FACTORIZABLE MODULE ALGEBRAS, CANONICAL BASES, AND CLUSTERS

by

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DISSERTATION ABSTRACT

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The present dissertation consists of four interconnected projects. In the first, we introduce and study what we call *factorizable* module algebras. These are $U_q(\mathfrak{g})$-module algebras A which factor, potentially after localization, as the tensor product of the subalgebra A^+ of highest weight vectors of A and a copy of the quantum coordinate algebra $\mathcal{A}_q[U]$, where U is a maximal unipotent subgroup of G, a semisimple Lie group whose Lie algebra is $\mathfrak{g}$.

The class of factorizable module algebras is surprisingly rich, in particular including the quantum coordinate algebras $\mathcal{A}_q[Mat_{m,n}]$, $\mathcal{A}_q[G]$ and $\mathcal{A}_q[G/U]$. It is closed under the braided tensor product and, moreover, the subalgebra A^+ of each such A is naturally a module algebra over the quantization of $\mathfrak{g}^*$, the Lie algebra of the Poisson dual group G^*.

The aforementioned examples of factorizable module algebras all possess dual canonical bases which behave nicely with respect to factorization $A = A^+ \otimes \mathcal{A}_q[U]$. We expect the same is true for many other members of this class, including braided tensor products of such. To facilitate such a construction in tensor products, we propose an axiomatic framework of based modules which, in particular, vastly generalizes Lusztig's notion of based modules. We argue that all of the aforementioned $U_q(\mathfrak{g})$-

module algebras (and many others) with their dual canonical bases are included, along with their tensor products.

One of the central objects of study emerging from our generalization of Lusztig's based modules is a new (very canonical) basis $\mathcal{B}^{\diamond n}$ in the n-th braided tensor power $\mathcal{A}_q[G/U]$. We argue (yet conjecturally) that $\mathcal{A}_q[G/U]^{\underline{\otimes} n}$ has a quantum cluster structure and conjecture that the expected cluster structure structure on $\mathcal{A}_q[G/U]^{\underline{\otimes} n}$ is completely controlled by the *real* elements of our canonical basis $\mathcal{B}^{\diamond n}$.

Finally, in order to partially explain the monoidal structures appearing above, we provide an axiomatic framework to construct examples of bialgebroids of Sweedler type. In particular, we describe a bialgebroid structure on $\mathfrak{u}_q(\mathfrak{g}) \rtimes \mathbb{Q}C_2$, where $\mathfrak{u}_q(\mathfrak{g})$ is the small quantum group and C_2 is the cyclic group of order two.

This dissertation contains previously published co-authored material.

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For my parents.

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CHAPTER I

INTRODUCTION

This dissertation consists of four interconnected projects, the first of which is based on the published paper [6], co-authored with Arkady Berenstein. Material from [6] appears in Chapter III.

1.1. Factorizable Module Algebras

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. We say that a $U_q(\mathfrak{g})$-module algebra A is *factorizable* over a $U_q(\mathfrak{g})$-equivariant subalgebra A_0 if the restriction of the multiplication map of A, $\mu : A^+ \otimes A_0 \to A$, is an isomorphism of vector spaces. We will focus on the case when $A_0 = \mathcal{A}_q[U]$.

Theorem 1.1. Up to a localization, the n-fold braided tensor power of $\mathcal{A}_q[G/U]$, $A = \mathcal{A}_q[G/U]^{\underline{\otimes} n}$, is factorizable over $A_0 = \mathcal{A}_q[U]$ for any $n \geq 1$.

Factorizability of module algebras in this setting is easy to establish and reproduce.

Theorem 1.2. For any complex semisimple Lie algebra $\mathfrak{g}$, we have:

(a) (Theorems 3.9 & 3.12) Let A be a $U_q(\mathfrak{g})$-module algebra containing a $U_q(\mathfrak{g})$-module subalgebra isomorphic to $\mathcal{A}_q[U]$ and let A^+ denote the subalgebra of all highest weight vectors in A. Then the vector space $A' = A^+ \otimes \mathcal{A}_q[U]$ has the structure of a $U_q(\mathfrak{g})$-module algebra and is factorizable over $\mathcal{A}_q[U]$.

(b) The assignments $A \mapsto A'$ define a functor R_q from the category $\mathcal{A}_\mathfrak{g}^q$ of $U_q(\mathfrak{g})$-module algebras containing a $U_q(\mathfrak{g})$-module subalgebra isomorphic to $\mathcal{A}_q[U]$

to the category $\mathcal{C}_{\mathfrak{g}}^q$ of $U_q(\mathfrak{g})$-module algebras which are factorizable over $\mathcal{A}_q[U]$, a full subcategory of $\mathcal{A}_{\mathfrak{g}}^q$.

We can think of R_q as a "remembering" functor because it is right adjoint to the "forgetful" functor $F_q : \mathcal{C}_{\mathfrak{g}}^q \to \mathcal{A}_{\mathfrak{g}}^q$. Clearly, the composition $R_q \circ F_q$ is the identity functor on $\mathcal{C}_{\mathfrak{g}}^q$. In order to tensor multiply objects of these categories, we need to "trim" it a bit. Namely, we consider the full subcategory $\underline{\mathcal{A}}_{\mathfrak{g}}^q$ consisting of *weight* module algebras in $\mathcal{A}_{\mathfrak{g}}^q$ satisfying some additional mild conditions (see Section 3.1.). It turns out that $\underline{\mathcal{A}}_{\mathfrak{g}}^q$ has a natural *braided tensor product* which we denote by $\underline{\otimes}$ (see, e.g. [24] and Corollary 2.23 below). For $A, B \in \underline{\mathcal{A}}_{\mathfrak{g}}^q$, $A\underline{\otimes}B$ is naturally in $\underline{\mathcal{A}}_{\mathfrak{g}}^q$ with an embedding $\mathcal{A}_q[U] = 1\underline{\otimes}\mathcal{A}_q[U] \subset A\underline{\otimes}B$. This natural multiplication lacks a unit object.

Since $\mathcal{C}_{\mathfrak{g}}^q$ is a full subcategory of $\mathcal{A}_{\mathfrak{g}}^q$, we can define $\underline{\mathcal{C}}_{\mathfrak{g}}^q$ as the intersection of $\mathcal{C}_{\mathfrak{g}}^q$ and $\underline{\mathcal{A}}_{\mathfrak{g}}^q$.

Proposition 1.3. (Proposition 3.17) $\underline{\mathcal{C}}_{\mathfrak{g}}^q$ is closed under the braided tensor multiplication in $\underline{\mathcal{A}}_{\mathfrak{g}}^q$.

That is, the category $\underline{\mathcal{C}}_{\mathfrak{g}}^q$ of factorizable $U_q(\mathfrak{g})$-weight module algebras is "almost" monoidal but it lacks a unit object.

We can build factorizable module algebras over $\mathcal{A}_q[U]$ out of $U_q(\mathfrak{g}^*)$-module algebras, where $\mathfrak{g}^*$ is the dual Lie bialgebra of $\mathfrak{g}$ and all factorizable algebras are obtained this way.

Main Theorem 1.4. (Theorems 3.9 & 3.14) For any semisimple Lie algebra $\mathfrak{g}$, the assignments $A \mapsto A^+$ defines a functor P_q from $\mathcal{A}_{\mathfrak{g}}^q$ to the category $U_q(\mathfrak{g}^*)$-**ModAlg** of $U_q(\mathfrak{g}^*)$-module algebras. Moreover, the composition $P_q \circ F_q$ is an equivalence of categories $\mathcal{C}_{\mathfrak{g}}^q \widetilde{\to} U_q(\mathfrak{g}^*)$-**ModAlg**.

Remark 1.5. The theorem asserts that the assignment $A \mapsto A^+$ is the forgetful functor which "remembers almost everything."

The functor P_q from Theorem 1.4 is highly nontrivial: it involves a quite mysterious $U_q(\mathfrak{g}^*)$ action on A such that A^+ is a $U_q(\mathfrak{g}^*)$-equivariant subalgebra. Namely, the Cartan subalgebra action of $U_q(\mathfrak{g}^*)$ is inherited from that of $U_q(\mathfrak{g})$, but the action of the generators $F_{i,1}$ and $F_{i,2}$ of $U_q(\mathfrak{g}^*)$ is given by the formulas

$$F_{i,1} \triangleright a = F_i(a) - \frac{x_i a - K_i^{-1}(a) x_i}{q_i - q_i^{-1}}, \quad F_{i,2} \triangleright a = \frac{x_i K_i^{-1}(a) - a x_i}{q_i - q_i^{-1}}$$

for $a \in A^+$, where K_i is the i-th Cartan generator of $U_q(\mathfrak{g})$, $q_i = q^{d_i}$, and x_i is the i-th generator of $\mathcal{A}_q[U] \subset A$.

1.2. Based Module Algebras

The search for "good" bases plays a central role in representation theory, especially that of quantized enveloping algebras. Many $U_q(\mathfrak{g})$-module algebras (such as $\mathcal{A}_q[Mat_{m,n}]$, $\mathcal{A}_q[G]$, and $\mathcal{A}_q[G/U]$) possess dual canonical bases, which behave nicely with respect to the factorization of Chapter III. The goal of Chapter IV is to generalize Lusztig's notion of based modules in [23] to include these examples and find dual canonical bases in the braided tensor products of such module algebras.

We use the key idea of Lusztig (and others) that "good" bases should consist of elements fixed by an antilinear involution of the underlying module. We therefore introduce a category of *barred modules* $U_q(\mathfrak{g})$-**BarMod**, the objects of which are pairs $(M, {}^-)$, where M is a $U_q(\mathfrak{g})$-weight module and ${}^- : M \to M$ is a compatible antilinear involution, which we hereafter call a *bar*.

Theorem 1.6. (Theorem 4.4) $U_q(\mathfrak{g})$-**BarMod** is monoidal.

The preceding theorem is very similar to theorems that can be found in [23]. The key difference is in our choice of the bar on the tensor product. Namely, Lusztig defines it by $\overline{m \otimes m'} := \Theta(\overline{m} \otimes \overline{m'})$, where we define it by $\overline{m \otimes m'} := \mathcal{R}_{2,1}(\overline{m} \otimes \overline{m'})$. Here Θ is his *quasi-R-matrix* and $\mathcal{R}_{2,1}$ is the opposite of the *universal R-matrix*. The effect of this choice is best seen on an appropriately defined category of *barred module algebras* $U_q(\mathfrak{g})$-**BarModAlg**. The objects of $U_q(\mathfrak{g})$-**BarModAlg** are barred modules $(A, {}^-)$ such that the bar is a $\mathbb{Q}$-algebra anti-involution.

Theorem 1.7. (Theorem 4.8) $U_q(\mathfrak{g})$-**BarModAlg** is monoidal, where we take the braided tensor product of the underlying module algebras.

Coming from the point of view of quantum cluster algebras, the axiom that the bar is an algebra anti-involution is very natural. We therefore view the reproduction of this property in braided tensor products to be a major advantage of our construction.

With a natural way for bars to reproduce in tensor products, we now turn our attention to our actual goal: "good" bases. In Definition 4.10, we give an alternate definition of based modules, vastly generalizing that of [23]. Definition 4.20 gives a version for module algebras. The dual canonical bases of $\mathcal{A}_q[Mat_{m,n}]$, $\mathcal{A}_q[G]$, $\mathcal{A}_q[G/U]$, and other quantized coordinate rings all fit within this framework, even some which are not locally finite. With this definition, we play the balancing act of generalizing enough to include our examples, but retaining enough structure to use established strategies.

Main Theorem 1.8. (Corollary 4.16, Theorem 4.23) The categories of based modules and based module algebras are monoidal.

The preceding theorem gives a way to find "good" bases in tensor products of modules and module algebras containing "good" bases. In particular, it results in a new (very canonical) basis in each braided tensor power of $\mathcal{A}_q[G/U]$, a central object of study for us. We conjecture that upper global crystal bases also reproduce under this process.

1.3. Quantum Cluster Algebras

It very often happens that quantum cluster algebras have a natural grading by an Abelian group (not necessarily $\mathbb{Z}$). For instance, if a quantum cluster algebra is also a $U_q(\mathfrak{g})$-weight module algebra with homogeneous cluster variables (like $\mathcal{A}_q[Mat_{m,n}]$, $\mathcal{A}_q[G]$, and $\mathcal{A}_q[G/U]$), then it is graded by the weight lattice $\mathcal{P}$.

In our study of factorizable module algebras, we observe that it sometimes happens that a $U_q(\mathfrak{g})$-weight module algebra can be written as a *graded tensor product* of subalgebras. Specifically, the braided tensor products $\mathcal{A}_q[U]\underline{\otimes}\mathcal{A}_q[U]$ and $\mathcal{A}_q[B]\underline{\otimes}\mathcal{A}_q[B]$ are isomorphic to the graded tensor products of the same algebras. In particular, both factors are quantum cluster algebras and the factors skew-commute, depending on the grading. This naturally leads to the idea of a monoidal structure on some category of quantum cluster algebras that happen to be nicely graded.

In Chapter V, we formalize the notion of a graded quantum cluster algebra. As in the case of regular quantum cluster algebras, these are defined by specifying certain initial data and "mutating" to generate a subalgebra of a division algebra. The key feature is that these structures reproduce under graded tensor products.

Main Theorem 1.9. (Theorem 5.17) The graded tensor product of graded quantum cluster algebras is naturally a graded quantum cluster algebra.

Corollary 1.10. (Corollary 5.18) For any $n \geq 1$, $\mathcal{A}_q[U]^{\underline{\otimes} n}$ and $\mathcal{A}_q[B]^{\underline{\otimes} n}$ are graded quantum cluster algebras.

Unlike with $\mathcal{A}_q[U]$ and $\mathcal{A}_q[B]$, $\mathcal{A}_q[G/U]^{\underline{\otimes} n}$ is not isomorphic the n-fold graded tensor product. Nevertheless, we make the following conjecture.

Conjecture 1.11. For any $n \geq 1$, the n-fold braided tensor product $\mathcal{A}_q[G/U]^{\underline{\otimes} n}$ is a graded quantum cluster algebra.

The preceding conjecture is well-known to be true for $G = SL_2$, since $\mathcal{A}_q[SL_2/U] \cong \mathcal{A}_q[Mat_{2,1}]$ and therefore, $\mathcal{A}_q[SL_2/U]^{\underline{\otimes} n} \cong \mathcal{A}_q[Mat_{2,1}]^{\underline{\otimes} n} \cong \mathcal{A}_q[Mat_{2,n}]$. In Chapter V, we outline a program, so to speak, for proving the conjecture in the general setting, then implement the program to prove the conjecture in the case $G = SL_3$ and $n = 2$.

1.4. Bialgebroids

It is well-known that if A is a bialgebra over a commutative ring $\mathbb{K}$, the category A-**Mod** of A-modules is closed under tensor products over $\mathbb{K}$. In the dual setting, the same can be done in the category A-**Comod** of comodules over A. It is a little less known that to achieve the same outcome, bialgebras can be generalized to what is known as a *bialgebroid*.

Early nontrivial examples of bialgebroids were introduced by Sweedler in [25] under the name $\times_A$-bialgebras and as a slight generalization of the usual notion of bialgebra over a commutative ring A. A few years later, Takeuchi generalized Sweedler's $\times_A$-bialgebras in [26] to include noncommutative A. Later still, Lu defined bialgebroids in [22], which are equivalent to Takeuchi's $\times_A$-bialgebras.

We will focus on a class of bialgebroids over a commutative ring $\mathbb{K}$, which we call *bialgebroids over* $\mathbb{K}$ *of Sweedler type*. These are actually equivalent to Sweedler's $\times_{\mathbb{K}}$-bialgebras, but we call them by a different name to avoid confusion with Takeuchi's generalization. In Chapter VI, we construct a class of bialgebroids by "crossing" a bialgebra or Hopf algebra A with a bialgebroid H acting on A in a certain way. The construction which follows is the chapter's main result.

Main Theorem 1.12. (Theorem 6.5) Let B be a cocommutative bialgebroid over $\mathbb{K}$ of Sweedler type and let A be a B-module and a $\mathbb{K}$-bialgebra, such that m_A, η_A, and ε_A are B-module homomorphisms. Suppose further that $R : B \to A \otimes_{\mathbb{K}} A$ is a (left) $\mathbb{K}$-linear map such that

1. $R(1) = 1 \otimes 1 \qquad (\in A \otimes_{\mathbb{K}} A)$
2. $R(bb') = R(b_{(1)})(b_{(2)} \triangleright R(b')) \qquad (\in A \otimes_{\mathbb{K}} A)$
3. $R(b_{(1)})(b_{(2)} \triangleright \Delta_A(a)) = \Delta_A(b_{(1)} \triangleright a)R(b_{(2)}) \qquad (\in A \otimes_{\mathbb{K}} A)$
4. $(\Delta_A \otimes \mathrm{id}_A)(R(b_{(1)}))R_{12}(b_{(2)}) = (\mathrm{id}_A \otimes \Delta_A)(R(b_{(1)}))R_{23}(b_{(2)}) \quad (\in A \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} A)$
5. $m_A \circ ((\eta_A \circ \varepsilon_A) \otimes \mathrm{id}_A) \circ R = \eta_A \circ \varepsilon_B = m_A \circ (\mathrm{id}_A \otimes (\eta_A \circ \varepsilon_A)) \circ R \qquad (\in A)$

where we use the notation $R_{ij}(b)$ $(1 \leq i < j \leq 3)$ to denote that $R(b)$ appears in the i^{th} and j^{th} places of the three-fold tensor and we also use sumless Sweedler notation to write $\Delta_B(b) = b_{(1)} \otimes b_{(2)}$. Then the $\mathbb{K}$-ring $A \rtimes B$ may be given the structure of a bialgebroid over $\mathbb{K}$ of Sweedler type via the additional assignments

$$\Delta(a \bullet b) = (a_{(1)}(b_{(1)})^{(1)} \bullet b_{(2)}) \otimes (a_{(2)}(b_{(1)})^{(2)} \bullet b_{(3)}), \qquad \varepsilon(a \bullet b) = \varepsilon_A(a)\varepsilon_B(b)$$

where we use Sweedler-like notation to write $R(b) = b^{(1)} \otimes b^{(2)}$.

Roughly speaking, this construction explains the monoidality of barred modules in Chapter IV. Namely, $U_q(\mathfrak{g}) \rtimes \mathbb{Q}C_2$ is *almost* a bialgebroid of Sweedler type over $\mathbb{Q}(q^{\frac{1}{2d}})$. To make this precise, one passes to the small quantum group $\mathfrak{u}_q(\mathfrak{g})$, which has a true quasitriangular structure. In fact, it was this very example that provided the inspiration for our construction, to which the use of R for the map $B \to A \otimes_{\mathbb{K}} A$ is an homage.

CHAPTER II

PRELIMINARIES

2.1. Fundamental Algebraic Objects

Fix a commutative ring with identity $\mathbb{K}$. Denote the category of left $\mathbb{K}$-modules by $\mathbb{K}$-**Mod** and the category of $\mathbb{K}$-bimodules by $\mathbb{K}$-**Bimod**. We use the symbols $\otimes_{\mathbb{K}}$ and ${}_{\mathbb{K}}\otimes_{\mathbb{K}}$ for their respective natural tensor products. Of course, for $\mathbb{K}$-bimodules U and V, there is a canonical way to consider $U \otimes_{\mathbb{K}} V$ as a left $\mathbb{K}$-module, but two (possibly distinct) right $\mathbb{K}$-module structures: $(u \otimes v)k = (uk) \otimes v$ and $(u \otimes v)k = u \otimes (vk)$. The following definition, due to Sweedler [25], formalizes the left $\mathbb{K}$-submodule on which these two right actions agree and hence on which there is a canonical right action.

Definition 2.1. Let U and V be $\mathbb{K}$-bimodules. Define

$$U \times_{\mathbb{K}} V := \left\{ \sum_i u_i \otimes v_i \in U \otimes_{\mathbb{K}} V \;\middle|\; \sum_i u_i \otimes (v_i k) = \sum_i (u_i k) \otimes v_i \ \forall k \in \mathbb{K} \right\}.$$

For an element $x \in U \times_{\mathbb{K}} V$, we will sometimes use Sweedler-like notation and write $x = x_{[1]} \otimes x_{[2]}$ since $U \times_{\mathbb{K}} V$ is not necessarily spanned by pure tensors.

It's worth mentioning here that if the left and right actions of $\mathbb{K}$ on U and V coincide, then $U \times_{\mathbb{K}} V = U \otimes_{\mathbb{K}} V$. Now given $\mathbb{K}$-bimodules U and V, $U {}_{\mathbb{K}}\otimes_{\mathbb{K}} V$ and $U \times_{\mathbb{K}} V$ each have a canonical $\mathbb{K}$-bimodule structure. Unless otherwise specified, we will also give $U \otimes_{\mathbb{K}} V$ the following $\mathbb{K}$-bimodule structure: $k(u \otimes v) = (ku) \otimes v$ and $(u \otimes v)k = u \otimes (vk)$ for $k \in \mathbb{K}$, $u \in U$, and $v \in V$. Namely, in all tensor products which follow, the left action of $\mathbb{K}$ is given by the left action on the left-most factor and the right action of $\mathbb{K}$ is given by the right action on the right-most factor, assuming these make sense. Also, given any right $\mathbb{K}$-module U, $\mathbb{K}$-bimodule V, and left $\mathbb{K}$-module W, there is a natural isomorphism of abelian groups $(U_{\mathbb{K}}\otimes_{\mathbb{K}}V)_{\mathbb{K}}\otimes_{\mathbb{K}}W \to U_{\mathbb{K}}\otimes_{\mathbb{K}}(V_{\mathbb{K}}\otimes_{\mathbb{K}}W)$, $(u\otimes v)\otimes w \mapsto u\otimes(v\otimes w)$, for which we will either omit notation or simply denote $\cong$. We will similarly denote the natural isomorphisms $\mathbb{K}_{\mathbb{K}}\otimes_{\mathbb{K}}V \to V$, $k\otimes v\mapsto kv$ and $U_{\mathbb{K}}\otimes_{\mathbb{K}}\mathbb{K} \to U$, $u\otimes k \mapsto uk$.

Definition 2.2. A $\mathbb{K}$-*ring* is a monoid object in $\mathbb{K}$-**Bimod**. In other words, it is a triple (A, m, η), where A is a $\mathbb{K}$-bimodule and $m : A_{\mathbb{K}}\otimes_{\mathbb{K}}A \to A$ and $\eta : \mathbb{K} \to A$ are homomorphisms of $\mathbb{K}$-bimodules such that the following diagrams commute.

$$\begin{tikzcd}
(A_{\mathbb{K}}\otimes_{\mathbb{K}}A)_{\mathbb{K}}\otimes_{\mathbb{K}}A \arrow[r, "m\otimes \mathrm{id}_A"] \arrow[dd, "\cong"'] & A_{\mathbb{K}}\otimes_{\mathbb{K}}A \arrow[dr, "m"] & \\
 & & A \\
A_{\mathbb{K}}\otimes_{\mathbb{K}}(A_{\mathbb{K}}\otimes_{\mathbb{K}}A) \arrow[r, "\mathrm{id}_A\otimes m"'] & A_{\mathbb{K}}\otimes_{\mathbb{K}}A \arrow[ur, "m"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{K}_{\mathbb{K}}\otimes_{\mathbb{K}}A \arrow[r, "\eta\otimes\mathrm{id}_A"] \arrow[dr, "\cong"'] & A_{\mathbb{K}}\otimes_{\mathbb{K}}A \arrow[d, "m"] \\
 & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A_{\mathbb{K}}\otimes_{\mathbb{K}}\mathbb{K} \arrow[r, "\mathrm{id}_A\otimes\eta"] \arrow[dr, "\cong"'] & A_{\mathbb{K}}\otimes_{\mathbb{K}}A \arrow[d, "m"] \\
 & A
\end{tikzcd}$$

A homomorphism between $\mathbb{K}$-rings (A, m_1, η_1) and (A_2, m_2, η_2) is a homomorphism of $\mathbb{K}$-bimodules $f : A_1 \to A_2$ so that the following diagrams commute.

$$\begin{tikzcd}
A_{1\mathbb{K}}\otimes_{\mathbb{K}}A_1 \arrow[r, "m_1"] \arrow[d, "f\otimes f"'] & A_1 \arrow[d, "f"] \\
A_{2\mathbb{K}}\otimes_{\mathbb{K}}A_2 \arrow[r, "m_2"'] & A_2
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{K} \arrow[dl, "\eta_1"'] \arrow[dr, "\eta_2"] & \\
A_1 \arrow[rr, "f"] & & A_2
\end{tikzcd}$$

A $\mathbb{K}$-ring on which the two actions of $\mathbb{K}$ agree is called a $\mathbb{K}$-*algebra* or an *algebra over* $\mathbb{K}$ and homomorphisms of $\mathbb{K}$-algebras are simply homomorphisms of $\mathbb{K}$-rings.

It is sometimes more useful to define a $\mathbb{K}$-ring as a pair (A, η), where A is a ring and $\eta : \mathbb{K} \to A$ is a ring homomorphism. For instance, in this language, it is easy to state the difference between a $\mathbb{K}$-ring and a $\mathbb{K}$-algebra: a $\mathbb{K}$-algebra is a $\mathbb{K}$-ring in which the image of η is central. Additionally, we can easily define a $\mathbb{K}$-ring A to be *commutative* if it is a commutative ring, i.e. if $ba = ab$ for all $a, b \in A$ (which implies that A is a $\mathbb{K}$-algebra). In practice, we will go back and forth between these two definitions of $\mathbb{K}$-ring, using the one that suits each circumstance. The following lemma makes use of our original definition.

Lemma 2.3. If (A_1, m_1, η_1) and (A_2, m_2, η_2) are $\mathbb{K}$-rings, then so is $(A_1 \times_{\mathbb{K}} A_2, m, \eta)$, where

$$m((x_{[1]} \otimes x_{[2]}) \otimes (y_{[1]} \otimes y_{[2]})) = m_1(x_{[1]} \otimes y_{[1]}) \otimes m_2(x_{[2]} \otimes y_{[2]})$$

and $\eta(k) = \eta_1(k) \otimes \eta_2(1)$.

Since every $\mathbb{K}$-ring is a priori a ring, we may consider its modules. Using η, they are automatically $\mathbb{K}$-modules, which leads to the following formal definition of modules over a $\mathbb{K}$-ring.

Definition 2.4. Let A be a $\mathbb{K}$-ring. A pair $(V, \triangleright)$ is called a (left) A-module if V is a left $\mathbb{K}$-module, $\triangleright : A_{\mathbb{K}}\otimes_{\mathbb{K}}V \to V$ is a (left) $\mathbb{K}$-linear map, and the following diagrams commute.

$$\begin{tikzcd}
\mathbb{K}_{\mathbb{K}}\otimes_{\mathbb{K}}V \arrow[r, "\eta\otimes\mathrm{id}_V"] \arrow[dr, "\cong"'] & A_{\mathbb{K}}\otimes_{\mathbb{K}}V \arrow[d, "\triangleright"] \\
& V
\end{tikzcd}
\qquad
\begin{tikzcd}
(A_{\mathbb{K}}\otimes_{\mathbb{K}}A)_{\mathbb{K}}\otimes_{\mathbb{K}}V \arrow[r, "m\otimes\mathrm{id}_V"] \arrow[dd, "\cong"'] & A_{\mathbb{K}}\otimes_{\mathbb{K}}V \arrow[dr, "\triangleright"] & \\
& & V \\
A_{\mathbb{K}}\otimes_{\mathbb{K}}(A_{\mathbb{K}}\otimes_{\mathbb{K}}V) \arrow[r, "\mathrm{id}_A\otimes\triangleright"'] & A_{\mathbb{K}}\otimes_{\mathbb{K}}V \arrow[ur, "\triangleright"'] &
\end{tikzcd}$$

A homomorphism between A-modules $(U, \triangleright_U)$ and $(V, \triangleright_V)$ is a $\mathbb{K}$-linear map $f : U \to V$ such that the following diagram commutes.

$$\begin{tikzcd}
A_{\mathbb{K}}\otimes_{\mathbb{K}}U \arrow[r, "\mathrm{id}_A\otimes f"] \arrow[d, "\triangleright_U"'] & A_{\mathbb{K}}\otimes_{\mathbb{K}}V \arrow[d, "\triangleright_V"] \\
U \arrow[r, "f"] & V
\end{tikzcd}$$

Given a $\mathbb{K}$-ring A, A-**Mod** is the category whose objects are A-modules and whose morphisms are homomorphisms of A-modules. **Mod**-A is defined analogously.

Definition 2.5. A $\mathbb{K}$-*coring* is a comonoid object in $\mathbb{K}$-**Bimod**. In other words, it is a triple (C, Δ, ε), where C is a $\mathbb{K}$-bimodule, $\Delta : C \to C_{\mathbb{K}}\otimes_{\mathbb{K}}C$ and $\varepsilon : C \to \mathbb{K}$ are homomorphisms of $\mathbb{K}$-bimodules, and the following diagrams commute.

$$\begin{tikzcd}
 & C_{\mathbb{K}}\otimes_{\mathbb{K}}C \arrow[r, "\Delta\otimes\mathrm{id}_C"] & (C_{\mathbb{K}}\otimes_{\mathbb{K}}C)_{\mathbb{K}}\otimes_{\mathbb{K}}C \arrow[dd, "\cong"] \\
C \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & \\
 & C_{\mathbb{K}}\otimes_{\mathbb{K}}C \arrow[r, "\mathrm{id}_C\otimes\Delta"'] & C_{\mathbb{K}}\otimes_{\mathbb{K}}(C_{\mathbb{K}}\otimes_{\mathbb{K}}C)
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[dr, "\cong"'] & C_{\mathbb{K}}\otimes_{\mathbb{K}}C \arrow[d, "\varepsilon\otimes\mathrm{id}_C"] \\
 & \mathbb{K}_{\mathbb{K}}\otimes_{\mathbb{K}}C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[dr, "\cong"'] & C_{\mathbb{K}}\otimes_{\mathbb{K}}C \arrow[d, "\mathrm{id}_C\otimes\varepsilon"] \\
 & C_{\mathbb{K}}\otimes_{\mathbb{K}}\mathbb{K}
\end{tikzcd}$$

A homomorphism between $\mathbb{K}$-corings $(C_1, \Delta_1, \varepsilon_1)$ and $(C_2, \Delta_2, \varepsilon_2)$ is a homomorphism of $\mathbb{K}$-bimodules $f : C_1 \to C_2$ so that the following diagrams commute.

$$\begin{tikzcd}
C_1 \arrow[r, "\Delta"] \arrow[d, "f"'] & C_{1\mathbb{K}}\otimes_{\mathbb{K}}C_1 \arrow[d, "f\otimes f"] \\
C_2 \arrow[r, "\Delta_2"'] & C_{2\mathbb{K}}\otimes_{\mathbb{K}}C_2
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[rr, "f"] \arrow[dr, "\varepsilon_1"'] & & C_2 \arrow[dl, "\varepsilon_2"] \\
 & \mathbb{K} &
\end{tikzcd}$$

A $\mathbb{K}$-coring on which the two actions of $\mathbb{K}$ agree is called a $\mathbb{K}$-*coalgebra* or a *coalgebra over* $\mathbb{K}$. A $\mathbb{K}$-coalgebra is called *cocommutative* if $\Delta^{op} := \tau \circ \Delta = \Delta$, where $\tau : A \otimes_{\mathbb{K}} A \to A \otimes_{\mathbb{K}} A$ is the $\mathbb{K}$-linear map such that $\tau(a \otimes b) = b \otimes a$ for all $a, b \in A$.

Analogous to modules over a $\mathbb{K}$-ring, $\mathbb{K}$-corings have comodules.

Definition 2.6. Let C be a $\mathbb{K}$-coring. A pair (V, δ) is called a left C-comodule if V is a left $\mathbb{K}$-module, $\delta : V \to C_{\mathbb{K}}\otimes_{\mathbb{K}}V$ is a $\mathbb{K}$-linear map, and the following diagrams commute.

$$
\begin{tikzcd}
V \arrow[r, "\delta"] \arrow[dr, "\cong"'] & C_{\mathbb{K}}\otimes_{\mathbb{K}}V \arrow[d, "\varepsilon\otimes\mathrm{id}_V"] \\
& \mathbb{K}_{\mathbb{K}}\otimes_{\mathbb{K}}V
\end{tikzcd}
\qquad
\begin{tikzcd}
& C_{\mathbb{K}}\otimes_{\mathbb{K}}V \arrow[r, "\Delta\otimes\mathrm{id}_V"] & (C_{\mathbb{K}}\otimes_{\mathbb{K}}C)_{\mathbb{K}}\otimes_{\mathbb{K}}V \arrow[d, "\cong"] \\
V \arrow[ur, "\delta"] \arrow[dr, "\delta"'] & & \\
& C_{\mathbb{K}}\otimes_{\mathbb{K}}V \arrow[r, "\mathrm{id}_C\otimes\delta"'] & C_{\mathbb{K}}\otimes_{\mathbb{K}}(C_{\mathbb{K}}\otimes_{\mathbb{K}}V)
\end{tikzcd}
$$

A homomorphism between C-comodules (U, δ_U) and (V, δ_V) is a $\mathbb{K}$-linear map $f : U \to V$ such that the following diagram commutes.

$$
\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "\delta_U"'] & V \arrow[d, "\delta_V"] \\
C_{\mathbb{K}}\otimes_{\mathbb{K}}U \arrow[r, "\mathrm{id}_C\otimes f"'] & C_{\mathbb{K}}\otimes_{\mathbb{K}}V
\end{tikzcd}
$$

Right C-comodules are defined analogously.

Definition 2.7. A tuple $(B, m, \eta, \Delta, \varepsilon)$ is called a *bialgebroid of Sweedler type over* $\mathbb{K}$ if the following conditions hold.

1. (B, m, η) is a $\mathbb{K}$-ring.

2. (B, Δ, ε) is a $\mathbb{K}$-coalgebra such that $\Delta(B) \subset B \times_{\mathbb{K}} B$.

3. The corestriction of Δ to $B \times_{\mathbb{K}} B$ is a homomorphism of $\mathbb{K}$-rings.

4. $\varepsilon(1) = 1$.

5. For $b, b' \in B$, $\varepsilon(bb') = \varepsilon(b\varepsilon(b'))$.

A homomorphism between bialgebroids of Sweedler type over $\mathbb{K}$ $(B_1, \eta_1, \Delta_1, \varepsilon_1)$ and $(B_2, \eta_2, \Delta_2, \varepsilon_2)$ is a (left and right) $\mathbb{K}$-linear map $B_1 \to B_2$ which is a homomorphism of $\mathbb{K}$-rings and $\mathbb{K}$-coalgebras.

A bialgebroid of Sweedler type is called *commutative* (resp. *cocommutative*) if the associated $\mathbb{K}$-ring (resp. $\mathbb{K}$-coalgebra) is commutative (resp. cocommutative) and one on which the two actions of $\mathbb{K}$ agree is called a $\mathbb{K}$-*bialgebra* or a *bialgebra over* $\mathbb{K}$.

Here, as usual, we suppress m and η, writing bb' for $m(b \otimes b')$ and k for $\eta(k)$. We will also use sumless Sweedler notation, writing $\Delta(b) = b_{(1)} \otimes b_{(2)}$. In practice, we will denote a bialgebroid of Sweedler type $(B, m, \eta, \Delta, \varepsilon)$ by B, with the structure maps implied.

