MANIFOLDS WITH INDEFINITE METRICS WHOSE SKEW-SYMMETRIC CURVATURE OPERATOR HAS CONSTANT EIGENVALUES

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Relative to a non-degenerate metric of signature (p, q), an algebraic curvature tensor is said to be IP if the associated skew-symmetric curvature operator $R(\pi)$ has constant eigenvalues and if the kernel of $R(\pi)$ has constant dimension on the Grassmanian of non-degenerate oriented 2-planes. A pseudo-Riemannian manifold with a non-degenerate indefinite metric of signature (p, q) is said to be IP if the curvature tensor of the Levi-Civita connection is IP at every point; the eigenvalues are permitted to vary with the point. In the Riemannian setting $(p, q) = (0, m)$, the work of Gilkey, Leahy, and Sadofsky and the work of Ivanov and Petrova have classified the IP metrics and IP algebraic curvature tensors if the dimension is at least 4 and if the dimension is not 7. We use techniques from algebraic topology and from differential geometry to extend some of their results to the Lorentzian setting $(p, q) = (1, m-1)$ and to the setting of metrics of signature $(p, q) = (2, m-2)$.

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CHAPTER I

INTRODUCTION

In differential geometry, the Riemann curvature tensor carries crucial geometric information about the manifold. Because the full curvature tensor is quite complicated, one often uses the curvature tensor to define natural endomorphisms of the tangent bundle. The Jacobi, the Ricci, the Stanilov, the Szabó, and the skew-symmetric curvature operators are such examples; we refer to §1.5 for further details. Assume that one of these operators has constant eigenvalues on the appropriate domain; one wants to determine the possible underlying geometries. We shall focus on the skew-symmetric curvature operator in the pseudo-Riemannian setting in this thesis.

§1.1 Algebraic Curvature Tensors

Let M be a smooth connected manifold of dimension m. We assume there is an indefinite nondegenerate metric g_M on the tangent bundle TM. Fix a point P on the manifold M and let $V := T_PM$. The metric g_M induces a nondegenerate symmetric bilinear form on V. We can choose an orthonormal basis $\{v_i\}$ for V so that $g_M(v_i, v_j) = 0$ for $i \neq j$, so that $g_M(v_i, v_i) = -1$ for $i \leq p$, and so that $g_M(v_i, v_i) = 1$ for $i > p$. Let $q = m - p$ be the complementary index; the metric g_M is said to have *signature* (p, q); this is independent of the choices made. We shall suppose henceforth that $p \leq q$ since we can always replace g_M by $-g_M$ and reverse the roles of p and of q. M is called a pseudo-Riemannian manifold.

Let $O(p,q)$ be the group of all linear maps from V to V which preserve g_M and let $\mathfrak{so}(p,q)$ be the associated Lie algebra. We have:

$$O(p,q) = \{A \in \mathrm{End}(V) : g_M(Au, Av) = g_M(u,v)\ \forall u,v \in V\},\ \text{and}$$
$$\mathfrak{so}(p,q) = \{A \in \mathrm{End}(V) : g_M(Au, v) + g_M(u, Av) = 0\ \forall u,v \in V\}.$$

1.1.1 *The Riemann curvature tensor.* Let ∇ be the Levi-Civita connection on TM and let the *associated curvature operator* R be defined by the identity:

$$R(x,y) := \nabla_x\nabla_y - \nabla_y\nabla_x - \nabla_{[x,y]}.$$

Then $R : T_PM \otimes_{\mathbb{R}} T_PM \to \mathrm{End}(T_PM)$ has the curvature symmetries:

$$\begin{aligned} &R(x,y) = -R(y,x),\\ &g_M(R(x,y)z,w) = g_M(R(z,w)x,y),\ \text{and}\\ &R(x,y)z + R(y,z)x + R(z,x)y = 0. \end{aligned} \tag{1.1.1.a}$$

The equations displayed in (1.1.1.a) imply $g_M(R(x,y)z,w) = -g_M(R(x,y)w,z)$. Thus in particular, we have that $R(x,y) \in \mathfrak{so}(p,q)$.

1.1.2 *Algebraic curvature tensors.* We now go to a more general framework by studying a purely algebraic problem and working with algebraic curvature tensors - once the algebraic structure of these tensors has been investigated, we will then study the corresponding geometric questions. We shall say that $R \in \otimes^4(T_PM)$ is an *algebraic curvature tensor* if the equations displayed in (1.1.1.a) are satisfied. We note that the Riemann curvature tensor R of a manifold (M, g_M) defines an algebraic curvature tensor on T_PM for every P in M; conversely, given a metric g_P on T_PM and an algebraic curvature tensor R_P, there exists the germ of a metric $\tilde{g}_M$ on M extending g_P so that R_P is the curvature tensor of $\tilde{g}_M$ at P. Consequently we conclude that every algebraic curvature tensor is *geometrically realizable at P.*

Thus the study of algebraic curvature tensors is important in differential geometry. We refer to Gilkey [44] and Osserman [72] for expository accounts of this field and for a more detailed bibliography than can be presented here.

1.1.3 *Definition.* Let $\mathbb{R}^{p,q}$ be the vector space of real $(p+q)$-tuples of the form $x = (x_1, ..., x_p, x_{p+1}, ..., x_{p+q})$ with the nondegenerate symmetric bilinear form g

$$g(x,y) := -\sum_{i=1}^{p} x_i y_i + \sum_{i=p+1}^{p+q} x_i y_i \text{ and } |x|^2 := g(x,x).$$

By choosing a suitable orthonormal basis we may identify (V, g_M) with $(\mathbb{R}^{p,q}, g)$. Let π be a 2-plane in $\mathbb{R}^{p,q}$. We say π is *nondegenerate* if the restriction of g to π is nondegenerate. Let $\{x, y\}$ be a basis for π; π is nondegenerate if and only if $g(x,x)g(y,y) - g(x,y)^2 \neq 0$. We say that π is a 2-plane of type $(0,2)$, $(1,1)$, or $(2,0)$ if the restriction of g to π has this signature. Let $Gr^+_{(r,s)}(\mathbb{R}^{p,q})$ be the manifold of nondegenerate oriented 2-planes of type (r,s) in $\mathbb{R}^{p,q}$ where $r+s=2$. Let $Gr^+_2(\mathbb{R}^{p,q})$ be the manifold of nondegenerate oriented 2-planes in $\mathbb{R}^{p,q}$. Let $\dot\sqcup$ denote the disjoint union. We shall need the following decomposition later

$$Gr^+_2(\mathbb{R}^{p,q}) = Gr^+_{(0,2)}(\mathbb{R}^{p,q}) \,\dot\sqcup\, Gr^+_{(1,1)}(\mathbb{R}^{p,q}) \,\dot\sqcup\, Gr^+_{(2,0)}(\mathbb{R}^{p,q}).$$

§1.2 IP Algebraic Curvature Tensors and IP Metrics

1.2.1 *The skew-symmetric curvature operator.* Let $\{x, y\}$ be an oriented basis for $\pi \in Gr^+_2(\mathbb{R}^{p,q})$. We define the *skew-symmetric curvature operator* $R(\pi)$ by

$$R(\pi) := |g(x,x)g(y,y) - g(x,y)^2|^{-\frac{1}{2}} R(x,y);$$

$R(\pi)$ is independent of the particular basis chosen. This operator was introduced in the Riemannian context by Ivanova and Stanilov [61].

1.2.2 *Definition.* An algebraic curvature tensor R is said to be *IP* if (a) $R(\pi)$ has constant eigenvalues on all $\pi \in Gr_2^+(\mathbb{R}^{p,q})$ and (b) $\dim \operatorname{Ker} R(\pi)$ is constant on all $\pi \in Gr_2^+(\mathbb{R}^{p,q})$. A metric g_M on a manifold M is said to be *IP* if $R(\pi)$ is IP at every point $P \in M$; the eigenvalues are permitted to depend on $P \in M$.

1.2.3 *Remark:* In Definition 1.2.2, for $p > 0$, we do not have an orthogonal direct sum decomposition of $\mathbb{R}^{p,q}$ into $\operatorname{Ker} R(\pi)$ and $\operatorname{Range} R(\pi)$. This phenomenon is caused by the Jordan normal form associated with the zero eigenvalues of $R(\pi)$. So $R(\pi)$ having constant eigenvalues on all $\pi \in Gr_2^+(\mathbb{R}^{p,q})$ does not imply $\operatorname{rank} R(\pi)$ is constant on all $\pi \in Gr_2^+(\mathbb{R}^{p,q})$. This is a crucial distinction between the Riemannian setting and the pseudo-Riemannian setting. But by condition (b), we have $\dim \operatorname{Range} R(\pi) = p + q - \dim \operatorname{Ker} R(\pi)$ is constant on all $\pi \in Gr_2^+(\mathbb{R}^{p,q})$. Thus $\operatorname{rank} R := \operatorname{rank} R(\pi)$ is a well defined constant on all $\pi \in Gr_2^+(\mathbb{R}^{p,q})$. A precise replacement of condition (a) in Definition 1.2.2 is given in Theorem 2.1.1, this uses unpublished work of Gilkey.

IP algebraic curvature tensors and IP metrics were first studied by Ivanov and Petrova [59] in the context of four dimensional Riemannian geometry. Subsequently Gilkey [45], and Gilkey, Leahy and Sadofsky [48] classified the IP algebraic curvature tensors and IP metrics in the Riemannian setting except in dimension 7; some partial results regarding dimension 7 can be found in Gilkey and Semmelman [49].

1.2.4 *Definition.* Let C be a nonzero constant and let ϕ be a linear map of $\mathbb{R}^{p,q}$. (C, ϕ) is said to be an *admissible pair* if $\phi^2 = \varepsilon \cdot \mathrm{id}$ and if $g(\phi(u), \phi(v)) = \varepsilon \cdot g(u, v)$ where $\varepsilon = \pm 1$. If $\varepsilon = 1$, then ϕ is said to be an *unipotent* (*of order* 2) *isometry*; if $\varepsilon = -1$, then ϕ is said to be a *unipotent* (*of order* 4) *para-isometry*. If (C, ϕ) is an admissible pair, we define

$$R_{C,\phi}(x, y)z := C\{g(\phi(y), z)\phi(x) - g(\phi(x), z)\phi(y)\}.$$

We remark that $\varepsilon = -1$ is only possible when $p = q$. Later in §1.4, Theorem D asserts that $R_{C,\phi}$ is an IP algebraic curvature tensor.

1.2.5 *Note:* If ϕ is the identity map, then $R_C := R_{C,\phi}$ has constant sectional curvature C since the sectional curvature $\mathcal{K}(\pi, R_C)$ is given by

$$\mathcal{K}(\pi, R_C) := \frac{R_C(x,y,y,x)}{g(x,x)g(y,y) - g(x,y)^2} = C.$$

1.2.6 *Constant sectional curvature manifolds.* Let $r > 0$. Let

$$S_r(\mathbb{R}^{p,q}) := \{v \in \mathbb{R}^{p,q} : |v|^2 = r^2\}, \text{ and}$$

$$H_r(\mathbb{R}^{p+1,q-1}) := \{v \in \mathbb{R}^{p+1,q-1} : |v|^2 = -r^2\}$$

be the pseudo-Riemannian spheres and the hyperbolic spaces. These are complete pseudo-Riemannian manifolds of signature (p, q) which have constant sectional curvatures r^{-2} and $-r^{-2}$ respectively; we refer to Wolf [90] and O'Neill [71] for further details. The following theorem characterizes constant sectional curvature manifolds in the pseudo-Riemannian setting up to local isometry; we refer to [90] for the proof of the theorem.

1.2.7 Theorem. *Let M be a pseudo-Riemannian manifold of signature (p, q) with $p + q \geq 2$. Let $\mathcal{K}$ be a real number. The following conditions are equivalent.*

(1) *M has constant sectional curvature $\mathcal{K}$.*

(2) *If $x \in M$, then there exist local coordinates $\{x_i\}$ on a neighborhood of x in which the metric is given by*

$$ds^2 = \frac{\sum_i \varepsilon_i \cdot dx_i^2}{\{1 + \frac{\mathcal{K}}{4}\sum_i \varepsilon_i \cdot x_i^2\}^2}, \text{ where } \varepsilon_i = \pm 1.$$

(3) *If $x \in M$, then x has a neighborhood which is isometric to an open set on some $S_r(\mathbb{R}^{p,q})$ if $\mathcal{K} > 0$, $\mathbb{R}^{p,q}$ if $\mathcal{K} = 0$, $H_r(\mathbb{R}^{p,q})$ if $\mathcal{K} < 0$.*

We shall need another well known result about the pseudo-Riemannian spheres later in Chapter Five. We omit the proof of the theorem; again we refer to [90, 71] for details.

1.2.8 Theorem. *Let $p \geq 0$ and $q \geq 1$. We have that $S(\mathbb{R}^{p,q}) := S_1(\mathbb{R}^{p,q})$ is diffeomorphic to $\mathbb{R}^p \times S^{q-1}$.*

§1.3 The Classification of IP Manifolds in the Riemannian Setting

In this section, we review previous work of [45], [48], and [59] on the classification of IP algebraic curvature tensors and IP metrics in the Riemannian setting. The following result classifies IP algebraic curvature tensors in the Riemannian setting if $m = 5, 6$ or if $m \geq 8$:

1.3.1 Theorem (Gilkey [45], Gilkey, Leahy and Sadofsky [48]) *Let R be an IP algebraic curvature tensor. Assume that $(p, q) = (0, m)$. Let $m \geq 5$.*

(1) *If $m \neq 7$, then* $\operatorname{rank} R \leq 2$.

(2) *If* $\operatorname{rank} R = 2$*, then there exists an admissible pair (C, ϕ) with ϕ an unipotent (of order 2) isometry of $\mathbb{R}^{0,m}$ so that $R = R_{C,\phi}$.*

The four dimensional case is exceptional. We have the following classification in the Riemannian setting if $m = 4$:

1.3.2 Theorem (Ivanov and Petrova [59]) *Let R be an IP algebraic curvature tensor. Assume that $(p, q) = (0, 4)$.*

(1) *If* $\operatorname{rank} R = 2$*, then there exists an admissible pair (C, ϕ) with ϕ an unipotent (of order 2) isometry of $\mathbb{R}^{0,4}$ so that $R = R_{C,\phi}$.*

(2) *If* $\operatorname{rank} R = 4$*, then R is equivalent to a nonzero multiple of the "exotic"* rank 4 *tensor:*

$$R_{1212} = 2,\ R_{1313} = 2,\ R_{1414} = -1, R_{2424} = 2, R_{2323} = -1,$$
$$R_{3434} = 2,\ R_{1234} = -1,\ R_{1324} = 1,\ R_{1423} = 2.$$

Theorems 1.3.1 and 1.3.2 classify the IP algebraic curvature tensors if $m \geq 4$ and if $m \neq 7$. The corresponding classification of IP metrics is provided by the following result:

1.3.3 Theorem (Gilkey [45], Gilkey, Leahy and Sadofsky [48]; Ivanov and Petrova [59]) *Let M be an IP Riemannian manifold of dimension m. Assume $m \geq 4$. If*

$m = 7$, *we further assume* $\operatorname{rank} R = 2$. *Exactly one and only one of the following assertions is valid for* M*:*

(1) M *has constant sectional curvature.*

(2) M *is locally a warped product:*

$$ds_M^2 = dt^2 + f(t)ds_N^2$$

of an interval I *with a Riemannian manifold* N *of dimension* $m-1$ *which has constant sectional curvature* $\mathcal{K} \neq 0$. *Furthermore, the warping function* f *is given by*

$$f(t) = \mathcal{K}t^2 + At + B,$$

where A *and* B *are auxiliary constants so that* $4\mathcal{K}B - A^2 \neq 0$ *and that* $f(t) > 0$ *is a smooth function defined on* I.

We sketch the proofs of Theorem 1.3.1 and Theorem 1.3.3 in the following three steps for $m \neq 4$; the case $m = 4$ does not follow this pattern and is handled separately.

Step 1: (Algebraic topology) Let R be an IP algebraic curvature tensor. Let $W_0(R(\pi))$ and $W_1(R(\pi))$ be the kernel and the range of $R(\pi)$ for $\pi \in Gr_2^+(\mathbb{R}^{0,m})$. Since $R(\pi)$ has constant rank on $Gr_2^+(\mathbb{R}^{0,m})$, $W_i(R(\pi))$ define vector bundles over $Gr_2^+(\mathbb{R}^{0,m})$. Since $R(-\pi) = -R(\pi)$, $W_i(R(-\pi)) = W_i(R(\pi))$. Thus these bundles descend to define vector bundles V_i over the unoriented Grassmannian $Gr_2(\mathbb{R}^{0,m})$ and over the real projective space $\mathbb{RP}^{m-2} \subset Gr_2(\mathbb{R}^{0,m})$. The cohomology algebras of $Gr_2(\mathbb{R}^{0,m})$ and $\mathbb{RP}^{m-2}$ and the K-theory of $\mathbb{RP}^{m-2}$ play an important role in the analysis; this uses work of Adams [1] and Borel [18]. One studies the Stiefel-Whitney classes of the bundles V_i to show that $\dim V_1 = \dim W_1(R(\pi)) \leq 2$ if $m = 5$, $m = 6$, or $m \geq 9$; this restricts the eigenspace structure and shows that $R(\pi)$ has rank 2 if $R \neq 0$. The cases $m = 7, 8$ are exceptional, but some information on the eigenspace structure can be obtained.

Step 2: (Linear algebra) The map $R(\pi)$ takes values in $\mathfrak{so}(m)$. By Step 1, if R is nontrivial, we may assume that $R(\pi)$ has constant rank 2; Theorem 1.3.1 can then be established using fairly standard techniques; the fact that R has rank 2 is crucial to these arguments. We shall give a different proof from that given in [48] in chapter V, as the proof given in [48] does not extend to the pseudo-Riemannian setting.

Step 3: (Differential geometry) Let R be an IP metric. One uses Theorem 1.3.1 to construct an isometry ϕ of the tangent bundle with ϕ^2 the identity. Let $\mathcal{F}_\pm$ be the distributions defined by the ± 1 eigenspaces of ϕ; these are orthogonal. One uses the second Bianchi identity to show these distributions are integrable and to show that one of them has dimension 1. Theorem 1.3.3 then follows.

1.3.4 *Remark:* Theorem 1.3.1 and Theorem 1.3.3 show that not every IP algebraic curvature tensor is geometrically realizable by an IP metric; $R_{C,\phi}$ is geometrically realizable by an IP metric which does not have constant sectional curvature if and only if one of the eigenspaces of ϕ has dimension 1.

§1.4 Main Results of the Thesis

The results discussed in §1.3 are in the Riemannian setting where $(p,q) = (0,m)$; the fact that the metric in question is positive definite is used at several crucial points in the argument. We shall extend these results to the Lorentzian setting $(p,q) = (1, m-1)$ if $m \geq 10$. We shall also obtain some partial results in the higher signature setting.

1.4.1 *Definition.* Let $W_0(R(\pi)) := \operatorname{Ker} R(\pi)$ and let $W_1(R(\pi)) := \operatorname{Range} R(\pi)$. An algebraic curvature tensor R is said to be *spacelike* (or *timelike*) if $W_1(R(\pi))$ is spacelike (or timelike) for every spacelike 2-plane π. If R is a rank 2 IP algebraic curvature tensor, then R is said to be *mixed* if $W_1(R(\pi))$ is of type $(1,1)$ for every spacelike 2-plane π; R is said to be *null* if $W_1(R(\pi))$ is a degenerate 2-plane for every spacelike 2-plane π and $R(\pi)$ has only the zero eigenvalue. We note that in the Lorentzian setting a degenerate 2-plane is spanned by a spacelike vector and a null vector. We shall use this fact later in chapter IV.

We can now state the seven main results of the thesis.

Theorem A. *Let R be an IP algebraic curvature tensor on $\mathbb{R}^{p,q}$.*

(1) *If $p = 1$ and if $q \geq 9$, then* $\operatorname{rank} R \leq 2$.

(2) *If $p = 2$ and if $q \geq 11$, then* $\operatorname{rank} R \leq 4$. *Furthermore, if q and $2 + q$ are not powers of 2, then* $\operatorname{rank} R \leq 2$.

(3) *There exists a rank 4 IP algebraic curvature tensor if $(p, q) = (2, 2)$.*

Theorem A bounds the rank of an IP algebraic curvature tensor. In the rank 2 Lorentzian setting, we have a trichotomy:

Theorem B. *Let R be a rank 2 Lorentzian IP algebraic curvature tensor and let $m \geq 4$. Exactly one and only one of the following assertions is valid for R:*

(1) *For all $\pi \in Gr^+_{(0,2)}(\mathbb{R}^{1,m-1})$, we have that $W_1(R(\pi))$ is spacelike and that $R(\pi)$ has two nontrivial purely imaginary eigenvalues. Thus R is spacelike.*

(2) *For all $\pi \in Gr^+_{(0,2)}(\mathbb{R}^{1,m-1})$, we have that $W_1(R(\pi))$ is of type $(1,1)$ and that $R(\pi)$ has two nontrivial real eigenvalues. Thus R is mixed.*

(3) *For all $\pi \in Gr^+_{(0,2)}(\mathbb{R}^{1,m-1})$, we have that $W_1(R(\pi))$ is degenerate with a positive semi-definite metric and that $R(\pi)$ has only the zero eigenvalue. Thus R is null.*

Theorem B shows the trichotomy of rank 2 Lorentzian IP algebraic curvature tensors. The following theorem asserts that most rank 2 Lorentzian IP algebraic curvature tensors are spacelike.

Theorem C. *Assume that $m \geq 4$. Let R be a rank 2 Lorentzian IP algebraic curvature tensor.*

(1) *If R is mixed, then $m = 4$, 5, 8, or 9.*

(2) *If R is null, then $m = 5$ or 9.*

We have the following classification of rank 2 IP algebraic curvature tensors which are spacelike or timelike with certain dimensional restraint.

Theorem D.

(1) *If (C, ϕ) is an admissible pair, then $R_{C,\phi}$ is a rank 2 IP algebraic curvature tensor which is spacelike if $\varepsilon = 1$ and timelike if $\varepsilon = -1$.*

(2) *Let R be an IP algebraic curvature tensor on $\mathbb{R}^{p,q}$. Suppose that $q = 6$ or that $q \geq 9$. Suppose that R is spacelike or timelike and that R has rank 2. Then there exists an admissible pair (C, ϕ) so that $R = R_{C,\phi}$.*

Let ϕ be an unipotent (of order 2) isometry of $\mathbb{R}^{p,q}$. Let $r_\pm(\phi)$ be the associated dimensions of the ± 1 eigenspaces of ϕ. The following theorem shows that not every IP algebraic curvature tensor is geometrically realizable by an IP metric:

Theorem E. *Assume $m \geq 4$. If (M, g_M) is an IP pseudo-Riemannian manifold and if the curvature tensor R at a point $P \in M$ is given by $R_{C,\phi}$ for some admissible (C, ϕ), then $r_+(\phi) \leq 1$ or $r_-(\phi) \leq 1$.*

We now generalize the construction of IP metrics given in Theorem 1.3.3.

Theorem F. *Let $\varepsilon = \pm 1$. Let $I \subset \mathbb{R}$ be a connected open interval. Let N be the germ of a pseudo-Riemannian manifold of constant sectional curvature $\mathcal{K} \neq 0$. Let A and B be auxiliary constants so that $4\mathcal{K}B - \varepsilon A^2 \neq 0$ and that $f_\varepsilon(t) := \varepsilon \mathcal{K} t^2 + At + B$ is a smooth nonzero function on I. Let $M := I \times N$ and let $g_M := \varepsilon dt^2 + f_\varepsilon(t) g_N$. Then g_M is a rank 2 IP metric on M.*

As a consequence of Theorems D, E, and F, we have the following classification of IP algebraic curvature tensors and rank 2 IP metrics in the Lorentzian setting for $m \geq 10$.

Theorem G. *Assume that $m \geq 10$.*

(1) *Let R be an IP algebraic curvature tensor on $\mathbb{R}^{1,m-1}$. R is nontrivial if and only if there exists an admissible pair (C, ϕ) with ϕ an unipotent (of order 2) isometry of $\mathbb{R}^{1,m-1}$ so that $R = R_{C,\phi}$.*

(2) *If g_M is a rank 2 Lorentzian IP metric, then exactly one and only one of the following assertions is valid for g_M:*

(2a) *g_M is a metric of constant sectional curvature $C \neq 0$.*

(2b) *g_M is locally isometric to a warped product metric of the form given in Theorem F.*

1.4.2 *Outline of the thesis.* In chapter II, we prepare the necessary background material from analysis and algebraic topology for our later studies. In §2.1, we use unpublished work of Gilkey to prove Theorem 2.1.1. We also establish a technical lemma relating the two Lie algebras $\mathfrak{so}(p,q)$ and $\mathfrak{so}(p+q)$. In §2.2, we study the topology of the Grassmannians $Gr^{+}_{(r,s)}(\mathbb{R}^{p,q})$ and $Gr_{(r,s)}(\mathbb{R}^{p,q})$. In §2.3, we define the Stiefel-Whitney classes of a real vector bundle and introduce some results from of K-theory. We recall the calculation of $\widetilde{KO}(\mathbb{RP}^n)$ due to Adams [1]. This will play a crucial role in bounding the rank of IP algebraic curvature tensors. In §2.4, we recall the calculation of $H^*(Gr_2(\mathbb{R}^n); \mathbb{Z}_2)$ due to Borel [18]. We also introduce the Steenrod squares. In §2.5, we introduce the splitting principle and apply this principle to prove some technical lemmas which are used to determine the possible forms of the Stiefel-Whitney classes of certain vector bundles. In §2.6, we establish two important lemmas. The first lemma determines for what values of q, there exists a nonsingular bilinear map from $\mathbb{R}^q \times \mathbb{R}^q$ to $\mathbb{R}^{q+1}$. The second lemma is a continuity result which is needed later in chapter V.

In chapter III, we prove Theorem A by bounding the rank of IP algebraic curvature tensors in some cases. In §3.1-3.2, to study the rank, we introduce certain vector bundles over the Grassmannians and the real projective spaces so that they encode much of the information about $R(\pi)$. Our approach is analogous to Step 1 in §1.3. The works of Adams [1], Borel [18], Gilkey, Leahy and Sadofsky[48], and Stong [84] play important roles in our discussion. This will prove Theorem A (1) and the first part of Theorem A (2). In §3.3-3.4, we complete the proof of Theorem A (2). In §3.5, we investigate some lower dimensional cases in the Lorentzian setting. In §3.6, we prove Theorem A (3).

In chapter IV, we prove Theorems B and C. In §4.1, we establish the trichotomy of rank 2 Lorentzian IP algebraic curvature tensors; this proves Theorem B. In §4.2, we assume R is mixed or null and use the first lemma established in §2.6 to show that $q = 3$, $q = 4$, $q = 7$, or $q = 8$. Once again algebraic topology plays a crucial role in our analysis. This proves Theorem C (1). In §4.3, we complete the proof of Theorem C by ruling out the exceptional cases $q = 3$ and $q = 7$ (i.e. $m = 4$ or $m = 8$) if R is null.

In chapter V, we prove Theorems D and G (1). This chapter serves an analogous role in Step 2 of §1.3. In §5.1, we begin with some algebraic preliminaries. In §5.2, we prove the "common axis" lemma and then construct the admissible pair (C, ϕ) so $R = R_{C,\phi}$. In §5.3, we prove Theorems D and G (1).

In chapter VI, we prove Theorems E and F and we complete the proof of Theorem G. Our approach is analogous to Step 3 in §1.3. In §6.1, we prove Theorem E. We follow the argument given by Gilkey, Leahy and Sadofsky [48]; the second Bianchi identity enters at a crucial stage of the argument. In §6.2, we generalize the warped product construction of Gilkey, Leahy and Sadofsky, and of Ivanov and Petrova to higher signatures to prove Theorem F. In §6.3, we first show any C-ϕ type metric is a warped product of an interval with a metric of constant sectional curvature. We subsequently complete the proof of Theorem G in the seven steps:

Step 1: Theorem A (1) implies the associated Lorentzian algebraic curvature tensor R has rank at most 2.

Step 2: Theorem B implies either R is spacelike or R is mixed or R is null.

Step 3: Theorem C shows R is not mixed or null. Thus R is spacelike.

Step 4: Theorem D shows R is of C-ϕ type.

Step 5: By replacing ϕ by $-\phi$ if necessary, we may suppose $r_+ \leq r_-$. By Theorem E, either $r_+ = 0$ or $r_+ = 1$. If $r_+ = 0$, then M^m has constant sectional curvature C. We therefore assume that $r_+ = 1$. Thus g_M is a metric of C-ϕ type.

Step 6: By the technical lemma at the beginning of §6.3, any C-ϕ type metric is a warped product of an interval with a metric of constant sectional curvature.

Step 7: Theorem F shows if (M, g_M) is a warped product metric of an interval with a metric of constant sectional curvature, then (M, g_M) has the desired form. This completes the classification. In §6.4, we discuss the orthogonal equivalence of the curvature tensors $R_{C,\phi}$.

1.4.3 *Future research.* We have classified the IP algebraic curvature tensors and IP metrics in the Lorentzian setting if $m \geq 10$. We plan to use the second Bianchi identity to study the appropriate integrability results and prove every rank 2 IP metric is locally isometric to one of the metrics constructed in Theorem F in the higher signature setting. The possible existence of rank 2 mixed or null Lorentzian IP algebraic curvature tensors still needs to be explored in certain exceptional dimensions. We also will pursue the classification of IP algebraic curvature tensors in higher signatures. We will study whether or not there exist "exotic" IP algebraic curvature tensor of rank 4 when $(p, q) = (1, 3)$. We will also study whether or not there exist "exotic" IP algebraic curvature tensor of rank 4 arising from the unipotent (of order 4) para-isometry when $(p, q) = (2, 2)$.

§1.5 Other Operators

We conclude chapter I by giving a brief summary of some related results. We follow the discussion given in [44] on these topics.

1.5.1 *The Jacobi operator.* Let R be the curvature of a connected Riemannian manifold M of dimension m. If x is a unit tangent vector, let $J_R(x) : y \to R(y,x)x$ be the Jacobi operator. The Jacobi operator is an essential ingredient in the study of Jacobi vector fields, geodesic sprays and conjugate points. If M is a local 2-point homogeneous space, then the local isometries of M act transitively on the bundle of unit tangent vectors so the Jacobi operator has constant eigenvalues. Osserman conjectured [72] that the converse might hold. Chi [25] showed this to be the case if m is odd, if $m \equiv 2 \bmod 4$, or if $m = 4$; the case $m = 4k + 4$ for $k \geq 1$ remains open in this conjecture. Recently Rakić [76] has established a duality result showing:

1.5.2 Theorem (Rakić) *Let R be an Osserman algebraic curvature tensor and let x and y be unit vectors. If $J_R(x)y = \lambda y$, then $J_R(y)x = \lambda x$.*

There is an analogous duality for the skew-symmetric curvature operator as we shall see in Remark 5.3.3 in chapter V.

It is also known that a Lorentzian Osserman algebraic curvature tensor has constant sectional curvature, we refer to Blažić, Bokan, Gilkey and Rakić [9], and Garcia-Rio, Vázquez-Abal and Vázquez-Lorenzo [38]. The situation in the higher signature setting is much more complicated. For example, there exist Osserman pseudo-Riemannian metrics which are not homogeneous; see the survey article [10] for further details.

1.5.3 *The Stanilov operator.* Ivanova and Stanilov [61] defined a higher order generalization of the Jacobi operator. Let $Gr_p(\mathbb{R}^m)$ be the Grassmannian of unoriented p-planes in $\mathbb{R}^m$. We define:

$$J_{R;p}(\pi) = \int_{x\in\pi:|x|=1} J_R(x)\, dx.$$

Let $\{x_i\}_{i=1}^{p}$ be an orthonormal basis for π. Then, modulo a suitable normalizing constant which plays no role, we have that

$$J_{R;p}(\pi) := \sum_i J_R(x_i).$$

This sum is independent of the orthonormal basis chosen. An algebraic curvature tensor R is said to be *p-Osserman* if the eigenvalues of $J_{R;p}$ are constant on $Gr_p(\mathbb{R}^m)$; similarly, a Riemannian manifold (M^m, g) is said to be *p-Osserman* if the eigenvalues of $J_{R;p}$ are constant on $Gr_p(TM^m)$. If R is p-Osserman, then R is Einstein and $(m-p)$-Osserman; see [47] for details. One has a complete classification result [43].

1.5.4 Theorem (Gilkey [43]) *Let* $2 \le p \le m-2$.

(1) *Let R be a p-Osserman algebraic curvature tensor. If m is odd, then R has constant sectional curvature. If m is even, then either R has constant sectional curvature or there exists an almost complex structure c on $\mathbb{R}^m$ so that $R = \lambda_c R_c$ with $R_c(x,y)z := g(y,cz)cx - g(x,cz)cy - 2g(x,cy)cz$.*

(2) *Let (M^m, g) be a p-Osserman Riemannian manifold. Then (M^m, g) has constant sectional curvature.*

1.5.5 *The Szabó operator.* If x is a unit tangent vector, then the Szabó Operator is defined by $S_R(x) : y \to (\nabla_x R)(y,x)x$. This operator is self-adjoint. Szabó [85] proved the following result:

1.5.6 Theorem (Szabó [85])

(1) *If S_R has constant eigenvalues on S^{m-1}, then $\nabla R = 0$.*

(2) *Let (M^m, g) be a Riemannian metric so that S_R has constant eigenvalues on $S(TM^m)$. Then (M^m, g) is a local symmetric space.*

CHAPTER II

SOME ANALYTICAL AND TOPOLOGICAL BACKGROUND MATERIAL

In chapter II, we present some basic background material and prove some basic lemmas we shall need later. Here is a brief outline of chapter II. In §2.1, we follow the argument due to Gilkey to show that for R IP, the eigenvalues of $R(\pi)$ are independent of the plane type of π. This permits us to change the domain of R. In Lemma 2.1.2, we show there exists a rank preserving linear isomorphism between the two Lie algebras $\mathfrak{so}(p,q)$ and $\mathfrak{so}(p+q)$. So from the rank point of view, this permits us to change the range of R. We shall need these facts in chapter III when we rephrase the problem in the language of vector bundles. In §2.2, we study the topology of the Grassmannians $Gr^+_{(r,s)}(\mathbb{R}^{p,q})$ and $Gr_{(r,s)}(\mathbb{R}^{p,q})$. In §2.3, we define the Stiefel-Whitney classes of a real vector bundle. We recall the calculation of the real K-theory groups of $\mathbb{RP}^n$ due to Adams [1]. In §2.4, we recall the work of Borel [18] on $H^*(Gr_2(\mathbb{R}^n);\mathbb{Z}_2)$. We also introduce the Steenrod squares. In §2.5, we introduce a very important computational tool for characteristic classes in Theorem 2.5.5 (The splitting principle). Lemmas 2.5.7, 2.5.8, and 2.5.9 are applications of the splitting principle. In §2.6, we establish two important technical lemmas which are needed in chapter IV and chapter V.

§2.1 The Eigenvalues of IP Algebraic Curvature Tensors

2.1.1 Theorem. *Let R be an algebraic curvature tensor. The following conditions are equivalent:*

(1) *R has constant eigenvalues on all $\pi \in Gr^+_{(0,2)}(\mathbb{R}^{p,q})$.* (Assume $q \geq 2$).

(2) *R has constant eigenvalues on all $\pi \in Gr^+_{(1,1)}(\mathbb{R}^{p,q})$.* (Assume $p \geq 1$ and $q \geq 1$).

(3) *R has constant eigenvalues on all $\pi \in Gr^+_{(2,0)}(\mathbb{R}^{p,q})$.* (Assume $p \geq 2$).

Furthermore R has constant eigenvalues on the set of nondegenerate 2-planes.

Proof. The following argument is due to Gilkey. Let

$$F_2(\mathbb{R}^{p,q}) := \{(u,v) \in \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} : g(u,u)g(v,v) - g(u,v)^2 \neq 0\}$$

be the set of frames for the nondegenerate 2-planes in $\mathbb{R}^{p,q}$. We can decompose:

$$Gr^+_2(\mathbb{R}^{p,q}) = Gr^+_{(0,2)}(\mathbb{R}^{p,q}) \,\dot{\sqcup}\, Gr^+_{(1,1)}(\mathbb{R}^{p,q}) \,\dot{\sqcup}\, Gr^+_{(2,0)}(\mathbb{R}^{p,q}), \text{ and}$$

$$F_2(\mathbb{R}^{p,q}) = F_{(0,2)}(\mathbb{R}^{p,q}) \,\dot{\sqcup}\, F_{(1,1)}(\mathbb{R}^{p,q}) \,\dot{\sqcup}\, F_{(2,0)}(\mathbb{R}^{p,q}).$$

The frames in $F_{(r,s)}(\mathbb{R}^{p,q})$ span oriented 2-planes of type (r,s) in $Gr^+_{(r,s)}(\mathbb{R}^{p,q})$. If $(u,v) \in F_2(\mathbb{R}^{p,q})$, let $\pi(u,v)$ be the oriented 2-plane spanned by u and v. The map $\pi : F_2(\mathbb{R}^{p,q}) \to Gr^+_2(\mathbb{R}^{p,q})$ is a principal bundle with structure group $\mathrm{GL}^+_2(\mathbb{R})$. Let

$$\alpha(u,v) := R^2(\pi(u,v)) = \frac{R^2(u,v)}{g(u,u)g(v,v) - g(u,v)^2}.$$

The eigenvalues of $R(\pi)$ are constant on $Gr^+_2(\mathbb{R}^{p,q})$ if and only if the eigenvalues of $\alpha(u,v)$ are constant on $F_2(\mathbb{R}^{p,q})$. Let $C_i(u,v) := \mathrm{Tr}\{\alpha(u,v)^i\}$; the eigenvalues of $\alpha(u,v)$ are constant on $F_2(\mathbb{R}^{p,q})$ if and only if the functions $C_i(u,v)$ are constant on $F_2(\mathbb{R}^{p,q})$.

We complexify and extend the tensors R and g to the tensors R_c and g_c which are complex and multilinear. The role of (p,q) of course disappears once we complexify. We use R_c to define an associated curvature operator

$$R_c(z_1, z_2) : \otimes^2(\mathbb{C}^{p+q}) \to M_{p+q}(\mathbb{C})$$

which satisfies the defining identity:

$$g_c(R_c(z_1, z_2)z_3, z_4) = R_c(z_1, z_2, z_3, z_4).$$

We complexify to define

$$F_2^{\mathbb{C}} := \{(z_1, z_2) \in \mathbb{C}^{p+q} \times \mathbb{C}^{p+q} : g_c(z_1, z_1)g_c(z_2, z_2) - g_c(z_1, z_2)^2 \neq 0\}, \text{ and}$$
$$\alpha_c(z_1, z_2) := \frac{R_c^2(z_1, z_2)}{g_c(z_1, z_1)g_c(z_2, z_2) - g_c(z_1, z_2)^2} \text{ on } F_2^{\mathbb{C}}.$$

We note that $F_2^{\mathbb{C}}$ is a nonempty connected open dense subset of the complex vector space $\mathbb{C}^{p+q} \times \mathbb{C}^{p+q} = \mathbb{C}^{2(p+q)}$; we refer to Gunning and Rossi [55] for details. If $\pi \subset \mathbb{C}^{p+q}$ is closed under addition and under scalar multiplication by $\mathbb{R}$ and if π has $\mathbb{R}$ dimension 2, then π is said to be a *real 2-plane in* $\mathbb{C}^{p+q}$.

