COHOMOLOGY OF THE ORLIK-SOLOMON ALGEBRAS

by

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The Orlik-Solomon algebra of a hyperplane arrangement first appeared from the Brieskorn and Orlik-Solomon theorems as the cohomology of the complement of this arrangement (if the ground field is complex). Later, it was discovered that this algebra plays an important role in many other problems. In particular, define the cohomology of an Orlik-Solomon algebra as that of the complex formed by its homogeneous components with the differential defined via multiplication by an element of degree one. Cohomology of the Orlik-Solomon algebra is mostly studied in dimension one, and very little is known about the higher dimensions. We study this cohomology in higher dimensions.

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CHAPTER I

INTRODUCTION

The theory of hyperplane arrangements is an area of mathematics with applications in algebra, combinatorics, topology, analysis (hypergeometric functions), and physics (KZ-equations). The allure of hyperplane arrangements lies both in the straightforward definitions needed to begin studying the topic but, more importantly, in the ability to pose interesting, yet understandable, problems and examples. We therefore begin our discussion with two motivating examples.

EXAMPLE 1.1. *It is not a difficult task to determine that removing* n *distinct* points from the real line leaves $n+1$ regions. However, by raising the dimension just one, determining the number of regions which remain in the plane after removing n lines is dependent on the lines themselves and not merely n. For instance, removing the collection of lines in $\mathbb{R}^2$ given by $\{x = 0, y = 0, x + y = 0\}$ leaves 6 regions. But the collection $\{x = 0, y = 0, x + y = 1\}$ leaves 7 regions when removed from the plane. This question, of course, can be raised to any dimension: given a collection of codimension one affine spaces in $\mathbb{R}^\ell$, how many regions are left when this collection is removed from $\mathbb{R}^\ell$?

In Example 1.1, we considered a finite collection of affine subspaces of codimension one in $\mathbb{R}^\ell$. More generally, we can take F to be be any field and define the same notion.

DEFINITION 1.2. Let F be a field. A hyperplane is an affine subspace of *codimension one in* F^ℓ. A hyperplane arrangement *in* F^ℓ *is a finite collection of* hyperplanes in F^ℓ, written $\mathcal{A} = \{H_1, \ldots, H_n\}$.

EXAMPLE 1.3. We now switch our attention to an arrangement of hyperplanes in $\mathbb{C}^\ell$. In Example 1.1, we considered the space obtained by removing the hyperplanes from $\mathbb{R}^\ell$. Similarly, we define the complement space $M := \mathbb{C}^\ell \setminus \cup_{i=1}^n H$. Momentarily, let $\ell = 1$ and we see the hyperplanes of $\mathbb{C}$ are points in the complex plane (the hyperplanes have complex codimension one); hence, M is path connected. In general, for any hyperplane arrangement in $\mathbb{C}^\ell$ with $\ell \geq 1$, we have M is a path connected space. So, the question of the number of connected components of M is a trivial question. However, one can consider the cohomology algebra with coefficients in a commutative ring $\mathcal{K}$, denoted $H^*(M, \mathcal{K})$ and ask the question: can $H^*(M, \mathcal{K})$ be represented by generators and relations related to the collection of hyperplanes?

Allowing Example 1.1 to guide and motivate us, it is apparent the intersections of the hyperplanes play an important role as to the number of components of the complement space; in fact, the pattern of intersections of the hyperplanes is the determining factor. It is also apparent in Example 1.3 that the pattern of intersections of the hyperplanes is pivotal to understanding $H^*(M, \mathcal{K})$. Encoding the pattern of intersections of the hyperplanes in a combinatorial object is the purpose of the following definition, given first by Zaslavsky in [14].

DEFINITION 1.4. Let $\mathcal{A}$ be an arrangement of hyperplanes in $V = F^\ell$. We define the partially ordered set $L(\mathcal{A})$ with objects given by $\cap_{H\in\mathcal{B}} H$ for $\mathcal{B} \subseteq \mathcal{A}$ and $\cap_{H\in\mathcal{B}} H \neq \emptyset$; order the objects of $L(\mathcal{A})$ opposite to inclusion. Notice $\emptyset \subseteq \mathcal{A}$ gives $V \in L(\mathcal{A})$ with $V \leq X$ for all $X \in L(\mathcal{A})$. For $X \in L(\mathcal{A})$, we define $\operatorname{rank}(X) := \operatorname{codim} X$. We define $\operatorname{rank}(\mathcal{A}) := \max_{X\in L(\mathcal{A})} \operatorname{rank}(X)$.

In Example 1.3, we considered the complement of the hyperplanes in $\mathbb{C}^\ell$ and denoted this space M. The problem of expressing $H^*(M, \mathcal{K})$ in terms of generators and relations was first studied by Arnold [2] in the case $\mathcal{A}$ was the braid arrangement and $\mathcal{K} = \mathbb{C}$; that is, $\mathcal{A}$ was the collection of hyperplanes $\{x_i - x_j : 1 \leq i < j \leq \ell\}$.

This problem was later studied by Brieskorn [4] for an arbitrary arrangement. Orlik and Solomon [11] have found a purely algebraic characterization of $H^*(M,\mathcal{K})$.

These results can be briefly summarized as follows. An algebra $A(\mathcal{A})$ (referred to as the Orlik-Solomon algebra) over $\mathcal{K}$ is constructed in terms of generators and relations using only $L(\mathcal{A})$. This is a graded algebra with $A(\mathcal{A}) \cong H^*(M,\mathcal{K})$. Hence, in Example 1.3, $H^*(M,\mathcal{K})$ can be determined by $L(\mathcal{A})$.

The Orilik-Solomon algebra $A(\mathcal{A})$ can also be used to answer the question posed in Example 1.1. Zaslavsky has proven in [15] for a hyperplane arrangement in $\mathbb{R}^\ell$, the number of regions of the complement space is the sum of the dimensions of the homogeneous components of $A(\mathcal{A})$; that is, $\sum_{i=1}^{\ell} \dim A_i(\mathcal{A})$.

The answers to the questions posed in Example 1.3 and Example 1.1 are important results in that topological invariants of the complement space were expressed in term of combinatorics. Indeed, a central question in the theory of hyperplane arrangements is the problem of expressing topological invariants of the complement space in terms of combinatorics. In this manner, it is a natural question then to consider a generalization of $H^*(M,\mathcal{K})$ to cohomology with local coefficients.

For $a \in A_1(\mathcal{A})$, one can define a local coefficient system $\mathcal{L}(a)$. It turns out that $H^*(M,\mathcal{L}(a))$ relates closely to the cohomology of the Orlik-Solomon algebra. The connection between $H^*(M,\mathcal{L}(a))$ and the cohomology of the Orlik-Solomon algebra has been studied in many papers, for instance [8].

The cohomology of the Orlik-Solomon algebra is defined below. For a hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$, we let $\{a_i : H_i \in \mathcal{A}\}$ denote a basis for $A_1(\mathcal{A})$. This basis is discussed in Chapter III.

DEFINITION 1.5. We construct a cochain complex on the graded linear space $A(\mathcal{A})$ as follows. Let $a \in A_1(\mathcal{A})$ with $a = \sum_{i=1}^{n} \lambda_i a_i$ for $\lambda_i \in \mathcal{K}$. Multiplication by

a giving the differential $d_k : A_k(\mathcal{A}) \xrightarrow{a\cdot} A_{k+1}(\mathcal{A})$ forms a complex $(A(\mathcal{A}), a)$. The cohomology of this complex is said to be the cohomology of the Orlik-Solomon algebra and is denoted $H^*(A(\mathcal{A}), a)$.

Recently, there have been many results concerning $\dim H^1(A(\mathcal{A}), a)$. In the case char $\mathcal{K} = 0$, it has been shown in [8] that $\dim H^1(A(\mathcal{A}), a)$ can be determined by a particular set of elements from $L(\mathcal{A})$.

However, little is known about the higher dimensions $H^p(A(\mathcal{A}), a)$ for $p > 1$ [13], and this is what our work is devoted to.

Here is an outline of the thesis.

We begin Chapter II by discussing basic constructions and notions of arrangements. We define some of them here as these definitions are needed for the statements of the main theorems.

DEFINITION 1.6. A hyperplane arrangement $\mathcal{A}$ is central if $\cap_{H \in \mathcal{A}} H \neq \emptyset$.

DEFINITION 1.7. Let $\mathcal{A}_1$ be an arrangement in $V_1 \cong F^\ell$, and let $\mathcal{A}_2$ be an arrangement in $V_2 \cong F^k$. Let $V = V_1 \oplus V_2$. Define the product arrangement by

$$\mathcal{A}_1 \times \mathcal{A}_2 = \{H_1 \oplus V_2 : \ H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 : \ H_2 \in V_2\}.$$

DEFINITION 1.8. Let $\mathcal{A}$ be an arrangement in V. We say $\mathcal{A}$ is reducible if it is linearly isomorphic to a product of two nontrivial arrangements.

In Chapter III, the Orlik-Solomon algebra is defined. The definition of $A(\mathcal{A})$ is presented here as can be found in [12].

DEFINITION 1.9. Let $\mathcal{A} = \{H_1, ..., H_n\}$ be a hyperplane arrangement in $V = F^\ell$ for some field F. We fix an order on $\mathcal{A}$; that is, for hyperplanes H_i and H_j in $\mathcal{A}$, we have $H_i < H_j$ if and only if $i < j$.

Let $\mathcal{K}$ be a commutative ring. Let E_1 be the linear space over $\mathcal{K}$ on n generators. Let $E(\mathcal{A}) := \Lambda(E_1)$ be the exterior algebra on E_1. We have $E(\mathcal{A}) = \bigoplus_{p\geq 0} E_p$ is a graded algebra over $\mathcal{K}$. The standard $\mathcal{K}$-basis for E_p is given by

$$\{e_{i_1}\cdots e_{i_p} :\ 1 \leq i_1 < \ldots < i_p \leq p\}.$$

Any ordered subset $S = \{H_{i_1}, ..., H_{i_p}\}$ of $\mathcal{A}$ corresponds to an element $e_S := e_{i_1} \cdots e_{i_p}$ in $E(\mathcal{A})$.

DEFINITION 1.10. We define the map $\partial :\ E(\mathcal{A}) \to E(\mathcal{A})$ via the usual differential. That is,

$$\partial(1) :=0,$$

$$\partial(e_i) :=1,$$

$$\text{and for } p \geq 2,\ \partial(e_{i_1}\cdots e_{i_p}) := \sum_{k=1}^{p}(-1)^{k-1}e_{i_1}\cdots \hat{e}_{i_k}\cdots e_{i_p}.$$

DEFINITION 1.11. Let $S = \{H_{i_1}, ..., H_{i_p}\}$ be a subset of $\mathcal{A}$. We say S is dependent if $\cap S \neq \emptyset$ and $\text{rank}(\cap S) < |S|$.

DEFINITION 1.12. We define $I(\mathcal{A})$ to be the ideal of $E(\mathcal{A})$ which is generated by

$$\{\partial(e_S) :\ S \text{ is dependent }\} \cup \{e_S :\ \cap S = \emptyset\}.$$

DEFINITION 1.13. The Orlik-Solomon algebra, $A(\mathcal{A})$, is defined as

$$A(\mathcal{A}) := E(\mathcal{A})/I(\mathcal{A}).$$

Let $\pi :\ E(\mathcal{A}) \to A(\mathcal{A})$ be the canonical projection. We write a_S to represent the image of e_S under π.

In Chapter III, a linear basis for $A(\mathcal{A})$ is defined. We show this basis can be obtained as normal forms to a Gröbner basis for $I(\mathcal{A})$. We give conditions for when

$I(\mathcal{A})$ has a quadratic Gröbner basis; this is dependent not only on $\mathcal{A}$ but on the order of the hyperplanes in $\mathcal{A}$. In this case, we say $\mathcal{A}$ is quadratic with respect to the order.

In the last section of Chapter III, we define the cohomology of the Orlik-Solomon algebra (see Definition 1.5) and recall some results. For a central hyperplane arrangement $\mathcal{A}$ and $\sum_{i=1}^{n} \lambda_i a_i$ with $\sum_{i=1}^{n} \lambda_i \neq 0$, we have $H^*(A(\mathcal{A}), a) = 0$, see [13]. Therefore, we may assume for $\mathcal{A}$ central that $\sum_{i=1}^{n} \lambda_i = 0$.

For char $\mathcal{K} = 0$, it has been shown in [8] that $\dim H^1(A(\mathcal{A}), a)$ is determined by the set

$$\mathcal{X}(a) := \{X \in L(\mathcal{A}) : \ \operatorname{rank}(X) = 2, |X| > 2, \sum_{H_i < X} \lambda_i = 0, \sum_{H_i < X} \lambda_i a_i \neq 0\}.$$

It would be interesting to know whether $\dim H^p(A(\mathcal{A}), a)$ is determined combinatorially and, if so, whether $\mathcal{X}(a)$ determines $\dim H^p(A(\mathcal{A}), a)$ for any p. Towards this end, we proceed by determining when $H^*(A(\mathcal{A}), a) = 0$.

This problem is a particular case of a more general problem of skew commutative algebras, i.e. studying modules over an exterior algebra E, see [1]. If M is such a module, then $a \in E_1$ is said to be regular on M if and only if

$$H^*(M, a) = \{x \in M; \ ax = 0\}/aM = 0;$$

otherwise, a is said to be singular. The set of all singular elements is called a singular variety of M, denoted $\operatorname{Sing}(M)$. So we will compute $\operatorname{Sing}(A(\mathcal{A}))$ as $E(\mathcal{A})$-modules.

In Chapter IV, we let $\mathcal{K} = \mathbb{R}$ or $\mathbb{C}$ and establish a necessary and sufficient condition for $H^*(A(\mathcal{A}), a) = 0$. We show $H^*(A(\mathcal{A}), a) = 0$ if and only if $H^\ell(A(\mathcal{A}), a) = 0$, where $\operatorname{rank}(\mathcal{A}) = \ell$. The following theorem, which is one of the main results of this paper, gives a necessary and sufficient condition for $H^\ell(A(\mathcal{A}), a) = 0$.

THEOREM 4.3.11. Let $\mathcal{A}$ be an affine ℓ-arrangement. We may write

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k \times \mathcal{B},$$

where $\mathcal{A}_j$ are each central and $\mathcal{B}$ not central and they contain no proper central factors. Let $a \in A_1(\mathcal{A})$. We have $\dim H^\ell(A(\mathcal{A}), a) \neq 0$ if and only if $\sum_{H_i \in \mathcal{A}_j} \lambda_i = 0$ for all j.

In Chapter V, we need more definitions (see the chapter for more details). In particular, we deal with a famous class of arrangements called supersolvable arrangements. We define supersolvable arrangements here, see §2.2 and §3.2 for examples and some equaivalent definitions.

Assume $\mathcal{A}$ is central. A pair $(X, Y) \in L(\mathcal{A}) \times L(\mathcal{A})$ is called a modular pair if for all $Z \in L(\mathcal{A})$ with $Z \leq Y$

$$Z \vee (X \wedge Y) = (Z \vee X) \wedge Y.$$

An element $X \in L(\mathcal{A})$ is called modular if (X, Y) is a modular pair for all $Y \in L(\mathcal{A})$. We call $\mathcal{A}$ supersolvable if $L(\mathcal{A})$ has a maximal chain of modular elements

$$V = X_0 < X_1 < \cdots < X_\ell = \cap_{H \in \mathcal{A}} H.$$

If $\mathcal{A}$ is supersolvable, we say the order on the hyperplanes respects the supersolvable structure if for a maximal modular chain

$$V = X_0 < X_1 < \cdots < X_\ell = \cap_{H \in \mathcal{A}} H$$

in $L(\mathcal{A})$ we have

1. X_1 is the smallest hyperplane, i.e. $X_1 = H_1$
2. For $i > 1$, we have $X_i = \cap_{j=1}^{n_i} H_j$ and if a hyperplane $H < X_i$ then $H \in \{H_1, \ldots, H_{n_i}\}$.

For $\mathcal{A}$ supersolvable, if the order respects the supersolvable structure then the respective Gröbner basis is quadratic. We use this characterization throughout Chapter V. The following is an assumption maintained thoughout Chapter V.

CONDITION A. Let $\mathcal{A}$ be a hyperplane arrangement with $\cap_{i=1}^{n} H_i \neq \emptyset$, and assume $\mathcal{A}$ is supersolvable. Fix $X \in L(\mathcal{A})$ with $\mathrm{rank}(X) = 2$ and X a member of a maximal modular chain in $L(\mathcal{A})$. Fix an order on the hyperplanes so that the order respects the supersolvable structure.

We consider $a \in A_1(\mathcal{A})$ so $a = \sum_{H_i < X} \lambda_i a_i$. Again, we assume $a \neq 0$ and $\sum_{i=1}^{n} \lambda_i = 0$. We call such an a *concentrated* under X.

We show $\dim H^k(A(\mathcal{A}), a)$ is determined combinatorially by another main result of this paper.

THEOREM 5.1.11. Let $\mathcal{A}$ and $X \in L(\mathcal{A})$ be as in Condition A. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under X. Then we can compute the Hilbert series for $H^*(A(\mathcal{A}), a)$ in terms of the Hilbert series for $A(\mathcal{A})$ as follows:

$$H(H^*(A(\mathcal{A}), a), t) = \frac{t(n_X - 2)}{1 + t(n_X - 1)} H(A(\mathcal{A}), t).$$

In §5.2, we study the kernel, $Z(a) = \oplus Z_i(a)$, of the chain complex $(A(\mathcal{A}), a)$ as an ideal of $A(\mathcal{A})$. We do this with the idea in mind that if $Z_k(a) = A_k(\mathcal{A}) \cdot Z_1(a)$, then $\mathcal{X}(a)$ together with $\dim A_k(\mathcal{A})$ will determine $\dim Z_k(a)$. We show in the case $\mathcal{A}$ and $X \in L(\mathcal{A})$ satisfy Condition A with a concentrated under X, this result holds, except for the top dimension. This is given in the following result.

THEOREM 5.2.9. Suppose $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ satisfy Condition A. Suppose $\ell \geq 3$. Let $a \in A_1(\mathcal{A})$ be a nonzero element concentrated under X. We have $Z_k(a)$ is generated by $Z_1(a)$ for $k < \ell$.

In Chapter VI, we study $\dim H^2(A(\mathcal{A}), a)$. We let char $\mathcal{K} = 0$ and use the description of $\dim H^1(A(\mathcal{A}), a)$ in terms of $\mathcal{X}(a)$ as given in [8]. We begin by studying $H^2(A(\mathcal{A}), a)$ for $\operatorname{rank}(\mathcal{A}) = 3$. To do this, we demonstrate a relationship between $Z_1(a)$ and $Z_2(a)$. In particular, we prove

THEOREM 6.1.15. Let $\mathcal{A}$ be a central rank three hyperplane arrangement. We have

$$\dim Z_2(a) = \dim Z_1(a) + |\{jk : 1 < j < k \leq n, \operatorname{rank}(H_\alpha \cap H_j \cap H_k) = 3 \text{ for } \alpha < j\}|.$$

We then use this description to study $H^2(A(\mathcal{A}), a)$ for $\operatorname{rank} \mathcal{A} \geq 3$. For $X \in L(\mathcal{A})$ and $a \in A_1(\mathcal{A})$, we define $a(X) = \sum_{H_i < X} \lambda_i a_i$. Similar to the definition of $\mathcal{X}(a)$, we define the set

$$\mathcal{S}(a) := \{X \in L(\mathcal{A});\ \operatorname{rank}(X) = 3, |X| > 3, \sum_{H_i < X} \lambda_i = 0, a(X) \neq 0\}.$$

In determining $\dim Z_1(a)$, it is said that $\mathcal{X}(a)$ is affine to describe a particular situation. In particular, $\mathcal{X}(a)$ affine means $\dim Z_1(a)$ may be greater than one; whereas, $\mathcal{X}(a)$ is not affine means $\dim Z_1(a) = 1$.

THEOREM 6.2.9. Let $\mathcal{A}$ be a central hyperplane arrangement. Let $a, b \in A_1(\mathcal{A})$ with

$$a = \sum_{i=1}^{n} \lambda_i a_i,\ b = \sum_{i=1}^{n} \sigma_i a_i.$$

Suppose $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \sigma_i = 0$. In addition, suppose the following criteria are satisfied:

1. We have $\mathcal{S}(a) = \mathcal{S}(b)$.
2. We have $\mathcal{X}(a) = \mathcal{X}(b)$.
3. For $X \in L(\mathcal{A}) \setminus \mathcal{S}(a)$ with $\operatorname{rank}(X) = 3$, we have $a(X) = b(X) = 0$.
4. For $X \in \mathcal{S}(a) = \mathcal{S}(b)$, we have $\mathcal{X}(a(X))$ is affine (hence, $\mathcal{X}(b(X))$ is affine).

Then $\dim H^2(a) = \dim H^2(b)$.

We give plenty of examples in Chapter VI which demonstrate the various results of the chapter.

CHAPTER II

AN ARRANGEMENT OF HYPERPLANES AND ITS LATTICE

In this chapter, we define an arrangement of hyperplanes and a partially ordered set associated to an arrangement. In §2.1, we define an arrangement of hyperplanes and discuss some basic constructions. We show that coning and deconing are mutually inverse. In §2.2, we discuss the combinatorics of a hyperplane arrangement by defining the partially ordered set $L(\mathcal{A})$. We discuss properties of $L(\mathcal{A})$ and consider $L(\mathcal{A})$ for product arrangements.

We establish the following conventional notations to be used throughout this paper. Let F be a field. Let $V = F^\ell$ be a finite dimensional linear space over F. Let V^* be the dual space of V.

§2.1 Arrangements of Hyperplanes

In this section, basic constructions such as products of arrangements, deletion and restriction, and coning and deconing are discussed, see [12].

DEFINITION 2.1.1. A hyperplane is an affine subspace in V of codimension one. A hyperplane arrangement is a finite collection of hyperplanes in V. For a hyperplane arrangement, we write $\mathcal{A} = \{H_1, \dots, H_n\}$, with hyperplanes $H_i \subset V$. We write $|\mathcal{A}| = n$.

DEFINITION 2.1.2. An arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ is central if $\cap H_i \neq \emptyset$. We call an arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ affine if either $\cap H_i \neq \emptyset$ or $\cap H_i = \emptyset$.

Fix a basis $\{x_1, \dots, x_\ell\}$ for V^* over F. Let S be the symmetric algebra of V^*. Choose a basis $\{e_1, \dots, e_\ell\}$ in V and let $\{x_1, \dots, x_\ell\}$ be the dual basis in V^* so that

$x_i(e_j) = \delta_{ij}$. We may identify S with the polynomial algebra in ℓ indeterminants over F; that is, $S = F[x_1, \dots, x_\ell]$. Each hyperplane $H \in \mathcal{A}$ is the kernel of a polynomial α_H of degree one defined up to a constant.

DEFINITION 2.1.3. A defining polynomial of $\mathcal{A}$ is $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$.

EXAMPLE 2.1.4. Let $\mathcal{A}$ be the arrangement given by $Q(\mathcal{A}) = x_1 \cdots x_\ell$. We call $\mathcal{A}$ the Boolean arrangement. Note that $\mathcal{A}$ is central.

EXAMPLE 2.1.5. Let $Q(\mathcal{A}) = \prod_{1 \leq i < j \leq \ell}(x_i - x_j)$. We call $\mathcal{A}$ the Braid arrangement. Note that $\mathcal{A}$ is central with the intersection of the hyperplanes given by $\cap H_i = \{x_1 = \cdots = x_\ell\}$.

EXAMPLE 2.1.6. Let F be a finite field of q elements. We can consider the arrangement given by $\mathcal{A} = \{$all hyperplanes of F^ℓ which pass through the origin$\}$.

EXAMPLE 2.1.7. Let $Q(\mathcal{A}) = xy(x + y + 1)$. We have that $\mathcal{A}$ is an affine arrangement which is not central.

DEFINITION 2.1.8. Let $\mathcal{A}_1$ be an arrangement in V_1, and let $\mathcal{A}_2$ be an arrangement in V_2. Let $V = V_1 \oplus V_2$. Define the product arrangement by

$$\mathcal{A}_1 \times \mathcal{A}_2 = \{H_1 \oplus V_2 : \ H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 : \ H_2 \in V_2\}.$$

DEFINITION 2.1.9. Let $\mathcal{A}$ be an arrangement in V. We say $\mathcal{A}$ is reducible if, after a change of coordinates, $(\mathcal{A}, V) = (\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2)$. Equivalently, after a linear change of variables if necessary, $Q(\mathcal{A}_1)$ and $Q(\mathcal{A}_2)$ have no common variables. In this case, we write $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$.

EXAMPLE 2.1.10. The Boolean arrangement $Q(\mathcal{A}) = x_1 \cdots x_\ell$ is a product of ℓ arrangements $Q(\mathcal{A}_i) = x_i$.

We now define deletion and restriction. This construction takes an arrangement $\mathcal{A}$, fixes a hyperplane $H_0 \in \mathcal{A}$, and then forms two arrangements $\mathcal{A}'$ and $\mathcal{A}''$ with the result $|\mathcal{A}'|, |\mathcal{A}''| < |\mathcal{A}|$. Because $|\mathcal{A}'|, |\mathcal{A}''| < |\mathcal{A}|$, deletion and restriction is an important construction which allows one to induct on $|\mathcal{A}|$.

DEFINITION 2.1.11. Let $\mathcal{A}$ be an arrangement in $V = F^\ell$. Let $H_0 \in \mathcal{A}$. We define the arrangements

$$\mathcal{A}' = \{H : \ H \in \mathcal{A} \setminus H_0\} \text{ in } V, \text{and}$$

$$\mathcal{A}'' = \{H_0 \cap H : \ H \in \mathcal{A} \text{ and } H \cap H_0 \neq \emptyset\} \text{ in } H_0 \cong F^{\ell-1}.$$

EXAMPLE 2.1.12. Let $Q(\mathcal{A}) = xy(x+y+z)(2x+y+z)z$. Fix H_0 to be given by $x = 0$. We have $\mathcal{A}'$ is given by $Q(\mathcal{A}') = y(x+y+z)(2x+y+z)z$, and $\mathcal{A}''$ is given by $Q(\mathcal{A}'') = y(y+z)z$ in $\{x_0 = 0\}$. Notice $H_0 \cap H$ may equal $H_0 \cap K$ for hyperplanes $H \neq K$.

We now discuss two operations; one operation (coning) will take an affine arrangement to a central arrangement. The other operation (deconing) will take a central arrangement to an affine arrangement. These operations are inverse to each other. We begin by discussing deconing; this will take a central arrangement in F^ℓ to an affine arrangement in $F^{\ell-1}$.

DEFINITION 2.1.13. Let $\mathcal{A}$ be a central arrangement in F^ℓ. We define the deconed arrangement $d\mathcal{A}$ in $F^{\ell-1}$. Fix $H_0 \in \mathcal{A}$. Choose coordinates so that $H_0 = \operatorname{Ker}(x_0)$. Let $Q(\mathcal{A}) \in F[x_0, x_1, \ldots, x_\ell]$ be a defining polynomial for $\mathcal{A}$. The defining polynomial $Q(d\mathcal{A})$ is obtained by substituting 1 for x_0 in $Q(\mathcal{A})$.

LEMMA 2.1.14. Let $\mathcal{A}$ be an arrangement given by $Q(\mathcal{A}) = \prod \alpha_i$. Fix $H_0 \in \mathcal{A}$. Let $\alpha_0 = 0$ be an equation for H_0. The deconed arrangement, $d\mathcal{A}$, is equivalent up to linear isomorphism to the arrangement in $\tilde{H}_0 := \{\alpha_0 = 1\}$ given by $\{H_i \cap \tilde{H}_0 : H_i \in \mathcal{A} \setminus \{H_0\}\}$.

PROOF. By Definition 2.1.13, $d\mathcal{A}$ is found by a linear change of coordinates via $\alpha_0 \mapsto x_0$ then substituting $x_0 = 1$ into $Q(\mathcal{A})$. This is equivalent (up to the change of coordinates) as intersecting the hyperplanes $H_i \in \mathcal{A} \setminus \{H_0\}$ with the space given by $\{\alpha_0 = 1\}$. □

EXAMPLE 2.1.15. Let $\mathcal{A}$ be given by $Q(\mathcal{A}) = xy(x+y)z$. By deconing about the hyperplane given by $y = 0$, we obtain $Q(d\mathcal{A}) = x(x+1)z$, an arrangement which is not central. However, if we decone about the hyperplane given by $z = 0$, we obtain $Q(d\mathcal{A}) = xy(x+y)$, a central arrangement.