Definition 2.8. If $(H, m, \eta, \Delta, \varepsilon)$ is a $\mathbb{K}$-bialgebra and $S : H \to H$ is a $\mathbb{K}$-linear map so that the following diagram commutes, then we say that $(H, m, \eta, \Delta, \varepsilon, S)$ is a Hopf algebra over $\mathbb{K}$.

$$\begin{tikzcd}
H \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \mathbb{K} \arrow[r, "\eta"] & H \\
H \otimes_{\mathbb{K}} H \arrow[rr, "{S\otimes \mathrm{id}_H,\ \mathrm{id}_H \otimes S}"'] & & H \otimes_{\mathbb{K}} H \arrow[u, "m"']
\end{tikzcd}$$

A homomorphism between Hopf algebras over $\mathbb{K}$ $(H_1, m_1, \eta_1, \Delta_1, \varepsilon_1, S_1)$ and $(H_2, m_2, \eta_2, \Delta_2, \varepsilon_2, S_2)$ is a homomorphism $f : H_1 \to H_2$ of the underlying $\mathbb{K}$-bialgebras so that $f \circ S_1 = S_2 \circ f$, i.e. the following diagram commutes.

$$\begin{tikzcd}
H_1 \arrow[r, "f"] \arrow[d, "S_1"'] & H_2 \arrow[d, "S_2"] \\
H_1 \arrow[r, "f"'] & H_2
\end{tikzcd}$$

As with previous objects, we generally omit the structure maps and simply denote a Hopf algebra over $\mathbb{K}$ $(H, m, \eta, \Delta, \varepsilon, S)$ simply by H. The following proposition is probably well-known, but a reference was not quickly found. We therefore include a proof here.

Proposition 2.9. Let $(H, m, \eta, \Delta, \varepsilon)$ be a $\mathbb{K}$-bialgebra and $\rho : H \to H$ a $\mathbb{K}$-algebra automorphism. Set $\Delta_\rho := (\rho \otimes \rho) \circ \Delta \circ \rho^{-1}$, and $\varepsilon_\rho := \varepsilon \circ \rho^{-1}$. Then $(H, m, \eta, \Delta_\rho, \varepsilon_\rho)$ is a $\mathbb{K}$-bialgebra. Furthermore, ρ is an isomomorphism of $\mathbb{K}$-bialgebras.

If, additionally, $(H, m, \eta, \Delta, \varepsilon, S)$ is a Hopf algebra over $\mathbb{K}$, then $(H, m, \eta, \Delta_\rho, \varepsilon_\rho, S_\rho)$ is a Hopf algebra over $\mathbb{K}$ (where $S_\rho = \rho \circ S \circ \rho^{-1}$) and ρ is an isomorphism of Hopf algebras over $\mathbb{K}$.

Proof. Since ρ, ρ^{-1}, Δ, and ε are algebra homomorphisms, so are Δ_ρ and ε_ρ. Further, for $h \in H$, we have

$$\begin{aligned}
\varepsilon_\rho(h^{(1)})h^{(2)} &= (\varepsilon \circ \rho^{-1})(\rho([\rho^{-1}(h)]_{(1)}))\rho([\rho^{-1}(h)]_{(2)}) \\
&= \varepsilon([\rho^{-1}(h)]_{(1)})\rho([\rho^{-1}(h)]_{(2)}) \\
&= \rho(\varepsilon([\rho^{-1}(h)]_{(1)})[\rho^{-1}(h)]_{(2)}) \\
&= \rho(\rho^{-1}(h)) \\
&= h
\end{aligned}$$

and

$$\begin{aligned}
S_\rho(h^{(1)})h^{(2)} &= (\rho \circ S \circ \rho^{-1})(\rho([\rho^{-1}(h)]_{(1)}))\rho([\rho^{-1}(h)]_{(2)}) \\
&= (\rho \circ S)([\rho^{-1}(h)]_{(1)})\rho([\rho^{-1}(h)]_{(2)}) \\
&= \rho(S([\rho^{-1}(h)]_{(1)})[\rho^{-1}(h)]_{(2)}) \\
&= \rho((\varepsilon \circ \rho^{-1})(h)) \\
&= (\varepsilon \circ \rho^{-1})(h) \\
&= \varepsilon_\rho(h),
\end{aligned}$$

where we write $\Delta(h') = h'_{(1)} \otimes h'_{(2)}$ and $\Delta_\rho(h') = (h')^{(1)} \otimes (h')^{(2)}$ in sumless Sweedler notation. Similarly, $h^{(1)}\varepsilon_\rho(h^{(2)}) = h$ and $h^{(1)}S_\rho(h^{(2)}) = \varepsilon_\rho(h)$ for all $h \in H$. Hence $(H, m, \eta, \Delta_\rho, \varepsilon_\rho, S_\rho)$ is a Hopf algebra. By design, it is immediate that ρ is an isomorphism of Hopf algebras. $\square$

The following theorem is a special case of a well-known theorem about bialgebroids (see, for instance, [10, Section 3.5] or [11, Section 31.7 (3)]). In the more general theorem, B-modules are naturally $\mathbb{K}$-bimodules, but in our setting these left and right actions coincide. Hence, we use $\otimes_{\mathbb{K}}$ instead of ${}_{\mathbb{K}}\otimes_{\mathbb{K}}$.

Theorem 2.10. If B is a bialgebroid of Sweedler type over $\mathbb{K}$, then B-**Mod** is monoidal with tensor product $\otimes_{\mathbb{K}}$ and unit object $\mathbb{K}$ (where $b \triangleright k = \varepsilon(bk)$).

Definition 2.11. Let B be a bialgebroid of Sweedler type over $\mathbb{K}$. A *B-module algebra* is a monoid object in B-**Mod**. In other words, it is a $\mathbb{K}$-algebra A such that the multiplication $m_A : A \otimes_{\mathbb{K}} A \to A$ and unit $\eta_A : \mathbb{K} \to A$ are homomorphisms of B-modules.

A homomorphism of B-module algebras is a B-module homomorphism $A \to A'$ which is also a $\mathbb{K}$-algebra homomorphism.

We denote by B-**ModAlg** the category whose objects are B-module algebras and whose morphisms are homomorphisms of B-module algebras.

The following lemma is an easy analogue of the standard theorem for module algebras over bialgebras.

Lemma 2.12. Let B be a bialgebroid of Sweedler type over $\mathbb{K}$ and let A be a B-module algebra. Then $A \rtimes B$ is a $\mathbb{K}$-ring with base abelian group $A \otimes_{\mathbb{K}} B$, and structure given by

$$\eta(k) = k \bullet 1 \qquad (a \bullet b)(a' \bullet b') = a(b_{(1)} \triangleright a') \bullet b_{(2)} b'.$$

In particular, if B is a $\mathbb{K}$-bialgebra, then $A \rtimes B$ is a $\mathbb{K}$-algebra.

2.2. Quantized Enveloping Algebras

Let I be a finite index set, $C = (c_{i,j})$ an $I \times I$ Cartan matrix (in particular, of finite type), $\mathcal{P}$ an integer lattice of dimension at least $|I|$, and $(\cdot, \cdot) : \mathcal{P} \times \mathcal{P} \to \frac{1}{d}\mathbb{Z}$ a symmetric pairing for some fixed $d \in \mathbb{Z}_{>0}$. Suppose $\{\alpha_i\}_{i\in I}$ and $\{\omega_i\}_{i\in I}$ are linearly independent subsets of $\mathcal{P}$, such that

$$(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}, \ \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = c_{i,j}, \ \frac{2(\alpha_i, \omega_j)}{(\alpha_i, \alpha_i)} = \delta_{i,j}, \text{ and } \frac{2(\alpha_i, \alpha)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}$$

for $i, j \in I$, and $\alpha \in \mathcal{P}$. Set $d_i := \frac{(\alpha_i,\alpha_i)}{2} \in \mathbb{Z}_{>0}$, $\mathcal{Q} := \sum_{i\in I} \mathbb{Z}\alpha_i$, $\mathcal{Q}^+ := \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$, and $\mathcal{P}^+ := \sum_{i\in I} \mathbb{Z}_{\geq 0}\omega_i$.

Convention 2.13. If C is of type A_n, then we take $\mathcal{P} = \mathbb{Z}^{n+1}$ with standard basis $\{\varepsilon_i\}_{i=1}^{n+1}$ and the standard pairing satisfying $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$ for $1 \leq i, j \leq n$. We set $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ and $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq n$.

Let $\mathfrak{g}$ be the semisimple complex Lie algebra with Cartan matrix C. $\mathfrak{g}$ has a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{h}$ is a Cartan subalgebra with $\dim \mathfrak{h} = |I|$. Let W be the Weyl group of $\mathfrak{g}$, i.e. the Coxeter group with Cartan matrix C, generated by the simple reflections $\{s_i \mid i \in I\}$. Given $w \in W$, we denote by $\ell(w)$ the smallest nonnegative integer such that there exists some $\mathbf{i} = (i_1, i_2, \ldots, i_{\ell(w)}) \in I^{\ell(w)}$ with $s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}} = w$ and call $\ell(w)$ the *length* of w. Such an $\mathbf{i}$ is called a reduced expression for w and we denote by $R(w)$ the set of all reduced expressions for w. W has a unique element of maximal length, which we denote $w_{\circ}$. Furthermore, the assignment $s_i(\alpha) = \alpha - \frac{2(\alpha_i,\alpha)}{(\alpha_i,\alpha_i)}\alpha_i$ defines an action of W on $\mathcal{P}$. This restricts to an action of W on $\mathcal{Q}$ and also defines actions on $\frac{1}{2}\mathcal{P}$ and $\frac{1}{2}\mathcal{Q}$. Set $\Phi := \{w(\alpha_i) \mid i \in I, \ w \in W\}$ and $\Phi^+ := \Phi \cap \mathcal{Q}^+$. Then $\Phi = \Phi^+ \cup (-\Phi^+)$.

2.2.1. $U_q(\mathfrak{g})$

Let $q^{\frac{1}{2d}}$ be an indeterminate. $U_q(\mathfrak{g})$ is the $\mathbb{Q}(q^{\frac{1}{2d}})$-algebra generated by elements $\{K_i^{\pm\frac{1}{2}}, E_i, F_i \mid i \in I\}$ subject to the relations

$$K_i^{\pm\frac{1}{2}}K_j^{\pm\frac{1}{2}} = K_j^{\pm\frac{1}{2}}K_i^{\pm\frac{1}{2}}; \qquad K_i^{\frac{1}{2}}K_i^{-\frac{1}{2}} = 1;$$

$$K_i^{\frac{1}{2}}E_jK_i^{-\frac{1}{2}} = q_i^{\frac{c_{i,j}}{2}}E_j; \quad K_i^{\frac{1}{2}}F_jK_i^{-\frac{1}{2}} = q_i^{-\frac{c_{i,j}}{2}}F_j;$$

$$E_iF_j - F_jE_i = \delta_{i,j}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}};$$

$$\sum_{k=0}^{1-c_{i,j}}(-1)^kE_i^{(k)}E_jE_i^{(1-c_{i,j}-k)} = 0 \text{ if } i \neq j;$$

$$\sum_{k=0}^{1-c_{i,j}}(-1)^kF_i^{(k)}F_jF_i^{(1-c_{i,j}-k)} = 0 \text{ if } i \neq j;$$

where $y_i^{(n)} = \frac{1}{[n]_{q_i}!}y_i^n$, $[n]_{q_i}! = [1]_{q_i}[2]_{q_i}\cdots[n]_{q_i}$, $[m]_{q_i} = \dfrac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}$, and $q_i = q^{d_i}$. It is a $\mathcal{Q}$-graded algebra with grading given on generators by $|K_i^{\pm\frac{1}{2}}| = 0$, $|E_i| = \alpha_i$, and $|F_i| = -\alpha_i$ for $i \in I$. $U_q(\mathfrak{g})$ is a Hopf algebra with comultiplication Δ, counit ϵ, and antipode S given on generators by

$$\Delta(K_i^{\pm\frac{1}{2}}) = K_i^{\pm\frac{1}{2}} \otimes K_i^{\pm\frac{1}{2}}; \qquad \epsilon(K_i^{\pm\frac{1}{2}}) = 1; \quad S(K_i^{\pm\frac{1}{2}}) = K_i^{\mp\frac{1}{2}};$$

$$\Delta(E_i) = E_i \otimes K_i^{\frac{1}{2}} + K^{-\frac{1}{2}} \otimes E_i; \quad \epsilon(E_i) = 0; \quad S(E_i) = -qE_i;$$

$$\Delta(F_i) = F_i \otimes K^{\frac{1}{2}} + K_i^{-\frac{1}{2}} \otimes F_i; \quad \epsilon(F_i) = 0; \quad S(F_i) = -q^{-1}F_i.$$

We'll denote by $U_q(\mathfrak{n}_+)$, $U_q(\mathfrak{n}_-)$, $\mathcal{K}$, $U_q(\mathfrak{b}_+)$, and $U_q(\mathfrak{b}_-)$ the $\mathbb{Q}(q^{\frac{1}{2d}})$-subalgebras of $U_q(\mathfrak{g})$ generated by $\{E_i \mid i \in I\}$, $\{F_i \mid i \in I\}$, $\{K_i^{\pm\frac{1}{2}} \mid i \in I\}$, $\{K_i^{\pm\frac{1}{2}}, E_i \mid i \in I\}$, and $\{K_i^{\pm\frac{1}{2}}, F_i \mid i \in I\}$, respectively. The last three are Hopf subalgebras of $U_q(\mathfrak{g})$.

Actually, $U_q(\mathfrak{g})$ has several different frequently used Hopf algebra structures in addition to the one given above. We describe these presently. In light of Proposition

2.9, we consider the $\mathbb{Q}(q^{\frac{1}{2d}})$-algebra automorphism $\sigma : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ satisfying $\sigma(K_i^{\pm\frac{1}{2}}) = K_i^{\pm\frac{1}{2}}$, $\sigma(E_i) = K_i^{-\frac{1}{2}} E_i$, and $\sigma(F_i) = F_i K_i^{\frac{1}{2}}$. We then have, for example,

$$\begin{aligned}
\Delta_\sigma(E_i) &= E_i \otimes K_i + 1 \otimes E_i, & \Delta_\sigma(F_i) &= F_i \otimes 1 + K_i^{-1} \otimes F_i,\\
\Delta_{\sigma^{-1}}(E_i) &= E_i \otimes 1 + K_i^{-1} \otimes E_i, & \Delta_{\sigma^{-1}}(F_i) &= F_i \otimes K_i + 1 \otimes F_i,\\
\Delta^{op}(E_i) &= E_i \otimes K_i^{-\frac{1}{2}} + K_i^{\frac{1}{2}} \otimes E_i, & \Delta^{op}(F_i) &= F_i \otimes K_i^{-\frac{1}{2}} + K_i^{\frac{1}{2}} \otimes F_i,\\
\Delta^{op}_\sigma(E_i) &= E_i \otimes 1 + K_i \otimes E_i, & \Delta^{op}_\sigma(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i,\\
\Delta^{op}_{\sigma^{-1}}(E_i) &= E_i \otimes K_i^{-1} + 1 \otimes E_i, & \Delta^{op}_{\sigma^{-1}}(F_i) &= F_i \otimes 1 + K_i \otimes F_i.
\end{aligned}$$

In [23], Lusztig defines automorphisms $T''_{i,1}$ of $U_q(\mathfrak{g})$ for each $i \in I$. We'll denote these simply by T_i. To simplify notation, we set

$$K_\alpha := \prod_{i \in I} K_i^{a_i}, \quad \text{for } \alpha = \sum_{i \in I} a_i \alpha_i \in \frac{1}{2}\mathcal{Q}.$$

Then for each $i \in I$, T_i is defined on generators by the following (with $j \in I \setminus \{i\}$ and $\alpha \in \frac{1}{2}\mathcal{Q}$):

$$T_i(K_\alpha) = K_{s_i(\alpha)}, \quad T_i(E_i) = -F_i K_i, \quad T_i(F_i) = -K_i^{-1} E_i,$$

$$T_i(E_j) = \sum_{k+\ell=-c_{i,j}} (-1)^k q_i^{-k} E_i^{(\ell)} E_j E_i^{(k)}, \quad T_i(F_j) = \sum_{k+\ell=-c_{i,j}} (-1)^k q_i^{k} F_i^{(k)} F_j F_i^{(\ell)}.$$

These automorphisms have the property that, given $i \in I$ and homogenous $u \in U_q(\mathfrak{g})$, $T_i(u)$ is homogeneous with graded degree $|T_i(u)| = s_i(|u|)$. Given $\mathbf{i} \in R(w_\circ)$ and $\alpha \in \Phi^+$, it is well-known (cf. [13, 8.1.D]) that there is a unique $k \in [1, \ell(w_\circ)]$ so that $\alpha = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$. We set

$$q_\alpha := q_{i_k}, \quad E_\alpha := T_{i_1} \cdots T_{i_{k-1}}(E_{i_k}), \quad \text{and} \quad F_\alpha := T_{i_1} \cdots T_{i_{k-1}}(F_{i_k}).$$

2.2.2. $U_q(\mathfrak{g}^*)$

$U_q(\mathfrak{g}^*)$ is the $\mathbb{Q}(q^{\frac{1}{2d}})$-algebra generated by elements $\{K_i^{\pm\frac{1}{2}}, F_{i,1}, F_{i,2} \mid i \in I\}$ subject to the relations

$$K_i^{\pm\frac{1}{2}} K_j^{\pm\frac{1}{2}} = K_j^{\pm\frac{1}{2}} K_i^{\pm\frac{1}{2}}, \qquad K_i^{\frac{1}{2}} K_i^{-\frac{1}{2}} = 1,$$

$$K_i^{\frac{1}{2}} F_{j,r} K_i^{-\frac{1}{2}} = q_i^{-\frac{c_{i,j}}{2}} F_{j,r}, \quad F_{i,1} F_{j,2} = F_{j,2} F_{i,1},$$

$$\sum_{k=0}^{1-c_{i,j}} (-1)^k F_{i,r}^{(k)} F_{j,r} F_{i,r}^{(1-c_{i,j}-k)} = 0 \text{ if } i \neq j.$$

It is a $\mathcal{Q}$-graded algebra with grading given on generators by $|K_i^{\pm\frac{1}{2}}| = 0$ and $|F_{i,r}| = -\alpha_i$ for $i \in I$ and $r \in \{1, 2\}$. $U_q(\mathfrak{g}^*)$ is a Hopf algebra with comultiplication Δ, counit ϵ, and antipode S given on generators by

$$\Delta(K_i^{\pm\frac{1}{2}}) = K_i^{\pm\frac{1}{2}} \otimes K_i^{\pm\frac{1}{2}}, \qquad \epsilon(K_i^{\pm\frac{1}{2}}) = 1, \quad S(K_i^{\pm\frac{1}{2}}) = K_i^{\mp\frac{1}{2}},$$

$$\Delta(F_{i,1}) = F_{i,1} \otimes K_i^{\frac{1}{2}} + K^{-\frac{1}{2}} \otimes F_{i,1}, \quad \epsilon(F_{i,1}) = 0, \quad S(F_{i,1}) = -q^{-1} F_{i,1},$$

$$\Delta(F_{i,2}) = F_{i,2} \otimes K^{-\frac{1}{2}} + K_i^{\frac{1}{2}} \otimes F_{i,2}, \quad \epsilon(F_{i,2}) = 0, \qquad S(F_i) = -q F_{i,2}.$$

Like $U_q(\mathfrak{g})$, $U_q(\mathfrak{g}^*)$ has multiple similar Hopf algebra structures. Consider the $\mathbb{Q}(q^{\frac{1}{2d}})$-algebra automorphism $\sigma_* : U_q(\mathfrak{g}^*) \to U_q(\mathfrak{g}^*)$ satisfying $\sigma_*(K_i^{\pm\frac{1}{2}}) = K_i^{\pm\frac{1}{2}}$ and $\sigma_*(F_{i,r}) = F_{i,r} K_i^{\frac{1}{2}}$. We then have, for example,

$$\Delta_{\sigma_*}(F_{i,1}) = F_{i,1} \otimes 1 + K_i^{-1} \otimes F_{i,1}, \qquad \Delta_{\sigma_*}(F_{i,2}) = F_{i,2} \otimes K_i^{-1} + 1 \otimes F_{i,2},$$

$$\Delta_{\sigma_*^{-1}}(F_{i,1}) = F_{i,1} \otimes K_i + 1 \otimes F_{i,1}, \qquad \Delta_{\sigma_*^{-1}}(F_{i,2}) = F_{i,2} \otimes 1 + K_i \otimes F_{i,2},$$

$$\Delta^{op}(F_{i,1}) = F_{i,1} \otimes K_i^{-\frac{1}{2}} + K_i^{\frac{1}{2}} \otimes F_{i,1}, \quad \Delta^{op}(F_{i,2}) = F_{i,2} \otimes K_i^{\frac{1}{2}} + K_i^{-\frac{1}{2}} \otimes F_{i,2},$$

$$\Delta^{op}_{\sigma_*}(F_{i,1}) = F_{i,1} \otimes K_i^{-1} + 1 \otimes F_{i,1}, \qquad \Delta^{op}_{\sigma_*}(F_{i,2}) = F_{i,2} \otimes 1 + K_i^{-1} \otimes F_{i,2},$$

$$\Delta^{op}_{\sigma_*^{-1}}(F_{i,1}) = F_{i,1} \otimes 1 + K_i \otimes F_{i,1}, \qquad \Delta^{op}_{\sigma_*^{-1}}(F_{i,2}) = F_{i,2} \otimes K_i + 1 \otimes F_{i,2}.$$

2.3. Modules over Quantized Enveloping Algebras

We make the universal assumption that, for a $U_q(\mathfrak{n}_+)$-module M, $m \in M$, and $i \in I$, there exists some $n \in \mathbb{Z}_{\geq 0}$ such that $E_i^n(m) = 0$. Some results hold without this assumption, but all examples that we will see have this property. For such a module M, we set $M^+ := \{m \in M \mid E_i(m) = 0 \ \forall i \in I\}$ and call M^+ the set of highest weight vectors. It is immediately obvious that if A is a $U_q(\mathfrak{g})$-module algebra, then A^+ is a $\mathcal{K}$-module subalgebra of A.

A $U_q(\mathfrak{g})$-module M is called *locally finite* if the cyclic submodule $U_q(\mathfrak{g})m$ is finite-dimensional over $\mathbb{Q}(q^{\frac{1}{2d}})$ for all $m \in M$. For each $\lambda \in \mathcal{P}^+$, there is a unique finite-dimensional (and hence locally finite) simple $U_q(\mathfrak{g})$-module V_λ, generated by an element $v_\lambda \in V_\lambda$ and such that

$$E_i(v_\lambda) = 0, \quad \text{and} \quad K_i^{\pm\frac{1}{2}}(v_\lambda) = q^{\pm\frac{1}{2}(\alpha_i,\lambda)}v_\lambda$$

for all $i \in I$.

A module M over $U_q(\mathfrak{g})$ or $U_q(\mathfrak{g}^*)$ is called a *weight module* if it is $\mathcal{P}$-graded and, for $i \in I$ and homogeneous $m \in M$, $K_i^{\pm\frac{1}{2}}(m) = q^{\pm\frac{1}{2}(\alpha_i,|m|)}m$. If, additionally, A is a module algebra over $U_q(\mathfrak{g})$ or $U_q(\mathfrak{g}^*)$ (respectively), we say that it is a *weight module algebra.* $U_q(\mathfrak{g})$-**WMod** and $U_q(\mathfrak{g}^*)$-**WMod** are the full subcategories of $U_q(\mathfrak{g})$-**Mod** and $U_q(\mathfrak{g}^*)$-**Mod**, respectively, whose objects are weight modules. Similarly, $U_q(\mathfrak{g})$-

WModAlg and $U_q(\mathfrak{g}^*)$-**WModAlg** are the full subcategories of $U_q(\mathfrak{g})$-**ModAlg** and $U_q(\mathfrak{g}^*)$-**ModAlg**, respectively, whose objects are weight module algebras.

Implicit when we speak of module algebras is a specific choice of bialgebra structure. Unless otherwise specified, we use those we have denoted with the standard notations, i.e. Δ and ϵ, rather than (for instance) Δ_σ or $\Delta_{\sigma_*^{-1}}$. However, there are natural relationships between the respective categories. Namely, we have the following theorem, which is probably well-known. In any case, the proof is trivial, so we omit it here.

Theorem 2.14. Let $(B, m, \eta, \Delta, \varepsilon)$ be a bialgebra over a commutative ring $\mathbb{K}$ and suppose $\rho : B \to B$ is a $\mathbb{K}$-algebra homomorphism. Writing B_ρ-**ModAlg** for the category of B-module algebras using Δ_ρ and ε_ρ in place of Δ and ε, there is an equivalence of categories

$$\mathscr{F}_\rho : B_\rho\text{-}\mathbf{ModAlg} \to B\text{-}\mathbf{ModAlg}$$
$$A \mapsto A_\rho$$
$$f \mapsto f,$$

where $A_\rho = A$ as $\mathbb{K}$-modules, but the action of B on A_ρ is given by $b \triangleright_\rho a := \rho(b) \triangleright a$.

It is clear that if A is a module algebra over $U_q(\mathfrak{g})$ or $U_q(\mathfrak{g}^*)$ (with any comultiplication and counit) and is also a weight module, then so is the corresponding A_{σ^s} or $A_{\sigma_*^s}$ for any $s \in \mathbb{Z}$. We will therefore sometimes prove results for $U_q(\mathfrak{g})_{\sigma^s}$-**WModAlg** or $U_q(\mathfrak{g}^*)_{\sigma_*^s}$-**WModAlg** for some $s \in \mathbb{Z}$, then translate results back to $U_q(\mathfrak{g})$-**WModAlg** or $U_q(\mathfrak{g}^*)$-**WModAlg** via the equivalence.

There are several main $U_q(\mathfrak{g})$-weight module algebras that will attract most of our attention: $\mathcal{A}_q[U]$, $\mathcal{A}_q[T]$, $\mathcal{A}_q[B]$, and $\mathcal{A}_q[G/U]$. We define them presently and present some of their properties, all of which roughly matches [4].

$\mathcal{A}_q[U]$ is isomorphic to $U_q(\mathfrak{n}_-)$ as a $\mathcal{Q}$-graded $\mathbb{Q}(q^{\frac{1}{2d}})$-algebra, but is written with generators $\{x_i \mid i \in I\}$, rather than $\{F_i \mid i \in I\}$. It has a $U_q(\mathfrak{g})$-weight module algebra structure determined by the assignments

$$K_i^{\mp\frac{1}{2}}(x_j) = q^{\pm\frac{1}{2}(\alpha_i,\alpha_j)}x_j, \quad E_i(x_j) = \delta_{i,j}, \quad \text{and} \quad F_i(x) = \frac{x_i K_i^{\frac{1}{2}}(x) - K_i^{-\frac{1}{2}}(x)x_i}{q_i - q_i^{-1}},$$

for $i, j \in I$ and $x \in \mathcal{A}_q[U]$.

$\mathcal{A}_q[T]$ is the commutative $\mathcal{K}$-module algebra with $\mathbb{Q}(q^{\frac{1}{2d}})$-basis $\{v_\lambda \mid \lambda \in \mathbb{Z}\mathcal{P}^+\}$, multiplication $v_\lambda v_\mu = v_{\lambda+\mu}$ for $\lambda, \mu \in \mathbb{Z}\mathcal{P}^+$, and $\mathcal{K}$-action $K_i^{\pm\frac{1}{2}}(v_\lambda) = q^{\pm\frac{1}{2}(\alpha_i,\lambda)}v_\lambda$. It is $\mathcal{P}$-graded by $|v_\lambda| = \lambda$.

$\mathcal{A}_q[B]$ is the graded tensor product $\mathcal{A}_q[U]\overline{\otimes}\mathcal{A}_q[T]$ as an algebra. Namely, it has the unique algebra structure on $\mathcal{A}_q[U]\otimes\mathcal{A}_q[T]$ satisfying $(1\otimes y)(x\otimes 1) = q^{(|x|,|y|)}x\otimes y$ for homogeneous $x \in \mathcal{A}_q[U]$ and $y \in \mathcal{A}_q[T]$. It is also a $U_q(\mathfrak{g})$-weight module algebra satisfying

$$\begin{aligned} K_i^{\pm\frac{1}{2}}(x\otimes v_\lambda) &= q^{\pm\frac{1}{2}(\alpha_i,\lambda+|x|)}x\otimes v_\lambda, \\ E_i(x\otimes v_\lambda) &= q^{\frac{1}{2}(\alpha_i,\lambda)}E_i(x)\otimes v_\lambda, \\ F_i(x\otimes v_\lambda) &= \frac{(x_i\otimes 1)K_i^{\frac{1}{2}}(x\otimes v_\lambda) - K_i^{-\frac{1}{2}}(x\otimes v_\lambda)(x_i\otimes 1)}{q_i - q_i^{-1}} \end{aligned}$$

for $i \in I$, $\lambda \in \mathbb{Z}\mathcal{P}^+$, and homogeneous $x \in \mathcal{A}_q[U]$. We will usually suppress the tensor symbol when writing elements of $\mathcal{A}_q[B]$.

$\mathcal{A}_q[G/U]$ is the $U_q(\mathfrak{g})$-module subalgebra of $\mathcal{A}_q[B]$ generated by $\{v_\lambda \mid \lambda \in \mathcal{P}^+\}$. It is locally finite and decomposes nicely as a $U_q(\mathfrak{g})$-module. Namely,

$$\mathcal{A}_q[G/U] = \bigoplus_{\lambda \in \mathcal{P}^+} U_q(\mathfrak{g}) v_\lambda$$

and for each $\lambda \in \mathcal{P}^+$ and $U_q(\mathfrak{g})v_\lambda \cong V_\lambda$, $v_\lambda \mapsto v_\lambda$.

Finally, for integers $m, n \geq 1$, $\mathcal{A}_q[Mat_{m,n}]$ is the $\mathbb{Q}(q^{\frac{1}{2d}})$-algebra generated by $\{x_{i,j} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$, subject to the relations

$$\begin{aligned}
&x_{i,\ell} x_{i,j} = q x_{i,j} x_{i,\ell} && (n \geq 2)\\
&x_{k,j} x_{i,j} = q x_{i,j} x_{k,j} && (m \geq 2)\\
&x_{k,j} x_{i,\ell} = x_{i,\ell} x_{k,j} && (m, n \geq 2)\\
&x_{k,\ell} x_{i,j} = x_{i,j} x_{k,\ell} + (q - q^{-1}) x_{i,\ell} x_{k,j} && (m, n \geq 2)
\end{aligned}$$

where $i < k$ and/or $j < \ell$ if both indices occur in the same equation. If $m \geq 2$, then $\mathcal{A}_q[Mat_{m,n}]$ is a $U_q(\mathfrak{g})$-module algebra with

$$\begin{aligned}
K_i^{\pm\frac{1}{2}}(x_{j,k}) &= q^{\delta_{i,j} - \delta_{i+1,j}} x_{j,k},\\
E_i(x_{j,k}) &= \delta_{i+1,j} x_{i,k},\\
F_i(x_{j,k}) &= \delta_{i,j} x_{i+1,k}.
\end{aligned}$$

2.4. Universal R-Matrices for $U_q(\mathfrak{g})$

Recall that, according to Proposition 2.9, $(U_q(\mathfrak{g}), m, \eta, \Delta_{\sigma^s}, \epsilon_{\sigma^s}, S_{\sigma^s})$ is a Hopf algebra for any $s \in \mathbb{Z}$.

Lemma 2.15. For any $s \in \mathbb{Z}$ and $i \in I$, $\sigma^s \circ T_i = T_i \circ \sigma^s \circ Ad(K_i^{s/2})$, where $Ad(K_i^{s/2})(u) = K_i^{s/2} u K_i^{-s/2}$ for $u \in U_q(\mathfrak{g})$.