Let $\{u_i, v_i\}$ for $i = 1, 2$ be two $\mathbb{R}$ bases for a real 2-plane π. We must show that if $(u_1, v_1) \in F_2^{\mathbb{C}}$, then $(u_2, v_2) \in F_2^{\mathbb{C}}$ and $\alpha_c(u_1, v_1) = \alpha_c(u_2, v_2)$. We argue as follows. Choose constants $a, b, c, d \in \mathbb{R}$ with $ad - bc \neq 0$ so $u_2 = au_1 + bv_1$ and so $v_2 = cu_1 + dv_1$. Since $(u_1, v_1) \in F_2^{\mathbb{C}}$, we have $g_c(u_1, u_1)g_c(v_1, v_1) - g_c(u_1, v_1)^2 \neq 0$. We compute:

$$\begin{aligned}
&g_c(u_2, u_2)g_c(v_2, v_2) - g_c(u_2, v_2)^2 \\
&= g_c(au_1 + bv_1, au_1 + bv_1)g_c(cu_1 + dv_1, cu_1 + dv_1) - g_c(au_1 + bv_1, cu_1 + dv_1)^2 \\
&= (ad - bc)^2\{g_c(u_1, u_1)g_c(v_1, v_1) - g_c(u_1, v_1)^2\} \neq 0.
\end{aligned}$$

Thus we have $(u_2, v_2) \in F_2^{\mathbb{C}}$. Similarly, we compute $\alpha_c(u_1, v_1) = \alpha_c(u_2, v_2)$.

We say that a real 2-plane π in $\mathbb{C}^{p+q}$ is *nondegenerate* if there exists a $\mathbb{R}$ basis $(u, v) \in F_2^{\mathbb{C}}$ for π; this is independent of the basis chosen as noted above. Note that not every real 2-plane is nondegenerate. We let $\alpha_c(\pi) := \alpha_c(u, v)$ be this common value. We extend C_i to the complexification by defining $C_i(u, v) := \operatorname{Tr}\{\alpha_c(u, v)^i\}$; C_i are holomorphic functions on $F_2^{\mathbb{C}}$. Note that

$$F_2^{\mathbb{C}} \cap \mathbb{R}^{p,q} = F_2(\mathbb{R}^{p,q}) = F_{(0,2)}(\mathbb{R}^{p,q}) \,\dot\sqcup\, F_{(1,1)}(\mathbb{R}^{p,q}) \,\dot\sqcup\, F_{(2,0)}(\mathbb{R}^{p,q}).$$

If $F_{(r,s)}(\mathbb{R}^{p,q})$ is nonempty and if C_i are constant on $F_{(r,s)}(\mathbb{R}^{p,q})$, then the holomorphic functions C_i are constant on the nonempty open subset $F_{(r,s)}(\mathbb{R}^{p,q})$ of $F_2^{\mathbb{C}}$. Since the nonempty subset $F_2^{\mathbb{C}}$ of $\mathbb{C}^{2(p+q)}$ is open and connected, the identity Theorem asserts that C_i are constant on the whole domain $F_2^{\mathbb{C}}$ and hence on $F_2(\mathbb{R}^{p,q})$. $\square$

For simplicity, we shall henceforth use $\mathfrak{so}(m)$ for $\mathfrak{so}(0,m)$ and use $\mathbb{R}^m$ for $\mathbb{R}^{0,m}$. We shall need the following technical result.

2.1.2 Lemma. *There exists a rank preserving linear isomorphism T from $\mathfrak{so}(p,q)$ to $\mathfrak{so}(p+q)$.*

Proof. As noted in §1.1.3, we can choose coordinates $x = (x_1, ..., x_p, x_{p+1}, ..., x_{p+q})$ on $\mathbb{R}^{p,q}$ so that

$$g(x,x) = -x_1^2 - ... - x_p^2 + x_{p+1}^2 + ... + x_{p+q}^2.$$

Let g_e be the standard Euclidean metric. Let

$$T(x_1, ..., x_p, x_{p+1}, ..., x_{p+q}) := (-x_1, ..., -x_p, x_{p+1}, ..., x_{p+q}).$$

T is self-adjoint with respect to the inner product g_e, T^2 is the identity. Furthermore we have $g(u,v) = g_e(u,Tv) = g_e(Tu,v)$.

The following assertions are equivalent:

(1) We have $A \in \mathfrak{so}(p,q)$.

(2) We have $g(Au,v) + g(u,Av) = 0$ for all u, v.

(3) We have $g_e(Au,Tv) + g_e(u,TAv) = 0$ for all u, v.

(4) We have $g_e(TAu,v) + g_e(u,TAv) = 0$ for all u, v.

(5) We have $TA \in \mathfrak{so}(p+q)$.

This chain of equivalences shows that the map $T : A \mapsto TA$ is a linear isomorphism between $\mathfrak{so}(p,q)$ and $\mathfrak{so}(p+q)$; since T is invertible, $\operatorname{rank} A = \operatorname{rank} TA$. $\square$

2.1.3 *Definition.* We say that a continuous map $R : Gr_2^+(\mathbb{R}^{p,q}) \to \mathfrak{so}(\mu,\nu)$ is *admissible* if $R(-\pi) = -R(\pi)$ and if $\operatorname{rank} R(\pi)$ is constant on $Gr_2^+(\mathbb{R}^{p,q})$. Similarly a continuous map $R : S^n \to \mathfrak{so}(\mu,\nu)$ is *admissible* if $R(-v) = -R(v)$ and if $\operatorname{rank} R(v)$ is constant on S^n. We let $\operatorname{rank} R(\pi) = r$ be this constant in this setting. Note that if R is an IP algebraic curvature tensor, then the map $\pi \to R(\pi)$ is an admissible map R from $Gr_2^+(\mathbb{R}^{p,q})$ to $\mathfrak{so}(p,q)$.

The following lemma is an immediate consequence of Lemma 2.1.2; it permits us to pass from the Lie algebra $\mathfrak{so}(p,q)$ to the Lie algebra $\mathfrak{so}(p+q)$.

2.1.4 Lemma.

(1) *The following assertions are equivalent:*

  (1a) *There is an admissible map from $Gr_2^+(\mathbb{R}^{p,q})$ to $\mathfrak{so}(p,q)$ of rank r.*

  (1b) *There is an admissible map from $Gr_2^+(\mathbb{R}^{p,q})$ to $\mathfrak{so}(p+q)$ of rank r.*

(2) *The following assertions are equivalent*

  (2a) *There is an admissible map from S^n to $\mathfrak{so}(p,q)$ of rank r.*

  (2b) *There is an admissible map from S^n to $\mathfrak{so}(p+q)$ of rank r.*

§2.2 The Topology of the Grassmannians

The oriented Grassmannian $Gr_2^+(\mathbb{R}^{p,q})$ and the corresponding unoriented Grassmannian $Gr_2(\mathbb{R}^{p,q})$ will play important roles in our study. We decompose

$$Gr_2^+(\mathbb{R}^{p,q}) = Gr_{(0,2)}^+(\mathbb{R}^{p,q}) \,\dot\sqcup\, Gr_{(1,1)}^+(\mathbb{R}^{p,q}) \,\dot\sqcup\, Gr_{(2,0)}^+(\mathbb{R}^{p,q}), \text{ and}$$
$$Gr_2(\mathbb{R}^{p,q}) = Gr_{(0,2)}(\mathbb{R}^{p,q}) \,\dot\sqcup\, Gr_{(1,1)}(\mathbb{R}^{p,q}) \,\dot\sqcup\, Gr_{(2,0)}(\mathbb{R}^{p,q}).$$

These spaces are noncompact if $p \neq 0$. We show in this section that $Gr_{(0,2)}(\mathbb{R}^{p,q})$ and $Gr_{(1,1)}(\mathbb{R}^{p,q})$ strongly deformation retract to compact submanifolds.

If $0 \neq x \in \mathbb{R}^p$ and if $0 \neq y \in \mathbb{R}^q$, we let

$$\pi(x,y) := \operatorname{Span}\{(x,0),(0,y)\} \in Gr_{(1,1)}(\mathbb{R}^{p,q}).$$

The map $(x, y) \to \pi(x, y)$ extends to define an inclusion

$$i : \mathbb{RP}^{p-1} \times \mathbb{RP}^{q-1} \to Gr_{(1,1)}(\mathbb{R}^{p,q}). \tag{2.2.0.a}$$

For $\pi \in Gr_{(1,1)}(\mathbb{R}^{p,q})$, let $S^{\pm}(\pi) := \{v \in \pi : g(v,v) = \pm 1\}$ denote the set of spacelike and timelike unit vectors. If v is a nonzero vector, we shall let $\langle v \rangle$ denote the associated point in projective space. Our first goal is to construct a retract $r : Gr_{(1,1)}(\mathbb{R}^{p,q}) \to \mathbb{RP}^{p-1} \times \mathbb{RP}^{q-1}$.

2.2.1 Lemma. *Let* $\pi \in Gr_{(1,1)}(\mathbb{R}^{p,q})$.

(1) *The function* $g_e(u,u)$ *on* $S^+(\pi)$ *is minimized by exactly two vectors* $\pm P^+$.

(2) *The function* $g_e(u,u)$ *on* $S^-(\pi)$ *is minimized by exactly two vectors* $\pm P^-$.

(3) *Let* $P^{\pm} = (x^{\pm}, y^{\pm})$. *The maps* $\psi^- : \pi \to \langle x^- \rangle$ *and* $\psi^+ : \pi \to \langle y^+ \rangle$ *are smooth maps from* $Gr_{(1,1)}(\mathbb{R}^{p,q})$ *to* $\mathbb{RP}^{p-1}$ *and* $\mathbb{RP}^{q-1}$.

(4) *The map* $r = \psi^- \times \psi^+$ *is a retract to the inclusion* i *defined in equation* (2.2.0.a).

Proof. Let g^π and g_e^π denote the restrictions of the indefinite metric g and the Euclidean metric g_e to π; these are nondegenerate quadratic forms and g_e^π is positive definite. We can diagonalize g^π with respect to g_e^π. We define an endomorphism A^π by $g^\pi(u,v) = g_e^\pi(A^\pi u, v)$. Then A^π is symmetric with respect to g_e^π; since g^π is indefinite, A^π has eigenvalues $\lambda_\pm^\pi$ which have opposite signs. We can therefore diagonalize A^π; this permits us to choose orthogonal unit vectors $v_\pm^\pi$ with respect to the metric g_e^π so that if $v = a_+ v_+^\pi + a_- v_-^\pi$, we have

$$g_e(v,v) = a_+^2 + a_-^2 \text{ and } g(v,v) = \lambda_+^\pi a_+^2 - \lambda_-^\pi a_-^2.$$

Thus $v \in S^{\pm}(\pi)$ if $(\sqrt{\lambda_+^\pi}a_+)^2 - (\sqrt{\lambda_-^\pi}a_-)^2 = \pm 1$; this identifies $S^{\pm}(\pi)$ with two hyperbolas in $\mathbb{R}^2$ and the points closest to the origin with respect to the Euclidean metric are $\pm P^{\pm} := \pm v_\pm^\pi / \sqrt{\lambda_\pm^\pi}$; assertions (1) and (2) now follow.

If $\pi \in Gr_{(1,1)}(\mathbb{R}^{p,q})$, let ρ_π be orthogonal projection with respect to the metric g_e on π and let $\mathcal{O}(\pi)$ be a sufficiently small neighborhood of π in $Gr_{(1,1)}(\mathbb{R}^{p,q})$ so that ρ_π is an isomorphism from τ to π for $\tau \in \mathcal{O}(\pi)$. We use the isomorphism ρ_π to pull back the metrics g^τ and g_e^τ to π and to regard them as a smoothly varying family of metrics on π parameterized by $\tau \in \mathcal{O}(\pi)$. Since the eigenvalues $\lambda_\pm^\tau$ have opposite signs, we can choose the diagonalizations and corresponding eigenvectors to be smooth functions of τ; pulling back these eigenvectors to τ using ρ_π^{-1} then shows that the vectors $P^\pm(\tau)$ can be chosen to vary smoothly with τ at least locally. The maps $\pi \to P^- \to x^-$ and $\pi \to P^+ \to y^+$ are smooth. This construction is well defined locally; globally, of course, there is no way to distinguish P from $-P$, i.e. x^- and y^+ can not be defined globally. However, this indeterminacy vanishes once we pass to the associated projective space, assertion (3) follows. The final assertion is an immediate consequence of the definitions we have given in (2.2.0.a). $\square$

2.2.2 *Remark:* We can also think of this process geometrically. Let $\mathcal{O}$ be a small open set in $Gr_{(1,1)}(\mathbb{R}^{p,q})$. Choose a frame $\{v_1, v_2\}$ for $\pi \in \mathcal{O}$ which is orthonormal to the reference metric g_e. This choice of frame permits us to view the metric g as a varying family of indefinite quadratic forms on $\mathbb{R}^2$ which varies smoothly and which is parameterized by $\mathcal{O}$. The equations $g^\pi(v,v) = \pm 1$ define smoothly varying families of hyperbolas and the unique points closed to the origin are the points $\pm P^\pm$ in question.

Let $Gr_2^+(\mathbb{R}^\nu) := Gr_2^+(\mathbb{R}^{0,\nu})$ and let $Gr_2(\mathbb{R}^\nu) := Gr_2(\mathbb{R}^{0,\nu})$; these are smooth closed manifolds. We use the canonical inclusions $\mathbb{R}^q \hookrightarrow \mathbb{R}^{p,q} = \mathbb{R}^p \oplus \mathbb{R}^q$ and $\mathbb{R}^p \hookrightarrow \mathbb{R}^{p,q} = \mathbb{R}^p \oplus \mathbb{R}^q$ to define canonical embeddings

$$Gr_2^+(\mathbb{R}^q) \hookrightarrow Gr_{(0,2)}^+(\mathbb{R}^{p,q}) \text{ and } Gr_2^+(\mathbb{R}^p) \hookrightarrow Gr_{(2,0)}^+(\mathbb{R}^{p,q}).$$

Let $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ act on $S^{p-1} \times S^{q-1} \subset \mathbb{R}^p \times \mathbb{R}^q \cong \mathbb{R}^{p,q}$. Let

$$\mathcal{S}(p,q) := (S^{p-1} \times S^{q-1})/\mathbb{Z}_2$$

be the quotient by the diagonal action of $\mathbb{Z}_2$; note that

$$\mathbb{RP}^{p-1} \times \mathbb{RP}^{q-1} = (S^{p-1} \times S^{q-1})/(\mathbb{Z}_2 \oplus \mathbb{Z}_2).$$

Let (u,v) be an element in $S^{p-1} \times S^{q-1}$. Let $\langle u,v\rangle$ denote the associated point in $\mathcal{S}(p,q)$. We can also embed $\mathcal{S}(p,q) \hookrightarrow Gr^+_{(1,1)}(\mathbb{R}^{p,q})$ by $\langle u,v\rangle \hookrightarrow \operatorname{Span}\{u,v\}$.

2.2.3 Theorem.

(1) *We have $Gr^+_2(\mathbb{R}^q)$ is a strong deformation retract of $Gr^+_{(0,2)}(\mathbb{R}^{p,q})$.*

(2) *We have $Gr_2(\mathbb{R}^q)$ is a strong deformation retract of $Gr_{(0,2)}(\mathbb{R}^{p,q})$.*

(3) *We have $\mathcal{S}(p,q)$ is a strong deformation retract of $Gr^+_{(1,1)}(\mathbb{R}^{p,q})$.*

(4) *We have $\mathbb{RP}^{p-1} \times \mathbb{RP}^{q-1}$ is a strong deformation retract of $Gr_{(1,1)}(\mathbb{R}^{p,q})$.*

(5) *We have $Gr^+_2(\mathbb{R}^p)$ is a strong deformation retract of $Gr^+_{(2,0)}(\mathbb{R}^{p,q})$.*

(6) *We have $Gr_2(\mathbb{R}^p)$ is a strong deformation retract of $Gr_{(2,0)}(\mathbb{R}^{p,q})$.*

Proof. Decompose $\mathbb{R}^{p+q} = \mathbb{R}^p \oplus \mathbb{R}^q$ and $z = x \oplus y$. Let $\Psi_s(z) := sx \oplus y$ define a linear deformation retract from $\mathbb{R}^{p+q}$ to $\mathbb{R}^q$. If $g(z,z) > 0$ and if $s \in [0,1]$, then

$$g(\Psi_s(z), \Psi_s(z)) = s^2 g((x,0),(x,0)) + g((0,y),(0,y)) > 0.$$

Thus if π is a spacelike 2-plane, then $\Psi_s(\pi)$ is a spacelike 2-plane for $s \in [0,1]$; the map Ψ_s provides the required strong deformation retract from $Gr^+_{(0,2)}(\mathbb{R}^{p,q})$ to $Gr^+_2(\mathbb{R}^q)$; assertion (1) follows. Reversing the orientation defines a $\mathbb{Z}_2$ structure on $Gr^+_2(\mathbb{R}^{p,q})$ so that

$$Gr_2(\mathbb{R}^{p,q}) = Gr^+_2(\mathbb{R}^{p,q})/\mathbb{Z}_2.$$

Since the construction is equivariant with respect to this action, assertion (2) follows from assertion (1). Assertions (5) and (6) follow similarly.

Let the vectors $P^{\pm} = (x^{\pm}, y^{\pm})$ be as in Lemma 2.2.1 for $\pi \in Gr^{+}_{(1,1)}(\mathbb{R}^{p,q})$. Let

$$\Psi_s(\pi) := \operatorname{Span}\{(x^-, sy^-), (sx^+, y^+)\}.$$

The same argument as that given above shows that if $s \in [0,1]$, then

$$g((x^-, sy^-), (x^-, sy^-)) < 0 \text{ and } g((sx^+, y^+), (sx^+, y^+)) > 0.$$

Thus $g((x^-, sy^-), (x^-, sy^-))g((sx^+, y^+), (sx^+, y^+)) - g((x^-, sy^-), (sx^+, y^+))^2 < 0$. Consequently $\Psi_s(\pi)$ is a 2-plane of type $(1,1)$. Our construction is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ equivariant so the indeterminacy in the choice of $P^{\pm}$ plays no role and Ψ_s defines smooth maps on $Gr^{+}_{(1,1)}(\mathbb{R}^{p,q})$ and $Gr_{(1,1)}(\mathbb{R}^{p,q})$ providing the required strong deformation retract to $\mathcal{S}(p,q)$ and $\mathbb{RP}^{p-1} \times \mathbb{RP}^{q-1}$. $\square$

2.2.4 *Remark:* Let $F^{\mathrm{SO}}_{(1,1)}(\mathbb{R}^{p,q})$ be the set of pairs $\{(u,v) \in \mathbb{R}^{p,q} \times \mathbb{R}^{p,q}\}$ so that $g(u,u) = -1$, that $g(v,v) = 1$, and that $g(u,v) = 0$. If $(u,v) \in F^{\mathrm{SO}}_{(1,1)}(\mathbb{R}^{p,q})$, let $\pi(u,v) := \operatorname{Span}\{u,v\}$. Then the map $\pi : F^{\mathrm{SO}}_{(1,1)}(\mathbb{R}^{p,q}) \to Gr^{+}_{(1,1)}(\mathbb{R}^{p,q})$ is a fiber bundle with structure group $F = SO(1,1) = \mathbb{R} \times \mathbb{Z}_2$. Since $F^{\mathrm{SO}}_{(1,1)}(\mathbb{R}^{p,q})$ has a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ equivariant deformation retract to $S^{p-1} \times S^{q-1}$; this provides another way to see that $Gr^{+}_{(1,1)}(\mathbb{R}^{p,q})$ is homotopy equivalent to $(S^{p-1} \times S^{q-1})/\mathbb{Z}_2 = \mathcal{S}(p,q)$.

§2.3 The Stiefel-Whitney Classes and $\widetilde{KO}(\mathbb{RP}^n)$

2.3.1 *The Stiefel-Whitney classes.* Let E be a real vector bundle over a topological space B. Let $w(E)$ be the total Stiefel-Whitney class of E; $w(E)$ is characterized by the following properties:

(1) We may decompose $w(E) = 1 + w_1(E) + w_2(E) + \ldots$ for $w_i \in H^i(B; \mathbb{Z}_2)$.

(2) We have $w_i(E) = 0$ for $i > \dim(E)$.

(3) We have $w(E \oplus F) = w(E)w(F)$ i.e. $w_k(E \oplus F) = \sum_{i+j=k} w_i(E)w_j(F)$.

(4) If E is a trivial bundle, then $w(E) = 1$.

(5) If $\mathbb{L}$ is the classifying real line bundle over $\mathbb{RP}^n$, then $u := w_1(\mathbb{L})$ generates $H^1(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2$.

(6) We have w is natural with respect to pullback, i.e. $w(f^*E) = f^*w(E)$.

If E is a real vector bundle over a topological space B, let $[E]$ denote the corresponding element in the reduced real K-theory group $\widetilde{KO}(B)$. The following lemma calculating $\widetilde{KO}(\mathbb{RP}^n)$ follows from work of Adams [1].

2.3.2 Lemma. *Let $\mathbb{L}$ be the classifying real line bundle over $\mathbb{RP}^n$, see equation* (2.4.1.a) *below.*

(1) *Let $u := w_1(\mathbb{L})$. We then have $H^*(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2[u]/(u^{n+1} = 0)$.*

(2) *The elements* $[1]$ *and* $[\mathbb{L}]$ *generate $KO(\mathbb{RP}^n)$.*

(3) *The element $[\mathbb{L}] - [1]$ has order $\rho(n) := 2^{\phi(n)}$ in $\widetilde{KO}(\mathbb{RP}^n)$ where $\phi(0) = 0$, $\phi(1) = 1$, $\phi(2) = 2$, $\phi(3) = 2$, $\phi(4) = 3$, $\phi(5) = 3$, $\phi(6) = 3$, $\phi(7) = 3$, and where $\phi(8k + \ell) = 4k + \phi(\ell)$ for $\ell > 0$.*

For $n \in \mathbb{N}$, let $j(n) := [\log_2 n]$, then $2^{j(n)} \le n < 2^{j(n)+1}$. We tabulate some values of $\phi(n)$, $j(n)$ and $\rho(n)$.

TABLE 1. Some Useful Data

n	2	3	4	5	6	7	8	9	10	11
$n+2$	4	5	6	7	8	9	10	11	12	13
$n+3$	5	6	7	8	9	10	11	12	13	14
$\phi(n)$	2	2	3	3	3	3	4	5	6	6
$j(n)$	1	1	2	2	2	2	3	3	3	3
$\rho(n)$	4	4	8	8	8	8	16	32	64	64

§2.4 $H^*(Gr_2(\mathbb{R}^n);\mathbb{Z}_2)$ and the Steenrod Squares

2.4.1 *Classifying bundles.* We define

$$\begin{aligned} \mathbb{L} :=&\{(\ell, z) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : z \in \ell\}, \\ \gamma_2 :=&\{(\pi, z) \in Gr_2(\mathbb{R}^n) \times \mathbb{R}^n : z \in \pi\} \text{ and} \\ L :=&Gr_2^+(\mathbb{R}^n) \times \mathbb{R}/(\pi, \lambda) \sim (-\pi, -\lambda). \end{aligned} \tag{2.4.1.a}$$

to be the classifying real line bundle over $\mathbb{RP}^n$, the classifying real 2-plane bundle over $Gr_2(\mathbb{R}^n)$ and the canonical real line bundle over $Gr_2(\mathbb{R}^n)$. Let $\mathrm{Vect}_r(B)$ denote the isomorphism classes of rank r real vector bundles over B. The following lemma is well known:

2.4.2 Lemma.

(1) *We have* $\pi_1\mathbb{RP}^n = \mathbb{Z}_2$ *for* $n > 1$ *and* $Vect_1(\mathbb{RP}^n) = \mathbb{Z}_2$ *is generated by* $\mathbb{L}$.

(2) *We have* $\pi_1 Gr_2(\mathbb{R}^n) = \mathbb{Z}_2$ *for* $n > 2$ *and* $Vect_1(Gr_2(\mathbb{R}^n)) = \mathbb{Z}_2$ *is generated by* L.

2.4.3 *Remark:* We note that the restriction of L to $\mathbb{RP}^{n-2} \subset Gr_2(\mathbb{R}^n)$ is the classifying line bundle $\mathbb{L}$ over $\mathbb{RP}^{n-2}$ thus L is nontrivial.

We define the natural inclusion $i : \mathbb{RP}^{n-2} \to Gr_2(\mathbb{R}^n)$ as follows. Let $v \in S^{n-2}$ and let $\langle v\rangle$ be the associated point in $\mathbb{RP}^{n-2} = S^{n-2}/\mathbb{Z}_2$. Choose the standard orthonormal basis $\{e_1, \dots, e_{n-1}, e_n\}$ for $\mathbb{R}^n$ so that $\mathbb{R}^{n-1} = \mathrm{Span}\{e_1, \dots, e_{n-1}\}$. We define

$$i(\langle v\rangle) := \mathrm{Span}\{v, e_n\} \in Gr_2(\mathbb{R}^n).$$

We define

$$\begin{aligned} x &:= w_1(\gamma_2) \in H^1(Gr_2(\mathbb{R}^n);\mathbb{Z}_2), \\ y &:= w_2(\gamma_2) \in H^2(Gr_2(\mathbb{R}^n);\mathbb{Z}_2), \text{ and} \\ u &:= w_1(\mathbb{L}) \in H^1(\mathbb{RP}^{n-2};\mathbb{Z}_2). \end{aligned}$$

Let $\gamma_2^\perp$ be the orthogonal complement of γ_2 and let $w_i^\perp := w_i(\gamma_2^\perp)$. Since $\gamma_2 \oplus \gamma_2^\perp$ is a trivial bundle of dimension n, we use $w(\gamma_2) = 1 + x + y$ to express

$$w(\gamma_2^\perp) = \sum_k w_k^\perp = (1 + x + y)^{-1} \in \mathbb{Z}_2[[x, y]].$$

Since $\dim(\gamma_2^\perp) = n - 2$, we see that $w_i^\perp = 0$ in $H^*(Gr_2(\mathbb{R}^n); \mathbb{Z}_2)$ for $i \geq n - 1$. These relations generate all relations in $H^*(Gr_2(\mathbb{R}^n); \mathbb{Z}_2)$; we refer to Borel [18] for the proof of the following Theorem:

2.4.4 Theorem. *We have*

$$H^*(Gr_2(\mathbb{R}^n); \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y]/w_i^\perp = 0 \textit{ for } i \geq n - 1.$$

We shall need the following technical lemma later in §3.3.

2.4.5 Lemma. *Let $i : \mathbb{RP}^{n-2} \to Gr_2(\mathbb{R}^n)$ be the natural inclusion. We have:*

(1) $i^*(\gamma_2) \cong \mathbb{L} \oplus 1$ *and* $i^*(L) \cong \mathbb{L}$.

(2) $i^*x = u$ *and* $i^*y = 0$.

Proof. We use equation (2.4.1.a) and the definition of a pullback bundle to see that

$$\begin{aligned} i^*(\gamma_2) =& \{(\langle v\rangle, (\pi, z)) \in \mathbb{RP}^{n-2} \times Gr_2(\mathbb{R}^n) \times \mathbb{R}^n : \pi = i(\langle v\rangle) = \mathrm{Span}\{v, e_n\} \\ & \text{and } z \in \pi\}. \end{aligned}$$

So the fiber over each point $\langle v\rangle \in \mathbb{RP}^{n-2}$ is precisely the 2-plane $\mathrm{Span}\{v, e_n\}$. On the other hand, since $\mathbb{L} = \{(\langle v\rangle, z) \in \mathbb{RP}^{n-2} \times \mathbb{R}^{n-1} : z \in \langle v\rangle\}$ and since $1 = \mathbb{RP}^{n-2} \times \mathbb{R}$, the fiber of the bundle $\mathbb{L} \oplus 1$ over each point $\langle v\rangle \in \mathbb{RP}^{n-2}$ is the 2-plane $\mathrm{Span}\{v\} \oplus \mathbb{R} \cong \mathrm{Span}\{v, e_n\}$. Thus $i^*(\gamma_2) \cong \mathbb{L} \oplus 1$. We use Remark 2.4.3 and Lemma 2.4.2 (2) to see that $i^*(L)$ is a nontrivial line bundle over $\mathbb{RP}^{n-2}$, so it has to be $\mathbb{L}$. We use assertion (1) and naturality to see that

$$i^*x = w_1(i^*(\gamma_2)) = w_1(\mathbb{L} \oplus 1) = w_1(\mathbb{L}) = u, \text{ and}$$

$$i^*y = w_2(i^*(\gamma_2)) = w_2(\mathbb{L} \oplus 1) = 0. \quad \square$$

2.4.6 *Steenrod Squares and the top Stiefel-Whitney class.* In this section, we use the total Steenrod square to establish a well known technical lemma (Lemma 2.5.8) about the top Stiefel-Whitney class of a real vector bundle, we refer to Glover, Homer, and Stong [52]. We first recall the properties of the Steenrod squares Sq^i from Steenrod and Epstein [83].

2.4.7 Theorem. *Let B be a topological space.*

(1) *For all integers $i \geq 0$ and $n \geq 0$, there exists a natural transformation of functors which is a homomorphism $Sq^i : H^n(B;\mathbb{Z}_2) \to H^{n+i}(B;\mathbb{Z}_2)$.*

(2) $Sq^0 = 1$.

(3) *If* $\dim x = n$, *then* $Sq^n x = x^2$.

(4) *If* $i > \dim x$, *then* $Sq^i x = 0$.

(5) *(Cartan formula)* $Sq^k(x \cdot y) = \sum_{j=0}^{k} Sq^j x \cdot Sq^{k-j} y$.

(6) Sq^1 *is the Bockstein homomorphism β of the coefficient sequence*

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0.$$

(7) *(Adem relations) Let $\binom{m}{n}$ denote the number m choose n. If $0 < a < 2b$, then*

$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \left\{ \binom{b-1-j}{a-2j} \mod 2 \right\} Sq^{a+b-j} Sq^j.$$

§2.5 The Splitting Principle

A very useful tool in determining polynomial relations between characteristic classes is the splitting principle. This section is devoted to the discussion of the splitting principle and its various applications useful to our studies. We first introduce the following notational conventions. We follow the setup given in Bott and Tu [19].

2.5.1 *Notational conventions.* Let $K := \mathbb{R}$ or $\mathbb{C}$. Let V be a vector space over K. Let $\mathbb{P}(V)$ be the set of all 1 dimensional K subspaces of V. Let $\mathcal{P} := \mathbb{P}(V) \times V$ be the product bundle. Let $S_{\mathcal{P}} := \{(\ell, v) \in \mathcal{P} : v \in \ell\}$ be the canonical subbundle. Let $Q_{\mathcal{P}}$ be the canonical quotient bundle defined by the short exact sequence

$$0 \to S_{\mathcal{P}} \hookrightarrow \mathcal{P} \to Q_{\mathcal{P}} \to 0.$$

2.5.2 *Projective bundles and flag manifolds.* Let $\pi : E \to B$ be a K vector bundle with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, K)$. We define the *projectivization of* E by the fiber bundle $\rho : \mathbb{P}(E) \to B$, whose fiber over each $x \in B$ is $\mathbb{P}(E_x)$ and whose transition functions $\bar{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{PGL}(n, K)$ are induced by $g_{\alpha\beta}$. So a point in $\mathbb{P}(E)$ is a line ℓ_x in the fiber E_x. By definition, we have $\rho^* E \subset \mathbb{P}(E) \times E$ whose fiber over the point $\ell_x \in \mathbb{P}(E)$ is E_x, i.e. $(\rho^* E)_{\ell_x} = E_x$. The restriction of $\rho^* E$ to each fiber $\rho^{-1} x = \mathbb{P}(E)_x$ is the trivial bundle $\mathbb{P}(E)_x \times E_x$. The subbundle $S_E := \{(\ell_x, v) \in \rho^* E : v \in \ell_x\}$ is a line bundle; its fiber over each point $\ell_x \in \mathbb{P}(E)$ contains all the vectors in ℓ_x.

2.5.3 *Example:* Let $\pi : L \to B$ be a line bundle, we then have $\mathbb{P}(L) = B$ and $\rho^* L = S = L$.

We now construct a space $F(E)$ called the flag manifold together with a map $\sigma : F(E) \to B$ called the splitting map so that $\sigma^* E$ is a sum of line bundles. We proceed inductively on $\dim E$.

(1) If $\dim E = 1$, then E is a line bundle. Our construction is completed by Example 2.5.3.

(2) If $\dim E = 2$, we use the projectivization of E discussed in §2.5.2 to see that $\rho^* E = S_E \oplus Q_E$ over $\mathbb{P}(E)$. We set $\sigma := \rho$ and $F(E) := \mathbb{P}(E)$ to complete the construction.

(3) In general, at every next step, we projectivize the previously obtained quotient bundle Q_E to split off a new line bundle, so eventually all that remains is a sum of line bundles. We now set σ to be the composition of all these ρ's and set $F(E)$ to be the projectivization of the last quotient bundle.

The fact that the map $\sigma^* : H^*(B;\mathbb{Z}_2) \to H^*(F(E);\mathbb{Z}_2)$ is a ring monomorphism follows from the Theorem below. The detailed proof is omitted, we refer to [19] for the argument.

2.5.4 Theorem (Leray-Hirsch) *Let K be a principal ideal domain. Let $\pi : E \to B$ be a fiber bundle with fiber F of finite type. If there are globally defined cohomology classes $\{a_1, ..., a_r\}$ on E whose restriction to each fiber freely generate the cohomology of the fiber as a K-module, then $H^*(E;K)$ is a free $H^*(B;K)$-module with basis $\{a_1, ..., a_r\}$.*

2.5.5 Theorem (The splitting principle) *Let E be a real vector bundle over B. There exists a splitting map $\sigma : F(E) \to B$ so that $\sigma^* E$ is a sum of line bundles and $\sigma^* : H^*(B;\mathbb{Z}_2) \to H^*(F(E);\mathbb{Z}_2)$ is a ring monomorphism.*

2.5.6 *Remark:* A more general version of the Leray-Hirsch Theorem can be found in Husemoller [57].

The following three lemmas are needed in §3.1 and §3.3.

2.5.7 Lemma. *Let L be defined in equation* (2.4.1.a). *Let U be a real 4-plane bundle over $Gr_2(\mathbb{R}^n)$ so that $U \otimes L$ is isomorphic to U. Then*

$$x^4 + x^3 \cdot w_1(U) + x^2 \cdot (w_1(U) + w_2(U)) + x \cdot (w_1(U) + w_3(U)) = 0.$$

Proof. We use Theorem 2.5.5 (the splitting principle) to see that $\sigma^*(U) = \bigoplus_{i=1}^4 L_i$ and that $\sigma^* : H^*(Gr_2(\mathbb{R}^n);\mathbb{Z}_2) \to H^*(F(U);\mathbb{Z}_2)$ is a ring monomorphism. Let $s := \sigma^*(x)$. Let $s_i := w_1(L_i)$. Let $\tilde{w}_i := \sigma^*(w_i(U)) = w_i(\sigma^*(U))$. Since $U \cong U \otimes L$ and since the Stiefel-Whitney classes are natural, we have:

$$w(U) = w(U \otimes L) \text{ and } \sigma^*(w(U)) = w(\sigma^*(U)) = \prod_{i=1}^4 (1 + s_i).$$

Consequently we have that:

$$\prod_{i=1}^{4}(1+s_i) = \sigma^*(w(U)) = \sigma^*(w(U\otimes L)) = w(\sigma^*(U\otimes L)) = \prod_{i=1}^{4}(1+s_i+s).$$

We expand this identity to see:

$$\begin{aligned}\prod_{i=1}^{4}(1+s_i+s) =& \prod_{i=1}^{4}(1+s_i) + s\cdot\left(\sum_i s_i + \sum_{i<j<k} s_is_js_k\right)\\ &+ s^2\cdot\left(\sum_i s_i + \sum_{i<j} s_is_j\right) + s^3\cdot\left(\sum_i s_i\right) + s^4\\ =& \prod_{i=1}^{4}(1+s_i) + s^4 + s^3\cdot\tilde{w}_1 + s^2\cdot(\tilde{w}_1+\tilde{w}_2) + s\cdot(\tilde{w}_1+\tilde{w}_3).\end{aligned}$$

Thus $s^4 + s^3\cdot\tilde{w}_1 + s^2\cdot(\tilde{w}_1+\tilde{w}_2) + s\cdot(\tilde{w}_1+\tilde{w}_3) = 0$. Since σ^* is injective, the assertion now follows. $\square$

2.5.8 Lemma. *Let B be a topological space. Let E be a real vector bundle over B of dimension m. Let $w_m(E)$ be the top Stiefel-Whitney class of E. Let $w(E)$ be the total Stiefel-Whitney class of E. We have $Sq(w_m(E)) = w(E)\cdot w_m(E)$.*

Proof. By Theorem 2.5.5, it suffices to verify the assertion for sums of line bundles. Furthermore, by Theorem 2.4.7, Sq is a ring homomorphism, we may reduce to the case of a single line bundle λ. We compute:

$$\begin{aligned}Sq(w_1(\lambda)) =& w_1(\lambda) + Sq^1(w_1(\lambda)) = w_1(\lambda) + (w_1(\lambda))^2\\ =&(1+w_1(\lambda))\cdot w_1(\lambda) = w(\lambda)\cdot w_1(\lambda). \quad \square\end{aligned}$$

2.5.9 Lemma. *Let L be the nontrivial line bundle over $Gr_2(\mathbb{R}^n)$ and let γ_2 be the classifying 2-plane bundle over $Gr_2(\mathbb{R}^n)$ defined in equation (2.4.1.a). We have*

$$Sq(x) = (1+x)x \text{ and } Sq(y) = (1+x+y)y.$$

Proof. We apply Lemma 2.5.8 to L and γ_2 respectively; the result now follows. $\square$

§2.6 Two Important Lemmas

We say a bilinear map $\Phi : \mathbb{R}^a \times \mathbb{R}^b \to \mathbb{R}^c$ is *nonsingular* if $\Phi(x, y) = 0$ implies either $x = 0$ or $y = 0$ or $x = y = 0$.

2.6.1 Lemma. *Assume that $q \geq 3$. If there exists a nonsingular bilinear map $\Phi : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^{q+1}$, then $q = 3,\ 4,\ 7,$ or 8.*

Proof. Let $q \cdot \mathbb{L}$ be q copies of the classifying line bundle and let $(q+1) \cdot 1$ be $q+1$ copies of the trivial line bundle over $\mathbb{RP}^{q-1}$. In other words,

$$q \cdot \mathbb{L} \cong S^{q-1} \times \mathbb{R}^q/(x, y) \sim (-x, -y) \text{ and}$$

$$(q+1) \cdot 1 \cong S^{q-1} \times \mathbb{R}^{q+1}/(x, y) \sim (-x, y).$$

We observe $(-x, \Phi(-x, -y)) = (-x, \Phi(x, y))$ so the following gluing relations are preserved under Φ:

$$\begin{array}{ccc} (x, y) & \xrightarrow{\Phi} & (x, \Phi(x, y)) \\ \wr & \circ & \wr \\ (-x, -y) & \xrightarrow{\Phi} & (-x, \Phi(x, y)). \end{array}$$

Hence, Φ extends to a linear injective map from $q \cdot \mathbb{L}$ to $(q+1) \cdot 1$. Consequently, we have a short exact sequence

$$0 \to q \cdot \mathbb{L} \xrightarrow{\Phi} (q+1) \cdot 1 \to \{(q+1) \cdot 1\}/\Phi\{q \cdot \mathbb{L}\} \to 0. \tag{2.6.1.a}$$

The quotient in (2.6.1.a) is a 1 dimensional line bundle $\tilde{L}$ over $\mathbb{RP}^{q-1}$. Since any short exact sequence of line bundles splits, we have a decomposition:

$$(q+1) \cdot 1 = q\mathbb{L} \oplus \tilde{L}, \tag{2.6.1.b}$$

where $\tilde{L} = \{(q+1) \cdot 1\}/\Phi\{q \cdot \mathbb{L}\}$. Since by Lemma 2.4.2 (1), there are exactly two distinct line bundles over $\mathbb{RP}^{q-1}$, either $\tilde{L} = 1$ or $\tilde{L} = \mathbb{L}$. We distinguish these two cases in Equation (2.6.1.b).

Case 1. If $\tilde{L} = 1$, then we have $q([\mathbb{L}] - [1]) = 0$ in $\widetilde{KO}(\mathbb{RP}^{q-1})$. This implies that $\rho(q-1)$ divides q. We use Table (2.3.2.a) to see that $q = 4$ or $q = 8$; once $q \geq 10$, the powers of 2 grow too rapidly to permit this divisibility to occur.

