Three-dimensional Fractals

By
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A THESIS

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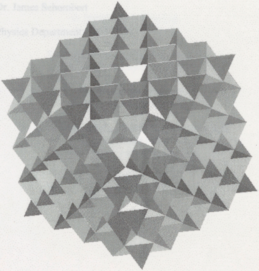

APPROVED: [redacted] [redacted] DATE: 23 Sep 01

Dr. James Schombert

Physics Department

This paper is dedicated to my Lord and Master

My Beloved

Avatar Meher Baba

When I awake I see you
When I sleep I dream you

In my ignorance of you I do suffer
In my knowledge of you I find bliss

You are my everything
I am your nothing

My beloved how you tease me
My beloved how you bless me

You give me fractals to study your way
You give me fractals to beguile my day

In fractals you are hiding
In fractals you are seen

You have shown me your fingerprint
I am entrapped by its freedom

If only I could give them up
If only you would let me

To leave understanding behind me is what I want
Only then would I Know
Just how much you really love me

KBJ

Acknowledgement:

My family has been a great support to me while I have been in school and working on my thesis. I want to thank my father, Irvin Jacob, for his financial and moral support. He has been an excellent source of philosophical inquiry for me. I want to thank my mother, Karen Jacob, for her financial and moral support. She has been a source of stability and comfort for me. I want to thank my brother, Benjamin Jacob, for challenging me to excel. He has long been a major role model for me and he is my best friend. I want most to thank my wife, Paritha Jacob, for the joy she brings to my life. She has been a great source of motivation and inspiration for me whenever I had none.

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Thesis Statement:

To Plato, the five perfect solids represented archetypes or ideals of reality. People and groups through the millennia have viewed these shapes with awe and wonder. They have gone so far as to ascribe mystical and magical properties to these objects. I too am struck with wonder and a deep appreciation for these shapes because of their unique properties of symmetry and form. Over the course of the last ten years I have entered into thought experiments where I have applied fractal processes onto some these shapes. As a result, I have come to believe that these solids are not just pure Euclidean volumes. I think they contain the property of being not only fractal initiators but that they are themselves complete fractals.

This paper is a treatment of three-dimensional fractals. I will begin by introducing some basic concepts of fractals. Then I will introduce two three-dimensional fractals, the tetrahedron fractal and the octahedron fractal, and analyze them in full detail. Both of these fractals have been studied in previous published articles. There are some important new insights about these fractals that will be presented here. Other three dimensional fractals will be presented. These will not be fully analyzed, but used mainly to discuss techniques in construction of three-dimensional fractals. This is intended to give ideas and methods of creating an infinite number of different three-dimensional fractals.

Definitions:

Dodecahedron – A three-dimensional object with 20 vertex points, 30 edge lines, and 12 pentagonal surfaces. See Figure 5 next page.

Fractal – An object constructed after an infinite number of iterations containing the property of self-similarity over an arbitrary magnification or scaling.

Fractal Dimension – A value assigned to a fractal that characterizes the fractal's non-integer dimensionality. This is determined by solving the fractal dimension equation.

Hexahedron – A three-dimensional object with 8 vertex points, 12 edge lines, and 6 square surfaces. See Figure 3 next page. Commonly known as a cube.

Generator – A geometric function that operates on the initiator and each successive iteration level.

Icosahedron – A three-dimensional object with 12 vertex points, 30 edge lines, and 20 triangular surfaces. See Figure 4 next page.

Initiator – The initial geometric input shape that is operated on by the generator resulting in the first iteration. Also the zeroth iteration of a fractal process.

Iteration – Taking the result of a function that is operated on and feeding this result back into the same function for the subsequent result.

Octahedron – A three-dimensional object with 6 vertex points, 12 edge lines, and 8 triangular surfaces. See Figure 2 next page.

Self-similarity – An invariance of form with respect to scaling; in other words, an invariance not with additive translations, but invariance with multiplicative changes of scale.

Tetrahedron – A three-dimensional object with 4 vertex points, 6 edge lines, and 4 triangular surfaces. See Figure 1 next page.

Perfect Solids

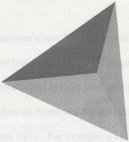


Figure 1: Tetrahedron

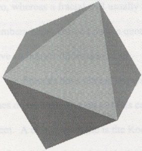


Figure 2: Octahedron

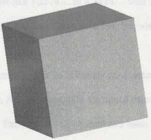


Figure 3: Hexahedron (Cube)

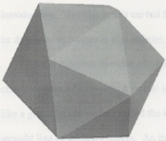


Figure 4: Icosahedron

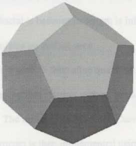


Figure 5: Dodecahedron

Introduction to Fractals:

In recent years there has been a new development in math and science called Chaos theory. This new way of looking at the world has generated interest in objects that are defined as fractals. The word fractal derives its meaning from what are called fractal dimensions. A fractal is therefore an object that has a fraction or non-integer number corresponding to its dimensional value. For example, a circular shaped surface is an object that requires two dimensions to be described and thus has a dimensional value of two, whereas a fractal will usually have a value like 1.2578… or 2.3454… with these numbers being derived from a quotient of two numbers. There are a few fractals that do have an integer dimension value.

Fractals have almost always shown themselves to be infinitely convoluted curved lines or surfaces. Because of this convolution, fractal slopes defy the use of calculus on them. A classic example is the Koch Curve. This fractal will be used as a basic example for fractal ideas throughout the introduction. This fractal has a dimension of 1.26186… The method for calculating this number will be shown at the conclusion of this introduction. The reason this curved line is a fractal is because its length is infinite, and its degree of curvature is also infinite, yet it contains no surface area.

As in calculus, the idea of taking a limit is used to determine quantitative characteristics of a fractal. A fractal is produced by first starting with an initial condition like a given line or surface called the initiator. The initiator for the Koch Curve is a straight line of arbitrary length. An iteration process is then implemented using what is called a generator. Here, a given line with a unit length is changed into a line that is broken into thirds. The middle third line segment is replaced with two sides of an equilateral triangle as shown in figure 6.

Figure 6: Koch Curve Generator and First Iteration

This shape is the generator shape and the process of creating it is called iteration. This process is then repeated on all of the four new line segments and has now been iterated twice as shown in figure 7.

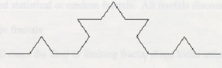


Figure 7: Koch Curve Second Iteration

By iterating an infinite number of times a fractal is formed. The actual fractal curve would be impossible to completely picture since there is a finite limit to the level of resolution that humans can perceive. Still, the infinite iteration of the Koch Curve would look something close to figure 8.

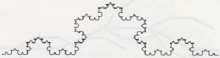


Figure 8: Koch Curve Fractal

In math, a value produced from an infinite number of operations is called a limit. Therefore the limit of the infinite iteration process or the sum of all the iterations is the fractal itself. Often the whole process is called the fractal process and the limit is called the fractal shape or just fractal. This type of fractal is called an exact or deterministic fractal because its generator can be completely known by mathematics. Hence all subsequent iterations and the final limit are also completely determined. The other type of fractal process where the generator is governed by probabilities or statistics and yields what are aptly called statistical or random fractals. All fractals discussed in this paper will be deterministic fractals.

In almost all cases the final or limiting fractal shape looks infinitely complex. Another "simple" example of a fractal and its first few iterations of the fractal process are shown in figures 9 through12.