Proof. It suffices to check when $s = 1$ and since the compositions are algebra homomorphisms, it also suffices to check this case on generators. We do this explicitly, assuming $j \in I \setminus \{i\}$, except for when showing

$$(\sigma \circ T_i)(K_j^{\pm\frac{1}{2}}) = (T_i \circ \sigma \circ Ad(K_i^{\frac{1}{2}}))(K_j^{\pm\frac{1}{2}}),$$

where we allow $j = i$.

$$\begin{aligned}
(\sigma \circ T_i)(K_j^{\pm\frac{1}{2}}) &= \sigma(K_{\pm\frac{1}{2}\alpha_j \mp \frac{c_{i,j}}{2}\alpha_i}) \\
&= K_{\pm\frac{1}{2}\alpha_j \mp \frac{c_{i,j}}{2}\alpha_i} \\
&= T_i(K_j^{\pm\frac{1}{2}}) \\
&= (T_i \circ \sigma \circ Ad(K_i^{\frac{1}{2}}))(K_j^{\pm\frac{1}{2}})
\end{aligned}$$

$$\begin{aligned}
(\sigma \circ T_i)(E_i) &= \sigma(-F_i K_i) \\
&= -F_i K_i^{\frac{3}{2}} \\
&= -q_i K_i^{\frac{1}{2}} F_i K_i \\
&= T_i(q_i K_i^{-\frac{1}{2}} E_i) \\
&= (T_i \circ \sigma)(q_i E_i) \\
&= (T_i \circ \sigma \circ Ad(K_i^{\frac{1}{2}}))(E_i)
\end{aligned}$$

$$
\begin{aligned}
(\sigma \circ T_i)(F_i) &= \sigma(-K_i^{-1}E_i) \\
&= -K_i^{-\frac{3}{2}}E_i \\
&= -q_i^{-1}K_i^{-1}E_iK_i^{-\frac{1}{2}} \\
&= T_i(q_i^{-1}F_iK_i^{\frac{1}{2}}) \\
&= (T_i \circ \sigma)(q_i^{-1}F_i) \\
&= (T_i \circ \sigma \circ Ad(K_i^{\frac{1}{2}}))(F_i)
\end{aligned}
$$

$$
\begin{aligned}
(\sigma \circ T_i)(E_j) &= \sigma\left(\sum_{k+\ell=-a_{i,j}} (-1)^k q_i^{-\ell} E_i^{(k)} E_j E_i^{(\ell)}\right) \\
&= \sum_{k+\ell=-a_{i,j}} (-1)^k q_i^{-\ell} (K_i^{-\frac{1}{2}}E_i)^{(k)} (K_j^{-\frac{1}{2}}E_j)(K_i^{-\frac{1}{2}}E_i)^{(\ell)} \\
&= \sum_{k+\ell=-a_{i,j}} (-1)^k q_i^{-\ell} (q_i^{\frac{1}{2}[k(k-1)+\ell(\ell-1)+a_{i,j}(k+\ell)+2k\ell]} K_{-\frac{1}{2}\alpha_j+\frac{c_{i,j}}{2}\alpha_i}) E_i^{(k)} E_j E_i^{(\ell)} \\
&= q_i^{\frac{1}{2}a_{i,j}} K_{-\frac{1}{2}\alpha_j+\frac{c_{i,j}}{2}\alpha_i} \sum_{k+\ell=-a_{i,j}} (-1)^k q_i^{-\ell} E_i^{(k)} E_j E_i^{(\ell)} \\
&= T_i(q_i^{\frac{1}{2}a_{i,j}} K_j^{-\frac{1}{2}} E_j) \\
&= (T_i \circ \sigma)(q_i^{\frac{1}{2}a_{i,j}} E_j) \\
&= (T_i \circ \sigma \circ Ad(K_i^{\frac{1}{2}}))(E_j)
\end{aligned}
$$

$$
\begin{aligned}
(\sigma \circ T_i)(F_j) &= \sigma\left(\sum_{k+\ell=-a_{i,j}} (-1)^k q_i^\ell F_i^{(\ell)} F_j F_i^{(k)}\right) \\
&= \sum_{k+\ell=-a_{i,j}} (-1)^k q_i^\ell (F_i K_i^{\frac{1}{2}})^{(\ell)} (F_j K_j^{\frac{1}{2}})(F_i K_i^{\frac{1}{2}})^{(k)} \\
&= \sum_{k+\ell=-a_{i,j}} (-1)^k q_i^\ell F_i^{(\ell)} F_j F_i^{(k)} (q_i^{-\frac{1}{2}[k(k-1)+\ell(\ell-1)+a_{i,j}(k+\ell)+2k\ell]} K_{\frac{1}{2}\alpha_j - \frac{c_{i,j}}{2}\alpha_i}) \\
&= q_i^{-\frac{1}{2}a_{i,j}} \sum_{k+\ell=-a_{i,j}} (-1)^k q_i^\ell F_i^{(\ell)} F_j F_i^{(k)} K_{\frac{1}{2}\alpha_j - \frac{c_{i,j}}{2}\alpha_i} \\
&= T_i(q_i^{-\frac{1}{2}a_{i,j}} F_j K_j^{\frac{1}{2}}) \\
&= (T_i \circ \sigma)(q_i^{-\frac{1}{2}a_{i,j}} F_j) \\
&= (T_i \circ \sigma \circ Ad(K_i^{\frac{1}{2}}))(F_j)
\end{aligned}
$$

□

Corollary 2.16. Let $s \in \mathbb{Z}$, $k \in \mathbb{Z}_{\geq 0}$, $\mathbf{i} \in I^k$, and $j \in I$. Set $\alpha = s_{i_1} s_{i_2} \cdots s_{i_k}(\alpha_j)$. Then

$$
\begin{aligned}
&(\sigma^s \circ T_{i_1} \circ \cdots \circ T_{i_k})(E_j) \otimes (\sigma^s \circ T_{i_1} \circ \cdots \circ T_{i_k})(F_j) \\
&\qquad = K_\alpha^{-s/2} (T_{i_1} \circ \cdots \circ T_{i_k})(E_j) \otimes (T_{i_1} \circ \cdots \circ T_{i_k})(F_j) K_\alpha^{s/2}.
\end{aligned}
$$

Proof. Once again, it suffices to check for $s = 1$. To do so, we simply compute $(\sigma \circ T_{i_1} \circ \cdots \circ T_{i_k})(E_j)$ and $(\sigma \circ T_{i_1} \circ \cdots \circ T_{i_k})(F_j)$.

$$
\begin{aligned}
(\sigma \circ T_{i_1} \circ \cdots \circ T_{i_k})(E_j) &= (T_{i_1} \circ Ad(K_{i_1}^{\frac{1}{2}}) \circ \cdots \circ T_{i_k} \circ Ad(K_{i_k}^{\frac{1}{2}}) \circ \sigma)(E_j) \\
&= (T_{i_1} \circ Ad(K_{i_1}^{\frac{1}{2}}) \circ \cdots \circ T_{i_k} \circ Ad(K_{i_k}^{\frac{1}{2}}))(K_j^{-\frac{1}{2}} E_j) \\
&= q^{\frac{1}{2}\sum_{m=1}^k (\alpha_m, s_{i_{m+1}} \cdots s_{i_k}(\alpha_j))} (T_{i_1} \circ \cdots \circ T_{i_k})(K_j^{-\frac{1}{2}} E_j) \\
&= q^{\frac{1}{2}\sum_{m=1}^k (\alpha_m, s_{i_{m+1}} \cdots s_{i_k}(\alpha_j))} K_\alpha^{-\frac{1}{2}} (T_{i_1} \circ \cdots \circ T_{i_k})(E_j)
\end{aligned}
$$

$$
\begin{aligned}
(\sigma \circ T_{i_1} \circ \cdots \circ T_{i_k})(F_j) &= (T_{i_1} \circ Ad(K_{i_1}^{\frac{1}{2}}) \circ \cdots \circ T_{i_{k-1}} \circ Ad(K_{i_k}^{\frac{1}{2}}) \circ \sigma)(F_j) \\
&= (T_{i_1} \circ Ad(K_{i_1}^{\frac{1}{2}}) \circ \cdots \circ T_{i_k} \circ Ad(K_{i_k}^{\frac{1}{2}}))(F_j K_j^{\frac{1}{2}}) \\
&= q^{-\frac{1}{2}\sum_{m=1}^{k}(\alpha_m, s_{i_{m+1}} \cdots s_{i_k}(\alpha_j))}(T_{i_1} \circ \cdots \circ T_{i_k})(F_{i_k} K_{i_k}^{\frac{1}{2}}) \\
&= q^{-\frac{1}{2}\sum_{m=1}^{k}(\alpha_m, s_{i_{m+1}} \cdots s_{i_k}(\alpha_j))}(T_{i_1} \circ \cdots \circ T_{i_k})(F_{i_k}) K_{\alpha}^{\frac{1}{2}}
\end{aligned}
$$

The result is now clear. $\square$

For any $r \in \frac{1}{2}\mathbb{Z}$ and $U_q(\mathfrak{g})$-weight modules M_1 and M_2, we have an invertible $\mathbb{Q}(q^{\frac{1}{2d}})$-linear map $\Pi^r : M_1 \otimes M_2 \to M_1 \otimes M_2$, defined by

$$\Pi^r(m_1 \otimes m_2) = q^{r(|m_1|,|m_2|)} m_1 \otimes m_2$$

for homogeneous $m_1 \in M_1$ and $m_2 \in M_2$. We extend this definition to $n \geq 2$ tensorands by assigning $\Pi^r_{i,j}(m_1 \otimes m_2 \otimes \cdots \otimes m_n) = q^{r(|m_i|,|m_j|)} m_1 \otimes m_2 \otimes \cdots \otimes m_n$ for $1 \leq i < j \leq n$ and $m_1, \ldots, m_n$ homogeneous elements of $U_q(\mathfrak{g})$-weight modules $M_1, \ldots, M_n$, respectively.

Remark 2.17. In what follows, we often consider elements of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ (or infinite sums of such) as operators on tensor products of two arbitrary $U_q(\mathfrak{g})$-weight modules. Hence, in order to make sense of some equalities, they must be applied to elements of such weight modules. This is especially true when we apply maps to infinite sums of elements of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$.

Lemma 2.18. Given $n \geq 2$, $1 \leq i < j \leq n$, $r \in \frac{1}{2}\mathbb{Z}$ and homogeneous $u_1, \ldots, u_n \in U_q(\mathfrak{g})$,

$$\Pi^r_{i,j} \circ (u_1 \otimes \cdots \otimes u_n) \circ \Pi^{-r}_{i,j} = u_1 \otimes \cdots \otimes K_{r|u_j|} u_i \otimes \cdots \otimes u_j K_{r|u_i|} \otimes \cdots \otimes u_n.$$

Proof. Let $M_1, \ldots, M_n$ be $U_q(\mathfrak{g})$-weight modules with homogeneous elements $m_k \in M_k$ for each $1 \leq k \leq n$. We directly compute:

$$\begin{aligned}
&\Pi^r_{i,j}((u_1 \otimes \cdots \otimes u_n)(\Pi^{-r}_{i,j}(m_1 \otimes \cdots \otimes m_n))) \\
&\quad = q^{-r(|m_i|,|m_j|)}\Pi^r_{i,j}((u_1 \otimes \cdots \otimes u_n)(m_1 \otimes m_2)) \\
&\quad = q^{-r(|m_i|,|m_j|)}\Pi^r_{i,j}(u_1(m_1) \otimes \cdots \otimes u_n(m_n)) \\
&\quad = q^{r(|u_i|+|m_i|,|u_j|+|m_j|)-r(|m_i|,|m_j|)}u_1(m_1) \otimes \cdots \otimes u_n(m_n) \\
&\quad = q^{(r|u_j|,|u_i|+|m_i|)+(r|u_i|,|m_j|)}u_1(m_1) \otimes \cdots \otimes u_n(m_n) \\
&\quad = u_1(m_1) \otimes \cdots \otimes K_{r|u_j|}u_i(m_i) \otimes \cdots \otimes u_j K_{r|u_i|}(m_j) \otimes \cdots \otimes u_n(m_n) \\
&\quad = (u_1 \otimes \cdots \otimes K_{r|u_j|}u_i \otimes \cdots \otimes u_j K_{r|u_i|} \otimes \cdots \otimes u_n)(m_1 \otimes \cdots \otimes m_n). \qquad \square
\end{aligned}$$

Let Θ be as in [23, 4.1.2]. We view it simultaneously as an infinite sum of elements of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ (i.e. in some completion) and as an invertible operator on any tensor product of two $U_q(\mathfrak{g})$-weight modules (recalling that E_i acts finitely so that this is well-defined). Note that its inverse operator can be viewed similarly. We also write $\Theta_{2,1}$ for the operator $\tau \circ \Theta \circ \tau$ or, equivalently, the element $\tau(\Theta)$, where τ is the "flip" linear map $x \otimes y \mapsto y \otimes x$. $\Theta^{-1}_{2,1}$ is similar.

Corollary 2.19. For all $s \in \mathbb{Z}$, $(\sigma^s \otimes \sigma^s)(\Theta^{-1}_{2,1}) = \Pi^{\frac{s}{2}} \circ \Theta^{-1}_{2,1} \circ \Pi^{-\frac{s}{2}}$.

Proof. Note that $\Theta^{-1}_{2,1}$ is multiplicative:

$$\Theta^{-1}_{2,1} = \overleftarrow{\prod_{\alpha \in \Phi^+}} \left(\sum_{k=0}^{\infty} q_\alpha^{\frac{1}{2}k(k-1)} \frac{(q_\alpha - q_\alpha^{-1})^k}{[k]_{q_\alpha}!} E^k_\alpha \otimes F^k_\alpha \right)$$

where q_α, E_α, and F_α are as in 2.2.1. (cf. [13, Theorem 8.3.9]). Here E_α and F_α, as well as the order in which the product is taken depend on the choice of a reduced expression for w_{o}, but $\Theta^{-1}_{2,1}$ does not.

Since $\sigma^s \otimes \sigma^s$ and conjugation by $\Pi^{s/2}$ are algebra homomorphisms, it therefore suffices to show that

$$(\sigma^s \otimes \sigma^s)(E_\alpha^k \otimes F_\alpha^k) = \Pi^{\frac{s}{2}} \circ (E_\alpha^k \otimes F_\alpha^k) \circ \Pi^{-\frac{s}{2}}.$$

Now applying Corollary 2.16 and Lemma 2.18, we see

$$\begin{aligned}
\sigma^s(E_\alpha^n) \otimes \sigma^s(F_\alpha^n) &= (K_\alpha^{-s/2} E_\alpha)^n \otimes (F_\alpha K_\alpha^{s/2})^n \\
&= (q_\alpha^{sn(n-1)/2} K_\alpha^{-ns/2} E_\alpha^n) \otimes (q_\alpha^{-sn(n-1)/2} F_\alpha^n K_\alpha^{ns/2}) \\
&= K_\alpha^{-ns/2} E_\alpha^n \otimes F_\alpha^n K_\alpha^{ns/2} \\
&= \Pi^{\frac{s}{2}} \circ (E_\alpha^n \otimes F_\alpha^n) \circ \Pi^{-\frac{s}{2}}.
\end{aligned}$$

□

Lemma 2.20. Suppose $\Pi_{i,j}^r \circ z \circ \Pi_{i,j}^{-r} = z'$ for some $1 \le i < j \le n$, $r \in \frac{1}{2}\mathbb{Z}$, and $z, z' \in U_q(\mathfrak{g})^{\otimes n}$ with $n \ge 2$. Then $\Pi_{i,j}^r \circ (\sigma^s \otimes \cdots \otimes \sigma^s)(z) \circ \Pi^{-r} = (\sigma^s \otimes \cdots \otimes \sigma^s)(z')$ for all $s \in \mathbb{Z}$.

Proof. It suffices to show the result when $z = u_1 \otimes \cdots \otimes u_n$ for $u_k \in U_q(\mathfrak{g})$ homogeneous. Then by Lemma 2.18, $z' = u_1 \otimes \cdots \otimes K_{r|u_j|} u_i \otimes \cdots \otimes u_j K_{r|u_i|} \otimes \cdots \otimes u_n$. So since $|\sigma^s(u_k)| = |u_k|$ for $1 \le k \le n$, we have

$$\begin{aligned}
&\Pi_{i,j}^r \circ (\sigma^s \otimes \cdots \otimes \sigma^s)(z) \circ \Pi_{i,j}^{-r} \\
&\quad = \Pi_{i,j}^r \circ (\sigma^s(u_1) \otimes \cdots \otimes \sigma^s(u_n)) \circ \Pi^{-r} \\
&\quad = \sigma^s(u_1) \otimes \cdots \otimes K_{r|u_j|} \sigma^s(u_i) \otimes \cdots \otimes \sigma^s(u_j) K_{r|u_i|} \otimes \cdots \otimes \sigma^s(u_n) \\
&\quad = (\sigma^s \otimes \cdots \otimes \sigma^s)(u_1 \otimes \cdots \otimes K_{r|u_j|} u_i \otimes \cdots \otimes u_j K_{r|u_i|} \otimes \cdots \otimes u_n) \\
&\quad = (\sigma^s \otimes \cdots \otimes \sigma^s)(z').
\end{aligned}$$

□

Set ${}_s\mathcal{R} := \Pi \circ (\sigma^{s-1} \otimes \sigma^{s-1})(\Theta_{2,1}^{-1})$. In what follows, we apply ${}_s\mathcal{R}$ to tensor products in various ways. Given a tensor product of three $U_q(\mathfrak{g})$-weight modules $M_1 \otimes M_2 \otimes M_3$ and $1 \leq i < j \leq 3$, we write ${}_s\mathcal{R}_{i,j}$ for the application of ${}_s\mathcal{R}$ to the i-th and j-th tensorands. Given a similar tensor product, we write ${}_s\mathcal{R}_{1,23}$ for the application of ${}_s\mathcal{R}$ to the iterated tensor product $M_1 \otimes (M_2 \otimes M_3)$. Similarly, we write ${}_s\mathcal{R}_{12,3}$ for the application to $(M_1 \otimes M_2) \otimes M_3$. We write $\Pi^r_{12,3}$ and $\Pi^r_{1,23}$ for the application of Π^r to the respective tensor product. In short, the subscripts tell which tensorands it acts on, in what order, and whether to use the opposite comultiplication or not. We can do this with an arbitrary number of tensorands.

Given a permutation $\rho \in S_n$ (denoted in cycle notation), we will write τ_ρ for the linear permutation of factors $M_1 \otimes \cdots \otimes M_n \mapsto M_{\rho(1)} \otimes \cdots \otimes M_{\rho(n)}$. In a slight departure from this notation, if $n = 2$, we write τ in place of $\tau_{(12)}$.

Theorem 2.21. $\tau \circ {}_s\mathcal{R}$ is a braiding for the monoidal category of $U_q(\mathfrak{g})_{\sigma^s}$-weight module algebras.

Proof. $\tau \circ {}_s\mathcal{R}$ is clearly a linear isomorphism with inverse ${}_s\mathcal{R}^{-1} \circ \tau$, where ${}_s\mathcal{R}^{-1} = (\sigma^{s-1} \otimes \sigma^{s-1})(\Theta_{2,1}) \circ \Pi^{-1}$. So it remains to show that $\tau \circ {}_s\mathcal{R}$ is a natural homomorphism of $U_q(\mathfrak{g})$-modules and satisfies the hexagon axioms.

We first show that $\tau \circ {}_s\mathcal{R}$ is a homomorphism of $U_q(\mathfrak{g})$-modules. To do so, we must show $(\tau \circ {}_s\mathcal{R}) \circ \Delta_{\sigma^s}(u) = \Delta_{\sigma^s}(u) \circ (\tau \circ {}_s\mathcal{R})$ for all $u \in U_q(\mathfrak{g})$ or, equivalently, ${}_s\mathcal{R} \circ \Delta_{\sigma^s}(u) \circ {}_s\mathcal{R}^{-1} = \Delta^{op}_{\sigma^s}(u)$.

According to the proof of [23, Theorem 32.1.5], $\Theta \circ \Pi^{-1} \circ \Delta_\sigma(u') = \Delta^{op}_\sigma(u')\Theta \circ \Pi^{-1}$ for all $u' \in U_q(\mathfrak{g})$. This is equivalent to $\Pi \circ \Theta_{2,1}^{-1}\Delta_\sigma(u')\Theta_{2,1} \circ \Pi^{-1} = \Delta^{op}_\sigma(u')$ and according to Lemma 2.18, this implies that for any $s' \in \mathbb{Z}$,

$$\Pi \circ (\sigma^{s'} \otimes \sigma^{s'})(\Theta_{2,1}^{-1}\Delta_\sigma(u')\Theta_{2,1}) \circ \Pi^{-1} = (\sigma^{s'} \otimes \sigma^{s'})(\Delta^{op}_\sigma(u')).$$

Combining all of this, we see that for $u \in U_q(\mathfrak{g})$ and $s \in \mathbb{Z}$,

$$\begin{aligned}
{}_s\mathcal{R} \circ \Delta_{\sigma^s}(u) \circ {}_s\mathcal{R}^{-1} &= \Pi \circ (\sigma^{s-1} \otimes \sigma^{s-1})(\Theta_{2,1}^{-1} \Delta_\sigma(\sigma^{1-s}(u))\Theta_{2,1}) \circ \Pi^{-1} \\
&= (\sigma^{s-1} \otimes \sigma^{s-1})(\Delta_\sigma^{op}(\sigma^{1-s}(u))) \\
&= \Delta_{\sigma^s}^{op}(u).
\end{aligned}$$

To see that $\tau \circ {}_s\mathcal{R}$ is natural, let $f : M_1 \to N_1$ and $g : M_2 \to N_2$ be homomorphisms of $U_q(\mathfrak{g})$-weight modules. Then it is clear that $(\sigma^{s-1} \otimes \sigma^{s-1})(\Theta_{2,1}^{-1})$ and $f \otimes g$ commute. And since f and g preserve grading on the factors, Π commutes with $f \otimes g$ as well. Therefore ${}_s\mathcal{R}$ and $f \otimes g$ commute. It follows that

$$\tau \circ {}_s\mathcal{R} \circ (f \otimes g) = (g \otimes f) \circ \tau \circ {}_s\mathcal{R}$$

and so $\tau \circ {}_s\mathcal{R}$ is natural.

In order to show the hexagon axioms, we must verify that

$$\tau_{(123)} \circ {}_s\mathcal{R}_{12,3} = \tau_{(12)} \circ {}_s\mathcal{R}_{1,2} \circ \tau_{(23)} \circ {}_s\mathcal{R}_{2,3} \quad \text{and}$$

$$\tau_{(132)} \circ {}_s\mathcal{R}_{1,23} = \tau_{(23)} \circ {}_s\mathcal{R}_{2,3} \circ \tau_{(12)} \circ {}_s\mathcal{R}_{1,2},$$

or equivalently, ${}_s\mathcal{R}_{12,3} = {}_s\mathcal{R}_{1,3} \circ {}_s\mathcal{R}_{2,3}$ and ${}_s\mathcal{R}_{1,23} = {}_s\mathcal{R}_{1,3} \circ {}_s\mathcal{R}_{1,2}$. We'll show that ${}_s\mathcal{R}_{1,3}^{-1} \circ {}_s\mathcal{R}_{1,23} \circ {}_s\mathcal{R}_{1,2}^{-1} = 1 \otimes 1 \otimes 1$ and omit the proof that ${}_s\mathcal{R}_{1,3}^{-1} \circ {}_s\mathcal{R}_{12,3} \circ {}_s\mathcal{R}_{2,3}^{-1} = 1 \otimes 1 \otimes 1$ as it will be very similar. According to [23, Theorem 32.2.4],

$$\begin{aligned}
(\Delta_\sigma^{op} \otimes 1)(\Theta) \circ \Pi_{12,3}^{-1} \circ \tau_{(132)} &= \Theta_{2,3} \circ \Pi_{2,3}^{-1} \circ \tau_{(23)} \circ \Theta_{1,2} \circ \Pi_{1,2}^{-1} \circ \tau_{(12)} \\
&= \Theta_{2,3} \circ \Pi_{2,3}^{-1} \circ \Theta_{1,3} \circ \Pi_{1,3}^{-1} \circ \tau_{(132)}.
\end{aligned}$$

Hence $(1 \otimes \Delta_\sigma)(\Theta_{2,1}) \circ \Pi_{1,23}^{-1} = \Theta_{2,1} \circ \Pi_{1,2}^{-1} \circ \Theta_{3,1} \circ \Pi_{1,3}^{-1}$ and so

$$\Pi_{1,23} \circ (1 \otimes \Delta_\sigma)(\Theta_{2,1}^{-1}) = \Pi_{1,3} \circ \Theta_{3,1}^{-1} \circ \Pi_{1,2} \circ \Theta_{2,1}^{-1}.$$

It's clear that $\Pi_{1,3}^{-1} \circ \Pi_{1,23} = \Pi_{1,2}$, so we see

$$\Pi_{1,2} \circ (1 \otimes \Delta_\sigma)(\Theta_{2,1}^{-1})\Theta_{2,1} \circ \Pi_{1,2}^{-1} = \Theta_{3,1}^{-1}.$$

We're now ready to compute.

$$\begin{aligned}
&{}_s\mathcal{R}_{1,3}^{-1} \circ {}_s\mathcal{R}_{1,23} \circ {}_s\mathcal{R}_{1,2}^{-1} \\
&\quad = (\sigma^{s-1} \otimes \sigma^{s-1} \otimes \sigma^{s-1})(\Theta_{3,1}) \circ \Pi_{1,3}^{-1} \circ \Pi_{1,23} \circ (1 \otimes \Delta_{\sigma^s})((\sigma^{s-1} \otimes \sigma^{s-1})(\Theta_{2,1}^{-1})) \\
&\qquad \circ (\sigma^{s-1} \otimes \sigma^{s-1} \otimes \sigma^{s-1})(\Theta_{2,1}) \circ \Pi_{1,2}^{-1} \\
&\quad = (\sigma^{s-1} \otimes \sigma^{s-1} \otimes \sigma^{s-1})(\Theta_{3,1}) \circ \Pi_{1,2} \circ ((\sigma^{s-1} \otimes \sigma^{s-1} \otimes \sigma^{s-1}) \circ (1 \otimes \Delta_\sigma))(\Theta_{2,1}^{-1}) \\
&\qquad \circ (\sigma^{s-1} \otimes \sigma^{s-1} \otimes \sigma^{s-1})(\Theta_{2,1}) \circ \Pi_{1,2}^{-1} \\
&\quad = (\sigma^{s-1} \otimes \sigma^{s-1} \otimes \sigma^{s-1})(\Theta_{3,1}) \\
&\qquad \circ \Pi_{1,2} \circ (\sigma^{s-1} \otimes \sigma^{s-1} \otimes \sigma^{s-1})((1 \otimes \Delta_\sigma)(\Theta_{2,1}^{-1})\Theta_{2,1}) \circ \Pi_{1,2}^{-1} \\
&\quad = (\sigma^{s-1} \otimes \sigma^{s-1} \otimes \sigma^{s-1})(\Theta_{3,1})(\sigma^{s-1} \otimes \sigma^{s-1} \otimes \sigma^{s-1})((\Theta_{3,1})^{-1}) \\
&\quad = 1 \otimes 1 \otimes 1 \qquad \square
\end{aligned}$$

${}_s\mathcal{R}$ is called the universal R-matrix for $U_q(\mathfrak{g})_{\sigma^s}$. It behaves like a true R-matrix, satisfying the quantum Yang-Baxter equation, for instance.

Corollary 2.22. ${}_s\mathcal{R}$ satisfies ${}_s\mathcal{R}_{1,2} \circ {}_s\mathcal{R}_{1,3} \circ {}_s\mathcal{R}_{2,3} = {}_s\mathcal{R}_{2,3} \circ {}_s\mathcal{R}_{1,3} \circ {}_s\mathcal{R}_{1,2}$.

Proof. As in the proof for usual R-matrices, we compute ${}_s\mathcal{R}_{1,32}$ two different ways. We will, however, need the obvious fact that ${}_s\mathcal{R}$ preserves degree, yielding

$$\Pi^r_{1,23} \circ {}_s\mathcal{R}_{2,3} = {}_s\mathcal{R}_{2,3} \circ \Pi^r_{1,23}$$

for all $r \in \frac{1}{2}\mathbb{Z}$. Then

$$\begin{aligned}
{}_s\mathcal{R}_{1,32} &= \tau_{(23)} \circ {}_s\mathcal{R}_{1,23} \circ \tau_{(23)} \\
&= \tau_{(23)} \circ {}_s\mathcal{R}_{1,3} \circ {}_s\mathcal{R}_{1,2} \circ \tau_{(23)} \\
&= {}_s\mathcal{R}_{1,2} \circ {}_s\mathcal{R}_{1,3}
\end{aligned}$$

and

$$\begin{aligned}
{}_s\mathcal{R}_{1,32} &= \Pi_{1,23} \circ ((1 \otimes \Delta^{op}_{\sigma^s}) \circ (\sigma^{s-1} \otimes \sigma^{s-1}))(\Theta^{-1}_{2,1}) \\
&= \Pi_{1,23} \circ {}_s\mathcal{R}_{2,3} \circ ((1 \otimes \Delta_{\sigma^s}) \circ (\sigma^{s-1} \otimes \sigma^{s-1}))(\Theta^{-1}_{2,1}) \circ {}_s\mathcal{R}^{-1}_{2,3} \\
&= {}_s\mathcal{R}_{2,3} \circ \Pi_{1,23} \circ ((1 \otimes \Delta_{\sigma^s}) \circ (\sigma^{s-1} \otimes \sigma^{s-1}))(\Theta^{-1}_{2,1}) \circ {}_s\mathcal{R}^{-1}_{2,3} \\
&= {}_s\mathcal{R}_{2,3} \circ {}_s\mathcal{R}_{1,23} \circ {}_s\mathcal{R}^{-1}_{2,3} \\
&= {}_s\mathcal{R}_{2,3} \circ {}_s\mathcal{R}_{1,3} \circ {}_s\mathcal{R}_{1,2} \circ {}_s\mathcal{R}^{-1}_{2,3}
\end{aligned}$$

Combining these, we find

$${}_s\mathcal{R}_{1,2} \circ {}_s\mathcal{R}_{1,3} \circ {}_s\mathcal{R}_{2,3} = {}_s\mathcal{R}_{1,32} \circ {}_s\mathcal{R}_{2,3} = {}_s\mathcal{R}_{2,3} \circ {}_s\mathcal{R}_{1,3} \circ {}_s\mathcal{R}_{1,2}. \qquad \square$$

As a result of the braiding $\tau \circ {}_s\mathcal{R}$, algebra objects in $U_q(\mathfrak{g})_{\sigma^s}$-**WMod** (that is, weight module algebras) reproduce under tensor products. Namely, we have the following.

Corollary 2.23. If A and B are $U_q(\mathfrak{g})_{\sigma^s}$-weight module algebras, then the $U_q(\mathfrak{g})_{\sigma^s}$-weight module $A \otimes B$ may be given the structure of a weight module algebra via

$$(a\otimes 1)(a'\otimes b') = aa'\otimes b', \quad (a\otimes b)(1\otimes b') = a\otimes bb', \quad \text{and} \quad (1\otimes b)(a'\otimes 1) = {}_s\mathcal{R}_{2,1}(a'\otimes b)$$

for $a \in A$ and $b \in B$.

The $U_q(\mathfrak{g})_{\sigma^s}$-weight module algebra in the preceding corollary is called the *braided tensor product* of A and B and is denoted $A\underline{\otimes}B$.

Example 2.24. Given integers $m \geq 2$ and $n_1, n_2 \geq 1$, the assignment

$$x_{i,j} \mapsto \begin{cases} x_{i,j} \otimes 1 & \text{if } 1 \leq j \leq n_1 \\ 1 \otimes x_{i,j-n_1} & \text{if } n_1 + 1 \leq j \leq n_1 + n_2 \end{cases}$$

gives rise to an isomorphism of $U_q(\mathfrak{sl}_m)$-weight module algebras

$$\mathcal{A}_q[Mat_{m,n_1+n_2}] \tilde{\to} \mathcal{A}_q[Mat_{m,n_1}]\underline{\otimes}\mathcal{A}_q[Mat_{m,n_2}].$$

2.5. Quantum Cluster Algebras

In this section, we largely follow [16]. All definitions and results can either be found there or are equivalent to ones found there.

For integers $m \leq n$, we'll write $[m, n]$ to denote the set $\{m, m+1, ..., n\}$. Given positive integers m, n, p, and s, as well as $A \in Mat_{m,n}(\mathbb{Z})$, $B \in Mat_{n,p}(\Bbbk)$, and $C \in Mat_{p,s}(\mathbb{Z})$ for any commutative ring $\Bbbk$, we define an $m \times s$ matrix ${}^{A}B^{C}$ with entries in $\Bbbk$ by setting the (i, j)-th entry to be

$$\prod_{k,\ell} b_{k,\ell}^{a_{i,k}c_{\ell,j}}$$

whenever this makes sense.

For $N \in \mathbb{Z}_{\geq 0}$, elements of $\mathbb{Z}^N$ will be thought of as column vectors with the standard basis elements e_i for $i \in [1, N]$. Given $\mathbf{c} = \sum_{i=1}^{N} c_i e_i \in \mathbb{Z}^N$, we write

$$[\mathbf{c}]_+ := \sum_{i=1}^{N} \max(c_i, 0)e_i \quad \text{and} \quad [\mathbf{c}]_- := \sum_{i=1}^{N} \max(-c_i, 0)e_i.$$

Fix a field $\mathbb{K}$ and $N \in \mathbb{Z}_{\geq 0}$.

A matrix $\mathbf{q} \in Mat_{N,N}(\mathbb{K})$ is called *multiplicatively skew-symmetric* if $q_{i,i} = 1$ and $q_{i,j}q_{j,i} = 1$ for $i, j, \in [1, N]$. Such a matrix yields a quantum torus, i.e. the $\mathbb{K}$-algebra $\mathcal{T}_{\mathbf{q}}$ generated by $\{X_i^{\pm 1} \mid i \in [1, N]\}$ subject to the relation $X_i X_j = q_{i,j} X_j X_i$ for all $i, j \in [1, N]$. Given a unital subring $\Bbbk$ of $\mathbb{K}$ containing all $q_{i,j}$, we denote by $\mathcal{T}_{\mathbf{q}}^{\Bbbk}$ the $\Bbbk$-subalgebra of $\mathcal{T}_{\mathbf{q}}$ generated by all $X_i^{\pm 1}$. Equivalently, $\mathcal{T}_{\mathbf{q}}^{\Bbbk}$ could be defined as the $\Bbbk$ algebra on the same generators and relations as $\mathcal{T}_{\mathbf{q}}$.

Given a multiplicatively skew-symmetric matrix $\mathbf{q} \in Mat_{N,N}(\mathbb{K})$, denote by $\mathbf{q}^{\cdot 2}$ the multiplicatively skew-symmetric matrix whose (i, j)-th entry is $q_{i,j}^2$. $\mathcal{T}_{\mathbf{q}^{\cdot 2}}$ then has a distinguished $\mathbb{K}$-basis $\{X^{(\mathbf{c})} \mid \mathbf{c} \in \mathbb{Z}^N\}$, where

$$X^{(\mathbf{c})} := \mathcal{S}_{\mathbf{q}}(\mathbf{c}) X_1^{c_1} X_2^{c_2} \cdots X_N^{c_N} := \left(\prod_{1 \leq i < j \leq N} q_{i,j}^{-c_i c_j} \right) X_1^{c_1} X_2^{c_2} \cdots X_N^{c_N}.$$

Definition 2.25. Let $\mathcal{F}$ be a division algebra over $\mathbb{K}$. A map $M : \mathbb{Z}^N \to \mathcal{F}$ is called a *toric frame* if there exists a multiplicatively skew-symmetric matrix $\mathbf{q} \in Mat_{N,N}(\mathbb{K})$ such that

1. The assignment $\varphi_M(X_i) = M(e_i)$ for $i \in [1, N]$ defines an algebra embedding $\varphi_M : \mathcal{T}_{\mathbf{q}^{\cdot 2}} \hookrightarrow \mathcal{F}$ such that $\mathcal{F} = Frac(\varphi_M(\mathcal{T}_{\mathbf{q}^{\cdot 2}}))$.

2. For all $\mathbf{c} \in \mathbb{Z}^N$, $M(\mathbf{c}) = \varphi(X^{(\mathbf{c})})$.

Given a toric frame $M : \mathbb{Z}^N \to \mathcal{F}$, the matrix $\mathbf{q}$ appearing in the definition is easily recovered via the formula

$$q_{i,j} = M(e_i)M(e_j)M(e_i + e_j)^{-1}$$

for $1 \leq i < j \leq N$, then setting $q_{j,i} = q_{i,j}^{-1}$. We write $\mathbf{q}(M)$ to denote this matrix.

Fix $\mathbf{ex} \subset [1, N]$. By an "$N \times \mathbf{ex}$" matrix, we mean an $N \times |\mathbf{ex}|$ matrix with columns indexed by $\mathbf{ex}$.

Definition 2.26. Given a multiplicatively skew-symmetric matrix $\mathbf{q} \in Mat_{N,N}(\mathbb{K})$ and an $N \times \mathbf{ex}$ integer matrix $\tilde{B}$ such that the $\mathbf{ex} \times \mathbf{ex}$ submatrix of $\tilde{B}$ is skew-symmetrizable, we say the pair $(\mathbf{q}, \tilde{B})$ is *compatible* if

1. $\displaystyle\prod_{k=1}^{N} q_{k,i}^{b_{k,j}} = 1$ for all $i \in [1, N]$ and $j \in \mathbf{ex}$ with $i \neq j$ and

2. $\displaystyle\prod_{k=1}^{N} q_{k,j}^{b_{k,j}}$ is not a root of unity for any $j \in \mathbf{ex}$.

Given a compatible pair $(\mathbf{q}, \tilde{B})$, some $k \in \mathbf{ex}$, and $\epsilon \in \{+, -\}$, E_ϵ and F_ϵ are the $N \times N$ and $\mathbf{ex} \times \mathbf{ex}$ integer matrices with entries given by

$$(E_\epsilon)_{i,j} = \begin{cases} \delta_{i,j} & \text{if } j \neq k \\ -1 & \text{if } i = j = k \\ \max(-\epsilon b_{i,k}, 0) & \text{if } i \neq j = k \end{cases} \qquad \text{(Equation 2.1.)}$$

$$(F_\epsilon)_{i,j} = \begin{cases} \delta_{i,j} & \text{if } i \neq k \\ -1 & \text{if } i = j = k \cdot \\ \max(\epsilon b_{k,j}, 0) & \text{if } j \neq i = k \end{cases} \qquad \text{(Equation 2.2.)}$$

We can now define mutation of compatible pairs. Namely, we set

$$\mu_k(\mathbf{q}, \tilde{B}) = (\mu_k(\mathbf{q}), \mu_k(\tilde{B})) := \left({}^{E_\epsilon^T}\mathbf{q}^{E_\epsilon}, E_\epsilon \tilde{B} F_\epsilon\right).$$

Proposition 2.27. [16, Proposition 2.6] Let $(\mathbf{q}, \tilde{B})$ be a compatible pair. Then $\mu_k(\mathbf{q}, \tilde{B})$ is a compatible pair which does not depend on the choice of ϵ.

Definition 2.28. A *quantum seed* of a division algebra $\mathcal{F}$ is a pair $(M, \tilde{B})$, where $M : \mathbb{Z}^N \to \mathcal{F}$ is a toric frame and $\tilde{B}$ is an $N \times \mathbf{ex}$ integer matrix so that $(\mathbf{q}(M), \tilde{B})$ is a compatible pair.

Proposition 2.29. Given a quantum seed $(M, \tilde{B})$ and some $k \in \mathbf{ex}$, there is a unique toric frame $\mu_k(M)$ of $\mathcal{F}$ so that

$$\mu_k(M)(e_i) = \begin{cases} M([b^k]_+ - e_k) + M([b^k]_- - e_k) & \text{if } i = k \\ M(e_i) & \text{else} \end{cases},$$

where b^k denotes the column of $\tilde{B}$ indexed by k. Furthermore, $(\mu_k(M), \mu_k(\tilde{B}))$ is a quantum seed.