Case 2. If $\tilde{L} = \mathbb{L}$, then we have $(q+1)([\mathbb{L}] - [1]) = 0$ in $\widetilde{KO}(\mathbb{RP}^{q-1})$. This implies that $\rho(q-1)$ divides $q+1$. We use Table (2.3.2.a) to see that $q = 3$ or $q = 7$; again once $q \geq 10$, the powers of 2 grow too rapidly to permit this divisibility to occur. $\square$

2.6.2 Lemma. *Let X be a topological space.*

(1) *Let $A : X \to M_n(\mathbb{R})$ (the set of all $n \times n$ real matrices) be a continuous map. Assume* $\dim \operatorname{Ker} A = k$ *is constant. Then $x \to \operatorname{Ker} A(x)$ is a continuous map from X to $Gr_k(\mathbb{R}^n)$.*

(2) *Let $\pi_i : X \to Gr_2(\mathbb{R}^n)$ be continuous maps. Assume* $\dim(\pi_1(x) \cap \pi_2(x)) = 1$ *for all x. Then the map $x \to \pi_1(x) \cap \pi_2(x)$ is a continuous map from X to* $Gr_1(\mathbb{R}^n) = \mathbb{RP}^{n-1}$.

Proof. The first assertion is well known; we refer to Atiyah [2]. Let $\rho_i(x)$ be orthogonal projection on $\pi_i(x)$. Let I be the $n \times n$ identity matrix. We define

$$A(x) := 2 \cdot I - \rho_1(x) - \rho_2(x).$$

If $\lambda \in \pi_1(x) \cap \pi_2(x)$, then $A(x)\lambda = 2\lambda - \lambda - \lambda = 0$. Conversely, suppose that $\lambda \neq 0$ satisfies the equation $A(x)\lambda = 0$. We then have $2\lambda = \rho_1(x)\lambda + \rho_2(x)\lambda$. Since $\rho_i(x)$ is an orthogonal projection, $|\rho_i(x)\lambda| \leq |\lambda|$. Consequently, we have

$$2|\lambda| \leq |\rho_1(x)\lambda| + |\rho_2(x)\lambda| \leq |\lambda| + |\lambda| = 2|\lambda|.$$

This shows that $|\rho_i(x)\lambda| = |\lambda|$. Since $\rho_i(x)$ is an orthogonal projection, this shows $\rho_i(x)\lambda = \lambda$ and thus $\lambda \in \pi_i(x)$. Thus $\operatorname{Ker} A = \pi_1(x) \cap \pi_2(x)$ and the second assertion follows from the first. $\square$

CHAPTER III

BOUNDING THE RANK OF IP ALGEBRAIC CURVATURE TENSORS

In chapter III, we prove Theorem A by bounding the rank of IP algebraic curvature tensors. Here is a brief outline to chapter III. In §3.1, we list the main results of this chapter and use these results to prove the first two assertions of Theorem A. In §3.2, we prove Theorem 3.1.1. In §3.2-3.4, we prove Theorem 3.1.2. In §3.5, we prove Theorem 3.1.3. We postpone the proof of Theorem A (3) until §3.6 as the techniques of proof are quite different from the topological ones that will be used to prove the results cited above. We also establish some additional low dimensional results using similar techniques.

§3.1 Proof of Theorem A (1) and (2)

We shall use techniques from algebraic topology to prove the following results:

3.1.1 Theorem.

(1) *Let $R : S^n \to \mathfrak{so}(n+2)$ be admissible. Assume $n \geq 9$. We have that* $\operatorname{rank} R \leq 2$.

(2) *Let $R : S^n \to \mathfrak{so}(n+3)$ be admissible. Assume $n \geq 10$.*

(2a) *If n is even, then* $\operatorname{rank} R \leq 2$.

(2b) *If n is odd, then* $\operatorname{rank} R \leq 4$.

3.1.2 Theorem. *Let $R : Gr_2^+(\mathbb{R}^q) \to \mathfrak{so}(q+2)$ be an admissible map of rank 4. Let $q \geq 12$ and let q be even. Then either q is a power of 2 or $2+q$ is a power of 2.*

3.1.3 Theorem. *Let R be an IP algebraic curvature tensor on $\mathbb{R}^{p,q}$.*

3.1.3 Theorem. *Let R be an IP algebraic curvature tensor on $\mathbb{R}^{p,q}$.*

(1) *If $p = 1$ and if $q = 5$, then* $\operatorname{rank} R \leq 2$.

(2) *If $p = 1$ and if $q = 9$, then* $\operatorname{rank} R \leq 2$.

We now use these results to prove Theorem A (1) and (2) as follows.

Proof of Theorem A (1). Let $p = 1$ and let $q \geq 9$. Let R be an IP algebraic curvature tensor on $\mathbb{R}^{1,q}$. Then R defines an admissible map from $Gr^+(\mathbb{R}^{1,q}) \to \mathfrak{so}(1,q)$ of rank r. We wish to show $r \leq 2$. If $q = 9$, we use Theorem 3.1.3 (2) to see $r \leq 2$. We may therefore assume $q \geq 10$. We use Lemma 2.1.4 to construct an admissible map $\tilde{R} : Gr^+(\mathbb{R}^{1,q}) \to \mathfrak{so}(1+q)$ of rank r. We use the $\mathbb{Z}_2$ equivariant embedding $S^{q-1} \to Gr_{(1,1)}(\mathbb{R}^{p,q})$ discussed in chapter II to construct an admissible map $\tilde{R} : S^{q-1} \to \mathfrak{so}(q+1)$ of rank r. Theorem 3.1.1 (1) then implies $r \leq 2$ as desired since $q \geq 10$ implies that $q - 1 \geq 9$. $\square$

Proof of Theorem A (2). Let $p = 2$ and let $q \geq 11$. Let R be an IP algebraic curvature tensor on $\mathbb{R}^{2,q}$. By Lemma 2.1.4, R defines an admissible map from $Gr^+(\mathbb{R}^{2,q}) \to \mathfrak{so}(2,q)$ of rank r. Again, we use Lemma 2.1.4 to construct an admissible map $\tilde{R} : Gr^+(\mathbb{R}^{2,q}) \to \mathfrak{so}(2+q)$ of rank r. Again, we use the $\mathbb{Z}_2$ equivariant embedding $S^{q-1} \to Gr^+(\mathbb{R}^{2,q})$ to construct an admissible map $\tilde{R} : S^{q-1} \to \mathfrak{so}(q+2)$ of rank r. Since $q - 1 \geq 10$, Theorem 3.1.1 (2) shows that $r \leq 4$. Furthermore in the exceptional case that $r = 4$, we may conclude that $q - 1$ is odd and hence q is even. We now suppose $r = 4$ and q even. We use the $\mathbb{Z}_2$ equivariant embedding of $Gr_2^+(\mathbb{R}^q)$ in $Gr_{(0,2)}^+(\mathbb{R}^q)$ to construct an admissible map $\tilde{R} : Gr_2^+(\mathbb{R}^q) \to \mathfrak{so}(2+q)$ of rank 4. We use Theorem 3.1.2 to see that q or $q + 2$ is a power of 2. $\square$

3.1.4 *Remark:* We construct rank 2 and rank 4 admissible maps to show Theorem 3.1.1 is sharp as follows. Let $\{e_i\}$ be the standard orthonormal basis for $\mathbb{R}^{n+3}$ relative to the standard Euclidean inner product g. Let $\{e_1, ..., e_\nu\}$ be the standard orthonormal basis for $\mathbb{R}^\nu$. If $\{v_1, v_2\}$ is an orthonormal set, let $\mathcal{R}_{v_1,v_2}$ be the rotation which sends v_1 to v_2 and which is zero on the orthogonal complement, i.e.

$$\mathcal{R}_{v_1,v_2} : w \to g(w, v_1)v_2 - g(w, v_2)v_1.$$

(1) Let $R_2(v) := \mathcal{R}_{v,e_{n+3}}$. Then we have $R_2 : S^{n+1} \to \mathfrak{so}(n+3)$ is an admissible map with rank 2 and assertions (1) and (2a) are sharp.

(2) Let $\mathcal{J}$ be a complex structure on $\mathbb{R}^{n+3}$ for $n+3$ even. Let

$$R_4(v) := R_2(v) + \mathcal{R}_{\mathcal{J}v,e_{n+2}}.$$

Then we have $R_4 : S^n \to \mathfrak{so}(n+3)$ is an admissible map with rank 4 and assertion (2b) is sharp.

3.1.5 *Remark:* We do not know if Theorem 3.1.1 is sharp; we do not know if there exist rank 4 admissible maps in this setting.

§3.2 Bounding the Rank of IP Algebraic Curvature Tensors

3.2.1 *Notational conventions.* Let R be an admissible map from S^n to $\mathfrak{so}(m)$. Let $V_0(R(v))$ and $V_1(R(v))$ be the kernel and range of $R(v)$ for $v \in S^n$. Since $R(v)$ has constant rank on S^n, $V_i(R(v))$ define vector bundles over S^n. Let $m \cdot 1$ be m copies of the trivial line bundle over S^n. We then have an orthogonal direct sum decomposition:

$$V_0 \oplus V_1 = m \cdot 1. \tag{3.2.1.a}$$

Since R takes values in $\mathfrak{so}(m)$, $V_0(R(v)) \cap V_1(R(v)) = \{0\}$ for all $v \in S^n$. This would not be the case if we were dealing with maps to $\mathfrak{so}(p,q)$ for $pq \neq 0$ which is why Lemma 2.1.4 will be useful in our future development. Since $R(-v) = -R(v)$, the vector bundles V_i descend to define vector bundles U_i over projective space $\mathbb{RP}^n$. Let $v_1 \in S^n$, let $v_2 \in V_1(R(v_1))$, and let $\lambda \in \mathbb{R}$. Since $R(-v_1) = -R(v_1)$, we have

$$R(-v_1)v_2 \otimes \lambda = -R(v_1)v_2 \otimes \lambda = R(v_1)v_2 \otimes (-\lambda).$$

Thus the following gluing relations are preserved:

$$\begin{array}{ccc} (v_1, v_2 \otimes \lambda) & \stackrel{R(v_1)}{\longrightarrow} & (v_1, R(v_1)v_2 \otimes \lambda) \\ \wr & \circ & \wr \\ (-v_1, v_2 \otimes \lambda) & \stackrel{R(-v_1)}{\longrightarrow} & (-v_1, R(v_1)v_2 \otimes (-\lambda)). \end{array}$$

We note that the left column of the diagram gives rise to the bundle $U_1 \otimes 1 \cong U_1$ over $\mathbb{RP}^n$, whereas the right column gives rise to the bundle $U_1 \otimes \mathbb{L}$ over $\mathbb{RP}^n$. Thus R descends to define an isomorphism between U_1 and $U_1 \otimes \mathbb{L}$. We decompose $[U_i]$ in $\widetilde{KO}(\mathbb{RP}^n)$ in the form:

$$[U_i] = a_i([\mathbb{L}] - [1]) + \dim(U_i)[1];$$

in this expression, the integer a_i is well defined modulo $\rho(n)$. Let $j(n)$ be defined in §2.4, then $2^{j(n)} \leq n < 2^{j(n)+1}$. We shall need the following technical lemma. 3.2.2 Lemma. *Let $R : S^n \to \mathfrak{so}(m)$ be admissible. Let U_1 be the associated bundle defined over $\mathbb{RP}^n$.*

(1) *We have* $2a_1 \equiv \dim(U_1) \mod \rho(n)$.

(2) *We have* $a_0 + a_1 \equiv 0 \mod 2^{j(n)+1}$.

(3) *If $n \geq 9$, then $j(n) + 2 \leq \phi(n)$ and* $a_1 \equiv \frac{1}{2}\dim(U_1) \mod 2^{j(n)+1}$.

Proof. By definition we have

$$\begin{aligned} [U_1] =& a_1([\mathbb{L}] - [1]) + \dim(U_1)[1] \text{ and} \\ [U_1 \otimes \mathbb{L}] =& (\dim(U_1) - a_1)([\mathbb{L}] - [1]) + \dim(U_1)[1]. \end{aligned}$$

Since U_1 is isomorphic to $U_1 \otimes \mathbb{L}$, we may equate the coefficients of $([\mathbb{L}] - [1])$ mod $\rho(n)$ in these expressions to prove assertion (1).

The orthogonal direct sum decomposition (3.2.1.a) descends to show $U_0 \oplus U_1 \cong m \cdot 1$. Consequently

$$1 = w(U_0)w(U_1) = (1+u)^{a_0}(1+u)^{a_1} = (1+u)^{a_0+a_1}. \tag{3.2.2.a}$$

Let $a_0 + a_1 \equiv \alpha + \beta \cdot 2^{j(n)} \mod 2^{j(n)+1}$ for $0 \le \alpha < 2^{j(n)}$ and $\beta = 0$, or 1. Since $2^{j(n)} < n$, all the coefficients of u^ℓ in equation (3.2.2.a) vanish for $\ell \le 2^{j(n)}$, so $\alpha = \beta = 0$; assertion (2) follows. We use Table (2.3.2.a) to see that $j(n) + 2 \le \phi(n)$ for $9 \le n \le 11$. The function ϕ is growing roughly linearly and the function j is growing logarithmically; hence assertion (3) follows. $\square$

3.2.3 *Proof of Theorem* 3.1.1. The first assertion of Theorem 3.1.1 follows from work of Gilkey, Leahy and Sadofsky [48]. We adopt the argument given by Gilkey, Leahy and Sadofsky to prove the remaining assertions.

We set $m := n + 3$. Let $j := j(n)$. Let $u_i := \dim(U_i)$. Assume $u_1 \ge 2$. We use Lemma 3.2.2 to choose integers $0 < \bar{a}_0 \le 2^{j+1}$, and $\bar{a}_1 = 2^{j+1} - \bar{a}_0$ so $0 \le \bar{a}_1 < 2^{j+1}$ such that $w(U_i) = (1+u)^{\bar{a}_i}$. We have the basic properties:

(1) $\bar{a}_0 + \bar{a}_1 = 2^{j+1}$.

(2) $u_0 + u_1 = n + 3$.

(3) $u_1 = 2\bar{a}_1$.

Now if $\bar{a}_0 \le n$, then $x^{\bar{a}_0}$ survives in $w(U_0)$ and hence $u_0 \ge \bar{a}_0$. Consequently

$$2^{j+1} + 2 \ge n + 3 = u_0 + u_1 \ge \bar{a}_0 + 2\bar{a}_1 = 2^{j+1} + \bar{a}_1.$$

Thus $\bar{a}_1 \le 2$ and $u_1 = 2\bar{a}_1 \le 4$. If $u_1 = 4$, i.e. $\bar{a}_1 = 2$, then all the inequalities must have been equalities, thus $2^{j+1} + 2 = n + 3$ and n is odd. We may therefore assume $\bar{a}_0 > n \ge 2^j$. Let α_ν, $\tilde{\alpha}_\nu$, and β_ν be the coefficients of 2^ν in the 2-adic expansions of $\bar{a}_0$, $\bar{a}_1 - 1$ and n. Then α_ν, $\tilde{\alpha}_\nu$, and β_ν are 0 or 1. Since $\bar{a}_0 + \bar{a}_1 = 2^{j+1}$, we must have $\tilde{\alpha}_\nu + \alpha_\nu = 1$. Thus

$$\bar{a}_0 = 1 \cdot 2^j + \alpha_{j-1}2^{j-1} + ... + \alpha_0$$

$$\bar{a}_1 = 0 \cdot 2^j + \tilde{\alpha}_{j-1}2^{j-1} + ... + \tilde{\alpha}_0 + 1$$

$$n \ = 1 \cdot 2^j + \beta_{j-1}2^{j-1} + ... + \beta_0.$$

If all the $\alpha_\nu = 1$, then $\bar{a}_1 = 1$ so $u_1 = 2$ and we are done. Thus $\alpha_\nu = 0$ for some $0 \leq \nu \leq j-1$. Choose k maximal so that $\alpha_k = 0$. Expand

$$\bar{a}_0 = 1 \cdot 2^j + ... + 1 \cdot 2^{k+1} + 0 \cdot 2^k + \alpha_{k-1}2^{k-1} + ... + \alpha_0$$

$$\bar{a}_1 = 0 \cdot 2^j + ... + 0 \cdot 2^{k+1} + 1 \cdot 2^k + \tilde{\alpha}_{k-1}2^{k-1} + ... + \tilde{\alpha}_0 + 1$$

$$n \ = 1 \cdot 2^j + ... + \beta_{k+1}2^{k+1} + \beta_k 2^k + \beta_{k-1}2^{k-1} + ... + \beta_0.$$

Let

$$n_{k+1} := 2^j + \beta_{j-1}2^{j-1} + ... + \beta_{k+1}2^{k+1} \leq n.$$

We use Lemma A.1 in Appendix A to see that $x^{n_{k+1}}$ survives in $w(U_0)$, this implies $u_0 \geq n_{k+1}$. We estimate:

$$u_0 \geq n_{k+1}$$

$$u_1 = 2\bar{a}_1 \geq 2 \cdot 2^k + 2 = 2^k + 2^{k-1} + ... + 2^0 + 3$$

$$n + 3 = u_0 + u_1 \geq n_{k+1} + 2^k + 2^{k-1} + ... + 2^0 + 3 \geq n + 3.$$

Thus all of these inequalities must have been equalities; we now have:

(3.2.3.a) $\qquad u_0 = n_{k+1},\ \bar{a}_1 = 2^k + 1,\ \text{and } n = n_{k+1} + 2^k + 2^{k-1} + ... + 2^0.$

If $k = 0$, then $\bar{a}_2 = 2$ so $u_1 = 4$. Furthermore n is odd. Thus we assume $k \geq 1$ and express:

$$\bar{a}_0 = 1 \cdot 2^j + ... + 1 \cdot 2^{k+1} + 0 \cdot 2^k + 1 \cdot 2^{k-1} + ... + 1 \cdot 2^0$$

$$\bar{a}_1 = 0 \cdot 2^j + ... + 0 \cdot 2^{k+1} + 1 \cdot 2^k + 0 \cdot 2^{k-1} + ... + 0 \cdot 2^0 + 1$$

$$n \ = 1 \cdot 2^j + ... + \beta_{k+1}2^{k+1} + 1 \cdot 2^k + 1 \cdot 2^{k-1} + ... + 1 \cdot 2^0.$$

This shows that $n_{k+1} + 1 \leq n$ so $x^{n_{k+1}+1}$ survives in $w(U_0)$ and hence we have $u_0 \geq n_{k+1} + 1 > n_{k+1}$; this contradicts equation (3.2.3.a). Thus $k = 0$, and this completes the proof. $\square$

The following is an immediate consequence of the proof we have given of Theorem 3.1.1 since $\bar{a}_1 = \frac{1}{2}u_1 = 2$.

3.2.4 Corollary. *Assume $n \geq 10$. Let $R : S^n \to \mathfrak{so}(n+3)$ be admissible. If $\dim(U_1) = 4$, then we have $w(U_1) = 1 + u^2$.*

Let $R : Gr_2^+(\mathbb{R}^{m-2}) \to \mathfrak{so}(m)$ be a rank 4 admissible map. Let $\tilde{W}_0(R(\pi))$ and $\tilde{W}_1(R(\pi))$ be the kernel and range of $R(\pi)$ for $\pi \in Gr_2^+(\mathbb{R}^{m-2})$. Since $R(\pi)$ has constant rank on $Gr_2^+(\mathbb{R}^{m-2})$, $\tilde{W}_i(R(\pi))$ define vector bundles over the oriented Grassmannian $Gr_2^+(\mathbb{R}^{m-2})$; we have that $\dim \tilde{W}_0 = m - 4$, that $\dim \tilde{W}_1 = 4$, and that $\tilde{W}_0 \oplus \tilde{W}_1$ is a trivial bundle of dimension m. Since $R(-\pi) = -R(\pi)$, $\tilde{W}_i(R(-\pi)) = \tilde{W}_i(R(\pi))$. Thus these bundles descend to define vector bundles W_i over the unoriented Grassmannian $Gr_2(\mathbb{R}^{m-2})$ and $W_0 \oplus W_1 = m \cdot 1$. Let L be the nontrivial real line bundle over $Gr_2(\mathbb{R}^{m-2})$ defined in equation (2.4.1.a). We have that R induces an isomorphism from $W_1 \otimes L$ to W_1. We use Theorem 2.4.4 and Lemma 2.4.5 to study the Stiefel-Whitney classes of the bundle W_1.

3.2.5 Lemma. *Assume $m \geq 11$. Let $R : Gr_2^+(\mathbb{R}^{m-2}) \to \mathfrak{so}(m)$ be a rank 4 admissible map. There exist integers S, T, and U taking values in $\{0, 1\}$ so that*

$$w(W_1) = 1 + x^2 + S(y + xy) + Tx^2y + Uy^2.$$

Proof. Let $i : \mathbb{RP}^{m-4} \to Gr_2(\mathbb{R}^{m-2})$ be the natural inclusion discussed in §2.5. Let U_i be the restriction of W_i to $\mathbb{RP}^{m-4}$. We use Corollary 3.2.4 to see that $i^*(w(W_1)) = w(U_1) = 1 + u^2$. Lemma 2.4.5 shows that the coefficients of x, x^3, and x^4 in $w(W_1)$ are zero while the coefficient of x^2 is 1, so $w_1(W_1) = 0$. By Theorem

2.4.4, x and y generate $H^*(Gr_2(\mathbb{R}^{m-2}); \mathbb{Z}_2)$. Consequently, there exist constants S, Q, T, and U so that

$$w(W_1) = 1 + P(x,y) \text{ for } P(x,y) := x^2 + Sy + Qxy + Tx^2y + Uy^2.$$

We use Lemma 2.5.7 with $U = W_1$ to see that $x^4 + x^2 \cdot w_2(W_1) + x \cdot w_3(W_1) = 0$, i.e. we have that $x^4 + x(Qxy) + x^2(x^2 + Sy) = 0$ so $S = Q$. □

§3.3 A Technical Lemma

3.3.1 Lemma. *Assume $m \geq 11$. Let $R : Gr_2^+(\mathbb{R}^{m-2}) \to \mathfrak{so}(m)$ be a rank 4 admissible map. We have $w(W_1) = 1 + P_i$ for $i = 2$, 3, or 4; where $P_2 = x^2$, $P_3 = x^2 + y^2$, and $P_4 = x^2 + y + xy$.*

Proof. In Lemma 3.2.5, we showed $w(W_1) = 1 + x^2 + S(y + xy) + Tx^2y + Uy^2$. The top Stiefel-Whitney class of W_1 is $w_4(W_1) = Tx^2y + Uy^2$. We consider the following cases:

Case 1. Suppose $(T, U) \neq (0, 0)$. Since Sq is a ring homomorphism, we apply Lemma 2.5.9 to see that

$$\begin{aligned} Sq(w_4(W_1)) =& T(1+x)^2x^2(1+x+y)y + U(1+x+y)^2y^2 \\ =& Tx^2y + Tx^4y + Tx^3y + Tx^5y + Tx^2y^2 \\ &+ Tx^4y^2 + Uy^2 + Ux^2y^2 + Uy^4. \end{aligned} \tag{3.3.1.a}$$

We apply Lemma 2.5.8 to see that:

$$\begin{aligned} Sq(w_4(W_1)) =& (1 + x^2 + S(y+xy) + Tx^2y + Uy^2)(Tx^2y + Uy^2) \\ =& Tx^2y + Tx^4y + TSx^2y^2 + TSx^3y^2 + USxy^3 \\ &+ USy^3 + Uy^2 + Ux^2y^2 + Uy^4 + Tx^4y^2. \end{aligned} \tag{3.3.1.b}$$

Since $m \geq 11$, there are no relations in $H^k(Gr_2(\mathbb{R}^{m-2}); \mathbb{Z}_2)$ for $k \leq 7$. We compare the coefficients of x^3y in equations (3.3.1.a) and (3.3.1.b) to see $T = 0$. Since $(T, U) \neq (0, 0)$, we have $U = 1$. We compare the coefficients of xy^3 in equations (3.3.1.a) and (3.3.1.b) to see $S = 0$.

Case 2. Suppose $(T, U) = (0, 0)$. Then $S = 0$ or $S = 1$ is automatic in $\mathbb{Z}_2$. Our assertion now follows. $\square$

3.3.2 *Additional notation.* Let $\mathcal{V}^i := (1 + P_i)^{-1}$ be the corresponding formal power series in the formal power series ring $\mathbb{Z}_2[[x, y]]$ defined by P_i which were listed in Lemma 3.3.1. Let $\mathcal{V}^i_k$ be the kth degree homogeneous terms in the corresponding expansions. For clarity, we now tabulate these expressions as follows:

TABLE 2. Possible Choises for $\mathcal{V}$

$w(\gamma_2^\perp) = (1 + x + y)^{-1}$	$\mathcal{V}^2 = (1 + x^2)^{-1}$
$\mathcal{V}^3 = (1 + x^2 + y^2)^{-1}$	$\mathcal{V}^4 = (1 + x^2 + y + xy)^{-1}$

§3.4 Rank 4 Admissible Maps in the $(2, m-2)$ Setting

In this section, we work in the setting $(p, q) = (2, m-2)$ with $p = 2$ and $q \geq 10$. We have the natural embedding S^{q-1} in $Gr^+_{(1,1)}(\mathbb{R}^{2,q})$. If $R : Gr^+_2(\mathbb{R}^{2,q}) \to \mathfrak{so}(2+q)$ is an admissible map, then the restriction of R to S^{q-1} defines an admissible map from S^{q-1} to $\mathfrak{so}(2+q)$. By Theorem 3.1.1 (2), we have $\operatorname{rank} R \leq 4$ and $\operatorname{rank} R = 4$ only if q is even, so m is also even. Suppose there exists a rank 4 admissible map R from $Gr^+_2(\mathbb{R}^{2,q})$ to $\mathfrak{so}(2+q)$. We use the $\mathbb{Z}_2$ equivariant embedding of $Gr^+_2(\mathbb{R}^q)$ into $Gr^+_2(\mathbb{R}^{2,q})$ discussed in §2.2 to extend R to a rank 4 admissible map from $Gr^+_2(\mathbb{R}^q)$ to $\mathfrak{so}(2+q)$.

We adopt the notational conventions established in §2.4.1 and §3.3.2. Since $\dim W_0 = q-2$, $\mathcal{V}^i_k = 0$ in $H^*(Gr_2(\mathbb{R}^q); \mathbb{Z}_2)$ for $k \geq q-1$ and $i = 2, 3$ or 4. We now study the relationship between q and $\mathcal{V}^i$.

3.4.1 Lemma. *Assume $p = 2$ and $q \geq 10$ even. Let $R : Gr^+_2(\mathbb{R}^q) \to \mathfrak{so}(q+2)$ be a rank 4 admissible map. If $w(W_0) = \mathcal{V}^2$, then q is a power of 2.*

Proof. Since R has rank 4, $\dim W_1 = 4$ and $\dim W_0 = q - 2$. Thus $\mathcal{V}^2_q$ vanishes in $H^q(Gr_2(\mathbb{R}^q); \mathbb{Z}_2)$; we may express $\mathcal{V}^2_q = \alpha w^\perp_q + \beta x \cdot w^\perp_{q-1}$ in $\mathbb{Z}_2[x,y]$ for $\alpha, \beta = 0$ or 1. We have $\mathcal{V}^2 = \sum_{i \geq 0} x^{2i}$. Since q is even, $\mathcal{V}^2_q = x^q$. We consider the following cases:

Case 1. Suppose $(\alpha, \beta) = (0,0)$. This implies $\mathcal{V}^2_q = 0$ which is false.

Case 2. Suppose $(\alpha, \beta) = (1,0)$. This implies $\mathcal{V}^2_q = w^\perp_q$. Since there is no $y^{q/2}$ term in $\mathcal{V}^2_q$, and since $w^\perp_q$ contains the term $y^{q/2}$, this is not possible.

Case 3. Suppose $(\alpha, \beta) = (1,1)$. This implies $\mathcal{V}^2_q = w^\perp_q + x \cdot w^\perp_{q-1}$. Since there is no x^q term in $w^\perp_q + x \cdot w^\perp_{q-1}$, and since $\mathcal{V}^2_q$ contains the term x^q, this is not possible.

Case 4. Suppose $(\alpha, \beta) = (0,1)$. This implies $\mathcal{V}^2_q = x \cdot w^\perp_{\bar{q}}$. Since $\mathcal{V}^2_q$ has only even powers of x, this can happen only if $w^\perp_{q-1} = x^{q-1}$. We use Lemma A.2 in Appendix A to see that $w^\perp_{q-1} = x^{q-1}$ in $H^*(Gr_2(\mathbb{R}^q); \mathbb{Z}_2)$ if and only if q is a power of 2. □

3.4.2 Lemma. *Assume $p = 2$ and $q \geq 10$ even. Let $R : Gr^+_2(\mathbb{R}^q) \to \mathfrak{so}(q+2)$ be a rank 4 admissible map. Then $w(W_0) \neq \mathcal{V}^3$.*

Proof. If Q_k is a homogeneous polynomial in x, y of degree k, then we can expand

$$\begin{aligned} Q_k =& C_1(Q_k)x^k + C_2(Q_k)x^{k-2}y + C_3(Q_k)x^{k-4}y^2 \\ &+ C_4(Q_k)x^{k-6}y^3 + C_5(Q_k)x^{k-8}y^4 + \ldots \end{aligned}$$

We set $C_i(Q_k) := 0$ if $i < 0$ or $k < 0$. Let

$$\vec{C}(Q_k) := (C_1(Q_k)C_2(Q_k)C_3(Q_k)C_4(Q_k)C_5(Q_k)) \in \mathbb{Z}_2^5$$

be the first five coefficients in this expansion. In the expansion of $w^\perp_k$, we have $x^{k-2\nu}y^\nu = x \cdot x^{k-2\nu-1}y^\nu + y \cdot x^{k-2\nu}y^{\nu-1}$ where the term $x^{k-2\nu-1}y^\nu$ comes from $w^\perp_{k-1}$ and the term $x^{k-2\nu}y^{\nu-1}$ comes from $w^\perp_{k-2}$. In the expansion of $\mathcal{V}^3_k$, we have $x^{k-2\nu}y^\nu = x^2 \cdot x^{k-2\nu-2}y^\nu + y^2 \cdot x^{k-2\nu}y^{\nu-2}$ where the term $x^{k-2\nu-2}y^\nu$ comes from

$\mathcal{V}^3_{k-2}$ and the term $x^{k-2\nu}y^{\nu-2}$ comes from $\mathcal{V}^3_{k-4}$. Thus we have the following recursion relations:

$$C_i(w_k^\perp) = C_i(w_{k-1}^\perp) + C_{i-1}(w_{k-2}^\perp) \text{ and}$$
$$C_i(\mathcal{V}_k^3) = C_i(\mathcal{V}_{k-2}^3) + C_{i-2}(\mathcal{V}_{k-4}^3). \tag{3.4.2.a}$$

We tabulate $\vec{C}(w_k^\perp)$ and $\vec{C}(\mathcal{V}_k^3)$ for the following values of k

TABLE 3. The Periodicities of $\vec{C}(w_k^\perp)$ and $\vec{C}(\mathcal{V}_k^3)$

k	$\vec{C}(w_k^\perp)$	$\vec{C}(\mathcal{V}_k^3)$	$k+16$	$\vec{C}(w_{k+16}^\perp)$	$\vec{C}(\mathcal{V}_{k+16}^3)$
9	10101	00000	25	10101	00000
10	11011	10001	26	11011	10001
11	10001	00000	27	10001	00000
12	11100	10100	28	11100	10100
13	10100	00000	29	10100	00000
14	11010	10000	30	11010	10000
15	10000	00000	31	10000	00000
16	11101	10101	32	11101	10101
17	10101	00000	33	10101	00000
18	11011	10001	34	11011	10001
19	10001	00000	35	10001	00000
20	11100	10100	36	11100	10100
21	10100	00000	37	10100	00000
22	11010	10000	38	11010	10000
23	10000	00000	39	10000	00000
24	11101	10101	40	11101	10101
25	10101	00000	41	10101	00000

The recursion relations given in equation (3.4.2.a) imply $\vec{C}(w_k^\perp)$ and $\vec{C}(\mathcal{V}_k^3)$ are periodic with period 16 for all values of $k \geq 9$.

Since R has rank 4, $\dim W_1 = 4$ and $\dim W_0 = q-2$. Thus $\mathcal{V}_q^3$ vanishes in $H^q(Gr_2(\mathbb{R}^q);\mathbb{Z}_2)$; we may express $\mathcal{V}_q^3 = \alpha w_q^\perp + \beta x \cdot w_{q-1}^\perp$ for $\alpha, \beta = 0$ or 1. We use Table (3) to tabulate these values:

TABLE 4. The Elimination of $\mathcal{V}^3$

q	0	$w_q^\perp$	$x \cdot w_{q-1}^\perp$	$w_q^\perp + x \cdot w_{q-1}^\perp$	$\mathcal{V}_q^3$
10	00000	11011	10101	01110	10001
12	00000	11100	10001	01101	10100
14	00000	11010	10100	01110	10000
16	00000	11101	10000	01101	10101
18	00000	11011	10101	01110	10001
20	00000	11100	10001	01101	10100
22	00000	11010	10100	01110	10000
24	00000	11101	10000	01101	10101
26	00000	11011	10101	01110	10001
28	00000	11100	10001	01101	10100
30	00000	11010	10100	01110	10000
32	00000	11101	10000	01101	00101

By comparing the data from each column, we can rule out $w(W_0) = \mathcal{V}^3$ as required. $\square$

Lemma A.3 in Appendix A due to Stong [84] is needed for the case $w(W_0) = \mathcal{V}^4$.

3.4.3 Lemma. *Assume $p = 2$ and $q \geq 10$. Let $R : Gr_2^+(\mathbb{R}^q) \to \mathfrak{so}(q+2)$ be a rank 4 admissible map. If $w(W_0) = \mathcal{V}^4$, then $2 + q$ is a power of 2.*

Proof. We note that $w(W_1) = 1 + x^2 + y + xy = (1+x)(1+x+y)$. We apply Theorem 2.5.5 to γ_2 to see that

(1) $\sigma^*(\gamma_2) = L_1 \oplus L_2$.

(2) $\sigma^* : H^*(Gr_2(\mathbb{R}^q); \mathbb{Z}_2) \to H^*(F(\gamma_2); \mathbb{Z}_2)$ is a ring monomorphism.

(3) $\sigma^*(x) = u_1 + u_2$, where $u_i = w_1(L_i)$ for $i = 1, 2$.

(4) $\sigma^*(y) = u_1 \cdot u_2$, where $u_i = w_1(L_i)$ for $i = 1, 2$.

Consequently, we have:

$$\begin{aligned} \sigma^*(w(\gamma_2)) =& (1+u_1)\cdot(1+u_2), \text{ and} \\ \sigma^*(w(W_1)) =& \sigma^*((1+x)(1+x+y)) = (1+u_1)\cdot(1+u_2)\cdot(1+u_1+u_2). \end{aligned}$$

We now compute:

$$(\sigma^* w(\gamma^\perp))_n = \textstyle\sum_{i+j=n} u_1^i \cdot u_2^j, \text{ and}$$
$$(\sigma^* \mathcal{V}^4)_n = \textstyle\sum_{a+b+c=n} (u_1+u_2)^a \cdot u_1^b \cdot u_2^c.$$

Since $\mathcal{V}^4_{q-1} = 0$ in $H^{q-1}(Gr_2(\mathbb{R}^q); \mathbb{Z}_2)$, we have

$$0 = (\sigma^* \mathcal{V})^4_{q-1} = \textstyle\sum_{a+b+c=q-1} (u_1+u_2)^a \cdot u_1^b \cdot u_2^c \in \mathbb{Z}_2[u_1, u_2].$$

Lemma A.3 in Appendix A now shows $(q-1)+3 = 2+q$ is a power of 2 as required. □

3.4.4 *Proof of Theorem* 3.1.2. Theorem 3.1.2 now follows from Lemmas 3.3.1, 3.4.1, 3.4.2, and 3.4.3. □

§3.5 Some Low Dimensional Results

We now investigate some lower dimensional cases in the Lorentzian setting.

3.5.1 *Proof of Theorem* 3.1.3. Let R be an IP algebraic curvature tensor. First we assume $(p, q) = (1, 5)$. Then R defines an admissible map from $Gr_2(\mathbb{R}^{1,5})$ to $\mathfrak{so}(1,5)$. We use Lemma 2.1.4 to construct an admissible map $\tilde{R} : Gr_2(\mathbb{R}^{1,5}) \to \mathfrak{so}(6)$ of the same rank. Since $Gr_2(\mathbb{R}^{1,5}) = Gr_{(0,2)}(\mathbb{R}^{1,5}) \dot\sqcup Gr_{(1,1)}(\mathbb{R}^{1,5})$ and since $Gr_{(0,2)}(\mathbb{R}^{1,5})$ strongly deformation retracts to $Gr_2(\mathbb{R}^5)$, we have the following commutative diagram:

$$\begin{array}{ccc} Gr_2(\mathbb{R}^5) & \xrightarrow{R} & \mathfrak{so}(6) \\ \uparrow i & \circ & \uparrow \\ \mathbb{RP}^3 & \xrightarrow{=} & \mathbb{RP}^3 \end{array}$$

We adopt the notational conventions established in §2.4.1. We have $W_1 \oplus W_0 = 6 \cdot 1$ and $W_1 \otimes L \cong W_1$. We must rule out the possibilities of having $\dim W_1 = 6$ or of having $\dim W_1 = 4$.

Case 1. Suppose $\dim W_1 = 6$ and $\dim W_0 = 0$. Then we have $6 \cdot ([\mathbb{L}] - 1) = 0$ in $\widetilde{KO}(\mathbb{RP}^3)$. This implies 6 divides $2^{\phi(3)} = 4$, which is false.

Case 2. Suppose $\dim W_1 = 4$ and $\dim W_0 = 2$. We apply Corollary 3.2.4 to see that $w_1(W_1) = 0$. Since $\dim W_0 = 2$, $w(W_0) = 1 + bx^2 + cy$. We use the relation $w(W_1) = w(W_0)^{-1}$ to see that

$$w_1(W_1) = 0, w_2(W_1) = bx^2 + cy, w_3(W_1) = 0, \text{ and } w_4(W_1) = bx^4 + cy^2.$$

We apply Lemma 2.5.7 to see that

$$x^4 + x^2(bx^2 + cy) = (1 + b)x^4 + cx^2y = 0.$$

We apply Theorem 2.4.4 to see that $0 = w_4^\perp = x^4 + x^2y + y^2$, so we must have $(1 + b)x^4 + cx^2y = \varepsilon(x^4 + x^2y + y^2)$ for $\varepsilon = 0$ or 1. Thus $\varepsilon = 0$, $b = 1$, and $c = 0$. Consequently, we have:

$$\text{(3.5.1.a)} \qquad w(W_0) = 1 + x^2 \text{ and } w(W_1) = 1 + x^2 + x^4.$$

Since $w(W_0) \cdot w(W_1) = 1$, equation (3.5.1.a) implies x^6 belongs to the ideal generated by the elements $\{w_4^\perp = x^4 + x^2y + y^2, w_5^\perp = x^5 + xy^2, w_6^\perp = x^6 + x^4y + y^3\}$. So we must be able to express x^6 as a nontrivial linear combination of $w_4^\perp$, $w_5^\perp$ and $w_6^\perp$, but this is not possible and hence proves assertion (1).