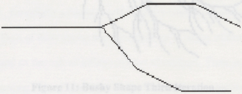


Figure 9: Bushy Shape Generator and First Iteration

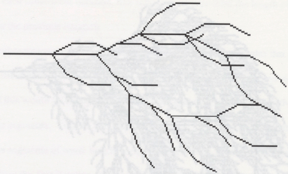


Figure 10: Bushy Shape Second Iteration

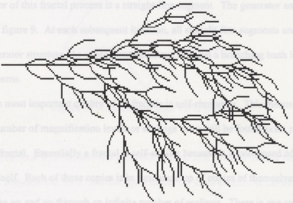


Figure 11: Bushy Shape Third Iteration

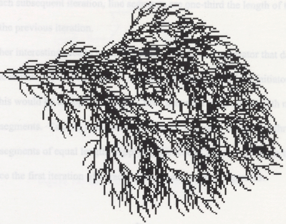


Figure 12: Bushy Shape Fourth Iteration

The initiator of this fractal process is a straight-line segment. The generator and first iteration is figure 9. At each subsequent iteration, all straight-line segments are replaced by the generator structure. As subsequent iterations develop a branching bush like structure forms.

The most important quality of all fractals is self-similarity. This means that at an arbitrary number of magnification levels or scalings there will be found exact replicas of the whole fractal. Essentially a fractal is self-similar because it is composed of smaller copies of itself. Each of these copies is in turn made up of copies of themselves. This process goes on and on through an infinite number of scalings. There is one restriction with this. Each scaling of the fractal must be done in relation to a particular iteration level. The Koch curve, for example, has a self-similarity scaling level of three due to the fact that at each subsequent iteration, line segments are one-third the length of the line segments in the previous iteration.

Another interesting idea in creating fractals is to have a generator that does not operate on the initiator as one unit, but on multiple similar pieces of the initiator. A good example of this would be to apply the Koch Curve generator to an object with multiple straight-line segments. Take an equilateral triangle for instance. It contains three straight-line segments of equal length. Applying the Koch Curve generator to this shape would produce the first iteration of a fractal process as shown in figure 13.

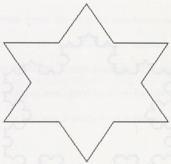


Figure 13: Koch Snowflake First Iteration

Unlike the previous fractals, here the generator and the first iteration are not the same. The second iteration and third iteration again repeat the same process as before by applying the generator shape to all the new straight line segments created by the previous iteration as shown in figures 14 and 15.

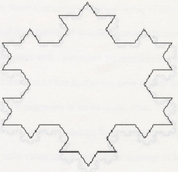


Figure 14: Koch Snowflake Second Iteration

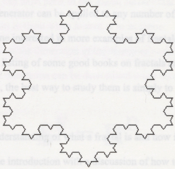


Figure 15: Koch Snowflake Third Iteration

When this process is taken to the infinite limit the fractal shape called the Koch Snowflake Fractal is formed as shown in figure 16.

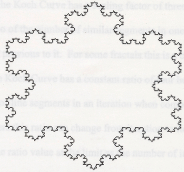


Figure 16: Koch Snowflake Fractal

The possibilities for different fractal shapes in the two-dimensional realm are literally limitless. Any type of generator can be applied to any number of different initiator combinations. For anyone interested in more examples of fractals, in the bibliography section of this paper is a listing of some good books on fractals and chaos theory for further study. Of course, the best way to study them is simply to create and discover them for one's self.

Now with the understanding of what a fractal is and how it is formed, it is important to conclude the introduction with a discussion of how to calculate the fractal dimension of a fractal. The first thing to determine is what number of dimensions are the parameters of an object changing. For a line, the change is one of length and therefore in one dimension. For a surface, the change is one of area and therefore in two dimensions. The Koch Curve has a parameter that changes length and therefore has a parameter dimension of one. Next, it is important to know the scaling factor of the fractal. This is found by looking at the relative size or area of similar segments in subsequent iterations.

As mentioned before, the Koch Curve has a scaling factor of three. The third and final piece needed is the ratio of the number of similar segments in one iteration with the number in the iteration previous to it. For some fractals this is a constant and therefore easy to determine. The Koch Curve has a constant ratio of four because there is always four times the number of line segments in an iteration when compared to the iteration previous to it. For others this ratio can change from iteration to iteration. In this case the thing to do is to take the ratio value at the limit as the number of iterations goes to infinity. This will give a limit ratio and thus an overall fractal dimension. It is important to look closely at the fractal process of these fractals with changing ratios. Even though the overall fractal dimension is some particular number, as will be seen with the octahedron fractal, there can exist parts of the fractal surface that have a distinctly different fractal dimension to them when looked at in isolation.

With information on the dimension of the changing parameter, scaling factor, and segment ratio, the fractal dimension can be determined using equation 1:

$$\mathbf{S^F = R^D}$$

Equation 1: Base Fractal Dimension Equation

In this equation R is defined as the segment ratio, D is defined as the changing parameter dimension, S is defined as the scaling factor, and F is defined as the fractal dimension of the object. What this equation says is that the segment ratio to the parameter dimension power, in which it propagates, is equal to the scaling factor to the fractal dimension power, in which it propagates. The motivation behind this equation is to relate the dimension in which the segment ratio operates to the dimension in which the scaling factor operates. To find the fractal dimension this equation is solved for F as shown here with the solution as equation 2:

$$S^F = R^D$$

$$Ln(S^F) = Ln(R^D)$$

$$F*Ln(S) = D*Ln(R)$$

$$F = D*Ln(R)/Ln(S)$$

Equation 2: Fractal Dimension Equation

Equation 2 is the Fractal Dimension Equation used in this paper to determine all fractal dimensions. There are other methods to determine the fractal dimension of a fractal shape. They all lead to the same answer and contain basically the same ingredients. This one is preferred here because of the intuitive nature behind the original base equation. Also, the concept of fractional dimension is well displayed in this equation. It is primarily a quotient of two logarithmic numbers creating a fraction. Calculating the fractal dimension of the Koch Curve as good example of the use of the Fractal Dimension Equation. This is shown in equation 3:

$$D = 1, R = 4, S = 3$$

$$F = 1 * Ln(4)/Ln(3)$$

$$= 1.2618595071429...$$

Equation 3: Fractal dimension of the Koch Curve

The importance of fractal dimension is twofold. First it allows for two different fractals to be compared in terms their respective scaling factors, segment ratios, and parameter dimensions. This gives a very quantitative characteristic to the fractal that will differentiate different fractal processes from one another. Second it allows for a fractal to be compared with non-fractal shapes with respect to their overall dimension. This gives a very qualitative characteristic to the fractal, which can differentiate fractal processes from non-fractal processes. As will be seen with both the tetrahedron fractal and the octahedron fractal, the determination and value of a fractal dimension plays a very important role in analyzing shape and structural aspects of a fractal.

Three-dimensional fractals can be constructed in the same way that two-dimensional fractals are. The reason they are called three-dimensional fractals is not because their fractal dimension is three. It is because the fractal shape is embedded in three dimensions that they are called three-dimensional fractals. The fractal dimension is determined with the same formula. Instead of the parameter dimension (D) being one as in most two-dimensional fractals, it is two with three-dimensional fractals. This means that what is changing with each iteration is a surface and not a line. When the segment ratio (R) is determined, each segment is now a polygon shape like a triangle or a square. The scaling factor (S) will now be a relation between the surface area of one segment to the surface area of a similar segment in a successive iteration.

Two three-dimensional fractals of special interest are the tetrahedron fractal and the octahedron fractal. Both of these fractals are interesting because unlike most fractals, these two seem to form a very simple outer shape of a cube. Figure 17 shows the relationship between a tetrahedron, an octahedron, and the cube they form as fractals.