Now, similarly to for compatible pairs, we define the mutation of a quantum seed in the direction $k \in \mathbf{ex}$ to be $\mu_k(M, \tilde{B}) := (\mu_k(M), \mu_k(\tilde{B}))$.

Proposition 2.30. [16, Corollary 2.11] For all quantum seeds $(M, \tilde{B})$ of $\mathcal{F}$ and $k \in \mathbf{ex}$, we have $\mathbf{q}(\mu_k(M)) = \mu_k(\mathbf{q}(M))$ and $\mu_k^2(M, \tilde{B}) = (M, \tilde{B})$.

In light of the preceding proposition, two quantum seed will be called *mutation-equivalent* if they can be obtained, one from another, by a sequence of mutations. The fact that mutation is involutive guarantees that this is a well-defined equivalence relation.

Definition 2.31. Given $\mathbf{inv} \subset [1, N] \setminus \mathbf{ex}$ and any unital subring $\Bbbk$ of $\mathbb{K}$ containing $q_{i,j}(M)$ for all $i, j \in [1, N]$, we define the *quantum cluster algebra* $\Bbbk(M, \tilde{B}, \mathbf{inv})$ to be the unital $\Bbbk$-subalgebra of $\mathcal{F}$ generated by $M(e_i)^{-1}$ with $i \in \mathbf{inv}$ and by $M'(e_i)$ with $i \in I$ for all quantum seeds $(M', \tilde{B}')$ of $\mathcal{F}$ which are mutation-equivalent to $(M, \tilde{B})$. For $\mathbf{c} \in \mathbb{Z}^N$ with $c_i \geq 0$ if $i \notin \mathbf{inv}$, we call $M'(\mathbf{c})$ a *quantum cluster monomial.*

Theorem 2.32 (Quantum Laurent Phenomenon)**.** For all quantum seeds $(M, \tilde{B})$ of a division algebra $\mathcal{F}$, subrings $\Bbbk$ containing $q_{i,j}(M)$ for all $i, j \in [1, N]$, and subsets $\mathbf{inv} \subset [1, N] \setminus \mathbf{ex}$, we have the inclusion

$$\Bbbk(M, \tilde{B}, \mathbf{inv}) \subset \varphi_M(\mathcal{T}^{\Bbbk}_{\mathbf{q}(M)^{\cdot 2}}).$$

Since the triple $(\mathbf{q}(M), \tilde{B}, \mathbf{inv})$ determines the $\Bbbk$-algebra $\Bbbk(M, \tilde{B}, \mathbf{inv})$ up to isomorphism, we will write $\Bbbk(\mathbf{q}(M), \tilde{B}, \mathbf{inv})$ in place of $\Bbbk(M, \tilde{B}, \mathbf{inv})$, when convenient.

2.6. Single-Parameter Quantum Cluster Algebras

We retain the assumptions of the previous section that K is a field, $N \in \mathbb{Z}_{\geq 0}$, and $\mathbf{ex} \subset [1, N]$.

Suppose $q \in \mathbb{K}^\times$ is not a root of unity and has a specified $2d$-th root $q^{\frac{1}{2d}}$. We then have an altered definition of compatible pair.

Definition 2.33. Let Λ be an $N \times N$ skew-symmetric integer matrix and $\tilde{B}$ an $N \times \mathbf{ex}$ integer matrix. The pair $(\Lambda, \tilde{B})$ is *compatible* if for each $i \in \mathbf{ex}$ and $j \in [1, N]$,

$$\sum_{k=1}^{N} b_{k,i}\lambda_{k,j} = \delta_{i,j}d_i$$

for some $d_j \in \mathbb{Z}_{\geq 0}$.

If $(\Lambda, \tilde{B})$ is a compatible pair, then the $\mathbf{ex} \times \mathbf{ex}$ submatrix of $\tilde{B}$ is automatically skew-symmetrizable and we see that $((q^{\frac{1}{2}}I_N)^\Lambda, \tilde{B})$ is a compatible pair in the sense of Section 2.5.. Furthermore, it is clear that compatible pairs of this type are preserved under mutation with $\mu_k((q^{\frac{1}{2}}I_N)^\Lambda) = (q^{\frac{1}{2}}I_N)^{E_\epsilon^T \Lambda E_\epsilon}$. We therefore write $\mu_k(\Lambda) := E_\epsilon^T \Lambda E_\epsilon$ (which matches the original definition in [8]). Since q is not a root of unity, $\mu_k(\Lambda)$ is independent of sign. We write $\mathbf{q}_{\Lambda,q} := (q^{\frac{1}{2}}I_N)^\Lambda$ and note that $\mu_k(\mathbf{q}_{\Lambda,q}) = \mathbf{q}_{\mu_k(\Lambda),q}$.

Definition 2.34. A quantum seed $(M, \tilde{B})$ is called a *single parameter quantum seed* if $\mathbf{q}(M) = \mathbf{q}_{\Lambda,q}$ for some q as above and skew-symmetric matrix $\Lambda \in Mat_{N,N}(\mathbb{Z})$ so that $(\Lambda, \tilde{B})$ is a compatible pair.

A single parameter quantum seed is a priori a quantum seed. Therefore, there is no need to define single parameter quantum cluster algebras except to say that they are quantum cluster algebras coming from a single parameter quantum seed. However, when convenient, we will denote $\mathbb{K}(\mathbf{q}_{\Lambda,q}, \tilde{B}, \mathbf{inv})$ instead by $\mathbb{K}_q(\Lambda, \tilde{B}, \mathbf{inv})$.

Remark 2.35. The assumption that q is not a root of unity implies that Λ can be recovered from $\mathbf{q}_{\Lambda,q}$, assuming $q^{\frac{1}{2}}$ is known. Given a toric frame $M : \mathbb{Z}^N \to \mathcal{F}$ which is part of a single parameter quantum seed, we may therefore write $\Lambda(M)$ for the unique skew-symmetric $N \times N$ integer matrix so that $\mathbf{q}(M) = \mathbf{q}_{\Lambda(M),q}$.

CHAPTER III

FACTORIZABLE MODULE ALGEBRAS

This chapter contains material which originally appeared in the paper [6], which was co-authored with Arkady Berenstein. We developed the results over the course of many meetings and, as is often the case with collaborative work, it is difficult to attribute specific ideas either to Berenstein or myself.

For the sake of convenience, we will use the comultiplication Δ_σ, counit ϵ_σ, and antipode S_σ for $U_q(\mathfrak{g})$ throughout this chapter. Therefore, we also use the universal R-matrix ${}_1\mathcal{R} = \Pi \circ \Theta_{2,1}^{-1}$. However, again for convenience, we denote them by Δ, ϵ, S, and $\mathcal{R}$ unless stated otherwise.

3.1. Definitions, Notation, and Results

In this section, we will recall and introduce the relevant definitions and notation necessary to present our main results of the chapter, which will also be included.

First, suppose M is a $U_q(\mathfrak{b}_+)$-module. For each $i \in I$ and $x \in M \setminus \{0\}$, set $\ell_i(x) = \max\{\ell \in \mathbb{Z}_{\geq 0} \mid E_i^\ell(x) \neq 0\}$ and $E_i^{(top)}(x) = E_i^{(\ell_i(x))}(x)$. Given $\mathbf{i} \in I^m$ for some $m \geq 0$ and $x \in M \setminus \{0\}$, we also use the shorthand

$$E_{\mathbf{i}}^{(top)}(x) = E_{i_m}^{(top)} E_{i_{m-1}}^{(top)} \cdots E_{i_1}^{(top)}(x)$$

and define $\nu_{\mathbf{i}} : M \setminus \{0\} \to \mathbb{Z}_{\geq 0}^m$, $x \mapsto (a_1, a_2, \ldots, a_m)$ by the following:

$$a_k = \ell_{i_k}(E_{i_{k-1}}^{(top)} E_{i_{k-2}}^{(top)} \cdots E_{i_1}^{(top)}(x)).$$

Lastly, for $\mathbf{j} = (j_1, \ldots, j_m) \in \mathbb{Z}_{\geq 0}^m$, we set $E_{\mathbf{i}}^{(\mathbf{j})} := E_{i_m}^{(j_m)} E_{i_{m-1}}^{(j_{m-1})} \cdots E_{i_1}^{(j_1)}$.

Definition 3.1. Let A be a $U_q(\mathfrak{b}_+)$-module algebra, $w \in W$, and $\mathbf{i} \in R(w)$. If $E_{\mathbf{i}}^{(top)}(x) \in A^+$ for all $x \in A \setminus \{0\}$, then we say A is $\mathbf{i}$-*adapted*. We say a basis $\mathcal{B}$ for A is an $\mathbf{i}$-*adapted basis* if

1. $E_{\mathbf{i}}^{(top)}(b) = 1$ for all $b \in \mathcal{B}$.
2. The restriction of $\nu_{\mathbf{i}}$ to $\mathcal{B}$ is an injective map $\mathcal{B} \hookrightarrow \mathbb{Z}_{\geq 0}^m$, where m is the length of w.

If there exists any $w \in W$ and $\mathbf{i} \in R(w)$ so that A is $\mathbf{i}$-adapted, then we say more generally that A is *adapted*.

Remark 3.2. Our notion of an $\mathbf{i}$-adapted $U_q(\mathfrak{b}_+)$-module algebra is different than P. Caldero's notion of adapted algebra in [14], though they do have some examples in common. On the other hand, our notion of $\mathbf{i}$-adapted basis is stronger than the similar notion of an adapted basis for $(A, \nu_{\mathbf{i}})$ as in [18].

It turns out that if A_0 possesses an $\mathbf{i}$-adapted basis for some $\mathbf{i} \in R(w)$ and is a "large enough" $U_q(\mathfrak{b}_+)$-module subalgebra of A, then A is factorizable over A_0. The following theorem makes this precise.

Theorem 3.3. Let A be a $U_q(\mathfrak{b}_+)$-module algebra. Suppose A_0 is a $U_q(\mathfrak{b}_+)$-module subalgebra of A possessing an $\mathbf{i}$-adapted basis $\mathcal{B}$ for some reduced $\mathbf{i}$. Then:

1. The restriction $\mu : A^+ \otimes A_0 \to A$ of the multiplication in A is an injective homomorphism of $U_q(\mathfrak{b}_+)$-modules.
2. The map μ is an isomorphism if and only if A is $\mathbf{i}$-adapted and $\nu_{\mathbf{i}}(A \setminus \{0\}) = \nu_{\mathbf{i}}(A_0 \setminus \{0\})$.

We will prove Theorem 3.3 in Section 3.2.1.. Theorem 3.3 demonstrates a close relationship between being **i**-adapted and being factorizable over a $U_q(\mathfrak{b}_+)$-module subalgebra possessing an **i**-adapted basis. The following theorem explores this relationship from a different angle.

Theorem 3.4. Let A be an **i**-adapted $U_q(\mathfrak{b}_+)$-module algebra for some reduced **i** and suppose A_0 is a $U_q(\mathfrak{b}_+)$-module subalgebra of A. Then $\mu : A^+ \otimes A_0 \to A$ as in Theorem 3.3 is an isomorphism of $U_q(\mathfrak{b}_+)$-modules if and only if A_0 possesses an **i**-adapted basis and $\nu_{\mathbf{i}}(A_0 \setminus \{0\}) = \nu_{\mathbf{i}}(A \setminus \{0\})$.

Theorem 3.4 is proved in Section 3.2.2.. We now restrict our focus to a specific $U_q(\mathfrak{g})$-module algebra, namely $\mathcal{A}_q[U]_{\sigma^{-1}}$. Recall that, as a $\mathbb{Q}(q^{\frac{1}{2d}})$-algebra, $\mathcal{A}_q[U]$ is generated by the set $\{x_i \mid i \in I\}$, subject to the quantum Serre relations:

$$\sum_{k=0}^{1-c_{i,j}} (-1)^k x_i^{(k)} x_j x_i^{(1-c_{i,j}-k)} = 0 \text{ if } i \neq j.$$

The $U_q(\mathfrak{g})$-module structure on $\mathcal{A}_q[U]_{\sigma^{-1}}$ (denoted just $\mathcal{A}_q[U]$ for the rest of the chapter) is summarized in the following equations:

$$\begin{aligned} K_i^{\pm\frac{1}{2}}(x_j) &= q_i^{\mp\frac{c_{i,j}}{2}} x_j \quad \text{for all } i, j \in I \\ E_i(x_j) &= \delta_{i,j} \quad \text{for all } i, j \in I \\ F_i(x) &= \frac{x_i x - K_i^{-1}(x) x_i}{q_i - q_i^{-1}} \quad \text{for all } i \in I \text{ and } x \in \mathcal{A}_q[U]. \end{aligned}$$

Of course, the actions of $K_i^{\pm\frac{1}{2}}$ and E_i must be extended to all of $\mathcal{A}_q[U]$ by the rules

$$K_i^{\pm\frac{1}{2}}(xx') = K_i^{\pm\frac{1}{2}}(x)K_i^{\pm\frac{1}{2}}(x') \quad \text{for all } i \in I \text{ and } x, x' \in \mathcal{A}_q[U]$$

$$E_i(xx') = E_i(x)K_i(x') + xE_i(x') \quad \text{for all } i \in I, \text{ and } x, x' \in \mathcal{A}_q[U].$$

Berenstein and Zelevinsky observed in [7, Proposition 3.5] that $\mathcal{A}_q[U]$ possesses a basis $\mathcal{B}^{dual}$ such that, for $\mathbf{i} \in R(w_\circ)$, the restriction of $\nu_{\mathbf{i}}$ to $\mathcal{B}^{dual}$ is injective. Note that they use the notation $\mathcal{A}$ in place of $\mathcal{A}_q[U]$ and view it only as a $U_q(\mathfrak{n}_+)$-module. As hinted by the notation, $\mathcal{B}^{dual}$ is the so-called *dual canonical basis*. In Section 3.2.3., we prove the following proposition.

Proposition 3.5. Given any $\mathbf{i} \in R(w_\circ)$, $\mathcal{B}^{dual}$ is an $\mathbf{i}$-adapted basis for $\mathcal{A}_q[U]$.

Remark 3.6. Based on the recent paper [20], we expect that the dual canonical basis $\mathcal{B}^{dual} \cap U_q(w)$ in each quantum Schubert cell $U_q(w)$ is $\mathbf{i}$-adapted for any reduced word $\mathbf{i}$ for w.

Combining Proposition 3.5 with Theorem 3.3, we are led to the following corollary, though it does still require some proof.

Corollary 3.7. Let A be a $U_q(\mathfrak{g})$-module algebra containing $\mathcal{A}_q[U]$ as a $U_q(\mathfrak{g})$-module subalgebra. If there exists a $U_q(\mathfrak{g})$-module algebra A' containing A as a $U_q(\mathfrak{g})$-module subalgebra, such that A' is generated by $(A')^+$ as a $U_q(\mathfrak{g})$-module algebra, then $\mu : A^+ \otimes \mathcal{A}_q[U] \to A$ as in Theorem 3.3 is an isomorphism of $U_q(\mathfrak{b}_+)$-modules.

Corollary 3.7 will be proved in Section 3.2.4. and provides us with the means to prove Theorem 1.1, which we do in Section 3.2.5..

Example 3.8. Recall that $\mathcal{A}_q[Mat_{3,2}]$ is generated by $\{x_{i,j} \mid 1 \leq i \leq 3,\ 1 \leq j \leq 2\}$, subject to relations

$$x_{k,j}x_{i,j} = qx_{i,j}x_{k,j} \text{ if } i < k,$$

$$x_{i,k}x_{i,j} = qx_{i,j}x_{i,k} \text{ if } j < k,$$

$$x_{k,\ell}x_{i,j} = x_{i,j}x_{k,\ell} \text{ if } i < k \text{ and } j > \ell,$$

$$x_{k,\ell}x_{i,j} = x_{i,j}x_{k,\ell} + (q - q^{-1})x_{i,\ell}x_{k,j} \text{ if } i < k \text{ and } j < \ell.$$

Then

$$\begin{aligned}&\mathcal{A}_q[Mat_{3,2}][x_{1,1}^{-1}, \Delta_2^{-1}] \\ &\quad\cong \mathcal{A}_q\left[x_{1,1}^{\pm 1}, x_{1,2}, \Delta_2^{\pm 1}\right] \otimes \mathcal{A}_q\left[x_{1,1}^{-1}x_{2,1}, x_{1,1}^{-1}x_{3,1}, \Delta_2^{-1}(x_{1,1}x_{3,2} - q^{-1}x_{1,2}x_{3,1})\right],\end{aligned}$$

where $\Delta_2 = x_{1,1}x_{2,2} - q^{-1}x_{1,2}x_{2,1}$ and $\mathcal{A}_q[-]$ denotes the subalgebra of $\mathcal{A}_q[Mat_{3,2}][x_{1,1}^{-1}, \Delta_2^{-1}]$ generated by those elements appearing inside the brackets.

The natural action of $U_q(\mathfrak{sl}_3)$ extends to the localized algebra and a short examination verifies that

$$\left(\mathcal{A}_q[Mat_{3,2}][x_{1,1}^{-1}, \Delta_2^{-1}]\right)^+ = \mathcal{A}_q\left[x_{1,1}^{\pm 1}, x_{1,2}, \Delta_2^{\pm 1}\right] \quad \text{and}$$

$$\mathcal{A}_q\left[x_{1,1}^{-1}x_{2,1}, x_{1,1}^{-1}x_{3,1}, \Delta_2^{-1}(x_{1,1}x_{3,2} - q^{-1}x_{1,2}x_{3,1})\right] \cong \mathbb{C}[U],$$

where the isomorphism is an isomorphism of $U_q(\mathfrak{sl}_3(\mathbb{C}))$-module algebras and the generators x_1 and x_2 of $\mathcal{A}_q[U]$ are mapped to by $x_{1,1}^{-1}x_{2,1}$ and $\Delta_2^{-1}(x_{1,1}x_{3,2} - q^{-1}x_{1,2}x_{3,1})$, respectively.

There are two families of quantities that arose in the proof of Corollary 3.7:

$$x_i a - K_i(a)x_i \quad \text{and} \quad F_i(a) + x_i\frac{K_i^{-2}(a) - a}{q_i - q_i^{-1}}$$

where $i \in I$ and $a \in A'$. These quantities are equally valid to consider for $a \in A$, without the assumed presence of A'. If $a \in A^+$, then both of these quantities are also in A^+. It is therefore natural to ask what relations the families of operators $L_i - R_i K_i$ and $F_i + L_i \frac{K_i^{-2}-1}{q_i - q_i^{-1}}$ satisfy, where L_i (respectively R_i) represents left (respectively right) multiplication by x_i. Or to put it another way, do these operators indicate the action of a known algebra which is somehow related to $U_q(\mathfrak{g})$? We can answer in the affirmative. It can be proved that both of the families of operators observed in fact satisfy the quantum Serre relations and the two families "almost" commute with each other. This resembles an action of the Hopf algebra $U_q(\mathfrak{g}^*)$

After some tweaking and combining of our operators with the inherited Cartan action, we see that our operators really do indicate the presence of a $U_q(\mathfrak{g}^*)$-module algebra structure. The following theorem summarizes this and is proved in Section 3.2.6..

Theorem 3.9. Let A be a $U_q(\mathfrak{g})$-module algebra containing $\mathcal{A}_q[U]$ as a $U_q(\mathfrak{g})$-module subalgebra. Then A is a $U_q(\mathfrak{g}^*)$-module algebra with action given by

$$K_i^{\pm\frac{1}{2}} \triangleright a = K_i^{\pm\frac{1}{2}}(a), \quad F_{i,1} \triangleright a = F_i(a) - \frac{x_i a - K_i^{-1}(a) x_i}{q_i - q_i^{-1}}, \quad F_{i,2} \triangleright a = \frac{x_i K_i^{-1}(a) - a x_i}{q_i - q_i^{-1}}.$$

In particular, the subalgebra A^+ is preserved by this action of $U_q(\mathfrak{g}^*)$ and is therefore a $U_q(\mathfrak{g}^*)$-module subalgebra.

Theorem 3.9 is in some sense a statement about the existence of a functor. To make this precise, we introduce a category whose objects bear properties similar to those found in Theorem 3.3.

Definition 3.10. Let $\mathcal{C}_{\mathfrak{g}}^q$ be the category whose objects consist of pairs (A, φ_A), where

– A is an adapted $U_q(\mathfrak{g})$-module algebra such that $\nu_{\mathbf{i}}(A \setminus \{0\}) = \nu_{\mathbf{i}}(\mathcal{A}_q[U] \setminus \{0\})$ for all $\mathbf{i} \in R(w_\mathrm{o})$.

– $\varphi_A : \mathcal{A}_q[U] \hookrightarrow A$ is an embedding of $U_q(\mathfrak{g})$-module algebras.

A morphism $(A, \varphi_A) \to (B, \varphi_B)$ in $\mathcal{C}^q_{\mathfrak{g}}$ is a homomorphism of $U_q(\mathfrak{g})$-module algebras $\psi : A \to B$ such that $\psi \circ \varphi_A = \varphi_B$.

Given a homomorphism of $U_q(\mathfrak{g})$-module algebras $\psi : A \to B$, it follows that $\psi(A^+) \subseteq B^+$, so $\psi|_{A^+}$ may be thought of as a map of $\mathbb{Q}(q^{\frac{1}{2d}})$-algebras $A^+ \to B^+$. If ψ is a morphism in $\mathcal{C}^q_{\mathfrak{g}}$, $(A, \varphi_A) \to (B, \varphi_B)$, then actually $\psi|_{A^+}$ is a homomorphism of $U_q(\mathfrak{g}^*)$-module algebras. As a consequence, we have the following corollary.

Corollary 3.11. There is a functor $(-)^+ : \mathcal{C}^q_{\mathfrak{g}} \to U_q(\mathfrak{g}^*)$-**ModAlg** (denoted $P_q \circ F_q$ in Section 1.1.) which assigns to an object (A, φ_A) of $\mathcal{C}^q_{\mathfrak{g}}$ its subalgebra of highest weight vectors A^+, equipped with the $U_q(\mathfrak{g}^*)$-module algebra structure of Theorem 3.9. The functor $(-)^+$ is given on morphisms by restriction.

Theorem 3.3 strongly suggests that $(-)^+$ might actually be an equivalence of categories. In fact, this is the case, but in order to describe a quasi-inverse, we need the following theorem which describes a $U_q(\mathfrak{g})$-module algebra structure on $A \otimes \mathcal{A}_q[U]$ if A is a $U_q(\mathfrak{g}^*)$-module algebra.

Theorem 3.12. If A is a $U_q(\mathfrak{g}^*)$-module algebra, then $A \otimes \mathcal{A}_q[U]$ has the structure of a $U_q(\mathfrak{g})$-module algebra determined by:

$$
\begin{aligned}
(1 \otimes x_i)(a \otimes 1) &= K_i(a) \otimes x_i + (q_i - q_i^{-1})F_{i,2}K_i(a) \otimes 1, \\
K_i^{\pm\frac{1}{2}} \triangleright (a \otimes x) &= K_i^{\pm\frac{1}{2}}(a) \otimes K_i^{\pm\frac{1}{2}}(x), \\
E_i \triangleright (a \otimes x) &= a \otimes E_i(x), \\
F_i \triangleright (a \otimes x) &= (F_{i,1}(a) + F_{i,2}K_i(a)) \otimes x, \\
&\quad + \frac{K_i(a) - K_i^{-1}(a)}{q_i - q_i^{-1}} \otimes x_i x + K_i^{-1}(a) \otimes F_i(x).
\end{aligned}
$$

Theorem 3.12 will be proved in Section 3.2.6.. Since the action of each E_i is completely described on the $\mathcal{A}_q[U]$ factor and $\mathcal{A}_q[U]$ is adapted, we have the following corollary.

Corollary 3.13. There is a functor $(-) \otimes \mathcal{A}_q[U] : U_q(\mathfrak{g}^*)\text{-}\mathbf{ModAlg} \to \mathcal{C}_{\mathfrak{g}}^q$ which assigns to a $U_q(\mathfrak{g}^*)$-module algebra A, the pair $(A \otimes \mathcal{A}_q[U], 1 \otimes \mathrm{id})$, where $A \otimes \mathcal{A}_q[U]$ is given the $U_q(\mathfrak{g})$-module structure of Theorem 3.12. The functor $(-) \otimes \mathcal{A}_q[U]$ is given on morphisms by $\psi \mapsto \psi \otimes \mathrm{id}$.

The following theorem says that $(-) \otimes \mathcal{A}_q[U]$ is the promised quasi-inverse for $(-)^+$.

Theorem 3.14. The functors

$$
(-)^+ : \mathcal{C}_{\mathfrak{g}}^q \to U_q(\mathfrak{g}^*)\text{-}\mathbf{ModAlg} \quad \text{and} \quad (-) \otimes \mathcal{A}_q[U] : U_q(\mathfrak{g}^*)\text{-}\mathbf{ModAlg} \to \mathcal{C}_{\mathfrak{g}}^q
$$

are quasi-inverses of each other and thus provide equivalences of categories.

Theorem 3.14 is proved in section 3.2.7.. We define a subcategory of $\mathcal{C}_{\mathfrak{g}}^q$ on which the braided tensor product of $U_q(\mathfrak{g})$-weight module algebras can be used to create another tensor product.

Definition 3.15. Let $\underline{\mathcal{C}}_{\mathfrak{g}}^q$ be the full subcategory of $\mathcal{C}_{\mathfrak{g}}^q$ whose objects consist of pairs (A, φ_A), where A is additionally assumed to be a $U_q(\mathfrak{g})$-weight module algebra.

The following proposition is then clear.

Proposition 3.16. The functors $(-)^+$ and $(-) \otimes \mathcal{A}_q[U]$ restrict to equivalences between $\underline{\mathcal{C}}_{\mathfrak{g}}^q$ and $U_q(\mathfrak{g}^*)$-**WModAlg**.

If (A, φ_A) and (B, φ_B) are objects of $\underline{\mathcal{C}}_{\mathfrak{g}}^q$, then we already saw that $A \underline{\otimes} B$ is also a $U_q(\mathfrak{g})$-weight module algebra. Furthermore, it is obvious that $1 \otimes \varphi_B$ and $\varphi_A \otimes 1$ are injections $\mathcal{A}_q[U] \hookrightarrow A \underline{\otimes} B$. However, it is not immediately obvious that $A \underline{\otimes} B$ is adapted with $\nu_{\mathbf{i}}(A \underline{\otimes} B \setminus \{0\}) = \nu_{\mathbf{i}}(\mathcal{A}_q[U] \setminus \{0\})$ for all $\mathbf{i} \in R(w_\circ)$. Nevertheless, this is the case, which the following proposition asserts.

Proposition 3.17. If (A, φ_A) and (B, φ_B) are objects of $\underline{\mathcal{C}}_{\mathfrak{g}}^q$, then $(A \underline{\otimes} B, 1 \otimes \varphi_B)$ and $(A \underline{\otimes} B, \varphi_A \otimes 1)$ are objects of $\underline{\mathcal{C}}_{\mathfrak{g}}^q$ as well.

Proposition 3.17 is proved in Section 3.2.8.. Proposition 3.16 allows us to turn Proposition 3.17 into a statement about $U_q(\mathfrak{g}^*)$-module algebras. We define two "fusion" products on the category of $U_q(\mathfrak{g}^*)$-weight module algebras, namely the following:

$$A * B := ((A \otimes \mathcal{A}_q[U]) \underline{\otimes} (B \otimes \mathcal{A}_q[U]), 1 \otimes 1 \otimes 1 \otimes \mathrm{id})^+,$$
$$A \star B := ((A \otimes \mathcal{A}_q[U]) \underline{\otimes} (B \otimes \mathcal{A}_q[U]), 1 \otimes \mathrm{id} \otimes 1 \otimes 1)^+.$$

These fusion products are associative, but not monoidal due to the easy observation that there is no unit object. The reader may be bothered that the objects $(A \underline{\otimes} B, 1 \otimes \varphi_B)$ and $(A \underline{\otimes} B, \varphi_A \otimes 1)$ are not (necessarily at least) isomorphic despite having equal underlying $U_q(\mathfrak{g})$-module algebras. An attempt to force a common quotient leads to the discovery of an interesting $U_q(\mathfrak{g}^*)$-module algebra structure on $A \otimes B$ if A and B are $U_q(\mathfrak{g}^*)$-weight module algebras.

Proposition 3.18. Let A and B be $U_q(\mathfrak{g}^*)$-weight module algebras. Then the $\mathbb{Q}(q^{\frac{1}{2d}})$-vector space $A \otimes B$ has the structure of a $U_q(\mathfrak{g}^*)$-weight module algebra satisfying the following equations

$$
\begin{aligned}
(a \otimes b)(a' \otimes b') &= q^{(|a'|,|b|)} aa' \otimes bb' \\
K_i^{\pm\frac{1}{2}} \triangleright (a \otimes b) &= K_i^{\pm\frac{1}{2}}(a) \otimes K_i^{\pm\frac{1}{2}}(b) \\
F_{i,1} \triangleright (a \otimes b) &= K_i^{-1}(a) \otimes F_{i,1}(b) \\
F_{i,2} \triangleright (a \otimes b) &= F_{i,2}(a) \otimes K_i^{-1}(b).
\end{aligned}
$$

for $i \in I$ and weight vectors $a, a' \in A$ and $b, b' \in B$.

Proposition 3.18 is proved in Section 3.2.9. and induces a fusion product on $\underline{\mathcal{C}}_{\mathfrak{g}}^q$:

$$(A, \varphi_A) \diamond (B, \varphi_B) := (((A, \varphi_A)^+ \otimes (B, \varphi_B)^+) \otimes \mathcal{A}_q[U], 1 \otimes 1 \otimes \mathrm{id}).$$

Just like for $*$ and $\star$, there is no unit object for $\diamond$, so it is not a monoidal tensor product.

3.2. Proofs

In many proofs, we will use the fact that $\mathbb{Z}_{\geq 0}^m$ is well-ordered by the lexicographic order. For given $w \in W$, $\mathbf{i} \in R(w)$, and $U_q(\mathfrak{b}_+)$-module M, we have that $\nu_{\mathbf{i}}(M)$ is well-ordered, allowing us to induct on $\nu_{\mathbf{i}}(x)$ for $x \in M$.

3.2.1. Proof of Theorem 3.3

For $\mathbf{j} \in \nu_{\mathbf{i}}(\mathcal{B})$, let $b_{\mathbf{j}}$ be the unique element of $\mathcal{B}$ such that $\nu_{\mathbf{i}}(b_{\mathbf{j}}) = \mathbf{j}$.

1. We first observe that since A is a $U_q(\mathfrak{g})$-module algebra and A^+ and A_0 are $U_q(\mathfrak{b}_+)$-submodules, μ is a homomorphism of $U_q(\mathfrak{b}_+)$-modules. Hence we simply show that μ is injective. Now each nonzero element $a \in A^+ \otimes A_0$ can be written

$$a = \sum_{k=1}^{n} a_k \otimes b_{\mathbf{j}_k}$$

for some $n > 0$, $a_k \in A^+ \setminus \{0\}$, and $\mathbf{j}_k \in \nu_{\mathbf{i}}(A_0 \setminus \{0\})$. We may assume $\mathbf{j}_k < \mathbf{j}_l$ if $1 \leq k < l \leq n$ so that

$$E_{\mathbf{i}}^{\mathbf{j}_l}(b_{\mathbf{j}_k}) = \begin{cases} 0 & \text{if } k < l \\ 1 & \text{if } k = \ell \end{cases}.$$

Suppose for the sake of contradiction that $\mu(a) = 0$. Then

$$0 = E_{\mathbf{i}}^{(\mathbf{j}_n)}(\mu(a)) = \mu\left(E_{\mathbf{i}}^{(\mathbf{j}_n)}(a)\right) = \mu(a_n \otimes 1) = a_n$$

which is a contradiction. Hence $\mu(a) = 0$ if and only if $a = 0$, showing that μ is injective.

2. ($\Rightarrow$) Suppose μ is an isomorphism. Given nonzero $a \in A$, write

$$a = \mu\left(\sum_{k=1}^{n} a_k \otimes b_{\mathbf{j}_k}\right)$$

as in (1). Then, since μ is injective, it is clear that $\nu_{\mathbf{i}}(a) = \mathbf{j}_n = \nu_{\mathbf{i}}(b_{\mathbf{j}_n})$, showing that $\nu_{\mathbf{i}}(A \setminus \{0\}) \subseteq \nu_{\mathbf{i}}(A_0 \setminus \{0\})$. But since $A_0 \subseteq A$, it follows that

$\nu_{\mathbf{i}}(A \setminus \{0\}) = \nu_{\mathbf{i}}(A_0 \setminus \{0\})$. Also, as seen above

$$E_{\mathbf{i}}^{(top)}(a) = E_{\mathbf{i}}^{(\mathbf{j}_n)}(a) = \mu(a_n \otimes 1) = a_n \in A^+,$$

so we see that A is $\mathbf{i}$-adapted.

($\Leftarrow$) Suppose that A is $\mathbf{i}$-adapted and $\nu_{\mathbf{i}}(A \setminus \{0\}) = \nu_{\mathbf{i}}(A_0 \setminus \{0\})$. By (1), we already know that μ is an injective $U_q(\mathfrak{b}_+)$-module homomorphism. Hence we simply use induction to show that μ is surjective. We first note that since A is $\mathbf{i}$-adapted, if $\nu_{\mathbf{i}}(a) = (0, 0, \ldots, 0)$, then $a = E_{\mathbf{i}}^{(top)}(a) \in A^+$. In other words,

$$\{a \in A \setminus \{0\} \mid \nu_{\mathbf{i}}(a) = (0, 0, \ldots, 0)\} = A^+ \setminus \{0\} \subset \mu(A^+ \otimes A_0).$$

Let $a \in A \setminus \{0\}$ and suppose $a' \in \mu(A^+ \otimes A_0)$ for all $a' \in A \setminus \{0\}$ such that $\nu_{\mathbf{i}}(a') < \nu_{\mathbf{i}}(a)$. We have

$$E_{\mathbf{i}}^{(\nu_{\mathbf{i}}(a))}(a - \mu(E_{\mathbf{i}}^{(top)}(a) \otimes b_{\nu_{\mathbf{i}}(a)})) = 0.$$

Hence either $a - \mu(E_{\mathbf{i}}^{(top)}(a) \otimes b_{\nu_{\mathbf{i}}(a)}) = 0$ or $\nu_{\mathbf{i}}(a - \mu(E_{\mathbf{i}}^{(top)}(a) \otimes b_{\nu_{\mathbf{i}}(a)})) < \nu_{\mathbf{i}}(a)$. In the former case, $a \in \mu(A^+ \otimes A_0)$. In the latter case,

$$a - \mu(E_{\mathbf{i}}^{(top)}(a) \otimes b_{\nu_{\mathbf{i}}(a)}) \in \mu(A^+ \otimes A_0)$$

and so

$$a = (a - \mu(E_{\mathbf{i}}^{(top)}(a) \otimes b_{\nu_{\mathbf{i}}(a)})) + \mu(E_{\mathbf{i}}^{(top)}(a) \otimes b_{\nu_{\mathbf{i}}(a)}) \in \mu(A^+ \otimes A_0).$$

So we have shown that $a \in \mu(A^+ \otimes A_0)$. By induction, μ is surjective. Hence μ is an isomorphism. $\square$

3.2.2. Proof of Theorem 3.4

($\Leftarrow$) Suppose A_0 possesses an $\mathbf{i}$-adapted basis and $\nu_{\mathbf{i}}(A_0 \setminus \{0\}) = \nu_{\mathbf{i}}(A \setminus \{0\})$. Then by Theorem 3.3, $\mu : A^+ \otimes A_0 \to A$ is an isomorphism of $U_q(\mathfrak{b}_+)$-modules.