Next we assume $(p, q) = (1, 9)$. Then R defines an admissible map from $Gr_2(\mathbb{R}^{1,9})$ to $\mathfrak{so}(1, 9)$. We use Lemma 2.1.4 to construct an admissible map $\tilde{R}$ from $Gr_2(\mathbb{R}^{1,9})$ to $\mathfrak{so}(10)$ of the same rank. Since $i : \mathbb{RP}^8 \to Gr_{(1,1)}(\mathbb{R}^{1,9})$ and since $Gr_2(\mathbb{R}^{1,9}) = Gr_{(0,2)}(\mathbb{R}^{1,9}) \sqcup Gr_{(1,1)}(\mathbb{R}^{1,9})$, we have the following commutative diagram:

$$\begin{array}{ccc} Gr_{(1,1)}(\mathbb{R}^{1,9}) & \xrightarrow{R} & \mathfrak{so}(10) \\ \uparrow i & \circ & \uparrow \\ \mathbb{RP}^8 & \xrightarrow{=} & \mathbb{RP}^8 \end{array}$$

As before, we let W_i be the associated vector bundles over $Gr_2(\mathbb{R}^{1,9})$ and hence restrict to define vector bundles U_i over $\mathbb{RP}^8$. Furthermore, we have $U_1 \oplus U_0 = 10 \cdot 1$ and $U_1 \otimes \mathbb{L} \cong U_1$. We must rule out the possibilities of having $\dim U_1 = 10$, having $\dim U_1 = 8$, having $\dim U_1 = 6$, and having $\dim U_1 = 4$. We use Lemma 3.2.2 to see that $2a_1 \equiv \dim U_1 \mod \rho(8) = 16$, so $a_1 \equiv \frac{1}{2}\dim U_1 \mod 8$.

Case 1. Suppose $\dim U_1 = 10$ and $\dim U_0 = 0$. Then we have $10 \cdot ([\mathbb{L}] - 1) = 0$ in $\widetilde{KO}(\mathbb{RP}^8)$. This implies 10 divides $2^{\phi(8)} = 16$, which is false.

Case 2. Suppose $\dim U_1 = 8$ and $\dim U_0 = 2$. Either $a_1 = 4$ and $a_0 = 12$, or $a_1 = 12$ and $a_0 = 4$.

(2.1) If $a_1 = 4$ and $a_0 = 12$, then $w(U_0) = (1+u)^{12} = (1+u^4)^3 = 1 + u^4 + u^8$ in $H^*(\mathbb{RP}^8; \mathbb{Z}_2)$. But this contains u^4, which is false.

(2.2) If $a_1 = 12$ and $a_0 = 4$, then $w(U_0) = (1+u)^4 = 1 + u^4$ in $H^*(\mathbb{RP}^8; \mathbb{Z}_2)$. But this contains u^4, which is false.

Case 3. Suppose $\dim U_1 = 6$ and $\dim U_0 = 4$. Either $a_1 = 3$ and $a_0 = 13$, or $a_1 = 11$ and $a_0 = 5$.

(3.1) If $a_1 = 3$ and $a_0 = 13$, then $w(U_0) = (1+u)^{13}$ in $H^*(\mathbb{RP}^8; \mathbb{Z}_2)$. But this contains u^5, which is false.

(3.2) If $a_1 = 11$ and $a_0 = 5$, then $w(U_0) = (1+u)^5 = 1 + u + u^4 + u^5$ in $H^*(\mathbb{RP}^8; \mathbb{Z}_2)$. But this contains u^5, which is false.

Case 4. Suppose $\dim U_1 = 4$ and $\dim U_0 = 6$. Either $a_1 = 2$ and $a_0 = 14$, or $a_1 = 10$ and $a_0 = 6$.

(4.1) If $a_1 = 2$ and $a_0 = 14$, then $w(U_0) = (1+u)^{14} = (1+u^2)^7$ in $H^*(\mathbb{RP}^8; \mathbb{Z}_2)$. But this contains u^8, which is false.

(4.2) If $a_1 = 10$ and $a_0 = 6$, then $w(U_1) = (1+u)^{10} = (1+u^2)^5$ in $H^*(\mathbb{RP}^8; \mathbb{Z}_2)$. But this contains u^8, which is false. Our assertion now follows. $\square$

§3.6 Rank 4 IP Algebraic Curvature Tensors in the (2,2) Setting

In Theorem 1.3.2, Ivanov-Petrova exhibited a family of "exotic" rank 4 IP algebraic curvature tensor in the 4 dimensional Riemannian setting. By Theorem 1.3.3, this algebraic curvature tensor is not geometrically realizable by an IP metric. In this section, we give similar constructions of some rank 4 IP algebraic curvature tensor in the signature $(2,2)$ case.

3.6.1 *Proof of Theorem A* (3). Let $R \in \otimes^4(\mathbb{R}^4)$ be the "exotic" rank 4 tensor given in Theorem 1.3.2. We have that R satisfies the curvature identities relative to the standard real-valued positive definite metric g on $\mathbb{R}^4$. We complexify and extend the tensors R and g to the tensors R_c and g_c which are complex and multilinear. Let $\{e_i\}$ be the usual $\mathbb{R}$ basis for $\mathbb{R}^4$ and let

$$f_1 := \sqrt{-1}e_1,\ f_2 := \sqrt{-1}e_2,\ f_3 := e_3,\ f_4 := e_4$$

be a $\mathbb{R}$ basis for

$$H := \operatorname{span}_{\mathbb{R}}\{f_1, f_2, f_3, f_4\} \subset \mathbb{C}^4.$$

Let $\hat{R}$ and $\hat{g}$ be the restrictions of R_c and g_c to H. We note that $\hat{g}$ is a *real* metric of signature $(-,-,+,+)$ and that $\hat{R}$ is a *real* 4 tensor. We use Theorem 2.1.1 to see that $\hat{R}^2(\pi)$ has constant eigenvalues on $Gr_2(\mathbb{R}^{2,2})$ and hence $\hat{R}$ is IP. This constructs an IP algebraic curvature tensor of rank 4 for a metric of type $(2,2)$. We compute the nonvanishing components of $\hat{R}$ to be:

$$\hat{R}_{1212} = 2, \hat{R}_{1313} = -2, \hat{R}_{1414} = 1, \hat{R}_{2323} = 1, \hat{R}_{2424} = -2,$$

$$\hat{R}_{3434} = 2, \hat{R}_{1234} = 1, \hat{R}_{1324} = -1, \hat{R}_{1423} = -2.$$

Consequently, $\hat{R}$ is a rank 4 IP algebraic curvature tensor of signature $(2,2)$. $\square$

3.6.2 *Remark:* We can now give a more explicit construction of this tensor. Let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be the standard orthonormal basis for $\mathbb{R}^4$ relative to the metric g

of signature $(2,2)$, i.e. $g(\xi_1,\xi_1) = g(\xi_2,\xi_2) = -1$, $g(\xi_3,\xi_3) = g(\xi_4,\xi_4) = +1$, and $g(\xi_i,\xi_j) = 0$ for $i \neq j$. Let $\mathcal{J} : \mathbb{R}^{2,2} \to \mathbb{R}^{2,2}$ be the map

$$\mathcal{J}\xi_1 = \xi_4, \mathcal{J}\xi_2 = -\xi_3, \mathcal{J}\xi_3 = -\xi_2, \text{ and } \mathcal{J}\xi_4 = \xi_1.$$

Then $\mathcal{J}^2 = 1$, so $\mathcal{J}$ is a unitary paracomplex structure on $\mathbb{R}^{2,2}$. We define the algebraic curvature tensors R_0 and $R_{\mathcal{J}}$ of $\otimes^4(\mathbb{R}^{2,2})$ by

$$R_0(x,y)z := g(y,z)x - g(x,z)y, \text{ and}$$

$$R_{\mathcal{J}}(x,y)z := g(y,\mathcal{J}z)\mathcal{J}x - g(x,\mathcal{J}z)\mathcal{J}y - 2g(x,\mathcal{J}y)\mathcal{J}z.$$

Let a_1, a_2 be nonzero constants so $a_2 + 2a_1 = 0$. Then the algebraic curvature tensor $\hat{R} := -a_2 R_0 - a_1 R_{\mathcal{J}}$ is the rank 4 IP algebraic curvature tensor of signature $(2,2)$ given in Theorem 3.6.1.

CHAPTER IV

LORENTZIAN IP ALGEBRAIC CURVATURE TENSORS

In chapter IV, we prove Theorems B and C. We let R be a nontrivial Lorentzian IP algebraic curvature tensor. By Theorem A, R has rank 2 if $q \geq 9$. Here is a brief outline to chapter IV. In §4.1, we establish the trichotomy of rank 2 Lorentzian IP algebraic curvature tensors; this proves Theorem B. Lemma 4.1.2 is the primary technical tool we will use in the proof of Theorem B; the proof is a fairly straightforward computation. In §4.2, we assume R is mixed or null and use Lemma 2.6.1 to show that $q = 3$, $q = 4$, $q = 7$, or $q = 8$. Thus once again algebraic topology plays a crucial role in our analysis. This completes the proof of assertion (1) of Theorem C. In §4.3, we complete the proof of assertion (2) of Theorem C by ruling out the exceptional cases $q = 3$ and $q = 7$ (i.e. $m = 4$ or $m = 8$) if R is null. In the proof of Lemma 2.6.1, we constructed a line bundle $\tilde{L}$; if $\tilde{L}$ was trivial, then $q = 4$ or $q = 8$ while if $\tilde{L}$ was nontrivial, then $q = 3$ or $q = 7$. Thus to show $q = 4$ or $q = 8$, it suffices to prove that the line bundle $\tilde{L}$ constructed in the proof of Lemma 2.6.1 is the trivial line bundle. This is done by constructing a "universal axis".

§4.1 The Trichotomy of Lorentzian IP Algebraic Curvature Tensors

We now begin our preparations for the proof of Theorem B. We first establish some notational conventions.

4.1.1 *Notational conventions.* Let $T \in \mathfrak{so}(1, q)$, then $(\operatorname{Ker} T)^{\perp} = \operatorname{Range} T$. We set:

$W_1(T) := (\operatorname{Ker} T)^\perp = \operatorname{Range} T.$

$\mathfrak{so}_2(n) := \{T \in \mathfrak{so}(n) : \operatorname{rank}(T) = 2\}.$

$\mathfrak{so}_2(1,q) := \{T \in \mathfrak{so}(1,q) : \operatorname{rank}(T) = 2\}.$

$\mathfrak{so}_2^{\mathcal{E}}(1,q) := \{T \in \mathfrak{so}_2(1,q) : \operatorname{Spec}(T) \neq \{0\}\}.$

$\mathfrak{so}_2^{\mathcal{N}}(1,q) := \{T \in \mathfrak{so}_2(1,q) : \operatorname{Spec}(T) = \{0\}\}.$

It is clear that $\mathfrak{so}_2(1,q) = \mathfrak{so}_2^{\mathcal{E}}(1,q) \sqcup \mathfrak{so}_2^{\mathcal{N}}(1,q)$.

4.1.2 Lemma. *Let $T \in \mathfrak{so}_2(1,q)$ and let ξ be a unit timelike vector in $\mathbb{R}^{1,q}$. Then*

(1) *There exists an orthonormal basis $\{e_i\}$ for $\mathbb{R}^{1,q}$ so that $\xi = e_1$ and there exists real numbers λ_1 and t_1 with $\lambda_1^2 + t_1^2 \neq 0$, so that T has the form*

$$T = \begin{pmatrix} 0 & t_1 & 0 & 0 & \dots & 0 \\ t_1 & 0 & \lambda_1 & 0 & \dots & 0 \\ 0 & -\lambda_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Furthermore, the characteristic polynomial of T is given by

$$\det(\lambda - T) = \lambda^{q-1}[\lambda^2 + \lambda_1^2 - t_1^2].$$

(2) *If $t_1^2 = \lambda_1^2$, then we have:*

(2a) $T \in \mathfrak{so}_2^{\mathcal{N}}(1,q)$.

(2b) $T^2 \neq 0$ *but* $T^3 = 0$.

(2c) $\operatorname{Range} T = W_1(T) = \operatorname{Span}\{T\xi, T^2\xi\}$.

(2d) *$T\xi$ is spacelike, $T^2\xi$ is null, and $W_1(T)$ is a degenerate 2-plane.*

(3) *Let $T = \begin{pmatrix} 0 & \vec{t} \\ (\vec{t})^t & C \end{pmatrix} \in \mathfrak{so}_2^{\mathcal{N}}(1,q)$. We have $C \in \mathfrak{so}_2(q)$ and the eigenvalues of C are $\{0, \pm\sqrt{-1}|\vec{t}|\}$.*

(4) *If $t_1^2 \neq \lambda_1^2$, then we have:*

(4a) $T \in \mathfrak{so}_2^{\xi}(1,q)$.

(4b) $\mathrm{Range}\, T = W_1(T)$ *is a nondegenerate* 2*-plane.*

(4c) *If* $\lambda_1^2 > t_1^2$, *then* $W_1(T)$ *is spacelike.*

(4d) *If* $\lambda_1^2 < t_1^2$, *then* $W_1(T)$ *is of type* $(1,1)$.

(4e) $T|_{\mathrm{Range}\, T}$ *is invertible.*

Proof. Let ξ be a unit timelike vector in $\mathbb{R}^{1,q}$ and let $T \in \mathfrak{so}_2(1,q)$. We choose an orthonormal basis for $\mathbb{R}^{1,q}$ so that $\xi = (1,0,...,0)$. Relative to this basis, T has the form

$$T = \begin{pmatrix} 0 & \vec{x} \\ (\vec{x})^t & S \end{pmatrix}.$$

In this expression, $\vec{x} \in \mathbb{R}^q$, $(\vec{x})^t$ represents the transposed column vector, and S is a $q \times q$ skew-symmetric matrix. We further normalize the choice of basis for $\mathbb{R}^{1,q}$ so that S has the form

$$S = \begin{pmatrix} \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} & & & & & \\ & \ddots & & & & \\ & & \begin{pmatrix} 0 & \lambda_k \\ -\lambda_k & 0 \end{pmatrix} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

Since $\mathrm{rank}\, T = 2$, at most one of the blocks of $\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}$ can be nontrivial. Thus, we may assume T has the form

$$T = \begin{pmatrix} 0 & t_1 & t_2 & t_3 & \dots & t_q \\ t_1 & 0 & \lambda_1 & 0 & \dots & 0 \\ t_2 & -\lambda_1 & 0 & 0 & \dots & 0 \\ t_3 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_q & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Let $\vec{x} := (t_1,...,t_q)$. Since $T \neq 0$, we have $|\vec{x}|^2 + \lambda_1^2 \neq 0$.

Suppose that $\lambda_1 = 0$. We then have

$$T = \begin{pmatrix} 0 & \vec{x} \\ (\vec{x})^t & 0 \end{pmatrix} \text{ and } T^2 = \begin{pmatrix} |\vec{x}|^2 & 0 \\ 0 & (t_i t_j) \end{pmatrix}.$$

Since $|\vec{x}|^2 + \lambda_1^2 \neq 0$, $\vec{x} \neq \vec{0}$, so $|\vec{x}|^2$ is an eigenvalue of T^2. Thus, T is not nilpotent. Moreover, since $\vec{x} \neq \vec{0}$, we may further normalize the basis chosen for $\mathbb{R}^q$ so that $\vec{x} = (t_1, 0, ..., 0)$. Relative to this basis, T has the desired form given in (1).

Suppose that $\lambda_1 \neq 0$. If $t_i \neq 0$, for $i \geq 3$, then $\operatorname{rank} T \geq 3$, which is false. Let $\vec{x} = (t_1, t_2, 0, ..., 0)$. We then have:

$$\text{(4.1.2.a)} \qquad T = \begin{pmatrix} 0 & t_1 & t_2 & 0 & \dots & 0 \\ t_1 & 0 & \lambda_1 & 0 & \dots & 0 \\ t_2 & -\lambda_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

We further normalize the basis chosen for $\mathbb{R}^q$ so that $\vec{x} = (t_1, 0, ..., 0)$ and put T in the form given in (1).

We complete the proof of (1) by calculating the characteristic polynomial of T

$$\text{(4.1.2.b)} \qquad \begin{aligned} \det(\lambda - T) &= \lambda^{q-2} \det \begin{pmatrix} \lambda & -t_1 & 0 \\ -t_1 & \lambda & -\lambda_1 \\ 0 & \lambda_1 & \lambda \end{pmatrix} \\ &= \lambda^{q-2} \left[\lambda \det \begin{pmatrix} \lambda & -\lambda_1 \\ \lambda_1 & \lambda \end{pmatrix} + t_1 \det \begin{pmatrix} -t_1 & -\lambda_1 \\ 0 & \lambda \end{pmatrix} \right] \\ &= \lambda^{q-1} [\lambda^2 + \lambda_1^2 - t_1^2]. \end{aligned}$$

We now prove assertion (2). Suppose that $\lambda_1^2 = t_1^2$. We use equation (4.1.2.b) to see that $T \in \mathfrak{so}_2^{\mathcal{N}}(1, q)$. This proves assertion (2a). We use assertion (1) and the fact that $t_1 \neq 0$ to see that

$$\operatorname{Range} T = W_1(T) = \operatorname{Span}\left\{ \begin{pmatrix} 0 \\ t_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, t_1 \begin{pmatrix} t_1 \\ 0 \\ -\lambda_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\} = \operatorname{Span}\{T\xi, T^2\xi\}.$$

This proves assertion (2c). Since $T^3\xi = 0$, $g(T^2\xi, T^2\xi) = -g(T\xi, T^3\xi) = 0$. Also, $g(T\xi, T\xi) = t_1^2 > 0$; assertions (2b) and (2d) now follow.

We now prove assertion (3). By assertion (1), we can normalize the form of T by choosing a suitable orthonormal basis for $\mathbb{R}^q$. This means we can find $h \in \mathrm{O}(q)$ so that

$$\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} T \begin{pmatrix} 1 & 0 \\ 0 & h^{-1} \end{pmatrix} = \begin{pmatrix} 0 & |\vec{t}| & 0 & 0 & \dots & 0 \\ |\vec{t}| & 0 & \lambda_1 & 0 & \dots & 0 \\ 0 & -\lambda_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Since $T \in \mathfrak{so}_2^{\mathcal{N}}(1,q)$, by assertion (2) we have $\lambda_1^2 = |\vec{t}|^2$. Thus

$$hCh^{-1} = \begin{pmatrix} 0 & \lambda_1 & 0 & \dots & 0 \\ -\lambda_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Thus $\mathrm{Spec}(hCh^{-1}) = \{0, \pm\sqrt{-1}|\vec{t}|\}$; the same holds for C as required.

We now complete the proof of the lemma. Suppose that $\lambda_1^2 \neq t_1^2$. We use equation (4.1.2.b) to see that $T \in \mathfrak{so}_2^{\mathcal{E}}(1,q)$; this proves assertion (4a). Suppose $t_1 = 0$; since not both λ_1 and t_1 can vanish, $\lambda_1 \neq 0$. We use assertion (1) to see:

$$\mathrm{Range}\, T = W_1(T) = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ \lambda_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\lambda_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}.$$

These are orthogonal spacelike vectors and hence $W_1(T)$ is a spacelike 2-plane.

Suppose $t_1 \neq 0$, by assertion (1) we have

$$\mathrm{Range}\, T = W_1(T) = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ t_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} t_1 \\ 0 \\ -\lambda_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}.$$

These are orthogonal, linearly independent vectors. If $\lambda_1^2 > t_1^2$, then both vectors are spacelike, and $W_1(T)$ is a spacelike 2-plane. If $\lambda_1^2 < t_1^2$, then the first vector is spacelike and the second vector is timelike, and $W_1(T)$ is a 2-plane of type $(1,1)$; this proves assertions (4b), (4c) and (4d).

Since

$$T\begin{pmatrix}0\\1\\0\\0\\\vdots\\0\end{pmatrix}=\begin{pmatrix}t_1\\0\\-\lambda_1\\0\\\vdots\\0\end{pmatrix}\text{ and }T\begin{pmatrix}t_1\\0\\-\lambda_1\\0\\\vdots\\0\end{pmatrix}=\begin{pmatrix}0\\t_1^2-\lambda_1^2\\0\\0\\\vdots\\0\end{pmatrix};$$

assertion (4e) follows. $\square$

4.1.3 *Proof of Theorem B.* Let $\mathcal{S}$, $\mathcal{M}$, and $\mathcal{N}$ be the set of oriented spacelike, mixed, and null 2-planes in $\mathbb{R}^{1,q}$. We can decompose

$$Gr_2^+(\mathbb{R}^{1,q}) = \mathcal{S} \,\dot\sqcup\, \mathcal{M} \,\dot\sqcup\, \mathcal{N}.$$

Note that $\mathcal{S}$ and $\mathcal{M}$ are open subsets of $Gr_2^+(\mathbb{R}^{1,q})$ while $\mathcal{N}$ is a closed subset of $Gr_2^+(\mathbb{R}^{1,q})$.

Suppose that R is a rank 2 Lorentzian IP algebraic curvature tensor. If $R(\pi)$ is nilpotent for any π in $Gr_{(0,2)}^+(\mathbb{R}^{1,q})$, then we may use Lemma 4.1.2 to see that $W_1(R(\pi))$ is spanned by a spacelike vector and a null vector, hence is degenerate; conversely if $W_1(R(\pi))$ is degenerate, since we work in the Lorentzian setting, $W_1(R(\pi))$ is spanned by a spacelike vector and a null vector, then necessarily $R(\pi)$ is nilpotent. Since the eigenvalues of $R(\pi)$ are constant on $Gr_{(0,2)}^+(\mathbb{R}^{1,q})$, alternative (3) holds. Thus if alternative (3) fails, $R(\pi)$ is not nilpotent and the eigenvalues of $R(\pi)$ are nontrivial for any $\pi \in Gr_{(0,2)}^+(\mathbb{R}^{1,q})$. Since $Gr_{(0,2)}^+(\mathbb{R}^{1,q})$ is connected and since $Gr_2^+(\mathbb{R}^{1,q}) \setminus \mathcal{N} = \mathcal{S} \,\dot\sqcup\, \mathcal{M}$, this implies either that $W_1(R(\pi)) \in \mathcal{S}$ for every

$\pi \in Gr^+_{(0,2)}(\mathbb{R}^{1,q})$, in which alternative (1) holds, or that $W_1(R(\pi)) \in \mathcal{M}$ for every $\pi \in Gr^+_{(0,2)}(\mathbb{R}^{1,q})$, in which case alternative (2) holds. □

4.1.4 *Remark:* In the proof of Theorem B, we could also use the fact that the eigenvalues of $R(\pi)$ are $\{0, \pm\sqrt{t_1^2 - \lambda_1^2}\}$ on $Gr_2^+(\mathbb{R}^{1,q})$ to obtain the desired trichotomy. We further remark that cases (2) and (3) in Theorem B can only arise for special values of m; we can eliminate most values of m on an a-priori basis. This will be made clear in the next section.

§4.2 Most Lorentzian IP Algebraic Curvature Tensors are Spacelike

4.2.1 Theorem. *Let R be a rank 2 Lorentzian IP algebraic curvature tensor and let $q \geq 3$. If R is not spacelike, then $q = 3$, 4, 7, or 8.*

Proof. Suppose that R is not spacelike, by Theorem B, R is either mixed or null. Fix a unit timelike vector ξ and decompose $\mathbb{R}^{1,q} = \text{Span}\{\xi\} \oplus \xi^\perp$. Let $\{x, y\} \subset \xi^\perp$ be a spacelike orthogonal set with $x \neq 0$ and $y \neq 0$. Since $R(x, y)$ is skew, $R(x, y)\xi \in \xi^\perp$. If $R(x, y)\xi = 0$, then

$$0 = g(R(x, y)\xi, \alpha) = -g(\xi, R(x, y)\alpha) \text{ for all } \alpha$$

and hence $W_1(R(x, y)) \subset \xi^\perp = \mathbb{R}^q$ is spacelike which is false. We define a bilinear map Φ from $\mathbb{R}^q \times \mathbb{R}^q$ to $\mathbb{R}^{q+1} = \mathbb{R} \oplus \mathbb{R}^q$ by

$$\Phi(x, y) := g(x, y) \oplus R(x, y)\xi.$$

Suppose $x \neq 0$ and $y \neq 0$. If $\Phi(x, y) = 0$, then $g(x, y) = 0$ so $x \perp y$. Furthermore $R(x, y)\xi = 0$. Thus $R(\frac{x}{|x|}, \frac{y}{|y|})\xi = 0$ which is false as $\{\frac{x}{|x|}, \frac{y}{|y|}\}$ is a spacelike orthonormal subset of $\mathbb{R}^q$. Thus we may apply Lemma 2.6.1 to Φ and complete the proof. □

4.2.2 *Remark:* Theorem 4.2.1 completes the proof of Theorem C (1). Furthermore, if R is null, to prove Theorem (2), we need only to eliminate the cases $m = 4$ or $m = 8$. This however requires a surprisingly detailed investigation, so we shall begin our discussion in the next section.

§4.3 Rank 2 Null Lorentzian IP Algebraic Curvature Tensors

In §4.2, we showed that if R was a rank 2 null Lorentzian IP algebraic curvature tensor, then $q = 3$, 4, 7, or 8. This used Lemma 2.6.1. In the proof of Lemma 2.6.1, we constructed a line bundle $\tilde{L}$ and showed that if $\tilde{L}$ was trivial, then $q = 4$ or $q = 8$. We will complete the proof of Theorem C (2) by showing that $\tilde{L}$ is in fact trivial. This will be done by constructing an "universal axis".

We begin our observation with the following somewhat paradoxical observation that poses a significant epistemological difficulty.

4.3.1 Lemma. *If $N_1, N_2 \in \mathbb{R}^{1,q}$ are null vectors, then N_1 and N_2 are linearly dependent if and only if they are orthogonal.*

Proof. Let $N_1, N_2 \in \mathbb{R}^{1,q}$ be two nonzero null vectors. Let ξ be a unit timelike vector. We express $N_i = a_i\xi + s_i$ where $s_i \perp \xi$ are spacelike vectors. Since N_i are null, $-a_i^2 + |s_i|^2 = 0$. By replacing N_i by $\frac{N_i}{a_i}$, we may assume $a_i = 1$, and thus $N_i = \xi + \tilde{s}_i$ for $\tilde{s}_i \perp \xi$ a unit spacelike vector. Then $g(N_1, N_2) = g(\tilde{s}_1, \tilde{s}_2) - 1$. So $g(N_1, N_2) = 0$ if and only if $g(\tilde{s}_1, \tilde{s}_2) = 1$. Since $\tilde{s}_i$ are unit spacelike vectors and since the metric g is positive definite on $\xi^\perp$, by the Cauchy-Schwarz inequality, we have $g(\tilde{s}_1, \tilde{s}_2) = 1$ if and only if $\tilde{s}_1 = \tilde{s}_2$. $\square$

4.3.2 *Remark:* This is a crucial point at which we use the Lorentzian assumption, this fails for higher signatures if $p \geq 2$ and $q \geq 2$. It is also worth noticing that for any unit timelike vector ξ,

$$W_1(R(\pi)) = \mathrm{Span}\{R(\pi)\xi, R^2(\pi)\xi\}.$$

This observation played a crucial role in proving Lemma B.1 in Appendix B. Furthermore, we will show shortly in Lemma 4.3.4 that the null vectors $R^2(\pi)\xi$ are universal.

We now recall several results proved by Gilkey, Leahy and Sadofsky [48] in the Riemannian setting. Let g be the Euclidean metric on $\mathbb{R}^n$. If $T \in \mathfrak{so}(n)$, then $g(T\xi, \eta) = -g(\xi, T\eta)$. So we may define $\omega(T) \in \Lambda^2(\mathbb{R}^n)$ by

$$\omega(T)(\xi, \eta) := g(T\xi, \eta).$$

If $e = \{e_1, ..., e_n\}$ is an orthonormal basis for $\mathbb{R}^n$, we define

$$T^e_{ij}(z) := g(e_i, z)e_j - g(e_j, z)e_i.$$

Geometrically, this means T^e_{ij} is a rotation through an angle of $\frac{\pi}{2}$ in the oriented plane spanned by $\{e_i, e_j\}$. Note $\{T^e_{ij}\}_{i<j}$ is an orthonormal basis for $\mathfrak{so}(n)$ with respect to the Killing metric $(T_1, T_2) := -\frac{1}{2}\operatorname{Tr}(T_1T_2)$. The following lemma gives an alternative characterization of the conical subset $\mathfrak{so}_2(n)$ of $\mathfrak{so}(n)$.

4.3.3 Lemma.

(1) $\mathfrak{so}_2(n) = \{T \in \mathfrak{so}(n) : \omega(T) \wedge \omega(T) = 0\}$.

(2) *Let* $T : \mathbb{R}^2 \to \mathfrak{so}(n)$ *be a* 1-1 *linear map. Assume* $T(f) \in \mathfrak{so}_2(n)$ *for all* $f \neq 0$. *Then there exist a basis* $\{f_1, f_2\}$ *for* $\mathbb{R}^2$ *and an orthonormal basis* $e = \{e_1, ...e_n\}$ *for* $\mathbb{R}^n$ *so that* $T(f_1) = T^e_{12}$ *and that* $T(f_2) = T^e_{13}$.

Proof. We use the proofs of Lemma 2.1 and Lemma 2.2 given in Gilkey, Leahy and Sadofsky [48]. Let $\{\xi_a, \eta_a\}$ be an orthonormal basis for $W_1(T) := \operatorname{Range} T$ so that $T\xi_a = \lambda_a\eta_a$ and $T\eta_a = -\lambda_a\xi_a$ for $\lambda_a > 0$. We use the metric to identify $\mathbb{R}^n$ with the dual vector space $(\mathbb{R}^n)^*$. We then may express

$$\omega(T) = \textstyle\sum_{1\leq a\leq \operatorname{rank}(T)} \lambda_a\xi_a \wedge \eta_a.$$

Consequently, T has rank 2 if and only if $\omega(T) \wedge \omega(T) = 0$; this proves assertion (1). To prove assertion (2), we pull back the Killing form on $\mathfrak{so}(n)$ to define a positive definite inner product on $\mathbb{R}^2$ by

$$(f, \tilde{f}) := -\frac{1}{2} \operatorname{Tr}(T(f)T(\tilde{f})).$$

This allows us to assume T is an isometry. Let $\{f_1, f_2\}$ be an orthonormal basis for $\mathbb{R}^2$ with respect to this inner product, then $\{T(f_1), T(f_2)\}$ is an orthonormal set in $\mathfrak{so}(n)$. Choose a unit vector $e_1 \in \operatorname{Range}(T(f_1))$. Let $e_2 = T(f_1)e_1$. Then $\{e_1, e_2\}$ is an orthonormal set which we may complete to a basis e for $\mathbb{R}^n$. We will further normalize the choice of e_3 presently. We expand $T(f_2) = \sum_{k<l} a^e_{kl} T^e_{kl}$. Let $\xi = \xi_1 f_1 + \xi_2 f_2$. Since T is an isometry, we see that

$$\xi_1^2 + \xi_2^2 = |T(\xi)|^2 = (\xi_1 + a^e_{12}\xi_2)^2 + \xi_2^2 \textstyle\sum_{k<l,(k,l)\neq(1,2)} (a^e_{kl})^2. \tag{4.3.3.a}$$

We use equation (4.3.3.a) to see that $a^e_{12} = 0$ and that $\sum_{k<l,(k,l)\neq(1,2)} (a^e_{kl})^2 = 1$. Thus

$$T(\xi) = \xi_1 T^e_{12} + \xi_2 \textstyle\sum_{(k,l)\neq(1,2)} a^e_{kl} T^e_{kl}. \tag{4.3.3.b}$$

By assertion (1), $\omega(T(\xi)) \wedge \omega(T(\xi))(e_1, e_2, e_i, e_j) = 0$. Let $2 < i < j$, we compute:

$$\begin{aligned} 0 =& \omega(T(\xi)) \wedge \omega(T(\xi))(e_1, e_2, e_i, e_j) \\ =& g(T(\xi)e_1, e_2) g(T(\xi)e_i, e_j) - g(T(\xi)e_1, e_i) g(T(\xi)e_2, e_j) \\ &+ g(T(\xi)e_1, e_j) g(T(\xi)e_2, e_i) \\ =& \xi_1\xi_2 a^e_{ij} + \xi_2^2(-a^e_{1i}a^e_{2j} + a^e_{1j}a^e_{2i}). \end{aligned}$$

This shows that

$$a^e_{ij} = 0 \text{ and } a^e_{1i}a^e_{2j} = a^e_{1j}a^e_{2i}. \tag{4.3.3.c}$$

By equations (4.3.3.b) and (4.3.3.c), $T(f_2) = \sum_{2<i}(a^e_{1i}T^e_{1i} + a^e_{2i}T^e_{2i})$. By assumption $T(f_2) \neq 0$, so either $a^e_{1i} \neq 0$ or $a^e_{2i} \neq 0$ for some $i > 2$. By interchanging e_1 and e_2 if necessary, we may suppose $a^e_{1i} \neq 0$ for some $i > 2$. We replace e_3 by a suitable multiple of $\sum_{2<i} a^e_{1i}e_i$ to choose the basis so $a^e_{13} \neq 0$ and $a^e_{1i} = 0$ for $i > 3$. Then we have

$$T(f_2) = a^e_{13}T^e_{13} + \textstyle\sum_{2<i} a^e_{2i}T^e_{2i}.$$

We use equation (4.3.3.c) to see that $a^e_{13}a^e_{2i} = a^e_{1i}a^e_{23}$ for $i > 3$. Thus, $a^e_{1i} = 0$ implies $a^e_{2i} = 0$ for $i > 3$ and

$$T(f_2) = a^e_{13}T^e_{13} + a^e_{23}T^e_{23}.$$

Set $\tilde{e}_1 := a^e_{13}e_1 + a^e_{23}e_2 = T(f_2)e_3$. Since T is an isometry, $(a^e_{13})^2 + (a^e_{23})^2 = 1$. Set $\tilde{e}_2 := T(f_1)\tilde{e}_1$, $\tilde{e}_3 := T(f_2)\tilde{e}_1$, and complete the remaining basis vectors arbitrarily. It follows that

$$T(f_1)\tilde{e}_1 = \tilde{e}_2\ ,\ T(f_1)\tilde{e}_2 = -\tilde{e}_1;$$

$$T(f_2)\tilde{e}_1 = \tilde{e}_3\ ,\ T(f_2)\tilde{e}_3 = -\tilde{e}_1.$$

Relative to the basis $\tilde{e}$, we have $T(f_1) = T^{\tilde{e}}_{12}$ and $T(f_2) = T^{\tilde{e}}_{13}$ as desired. $\square$

We now return to the Lorentzian setting and continue with our preparations for the proof of Theorem C (2). Let R be a rank 2 null IP Lorentzian algebraic curvature tensor. Let $x \in \mathbb{R}^{1,q}$ be a nonnull vector. Let $\mathcal{H}$ be a maximal spacelike hyperplane orthogonal to x. If $0 \neq y, z \in \mathcal{H}$, by Lemma 4.1.2, $R(x,y)\xi$ and $R(x,z)\xi$ are nontrivial spacelike vectors. We introduce a new positive definite inner product $h = h_{x,\xi}$ on $\mathcal{H}$ by

$$h(y,z) := g(R(x,y)\xi, R(x,z)\xi).$$

4.3.4 Lemma. *Let $q \geq 3$. Let R be a rank 2 null IP Lorentzian algebraic curvature tensor. Let $x \in \mathbb{R}^{1,q}$ be a nonnull vector. Let $\mathcal{H}$ be a spacelike hyperplane* *perpendicular to x and let $h = h_{x,\xi}$ be the positive definite inner product on $\mathcal{H}$. There is a nonzero null vector N_R determined by x and ξ so that $R^2(x,y)\xi = h(y,y)N_R$, for all $0 \neq y \in \mathcal{H}$.*

Proof. We proceed as follows.

(1) Fix $y \in \mathcal{H}$ with $h(y,y) = 1$. Let $N_R := R^2(x,y)\xi$. Let $0 \neq w \in \mathcal{H}$. If y and w are linearly dependent, then $R^2(x,w)\xi = h(w,w)N_R$. We therefore assume y and w are linearly independent. Choose z so $\{y,z\}$ forms an orthonormal set with respect to the inner product h and so $w \in \operatorname{Span}\{y,z\}$. We then have $|R(x,y)\xi|^2 = 1$, $|R(x,z)\xi|^2 = 1$, and $R(x,y)\xi \perp R(x,z)\xi$.

(2) Let $R(x,y)\xi = \begin{pmatrix} 0 \\ \vec{t} \end{pmatrix}$ and $R(x,z)\xi = \begin{pmatrix} 0 \\ \vec{s} \end{pmatrix}$. Then

$$R(x,y) = \begin{pmatrix} 0 & \vec{t} \\ (\vec{t})^t & C_1 \end{pmatrix} \text{ and } R(x,z) = \begin{pmatrix} 0 & \vec{s} \\ (\vec{s})^t & C_2 \end{pmatrix}.$$

(3) For $0 \leq \theta \leq \pi$, let

$$\begin{aligned} R(\pi(\theta)) :=& R(x, \cos(\theta)y + \sin(\theta)z) \\ =& \begin{pmatrix} 0 & \cos(\theta)\vec{t} + \sin(\theta)\vec{s} \\ \cos(\theta)\vec{t} + \sin(\theta)\vec{s} & \cos(\theta)C_1 + \sin(\theta)C_2 \end{pmatrix} \in \mathfrak{so}_2^{\mathcal{N}}(1,q). \end{aligned}$$

Since $h(y,y) = h(z,z) = 1$ and $h(y,z) = 0$, $\{\vec{t}, \vec{s}\}$ is an orthonormal spacelike set. Thus we have $|\cos(\theta)\vec{t} + \sin(\theta)\vec{s}| = 1$. We apply Lemma 4.1.2 (3) to see that $(\cos(\theta)C_1 + \sin(\theta)C_2) \in \mathfrak{so}_2(q)$ for $0 \leq \theta \leq \pi$ and that the eigenvalues of $(\cos(\theta)C_1 + \sin(\theta)C_2)$ are $\{0, \pm\sqrt{-1}\}$. Thus, by Lemma 4.3.3, we can choose a basis e for $\mathbb{R}^{m-1}$ so that

$$C_1 = T_{12}^e = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and

$$C_2 = T^e_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(4) We use equation (4.1.2.a) to choose vector $\vec{t} = (t_1, t_2, 0, 0, ..., 0)$ and vector $\vec{s} = (s_1, 0, s_3, 0, ..., 0)$ so that $t_1^2 + t_2^2 = s_1^2 + s_3^2 = 1$, that

$$R(x,y) = \begin{pmatrix} 0 & t_1 & t_2 & 0 & 0 & \dots & 0 \\ t_1 & 0 & 1 & 0 & 0 & \dots & 0 \\ t_2 & -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

and that

$$R(x,z) = \begin{pmatrix} 0 & s_1 & 0 & s_3 & 0 & \dots & 0 \\ s_1 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ s_3 & -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

(5) Since $R(x,y)\xi = (0, t_1, t_2, 0, ..., 0)^t \perp (0, s_1, 0, s_3, 0, ..., 0)^t = R(x,z)\xi$, we see that $t_1 s_1 = 0$.

(6) By Lemma B.1 in Appendix B, we have $\dim[W_1(R(x,y)) + W_1(R(x,z))] = 3$. Thus,

$$\begin{aligned} &W_1(R(x,y)) + W_1(R(x,z)) \\ &= \mathrm{Span}\{(s_1, 0, 0, -1, \vec{0})^t, (t_1, 0, -1, 0, \vec{0})^t, (s_3, 1, 0, 0, \vec{0})^t\}. \end{aligned}$$

Since the vector $(t_2, 1, 0, ..., 0)^t \in W_1(R(x,y)) + W_1(R(x,z))$, we must have $s_3 = t_2$. From the relation $t_1^2 + t_2^2 = s_1^2 + s_3^2 = 1$, it follows that $t_1^2 = s_1^2$. Since $t_1 s_1 = 0$, we have $t_1 = s_1 = 0$. Also, we have $s_3 = t_2 = \pm 1$.

(7) Without loss of generality, we may rescale w so that $h(w,w) = 1$. Since w belongs to $\mathrm{Span}\{y,z\}$, we may write $w = \cos(\theta)y + \sin(\theta)z$. Thus

$$\begin{aligned} R^2(x,w)\xi &= \cos^2(\theta)(1,t_2,0,0,\vec{0})^t + \sin^2(\theta)(1,s_3,0,0,\vec{0})^t \\ &\quad + \cos(\theta)\sin(\theta)R(x,z)(0,0,t_2,0,\vec{0})^t \\ &\quad + \cos(\theta)\sin(\theta)R(x,y)(0,0,0,s_3,\vec{0})^t \\ &= (\cos^2(\theta)+\sin^2(\theta))(1,t_2,0,0,\vec{0})^t \\ &= R^2(x,y)\xi. \quad \square \end{aligned}$$

We use Lemma 4.3.4 to establish a crucial result.