REMARK 2.1.16. Example 2.1.15 demonstrates the deconed arrangement depends upon the choice of hyperplane about which one decones.

DEFINITION 2.1.17. Let $f, g \in K[x_1, \dots, x_\ell]$. We define f homogenized about the factor g to be $\tilde{f} := g^{\deg(f)} f(x_1/g, \dots, x_\ell/g)$.

EXAMPLE 2.1.18. Let $f = x(y+1)$. We have f homogenized about z given by $x(y+z)$. Moreover, f homogenized about $z-1$ is given by $x(y+z-1)$.

DEFINITION 2.1.19. Let $\mathcal{A}$ be an affine arrangement in F^ℓ. We define the central arrangement, $c\mathcal{A}$, in $F^{\ell+1}$ as follows. Let $Q' \in F[x_0, x_1, \dots, x_\ell]$ be the polynomial $Q(\mathcal{A})$ homogenized about the factor x_0, and define $Q(c\mathcal{A}) = x_0 Q'$. Note that $|c\mathcal{A}| = |\mathcal{A}| + 1$.

LEMMA 2.1.20. Let $\mathcal{A}$ be an arrangement given by $Q(\mathcal{A})$. As in Definition 2.1.19, consider the arrangement $c\mathcal{A}$. Let $\{e_1, \dots, e_\ell\}$ be a basis for V over F. Consider $F^{\ell+1}$ with the basis $\{e_0, e_1, \dots, e_\ell\}$. Let H_0 be a hyperplane in $F^{\ell+1}$ with defining equation $\alpha_0 = 0$ for $\alpha_0 \in F[x_0, x_1, \dots, x_\ell] \setminus F[x_1, \dots, x_\ell]$. Up to linear isomorphism, $c\mathcal{A}$ is equivalent to the arrangement obtained by homogenizing the polynomial $Q(\mathcal{A})$ with the parameter α_0 and adding the factor α_0.

PROOF. Since $\alpha_0 \in F[x_0, x_1, \dots, x_\ell] \setminus F[x_1, \dots, x_\ell]$, the linear change of coordinates given by $\alpha_0 \mapsto x_0$ is a linear isomorphism. □

REMARK 2.1.21. In Definition 2.1.19, we can describe the hyperplanes of $c\mathcal{A}$ geometrically. For $H \in \mathcal{A}$, let the coned hyperplane cH in $F^{\ell+1}$ be given by the linear span of H_i and the origin. Then $c\mathcal{A} = \{H_0, cH : H \in \mathcal{A} \text{ and } H_0 = \operatorname{Ker}(x_0)\}$.

We construct a similar geometric interpretation as in Lemma 2.1.20 when coning about $H_0 = \text{Ker}(\alpha_0)$ with $\alpha_0 \in F[x_0, x_1, \dots, x_\ell] \setminus F[x_1, \dots, x_\ell]$. We consider $V \subset F^{\ell+1}$ as the hyperplane $\{(\gamma_0, \gamma_1, \dots, \gamma_\ell) \in F^{\ell+1} : \alpha_0(\gamma_0, \gamma_1, \dots, \gamma_\ell) = 1\}$. In this fashion, H_i can be considered as a subset of $F^{\ell+1}$. Since $H_0 \neq V$, we have $H_i \cap H_0 = T_i \neq \emptyset$. For $H_i \in \mathcal{A}$, we define the coned hyperplane in $F^{\ell+1}$, written cH_i, to be given by the linear span of H_i and T_i in $F^{\ell+1}$. Then the coned arrangement in $F^{\ell+1}$ is given by $c\mathcal{A} = \{H_0, cH_i : H_i \in \mathcal{A}\}$.

REMARK 2.1.22. Unlike the deconing construction, Lemma 2.1.20 shows the coned arrangement does not depend upon the choice of hyperplane about which one cones.

EXAMPLE 2.1.23. Let $\mathcal{A}$ be given by $Q(\mathcal{A}) = x(x+1)y$. By coning about the hyperplane given by $z = 0$, we obtain $Q(c\mathcal{A}) = x(x+z)yz$. By coning about the hyperplane given by $x+z+1 = 0$, we obtain $Q(c\mathcal{A}) = x(x+x+z+1)y(x+z+1)$. Notice by the linear change of coordinates $x+y+1 \mapsto z$, these arrangements are equivalent.

PROPOSITION 2.1.24. The coning and deconing are inverse operations in the following sense:

1. Let $\mathcal{A}$ be an arrangement. Fix $H_0 \in \mathcal{A}$. Let $d\mathcal{A}$ represent the arrangement deconed about H_0. Then by coning about x_0, we have $c(d\mathcal{A})$ is $\mathcal{A}$.
2. Let $\mathcal{A}$ be an arrangement. Let $c\mathcal{A}$ denote the coned arrangement about x_0 as given in Definition 2.1.19. If $c\mathcal{A}$ is deconed about x_0, then $d(c\mathcal{A})$ is $\mathcal{A}$.

PROOF. The proposition follows from Lemma 2.1.14 and LEMMA 2.1.20. □

§2.2 Combinatorics of Hyperplane Arrangements

In this section, we associate to each arrangement a combinatorial object, $L(\mathcal{A})$. Properties of $L(\mathcal{A})$ are discussed which make $L(\mathcal{A})$ a matroid in the case $\mathcal{A}$ is central. We also prove $L(\mathcal{A}_1 \times \mathcal{A}_2)$ is a product of $L(\mathcal{A}_1)$ and $L(\mathcal{A}_2)$.

DEFINITION 2.2.1. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of hyperplanes. Let $L(\mathcal{A})$ be the partially ordered set with objects given by

$$\{\cap_{H \in \mathcal{B}} H : \; \mathcal{B} \subseteq \mathcal{A} \text{ and } \cap_{H \in \mathcal{B}} H \neq \emptyset\};$$

the objects of $L(\mathcal{A})$ are ordered opposite to inclusion.

DEFINITION 2.2.2. Let $(P, <)$ be a partially ordered set; let $X, Y \in P$. The join of X and Y is given by $X \vee Y := \inf\{Z \in P : \; Z \geq X \text{ and } Z \geq Y\}$. The meet of X and Y is given by $X \wedge Y := \sup\{Z \in P : \; Z \leq X \text{ and } Z \leq Y\}$. If $X \vee Y$ and $X \wedge Y$ exists in P for all $X, Y \in P$, then P is a lattice.

DEFINITION 2.2.3. Let $(P, <)$ be a partially ordered set with $V \in P$ so that $V \leq X$ for all $X \in P$. We say P is a ranked and write $\text{rank}(X) = p$ if for any $X \in P$ and any two maximal chains $V = X_0 < X_1 < \cdots < X_r = X$ and $V = Y_0 < Y_1 \cdots < Y_s = X$ we have $r = s = p$.

DEFINITION 2.2.4. For $X \in L(\mathcal{A})$, define $\text{rank}(X) := \text{codim } X$. For $X \in L(\mathcal{A})$ with $\text{rank}(X) = p$, we write $X \in L(p, \mathcal{A})$. For the rank of an arrangement, we define $\text{rank}(\mathcal{A}) := \max_{X \in L(\mathcal{A})} \text{rank}(X)$.

DEFINITION 2.2.5. Let $\mathcal{A}$ be an arrangement of hyperplanes. We call $H \in \mathcal{A}$ at atom. Notice $\text{rank}(H) = 1$ for all $H \in \mathcal{A}$.

PROPOSITION 2.2.6. Let $\mathcal{A}$ be an arrangement. We have

1. $L(\mathcal{A})$ is atomic; that is, each $X \in L(\mathcal{A}) \setminus V$ is a join of hyperplanes.

2. $L(\mathcal{A})$ is ranked via codimension; that is, for each $X \in L(\mathcal{A})$, the length of any maximal chain $V = X_0 < X_1 < \cdots < X_p = X$ is equal to codim X.

3. If $\mathcal{A}$ is central, then $L(\mathcal{A})$ is semi-modular; that is, for any $X, Y \in L(\mathcal{A})$ we have $\text{rank}(X) + \text{rank}(Y) \geq \text{rank}(X \wedge Y) + \text{rank}(X \vee Y)$.

PROOF. This is adapted from Lemma 2.3 in [12].

Property (1) follows from the definition of $L(\mathcal{A})$.

To verify property (2), fix $X \in L(\mathcal{A})$. Consider a maximal chain in $L(\mathcal{A})$ given by $V = X_0 < X_1 < \cdots < X_p = X$. Since the inequalities are strict, we have codim $X \geq p$. For a hyperplane $H_j < X$, notice $X_i \cap H_j = X_i$ if $X_i \subseteq H_j$, and $X_i \cap H_j = X_{i+1}$ if $X_i \not\subseteq H_j$. Therefore, the codim X_{i+1} in X_i is one.

To verify property (3), first notice $\dim(X+Y) + \dim(X \cap Y) = \dim X + \dim Y$ for $X, Y \in L(\mathcal{A})$. Since $X + Y \subseteq X \wedge Y$, we have $\dim(X+Y) \leq \dim(X \wedge Y)$. Hence, $\text{rank}(X) + \text{rank}(Y) \geq \text{rank}(X \wedge Y) + \text{rank}(X \vee Y)$. □

DEFINITION 2.2.7. A lattice which is atomic, ranked, and semi-modular is a matroid.

EXAMPLE 2.2.8. If $\mathcal{A}$ is a central hyperplane arrangement, then $L(\mathcal{A})$ is a matroid.

DEFINITION 2.2.9. Let P and P' be two partially ordered sets. Then $P \times P'$ is a partially ordered set defined by $(a, b) \leq (\alpha, \beta)$ if and only if $a \leq \alpha$ (in P) and $b \leq \beta$ (in P').

DEFINITION 2.2.10. Let P and Q be two partially ordered sets. We say P is isomorphic to Q if there exists an order preserving bijection $\pi : P \to Q$.

PROPOSITION 2.2.11. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two arrangements with $\mathcal{A}_1$ an arrangement in V_1 and $\mathcal{A}_2$ an arrangement in V_2. The partially ordered set $L(\mathcal{A}_1) \times L(\mathcal{A}_2)$ is isomorphic to the partially ordered set $L(\mathcal{A}_1 \times \mathcal{A}_2)$.

PROOF. The statement of this proposition can be found in Proposition 2.14 in [12].

Define $\theta : L(\mathcal{A}_1) \times L(\mathcal{A}_2) \to L(\mathcal{A}_1 \times \mathcal{A}_2)$ via $\theta(X,Y) := X \oplus Y$.

First, notice $X \oplus Y \in L(\mathcal{A}_1 \times \mathcal{A}_2)$. Since $X \in L(\mathcal{A}_1)$, there exists $\mathcal{B}_1 \subseteq \mathcal{A}_1$ so that $X = \cap\{H_i : H_i \in \mathcal{B}_1\}$. Similarly, $Y = \cap\{K_i : K_i \in \mathcal{B}_2\}$ for some $\mathcal{B}_2 \subseteq \mathcal{A}_2$. Hence $X \oplus Y = (\cap_{H_i \in \mathcal{B}_1}\{H_i \oplus V_2\}) \cap (\cap_{K_i \in \mathcal{B}_2}\{V_1 \oplus K_i\})$ as required to verify $X \oplus Y \in L(\mathcal{A}_1 \times \mathcal{A}_2)$.

Now, θ is surjective. An element in $L(\mathcal{A}_1 \times \mathcal{A}_2)$ is the intersection of hyperplanes in $V_1 \oplus V_2$; hence, it has the form $(\cap_{H_i \in \mathcal{B}_1}\{H_i \oplus V_2\}) \cap (\cap_{K_i \in \mathcal{B}_2}\{V_1 \oplus K_i\})$ for some $\mathcal{B}_1 \subseteq \mathcal{A}_1$ and $\mathcal{B}_2 \subseteq \mathcal{A}_2$. Thus

$$\begin{aligned}\theta(\cap_{H_i \in \mathcal{B}_1} H_i, \cap_{K_i \in \mathcal{B}_2} K_i) =& (\cap_{H_i \in \mathcal{B}_1} H_i) \oplus (\cap_{K_i \in \mathcal{B}_2} K_i)\\ =& (\cap_{H_i \in \mathcal{B}_1}\{H_i \oplus V_2\}) \cap (\cap_{K_i \in \mathcal{B}_2}\{V_1 \oplus K_i\}).\end{aligned}$$

Also, θ is injective since $X \oplus Y = X' \oplus Y'$ implies $X = X'$ and $Y = Y'$.

Finally, θ preserves the order of the lattices. Suppose $(X,Y) \leq (X',Y')$ in $L(\mathcal{A}_1) \times L(\mathcal{A}_2)$. Then $X \leq X'$ and $Y \leq Y'$ which implies $X' \subseteq X$ and $Y' \subseteq Y$. Hence, $X' \oplus Y' \subseteq X \oplus Y$ in $L(\mathcal{A}_1 \times \mathcal{A}_2)$. □

We now define a particular central subarrangement which will be used in later chapters.

DEFINITION 2.2.12. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a hyperplane arrangement. Fix $X \in L(\mathcal{A})$. Define

$$\mathcal{A}_X := \{H_i : H_i \in \mathcal{A} \text{ and } H_i \leq X\}.$$

Notice $\mathcal{A}_X$ is a central subarrangement of $\mathcal{A}$ with $\text{rank}(\mathcal{A}_X) = \text{rank}(X)$. We write $|X|$ to denote $|\mathcal{A}_X|$.

EXAMPLE 2.2.13. Let $Q(\mathcal{A}) = x(x+1)y$; order the hyperplanes as they are written. Fix $X \in L(\mathcal{A})$ to be given by $H_1 \cap H_3$. Then $Q(\mathcal{A}_X) = xy$.

The following definitions are standard definitions for lattices in general and will be used in later chapters.

DEFINITION 2.2.14. Let $\mathcal{A}$ be a central hyperplane arrangement. A pair $(X, Y) \in L(\mathcal{A}) \times L(\mathcal{A})$ is called a modular pair if for all $Z \in L(\mathcal{A})$ with $Z \leq Y$

$$Z \vee (X \wedge Y) = (Z \vee X) \wedge Y.$$

DEFINITION 2.2.15. Let $\mathcal{A}$ be a central hyperplane arrangement. An element $X \in L(\mathcal{A})$ is called modular if (X, Y) is a modular pair for all $Y \in L(\mathcal{A})$.

DEFINITION 2.2.16. Let $\mathcal{A}$ be a central hyperplane arrangement in V. Let $\operatorname{rank}(\mathcal{A}) = \ell$. We call $\mathcal{A}$ supersolvable if $L(\mathcal{A})$ has a maximal chain of modular elements

$$V = X_0 < X_1 < \cdots < X_\ell = \cap_{H \in \mathcal{A}} H.$$

EXAMPLE 2.2.17. The Boolean arrangement $Q(\mathcal{A}) = \prod_{i=1}^{\ell} x_i$ is supersolvable as all the elements in $L(\mathcal{A})$ are modular.

EXAMPLE 2.2.18. The arrangement given by

$$Q(\mathcal{A}) = x(x-y)(x+y)y(x-z)(x+z)(y+z)(y-z)z$$

is supersolvable as a maximal chain of modular elements is given by

$$V < \{x = 0\} < \{x = y = 0\} < \{0\}.$$

CHAPTER III

ORLIK-SOLOMON ALGEBRAS AND THEIR COHOMOLOGY

In this chapter, we define the Orlik-Solomon algebras and their cohomology. In §3.1, we define the Orlik-Solomon algebras and discuss a linear basis for such an algebra. In §3.2, we demonstrate the relationship between the basis found in §3.1 with a Gröbner basis. In §3.3, we define the cohomology of an Orlik-Solomon algebra and discuss some results on the dimension of the first cohomology group.

§3.1 The Orlik-Solomon Algebra and the Broken Circuit Basis

In this section, we define the Orlik-Solomon algebra and a linear basis for this algebra, referred to as the broken circuit basis; see Chapter 3 in [12]. The Orlik-Solomon algebra is a factor algebra of the exterior algebra by an ideal $I(\mathcal{A})$. In §3.2, we show the relationship between the broken circuit basis and a Gröbner basis for $I(\mathcal{A})$.

Let $\mathcal{A} = \{H_1, ..., H_n\}$ be a hyperplane arrangement in $V = F^\ell$ for some field F. We fix an order on $\mathcal{A}$; that is, for hyperplanes H_i and H_j in $\mathcal{A}$, we have $H_i < H_j$ if and only if $i < j$.

We begin by defining the Orlik-Solomon algebra.

Let $\mathcal{K}$ be a commutative ring. Let E_1 be the linear space over $\mathcal{K}$ on n generators, $e_1, \ldots, e_n$. Let $E(\mathcal{A}) := \Lambda(E_1)$ be the exterior algebra on E_1. We have $E(\mathcal{A}) = \bigoplus_{p\geq 0} E_p$ is a graded algebra over $\mathcal{K}$. The standard $\mathcal{K}$-basis for E_p is given by

$$\{e_{i_1} \cdots e_{i_p} : \ 1 \leq i_1 < \ldots < i_p \leq p\}.$$

Any ordered subset $S = \{H_{i_1}, ..., H_{i_p}\}$ of $\mathcal{A}$ corresponds to an element $e_S := e_{i_1} \cdots e_{i_p}$ in $E(\mathcal{A})$.

We define the map $\partial : E(\mathcal{A}) \to E(\mathcal{A})$ via the usual differential. That is,

$$\partial(1) := 0,$$

$$\partial(e_i) := 1,$$

$$\text{and for } p \geq 2,\ \partial(e_{i_1} \cdots e_{i_p}) := \sum_{k=1}^{p} (-1)^{k-1} e_{i_1} \cdots \hat{e}_{i_k} \cdots e_{i_p}.$$

EXAMPLE 3.1.1. As an example of the differential on the exterior algebra, we have $\partial(e_1 \cdot e_2 \cdot e_3) = e_2 \cdot e_3 - e_1 \cdot e_3 + e_1 \cdot e_2$.

DEFINITION 3.1.2. Let $S = \{H_{i_1}, ..., H_{i_p}\}$ be a subset of $\mathcal{A}$. We say S is dependent if $\cap S \neq \emptyset$ and $\text{rank}(\cap S) < |S|$. Equivalently, S is dependent if polynomials $\alpha_{i_k} \in F[x_1, \ldots, x_l]$ defining the hyperplanes H_{i_k} are linearly dependent.

DEFINITION 3.1.3. We define $I(\mathcal{A})$ to be the ideal of $E(\mathcal{A})$ which is generated by the elements

$$\{\partial(e_S) :\ S \text{ is dependent }\} \cup \{e_S :\ \cap_{H \in S} H = \emptyset\}.$$

DEFINITION 3.1.4. The Orlik-Solomon algebra, $A(\mathcal{A})$, is defined as

$$A(\mathcal{A}) := E(\mathcal{A})/I(\mathcal{A}).$$

Let $\pi :\ E(\mathcal{A}) \to A(\mathcal{A})$ be the canonical projection. We write a_S to represent the image of e_S under π.

We demonstrate that $A(\mathcal{A})$ is a free graded $\mathcal{K}$-module by defining the broken circuit basis for $A(\mathcal{A})$. By Theorem 3.1.6 to follow, this is indeed a basis for $A(\mathcal{A})$.

DEFINITION 3.1.5. Let $S = \{H_{i_1}, ..., H_{i_p}\}$ be an ordered subset of $\mathcal{A}$ with $i_1 < \cdots < i_p$. We say a_S is basic in $A_p(\mathcal{A})$ if

1. S is independent, and
2. For any $1 \leq k \leq p$, there does not exist a hyperplane $H \in \mathcal{A}$ so that $H < H_{i_k}$ with $\{H, H_{i_k}, H_{i_{k+1}}, ..., H_{i_p}\}$ dependent.

The set of $\{a_S\}$ with S as above form the broken circuit basis for $A(\mathcal{A})$, whose name is justified by the following theorem.

THEOREM 3.1.6. As a $\mathcal{K}$-module, $A(\mathcal{A})$ is a free, graded module. The broken circuit basis forms a basis for $A(\mathcal{A})$.

PROOF. This is proven in Theorem 3.55 in [12]. □

The following two examples demonstrate the use of the broken circuit basis for computing $\dim A_p(\mathcal{A})$.

EXAMPLE 3.1.7. Let $\mathcal{A}$ be a central generic arrangement; this means for any collection $\{H_{i_1}, \ldots, H_{i_p}\} \subseteq \mathcal{A}$ with $p < \ell$, we have $\{H_{i_1}, \ldots, H_{i_p}\}$ is independent. Hence, for $p \leq \ell$, there are no dependencies, so $\dim\ A_p(\mathcal{A}) = \dim\ E_p = \binom{n}{p}$ for $p < \ell$. For $p = \ell$, any $S \subseteq \{1, 2, ...n\}$ with $|S| = \ell + 1$ is dependent, so $A_\ell(\mathcal{A})$ has a broken circuit basis of $\{a_{1S} : S \subset \{2, 3, ..., n\}$ with $|S| = \ell - 1\}$. Hence, $\dim A_\ell(\mathcal{A}) = \binom{n-1}{\ell-1}$.

EXAMPLE 3.1.8. Let $\dim\ V = \ell$, and let $\mathcal{A}$ be the braid arrangement in V given by $Q(\mathcal{A}) = \prod_{1 \leq i < j \leq \ell} (x_i - x_j)$. Let H_{ij} correspond to the hyperplane given by $x_i - x_j = 0$. Order the hyperplanes lexicographically; that is, $H_{ij} < H_{mn}$ if either $i < m$ or $i = m$ and $j < n$. We will write $a_{H_{ij}} = a_{ij}$ in $A_1(\mathcal{A})$.

In order to compute $\dim\ A_p(\mathcal{A})$, we need to describe the elements of the broken circuit basis in $A_p(\mathcal{A})$. Let $a := a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_p j_p}$ be an element of the broken circuit basis in $A_p(\mathcal{A})$. By definition of the hyperplanes, we have $i_k < j_k$.

Suppose $j_1 = j_2$. Without loss of generality, we may assume $i_1 < i_2$. Then $\{H_{i_1 j_1}, H_{i_2 j_2}, H_{i_1 i_2}\}$ is dependent with $H_{i_1 i_2}$ being minimal in the set; this contradicts the assumption a is in the broken circuit basis. In a similar fashion, we have

$j_1 < j_2 < \cdots < j_p$. Moreover, if $i_1 = i_2$, then $\{H_{i_1j_1}, H_{i_2j_2}, H_{j_1,j_2}\}$ is dependent; but the minimal element of this set is $H_{i_1j_1}$. Therefore, a is still an element of the broken circuit basis. Hence, there are no restrictions on i_k other than $j_k > i_k$.

It is now just a matter of counting the possibilities we have for $\{i_1j_1, ..., i_pj_p\}$ with the restrictions $j_1 < j_2 < \cdots < j_p$ and $i_k < j_k$ for $k = 1, ..., p$.

Fix $j_1, ..., j_p$. There are $\ell - j_k$ choices for i_k for each $k = 1, ..., p$. Thus,

$$\begin{aligned} \dim\ A_p(\mathcal{A}) &= \sum_{i_p=1+i_{p-1}}^{\ell-1} \cdots \sum_{i_2=1+i_1}^{\ell-p+1} \sum_{i_1=1}^{\ell-p} (\prod_{k=1}^{p} (\ell - j_k)) \\ &= \sum_{1 \leq j_1 < j_2 < \cdots < j_p \leq \ell-1} j_1 j_2 \cdots j_p. \end{aligned}$$

As usual, if $p = 0$, then this sum is taken to be 1.

The dimensions of $A_1(\mathcal{A})$ and $A_2(\mathcal{A})$ can be easily simplified. Obviously, we have $\dim A_1(\mathcal{A}) = \binom{\ell}{2}$. For the dimension of $A_2(\mathcal{A})$, consider circuits with three hyperplanes. Any such circuit must be of the form $\{H_{ij}, H_{ik}, H_{jk} : \ i < j < k\}$. There are $\binom{\ell}{3}$ of these circuits. Hence, dim $A_2(\mathcal{A}) =$ dim $E_2 - \binom{\ell}{3}$. Using the fact $n = \binom{\ell}{2}$, we arrive at dim $A_2(\mathcal{A}) = \frac{\ell(\ell-1)(\ell-2)(3\ell-1)}{24}$.

DEFINITION 3.1.9. For the algebra $A(\mathcal{A})$, we define

$$\mathrm{Poin}(A(\mathcal{A}), t) := \sum_{p \geq 0} \dim A_p(\mathcal{A}) t^p$$

$$\chi(A(\mathcal{A})) := \mathrm{Poin}(A(\mathcal{A}), -1) = \sum_{p \geq 0} (-1)^p \dim A_p(\mathcal{A}).$$

From Theorem 3.68 in [12], we have Poin$(A(\mathcal{A}), t)$ depends only on $L(\mathcal{A})$. Let $H_0 \in \mathcal{A}$, and consider the deconed arrangement, $d\mathcal{A}$, obtained by deconing about H_0. From Corollary 2.58 in [12], we have Poin$(A(\mathcal{A}), t) = (1 + t)$Poin$(A(d\mathcal{A}), t)$. Hence, as in Proposition 2.7 of [13], $\chi(A(d\mathcal{A}))$ depends only on $\mathcal{A}$ and not on the choice of hyperplane about which one decones.

§3.2 A Gröbner Basis for $I(\mathcal{A})$

In this section, we establish the relationship between the broken circuit basis and a Gröbner basis for the ideal $I(\mathcal{A})$.

We now establish some definitions and notations regarding Gröbner bases. These are standard notations and results which can be found in [7].

Let V be a module over a commutative ring $\mathcal{K}$. Let $B \subset V$ be a $\mathcal{K}$-basis. Suppose B is ordered with $<$; this means the order is linear and that $(B, <)$ is well ordered.

DEFINITION 3.2.1. Let $v \in V$. Since B is a $\mathcal{K}$-basis, we can write $v = \sum_{b_i \in B} \alpha_i b_i$ for $\alpha_i \in \mathcal{K}$ and $b_i \in B$. Since B is ordered and there are only finitely many nonzero terms in the summation, there is a maximal element $b_i \in B$ with $\alpha_i \neq 0$; say this element is b_1. We define $\mathrm{Tip}(v) := b_1$.

DEFINITION 3.2.2. Let $W \subseteq V$. We define $\mathrm{Tip}\, W := \{\mathrm{Tip}(w) : \; w \in W\}$. Define the non-tips of W to be $NT(W) := B \setminus \mathrm{Tip}\, W$.

THEOREM 3.2.3. Let V be a module over $\mathcal{K}$ with an ordered basis $(B, <)$. Let $W \subseteq V$ be a submodule of V with the condition:

(*) for any $w \in W$, there exists $w' \in W$ such that

1. $\mathrm{Tip}(w) = \mathrm{Tip}(w')$ and
2. $w' = \mathrm{Tip}(w') + \sum \gamma_i b_i$, for $\gamma_i \in \mathcal{K}$ and $b_i \in B \setminus \{\mathrm{Tip}(w')\}$.

Then $V = W \oplus \langle NT(W) \rangle$.

PROOF. We begin by showing $W \cap \langle NT(W) \rangle = 0$. Let $v \in W \cap \langle NT(W) \rangle$. We have $\mathrm{Tip}(v) \in \mathrm{Tip}\, W$ since $v \in W$. But $v \in \langle NT(W) \rangle$ implies $\mathrm{Tip}(v) \in NT(W)$. Hence, $v = 0$ as required.

Suppose $W + \langle NT(W) \rangle \neq V$. Choose $v \in V \setminus (W + \langle NT(W) \rangle)$ with $\mathrm{Tip}(v)$ minimal; that is, $\mathrm{Tip}(v) \leq \mathrm{Tip}(w)$ for any $w \in V \setminus (W + \langle NT(W) \rangle)$. Let $0 \neq \alpha \in \mathcal{K}$ so that $v = \alpha\, \mathrm{Tip}(v) + \sum \alpha_i b_i$ for $\alpha_i \in \mathcal{K}$ and $b_i \in B \setminus \{\mathrm{Tip}(v)\}$.