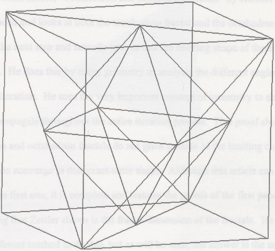


Figure 17: Octahedron Inscribed in a Tetrahedron Inscribed in a Cube

It is this cube that is the same limiting shape for both of these fractals.

There are two published articles that have been found regarding the tetrahedron and octahedron fractals. The first article, "A Fractal Excursion" by Dane R. Camp [1], is a very nice introduction to what a fractal is and how a fractal is created. Camp goes into a basic description of the tetrahedron fractal and briefly mentions the octahedron fractal. His focus is on the observation of the convergence and divergence of different properties of a fractal. He shows that the tetrahedron fractal converges to a finite volume in three dimensions yet diverges to infinite surface area in two dimensions. He also shows that the limiting volume of the tetrahedron fractal is cubic. Even though this article is designed primarily for high school students to motivate interest in math, it serves as a good beginning to the exploration of three-dimensional fractals.

The second article, "Tetrahedron and octahedron fractals" by Herbert Zeitler [2], is more advanced and looks at both the tetrahedron fractal and the octahedron fractal. Zeitler takes the next step and actually proves that the limiting shape of these fractals is that of a cube. He does this by using geometry to analysis the different angles that are formed from iteration. He uses the very important concept of symmetry to show that these angles propagate throughout the entire iteration process. This proof shows that both the tetrahedron and octahedron fractals do not grow outside of the limiting cube shape and that they do converge to that exact outer shape. Although this article can be seen as building on the first one, it is complete and restates the proofs of the first paper. Another important thing that Zeitler shows is the fractal dimension of the fractals. He uses a somewhat different method to do this, but as will be seen, the answer is the same. Interestingly, he shows that both of these fractals have the same dimensional value of 2.58496…

Tetrahedron Fractal:

The tetrahedron fractal is a special fractal for me. About ten years ago I was first beginning to study fractals. I had learned about the Koch Curve and its application to a triangle forming the Koch Snowflake. I thought it would be interesting to try this in a three-dimensional way. Instead of iterating a line, I would iterate a surface. Keeping with the theme of triangles, I thought a good candidate initiator shape would be the tetrahedron as shown in figure 18.

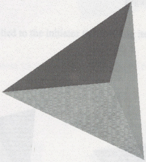


Figure 17: Iteration 0 (Tetrahedron)

I envisioned each triangular surface iterating to make a new tetrahedron shape at its center as shown in figure 19 and 20.

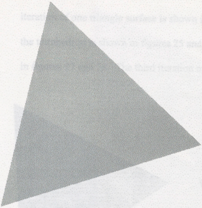


Figure 19: Triangle

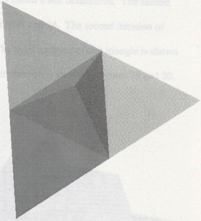
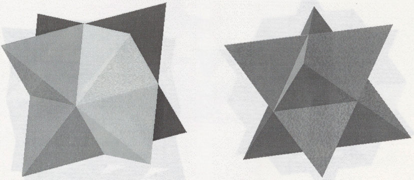


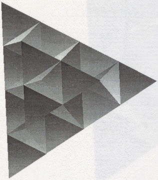
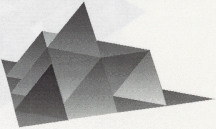
Figure 20: Generator Shape

When this generator is applied to the initiator the resulting iteration is as shown in figure 21 and 22.

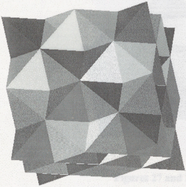
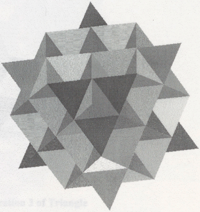


Figures 21 and 22: Iteration 1 (Star Octahedron)

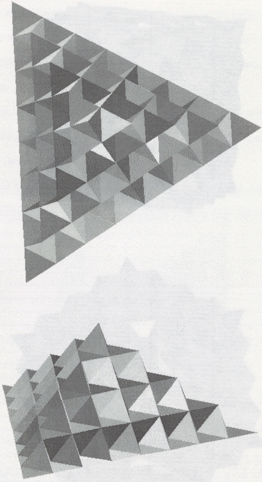
This is the exact same shape as what is commonly called a star octahedron. The second iteration of one triangle surface is shown in figures 23 and 24. The second iteration of the tetrahedron is shown in figures 25 and 26. The third iteration of one triangle is shown in figures 27 and 28. The third iteration of the tetrahedron is shown in figures 29 and 30.

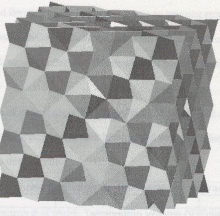
Figures 23 and 24: Iteration 2 of Triangle

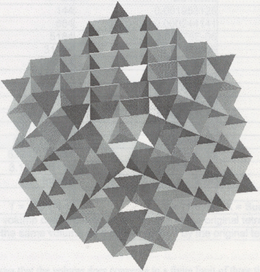



Figures 25 and 26: Iteration 2 of Tetrahedron



Figures 27 and 28: Iteration 3 of Triangle





Figures 29 and 30: Iteration 3 of Tetrahedron

The first time I realized that the outer limiting shape was a cube was after I had completed the second iteration of the fractal! This was an amazing epiphany for me and that is how my interest in three-dimensional fractals started. Since then, I have created several other three-dimensional fractals. The tetrahedron fractal will always be my favorite though.

The first item of interest about the tetrahedron fractal is to look at how the volume contained by the surface changes over each iteration and what the volume is at the fractal limit. Table 1 shows this analysis.

Table 1: Analysis of the Volume Evolution for the Tetrahedron Fractal

Iteration (N)	Addl. Tetrahedron # (T)	Addl. Tetrahedron Volume (V)	Total Structure Volume (X)
0	1	1	1
1	4	0.125	1.5
2	24	0.015625	1.875
3	144	0.001953125	2.15625
4	864	0.000244141	2.3671875
5	5184	3.05176E-05	2.525390625
6	31104	3.8147E-06	2.644042969
7	186624	4.76837E-07	2.733032227
8	1119744	5.96046E-08	2.79977417
9	6718464	7.45058E-09	2.849830627
10	40310784	9.31323E-10	2.887372971
11	241864704	1.16415E-10	2.915529728
12	1451188224	1.45519E-11	2.936647296
13	8707129344	1.81899E-12	2.952485472
14	52242776064	2.27374E-13	2.964364104
15	3.13457E+11	2.84217E-14	2.973273078
Infinity	Infinity	Zero	Three
Formulas:	T = 4*6^(N-1)	V = (1/8)^N	X = Sum of T*V over all N
	Fractal volume converges to 3x the volume of the original tetrahedron.		
	This is the same volume as a cube inscribed by the original tetrahedron.		

The analysis shows that the volume does converge to a finite limit of three times the volume of the original tetrahedron. This is exactly the volume of a cube that has the generator tetrahedron shape inscribed inside it (see back to figure 17). As has already been mentioned, the actual outer shape limit of the tetrahedron fractal has been proven to be a cube by Zeitler [2].

Another item of interest that can be learned about the tetrahedron fractal is what the surface area of the fractal limit is. Table 2 shows an analysis of the surface area.