4.3.5 Lemma. *Let $q \geq 3$. Let R be a rank 2 null IP Lorentzian algebraic curvature tensor. Let $\mathcal{H}$ be any spacelike hyperplane of dimension q. Then we have that $\bigcap_{\pi\in Gr_2^+(\mathcal{H})} W_1(R(\pi))$ is a nontrivial 1 dimensional null line.*

Proof. Let $\xi \in \mathcal{H}^\perp$ be a unit timelike vector. Let $\pi_1, \pi_2 \in \mathcal{H}$. Fix π_1 and we let $N_R := R^2(\pi_1)\xi$. Suppose that $\pi_1 \cap \pi_2 \neq \{0\}$. We can choose bases so that $\pi_1 = \mathrm{Span}\{x,y\}$ and that $\pi_2 = \mathrm{Span}\{x,z\}$. We use Lemma 4.3.4 to see $R^2(\pi_2)\xi$ is a nontrivial multiple of N_R. If $\pi_1 \cap \pi_2 = \{0\}$, then we can choose bases so $\pi_1 = \mathrm{Span}\{x_1,y_1\}$ and $\pi_2 = \mathrm{Span}\{x_2,y_2\}$. Let $\pi_3 := \mathrm{Span}\{x_1,x_2\}$. Again we use Lemma B.1 in Appendix B to see that $\dim(\pi_1\cap\pi_3) = \dim(\pi_2\cap\pi_3) = 1$. So the nonzero null vectors $R^2(\pi_1)\xi$ and $R^2(\pi_2)\xi$ are both nontrivial multiples of $R^2(\pi_3)\xi$. $\square$

4.3.6 *Remark:* We may call such N_R a universal null vector for R. We now return to complete the proof of Theorem C (2).

4.3.7 *Proof of Theorem C* (2). Let R be a rank 2 null Lorentzian algebraic curvature tensor with $q \geq 3$. We have shown in Theorem 4.2.1 that $q = 3, 4, 7$ or 8. We now use Lemma 4.3.5 to eliminate the cases $q = 3$ or $q = 7$ as follows. Choose a nonzero null vector $N_R \in \mathbb{R}^{1,q}$ so $N_R \in \bigcap_{\pi \in Gr_2^+(\mathbb{R}^q)} W_1(R(\pi))$. By Lemma 4.1.2 (2), N_R is a nontrivial multiple of $R^2(x,y)\xi$ for all pairs of linearly independent spacelike vectors $\{x,y\} \subset \xi^\perp \cong \mathbb{R}^q$. So N_R is perpendicular to $R(x,y)\xi$ for all pairs of linearly independent spacelike vectors $\{x,y\} \subset \xi^\perp \cong \mathbb{R}^q$. This implies $0 \oplus N_R \in \mathbb{R}^{q+1}$ is perpendicular to the range of $\Phi(x,y)$ for all $x \in S^{q-1}$. Thus $0 \oplus N_R$ projects to define a nonvanishing global section to the quotient line bundle $\tilde{L}$ over $\mathbb{RP}^{q-1}$. Hence $\tilde{L} = 1$ and thus $q = 4$ or $q = 8$. $\square$

CHAPTER V

CLASSIFICATION OF RANK TWO SPACELIKE IP ALGEBRAIC CURVATURE TENSORS

In chapter V, we prove Theorem D by classifying rank 2 spacelike or timelike IP algebraic curvature tensor for $q = 6$ or $q \geq 9$. We complete the proof of Theorem D by showing that any rank 2 spacelike or timelike IP algebraic curvature tensor has the form $R = R_{C,\phi}$ for an admissible pair (C, ϕ). Our crucial task is to build the map ϕ. If x is a unit spacelike vector, we will show that $\cap_{y \perp x, |y|=1} W_1(R(x, y)) = \mathcal{L}(x)$ is a line. This defines a line bundle $\mathcal{L}$ over the set of unit spacelike vectors. We will show that this line bundle is trivial and choose a global unit section ϕ to $\mathcal{L}$. We will then show that ϕ extends to a linear map of $\mathbb{R}^{p,q}$ that is an isometry if R is spacelike and a para-isometry if R is timelike. It will then follow that $R = R_{C,\phi}$ for some $C \neq 0$. We will use the Bianchi identities to show $\phi^2 = \mathrm{id}$ if R is spacelike and that $\phi^2 = -\mathrm{id}$ if R is timelike. Here is a brief outline to chapter V. In §5.1, we begin our study with some algebraic preliminaries. In §5.2, we first construct the line bundle $\mathcal{L}$, then show $\mathcal{L}$ is trivial. We subsequently construct ϕ and show it has the required properties. In §5.3, we prove Theorem D and Theorem G (1).

§5.1 Linear Algebra Technical Lemmas

The following technical lemma is needed to simplify some later calculations.

5.1.1 Lemma.

(1) *Let π_i be spacelike p-planes for $i = 1, 2$. There exist orthonormal bases $\{u_\nu\}$ and $\{v_\nu\}$ for π_1 and π_2 respectively so that $g(u_\nu, v_\mu) = 0$ for $\nu \neq \mu$.*

(2) *Let T be a rank 2 spacelike IP algebraic curvature tensor with eigenvalues $\{0, \pm\sqrt{-1}\}$. Then T induces a unitary almost complex structure on $W_1(T)$.*

Proof. (1) Let ρ_i be orthogonal projection on π_i for $i = 1, 2$. If $\{v_\nu\}$ is an orthonormal basis for π_2, then for any v, we have $\rho_2(v) = \sum_\nu g(v, v_\nu)v_\nu$. Define a symmetric bilinear form on π_1 by

$$h(\xi, \eta) := g(\rho_2(\xi), \rho_2(\eta)).$$

Since π_1 is spacelike, the metric on π_1 is positive definite. We can diagonalize h with respect to this metric to find an orthonormal basis $\{u_\nu\}$ for π_1 so that $h(u_\nu, u_\mu) = 0$ for $\nu \neq \mu$. Thus $g(\rho_2(u_\nu), \rho_2(u_\mu)) = 0$ for $\nu \neq \mu$, and $g(\rho_2(u_\nu), \rho_2(u_\nu)) = \lambda_\nu$. Let $\mathcal{I}$ be the set of all ν so that $\rho_2(u_\nu) \neq 0$. For $\nu \in \mathcal{I}$, let $v_\nu := \frac{\rho_2(u_\nu)}{|\rho_2(u_\nu)|}$. Note that if $\nu \notin \mathcal{I}$ then we have $\rho_2(u_\nu) = 0$, i.e. $\lambda_\nu = 0$. We extend $\{v_\nu\}_{\nu\in\mathcal{I}}$ to a full orthonormal basis for π_2. We check that the bases $\{u_\nu\}$ and $\{v_\mu\}$ satisfy the conclusions of (1) by checking

$$g(u_\nu, v_\mu) = g(\rho_2(u_\nu), v_\mu) = g(\sqrt{\lambda_\nu}v_\nu, v_\mu) = 0 \text{ for } \mu \neq \nu.$$

(2) Assume that T is a rank 2 spacelike IP algebraic curvature tensor. Since $W_1(T)$ is spacelike, we may decompose $\mathbb{R}^{p,q} = W_1(T) \oplus W_1(T)^\perp$. Since $W_1(T) = \text{Range}(T)$, $W_1(T)$ is preserved by T. As T is skew-symmetric, T vanishes on $W_1(T)^\perp$. The eigenvalues of T^2 are $\{0, -1\}$. Since the eigenvalue -1 has multiplicity 2, $T^2 = -1$ on $W_1(T)$. Thus T defines a unitary almost complex structure on $W_1(T)$. $\square$

5.1.2 Lemma. *Let R be a rank 2 spacelike IP algebraic curvature tensor. Let $\{x, y, z\}$ be an orthonormal set of spacelike vectors. Then there exists an orthonormal set of spacelike vectors $\{\alpha, \beta, \gamma\}$ so that*

$$W_1(R(x,y)) = \mathrm{Span}\{\alpha, \beta\} \text{ and } W_1(R(x,z)) = \mathrm{Span}\{\alpha, \gamma\}.$$

Proof. We adopt the notation used to prove Lemma B.1 in Appendix B to see that $\dim[W_1(T_1) \cap W_1(T_2)] = 1$. Let $\alpha \in W_1(T_1) \cap W_1(T_2)$ be a unit spacelike vector. By rescaling we may assume R has eigenvalues $\{0, \pm\sqrt{-1}\}$. Let $\beta := T_1\alpha$ and let $\gamma := T_2\alpha$. Then we have that $\{\alpha, \beta, \gamma\}$ are linearly independent. Furthermore, we see that $W_1(T_1) + W_1(T_2) = \mathrm{Span}\{\alpha, \beta, \gamma\}$, that $\alpha \perp \beta$, and that $\alpha \perp \gamma$. We compute:

$$\begin{aligned} T_2\beta &= g(T_2\beta, \alpha)\alpha + g(T_2\beta, \gamma)\gamma = -g(\beta, T_2\alpha)\alpha - g(\beta, T_2\gamma)\gamma \\ &= -g(\beta, \gamma)\alpha + g(\beta, \alpha)\gamma = -g(\beta, \gamma)\alpha. \\ T_1\gamma &= g(T_1\gamma, \alpha)\alpha + g(T_1\gamma, \beta)\beta = -g(\gamma, T_1\alpha)\alpha - g(\gamma, T_1\beta)\beta \\ &= -g(\gamma, \beta)\alpha + g(\gamma, \alpha)\beta = -g(\beta, \gamma)\alpha. \end{aligned}$$

For $\theta \in [0, \pi]$, let $\pi(\theta) := \mathrm{Span}\{\alpha, \cos(\theta)\beta + \sin(\theta)\gamma\}$. Then

$$\begin{aligned} R(\pi(\theta))\alpha &= (\cos(\theta)T_1 + \sin(\theta)T_2)\alpha = \cos(\theta)\beta + \sin(\theta)\gamma, \\ R(\pi(\theta))\beta &= (\cos(\theta)T_1 + \sin(\theta)T_2)\beta = -\{\cos(\theta) + g(\beta,\gamma)\sin(\theta)\}\alpha, \\ R(\pi(\theta))\gamma &= (\cos(\theta)T_1 + \sin(\theta)T_2)\gamma = -\{g(\beta,\gamma)\cos(\theta) + \sin(\theta)\}\alpha. \end{aligned}$$

Thus relative to the basis $\{\alpha, \beta, \gamma\}$, $R(\pi(\theta))$ has the form:

$$R(\pi(\theta)) = \begin{pmatrix} 0 & -\{\cos(\theta) + g(\beta,\gamma)\sin(\theta)\} & -\{g(\beta,\gamma)\cos(\theta) + \sin(\theta)\} \\ \cos(\theta) & 0 & 0 \\ \sin(\theta) & 0 & 0 \end{pmatrix}.$$

Let $\chi_\theta(\lambda)$ be the characteristic polynomial of $R(\pi(\theta))$ acting on the space spanned by $\{\alpha, \beta, \gamma\}$; this space is $R(\pi(\theta))$ invariant and containing the range of $R(\pi(\theta))$.

Thus since R is IP, $\chi_\theta(\lambda)$ is independent of θ and $\chi_\theta(\lambda)$ must have roots $\{0, \pm\sqrt{-1}\}$ for all $\theta \in [0, \pi]$. We compute:

$$\begin{aligned}\chi_\theta(\lambda) &= \det \begin{pmatrix} \lambda & \{\cos(\theta) + g(\beta,\gamma)\sin(\theta)\} & \{g(\beta,\gamma)\cos(\theta) + \sin(\theta)\} \\ -\cos(\theta) & \lambda & 0 \\ -\sin(\theta) & 0 & \lambda \end{pmatrix} \\ &= \lambda^3 + \lambda\{1 + 2g(\beta,\gamma)\sin(\theta)\cos(\theta)\}.\end{aligned}$$

Since $\operatorname{Spec}(R(\pi(\theta)))$ is independent of θ by assumption, $2g(\beta,\gamma)\sin(\theta)\cos(\theta) \equiv 0$ for all $\theta \in [0, \pi]$. Thus we must have $g(\beta,\gamma) = 0$. Our assertion now follows. $\square$

5.1.3 Corollary. *Let R be a rank 2 spacelike IP algebraic curvature tensor. Fix a unit spacelike vector x. Then for any spacelike vector $y, z \perp x$, we have*

$$\operatorname{Tr}(R(x,y)R(x,z)) = -2g(y,z).$$

Proof. Let $\{x, y, z\}$ be an orthonormal set. We adopt the notation used to prove Lemma 5.1.2 to see that

$$T_1\alpha = \beta,\ T_2\alpha = \gamma,\ T_1\beta = -\alpha,\ T_2\beta = 0,\ T_1\gamma = 0,\ T_2\gamma = -\alpha;$$

$$T_1T_2\alpha = T_1\gamma = 0,\ T_1T_2\beta = 0,\ \text{and}\ \ T_1T_2\gamma = T_1(-\alpha) = -\beta.$$

Hence $\operatorname{Tr}(T_1T_2) = 0$. Thus $\operatorname{Tr}(R(x,y)R(x,z)) = -2g(y,z)$ in this special situation. More generally, we use multilinearity to see that for any spacelike vectors $y, z \perp x$, we have $\operatorname{Tr}(R(x,y)R(x,z)) = -2g(y,z)$. $\square$

§5.2 The "Common Axis" Lemma and Its Consequences

We assume $q \geq 6$ henceforth. In this section, we prove the "common axis" lemma and then construct admissible pair (C, ϕ) so that $R = R_{C,\phi}$. We introduce the following definition.

5.2.1 *Definition.* If H is a linear subspace of $\mathbb{R}^{p,q}$, let $S(H) := \{v \in H : |v| = 1\}$ and let $\mathbb{P}(H) := S(H)/\mathbb{Z}_2$ be the associated projective space. If $x \in S(\mathbb{R}^{p,q})$, we set

$$P(x) := \{\pi \in Gr^+_{(0,2)}(\mathbb{R}^{p,q}) : x \in \pi\} \text{ and } \mathcal{L}(x) := \cap_{\pi \in P(x)} W_1(R(\pi)).$$

We now show that $\mathcal{L}(x)$ is 1 dimensional; this line will be called the "common axis" defined by x.

5.2.2 Lemma. *Let R be a rank 2 spacelike (or timelike) IP algebraic curvature tensor. Suppose that $q \geq 6$. If $x \in S(\mathbb{R}^{p,q})$, then* $\dim \mathcal{L}(x) = 1$.

Proof. Suppose R is spacelike. Fix $x \in S(\mathbb{R}^{p,q})$. Let H be any spacelike subspace of $\mathbb{R}^{p,q}$ which contains x with $\dim H \geq 3$. Let

$$P(x,H) := \{\pi \in Gr_2^+(H) : x \in H\} \subset P(x), \text{ and}$$
$$\mathcal{L}(x,H) := \cap_{\pi \in P(x,H)} W_1(R(\pi)) \supseteq \mathcal{L}(x).$$

We first establish that $\dim \mathcal{L}(x,H) = 1$. We use Lemma B.1 in Appendix B to see that $\dim \mathcal{L}(x,H) \leq 1$. Suppose that $\dim \mathcal{L}(x,H) = 0$; we shall argue for a contradiction. We may suppose without loss of generality that H is maximal so that $\dim H = q \geq 6$. Let $\tilde{H} := H \cap x^\perp$. If $y \in S(\tilde{H})$, let

$$\pi(y) := \operatorname{span}\{x, y\} \text{ and } \sigma(y) := W_1(R(\pi(y))).$$

Let $\tilde{y} \in S(\tilde{H})$. By Lemma B.1 in Appendix B, if $y \neq \pm y$, then $L(y, \tilde{y}) := \sigma(y) \cap \sigma(\tilde{y})$ is a line. Since $\dim \mathcal{L}(x,H) = 0$, there must exist unit vectors $\{y_i\}$ in $\tilde{H}$ so that

$$\sigma(y_1) \cap \sigma(y_2) \cap \sigma(y_3) = \{0\}.$$

Let $E := \sigma(y_1) + \sigma(y_2)$. We use Lemma B.1 in Appendix B to see that E is a 3-plane. Moreover, since $L(y_1, y_3) = \sigma(y_1) \cap \sigma(y_3)$ and $L(y_2, y_3) = \sigma(y_2) \cap \sigma(y_3)$, $L(y_1, y_3) \cap L(y_2, y_3) = \sigma(y_1) \cap \sigma(y_2) \cap \sigma(y_3) = \{0\}$, so $L(y_1, y_3)$ and $L(y_2, y_3)$ are different lines contained in $\sigma(y_3)$. Thus

$$\sigma(y_3) = \operatorname{span}\{L(y_1, y_3), L(y_2, y_3)\} \subset E.$$

Let $y \in S(\tilde{H})$ and suppose $y \neq \pm y_i$ for $i = 1, 2, 3$. If $L(y_1, y) = L(y_2, y) = L(y_3, y)$, then these three lines coincide and

$$L(y_3, y) \subset \sigma(y_1) \cap \sigma(y_2) \cap \sigma(y_3) = \{0\} \text{ which is false.}$$

Thus at least two of these lines are different so

$$\sigma(y) = \operatorname{span}\{L(y_1, y), L(y_2, y), L(y_3, y)\} \subset \sigma(y_1) + \sigma(y_2) + \sigma(y_3) = E.$$

Now we have a well defined continuous map $\sigma : \mathbb{P}(\tilde{H}) \to Gr_2(E)$ which is injective. This is impossible for dimensional reasons; because $q \geq 6$ we have

$$\dim \mathbb{P}(\tilde{H}) = q - 2 > \dim Gr_2(E) = 2.$$

The argument given above shows that $\dim \mathcal{L}(x, H) = 1$. To complete the proof, we need only show that $\dim \mathcal{L}(x) = 1$. Suppose that $\dim \mathcal{L}(x) = 0$. We must then have planes $\pi_i \in P(x)$ so that

$$\text{(5.2.2.a)} \qquad W_1(R(\pi_1)) \cap W_1(R(\pi_2)) \cap W_1(R(\pi_3)) = \{0\}.$$

Let $\{x, y_i\}$ be an orthonormal basis for π_i. Let $H(x)$ be a maximal spacelike subspace containing x. Because $q = \dim H \geq 6$, we can find an orthonormal set

$$\{z_1, z_2\} \subset H(x) \cap x^\perp \cap y_1^\perp \cap y_2^\perp \cap y_3^\perp.$$

Let $H_i := \operatorname{span}\{x, y_i, z_1, z_2\}$. This is a spacelike set for $i = 1, 2, 3$. Consequently the argument given above shows $W_1(R(x, z_1)) \cap W_1(R(x, z_2)) \subset W_1(R(x, y_i))$ for $i = 1, 2, 3$. This contradicts equation (5.2.2.a) and our assertion follows. We argue similarly if R is timelike. $\square$

Let R be a rank 2 spacelike IP algebraic curvature tensor. Since $\mathcal{L}(x) = \mathcal{L}(-x)$, we may use Lemma 5.2.2 to define a map

$$\mathcal{L} : Gr_{(0,1)}(\mathbb{R}^{p,q}) \to Gr_{(0,1)}(\mathbb{R}^{p,q})$$

from the set of spacelike lines to the set of spacelike lines; we showed in Lemma 2.6.2 that this map is continuous.

5.2.3 Lemma. *Let $q = 6$ or $q \geq 9$. Let R be a rank 2 spacelike IP algebraic curvature tensor. Let $\tau_i \in Gr_{(0,1)}(\mathbb{R}^{p,q})$ for $i = 1, 2$.*

(1) *If $\mathcal{L}(\tau_1) = \mathcal{L}(\tau_2)$, then $\tau_1 = \tau_2$.*

(2) *If $\tau_1 \perp \tau_2$, then $\mathcal{L}(\tau_1) \perp \mathcal{L}(\tau_2)$. Furthermore, if x_i are unit spacelike vectors spanning the lines τ_i, then $W_1(R(x_1, x_2)) = \mathcal{L}(\tau_1) \oplus \mathcal{L}(\tau_2)$ is an orthogonal direct sum decomposition.*

Proof. Let τ_1 and τ_2 be distinct lines. To establish assertion (1), we suppose that $\mathcal{L}(\tau_1) = \mathcal{L}(\tau_2)$ and argue for a contradiction. We first show in Step 1 that for any $\tau \in Gr_{(0,1)}(\mathbb{R}^{p,q})$, $\mathcal{L}(\tau) = \mathcal{L}(\tau_1)$; let $\mathcal{L}$ be this "universal common axis". We then use topological methods to derive the desired contradiction in Step 2.

Step 1. Let x_i be unit spacelike vectors spanning the lines $\tau_i \in Gr_{(0,1)}(\mathbb{R}^{p,q})$ for $i = 1, 2, 3$. Suppose $\mathcal{L}(\tau_1) = \mathcal{L}(\tau_2)$. We wish to show $\mathcal{L}(\tau_3) = \mathcal{L}(\tau_1)$ so $\mathcal{L}$ is the "universal common axis". Because $q \geq 5$, we may choose an orthonormal spacelike subset $\{y_1, y_2\}$ of $\mathbb{R}^{p,q}$ so that $y_i \perp x_j$ for $i = 1, 2$ and $j = 1, 2, 3$. We use Lemma B.1 in Appendix B to see that

$$W_1(R(x_1, y_1)) \cap W_1(R(x_1, y_2)) = \mathcal{L}(x_1) = \mathcal{L}(x_2) = W_1(R(x_2, y_1)) \cap W_1(R(x_2, y_2).$$

We use Lemma B.2 in Appendix B to see that

$$\mathcal{L}(x_1) \subset W_1(R(x_1, y_i)) \cap W_1(R(x_2, y_i)) = \mathcal{L}(y_i)$$

so $\mathcal{L}(y_i) = \mathcal{L}(x_1)$ for $i = 1, 2$. Thus $\mathcal{L}(x_1) \subset W_1(R(x_3, y_i))$ for $i = 1, 2$. Since $\{x_3, y_1, y_2\}$ is an orthonormal set, we see that $\mathcal{L}$ is a "common axis" by checking:

$$\mathcal{L}(x_1) \subset W_1(R(x_3, y_1)) \cap W_1(R(x_3, y_2)) = \mathcal{L}(x_3).$$

Step 2. Let H be a maximal spacelike subspace of $\mathbb{R}^{p,q}$; we then have that $H^\perp$ is a maximal timelike subspace and that $\mathbb{R}^{p,q} = H \oplus H^\perp$. Let ρ_H be orthogonal projection of $\mathbb{R}^{p,q}$ onto H. Let $\{y_1, y_2\}$ be any orthonormal subset of H and suppose that $\rho_H R(y_1, y_2)\mathcal{L} = 0$. Since $\mathcal{L} \subset W_1(R(y_1, y_2))$, $R(y_1, y_2)\mathcal{L}$ is a spacelike line. But $\rho_H R(y_1, y_2)\mathcal{L} = 0$ implies $R(y_1, y_2)\mathcal{L} \subset H^\perp$ so $R(y_1, y_2)\mathcal{L}$ is timelike. This is false. Thus $\rho_H R(y_1, y_2)\mathcal{L} \neq 0$. Let λ be a unit vector in $\mathcal{L}$. We now define a bilinear map

$$\Phi : H \times H \to \mathbb{R} \oplus H \text{ by } \Phi(h_1, h_2) := g(h_1, h_2) \oplus \rho_H R(h_1, h_2)\lambda.$$

We show Φ is nonsingular as follows. Suppose $h_1 \neq 0$ and $h_2 \neq 0$. If $\Phi(h_1, h_2) = 0$, then $g(h_1, h_2) = 0$ so $h_1 \perp h_2$. Furthermore $\rho_H R(h_1, h_2)\lambda = 0$. It follows that $\rho_H R(\frac{h_1}{|h_1|}, \frac{h_2}{|h_2|})\lambda = 0$ which is false as $\{\frac{h_1}{|h_1|}, \frac{h_2}{|h_2|}\}$ is an orthonormal subset of H. We apply Lemma 2.6.1 to $H = \mathbb{R}^q$ to complete the proof of assertion (1).

We clear the previous notation to prove assertion (2). Let $\{x_1, x_2\}$ be an orthonormal set of spacelike vectors. Since $q \geq 3$, we may choose a third unit spacelike vector x_3 which is perpendicular to x_1 and x_2. Let λ_i be unit vectors in $\mathcal{L}(x_i)$. We will show $\lambda_1 \perp \lambda_2$. Since $\{\lambda_1, \lambda_2\} \subset W_1(R(x_1, x_2))$, this will then imply $W_1(R(x_1, x_2)) = \mathcal{L}(x_1) \oplus \mathcal{L}(x_2)$ is an orthogonal direct sum decomposition.

We choose $\{v_1, \lambda_3\}$ and $\{v_2, \lambda_3\}$ to be orthonormal bases for the spacelike 2-planes $W_1(R(x_1, x_3))$ and $W_1(R(x_2, x_3))$ respectively. By Lemma 5.1.2, these two planes meet at right angles, $v_1 \perp v_2$ so $\{v_1, v_2, \lambda_3\}$ is an orthonormal set. Since $\lambda_1 \in W_1(R(x_1, x_3))$ and $\lambda_2 \in W_1(R(x_2, x_3))$, we may choose angles θ_i so that

$$\lambda_1 = \cos(\theta_1)\lambda_3 + \sin(\theta_1)v_1 \text{ and } \lambda_2 = \cos(\theta_2)\lambda_3 + \sin(\theta_2)v_2.$$

As $x_i \neq \pm x_3$, $\sin(\theta_i) \neq 0$ for $i = 1, 2$ by assertion (1). We define $\tilde{\lambda}_i \in W_1(R(x_i, x_3))$ with $\tilde{\lambda}_i \perp \lambda_i$ by:

$$\tilde{\lambda}_1 := -\sin(\theta_1)\lambda_3 + \cos(\theta_1)v_1 \text{ and } \tilde{\lambda}_2 := -\sin(\theta_2)\lambda_3 + \cos(\theta_2)v_2.$$

We use the fact that $\lambda_1 \in W_1(R(x_1, x_3)) \cap W_1(R(x_1, x_2))$, the fact that these two planes are perpendicular, and the fact that $\tilde{\lambda}_1 \perp \lambda_1$ to see $\tilde{\lambda}_1 \perp W_1(R(x_1, x_2))$, so in particular $\tilde{\lambda}_1 \perp \lambda_2$. Since $\{v_1, v_2, \lambda_3\}$ is an orthonormal set,

$$0 = g(\tilde{\lambda}_1, \lambda_2) = -\sin(\theta_1)\cos(\theta_2).$$

Since $\sin(\theta_1) \neq 0$, we have $\cos(\theta_2) = 0$ and thus $\lambda_1 = \pm v_1$. A similar argument shows that $\lambda_2 = \pm v_2$. $\square$

The map $x \to \mathcal{L}(x)$ is a continuous map from $S(\mathbb{R}^{p,q})$ to $Gr_{(0,1)}(\mathbb{R}^{p,q})$. By Theorem 1.2.8 $S(\mathbb{R}^{p,q})$ is simply connected, so we can lift this map to a map $\phi : S(\mathbb{R}^{p,q}) \to S(\mathbb{R}^{p,q})$. We extend ϕ radially to the set of all spacelike vectors in $\mathbb{R}^{p,q}$ by defining

$$\phi(0) := 0 \text{ and } \phi(x) := |x| \cdot \phi(|x|^{-1}x) \text{ if } |x| > 0.$$

We use Lemma 5.2.3 to show:

5.2.4 Lemma. *Let $q = 6$ or $q \geq 9$. Let R be a rank 2 spacelike IP algebraic curvature tensor.*

(1) *If $\pi \in Gr_{(0,2)}(\mathbb{R}^{p,q})$, then $\phi|_\pi$ is a linear isometric embedding.*

(2) *We may extend ϕ to a linear isometry of $\mathbb{R}^{p,q}$.*

Proof. Let $\{x, y\}$ be an orthonormal basis for a spacelike 2-plane π. Choose z so $\{x, y, z\}$ is an orthonormal set. Let $T_1 := R(x, z)$ and $T_2 := R(y, z)$. For $\theta \in [0, 2\pi]$, let $\pi(\theta) := \text{Span}\{\cos(\theta)x + \sin(\theta)y, z\}$. Then we have

$$R(\pi(\theta)) = R(\cos(\theta)x + \sin(\theta)y, z) = \cos(\theta)T_1 + \sin(\theta)T_2.$$

Since $\{\cos(\theta)x + \sin(\theta)y, z\}$ is an orthonormal basis for $\pi(\theta)$, we may use Lemma 5.2.3 to see that

$$W_1(R(\pi(\theta))) = \mathcal{L}(\cos(\theta)x + \sin(\theta)y) \oplus \mathcal{L}(z)$$

is an orthogonal direct sum decomposition. On the other hand, by rescaling we may assume R has eigenvalues $\{0, \pm\sqrt{-1}\}$, so by Lemma 5.1.1, $R(\pi(\theta))$ is a $90°$ rotation in $W_1(R(\pi(\theta))) = \text{Range}(R(\pi(\theta)))$. Thus

$$\begin{aligned} \mathcal{L}(\cos(\theta)x + \sin(\theta)y) &= R(\pi(\theta))\mathcal{L}(z) \\ &= R(\pi(\theta))\phi(z) \cdot \mathbb{R} \\ &= (\cos(\theta)T_1 + \sin(\theta)T_2)\phi(z) \cdot \mathbb{R}. \end{aligned}$$

Thus $\phi(\cos(\theta)x + \sin(\theta)y) = \epsilon(\theta)(\cos(\theta)T_1 + \sin(\theta)T_2)\phi(z)$ for any θ with $\epsilon(\theta) = \pm 1$. By continuity, the choice of ϵ is independent of θ. Therefore $\phi(x) = \epsilon T_1\phi(z)$ and $\phi(y) = \epsilon T_2\phi(z)$, so we have the identity:

$$\phi(\cos(\theta)x + \sin(\theta)y) = \cos(\theta)\phi(x) + \sin(\theta)\phi(y).$$

It now follows that $\phi(-x) = -\phi(x)$ so $\phi(\lambda x) = \lambda\phi(x)$ for all $\lambda \in \mathbb{R}$. Consequently for any λ and θ we have:

$$\phi(\lambda\cos(\theta)x + \lambda\sin(\theta)y) = \lambda\cos(\theta)\phi(x) + \lambda\sin(\theta)\phi(y).$$

This shows that the map ϕ is linear on π. Since $|\phi(z)| = |z|$ for any spacelike vectors, ϕ is an isometric embedding of π; this proves assertion (1) .

We extend ϕ to all vectors in $\mathbb{R}^{p,q}$ as follows. Let $v \in \mathbb{R}^{p,q}$. Choose z spacelike with $z \perp v$ so that $|z|^2 > |v|^2$. Then z and $z + v$ are spacelike so

$$\phi_z(v) := \phi(z + v) - \phi(z) \text{ is well defined.}$$

If v is spacelike, then $\mathrm{Span}\{z, v\}$ is spacelike and hence $\phi_z(v) = \phi(v)$. We check this is independent of the choice of z as follows. Suppose z_i are spacelike vectors with $z_i \perp v$ and $|z_i|^2 > |v|^2$. Since $q \geq 6$, we may choose w spacelike with $w \perp \{z_1, z_2, v\}$ and $|w|$ large. Since the planes $\{z_i + v, w\}$, $\{z_i, w\}$, $\{v + w, z_i\}$ are spacelike we may use assertion (1) to see that $\phi_{z_i}(v)$ is independent of the choice of z_i by computing:

$$\begin{aligned}
\phi_{z_i}(v) =& \phi(z_i + v) - \phi(z_i)\\
=& \phi(v + z_i) + \phi(w) - \phi(z_i) - \phi(w)\\
=& \phi(v + z_i + w) - \phi(z_i + w)\\
=& \phi(v + w + z_i) - \phi(w + z_i)\\
=& \phi(v + w) + \phi(z_i) - \phi(w) - \phi(z_i)\\
=& \phi(v + w) - \phi(v).
\end{aligned}$$

The proof that ϕ is linear is similar. Let $\{v_1, v_2\}$ be given. Choose z spacelike with $z \perp \{v_1, v_2\}$. We may then argue if ε is sufficiently large that:

$$\begin{aligned}
\phi(v_1 + v_2) =& \phi(v_1 + v_2 + \varepsilon z) - \phi(\varepsilon z)\\
=& \phi(v_1 + v_2 + \varepsilon z) - \phi(v_2 + \varepsilon z)\\
& + \phi(v_2 + \varepsilon z) - \phi(\varepsilon z)\\
=& \phi(v_1) + \phi(v_2).
\end{aligned}$$

Let $\lambda_1 \neq 0$. We complete the proof that ϕ is linear by checking that:

$$\begin{aligned}
\phi(\lambda_1 v_1) =& \phi(\lambda_1 v_1 + \lambda_1 \varepsilon z) - \phi(\lambda_1 \varepsilon z)\\
=& \lambda_1 \phi(v_1 + \varepsilon z) - \lambda_1 \phi(\varepsilon z)\\
=& \lambda_1 \phi(v_1).
\end{aligned}$$

Let $Q_\phi(x) := |\phi(x)|^2 - |x|^2$. This is a quadratic function on $\mathbb{R}^{p,q}$ as ϕ is linear. Furthermore, $Q_\phi(x)$ vanishes by construction if x is a spacelike vector. Thus Q_ϕ

vanishes on the nonempty open set of nonzero spacelike vectors. Thus all the partial derivatives of Q_ϕ vanish on this open set. Since Q_ϕ is quadratic, Q_ϕ vanishes identically. Hence ϕ is an isometry. $\square$

The timelike case is similar. The domain and the range have been decoupled to this point; thus the sign of the target metric is irrelevant. We say that a map $\phi : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ is an para-isometry if we have that $g(\phi(v), \phi(v)) = -g(v, v)$; this necessarily implies that $p = q$. The proof given of Lemma 5.2.4 extends immediately to establish the following assertion:

5.2.5 Lemma. *Let $q = 6$ or $q \geq 9$. Let R be a rank 2 timelike IP algebraic curvature tensor.*

(1) *If $\pi \in Gr_{(0,2)}(\mathbb{R}^{p,q})$, then $\phi|_\pi$ is a linear para-isometric embedding.*

(2) *We may extend ϕ to a linear para-isometry of $\mathbb{R}^{p,q}$.*

§5.3 Classification of Rank 2 Spacelike IP Algebraic Curvature Tensors

We recall some notation from §1.2.4. Let (C, ϕ) be an admissible pair, we define:

$$R_{C,\phi}(x, y) : z \to C\{g(\phi(y), z)\phi(x) - g(\phi(x), z)\phi(y)\}.$$

Recall that ϕ is unipotent (of order 2) if $\phi^2 = \mathrm{id}$ and that ϕ is unipotent (of order 4) if $\phi^2 = -\mathrm{id}$.

We now consider a special case. Let $\mathcal{R} := R_{1,\mathrm{id}}$. Then we have

$$\mathcal{R}(x, y) : z \to g(y, z)x - g(x, z)y. \tag{5.3.0.a}$$

5.3.1 *Proof of Theorem D.* We prove assertion (1) of Theorem D as follows. we first assume that ϕ is an unipotent (of order 2) isometry of $\mathbb{R}^{p,q}$. Let $\pi = \mathrm{Span}\{x, y\}$ be an oriented spacelike 2-plane. From equation (5.3.0.a), we see that $\mathcal{R}$ preserves the 2-plane π; $\mathcal{R}(x,y) : y \mapsto x$ and $\mathcal{R}(x,y) : x \mapsto -y$. It vanishes on $\pi^\perp$. Thus $\mathcal{R}$ is IP of rank 2. More generally, for any $C \neq 0$, since ϕ is an isometry and since $\phi^2 = \mathrm{id}$, we have $R_{C,\phi}(\pi) = C\mathcal{R}(\phi\pi)$ and hence $R_{C,\phi}$ is IP of rank 2 for $C \neq 0$. We now verify that $R_{C,\phi}$ satisfies the curvature identities. It is immediate that $R_{C,\phi}(x,y) = -R_{C,\phi}(y,x)$. Since ϕ is an isometry and since $\phi^2 = \mathrm{id}$, we have $g(\phi(u),v) = g(u,\phi(v))$. Thus we may check that $R_{C,\phi}$ satisfies the second curvature identity by computing that:

$$\begin{aligned} g(R_{C,\phi}(x,y)z,w) =& C\{g(\phi(y),z)g(\phi(x),w) - g(\phi(x),z)g(\phi(y),w)\} \\ =& C\{g(y,\phi(z))g(x,\phi(w)) - g(x,\phi(z))g(y,\phi(w))\} \\ =& g(R_{C,\phi}(z,w)x,y). \end{aligned}$$

We may also verify that the Bianchi identities are satisfied by computing:

$$\begin{aligned} & R_{C,\phi}(x,y)z + R_{C,\phi}(y,z)x + R_{C,\phi}(z,x)y \\ =& C\{g(\phi(y),z)\phi(x) - g(\phi(x),z)\phi(y) + g(\phi(z),x)\phi(y) - g(\phi(y),x)\phi(z) \\ & + g(\phi(x),y)\phi(z) - g(\phi(z),y)\phi(x)\} \\ =& 0. \end{aligned}$$

We now consider ϕ is an unipotent (of order 4) para-isometry. For any $C \neq 0$, we still have $R_{C,\phi}(\pi) = C\mathcal{R}(\phi\pi)$ and hence $R_{C,\phi}$ is IP of rank 2. We now verify that $R_{C,\phi}$ satisfies the curvature identities. It is immediate that $R_{C,\phi}(x,y) = -R_{C,\phi}(y,x)$. Since ϕ is a para-isometry and since $\phi^2 = -\mathrm{id}$, we have

$$g(\phi(u),v) = -g(\phi^2(u),\phi(v)) = -g(-u,\phi(v)) = g(u,\phi(v)).$$

Thus we may check that $R_{C,\phi}$ satisfies the second curvature identity by computing that:

$$\begin{aligned} g(R_{C,\phi}(x,y)z,w) =& C\{g(\phi(y),z)g(\phi(x),w) - g(\phi(x),z)g(\phi(y),w)\} \\ =& C\{g(y,\phi(z))g(x,\phi(w)) - g(x,\phi(z))g(y,\phi(w))\} \\ =& g(R_{C,\phi}(z,w)x,y). \end{aligned}$$

We may also verify that the Bianchi identities are satisfied by computing:

$$\begin{aligned}
&R_{C,\phi}(x,y)z + R_{C,\phi}(y,z)x + R_{C,\phi}(z,x)y\\
=&C\{g(\phi(y),z)\phi(x) - g(\phi(x),z)\phi(y) + g(\phi(z),x)\phi(y) - g(\phi(y),x)\phi(z)\\
&+ g(\phi(x),y)\phi(z) - g(\phi(z),y)\phi(x)\}\\
=&0.
\end{aligned}$$

We now prove assertion (2) of Theorem D. We use Lemma 5.2.4 to define a linear map ϕ on $\mathbb{R}^{p,q}$ so that $\phi(x) \in \mathcal{L}(x)$ for any unit spacelike vector x. If R is spacelike, then ϕ is an isometry; if R is timelike, then ϕ is a para-isometry. Assume R has eigenvalues $\{0, \pm C\sqrt{-1}\}$ for some constant $C \neq 0$. By rescaling, we may assume that $C = 1$. Let $\{x, y\}$ be an oriented orthonormal basis for a spacelike 2-plane π. Since $\{\phi(x), \phi(y)\}$ is an orthonormal basis for $W_1(R(\pi)) = \phi(\pi)$ and since $R(\pi)$ is an almost complex structure on $W_1(R(\pi))$,

$$R(\pi)\phi(y) = \varepsilon(\pi)\phi(x) \text{ and } R(\pi)\phi(x) = -\varepsilon(\pi)\phi(x)$$

where $\varepsilon(\pi) = \pm 1$. Since $Gr^+_{(0,2)}(\mathbb{R}^{p,q})$ is connected and ε is continuous, ε is independent of π. Again, by rescaling R if necessary, we may suppose that $\varepsilon \equiv +1$. Thus

$$\text{(5.3.1.a)} \qquad R(x,y) : z \to g(\phi(y),z)\phi(x) - g(\phi(x),z)\phi(y) \text{ for all } z \in \mathbb{R}^{p,q}.$$

Since both sides of this identity are bilinear and skew-symmetric in (x, y), this identity holds as long as $\{x, y\}$ spans a spacelike 2-plane. Since the identity is trilinear and holds on a nonempty open set of $(\mathbb{R}^{p,q})^3$, it holds identically for all (x, y, z); the argument is the same as that given using partial derivatives to show that ϕ was quadratic in the proof of Lemma 5.2.4 and is therefore omitted.