Suppose $\text{Tip}(v) \in NT(W)$. We construct an element with a smaller tip by considering $v - \alpha\,\text{Tip}(v)$. Then $\text{Tip}(v - \alpha\,\text{Tip}(v)) < \text{Tip}(v)$; hence, $v - \alpha\,\text{Tip}(v) \in W + \langle NT(W)\rangle$. This implies $v - \alpha\,\text{Tip}(v) = w + n$ for $w \in W$ and $n \in \langle NT(W)\rangle$. We solve the equation for v to see that

$$v = w + (n + \alpha\,\text{Tip}(v)) \in W + \langle NT(W)\rangle.$$

This is a contradiction to the choice of v.

Suppose $\text{Tip}(v) \in \text{Tip}\,W$. Then there exists $w \in W$ so that $\text{Tip}(v) = \text{Tip}(w)$. By the condition (*) on W, we may assume $w = \text{Tip}(w) + \sum \gamma_i b_i$ for $\gamma_i \in \mathcal{K}$ and $b_i \in B \setminus \{\text{Tip}(w)\}$. Then $\text{Tip}(v - \alpha w) < \text{Tip}(v)$; hence, by the choice of v, $v - \alpha w \in W + \langle NT(W)\rangle$. This implies $v - \alpha w = w' + n$ for $w' \in W$ and $n \in \langle NT(W)\rangle$. By solving for v, we have $v = (w' + \alpha w) + n \in W + \langle NT(W)\rangle$, a contradiction. □

COROLLARY 3.2.4. Let V be a vector space over a field $\mathcal{K}$ with an ordered basis $(B, <)$. If $W \subseteq V$ is a subspace of V, then $V = W \oplus \langle NT(W)\rangle$.

PROOF. It will suffice to show W satisfies condition (*) as given in Theorem 3.2.3. Let $w \in W$. Then we have that $w = \gamma\,\text{Tip}(w) + \sum \gamma_i b_i$ for $0 \neq \gamma, \gamma_i \in \mathcal{K}$ and that $b_i \in B \setminus \{\text{Tip}(w)\}$. Since W is a subspace of V and $\mathcal{K}$ is a field, we have $\gamma^{-1}w \in W$, and we take $w' := \gamma^{-1}w$. □

DEFINITION 3.2.5. Given a module V over $\mathcal{K}$ with an ordered basis $(B, <)$ and a submodule $W \subseteq V$, we define $\mathcal{G} \subset W$ to be a Gröbner basis of W if $\text{Tip}\,\mathcal{G} = \text{Tip}\,W$.

EXAMPLE 3.2.6. Let V be a 4-dimensional vector space over a field $\mathcal{K}$ with an ordered basis defined by $(B, <) := \{b_1 > b_2 > b_3 > b_4\}$. Let W be the 3-dimensional linear subspace of V generated by the set $\mathcal{H} := \{b_1 - b_2, b_1 - b_3, b_1 - b_4\}$. Consider $\mathcal{G} := \{b_1 - b_2, b_2 - b_3, b_3 - b_4\}$. Then $\text{Tip}\,\mathcal{G} = \{b_1, b_2, b_3\} = \text{Tip}\,W$; hence, $\mathcal{G}$ is a Gröbner basis of W. However, if we consider $\mathcal{H}$, then $\text{Tip}\,\mathcal{H} = \{b_1\} \neq \text{Tip}\,W$; hence, $\mathcal{H}$ is not a Gröbner basis for W.

We now define Gröbner bases in algebras. Again, these are standard and can be found in [7] for the case R is commutative.

Let R be a $\mathcal{K}$-algebra and let B be a $\mathcal{K}$-basis of R. Suppose $(B, <)$ is well ordered; that is, the order is linear and any subset $C \subsetneq B$ has a minimal element $c \in C$.

EXAMPLE 3.2.7. Consider the exterior algebra on n generators, $E(\mathcal{A})$, with the standard basis $B = \{e_{i_1} \cdots e_{i_p} : 1 \leq i_1 < \cdots < i_p \leq p\}$. We can give B the degree lexicographic (DegLex) order. That is,

- if $p < q$, then $e_{i_1} \cdots e_{i_p} < e_{j_1} \cdots e_{j_q}$,
- if $k_0 = \min\{k : i_k \neq j_k\}$ with $i_{k_0} < j_{k_0}$, then $e_{i_1} \cdots e_{i_p} < e_{j_1} \cdots e_{j_p}$.

Then B is a $\mathcal{K}$-basis of $E(\mathcal{A})$ and with respect to DegLex, $(B, <)$ is well ordered.

DEFINITION 3.2.8. Let R be a $\mathcal{K}$-algebra, and let B be a $\mathcal{K}$-basis of R. Let $(B, <)$ be well ordered. We say B is monomial if for any $b, b' \in B$ we have $\text{Tip}(b'b), \text{Tip}(b'b) \in B$ unless they are zero.

DEFINITION 3.2.9. Consider $E(\mathcal{A})$ with the well ordered basis $(B, <)$ given in Example 3.2.7. Then B is monomial.

DEFINITION 3.2.10. Let R be a $\mathcal{K}$-algebra and let B be a $\mathcal{K}$-basis of R. Let $(B, <)$ be well ordered, and let B be monomial. We say the order $(B, <)$ is monomial if the following are satisfied:

1. Let $b_1, b_2, c \in B$ with $b_1 > b_2$. If $cb_i \neq 0$ for $i = 1, 2$, then $\text{Tip}(cb_1) > \text{Tip}(cb_2)$ and $\text{Tip}(b_1 c) > \text{Tip}(b_2 c)$.
2. If $1 \in B$, then $1 < b$ for all $1 \neq b \in B$. If $1 \notin B$, then for all $b, b' \in B$ we have $\text{Tip}(bb') > b, b'$ and $\text{Tip}(b'b) > b, b'$ unless zero appears.

EXAMPLE 3.2.11. Consider the exterior algebra $E(\mathcal{A})$ with the standard basis B ordered with the DegLex order as in Example 3.2.7. Then $(B, <)$ is monomial.

DEFINITION 3.2.12. Let R be a $\mathcal{K}$-algebra, and let B be a $\mathcal{K}$-basis of R. Let $(B,<)$ be well ordered and monomial. Let $\mathcal{G} \subseteq R$. Let $(\text{Tip}\,\mathcal{G}) \subseteq B$ be defined by the smallest set containing $\text{Tip}\,\mathcal{G}$ so that the following holds:

for any $g \in (\text{Tip } \mathcal{G})$ and any $b \in B$, we have either $\text{Tip}(bg), \text{Tip}(gb) \in (\text{Tip } \mathcal{G})$ or $bg = 0$.

DEFINITION 3.2.13. Let R be a $\mathcal{K}$-algebra, and let B be a $\mathcal{K}$-basis for R. Let $(B,<)$ be well ordered and monomial. Let $I \triangleleft R$. Let $\mathcal{G} \subseteq I$. We say $\mathcal{G}$ is a Gröbner basis for I if $(\text{Tip } \mathcal{G}) = \text{Tip}\, I$.

DEFINITION 3.2.14. Let R be a $\mathcal{K}$-algebra, and let B be a $\mathcal{K}$-basis for R. Let $(B,<)$ be well ordered and monomial. Let $I \triangleleft R$. Define $NT(I) := B \setminus (\text{Tip}\, I)$.

THEOREM 3.2.15. Let R be a $\mathcal{K}$-algebra, and let B be a $\mathcal{K}$-basis of R. Let $(B,<)$ be well ordered and monomial. Let $I \triangleleft R$. If $\mathcal{K}$ is a field, then $R = I \oplus \langle NT(I) \rangle$ as $\mathcal{K}$-modules. Moreover, $NT(I)$ is a $\mathcal{K}$-basis for R/I.

PROOF. The statement $R = I \oplus \langle NT(I) \rangle$ as $\mathcal{K}$-modules follows from Corollary 3.2.4. Let $\pi : R \to \langle NT(I) \rangle$ be the canonical projection. It follows that $NT(I)$ is a $\mathcal{K}$-basis for R/I. □

DEFINITION 3.2.16. Let R be a $\mathcal{K}$-algebra, and let B be a $\mathcal{K}$-basis of R. Let $(B,<)$ be well ordered and monomial. Let $\mathcal{G} \subseteq R$. We say $lc(\mathcal{G}) = 1$ if the following holds:

for any $g \in \mathcal{G}$ with $g = \gamma\,\text{Tip}(g) + \sum \gamma_i b_i$ for $0 \neq \gamma, \gamma_i \in \mathcal{K}$ and $b_i \in B \setminus \{\text{Tip}(g)\}$, we have $\gamma = 1$.

THEOREM 3.2.17. Let R be a $\mathcal{K}$-algebra, and let B be a $\mathcal{K}$-basis for R. Let $(B,<)$ be well ordered and monomial. Let $I \triangleleft R$ with $I = (\mathcal{G})$ as an ideal in R. Suppose $lc(\mathcal{G}) = 1$. Then $\mathcal{G}$ is a Gröbner basis of I if and only if $R = I \oplus \langle NT(\mathcal{G}) \rangle$ as $\mathcal{K}$-modules.

PROOF. Suppose $\mathcal{G}$ is a Gröbner basis of I. Then $\operatorname{Tip} I = (\operatorname{Tip} \mathcal{G})$ by Definition 3.2.13. Hence, $NT(\mathcal{G}) = NT(I)$. Since $lc(\mathcal{G}) = 1$, $R = I \oplus \langle NT(\mathcal{G})\rangle$ follows from Theorem 3.2.3.

Suppose $R = I \oplus \langle NT(\mathcal{G})\rangle$. We need to show $\operatorname{Tip} I = (\operatorname{Tip} \mathcal{G})$.

Let $g \in \operatorname{Tip} \mathcal{G}$ and $b \in B$ so that $\operatorname{Tip}(bg) \neq 0$. Since $g \in \operatorname{Tip} \mathcal{G}$, there exists $h \in \mathcal{G}$ so that $\operatorname{Tip}(h) = g$. Since $h \in \mathcal{G}$ and I is generated by $\mathcal{G}$, we have $h \in I$. Hence, $bh \in I$ and $\operatorname{Tip}(bh) \in \operatorname{Tip} I$. Since the order is monomial, $\operatorname{Tip}(bh) = \operatorname{Tip}(bg)$ or $bg = 0$. Therefore, $\operatorname{Tip}(bg) \in \operatorname{Tip} I$.

Let $g \in \operatorname{Tip} I$. Then there exists $h \in I$ so that $\operatorname{Tip}(h) = g$. Since B is a linear basis for R over $\mathcal{K}$, we have $h = \sum \alpha_i b_i \operatorname{Tip}(g_i) + \sum \beta_i n_i$ for $\alpha_i, \beta_i \in \mathcal{K}$, $b_i \in B$, $g_i \in \mathcal{G}$, and $n_i \in NT(\mathcal{G})$. Since $R = I \oplus \langle NT(\mathcal{G})\rangle$ and $h \in I$, we must have $\beta_i = 0$ for all β_i. Hence $g = \operatorname{Tip}(h) \in (\operatorname{Tip} \mathcal{G})$ as required. □

We now apply this theory to the Orlik-Solomon algebra $A(\mathcal{A})$. Recall that for any set of ordered hyperplanes $S = \{H_{i_1}, \ldots, H_{i_p}\}$, we have $e_S = e_{i_1} \cdots e_{i_p} \in E(\mathcal{A})$.

THEOREM 3.2.18. Let $A(\mathcal{A})$ be the Orlik-Solomon algebra. Let B be the standard basis for $E(\mathcal{A})$ with the DegLex order. Let

$$\mathcal{G} = \{\partial(e_S) : \ S \text{ is dependent}\} \cup \{e_S : \ \cap_{H\in S} H = \emptyset\}.$$

$NT(\mathcal{G})$ is a linear basis for $A(\mathcal{A})$.

PROOF. By definition, $\mathcal{G}$ generates $I(\mathcal{A})$ as an ideal in $E(\mathcal{A})$. Also, $lc(\mathcal{G}) = 1$.

We show $\mathcal{G}$ is a Gröbner basis of $I(\mathcal{A})$.

Let $\operatorname{Tip}(bg) \in (\operatorname{Tip} \mathcal{G})$ for $b \in B$ and $g = \operatorname{Tip}(h)$ for $h \in \mathcal{G}$. Since $\mathcal{G}$ generates $I(\mathcal{A})$, $h \in I(\mathcal{A})$. Since $I(\mathcal{A})$ is an ideal, $bh \in I(\mathcal{A})$, so $\operatorname{Tip}(bh) \in \operatorname{Tip} I(\mathcal{A})$. But $\operatorname{Tip}(bh) = \operatorname{Tip}(bg)$.

Let $g \in \operatorname{Tip} I(\mathcal{A})$. Then $g = e_S$ for $S = \{H_{i_1}, \ldots, H_{i_k}\} \subseteq \mathcal{A}$. We consider different cases for S.

If $\cap_{H\in S} H = \emptyset$, then $e_S \in \operatorname{Tip}\mathcal{G}$.

Suppose $\cap_{H\in S} H \neq \emptyset$ for the remainder of the proof.

If S is dependent, then let $H := \min S$. Then $e_{S\setminus\{H\}} \in \operatorname{Tip}\mathcal{G}$. We then have $g = \operatorname{Tip}(e_H e_{S\setminus\{H\}}) \in (\operatorname{Tip}\mathcal{G})$.

Suppose S is independent. If there exists H_0 with $H_0 < \min S$ and $\{H_0\} \cup S$ is dependent, then by definition of $\mathcal{G}$ we have $g = e_S \in \operatorname{Tip}\mathcal{G}$.

Suppose S is independent, and suppose there does not exist $H_0 < \min S$ so that $\{H_0\} \cup S$ is dependent. Then $e_S \in NT(\mathcal{G})$.

We may apply Theorem 3.2.17 to conclude $\mathcal{G}$ is a Gröbner basis for I and $\langle NT(\mathcal{G})\rangle$ is a $\mathcal{K}$-basis for $A(\mathcal{A})$. □

We now consider the case that $\mathcal{A}$ is central and give a characterization of when $\operatorname{Tip}\mathcal{G}$ is generated by elements of degree two; that is, any element $g \in \operatorname{Tip}\mathcal{G}$ may be written as $\operatorname{Tip}(e_S e_T)$ for $|T| = 2$

DEFINITION 3.2.19. A Gröbner basis $\mathcal{G}$ is quadratic if for any $g \in \operatorname{Tip}\mathcal{G}$, there exists $h \in \mathcal{G}$ so that $\deg(h) = 2$ and $g = \operatorname{Tip}(bh)$ or $g = \operatorname{Tip}(hb)$ for some $b \in B$.

DEFINITION 3.2.20. A subset $S := \{H_{i_1}, \dots, H_{i_k}\} \subseteq \mathcal{A}$ is minimally dependent means S is dependent but $\{H_{i_1}, \dots, \hat{H}_{i_p}, \dots, H_{i_k}\}$ is independent for all $1 \leq p \leq k$.

DEFINITION 3.2.21. Let $\mathcal{A}$ be a central hyperplane arrangement. Order the hyperplanes via $<$. Let

$$BC := \{S \subseteq \mathcal{A} : \text{ there is } H < \min S \text{ so that } \{H\} \cup S \text{ is minimally dependent}\}.$$

We say $\mathcal{A}$ is quadratic with respect to $<$ to mean for $S \in BC$, there exists $T \in BC$ with $T \subseteq S$ and $|T| = 2$.

PROPOSITION 3.2.22. Let $\mathcal{A}$ be a central hyperplane arrangement. If $\mathcal{A}$ is quadratic under an order $<$ of the hyperplanes, then $\operatorname{Tip} I(\mathcal{A})$ is generated by elements of degree two, i.e. $\mathcal{G}$ is a quadratic Gröbner basis.

PROOF. Let $S \subseteq \mathcal{A}$ be dependent. Let $R \subset S$ be minimally dependent. Fix $H_0 := \min R$; let $\tilde{R} := R \setminus \{H_0\}$. Then $\tilde{R} \in BC$. Since $\mathcal{A}$ is quadratic, there exists $T \in BC$ with $T \subseteq \tilde{R}$ and $|T| = 2$. Then $e_T \in \mathrm{Tip}\,\mathcal{G}$ with degree two. Moreover, $e_{S \setminus \min S} = \mathrm{Tip}(e_{S\setminus(T \cup \min S)} \cdot e_T)$ as required. □

Recall a central hyperplane arrangement $\mathcal{A}$ is called supersolvable if $L(\mathcal{A})$ has a maximal chain of modular elements

$$V = X_0 < X_1 < \cdots < X_\ell = \cap_{H \in \mathcal{A}} H.$$

DEFINITION 3.2.23. Let $\mathcal{A}$ be a central hyperplane arrangement with order $<$ on the hyperplanes. If $\mathcal{A}$ is supersolvable, we say the order on the hyperplanes respects the supersolvable structure if for a maximal modular chain

$$V = X_0 < X_1 < \cdots < X_\ell = \cap_{H \in \mathcal{A}} H$$

in $L(\mathcal{A})$ we have

1. X_1 is the smallest hyperplane, i.e. $X_1 = H_1$
2. For $i > 1$, we have $X_i = \cap_{j=1}^{n_i} H_j$ and if a hyperplane $H < X_i$ then $H \in \{H_1, \ldots, H_{n_i}\}$.

THEOREM 3.2.24. (Björner and Ziegler [3]) Let $\mathcal{A}$ be a central hyperplane arrangement. $\mathcal{A}$ is supersolvable if and only if $\mathcal{A}$ is quadratic under an order that respects the supersolvable structure.

PROOF. This is Theorem 2.8 in [3]. □

EXAMPLE 3.2.25. This example illustrates the importance of the choice of order on the hyperplanes. Let $Q(\mathcal{A}) = x(x-y)(x+y)y(x-z)(x+z)(y+z)(y-z)z$; order the hyperplanes as they are written. Then $\mathcal{A}$ is supersolvable; see Example 2.2.18. Under the current order, we see the indices for the broken circuit basis for $A_2(\mathcal{A})$ are

$\{12,13,14,15,16,17,18,19,25,26,27,28,29,35,36,37,38,39,45,46,47,48,49\}$. We can check to see that $\mathcal{A}$ is quadratic with this order. Notice the element $H_1 \cap H_2 \cap H_3 \cap H_4 \in L(\mathcal{A})$ is modular and part of a maximal modular chain in $L(\mathcal{A})$. However, if $Q(\mathcal{A}) = (x-y)(x-z)(y-z)x(x+y)y(x+z)(y+z)z$ with the hyperplanes ordered as they are written, then the indices for the broken circuit basis for $A_2(\mathcal{A})$ are $\{12,13,14,15,16,17,18,19,24,25,26,27,28,29,34,35,36,37,38,39,48,59,67\}$. We also have $\mathcal{A}$ is not quadratic under this order because $S = \{H_1, H_2, H_4, H_8\}$ is minimally dependent so $\{H_2, H_4, H_8\} \in BC$. However, $\{H_2, H_4\}, \{H_2, H_8\}, \{H_4, H_8\} \notin BC$. Notice the element $H_1 \cap H_2 \cap H_3 \in L(\mathcal{A})$ is not modular.

§3.3 Cohomology of the Orlik-Solomon Algebras and $\dim H^1(A(\mathcal{A}), a)$

In this section, we define the *cohomology* of the Orlik-Solomon algebra and discuss recent results from the literature on $\dim H^1(A(\mathcal{A}), a)$. We refer to [8] for expository accounts on this subject and for a more detailed bibliography than will be presented here.

Let $\mathcal{A}$ be an arrangement, and let $A(\mathcal{A})$ be the Orlik-Solomon algebra. By §3.1, we have that $A(\mathcal{A}) = \oplus A_p(\mathcal{A})$.

DEFINITION 3.3.1. We construct a cochain complex on the homogeneous components of $A(\mathcal{A})$ as follows. Let $a \in A_1(\mathcal{A})$ with $a = \sum_{i=1}^{n} \lambda_i a_i$ for $\lambda_i \in \mathcal{K}$. Multiplication by a giving the differential $d_k : A_k(\mathcal{A}) \xrightarrow{a\cdot} A_{k+1}(\mathcal{A})$ forms a complex $(A(\mathcal{A}), a)$. The cohomology of this complex is said to be the cohomology of the Orlik-Solomon algebra and is denoted $H^*(A(\mathcal{A}), a)$.

THEOREM 3.3.2. Let $\mathcal{A}$ be a central hyperplane arrangement. Let $a = \sum_{i=1}^{n} \lambda_i a_i$ for $\lambda_i \in \mathcal{K}$. If $\sum_{i=1}^{n} \lambda_i \neq 0$, then $H^*(A(\mathcal{A}), a) = 0$.

PROOF. This is given in Proposition 2.1 in [13]. □

EXAMPLE 3.3.3. Let $Q(\mathcal{A}) = xy(x+y)$; let $a = a_1 - a_2$. Considering $H^1(A(\mathcal{A}), a)$, we see that $b := a_1 - a_3$ is in the kernel of d_1 but not in the image of d_0. Hence, $0 \neq [b] \in H^1(A(\mathcal{A}), a)$.

DEFINITION 3.3.4. Let $X \in L(\mathcal{A})$. Let $a \in A_1(\mathcal{A})$ with $a = \sum_{i=1}^{n} \lambda_i a_i$. We define

$$a(X) := \sum_{H_i \leq X} \lambda_i a_i.$$

The following results regarding $\dim H^1(A(\mathcal{A}), a)$ are from [8]. The results are presented here in a simplified version for our purposes.

DEFINITION 3.3.5. Let

$$\mathcal{X}(a) := \{X \in L(2, \mathcal{A}) : \ |X| > 2, a(X) \neq 0, \sum_{H_i < X} \lambda_i = 0\}.$$

DEFINITION 3.3.6. Let $I(a) \subset \{1, \dots n\}$ be defined as follows. We have $i \in I(a)$ if

(i) $H_i < X$ for some $X \in \mathcal{X}(a)$, and

(ii) if $\lambda_i = 0$, then there does not exist $\lambda_j \neq 0$ for which H_i, H_j are not in any $X \in \mathcal{X}(a)$.

DEFINITION 3.3.7. Let Γ be the graph with vertices $i \in I(a)$ and edges defined as follows. Define an edge from i to j if $H_i \vee H_j \notin \mathcal{X}(a)$. We then have a partition of $I(a)$ via the path components of Γ; let Π be the partition of Γ into its connected components.

DEFINITION 3.3.8. The incidence matrix J is the $|\mathcal{X}(a)| \times |I(a)|$ matrix with $J_{X,i} = 1$ if $H_i < X$ and zero otherwise.

Let E be the $|I(a)| \times |I(a)|$ matrix with ones in every entry. Let $Q = J^t J - E$. Decompose Q into the direct sum of its principle indecomposable submatrices so that $Q = \bigoplus_{K \in \Pi} Q_K$.

DEFINITION 3.3.9. A matrix M over $\mathbb{R}$ is affine if it is positive semidefinite and its null space is spanned by a positive vector, meaning all coordinates are positive. A matrix M is indefinite if there exists a vector $u > 0$ so that $Mu < 0$.

THEOREM 3.3.10. Let char $\mathcal{K} = 0$. For an arrangement $\mathcal{A}$, there are only two possibilities:

1. For each K, we have Q_K is either affine or has only the zero vector for its kernel. In this case, we say $\mathcal{X}(a)$ is affine.
2. There exists an unique K_0 so that Q_{K_0} is indefinite and for all other K we have that Q_K has only the zero vector for its kernel. In this case, we say $\mathcal{X}(a)$ is indefinite.

PROOF. This is given in Proposition 2.2 in [8]. □

THEOREM 3.3.11. Let char $\mathcal{K} = 0$. We have the following:

1. If $\mathcal{X}(a)$ is affine, then $Z_1(a) = \text{Ker } J \cap \{\sum_{i \in I(a)} x_i = 0\} \cap \{x_i = 0 : \text{ if } i \notin I(a)\}$.
2. If $\mathcal{X}(a)$ is indefinite or $\mathcal{X}(a) = \emptyset$, then $\dim Z_1(a) = 1$.

PROOF. This is given in Theorem 3.4 in [8]. □

EXAMPLE 3.3.12. Let char $\mathcal{K} = 0$. Let $\mathcal{A}$ be the arrangement given by $Q(\mathcal{A}) = xy(x+y)$; order the hyperplanes as they are written. Let $a := a_1 - a_2 \in A_1(\mathcal{A})$. We compute $\mathcal{X}(a) = \{H_1 \cap H_2 \cap H_3\}$, $I(a) = \{1, 2, 3\}$, and $\Pi = \{\{1, 2, 3\}\}$. Moreover, the matrix $J = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ gives Q to be the 3×3 matrix of zeros. Since Q is affine, Theorem 3.3.11 gives us that $Z_1(a) = \text{Ker } J \cap \{\sum_{i \in I(a)} x_i = 0\} = \{\sum_{i \in I(a)} x_i = 0\}$. Hence, $Z_1(a) = \{\sum_{i=1}^{3} x_i a_i : \ x_1 + x_2 + x_3 = 0\}$. Therefore, $\dim Z_1(a) = 2$.

EXAMPLE 3.3.13. Let char $\mathcal{K} = 0$. Let $\mathcal{A}$ be the arrangement given by $Q(\mathcal{A}) = xy(x+y)(x+y+z)z$; order the hyperplanes as they are written. Let $a := a_1 - a_2 + a_4 - a_5$. We compute

$$\mathcal{X}(a) = \{H_1 \cap H_2 \cap H_3, H_3 \cap H_4 \cap H_5\},$$

$$I(a) = \{1, 2, 3, 4, 5\}, \text{and}$$

$$\Pi = \{\{3\}, \{1, 2, 4, 5\}\}.$$

The matrix

$$J = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

gives us

$$Q = \begin{pmatrix} 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

Since Q is indefinite, Theorem 3.3.11 gives us $\dim Z_1(a) = 1$.

EXAMPLE 3.3.14. Let char $\mathcal{K} = 0$. Let $\mathcal{A}$ be the arrangement given by $Q(\mathcal{A}) = xyz(x-y)(x-z)(y-z)(x+y)$; order the hyperplanes as they are written. Let $a := a_1 - a_2 - a_5 + a_6$. We compute

$$\mathcal{X}(a) = \{H_1 \cap H_2 \cap H_4 \cap H_7, H_1 \cap H_3 \cap H_5, H_2 \cap H_3 \cap H_6, H_4 \cap H_5 \cap H_6\},$$

$$I(a) = \{1, 2, 3, 4, 5, 6\}, \text{and}$$

$$\Pi = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\}.$$

The matrix

$$J = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

gives us

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since Q is affine, Theorem 3.3.11 and some linear algebra gives us $\dim Z_1(a) = 2$.

EXAMPLE 3.3.15. Let char $\mathcal{K} = 0$. Let $\mathcal{A}$ be the arrangement given by $Q(\mathcal{A}) = xyzw(x+y)$; order the hyperplanes as they are written. Let $a := a_1 - a_2$, and let $b := a_1 - a_2 + a_3 - a_4$. By computing, we have

$$\mathcal{X}(a) = \mathcal{X}(b) = \{H_1 \cap H_2 \cap H_5\},$$

$$I(a) = \{1, 2, 5\}, \text{ and}$$

$$I(b) = \{1, 2\}.$$

Therefore, $\dim Z_1(a) = 2$ and $\dim Z_1(b) = 1$.

CHAPTER IV

THE VANISHING OF $H^*(A(\mathcal{A}), a)$

In this chapter, our main goal is to establish a necessary and sufficient condition for the vanishing of $H^*(A(\mathcal{A}), a)$. In §4.1, we employ tools from operator theory to prove the upper semicontinuity of the map $t \mapsto \dim H^p(A(\mathcal{A}), t)$ for any $p \geq 0$ and for any $t \in A_1(\mathcal{A})$. In §4.2, we analyze tensor products in the category of graded commutative algebras in order to express the cohomology of a reducible arrangement in terms of the cohomology of each factor of the arrangement. In §4.3, we apply results discussed in §4.1 and §4.2 to achieve the goal.

§4.1 The Upper Semicontinuity of $t \mapsto \dim H^p(A(\mathcal{A}), t)$

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement. Let $\mathcal{K} = \mathbb{C}$ or $\mathbb{R}$. Let $A(\mathcal{A})$ be the Orlik-Solomon algebra over $\mathcal{K}$.

