P_3d_3^-$$

st

1) $SAB - 2SAM + d_1^- - d_1^+ = 0$

2) $SAB + d_2^- = 80$

3) $SAM + d_3^- = 60$

4) $SAB + SAM + d_4^- - d_4^+ = 130$

5) $SAB, SAM, d_1^-, d_2^-, d_3^-, d_4^-, d_1^+, d_4^+ \geqslant 0$

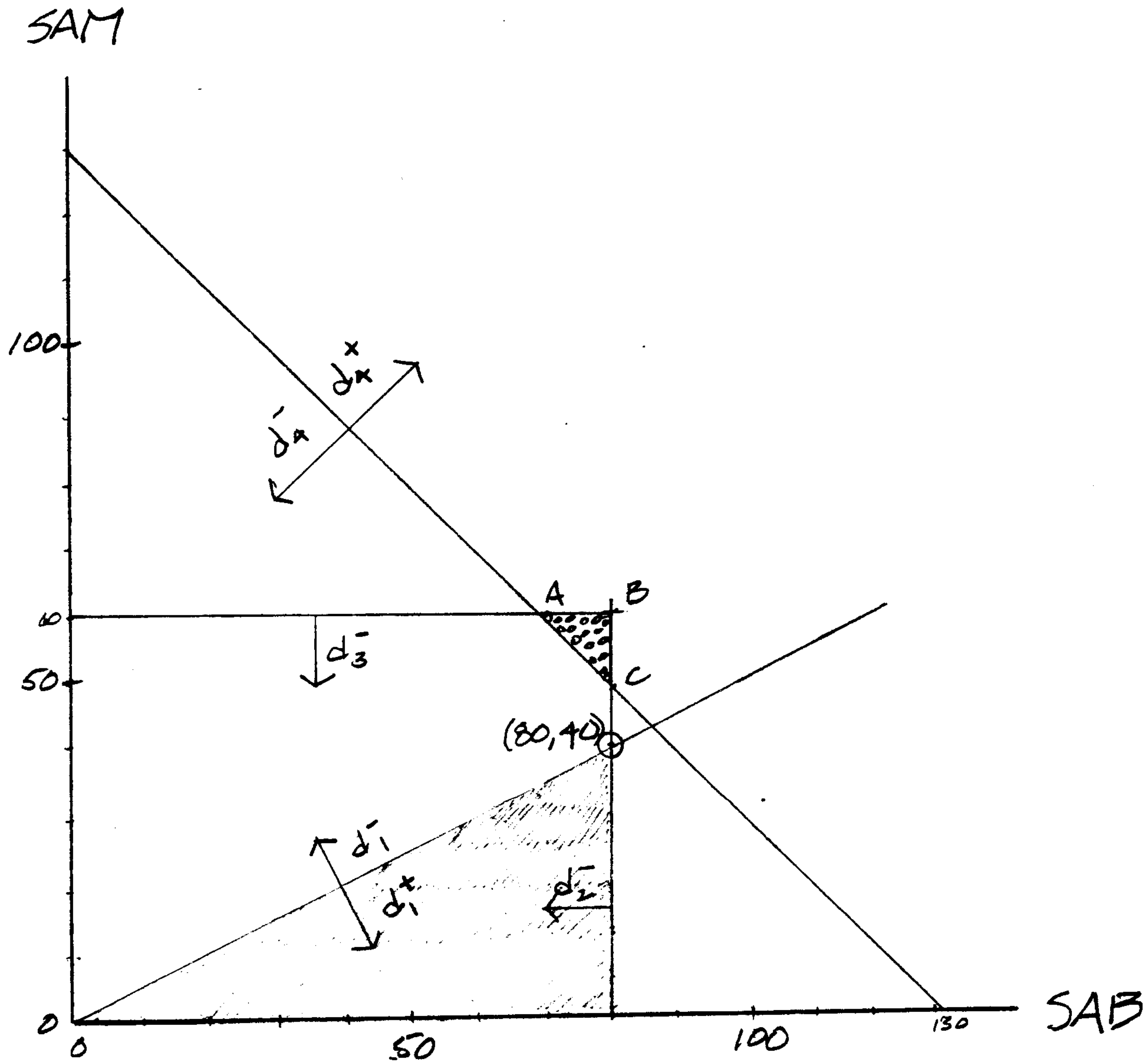

Figure 2.5

The shaded area fulfills the first priority given the constraints that SAB $\leqslant$ 80 and SAM $\leqslant$ 60. The second and third constraints make this condition necessary because d_2^+ and d_3^+ are omitted. It now will not be possible to set d_4^- equal to zero, however d_4^- can be minimized at (80,40). This also is the point that maximizes profit. So 80 Sabras and 40 Samarias will be manufactured with $d_1^- = d_1^+ = d_2^- = d_4^+ = 0$, $d_3^- = 20$ and $d_4^- = 10$. Goal programming allows a solution for the infeasible linear programming problem.

If the average gas mileage requirement were only a voluntary government program, then the objective function might be reordered:

$$\text{Min } Z = P_1 d_4^- + P_2 d_1^- + P_3 d_2^- + 2P_3 d_3^-.$$

Now the first priority is to produce above the line SAB + SAM = 130. But because SAB $\leqslant$ 80 and SAM $\leqslant$ 60, this leaves the region filled in above with the circles: ΔABC. It will not be possible to average 30 mph. The third goal, maximization of profit, will force the solution to B(80,60). At this point, $d_1^+ = d_2^- = d_3^- = 0$, $d_4^+ = 10$ and $d_1^- = 20$. Eighty Sabras and 60 Samarias are produced.

Models are, by their very nature, simplified forms of reality. Goal programming models are both more realistic and more complex than linear programming models. Yet the goal programming technique remains simple enough to be applied to most of the situations in which linear programming has been used. In addition, goal programming also can be used in many instances in which linear programming cannot--the Israeli Motor Company, for example. Goal programming is the superior method for organizations with multiple, conflicting goals.