Table 2: Analysis of the Surface Area Evolution for the Tetrahedron Fractal			
Iteration (N)	Triangle Surface # (T)	Triangle Area (A)	Total Structure Surface Area (X)
0	4	1	4
1	24	0.25	6
2	144	0.0625	9
3	864	0.015625	13.5
4	5184	0.00390625	20.25
5	31104	0.000976563	30.375
6	186624	0.000244141	45.5625
7	1119744	6.10352E-05	68.34375
8	6718464	1.52588E-05	102.515625
9	40310784	3.8147E-06	153.7734375
10	241864704	9.53674E-07	230.6601563
11	1451188224	2.38419E-07	345.9902344
12	8707129344	5.96046E-08	518.9853516
13	52242776064	1.49012E-08	778.4780273
14	3.13457E+11	3.72529E-09	1167.717041
15	1.88074E+12	9.31323E-10	1751.575562
Infinity	Infinity	Zero	Infinity
Formulas:	T = 4*6^N	A = (1/4)^N	X = T*A
	Fractal surface area diverges to infinity.		

The analysis shows that the surface does go to infinity, as one would expect for this fractal.

A third item of interest is the fractal dimension of the tetrahedron fractal. Table 3 shows an analysis of calculating the segment ratio for the tetrahedron fractal and the subsequent value for the fractal dimension.

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Table 3: Analysis of the Fractal Dimension for the Tetrahedron Fractal			
Iteration	# of Initial Surface Segments	# of Generated Surface Segments	Ratio
0	N/A	4	N/A
1	4	24	6
2	24	144	6
3	144	864	6
4	864	5184	6
5	5184	31104	6
6	31104	186624	6
7	186624	1119744	6
8	1119744	6718464	6
9	6718464	40310784	6
10	40310784	241864704	6
Infinity	Infinity	Infinity	Six
	Ratio of Edge Lengths: Initial to Generated = 2		
	Fractal Dim.= 2*ln (6) / ln (4) =	2.584962501	
	The tetrahedron fractal has a constant dimension on its entire surface.		

All the information shown so far about the tetrahedron fractal has already been published. Unfortunately, none of this shows what the actual shape of the fractal surface. Up until now there has been no analysis done on what the shape the fractal surface actually has. After examining this fractal for many years now I have discovered something very important and exciting about this fractal. The shape of a cube is indeed the outer limit to the fractal surface. There is also, completely contained inside the cube, another whole surface dynamic going on.

In order to completely understand the dynamics of a given three-dimensional fractal, one has to be able to completely describe the changes that occur between each successive iteration. To do this I use what I call the relational model. As the fractal process moves from one iteration to the next, individual surface segments can be paired up with other surface segments in reoccurring surface relations. Each distinct surface relation always produces the same new set of relations after a single iteration. If all the different surface relations and what they produce can be learned for a particular fractal, then the entire dynamic of that fractal surface can be known.

The initiator of the tetrahedron fractal has four similar equilateral surfaces. The subsequent iterated tetrahderonal surface structures on each of the original surfaces will touch each other at only at shared vertex points. They do not share edges or surfaces as can be seen in the previous figures 20 and 21. Because of this, the iteration of one surface will develop in complete isolation to the other three surfaces. Since this is the case, there is no surface relation other than a single surface relation for the generator of the fractal. After the first iteration there are twenty-four new equilateral triangle surfaces exactly one quarter the area of the four original triangular surfaces. Each surface is paired up with another one to make twelve identical angular relationships called double surface relations and one is shown in figure 31.

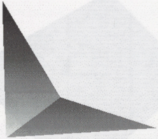
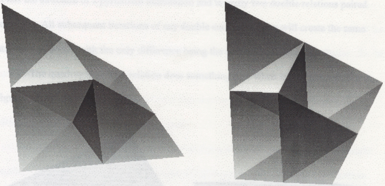


Figure 31: Double Surface Relation

Each of these paired surfaces will iterate in the exact same way as shown in figures 32 and 33.



Figures 32 and 33: Double Surface Relation Iteration

After this second iteration, there are now two more double surface relations created for each original double relation and a new relation is created. Two quadruple surface relations are formed as shown in figure 34.

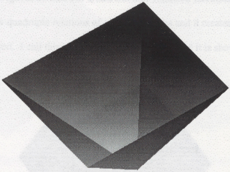
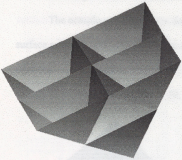
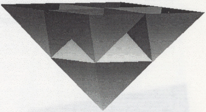


Figure 34: Quad Surface Relation

It has the structure of a pyramidal indentation and is really two double relations paired together. All subsequent iterations of any double surface relation will create the same structure as above with the only difference being the size of its surface area.

The quadruple surface relation does something new when it iterates as shown in figures 35 and 36.





Figures 35 and 36: Quad Surface Relation Iteration

It creates four new quadruple relations of one-quarter area and it creates a completely enclosed octahedron. I call this an octuple surface relation and it is shown in figure 37.

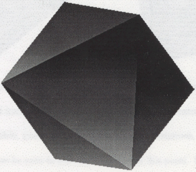
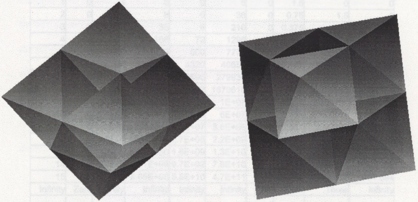


Figure 37: Octuple Surface Relation (Inside Surface of Octahedron)

The octuple surface relation is really two quadruple relations paired together. It is important to keep in mind that this octahedron is really inside out. The inward side of the octahedron surface is the actual "outer" iterating surface of the tetrahedron fractal. So the next iteration will be inward towards the octahedron's center. All subsequent iterations of any quadruple surface relation creates the same structure as above with the only difference being the size of its surface area.

The octuple surface relation does something new. It forms six enclosed octuple surface relations with one-quarter the area of the original, but it forms no new surface relations. All subsequent iterations of an octuple surface relation will repeat this process exactly as shown in figures 38 and 39.



Figures 38 and 39: Octuple Surface Relation Iteration

After the third iteration, no new surface relations are created. The fractal continues to form double, quadruple, and octuple surface relations. The evolution of the tetrahedron surface can be seen to have stabilized after this point. By being stabilized I mean that the limiting shape of the fractal surface can be completely determined. Since the shape of the surface is nothing more than triangular areas connected to other triangular areas, the shape is completely determined by the angular relationship between adjoining triangles. This is how the relational model works. Once the cycle of relation creating ends, and all the possible relations and what they make are known, understanding the surface dynamic of the fractal is done. Table 4 is an analysis of the surface relations for tetrahedron fractal.

Table 4: Analysis of the Evolution of Surface Relations for one Triangle on the Tetrahedron Fractal

Initial Surface Area = 1		Total # of:			Total # of	Total Surface Area Comprised by:			
Iteration	Single	Double	Quadruple	Octuple	Triangles	Single	Double	Quadruple	Octuple
0	1	0	0	0	1	1	0	0	0
1	0	3	0	0	6	0	1.5	0	0
2	0	6	6	0	36	0	0.75	1.5	0
3	0	12	36	6	216	0	0.375	2.25	0.75
4	0	24	168	72	1296	0	0.188	2.625	2.25
5	0	48	720	600	7776	0	0.094	2.8125	4.6875
6	0	96	2976	4320	46656	0	0.047	2.90625	8.4375
7	0	192	12096	28896	279936	0	0.023	2.953125	14.10938
8	0	384	48768	185472	1679616	0	0.012	2.976563	22.64063
9	0	768	195840	1161600	1E+07	0	0.006	2.988281	35.44922
10	0	1536	784896	7165440	6E+07	0	0.003	2.994141	54.66797
11	0	3072	3142656	4.4E+07	3.6E+08	0	0.001	2.99707	83.49902
12	0	6144	12576768	2.7E+08	2.2E+09	0	7E-04	2.998535	126.7471
13	0	12288	50319360	1.6E+09	1.3E+10	0	4E-04	2.999268	191.6199
14	0	24576	2.01E+08	9.7E+09	7.8E+10	0	2E-04	2.999634	288.9294
15	0	49152	8.05E+08	5.8E+10	4.7E+11	0	9E-05	2.999817	434.894
Infinity	Zero	Infinity	Infinity	Infinity	Infinity	Zero	Zero	Three	Infinity

The double relations evolve into the edges of the limiting cube.
This conforms with them comprising none of the fractal surface area at infinity.
The quadruple relations evolve into the finite surface area of the limiting cube.
This conforms with them comprising a finite surface area at infinity.
The octuple relations contain the infinite surface area of the fractal.