In this section, we show the function $A_1(\mathcal{A}) \to \mathbb{Z}$ given by $t \mapsto \dim H^p(A(\mathcal{A}), t)$ is upper semicontinuous in t for any p. We show this in the more general setting of finite dimensional vector spaces and hence begin by establishing some standard definitions and notational conventions, which can be found in [5].

Let V be an n-dimensional vector space over $\mathcal{K}$. Relative to a basis $\{b_1, \ldots, b_n\}$ for V over $\mathcal{K}$, for $v \in V$ we express $v = (v_1, \ldots, v_n)$ as $v = \sum_{i=1}^{n} v_i b_i$.

Since $\mathcal{K} = \mathbb{C}$ or $\mathbb{R}$, we define the standard Euclidean norm, $\|\cdot\|_V$, on V as

$$\|v\|_V = \|(v_1, \ldots, v_n)\|_V := \sqrt{v_1\bar{v}_1 + \cdots + v_n\bar{v}_n},$$

relative to the standard orthonormal basis $\{e_1, \dots, e_n\}$ for V.

With respect to the norm $\|\cdot\|_V$, we define the unit sphere in V by

$$S(V) := \{v \in V : \|v\|_V = 1\}.$$

We also have the corresponding standard inner product, $(\cdot,\cdot)$, defined on V as

$$(x, y) := x_1\bar{y}_1 + \cdots + x_n\bar{y}_n \text{ for } x, y \in V.$$

We define the orthogonal complement relative to the inner product. Let $X \subseteq V$ be a linear subspace. Then

$$X^\perp := \{v \in V : (v, x) = 0 \text{ for all } x \in X\}.$$

Let $X \subseteq V$ be a linear subspace and $v \in V$. We define the distance from v to X to be

$$\text{dist } (v, X) := \inf\{\|v - x\|_V : x \in X\}.$$

Note there exists an unique $x_0 \in X$ for which dist $(v, X) =$ dist (v, x_0).

For the remainder of this section, we fix two finite dimensional vector spaces V and W over $\mathcal{K}$. Fix $0 \neq A \in \text{Hom}_{\mathcal{K}}(V, W)$.

DEFINITION 4.1.1. Define

$$\gamma(A) := \inf\{\|Ah\|_W : h \in S(V) \cap (\text{Ker } A)^\perp\}.$$

LEMMA 4.1.2. If $0 \neq A \in \text{Hom}_{\mathcal{K}}(V, W)$, then $\gamma(A) > 0$.

PROOF. Clearly, $\gamma(A) \geq 0$. Suppose that $\gamma(A) = 0$. By definition of the infimum, there exists a sequence $\{h_j\} \subset S(V) \cap (\text{Ker } A)^\perp$ so that $\|Ah_j\|_W \to 0$. This implies $\lim_{j\to\infty} h_j \in \text{Ker } A \cap (\text{Ker } A)^\perp = \{0\}$. But $\|\cdot\|$ is continuous in the metric; hence, $\lim_{j\to\infty} \|h_j\|_V = 1$. This contradiction proves the lemma. □

LEMMA 4.1.3. If $A \in \mathrm{Hom}_{\mathcal{K}}(V, W)$ and if $h \in V$, then we have

$$\gamma(A) \cdot \mathrm{dist}\ (h, \mathrm{Ker}\, A) \leq \|Ah\|_W.$$

PROOF. Let $\rho : V \to (\mathrm{Ker}\, A)^{\perp}$ be orthogonal projection of V onto $(\mathrm{Ker}\, A)^{\perp}$. We relate the norm to the distance by noticing $\|\rho h\|_V = \mathrm{dist}\ (h, \mathrm{Ker}\, A)$. Hence,

$$\begin{aligned} \|Ah\|_W =& \|A\rho h\|_W \\ \geq& \gamma(A) \cdot \|\rho h\|_V \\ =& \gamma(A) \cdot \mathrm{dist}\ (h, \mathrm{Ker}\, A). \end{aligned}$$

The lemma now follows. □

LEMMA 4.1.4. If $V_1, V_2 \subseteq V$ are linear subspaces with $\dim V_1 > \dim V_2$, then there exists $0 \neq v_1 \in V_1$ so that $\|v_1\|_V = \mathrm{dist}\ (v_1, V_2)$.

PROOF. Let ρ_1 be the orthogonal projection of V onto V_1. We have the inequality $\dim \rho_1(V_2) \leq \dim V_2 < \dim V_1$, so $\rho_1(V_2)$ is a proper linear subspace of V_1. Take $0 \neq v_1 \in V_1 \cap (\rho_1(V_2))^{\perp}$. Then for any $v_2 \in V_2$, we have

$$\begin{aligned} 0 =& (\rho_1(v_2), v_1) \\ =& (v_2, \rho_1(v_1)) \\ =& (v_2, v_1). \end{aligned}$$

Thus, $v_1 \in V_2^{\perp}$. Consequently, $\|v_1\|_V = \mathrm{dist}\ (v_1, V_2)$. □

DEFINITION 4.1.5. Let $B \in \mathrm{Hom}_{\mathcal{K}}(V, W)$. The operator norm of B is defined to be

$$\|B\|_{\mathrm{op}} := \sup\{\|Bh\|_W : \ h \in S(V).\}$$

We note that for any $h \in V$, the inequality holds:

$$\|Bh\|_W \leq \|B\|_{\mathrm{op}} \|h\|_V.$$

PROPOSITION 4.1.6. If $B \in \mathrm{Hom}_{\mathcal{K}}(V, W)$ with $\|B\|_{\mathrm{op}} < \gamma(A)$ then

$$\dim \mathrm{Ker}(A+B) \leq \dim \mathrm{Ker}\, A.$$

PROOF. If $0 \neq h \in \mathrm{Ker}(A+B)$, then $Ah = -Bh$. By Lemma 4.1.3, we have

$$\begin{aligned}\gamma(A) \cdot \text{dist } (h, \mathrm{Ker}\, A) \leq & \|Ah\|_W \\ = & \|Bh\|_W \\ \leq & \|B\|_{op} \cdot \|h\|_V \\ < & \gamma(A) \cdot \|h\|_V.\end{aligned}$$

Thus, dist $(h, \mathrm{Ker}\, A) < \|h\|_V$, for all $0 \neq h \in \mathrm{Ker}(A+B)$. By Lemma 4.1.4, we have $\dim \mathrm{Ker}(A+B) \leq \dim \mathrm{Ker}\, A$. □

DEFINITION 4.1.7. Let $A \in \mathrm{Hom}_{\mathcal{K}}(V, W)$. We define the adjoint of A, denoted by $A^* \in \mathrm{Hom}_{\mathcal{K}}(W, V)$, by $(x, A^*y) := (Ax, y)$ for all $x \in V$ and for all $y \in W$.

LEMMA 4.1.8. If $A \in \mathrm{Hom}_{\mathcal{K}}(V, W)$, then $\mathrm{Ker}\, A^* = (\text{range } A)^{\perp}$.

PROOF. Let $y \in \mathrm{Ker}\, A^*$. Then $y \in (Ax)^{\perp}$ for any $x \in V$. Thus $y \in (\text{range } A)^{\perp}$.

Let $y \in (\text{range } A)^{\perp}$. Then for any $x \in V$, we have $0 = (Ax, y) = (x, A^*y)$. This implies $A^*y = 0$; hence, $y \in \mathrm{Ker}\, A^*$. □

PROPOSITION 4.1.9. Let $A, B \in \mathrm{Hom}_{\mathcal{K}}(V, W)$. If $\|B^*\|_{\mathrm{op}} < \gamma(A^*)$, then we have $\mathrm{rank}(A+B) \geq \mathrm{rank}\, A$.

PROOF. From Proposition 4.1.6, we have

$$\dim \mathrm{Ker}(A+B)^* = \dim \mathrm{Ker}(A^* + B^*) \leq \dim \mathrm{Ker}\, A^*.$$

Since $\dim \mathrm{Ker}\, A^* = \dim(\text{range } A)^{\perp} = \dim W - \mathrm{rank}\, A$, it follows that

$$\dim W - \mathrm{rank}(A+B) \leq \dim W - \mathrm{rank}(A). \quad \square$$

DEFINITION 4.1.10. Let X be a topological space. Let $f : X \to \mathbb{R}$ be a real-valued function; f is said to be upper semicontinuous if for any real number α the set $\{x \in X : f(x) < \alpha\}$ is open. Alternatively, for X a metric space we may define f to be upper semicontinuous at $x_0 \in X$ if

$$\limsup_{x \to x_0} f(x) \leq f(x_0).$$

Recall from Chapter III that for an arrangement $\mathcal{A}$, we have the Orlik-Solomon algebra $A(\mathcal{A})$ over $\mathcal{K}$. Moreover, this is a graded algebra $A(\mathcal{A}) = \oplus A_p(\mathcal{A})$. Since $\mathcal{K} = \mathbb{R}$ or $\mathbb{C}$, we have $A_p(\mathcal{A})$ is a finite dimensional vector space. For any $t \in A_1(\mathcal{A})$, let the map $t\cdot : A_p(\mathcal{A}) \to A_{p+1}(\mathcal{A})$ be given by multiplication by t. Let $Z_p(A(\mathcal{A}), t)$ denote the kernel of the map $t\cdot : A_p(\mathcal{A}) \to A_{p+1}(\mathcal{A})$; let $B_p(A(\mathcal{A}), t)$ denote the image of the map $t\cdot : A_p(\mathcal{A}) \to A_{p+1}(\mathcal{A})$.

LEMMA 4.1.11. If $\|t - t_0\|_{\mathcal{K}^n} \to 0$, then $\|t \cdot - t_0 \cdot\|_{\text{op}} \to 0$.

PROOF. We have $A_p(\mathcal{A})$ and $E_p(\mathcal{A})$ are finite dimensional vector spaces over $\mathcal{K}$. We use the standard basis for $E_p(\mathcal{A})$ given by $\{e_{i_1} \cdots e_{i_p} : 1 \leq i_1 < \ldots < i_p \leq p\}$. We use the broken circuit basis for $A_p(\mathcal{A})$. Then $\|\cdot\|_{E_p(\mathcal{A})}$ and $\|\cdot\|_{A_p(\mathcal{A})}$ are defined as previously.

It will suffice to show $\|(t - t_0)(v)\|_{A_p(\mathcal{A})} \to 0$ for any $v \in A_{p-1}(\mathcal{A})$. But the maps given by $(t - t_0)\cdot : E_{p-1}(\mathcal{A}) \to E_p(\mathcal{A})$ commute with the projection map $\pi : E_p(\mathcal{A}) \to A_p(\mathcal{A})$; that is,

$$\begin{array}{ccc} E_{p-1}(\mathcal{A}) & \xrightarrow{(t-t_0)\cdot} & E_p(\mathcal{A}) \\ \downarrow \pi & \circ & \downarrow \pi \\ A_{p-1}(\mathcal{A}) & \xrightarrow{(t-t_0)\cdot} & A_p(\mathcal{A}). \end{array}$$

Hence, it will suffice to show $\|(t - t_0) \cdot v\|_{E_p(\mathcal{A})} \to 0$ for any $v \in E_{p-1}(\mathcal{A})$.

We need only show that $\|(t - t_0) \cdot b\|_{E_p(\mathcal{A})} \to 0$ for any standard basis element $b \in E_{p-1}(\mathcal{A})$. We write $b = e_{i_1} \cdots e_{i_{p-1}}$, where $1 \le i_1 < \ldots < i_{p-1} \le n$. Then

$$\begin{aligned}\|(t - t_0) \cdot b\|^2_{E_p(\mathcal{A})} =& \|(t - t_0) \cdot e_{i_1} \cdots e_{i_p}\|^2_{E_p(\mathcal{A})} \\ =& \|\left(\sum_{i=1}^{n} (t^i - t_0^i) e_i\right) e_{i_1} \cdots e_{i_{p-1}}\|^2_{E_p} \\ =& \sum_{i \neq i_1, \ldots, i_{p-1}} (t^i - t_0^i)^2\end{aligned}$$

As $\|t - t_0\|_{\mathcal{K}^n} \to 0$, we have $\|(t - t_0) \cdot b\|^2_{E_p(\mathcal{A})} \to 0$ as required. □

THEOREM 4.1.12. Let $\mathcal{A}$ be a hyperplane arrangement with n hyperplanes. Let $A(\mathcal{A})$ be the Orlik-Solomon algebra on $\mathcal{A}$ over the field $\mathcal{K}$, where $\mathcal{K}$ is either $\mathbb{C}$ or $\mathbb{R}$. The function $t \mapsto \dim H^p(A(\mathcal{A}), t)$ from $A_1(\mathcal{A})$ to $\mathbb{Z}$ is upper semicontinuous.

PROOF. We first identify $A_1(\mathcal{A})$ with $\mathcal{K}^n$.

The result clearly holds for $t_0 = 0 \in A_1(\mathcal{A})$. That is,

$$\limsup_{t \to 0} \dim H^p(A(\mathcal{A}), t) \le \dim H^p(A(\mathcal{A}), 0) = \dim A_p(\mathcal{A}).$$

Fix $0 \neq t_0 \in \mathcal{K}^n$. Let $\epsilon = \min\{\gamma(t_0 \cdot), \gamma(t_0 \cdot^*)\}$. By Lemma 4.1.2, $\epsilon > 0$. As $t \to t_0$ in $\mathcal{K}^n$, by Lemma 4.1.11, we have $\|t \cdot - t_0 \cdot\|_{\text{op}} \to 0$. Hence, there exists $\delta > 0$ so that $\|t \cdot - t_0 \cdot\|_{\text{op}} < \epsilon$ whenever $\|t - t_0\|_{\mathcal{K}^n} < \delta$.

Consequently, we use Proposition 4.1.6, Proposition 4.1.9, and Lemma 4.1.11 to see $\dim \operatorname{Ker}(A(\mathcal{A}), t) \le \dim \operatorname{Ker}(A(\mathcal{A}), t_0)$ and $\operatorname{rank}(A(\mathcal{A}), t) \ge \operatorname{rank}(A, t_0)$. Thus,

$$\begin{aligned}\dim H^p(A(\mathcal{A}), t) =& \dim Z_p(A(\mathcal{A}), t) - \dim B_{p-1}(A(\mathcal{A}), t) \\ \le& \dim Z_p(A(\mathcal{A}), t_0) - \dim B_{p-1}(A(\mathcal{A}), t_0) \\ =& \dim H^p(A(\mathcal{A}), t_0).\end{aligned}$$

The assertion now follows. □

§4.2 Tensor Products in the Category of Graded Commutative Algebras

Let $\mathcal{K}$ be a commutative ring. We introduce the following definitions and notational conventions, as can be found in [9].

DEFINITION 4.2.1. (Tensor Product of Modules) Let M and N be $\mathcal{K}$-modules. The tensor product $M \otimes N$ is the abelian group with generators being all symbols $m \otimes n$ for $m \in M$ and $n \in N$ subject to the relations ($k \in \mathcal{K}$)

(i) $(m + m', n) - (m, n) - (m', n)$

(ii) $(m, n + n') - (m, n) - (m, n')$

(iii) $(km, n) - (m, kn)$.

There exists a bilinear map $\phi : M \times N \to M \otimes N$ so that $\phi(m, n) = m \otimes n$. We have the following universal property. Let A be a $\mathcal{K}$-module. For any bilinear homomorphism $f : M \times N \to A$, there exists an unique $\hat{f} : M \otimes N \to A$ so that $f(m, n) = \hat{f}(m \otimes n)$.

DEFINITION 4.2.2. (Graded Module) We say M is a graded $\mathcal{K}$-module if there is a family of $\mathcal{K}$-modules $\{M_n\}_{n \geq 0}$ so that $M = \bigoplus_{n \geq 0} M_n$. For $m \in M_n$, we write $\deg(m) = n$.

DEFINITION 4.2.3. (Tensor Product of Graded Modules) Let M and N be graded $\mathcal{K}$-modules. The tensor product $M \otimes N$ is the graded module given by

$$(M \otimes N)_n = \bigoplus_{p+q=n} M_p \otimes N_q. \qquad (4.2.3.a)$$

Let A be a graded $\mathcal{K}$-module. Let $f : M \times N \to A$ be any bilinear graded homomorphism, there exists an unique graded homomorphism $\hat{f} : M \otimes N \to A$ so that $f(m, n) = \hat{f}(m \otimes n)$.

DEFINITION 4.2.4. (Graded Commutative Algebra) M is said to be a graded commutative $\mathcal{K}$-algebra if the following are satisfied:

1. M is a graded $\mathcal{K}$-module.
2. There is an associative multiplication in M so that $M_pM_q \subseteq M_{p+q}$.
3. (Commutative) For homogeneous elements $a, b \in M$ we have

$$ab = (-1)^{\deg(a)\cdot\deg(b)}\ ba.$$

DEFINITION 4.2.5. (Tensor Product of Graded Commutative Algebras) Let M and N be graded commutative $\mathcal{K}$-algebras. The tensor product $M \otimes N$ is the graded commutative $\mathcal{K}$-algebra given by

1. $M \otimes N$ is a graded $\mathcal{K}$-module defined in (4.2.3.a).
2. Multiplication is defined by $(m \otimes n)(m' \otimes n') := (-1)^{\deg(n)\deg(m')} mm' \otimes nn'$.

Note: One can check that this multiplication is commutative.

We have the following universality description of $M \otimes N$. Let A be graded commutative $\mathcal{K}$-algebra. Let $f:\ M \times N \to A$ be a bilinear graded homomorphism with

$$f((m,n)(m',n')) = (-1)^{\deg(n)\deg(m')} f(m,n)f(m',n').$$

There exists an unique $\hat{f}:\ M \otimes N \to A$ so that $f(m,n) = \hat{f}(m \otimes n)$.

EXAMPLE 4.2.6. Let $\mathcal{A}_i$ be arrangements. Let $A(\mathcal{A}_i)$ denote the Orlik-Solomon algebra on the arrangement $\mathcal{A}_i$ over the commutative ring $\mathcal{K}$. Then $A(\mathcal{A}_i)$ is a graded commutative algebra over $\mathcal{K}$. Hence, we have defined $\bigotimes_i A(\mathcal{A}_i)$.

We recall the product arrangement as defined in Chapter II. Let $\mathcal{A}_1$ be an arrangement in V_1, and let $\mathcal{A}_2$ be an arrangement in V_2. If $V = V_1 \oplus V_2$, then we put

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 = \{H \oplus V_2 :\ H \in \mathcal{A}_1\} \cup \{V_1 \oplus H :\ H \in \mathcal{A}_2\}.$$

We recall the broken circuit basis for the Orlik-Solomon algebra $A(\mathcal{A})$. Let $S = \{H_{i_1}, ..., H_{i_p}\}$ be an ordered subset of $\mathcal{A}$ with $i_1 < \cdots < i_p$. We say a_S is basic in $A_p(\mathcal{A})$ if

1. S is independent, and

2. For any $1 \leq k \leq p$, there does not exist a hyperplane $H \in \mathcal{A}$ so that $H < H_{i_k}$ with $\{H, H_{i_k}, H_{i_{k+1}}, ..., H_{i_p}\}$ dependent.

LEMMA 4.2.7. If $a_\gamma \in A_p(\mathcal{A}_1)$ and $a_\sigma \in A_q(\mathcal{A}_2)$ are basic, then $a_\gamma a_\sigma$ is basic in $A(\mathcal{A}_1 \times \mathcal{A}_2)$.

PROOF. Order the hyperplanes in $\mathcal{A}_1 \times \mathcal{A}_2$ via

1. $H \oplus V_2 < K \oplus V_2$ if $H < K$ in $\mathcal{A}_1$

2. $V_1 \oplus H < V_1 \oplus K$ if $H < K$ in $\mathcal{A}_2$

3. $H \oplus V_2 < V_1 \oplus K$ if $H \in \mathcal{A}_1$ and $K \in \mathcal{A}_2$

Let $a_\gamma \in A_p(\mathcal{A}_1)$ and $a_\sigma \in A_q(\mathcal{A}_2)$ be basic (i.e. in the broken circuit basis). Suppose $\mathcal{A}_1$ is an arrangement in V_1 and $\mathcal{A}_2$ is an arrangement in V_2. Suppose $a_\gamma a_\sigma$ is not basic in $A(\mathcal{A}_1 \times \mathcal{A}_2)$. By definition of the broken circuit basis, there are only two possibilities. Suppose $(\cap_{H\in\gamma}(H \oplus V_2)) \cap (\cap_{H\in\sigma}(V_1 \oplus H)) = \emptyset$. This happens only if $\cap_{H\in\gamma} H = \emptyset$ or $\cap_{H\in\sigma} H = \emptyset$. This is not possible since a_σ and a_γ are basic.

Suppose there exists a hyperplane $H \in \mathcal{A}_1 \times \mathcal{A}_2$ and a subset ρ of $\gamma \cup \sigma$ with $H < \rho$ so that $\{H, \rho\}$ is dependent. But this implies the linear functionals defining the hyperplanes are linearly dependent. Since $Q(\mathcal{A}_1)$ and $Q(\mathcal{A}_2)$ have no common variables, this implies H is dependent upon $\rho \cap \gamma$ or $\rho \cap \sigma$. This contradicts the fact a_γ and a_σ are basic. Our assertion now follows. □

LEMMA 4.2.8. For the product arrangement, we have $A(\mathcal{A}_1 \times \mathcal{A}_2) \cong A(\mathcal{A}_1) \otimes A(\mathcal{A}_2)$.

PROOF. We define the map $\phi: A(\mathcal{A}_1) \times A(\mathcal{A}_2) \rightarrow A(\mathcal{A}_1 \times \mathcal{A}_2)$ on the generators by $\phi(a_\gamma, a_\sigma) := a_\gamma a_\sigma$, and we extend the map ϕ bilinearly.

Moreover, we have

$$\begin{aligned}\phi((a_\gamma, a_\sigma)\cdot(a_{\gamma'}, a_{\sigma'})) &= \phi(a_\gamma a_{\gamma'}, a_\sigma a_{\sigma'})\\ &= a_\gamma a_{\gamma'} a_\sigma a_{\sigma'}\\ &= (-1)^{\deg(\sigma)\cdot\deg(\gamma')} a_\gamma a_\sigma a_{\gamma'} a_{\sigma'}\\ &= (-1)^{\deg(\sigma)\cdot\deg(\gamma')} \phi(a_\gamma, a_\sigma)\cdot\phi(a_{\gamma'}, a_{\sigma'}).\end{aligned}$$

By the universal mapping property, there exists

$$\hat{\phi}:\ A(\mathcal{A}_1)\otimes A(\mathcal{A}_2)\to A(\mathcal{A}_1\times A_2)$$

so that $\phi = \hat{\phi}\pi$, where $\pi : A(\mathcal{A}_1)\times A(\mathcal{A}_2) \to A(\mathcal{A}_1)\otimes A(\mathcal{A}_2)$ is the canonical projection. Now, $\hat{\phi}$ is clearly surjective. All that remains is to verify injectivity.

Let $a_\gamma \in A_p(\mathcal{A}_1)$ and $a_\sigma \in A_q(\mathcal{A}_2)$ be basic. Suppose $\hat{\phi}(\sum \alpha_i a_{\gamma_i}\otimes a_{\sigma_i}) = 0$. Then by the linearity of $\hat{\phi}$, we have $\sum \alpha_i a_{\gamma_i} a_{\sigma_i} = 0$. Since a_{γ_i} and a_{σ_i} are basic in $A(\mathcal{A}_1)$ and $A(\mathcal{A}_2)$, we have $a_{\gamma_i} a_{\sigma_i}$ is basic in $A(\mathcal{A}_1\times\mathcal{A}_2)$. Hence, we must have $\alpha_i = 0$ for each i. □

Suppose $\mathcal{A} = \mathcal{A}_1\times\mathcal{A}_2$. By Lemma 4.2.7, $A_1(\mathcal{A})$ can be identified with the linear space $A_1(\mathcal{A}_1)\oplus A_1(\mathcal{A}_2)$. Let $a\in A_1(\mathcal{A})$. We may express $a = a_1 + a_2$ uniquely for $a_1\in A_1(\mathcal{A})$ and $a_2\in A_1(\mathcal{A}_2)$. For the chain complexes $(A(\mathcal{A}_1), a_1)$ and $(A(\mathcal{A}_2), a_2)$, we recall tensor products of chain complexes; see [10].

Let the differential (multiplication by a_i) for the complex $(A(\mathcal{A}_i), a_i)$ be denoted d_i for $i = 1, 2$. The differential for the chain complex $(A(\mathcal{A}_1), a_1)\otimes(A(\mathcal{A}_2), a_2)$, written $d_1\otimes d_2$, is defined on generators as

$$(d_1\otimes d_2)(a_\gamma\otimes a_\sigma) := a_1 a_\gamma\otimes a_\sigma + (-1)^{\deg(a_\gamma)} a_\gamma\otimes a_2 a_\sigma.$$

LEMMA 4.2.9. Let $\mathcal{A} = \mathcal{A}_1\times\mathcal{A}_2$. Let $a\in A_1(\mathcal{A})$ with $a = a_1 + a_2$ for $a_1\in A_1(\mathcal{A}_1)$ and for $a_2\in A_1(\mathcal{A}_2)$. As chain complexes, $(A(\mathcal{A}_1\times\mathcal{A}_2), a) = (A(\mathcal{A}_1)\otimes A(\mathcal{A}_2), d_1\otimes d_2)$

PROOF. From Lemma 4.2.8, we have $A(\mathcal{A}_1 \times \mathcal{A}_2) = A(\mathcal{A}_1) \otimes A(\mathcal{A}_2)$. For a basic element $a_\rho \in A_p(\mathcal{A}_1 \times \mathcal{A}_2)$, we may write $a_\rho = a_\gamma \cdot a_\sigma$ with $a_\gamma \in A_m(\mathcal{A}_1)$ and $a_\sigma \in A_n(\mathcal{A}_2)$ and $m + n = p$. Hence, multiplying by a in the chain complex $A(\mathcal{A}_1 \times \mathcal{A}_2)$ gives the differential defined on generators as

$$\begin{aligned} a \cdot a_\rho =& (a_1 + a_2)(a_\gamma \cdot a_\sigma) \\ =& a_1 a_\gamma \cdot a_\sigma + (-1)^{\deg(a_\gamma)} a_\gamma \cdot a_2 a_\sigma. \end{aligned}$$

The result follows immediately. □

THEOREM 4.2.10. Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a product arrangement. Let $a \in A_1(\mathcal{A}_1 \times \mathcal{A}_2)$. Write $a = a_1 + a_2$ for $a_1 \in A_1(\mathcal{A}_1)$ and $a_2 \in A_1(\mathcal{A}_2)$. Let $\mathcal{K}$ be a field. We have:

$$H^m(A(\mathcal{A}_1 \times \mathcal{A}_2), a) = \bigoplus_{p+q=m} H^p(A(\mathcal{A}_1), a_1) \otimes H^q(A(\mathcal{A}_2), a_2).$$

PROOF. By Lemma 4.2.9, this is a direct application of the Künneth Formula (see [9]) to the cochain complex $(A(\mathcal{A}_1) \otimes A(\mathcal{A}_2), d_1 \otimes d_2)$. □

§4.3 $H^*(A(\mathcal{A}), a)$

In this section, we use the results of §4.1 and §4.2 to establish necessary and sufficient conditions for $H^*(A(\mathcal{A}), a) = 0$.

Let $\mathcal{A}$ be an affine arrangement. We may write

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k \times \mathcal{B},$$

where $\mathcal{A}_i$ are each central and $\mathcal{B}$ is not central. Moreover, we may assume each $\mathcal{A}_i$ contains no proper central factors and $\mathcal{B}$ contains no central factors; otherwise, we would decompose the arrangement further. For $a \in A_1(\mathcal{A})$, write $a = a_1 + \cdots + a_k + a_{\mathcal{B}}$ for $a_j \in A_1(A_j)$ and $a_{\mathcal{B}} \in A_1(\mathcal{B})$.

EXAMPLE 4.3.1. Let $Q(\mathcal{A}) = x(x-1)y(y-1)$. Atlhough $\mathcal{A}$ is a product of affine arrangements, $\mathcal{A}$ contains no central factors. Hence, $\mathcal{A} = \mathcal{B}$ in this case.