($\Rightarrow$) Suppose $\mu : A^+ \otimes A_0 \to A$ is an isomorphism of $U_q(\mathfrak{b}_+)$-modules. Hence we must have $(A_0)^+ = \mathbb{Q}(q^{\frac{1}{2d}})$ or else μ would fail to be injective. Also, since A is $\mathbf{i}$-adapted, A_0 is as well. Now for each $\mathbf{j} \in \nu_{\mathbf{i}}(A_0 \setminus \{0\})$ choose $b_{\mathbf{j}} \in A_0 \setminus \{0\}$ such that $E_{\mathbf{i}}^{(top)}(b_{\mathbf{j}}) = 1$ and $\nu_{\mathbf{i}}(b_{\mathbf{j}}) = \mathbf{j}$ (note that $b_{(0,\cdots,0)} = 1$). We claim that $\mathcal{B} = \{b_{\mathbf{j}} \mid \mathbf{j} \in \nu_{\mathbf{i}}(A_0 \setminus \{0\})\}$ is an $\mathbf{i}$-adapted basis for A_0. To prove that $\mathcal{B}$ is linearly independent and spans A_0, we mimic the proofs that μ is injective and surjective (respectively) in Theorem 3.3. Suppose

$$\sum_{k=1}^{n} r_k b_{\mathbf{j}_k} = 0$$

for some $r_k \in \mathbb{Q}(q^{\frac{1}{2d}})$ and $\mathbf{j}_k \in \nu_{\mathbf{i}}(A_0 \setminus \{0\})$ such that $\mathbf{j}_k < \mathbf{j}_l$ if $k < l$. Then

$$0 = E_{\mathbf{i}}^{(\mathbf{j}_n)} \left(\sum_{k=1}^{n} r_k b_{\mathbf{j}_k} \right) = r_n.$$

By induction, each $r_k = 0$. It follows that $\mathcal{B}$ is linearly independent. Note that

$$\{a \in A_0 \setminus \{0\} \mid \nu_{\mathbf{i}}(a) = (0, 0, \ldots, 0)\} = (A_0)^+ \setminus \{0\} = \mathbb{Q}(q^{\frac{1}{2d}})^{\times} \subseteq \operatorname{span}_{\mathbb{Q}(q^{\frac{1}{2d}})}(\mathcal{B}).$$

Let $a \in A_0 \setminus \{0\}$ and suppose $a' \in \operatorname{span}_{\mathbb{Q}(q^{\frac{1}{2d}})}(\mathcal{B})$ for all $a' \in A_0 \setminus \{0\}$ such that $\nu_{\mathbf{i}}(a') < \nu_{\mathbf{i}}(a)$. We have

$$E_{\mathbf{i}}^{(\nu_{\mathbf{i}}(a))}(a - E_{\mathbf{i}}^{(top)}(a)b_{\nu_{\mathbf{i}}(a)}) = 0.$$

Hence either $a - E_{\mathbf{i}}^{(top)}(a)b_{\nu_{\mathbf{i}}(a)} = 0$ or $\nu_{\mathbf{i}}(a - E_{\mathbf{i}}^{(top)}(a)b_{\nu_{\mathbf{i}}(a)}) < \nu_{\mathbf{i}}(a)$. In the former case $a \in \operatorname{span}_{\mathbb{Q}(q^{\frac{1}{2d}})}(\mathcal{B})$. In the latter case $a - E_{\mathbf{i}}^{(top)}(a)b_{\nu_{\mathbf{i}}(a)} \in \operatorname{span}_{\mathbb{Q}(q^{\frac{1}{2d}})}(\mathcal{B})$ and so

$$a = (a - E_{\mathbf{i}}^{(top)}(a)b_{\nu_{\mathbf{i}}(a)}) + E_{\mathbf{i}}^{(top)}(a)b_{\nu_{\mathbf{i}}(a)} \in \operatorname{span}_{\mathbb{Q}(q^{\frac{1}{2d}})}(\mathcal{B}).$$

So we have shown that $a \in \operatorname{span}_{\mathbb{Q}(q^{\frac{1}{2d}})}(\mathcal{B})$. By induction, $\mathcal{B}$ spans A_0. Hence we have shown that $\mathcal{B}$ is a basis for A_0. By construction, it is in fact an $\mathbf{i}$-adapted basis for A_0.

In light of $\mathcal{B}$'s existence, a typical element of A is of the form $\mu\left(\sum_{k=1}^{n} a_k \otimes b_{\mathbf{j}_k}\right)$ for some $a_k \in A^+$ and $\mathbf{j}_k \in \nu_{\mathbf{i}}(A_0 \setminus \{0\})$. It is now clear that

$$\nu_{\mathbf{i}}\left(\mu\left(\sum_{k=1}^{n} a_k \otimes b_{\mathbf{j}_k}\right)\right) = \max\{\mathbf{j}_k \mid k = 1, \ldots, n\}$$

so that $\nu_{\mathbf{i}}(A_0 \setminus \{0\}) \supseteq \nu_{\mathbf{i}}(A \setminus \{0\})$. Since $A_0 \subseteq A$, we have $\nu_{\mathbf{i}}(A_0 \setminus \{0\}) \subseteq \nu_{\mathbf{i}}(A \setminus \{0\})$ and so $\nu_{\mathbf{i}}(A_0 \setminus \{0\}) = \nu_{\mathbf{i}}(A \setminus \{0\})$. $\square$

3.2.3. Proof of Proposition 3.5

We have already observed that for any $\mathbf{i} \in R(w_{\mathrm{o}})$, the restriction of $\nu_{\mathbf{i}}$ to $\mathcal{B}^{dual}$ is an injective map $\mathcal{B}^{dual} \hookrightarrow \mathbb{Z}_{\geq 0}^m$, where m is the length of w_{o}. Hence it suffices to show that $E_{\mathbf{i}}^{(top)}(b) = 1$ for all $\mathbf{i} \in R(w_{\mathrm{o}})$ and $b \in \mathcal{B}^{dual}$. To do this we need the following lemma.

Lemma 3.19. Given $w \in W$ and $\mathbf{i}, \mathbf{i}' \in R(w)$, $E_{\mathbf{i}}^{(top)}(b) = E_{\mathbf{i}'}^{(top)}(b)$ for all $b \in \mathcal{B}^{dual}$.

Proof. Now $\mathcal{A}_q[U]$ factors as the product of two subalgebras:

$$\mathcal{A}_q[U] = (\mathcal{A}_q[U]_{>w})(\mathcal{A}_q[U]_{\leq w})$$

(see [19], for example, where they are respectively denoted $\mathbf{U}_q^-(> w, -1)$ and $\mathbf{U}_q^-(\leq w, -1)$). In fact, these subalgebras can be described explicitly as follows. For any reduced word $\mathbf{i} \in R(w_\mathrm{o})$ such that $s_{i_1} \cdots s_{i_k} = w$, consider elements $X_1, \cdots, X_m$ as in [3, Section 4], where m is the length of w_o. This choice guarantees that monomials $X^{\mathbf{a}} = X_1^{a_1} \cdots X_m^{a_m}$ for $\mathbf{a} \in \mathbb{Z}_{\geq 0}^m$ form a basis for $\mathcal{A}_q[U]$. It follows that those $X^{\mathbf{a}}$ with $a_\ell = 0$ for $\ell > k$ form a basis for $\mathcal{A}_q[U]_{\leq w}$ and those $X^{\mathbf{a}}$ with $a_\ell = 0$ for $\ell \leq k$ form a basis for $\mathcal{A}_q[U]_{>w}$. Since $X_1 = x_{i_1}$ and these two subalgebras are orthogonal with respect to Lusztig's pairing (under which multiplication by x_i and action by E_i are adjoint), we obtain the following well-known fact:

$$E_i(\mathcal{A}_q[U]_{>w}) = 0$$

for any $i \in I$ such that $\ell(s_i w) < \ell(w)$. In particular, this implies that

$$E_i(\mathcal{A}_q[U]_{>w_{i,j}}) = E_j(\mathcal{A}_q[U]_{>w_{i,j}}) = 0,$$

where $w_{i,j}$ is the longest element in the subgroup generated by s_i and s_j.

It is well-known that any two reduced words for a fixed $w \in W$ are related by a series of rank two relations. Hence it suffices to show the lemma when $\mathbf{i}$ and $\mathbf{i}'$ differ by a single rank two relation. But it is also well-known that $E_j^{(top)}(b) \in \mathcal{B}^{dual}$ for all $j \in I$ and $b \in \mathcal{B}^{dual}$. For any $\mathbf{j} \in R(w)$, the operator $E_{\mathbf{j}}^{(top)}$ is by definition just the composition of operators $E_{j_\ell}^{(top)} \cdots E_{j_2}^{(top)} E_{j_1}^{(top)}$, where ℓ is the length of w. This reduces the problem to the case when w is the longest element of a rank two parabolic subgroup of W. We will therefore assume for the rest of the proof that $\mathbf{i} = (i, j, \ldots)$ and $\mathbf{i}' = (j, i, \ldots)$ the only two distinct reduced words for $w_{i,j}$. An explicit (and apparently well-known) computation verifies that

$$E_{\mathbf{i}}^{(top)}(\mathcal{A}_q[U]_{\leq w_{i,j}} \cap \mathcal{B}^{dual}) = E_{\mathbf{i}'}^{(top)}(\mathcal{A}_q[U]_{\leq w_{i,j}} \cap \mathcal{B}^{dual}) = 1.$$

According to [19, Theorem 3.14], for each $b \in \mathcal{B}^{dual}$, there exist $b' \in \mathcal{A}_q[U]_{>w_{i,j}} \cap \mathcal{B}^{dual}$, $b'' \in \mathcal{A}_q[U]_{\leq w_{i,j}} \cap \mathcal{B}^{dual}$, and $\xi \in \mathcal{A}_q[U]$ such that $\nu_{\mathbf{i}}(\xi) < \nu_{\mathbf{i}}(b)$, $\nu_{\mathbf{i}'}(\xi) < \nu_{\mathbf{i}'}(b)$, and

$$b'b'' = b + \xi.$$

Hence

$$E_{\mathbf{i}}^{(top)}(b) = E_{\mathbf{i}}^{(top)}(b + \xi) = E_{\mathbf{i}}^{(top)}(b'b'') = b'E_{\mathbf{i}}^{(top)}(b'') = b'.$$

Likewise, $E_{\mathbf{i}'}^{(top)}(b) = b'$, so the lemma is proved. □

In light of Lemma 3.19, given $w \in W$ and $b \in \mathcal{B}^{dual}$, we may unambiguously define $E_w^{(top)}(b) := E_{\mathbf{i}}^{(top)}(b)$ for any $\mathbf{i} \in R(w)$. Now given $j \in I$, there exists some $\mathbf{i} \in R(w_\circ)$ such that if $\mathbf{i} = (i_1, \ldots, i_m)$, then $i_m = j$. It follows that $E_j(E_{w_\circ}^{(top)}(b)) = 0$ for all $j \in I$, i.e. $E_{w_\circ}^{(top)}(b) \in (\mathcal{A}_q[U])^+ = \mathbb{Q}(q^{\frac{1}{2d}})$. Since $E_j^{(top)}(b) \in \mathcal{B}^{dual}$ for all $j \in I$ and $b \in \mathcal{B}^{dual}$, it follows that $E_{w_\circ}^{(top)}(b) = 1$ for $b \in \mathcal{B}^{dual}$. □

3.2.4. Proof of Corollary 3.7

Let $\mathbf{i} \in R(w_\circ)$. As previously remarked, $\mathcal{A}_q[U]$ possesses an $\mathbf{i}$-adapted basis. Then Theorem 3.3 (1) says that $\mu' : (A')^+ \otimes \mathcal{A}_q[U] \to A'$ is an injective homomorphism of $U_q(\mathfrak{b}_+)$-modules.

We now show that $\mu'((A')^+ \otimes \mathcal{A}_q[U])$ is a $U_q(\mathfrak{g})$-module subalgebra of A' and hence is equal to A' by the assumption that $(A')^+$ generates A'. To see that $\mu'((A')^+ \otimes \mathcal{A}_q[U])$ is a subalgebra of A', it suffices to show that $x_i a \in \mu'((A')^+ \otimes \mathcal{A}_q[U])$ for $i \in I$ and $a \in (A')^+$. For this, we observe that

$$E_i(x_j a - K_j(a)x_j) = \delta_{ij} K_i(a) - \delta_{ij} K_j(a) = 0$$

and hence $x_j a - K_j(a) x_j \in (A')^+$. Then

$$x_i a = K_i(a) x_i + (x_i a - K_i(a) x_i) = \mu'(K_i(a) \otimes x_i + (x_i a - K_i(a) x_i) \otimes 1) \in \mu'((A')^+ \otimes \mathcal{A}_q[U]).$$

So $\mu'((A')^+ \otimes \mathcal{A}_q[U])$ is a subalgebra of A'.

Now we need to show that $\mu'((A')^+ \otimes \mathcal{A}_q[U])$ is closed under the action of $U_q(\mathfrak{g})$. By Theorem 3.3 (1), μ' is $U_q(\mathfrak{b}_+)$-equivariant and hence it suffices to show that $\mu'((A')^+ \otimes \mathcal{A}_q[U])$ is closed under the action of F_i for $i \in I$. Observe that for $a \in (A')^+$ and $x \in \mathcal{A}_q[U]$, $\mu'(a \otimes x) = ax$, so we will simply compute the action of F_i on such an element. However, before doing so, we note that for $a \in (A')^+$ and $i, j \in I$, we have

$$E_i \left(F_j(a) + x_j \frac{K_j^{-2}(a) - a}{q_j - q_j^{-1}} \right) = \delta_{ij} \frac{K_i(a) - K_i^{-1}(a)}{q_i - q_i^{-1}} + \delta_{ij} \frac{K_i^{-1}(a) - K_i(a)}{q_i - q_i^{-1}} = 0,$$

showing that $F_j(a) + x_j \dfrac{K_j^{-2}(a) - a}{q_i - q_i^{-1}} \in (A')^+$. Now we compute:

$$
\begin{aligned}
F_i(ax) &= F_i(a)x + K_i^{-1}(a)F_i(x) \\
&= \left(F_i(a) + x_i\frac{K_i^{-2}(a)-a}{q_i-q_i^{-1}}\right)x - x_i\frac{K_i^{-2}(a)-a}{q_i-q_i^{-1}}x + K_i^{-1}(a)F_i(x) \\
&= \mu'\left(\left(F_i(a) + x_i\frac{K_i^{-2}(a)-a}{q_i-q_i^{-1}}\right)\otimes x + K_i^{-1}(a)\otimes F_i(x)\right) \\
&\quad - \mu'(1\otimes x_i)\mu'\left(\frac{K_i^{-2}(a)-a}{q_i-q_i^{-1}}\otimes x\right) \\
&\in \mu'((A')^+\otimes \mathcal{A}_q[U]).
\end{aligned}
$$

So $\mu'((A')^+ \otimes \mathcal{A}_q[U])$ is closed under the action of $U_q(\mathfrak{g})$ and we may conclude that $\mu'((A')^+ \otimes \mathcal{A}_q[U]) = A'$. Hence μ' is an isomorphism. By Theorem 3.3 (2), this implies that A' is $\mathbf{i}$-adapted and $\nu_{\mathbf{i}}(A' \setminus \{0\}) = \nu_{\mathbf{i}}(\mathcal{A}_q[U] \setminus \{0\})$. We deduce that A is $\mathbf{i}$-adapted and

$$\nu_{\mathbf{i}}(\mathcal{A}_q[U] \setminus \{0\}) \subseteq \nu_{\mathbf{i}}(A \setminus \{0\}) \subseteq \nu_{\mathbf{i}}(A' \setminus \{0\}) = \nu_{\mathbf{i}}(\mathcal{A}_q[U] \setminus \{0\}),$$

i.e. $\nu_{\mathbf{i}}(\mathcal{A}_q[U] \setminus \{0\}) = \nu_{\mathbf{i}}(A \setminus \{0\})$. Hence applying Theorem 3.3 (2) again, μ is an isomorphism. $\square$

3.2.5. Proof of Theorem 1.1

Before showing the factorizability of $\mathcal{A}_q[G/U]$ after localization, we recall the definition of right Ore sets (which allow for Ore localizations and are sometimes also called right denominator sets) for the reader's convenience.

Definition 3.20. Let R be any unital ring. A submonoid $\mathcal{S} \subset R \setminus \{0\}$ is called a *right Ore set* if the following conditions are satisfied for $r \in R$ and $s \in \mathcal{S}$:

1. $r\mathcal{S} \cap sR \neq \varnothing$.

2. If $sr = 0$, then $\exists s' \in \mathcal{S}$ such that $rs' = 0$.

Recall (see, e.g., [17]) that an element p of a ring R is *normal* if $pR = Rp$. It is immediate (and well-known) that for any ring R, any submonoid $\mathcal{S} \subset R \setminus \{0\}$ consisting of normal elements that aren't zero-divisors is automatically both right and left Ore. In what follows, we will refer to these as *normal* Ore sets. In particular, $\mathcal{S} = \{v_\lambda \mid \lambda \in \mathcal{P}^+\} \subset \mathcal{A}_q[G/U]$ is a normal Ore set and the Ore localization $(\mathcal{A}_q[G/U])[\mathcal{S}^{-1}]$ is isomorphic to $\mathcal{A}_q[B]$ as $U_q(\mathfrak{g})$-module algebras. The following lemmas allow us to create normal Ore sets in the n-fold braided tensor product $\mathcal{A}_q[G/U]^{\underline{\otimes} n}$.

Lemma 3.21. Let $\Bbbk$ be any field and suppose A and B are $\Bbbk$-algebras such that the $\Bbbk$-vector space $A \otimes_\Bbbk B$ has the structure of a $\Bbbk$-algebra satisfying

$$(a \otimes 1)(a' \otimes b') = aa' \otimes b', \ (a \otimes b)(1 \otimes b') = a \otimes bb'$$

for $a, a' \in A$ and $b, b' \in B$. If $\mathcal{S}$ is a normal Ore set in B such that

$$(1 \otimes s)((A \setminus \{0\}) \otimes 1) = ((A \setminus \{0\}) \otimes 1)(1 \otimes s)$$

for $s \in \mathcal{S}$, then $1 \otimes \mathcal{S} := \{1 \otimes s \mid s \in \mathcal{S}\}$ is a normal Ore set in $A \otimes_\Bbbk B$.

Proof. It is clear that $1 \otimes \mathcal{S}$ is a multiplicative set containing $1 \otimes 1$ and that

$$(1 \otimes s)(A \otimes_\Bbbk B) = (A \otimes_\Bbbk B)(1 \otimes s)$$

for $s \in \mathcal{S}$, so we simply show that $1 \otimes \mathcal{S}$ does not contain any zero-divisors. Fix $s \in \mathcal{S}$. Now an arbitrary nonzero element $x \in A \otimes_\Bbbk B$ can be written in the form

$x = \sum\limits_{k=1}^{n} a_k \otimes b_k$ for some $a_k \in A \setminus \{0\}$ and $b_k \in B \setminus \{0\}$. We may assume that $\{b_k\}_{k=1}^n$ is a linearly independent set. Since s is not a zero-divisor in B, it follows that $\{sb_k\}_{k=1}^n$ is a linearly independent set, as is $\{b_k s\}_{k=1}^n$. Also, by assumption, for each $k = 1, \ldots, n$, there exists $a'_k \in A \setminus \{0\}$ such that $(1 \otimes s)(a_k \otimes 1) = a'_k \otimes s$. Then

$$(1 \otimes s)x = (1 \otimes s)\left(\sum_{k=1}^{n} a_k \otimes b_k\right) = \sum_{k=1}^{n} a'_k \otimes sb_k \neq 0,$$

$$x(1 \otimes s) = \left(\sum_{k=1}^{n} a_k \otimes b_k\right)(1 \otimes s) = \sum_{k=1}^{n} a_k \otimes b_k s \neq 0.$$

Since x was an arbitrary element of $A \otimes_{\Bbbk} B$, we have shown that $1 \otimes s$ is not a zero-divisor in $A \otimes_{\Bbbk} B$. □

Lemma 3.22. Let A and B be $U_q(\mathfrak{g})$-weight module algebras and let $\mathcal{S}$ be a normal Ore set in B consisting of highest weight vectors. Then $1 \otimes \mathcal{S}$ is a normal Ore set in the braided tensor product $A \underline{\otimes} B$.

Proof. In light of Lemma 3.21, it suffices to show that

$$(1 \otimes s)((A \setminus \{0\}) \otimes 1) = ((A \setminus \{0\}) \otimes 1)(1 \otimes s)$$

for $s \in \mathcal{S}$. Since $\mathcal{S}$ consists of highest weight vectors in B, we have the commutation relation

$$(1 \otimes s)(a \otimes 1) = q^{(|a|,|s|)} a \otimes s$$

for weight vectors $a \in A$ and $s \in \mathcal{S}$ of weight $|a|$ and $|s|$, respectively. Let us denote $q_{s,a} := q^{(|a|,|s|)}$. Now an arbitrary nonzero element $a \in A \setminus \{0\}$ is of the form $\sum_{k=1}^{n} a_k$, where each $a_k \in A \setminus \{0\}$ is a weight vector. We may assume $|a_k| \neq |a_l|$ if $k \neq l$. Then for $s \in \mathcal{S}$,

$$\sum_{k=1}^{n} q_{s,a_k} a_k \neq 0 \quad \text{and} \quad \sum_{k=1}^{n} q_{s,a_k}^{-1} a_k \neq 0.$$

Therefore since

$$(1 \otimes s)\left(\left(\sum_{k=1}^{n} a_k\right) \otimes 1\right) = \left(\left(\sum_{k=1}^{n} q_{s,a_k} a_k\right) \otimes 1\right)(1 \otimes s) \quad \text{and}$$
$$\left(\left(\sum_{k=1}^{n} a_k\right) \otimes 1\right)(1 \otimes s) = (1 \otimes s)\left(\left(\sum_{k=1}^{n} q_{s,a_k}^{-1} a_k\right) \otimes 1\right),$$

it follows that $(1 \otimes s)((A \setminus \{0\}) \otimes 1) = ((A \setminus \{0\}) \otimes 1)(1 \otimes s)$ for $s \in \mathcal{S}$ and so the lemma is proven. □

By Lemma 3.22 and induction, $\mathcal{S}' := 1 \otimes \cdots \otimes 1 \otimes \mathcal{S}$ is a normal Ore set in $\mathcal{A}_q[G/U]^{\underline{\otimes} n}$. Furthermore, it is clear that

$$\mathcal{A}_q[G/U]^{\underline{\otimes} n}[\mathcal{S}'^{-1}] \cong \mathcal{A}_q[G/U]^{\underline{\otimes}(n-1)} \underline{\otimes} \mathcal{A}_q[B]$$

as $U_q(\mathfrak{g})$-module algebras and $\mathcal{A}_q[G/U]^{\underline{\otimes}(n-1)} \underline{\otimes} \mathcal{A}_q[B]$ is generated by

$$(\mathcal{A}_q[G/U]^{\underline{\otimes}(n-1)} \underline{\otimes} \mathcal{A}_q[B])^{+}$$

as a $U_q(\mathfrak{g})$-module algebra. We now have an embedding of $U_q(\mathfrak{g})$-module algebras

$$\mathcal{A}_q[U] \hookrightarrow \mathcal{A}_q[B] \subset \mathcal{A}_q[G/U]^{\underline{\otimes}(n-1)} \underline{\otimes} \mathcal{A}_q[B].$$

Then by Corollary 3.7, $\mathcal{A}_q[G/U]^{\underline{\otimes}(n-1)} \underline{\otimes} \mathcal{A}_q[B] \cong (\mathcal{A}_q[G/U]^{\underline{\otimes} n})[\mathcal{S}'^{-1}]$ is factorizable over $\mathcal{A}_q[U]$. □

3.2.6. Proofs of Theorems 3.9 and 3.12

Let H be a Hopf algebra with invertible antipode (e.g. $H = \mathcal{K}$). We will refer to Yetter-Drinfeld modules of various kinds: ${}^H_H\mathcal{YD}$, ${}_H\mathcal{YD}^H$, and $\mathcal{YD}^H_H$ (see, e.g., [12, Section 2]). The side of the subscript denotes the side on which H will act, while the side of the superscript denotes the side on which H will coact. We use sumless Sweedler notation to write left coactions $x \mapsto x^{(-1)} \otimes x^{(0)}$ and right coactions $x \mapsto x^{(0)} \otimes x^{(1)}$. To distinguish the structure maps of a Nichols algebra (a Hopf algebra in the appropriate Yetter-Drinfeld category, see for example [2]) from those of H, we underline them. For instance, we write the braided comultiplication $\underline{\Delta}(b) = \underline{b}_{(1)} \otimes \underline{b}_{(2)}$.

We start with some results that will play key roles in the proofs of the Theorems 3.9 and 3.12.

Theorem 3.23. Let A be a left H-module algebra and suppose $V \in {}^H_H\mathcal{YD}$ is such that the Nichols algebra $\mathcal{B}(V)$ is a left H-module subalgebra of A, where ${}^H_H\mathcal{YD}$ is the category of left-left Yetter-Drinfeld modules over H. Then A can be given a left $\mathcal{B}(V)$-module structure via

$$v \triangleright a = va - (v^{(-1)}(a))v^{(0)}.$$

Proof. Consider the Hopf algebra $\widetilde{H} := \mathcal{B}(V) \rtimes H$, where

$$\Delta(u) = \underline{u}_{(1)}(\underline{u}_{(2)})^{(-1)} \otimes (\underline{u}_{(2)})^{(0)}$$

and $S(u) = S(u^{(-1)})\underline{S}(u^{(0)})$ for $u \in \mathcal{B}(V)$. Then $\widetilde{H}$ can naturally be considered as a subalgebra of $\widetilde{A} := A \rtimes H$. Hence $\widetilde{A}$ is an $\widetilde{H}$-module algebra under the adjoint action:

$$\widetilde{h} \triangleright \widetilde{a} = \widetilde{h}_{(1)}\widetilde{a}S(\widetilde{h}_{(2)}).$$

We observe that A is preserved under the restriction of this action to $\mathcal{B}(V)$ (note that for convenience we will write, e.g., a instead of $a \otimes 1$):

$$\begin{aligned}
u \triangleright a &= u_{(1)}aS(u_{(2)}) \\
&= \underline{u}_{(1)}(\underline{u}_{(2)})^{(-1)}aS(((\underline{u}_{(2)})^{(0)})^{(-1)})\underline{S}(((\underline{u}_{(2)})^{(0)})^{(0)}) \\
&= \underline{u}_{(1)}(\underline{u}_{(2)})^{(-2)}aS((\underline{u}_{(2)})^{(-1)})\underline{S}((\underline{u}_{(2)})^{(0)}) \\
&= \underline{u}_{(1)}((\underline{u}_{(2)})^{(-3)}(a))(\underline{u}_{(2)})^{(-2)}S((\underline{u}_{(2)})^{(-1)})\underline{S}((\underline{u}_{(2)})^{(0)}) \\
&= \underline{u}_{(1)}((\underline{u}_{(2)})^{(-2)}(a))\epsilon((\underline{u}_{(2)})^{(-1)})\underline{S}((\underline{u}_{(2)})^{(0)}) \\
&= \underline{u}_{(1)}((\underline{u}_{(2)})^{(-1)}(a))\underline{S}((\underline{u}_{(2)})^{(0)}) \in A
\end{aligned}$$

for $u \in \mathcal{B}(V)$ and $a \in A$. In fact, it is clear that A has become a left $\widetilde{H}$-module algebra. Now computing the given action for $v \in V$ and $a \in A$, we find

$$v \triangleright a = \underline{v}_{(1)}((\underline{v}_{(2)})^{(-1)}(a))\underline{S}((\underline{v}_{(2)})^{(0)}) = va + v^{(-1)}(a)\underline{S}(v^{(0)}) = va - v^{(-1)}(a)v^{(0)},$$

as required. The second and third equalities follow from the fact that every element of V is a primitive element of the braided Hopf algebra $\mathcal{B}(V)$. $\square$

Of course, Theorem 3.23 has a natural counterpart with "left" replaced by "right".

Theorem 3.24. Let A be a right H-module algebra and suppose $V \in \mathcal{YD}_H^H$ is such that the Nichols algebra $\mathcal{B}(V)$ is a right H-module subalgebra of A. Then A can be given a right $\mathcal{B}(V)$-module structure via

$$a \blacktriangleleft v = av - v^{(0)}((a)v^{(1)}).$$

Given any ring R, a right R-module is naturally a left R^{op}-module, giving us the following obvious corollary.

Corollary 3.25. In the assumptions of Theorem 3.24, if H is commutative, then A can be given a left $\mathcal{B}(V)^{op}$-module structure via

$$v \blacktriangleright a = av - v^{(0)}(v^{(1)}(a)).$$

Remark 3.26. If A is a $(\mathcal{B}(V) \rtimes H)$-module algebra (e.g. Theorem 3.23), then we can form the *braided cross product* $A \underline{\rtimes} \mathcal{B}(V)$ which, as a vector space, is just $A \otimes \mathcal{B}(V) \subset A \rtimes (\mathcal{B}(V) \rtimes H)$ and it is a subalgebra. Furthermore, it is an H-module algebra. We note that if A is additionally a $\mathcal{B}(V)$-module algebra in ${}^H_H\mathcal{YD}$, then our definition of $A \underline{\rtimes} \mathcal{B}(V)$ matches that of $A \rtimes \mathcal{B}(V)$. However, we don't require that A is even an H-comodule, which is why we use a different notation. Similarly, we can form the braided tensor product $A \underline{\otimes} \mathcal{B}(V)$ (which is an H-module algebra) even if A is an H-module algebra and is not in ${}^H_H\mathcal{YD}$, simply satisfying $(1 \otimes v)(a \otimes 1) = (v^{(-1)} \triangleright a) \otimes v^{(0)}$. This corresponds to the braided cross product $A \underline{\rtimes} \mathcal{B}(V)$, where $\mathcal{B}(V) \rtimes H$ acts on A by the "trivial" action: $(u \otimes h) \triangleright a = \underline{\epsilon}(u) h \triangleright a \quad \text{for } u \in \mathcal{B}(V),\ h \in H,\ a \in A$.

Theorem 3.27. Let $V \in {}^H_H\mathcal{YD}$ and suppose A is an H-module algebra containing $\mathcal{B}(V)$ as an H-module subalgebra. Then the linear map

$$\tau := (\mu_A \otimes id) \circ (id \otimes \iota \otimes id) \circ (id \otimes \underline{\Delta}) : A \underline{\rtimes} \mathcal{B}(V) \to A \underline{\otimes} \mathcal{B}(V)$$

is an H-module algebra isomorphism with inverse

$$\tau^{-1} = (\mu_A \otimes id) \circ (id \otimes \iota \otimes id) \circ (id \otimes \underline{S} \otimes id) \circ (id \otimes \underline{\Delta})$$

, where $\iota : \mathcal{B}(V) \to A$ is the inclusion and the implied $\mathcal{B}(V) \rtimes H$ action on A is that of Theorem 3.23.

Proof. We first verify that τ and τ^{-1} are truly mutually inverse (and hence that we are justified in using the name τ^{-1}). For $a \in A$ and $b \in \mathcal{B}(V)$, we directly compute

$$\begin{aligned}
(\tau \circ \tau^{-1})(a \otimes b) &= \tau(a\underline{S}(\underline{b}_{(1)}) \otimes \underline{b}_{(2)}) \\
&= aS(\underline{b}_{(1)})\underline{b}_{(2)} \otimes \underline{b}_{(3)} \\
&= a\underline{\epsilon}(\underline{b}_{(1)}) \otimes \underline{b}_{(2)} \\
&= a \otimes \underline{\epsilon}(\underline{b}_{(1)})\underline{b}_{(2)} = a \otimes b,
\end{aligned}$$

$$\begin{aligned}
(\tau^{-1} \circ \tau)(a \otimes b) &= \tau^{-1}(a\underline{b}_{(1)} \otimes \underline{b}_{(2)}) \\
&= a\underline{b}_{(1)}\underline{S}(\underline{b}_{(2)}) \otimes \underline{b}_{(3)} \\
&= a\underline{\epsilon}(\underline{b}_{(1)}) \otimes \underline{b}_{(2)} \\
&= a \otimes \underline{\epsilon}(\underline{b}_{(1)})\underline{b}_{(2)} = a \otimes b.
\end{aligned}$$

Since $\tau \circ \tau^{-1}$ and $\tau^{-1} \circ \tau$ act as the identity on pure tensors, they are both the identity homomorphism. Hence τ and τ^{-1} are mutually inverse. We conclude by verifying that τ is actually a homomorphism of algebras (and hence that τ^{-1} is as well). For $v \in V$, $a, a' \in A$, and $b \in \mathcal{B}(V)$, we have

$$
\begin{aligned}
\tau(a\otimes v)\tau(a'\otimes b) &= (av\otimes 1 + a\otimes v)(a'\underline{b}_{(1)}\otimes \underline{b}_{(2)})\\
&= ava'\underline{b}_{(1)}\otimes \underline{b}_{(2)} + a(v^{(-1)}(a'\underline{b}_{(1)}))\otimes v^{(0)}\underline{b}_{(2)}\\
&= ava'\underline{b}_{(1)}\otimes \underline{b}_{(2)} + a(v^{(-2)}(a'))(v^{(-1)}(\underline{b}_{(1)}))\otimes v^{(0)}\underline{b}_{(2)}\\
&= ava'\underline{b}_{(1)}\otimes \underline{b}_{(2)} - a(v^{(-1)}(a'))v^{(0)}\underline{b}_{(1)}\otimes \underline{b}_{(2)}\\
&\quad + a(v^{(-1)}(a'))v^{(0)}\underline{b}_{(1)}\otimes \underline{b}_{(2)} + a(v^{(-2)}(a'))(v^{(-1)}(\underline{b}_{(1)}))\otimes v^{(0)}\underline{b}_{(2)}\\
&= \tau(ava'\otimes b - a(v^{(-1)}(a'))v^{(0)}\otimes b + a(v^{(-1)}(a'))\otimes v^{(0)}b)\\
&= \tau(a(va' - (v^{(-1)}(a'))v^{(0)})\otimes b + a(v^{(-1)}(a'))\otimes v^{(0)}b)\\
&= \tau(a(v\triangleright a')\otimes b + a(v^{(-1)}(a'))\otimes v^{(0)}b)\\
&= \tau((a\otimes v)(a'\otimes b)).
\end{aligned}
$$

It is clear that

$$\{b\in\mathcal{B}(V)\mid \tau(a\otimes b)\tau(a'\otimes b') = \tau((a\otimes b)(a'\otimes b'))\ \forall a,a'\in A, b'\in\mathcal{B}(V)\}$$

is a subalgebra of $A\underline{\rtimes}\mathcal{B}(V)$. We have shown it contains V, so it must be equal to $\mathcal{B}(V)$. Now, since pure tensors span $A\underline{\rtimes}\mathcal{B}(V)$ and τ is a linear map, it follows that τ respects multiplication. The theorem is proved. □

Corollary 3.28. Let $V\in {}^H_H\mathcal{YD}$. Then there are injective H-module algebra homomorphisms $\mathcal{B}(V)\to\mathcal{B}(V)\underline{\rtimes}\mathcal{B}(V)$ and $\mathcal{B}(V)\to\mathcal{B}(V)\underline{\otimes}\mathcal{B}(V)$ given by $v\mapsto 1\otimes v - v\otimes 1$ and $v\mapsto 1\otimes v + v\otimes 1$, respectively, for $v\in V$.