We now study ϕ^2. Let $\{x, y\}$ be an orthonormal subset of $\mathbb{R}^{p,q}$ which spans a spacelike 2-plane π. We apply the Gram-Schmidt process to $\{x, y\}$ to extend to a full orthonormal basis $\{x, y, e_i\}$ for $\mathbb{R}^{p,q}$. Since ϕ is either an isometry or a para-isometry, $\phi(x) \perp \phi(e_i)$ and $\phi(y) \perp \phi(e_i)$ for all i. We use the second curvature symmetry and equation (5.3.1.a) to compute:

$$\begin{aligned} g(R(\phi(x), \phi(y))x, e_i) =& g(R(x, e_i)\phi(x), \phi(y)) \\ =& g(\phi(e_i), \phi(x))g(\phi(x), \phi(y)) \\ & - g(\phi(x), \phi(x))g(\phi(e_i), \phi(y)) \\ =& 0. \end{aligned}$$

Since $g(R(\phi(x), \phi(y))x, x) = 0$, we have $R(\phi(x), \phi(y))x = \lambda y$ for some λ. We show that $\lambda = -1$ by computing:

$$\lambda = g(R(\phi(x), \phi(y))x, y) = g(R(x, y)\phi(x), \phi(y)) = -1.$$

This shows that $x \in W_1(R(\phi(x), \phi(y))$, so consequently $x \in \mathcal{L}(\phi(x))$. Thus we have $\phi(\phi(x)) = \varepsilon(x)x$ where $\varepsilon(x) = \pm 1$; again, continuity implies ε is independent of x.

Let $g(\phi(u), \phi(v)) = \delta g(u, v)$ and $\phi^2 = \varepsilon \operatorname{id}$. Let $\{x, y\}$ be an orthonormal spacelike set so that $x \perp \phi(y)$. We show $\varepsilon = \delta$ by computing:

$$\begin{aligned} 0 =& R(x, y)\phi(x) + R(\phi(x), x)y + R(y, \phi(x))x \\ =& C\{g(\phi(y), \phi(x))\phi(x) - g(\phi(x), \phi(x))\phi(y) + g(\phi(x), y)\phi^2(x) \\ & - g(\phi^2(x), y)\phi(x) + g(\phi^2(x), x)\phi(y) - g(\phi(y), x)\phi^2(x)\} \\ =& C\{-\delta\phi(y) + \varepsilon\phi(y)\}. \quad \square \end{aligned}$$

5.3.2 *Proof of Theorem G* (1). Let $m \geq 10$. Let R be a nontrivial Lorentzian IP algebraic curvature tensor on $\mathbb{R}^{1,m-1}$. Theorem A (1) implies $\operatorname{rank} R = 2$. We use Theorem B to see that either R is spacelike or R is mixed or R is null. We use Theorem C to see that R is not mixed or null. Thus R is spacelike. We use Theorem D to see that $R = R_{C,\phi}$ for an admissible pair (C, ϕ) with ϕ an unipotent (of order 2) isometry of $\mathbb{R}^{1,m-1}$. $\square$

5.3.3 *Remark:* The classification of rank 2 spacelike IP algebraic curvature tensors exhibits an analogue of the Rakić Duality in this setting: Let $q = 6$ or $q \geq 9$. Let R be a rank 2 spacelike IP algebraic curvature tensor. Let π and σ be two spacelike 2-planes in $\mathbb{R}^{p,q}$. We have $R(\pi)\sigma \subseteq \sigma$ if and only if $R(\sigma)\pi \subseteq \pi$.

CHAPTER VI

SOME EXAMPLES OF PSEUDO-RIEMANNIAN IP MANIFOLDS

In chapter VI, we prove Theorems E and F and we complete the proof of Theorem G. We shall henceforth assume $m \geq 10$. In §6.1, we generalize the argument given by Gilkey, Leahy and Sadofsky [48] to prove Theorem E. In §6.2, we generalize the warped product construction of Gilkey, Leahy and Sadofsky, and of Ivanov and Petrova to higher signatures to prove Theorem F. In §6.3, we first show that any C-ϕ type metric is a warped product of an interval with a metric of constant sectional curvature. We subsequently use the seven steps outlined in §1.4.2 to complete the proof of Theorem G. In §6.4, we discuss the orthogonal equivalence of the curvature tensors $R_{C,\phi}$.

§6.1 The Geometric Realizability of IP Algebraic Curvature Tensors

6.1.1 *Definition.* A metric g_M is said to be C-ϕ type if there exists a smooth nonzero function $C(x)$ on M and if there exists a smooth section ϕ to the bundle of unipotent (of order 2) isometries or unipotent (of order 4) para-isometries of the tangent bundle so that $R_{g_M} = R_{C,\phi}$ at each point of M. We shall focus on the case where ϕ is an unipotent (of order 2) isometry.

In Lemma 6.1.2, we show any unipotent (of order 2) isometry ϕ induces an orthogonal direct sum decomposition $\mathbb{R}^{p,q} = E_+ \oplus E_-$ into the ± 1 eigenspaces of ϕ. In Lemma 6.1.6, we give a geometric realization of this tensor. In Theorem E, we show $R_{C,\phi}$ is not geometrically realizable by a C-ϕ type IP metric if $\dim E_+ > 1$ and if $\dim E_- > 1$.

6.1.2 Lemma. *Let ϕ be an unipotent (of order 2) isometry of $\mathbb{R}^{p,q}$. There exists an orthonormal basis for $\mathbb{R}^{p,q}$ which diagonalizes ϕ.*

Proof. Let $\phi \in \mathrm{O}(p,q)$ with $\phi^2 = \mathrm{id}$; then necessarily $\phi = \phi^*$. Let $E_\pm$ be the ± 1 eigenspaces of ϕ. For any $x \in \mathbb{R}^{p,q}$, we can write

$$x = \tfrac{1}{2}(x + \phi x) + \tfrac{1}{2}(x - \phi x).$$

Since $\frac{1}{2}(x \pm \phi x) \in E_\pm$, we have $\mathbb{R}^{p,q} = E_+ + E_-$. If $x_\pm \in E_\pm$, then we have:

$$g(x_+, x_-) = g(\phi x_+, x_-) = g(x_+, \phi x_-) = -g(x_+, x_-).$$

Thus $E_+ \perp E_-$. So $\mathbb{R}^{p,q} = E_+ \oplus E_-$ is an orthogonal direct sum decomposition. Let $g_\pm := g|_{E_\pm}$. Since $E_+ \perp E_-$, we have $g = g_+ \oplus g_-$. Since the metric g is nondegenerate, the metrics $g_\pm$ are nondegenerate on $E_\pm$. Consequently, we can find bases diagonalizing $g_\pm$ and ϕ. $\square$

6.1.3 *Definition.* We say that $\{e_i\}$ is a normalized orthonormal basis for $\mathbb{R}^{p,q}$, if

$$g(e_i, e_i) = \varepsilon_i e_i \text{ and } \phi(e_i) = \delta_i e_i \text{ where } \varepsilon_i = \pm 1,\ \delta_i = \pm 1.$$

We omit the proof of the following lemma as it is an immediate algebraic consequence of the definitions given above.

6.1.4 Lemma. *Let $\{e_i\}$ be a normalized orthonormal basis. Then we have :*

$$R_{C,\phi}(e_i, e_j, e_j, e_i) = C\varepsilon_i\varepsilon_j\delta_i\delta_j \text{ for } i \neq j,$$

$$R_{C,\phi}(e_i, e_j, e_k, e_\ell) = 0 \text{ for } (i,j) \neq (k,\ell) \text{ and } (i,j) \neq (\ell,k).$$

$$R_{C,\phi}(e_i, e_j)e_j = C\varepsilon_j\delta_i\delta_j e_i \text{ for } i \neq j$$

$$R_{C,\phi}(e_i, e_j)e_k = 0 \text{ for } i \neq k \text{ and } j \neq k.$$

We use Lemma 6.1.2 to diagonalize ϕ and define an orthogonal direct sum decomposition $\mathbb{R}^{p,q} = E_+ \oplus E_-$, where $E_\pm$ are the ± 1 eigenspaces of ϕ. The restrictions of g to $E_\pm$ determine nondegenerate metrics $g_\pm$ of signatures $(p_\pm, q_\pm)$ and permit us to further decompose $E_\pm$ according to $g_\pm$. Thus we have the orthogonal direct sum decomposition $\mathbb{R}^{p,q} = E_{++} \oplus E_{+-} \oplus E_{-+} \oplus E_{--}$. The notation is chosen so that $\phi = +1$ on $E_{++} \oplus E_{+-}$, so that $\phi = -1$ on $E_{-+} \oplus E_{--}$, so that g is positive definite on $E_{++} \oplus E_{-+}$, and so that g is negative definite on $E_{+-} \oplus E_{--}$.

6.1.5 *Definition.* Let

$$r_\pm := \dim E_\pm,\ p_+ := \dim E_{++},\ q_+ := \dim E_{+-},$$

$$p_- := \dim E_{-+},\ \text{and } q_- := \dim E_{--}.$$

Then $p_+ + q_+ = r_+,\ \ p_- + q_- = r_-,\ \ p_+ + p_- = q,$ and $q_+ + q_- = p$.

The following lemma gives a geometric realization of $R_{C,\phi}$.

6.1.6 Lemma. *Let ϕ be an unipotent (of order 2) isometry of $\mathbb{R}^{p,q}$. Choose a normalized orthonormal basis $\{e_i\}$ and introduce coordinates $x = \sum_i x_i e_i$ on $\mathbb{R}^{p,q}$. Let*

$$ds^2_{C,\phi} := \textstyle\sum_i \left\{\varepsilon_i\left(1 - \frac{C}{2}\sum_{j\neq i}\varepsilon_j\delta_i\delta_j x_j^2\right)dx_i^2\right\}.$$

This defines a nondegenerate metric of signature (p,q) near the origin so that the coordinate frame $e_i := \{\frac{\partial}{\partial x_i}\}$ is a normalized orthonormal basis at the origin. We have that $R(0) = R_{C,\phi}$.

Proof. Let $\{\partial_i := \frac{\partial}{\partial x_i} : 1 \le i \le p+q\}$ be the standard coordinate frame on $\mathbb{R}^{p,q}$. Let

$$g_{ij/k} := \partial_k g(\partial_i, \partial_j) \text{ and } g_{ij/kl} := \partial_l\partial_k g(\partial_i, \partial_j).$$

We compute:

(1) $g(\partial_i, \partial_i)(0) = \varepsilon_i$, where $\varepsilon_i = \pm 1$.

(2) $g(\partial_i, \partial_j)(0) = 0$ for $i \neq j$.

(3) $g_{ij/k}(0) = 0$ for $1 \le i, j \le p+q$.

Relative to this coordinate frame, we have:

$$\Gamma_{ijk} = \tfrac{1}{2}(g_{jk/i} + g_{ik/j} - g_{ij/k}).$$

Thus $\Gamma_{ijk}(0) = 0$. We compute:

$$\begin{aligned} R_{ijk\ell}(0) =& g((\nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\partial_j}\nabla_{\partial_i})\partial_k, \partial_\ell)(0) \\ =& g(\partial_i\Gamma_{jk\ell} - \partial_j\Gamma_{ik\ell})(0) \\ =& \tfrac{1}{2}\big[g_{j\ell/ik}(0) + g_{k\ell/ij}(0) - g_{jk/i\ell}(0) \\ & - g_{i\ell/jk}(0) - g_{k\ell/ij}(0) + g_{ik/j\ell}(0)\big] \\ =& \tfrac{1}{2}\big[g_{ik/j\ell}(0) + g_{j\ell/ik}(0) - g_{jk/i\ell}(0) - g_{i\ell/jk}(0)\big]. \end{aligned} \tag{6.1.6.a}$$

We use the definition and (6.1.6.a) to compute

$$\begin{aligned} &R_{ijk\ell}(0) = 0 \text{ for } (i,j) \neq (k,\ell) \text{ and } (i,j) \neq (\ell,k); \\ &R_{ijji}(0) = -\tfrac{1}{2}[g_{ii/jj}(0) + g_{jj/ii}(0)] = C\varepsilon_i\varepsilon_j\delta_i\delta_j. \quad \square \end{aligned}$$

6.1.7 *Remark:* This metric need not be IP away from the origin.

We now introduce contraction of tensors which is needed later in our discussion.

6.1.8 *Definition.* Let the natural map $c : \otimes^4(T^*M) \to \otimes^2(T^*M)$ be defined on pure tensors by

$$c(\omega_1 \otimes \omega_2 \otimes \omega_3 \otimes \omega_4) := g(\omega_2, \omega_3)\omega_1 \otimes \omega_4.$$

Since this map is bilinear, it extends to a map on the whole tensor product.

6.1.9 Lemma. *Contraction commutes with covariant differentiation.*

Proof. We compute

$$
\begin{aligned}
&c(\nabla_{e_k}(\omega_1 \otimes \omega_2 \otimes \omega_3 \otimes \omega_4)) \\
=&c(\nabla_{e_k}\omega_1 \otimes \omega_2 \otimes \omega_3 \otimes \omega_4 + \omega_1 \otimes \nabla_{e_k}\omega_2 \otimes \omega_3 \otimes \omega_4 \\
&+ \omega_1 \otimes \omega_2 \otimes \nabla_{e_k}\omega_3 \otimes \omega_4 + \omega_1 \otimes \omega_2 \otimes \omega_3 \otimes \nabla_{e_k}\omega_4) \\
=&g(\omega_2, \omega_3)\nabla_{e_k}\omega_1 \otimes \omega_4 + g(\omega_2, \omega_3)\omega_1 \otimes \nabla_{e_k}\omega_4 \\
&+ g(\nabla_{e_k}\omega_2, \omega_3)\omega_1 \otimes \omega_4 + g(\omega_2, \nabla_{e_k}\omega_3)\omega_1 \otimes \omega_4 \\
=&g(\omega_2, \omega_3)\nabla_{e_k}\omega_1 \otimes \omega_4 + g(\omega_2, \omega_3)\omega_1 \otimes \nabla_{e_k}\omega_4 \\
&+ e_k\{g(\omega_2, \omega_3)\}\omega_1 \otimes \omega_4 \\
=&\nabla_{e_k}c(\omega_1 \otimes \omega_2 \otimes \omega_3 \otimes \omega_4). \quad \square
\end{aligned}
$$

6.1.10 *Notational conventions.* Let g be a rank 2 C-ϕ type IP metric. Let R be the curvature tensor of g. Suppose there exist $C \neq 0$ and ϕ an unipotent (of order 2) isometry of $\mathbb{R}^{p,q}$ so that $R = R_{C,\phi}$. Let indices i, j etc. range from 1 through $m = p + q$. Let the roman indices a, b, etc. range from 1 through r_+. Let the greek indices α, β range from $r_+ + 1$ through m. We use Lemma 6.1.2 to choose a local frame e diagonalizing ϕ so that $\phi e_a = e_a$ and that $\phi e_\alpha = -e_\alpha$. Let $\phi_{ij} := g(\phi e_i, e_j)$. Let $\phi_{ij;k}$, $R_{ijk\ell}$, and $R_{ijk\ell;n}$ be the components of $\nabla\phi$, R, and ∇R. Let $\mathcal{F}_\pm$ be the distributions defined by the ± 1 eigenspaces of ϕ. Then $\{e_a\}$ span $\mathcal{F}_+$ and $\{e_\alpha\}$ span $\mathcal{F}_-$.

We adopt arguments of Gilkey, Leahy and Sadofsky [48] to establish the following technical lemma which we shall need later.

6.1.11 Lemma. *Let $m \geq 4$. Let g be a C-ϕ type IP metric of rank 2. Then*

(1) $R_{ijk\ell;n} = C_{;n}(\phi_{i\ell}\phi_{jk} - \phi_{ik}\phi_{j\ell}) + C(\phi_{i\ell;n}\phi_{jk} + \phi_{i\ell}\phi_{jk;n} - \phi_{ik;n}\phi_{j\ell} - \phi_{ik}\phi_{j\ell;n})$.

(2) *We have $\phi_{ij;k} = \phi_{ji;k}$ for any i, j, and k.*

(3) *We have $\phi_{ab;i} = 0$ and $\phi_{\alpha\beta;i} = 0$ for any a, b, α, β, and i.*

(4) *If i, j, and k are distinct, then* $\phi_{ij;k} = \phi_{ik;j}$.

(5) *If* $a \neq b$, *then* $\phi_{a\alpha;b} = 0$; *if* $\alpha \neq \beta$, *then* $\phi_{a\alpha;\beta} = 0$.

(6) *The Christoffel symbols* $\Gamma_{ia\alpha} = \frac{1}{2}\phi_{a\alpha;i}$.

(7) *The distributions* $\mathcal{F}_\pm$ *are integrable.*

(8) *If there exists* $\alpha \neq \beta$, *then*

$$C_{;a} = -C\{\varepsilon_\beta \phi_{\beta a;\beta} + \varepsilon_\alpha \phi_{\alpha a;\alpha}\} \text{ and } C_{;\beta} = -C\varepsilon_a \phi_{a\beta;a}.$$

(9) *If there exists* $a \neq b$, *then*

$$C_{;\alpha} = C\{\varepsilon_a \phi_{\alpha a;a} + \varepsilon_b \phi_{\alpha b;b}\} \text{ and } C_{;b} = C\varepsilon_\alpha \phi_{\alpha b;\alpha}.$$

(10) *If* $r_- \geq 3$, *then* $C_{;\alpha} = 0$. *If* $r_+ \geq 3$, *then* $C_{;a} = 0$.

(11) *Either* $r_+ \leq 1$ *or* $r_- \leq 1$.

Proof. We covariantly differentiate the identity $R_{ijk\ell} = C(\phi_{i\ell}\phi_{jk} - \phi_{ik}\phi_{j\ell})$ to see

$$\begin{aligned} R_{ijk\ell;n} =& \nabla_{e_n}(C(\phi_{i\ell}\phi_{jk} - \phi_{ik}\phi_{j\ell})) \\ =& C_{;n}(\phi_{i\ell}\phi_{jk} - \phi_{ik}\phi_{j\ell}) + C(\phi_{i\ell;n}\phi_{jk} + \phi_{i\ell}\phi_{jk;n} - \phi_{ik;n}\phi_{j\ell} - \phi_{ik}\phi_{j\ell;n}). \end{aligned}$$

Assertion (1) follows. Since $\phi \in \mathrm{O}(p,q)$ with $\phi^2 = \mathrm{id}$, ϕ is necessarily self-adjoint so assertion (2) holds. To prove assertion (3), we consider the 4 cotensor $\Phi \in \otimes^4(T^*M)$ defined by $\Phi(x,y,z,w) := g(\phi x, y)g(\phi z, w)$ and compute:

$$\begin{aligned} \Phi =& \textstyle\sum_{i,j,k,\ell} \phi_{ij}\phi_{k\ell} e^i \otimes e^j \otimes e^k \otimes e^\ell \\ c\Phi =& \textstyle\sum_{i,\ell} \sum_{j,k} g^{jk}\phi_{ij}\phi_{k\ell} e^j \otimes e^\ell \\ =& \textstyle\sum_{i,\ell} \sum_{j,k} \varepsilon_j \varepsilon_i \delta_{jk} \delta_i \delta_{ij} \varepsilon_k \delta_k \delta_{k\ell} e^i \otimes e^\ell \\ =& \textstyle\sum_{i,j} \varepsilon_i \delta_{ij} e^i \otimes e^j \\ =& \textstyle\sum_{i,j} g_{ij} e^i \otimes e^j. \end{aligned}$$

Thus $c\Phi = g$. Since $\nabla g = 0$, we have $\nabla_{e_n} c\Phi = \nabla_{e_n} g = 0$ and hence by Lemma 6.1.9 we see that $c\nabla_{e_n}\Phi = 0$. We have

$$\nabla_{e_n}\Phi = (\phi_{ij;n}\phi_{k\ell} + \phi_{ij}\phi_{k\ell;n}) e^i \otimes e^j \otimes e^k \otimes e^\ell.$$

Since $\phi_{ij} = 0$ for $i \neq j$ and $\phi_{k\ell} = 0$ for $k \neq \ell$, we use the relation $c\nabla_{e_n}\Phi = 0$ to see

$$\begin{aligned}0 =&\varepsilon_\ell \phi_{i\ell;n}\phi_{\ell\ell} + \varepsilon_i \phi_{ii}\phi_{i\ell;n}\\ =&\varepsilon_\ell\varepsilon_\ell\delta_\ell\phi_{i\ell;n} + \varepsilon_i\varepsilon_i\delta_i\phi_{i\ell;n}\\ =&(\delta_\ell + \delta_i)\phi_{i\ell;n}.\end{aligned}$$

Assertion (3) now follows. We use the second Bianchi identity:

$$R_{ijk\ell;n} + R_{ij\ell n;k} + R_{ijnk;\ell} = 0$$

to prove assertion (4). Let $\{i, j, \ell, n\}$ be distinct indices, this is possible as $m \geq 4$. We use assertion (1) and the fact that $\phi_{i\ell} = \phi_{ij} = \phi_{j\ell} = 0$ to compute:

$$\begin{aligned}R_{ijj\ell;n} =&C_{;n}(\phi_{i\ell}\phi_{jj} - \phi_{ij}\phi_{j\ell})\\ &+ C(\phi_{i\ell;n}\phi_{jj} + \phi_{i\ell}\phi_{jj;n} - \phi_{ij;n}\phi_{j\ell} - \phi_{ij}\phi_{j\ell;n})\\ =&C\phi_{i\ell;n}\phi_{jj}.\\ R_{ij\ell j;n} =& - R_{ijj\ell;n} = -C\phi_{i\ell;n}\phi_{jj}.\\ R_{ijk\ell;j} =&C_{;j}(\phi_{i\ell}\phi_{jk} - \phi_{jk}\phi_{i\ell})\\ &+ C(\phi_{i\ell;j}\phi_{jk} + \phi_{i\ell}\phi_{jk;j} - \phi_{jk;j}\phi_{i\ell} - \phi_{jk}\phi_{i\ell;j})\\ =&0.\end{aligned}$$

We relabel the indices at this point. Let $\{i, j, k\}$ be distinct. Since $m \geq 4$, we may choose ℓ so $\{i, j, k, \ell\}$ are distinct indices. We now use the second Bianchi identity with $(i, j, k, \ell, n) = (i, \ell, \ell, j, k)$ to see

$$0 = R_{i\ell\ell j;k} + R_{i\ell jk;\ell} + R_{i\ell k\ell;j} = C(\phi_{ij;k} - \phi_{ik;j})\phi_{\ell\ell}.$$

Thus $\phi_{ij;k} = \phi_{ik;j}$ and assertion (4) holds. We use assertions (3) and (4) to see that if $a \neq b$, then

$$\phi_{a\alpha;b} = \phi_{ab;\alpha} = 0.$$

Similarly, if $\alpha \neq \beta$, we may use assertions (2), (3), and (4) to compute:

$$\phi_{a\alpha;\beta} = \phi_{\alpha a;\beta} = \phi_{\alpha\beta;a} = 0.$$

Assertion (5) follows. We prove assertion (6) by computing:

$$\begin{aligned}
\phi_{ij;k} :=&(\nabla_{e_k}\phi)(e_i, e_j) = e_k\phi(e_i, e_j) - \phi(\nabla_{e_k}e_i, e_j) - \phi(e_i, \nabla_{e_k}e_j)\\
=&-\textstyle\sum_\ell \Gamma_{ki}{}^\ell\phi(e_\ell, e_j) - \sum_\ell \Gamma_{kj}{}^\ell\phi(e_i, e_\ell)\\
=&-\textstyle\sum_\ell \Gamma_{ki}{}^j\delta_j g_{\ell j} - \sum_\ell \Gamma_{kj}{}^\ell\delta_i g_{\ell i}\\
=&-\Gamma_{kij}\delta_j - \Gamma_{kji}\delta_i\\
=&\Gamma_{kij}(\delta_i - \delta_j).
\end{aligned}$$

It now follows that $\phi_{a\alpha;k} = \Gamma_{ka\alpha}(\delta_a - \delta_\alpha) = 2\Gamma_{ka\alpha}$, thus $\Gamma_{ka\alpha} = \frac{1}{2}\phi_{a\alpha;k}$. Note this also provides another check that $\phi_{ab;k} = 0$ and $\phi_{\alpha\beta;k} = 0$ as $\delta_i - \delta_j = 0$ if $(i, j) = (a, b)$ or $(i, j) = (\alpha, \beta)$.

We now prove assertion (7), we set $\Pi_\pm := \frac{1}{2}(1 \pm \phi)$ to be orthogonal projection on $\mathcal{F}_\pm$. To show $\mathcal{F}_+$ is integrable, we must show $g([e_a, e_b], e_\alpha) = 0$. We compute:

$$g([e_a, e_b], e_\alpha) = g(\nabla_{e_a}e_b - \nabla_{e_b}e_a, e_\alpha) = \Gamma_{ab\alpha} - \Gamma_{ba\alpha} = \tfrac{1}{2}(\phi_{b\alpha;a} - \phi_{a\alpha;b}).$$

This vanishes trivially if $a = b$; we use assertion (5) to see this still vanishes if $a \neq b$. The argument is the same to show $\mathcal{F}_-$ is integrable where we also use the symmetry given in assertion (2).

We now prove assertion (8). Let $\alpha \neq \beta$. We use assertions (1) and (3) to compute

$$\begin{aligned}
R_{\alpha\beta\beta\alpha;a} =&C_{;a}(\phi_{\alpha\alpha}\phi_{\beta\beta} - \phi_{\alpha\beta}\phi_{\beta\alpha})\\
&+ C(\phi_{\alpha\alpha;a}\phi_{\beta\beta} + \phi_{\alpha\alpha}\phi_{\beta\beta;a} - \phi_{\alpha\beta;a}\phi_{\alpha\beta} - \phi_{\alpha\beta}\phi_{\beta\alpha;a})\\
=&C_{;a}\phi_{\alpha\alpha}\phi_{\beta\beta} = \varepsilon_\alpha\varepsilon_\beta C_{;a}.\\
R_{\alpha\beta\alpha a;\beta} =&C_{;\beta}(\phi_{\alpha a}\phi_{\beta\alpha} - \phi_{\alpha\alpha}\phi_{\beta a})\\
&+ C(\phi_{\alpha a;\beta}\phi_{\beta\alpha} + \phi_{\alpha a}\phi_{\beta\alpha;\beta} - \phi_{\alpha\alpha;\beta}\phi_{\beta a} - \phi_{\alpha\alpha}\phi_{\beta a;\beta})\\
=& - C\phi_{\alpha\alpha}\phi_{\beta a,\beta} = \varepsilon_\alpha C\phi_{\beta a;\beta}.\\
R_{\alpha\beta a\beta;\alpha} =&C_{;\alpha}(\phi_{\alpha\beta}\phi_{\beta a} - \phi_{\alpha a}\phi_{\beta\beta})\\
&+ C(\phi_{\alpha\beta;\alpha}\phi_{\beta a} + \phi_{\alpha\beta}\phi_{\beta a;\alpha} - \phi_{\alpha a;\alpha}\phi_{\beta\beta} - \phi_{\alpha a}\phi_{\beta\beta;\alpha})\\
=& - C\phi_{\beta\beta}\phi_{\alpha a,\alpha} = \varepsilon_\beta C\phi_{\alpha a;\alpha}.\\
R_{\alpha a\beta a;\alpha} =&C_{;\alpha}(\phi_{\alpha a}\phi_{a\beta} - \phi_{\alpha\beta}\phi_{aa})\\
&+ C(\phi_{\alpha a;\alpha}\phi_{a\beta} + \phi_{\alpha a}\phi_{a\beta;\alpha} - \phi_{\alpha\beta;\alpha}\phi_{aa} - \phi_{\alpha\beta}\phi_{aa;\alpha})\\
=&0.
\end{aligned}$$

We now use the second Bianchi identity with $(i,j,k,\ell,n) = (\alpha,\beta,\beta,\alpha,a)$ to see

$$0 = R_{\alpha\beta\beta\alpha;a} + R_{\alpha\beta\alpha a;\beta} + R_{\alpha\beta a\beta;\alpha} = \varepsilon_\alpha\varepsilon_\beta C_{;a} + \varepsilon_\alpha C\phi_{\beta a;\beta} + \varepsilon_\beta C\phi_{\alpha a;\alpha}.$$

Thus $C_{;a} = -C\{\varepsilon_\beta\phi_{\beta a;\beta} + \varepsilon_\alpha\phi_{\alpha a;\alpha}\}$. We relabel the indices to see

$$R_{\alpha aa\alpha;\beta} = \varepsilon_\alpha\varepsilon_a C_{;\beta} \text{ and } R_{\alpha a\alpha\beta;a} = \varepsilon_\alpha C\phi_{a\beta;a}.$$

We now use the second Bianchi identity with $(i,j,k,\ell,n) = (\alpha,a,a,\alpha,\beta)$ to see

$$0 = R_{\alpha aa\alpha;\beta} + R_{\alpha a\alpha\beta;a} + R_{\alpha a\beta a;\alpha} = \varepsilon_\alpha\varepsilon_a C_{;\beta} + \varepsilon_\alpha C\phi_{a\beta;a}.$$

Assertion (8) now follows. We replace ϕ by $-\phi$ and interchange the roles of the greek and roman indices to derive assertion (9) from assertion (8).

To prove assertion (10), we suppose $r_- \geq 3$. Choose α, β, and γ distinct, we compute:

$$\begin{aligned}
R_{\gamma\beta\beta\gamma;\alpha} =& C_{;\alpha}(\phi_{\gamma\gamma}\phi_{\beta\beta} - \phi_{\gamma\beta}\phi_{\beta\gamma}) \\
&+ C(\phi_{\gamma\gamma;\alpha}\phi_{\beta\beta} + \phi_{\gamma\gamma}\phi_{\beta\beta;\alpha} - \phi_{\gamma\beta;\alpha}\phi_{\beta\gamma} - \phi_{\gamma\beta}\phi_{\beta\gamma;a}) \\
=& C_{;\alpha}\phi_{\gamma\gamma}\phi_{\beta\beta} = \varepsilon_\gamma\varepsilon_\beta C_{;\alpha}. \\
R_{\gamma\beta\gamma\alpha;\beta} =& C_{;\beta}(\phi_{\gamma\alpha}\phi_{\beta\gamma} - \phi_{\gamma\gamma}\phi_{\beta\alpha}) \\
&+ C(\phi_{\gamma\alpha;\beta}\phi_{\beta\gamma} + \phi_{\gamma\alpha}\phi_{\beta\gamma;\beta} - \phi_{\gamma\gamma;\beta}\phi_{\beta\alpha} - \phi_{\gamma\gamma}\phi_{\beta\alpha;\beta}) \\
=& 0. \\
R_{\gamma\beta\alpha\beta;\gamma} =& C_{;\gamma}(\phi_{\gamma\beta}\phi_{\beta\alpha} - \phi_{\gamma\alpha}\phi_{\beta\alpha}) \\
&+ C(\phi_{\gamma\beta;\gamma}\phi_{\beta\alpha} + \phi_{\gamma\beta}\phi_{\beta\alpha;\gamma} - \phi_{\gamma\alpha;\gamma}\phi_{\beta\alpha} - \phi_{\gamma\alpha}\phi_{\beta\alpha;\gamma}) \\
=& 0.
\end{aligned}$$

We now use the second Bianchi identity with $(i,j,k,\ell,n) = (\gamma,\beta,\beta,\gamma,\alpha)$ to see

$$0 = R_{\gamma\beta\beta\gamma;\alpha} + R_{\gamma\beta\gamma\alpha;\beta} + R_{\gamma\beta\alpha\beta;\gamma} = \varepsilon_\gamma\varepsilon_\beta C_{;\alpha}.$$

Thus $C_{;\alpha} = 0$. Similarly if $r_+ \geq 3$, then we have $C_{;a} = 0$.

To prove assertion (11), we suppose $r_+ \geq 2$ and $r_- \geq 2$. We show $\nabla C = 0$ and $\nabla\phi = 0$ as follows. For fixed a, since $r_- \geq 2$, we may choose $\alpha \neq \beta$. We use assertions (2), (4), and (8) to see that

$$C_{;a} = -C\{\varepsilon_\beta\phi_{\beta a;\beta} + \varepsilon_\alpha\phi_{\alpha a;\alpha}\} = -C\{\varepsilon_\beta\phi_{\beta\beta;a} + \varepsilon_\alpha\phi_{\alpha\alpha;a}\} = 0.$$

Likewise, for fixed α, since $r_+ \geq 2$, we may choose $a \neq b$. We use assertions (2), (4), and (9) to see that $C_{;\alpha} = 0$. Thus $\nabla C = 0$. Moreover, we use assertions (3) and (5) to see that $\phi_{ij;k} = 0$ for all i, j, k. Thus $\nabla\phi = 0$. Consequently, we use assertion (6) to see that $\Gamma_{ia\alpha} = \frac{1}{2}\phi_{a\alpha;i} = 0$. Thus the distribution $\mathcal{F}_+$ is parallel and

$$0 = g(R(e_\alpha, e_a)e_a, e_\alpha) = C\varepsilon_a\varepsilon_\alpha\delta_a\delta_\alpha = -C\varepsilon_a\varepsilon_\alpha.$$

So $C = 0$ which is false. This completes the proof. $\square$

6.1.12 *Proof of Theorem E.* Assume $m \geq 10$. Let (M, g_M) be an IP pseudo-Riemannian manifold. Suppose the curvature tensor R at $P \in M$ is of C-ϕ type, we apply Lemma 6.1.11 to see that $r_+(\phi) \leq 1$ or $r_-(\phi) \leq 1$. $\square$

Theorem 1.3.3 constructed warped product metrics $ds_M^2 = dt^2 + f(t)ds_N^2$ on the product between an interval $I \subset \mathbb{R}$ and a Riemannian manifold N of constant sectional curvature $\mathcal{K}$ which are IP. Furthermore, the warping function $f(t)$ takes the form $f(t) = \mathcal{K}t^2 + At + B$, where A, B are auxilliary constants. Notice this construction corresponds to the case $r_+ = m - 1$ and $r_- = 1$ in Theorem E. Let R be the associated algebraic curvature tensor; $R(\pi)$ has constant eigenvalues $\{0, \pm\sqrt{-1}C\}$ where $C = \frac{4\mathcal{K}B - A^2}{4f^2}$. If $4\mathcal{K}B - A^2 = 0$, then this metric is flat. We therefore assume that $4\mathcal{K}B - A^2 \neq 0$. We now generalize the construction of Gilkey, Leahy and Sadofsky, and of Ivanov and Petrova to higher signatures. Topological suspension is a way of increasing the dimension. We introduce an analogous construction in the next section.

§6.2 Constructing Rank 2 IP Metrics Via Suspension

In §6.1, we have shown that $R_{C,\phi}$ is not geometrically realizable by a C-ϕ type IP metric if $\dim E_+ > 1$ and if $\dim E_- > 1$. Conversely, in Theorem F we use a warped product construction to give a C-ϕ type geometric realization of $R_{C,\phi}$ by an IP metric if $\dim E_+ \leq 1$ or if $\dim E_- \leq 1$. For clarity, we change our notation slightly at this point:

6.2.1 *Definition.* Let $x = (x_1, ..., x_p, x_{p+1}, ..., x_{p+q})$ be the usual coordinate on $\mathbb{R}^{p,q}$ so that the standard metric takes the form given in §1.1.3:

$$g_{(p,q)}(x, y) = -\textstyle\sum_{i=1}^{p} x_i y_i + \sum_{i=p+1}^{p+q} x_i y_i.$$

Let

$$(6.2.1.a)\qquad \Sigma_\varepsilon g_{(p,q)}((t,x),(\tilde{t},\tilde{x})) := \varepsilon\cdot t\tilde{t} + g_{(p,q)}(x,\tilde{x}) \text{ where } \varepsilon = \pm 1$$

be the suspension of the metric $g_{(p,q)}$. We let $\Sigma_\varepsilon \mathbb{R}^{p,q}$ be $\mathbb{R}^{p+q+1}$ with this metric. Note that $\Sigma_+ g_{(p,q)}$ is a metric of signature $(p, q+1)$, and that $\Sigma_- g_{(p,q)}$ is a metric of signature $(p+1, q)$. The first coordinate plays a distinguished role in our investigations. Let ϕ be an unipotent (of order 2) isometry of $\mathbb{R}^{p,q}$. We set

$$(6.2.1.b)\qquad \Sigma_\varepsilon(\phi) := \begin{pmatrix} \varepsilon & 0 \\ 0 & \phi \end{pmatrix}.$$

For $C \neq 0$, we have defined $R_{C,\phi}(x,y)z := C\{g(\phi(y),z)\phi(x) - g(\phi(x),z)\phi(y)\}$. We now define

$$(6.2.1.c)\qquad \Sigma_\varepsilon R_{C,\phi} := R_{C,\Sigma_\varepsilon(\phi)}.$$

By Theorem D, $\Sigma_\varepsilon R_{C,\phi}$ are IP algebraic curvature tensors. Similarly, we suspend a metric on a manifold $N(p,q)$ using a warped product construction.

6.2.2 *Definition.* Let $ds^2_{N(p,q)}$ be a metric of constant sectional curvature $\mathcal{K}$ on a manifold $N(p,q)$ of signature (p,q). Let $f_\varepsilon(t)$ be nonzero smooth real-valued functions defined on a connected interval $I \subset \mathbb{R}$. Let

$$(6.2.2.a)\qquad \Sigma_\varepsilon^{f_\varepsilon} ds^2_{N(p,q)} := \varepsilon dt^2 + f_\varepsilon(t) ds^2_{N(p,q)}$$

define warped product metrics of signatures $(p, q+1)$ and $(p+1, q)$ on $I \times N(p,q)$.

Let $N(p,q)$ be a manifold of signature (p,q) which has constant sectional curvature $\mathcal{K}$. We now determine the necessary and sufficient condition of the warping functions $f_\varepsilon(t)$ so that the resulting suspended metrics are nonconstant sectional curvature IP metrics of rank 2. Before beginning the proof of Theorem F, we establish a technical lemma. Fix a point P of $N(p,q)$. We choose local coordinates $x=(x^1,...,x^{p+q})$ on $N(p,q)$ so that

$$g_{ii}(P)=\varepsilon_i=\pm 1,\ g_{ij}(P)=0 \text{ for } i\neq j, \text{ and } g_{ij/k}(P)=0.$$

We let indices i,j,k,ℓ range over 1 through $p+q$ and index the coordinate frames $\{\partial_i := \frac{\partial}{\partial x^i}\}$ and $\{dx^i\}$ for the tangent and cotangent bundles of $N(p,q)$. Let $\partial_0 := \frac{\partial}{\partial t}$. These are not orthonormal frames. Let g,∇,Γ and R be defined by the metric on $N(p,q)$.