We recall the deconed arrangement from Chapter II. Suppose $\mathcal{A}$ is central. We form the deconed arrangement $d\mathcal{A}$ as follows. Let α_i be the functional corresponding to H_i. Without loss of generality, we may assume $\alpha_1 = x_1$. Decone at $\alpha_1 = x_1$ by setting $x_1 = 1$.

LEMMA 4.3.2. If $\mathcal{A}$ is a central hyperplane arrangement and $\mathcal{A}$ contains no proper central factor, then $d\mathcal{A}$ contains no central factor.

PROOF. Suppose $d\mathcal{A}$ contains a central factor. There exist subarrangements $\mathcal{C}_1$ and $\mathcal{C}_2$ of $d\mathcal{A}$ so that $d\mathcal{A} = \mathcal{C}_1 \times \mathcal{C}_2$. Moreover, we may assume $\mathcal{C}_1$ is central. Since $\mathcal{C}_1$ is central, by taking a linear change of coordinates if necessary, we may assume the hyperplanes of $\mathcal{C}_1$ pass through the origin; i.e. we are assuming the defining equation $Q(\mathcal{C}_1)$ consists of linear functionals. Then by coning, we obtain $\mathcal{A}$. Since the defining equation of $\mathcal{C}_1$ is unaffected by coning the arrangement $d\mathcal{A}$, we have constructed a central factor of $\mathcal{A}$. This contradicts the assumption that $\mathcal{A}$ contains no central factors. □

EXAMPLE 4.3.3. To demonstrate the proof of Lemma 4.3.2, we consider an arrangement $\mathcal{A}$ where both $\mathcal{A}$ and $d\mathcal{A}$ contain a central factor. Let $Q(\mathcal{A}) = z(x - z)(y-z)(w-z)w$. When deconing at $z = 1$, we have $Q(d\mathcal{A}) = (x-1)(y-1)(w-1)w$. Take $Q(\mathcal{C}_1) = (x-1)(y-1)$ and $Q(\mathcal{C}_2) = (w-1)w$. Let $\hat{x} = x - 1$ and $\hat{y} = y - 1$. We have $Q(d\mathcal{A}) = \hat{x}\hat{y}(w-1)w$. When coning, we have $Q((d\mathcal{A})^c) = z\hat{x}\hat{y}(w-z)w$. By taking the linear change of coordinate $\hat{x} = x - z$ and $\hat{y} = y - z$, we see that $\mathcal{A}$ and $(d\mathcal{A})^c$ are linearly isomorphic.

We recall the Euler characteristic of an arrangement $\mathcal{A}$. Let $\operatorname{rank}(\mathcal{A}) = \ell$. The Euler characteristic is given by

$$\chi(\mathcal{A}) = \sum_{i=1}^{\ell}(-1)^i \dim A_i(\mathcal{A}).$$

We also note that $\chi(d\mathcal{A})$ depends only on $L(\mathcal{A})$.

Let $\mathcal{A}$ be an arrangement. Let $H_0 \in \mathcal{A}$. We recall the arrangements given by deletion and restriction

$$\mathcal{A}' = \{H : \ H \in \mathcal{A} \setminus H_0\}, \text{and}$$

$$\mathcal{A}'' = \{H_0 \cap H : \ H \in \mathcal{A} \text{ and } H \cap H_0 \neq \emptyset\}.$$

Recall $r(\mathcal{A}) = \max_{X \in L(\mathcal{A})} \operatorname{rank}(X)$.

We need the following lemmas and proposition, established in [6].

LEMMA 4.3.4. Let $\mathcal{A}$ be an affine arrangement with $r(\mathcal{A}) > 1$. If $\mathcal{A}$ does not contain a central factor, then for any distinguished hyperplane $H_0 \in \mathcal{A}$ either $\mathcal{A}'$ or A'' does not contain a central factor.

PROOF. We refer to the proof given in Lemma to Theorem II in [6]. □

We define

$$\beta(\mathcal{A}) := (-1)^{r(\mathcal{A})}\chi(\mathcal{A}).$$

LEMMA 4.3.5. Let $\mathcal{A}$ be an arrangement with $|\mathcal{A}| > 1$. If $\mathcal{A}$ is not central, then there exists $H_0 \in \mathcal{A}$ so that $\operatorname{rank}(\mathcal{A}') = \operatorname{rank}(\mathcal{A})$. With respect to H_0, we have the equality $\beta(\mathcal{A}) = \beta(\mathcal{A}') + \beta(\mathcal{A}'')$. If $\mathcal{A}$ is central, then this inequality holds for any $H \in \mathcal{A}$.

PROOF. Suppose $\mathcal{A}$ is not central. Then there exists a maximal element $T \in L(\mathcal{A})$ and a hyperplane $H_0 \not< T$. Hence, T is a maximal element in $L(\mathcal{A}')$. Since $\chi(A) = \chi(\mathcal{A}') - \chi(\mathcal{A}'')$ by Theorem 2.56 in [12], we have

$$(-1)^{r(\mathcal{A})}\chi(\mathcal{A}) = (-1)^{r(\mathcal{A})}\chi(\mathcal{A}') - (-1)^{r(\mathcal{A})}\chi(\mathcal{A}'').$$

We have $r(\mathcal{A}'') = r(\mathcal{A}) - 1$ and $\operatorname{rank}(\mathcal{A}') = \operatorname{rank}(\mathcal{A})$, so

$$\beta(\mathcal{A}) = \beta(\mathcal{A}') + \beta(\mathcal{A}'').$$

If $\mathcal{A}$ is central, then $\mathcal{A}'$ and $\mathcal{A}''$ are both central, so $\chi(\mathcal{A}) = \chi(\mathcal{A}') = \chi(\mathcal{A}'') = 0$. □

LEMMA 4.3.6. If $\mathcal{A}$ is an arrangement, then $\beta(\mathcal{A}) \geq 0$.

PROOF. We induct on $|\mathcal{A}|$.

If $\mathcal{A} = \emptyset$, then $\beta(\mathcal{A}) = 1$. If $|\mathcal{A}| = 1$, then $\beta(\mathcal{A}) = 0$.

Assume $\beta(\mathcal{B}) \geq 0$ for all arrangements $\mathcal{B}$ with $|\mathcal{B}| < k$. Suppose $|\mathcal{A}| = k > 1$. By Lemma 4.3.5, we have $\beta(\mathcal{A}) = \beta(\mathcal{A}') + \beta(\mathcal{A}'')$ for some hyperplane $H_0 \in \mathcal{A}$. By the induction hypothesis, we have $\beta(\mathcal{A}'), \beta(\mathcal{A}'') \geq 0$. We therefore have $\beta(\mathcal{A}) \geq 0$ as required. □

PROPOSITION 4.3.7. Let $\mathcal{A}$ be an affine arrangement. We have $\chi(\mathcal{A}) \neq 0$ if and only if $\mathcal{A}$ contains no central factors.

PROOF. Suppose $\mathcal{A}$ contains a central factor; that is, $\mathcal{A} = \mathcal{B} \times \mathcal{C}$, where $\mathcal{B}$ is central. Then $\chi(\mathcal{A}) = \chi(\mathcal{B})\chi(\mathcal{C})$; see Lemma 2.50 in [12]. Since $\mathcal{B}$ is central, we have $\chi(\mathcal{B}) = 0$; see Proposition 2.51 in [12]. Hence, $\chi(\mathcal{A}) = 0$.

Suppose $\mathcal{A}$ contains no central factors. We want to show $\chi(\mathcal{A}) \neq 0$. It will suffice to show $\beta(\mathcal{A}) \neq 0$. We proceed by induction. Suppose $|\mathcal{A}| = 2$, then $\mathcal{A}$ consists of two hyperplanes which don't intersect; hence, $\chi(\mathcal{A}) = -1$.

Suppose for any $\mathcal{B}$ with $|\mathcal{B}| < k$ $(k > 1)$ for which $\mathcal{B}$ contains no central factors, we have $\beta(\mathcal{B}) \neq 0$ (hence, $\chi(\mathcal{B}) \neq 0$). Fix $H_0 \in \mathcal{A}$ so that $\beta(\mathcal{A}) = \beta(\mathcal{A}') + \beta(\mathcal{A}'')$. We apply Lemma 4.3.6 to see that $\mathcal{A}'$ or $\mathcal{A}''$ contains no central factors. By the induction hypothesis, $\beta(\mathcal{A}') > 0$ or $\beta(\mathcal{A}'') > 0$. Since $\beta(\mathcal{A}) = \beta(\mathcal{A}') + \beta(\mathcal{A}'')$ and $\beta(\mathcal{A}'), \beta(\mathcal{A}'') \geq 0$ with at least one positive, we have $\beta(\mathcal{A}) > 0$. Hence, $\chi(\mathcal{A}) \neq 0$ as required. □

THEOREM 4.3.8. Let $\mathcal{A}$ be an affine arrangement. We may write

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k \times \mathcal{B},$$

where $\mathcal{A}_j$ are each central and $\mathcal{B}$ not central and they contain no proper central factors. For $a \in A_1(\mathcal{A})$, write $a = a_1 + \cdots + a_k + a_{\mathcal{B}}$ for $a_j \in A_1(A_j)$ and $a_{\mathcal{B}} \in A_1(\mathcal{B})$. We have $H^*(A(\mathcal{A}), a) \neq 0$ if and only if $\sum_{H_i \in \mathcal{A}_j} \lambda_i = 0$ for all $1 \leq j \leq k$.

PROOF. We use the Künneth Formula from Theorem 4.2.10; that is,

$$H^m(A(\mathcal{A}), a) =$$

$$\bigoplus_{p_1 + \cdots p_k + p_{k+1} = m} H^{p_1}(A(\mathcal{A}_1), a_1) \otimes \cdots \otimes H^{p_k}(A(\mathcal{A}_k), a_k) \otimes H^{p_{k+1}}(A(\mathcal{B}), a_{\mathcal{B}}).$$

Suppose $\sum_{A_1(\mathcal{A}_j)} \lambda_i \neq 0$ for some j. We have $H^*(\mathcal{A}_j, a_j) = 0$ since $\mathcal{A}_j$ is central; we refer to [13]. By the Künneth Formula, it follows that $H^*(\mathcal{A}, a) = 0$.

Suppose $\sum_{A_1(\mathcal{A}_j)} \lambda_i = 0$ for all j. By the Künneth Formula, it will suffice to show $H^*(A(\mathcal{A}_i), a_i) \neq 0$ and $H^*(A(\mathcal{B}), a_{\mathcal{B}}) \neq 0$. Since $\mathcal{B}$ contains no central factors, we have $\chi(\mathcal{B}) \neq 0$. Hence, $H^*(A(\mathcal{B}), a_{\mathcal{B}}) \neq 0$.

Take $\tilde{a} = \sum_{i=2}^{n} \lambda_i a_i$. We consider the chain complex formed by multiplication by $\tilde{a}$, $(d\mathcal{A}, \tilde{a})$; here, $d\mathcal{A}$ is $\mathcal{A}$ deconed at H_1. Since we have the short exact sequences, see [13]

$$0 \to H^{p-1}(A(d\mathcal{A}_i), \tilde{a}) \to H^p(A(\mathcal{A}_i), a) \to H^p(A(d\mathcal{A}_i), \tilde{a}) \to 0,$$

it will suffice to show $H^*(A(d\mathcal{A}_i), \tilde{a}) \neq 0$. But by Lemma 4.3.2, $d\mathcal{A}_i$ contains no central factors, so by Proposition 4.3.7 $\chi(d\mathcal{A}_i) \neq 0$; hence, $H^*(A(d\mathcal{A}_i), \tilde{a}) \neq 0$ as required. □

We recall the following theorem from [13].

THEOREM 4.3.9. (Yuzvinsky [13]) Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arbitrary arrangement with rank$(\mathcal{A}) = \ell$. Let $a = \sum_{i=1}^{n} \lambda_i a_i \in A_1(\mathcal{A})$ and satisfy the condition $\sum_{X \subset H_i} \lambda_i \neq 0$ for all $X \in L(\mathcal{A})$ such that $\chi(dA(\mathcal{A}_X)) \neq 0$. Then $H^p(A(\mathcal{A}), a) = 0$ for every $p < \ell$.

We use Theorem 4.3.9 in conjunction with the upper semicontinuity of the map $t \mapsto \dim H^p(A(\mathcal{A}), t)$ discussed in §4.1 to establish conditions under which $\dim H^\ell(A(\mathcal{A}), a) \neq 0$ for an affine ℓ-arrangement $\mathcal{A}$.

THEOREM 4.3.10. Let $\mathcal{A}$ be an arbitrary ℓ-arrangement with $|\mathcal{A}| = n$. Suppose $\mathcal{A}$ contains no central factors. Let $a = \sum_{i=1}^{n} \lambda_i a_i \in A_1(\mathcal{A})$. We have:

$$H^\ell(A(\mathcal{A}), a) \neq 0.$$

PROOF. If $a = 0$, then $H^\ell(A(\mathcal{A}), a) = Z_\ell(A(\mathcal{A}), a) = A_\ell(\mathcal{A}) \neq 0$. Hence, we may assume $a \neq 0$ for the remainder of the proof.

Let

$$S = \{X \in L(\mathcal{A}) : \sum_{j \in X} \lambda_j \neq 0\}.$$

Since $a \neq 0$, we have $S \neq \emptyset$. We define

$$\kappa := \min \{|\sum_{j \in X} \lambda_j| : X \in S\}.$$

Since $S \neq \emptyset$, $\kappa > 0$.

We now construct a sequence $\alpha_i \in F^n$ so that $(\alpha_i)_j \to \lambda_j$ as $i \to \infty$. For $i \in \mathbb{N}$, define

$$(\alpha_i)_j := \lambda_j + \frac{\kappa}{n \cdot 2^i}.$$

It is clear that $(\alpha_i)_j \to \lambda_j$ as $i \to \infty$. Moreover, we now show $\sum_{j \in X} (\alpha_i)_j \neq 0$ for any $X \in L(\mathcal{A})$ and any $i \in \mathbb{N}$.

Fix $X \in L(\mathcal{A})$. If $\sum_{j \in X} \lambda_j = 0$, then $\sum_{j \in X} (\alpha_i)_j \neq 0$ since $\kappa > 0$.

Suppose $\sum_{j \in X} \lambda_j \neq 0$. If $\sum_{j \in X} \lambda_j > 0$, then $\sum_{j \in X} (\alpha_i)_j > 0$ since $\frac{\kappa}{n \cdot 2^i} > 0$.

Suppose $\sum_{j\in X}\lambda_j < 0$. Then

$$\begin{aligned}\sum_{j\in X}(\alpha_i)_j =& \sum_{j\in X}(\lambda_j + \frac{\kappa}{n\cdot 2^i})\\ =&(\sum_{j\in X}\lambda_j) + \frac{|X|\cdot\kappa}{n\cdot 2^i}\\ \leq&(\sum_{j\in X}\lambda_j) + \frac{\kappa}{2^i}\\ <&0,\end{aligned}$$

where the last inequality is true because of the definition of κ.

Therefore, for any $i \in \mathbb{N}$, we have α_i satisfies the condition of Theorem 4.3.9 ensuring that $\dim H^p(A(\mathcal{A}), \alpha_i) = 0$ for $p < \ell$. Since $\mathcal{A}$ contains no central factors, we have $H^*(A(\mathcal{A}), \alpha_i) \neq 0$; hence, $\dim H^\ell(A(\mathcal{A}), \alpha_i) \neq 0$. By Theorem 4.1.12, the function $t \mapsto \dim H^\ell(A(\mathcal{A}), t)$ is upper semicontinuous in t; therefore, $\dim H^\ell(A(\mathcal{A}), a) > 0$. □

THEOREM 4.3.11. Let $\mathcal{A}$ be an affine ℓ-arrangement. We may write

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k \times \mathcal{B},$$

where $\mathcal{A}_j$ are each central and $\mathcal{B}$ not central and they contain no proper central factors. Let $a \in A_1(\mathcal{A})$. We have $\dim H^\ell(A(\mathcal{A}), a) \neq 0$ if and only if $\sum_{H_i\in\mathcal{A}_j}\lambda_i = 0$ for all j.

PROOF. Let $\mathcal{A}_j$ be an ℓ_j-arrangement, and let $\mathcal{B}$ be an $\ell_{\mathcal{B}}$-arrangement. Then $\mathcal{A}$ is an $(\ell_{\mathcal{B}} + \sum_{i=1}^{k}\ell_j)$-arrangement. Since we have the short exact sequences

$$0 \to H^{p-1}(A(d\mathcal{A}_i), \tilde{a}) \to H^p(A(\mathcal{A}_i), a) \to H^p(A(d\mathcal{A}_i), \tilde{a}) \to 0$$

and the Künneth Formula

$$H^m(A(\mathcal{A}), a) =$$

$$\bigoplus_{p_1+\cdots p_k+p_{k+1}=m} H^{p_1}(A(\mathcal{A}_1), a_1) \otimes \cdots \otimes H^{p_k}(A(\mathcal{A}_k), a_k) \otimes H^{p_{k+1}}(A(\mathcal{B}), a_{\mathcal{B}}),$$

it will suffice to show $H^{\ell_j - 1}(A(d\mathcal{A}_j), \tilde{a}_j) \neq 0$ and $H^{\ell_{\mathcal{B}}}(A(\mathcal{B}), a_{\mathcal{B}}) \neq 0$. This result was established in Theorem 4.3.10. □

THEOREM 4.3.12. Let $\mathcal{A}$ be an arrangement with $\ell = \text{rank}(\mathcal{A})$. Fix $a \in A_1(\mathcal{A})$. Then $H^*(A(\mathcal{A}), a) = 0$ if and only if $H^\ell(A(\mathcal{A}), a) = 0$.

PROOF. This follows immediately from Theorem 4.3.11 and Theorem 4.3.8. □

CHAPTER V

THE DIMENSION OF $H^k(A(\mathcal{A}), a)$ FOR A SPECIAL CASE

In this chapter, we determine the dimension of $H^k(A(\mathcal{A}), a)$ while imposing special conditions on a and $\mathcal{A}$. In particular, we require $\mathcal{A}$ to be supersolvable. In §5.1, we determine the dimension of $Z_k(a)$ for this special case and compute the Hilbert series for $H^*(A(\mathcal{A}), a)$ in terms of the Hilbert series for $A(\mathcal{A})$. In §5.2, we study the ideal $Z(a) = \oplus Z_k(a)$ under the same conditions and show $Z_k(a) = A_{k-1}(\mathcal{A}) \cdot Z_1(a)$ for $k < \ell$. In §5.3, we consider examples illustrating the results from the first two sections.

Throughout this chapter, we maintain the following assumption.

CONDITION A. Let $\mathcal{A}$ be a central hyperplane arrangement, and assume $\mathcal{A}$ is supersolvable. Fix $X \in L(\mathcal{A})$ with rank$(X) = 2$ and X a member of a maximal modular chain in $L(\mathcal{A})$. Fix an order on the hyperplanes so that the order respects the supersolvable structure. Then we have $\mathcal{A}_X = \{H_1, \dots, H_{n_X}\}$.

Recall from §3.2 that $\mathcal{A}$ satisfying Condition A implies $\mathcal{A}$ is quadratic under this order.

§5.1 The Dimension of $Z_k(a)$ for a Special Case

Let $\mathcal{A} = \{H_1, ..., H_n\}$ be a central hyperplane arrangement in V. The lattice, $L(\mathcal{A})$, of subspace intersections formed by the hyperplanes is ranked (via codimension) and atomic; see chapter II. This allows us to discuss the rank of each element from the lattice and to associate to it the hyperplanes which contain it. The following notational conventions are maintained throughout the chapter.

NOTATIONAL CONVENTIONS:

1. For $X \in L(\mathcal{A})$, we write $i \in X$ to mean X is contained in the hyperplane H_i.
2. For $X \in L(\mathcal{A})$, we write $X = \{i_1, ..., i_p\}$ to mean

   (i) X is the intersection of the hyperplanes $\{H_{i_1}, ..., H_{i_p}\}$,

   (ii) if $X \subseteq H$ then $H \in \{H_{i_1}, ..., H_{i_p}\}$.
3. If $\text{rank}(X) = p$, then we write $X \in L(p, \mathcal{A})$.

We recall the Orlik-Solomon algebra for the central case. Let $\mathcal{K}$ be a field. Let E_1 be the linear space over $\mathcal{K}$ on n generators. Let $E(\mathcal{A}) := \Lambda(E_1)$ be the exterior algebra on E_1. We have that any ordered subset $S = \{H_{i_1}, ..., H_{i_p}\}$ of $\mathcal{A}$ corresponds to an element $e_S = e_{i_1} \cdots e_{i_p}$ in $E(\mathcal{A})$. We say S is dependent if $\text{rank}(\cap S) < |S|$. We define the map $\partial : \ E(\mathcal{A}) \to E(\mathcal{A})$ via the usual differential. That is,

$$\partial(1) := 0,$$

$$\partial(e_H) := 1,$$

$$\text{and for } p \geq 2, \ \partial(e_{H_1} \cdots e_{H_p}) := \sum_{k=1}^{p} (-1)^{k-1} e_{H_1} \cdots \hat{e}_{H_k} \cdots e_{H_p}.$$

We define $I(\mathcal{A})$ to be the ideal of $E(\mathcal{A})$ which is generated by $\partial(e_S)$ for all dependent S. The Orlik-Solomon algebra is defined as $A(\mathcal{A}) := E(\mathcal{A})/I(\mathcal{A})$.

We have $A(\mathcal{A})$ is a free graded $\mathcal{K}$-module. We recall the broken circuit basis for $A_p(\mathcal{A})$. Fix an order on $\mathcal{A}$. Consider an ordered subset $S = \{H_{i_1}, ..., H_{i_p}\}$ of $\mathcal{A}$ with $1 \leq i_1 < \cdots < i_p \leq n$. Then a_S is basic in A_p if

1. S is independent, and
2. For any $1 \leq k \leq p$, there does not exist a hyperplane $H \in \mathcal{A}$ so that $H < H_{i_k}$ and $\{H, H_{i_k}, H_{i_{k+1}}, ..., H_{i_p}\}$ is dependent.

Let $B_p := \{(i_1, \ldots, i_p) : \ a_{i_1} \cdots a_{i_p}$ is in the broken circuit basis for $A_p(\mathcal{A})\}$.

We recall the cohomology of the Orlik-Solomon algebra from Chapter III. We construct a cochain complex on the homogeneous components of $A(\mathcal{A})$ as follows.

Let $a \in A_1(\mathcal{A})$. Multiplication by a giving the differential $d_k : A_k \xrightarrow{a\cdot} A_{k+1}$ forms a complex $(A(\mathcal{A}), a)$.

DEFINITION 5.1.1. Let M_k be the matrix of the map $d_k : A_k \xrightarrow{a\cdot} A_{k+1}$ in the broken circuit basis.

DEFINITION 5.1.2. Let $X \in L(2, \mathcal{A})$. Let a be a nonzero element of $A_1(\mathcal{A})$; write $a = \sum_{i=1}^{n} \lambda_i a_i$. Assume $\lambda_i = 0$ for $i \notin X$ and $\sum_{i=1}^{n} \lambda_i = 0$. In this case, we say a is concentrated under X.

In the setting of Definition 5.1.1 and Definition 5.1.2, M_k is a $|B_{k+1}| \times |B_k|$ matrix. We compute the rank of M_k by considering the span of the column space of M_k. Let $X = \{1, ..., n_X\} \in L(2, \mathcal{A})$. We need to consider the types of basic elements of A_k. Let $\vec{j} = \{j_1, ..., j_p\}$ be a subset of $\vec{n}$. For $\mathcal{A}$ satisfying Condition A, we have the following types of elements from B_k.

1. $S = (\alpha, \vec{j})$ for $\vec{j} \in B_{k-1}$ and $\vec{j} \subseteq \{n_X + 1, ..., n\}$ and $\alpha \in \{1, ..., n_X\}$.
2. $S = (1, \vec{j})$ for $j_1 \in \{2, ..., n_X\}$ and $\vec{j} \in B_{k-1}$.
3. $S = \vec{j}$ for $\vec{j} \subseteq \{n_X + 1, ..., n\}$ and $\vec{j} \in B_k$.

LEMMA 5.1.3. Let $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ be as in Condition A. Let $1 < k < \ell$. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under X. Fix $\vec{j} \subseteq \{n_X + 1, ..., n\}$ and $\vec{j} \in B_{k-1}$. Then the set of columns of M_k labeled by $1\vec{j}, 2\vec{j}, ..., n_X\vec{j}$ are the same. If $k = 1$, then the columns of M_k labeled by $1, 2, \ldots, n_X$ are the same.

PROOF. Fix $\vec{j} \subseteq \{n_X + 1, ..., n\}$ and $\vec{j} \in B_{k-1}$. Notice $(\alpha, \vec{j}) \in B_k$ for any $\alpha \in \{1, ..., n_X\}$. For $\alpha \in \{1, ..., n_X\}$, we have

$$a \cdot a_{\alpha\vec{j}} = \sum_{i<\alpha} \lambda_i a_{i\alpha\vec{j}} - \sum_{i=\alpha+1}^{n_X} \lambda_i a_{\alpha i\vec{j}}.$$

If $\alpha = 1$, then we have $a \cdot a_{\alpha\vec{j}} = -\sum_{i=2}^{n_X} \lambda_i a_{1i\vec{j}}$. If $\alpha > 1$, then we have

$$a \cdot a_{\alpha\vec{j}} = \lambda_1 a_{1\alpha\vec{j}} + \sum_{1<i<\alpha} \lambda_i a_{i\alpha\vec{j}} - \sum_{i=\alpha+1}^{n_X} \lambda_i a_{\alpha i\vec{j}}.$$

However,

$$a_{i\alpha\vec{j}} = a_{1\alpha\vec{j}} - a_{1i\vec{j}},$$

$$a_{\alpha ij} = a_{1ij} - a_{1\alpha j}, \text{ and}$$

$$\sum_{i=1}^{n} \lambda_i = 0$$

implies $a \cdot a_{\alpha\vec{j}} = -\sum_{2\le\alpha\le n_X} \lambda_i a_{1i\vec{j}}$. Therefore, the $\alpha\vec{j}$ columns are the same for any $1 \le \alpha \le n_X$ as required. Since $\mathcal{A}$ is quadratic under this order, $a_{1i\vec{j}} \neq 0$. That is, if $\{H_1, H_i, H_{\vec{j}}\}$ is dependent, then $\{H_i, H_{\vec{j}}\}$ is minimally dependent since $\vec{j} \in B_{k-1}$. Hence, $\{H_i, H_{j_k}\}$ is minimally dependent for some j_k. But this implies $H_{j_k} \in X$, a contradiction.

Notice that in the case $k = 1$, the same proof works. □

In light of the above theorem, we define

$$|\vec{j} \in B_0 : \ \vec{j} \subseteq \{n_X + 1, ..., n\}| := 1$$

for ease in computations.

LEMMA 5.1.4. Let $\mathcal{A}$ be a central hyperplane arrangement with $\text{rank}(\mathcal{A}) = \ell$. Let $0 < k < \ell$. Let $X = \{1, ..., n_X\}$ be in $L(2, \mathcal{A})$. Let $0 \neq a \in A_1$ be concentrated under X. Fix $\vec{j} \in B_{k-1}$ with $j_1 \in \{2, \ldots, n_X\}$. The column of M_k labeled by $1\vec{j}$ is the zero column.

PROOF. This is immediate since any three elements under X are dependent; in particular, we have

$$a \cdot a_{1\vec{j}} = \sum_{i=1}^{n_X} \lambda_i a_i a_{1\vec{j}} = 0. \ \square$$

LEMMA 5.1.5. Let $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ be as in Condition A. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under X. Let $0 < k < \ell$. The set of columns given by $\vec{j}$ for $\vec{j} \subseteq \{n_X + 1, ..., n\}$ and $\vec{j} \in B_k$ are linearly independent.