Three main qualities to the fractal surface can be seen. The double surface relations are always created along an edge of the limiting cube shape. At infinity they actually create this edge. The combined surface area of all the double surface relations goes to zero. This makes sense since the edges of the cube have no area. The quadruple surface relations are always created along a surface of the limiting cube. At infinity they actually make this surface. The combined area of all quadruple surface relations is a finite value. This makes sense since it the surface of the cube has a finite area. The octuple surface relation is always created within the general volume of the cube. There are an infinite number of these enclosed octahedral shaped surface regions. Even though the combined volume contained within all the octuple surface relations goes to zero as iteration goes to infinity, their combined inward surface area does not. The sum total of the surface area of all these infinitely small surface nodes goes to infinity. This is where the infinite surface area of the fractal is "hidden". This inward iterating structure can be seen as an entire fractal in itself. This inward iterating fractal has the same fractal dimension as it's parent, the tetrahedron fractal. With this information the surface of the tetrahedron fractal is completely described and hence the fractal itself is completely determined.

Octahedron Fractal:

The octahedron fractal, not surprisingly, has an initiator shape of an octahedron as shown in figure 40.

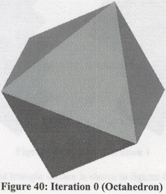


Figure 40: Iteration 0 (Octahedron)

The generator operates on triangular surfaces as before except that instead of tetrahedrons being generated, octahedrons are generated as shown in figures 41 and 42.

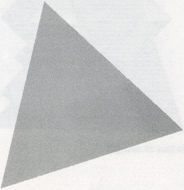


Figure 41: Triangle

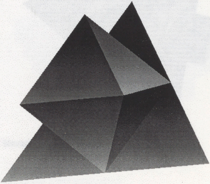
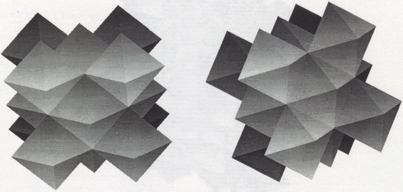


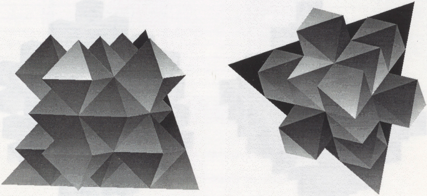
Figure 42: Generator Shape

When this generator is applied to the initiator the resulting iteration is as shown in figure 43 and 44.



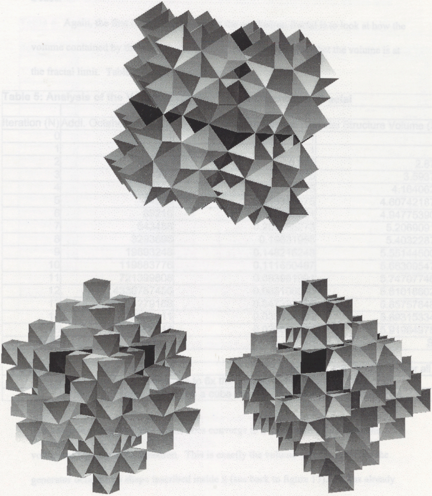
Figures 43 and 44: Iteration 1

The second iteration of one triangle surface is shown in figures 45 and 46.



Figures 45 and 46: Iteration 2 of Triangle

The second iteration of the octahedron is shown in figures 47, 48, and 49.



Figures 47, 48, and 49: Iteration 2 of Octahedron

After the second iteration it is easy to see why the limiting shape for this fractal process is a cube.

Again, the first item of interest about the octahedron fractal is to look at how the volume contained by the surface changes over each iteration and what the volume is at the fractal limit. Table 5 shows this analysis.

Table 5: Analysis of the Volume Evolution for the Octahedron Fractal			
Iteration (N)	Addl. Octahedron # (O)	Addl. Tetrahedron Volume (V)	Total Structure Volume (X)
0	1	1	1
1	8	1	2
2	56	0.875	2.875
3	368	0.71875	3.59375
4	2336	0.5703125	4.1640625
5	14528	0.443359375	4.607421875
6	89216	0.340332031	4.947753906
7	543488	0.259155273	5.20690918
8	3293696	0.19631958	5.40322876
9	19893248	0.148216248	5.551445007
10	119883776	0.111650467	5.663095474
11	721399808	0.083981991	5.747077465
12	4336787456	0.063108563	5.810186028
13	26054279168	0.047392458	5.857578486
14	1.5646E+11	0.035574861	5.893153347
15	9.39296E+11	0.026696404	5.919849752
Infinity	Infinity	Zero	Six
Formulas:	O = 2*6^N-4^N	V = (1/8)^N	X = Sum of T*V over all N
	Fractal volume converges to 6x the volume of the original octahedron.		
	This is the same volume as a cube inscribed by the original octahedron.		

The analysis shows that the volume does converge to a finite limit of six times the volume of the original octahedron. This is exactly the volume of a cube that has the generator octahedron shape inscribed inside it (see back to figure 17). As has already been mentioned, the actual outer shape limit of the octahedron fractal has been proven to be a cube by Zeitler [2].

The next item of interest that can be learned about the octahedron fractal is what the surface area of the fractal limit is. Table 6 shows an analysis of the surface area.

Table 6: Analysis of the Surface Area Evolution for the Octahedron Fractal

Iteration (N)	Triangle Surface # (T)	Triangle Area (A)	Total Structure Surface Area (X)
0	8	1	8
1	80	0.25	20
2	608	0.0625	38
3	4160	0.015625	65
4	27008	0.00390625	105.5
5	170240	0.000976563	166.25
6	1054208	0.000244141	257.375
7	6456320	6.10352E-05	394.0625
8	39262208	1.52588E-05	599.09375
9	237670400	3.8147E-06	906.640625
10	1434411008	9.53674E-07	1367.960938
11	8640020480	2.38419E-07	2059.941406
12	51974340608	5.96046E-08	3097.912109
13	3.12383E+11	1.49012E-08	4654.868164
14	1.87644E+12	3.72529E-09	6990.302246
15	1.12673E+13	9.31323E-10	10493.45337
Infinity	Infinity	Zero	Infinity
Formulas:	T = 4*(6^(N+1)-4^(N+1))	A = (1/4)^N	X = T*A
	Fractal surface area diverges to infinity.		

Like the tetrahedron fractal the analysis shows that the surface does go to infinity, as is expected.

A third item of interest is the fractal dimension of the octahedron fractal. Table 7 shows an analysis of calculating the segment ratio for the octahedron fractal and the subsequent value for the fractal dimension.