Proof. Let τ be as in Theorem 3.27, where $A = \mathcal{B}(V)$. Restrict τ^{-1} to $\mathcal{B}(V)\cong 1\underline{\otimes}\mathcal{B}(V)\subset\mathcal{B}(V)\underline{\otimes}\mathcal{B}(V)$ and observe that $\tau^{-1}(1\otimes v) = 1\otimes v - v\otimes 1$ for $v\in V$.

Similarly, restrict τ to $\mathcal{B}(V) \cong 1\underline{\rtimes}\mathcal{B}(V) \subset \mathcal{B}(V)\underline{\rtimes}\mathcal{B}(V)$ and note that $\tau(1 \otimes v) = 1 \otimes v + v \otimes 1$ for $v \in V$. $\square$

Theorem 3.29. Let $V \in {}^H_H\mathcal{YD}$ and set $\widetilde{H} = \mathcal{B}(V) \rtimes H$. Let A be a left $\widetilde{H}$-module algebra and suppose A contains an $\widetilde{H}$-module subalgebra isomorphic to $\mathcal{B}(V)$ with the "adjoint" action:

$$(u \otimes h) \triangleright u' = \underline{u}_{(1)}([(\underline{u}_{(2)})^{(-1)}h](u'))\underline{S}((\underline{u}_{(2)})^{(0)}) \quad \text{for } u, u' \in \mathcal{B}(V),\ h \in H.$$

Then there is a left $\mathcal{B}(V)$ action $\blacktriangleright$ on A given by

$$v \blacktriangleright a = (v \triangleright a) - [va - (v^{(-1)}(a))v^{(0)}] \quad \text{for } v \in V,\ a \in A.$$

Proof. We first observe that $\mathcal{B}(V)\underline{\rtimes}\mathcal{B}(V)$ is an H-module subalgebra of $A\underline{\rtimes}\mathcal{B}(V)$. By Corollary 3.28, the elements $1 \otimes v - v \otimes 1 \in A\underline{\rtimes}\mathcal{B}(V)$ $(v \in V)$ generate an H-module algebra isomorphic to $\mathcal{B}(V)$. Then by Theorem 3.23, we can define an action of $\mathcal{B}(V)$ on $A\underline{\rtimes}\mathcal{B}(V)$ by

$$v \cdot (a \otimes u) = (1 \otimes v - v \otimes 1)(a \otimes u) - [v^{(-1)}(a \otimes u)](1 \otimes v^{(0)} - v^{(0)} \otimes 1).$$

Now we need only observe that this action preserves $A = A\underline{\rtimes}1 \subset A\underline{\rtimes}\mathcal{B}(V)$ and acts in the prescribed manner:

$$\begin{aligned} v \cdot (a \otimes 1) &= (1 \otimes v - v \otimes 1)(a \otimes 1) - [v^{(-1)}(a \otimes 1)](1 \otimes v^{(0)} - v^{(0)} \otimes 1) \\ &= (v \triangleright a) \otimes 1 + (v^{(-1)}(a)) \otimes v^{(0)} - va \otimes 1 \\ &\quad - (v^{(-1)}(a)) \otimes v^{(0)} + (v^{(-1)}(a))v^{(0)} \otimes 1 \\ &= [(v \triangleright a) - [va - (v^{(-1)}(a))v^{(0)}]] \otimes 1. \end{aligned}$$

$\square$

Theorem 3.30. Let $V \in {}^H_H\mathcal{YD}$ and set $\widetilde{H} = \mathcal{B}(V) \rtimes H$. Let A be a left $\widetilde{H}$-module algebra and suppose A contains an $\widetilde{H}$-module subalgebra isomorphic to $\mathcal{B}(V)$ with the trivial action:

$$(u \otimes h) \triangleright u' = [\underline{\epsilon}(u)h](u') \quad \text{for } u, u' \in \mathcal{B}(V),\ h \in H.$$

Then there is a left $\mathcal{B}(V)$ action $\blacktriangleright$ on A given by

$$v \blacktriangleright a = (v \triangleright a) + [va - (v^{(-1)}(a))v^{(0)}] \quad \text{for } v \in V,\ a \in A.$$

Proof. By Corollary 3.28, the elements $1 \otimes v + v \otimes 1 \in A \underline{\rtimes} \mathcal{B}(V)$ $(v \in V)$ generate an H-module algebra isomorphic to $\mathcal{B}(V)$. Then by Theorem 3.23, we can define an action of $\mathcal{B}(V)$ on $A \underline{\rtimes} \mathcal{B}(V)$ by

$$v \cdot (a \otimes u) = (1 \otimes v + v \otimes 1)(a \otimes u) - [v^{(-1)}(a \otimes u)](1 \otimes v^{(0)} + v^{(0)} \otimes 1).$$

Now we need only observe that this action preserves $A = A \underline{\rtimes} 1 \subset A \underline{\rtimes} \mathcal{B}(V)$ and acts in the prescribed manner:

$$\begin{aligned} v \cdot (a \otimes 1) &= (1 \otimes v + v \otimes 1)(a \otimes 1) - [v^{(-1)}(a \otimes 1)](1 \otimes v^{(0)} + v^{(0)} \otimes 1) \\ &= (v \triangleright a) \otimes 1 + (v^{(-1)}(a)) \otimes v^{(0)} + va \otimes 1 \\ &\quad - (v^{(-1)}(a)) \otimes v^{(0)} - (v^{(-1)}(a))v^{(0)} \otimes 1 \\ &= [(v \triangleright a) + [va - (v^{(-1)}(a))v^{(0)}]] \otimes 1. \end{aligned}$$

□

Theorem 3.31. Let $\hat{A}$ be a left H-module algebra and suppose that $V_1 \in {}^H_H\mathcal{YD}$ and $V_2 \in {}_H\mathcal{YD}^H$ are H-submodules of $\hat{A}$. For $v_i \in V_i$, define the following actions on $\hat{A}$:

$$v_1 \triangleright a = v_1 a - v_1^{(-1)}(a)v_1^{(0)} \quad v_2 \blacktriangleright a = v_2^{(0)}v_2^{(1)}(a) - av_2.$$

If

(1) $v_1 \triangleright v_2 = v_2 \blacktriangleright v_1 = 0$ and

(2) $v_1^{(-2)}(v_2^{(0)})v_1^{(-1)}(v_2^{(1)}(a))v_1^{(0)} = v_2^{(0)}v_2^{(1)}(v_1^{(-1)}(a))v_2^{(2)}(v_1^{(0)})$

for all $v_i \in V_i$ and $a \in \hat{A}$, then $v_1 \triangleright (v_2 \blacktriangleright a) = v_2 \blacktriangleright (v_1 \triangleright a)$ for $v_i \in V_i$ and $a \in \hat{A}$.

Proof. We first note that if $v_1 \triangleright v_2 = v_2 \blacktriangleright v_1 = 0$, then

$$v_1 v_2 = v_1^{(-1)}(v_2)v_1^{(0)} = v_2^{(0)}v_2^{(1)}(v_1)$$

and for $a, b \in A$, we have $v_1 \triangleright (ab) = (v_1 \triangleright a)b + v_1^{(-1)}(a)(v_1^{(0)} \triangleright b)$ and $v_2 \blacktriangleright (ab) = (v_2^{(0)} \blacktriangleright a)v_2^{(1)}(b) + a(v_2 \triangleright b)$. Now we simply compute:

$$\begin{aligned}
v_1 \triangleright (v_2 \blacktriangleright a) &= v_1 \triangleright (v_2^{(0)}v_2^{(1)}(a) - av_2) \\
&= (v_1 \triangleright v_2^{(0)})v_2^{(1)}(a) + v_1^{(-1)}(v_2^{(0)})(v_1^{(0)} \triangleright v_2^{(1)}(a)) \\
&\quad - (v_1 \triangleright a)v_2 - v_1^{(-1)}(a)(v_1^{(0)} \triangleright v_2) \\
&= v_1^{(-1)}(v_2^{(0)})(v_1^{(0)} \triangleright v_2^{(1)}(a)) - (v_1 \triangleright a)v_2 \\
&= v_1^{(-1)}(v_2^{(0)})(v_1^{(0)}v_2^{(1)}(a) - (v_1^{(0)})^{(-1)}(v_2^{(1)}(a))(v_1^{(0)})^{(0)}) \\
&\quad - (v_1 a - v_1^{(-1)}(a)v_1^{(0)})v_2 \\
&= v_1^{(-1)}(v_2^{(0)})v_1^{(0)}v_2^{(1)}(a) - v_1^{(-2)}(v_2^{(0)})v_1^{(-1)}(v_2^{(1)}(a))v_1^{(0)} \\
&\quad - v_1 a v_2 + v_1^{(-1)}(a)v_1^{(0)}v_2,
\end{aligned}$$

$$
\begin{aligned}
v_2 \blacktriangleright (v_1 \triangleright a) &= v_2 \blacktriangleright (v_1 a - v_1^{(-1)}(a) v_1^{(0)}) \\
&= (v_2^{(0)} \blacktriangleright v_1) v_2^{(1)}(a) + v_1 (v_2 \blacktriangleright a) - (v_2^{(0)} \blacktriangleright v_1^{(-1)}(a)) v_2^{(1)}(v_1^{(0)}) \\
&\quad + v_1^{(-1)}(a)(v_2 \triangleright v_1^{(0)}) \\
&= v_1 (v_2 \blacktriangleright a) - (v_2^{(0)} \blacktriangleright v_1^{(-1)}(a)) v_2^{(1)}(v_1^{(0)}) \\
&= v_1 (v_2^{(0)} v_2^{(1)}(a) - a v_2) - ((v_2^{(0)})^{(0)} (v_2^{(0)})^{(1)} (v_1^{(-1)}(a)) \\
&\quad - v_1^{(-1)}(a) v_2^{(0)}) v_2^{(1)}(v_1^{(0)}) \\
&= v_1 v_2^{(0)} v_2^{(1)}(a) - v_1 a v_2 - v_2^{(0)} v_2^{(1)} (v_1^{(-1)}(a)) v_2^{(2)}(v_1^{(0)}) \\
&\quad + v_1^{(-1)}(a) v_2^{(0)} v_2^{(1)}(v_1^{(0)}).
\end{aligned}
$$

Comparing terms, we see that the two quantities are indeed equal. $\square$

We are now ready to prove Theorem 3.9.

Proof of Theorem 3.9. Let $V = \operatorname{span}_{\mathbb{Q}(q^{\frac{1}{2d}})}\{F_i \mid i \in I\} \subset U_q(\mathfrak{b}_-)$. Then $V \in {}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$ with structure given by

$$K_i^{\pm\frac{1}{2}} \triangleright F_j = q_i^{\mp\frac{c_{i,j}}{2}} F_j; \quad \delta_L(F_i) = K_i^{-1} \otimes F_i.$$

Let $V' = V$ as a vector space, but $V' \in {}_{\mathcal{K}}\mathcal{YD}^{\mathcal{K}}$ with structure given by

$$K_i^{\pm\frac{1}{2}} \triangleright F_j = q_i^{\mp\frac{c_{i,j}}{2}} F_j; \quad \delta_R(F_i) = F_i \otimes K_i^{-1}.$$

Note that we can also consider V' as an object of $\mathcal{YD}_{\mathcal{K}}^{\mathcal{K}}$ since $\mathcal{K}$ is commutative. It is well-known (see, e.g., [1] or [23], though Lusztig never used the term "Nichols algebra") that the corresponding Nichols algebras are isomorphic to $U_q(\mathfrak{n}_-)$ as $\mathcal{K}$-module algebras in the obvious way, i.e. $F_i \mapsto F_i$.

By assumption, $\mathcal{A}_q[U] \subset A$, so there is a natural embedding of $U_q(\mathfrak{b}_-)$-module algebras $U_q(\mathfrak{n}_-) \hookrightarrow A$ given by $F_i \mapsto \frac{x_i}{q_i - q_i^{-1}}$. Theorems 3.23 and 3.29 then imply that there is a $U_q(\mathfrak{n}_-)$ action on A given by

$$F_i \triangleright a = F_i(a) - \frac{x_i a - K_i^{-1}(a) x_i}{q_i - q_i^{-1}},$$

matching the proposed action of $F_{i,1}$.

Now utilizing a slightly different embedding $U_q(\mathfrak{n}_-) \hookrightarrow A$, $F_i \mapsto -\frac{x_i}{q_i - q_i^{-1}}$, Corollary 3.25 gives another action of $U_q(\mathfrak{n}_-) \cong U_q(\mathfrak{n}_-)^{op}$ on A:

$$F_i \triangleright a = \frac{x_i K_i^{-1}(a) - a x_i}{q_i - q_i^{-1}},$$

matching the proposed action of $F_{i,2}$.

It is easily observed that we have made A into both a $\mathcal{B}(V) \rtimes \mathcal{K}$-module algebra and a $\mathcal{B}(V') \rtimes \mathcal{K}$-module algebra.

We now wish to show that the operators $F_{i,1}$ and $F_{j,2}$ commute. To do so, we construct the braided cross product $\hat{A} := A \underline{\rtimes} U_q(\mathfrak{n}_-)$, where the F_i act as $F_{i,1}$. As above, we define "clever" embeddings of V and V' into $\hat{A}$, namely $F_i \mapsto \frac{x_i}{q_i - q_i^{-1}} \otimes 1$ and $F_i \mapsto 1 \otimes F_i$, respectively. It is easily checked that the hypotheses of Theorem 3.31 are satisfied. Furthermore, the actions defined in Theorem 3.31 preserve A and match the actions of $F_{i,1}$ and $F_{j,2}$ on A, showing that the prescribed actions of $F_{i,1}$ and $F_{j,2}$ do, in fact, commute. $\square$

In light of Theorem 3.9, $\mathcal{A}_q[U]$ is a $U_q(\mathfrak{g}^*)$-module algebra with action given by

$$K_i^{\pm\frac{1}{2}} \triangleright x_j = q_i^{\mp\frac{c_{i,j}}{2}} x_j; \quad F_{i,1} \triangleright x_j = 0; \quad F_{i,2} \triangleright x_j = \frac{q_i^{c_{i,j}} x_i x_j - x_j x_i}{q_i - q_i^{-1}}.$$

We make $\mathcal{A}_q[U]$ into a $U_q(\mathfrak{g}^*)$-comodule algebra via the algebra homomorphism $\delta : \mathcal{A}_q[U] \to U_q(\mathfrak{g}^*) \otimes \mathcal{A}_q[U]$ given on generators by

$$\delta(x_i) = K_i \otimes x_i + (q_i - q_i^{-1})F_{i,2}K_i \otimes 1.$$

The fact that this gives a well-defined algebra homomorphism follows immediately from the following lemma, which can be deduced from the fact that in [23, 1.2.6], $r : \mathbf{f} \to \mathbf{f} \otimes \mathbf{f}$, $\theta_i \mapsto \theta_i \otimes 1 + 1 \otimes \theta_i$ is well-defined.

Lemma 3.32. Let R be any $\mathbb{Q}(q^{\frac{1}{2d}})$-algebra and suppose $\{y_i\}_{i\in I}, \{z_i\}_{i\in I} \subseteq R$ are two families of elements satisfying the quantum Serre relations. If

$$z_j y_i = q_i^{c_{i,j}} y_i z_j \quad \text{for } i, j \in I$$

then $\{y_i + z_i\}_{i\in I}$ also satisfies the quantum Serre relations.

It is easily checked that $(\mathrm{id} \otimes \delta) \circ \delta = (\Delta \otimes \mathrm{id}) \circ \delta$ and $(\epsilon \otimes \mathrm{id}) \circ \delta = \mathrm{id}$.

Proposition 3.33. The above action and coaction make $\mathcal{A}_q[U]$ into an algebra in the category ${}^{U_q(\mathfrak{g}^*)}_{U_q(\mathfrak{g}^*)}\mathcal{YD}$ of left-left Yetter-Drinfeld modules over $U_q(\mathfrak{g}^*)$.

Proof. We need only verify that the compatibility condition is satisfied, i.e. that

$$h_{(1)}x^{(-1)} \otimes (h_{(2)} \triangleright x^{(0)}) = (h_{(1)} \triangleright x)^{(-1)}h_{(2)} \otimes (h_{(1)} \triangleright x)^{(0)} \qquad \text{(Equation 3.1.)}$$

for $h \in U_q(\mathfrak{g}^*)$ and $x \in \mathcal{A}_q[U]$. It is easily checked that (Equation 3.1.) is satisfied for $h \in \{K_i^{\pm\frac{1}{2}}, F_{i,1}, F_{i,2} \mid i \in I\}$ and $x \in \{x_i \mid i \in I\}$. Suppose (Equation 3.1.) is satisfied for some $x, x' \in \mathcal{A}_q[U]$ and all $h \in \{K_i^{\pm\frac{1}{2}}, F_{i,1}, F_{i,2} \mid i \in I\}$. Then since $\Delta(\{K_i^{\pm\frac{1}{2}}, F_{i,1}, F_{i,2} \mid i \in I\}) \subset \{K_i^{\pm\frac{1}{2}}, F_{i,1}, F_{i,2} \mid i \in I\} \otimes \{K_i^{\pm\frac{1}{2}}, F_{i,1}, F_{i,2} \mid i \in I\}$, we observe:

$$
\begin{aligned}
&h_{(1)}(xx')^{(-1)} \otimes (h_{(2)} \triangleright (xx')^{(0)}) \\
&\quad = h_{(1)}x^{(-1)}(x')^{(-1)} \otimes (h_{(2)} \triangleright (x^{(0)}(x')^{(0)})) \\
&\quad = h_{(1)}x^{(-1)}(x')^{(-1)} \otimes (h_{(2)} \triangleright x^{(0)})(h_{(3)} \triangleright (x')^{(0)}) \\
&\quad = (h_{(1)} \triangleright x)^{(-1)}h_{(2)}(x')^{(-1)} \otimes (h_{(1)} \triangleright x)^{(0)}(h_{(3)} \triangleright (x')^{(0)}) \\
&\quad = (h_{(1)} \triangleright x)^{(-1)}(h_{(2)} \triangleright x')^{(-1)}h_{(3)} \otimes (h_{(1)} \triangleright x)^{(0)}(h_{(2)} \triangleright x')^{(0)} \\
&\quad = ((h_{(1)} \triangleright x)(h_{(2)} \triangleright x'))^{(-1)}h_{(3)} \otimes ((h_{(1)} \triangleright x)(h_{(2)} \triangleright x'))^{(0)} \\
&\quad = (h_{(1)} \triangleright (xx'))^{(-1)}h_{(2)} \otimes (h_{(1)} \triangleright (xx'))^{(0)}
\end{aligned}
$$

for $h \in \{K_i^{\pm\frac{1}{2}}, F_{i,1}, F_{i,2} \mid i \in I\}$. Hence we see that the set of all $x \in \mathcal{A}_q[U]$ such that (Equation 3.1.) holds for all $h \in \{K_i^{\pm\frac{1}{2}}, F_{i,1}, F_{i,2} \mid i \in I\}$ is a subalgebra of $\mathcal{A}_q[U]$ containing $\{x_i \mid i \in I\}$. Namely, (Equation 3.1.) holds for all $x \in \mathcal{A}_q[U]$ and $h \in \{K_i^{\pm\frac{1}{2}}, F_{i,1}, F_{i,2} \mid i \in I\}$. Now suppose (Equation 3.1.) holds for some $h, h' \in U_q(\mathfrak{g}^*)$ and all $x \in \mathcal{A}_q[U]$. Then we observe:

$$
\begin{aligned}
&(hh')_{(1)}x^{(-1)} \otimes ((hh')_{(2)} \triangleright x^{(0)}) \\
&\quad = h_{(1)}h'_{(1)}x^{(-1)} \otimes ((h_{(2)}h'_{(2)}) \triangleright x^{(0)}) \\
&\quad = h_{(1)}h'_{(1)}x^{(-1)} \otimes (h_{(2)} \triangleright (h'_{(2)} \triangleright x^{(0)})) \\
&\quad = h_{(1)}(h'_{(1)} \triangleright x)^{(-1)}h'_{(2)} \otimes (h_{(2)} \triangleright (h'_{(1)} \triangleright x)^{(0)}) \\
&\quad = (h_{(1)} \triangleright (h'_{(1)} \triangleright x))^{(-1)}h_{(2)}h'_{(2)} \otimes (h_{(1)} \triangleright (h'_{(1)} \triangleright x))^{(0)} \\
&\quad = ((h_{(1)}h'_{(1)}) \triangleright x)^{(-1)}h_{(2)}h'_{(2)} \otimes ((h_{(1)}h'_{(1)}) \triangleright x)^{(0)} \\
&\quad = ((hh')_{(1)} \triangleright x)^{(-1)}(hh')_{(2)} \otimes ((hh')_{(1)} \triangleright x)^{(0)}
\end{aligned}
$$

for $x \in \mathcal{A}_q[U]$. Hence we see that the set of all $h \in U_q(\mathfrak{g}^*)$ such that (Equation 3.1.) holds for all $x \in \mathcal{A}_q[U]$ is a subalgebra of $U_q(\mathfrak{g}^*)$ containing $\{K_i^{\pm\frac{1}{2}}, F_{i,1}, F_{i,2} \mid i \in I\}$. Namely, (Equation 3.1.) holds for all $x \in \mathcal{A}_q[U]$ and $h \in U_q(\mathfrak{g}^*)$. □

The following proposition is probably well-known, but a source was not quickly found, so we provide a proof here.

Proposition 3.34. Let H be a $\Bbbk$-bialgebra, A an H-module algebra, and B an algebra in ${}^H_H\mathcal{YD}$. Then the H-module $A \otimes_\Bbbk B$ is an H-module algebra with multiplication given by

$$(a \otimes b)(a' \otimes b') = a(b^{(-1)} \triangleright a') \otimes b^{(0)} b', \quad \text{for all } a, a' \in A,\ b, b' \in B,$$

where $\triangleright$ is the action of H and $\delta(b) = b^{(-1)} \otimes b^{(0)}$ is the coaction of H in sumless Sweedler notation.

Proof. We first show that $A \otimes_\Bbbk B$ is indeed an associative algebra under the prescribed multiplication. For $a, a', a'' \in A$ and $b, b', b'' \in B$, we have

$$\begin{aligned}
((a \otimes b)(a' \otimes b'))(a'' \otimes b'') &= (a(b^{(-1)} \triangleright a') \otimes b^{(0)} b)(a'' \otimes b'') \\
&= a(b^{(-1)} \triangleright a')((b^{(0)} b')^{(-1)} \triangleright a'') \otimes (b^{(0)} b)^{(0)} b'' \\
&= a(b^{(-1)} \triangleright a')((b^{(0)})^{(-1)} (b')^{(-1)} \triangleright a'') \otimes (b^{(0)})^{(0)} (b')^{(0)} b'' \\
&= a(b^{(-2)} \triangleright a')(b^{(-1)} (b')^{(-1)} \triangleright a'') \otimes b^{(0)} (b')^{(0)} b'' \\
&= a(b^{(-1)} \triangleright (a'((b')^{(-1)} \triangleright a''))) \otimes b^{(0)} (b')^{(0)} b'' \\
&= (a \otimes b)(a'((b')^{(-1)} \triangleright a'') \otimes (b')^{(0)} b'') \\
&= (a \otimes b)((a' \otimes b')(a'' \otimes b'')).
\end{aligned}$$

Hence the prescribed multiplication is associative. We now verify that $A \otimes_{\Bbbk} B$ is indeed an H-module algebra. For $h \in H$, $a, a' \in A$, and $b, b' \in B$, we have

$$\begin{aligned}
h \triangleright ((a \otimes b)(a' \otimes b')) &= h \triangleright (a(b^{(-1)} \triangleright a') \otimes b^{(0)}b') \\
&= (h_{(1)} \triangleright a)(h_{(2)}b^{(-1)} \triangleright a') \otimes (h_{(3)} \triangleright b^{(0)})(h_{(4)} \triangleright b') \\
&= (h_{(1)} \triangleright a)((h_{(2)} \triangleright b)^{(-1)}h_{(3)} \triangleright a') \otimes (h_{(2)} \triangleright b)^{(0)}(h_{(4)} \triangleright b') \\
&= (h_{(1)} \triangleright a)((h_{(2)} \triangleright b)^{(-1)} \triangleright (h_{(3)} \triangleright a')) \otimes (h_{(2)} \triangleright b)^{(0)}(h_{(4)} \triangleright b') \\
&= ((h_{(1)} \triangleright a) \otimes (h_{(2)} \triangleright b))((h_{(3)} \triangleright a') \otimes (h_{(4)} \triangleright b')) \\
&= (h_{(1)} \triangleright (a \otimes b))(h_{(2)} \triangleright (a' \otimes b')).
\end{aligned}$$

$$h \triangleright (1 \otimes 1) = (h_{(1)} \otimes 1) \otimes (h_{(2)} \triangleright 1) = \epsilon(h_{(1)}) \otimes \epsilon(h_{(2)}) = \epsilon(h_{(1)}\epsilon(h_{(2)})) \otimes 1 = \epsilon(h) \otimes 1.$$

The proposition is proved. □

Proof of Theorem 3.12. By Propositions 3.33 and 3.34 we may give $A \otimes \mathcal{A}_q[U]$ a $U_q(\mathfrak{g}^*)$-module algebra structure satisfying

$$\begin{aligned}
(1 \otimes x_i)(a \otimes 1) &= K_i(a) \otimes x_i + (q_i - q_i^{-1})F_{i,2}K_i(a) \otimes 1 \\
K_{i,1}^{\pm\frac{1}{2}} \triangleright (a \otimes x) &= K_i^{\pm\frac{1}{2}}(a) \otimes K_i^{\pm\frac{1}{2}}(x) \\
F_{i,1} \triangleright (a \otimes x) &= F_{i,1}(a) \otimes x + K_i^{-1}(a) \otimes F_{i,1}(x) \\
F_{i,2} \triangleright (a \otimes x) &= F_{i,2}(a) \otimes K_i^{-1}(x) + a \otimes F_{i,2}(x)
\end{aligned}$$

for $i \in I$, $a \in A$, and $x \in \mathcal{A}_q[U]$.

Now by Theorem 3.30, there is a left action of $U_q(\mathfrak{n}_-)$ on $A \otimes \mathcal{A}_q[U]$ given by

$$
\begin{aligned}
&F_i \rhd (a \otimes x) \\
&\quad = F_{i,1} \rhd (a \otimes x) + \frac{(1 \otimes x_i)(a \otimes x) - (K_i^{-1} \rhd (a \otimes x))(1 \otimes x_i)}{q_i - q_i^{-1}} \\
&\quad = F_{i,1}(a) \otimes x + \frac{K_i(a) \otimes x_i x + (q_i - q_i^{-1}) F_{i,2} K_i(a) \otimes x - K_i^{-1}(a) \otimes K_i^{-1}(x) x_i}{q_i - q_i^{-1}} \\
&\quad = (F_{i,1}(a) + F_{i,2} K_i(a)) \otimes x + \frac{K_i(a) \otimes x_i x - K_i^{-1}(a) \otimes K_i^{-1}(x) x_i}{q_i - q_i^{-1}} \\
&\quad = (F_{i,1}(a) + F_{i,2} K_i(a)) \otimes x + \frac{K_i(a) \otimes x_i x - K_i^{-1}(a) \otimes x_i x}{q_i - q_i^{-1}} \\
&\qquad + \frac{K_i^{-1}(a) \otimes x_i x - K_i^{-1}(a) \otimes K_i^{-1}(x) x_i}{q_i - q_i^{-1}} \\
&\quad = (F_{i,1}(a) + F_{i,2} K_i(a)) \otimes x + \frac{K_i(a) - K_i^{-1}(a)}{q_i - q_i^{-1}} \otimes x_i x + K_i^{-1}(a) \otimes F_i(x),
\end{aligned}
$$

matching the proposed action of F_i.

Furthermore, it is obvious that $E_i \rhd (a \otimes x) = a \otimes E_i(x)$ yields a well-defined action of $U_q(\mathfrak{n}_+)$ on $A \otimes \mathcal{A}_q[U]$. It is now straight-forward to check that

$$
\begin{aligned}
K_i^{\pm\frac{1}{2}} \triangleright (K_j^{\pm\frac{1}{2}} \triangleright (a \otimes x)) &= K_j^{\pm\frac{1}{2}} \triangleright (K_i^{\pm\frac{1}{2}} \triangleright (a \otimes x)) \\
K_i^{\frac{1}{2}} \triangleright (E_j \triangleright (K_i^{-\frac{1}{2}}((a \otimes x)))) &= q_i^{\frac{c_{i,j}}{2}} E_j \triangleright (a \otimes x), \\
K_i^{\frac{1}{2}} \triangleright (F_j \triangleright (K_i^{-\frac{1}{2}}((a \otimes x)))) &= q_i^{-\frac{c_{i,j}}{2}} F_j \triangleright (a \otimes x), \\
E_i \triangleright (F_j \triangleright (a \otimes x)) - F_j \triangleright (E_i \triangleright (a \otimes x)) &= \delta_{i,j}, \frac{K_i \triangleright (a \otimes x) - K_i^{-1} \triangleright (a \otimes x)}{q_i - q_i^{-1}},
\end{aligned}
$$

$$
\begin{aligned}
K_i^{\pm\frac{1}{2}}(1 \otimes 1) &= 1 \otimes 1, \\
E_i(1 \otimes 1) &= 0, \\
F_i(1 \otimes 1) &= 0.
\end{aligned}
$$

Hence we have given $A \otimes \mathcal{A}_q[U]$ the structure of a $U_q(\mathfrak{g})$-module. To see that it is in fact a module algebra, we need to check the following.

$$
\begin{aligned}
&K_i^{\pm\frac{1}{2}} \triangleright ((a \otimes x)(a' \otimes x')) \\
&\quad = (K_i^{\pm\frac{1}{2}} \triangleright (a \otimes x))(K_i^{\pm\frac{1}{2}} \triangleright (a' \otimes x')) && \text{(Equation 3.2.)} \\
&E_i \triangleright ((a \otimes x)(a' \otimes x')) \\
&\quad = (E_i \triangleright (a \otimes x))(K_i \triangleright (a' \otimes x')) + (a \otimes x)(E_i \triangleright (a' \otimes x')) && \text{(Equation 3.3.)} \\
&F_i \triangleright ((a \otimes x)(a' \otimes x')) \\
&\quad = (F_i \triangleright (a \otimes x))(a' \otimes x') + (K_i^{-1} \triangleright (a \otimes x))(F_i \triangleright (a' \otimes x')) && \text{(Equation 3.4.)}
\end{aligned}
$$

Rather than direct verification, we begin by observing that

$$h \triangleright ((a \otimes 1)z) = (h_{(1)} \triangleright (a \otimes 1))(h_{(2)} \triangleright z) \text{ and } h \triangleright ((a \otimes x_j)z) = (h_{(1)} \triangleright (a \otimes x_j))(h_{(2)} \triangleright z)$$

for $h \in \{K_i^{\pm\frac{1}{2}}, E_i, F_i \mid i \in I\}$, $j \in I$, $a \in A$, and $z \in A \otimes \mathcal{A}_q[U]$. Let Y be the set of all $x \in \mathcal{A}_q[U]$ so that $h \triangleright ((a \otimes x)z) = (h_{(1)} \triangleright (a \otimes x))(h_{(2)} \triangleright z)$ for all $a \in A$, $z \in A \otimes \mathcal{A}_q[U]$, and $h \in \{K_i^{\pm\frac{1}{2}}, E_i, F_i \mid i \in I\}$. Then Y is clearly a $\mathbb{Q}(q^{\frac{1}{2d}})$-vector space (containing 1 and x_j). We show that Y is closed under multiplication. Suppose $x, x' \in Y$, $a \in A$, and $z \in A \otimes \mathcal{A}_q[U]$. Then for $h \in \{K_i^{\pm\frac{1}{2}}, E_i, F_i \mid i \in I\}$,

$$
\begin{aligned}
h \triangleright ((a \otimes xx')z) &= h \triangleright ((a \otimes x)(1 \otimes x')z) \\
&= (h_{(1)} \triangleright (a \otimes x))(h_{(2)} \triangleright ((1 \otimes x')z)) \\
&= (h_{(1)} \triangleright (a \otimes x))(h_{(2)} \triangleright (1 \otimes x'))(h_{(3)}(z)) \\
&= (h_{(1)} \triangleright (a \otimes x)(1 \otimes x'))(h_{(2)}(z)) \\
&= (h_{(1)} \triangleright (a \otimes xx'))(h_{(2)}(z)).
\end{aligned}
$$

Hence $xx' \in Y$ and we have shown that Y is closed under multiplication. It follows that Y is a $\mathbb{Q}(q^{\frac{1}{2d}})$-subalgebra of $\mathcal{A}_q[U]$ containing x_j and hence is actually $\mathcal{A}_q[U]$ itself. Hence we have verified equations (Equation 3.2.), (Equation 3.3.), and (Equation 3.4.). It follows that the given structure makes $A \otimes \mathcal{A}_q[U]$ into a $U_q(\mathfrak{g})$-module algebra. $\square$

3.2.7. Proof of Theorem 3.14

We begin by constructing a natural isomorphism $\psi : (-)^+ \otimes \mathcal{A}_q[U] \Rightarrow \mathrm{id}_{\mathcal{C}_{\mathfrak{g}}^q}$. For every object (A, φ_A) of $\mathcal{C}_{\mathfrak{g}}^q$, set $\psi_{(A,\varphi_A)} := m_A \circ (\iota_A \otimes \varphi_A)$, where ι_A is the inclusion $A^+ \hookrightarrow A$ and $m_A : A \otimes A \to A$ is multiplication. As an abuse of notation, we will write ψ_A when context is clear. Since ψ_A is clearly a linear map, we check that it respects multiplication and is $U_q(\mathfrak{g})$-equivariant. One easily computes

$$\psi_A((a \otimes 1)(a' \otimes x')) = \psi_A(a \otimes 1)\psi_A(a' \otimes x') \text{ and}$$
$$\psi_A((a \otimes x_i)(a' \otimes x')) = \psi_A(a \otimes x_i)\psi_A(a' \otimes x').$$

Let $Y = \{x \in \mathcal{A}_q[U] \mid \psi_A((a \otimes x)z) = \psi_A(a \otimes x)\psi_A(z) \ \forall a \in A, z \in A \otimes \mathcal{A}_q[U]\}$. We have seen that $1, x_i \in Y$ for $i \in I$, so the computations

$$\begin{aligned}
\psi_A((a \otimes (x+y))z) &= \psi_A((a \otimes x)z + (a \otimes y)z) \\
&= \psi_A((a \otimes x)z) + \psi_A((a \otimes y)z) \\
&= \psi_A(a \otimes x)\psi_A(z) + \psi_A(a \otimes y)\psi_A(z) \\
&= (\psi_A(a \otimes x) + \psi_A(a \otimes y))\psi_A(z) \\
&= \psi_A(a \otimes (x+y))\psi_A(z),
\end{aligned}$$

$$
\begin{aligned}
\psi_A((a \otimes xy)z) &= \psi_A((a \otimes x)(1 \otimes y)z) \\
&= \psi_A(a \otimes x)\psi_A((1 \otimes y)z) \\
&= \psi_A(a \otimes x)\psi_A(1 \otimes y)\psi_A(z) \\
&= \psi_A((a \otimes x)(1 \otimes y))\psi_A(z) \\
&= \psi_A((a \otimes xy))\psi_A(z)
\end{aligned}
$$

show that Y is a subalgebra of $\mathcal{A}_q[U]$ containing a generating set. Hence $Y = \mathcal{A}_q[U]$, i.e. ψ_A is a homomorphism of algebras. Now we verify that ψ_A is $U_q(\mathfrak{g})$-equivariant. For $i \in I$, we have

$$
\begin{aligned}
K_i^{\pm\frac{1}{2}}(\psi_A(a \otimes x)) &= K_i^{\pm\frac{1}{2}}(a\varphi_A(x)) \\
&= K_i^{\pm\frac{1}{2}}(a)K_i^{\pm\frac{1}{2}}(\varphi_A(x)) \\
&= K_i^{\pm\frac{1}{2}}(a)\varphi_A(K_i^{\pm\frac{1}{2}}(x)) \\
&= \psi_A(K_i^{\pm\frac{1}{2}}(a) \otimes K_i^{\pm\frac{1}{2}}(x)) \\
&= \psi_A(K_i^{\pm\frac{1}{2}} \triangleright (a \otimes x)),
\end{aligned}
$$

$$
\begin{aligned}
E_i(\psi_A(a \otimes x)) &= E_i(a\varphi_A(x)) \\
&= E_i(a)K_i(\varphi_A(x)) + aE_i(\varphi_A(x)) \\
&= a\varphi_A(E_i(x)) \\
&= \psi_A(a \otimes E_i(x)) \\
&= \psi_A(E_i \triangleright (a \otimes x)),
\end{aligned}
$$

$$
\begin{aligned}
F_i(\psi_A(a \otimes x)) &= F_i(a\varphi_A(x)) \\
&= F_i(a)\varphi_A(x) + K_i^{-1}(a)F_i(\varphi_A(x)) \\
&= \left(F_i(a) - \frac{\varphi_A(x_i)a - K_i^{-1}(a)\varphi_A(x_i)}{q_i - q_i^{-1}}\right)\varphi_A(x) \\
&\quad + \frac{\varphi_A(x_i)a - K_i(a)\varphi_A(x_i)}{q_i - q_i^{-1}}\varphi_A(x) \\
&\quad + \frac{K_i(a) - K_i^{-1}(a)}{q_i - q_i^{-1}}\varphi_A(x_i)\varphi_A(x) + K_i^{-1}(a)\varphi_A(F_i(x)) \\
&= \left(F_{i,1}(a) + F_{i,2}K_i(a)\right)\varphi_A(x) + \frac{K_i(a) - K_i^{-1}(a)}{q_i - q_i^{-1}}\varphi(x_i x) \\
&\quad + K_i^{-1}(a)\varphi_A(F_i(x)) \\
&= \psi_A\left((F_{i,1}(a) + F_{i,2}K_i(a)) \otimes x + \frac{K_i(a) - K_i^{-1}(a)}{q_i - q_i^{-1}} \otimes x_i x\right) \\
&\quad \psi_A\left(+K_i^{-1}(a) \otimes F_i(x)\right) \\
&= \psi_A(F_i \triangleright (a \otimes x)).
\end{aligned}
$$

So ψ_A is a homomorphism of $U_q(\mathfrak{g})$-modules and thus a homomorphism of $U_q(\mathfrak{g})$-module algebras. By Theorem 3.3, ψ_A is an isomorphism of $U_q(\mathfrak{g})$-module algebras. Now $\psi_A \circ (1 \otimes \mathrm{id}) = m_A \circ (\iota_A \otimes \varphi_A) \circ (1 \otimes \mathrm{id}) = m_A \circ (1 \otimes \varphi_A) = \varphi_A$.