PROOF. This follows because $a_{i\vec{j}}$ is basic in $A_{k+1}(\mathcal{A})$ for $i \in \{1, ..., n_X\}$ since $\mathcal{A}$ is quadratic under this order. Indeed, if $a_{i\vec{j}}$ is not basic, then we have two cases. Let $S = \{H_{j_1}, \ldots, H_{j_k}\}$. If $\{H_i\} \cup T$ is dependent for any $T \subsetneq S$, then $a_{\vec{j}}$ is not basic, a contradiction. If there exists $H < H_i$ so that $\{H, H_i\} \cup S$ is dependent, then this set is minimally dependent since a_S is basic. Since $\mathcal{A}$ is quadratic, this implies $H_{j_k} < X$ for some k, a contradiction. □

THEOREM 5.1.6. Let $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ be as in Condition A. Let $0 < k < \ell$. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under X. We have

$$\operatorname{rank} d_k = \left|\left\{\vec{j} \in B_{k-1} : \vec{j} \subseteq \{n_X + 1, ..., n\}\right\}\right| + \left|\left\{\vec{j} \in B_k : \vec{j} \subseteq \{n_X + 1, ..., n\}\right\}\right|.$$

PROOF. Lemmas 5.1.3, 5.1.4, and 5.1.5 imply the rank d_k is the number of $1\vec{j}$ for $\vec{j} \subseteq \{n_X + 1, ..., n\}$ and $\vec{j} \in B_{k-1}$ and the number of $\vec{j}$ for $\vec{j} \subseteq \{n_X + 1, ..., n\}$ and $\vec{j} \in B_k$.

Notice in the case that $k = 0$, we have $\operatorname{rank} d_0 = 1$ since $a \neq 0$. □

THEOREM 5.1.7. Let $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ be as in Condition A. Let $0 < k < \ell$. Let $0 \neq a \in A_1$ be concentrated under X. We have $\dim Z_k(a) = (n_X - 1) \operatorname{rank} d_{k-1}$.

PROOF. We use Theorem 5.1.6 and calculate:

$$\begin{aligned}
\dim Z_k(a) &= \dim A_k - \operatorname{rank} d_k \\
&= |\{\vec{j} \in B_k\}| - \left|\left\{\vec{j} \in B_{k-1} : \vec{j} \subseteq \{n_X + 1, ..., n\}\right\}\right| \\
&\quad - \left|\left\{\vec{j} \in B_k : \vec{j} \subseteq \{n_X + 1, ..., n\}\right\}\right| \\
&= \left|\left\{\vec{j} \in B_k : j_1 \in \{1, ..., n_X\}\right\}\right| - \left|\left\{\vec{j} \in B_{k-1} : \vec{j} \subseteq \{n_X + 1, ..., n\}\right\}\right|.
\end{aligned}$$

Consider the first term above. Since $\mathcal{A}$ is quadratic, for any $\alpha \in X$ and $\vec{j} \in B_{k-2}$, we have $1\alpha\vec{j} \in B_k$. Hence,

$$\begin{aligned}
\left|\left\{\vec{j} \in B_k : j_1 \in \{1, ..., n_X\}\right\}\right| &= |\{\alpha\vec{j} \in B_k : \alpha \in X, \vec{j} \in B_{k-1}, j_1 > n_X\}| \\
&\quad + |\{1\alpha\vec{j} \in B_k : \alpha \in X, \vec{j} \in B_{k-2}, j_1 > n_X\}|.
\end{aligned}$$

Returning to our calculations, we now have

$$\begin{aligned}\dim Z_k(a) =&|\{\alpha\vec{j} \in B_k : \alpha \in X, \vec{j} \in B_{k-1}, j_1 > n_X\}| \\ &+ |\{1\alpha\vec{j} \in B_k : \alpha \in X, \vec{j} \in B_{k-2}, j_1 > n_X\}| - |\{\vec{j} \in B_{k-1} : j_1 > n_X\}|.\end{aligned}$$

Consider the first and third terms. Since $\mathcal{A}$ is quadratic, for any $\vec{j} \in B_{k-1}$ with $j_1 > n_X$, we have $\alpha\vec{j} \in B_k$ for any $\alpha \in X$. Hence, the sum of the first and third terms can be expressed as $(n_X - 1)|\{\vec{j} \in B_{k-1} : j_1 > n_X\}|$. The middle term as written above is $|\{1\alpha\vec{j} \in B_k : \alpha \in X, \vec{j} \in B_{k-2}, u_1 > n_X\}|$, and gives $n_X - 1$ choices for α. Hence, the middle term can be simplified to $(n_X - 1)|\{\vec{j} \in B_{k-2} : j_1 > n_X\}|$. Continuing with our calculations, we have

$$\begin{aligned}\dim Z_k(a) =&(n_X - 1) \cdot |\{\vec{j} \in B_{k-1} : j_1 > n_X\}| + (n_X - 1) \cdot |\{\vec{j} \in B_{k-2} : j_1 > n_X\}| \\ =&(n_X - 1) \ \operatorname{rank} d_{k-1}. \quad \square\end{aligned}$$

THEOREM 5.1.8. Let $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ be as in Condition A. Let $k < \ell$. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under X. Then

$$\dim H^k(A(\mathcal{A}), a) = (n_X - 2) \operatorname{rank} d_{k-1}.$$

PROOF. We use Theorems 5.1.6 and 5.1.7 to compute:

$$\begin{aligned}\dim H^k(A(\mathcal{A}), a) =& \dim Z_k(a) - \operatorname{rank} d_{k-1} \\ =&(n_X - 1) \operatorname{rank} d_{k-1} - \operatorname{rank} d_{k-1} \\ =&(n_X - 2) \operatorname{rank} d_{k-1}. \quad \square\end{aligned}$$

THEOREM 5.1.9. Let $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ be as in Condition A. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under X. Then for $0 < k < \ell$, we have

$$\dim H^k(A(\mathcal{A}), a) = (n_X - 2) \sum_{i=1}^{k} (-1)^{i-1} (n_X - 1)^{i-1} \dim A_{k-i},$$

and for $k = \ell$, we have

$$\dim H^\ell(A(\mathcal{A}), a) = \dim A_\ell + \sum_{i=1}^{\ell}(-1)^i(n_X - 1)^{i-1} \dim A_{\ell-i}.$$

PROOF. We consider the first statement. For $k = 1$, the statement clearly hold true as $\dim H^1(A(\mathcal{A}), a) = n_X - 2$. Fix $1 < k < \ell - 1$ and suppose the statement is true for $k - 1$. By Theorem 5.1.8, Theorem 5.1.7, and the induction hypothesis, we have

$$\begin{aligned}
\dim H^k(A(\mathcal{A}), a) =&(n_X - 2) \operatorname{rank} d_{k-1} \\
=&(n_X - 2)[\dim A_{k-1} - \dim Z_{k-1}(a)] \\
=&(n_X - 2) \dim A_{k-1} - (n_X - 2) \dim Z_{k-1}(a) \\
=&(n_X - 2) \dim A_{k-1} - (n_X - 1) \dim Z_{k-1}(a) + \dim Z_{k-1}(a) \\
=&(n_X - 2) \dim A_{k-1} - (n_X - 1) \dim Z_{k-1}(a) \\
&+ (n_X - 1) \operatorname{rank} d_{k-2} \\
=&(n_X - 2) \dim A_{k-1} - (n_X - 1) \dim H^{k-1}(A(\mathcal{A}), a) \\
=&(n_X - 2) \dim A_{k-1} \\
&- ((n_X - 1)(n_X - 2) \sum_{i=1}^{k-1}(-1)^{i-1}(n_X - 1)^{i-1} \dim A_{k-1-i} \\
=&(n_X - 2) \sum_{i=1}^{k}(-1)^{i-1}(n_X - 1)^{i-1} \dim A_{k-i}.
\end{aligned}$$

We now consider the second statement. We first prove for $1 \leq k < \ell$,

$$\dim Z_k(a) = \sum_{i=1}^{k}(-1)^{i-1}(n_X - 1)^i \dim A_{k-i}. \qquad (*)$$

For $k = 1$, (*) holds since $\dim Z_1(a) = n_X - 1$. Fix $1 < k < \ell$ and suppose (*) holds for $k-1$. Then

$$\begin{aligned}\dim Z_k(a) =&(n_X-1)\operatorname{rank} d_{k-1}\\ =&(n_X-1)(\dim A_{k-1}-\dim Z_{k-1}(a))\\ =&(n_X-1)\dim A_{k-1}-(n_X-1)\sum_{i=1}^{k-1}(-1)^{i-1}(n_X-1)^i\dim A_{k-1-i}\\ =&\sum_{i=1}^{k}(-1)^{i-1}(n_X-1)^i\dim A_{k-i}.\end{aligned}$$

Hence, (*) is true for all $1 \leq k < \ell - 1$ and we use it to prove the second statement of the theorem.

Indeed, we have the following which proves the theorem:

$$\begin{aligned}\dim H^\ell(A(\mathcal{A}),a) &= \dim A_\ell - \operatorname{rank} d_{\ell-1}\\ &= \dim A_\ell - \dim A_{\ell-1} + \dim Z_{\ell-1}(a)\\ &= \dim A_\ell - \dim A_{\ell-1} + \sum_{i=1}^{\ell-1}(-1)^{i-1}(n_X-1)^i\dim A_{\ell-1-i}\\ &= \dim A_\ell + \sum_{i=1}^{\ell}(-1)^i(n_X-1)^{i-1}\dim A_{\ell-i}. \quad \square\end{aligned}$$

DEFINITION 5.1.10. We define the Hilbert series of a graded algebra A over $\mathcal{K}$ to be

$$H(A,t) := \sum_{i=1}^{\infty}(\dim_{\mathcal{K}} A_i)t^i.$$

THEOREM 5.1.11. Let $\mathcal{A}$ and $X \in L(2,\mathcal{A})$ be as in Condition A. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under X. Then we can compute the Hilbert series for $H^*(A(\mathcal{A}),a)$ in terms of the Hilbert series for $A(\mathcal{A})$ as follows:

$$H(H^*(A(\mathcal{A}),a),t) = \frac{t(n_X-2)}{1+t(n_X-1)}H(A(\mathcal{A}),t).$$

PROOF. In the proof of Theorem 5.1.9, we have for $1 \leq k < \ell$

$$\dim H^k(A(\mathcal{A}),a) = (n_X-2)\dim A_{k-1} - (n_X-1)\dim H^{k-1}(A(\mathcal{A}),a).$$

So, the series holds for $k < \ell$.

We now check for $k = \ell$. For $k < \ell$, we have

$$\begin{aligned}\dim Z_k(a) =&(n_X - 1)\operatorname{rank} d_{k-1}\\ =&(n_X - 1)(\dim A_{k-1} - \dim Z_{k-1}(a)).\end{aligned}$$

Hence, we may use the series $\sum_{i=0}^{\infty} \frac{t(n_X-1)}{1+t(n_X-1)} H(A(\mathcal{A}), a)$ to compute $\dim Z_k(a)$ for $k < \ell$. Since $\dim H^\ell(A(\mathcal{A}), a) = \dim A_\ell - \dim A_{\ell-1} + \dim Z_{\ell-1}$, we find $\dim H^\ell(A(\mathcal{A}), a)$ by taking the coefficient of t^ℓ in the series $(1+t)H(A(\mathcal{A}), t) + \frac{t(n_X-1)}{1+t(n_X-1)} H(A(\mathcal{A}), a)$. By obtaining a common denominator and adding, we have $\dim H^\ell(A(\mathcal{A}), a)$ is given by the coefficient of t^ℓ in the series $\frac{t(n_X-2)}{1+t(n_X-1)} H(A(\mathcal{A}), t)$ as required. □

§5.2 The Ideal $Z(a) = \oplus Z_k(a)$ for a Special Case

We now consider $Z(a) = \oplus Z_k(a)$ as an ideal of $A(\mathcal{A})$. We endeavor to show that if $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ are as in Condition A with a concentrated under X, then we have $Z_k(a)$ is generated by $Z_1(a)$ (that is, $Z_k(a) = A_{k-1}(\mathcal{A}) \cdot Z_1(a)$) except in the top dimension ℓ.

We recall the following description of $Z_1(a)$ from Libgober and Yuzvinsky [8]. Let $\mathcal{A}$ be a central hyperplane arrangement. Let $x = \sum_{i=1}^{n} x_i a_i \in A_1(\mathcal{A})$. Then $x \in Z_1(a)$ if and only if the following conditions hold:

1. For every $Y \in L(2)$ with $|Y| > 2$ and $a(Y) \neq 0$ but $\sum_{i \in Y} \lambda_i = 0$, we have $\sum_{i \in Y} x_i = 0$.
2. For every other $Y \in L(2)$ and every pair $i < j$ from Y, we have $\lambda_i x_j - \lambda_j x_i = 0$.

We use this description to prove the following lemma.

LEMMA 5.2.1. Let $\mathcal{A}$ be a central hyperplane arrangement. Let $X = \{1, ..., n_X\}$ be in $L(2, \mathcal{A})$. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under X. If $z, w \in Z_1(a)$ and both nonzero, then $z \in Z_1(w)$ and $\dim(z \cdot A_1(\mathcal{A})) = \dim(w \cdot A_1(\mathcal{A}))$.

PROOF. Let $z, w \in Z_1(a)$. It will suffice to show $z \in Z_1(w)$. We show conditions (1) and (2) above hold for any $Y \in L(2, \mathcal{A})$. Let $Y \in L(2)$ with $Y = \{i_1, ..., i_k\}$. We consider the following three cases.

Case 1. Suppose $Y = X$. If $|X| > 2$, then since $z, w \in Z_1(a)$, $a(X) \neq 0$, and $\sum_{i \in X} \lambda_i = 0$, condition (1) gives $\sum_{i \in X} z_i = \sum_{i \in X} w_i = 0$ as required. If $|X| = 2$, then condition (2) together with $a(X) \neq 0$ gives $z_1 = -z_2$ and $w_1 = -w_2$; hence, $z_1 w_2 - z_2 w_1 = 0$ as required.

Case 2. Suppose $i_1 > n_X$. In this case, we have $a(Y) = 0$. It will suffice to show $z(Y)$ and $w(Y)$ are both zero. Since $a \neq 0$, we may assume without loss of generality that $\lambda_1 \neq 0$. Consider the element $W_j \in L(2)$ which contains $\{H_1, H_{i_j}\}$. Then $a(W_j) \neq 0$ and $\sum_{i \in W_j} \lambda_i = \lambda_1 \neq 0$. By condition (2), we have $z_{i_j} = w_{i_j} = 0$ for all $1 \leq j \leq k$.

Case 3. Suppose $i_1 \in X$. Then $\sum_{i \in Y} \lambda_i = \lambda_{i_1}$. If $\lambda_{i_1} \neq 0$, then by condition (2), $z_{i_j}, w_{i_j} = 0$ for all $j > 1$. Hence, $z_{i_j} w_{i_m} - z_{i_m} w_{i_j} = 0$ for any $H_{i_m}, H_{i_j} \in Y$. If $\lambda_{i_1} = 0$, then we follow the same approach as Case 2 to obtain $z(Y)$ and $w(Y)$ are linearly dependent. In particular, assume $\lambda_1 \neq 0$. Then consider W_j as defined previously, noting $W_1 = X$. We have $z_{i_j} = w_{i_j} = 0$ for all $2 \leq j \leq k$. Hence, $z(Y)$ and $w(Y)$ are linearly dependent. The lemma now follows. □

LEMMA 5.2.2. Let $\mathcal{A}$ be a central hyperplane arrangement. Let $X \in L(2, \mathcal{A})$ with $X = \{1, ..., n_X\}$. Let $0 \neq a \in A_1$ be concentrated under X. Assume $\lambda_1 \neq 0$. Then $Z_1(a)$ has a basis given by $\{a_1 - a_k\}$ for $2 \leq k \leq n_X$.

PROOF. By straightforward computation and the assumption $\sum_{i=1}^{n_X} \lambda_i = 0$, we have that $a_1 - a_k \in Z_1(a)$ for $2 \leq k \leq n_X$. Indeed, we compute

$$\begin{aligned} a \cdot (a_1 - a_k) &= (\sum_{i=1}^{n_X} \lambda_i a_i)(a_1 - a_k) \\ &= -\sum_{i=2}^{n_X} \lambda_i a_{1i} - \sum_{i<k} \lambda_i a_{ik} + \sum_{k<i<n_X} \lambda_i a_{ki}. \end{aligned}$$

Since $a_{ik} = a_{1k} - a_{1i}$ and $a_{ki} = a_{1i} - a_{1k}$, we substitute and have

$$a \cdot (a_1 - a_k) = -\sum_{i=1}^{n_X} \lambda_i a_{1k}$$
$$=0.$$

Obviously, $\{a_1 - a_k : 2 \le k \le n_X\}$ is a set of linearly independent elements from $A_1(\mathcal{A})$. Let $z \in Z_1(a)$. By the proof of Lemma 5.2.1, we have $z_i = 0$ for any $i > n_X$. Moreover, $\sum_{i=1}^{n_X} z_i = 0$ implies z is a linear combination of $\{a_1 - a_k : 2 \le k \le n_X\}$. □

THEOREM 5.2.3. Let $\mathcal{A}$ be a central hyperplane arrangement. Let $X = \{1, ..., n_X\} \in L(2, \mathcal{A})$. Let $0 \ne a \in A_1(\mathcal{A})$ be concentrated in $X \in L(2, \mathcal{A})$. We have the following description of $Z_1(a)$:

$$Z_1(a) = \left\{ \sum_{i=1}^{n} x_i a_i : \ x_j = 0 \text{ for } j \notin X, \sum_{i=1}^{n} x_i = 0 \right\}$$

PROOF. This follows immediately from Lemma 5.2.2. □

LEMMA 5.2.4. Let $\mathcal{A}$ be a central hyperplane arrangement. Let $X \in L(2, \mathcal{A})$ with $X = \{1, ..., n_X\}$. Let $0 \ne a \in A_1(\mathcal{A})$ be concentrated under X. Let z_i, z_k be basic elements of $Z_1(a)$ as given in Lemma 5.2.2. We have $A_1(\mathcal{A})z_i \cap A_1(\mathcal{A})z_k = 0$.

PROOF. Suppose $z_i = a_1 - a_i$ and $z_k = a_1 - a_k$. Let $\gamma \in A_1(\mathcal{A})$. Then by computation

$$z_i\gamma = \left(\sum_{j=1}^{n_X} \gamma_j\right) a_{1i} + \sum_{j > n_X} \gamma_j a_{1j} - \sum_{j > n_X} \gamma_j a_{ij}.$$

So, for $z_i\gamma = z_k\sigma$ with $\gamma, \sigma \in A_1(\mathcal{A})$, we have

$$\left(\sum_{j=1}^{n_X} \gamma_j\right) a_{1i} + \sum_{j > n_X} \gamma_j a_{1j} - \sum_{j > n_X} \gamma_j a_{ij} = \left(\sum_{j=1}^{n_X} \sigma_j\right) a_{1k} + \sum_{j > n_X} \sigma_j a_{1j} - \sum_{j > n_X} \sigma_j a_{kj}.$$

Since $i \ne k$, $\sum_{j=1}^{n_X} \gamma_j = \sum_{j=1}^{n_X} \sigma_j = 0$. Since $i \ne k$ and $n_X < j \le n$, a_{kj} and a_{ij} are distinct basic elements of $A_2(\mathcal{A})$; this forces $\sigma_j = \gamma_j = 0$ for $n_X < j \le n$. By Theorem 5.2.3, this implies $\gamma, \sigma \in Z_1(a)$ as required. □

THEOREM 5.2.5. Suppose $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ satisfy Condition A. Let $a \in A_1(\mathcal{A})$ be a nonzero element concentrated under X. We have $Z_2(a)$ is generated by $Z_1(a)$, i.e. $Z_2(a) = A_1(\mathcal{A}) \cdot Z_1(a)$.

PROOF. We follow the argument given in Theorem 5.1.7 and compute

$$\dim Z_2(a) = (n_X - 1)(n - n_X) + n_X - 1.$$

By using Lemma 5.2.1 and Lemma 5.2.4, we compute $\dim A_1(\mathcal{A}) \cdot Z_1(a)$ to be

$$(n_X - 1)(n - n_X + 1).$$

Since these two quantities are equal and we have the containment $A_1(\mathcal{A}) \cdot Z_1(a) \subseteq Z_2(a)$, the result now follows. □

LEMMA 5.2.6. Suppose $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ satisfy Condition A. Let $\ell \geq 3$. Let $a \in A_1(\mathcal{A})$ be a nonzero element concentrated under X. Let $\vec{j} \in B_k$ for $k < \ell$. Suppose $\gamma a_{\vec{j}} \in Z_k(a)$ for some $\gamma \in \mathcal{K}$. If $j_1 > n_X$, then $\gamma = 0$.

PROOF. Suppose $j_1 > n_X$. Since $\mathcal{A}$ is quadratic, $a_{\alpha\vec{j}} \in B_{k+1}$ for any $\alpha \in X$. Since $\gamma a_{\vec{j}} \in Z_k(a)$, we must have $\gamma = 0$. □

LEMMA 5.2.7. Suppose $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ satisfy Condition A. Let $a \in A_1(\mathcal{A})$ be a nonzero element concentrated under X. Let $\vec{j} \in B_k$ for $2 \leq k < \ell$. Suppose $a_{\vec{j}} \in Z_k(a)$. If $j_1 = 1$ and $j_2 \in X$, then $a_{\vec{j}} \in A_1(\mathcal{A}) \cdot Z_1(a)$.

PROOF. Without loss of generality, we may assume $\lambda_1 \neq 0$. Suppose $j_1 = 1$ and $j_2 \in X$. Then $(a_1 - a_\alpha)a_{1j_2} = 0$ for all $2 \leq \alpha \leq n_X$. Hence, $a_{1j_2} \in Z_2(a)$, and by Theorem 5.2.5, $Z_2(a)$ is generated by $Z_1(a)$. Thus, $a_{\vec{j}} \in A_1(\mathcal{A}) \cdot Z_1(a)$. □

LEMMA 5.2.8. Suppose $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ satisfy Condition A. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under X. Let $\vec{j}' \in B_{k-1}$ with $\vec{j}' \cap X = \emptyset$. If $\sum_{\alpha=1}^{n_X} \gamma_\alpha a_{\alpha\vec{j}'} \in Z_k(a)$ for $k < \ell$ and $\gamma_\alpha \in \mathcal{K}$, then $\sum_{\alpha=1}^{n_X} \gamma_\alpha a_{\alpha\vec{j}'} \in A_{k-1}(\mathcal{A}) \cdot Z_1(a)$.

PROOF. Suppose $\vec{j} \cap X \neq \emptyset$ with $j_1 \in X$ and $j_2 \notin X$. Then $\vec{j}' := \{j_2, ..., j_k\}$ is in B_{k-1}. Since $\mathcal{A}$ is quadratic, we have $a_{\alpha\vec{j}'} \in B_k$ for any $\alpha \in X$. Assume $\lambda_1 \neq 0$. By Lemma 5.2.2, we may express a as $a = \sum_{\alpha=2}^{n_X} c_\alpha(a_1 - a_\alpha)$. By computing,

$$aa_{\vec{j}} = \left(\sum_{\alpha=2}^{n_X} c_\alpha\right) a_{1\vec{j}} - \sum_{\alpha=2}^{n_X} c_\alpha a_{\alpha\vec{j}}.$$

But $\alpha\vec{j}$ begins with αj_1 for $2 \leq \alpha \leq n_X$. For $j_1 = 1$, we have

$$aa_{\vec{j}} = \sum_{\alpha=2}^{n_X} c_\alpha a_{1\alpha\vec{j}'}.$$

If $j_1 \neq 1$, then $a_{\alpha\vec{j}}$ is not basic and we have $a_{\alpha\vec{j}} = a_{1\vec{j}} - a_{1\alpha\vec{j}'}$; but we still obtain

$$aa_{\vec{j}} = \sum_{\alpha=2}^{n_X} c_\alpha a_{1\alpha\vec{j}'}.$$

Fix $\vec{j}' \in B_{k-1}$ with $\vec{j}' \cap X = \emptyset$. For any $\alpha \in X$, we have $\alpha\vec{j}' \in B_k$. Let $\gamma_\alpha \in \mathcal{K}$ so that $\sum_{\alpha=1}^{n_X} \gamma_\alpha a_{\alpha\vec{j}'} \in Z_k(a)$ as in the assumption of the lemma. We have

$$a\left(\sum_{\alpha=1}^{n_X} \gamma_\alpha a_{\alpha\vec{j}'}\right) = \sum_{\alpha=1}^{n_X} \gamma_\alpha \left(\sum_{i=2}^{n_X} c_i a_{1i\vec{j}'}\right) = \sum_{i=2}^{n_X} \left(\sum_{\alpha=1}^{n_X} \gamma_\alpha\right) c_i a_{1i\vec{j}'}$$

Since $\sum_{\alpha=1}^{n_X} \gamma_\alpha a_{\alpha\vec{j}'} \in Z_k(a)$, we have $\sum_{\alpha=1}^{n_X} \gamma_\alpha = 0$. Hence, $\sum_{\alpha=1}^{n_X} \gamma_\alpha a_\alpha \in Z_1(a)$ by Theorem 5.2.3, so $\sum_{\alpha=1}^{n_X} \gamma_\alpha a_{\alpha\vec{j}'}$ is generated by $Z_1(a)$. □

THEOREM 5.2.9. Suppose $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ satisfy Condition A. Suppose $\ell \geq 3$. Let $a \in A_1(\mathcal{A})$ be a nonzero element concentrated under X. We have $Z_k(a)$ is generated by $Z_1(a)$ for $k < \ell$.

PROOF. Theorem 5.1.8 shows $Z_2(a)$ is generated by $Z_1(a)$. Let $\gamma \in Z_k(a)$ for $k \geq 3$. Then $\gamma = \sum \gamma_{\vec{j}} a_{\vec{j}}$ for $\vec{j} \in B_k$. We now decompose γ by considering different types of $\vec{j}$. There are three possibilities for $\vec{j}$.

1. Suppose $j_1 > n_X$. Then by Lemma 5.2.6, we have $\gamma_{\vec{j}} = 0$.
2. Suppose $j_1 = 1$ and $j_2 \in X$. Then by Lemma 5.2.7, we have $a_{\vec{j}}$ is generated by $Z_1(a)$.
3. Suppose $j_1 \in X$ and $j_2 \notin X$. Then $\vec{j}' = \{j_2, ..., j_k\}$ is in B_{k-1}. We have

$$\sum_{\alpha=1}^{n_X} \gamma_{\alpha\vec{j}'} a_{\alpha\vec{j}'} \in Z_k(a).$$

By Lemma 5.2.8, this implies $\sum_{\alpha=1}^{n_X} \gamma_{\alpha\vec{j}'} a_{\alpha\vec{j}'}$ is generated by $Z_1(a)$. Since each summand of γ is generated by $Z_1(a)$, this implies γ is generated by $Z_1(a)$. □

§5.3 Examples

In this section, we provide examples demonstrating the results of the previous two sections and examples where dropping hypotheses cause the results to fail.

EXAMPLE 5.3.1. Let $Q(\mathcal{A}) = x(x-y)(x+y)y(x-z)(x+z)(y+z)(y-z)z$; order the hyperplanes as they are written. Then $\mathcal{A}$ is supersolvable and the order respects the supersolvable structure. Let a be concentrated under $X = \{1,2,3,4\} \in L(2, \mathcal{A})$. The indices for the broken circuit basis for $A_2(\mathcal{A})$ are

$$\{12, 13, 14, 15, 16, 17, 18, 19, 25, 26, 27, 28, 29, 35, 36, 37, 38, 39, 45, 46, 47, 48, 49\}.$$

Checking Theorem 5.1.11, we see

$$\begin{aligned}\frac{t(n_X-2)}{1+t(n_X-1)} H(A(\mathcal{A}), t) =& \frac{2t}{1+3t}(1+9t+23t^2+15t^3) \\ =&(2t)(t+1)(5t+1) \\ =&10t^3+12t^2+2t\end{aligned}$$

We now check the dimensions of $H^k(A(\mathcal{A}), a)$ by computing

$$\dim Z_1(a) = 3 \text{ and } \operatorname{rank} d_1 = 6,$$

$$\dim Z_2(a) = 18 \text{ and } \operatorname{rank} d_2 = 23 - 18 = 5.$$

Therefore, the dimensions of $H^k(A(\mathcal{A}), a)$ match the Hilbert series above.

Moreover, $\dim Z_2(a) = 18$ and $\dim A_1 \cdot Z_1(a) = 18$, so $Z_2(a) = A_1 \cdot Z_1(a)$.