Table 7: Analysis of the Fractal Dimension for the Octahedron Fractal			
Iteration	# of Initial Surface Segments	# of Generated Surface Segments	Ratio
0	N/A	8	N/A
1	8	80	10
2	80	608	7.6
3	608	4160	6.8421
4	4160	27008	6.4923
5	27008	170240	6.3033
6	170240	1054208	6.1925
7	1054208	6456320	6.1243
8	6456320	39262208	6.0812
9	39262208	237670400	6.0534
10	237670400	1434411008	6.0353
11	1434411008	8640020480	6.0234
12	8640020480	51974340608	6.0155
13	51974340608	3.12383E+11	6.0103
14	3.12383E+11	1.87644E+12	6.0069
15	1.87644E+12	1.12673E+13	6.0046
Infinity	Infinity	Infinity	Six
	The ratio is not constant over a finite number of iterations.		
	The ratio does converge at infinity to 6.		
	This is the same ratio as for the tetrahedron fractal,		
	therefore the overall Fractal Dim. = 2.584962501		

The octahedron fractal is an example of a fractal that has a changing segment ratio. The limiting value as iteration goes to infinity shows that this segment ratio does converge to a finite value of six. This is the same ratio as with the tetrahedron fractal. With the same segment ratio and same scaling factor of four, it is certain that the octahedron fractal has the same overall fractal dimension as the tetrahedron fractal. But, unlike the constant segment ratio of the tetrahedron fractal, the changing segment ratio of the octahedron fractal does not certify that the fractal dimension is constant throughout the entire fractal surface. It only shows that there is a particular overall fractal dimension. This is a very important distinction that will have an effect on what is learned through the relational model when it is applied to the octahedron fractal.

The initiator of the octahedron fractal has eight similar equilateral surfaces. Like the tetrahedron fractal, the subsequent iterated octahedral surface structures on each of the original surfaces will touch each other at only at shared vertex points. They will never share edges or surfaces. Because of this, the iteration of one surface will develop in complete isolation to the other seven surfaces. Since, this is the case there is no surface relation other than a single surface relation for the generator of the fractal.

After the first iteration, each original triangular surface of the original octahedron has generated one new octahedron in its center. There are two types of surface relations created here. There are four new non-paired or single surface relations and three double surface relations. These double surface relations are not the same as for the tetrahedron fractal. The angle between the surfaces is different. On subsequent iterations, the non-paired surfaces will duplicate the result of the first iteration. This double surface relation is shown in figure 50.

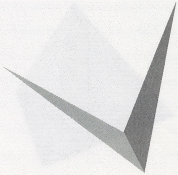
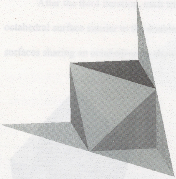
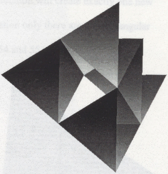


Figure 50: Double Surface Relation

After the second iteration, the double surface relations will share in creating one new octahedron surface as in figure 51 and 52.





Figures 51 and 52: Double Surface Relation Iteration

This is different than in the tetrahedron fractal where all triangular surfaces always create their own new tetrahedron. As can be seen in the figure, iterating a double surface relation, creates two single surface relations, creates two double surface relations, and creates two of a new relation called the triple surface relation as shown in figure 53.

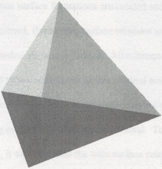
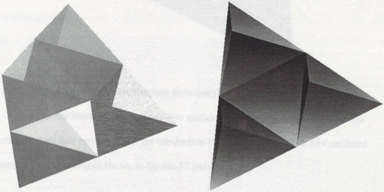


Figure 53: Triple Surface Relation

All subsequent double surface relations will create the exact same set of relations with each triangular surface being one quarter the area of the previous iteration.

After the third iteration, each triple surface relation will create exactly one new octahedral surface similar to the double surface relation only there are three triangular surfaces sharing an octahedron as shown in figure 54 and 55.



Figures 54 and 55: Triple Surface Relation Iteration

Again, some previous surface formations are created and a new one also. One single surface relation is formed, three triple surface relations are formed, and one completely enclosed tetrahedral surface is formed. All subsequent triple surface relations form the same number and type of structures as this original one. Like the enclosed octahedron of the tetrahedron fractal, the inward surface of the enclosed tetrahedron is really the "outer" evolving surface of the octahedron fractal. To distinguish this from the quadruple surface relation, it will be called the tetra surface relation as shown in figure 56.

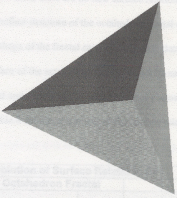
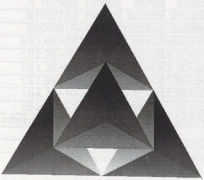
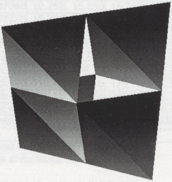


Figure 56: Tetra Surface Relation (Inside Surface of Tetrahedron)

After the fourth iteration, these four surfaces will all share in forming one new octahedral surface enclosed with the tetrahedron shape. This forms four new enclosed tetra surface relations as shown in figures 57 and 58.





Figures 57 and 58: Tetra Surface Relation Iteration

Like the tetrahedron fractal, there are no new surface relations created after the third iteration. Again the surface structure of the octahedron fractal can be said to have stabilized and the limiting shape of the fractal can be completely determined. Unlike the tetrahedron fractal, the surface of the octahedron fractal cannot be easily divided into two regions that of non-enclosed areas forming the outer cube and enclosed areas containing the infinite fractal area. Table 8 shows an analysis of the surface relations of the octahedron fractal.

Table 8: Analysis of the Evolution of Surface Relations for one Triangle of the Octahedron Fractal

Initial Surface Area = 1			Total # of:		Total # of	Total Surface Area Comprised by:			
Iteration	Single	Double	Triple	Tetra	Triangles	Single	Double	Triple	Tetra
0	1	0	0	0	1	1	0	0	0
1	4	3	0	0	10	1	1.5	0	0
2	22	18	6	0	76	1.375	2.25	1.125	1.5
3	130	102	54	6	520	2.0313	3.1875	2.5313	3.375
4	778	594	366	78	3376	3.0391	4.6406	4.2891	5.71875
5	4666	3522	2286	678	21280	4.5566	6.8789	6.6973	8.92969
6	27994	21042	13902	4998	131776	6.8345	10.274	10.182	13.5762
7	167962	126066	83790	33894	807040	10.252	15.389	15.342	20.4565
8	1007770	756018	503502	219366	4907776	15.377	23.072	23.048	30.7313
9	6046618	4535346	3022542	1380966	3E+07	23.066	34.602	34.59	46.1203
10	3.6E+07	2.7E+07	1.8E+07	8546406	1.8E+08	34.599	51.9	51.894	69.1922
11	2.2E+08	1.6E+08	1.1E+08	5.2E+07	1.1E+09	51.899	77.849	77.846	103.794
12	1.3E+09	9.8E+08	6.5E+08	3.2E+08	6.5E+09	77.848	116.77	116.77	155.694
13	7.8E+09	5.9E+09	3.9E+09	1.9E+09	3.9E+10	116.77	175.16	175.16	233.543
14	4.7E+10	3.5E+10	2.4E+10	1.2E+10	2.3E+11	175.16	262.74	262.74	350.315
15	2.8E+11	2.1E+11	1.4E+11	7E+10	1.4E+12	262.74	394.1	394.1	525.472
Infinity	Infinity	Infinity	Infinity	Infinity	Infinity	Infinity	Infinity	Infinity	Infinity
	All relations contain the infinite surface area of the fractal								

All surface relations contain an infinite surface area at the fractal limit. Except for the tetra surface relation, there is no way to isolate what type of relation is building what particular part of the limiting cube shape.