Hence ψ_A is a morphism of $\mathcal{C}_{\mathfrak{g}}^q$. To show that ψ_A is an isomorphism in $\mathcal{C}_{\mathfrak{g}}^q$, we make the following easy observation.

Lemma 3.35. A morphism between objects of $\mathcal{C}_{\mathfrak{g}}^q$ is an isomorphism if and only if the underlying homomorphism of $U_q(\mathfrak{g})$-module algebras is an isomorphism.

Proof. It is clear that the homomorphism of $U_q(\mathfrak{g})$-module algebras which underlies an isomorphism between objects of $\mathcal{C}_{\mathfrak{g}}^q$ is actually an isomorphism, so we simply show the converse. Let (A, φ_A) and (B, φ_B) be objects of $\mathcal{C}_{\mathfrak{g}}^q$ and $\xi : A \to B$ a morphism between them such that ξ is an isomorphism of $U_q(\mathfrak{g})$-module algebras.

Then $\xi \circ \varphi_A = \varphi_B$. Hence we have $\xi^{-1} \circ \varphi_B = \xi^{-1} \circ \xi \circ \varphi_A = \varphi_A$ and so ξ^{-1} is a morphism $(B, \varphi_B) \to (A, \varphi_A)$. Thus ξ is an isomorphism in $\mathcal{C}_{\mathfrak{g}}^q$. $\square$

Hence ψ_A is actually an isomorphism in $\mathcal{C}_{\mathfrak{g}}^q$. If we can show that $\psi := (\psi_A)_{(A,\varphi_A)\in\mathcal{C}_{\mathfrak{g}}^q}$ is a natural transformation between $(-)^+ \otimes \mathcal{A}_q[U]$ and $\mathrm{id}_{\mathcal{C}_{\mathfrak{g}}^q}$, then we will have shown that it is a natural isomorphism. Let (A, φ_A) and (B, φ_B) be objects of $\mathcal{C}_{\mathfrak{g}}^q$ and $\xi : A \to B$ a morphism. Then

$$
\begin{aligned}
\psi_B \circ (\xi|_{A^+} \otimes \mathrm{id}) &= m_B \circ (\iota_B \otimes \varphi_B) \circ (\xi|_{A^+} \otimes \mathrm{id}) \\
&= m_B \circ (\xi|_{A^+} \otimes \varphi_B) \\
&= m_B \circ (\xi|_{A^+} \otimes (\xi \circ \varphi_A)) \\
&= m_B \circ (\xi \otimes \xi) \circ (\iota_A \otimes \varphi_A) \\
&= \xi \circ m_A \circ (\iota_A \otimes \varphi_A) \\
&= \xi \circ \psi_A.
\end{aligned}
$$

Hence $\psi : (-)^+ \otimes \mathcal{A}_q[U] \Rightarrow \mathrm{id}_{\mathcal{C}_{\mathfrak{g}}^q}$ is a natural transformation and therefore a natural isomorphism.

Now for every $U_q(\mathfrak{g}^*)$-module A, let $\eta_A = \mathrm{id} \otimes 1 : A \to (A \otimes \mathcal{A}_q[U])^+$. Then η_A is obviously an injective homomorphism of algebras. We need to show that η_A is a homomorphism of $U_q(\mathfrak{g}^*)$-module algebras, namely that η_A respects the action of $U_q(\mathfrak{g}^*)$. So we make the following computations.

$$
K_i^{\pm\frac{1}{2}}(\eta_A(a)) = K_i^{\pm\frac{1}{2}}(a \otimes 1) = K_i^{\pm\frac{1}{2}}(a) \otimes 1 = \eta_A(K_i^{\pm\frac{1}{2}}(a)),
$$

$$
\begin{aligned}
F_{i,1}(\eta_A(a)) &= F_{i,1}(a \otimes 1) \\
&= F_i(a \otimes 1) - \frac{(1 \otimes x_i)(a \otimes 1) - K_i^{-1}(a \otimes 1)(1 \otimes x_i)}{q_i - q_i^{-1}} \\
&= (F_{i,1}(a) + F_{i,2}K_i(a)) \otimes 1 + \frac{K_i(a) - K_i^{-1}(a)}{q_i - q_i^{-1}} \otimes x_i \\
&\quad - \frac{K_i(a) \otimes x_i + F_{i,2}K_i(a) \otimes 1 - K_i^{-1}(a) \otimes x_i}{q_i - q_i^{-1}} \\
&= F_{i,1}(a) \otimes 1 \\
&= \eta_A(F_{i,1}(a)),
\end{aligned}
$$

$$
\begin{aligned}
F_{i,2}(\eta_A(a)) &= F_{i,2}(a \otimes 1) \\
&= \frac{(1 \otimes x_i)K_i^{-1}(a \otimes 1) - (a \otimes 1)(1 \otimes x_i)}{q_i - q_i^{-1}} \\
&= \frac{(1 \otimes x_i)(K_i^{-1}(a) \otimes 1) - a \otimes x_i}{q_i - q_i^{-1}} \\
&= \frac{K_iK_i^{-1}(a) \otimes x_i + (q_i - q_i^{-1})F_{i,2}K_iK_i^{-1}(a) \otimes 1 - a \otimes x_i}{q_i - q_i^{-1}} \\
&= F_{i,2}(a) \otimes 1 \\
&= \eta_A(F_{i,2}(a)).
\end{aligned}
$$

Hence η_A respects the action of $U_q(\mathfrak{g})$. Our last step is to show that η_A is surjective. Given an arbitrary element $\sum_{k=1}^{n} a_k \otimes x_{\mathbf{j}_k} \in (A \otimes \mathcal{A}_q[U])^+$ with $\mathbf{j}_k < \mathbf{j}_l$ if $k < l$, we have

$$\sum_{k=1}^{n} a_k \otimes x_{\mathbf{j}_k} = E_{\mathbf{i}}^{(top)} \left(\sum_{k=1}^{n} a_k \otimes x_{\mathbf{j}_k} \right) = a_n \otimes 1.$$

Hence $(A \otimes \mathcal{A}_q[U])^+ = A \otimes \mathbb{Q}(q^{\frac{1}{2d}})$, so η_A is surjective and therefore an isomorphism. One easily checks that $\eta := (\eta_A)_{A \in U_q(\mathfrak{g}^*)-\mathbf{ModAlg}}$ is a natural transformation. Since each η_A is an isomorphism, η is a natural isomorphism

$$\eta : \mathrm{id}_{U_q(\mathfrak{g}^*)-\mathbf{ModAlg}} \Rightarrow (- \otimes \mathcal{A}_q[U])^+.$$

We have now shown that

$$(-)^+ \otimes \mathcal{A}_q[U] \cong \mathrm{id}_{\mathcal{C}^q_{\mathfrak{g}}} \quad \text{and} \quad (- \otimes \mathcal{A}_q[U])^+ \cong \mathrm{id}_{U_q(\mathfrak{g}^*)-\mathbf{ModAlg}},$$

so $\mathcal{A}_q[U] \otimes -$ and $(-)^+$ are quasi-inverse equivalences of categories. $\square$

3.2.8. Proof of Proposition 3.17

We know by Theorem 3.3 that $A \cong A^+ \otimes \mathcal{A}_q[U]$ and $B \cong B^+ \otimes \mathcal{A}_q[U]$ and Theorem 3.14 says this is an isomorphism of $U_q(\mathfrak{g})$-module algebras. We now consider the map

$$\mu_L : (A \underline{\otimes} B)^+ \otimes [\varphi_A(\mathcal{A}_q[U]) \otimes \mathbb{Q}(q^{\frac{1}{2d}})] \to A \underline{\otimes} B \cong (A^+ \otimes \mathcal{A}_q[U]) \underline{\otimes} (B^+ \otimes \mathcal{A}_q[U])$$

as in Theorem 3.3. As in the proof of Corollary 3.7 (Section 3.2.4.),

$$\mu_L((A \underline{\otimes} B)^+ \otimes [\varphi_A(\mathcal{A}_q[U]) \otimes \mathbb{Q}(q^{\frac{1}{2d}})])$$

is a subalgebra of $A \underline{\otimes} B$. Since

$$A^+ \otimes \mathbb{Q}(q^{\frac{1}{2d}}), \quad \mathbb{Q}(q^{\frac{1}{2d}}) \otimes B^+, \quad \text{and} \quad \{\varphi_A(x_i) \otimes 1 - 1 \otimes \varphi_B(x_i) \mid i \in I\}$$

are all contained in $(A \underline{\otimes} B)^+$ and $\{\varphi_A(x_i) \otimes 1 \mid i \in I\} \subseteq \varphi_A(\mathcal{A}_q[U]) \otimes \mathbb{Q}(q^{\frac{1}{2d}})$, it follows that $\mu_L((A \underline{\otimes} B)^+ \otimes [\varphi_A(\mathcal{A}_q[U]) \otimes \mathbb{Q}(q^{\frac{1}{2d}})])$ contains all of these sets. Hence $\mu_L((A \underline{\otimes} B)^+ \otimes [\varphi_A(\mathcal{A}_q[U]) \otimes \mathbb{Q}(q^{\frac{1}{2d}})])$ contains a generating set of $A \underline{\otimes} B$. Being a subalgebra, it follows that $\mu_L((A\underline{\otimes}B)^+ \otimes [\varphi_A(\mathcal{A}_q[U]) \otimes \mathbb{Q}(q^{\frac{1}{2d}})]) = A\underline{\otimes}B$, i.e. μ_L is surjective. Hence μ_L is an isomorphism and by Theorem 3.3, $A\underline{\otimes}B$ is adapted and $\nu_{\mathbf{i}}(A\underline{\otimes}B \setminus \{0\}) = \nu_{\mathbf{i}}(\mathcal{A}_q[U] \setminus \{0\})\ \forall \mathbf{i} \in R(w_\circ)$. Since $A\underline{\otimes}B$ is a $U_q(\mathfrak{g})$-weight module algebra and $1 \otimes \varphi_B$ and $\varphi_A \otimes 1$ are injections, the proposition follows. □

3.2.9. Proof of Proposition 3.18

The vector space $A \otimes B$ is naturally viewed as a subspace of $A * B$ (or $A \star B$ if you prefer) via $a \otimes b \mapsto (a \otimes 1) \otimes (b \otimes 1)$. In fact, this subspace is actually a subalgebra since

$$((a \otimes 1) \otimes (b \otimes 1))((a' \otimes 1) \otimes (b' \otimes 1)) = q^{(|a'|,|b|)}(aa' \otimes 1) \otimes (bb' \otimes 1)$$

for weight vectors $a, a' \in A$ and $b, b' \in B$ of weight $|a|, |a'|, |b|$, and $|b'|$, respectively. Hence we may equip $A \otimes B$ with this multiplication.

By design, the prescribed actions of K_i and $F_{i,1}$ on $A \otimes B$ match those on $(A \otimes \mathbb{Q}(q^{\frac{1}{2d}})) \otimes (B \otimes \mathbb{Q}(q^{\frac{1}{2d}})) \subset A * B$, while the prescribed actions of $K_i^{\pm\frac{1}{2}}$ and $F_{i,2}$ match those on $(A \otimes \mathbb{Q}(q^{\frac{1}{2d}})) \otimes (B \otimes \mathbb{Q}(q^{\frac{1}{2d}})) \subset A \star B$. A straightforward check verifies that $F_{i,1} \triangleright (F_{j,2} \triangleright (a \otimes b)) = F_{j,2} \triangleright (F_{i,1} \triangleright (a \otimes b))$ for $a \in A,\ b \in B$, and $i, j \in I$, so it follows that the prescribed action of $U_q(\mathfrak{g}^*)$ on $A \otimes B$ is well-defined and compatible with multiplication. □

3.3. Translation to the Symmetric Coproduct Setting

The choice to use Δ_σ and Δ_{σ_*} throughout this chapter (rather than the more symmetric comultiplications) allowed for our specific methods of proof. However, we prefer Δ as presented in Chapter II. We therefore need to translate the language and properties.

Recall from Section 2.3. that $U_q(\mathfrak{g})_\sigma$-**WModAlg** is equivalent to $U_q(\mathfrak{g})$-**WModAlg** and $U_q(\mathfrak{g}^*)_{\sigma_*}$-**WModAlg** is equivalent to $U_q(\mathfrak{g}^*)$-**WModAlg** via very explicit equivalences of categories. We therefore define a category $\underline{\tilde{\mathcal{C}}}^q_{\mathfrak{g}}$ simply be the category analogous to $\underline{\mathcal{C}}^q_{\mathfrak{g}}$, but with $U_q(\mathfrak{g})$-weight module algebras in place of $U_q(\mathfrak{g})_\sigma$-weight module algebras. We then obtain an equivalence of categories $(-)^+ : \underline{\tilde{\mathcal{C}}}^q_{\mathfrak{g}} \to U_q(\mathfrak{g}^*)$-**WModAlg** as the composition

$$\underline{\tilde{\mathcal{C}}}^q_{\mathfrak{g}} \stackrel{\tilde{\mathscr{F}}_{\sigma^{-1}}}{\to} \underline{\mathcal{C}}^q_{\mathfrak{g}} \stackrel{(-)^+}{\to} U_q(\mathfrak{g}^*)_{\sigma_*}\text{-}\mathbf{WModAlg} \stackrel{\mathscr{F}_{\sigma_*}}{\to} U_q(\mathfrak{g}^*)\text{-}\mathbf{WModAlg},$$

where $\tilde{\mathscr{F}}_{\sigma^{-1}}$ is the obvious equivalence of categories induced by the equivalence $\mathscr{F}_{\sigma^{-1}} : U_q(\mathfrak{g})$-**WModAlg** $\to U_q(\mathfrak{g})_\sigma$-**WModAlg**. Then for (A, φ_A) in $\underline{\tilde{\mathcal{C}}}^q_{\mathfrak{g}}$, we have the following action of $U_q(\mathfrak{g}^*)$ on A^+:

$$\begin{aligned} K_i^{\pm\frac{1}{2}} \triangleright a &= K_i^{\pm\frac{1}{2}}(a), \\ F_{i,1} \triangleright a &= F_i(a) - \frac{x_i K_i^{\frac{1}{2}}(a) - K_i^{-\frac{1}{2}}(a) x_i}{q_i - q_i^{-1}}, \\ F_{i,2} \triangleright a &= \frac{x_i K_i^{-\frac{1}{2}}(a) - K_i^{\frac{1}{2}}(a) x_i}{q_i - q_i^{-1}} \end{aligned}$$

for $i \in I$ and $a \in A^+$, where we write x_i in place of $\varphi_A(x_i)$.

The quasi-inverse for $(-)^+$ is obtained similarly. Namely, we define

$$(-) \otimes \mathcal{A}_q[U] : U_q(\mathfrak{g}^*)\text{-}\mathbf{WModAlg} \to \underline{\tilde{\mathcal{C}}}^q_{\mathfrak{g}}$$

to be the composition

$$U_q(\mathfrak{g}^*)\text{-}\mathbf{WModAlg} \overset{\mathscr{F}_{\sigma_*^{-1}}}{\to} U_q(\mathfrak{g}^*)_{\sigma_*}\text{-}\mathbf{WModAlg} \xrightarrow{(-)\otimes\mathcal{A}_q[U]} \underline{\mathcal{C}}^q_{\mathfrak{g}} \overset{\tilde{\mathscr{F}}_\sigma}{\to} \underline{\tilde{\mathcal{C}}}^q_{\mathfrak{g}},$$

where $\tilde{\mathscr{F}}_\sigma$ is the obvious equivalence of categories induced by the equivalence $\mathscr{F}_\sigma : U_q(\mathfrak{g})_\sigma\text{-}\mathbf{WModAlg} \to U_q(\mathfrak{g})\text{-}\mathbf{WModAlg}$. Then for A in $U_q(\mathfrak{g})\text{-}\mathbf{WModAlg}$, we have

$$\begin{aligned}
(1\otimes x_i)(a\otimes 1) &= (K_i \triangleright a)\otimes x_i + (q_i - q_i^{-1})(F_{i,2}K_i^{\frac{1}{2}} \triangleright a)\otimes 1\\
K_i^{\pm\frac{1}{2}}(a\otimes x) &= (K_i^{\pm\frac{1}{2}} \triangleright a)\otimes K_i^{\pm\frac{1}{2}}(x),\\
E_i(a\otimes x) &= (K_i^{-\frac{1}{2}} \triangleright a)\otimes E_i(x),\\
F_i(a\otimes x) &= (F_{i,1}\triangleright a + F_{i,2}K_i \triangleright a)\otimes K_i^{\frac{1}{2}}(x)\\
&\quad + \frac{(K_i^{\frac{3}{2}} \triangleright a) - (K_i^{-\frac{1}{2}} \triangleright a)}{q_i - q_i^{-1}} \otimes x_i K_i^{\frac{1}{2}}(x) + (K_i^{\frac{1}{2}} \triangleright a)\otimes F_i(x)
\end{aligned}$$

for $i \in I$, $a \in A$, and $x \in \mathcal{A}_q[U]$.

Recall that $\mathcal{A}_q[B]$ is graded by $\mathcal{P}$ and factors as a $\mathbb{Q}(q^{\frac{1}{2d}})$-algebra into the graded tensor product $\mathcal{A}_q[U]\overline{\otimes}\mathcal{A}_q[T]$. In terms of how $U_q(\mathfrak{g}^*)$ acts on $\mathcal{A}_q[B]^+ \cong \mathcal{A}_q[T]$, this is equivalent to $F_{i,2} \triangleright \mathcal{A}_q[B]^+ = \{0\}$. Considering how $F_i \in U_q(\mathfrak{g})$ acts, we furthermore see that $F_{i,1} \triangleright \mathcal{A}_q[B]^+ = \{0\}$. In this sense $\mathcal{A}_q[B]$ can be viewed as a somewhat trivial module: it is the image under $(-)\otimes\mathcal{A}_q[U]$ of the "most trivial" nontrivial type of object in $U_q(\mathfrak{g}^*) - \mathbf{WModAlg}$, on which the Cartan subalgebra may act nontrivially, but the two Borel subalgebras act trivially.

Proposition 3.36. If $A, A' \in U_q(\mathfrak{g}^*) - \mathbf{WModAlg}$ are such that $F_{i,j} \triangleright A = \{0\}$ and $F_{i,2} \triangleright A' = \{0\}$ for all $i \in I$ and $j \in \{1, 2\}$, then

$$A \star A' \cong A \overline{\otimes} \mathcal{A}_q[U] \overline{\otimes} A'$$

as $\mathcal{K}$-module algebras and $F_{i,2} \triangleright (A \star A') = \{0\}$.

Proof. We begin with a linear map

$$\psi : A \overline{\otimes} \mathcal{A}_q[U] \overline{\otimes} A' \to (A \otimes \mathcal{A}_q[U]) \underline{\otimes} (A' \otimes \mathcal{A}_q[U])$$
$$a \otimes x \otimes a' \mapsto (a \otimes x_{[1]}) \otimes (K_{-|x_{[2]}|}(a') \otimes x_{[2]})$$

where we write $x \mapsto x_{[1]} \otimes x_{[2]}$ in sumless Sweedler-like notation for the embedding of $\mathcal{K}$-module algebras $\mathcal{A}_q[U] \hookrightarrow \mathcal{A}_q[U] \underline{\otimes} \mathcal{A}_q[U]$ given on generators by $x_i \mapsto 1 \otimes x_i - x_i \otimes 1$ and the existence of which is easily deduced from Corollary 3.28. It's clear that ψ is injective. We'll show that ψ is a homomorphism of $\mathcal{K}$-module algebras whose image is $[(A \otimes \mathcal{A}_q[U]) \underline{\otimes} (A' \otimes \mathcal{A}_q[U])]^+$, completing the proof.

We observe that ψ can be realized instead as the map

$$a \otimes x \otimes a' \mapsto ((a \otimes 1) \otimes (1 \otimes 1))((1 \otimes x_{[1]}) \otimes (1 \otimes x_{[2]}))((1 \otimes 1) \otimes (a' \otimes 1))$$

for $a \in A$, $a' \in A'$, and $x \in \mathcal{A}_q[U]$. So we see that, to show that ψ is an algebra homomorphism, it suffices to show that

$$((1 \otimes x_{[1]}) \otimes (1 \otimes x_{[2]}))((a \otimes 1) \otimes (1 \otimes 1))$$
$$= q^{(|x|,|a|)}((a \otimes 1) \otimes (1 \otimes 1))((1 \otimes x_{[1]}) \otimes (1 \otimes x_{[2]})), \qquad \text{(Equation 3.5.)}$$
$$((1 \otimes 1) \otimes (a' \otimes 1))((1 \otimes x_{[1]}) \otimes (1 \otimes x_{[2]}))$$
$$= q^{(|a'|,|x|)}((1 \otimes x_{[1]}) \otimes (1 \otimes x_{[2]}))((1 \otimes 1) \otimes (a' \otimes 1)), \qquad \text{(Equation 3.6.)}$$
$$((1 \otimes 1) \otimes (a' \otimes 1))((a \otimes 1) \otimes (1 \otimes 1))$$
$$= q^{(|a|,|a'|)}((a \otimes 1) \otimes (1 \otimes 1))((1 \otimes 1) \otimes (a' \otimes 1)) \qquad \text{(Equation 3.7.)}$$

for homogeneous $a \in A$, $a' \in A'$, and $x \in \mathcal{A}_q[U]$. (Equation 3.7.) is obvious from the definition of the braided tensor product, so we focus on the other two. For this, we note that since $x \mapsto x_{[1]} \otimes x_{[2]}$ is an algebra homomorphism, we may verify the required equations by checking on generators x_i, which satisfy $(x_i)_{[1]} \otimes (x_i)_{[2]} = 1 \otimes x_i - x_i \otimes 1$. We therefore compute.

$$
\begin{aligned}
&((1 \otimes 1) \otimes (1 \otimes x_i) - (1 \otimes x_i) \otimes (1 \otimes 1))((a \otimes 1) \otimes (1 \otimes 1)) \\
&\quad = q^{(|a|,-\alpha_i)}(a \otimes 1) \otimes (1 \otimes x_i) + q^{\frac{1}{2}(|a|,-\alpha_i)}(q_i - q_i^{-1})F_i(a \otimes 1) \otimes (1 \otimes 1) \\
&\qquad - ((K_i \triangleright a) \otimes x_i) \otimes (1 \otimes 1) - (q_i - q_i^{-1})((F_{i,2}K_i^{\frac{1}{2}} \triangleright a) \otimes 1) \otimes (1 \otimes 1) \\
&\quad = q^{(|a|,-\alpha_i)}(a \otimes 1) \otimes (1 \otimes x_i) \\
&\qquad + q^{\frac{1}{2}(|a|,-\alpha_i)}(q_i - q_i^{-1})((((F_{i,1} \triangleright a) + (F_{i,2}K_i \triangleright a)) \otimes 1) \otimes (1 \otimes 1) \\
&\qquad + q^{\frac{1}{2}(|a|,-\alpha_i)}(((K_i^{\frac{3}{2}} \triangleright a) - (K_i^{-\frac{1}{2}} \triangleright a)) \otimes x_i) \otimes (1 \otimes 1) \\
&\qquad - ((K_i \triangleright a) \otimes x_i) \otimes (1 \otimes 1) \\
&\quad = q^{(|a|,-\alpha_i)}(a \otimes 1) \otimes (1 \otimes x_i) - q^{(|a|,-\alpha_i)}(a \otimes x_i) \otimes (1 \otimes 1) \\
&\quad = q^{(|a|,-\alpha_i)}((a \otimes 1) \otimes (1 \otimes 1))((1 \otimes 1) \otimes (1 \otimes x_i) - (1 \otimes x_i) \otimes (1 \otimes 1))
\end{aligned}
$$

$$
\begin{aligned}
&((1 \otimes 1) \otimes (a' \otimes 1))((1 \otimes 1) \otimes (1 \otimes x_i) - (1 \otimes x_i) \otimes (1 \otimes 1)) \\
&\qquad = (1 \otimes 1) \otimes (a' \otimes x_i) - q^{(|a'|, -\alpha_i)} (1 \otimes x_i) \otimes (a' \otimes 1) \\
&\qquad = ((1 \otimes 1) \otimes (1 \otimes x_i))((1 \otimes 1) \otimes ((K_i^{-1} \triangleright a') \otimes 1)) \\
&\qquad\quad - (q_i - q_i^{-1})((1 \otimes 1) \otimes ((F_{i,2} K_i^{\frac{1}{2}} \triangleright a') \otimes 1)) \\
&\qquad\quad - q^{(|a'|, -\alpha_i)} (1 \otimes x_i) \otimes (a' \otimes 1) \\
&\qquad = q^{(|a'|, -\alpha_i)} ((1 \otimes 1) \otimes (1 \otimes x_i) - (1 \otimes x_i) \otimes (1 \otimes 1))((1 \otimes 1) \otimes (a' \otimes 1))
\end{aligned}
$$

So we see that ψ is an algebra homomorphism. It's now clear that ψ is in fact a homomorphism of $\mathcal{K}$-module algebras. It simply remains to show that the image of ψ is $[(A \otimes \mathcal{A}_q[U]) \underline{\otimes} (A' \otimes \mathcal{A}_q[U])]^+$.

Note that, since ψ is an algebra homomorphism, its image is automatically a subalgebra of $(A \otimes \mathcal{A}_q[U]) \underline{\otimes} (A' \otimes \mathcal{A}_q[U])$. And since the image of a pure tensor is a product of three things, two of which are obviously in $[(A \otimes \mathcal{A}_q[U]) \underline{\otimes} (A' \otimes \mathcal{A}_q[U])]^+$ and the image of ψ, we will check that $E_i((1 \otimes x_{[1]}) \otimes (1 \otimes x_{[2]})) = 0$ for $x \in \mathcal{A}_q[U]$. But again, since ψ is an algebra homomorphism, it suffices to check this equation on the generators. We have $E_i((1 \otimes 1) \otimes (1 \otimes x_i) - (1 \otimes x_i) \otimes (1 \otimes 1)) = 0$ for all $i \in I$, so the image of ψ is contained in $[(A \otimes \mathcal{A}_q[U]) \underline{\otimes} (A' \otimes \mathcal{A}_q[U])]^+$.

According to Theorem 3.4, the restriction of the multiplication μ of $(A \otimes \mathcal{A}_q[U]) \underline{\otimes} (A' \otimes \mathcal{A}_q[U])$ to

$$
[(A \otimes \mathcal{A}_q[U]) \underline{\otimes} (A' \otimes \mathcal{A}_q[U])]^+ \otimes [(\mathbb{Q}(q^{\frac{1}{2d}}) \otimes \mathcal{A}_q[U]) \underline{\otimes} (\mathbb{Q}(q^{\frac{1}{2d}}) \otimes \mathbb{Q}(q^{\frac{1}{2d}}))]
$$

is an isomorphism. It is clear that $(\mathcal{A}_q[U] \otimes \mathbb{Q}(q^{\frac{1}{2d}})) \underline{\otimes} (\mathbb{Q}(q^{\frac{1}{2d}}) \otimes \mathbb{Q}(q^{\frac{1}{2d}}))$ and the image of ψ generate $(A \otimes \mathcal{A}_q[U]) \underline{\otimes} (A' \otimes \mathcal{A}_q[U])$, implying that the image of ψ coincides exactly with $[(A \otimes \mathcal{A}_q[U])\underline{\otimes}(A' \otimes \mathcal{A}_q[U])]^+$. Otherwise, image of the restriction of μ to $\mathrm{im}(\psi) \otimes (\mathcal{A}_q[U] \otimes \mathbb{Q}(q^{\frac{1}{2d}}))\underline{\otimes}(\mathbb{Q}(q^{\frac{1}{2d}}) \otimes \mathbb{Q}(q^{\frac{1}{2d}}))$ would be a proper subalgebra.

With our new description of $A \star A'$, it is easily shown by checking on generators that $F_{i,2} \triangleright (A \star A') = \{0\}$. $\square$

The following corollary is then immediate.

Corollary 3.37. For any $n \geq 1$, we have isomorphisms of $\mathcal{K}$-module algebras:

$$\mathcal{A}_q[U]^{\underline{\otimes} n} \cong \mathcal{A}_q[U]^{\overline{\otimes} n}, \quad \mathcal{A}_q[B]^{\underline{\otimes} n} \cong \mathcal{A}_q[B]^{\overline{\otimes} n}.$$

CHAPTER IV

BASED MODULE ALGEBRAS

Given vector spaces V and V' over $\mathbb{Q}(q^{\frac{1}{2d}})$, we say a map $\varphi : V \to V'$ is *antilinear* if it is $\mathbb{Q}$-linear and

$$\varphi(q^{\frac{1}{2d}}v) = q^{-\frac{1}{2d}}\varphi(v)$$

for all $v \in V$. Following [23], we define an antilinear involution $^- : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ by

$$\overline{K_i^{\pm\frac{1}{2}}} = K_i^{\mp\frac{1}{2}}, \quad \overline{E_i} = E_i, \quad \text{and} \quad \overline{F_i} = F_i$$

for all $i \in I$.

4.1. Barred Module Algebras

Definition 4.1. A *barred module* is a pair $(M, ^-)$, where M is a $U_q(\mathfrak{g})$-module and $^- : M \to M$ is an anti-linear involution such that $\overline{u(m)} = \overline{u}(\overline{m})$ for each $u \in U_q(\mathfrak{g})$ and $m \in M$.

Example 4.2. Let V_λ be the irreducible finite dimensional $U_q(\mathfrak{g})$-module of highest weight $\lambda \in \Lambda^+$ and let v_λ be a highest weight vector. There is a unique anti-linear involution $^- : V_\lambda \to V_\lambda$ with $\overline{v_\lambda} = v_\lambda$ such that $(V_\lambda, ^-)$ is a barred module.

Definition 4.3. $U_q(\mathfrak{g})$-**BarMod** is the category whose objects are barred modules and whose morphisms are homomorphisms of the underlying $U_q(\mathfrak{g})$-modules which preserve the bar. Namely, a morphism $(M_1, ^-) \to (M_2, ^-)$ is a $U_q(\mathfrak{g})$-module homomorphism $\varphi : M_1 \to M_2$ such that $\overline{\varphi(m_1)} = \varphi(\overline{m_1})$ for all $m_1 \in M_1$.

Theorem 4.4. $U_q(\mathfrak{g})$-**BarMod** is monoidal with $(M_1, {}^-) \otimes (M_2, {}^-) := (M_1 \otimes M_2, {}^-)$, usual tensor product of $U_q(\mathfrak{g})$-module homomorphisms, and unit object $(\mathbb{Q}(q^{\frac{1}{2d}}), {}^-)$, where $M_1 \otimes M_2$ is considered a $U_q(\mathfrak{g})$-module in the usual way,

$$\overline{m_1 \otimes m_2} = \mathcal{R}_{2,1}(\overline{m_1} \otimes \overline{m_2})$$

for $m_1 \in M_1$ and $m_2 \in M_2$, and $\mathbb{Q}(q^{\frac{1}{2d}})$ is the trivial module with obvious bar. The associativity and left and right unit isomorphisms are the same as those for $U_q(\mathfrak{g})$-**WMod**.

Proof. We first show that $(M_1 \otimes M_2, {}^-)$ is indeed an object of $U_q(\mathfrak{g})$-**BarMod** as defined, namely that the bar is an anti-linear involution and $\overline{u(z)} = \overline{u}(\overline{z})$ for $u \in U_q(\mathfrak{g})$ and $z \in M_1 \otimes M_2$.

The bar is a composition of a linear map with an anti-linear map and is therefore anti-linear. To see that it is an involution, we observe that $({}^- \otimes {}^-) \circ \Pi^{\frac{1}{2}} \circ ({}^- \otimes {}^-) = \Pi^{-\frac{1}{2}}$ and according to [23, Corollary 4.1.3], $({}^- \otimes {}^-) \circ \Theta^{-1} \circ ({}^- \otimes {}^-) = \overline{\Theta^{-1}} = \Theta$. It follows that $({}^- \otimes {}^-) \circ \mathcal{R}_{2,1} \circ ({}^- \otimes {}^-) = \mathcal{R}_{2,1}^{-1}$. Therefore, the square of the bar is given by

$$\mathcal{R}_{2,1} \circ ({}^- \otimes {}^-) \circ \mathcal{R}_{2,1} \circ ({}^- \otimes {}^-) = \mathcal{R}_{2,1} \circ \mathcal{R}_{2,1}^{-1} = \mathrm{id}$$

and so the bar is an involution.