EXAMPLE 5.3.2. However, if $Q(\mathcal{A}) = (x-y)(x-z)(y-z)x(x+y)y(x+z)(y+z)z$ with the hyperplanes ordered as they are written, then the indices for the broken circuit basis for $A_2(\mathcal{A})$ are

$$\{12, 13, 14, 15, 16, 17, 18, 19, 24, 25, 26, 27, 28, 29, 34, 35, 36, 37, 38, 39, 48, 59, 67\}.$$

We also have $\mathcal{A}$ is not quadratic under this order because $S = \{H_1, H_2, H_4, H_8\}$ is minimally dependent but $|\{H_2, H_4, H_8\}| \neq 2$. Notice the element $H_1 \cap H_2 \cap H_3 \in L(\mathcal{A})$ is not modular. Even though $\mathcal{A}$ is supersolvable arrangement, we show the formulas derived earlier do not hold in this case because the order does not respect the supersolvable structure. Let a be concentrated under $\{1, 2, 3\} \in L(2, \mathcal{A})$. Then $\dim Z_2(a) = 17$ and $\operatorname{rank} d_1 = 7$, so $\dim Z_2(a) \neq 2 \cdot \operatorname{rank} d_1$.

Moreover, $\dim Z_2(a) = 17$ and $\dim A_1 \cdot Z_1(a) = 14$, so $Z_2(a) \neq A_1 \cdot Z_1(a)$.

EXAMPLE 5.3.3. Let $Q(\mathcal{A}) = xy(x+y)z(x+z)(y+z)(x+y+z)$. Then $\mathcal{A}$ is not supersolvable since no rank two element in $L(\mathcal{A})$ is modular. If we take a concentrated in $X = \{1, 2, 3\} \in L(2, \mathcal{A})$, then the previous formulas do not hold. The indices for the broken circuit basis for $A_2(\mathcal{A})$ are

$$\{12, 13, 14, 15, 16, 17, 24, 25, 26, 26, 34, 35, 36, 37, 56, 57\}.$$

We have

$$\dim Z_1(a) = 2 \text{ and } \operatorname{rank} d_1 = 5$$

$$\dim Z_2(a) = 12 \text{ and} \operatorname{rank} d_2 = 4.$$

Hence, $\dim H^1(A(\mathcal{A}), a) = 1, \dim H^2(A(\mathcal{A}), a) = 7, \dim H^3 = 6$. We therefore have

$$H(H^*(A(\mathcal{A}), a), t) = t + 7t^2 + 6t^3.$$

However, the series given in Theorem 5.1.11gives

$$\frac{t}{1+2t}(1 + 7t + 16t^2 + 10t^3)$$

and $1 + 7t + 16t^2 + 10t^3$ is not divisible by $1 + 2t$.

CHAPTER VI

THE DIMENSION OF $H^2(A(\mathcal{A}), a)$

In this chapter, we study the dimension of $H^2(A(\mathcal{A}), a)$ with char $\mathcal{K} = 0$. In §6.1, we construct a matrix description for $Z_2(A(\mathcal{A}), a)$ for the case rank$(\mathcal{A}) = 3$. In §6.2, we construct a matrix description of $Z_2(A(\mathcal{A}), a)$ for rank$(\mathcal{A}) \geq 3$.

§6.1 Dimension of $H^2(A(\mathcal{A}), a)$ For Rank Three Central Arrangements

Let $\mathcal{A} = \{H_1, ..., H_n\}$ be a hyperplane arrangement.

We recall the coned arrangement $c\mathcal{A}$ is formed as follows. Let $\{x_1, ..., x_\ell\}$ be a basis for V^*. Let cV^* have basis $\{x_0, x_1, ...x_\ell\}$. Then $c\mathcal{A}$ will be an arrangement in cV. Each $H \in \mathcal{A}$ can be identified to a linear functional $\alpha \in V^*$. Let α^h be the homogenization of α. We define $c\mathcal{A}$ to be the arrangement given by the functionals $\{x_0\} \cup \{\alpha^h | \alpha \in \mathcal{A}\}$.

Let $A = A(\mathcal{A})$, and let $cA = A(c\mathcal{A})$. We define maps

$$t : A \to cA \text{ via } t(a_S) := (-1)^{|S|} a_0 a_{cS}$$

$$s : cA \to A \text{ via } s(a_0 a_{cS}) = 0, s(a_{cS}) := a_S.$$

We want t and s to also be cochain maps. For this, we introduce $\bar{a} \in A_1(c\mathcal{A})$. Put $\lambda_0 := -\sum_{i=1}^{n} \lambda_i$. Let $\bar{a} := \sum_{i=0}^{n} \lambda_i a_i$. Then $(cA, \bar{a})$ is a cochain complex and we have the short exact sequence for any $p \geq 0$

$$0 \to H^{p-1}(A, a) \to H^p(cA, \bar{a}) \to H^p(A, a) \to 0. \qquad (6.1.0.a)$$

LEMMA 6.1.1. Let $0 \neq a \in A_1(\mathcal{A})$. We have $\dim H^1(A, a) = \dim H^1(cA, \bar{a})$.

PROOF. Take $p = 1$ in the short exact sequence (6.1.0.a). Since $0 \neq a$, we have $H^0(A) = 0$. The result is immediate. □

Suppose $\mathcal{A}$ is central. Recall we can reverse the coning process to form the deconed arrangement $d\mathcal{A}$ as follows. Let α_i be the functional corresponding to H_i. Without loss of generality, we may assume $\alpha_1 = x_1$. Decone at $\alpha_1 = x_1$ by setting $x_1 = 1$. Take $\tilde{a} = \sum_{i=2}^{n} \lambda_i a_i$, and consider the chain complex formed by multiplication of $\tilde{a}$, $(d\mathcal{A}, \tilde{a})$. Let $dA := A(d\mathcal{A})$. As in (6.1.0.a), we have the short exact sequence:

$$(6.1.1.a) \qquad 0 \to H^{p-1}(dA, \tilde{a}) \to H^p(A, a) \to H^p(dA, \tilde{a}) \to 0.$$

LEMMA 6.1.2. Let $\mathcal{A}$ be a central rank three arrangement. Let $a \in A_1(\mathcal{A})$. Let $\tilde{a} \in A_1(d\mathcal{A})$ be as defined in the paragraph following Lemma 6.1.1. We have $\dim H^2(dA, \tilde{a}) = \dim H^3(A, a)$.

PROOF. From the short exact sequence (6.1.1.a), we have

$$0 \to H^2(dA, \tilde{a}) \to H^3(A, a) \to H^3(dA, \tilde{a}) \to 0$$

Since $\operatorname{rank}(d\mathcal{A}) = 2$, we have $dA_3 = 0$, so $H^3(dA, \tilde{a}) = 0$. □

Recall for the algebra $A(\mathcal{A})$, we define

$$\operatorname{Poin}(A, t) := \sum_{p \geq 0} \dim A_p(\mathcal{A}) t^p$$

$$\chi(A) := \operatorname{Poin}(A, -1) = \sum_{p \geq 0} (-1)^p \dim A_p.$$

From [12], we have $\operatorname{Poin}(A(\mathcal{A}), t)$ depends only on $L(\mathcal{A})$. Also from [12], we have $\operatorname{Poin}(A(\mathcal{A}), t) = (1 + t)\operatorname{Poin}(A(d\mathcal{A}), t)$. Hence, $\chi(dA)$ depend only on $\mathcal{A}$, see [13]. This implies $\chi(A(d\mathcal{A}))$ does not depend on the choice of hyperplane about which one decones.

LEMMA 6.1.3. Let $\mathcal{A}$ be a rank three central hyperplane arrangement. Fix a nonzero $a \in A_1(\mathcal{A})$, where $a = \sum_{i=1}^{n} \lambda_i a_i$ and $\sum_{i=1}^{n} \lambda_i = 0$. We have

$$\dim H^3(A, a) = \chi(dA) + \dim H^1(A, a).$$

PROOF. From Lemma 6.1.2, we have $\dim H^3(A) = \dim H^2(dA)$. Let $\tilde{d}_1$ represent the linear map $dA_1 \to dA_2$ given by multiplication of $\tilde{a}$; let $\tilde{Z}_1$ be the kernel of $\tilde{d}_1$. Since $a \neq 0$ and $\sum_{i=1}^{n} \lambda_i = 0$, we have $\tilde{a} \neq 0$; hence, $\dim \tilde{Z}_1 = \dim H^1(dA, \tilde{a}) + 1$. We compute:

$$\begin{aligned}\dim H^2(dA, \tilde{a}) &= \dim dA_2 - \operatorname{rank} \tilde{d}_1 \\ &= \dim dA_2 + \dim \tilde{Z}_1 - \dim dA_1 \\ &= \dim dA_2 - \dim dA_1 + 1 + \dim H^1(dA, \tilde{a}) \\ &= \chi(dA) + \dim H^1(A, a). \ \square\end{aligned}$$

LEMMA 6.1.4. Let $\mathcal{A}$ be a rank three central hyperplane arrangement. Fix a nonzero $a \in A_1(\mathcal{A})$, where $a = \sum_{i=1}^{n} \lambda_i a_i$ and $\sum_{i=1}^{n} \lambda_i = 0$. We have

$$\dim H^2(A) = \dim H^1(A) + \dim H^3(A).$$

PROOF. From the short exact sequence (6.1.1.a), we have

$$0 \to H^1(dA, \tilde{a}) \to H^2(A, a) \to H^2(dA, \tilde{a}) \to 0.$$

By Lemma 6.1.1, we have $H^1(dA, \tilde{a}) \cong H^1(A, a)$. Thus, $H^2(dA, \tilde{a}) \cong H^3(A, a)$ follows from Lemma 6.1.2. $\square$

The following assertion is a consequence of Lemma 6.1.3 and Lemma 6.1.4.

THEOREM 6.1.5. If $\mathcal{A}$ is a rank three central hyperplane arrangement, then we have $\dim H^p(A, a)$ depends only on $\chi(dA)$ and $\dim H^1(A)$ for any p.

In order to study precisely how $\dim H^p(A)$ depends on $\chi(dA)$ and $\dim H^1(A)$, we use the broken circuit basis.

We have $A(\mathcal{A})$ is a free graded $\mathcal{K}$-module. We recall the broken circuit basis for $A_p(\mathcal{A})$. Fix an order on $\mathcal{A}$. Consider an ordered subset $S = \{H_{i_1}, ..., H_{i_p}\}$ of $\mathcal{A}$ with $i_1 < \cdots < i_p$. Then a_S is basic in A_p if

1. S is independent, and
2. For any $1 \leq k \leq p$, there does not exist a hyperplane $H \in \mathcal{A}$ so that $H < H_{i_k}$ and $\{H, H_{i_k}, H_{i_{k+1}}, ..., H_{i_p}\}$ is dependent.

DEFINITION 6.1.6. Let B_p denote the broken circuit basis for the linear space $A_p(\mathcal{A})$.

The following two lemmas are obvious by the definition of the broken circuit basis.

LEMMA 6.1.7. If $\mathcal{A}$ is a rank three central hyperplane arrangement, then A_3 has broken circuit basis $B_3 = \{a_{1ij} : \ a_{ij} \in B_2, i \geq 2\}$.

PROOF. Let $a_{ij} \in B_2$ with $i \geq 2$. By definition of the broken circuit basis, we have $\{H_1, H_i, H_j\}$ is independent. Indeed, if there exists $\alpha < i$ so that $\{H_\alpha, H_i, H_j\}$ is dependent, then this contradicts $a_{ij} \in B_2$. Hence, $a_{1ij} \in B_3$.

Suppose $a_{ijk} \in B_3$. If $i > 1$, then since $\mathcal{A}$ is rank three, we have the set $\{H_1, H_i, H_j, H_k\}$ is dependent. So, $i = 1$. Since $a_{1jk} \in B_3$, there does not exist $\alpha < j$ so that $\{H_\alpha, H_j, H_j\}$ is dependent. Hence, $a_{jk} \in B_2$. □

LEMMA 6.1.8. Let $\mathcal{A}$ be a central arrangement. We define

$$C_2 := \{a_{1i} : \ 2 \leq i \leq n\} \cup \{\partial(a_{1jk}) : \ a_{jk} \in B_2, j \geq 2\}.$$

Then C_2 is a basis for A_2.

PROOF. Let $a_{jk} \in B_2$ with $j \geq 2$. Since

$$B_2 = \{a_{1i} : 2 \leq i \leq n\} \cup \{a_{jk} : \ a_{jk} \in B_2, j \geq 2\}$$

is a basis for A_2 and $\partial(a_{1jk}) = a_{jk} - a_{1k} + a_{1j}$, the proof is immediate. □

Let $x \in A_2$. Using the basis C_2, there exist constants x_{1i} and x_{jk} so that

$$x = \sum_{i=2}^{n} x_{1i}a_{1i} + \sum_{jk \in B_2, j \geq 2} x_{jk}\partial(a_{1jk}). \tag{6.1.8.a}$$

LEMMA 6.1.9. Let $\mathcal{A}$ be a rank three central hyperplane arrangement. Let $a = \sum_{i=1}^{n} \lambda_i a_i$ be an element of $A_1(\mathcal{A})$. Suppose $\sum_{i=1}^{n} \lambda_i = 0$. Then $a \cdot \partial(a_{1jk}) = 0$.

PROOF. Since $a \in A_1(\mathcal{A})$, we have

$$\partial(a \cdot a_{1jk}) = \partial(a)a_{1jk} - a\partial(a_{1jk}).$$

But $\partial(a) = \sum_{i=1}^{n} \lambda_i = 0$. Moreover, $a_i a_{1jk} = 0$ for all $1 \leq i \leq n$. Since ∂ is linear, this implies $\partial(a \cdot a_{1jk}) = 0$ and the result follows. □

DEFINITION 6.1.10. Let $H_j, H_k \in \mathcal{A}$. Let $X_{jk} := \{i : \ H_j \cap H_k \subseteq H_i\}$.

THEOREM 6.1.11. Let $\mathcal{A}$ be a rank three central hyperplane arrangement. Let

$$a = \sum_{i=1}^{n} \lambda_i a_i \in A_1(\mathcal{A}) \text{ with } \sum_{i=1}^{n} \lambda_i = 0.$$

Let $x \in A_2$ be decomposed as in (6.1.8.a) using the basis C_2. In the product $a \cdot x$, the coefficient of a_{1jk} is given by

$$\Big(- \sum_{i \in X_{jk} \setminus \{k\}} \lambda_i \Big) x_{1k} + \lambda_k \Big(\sum_{i \in X_{jk} \setminus \{k\}} x_{1i} \Big).$$

PROOF. Using Lemma 6.1.9, we need only to compute $a \cdot \sum_{i=2}^{n} x_{1i}a_{1i}$.

$$\begin{aligned} a \cdot \sum_{i=2}^{n} x_{1i}a_{1i} &= \Big(\sum_{i=1}^{n} \lambda_i a_i \Big) \Big(\sum_{i=2}^{n} x_{1i}a_{1i} \Big) \\ &= - \sum_{i<j} \lambda_i x_{1j} a_{1ij} + \sum_{j<i} \lambda_i x_{1j} a_{1ji}. \end{aligned}$$

Suppose $a_{ij} \notin B_2$. Let α be minimal in X_{ij}. Then $a_{1ij} = a_{1\alpha j} - a_{1\alpha i}$. Using this, we compute the coefficient of a_{1jk} for $a_{jk} \in B_2, j \geq 2$ to be

$$\begin{aligned} &-\lambda_j x_{1k} + \lambda_k x_{1j} - \sum_{i<k, i \in X_{jk}} \lambda_i x_{1k} + \sum_{k<i, i \in X_{jk}} \lambda_k x_{1i} \\ &+ \sum_{i<k, i \in X_{jk}} \lambda_k x_{1i} - \sum_{k<i, i \in X_{jk}} \lambda_i x_{1k}. \end{aligned}$$

By combining like terms, the result follows. □

Suppose $\mathcal{A}$ is central. Then we can form the deconed arrangement $d\mathcal{A}$ as follows. Let α_i be the functional corresponding to H_i. Without loss of generality, we may assume $\alpha_1 = x_1$. Decone at $\alpha_1 = x_1$ by setting $x_1 = 1$.

We write $d\mathcal{A} = \{dH_2, ..., dH_n\}$, where dH_i denotes the hyperplane corresponding to the functional α_i where $x_1 = 1$. Denote the Orlik-Solomon algebra of $d\mathcal{A}$ by dA. We write dB_i to mean the broken circuit basis for dA.

LEMMA 6.1.12. Let $\mathcal{A}$ be a central arrangement. We have:

$$dB_2 = \{a_{jk} : \ j \geq 2, jk \in B_2\}.$$

PROOF. Suppose $a_{jk} \in B_2, j \geq 2$. To show $a_{jk} \in dB_2$, we need only check the intersection $dH_j \cap dH_k \neq \emptyset$. Since $a_{jk} \in B_2$ with $j \geq 2$, we have $\{H_1, H_j, H_k\}$ are independent; hence, $dH_j \cap dH_k \neq \emptyset$.

Suppose $a_{jk} \in dB_2$. Then by definition $a_{jk} \in B_2$. □

DEFINITION 6.1.13. Let $\mathcal{A}$ be a central arrangement. For $2 \leq j < k \leq n$, we set

$$Y_{jk} := \{i : \ 2 \leq i < n, H_j \cap H_k \subseteq H_i\}.$$

THEOREM 6.1.14. Let $\mathcal{A}$ be a rank three central hyperplane arrangement. In $A_1(d\mathcal{A})$, let

$$\tilde{a} := \sum_{i=2}^{n} \lambda_i a_i \text{ and } x := \sum_{i=2}^{n} x_i a_i.$$

Then in the product $\tilde{a} \cdot x \in A_2(d\mathcal{A})$, the coefficient of a_{jk} is

$$\Big(\sum_{i \in Y_{jk} \setminus \{k\}} \lambda_i\Big) x_k - \lambda_k \Big(\sum_{i \in Y_{jk} \setminus \{k\}} x_i\Big).$$

PROOF. By computing the product, we have:

$$\begin{aligned} a \cdot x &= \Big(\sum_{i=2}^{n} \lambda_i a_i\Big)\Big(\sum_{i=2}^{n} x_i a_i\Big) \\ &= \sum_{2 \le i < j \le n} (\lambda_i x_j - \lambda_j x_i) a_{ij}. \end{aligned}$$

Suppose $a_{ij} \notin dB_2$. If $dH_i \cap dH_j = \emptyset$, then $a_{ij} = 0$. Otherwise, let α be minimal in Y_{ij}. Then $a_{ij} = a_{\alpha j} - a_{\alpha i}$. Using this, we compute the coefficient of a_{jk} to be as required. □

THEOREM 6.1.15. Let $\mathcal{A}$ be a rank three central hyperplane arrangement. We have

$$\dim Z_2(a) = \dim Z_1(a) + |\{a_{jk} \in B_2(\mathcal{A}) : \ j > 1\}|.$$

PROOF. We apply Theorems 6.1.11 and 6.1.14 to see that

$$\dim Z_2(a) = \dim Z_1(\tilde{a}) + |\{a_{jk} \in B_2(\mathcal{A}) : \ j > 1\}|.$$

Furthermore by Lemma 6.1.1, we have $\dim Z_1(a) = \dim Z_1(\tilde{a})$. □

As a brief summary of the results thus far obtained, we decomposed

$$x = \sum_{i=2}^{n} x_{1i} a_{1i} + \sum_{a_{jk} \in B_2(\mathcal{A}), j>1} x_{jk}(a_{jk} - a_{1k} + a_{1j})$$

so we could show $\dim Z_2(\mathcal{A}) = \dim Z_1(\mathcal{A}) + |\{a_{jk} \in B_2(\mathcal{A}) : \ j > 1\}|$ for $\sum_{i=1}^{n} \lambda_i = 0$. But now we change the basis of $A_2(\mathcal{A})$ back to the broken circuit basis. We do this by noting

$$\sum x_{1i} a_{1i} + \sum x_{jk} a_{jk} = \sum (x_{1i} + \sum_j x_{ji} - \sum_j x_{ij}) a_{1i} + \sum x_{jk}(a_{jk} - a_{1k} + a_{1j}).$$

Moreover, we let a be arbitrary, dropping the condition $\sum_{i=1}^{n} \lambda_i = 0$. We do this so that we may obtain equations describing $x \in Z_2(a)$ in an arbitrary setting.

Let $a_{jk} \in B_2(\mathcal{A})$ with $j > 1$. For each fixed $a_{jk} \in B_2(\mathcal{A})$ with $j > 1$, we obtain the equation:

$$(\sum_{i=1}^{n} \lambda_i)x_{jk} - (\sum_{i \in X_{jk} \setminus \{k\}} \lambda_i)(\sum_{1 \leq i < k} x_{ik} - \sum_{i > k} x_{ki}) + \lambda_k \Big(\sum_{i \in X_{jk} \setminus \{k\}} (\sum_{1 \leq p < i} x_{pi} - \sum_{p > i} x_{ip}) \Big) = 0.$$

This equation can be simplified to

(6.1.15.a)
$$(\sum_{i \notin X_{jk}} \lambda_i)x_{jk} - (\sum_{i \in X_{jk} \setminus \{k\}} \lambda_i)(\sum_{i \notin X_{jk}} (x_{ik} - x_{ki})) + \lambda_k \Big(\sum_{i \in X_{jk} \setminus \{k\}} (\sum_{p \notin X_{jk}} (x_{pi} - x_{ip}) \Big) = 0.$$

In [8], $\dim Z_1(a)$ was found by encoding the structure of $\mathcal{A}$ and a into an incidence matrix. We will recall their construction and then proceed to use this construction to obtain a matrix description for $\dim Z_2(a)$. However, it will not be an incidence matrix; it will be a matrix with entries from $\{0, 1, -1\}$.

Recall from Chapter III the following notations and results established in [8]. Let

$$\chi(a) := \{X \in L(2) : \ |X| > 2, a(X) \neq 0, \sum_{i \in X} \lambda_i = 0\}$$

Let $I(a) \subset \vec{n}$ be defined as follows. We have $i \in I(a)$ if

(i) $H_i < X$ for some $X \in \mathcal{X}(a)$, and

(ii) if $\lambda_i = 0$, then there does not exist $\lambda_j \neq 0$ for which H_i, H_j are not in any $X \in \mathcal{X}(a)$.

In this setting, the incidence matrix J is the $|\chi(a)| \times |I(a)|$ matrix with $J_{X,i} = 1$ if $H_i < X$ and zero otherwise. The matrix J describes $\dim Z_1(a)$ for $a \neq 0$; see [8].

We say a matrix M is affine if it is positive semidefinite and its null space is spanned by a positive vector, meaning all coordinates are positive. We say a matrix M is indefinite if there exists a vector $u > 0$ so that $Mu < 0$.

Let $Q = J^t J$. Decompose Q into the direct sum of its principle submatrices so that $Q = \bigoplus_K Q_K$. Then by [8], we have only two possibilities

1. For each K, we have Q_K is either affine or has only the zero vector for its kernel. In this case, we say $\mathcal{X}(a)$ is affine. Since for $x \in Z_1(a)$ and $i \notin I(a)$, we have $x_i = 0$, we may assume $I(a) = \vec{n}$. Then $Z_1(a) = \operatorname{Ker} J \cap \{\sum_{i=1}^n x_i = 0\}$; we refer to [8].
2. There exists an unique K_0 so that Q_{K_0} is indefinite and for all other K Q_K has only the zero vector for its kernel. In this case, we say $\mathcal{X}(a)$ is indefinite. If $\mathcal{X}(a)$ is indefinite, then $\dim Z_1(a) = 1$ by [8].

In order to use the matrix J to describe the dimension of $Z_2(a)$, we first establish some technical lemmas.

LEMMA 6.1.16. Let $\mathcal{A}$ be a rank three central hyperplane arrangement. Let $x \in Z_1(\tilde{a})$ with $x = \sum_{i=2}^n x_i a_i$. We have $\hat{x} = (-\sum_{i=2}^n x_i)a_1 + \sum_{i=2}^n x_i a_i \in Z_1(a)$.

PROOF. We compute

$$\begin{aligned}
a\hat{x} =&\left(\sum_{i=1}^n \lambda_i a_i\right)\left((-\sum_{i=2}^n x_i)a_1 + \sum_{i=2}^n x_i a_i\right)\\
=&\sum_{i=2}^n \lambda_1 x_i a_{1i} + \sum_{i=2}^n(\sum_{j=2}^n x_j)\lambda_i a_{1i} + \sum_{2\le i<j\le n}(\lambda_i x_j - \lambda_j x_i)(a_{1j} - a_{1i})\\
=&\sum_{i=2}^n \left(\lambda_1 x_i + \lambda_i(\sum_{j=2}^n x_j) - \sum_{i<j}(\lambda_i x_j - \lambda_j x_i) + \sum_{i>j}(\lambda_j x_i - \lambda_i x_j)\right)a_{1i}\\
=&\sum_{i=2}^n(\sum_{j=1}^n \lambda_j)x_i a_{1i}\\
=&0. \quad \square
\end{aligned}$$

LEMMA 6.1.17. Let $\mathcal{A}$ be a rank three central hyperplane arrangement. Using the basis C_2, we decompose $x \in A_2(\mathcal{A})$ as in (6.1.8.a); that is,

$$x = \sum_{i=2}^{n} x_{1i}a_{1i} + \sum_{j>1} x_{jk}\partial a_{1jk}.$$

If $x \in Z_2(a)$, then

$$(-\sum_{i=2}^{n} x_{1i})a_1 + \sum_{i=2}^{n} x_{1i}a_i \in Z_1(a).$$

PROOF. We apply Theorems 6.1.11 and 6.1.14 to see $\sum_{i=2}^{n} x_{1i}a_{1i} \in Z_1(\tilde{a})$. Our conclusion now follows from Lemma 6.1.16. □

We will use the broken circuit basis instead of the basis C_2 of Lemma 6.1.8, and we will construct the matrix K similarly to the matrix J. We distinguish between the cases where $\mathcal{X}(a)$ is affine and $\mathcal{X}(a)$ is indefinite. We begin by establishing an analogue to $I(a)$.

DEFINITION 6.1.18. Let

$$\psi(a) := \{jk : \ a_{jk} \in B_2(\mathcal{A})\}$$

Let K be the $(|\mathcal{X}(a)| + n - |I(a)|) \times |\psi(a)|$ matrix constructed by using the matrix J. To do this, we notice the following for $X \in \mathcal{X}(a)$ via the change of base from C_2 to B_2.

1. For $1 < i \leq n$ and $H_i < X$, x_i for $x \in Z_1(a)$ corresponds to $x_{1i} + \sum x_{ji} - \sum x_{ij}$ for $x \in Z_2(a)$.
2. For $H_1 < X$, x_1 for $x \in Z_1(a)$ corresponds to $-\sum_{i=2}^{n} x_{1i}$ for $x \in Z_2(a)$.

For $\mathcal{X}(a)$ affine, the matrix K is given by the following for $jk \in \psi(a), X \in \mathcal{X}(a) \cup \{H_\alpha : \ \alpha \notin I(a)\}$:

$$K_{X,jk} = 1, \text{ if } H_k \le X \text{ but } H_j \not\le X$$
$$= -1, \text{ if } H_j \le X \text{ but } H_k \not\le X$$
$$= 0, \text{ otherwise.}$$

THEOREM 6.1.19. Let $\mathcal{A}$ be a rank three central arrangement. If $\mathcal{X}(a)$ is affine, then $Z_2(a) = \operatorname{Ker} K$. Hence,

$$\dim H^2(A, a) = \dim(\operatorname{Ker} K) - \operatorname{rank} d_1.$$

PROOF. Let $x \in Z_2(a)$ written as

$$x = \sum_{i=2}^{n} x_{1i}a_{1i} + \sum_{j>1} x_{jk}a_{jk}$$
$$= \sum_{i=2}^{n}(\sum_{j<i} x_{ji} - \sum_{j>i} x_{ij})a_{1i} + \sum_{j>1} x_{jk}\partial a_{1jk}.$$

By Lemma 6.1.17,

$$\check{x} := (-\sum_{i=2}^{n} x_{1i})a_1 + \sum_{i=2}^{n}(\sum_{j<i} x_{ji} - \sum_{j>i} x_{ij})a_i \in Z_1(a).$$

We have that $Z_1(a) \subseteq \operatorname{Ker} J$. Hence, $\check{x} \in \operatorname{Ker} J$. Fix $X \in \mathcal{X}(a)$. Since $\check{x} \in \operatorname{Ker} J$, we have $\sum_{i\in X} \check{x}_i = 0$; but this gives

$$0 = \sum_{i\in X} \check{x}_i = \sum_{i\in X, j\notin X} x_{ji} - \sum_{i\in X, j\notin X} x_{ij}$$

as required to verify $x \in \operatorname{Ker} K$.