Since the tetra surface relation is completely enclosed, it is not involved in creating the outer shape of the fractal. This enclosed iterating region can be seen as a fractal in itself. When looked as such the results are very surprising. The scaling factor for this new fractal is the same as its parent at four. The segment ratio is different though, with a value of four instead of six. This means the fractal dimension for this object is two! This really does make sense on a second look. At each iteration four, surface segments are being replaced with only four new surface segments. The value for the total surface area stays the constant for each iteration.

Question: Will the fractal dimension of the octahedron fractal change if none of the tetra surface relations are included in the calculation? Table 9 shows this analysis.

Table 9: Analysis of Fractal Dimension for the Octahedron Fractal not Including the Tetra Surface Relations

Iteration	# of Initial Surface Segments	# of Generated Surface Segments	Ratio
0	N/A	8	N/A
1	8	80	10
2	80	608	7.6
3	608	4136	6.802632
4	4136	26696	6.454545
5	26696	167528	6.275397
6	167528	1034216	6.173392
7	1034216	6320744	6.111629
8	6320744	38384744	6.072821
9	38384744	232146536	6.047885
10	232146536	1400225384	6.031645
11	1400225384	8430724712	6.020977
12	8430724712	50701813352	6.013933
13	50701813352	3.04681E+11	6.009266
14	3.04681E+11	1.82996E+12	6.006168
15	1.82996E+12	1.09873E+13	6.004107
Infinity	Infinity	Infinity	Six
	The ratio still converges to six at infinity.		

The answer is clearly no. The overall fractal dimension is not affected by the removal of the tetra surface relations. The fractal dimension of the octahedron fractal is the same with or without them. This is a very interesting result. Apparently, a fractal like the octahedron fractal can have regions where the fractal dimension is distinctly different from the overall fractal. These regions are like pockets of two-dimensional reality embedded within a higher fractal dimensional structure. These regions exist both at any finite iteration above three and at the infinite iteration.

This fractal has shown itself to have quite different qualities from that of the tetrahedron fractal. Without the use of the relational model, a complete understanding of the fractal surface dynamic would not be obtained. With it, the tetra surface relation is discovered and also a recognition that the single, double, and triple surface relations collectively form the fractal dimension and outer cube shape of the fractal. This information provides a complete description of the surface of the octahedron fractal and hence the fractal itself is completely determined.

Constructing Three-dimensional Fractals:

The tetrahedron fractal and the octahedron fractal are excellent examples of three-dimensional fractals. They give two different cases of the fractal dimension calculation process. They also are perfect candidates in showing how the relational model works. These two fractals, though, are only two of an infinitely large class called of fractals three-dimensional fractals. Just like with two-dimensional fractals, their three-dimensional cousins can be created by a limitless number of initiator shapes iterated on by all different kinds of generators.

One way to make a new fractal is to take one that is already known and change some aspect of its iteration process. This next fractal that I have discovered, I call the octahedron half fractal. It has the same initiator shape and generator shape as the octahedron fractal does. The difference is that at each iteration, only half the surfaces are iterated. The octahedron half fractal process is shown in figures 59, 60, and 61.

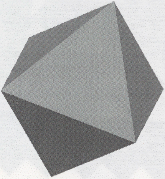


Figure 59: Iteration 0 (Generator Shape)

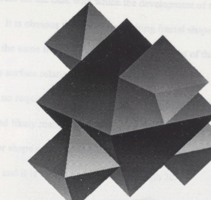


Figure 60: Iteration 1

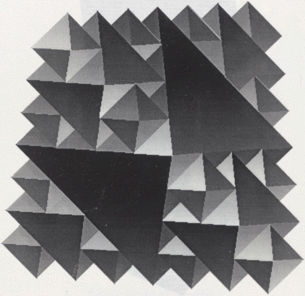


Figure 61: Iteration 2

This fractal is great for the ease with which the development of the fractal process can clearly be seen. It is obvious that the outer limiting fractal shape is that of a tetrahedron. This fractal has the same fractal dimension qualities as that of the octahedron fractal as well as the same surface relation qualities.

There is no requirement that the surface segment be triangular. Square surfaces could be used and likely many others. Here is a fractal where the initiator shape is a cube and the generator shape operates on each of the six square surfaces. I call this fractal the cube 2x2 fractal and it is shown in figures 62 through 70.

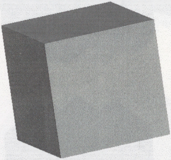


Figure 62: Iteration 0 (Initiator Shape)

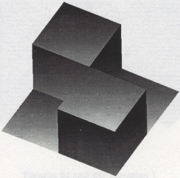
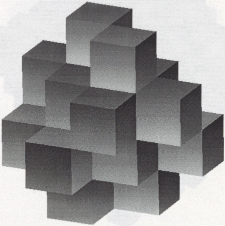
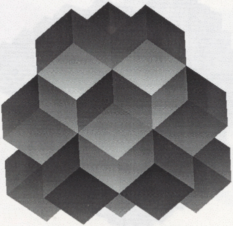
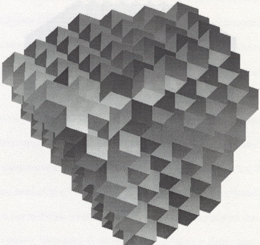


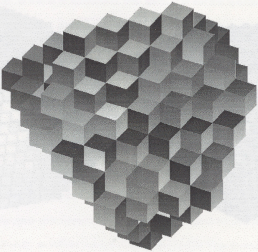
Figure 63: Generator Shape



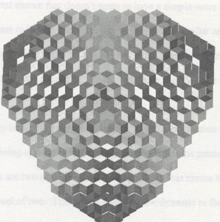


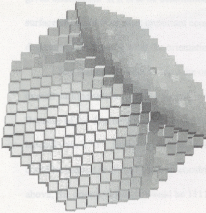
Figures 64 and 65: Iteration 1

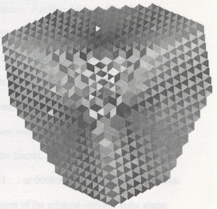




Figures 66 and 67: Iteration 2







Figures 68, 69, and 70: Iteration 3

This is the first fractal shown that doesn't seem to have a simple outer limiting shape. There are what appear to be four flat surfaces forming with a similar relationship to each other as the surfaces of a tetrahedron have to each other. The difference is that the "corners" of the tetrahedron are not only truncated, there is cavity like structure inside them.

Another very interesting aspect of this fractal is the orientation options available with the generator being applied to the initiator shape. When the generator is applied to a square surface there are two possible ways it can do this. This means that the generator has a degeneracy level of two. This brings a whole new dynamic to fractal construction.

First one has to decide whether to allow each square surface on the cube to be iterated with a random orientation or whether the orientation is determined within each given orientation. If it is to be determined, then whether or not it is symmetric to overall surface structure is another important consideration. Another degree of freedom this degeneracy provides is what the orientation will be for each iteration.

Say that one decides to have the orientation determined and symmetric for the overall surface structure. This overall symmetric orientation could be labeled with a "1". Because of the degeneracy, there is another symmetric orientation and this could be labeled with a "0". Each iteration could then use one or the other orientation. A code could be created to identify how to construct the fractal. The fractal in the figures shown above, the orientation code would be 11111111…. or 00000000…. since the orientation has been kept the same with respect to the corners of the original initiator cube shape. This produces literally an infinite number of different possible ways to construct a fractal using this generator.