6.2.3 Lemma. *Let $f_\varepsilon(t) := e^{2h_\varepsilon(t)}$. Let ${}^\varepsilon g, {}^\varepsilon\nabla, {}^\varepsilon\Gamma$ and ${}^\varepsilon R$ be defined by the suspended metrics $\Sigma_\varepsilon^{f_\varepsilon} ds^2_{N(p,q)}$ on $I\times N(p,q)$ given in equation* (6.2.2.a). *We have*

(1) ${}^\varepsilon R(\partial_i,\partial_j,\partial_k,\partial_\ell)(t,P)=e^{2h_\varepsilon}\{\mathcal{K}-\varepsilon\dot h_\varepsilon^2 e^{2h_\varepsilon}\}\varepsilon_i\varepsilon_j\big(\delta_{i\ell}\delta_{jk}-\delta_{ik}\delta_{j\ell}\big)$.

(2) ${}^\varepsilon R(\partial_i,\partial_0,\partial_0,\partial_j)(t,P)=-e^{2h_\varepsilon}\{\ddot h_\varepsilon+\dot h_\varepsilon^2\}\varepsilon_i\delta_{ij}$.

(3) *The curves $t\mapsto(t,x)$ are unit speed geodesics.*

Proof. We have $R(\partial_i,\partial_j,\partial_k,\partial_\ell)(t,P)=\mathcal{K}\varepsilon_i\varepsilon_j\{\delta_{i\ell}\delta_{jk}-\delta_{ik}\delta_{j\ell}\}$. Recall that

$$\Gamma_{uvw}=\tfrac{1}{2}(\partial_u g_{vw}+\partial_v g_{uw}-\partial_w g_{uv}) \text{ and } \Gamma_{uv}{}^x=g^{xy}\cdot\Gamma_{uvy}$$

where these indices range between 0 and $p+q$. We use these identities to see that for i,j,k,ℓ ranging over 1 through $p+q$

(1) We have ${}^\varepsilon\Gamma_{ijk}=\Gamma_{ijk}$ and ${}^\varepsilon\Gamma_{ij}{}^k=\Gamma_{ij}{}^k$.

(2) We have ${}^\varepsilon\Gamma_{ijk}(t,P)=\Gamma_{ijk}(t,P)=0$ and ${}^\varepsilon\Gamma_{ij}{}^k(t,P)=\Gamma_{ij}{}^k(t,P)=0$.

(3) We have ${}^\varepsilon\Gamma_{i0j}(t,P)=-{}^\varepsilon\Gamma_{ij0}(t,P)=\dot h_\varepsilon e^{2h_\varepsilon}\varepsilon_i\delta_{ij}$.

(4) We have ${}^\varepsilon\Gamma_{jk}{}^0(t,P)={}^\varepsilon g^{0u}\cdot{}^\varepsilon\Gamma_{jku}(t,P)=-\varepsilon\dot h_\varepsilon e^{2h_\varepsilon}\varepsilon_j\delta_{jk}$.

(5) We have ${}^{\varepsilon}\Gamma_{i0}{}^{\ell}(t,P) = {}^{\varepsilon}g^{\ell u} \cdot {}^{\varepsilon}\Gamma_{i0u}(t,P) = \dot{h}_{\varepsilon}\varepsilon_{\ell}\varepsilon_{i}\delta_{i\ell}$. We use these relations to prove assertion (1) by computing:

$$
\begin{aligned}
&{}^{\varepsilon}R(\partial_i,\partial_j,\partial_k,\partial_\ell)(t,P)\\
&= \{{}^{\varepsilon}g(({}^{\varepsilon}\nabla_{\partial_i}{}^{\varepsilon}\nabla_{\partial_j} - {}^{\varepsilon}\nabla_{\partial_j}{}^{\varepsilon}\nabla_{\partial_i})\partial_k,\partial_\ell)\}(t,P)\\
&= \{{}^{\varepsilon}g({}^{\varepsilon}\nabla_{\partial_i}({}^{\varepsilon}\Gamma_{jk}{}^{u}\cdot\partial_u),\partial_\ell) - {}^{\varepsilon}g({}^{\varepsilon}\nabla_{\partial_j}({}^{\varepsilon}\Gamma_{ik}{}^{u}\cdot\partial_u),\partial_\ell)\}(t,P)\\
&= \{(\partial_i{}^{\varepsilon}\Gamma_{jk}{}^{n} - \partial_j{}^{\varepsilon}\Gamma_{ik}{}^{n} + {}^{\varepsilon}\Gamma_{jk}{}^{0}\cdot{}^{\varepsilon}\Gamma_{i0}{}^{n} - {}^{\varepsilon}\Gamma_{ik}{}^{0}\cdot{}^{\varepsilon}\Gamma_{j0}{}^{n})\cdot{}^{\varepsilon}g(\partial_n,\partial_\ell)\}(t,P)\\
&= \{(\partial_i\Gamma_{jk}{}^{n} - \partial_j\Gamma_{ik}{}^{n} + {}^{\varepsilon}\Gamma_{jk}{}^{0}\cdot{}^{\varepsilon}\Gamma_{i0}{}^{n} - {}^{\varepsilon}\Gamma_{ik}{}^{0}\cdot{}^{\varepsilon}\Gamma_{j0}{}^{n})\cdot{}^{\varepsilon}g(\partial_n,\partial_\ell)\}(t,P)\\
&= \{e^{2h_\varepsilon}(R_{ijk\ell} + \varepsilon_\ell({}^{\varepsilon}\Gamma_{jk}{}^{0}\cdot{}^{\varepsilon}\Gamma_{i0}{}^{\ell} - {}^{\varepsilon}\Gamma_{ik}{}^{0}\cdot{}^{\varepsilon}\Gamma_{j0}{}^{\ell}))\}(t,P)\\
&= \{e^{2h_\varepsilon}(R_{ijk\ell} + \varepsilon_\ell[(-\varepsilon\dot{h}_\varepsilon^2)\varepsilon_j\delta_{jk}e^{2h_\varepsilon}\varepsilon_\ell\varepsilon_i\delta_{i\ell} - (-\varepsilon\dot{h}_\varepsilon^2)\varepsilon_i\delta_{ik}e^{2h_\varepsilon}\varepsilon_\ell\varepsilon_j\delta_{j\ell}])\}(t,P)\\
&= \{e^{2h_\varepsilon}(\mathcal{K} - \varepsilon\dot{h}_\varepsilon^2 e^{2h_\varepsilon})\varepsilon_i\varepsilon_j(\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell})\}(t,P).
\end{aligned}
$$

Since ${}^{\varepsilon}g({}^{\varepsilon}\nabla_{\partial_i}\partial_0,\partial_0) = \frac{1}{2}\partial_i{}^{\varepsilon}g(\partial_0,\partial_0) = 0$ and since

$$
{}^{\varepsilon}\Gamma_{i0j}(t,P) = {}^{\varepsilon}g({}^{\varepsilon}\nabla_{\partial_i}\partial_0,\partial_j)(t,P) = \{\dot{h}_\varepsilon e^{2h_\varepsilon}\varepsilon_i\delta_{ij}\}(t,P),
$$

we have: ${}^{\varepsilon}\nabla_{\partial_i}\partial_0(t,P) = \dot{h}_\varepsilon\partial_i(t,P)$. We prove assertion (2) by computing:

$$
\begin{aligned}
&{}^{\varepsilon}R(\partial_i,\partial_0,\partial_0,\partial_j)(t,P)\\
&= \{{}^{\varepsilon}g(({}^{\varepsilon}\nabla_{\partial_i}{}^{\varepsilon}\nabla_{\partial_0}\partial_0 - {}^{\varepsilon}\nabla_{\partial_0}{}^{\varepsilon}\nabla_{\partial_i}\partial_0),\partial_j)\}(t,P)\\
&= -\{{}^{\varepsilon}g({}^{\varepsilon}\nabla_{\partial_0}{}^{\varepsilon}\nabla_{\partial_i}\partial_0,\partial_j)\}(t,P)\\
&= -\{{}^{\varepsilon}g({}^{\varepsilon}\nabla_{\partial_0}(\dot{h}_\varepsilon\partial_i),\partial_j)\}(t,P)\\
&= -\{{}^{\varepsilon}g(\ddot{h}_\varepsilon\partial_i + \dot{h}_\varepsilon{}^{\varepsilon}\nabla_{\partial_0}\partial_i,\partial_j)\}(t,P)\\
&= \{-(\ddot{h}_\varepsilon + \dot{h}_\varepsilon^2){}^{\varepsilon}g(\partial_i,\partial_j)\}(t,P)\\
&= -\{e^{2h_\varepsilon}(\ddot{h}_\varepsilon + \dot{h}_\varepsilon^2)\}\varepsilon_i\delta_{ij}(t,P).
\end{aligned}
$$

We prove assertion (3) by computing:

$$
\begin{aligned}
{}^{\varepsilon}g({}^{\varepsilon}\nabla_{\partial_0}\partial_0, \partial_0) &= \tfrac{1}{2}\partial_0{}^{\varepsilon}g(\partial_0, \partial_0) = 0;\\
{}^{\varepsilon}g({}^{\varepsilon}\nabla_{\partial_0}\partial_0, \partial_i) &= -{}^{\varepsilon}g(\partial_0, {}^{\varepsilon}\nabla_{\partial_0}\partial_i)\\
&= -{}^{\varepsilon}g(\partial_0, {}^{\varepsilon}\nabla_{\partial_i}\partial_0)\\
&= -\tfrac{1}{2}\partial_i{}^{\varepsilon}g(\partial_0, \partial_0) = 0. \quad \square
\end{aligned}
$$

6.2.4 *Proof of Theorem F.* We begin with normalizing the coordinate frame $\{\partial_0 = \frac{\partial}{\partial t}, \partial_i = \frac{\partial}{\partial x^i}\}$ by setting $e_0 := \partial_0$ and $e_i := e^{-h_\varepsilon}\partial_i$ for $i \geq 1$. Since f_ε is a nonzero smooth function defined on a connected interval I, by replacing g_N by $-g_N$ if necessary, we may assume $f_\varepsilon > 0$ on I; so we may set $f_\varepsilon(t) = e^{2h_\varepsilon(t)}$. Thus we have

$$\dot{h}_\varepsilon = \tfrac{1}{2}\frac{d}{dt}(\ln f_\varepsilon) = \frac{\dot{f}_\varepsilon}{2f_\varepsilon} \text{ and } \ddot{h}_\varepsilon = \frac{2\ddot{f}_\varepsilon f_\varepsilon - 2\dot{f}_\varepsilon^2}{4f_\varepsilon^2}.$$

We use assertions (1) and (2) of Lemma 6.2.3 and normalize the bases to see that

$$
\begin{aligned}
{}^{\varepsilon}R(e_i, e_j, e_j, e_i)(t, P) &= (\mathcal{K}e^{-2h_\varepsilon} - \varepsilon\dot{h}_\varepsilon^2)\varepsilon_i\varepsilon_j\\
&= \left\{\frac{\mathcal{K}}{f_\varepsilon} - \varepsilon\left(\frac{\dot{f}_\varepsilon}{2f_\varepsilon}\right)^2\right\}\varepsilon_i\varepsilon_j\\
&= \frac{4\mathcal{K}f_\varepsilon - \varepsilon\dot{f}_\varepsilon^2}{4f_\varepsilon^2}\varepsilon_i\varepsilon_j;
\end{aligned}
\tag{6.2.4.a}
$$

$$
\begin{aligned}
{}^{\varepsilon}R(e_i, e_0, e_0, e_i)(t, P) &= -(\ddot{h}_\varepsilon + \dot{h}_\varepsilon^2)\varepsilon_i\varepsilon\\
&= -\left\{\frac{2\ddot{f}_\varepsilon f_\varepsilon - 2\dot{f}_\varepsilon^2}{4f_\varepsilon^2} + \left(\frac{\dot{f}_\varepsilon}{2f_\varepsilon}\right)^2\right\}\varepsilon_i\varepsilon\\
&= -\frac{2\ddot{f}_\varepsilon f_\varepsilon - \dot{f}_\varepsilon^2}{4f_\varepsilon^2}\varepsilon_i\varepsilon;
\end{aligned}
\tag{6.2.4.b}
$$

$$
{}^{\varepsilon}R(e_i, e_j, e_k, e_\ell)(t, P) = 0 \text{ for } (i,j) \neq (k,\ell) \text{ and } (i,j) \neq (\ell,k). \tag{6.2.4.c}
$$

Assume $f_\varepsilon(t) = \varepsilon\mathcal{K}t^2 + At + B$ for $4\mathcal{K}B - \varepsilon A^2 \neq 0$. We use equation (6.2.4.a) to see that

$$\begin{aligned} {}^{\varepsilon}R(e_i, e_j, e_j, e_i)(t, P) &= \frac{4\mathcal{K}(\varepsilon\mathcal{K}t^2 + At + B) - \varepsilon(2\varepsilon\mathcal{K}t + A)^2}{4f_\varepsilon^2}\varepsilon_i\varepsilon_j \\ &= \frac{4\mathcal{K}B - \varepsilon A^2}{4f_\varepsilon^2}\varepsilon_i\varepsilon_j. \end{aligned} \tag{6.2.4.d}$$

We use equation (6.2.4.b) to see that

$$\begin{aligned} {}^{\varepsilon}R(e_i, e_0, e_0, e_i)(t, P) &= -\frac{2(2\varepsilon\mathcal{K})(\varepsilon\mathcal{K}t^2 + At + B) - (2\varepsilon\mathcal{K}t + A)^2}{4f_\varepsilon^2}\varepsilon_i\varepsilon \\ &= -\frac{4\mathcal{K}B - \varepsilon A^2}{4f_\varepsilon^2}\varepsilon_i\varepsilon. \end{aligned} \tag{6.2.4.e}$$

Let $C_\varepsilon := \frac{4\mathcal{K}B - \varepsilon A^2}{4f_\varepsilon^2}$. Let

$$\Sigma_\varepsilon\phi(e_u) := \begin{cases} -e_0 & \text{if } u = 0, \\ e_u & \text{if } 1 \leq u \leq m. \end{cases}$$

We use equation (6.1.4) to see that ${}^{\varepsilon}R = R_{C_\varepsilon, \Sigma_\varepsilon(\phi)}$ are rank 2 IP algebraic curvature tensors and that the suspended metrics $\Sigma_\varepsilon^{f_\varepsilon} ds^2_{N(p,q)}$ are rank 2 IP metrics. Moreover, equations (6.2.4.d) and (6.2.4.e) imply the suspended metrics do not have constant sectional curvature.

Conversely, we assume the suspended metrics $\Sigma_\varepsilon^{f_\varepsilon} ds^2_{N(p,q)}$ are IP. We use equations (6.2.4.a) and (6.2.4.b) to see that

$$\frac{4\mathcal{K}f_\varepsilon - \varepsilon\dot{f}_\varepsilon^2}{4f_\varepsilon^2}\varepsilon_i\varepsilon_j = \sigma(\{-\frac{2\ddot{f}_\varepsilon f_\varepsilon - \varepsilon\dot{f}_\varepsilon^2}{4f_\varepsilon^2}\}\varepsilon_i\varepsilon) \text{ where } \sigma = \pm 1.$$

Case 1. Suppose $\sigma = 1$. Then

$$\frac{4\mathcal{K}f_\varepsilon - \varepsilon\dot{f}_\varepsilon^2}{4f_\varepsilon^2}\varepsilon_i\varepsilon_j = -\frac{2\ddot{f}_\varepsilon f_\varepsilon - \varepsilon\dot{f}_\varepsilon^2}{4f_\varepsilon^2}\varepsilon_i\varepsilon.$$

We compute the sectional curvature of the 2-plane $\pi_1 := \operatorname{Span}\{e_i, e_j\}$ to be

$$\frac{{}^{\varepsilon}R(e_i, e_j, e_j, e_i)}{g(e_i, e_i)g(e_j, e_j) - g(e_i, e_j)^2} = \frac{4\mathcal{K}f_\varepsilon - \varepsilon\dot{f}_\varepsilon^2}{4f_\varepsilon^2}.$$

We compute the sectional curvature of the 2-plane $\pi_2 := \mathrm{Span}\{e_i, e_0\}$ to be

$$\frac{{}^{\varepsilon}R(e_i, e_0, e_0, e_i)}{g(e_i, e_i)g(e_0, e_0) - g(e_i, e_0)^2} = -\frac{2\ddot{f}_\varepsilon f_\varepsilon - \varepsilon \dot{f}_\varepsilon^2}{4f_\varepsilon^2}.$$

Thus the metrics give constant sectional curvature.

Case 2. Suppose $\sigma = -1$. Then

$$\frac{4\mathcal{K}f_\varepsilon - \varepsilon \dot{f}_\varepsilon^2}{4f_\varepsilon^2}\varepsilon_i\varepsilon_j = \frac{2\ddot{f}_\varepsilon f_\varepsilon - \varepsilon \dot{f}_\varepsilon^2}{4f_\varepsilon^2}\varepsilon_i\varepsilon.$$

Since the nonzero eigenvalues of ${}^{\varepsilon}R(e_i, e_j)$ and ${}^{\varepsilon}R(e_i, e_0)$ are identical, we must have $\ddot{f}_\varepsilon = 2\varepsilon\mathcal{K}$; this implies $f_\varepsilon(t) = \varepsilon\mathcal{K}t^2 + At + B$. Furthermore, we compute the nonzero eigenvalues of ${}^{\varepsilon}R(\pi)$ are $\varepsilon\sqrt{-1}C_\varepsilon$ where $C_\varepsilon = \frac{4\mathcal{K}B - \varepsilon A^2}{4f_\varepsilon^2}$. Thus if $\Sigma_\varepsilon^{f_\varepsilon} ds^2_{N(p,q)}$ has rank 2, then $4\mathcal{K}B - \varepsilon A^2 \neq 0$; this proves Theorem F. $\square$

§6.3 Proof of Theorem G

In this section, we complete the proof of Theorem G. We begin with some notational conventions and a technical lemma. Let ϕ be an unipotent (of order 2) isometry of $\mathbb{R}^{p,q}$. We adopt the notational conventions established in §6.1.10. Let $y := (y^1, ..., y^{m-1})$ be local coordinates on a leaf of the foliation $\mathcal{F}_-$. We use $T(t, y) := \exp_t(te_1(y))$ to define local coordinates on M^m. We adopt arguments of Gilkey, Leahy and Sadofsky [48] to prove

6.3.1 Lemma. *Let $m \geq 4$. Let g be an IP metric of rank 2 which is C-ϕ type with ϕ an unipotent (of order 2) isometry. Let $R := R_{C,\phi}$. Assume $r_+ = 1$.*

(1) *For any α, $C_{;\alpha} = 0$, $C_{;1} = -2C\varepsilon_\alpha\phi_{1\alpha;\alpha}$, and $\Gamma_{\alpha 1\beta} = -\frac{1}{4}\varepsilon_\alpha\delta_{\alpha\beta}C^{-1}C_{;1}$.*

(2) *For fixed y_0, the curves $t \mapsto T(t, y_0)$ are unit speed geodesics in M^m which are leaves of the foliation $\mathcal{F}_+$.*

(3) *For fixed t_0, the hypersurfaces $T(t_0, y)$ are leaves of the foliation $\mathcal{F}_-$ and inherit metrics of constant sectional curvature.*

(4) *Locally the metric on M^m is given by $ds^2 = \varepsilon_1 dt^2 + f(t)ds_{\mathcal{K}}^2$ where $f(t)$ is a nonzero smooth function defined on a connected open interval $I \subset \mathbb{R}$ and $ds_{\mathcal{K}}^2$ is a metric of constant sectional curvature $\mathcal{K}$.*

Proof. Since $r_+ = 1$ and $m \geq 4$, $a = 1$ and $r_- \geq 3$. We use Lemma 6.1.11 (10) to see that $C_{;\alpha} = 0$. Since $r_- \geq 4$, we may choose α, β, γ distinct, and we use Lemma 6.1.11 (8) to see that

$$C_{;1} = -C\{\varepsilon_\beta \phi_{\beta 1;\beta} + \varepsilon_\alpha \phi_{\alpha 1;\alpha}\} = -C\{\varepsilon_\gamma \phi_{\gamma 1;\gamma} + \varepsilon_\alpha \phi_{\alpha 1;\alpha}\}, \text{ so } \varepsilon_\beta \phi_{\beta 1;\beta} = \varepsilon_\gamma \phi_{\gamma 1;\gamma}.$$

Thus $C_{;1} = -2C\varepsilon_\alpha \phi_{\alpha 1;\alpha}$. We use Lemma 6.1.11 (2), (5), and (6) to see that

$$\Gamma_{\alpha 1 \beta} = \tfrac{1}{2}\phi_{1\beta;\alpha} = \tfrac{1}{2}\delta_{\alpha\beta}\phi_{1\alpha;\alpha} = \tfrac{1}{2}\phi_{\alpha 1;\alpha} = -\tfrac{1}{4}\varepsilon_\alpha \delta_{\alpha\beta} C^{-1} C_{;1}.$$

Assertion (1) follows. Clearly $\Gamma_{111} = 0$ and by Lemma 6.1.11 (6), (8), and (10) we have

$$\Gamma_{11\alpha} = \tfrac{1}{2}\phi_{1\alpha;1} = C\varepsilon_1 C^{-1} C_{;\alpha} = 0.$$

This shows the integral curves for e_1 are unit speed geodesics; assertion (2) now follows. We now compute:

$$\partial_t g(\partial_t, \partial_\alpha^y) = g(\partial_t, \nabla_{\partial_t}\partial_\alpha^y) + g(\partial_\alpha^y, \nabla_{\partial_t}\partial_t) = g(\partial_t, \nabla_{\partial_t}\partial_\alpha^y) = \tfrac{1}{2}\partial_\alpha^y g(\partial_t, \partial_t) = 0.$$

Thus $\partial_t \perp \partial_\alpha^y$. this shows ∂_α^y span the perpendicular distribution $\mathcal{F}_-$ and the hypersurfaces $T(t_0, y)$ are leaves of the foliation $\mathcal{F}_-$. We need some additional notation at this point. Let X, Y be vector fields on the leaves of the foliation $\mathcal{F}_-$. Let $L(X,Y) := \nabla_X Y - \Pi_- \nabla_X Y$ be the normal component of $\nabla_X Y$. We have $L(e_\alpha, e_\beta) = g(L(e_\alpha, e_\beta), e_1)e_1 = \varepsilon_1 \Gamma_{\alpha\beta 1} e_1$. Let R_- be the associated curvature tensor of the induced metric on the leaves of $\mathcal{F}_-$. We use the Gauss-Codazzi equation to see that

$$\begin{aligned} R_-(e_\alpha, e_\beta, e_\gamma, e_\sigma) =& R(e_\alpha, e_\beta, e_\gamma, e_\sigma) + g(L(e_\alpha, e_\gamma), L(e_\beta, e_\sigma)) \\ & - g(L(e_\alpha, e_\sigma), L(e_\beta, e_\gamma)) \\ =& R(e_\alpha, e_\beta, e_\gamma, e_\sigma) - \varepsilon_1(\Gamma_{\beta\gamma 1}\Gamma_{\alpha 1\sigma} - \Gamma_{\alpha\gamma 1}\Gamma_{\beta 1\sigma}). \end{aligned}$$

We use assertion (1) to see $\Gamma_{\beta\gamma 1}\Gamma_{\alpha 1\sigma} - \Gamma_{\alpha\gamma 1}\Gamma_{\beta 1\sigma} = 0$; assertion (3) now follows. It remains to show that the metric g is locally a warped product. We express $\partial^y_\alpha = \sum_\gamma a_{\alpha\gamma} e_\gamma$. We compute:

$$\begin{aligned} g(\nabla_{\partial_t}\partial^y_\alpha, \partial^y_\beta) =& g(\nabla_{\partial^y_\alpha}\partial_t, \partial^y_\beta) \\ =& g(\textstyle\sum_\gamma a_{\alpha\gamma}\nabla_{e_\gamma}\partial_t, \partial^y_\beta) \\ =& \textstyle\sum_\gamma a_{\alpha\gamma} g(\nabla_{e_\gamma}\partial_t, \sum_\sigma a_{\beta\sigma} e_\sigma) \\ =& \textstyle\sum_{\gamma,\sigma} a_{\alpha\gamma}a_{\beta\sigma} g(\nabla_{e_\gamma}\partial_t, e_\sigma) \\ =& \textstyle\sum_{\gamma,\sigma} a_{\alpha\gamma}a_{\beta\sigma}\Gamma_{\gamma 1\sigma} \\ =& -\tfrac{1}{4}C^{-1}C_{;1}(\textstyle\sum_{\gamma,\sigma}\varepsilon_\gamma\delta_{\gamma\alpha}a_{\alpha\gamma}a_{\beta\sigma}) \\ =& -\tfrac{1}{4}C^{-1}C_{;1}(\textstyle\sum_{\gamma,\sigma}\varepsilon_\gamma a_{\alpha\gamma}a_{\beta\gamma}) \\ =& -\tfrac{1}{4}C^{-1}C_{;1}g_{\alpha\beta}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \partial_t g(\partial^y_\alpha, \partial^y_\beta) =& 2g(\nabla_{\partial^y_\alpha}\partial_t, \partial^y_\beta) \\ =& 2\Gamma_{\alpha 1\beta} = -\tfrac{1}{2}\varepsilon_\alpha\delta_{\alpha\beta}C^{-1}C_{;1} \\ =& -\tfrac{1}{2}C^{-1}C_{;1}g_{\alpha\beta}. \end{aligned}$$

Since $C^{-1}C_{;1}$ depends only on the parameter t, the metrics g is locally given by $ds^2 = \varepsilon_1 dt^2 + f(t)ds^2_K$. $\square$

6.3.2 *Proof of Theorem G* (2). We now use steps 1 through 7 outlined in §1.4.2 to complete the proof. $\square$

§6.4 The Orthogonal Equivalence of the Curvature Tensors $R_{C,\phi}$

We conclude this chapter by giving necessary and sufficient conditions in Theorem 6.4.5 that $R_{C,\phi}$ and $R_{\tilde{C},\tilde{\phi}}$ are orthogonally equivalent. We first introduce some additional notation.

6.4.1 *Definition.* If ψ is an isometry of $\mathbb{R}^{p,q}$ and if $R \in \otimes^4(\mathbb{R}^{p,q})$, then we define the *pull-back* 4-tensor $\psi^* R$ by

$$(\psi^* R)(x, y, z, w) := R(\psi(x), \psi(y), \psi(z), \psi(w)).$$

Note that R is an algebraic curvature tensor if and only if $\psi^* R$ is an algebraic curvature tensor; R is IP if and only if $\psi^* R$ is IP. This gives the natural action of the isometry group $\mathrm{O}(p, q)$ on these tensors. We say that R and $\tilde{R}$ are *orthogonally equivalent* if and only if there exists $\psi \in \mathrm{O}(p, q)$ so $\psi^* R = \tilde{R}$.

Both the Ricci operator and the Jacobi operator will play an important role in this section. We recall their definitions briefly.

6.4.2 *Definition.* Let ρ be the Ricci tensor defined by an algebraic curvature tensor R. We have:

$$\rho(x, y) := \mathrm{Tr}(z \mapsto R(z, x)y) = g^{ij} R(e_i, x, y, e_j).$$

This tensor is symmetric. In the Riemannian setting, $\rho(x, x)$ is the average sectional curvature of all the 2-planes containing x. Let $\tilde{\rho}$ be the associated endomorphism:

$$\tilde{\rho}(x) := g^{ij} \rho(x, e_i) e_j.$$

This is characterized by the identity:

$$g(\tilde{\rho}(x), y) := \rho(x, y).$$

The eigenvalues of $\tilde{\rho}$ are orthogonal invariants.

We recall from §1.5.1 the definition of the Jacobi operator $J(x) : y \mapsto R(y,x)x$. We also recall that $g(e_i, e_i) = \varepsilon_i e_i$ and $\phi(e_i) = \delta_i e_i$ where $\varepsilon_i = \pm 1$, $\delta_i = \pm 1$. We omit the proof of the following lemma as it is an immediate algebraic consequence of Lemma 6.1.4 and the above definitions.

6.4.3 Lemma. *Let $\{e_i\}$ be a normalized orthonormal basis. Let $R = R_{C,\phi}$ define $\rho, \tilde{\rho}$, and J.*

(1) *If $i \neq j$, then $\rho(e_i, e_j) = 0$. We have $\rho(e_i, e_i) = C\varepsilon_i\delta_i \sum_{j\neq i} \delta_j$.*

(2) *If $\delta_i = 1$, then $\tilde{\rho}(e_i) = C(r_+ - 1 - r_-)e_i$.*

(3) *If $\delta_i = -1$, then $\tilde{\rho}(e_i) = C(r_- - 1 - r_+)e_i$.*

(4) *If $i \neq j$, then $J(e_i)e_j = C\varepsilon_i\delta_i\delta_j e_j$. We have $J(e_i)e_i = 0$.*

Fix an idempotent isometry ϕ of $\mathbb{R}^{p,q}$. Let

$$\mathfrak{G} := \{x \in \mathbb{R}^{p,q} : |x|^2 = 1 \text{ and } J(x) \text{ has eigenvalues } \pm C \text{ on } x^\perp\},$$

$$\mathcal{S}(E_\pm) := \{x \in E_\pm : |x|^2 = 1\} \text{ and } \mathcal{N}(E_\pm) := \{x \in E_\pm : |x|^2 = 0\}.$$

We show the space $\mathfrak{G}$ is homotopy equivalent to $S^{p_+-1} \dot\sqcup S^{p_--1}$. Since the homotopy type of $\mathfrak{G}$ is an orthogonal invariant, the unordered pair (p_+, p_-) is also an orthogonal invariant of $R_{C,\phi}$.

6.4.4 Lemma.

(1) *The space $\mathfrak{G}$ is homeomorphic to $\mathcal{S}(E_+) \times \mathcal{N}(E_-) \dot\sqcup \mathcal{N}(E_+) \times \mathcal{S}(E_-)$.*

(2) *The space $\mathcal{N}(E_\pm)$ is contractible.*

(3) *The space $\mathcal{S}(E_+)$ is homeomorphic to $S^{p_+-1} \times \mathbb{R}^{q_+}$.*

(4) *The space $\mathcal{S}(E_-)$ is homeomorphic to $S^{p_--1} \times \mathbb{R}^{q_-}$.*

(5) *The space $\mathfrak{G}$ is homotopy equivalent to $S^{p_+-1} \dot\sqcup S^{p_--1}$.*

Proof. We decompose $\mathbb{R}^{p,q} = E_+ \oplus E_-$, and we identify $\mathbb{R}^{p,q}$ with the Cartesian product $E_+ \times E_-$. We first show $\mathfrak{G} \subseteq \mathcal{S}(E_+) \times \mathcal{N}(E_-) \dot\sqcup \mathcal{N}(E_+) \times \mathcal{S}(E_-)$. Let $x \in \mathfrak{G}$. Decompose $x = x_+ + x_-$ for $x_\pm \in E_\pm$. Since $|x|^2 = 1$, we have $|x_+|^2 + |x_-|^2 = 1$.

To show $x \in \mathcal{S}(E_+) \times \mathcal{N}(E_-) \dot{\cup} \mathcal{N}(E_+) \times \mathcal{S}(E_-)$, it suffices to show either x_+ or x_- is a null vector. Suppose the contrary. We choose a normalized orthonormal basis $\{e_i\}$ for $\mathbb{R}^{p,q}$ so that $x_+ = a_1 e_1$ and that $x_- = a_2 e_2$, where $a_1 \neq 0$ and $a_2 \neq 0$. Then we have:

$$(6.4.4.a) \qquad 1 = |x_+|^2 + |x_-|^2 = a_1^2 \varepsilon_1 + a_2^2 \varepsilon_2.$$

We compute: $J(x)e_3 = C\{g(\phi x, x)\phi e_3 - g(\phi e_3, x)\phi x\} = C\{\delta_3(a_1^2\varepsilon_1 - a_2^2\varepsilon_2)e_3\}$. Since $e_3 \perp x$ and since $x \in \mathfrak{G}$, we have:

$$(6.4.4.b) \qquad a_1^2\varepsilon_1 - a_2^2\varepsilon_2 = \pm 1.$$

If $a_1^2\varepsilon_1 + a_2^2\varepsilon_2 = 1$ and $a_1^2\varepsilon_1 - a_2^2\varepsilon_2 = -1$, then we add equations (6.4.4.a) and (6.4.4.b) to see that $2a_1^2\varepsilon_1 = 0$, so $a_1 = 0$ which is false. If $a_1^2\varepsilon_1 + a_2^2\varepsilon_2 = 1$ and $a_1^2\varepsilon_1 - a_2^2\varepsilon_2 = 1$, then we subtract equations (6.4.4.a) and (6.4.4.b) to see that $2a_2^2\varepsilon_2 = 0$, so $a_2 = 0$ which is false. Thus $\mathfrak{G} \subseteq \mathcal{S}(E_+) \times \mathcal{N}(E_-) \dot{\cup} \mathcal{N}(E_+) \times \mathcal{S}(E_-)$.

Next, we show $\mathfrak{G} \supseteq \mathcal{S}(E_+) \times \mathcal{N}(E_-) \dot{\cup} \mathcal{N}(E_+) \times \mathcal{S}(E_-)$. Suppose $x = x_+ + x_-$ where $|x_+|^2 = 1$ and x_- is a null vector; the other case is similar as one can replace ϕ by $-\phi$ to interchange the roles of r_+ and r_-. We choose a normalized orthonormal basis $\{e_i\}$ for $\mathbb{R}^{p,q}$ with $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $\varepsilon_3 = -1$; $\delta_1 = 1$, $\delta_2 = -1$ and $\delta_3 = -1$ so that $x = e_1 + a(e_2 + e_3)$ for some constant a. We complete the proof of assertion (1) by showing $x \in \mathfrak{G}$. If $i > 3$, then we use Lemma 6.4.3 to see that $J(x)e_i = C\delta_i e_i$. Hence we must show the eigenvalues of $J(x)$ on the three dimensional space spanned by $\{e_1, e_2, e_3\}$ are $\{0, -C, -C\}$. We have from the definition that

$$\begin{aligned}
J(x)y =& R(y,x)x = Cg(\phi(x), x)\phi(y) - Cg(\phi(y), x)\phi(x) \\
=& Cg(e_1 - a(e_2+e_3), e_1 + a(e_2+e_3))\phi(y) - Cg(\phi(y), x)\phi(x) \\
=& C\phi(y) - Cg(\phi(y), e_1 + ae_2 + ae_3)(e_1 - a(e_2+e_3)).
\end{aligned}$$

We compute the action of $J(x)$ to be:

$$\begin{aligned}
J(x)x &= 0,\\
J(x)(e_2+e_3) &= C(\phi(e_2)+\phi(e_3))\\
&\quad - Cg(\phi(e_2)+\phi(e_3), e_1+ae_2+ae_3)(e_1-ae_2-ae_3)\\
&= -C\{(e_2+e_3) - g(e_2+e_3, e_1+ae_2+ae_3)(e_1-ae_2-ae_3)\}\\
&= -C(e_2+e_3),\\
J(x)(e_2-e_3) &= C(\phi(e_2)-\phi(e_3))\\
&\quad - Cg(\phi(e_2)-\phi(e_3), e_1+ae_2+ae_3)(e_1-ae_2-ae_3)\\
&= C\{e_3-e_2) - g(e_3-e_2, e_1+ae_2+ae_3)(e_1-ae_2-ae_3)\}\\
&= C\{e_3-e_2) + 2a(e_1-a(e_2+e_3)\}\\
&= C\{e_3-e_2) + 2a(e_1+a(e_2+e_3) - 4a^2(e_2+e_3)\}\\
&= C\{2ax - 4a^2(e_2+e_3) - (e_2-e_3)\}.
\end{aligned}$$

Thus $J(x)$ is represented by an upper triangular matrix relative to this basis:

$$J(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -C & 0 \\ 2aC & -4a^2C & -C \end{pmatrix}. \tag{6.4.4.c}$$

We use matrix (6.4.4.c) to compute the characteristic polynomial:

$$\det(\lambda - J(x)) = \det \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda+C & 0 \\ -2aC & 4a^2C & \lambda+C \end{pmatrix} = \lambda(\lambda+C)^2. \tag{6.4.4.d}$$

We use equation (6.4.4.d) to see the eigenvalues of $J(x)$ are $\{0, -C, -C\}$. This completes the proof of assertion (1).

Let $I := [0,1]$. We prove assertion (2) by constructing the deformation retract $H_0 : \mathcal{N}(E_\pm) \times I \to \mathcal{N}(E_\pm)$ by $H_0(x,t) := (1-t)x$; geometrically speaking, we are sliding each null vector along the null cone to the origin. We prove assertions (3) and (4) by constructing a homeomorphism $H_\pm : S^{p_\pm - 1} \times \mathbb{R}^{q_\pm} \to \mathcal{S}(E_\pm)$ by

$$H_\pm(u,v) := (\sqrt{1+|v|^2}u, v).$$

The final assertion now follows. $\square$

We can now characterize the curvature tensors $R_{C,\phi}$ up to orthogonal equivalence in the following statement:

6.4.5 Theorem. *The following assertions are equivalent:*

(1) $R_{C,\phi}$ *and* $R_{\tilde{C},\tilde{\phi}}$ *are orthogonally equivalent.*

(2) $C = \tilde{C}$ *and* ϕ *is orthogonally equivalent to* $\pm\tilde{\phi}$.

Proof. Up to orthogonal equivalence, we see that ϕ is determined by the 4-tuple (p_+, q_+, p_-, q_-); $-\phi$ corresponds to (p_-, q_-, p_+, q_+) since we must interchange the roles of E_+ and E_-. We shall need to take this $\mathbb{Z}_2$ action into account. It is clear that assertion (2) implies assertion (1). To show that assertion (1) implies assertion (2), we must show that C is determined by orthogonal invariants of $R_{C,\phi}$ and that the tuple (p_+, q_+, p_-, q_-) is also determined by orthogonal invariants of $R_{C,\phi}$ up to the $\mathbb{Z}_2$ action described above.

By Lemma 6.4.3, $\tilde{\rho}_{C,\phi}$ has eigenvalues $\lambda_\pm := C\{r_\pm - 1 - r_\mp\}$, where $\lambda_\pm$ has multiplicity $r_\pm$. We distinguish two cases.

Case 1. Suppose the Ricci operator has two distinct eigenvalues $\lambda_\pm$. This implies that $r_+ \neq r_-$. By replacing ϕ by $-\phi$ if necessary, we may assume $r_+ > r_-$. Thus λ_+ is the eigenvalue with the greater multiplicity and is an orthogonal invariant. Since $E_\pm$ can be identified with the eigenspaces of $\tilde{\rho}_{C,\phi}$, the signature of the metric g restricted to $E_\pm$ is an orthogonal invariant. Thus the 4-tuple (p_+, q_+, p_-, q_-) is an orthogonal invariant. Finally, note that $\lambda_+ + \lambda_- = -2C$, so C is an orthogonal invariant.

Case 2. Suppose the Ricci operator has only one nonzero eigenvalue. This implies that $r_+ = r_-$. So $r_\pm = \frac{m}{2}$. We can not eliminate the $\mathbb{Z}_2$ ambiguity at this point. Note that $\lambda_+ = \lambda_- = -C$, so C is an orthogonal invariant. We apply Lemma 6.4.4 to determine the unordered pair (p_+, p_-). And we use the relations $q_+ = r_+ - p_+$, $p_- = q - p_+$, and $q_- = p - q_+$ to fill in the rest of the 4-tuple. $\square$

APPENDIX A

SOME COMBINATORIAL LEMMAS

A.1 Lemma. *Given 2-adic expansions* $a = \sum_{i=0}^{q} a_i 2^i$ *and* $b = \sum_{i=0}^{q} b_i 2^i$. *Then*

$$\binom{a}{b} \equiv \prod_{i=0}^{q} \binom{a_i}{b_i} \mod 2.$$

Proof. We define

$$\binom{0}{0} := 1, \ \binom{0}{1} := 0, \ \binom{1}{0} := 1, \text{ and } \binom{1}{1} := 1.$$

Since in the ring $\mathbb{Z}_2[x]$, $(1+x)^2 \equiv 1 + x^2 \mod 2$, using induction, we see that $(1+x)^{2^j} \equiv 1 + x^{2^j} \mod 2$. Thus,

$$(1+x)^a \equiv (1+x)^{\Sigma a_i 2^i} \equiv \prod_{i=0}^{q} (1+x^{2^i})^{a_i} \equiv \prod_{i=0}^{q} \left[\sum \binom{a_i}{t} x^{t2^i} \right] \mod 2$$

Notice the coefficient of the term x^b in this product is precisely given by $\prod_{i=0}^{q} \binom{a_i}{b_i}$. Our claim now follows. $\square$

A.2 Lemma. *In the cohomology ring*

$$H^*(Gr_2(\mathbb{R}^q); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2]/w_i^{\perp} = 0, \ \textit{for } i \geq q-1,$$

we have $w_{q-1}^{\perp} = w_1^{q-1}$ *if and only if* q *is a power of* 2.