Let $x \in \operatorname{Ker} K$, written as in the previous paragraph. Let $\check{x}$ be defined as in the previous paragraph. Since $x \in \operatorname{Ker} K$, we have

$$\sum_{i\in X, j\notin X} x_{ji} - \sum_{i\in X, j\notin X} x_{ij} = 0 \text{ for all } X \in \mathcal{X}(a).$$

This gives

$$\sum_{i \in X} (\sum x_{ji} - x_{ij}) = 0.$$

Hence, $\check{x} \in \operatorname{Ker} J$. Moreover, since $x \in \operatorname{Ker} K$, we have $\check{x}_i = 0$ for $i \notin I(a)$. Since $\mathcal{X}(a)$ is affine and the sum of the coefficients of $\check{x}$ is zero, we have $\check{x} \in Z_1(a)$. By Theorems 6.1.11 and 6.1.14, it now follows $x \in Z_2(a)$.

Therefore, $Z_2(a) = \operatorname{Ker} K$. □

EXAMPLE 6.1.20. Let $Q(\mathcal{A}) = xyz(x+y)$, ordered as they are written; let $a = a_1 - a_2$. Then $\mathcal{X}(a) = \{124\}$, I(a) ={1,2,4}, $\psi(a) = \{12, 13, 14, 23, 34\}$. The matrix K is given by

$$K = \begin{pmatrix} 0 & -1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 \end{pmatrix}.$$

And $\operatorname{rank} d_1 = 2$. So, $\dim H^2 = 4 - 2 = 2$, and this coincides with the results of Theorem 6.1.15. By direct computation, it is easily verified that $\operatorname{Ker} K = Z_2(a)$.

EXAMPLE 6.1.21. Let $Q(\mathcal{A}) = xy(x+y)(x+y+z)z$; order the hyperplanes as they are written. Let $a = a_1 - a_2$. We have $I(a) = \{1, 2, 3\} \neq \vec{n}$ and $\mathcal{X}(a)$ is affine. With $\psi(a) = \{12, 13, 14, 15, 24, 25, 34, 35\}$, we have

$$K = \begin{pmatrix} 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Hence, $\dim \operatorname{Ker} K = \dim Z_2(a) = 6$.

EXAMPLE 6.1.22. In the previous examples, it was enough to consider only the equations generated by $i \notin I(a)$. In this example, we must consider $X \in \mathcal{X}(a)$. Let $Q(\mathcal{A}) = xyz(x-y)(x-z)(y-z)(x+y)$; order the hyperplanes as they are written. Let $a := a_1 - a_2 - a_5 + a_6$. In Example 3.3.14, it was shown that $\mathcal{X}(a)$ is affine and $I(a) = \{1, 2, 3, 4, 5, 6\}$. Now, $\mathcal{X}(a) = \{1247, 135, 236, 456\}$ and $\psi(a) =$

$\{12, 13, 14, 15, 16, 17, 23, 25, 26, 34, 37, 45, 46, 57, 67\}$. Hence, the matrix K is given by

$$\begin{pmatrix} 0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 0 & -1 & 0 & -1 & -1 & 1 & 1 & 0 & -1 & -1 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

We can now see that $\dim \operatorname{Ker} K = 11 = \dim Z_2(a)$.

EXAMPLE 6.1.23. Let $Q(\mathcal{A}) = (x-y)(x-z)(y-z)x(x+y)y(x+z)(y+z)z$ and let $a = a_1 - a_2$. In Example 5.3.2, we computed $\dim Z_2(a) = 17$ but were unable to use the formulas of Chapter V as the order on the hyperplanes did not respect the supersolvable structure of $\mathcal{A}$. However, we can compute $\dim Z_2(a)$ by using the matrix K. We have $\mathcal{X}(a) = \{123\}$ and is affine, and $I(a) = \{1, 2, 3\}$. We compute $\psi(a)$ to be

$$\{12, 13, 14, 15, 16, 17, 18, 19, 24, 25, 26, 27, 28, 29, 34, 35, 36, 37, 38, 39, 48, 59, 67\}.$$

We label the rows of K by $\{123, 4, 5, 6, 7, 8, 9\}$. After computing, we have $\operatorname{rank} K = 6$. Hence, $\dim Z_2(a) = 23 - 6 = 17$ and the answer agrees with what we computed earlier.

We now consider the case where $\mathcal{X}(a)$ is indefinite or $\mathcal{X}(a) = \emptyset$; in this case, $\dim Z_1(a) = 1$. Hence, for any $x \in Z_1(a)$ we have that $x = \xi a$ for some $\xi \in \mathcal{K}$. In $Z_2(a)$, this corresponds to

$$\sum_{\alpha i \in B_2(\mathcal{A})} x_{\alpha i} - \sum_{i\alpha \in B_2(\mathcal{A})} x_{i\alpha} = \xi \lambda_i, \quad \text{for } 2 \leq i \leq n.$$

By treating ξ as a variable, we have a homogeneous system of equations describing $Z_2(a)$. Notice there are $n-1$ linearly independent equations in this system. Notice this is the same as the matrix K as done for the affine case for $I(a) = \{1\}$ except for the introduction of ξ.

DEFINITION 6.1.24. Let $\tilde{K}$ be the $(n-1) \times (|B_2|+1)$ with rows indexed by $\{2, \ldots, n\}$ and columns indexed by $\{jk : \ a_{jk} \in B_2(\mathcal{A})\} \cup \{\xi\}$ be the matrix given by

$$\begin{aligned} K_{\alpha,jk} &= 1, \text{ if } k = \alpha \\ &= -1, \text{ if } j = \alpha \\ &= 0, \text{ otherwise.} \\ K_{i,\xi} &= -\lambda_i \end{aligned}$$

THEOREM 6.1.25. Let $\mathcal{A}$ be a rank three central hyperplane arrangement. If $\mathcal{X}(a)$ is indefinite, then $Z_2(a) = \operatorname{Ker} \tilde{K}$.

PROOF. This is immediate by the discussion prior to Definition 6.1.24. □

EXAMPLE 6.1.26. Let $Q(\mathcal{A}) = xy(x+y)(x+y+z)z$; order the hyperplanes as they are written. Let $a = a - a_2 + a_4 - a_5$. In Example 3.3.13, it was shown that $\mathcal{X}(a)$ is indefinite. Now $\tilde{K}$ will be a matrix whose columns are indexed by $\{12, 13, 14, 15, 24, 25, 34, 35, \xi\}$ and whose rows are indexed by $\{2, 3, 4, 5\}$, giving

$$\tilde{K} = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

By elementary linear algebra, we see $\dim \operatorname{Ker} \tilde{K} = 5$; hence, $\dim Z_2(a) = 5$.

DEFINITION 6.1.27. Let

$$\begin{aligned} \hat{\mathcal{X}}(a) &:= \mathcal{X}(a) \cup \vec{n} \setminus I(a), \text{ if } \mathcal{X}(a) \text{ is affine,} \\ &:= \vec{n} \setminus \{1\}, \text{ if } \mathcal{X}(a) \text{ is indefinite.} \end{aligned}$$

Let

$$\begin{aligned} \hat{\psi}(a) &:= \{jk \in B_2(\mathcal{A})\}, \text{ if } \mathcal{X}(a) \text{ is affine,} \\ &:= \{jk \in B_2(\mathcal{A})\} \cup \{\xi\}, \text{ if } \mathcal{X}(a) \text{ is indefinite.} \end{aligned}$$

Let

$$\hat{K} := K, \text{ if } \mathcal{X}(a) \text{ is affine,}$$

$$:= \tilde{K}, \text{ if } \mathcal{X}(a) \text{ is indefinite..}$$

THEOREM 6.1.28. Let $\mathcal{A}$ be a rank three central hyperplane arrangement. If $0 \neq a \in A_1(\mathcal{A})$, then $\operatorname{Ker} \hat{K} = Z_2(a)$.

PROOF. By considering the two cases where $\mathcal{X}(a)$ is affine or $\mathcal{X}(a)$ is indefinite, the theorem follows immediately. □

§6.2 Dimension of $H^2(A, a)$ For Central Arrangements

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a central arrangement. Let $a = \sum_{i=1}^{n} \lambda_i a_i$ in $A_1(\mathcal{A})$. Since $\sum_{i=1}^{n} \lambda_i \neq 0$ implies $H^*(A, a) = 0$, refer to [13], we assume $\sum_{i=1}^{n} \lambda_i = 0$.

DEFINITION 6.2.1. Fix $X \in L(\mathcal{A})$. Then $a(X) = \sum_{H_i < X} \lambda_i a_i$. Similarly, for $x \in A_p(\mathcal{A})$, we have $x = \sum_{a_{\vec{j}} \in B_p} x_{\vec{j}} a_{\vec{j}}$. We define $x(X) := \sum_{a_{\vec{j}} \in B_p(\mathcal{A}_X)} x_{\vec{j}} a_{\vec{j}}$ in $A_1(\mathcal{A}_X)$.

THEOREM 6.2.2. Let $\mathcal{A}$ be a central arrangement. Let $a = \sum_{i=1}^{n} \lambda_i a_i$ in $A_1(\mathcal{A})$ with $\sum_{i=1}^{n} \lambda_i = 0$. We have $x \in Z_k(a)$ if and only if $x(X) \in Z_k(a(X))$ for all $X \in L(k+1)$.

PROOF. Let $a_{\vec{j}} \in B_{k+1}$. Let $X \in L(k+1)$ with $\vec{j} \subseteq X$. It will suffice to show the coefficient of $a_{\vec{j}}$ in the product ax is the same coefficient of $a_{\vec{j}}$ in the product $a(X)x(X)$.

Let $\vec{j} := (j_1, \ldots, j_{k+1})$. Let $\vec{j}^i := (j_1, \ldots, \hat{j}_i, \ldots, j_{k+1})$ for $i = 1, \ldots, k+1$. Since $\vec{j} \in B_{k+1}$, we have $\vec{j}^i \in B_k$ for all $1 \leq i \leq k+1$. We have three cases where $a_{\vec{j}}$ may have a nonzero coefficient as a product of an element from B_1 and an element from B_p.

1. We have $a_{j_i} a_{\vec{j}^i} = \pm a_{\vec{j}}$ by the definition of $E(\mathcal{A})$.
2. We have $a_\alpha a_{\vec{j}^1}$ for $\alpha \in X \setminus \vec{j}$ by the dependencies in the definition of A.
3. We have $a_\alpha a_{\vec{\ell}}$ for $\{\alpha, \vec{\ell}^i\} = \vec{j}^1$ and $\vec{\ell} \in B_k$ by the dependencies in the definition of A.

Since any of the three cases give the same result in $\mathcal{A}$ or $\mathcal{A}_X$, the result follows. □

Let $a = \sum_{i=1}^{n} \lambda_i a_i$ be in $A_1(\mathcal{A})$ so that $\sum_{i=1}^{n} \lambda_i = 0$. Let $x = \sum_{a_{jk} \in B_2} x_{jk} a_{jk}$ be in $Z_2(a)$. Then $x(X) \in Z_2(a(X))$ for all $X \in L(3)$.

Let $X \in L(3)$ so that $a(X) \neq 0$. Let 1_X denote the minimal element of X. We decomposed

$$x(X) = \sum_{i \in X \setminus \{1_X\}} x_{1_X i} a_{1_X i} + \sum_{a_{jk} \in B_2(\mathcal{A}_X), j > 1_X} x_{jk}(a_{jk} - a_{1_X k} + a_{1_X j})$$

so that we could show

$$\dim Z_2(\mathcal{A}_X) = \dim Z_1(\mathcal{A}_X) + |\{a_{jk} \in B_2(\mathcal{A}_X) : \quad j > 1_X\}|.$$

But now we change from the basis $C_2(\mathcal{A}_X)$ back to the broken circuit basis. We let $a_{jk} \in B_2(\mathcal{A}_X)$ with $j > 1_X$. Let $X_{jk} := \{i : \ H_j \cap H_k \subseteq H_i\}$. We obtain the equation:

$$\begin{aligned}(\sum_{i \in X} \lambda_i) x_{jk} - (\sum_{i \in X_{jk} \setminus \{k\}} \lambda_i)(\sum_{1_X \leq i < k} x_{ik} - \sum_{i > k} x_{ki}) \\ + \lambda_k \Big(\sum_{i \in X_{jk} \setminus \{k\}} (\sum_{1_X \leq p < i} x_{pi} - \sum_{p > i} x_{ip}) \Big) = 0.\end{aligned}$$

We can simplify this equation:

$$\begin{aligned}(\sum_{i \in X \setminus X_{jk}} \lambda_i) x_{jk} - (\sum_{i \in X_{jk} \setminus \{k\}} \lambda_i)(\sum_{i \in X \setminus X_{jk}} (x_{ik} - x_{ki})) \\ + \lambda_k (\sum_{i \in X_{jk} \setminus \{k\}} (\sum_{p \in X \setminus X_{jk}} (x_{pi} - x_{ip}))) = 0.\end{aligned} \tag{6.2.2.a}$$

The system of equations given by equation (6.2.2.a) for $X \in L(3)$ with $a(X) \neq 0$ describes $Z_2(a)$.

The image of d_1 should also be considered. Suppose $x = ay$ for some $y \in A_1(\mathcal{A})$. Then by computation, we have for each $jk \in B_2(\mathcal{A})$,

$$x_{jk} = \Big(\sum_{i \in X_{jk} \setminus \{k\}} \lambda_i \Big) y_k - \lambda_k \Big(\sum_{i \in X_{jk} \setminus \{k\}} y_i \Big). \tag{6.2.2.b}$$

DEFINITION 6.2.3. Let

$$\mathcal{S}(a) := \{X \in L(3) : \ a(X) \neq 0, \sum_{i \in X} \lambda_i = 0, |X| > 3\}.$$

THEOREM 6.2.4. Let $x \in Z_2(a)$.

(1) If $a_{jk} \notin B_2(\mathcal{A}_X)$ for any $X \in \mathcal{S}(a)$ and $a(X_{jk}) = 0$, then $x(X_{jk}) = 0$.

(2) If $a_{jk} \notin B_2(\mathcal{A}_X)$ for any $X \in \mathcal{S}(a)$ and $X_{jk} \notin \mathcal{X}(a)$ and $a(X_{jk}) \neq 0$, then the cohomology class $[x] \in H^2(a)$ is equivalent to a class $[w]$ where $w \in Z_2(a)$ and $w(H_\beta \vee H_i) = 0$ for any $H_\beta < X_{jk}$ and any $i \neq \beta$.

(3) Consider the set $\{X_1, \dots, X_m : \ X_i \in \mathcal{X}(a), \ X_i \not< Y$ for any $Y \in \mathcal{S}(a)\}$. Then the cohomology class $[x] \in H^2(a)$ is equivalent to a class $[w]$ where $w \in Z_2(a)$ and $w(X_i) = 0$ for any X_i in this set.

PROOF. We begin by showing (1). Suppose $a_{jk} \notin B_2(\mathcal{A}_X)$ for any $X \in \mathcal{S}(a)$. If $a(X_{jk}) = 0$, then we use equation (6.2.2.a) to see $x(X_{jk}) = 0$.

To show (2), let $\alpha \notin X_{jk}$. Let $X_\alpha \in L(3)$ contain $\{\alpha, j, k\}$. Notice $a(X_{jk}) \neq 0$, so we have $a(X_\alpha) \neq 0$ and $X_\alpha \notin \mathcal{S}(a)$. Thus, $H^*(A(\mathcal{A}_{X_\alpha}), a(X_\alpha)) = 0$; in particular, $H^2(a(X_\alpha)) = 0$. Hence, there exists $z_\alpha \in A_1(\mathcal{A}_{X_\alpha})$ so that $x(X_\alpha) = a(X_\alpha) z_\alpha$.

Since $\dim Z_1(a(X_{jk})) = 1$, we may assume $z_\alpha(X_{jk}) = z_{\alpha'}(X_{jk})$ for any $\alpha, \alpha' \notin X_{jk}$. That is, for $\alpha, \alpha' \notin X_{jk}$, we have $z_\alpha(X_{jk}) - z_{\alpha'}(X_{jk}) = c' a(X_{jk})$ for c' a constant. Hence, we may define $\hat{z}' := z' - c' a(X_{\alpha'})$. Then $\hat{z}'(X_{jk}) = z(X_{jk})$ and $a(X_{\alpha'}) \hat{z}_{\alpha'} = a(X_{\alpha'}) z_{\alpha'} = x(X_{\alpha'})$.

Therefore, we define $z \in A_1(\mathcal{A})$ via

$$z_i = (z_\alpha)_i \text{ if } H_i < X_\alpha.$$

Let $w = x - az \in Z_2(a)$. For $H_\beta < X_{jk}$ and $i \neq \beta$, we have $w(H_\beta \vee H_i) = 0$ as required.

To prove (3), we will proceed similarly as in (2) by constructing $z \in A_1(\mathcal{A})$ so that $x - az$ satisfies $x - az(X_i) = 0$. We will construct z recursively. Begin by noticing that if $|X_i \wedge X_j| = 0$ for all $i \neq j$, then the problem is solved easily. That is, for each X_i fix a hyperplane $H \not< X_i$. There exists $z_i(X_i \vee H)$ which satisfies $a(X_i \vee H)z(X_i \vee H) = x(X_i \vee H)$. Define $z \in A_1(\mathcal{A})$ to be

$$\begin{aligned} z_i =& (z_j)_i \text{ if } H_i < X_j \\ =& 0 \text{ otherwise.} \end{aligned}$$

Then by Equation 6.2.2.b, we have $a(X_i)z(X_i) = x(X_i)$ for each i.

We now assume there exists i, j so that $|X_i \wedge X_j| = 1$. Without loss of generality, assume $|X_1 \wedge X_2| = 1$. We now construct z recursively.

1. Begin with X_1 and X_2. Since $\operatorname{rank}(X_1 \vee X_2) = 3$, then there exists $z(X_1 \vee X_2)$ so that $a(X_1 \vee X_2)z(X_1 \vee X_2) = x(X_1 \vee X_2)$.
2. Suppose $\tilde{z}$ is defined so that $a\tilde{z}(X_i) = x(X_i)$ for all $i < k$.

   If $|X_k \wedge X_i| = 1$ for all $1 \leq i < k$, then notice $X_k < X_1 \vee X_2$. By our construction, $a(X_k)z(X_k) = x(X_k)$.

   If $|X_k \wedge X_i| \neq 1$ for some $1 \leq i < k$, then by equation 6.2.2.b, we have $|X_k| - 1$ degrees of freedom in choosing $z(X_k)$ so that $a(X_k)z(X_k) = x(X_k)$. We define z on X_k accordingly. Then $az(X_i) = x(X_i)$ for all $i \leq k$ as required. □

When studying $\dim Z_1(a)$, it was shown that $x_i = 0$ for any $i \notin I(a)$; thus, we assumed $\vec{n} = I(a)$. By Theorem 6.2.4, we may assume for any $a_{jk} \in B_2(\mathcal{A})$ there exists $X \in \mathcal{S}(a)$ so that $a_{jk} \in B_2(\mathcal{A}_X)$.

We use the matrix descriptions given earlier for $Z_2(a)$ for $\mathcal{A}_X$ with $X \in \mathcal{S}(a)$. Notice that in the case $\mathcal{X}(a(X))$ is not affine, we introduce ξ_X.

DEFINITION 6.2.5. Let

$$\Upsilon(a) := \{(Y, X)|\ X \in S, Y \in \hat{\chi}(a(X))\}.$$

Let

$$\Psi(a) := \bigcup_{X \in S} \hat{\psi}(a(X)).$$

The matrix K we obtain is a $|\Upsilon(a)| \times |\Psi(a)|$ matrix whose entries are

$$\begin{aligned} K_{(X,Y),jk} &= 1, \text{ if } H_k \leq X \text{ but } H_j \not\leq X \text{ and } a_{jk} \in B_2(\mathcal{A}_Y) \\ &= -1, \text{ if } H_j \leq X \text{ but } H_k \not\leq X \text{ and } a_{jk} \in B_2(\mathcal{A}_Y) \\ &= 0, \text{ otherwise.} \end{aligned}$$

THEOREM 6.2.6. Let $\mathcal{A}$ be central hyperplane arrangement. Let $a \in A_1(\mathcal{A})$ with $a = \sum_{i=1}^{n} \lambda_i a_i$ and $\sum_{i=1}^{n} \lambda_i = 0$. If $a(X) = 0$ for all $X \in L(3) \setminus \mathcal{S}(a)$, then $Z_2(a) = \operatorname{Ker} K \cap \{x_{jk} = 0 \text{ if } jk \notin \Psi(a)\}$.

PROOF. Let $x \in Z_2(a)$. If $X \in \mathcal{S}(a)$, then $x(X) \in Z_2(a(X))$ by Theorem 6.2.2. Hence, $x \in \operatorname{Ker} K$. By Theorem 6.2.4, we have $x_{jk} = 0$ if $jk \notin \Psi(a)$.

Let $x \in \operatorname{Ker} K \cap \{x_{jk} = 0 \text{ if } jk \notin \Psi(a)\}$. Then $x(X) \in Z_2(a(X))$ for all $X \in \mathcal{S}(a)$. If $X \in L(3) \setminus \mathcal{S}(a)$, then $a(X) = 0$ by assumption; hence, $x(X) \in Z_2(a(X))$. By Theorem 6.2.2, it follows that $x \in Z_2(a)$. □

EXAMPLE 6.2.7. Notice in the proof of Theorem 6.2.6, it suffices to show for $X \in L(3) \setminus \mathcal{S}(a)$, we have $x(X) \in Z_2(a(X))$. Suppose for any $X \in L(3) \setminus \mathcal{S}(a)$ with $a(X) \neq 0$ there exists $Y \in L(2, \mathcal{A}_X)$ with the following properties:

1. $Y \notin L(2, \mathcal{A}_Z)$ for all $Z \in \mathcal{S}(a)$,
2. $a(Y) = 0$, and

3. $|\mathcal{A}_X \setminus \mathcal{A}_Y| = 1$.

Then $a(X) \cdot x(X) = 0$. Hence, the result of Theorem 6.2.6 holds.

Let $Q(\mathcal{A}) = xyzw(x+y)$; order the hyperplanes as they are written. Let $a = a_1 - a_2$. Then $\Upsilon(a) = \{(125, 1235), (3, 1235), (125, 1245), (4, 1245)\}$ and $\Psi = \{12, 13, 14, 15, 23, 24, 35, 45\}$. The matrix K we obtain is

$$K = \begin{pmatrix} 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

Therefore, $\dim Z_2(a) = 6$.

EXAMPLE 6.2.8. Theorem 6.2.6 fails if the condition $a(X) = 0$ for $X \in L(3) \setminus \mathcal{S}(a)$ is dropped. Let $Q(\mathcal{A}) = xy(x+y)zw(w+x+y)$, and let $a = a_1 - a_2 + a_5 - a_6$. We compute $\dim H^2(\mathcal{A}, a)$ by first deconing the arrangement about the hyperplane given by $z = 0$. We obtain $Q(d\mathcal{A}) = xy(x+y)w(w+x+y)$. Order the hyperplanes as they are written. Then $\tilde{a} = a_1 - a_2 + a_4 - a_5$ and we consider the chain complex $(A(d\mathcal{A}), \tilde{a})$. In Example 3.3.13, we computed $\dim Z_1(\tilde{a}) = 1$. Hence, $\dim H^1(A(\mathcal{A}), a) = \dim H^1(A(d\mathcal{A}), \tilde{a}) = 0$. Since we have the short exact sequence

$$0 \to H^1(A(d\mathcal{A}), \tilde{a}) \xrightarrow{\cdot} H^2(A(\mathcal{A}), a) \to H^2(A(d\mathcal{A}), \tilde{a}) \to 0,$$

it will suffice to compute $H^2(A(d\mathcal{A}), \tilde{a})$. Since $d\mathcal{A}$ is central, we have $\dim Z_2(\tilde{a}) = 1 + 4 = 5$ by Theorem 6.1.15. Hence, $\dim H^2(A(\mathcal{A}), a) = \dim H^2(A(d\mathcal{A}), \tilde{a}) = 5 - 4 = 1$.

However, if we now compute the matrix K, we will have $|\Psi(a)| - \operatorname{rank}(K) - \operatorname{rank} d_1 \neq 1$.

We have the following:

$$\mathcal{X}(a) = \{123, 356\}, \ I(a) = \{1, 2, 3, 5, 6\}$$

$$\mathcal{S}(a) = \{1234, 12356, 3456\}$$

$$\Upsilon(a) = \{(123, 1234), (4, 1234), (2, 12356), (3, 12356), (5, 12356), (6, 12356), (356, 3456), (4, 3456)\}$$

$$\Psi(a) = \{12, 13, 14, 24, 34, 15, 16, 25, 26, 35, 36, 45, 46, \xi\}$$

where ξ is introduced because for $X = \{12356\}$ we have $\mathcal{X}(a(X))$ is indefinite. Notice $\{245\} \in L(3, \mathcal{A}) \setminus \mathcal{S}(a)$ and $a(\{245\}) \neq 0$. The matrix K is

$$\begin{pmatrix} 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \end{pmatrix}$$

Hence, $\operatorname{rank} K = 6$. But $|\Psi(a)| - \operatorname{rank} K = 13 - 6 = 8 \neq \dim Z_2(a)$.

THEOREM 6.2.9. Let $\mathcal{A}$ be a central hyperplane arrangement. Let $a, b \in A_1(\mathcal{A})$ with

$$a = \sum_{i=1}^{n} \lambda_i a_i, \ b = \sum_{i=1}^{n} \sigma_i a_i.$$

Suppose $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \sigma_i = 0$. In addition, suppose the following criteria are satisfied:

1. We have $\mathcal{S}(a) = \mathcal{S}(b)$.
2. We have $\mathcal{X}(a) = \mathcal{X}(b)$.
3. For $X \in L(3) \setminus \mathcal{S}(a)$, we have $a(X) = 0$. For $X \in L(3) \setminus \mathcal{S}(b)$, we have $b(X) = 0$.
4. For $X \in \mathcal{S}(a) = \mathcal{S}(b)$, we have $\mathcal{X}(a(X))$ is affine (hence, $\mathcal{X}(b(X))$ is affine).

Then $\dim H^2(a) = \dim H^2(b)$.

PROOF. In the matrix description given in Definition 6.2.5, both a and b will give the same matrix. Hence, $Z_2(a) = Z_2(b)$. Moreover, since $\mathcal{X}(a) = \mathcal{X}(b)$ and is affine, we have $\operatorname{rank} d_1(a)$ is equal to the image of $\operatorname{rank} d_1(b)$. Therefore, $\dim H^2(a) = \dim H^2(b)$. □

Relaxing the conditions slightly, we obtain the equality of $Z_2(a)$ and $Z_2(b)$ in the following theorem.

THEOREM 6.2.10. Let $\mathcal{A}$ be a central hyperplane arrangement. Let $a, b \in A_1(\mathcal{A})$ with

$$a = \sum_{i=1}^{n} \lambda_i a_i, \; b = \sum_{i=1}^{n} \sigma_i a_i.$$

Suppose $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \sigma_i = 0$. In addition, suppose the following criteria are satisfied:

1. We have $\mathcal{S}(a) = \mathcal{S}(b)$.
2. We have $\mathcal{X}(a(X)) = \mathcal{X}(b(X))$ for all $X \in \mathcal{S}(a)$.
3. For $X \in L(3) \setminus \mathcal{S}(a)$, we have $a(X) = 0$. For $X \in L(3) \setminus \mathcal{S}(b)$, we have $b(X) = 0$.
4. For $X \in \mathcal{S}(a) = \mathcal{S}(b)$, we have $\mathcal{X}(a(X))$ is affine (hence, $\mathcal{X}(b(X))$ is affine).

Then $Z_2(a) = Z_2(b)$.

PROOF. In the matrix description given in Definition 6.2.5, both a and b will give the same matrix. Hence, $Z_2(a) = Z_2(b)$. □

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