So far, all the fractals that have been looked at have generators that transform the surface area by first dividing up a surface segment into four smaller similar surface segments. The triangle was divided up into four similar triangles and the square was divided up into four similar squares. There is nothing to say that these surfaces can't be divided up in different ways. The triangle and square could just as well be divided up into 9 pieces, or 16, 25, 36, 49…N^2 pieces. Good examples of this are the next two fractals. They both have the generator iterating on the square surfaces of a cube. This time each square is divided up into 9 pieces. The first is called the cube 3x3 cross fractal and the second is called the cube 3x3 x fractal and are shown in figures 71 through 82.

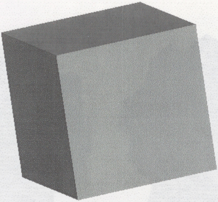


Figure 71: Iteration 0 (Initiator Shape)

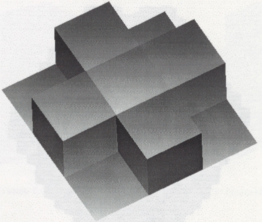
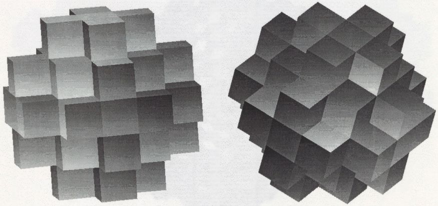
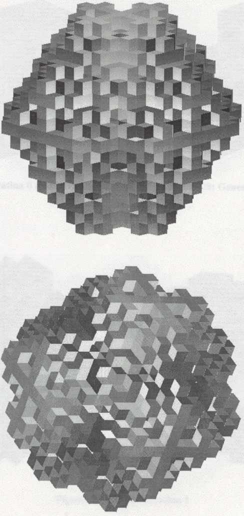


Figure 72: Generator Shape



Figures 73 and 74: Iteration 1



Figures 75 and 76: Iteration 2

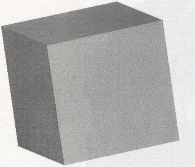


Figure 77: Iteration 0 (Initiator Shape)

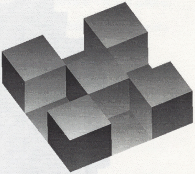
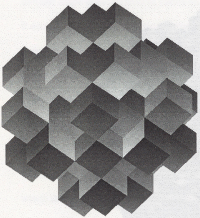
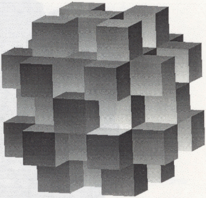
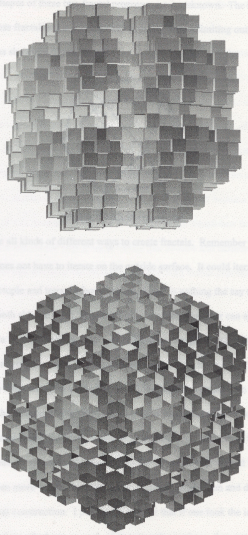


Figure 78: Generator Shape





Figures 79 and 80: Iteration 1



Figures 81 and 82: Iteration 2

What the final shapes of these two fractal processes are is unknown. The best guess on the cube 3x3 cross fractal is that it could very possibly form a limiting outer shape of a cuboctahderon as shown in figure 83.

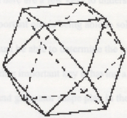


Figure 83: Cuboctahedron

There are all kinds of different ways to create fractals. Remember that the initiator shape does not have to iterate on the outside surface. It could iterate inwardly as it does for the octuple and tetra surface relations. There is nothing the say that it can't even iterate on both sides at once. Another idea is that the generator can apply any pattern of surface transformations. The two 3x3 fractals are good examples of two different patterns using the same 3x3 segment division. Also, there is no reason why tetrahedrons can only iterate on a tetrahedron. One could take octahedron initiator shape and use the tetrahedron generator shape on its outside surface or vice versus with the tetrahedron initiator shape and the octahedron generator. One could limit the surface which the generator iterates on the only the new surface segments formed by the previous iteration. An even more wild is the idea of bringing the icosahedron and dodecahedron into use for fractal construction. I personally suspect that if one took the icosahedron shape and iterated icosahedrons on each of its triangular surfaces, they would begin to form a fractal with an outer shape of a dodecahedron. Even the dodecahedron could be iterated upon. There would be substantial challenges here though due to the fact it has pentagonal surfaces and therefore it would be hard to keep a good symmetry.

There are all kinds of ways to create an infinite number of different three-dimensional fractal shapes. There are a few keys to understanding the dynamic of the fractal surface. The most important one is being able to solve the fractal dimension equation. In solving it one must be able to determine the scaling factor of the self-similarity in the fractal. Another important key is having a good understanding of the roles that the initiator shape and generator shape play in the construction of the fractal. These two initial conditions alone determine what the fractal will become. Lastly, using the relational model to investigate how the surface structure transforms from iteration to iteration. Without this model only general values like volume and surface area can be determined. With this model, a full picture of the evolving fractal process can be seen.

Conclusion:

There are a few important points about three-dimensional fractals that I wish to reiterate. First, there is a significant difference between two-dimensional fractals and three-dimensional fractals. With two-dimensional fractals, the generator operates on line segments. Even though a two-dimensional fractal contains no area, it is still embedded in a two-dimensional framework. With three-dimensional fractals, the generator operates on surface segments. Even though a three-dimensional fractal contains no volume, it is still embedded in a three-dimensional framework. The impact of this is that two-dimensional fractals can enclose an area and thus define a boundary line to the area. In contrast, three-dimensional fractals can enclose a volume and thus define a boundary surface to the volume. This allows three-dimensional fractals to model the three dimensional shapes of the human experience where two-dimensional fractals do not.

It important to keep in mind the difference between the surface area of a three-dimensional fractal and the surface growth of the three-dimensional fractal process. At each iteration, the fractal generator and the subsequent formation of surface relations determine the surface growth. The surface area is simply the sum total of all the surface segments of any given iteration level. If the surface growth is zero, as in the case of the octuple surface relation and tetra surface relation when seen as fractals, the surface area will be constant throughout the fractal process. If the surface growth is greater than zero, the surface area will increase exponentially to infinity at the fractal limit. The surface growth is what is used to determine the fractal dimension of a given fractal and not its surface area. Two fractals can have an infinite surface area and have very distinct surface growth rates.

The new method of investigating fractal shapes presented in this paper is the relational model. It is a powerful tool in the study of fractals. When applied to three-dimensional fractals, it discloses and sheds light on the dynamical nature of exactly how the surface shape evolves in the fractal process. Once the cycle of creating new surface relations is complete, all surface relations involved in forming the fractal shape are known. This allows for the limit shape of the fractal to be completely determined.

Three-dimensional fractals like the tetrahedron fractal, the octahedron fractal, and the octahedron half fractal are very exciting to study because they take the outer shape of other simple solids. By using the relational model, these three-dimensional fractals like expose their true fractal innards showing they are much more than their outer shape would indicate. Whether there are more three-dimensional fractals that have outer limiting shapes that are simple solids is not certain. One very likely candidate for this is the cube 3x3 cross fractal which may form a cuboctahedron. Hopefully this paper will spur further study into the fractal nature of the mystical and magical shapes that are the perfect solids.

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[2] Zeitler, Herbert. "Tetrahedron and octahedron fractals." *International Journal of Mathematical Education in Science and Technology* 29.3 (1998) : 329-341

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