Proof. Suppose $q = 2^s$ for some $s \geq 1$. We apply Lemma (4.1) in Monk's paper [70] to see that $w_{q-1}^{\perp} = w_1^{q-1}$.

Suppose $w_{q-1}^{\perp} = w_1^{q-1}$. Since $w_{q-1}^{\perp} = \sum_{a+2b=q-1}\left[\binom{a+b}{a} \mod 2\right] w_1^a w_2^b$, this implies:

$$\binom{q-b-1}{b} \text{ is even for all } 1 \leq b \leq \frac{q}{2} - 1.$$

Choose $j \in \mathbb{N}$ so that $2^j \leq q < 2^{j+1}$. We see that the 2-adic expansion of q is given by $q = 2^j + \sum_{i=0}^{j-1} a_i 2^i$ with the coefficients $a_i = 0$ or 1. Let $0 \leq N \leq j$ so that $N := \min\{0 \leq i \leq j : a_i = 1\}$ be the first nonzero index. Suppose $N < j$, then the 2-adic expansion of q reduces to $q = 2^j + \sum_{i=N+1}^{j-1} a_i 2^i + 2^N$. Choose $b = 2^N \leq \frac{q}{2} - 1$, we may express

$$\begin{aligned} q - b - 1 &= 2^j + a_{j-1}2^{j-1} + \dots + a_{N+1}2^{N+1} - 1 \\ &= \sum_{i=N+1}^{j-1} (a_i + 1)2^i + 2^N + 2^{N-1} + \dots + 2 + 1 \end{aligned}$$

By Lemma $A.1$, we have $\binom{q-b-1}{b} \equiv \prod_{i=0}^{j-1} 1 = 1 \mod 2$, which is false. Hence, $N = j$, i.e. q is a power of 2. $\square$

A.3 Lemma. *Let* $\phi_q := \sum_{a+b+c=q}(u_1 + u_2)^a {u_1}^b {u_2}^c$ *in* $\mathbb{Z}_2[u_1, u_2]$. *If* $\phi_q = 0$, *then* $q + 3 = 2^s$ *for some* s.

Proof. For the convenience of the reader, we reproduce the argument from Stong [84]. Suppose

$$0 = \phi_q = \sum_{a=0}^{q}(u_1 + u_2)^a \sum_{b+c=q-a} {u_1}^b {u_2}^c \in \mathbb{Z}_2[u_1, u_2]. \tag{A.3.a}$$

We multiply equation (A.3.a) by $(u_1 + u_2)$ to see that

$$0 = \phi_q = \sum_{a=0}^{q} \sum_{b+c=q-a} ({u_1}^{q+1-a} + {u_2}^{q+1-c}) \in \mathbb{Z}_2[u_1, u_2].$$

Hence,

$$0 = \text{ coefficient of } {u_1}^i \text{ in } \sum_{a=0}^{q}(u_1 + 1)^a({u_1}^{q+1-a} + 1).$$

But

$$\begin{aligned}
\sum_{a=0}^{q}(u_1+1)^a({u_1}^{q+1-a}+1) &= \sum_{a=0}^{q}{u_1}^{q+1}(\frac{u_1+1}{u_1})^a+\sum_{a=0}^{q}(u_1+1)^a \\
&= u_1^{q+1}\left\{\frac{1+\left(\frac{u_1+1}{u_1}\right)^{q+1}}{1+\left(\frac{u_1+1}{u_1}\right)}\right\}+\frac{1+(u_1+1)^{q+1}}{1+(u_1+1)} \\
&= u_1^{q+2}\left\{1+(\frac{u_1+1}{u_1})^{q+1}\right\}+\frac{1}{u_1}\{1+(u_1+1)^{q+1}\} \\
&= \frac{1}{u_1}\{u_1^2[u_1^{q+1}+(u_1+1)^{q+1}]+1+(u_1+1)^{q+1}\} \\
&= \frac{1}{u_1}\{u_1^{q+1}(1+u_1^2)+u_1^{q+3}+1\} \\
&= \frac{1}{u_1}\{(1+u_1)^{q+3}+1+u_1^{q+3}\} \\
&= \sum_{t=1}^{q+2}\binom{q+3}{t}u_1^{t-1}.
\end{aligned}$$

Thus $(1+u_1)^{q+3}\equiv 1+u_1^{q+3} \mod 2$, and so $q+3=2^s$ for some s. $\square$

APPENDIX B

THE INTERSECTION LEMMA

B.1 Lemma. *Assume* $q \geq 4$.

(1) *Let* R *be a rank* 2 *spacelike (or timelike) IP algebraic curvature tensor on* $\mathbb{R}^{p,q}$. *Let* $\{x_1, x_2, x_3\}$ *be vectors in* $\mathbb{R}^{p,q}$ *which span a spacelike* 3*-plane. Then*

$$\dim[W_1(R(x_1,x_2)) \cap W_1(R(x_1,x_3))] = 1 \textit{ and}$$

$$\dim[W_1(R(x_1,x_2)) + W_1(R(x_1,x_3))] = 3.$$

(2) *Let* R *be a rank* 2 *mixed Lorentzian IP algebraic curvature tensor. Let* $\{x_1, x_2, x_3\}$ *be vectors in* $\mathbb{R}^{1,q}$ *which span a spacelike* 3*-plane. Then*

$$\dim[W_1(R(x_1,x_2)) \cap W_1(R(x_1,x_3))] = 1 \textit{ and}$$

$$\dim[W_1(R(x_1,x_2)) + W_1(R(x_1,x_3))] = 3.$$

(3) *Let* R *be a rank* 2 *null IP Lorentzian algebraic curvature tensor. Let* $\{x, y, z\}$ *be an orthonormal set where* y *and* z *are spacelike vectors. Then*

$$\dim[W_1(R(x,y)) \cap W_1(R(x,z))] = 1 \textit{ and}$$

$$\dim[W_1(R(x,y)) + W_1(R(x,z))] = 3.$$

Proof. Before dealing with the general case, we first establish a special case of the Lemma. Let $\{x, y, z\}$ be an orthonormal subset of spacelike vectors in $\mathbb{R}^{p,q}$. Let

$$T_1 := R(x,y) \text{ and } T_2 := R(x,z).$$

We use the 1-parameter family $\pi(\theta) := \text{Span}\{x, \cos(\theta)y + \sin(\theta)z\}$ for $\theta \in [0, 2\pi]$ to prove our assertion in this special case. The fact that the characteristic polynomial of

$$R(\pi(\theta)) := \cos(\theta)T_1 + \sin(\theta)T_2$$

is independent of θ plays a crucial role in our discussion of this special case.

We have assumed that R does not change type. We first assume that R is spacelike. By rescaling R we may assume R has eigenvalues $\{0, \pm\sqrt{-1}\}$. Suppose that $\dim[W_1(T_1) \cap W_1(T_2)] = 0$. We then have $\dim[W_1(T_1) + W_1(T_2)] = 4$. By Lemma 5.1.1, we can find linearly independent unit spacelike vectors $\{u_1, u_2, v_1, v_2\}$ so

(1) $W_1(T_1) = \text{Span}\{u_1, u_2\}$ with $u_1 \perp u_2$; $T_1 u_1 = u_2$, and $T_1 u_2 = -u_1$.

(2) $W_1(T_2) = \text{Span}\{v_1, v_2\}$ with $v_1 \perp v_2$; $T_2 v_1 = v_2$, and $T_2 v_2 = -v_1$.

(3) $g(u_1, v_2) = 0$ and $g(u_2, v_1) = 0$.

Let $a := g(u_1, v_1)$ and $b := g(u_2, v_2)$. We compute:

$$\begin{aligned} T_1 v_1 &= g(T_1 v_1, u_1)u_1 + g(T_1 v_1, u_2)u_2 = -g(v_1, T_1 u_1)u_1 - g(v_1, T_1 u_2)u_2 \\ &= -g(v_1, u_2)u_1 + g(v_1, u_1)u_2 = a u_2. \end{aligned}$$

$$\begin{aligned} T_1 v_2 &= g(T_1 v_2, u_1)u_1 + g(T_1 v_2, u_2)u_2 = -g(v_2, T_1 u_1)u_1 - g(v_2, T_1 u_2)u_2 \\ &= -g(v_2, u_2)u_1 + g(v_2, u_1)u_2 = -b u_1. \end{aligned}$$

$$\begin{aligned} T_2 u_1 &= g(T_2 u_1, v_1)v_1 + g(T_2 u_1, v_2)v_2 = -g(u_1, T_2 v_1)v_1 - g(u_1, T_2 v_2)v_2 \\ &= -g(u_1, v_2)v_1 + g(u_1, v_1)v_2 = a v_2. \end{aligned}$$

$$\begin{aligned} T_2 u_2 &= g(T_2 u_2, v_1)v_1 + g(T_2 u_2, v_2)v_2 = -g(u_2, T_2 v_1)v_1 - g(u_2, T_2 v_2)v_2 \\ &= -g(u_2, v_2)v_1 + g(u_2, v_1)v_2 = -b v_1. \end{aligned}$$

Thus

$$R(\pi(\theta))u_1 = (\cos(\theta)T_1 + \sin(\theta)T_2)u_1 = \cos(\theta)u_2 + a\sin(\theta)v_2,$$

$$R(\pi(\theta))u_2 = (\cos(\theta)T_1 + \sin(\theta)T_2)u_2 = -\cos(\theta)u_1 - b\sin(\theta)v_1,$$

$$R(\pi(\theta))v_1 = (\cos\theta T_1 + \sin\theta T_2)v_1 = a\cos(\theta)u_2 + \sin(\theta)v_2,$$

$$R(\pi(\theta))v_2 = (\cos\theta T_1 + \sin\theta T_2)v_2 = -b\cos(\theta)u_1 - \sin(\theta)v_1.$$

Thus relative to the basis $\{u_1, u_2, v_1, v_2\}$, $R(\pi(\theta))$ has the form:

$$\text{(B.1.a)} \qquad R(\pi(\theta)) = \begin{pmatrix} 0 & -\cos(\theta) & 0 & -b\cos(\theta) \\ \cos(\theta) & 0 & a\cos(\theta) & 0 \\ 0 & -b\sin(\theta) & 0 & -\sin(\theta) \\ a\sin(\theta) & 0 & \sin(\theta) & 0 \end{pmatrix}.$$

Since $\operatorname{rank}(R(\pi(\theta))) = 2$ for all $\theta \in [0, \pi]$, the determinant of all 3×3 minors must vanish identically. Thus for all $\theta \in [0, \pi]$ we have that

$$\text{(B.1.b)} \qquad 0 \equiv \det \begin{pmatrix} 0 & -\cos(\theta) & 0 \\ \cos(\theta) & 0 & a\cos(\theta) \\ a\sin(\theta) & 0 & \sin(\theta) \end{pmatrix} = -\cos^2(\theta)\sin(\theta)(a^2 - 1).$$

$$\text{(B.1.c)} \qquad 0 \equiv \det \begin{pmatrix} 0 & -\cos(\theta) & -b\cos(\theta) \\ \cos(\theta) & 0 & 0 \\ 0 & -b\sin(\theta) & -\sin(\theta) \end{pmatrix} = \cos^2(\theta)\sin(\theta)(b^2 - 1).$$

We use equations (B.1.b) and (B.1.c) to see $a^2 = b^2 = 1$. Note the metric on $\operatorname{Span}\{u_1, u_2, v_1, v_2\}$ need not be positive definite, we can not apply the Cauchy-Schwarz inequality. Let $\chi_\theta(\lambda)$ be the characteristic polynomial of $R(\pi(\theta))$ acting on the space spanned by $\{u_1, u_2, v_1, v_2\}$; this space is $R(\pi(\theta))$ invariant and contains the range of $R(\pi(\theta))$. Thus $\chi_\theta(\lambda)$ is independent of θ as R is IP. We compute:

$$\begin{aligned}
\chi_\theta(\lambda) =&\det\begin{pmatrix} \lambda & \cos(\theta) & 0 & b\cos(\theta) \\ -\cos(\theta) & \lambda & -a\cos(\theta) & 0 \\ 0 & b\sin(\theta) & \lambda & \sin(\theta) \\ -a\sin(\theta) & 0 & -\sin(\theta) & \lambda \end{pmatrix} \\
=&\lambda\det\begin{pmatrix} \lambda & -\cos(\theta) & 0 \\ b\sin(\theta) & \lambda & \sin(\theta) \\ 0 & -\sin(\theta) & \lambda \end{pmatrix} \\
&-\cos(\theta)\det\begin{pmatrix} -\cos(\theta) & -a\cos(\theta) & 0 \\ 0 & \lambda & \sin(\theta) \\ -a\sin(\theta) & -\sin(\theta) & \lambda \end{pmatrix} \\
&-b\cos(\theta)\det\begin{pmatrix} -\cos(\theta)\lambda & -a\cos(\theta) & 0 \\ 0 & b\sin(\theta) & \lambda \\ -a\sin(\theta) & 0 & -\sin(\theta) \end{pmatrix} \\
=&\lambda[\lambda(\lambda^2+\sin^2(\theta))+a\cos(\theta)\lambda b\sin(\theta)] \\
&-\cos(\theta)[-\cos(\theta)(\lambda^2+\sin^2(\theta))+a\cos(\theta)a\sin^2(\theta)] \\
&-b\cos(\theta)[b\cos(\theta)\sin^2(\theta)-\lambda^2 a\sin(\theta)-a^2b\sin^2(\theta)\cos(\theta)] \\
=&\lambda^4+\lambda^2[\sin^2(\theta)+\cos^2(\theta)+2ab\sin(\theta)\cos(\theta)] \\
&+\sin^2(\theta)\cos^2(\theta)[1-a^2-b^2+a^2b^2] \\
=&\lambda^4+\lambda^2[1+2ab\sin(\theta)\cos(\theta)]+\sin^2(\theta)\cos^2(\theta)[1+a^2b^2-a^2-b^2].
\end{aligned}$$

Since $a^2 = b^2 = 1$, we have that $\chi_\theta(\lambda) = \lambda^4 + \lambda^2[1 + 2ab\sin(\theta)\cos(\theta)]$. The eigenvalues of the matrix (B.1.a) are $\{0, 0, \pm\sqrt{-1}\}$. So $\chi_\theta(\lambda) = \lambda^4 + \lambda^2$. This implies $2ab\sin(\theta)\cos(\theta) \equiv 0$ for all $\theta \in [0, \pi]$. This is not possible as $ab \neq 0$. This shows that

$$\dim[W_1(R(x,y)) \cap W_1(R(x,z))] \geq 1.$$

Suppose $\dim[W_1(R(x,y)) \cap W_1(R(x,z))] = 2$. Then $W_1(R(x,y)) = W_1(R(x,z))$. We use Lemma 5.1.1 to see that $T_1 = \pm T_2$. It follows that $R(\pi(\theta)) = 0$ for $\theta = \pm\frac{\pi}{4}$, which is false. Thus we have $\dim[W_1(R(x,y)) \cap W_1(R(x,z))] = 1$. This proves the special case if R is spacelike; the proof is similar if R is timelike.

We can now derive the general case from the special case discussed above. Let $\{x_1, x_2, x_3\}$ span a spacelike 3-plane. We must further normalize this basis. We apply the Gram-Schmidt process to define:

$$\tilde{x}_1 := \frac{x_1}{|x_1|},\ \tilde{x}_2 := \frac{x_2 - g(x_2, \tilde{x}_1)\tilde{x}_1}{|x_2 - g(x_2, \tilde{x}_1)\tilde{x}_1|},\ \tilde{x}_3 := \frac{x_3 - g(x_3, \tilde{x}_1)\tilde{x}_1}{|x_3 - g(x_3, \tilde{x}_1)\tilde{x}_1|}.$$

Then $\operatorname{span}\{x_1, x_2, x_3\} = \operatorname{span}\{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3\}$. Furthermore, there are nonzero constants c_2 and c_3 so that $R(x_1, x_2) = c_2 R(\tilde{x}_1, \tilde{x}_2)$ and $R(x_1, x_3) = c_3 R(\tilde{x}_1, \tilde{x}_3)$. Thus by replacing $\{x_i\}$ by $\{\tilde{x}_i\}$ if necessary, we may assume without loss of generality that

$$|x_i| = 1,\ x_1 \perp x_2, \text{ and } x_1 \perp x_3.$$

Again, we apply the Gram-Schmidt process. We define

$$w := \frac{x_3 - g(x_3, x_2)x_2}{|x_3 - g(x_3, x_2)x_2|}.$$

Since $x_3 \perp x_1$ and $x_2 \perp x_1$, we have that $\{x_1, x_2, w\}$ is an orthonormal set. Furthermore, we may expand $x_3 = \cos(\theta)x_2 + \sin(\theta)w$ for some $\theta \in [0, 2\pi]$. Since x_2 and x_3 are linearly independent, $\sin(\theta) \neq 0$. Let $T_1 := R(x_1, x_2)$ and $T_2 := R(x_1, w)$. We then have

$$T := R(x_1, x_3) = \cos(\theta)T_1 + \sin(\theta)T_2 \text{ and } T_2 = \csc(\theta)(T - \cos(\theta)T_1).$$

We compute:

$$\begin{aligned} &W_1(T) \subset W_1(T_1) + W_1(T_2),\ W_1(T_2) \subset W_1(T) + W_1(T_1), \text{ and} \\ &W_1(T) + W_1(T_2) = W_1(T_1) + W_1(T_2). \end{aligned}$$

We apply the special case to the orthonormal set $\{x_1, x_2, w\}$ to see that

$$3 = \dim(W_1(T_1) + W_1(T_2)) = \dim(W_1(T) + W_1(T_2)).$$

We next assume R is a rank 2 mixed Lorentzian algebraic curvature tensor. For clarity, we use τ_i to denote a unit timelike vector, and use σ_i to denote a unit spacelike vector which is orthogonal to τ_i so that $W_1(T_i) = \text{Span}\{\tau_i, \sigma_i\}$. Consequently, there exist nonzero constants a_i and b_i so $T_i\tau_i = a_i\sigma_i$ and $T_i\sigma_i = b_i\tau_i$ where $g(T_i\tau_i, \sigma_i) = a_i$ and $g(T_i\sigma_i, \tau_i) = -b_i$. Since T_i is skew-symmetric, we have that $g(T_i\tau_i, \sigma_i) = -g(\tau_i, T_i\sigma_i)$. Thus $a_i = b_i$, and T_i has eigenvalues $\{0, \pm c\}$. By rescaling R we may assume R has eigenvalues $\{0, \pm 1\}$. The skew-symmetric operator T_i defines a unitary paracomplex structure on $W_1(T_i))$, i.e. T is unitary and $T^2 = 1$. Suppose that $\dim[W_1(T_1) \cap W_1(T_2)] = 0$. We then have $\dim[W_1(T_1) + W_1(T_2)] = 4$. This enables us to find linearly independent unit vectors $\{\tau_1, \sigma_1, \tau_2, \sigma_2\}$ so

(1) $W_1(T_1) = \text{Span}\{\tau_1, \sigma_1\}$, $T_1\tau_1 = \sigma_1$, and $T_1\sigma_1 = \tau_1$.

(2) $W_1(T_2) = \text{Span}\{\tau_2, \sigma_2\}$, $T_2\tau_2 = \sigma_2$, and $T_2\sigma_2 = \tau_2$.

Since $\dim[\tau_1^\perp \cap W_1(T_2)] \geq (m-1) + 2 - m = 1$, we can choose $\sigma_2 \perp \tau_1$. Necessarily σ_2 is then spacelike; we normalize σ_2 to have unit length and set $\tau_2 := T_2\sigma_2$. Thus without loss of generality we may assume $\sigma_2 \perp \tau_1$. Let $A := g(\tau_1, \tau_2)$, $B := g(\tau_2, \sigma_1)$, and $C := g(\sigma_1, \sigma_2)$. We compute:

$$\begin{aligned} T_1\tau_2 &= g(T_1\tau_2, \sigma_1)\sigma_1 - g(T_1\tau_2, \tau_1)\tau_1 = -g(\tau_2, T_1\sigma_1)\sigma_1 + g(\tau_2, T_1\tau_1)\tau_1 \\ &= -g(\tau_2, \tau_1)\sigma_1 + g(\tau_2, \sigma_1)\tau_1 = -A\sigma_1 + B\tau_1. \end{aligned}$$

$$\begin{aligned} T_1\sigma_2 &= g(T_1\sigma_2, \sigma_1)\sigma_1 - g(T_1\sigma_2, \tau_1)\tau_1 = -g(\sigma_2, T_1\sigma_1)\sigma_1 + g(\sigma_2, T_1\tau_1)\tau_1 \\ &= -g(\sigma_2, \tau_1)\sigma_2 + g(\sigma_2, \sigma_1)\tau_1 = C\tau_1. \\ T_2\tau_1 &= g(T_2\tau_1, \sigma_2)\sigma_2 - g(T_2\tau_1, \tau_2)\tau_2 = -g(\tau_1, T_2\sigma_2)\sigma_2 + g(\tau_1, T_2\tau_2)\tau_2 \\ &= -g(\tau_1, \tau_2)\sigma_2 + g(\tau_1, \sigma_2)\tau_2 = -A\sigma_2. \\ T_2\sigma_1 &= g(T_2\sigma_1, \sigma_2)\sigma_2 - g(T_2\sigma_1, \tau_2)\tau_2 = -g(\sigma_1, T_2\sigma_2)\sigma_2 + g(\sigma_1, T_2\tau_2)\tau_2 \\ &= -g(\sigma_1, \tau_2)\sigma_2 + g(\sigma_1, \sigma_2)\tau_2 = -B\sigma_2 + C\tau_2. \end{aligned}$$

Thus

$$R(\pi(\theta))\tau_1 = (\cos(\theta)T_1 + \sin(\theta)T_2)\tau_1 = \cos(\theta)\sigma_1 - A\sin(\theta)\sigma_2,$$

$$R(\pi(\theta))\sigma_1 = (\cos(\theta)T_1 + \sin(\theta)T_2)\sigma_1 = \cos(\theta)\tau_1 + C\sin(\theta)\tau_2 - B\sin(\theta)\sigma_2,$$

$$R(\pi(\theta))\tau_2 = (\cos(\theta)T_1 + \sin(\theta)T_2)\tau_2 = B\cos(\theta)\tau_1 - A\cos(\theta)\sigma_1 + \sin(\theta)\sigma_2,$$

$$R(\pi(\theta))\sigma_2 = (\cos(\theta)T_1 + \sin(\theta)T_2)\sigma_2 = C\cos(\theta)\tau_1 + \sin(\theta)\tau_2.$$

Thus relative to the basis $\{\tau_1, \sigma_1, \tau_2, \sigma_2\}$, $R(\pi(\theta))$ has the form:

$$R(\pi(\theta)) = \begin{pmatrix} 0 & \cos(\theta) & B\cos(\theta) & C\cos(\theta) \\ \cos(\theta) & 0 & -A\cos(\theta) & 0 \\ 0 & C\sin(\theta) & 0 & \sin(\theta) \\ -A\sin(\theta) & -B\sin(\theta) & \sin(\theta) & 0 \end{pmatrix}.$$

Since $\operatorname{rank}(R(\pi(\theta))) = 2$ for all $\theta \in [0,\pi]$, all 3×3 minors must vanish identically. Thus for all $\theta \in [0,\pi]$ we have that

$$\text{(B.1.d)} \qquad 0 \equiv \det \begin{pmatrix} 0 & \cos(\theta) & B\cos(\theta) \\ \cos(\theta) & 0 & -A\cos(\theta) \\ 0 & C\sin(\theta) & 0 \end{pmatrix} = -\cos^2(\theta)\sin(\theta)BC.$$

$$\text{(B.1.e)} \qquad 0 \equiv \det \begin{pmatrix} 0 & \cos(\theta) & C\cos(\theta) \\ \cos(\theta) & 0 & 0 \\ 0 & C\sin(\theta) & \sin(\theta) \end{pmatrix} = \cos^2(\theta)\sin(\theta)(C^2 - 1).$$

$$\text{(B.1.f)} \quad 0 \equiv \det \begin{pmatrix} \cos(\theta) & 0 & -A\cos(\theta) \\ 0 & C\sin(\theta) & 0 \\ -A\sin(\theta) & -B\sin(\theta) & \sin(\theta) \end{pmatrix} = \sin^2(\theta)\cos(\theta)(CA^2 - C).$$

We use equations (B.1.d), (B.1.e), and (B.1.f) to see $A^2 = C^2 = 1$, and $B = 0$. Thus we have

$$\text{(B.1.g)} \qquad R(\pi(\theta)) = \begin{pmatrix} 0 & \cos(\theta) & 0 & C\cos(\theta) \\ \cos(\theta) & 0 & -A\cos(\theta) & 0 \\ 0 & C\sin(\theta) & 0 & \sin(\theta) \\ -A\sin(\theta) & 0 & \sin(\theta) & 0 \end{pmatrix}.$$

Let $\chi_\theta(\lambda)$ be the characteristic polynomial of $R(\pi(\theta))$ acting on the space spanned by $\{\tau_1, \sigma_1, \tau_2, \sigma_2\}$; this space is $R(\pi(\theta))$ invariant and containing the range of $R(\pi(\theta))$. Thus $\chi_\theta(\lambda)$ is independent of θ as R is IP. We compute:

$$
\begin{aligned}
\chi_\theta(\lambda) =& \det \begin{pmatrix} \lambda & -\cos(\theta) & 0 & -C\cos(\theta) \\ -\cos(\theta) & \lambda & A\cos(\theta) & 0 \\ 0 & -C\sin(\theta) & \lambda & -\sin(\theta) \\ A\sin(\theta) & 0 & -\sin(\theta) & \lambda \end{pmatrix} \\
=&\lambda \det \begin{pmatrix} \lambda & A\cos(\theta) & 0 \\ -C\sin(\theta) & \lambda & -\sin(\theta) \\ 0 & -\sin(\theta) & \lambda \end{pmatrix} \\
&+ \cos(\theta) \det \begin{pmatrix} -\cos(\theta) & A\cos(\theta) & 0 \\ 0 & \lambda & -\sin(\theta) \\ A\sin(\theta) & -\sin(\theta) & \lambda \end{pmatrix} \\
&+ C\cos(\theta) \det \begin{pmatrix} -\cos(\theta) & \lambda & A\cos(\theta) \\ 0 & -C\sin(\theta) & \lambda \\ A\sin(\theta) & 0 & -\sin(\theta) \end{pmatrix} \\
=&\lambda[\lambda(\lambda^2 - \sin^2(\theta)) - A\cos(\theta)(-\lambda C\sin(\theta))] \\
&+ \cos(\theta)[-\cos(\theta)(\lambda^2 - \sin^2(\theta)) - A\cos(\theta)(A\sin^2(\theta))] \\
&+ C\cos(\theta)[-\cos(\theta)(C\sin^2(\theta)) \\
&- \lambda(-\lambda A\sin(\theta)) + A\cos(\theta)(A\sin(\theta)C\sin(\theta))] \\
&= \lambda^4 + \lambda^2[-\sin^2(\theta) - \cos^2(\theta) + 2AC\sin(\theta)\cos(\theta)] \\
&+ \sin^2(\theta)\cos^2(\theta)(1 + A^2C^2 - A^2 - C^2) \\
=&\lambda^4 + \lambda^2[2AC\sin(\theta)\cos(\theta) - 1] + \sin^2(\theta)\cos^2(\theta)(1 + A^2C^2 - A^2 - C^2).
\end{aligned}
$$

Since $A^2 = C^2 = 1$, we have that $\chi_\theta(\lambda) = \lambda^4 + \lambda^2[2AC\sin(\theta)\cos(\theta) - 1]$. The eigenvalues of the matrix (B.1.g) are $\{0, 0, \pm 1\}$. So we must have $\chi_\theta(\lambda) = \lambda^4 - \lambda^2$. This implies $2AC\sin(\theta)\cos(\theta) \equiv 0$ for all $\theta \in [0, \pi]$. This is not possible as $AC \neq 0$. This shows that

$$\dim[W_1(R(x,y)) \cap W_1(R(x,z))] \geq 1.$$

We show similarly that $\dim[W_1(R(x,y)) \cap W_1(R(x,z))] \neq 2$. This completes the proof of analysis of the situation when we are dealing with a spacelike orthonormal set.

We can now derive the general case from the special case discussed above. Let $\{x_1, x_2, x_3\}$ span a spacelike 3-plane. We must further normalize this basis. We apply the Gram-Schmidt process to define:

$$\tilde{x}_1 := \frac{x_1}{|x_1|},\ \tilde{x}_2 := \frac{x_2 - g(x_2, \tilde{x}_1)\tilde{x}_1}{|x_2 - g(x_2, \tilde{x}_1)\tilde{x}_1|},\ \tilde{x}_3 := \frac{x_3 - g(x_3, \tilde{x}_1)\tilde{x}_1}{|x_3 - g(x_3, \tilde{x}_1)\tilde{x}_1|}.$$

Then $\operatorname{span}\{x_1, x_2, x_3\} = \operatorname{span}\{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3\}$. Furthermore, there are nonzero constants c_2 and c_3 so that $R(x_1, x_2) = c_2 R(\tilde{x}_1, \tilde{x}_2)$ and $R(x_1, x_3) = c_3 R(\tilde{x}_1, \tilde{x}_3)$. Thus by replacing $\{x_i\}$ by $\{\tilde{x}_i\}$ if necessary, we may assume without loss of generality that

$$|x_i| = 1,\ x_1 \perp x_2, \text{ and } x_1 \perp x_3.$$

Again, we apply the Gram-Schmidt process. We define

$$w := \frac{x_3 - g(x_3, x_2)x_2}{|x_3 - g(x_3, x_2)x_2|}.$$

Since $x_3 \perp x_1$ and $x_2 \perp x_1$, we have that $\{x_1, x_2, w\}$ is an orthonormal set. Furthermore, we may expand $x_3 = \cos(\theta)x_2 + \sin(\theta)w$ for some $\theta \in [0, 2\pi]$. Since x_2 and x_3 are linearly independent, $\sin(\theta) \neq 0$. Let $T_1 := R(x_1, x_2)$ and $T_2 := R(x_1, w)$. We then have

$$T := R(x_1, x_3) = \cos(\theta)T_1 + \sin(\theta)T_2 \text{ and } T_2 = \csc(\theta)(T - \cos(\theta)T_1).$$

We compute:

$$W_1(T) \subset W_1(T_1) + W_1(T_2),\ W_1(T_2) \subset W_1(T) + W_1(T_1), \text{ and}$$
$$W_1(T) + W_1(T_2) = W_1(T_1) + W_1(T_2).$$

The lemma now follows by applying the special case to the spacelike orthonormal set $\{x_1, x_2, w\}$.

Let R be a rank 2 null IP Lorentzian algebraic curvature tensor. Let $\{x, y, z\}$ be an orthonormal set where y and z are spacelike vectors. Note that

$$\pi(\theta) := \mathrm{Span}\{x, \cos\theta y + \sin\theta z\}$$

is nondegenerate for all $\theta \in [0, \pi]$ and the type of the plane $\pi(\theta)$ does not change. Since R has only the zero eigenvalue, for any $\theta \in [0, \pi]$, $R(\pi(\theta)) \in \mathfrak{so}_2^{\mathcal{N}}(1, q)$. Let $\xi \in \mathbb{R}^{1,q}$ be a unit timelike vector. We apply Lemma 4.1.2 (2c) to see that

$$W_1(R(x, y)) = \mathrm{Span}\{R(x, y)\xi, R^2(x, y)\xi\}$$

$$W_1(R(x, z)) = \mathrm{Span}\{R(x, z)\xi, R^2(x, z)\xi\}$$

We assume the lemma fails and argue for a contradiction.

Suppose that $W_1(R(x, y)) \cap W_1(R(x, z)) = \{0\}$. We use the identity

$$\begin{aligned}\dim[W_1(R(x, y))] + \dim[W_1(R(x, z))] &= \dim[W_1(R(x, y)) \cap W_1(R(x, z))]\\ &\quad + \dim[W_1(R(x, y)) + W_1(R(x, z))]\end{aligned}$$

to see that $\dim[W_1(R(x, y)) + W_1(R(x, z))] = 4$. Consequently, the vectors

$$\{R(x, y)\xi, R(x, z)\xi, R^2(x, y)\xi, R^2(x, z)\xi\}$$

are linearly independent. By Lemma 4.1.2 (2d), $R^2(x, y)\xi$ and $R^2(x, z)\xi$ are nonzero null vectors. We use Lemma 4.3.1 to see that the null vectors $\{R^2(x, y)\xi, R^2(x, z)\xi\}$ are linearly independent if and only if $g(R^2(x, y)\xi, R^2(x, z)\xi) \neq 0$. There exist constants α_i and β_i , for $i = 1, 2$ so that:

$$R(x, z)(R(x, y)\xi) = \alpha_1 R(x, z)\xi + \alpha_2 R^2(x, z)\xi,$$

$$R(x, z)(R^2(x, y)\xi) = \beta_1 R(x, z)\xi + \beta_2 R^2(x, z)\xi,$$

$$\alpha_1 := -\frac{g(R(x, y)\xi, R^2(x, z)\xi)}{|R(x, z)\xi|^2}, \text{ and}$$

$$\beta_1 := -\frac{g(R^2(x, y)\xi, R^2(x, z)\xi)}{|R(x, z)\xi|^2} \neq 0.$$

The coefficient β_1 is crucial. Consider the following system of equations:

$$R(x, y+z)(\xi) = R(x,y)\xi + R(x,z)\xi,$$

$$R(x, y+z)(R(x,y)\xi) = \alpha_1 R(x,z)\xi + R^2(x,y)\xi + \alpha_2 R^2(x,z)\xi, \text{ and}$$

$$R(x, y+z)(R^2(x,y)\xi) = \beta_1 R(x,z)\xi + \beta_2 R^2(x,z)\xi.$$

Relative to the basis $\{R(x,y)\xi, R(x,z)\xi, R^2(x,y)\xi, R^2(x,z)\xi\}$, we write the coefficient matrix associated with the above equations as:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & \alpha_1 & 1 & \alpha_2 \\ 0 & \beta_1 & 0 & \beta_2 \end{pmatrix}.$$

Notice this matrix contains the 3×3 submatrix whose determinant is

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 0 & \alpha_1 & 1 \\ 0 & \beta_1 & 0 \end{pmatrix} = -\beta_1 \neq 0.$$

Thus, $\operatorname{rank} R(x, y+z) \geq 3$, which is false.

Next we suppose that $\dim[W_1(R(x,y)) \cap W_1(R(x,z))] = 2$. This implies

$$\operatorname{Range} R(x,y) = \operatorname{Range} R(x,z).$$

We can express

$$R(x,z)\xi = \alpha R(x,y)\xi + \beta R^2(x,y)\xi.$$

Since $R(x,y)\xi$ is spacelike, $\alpha \neq 0$. So $R(x, z-\alpha y)\xi = \beta R^2(x,y)\xi$ is a null vector. This contradicts Lemma 4.1.2 (2d). Hence $\dim[W_1(R(x,y)) \cap W_1(R(x,z))] = 1$. The remaining assertion now follows. $\square$

We can now improve Lemma B.1 slightly by removing the restriction that $\operatorname{span}\{x_1, x_2, x_3\}$ is spacelike.

B.2 Lemma. *Let R be a rank 2 spacelike (or timelike) algebraic curvature tensor or let R be a rank 2 mixed Lorentzian algebraic curvature tensor. Let $\{x_1, x_2, x_3\}$*

be linearly independent vectors in $\mathbb{R}^{p,q}$ so that span$\{x_1, x_2\}$ and span$\{x_1, x_3\}$ are spacelike 2-planes. Then

$$\dim[W_1(R(x_1,x_2)) \cap W_1(R(x_1,x_3))] = 1 \text{ and}$$
$$\dim[W_1(R(x_1,x_2)) + W_1(R(x_1,x_3))] = 3.$$

Proof. Let $\pi_1 := \text{Span}\{x_1, x_2\}$ and $\pi_2 := \text{Span}\{x_1, x_3\}$. By Lemma 5.2.2, we have $\mathcal{L}(x_1) \subset W_1(R(\pi_1)) \cap W_1(R(\pi_2))$. Thus $\dim[W_1(R(x_1,x_2)) \cap W_1(R(x_1,x_3))] \geq 1$. Suppose the Lemma fails. Then $W_1(R(\pi_1)) \cap W_1(R(\pi_2))$ is 2 dimensional so

$$W_1(R(\pi_1)) = W_1(R(\pi_2)). \tag{B.2.a}$$

Let $\{x_1, y_i\}$ be an orthonormal basis for the spacelike 2-planes π_i. Let E be the span of $\{y_1, y_2\}$. If E is spacelike, then $\{x_1, y_1, y_2\}$ spans a spacelike 3-plane and (B.2.a) contradicts Lemma B.1. We distinguish two cases:

Case 1: Suppose that E is mixed. Choose a unit timelike vector $z \in E$ so that $z \perp y_1$. Then $\{y_1, z\}$ is an orthonormal basis for E. We express

$$y_2 = \cosh(\theta)y_1 + \sinh(\theta)z \text{ for some } \theta \text{ where } \sinh(\theta) \neq 0.$$

Let $T_1 := R(x_1, y_1)$ and $T_2 := R(x_1, z)$. By rescaling we may assume R has eigenvalues $\{0, \pm\sqrt{-1}\}$. The operators T_1 and $\cosh(\theta)T_1 + \sinh(\theta)T_2$ are rotations through 90° in the same subspace and they vanish on the same orthogonal complements. Thus $T_1 = \pm(\cosh(\theta)T_1 + \sinh(\theta)T_2)$ and thus T_2 is some nonzero multiple c of T_1. Let φ be any angle and let $y(\varphi) := \cosh(\varphi)y_1 + \sinh(\varphi)z$. Then $R(x_1, y(\varphi)) = (\cosh(\varphi) + c\sinh(\varphi))T_1$. The eigenvalues of $R(x_1, y(\varphi))$ are then dependent on φ which is false as $y(0) \perp x$ is a unit spacelike vector.

Case 2: Suppose that E is degenerate. Choose $0 \neq z \in E \cap y_1^{\perp}$. Since E is degenerate, z is null. We may express $y_2 = \epsilon(y_1 + dz)$ where $\epsilon = \pm 1$ and d is a nonzero constant. Let $T_1 := R(x_1, y_1)$ and $T_2 := R(x_1, z)$. By rescaling we may assume R has eigenvalues $\{0, \pm\sqrt{-1}\}$. The operators T_1 and $\epsilon(T_1 + dT_2)$ are rotations through $90°$ in the same subspace and they vanish on the same orthogonal complements. Thus $\pm T_1 = \epsilon(T_1 + dT_2)$ and thus again T_2 is some nonzero multiple $\tilde{d}$ of T_1. Let $\varphi \in \mathbb{R}$ and let $y(\varphi) := y_1 + \varphi z$. It is then clear that the eigenvalues of $R(x_1, y(\varphi)) = (1 + \varphi\tilde{d})T_1$ are independent of φ only if $\tilde{d} = 0$. Thus we conclude $R(x_1, z) = 0$. We express $z = e(w_1 + w_2)$ where w_1 and w_2 are unit spacelike and timelike vectors respectively. We then have $R(x_1, w_1) + R(x_1, w_2) = 0$. Let $w(t) := \cosh(t)w_1 + \sin h(t)w_2$ be a 1-parameter family of unit spacelike vectors. The eigenvalues of $R(x_1, w(t)) = (\cosh(t) - \sinh(t))R(x_1, w_1)$ are then dependent on t which is false. This completes the proof of the lemma. $\square$

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