The equality $\overline{u(z)} = \overline{u}(\overline{z})$ for $u \in U_q(\mathfrak{g})$ and $z \in M_1 \otimes M_2$ is equivalent to $\mathcal{R}_{2,1} \circ ({}^- \otimes {}^-) \circ \Delta(u) = \Delta(\overline{u}) \circ \mathcal{R}_{2,1} \circ ({}^- \otimes {}^-)$. We need the following observation, which is easily checked on generators:

$$({}^- \otimes {}^-)(\Delta^{op}(u)) = \Delta(\overline{u}) \text{ for all } u \in U_q(\mathfrak{g}).$$

Therefore we see that

$$\begin{aligned}
\mathcal{R}_{2,1} \circ (^- \otimes {}^-) \circ \Delta(u) \circ (^- \otimes {}^-) \circ \mathcal{R}_{2,1}^{-1} &= \tau \circ \mathcal{R} \circ (^- \otimes {}^-) \circ \Delta^{op}(u) \circ (^- \otimes {}^-) \circ \mathcal{R}^{-1} \circ \tau \\
&= \tau \circ \mathcal{R} \circ \overline{\Delta^{op}(u)} \circ \mathcal{R}^{-1} \circ \tau \\
&= \tau \circ \mathcal{R} \circ \Delta(\overline{u}) \circ \mathcal{R}^{-1} \circ \tau \\
&= \tau \circ \Delta^{op}(\overline{u}) \circ \tau \\
&= \Delta(\overline{u})
\end{aligned}$$

or equivalently $\mathcal{R}_{2,1} \circ (^- \otimes {}^-) \circ \Delta(u) = \Delta(\overline{u}) \circ \mathcal{R}_{2,1} \circ (^- \otimes {}^-)$, as desired.

We now show that if $f : M_1 \to N_1$ and $g : M_2 \to N_2$ are morphisms in $U_q(\mathfrak{g})$-**BarMod**, then $f \otimes g : M_1 \otimes M_2 \to N_1 \otimes N_2$ is as well. We observe that f and g preserve the weights of the factors and $f \otimes g$ is $\mathbb{Q}(q^{\frac{1}{2d}})$-linear, so $\Pi^{\frac{1}{2}} \circ (f \otimes g) = (f \otimes g) \circ \Pi^{\frac{1}{2}}$. Further, since f and g are $U_q(\mathfrak{g})$-module homomorphisms, $\Theta^{-1} \circ (f \otimes g) = (f \otimes g) \circ \Theta^{-1}$. So then $\mathcal{R}_{2,1} \circ (f \otimes g) = (f \otimes g) \circ \mathcal{R}_{2,1}$ and hence

$$^- \circ (f \otimes g) = \mathcal{R}_{2,1} \circ (^- \otimes {}^-) \circ (f \otimes g) = (f \otimes g) \circ \mathcal{R}_{2,1} \circ (^- \otimes {}^-) = (f \otimes g) \circ {}^-.$$

Therefore $f \otimes g$ is a morphism in $U_q(\mathfrak{g})$-**BarMod**.

We next show that the associativity isomorphism of $U_q(\mathfrak{g})$-**WMod** is indeed a morphism in $U_q(\mathfrak{g})$-**BarMod**. The bar on the iterated tensor product $(M_1 \otimes M_2) \otimes M_3$ is given by $\mathcal{R}_{3,12} \circ \mathcal{R}_{2,1} \circ (^- \otimes {}^- \otimes {}^-)$ while that of $M_1 \otimes (M_2 \otimes M_3)$ is given by $\mathcal{R}_{23,1} \circ \mathcal{R}_{3,2} \circ (^- \otimes {}^- \otimes {}^-)$. To show that they coincide, we therefore show that $\mathcal{R}_{3,12} \circ \mathcal{R}_{2,1} = \mathcal{R}_{23,1} \circ \mathcal{R}_{3,2}$. This can be checked directly:

$$
\begin{aligned}
\mathcal{R}_{3,12} \circ \mathcal{R}_{2,1} &= \tau_{(132)} \circ \mathcal{R}_{1,23} \circ \mathcal{R}_{3,2} \circ \tau_{(123)} \\
&= \tau_{(132)} \circ \mathcal{R}_{1,3} \circ \mathcal{R}_{1,2} \circ \mathcal{R}_{3,2} \circ \tau_{(123)} \\
&= \mathcal{R}_{3,2} \circ \mathcal{R}_{3,1} \circ \mathcal{R}_{2,1} \\
&= \tau_{(13)} \circ \mathcal{R}_{1,2} \circ \mathcal{R}_{1,3} \circ \mathcal{R}_{2,3} \circ \tau_{(13)} \\
&= \tau_{(13)} \circ \mathcal{R}_{2,3} \circ \mathcal{R}_{1,3} \circ \mathcal{R}_{1,2} \circ \tau_{(13)} \\
&= \mathcal{R}_{2,1} \circ \mathcal{R}_{3,1} \circ \mathcal{R}_{3,2} \\
&= \tau_{(123)} \circ \mathcal{R}_{1,3} \circ \mathcal{R}_{2,3} \circ \mathcal{R}_{2,1} \circ \tau_{(132)} \\
&= \tau_{(123)} \circ \mathcal{R}_{12,3} \circ \mathcal{R}_{2,1} \circ \tau_{(132)} \\
&= \mathcal{R}_{23,1} \circ \mathcal{R}_{3,2}.
\end{aligned}
$$

Finally, we show that the left unit isomorphism of $U_q(\mathfrak{g})$-**WMod** is a morphism in $U_q(\mathfrak{g})$-**BarMod** and omit the proof for the right unit isomorphism as it is nearly identical. Given a barred module M, $m \in M$, and $c \in \mathbb{Q}(q^{\frac{1}{2d}})$, we use the fact that $(\varepsilon \otimes \mathrm{id})(\Theta^{-1}) = 1$ to compute:

$$
\begin{aligned}
\overline{c \otimes m} &= \mathcal{R}_{2,1}(\overline{c} \otimes \overline{m}) \\
&= (\Pi^{\frac{1}{2}} \circ \Theta^{-1} \circ \Pi^{\frac{1}{2}})(\overline{c} \otimes \overline{m}) \\
&= (\Pi^{\frac{1}{2}} \circ \Theta^{-1})(\overline{c} \otimes \overline{m}) \\
&= (\Pi^{\frac{1}{2}} \circ (\varepsilon \otimes \mathrm{id})(\Theta^{-1}))(\overline{c} \otimes \overline{m}) \\
&= \Pi^{\frac{1}{2}}(\overline{c} \otimes \overline{m}) \\
&= \overline{c} \otimes \overline{m}
\end{aligned}
$$

It follows that the left unit isomorphism of $U_q(\mathfrak{g})$-**WMod** is a morphism in $U_q(\mathfrak{g})$-**BarMod**. $\square$

Morally speaking, the preceding theorem should be seen as an analogue of classical results about the monoidality of categories of modules over bialgebras since barred modules are (certain) $(U_q(\mathfrak{g}) \rtimes \mathbb{Q}C_2)$-modules. However, since $\mathbb{Q}(q^{\frac{1}{2d}})$ is not central in $U_q(\mathfrak{g}) \rtimes \mathbb{Q}C_2$, it cannot possibly be a bialgebra. This leads us naturally to the theory of bialgebroids, which we address in Chapter VI. However, the failure of $\mathcal{R}$ to be an element of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ forces us to address the objects at hand separately, though in fact, they were the inspiration for that theory.

Definition 4.5. A *barred module algebra* is a barred module $(A, ^-)$ such that additionally A is a $U_q(\mathfrak{g})$-module algebra and $^- : A \to A$ is a $\mathbb{Q}$-algebra anti-involution, i.e. $\overline{a \cdot a'} = \overline{a'} \cdot \overline{a}$ for all $a, a' \in A$.

Example 4.6. The unique anti-linear anti-involution on $\mathcal{A}_q[Mat_{m,n}]$ so that $\overline{x_{i,j}} = x_{i,j}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$ makes $(\mathcal{A}_q[Mat_{m,n}], ^-)$ into a barred module algebra.

Definition 4.7. $U_q(\mathfrak{g})$-**BarModAlg** is the category whose objects are barred module algebras and whose morphisms are homomorphisms of the underlying $U_q(\mathfrak{g})$-module algebras which preserve the bar. Namely, a morphism $(A_1, ^-) \to (A_2, ^-)$ is a $U_q(\mathfrak{g})$-module algebra homomorphism $\varphi : A_1 \to A_2$ such that $\overline{\varphi(a_1)} = \varphi(\overline{a_1})$ for all $a_1 \in A_1$.

Theorem 4.8. $U_q(\mathfrak{g})$-**BarModAlg** is monoidal with $(A_1, ^-) \otimes (A_2, ^-) := (A_1 \underline{\otimes} A_2, ^-)$ and unit object $(\mathbb{Q}(q^{\frac{1}{2d}}), ^-)$, where $A_1 \underline{\otimes} A_2$ is the braided tensor product, considered as a $U_q(\mathfrak{g})$-module algebra in the usual way, $\overline{a_1 \otimes a_2} = \mathcal{R}_{2,1}(\overline{a_1} \otimes \overline{a_2})$ for $a_1 \in A_1$ and $a_2 \in A_2$, and $\mathbb{Q}(q^{\frac{1}{2d}})$ is the trivial module algebra with obvious bar. The associativity and left and right unit isomorphisms are the same as those for $U_q(\mathfrak{g})$-**WModAlg**.

Proof. The proof of Theorem 4.4 carries through exactly here, with one exception: we must show that the bar on the tensor product is an algebra anti-involution. In other words, we must show that the maps ${}^{-} \circ m_{A\underline{\otimes}B} : (A \otimes B) \otimes (A \otimes B) \to A \otimes B$ and $m_{A\underline{\otimes}B} \circ \tau \circ ({}^{-} \otimes {}^{-}) : (A \otimes B) \otimes (A \otimes B) \to A \otimes B$ coincide for any barred module algebras $(A, {}^{-})$ and $(B, {}^{-})$, where m_A and m_B denote their multiplications, respectively, and $m_{A\underline{\otimes}B}$ denotes the multiplication of $A\underline{\otimes}B$. The former is given by the composition $\mathcal{R}_{2,1} \circ ({}^{-} \otimes {}^{-}) \circ (m_A \otimes m_B) \circ \tau_{(23)} \circ \mathcal{R}_{2,3}$ on $A \otimes B \otimes A \otimes B$, while the latter is given by $(m_A \otimes m_B) \circ \tau_{(23)} \circ \mathcal{R}_{2,3} \circ \tau_{(13)(24)} \circ \mathcal{R}_{2,1} \circ \mathcal{R}_{4,3} \circ ({}^{-} \otimes {}^{-} \otimes {}^{-} \otimes {}^{-})$. In order to show these are equal, we need the following identity:

$$\mathcal{R}_{2,1} \circ (m_A \otimes m_B) = (m_A \otimes m_B) \circ \mathcal{R}_{3,2} \circ \mathcal{R}_{4,2} \circ \mathcal{R}_{3,1} \circ \mathcal{R}_{4,1}.$$

This can be shown as follows, using the fact that m_A and m_B are $U_q(\mathfrak{g})$-module homomorphisms and therefore the linear maps $1 \otimes m_A : B \otimes (A \otimes A) \to B \otimes A$ and $m_B \otimes 1 : (B \otimes B) \otimes (A \otimes A) \to B \otimes (A \otimes A)$ commute with $\mathcal{R}$.

$$\begin{aligned}
\mathcal{R}_{2,1} \circ (m_A \otimes m_B) &= \tau \circ \mathcal{R} \circ (m_B \otimes m_A) \circ \tau_{(13)(24)} \\
&= \tau \circ (m_B \otimes 1) \circ \mathcal{R}_{12,3} \circ (1 \otimes m_A) \circ \tau_{(13)(24)} \\
&= \tau \circ (m_B \otimes 1) \circ \mathcal{R}_{1,3} \circ \mathcal{R}_{2,3} \circ (1 \otimes m_A) \circ \tau_{(13)(24)} \\
&= \tau \circ (m_B \otimes m_A) \circ \mathcal{R}_{1,34} \circ \mathcal{R}_{2,34} \circ \tau_{(13)(24)} \\
&= \tau \circ (m_B \otimes m_A) \circ \mathcal{R}_{1,4} \circ \mathcal{R}_{1,3} \circ \mathcal{R}_{2,4} \circ \mathcal{R}_{2,3} \circ \tau_{(13)(24)} \\
&= (m_A \otimes m_B) \circ \mathcal{R}_{3,2} \circ \mathcal{R}_{4,2} \circ \mathcal{R}_{3,1} \circ \mathcal{R}_{4,1}.
\end{aligned}$$

We can now finish the proof.

$$
\begin{aligned}
&\mathcal{R}_{2,1} \circ (^- \otimes {}^-) \circ (m_A \otimes m_B) \circ \tau_{(23)} \circ \mathcal{R}_{2,3} \\
&= \mathcal{R}_{2,1} \circ (m_A \otimes m_B) \circ \tau_{(12)(34)} \circ (^- \otimes {}^- \otimes {}^- \otimes {}^-) \circ \tau_{(23)} \circ \mathcal{R}_{2,3} \\
&= \mathcal{R}_{2,1} \circ (m_A \otimes m_B) \circ \tau_{(1243)} \circ (^- \otimes {}^- \otimes {}^- \otimes {}^-) \circ \mathcal{R}_{2,3} \\
&= (m_A \otimes m_B) \circ \mathcal{R}_{3,2} \circ \mathcal{R}_{4,2} \circ \mathcal{R}_{3,1} \circ \mathcal{R}_{4,1} \circ \tau_{(1243)} \circ (^- \otimes {}^- \otimes {}^- \otimes {}^-) \circ \mathcal{R}_{2,3} \\
&= (m_A \otimes m_B) \circ \mathcal{R}_{3,2} \circ \mathcal{R}_{4,2} \circ \mathcal{R}_{3,1} \circ \mathcal{R}_{4,1} \circ \tau_{(1243)} \circ \mathcal{R}_{2,3}^{-1} \circ (^- \otimes {}^- \otimes {}^- \otimes {}^-) \\
&= (m_A \otimes m_B) \circ \tau_{(23)} \circ \mathcal{R}_{2,3} \circ \tau_{(13)(24)} \circ \mathcal{R}_{2,1} \circ \mathcal{R}_{4,3} \circ \mathcal{R}_{2,3} \circ \mathcal{R}_{2,3}^{-1} \circ (^- \otimes {}^- \otimes {}^- \otimes {}^-) \\
&= (m_A \otimes m_B) \circ \tau_{(23)} \circ \mathcal{R}_{2,3} \circ \tau_{(13)(24)} \circ \mathcal{R}_{2,1} \circ \mathcal{R}_{4,3} \circ (^- \otimes {}^- \otimes {}^- \otimes {}^-) \qquad \square
\end{aligned}
$$

The beauty of this construction is that we really don't need to define the bar on the braided tensor product; it occurs naturally. Indeed, if we desire a bar that coincides with the bar on each factor and is an algebra anti-homomorphism, we have no choice but to set $\overline{a \otimes b} = (1 \otimes \bar{b})(\bar{a} \otimes 1)$. That this actually defines an anti-involution is a result of the compatibility of our particular choices of $\mathcal{R}$ and the bar on $U_q(\mathfrak{g})$.

Example 4.9. If we equip $\mathcal{A}_q[Mat_{m,n_1}]$, $\mathcal{A}_q[Mat_{m,n_2}]$, and $\mathcal{A}_q[Mat_{m,n_1+n_2}]$ with the bar of Example 4.6, we have

$$(\mathcal{A}_q[Mat_{m,n_1}], {}^-) \otimes (\mathcal{A}_q[Mat_{m,n_2}], {}^-) \cong (\mathcal{A}_q[Mat_{m,n_1+n_2}], {}^-).$$

4.2. Based Module Algebras

Let M be a vector space over $\mathbb{Q}(q^{\frac{1}{2d}})$ and $\mathcal{B}$ a basis for M. Given an element $m = \sum_{b \in \mathcal{B}} a_{m,b} b$ with $a_{m,b} \in \mathbb{Q}(q^{\frac{1}{2d}})$, we set

$$supp(m, \mathcal{B}) := \{b \in \mathcal{B} \mid a_{m,b} \neq 0\}$$

and call $supp(m, \mathcal{B})$ the *support of m with respect to $\mathcal{B}$*. When it is clear from context which basis is being used, we may use $supp(m)$ and call it simply the support of m.

If, additionally, M is a $U_q(\mathfrak{g})$-module and $\mathcal{B}$ is homogeneous, an *ascending string of length n in $\mathcal{B}$* is an n-tuple $(b_1, \ldots, b_n) \in \mathcal{B}^n$ such that there exists an $(n-1)$-tuple $(\beta_1, \ldots, \beta_{n-1}) \in (\Phi^+)^{n-1}$ such that $b_{i+1} \in supp(E_{\beta_i}(b_i), \mathcal{B})$ for $i = 1, \ldots, n-1$.

Since $\mathcal{B}$ is a homogenous basis, it is clear by weight considerations that the $(n-1)$-tuple $(\beta_1, \ldots, \beta_{n-1})$ is unique for each ascending string of length n in $\mathcal{B}$. It is also clear that any ascending string of maximal length must have a corresponding tuple of roots which consists of only simple roots.

Definition 4.10. Let M be a $U_q(\mathfrak{g})$-module and $\mathcal{B}$ a homogeneous basis of M. The pair $(M, \mathcal{B})$ is called a based module if the following hold.

- For $i \in I$ and $n \geq 1$, $E_i^{(n)}$ and $F_i^{(n)}$ preserve $\mathbb{Q}[q^{\frac{1}{2d}}, q^{-\frac{1}{2d}}]\mathcal{B}$
- Setting $\overline{b} = b$ for $b \in \mathcal{B}$ and extending anti-linearly makes $(M, ^-)$ into a barred module.
- There exists a function $\varepsilon_{\mathcal{B}} : \mathcal{B} \to \mathbb{Z}_{\geq 0}$ such that the following hold.
  * If $b' \in supp(E_i(b), \mathcal{B})$, then $\varepsilon_{\mathcal{B}}(b') \leq \varepsilon_{\mathcal{B}}(b) - 1$.
  * If $b' \in supp(F_i(b), \mathcal{B})$, then $\varepsilon_{\mathcal{B}}(b') \leq \varepsilon_{\mathcal{B}}(b) + 1$.
  * All ascending strings in $\mathcal{B}$ starting with $b \in \mathcal{B}$ have length at most $\varepsilon_{\mathcal{B}}(b)+1$.

Note that we don't require that $\varepsilon_{\mathcal{B}}(b)$ is the supremum of the lengths of ascending strings in $\mathcal{B}$, only that it be an upper bound.

Example 4.11. Let $\mathcal{B}_\lambda$ be Lusztig's canonical basis for the simple $U_q(\mathfrak{g})$-module V_λ and $\mathcal{B}_\lambda^{dual}$ its dual basis under the Shapovalov form. Then $(V_\lambda, \mathcal{B}_\lambda)$ and $(V_\lambda, \mathcal{B}_\lambda^{dual})$

are based modules. In fact, if $(M, \mathcal{B})$ is a based module in the sense of [23, 27.1.2], then it is a based module in our sense. For the simple modules V_λ, we may set $\varepsilon_{\mathcal{B}}(b) = ht(\lambda - |b|)$, where $\mathcal{B} = \mathcal{B}_\lambda$ or $\mathcal{B} = \mathcal{B}_\lambda^{dual}$ and $b \in \mathcal{B}$.

Given based modules $(M_1, \mathcal{B}_1)$ and $(M_2, \mathcal{B}_2)$, we define a binary relation on $\mathcal{B}_1 \times \mathcal{B}_2$ via $(b_1, b_2) > (b_1', b_2')$ if the following conditions hold:

– $|b_1'| + |b_2'| = |b_1| + |b_2|$, $|b_1'| < |b_1|$, and $|b_2'| > |b_2|$;

– $\varepsilon_{\mathcal{B}_1}(b_1') \leq \varepsilon_{\mathcal{B}_1}(b_1) + ht(|b_1| - |b_1'|)$ and $\varepsilon_{\mathcal{B}_2}(b_2') \leq \varepsilon_{\mathcal{B}_2}(b_2) - ht(|b_2'| - |b_2|)$.

This is clearly a partial order. It has the property that the length of chains in $\mathcal{B}_1 \times \mathcal{B}_2$ with top element (b_1, b_2) are bounded above by $\varepsilon_{\mathcal{B}_2}(b_2)$. Furthermore, it is clear that if $(b_1', b_2') \leq (b_1, b_2)$, then $\varepsilon_{\mathcal{B}_1}(b_1') + \varepsilon_{\mathcal{B}_2}(b_2') \leq \varepsilon_{\mathcal{B}_1}(b_1) + \varepsilon_{\mathcal{B}_2}(b_2)$. By Theorem 4.4, we have a bar-involution

$$\overline{m_1 \otimes m_2} = \mathcal{R}_{2,1}(\overline{m_1} \otimes \overline{m_2}),$$

which has the property that $\overline{u(m)} = \overline{u}(\overline{m})$ for $u \in U_q(\mathfrak{g})$ and $m \in M$. We write $b_1 * b_2 := q^{(|b_1|,|b_2|)/2} b_1 \otimes b_2$ for $b_1 \in \mathcal{B}_1$ and $b_2 \in \mathcal{B}_2$ and consider the basis

$$\mathcal{B}_1 * \mathcal{B}_2 := \{b_1 * b_2 \mid b_1 \in \mathcal{B}_1,\ b_2 \in \mathcal{B}_2\}$$

of $M_1 \otimes M_2$. It is clear based on our axioms of based module and [23, Corollary 24.1.6] (and since Lusztig's canonical basis elements are contained in ${}_{\mathcal{A}}\mathbf{f}$, which is generated by $\theta_i^{(n)}$) that we have

$$\overline{b_1 * b_2} - b_1 * b_2 \in \sum_{(b_1', b_2') < (b_1, b_2)} \mathbb{Q}[q^{\frac{1}{2d}}, q^{-\frac{1}{2d}}] b_1' * b_2'.$$

For the reader's convenience, we include a version of Lusztig's Lemma, which we shall use presently.

Lemma 4.12. [9, Theorem 1.1] Let A be a free $\mathbb{Z}[v, v^{-1}]$-module with a basis $\{E_a : a \in L\}$ indexed by a partially ordered set $(L, \prec)$ such that, for any $a \in L$, the lengths of chains in L with the top element a are bounded from above. Let $x \mapsto \bar{x}$ be a $\mathbb{Z}$-linear involution on A such that, for all $f \in \mathbb{Z}[v, v^{-1}]$ and $x \in A$, we have

$$\overline{fx} = \bar{f}\bar{x}, \text{ where } \bar{f}(v) = f(v^{-1}).$$

Suppose that

$$\bar{E}_a - E_a \in \bigoplus_{a' \prec a} \mathbb{Z}[v, v^{-1}]E_{a'} \quad (a \in L).$$

Then, for every $a \in L$, there exists a unique element $C_a \in A$ such that:

$$\bar{C}_a = C_a;$$

$$C_a - E_a \in \bigoplus_{a' \in L} v\mathbb{Z}[v]E'_a.$$

Moreover, the element C_a satisfies

$$C_a - E_a \in \bigoplus_{a' \prec a} v\mathbb{Z}[v]E_{a'},$$

hence the elements C_a, for $a \in L$ form a $\mathbb{Z}[v, v^{-1}]$-basis in A.

The proof of the preceding lemma still works with $\mathbb{Q}$ in place of $\mathbb{Z}$ and $q^{-\frac{1}{2d}}$ in place of v. Utilizing the lemma with A equal to the $\mathbb{Q}[q^{\frac{1}{2d}}, q^{-\frac{1}{2d}}]$-span of $\mathcal{B}_1 * \mathcal{B}_2$, then extending scalars, we obtain the following theorem.

Theorem 4.13. Given based modules $(M_1, \mathcal{B}_1)$ and $(M_2, \mathcal{B}_2)$, there is a unique bar-invariant basis, $\mathcal{B}_1 \diamond \mathcal{B}_2 = \{b_1 \diamond b_2 \mid (b_1, b_2) \in \mathcal{B}_1 \times \mathcal{B}_2\}$, for $M_1 \otimes M_2$ such that

$$b_1 \diamond b_2 - b_1 * b_2 \in \sum_{(b_1', b_2') \neq (b_1, b_2)} q^{-\frac{1}{2d}} \mathbb{Q}[q^{-\frac{1}{2d}}] b_1' * b_2'.$$

Furthermore, we have

$$b_1 \diamond b_2 - b_1 * b_2 \in \sum_{(b_1', b_2') < (b_1, b_2)} q^{-\frac{1}{2d}} \mathbb{Q}[q^{-\frac{1}{2d}}] b_1' * b_2'.$$

Theorem 4.14. If $(M_1, \mathcal{B}_1)$ and $(M_2, \mathcal{B}_2)$ are based modules, then $(M_1 \otimes M_2, \mathcal{B}_1 \diamond \mathcal{B}_2)$ is a based module.

Proof. For any $i \in I$ and $n \geq 1$, it is easy to deduce that $E_i^{(n)}$ and $F_i^{(n)}$ preserve $\mathbb{Q}[q^{\frac{1}{2d}}, q^{-\frac{1}{2d}}](\mathcal{B}_1 \diamond \mathcal{B}_2)$ from the formulas

$$\Delta(E_i^{(n)}) = \sum_{a=0}^{n} K_i^{(a-n)/2} E_i^{(a)} \otimes K_i^{a/2} E_i^{(n-a)}$$

and

$$\Delta(F_i^{(n)}) = \sum_{a=0}^{n} K_i^{(a-n)/2} F_i^{(a)} \otimes K_i^{a/2} F_i^{(n-a)}.$$

We have already seen that the bar we have defined on $M_1 \otimes M_2$ satisfies $\overline{b_1 \diamond b_2} = b_1 \diamond b_2$ and $\overline{u(m)} = \overline{u}(\overline{m})$. Our candidate for $\varepsilon_{\mathcal{B}_1 \diamond \mathcal{B}_2}(b_1 \diamond b_2)$ is $\varepsilon_{\mathcal{B}_1}(b_1) + \varepsilon_{\mathcal{B}_2}(b_2)$. We compute:

$$
\begin{aligned}
E_i(b_1 \diamond b_2) &= E_i\left(\sum_{(b_1',b_2')\leq(b_1,b_2)} a_{b_1,b_2}^{b_1',b_2'} q^{(|b_1'|,|b_2'|)/2} b_1' \otimes b_2'\right) \\
&= \sum_{(b_1',b_2')\leq(b_1,b_2)} a_{b_1,b_2}^{b_1',b_2'} \left(q^{(|b_1'|+\alpha_i,|b_2'|)/2} E_i(b_1') \otimes b_2' + q^{(|b_1'|,|b_2'|-\alpha_i)/2} b_1' \otimes E_i(b_2')\right) \\
&= \sum_{\substack{(b_1',b_2')\leq(b_1,b_2) \\ b_1''\in supp(E_i(b_1'),\mathcal{B}_1)}} a_{b_1,b_2}^{b_1',b_2'} q^{(|b_1'|+\alpha_i,|b_2'|)/2} c^i_{b_1',b_1''} b_1'' \otimes b_2' \\
&\quad + \sum_{\substack{(b_1',b_2')\leq(b_1,b_2) \\ b_2''\in supp(E_i(b_2'),\mathcal{B}_2)}} a_{b_1,b_2}^{b_1',b_2'} q^{(|b_1'|,|b_2'|-\alpha_i)/2} c^i_{b_2',b_2''} b_1' \otimes b_2'' \\
&= \sum_{\substack{(b_1',b_2')\leq(b_1,b_2) \\ b_1''\in supp(E_i(b_1'),\mathcal{B}_1)}} a_{b_1,b_2}^{b_1',b_2'} c^i_{b_1',b_1''} q^{(|b_1'|+\alpha_i-|b_1''|,|b_2'|)/2} q^{(|b_1''|,|b_2'|)/2} b_1'' \otimes b_2' \\
&\quad + \sum_{\substack{(b_1',b_2')\leq(b_1,b_2) \\ b_2''\in supp(E_i(b_2'),\mathcal{B}_2)}} a_{b_1,b_2}^{b_1',b_2'} c^i_{b_2',b_2''} q^{(|b_1'|,|b_2'|-\alpha_i-|b_2''|)/2} q^{(|b_1'|,|b_2''|)/2} b_1' \otimes b_2'' \\
&= \sum_{\substack{(b_1',b_2')\leq(b_1,b_2) \\ b_1''\in supp(E_i(b_1'),\mathcal{B}_1) \\ (b_1''',b_2''')\leq(b_1'',b_2')}} a_{b_1,b_2}^{b_1',b_2'} c^i_{b_1',b_1''} (a^{-1})_{b_1'',b_2'}^{b_1''',b_2'''} q^{(|b_1'|+\alpha_i-|b_1''|,|b_2'|)/2} b_1''' \diamond b_2''' \\
&\quad + \sum_{\substack{(b_1',b_2')\leq(b_1,b_2) \\ b_2''\in supp(E_i(b_2'),\mathcal{B}_2) \\ (b_1''',b_2''')\leq(b_1',b_2'')}} a_{b_1,b_2}^{b_1',b_2'} c^i_{b_2',b_2''} (a^{-1})_{b_1',b_2''}^{b_1''',b_2'''} q^{(|b_1'|,|b_2'|-\alpha_i-|b_2''|)/2} b_1''' \diamond b_2'''
\end{aligned}
$$

where we write

$$b_1 \diamond b_2 = \sum_{(b_1',b_2')\leq(b_1,b_2)} a_{b_1,b_2}^{b_1',b_2'} q^{(|b_1'|,|b_2'|)/2} b_1' \otimes b_2'$$

$$q^{(|b_1|,|b_2|)/2} b_1 \otimes b_2 = \sum_{(b_1',b_2')\leq(b_1,b_2)} (a^{-1})_{b_1,b_2}^{b_1',b_2'} b_1' \diamond b_2'$$

$$E_i(b) = \sum_{b'\in supp(E_i(b),\mathcal{B})} c^i_{b,b'} b'$$

for some $a_{b_1,b_2}^{b_1',b_2'}, (a^{-1})_{b_1,b_2}^{b_1',b_2'}, c_{b,b'}^i \in \mathbb{Q}[q^{\frac{1}{2d}}, q^{-\frac{1}{2d}}]$. It is now clear that if

$$b_1' \diamond b_2' \in supp(E_i(b_1 \diamond b_2), \mathcal{B}_1 \diamond \mathcal{B}_2),$$

then

$$\varepsilon_{\mathcal{B}_1}(b_1') + \varepsilon_{\mathcal{B}_2}(b_2') \leq \varepsilon_{\mathcal{B}_1}(b_1) + \varepsilon_{\mathcal{B}_2}(b_2) - 1.$$

A similar argument shows that if $b_1' \diamond b_2' \in supp(F_i(b_1 \diamond b_2), \mathcal{B}_1 \diamond \mathcal{B}_2)$, then

$$\varepsilon_{\mathcal{B}_1}(b_1') + \varepsilon_{\mathcal{B}_2}(b_2') \leq \varepsilon_{\mathcal{B}_1}(b_1) + \varepsilon_{\mathcal{B}_2}(b_2) + 1.$$

Furthermore, suppose an ascending string $(b_1 \diamond b_2, b_{1,2} \diamond b_{2,2}, \ldots, b_{1,n} \diamond b_{2,n})$ of length $n = \varepsilon_{\mathcal{B}_1}(b_1) + \varepsilon_{\mathcal{B}_2}(b_2) + 1$ exists. Then

$$\varepsilon_{\mathcal{B}_1}(b_{1,n}) + \varepsilon_{\mathcal{B}_2}(b_{2,n}) + 1 \leq n - (n-1) = 1.$$

Since $\varepsilon_{\mathcal{B}_1}(b_{1,n}), \varepsilon_{\mathcal{B}_2}(b_{2,n}) \in \mathbb{Z}_{\geq 0}$, this implies that $\varepsilon_{\mathcal{B}_1}(b_{1,n}) = \varepsilon_{\mathcal{B}_2}(b_{2,n}) = 0$, i.e. both $b_{1,n}$ and $b_{2,n}$ are highest weight vectors. In addition, this means that $b_{1,n} \diamond b_{2,n} = q^{(|b_{1,n}|,|b_{2,n}|)/2} b_{1,n} \otimes b_{2,n}$, which is clearly a highest weight vector. This means the ascending string cannot be extended. Hence every ascending string starting with $b_1 \diamond b_2$ has length at most $\varepsilon_{\mathcal{B}_1}(b_1) + \varepsilon_{\mathcal{B}_2}(b_2) + 1$. We conclude that the map $\varepsilon_{\mathcal{B}_1 \diamond \mathcal{B}_2}(b_1 \diamond b_2) := \varepsilon_{\mathcal{B}_1}(b_1) + \varepsilon_{\mathcal{B}_2}(b_2)$ satisfies the assumptions in the definition of based module. $\square$

The preceding theorem hints that there may be a monoidal category of based modules, which we define presently.

Definition 4.15. $U_q(\mathfrak{g})$-**BaseMod** is the category whose objects are based modules. Given objects $(M_1, \mathcal{B}_1)$ and $(M_2, \mathcal{B}_2)$, a morphism $(M_1, \mathcal{B}_1) \to (M_2, \mathcal{B}_2)$ is a morphism φ of the associated barred modules such that $\varphi(\mathcal{B}_1) \subset \mathcal{B}_2 \cup \{0\}$.

Corollary 4.16. $U_q(\mathfrak{g})$-**BaseMod** is monoidal with

$$(M_1, \mathcal{B}_1) \otimes (M_2, \mathcal{B}_2) := (M_1 \otimes M_2, \mathcal{B}_1 \diamond \mathcal{B}_2)$$

and unit object $(\mathbb{Q}(q^{\frac{1}{2d}}), \{1\})$, where $M_1 \otimes M_2$ is considered a $U_q(\mathfrak{g})$-module in the usual way, and $\mathbb{Q}(q^{\frac{1}{2d}})$ is the trivial module. The associativity and left and right unit isomorphisms are the same as those for $U_q(\mathfrak{g})$-**WMod**.

Proof. We've already shown that $(M_1 \otimes M_2, \mathcal{B}_{M_1} \diamond \mathcal{B}_{M_2})$ is a based module, so we now show that if $f : M_1 \to N_1$ and $g : M_2 \to N_2$ are morphisms $(M_1, \mathcal{B}_{M_1}) \to (N_1, \mathcal{B}_{N_1})$ and $(M_2, \mathcal{B}_{M_2}) \to (N_2, \mathcal{B}_{N_2})$ in $U_q(\mathfrak{g})$-**BaseMod**, respectively, then $f \otimes g : M_1 \otimes M_2 \to N_1 \otimes N_2$ is a morphism

$$(M_1 \otimes M_2, \mathcal{B}_{M_1} \diamond \mathcal{B}_{M_2}) \to (N_1 \otimes N_2, \mathcal{B}_{N_1} \diamond \mathcal{B}_{N_2}).$$

Keeping in mind our study of barred modules as well as how the diamond bases were defined, it suffices to show that $(f \otimes g)(\mathcal{B}_{M_1} \diamond \mathcal{B}_{M_2}) \subset (\mathcal{B}_{N_1} \diamond \mathcal{B}_{N_2}) \cup \{0\}$. We compute $(f \otimes g)(b_1 \diamond b_2)$ for $b_1 \in \mathcal{B}_{M_1}$ and $b_2 \in \mathcal{B}_{M_2}$, remembering that f and g preserve weight since they are $U_q(\mathfrak{g})$-module homomorphisms:

$$\begin{aligned}(f \otimes g)(b_1 \diamond b_2) &= (f \otimes g) \left(\sum_{(b_1', b_2') \leq (b_1, b_2)} a_{b_1, b_2}^{b_1', b_2'} q^{(|b_1'|, |b_2'|)/2} b_1' \otimes b_2' \right) \\ &= \sum_{(b_1', b_2') \leq (b_1, b_2)} a_{b_1, b_2}^{b_1', b_2'} q^{(|f(b_1')|, |g(b_2')|)/2} f(b_1') \otimes g(b_2')\end{aligned}$$

where $a_{b_1,b_2}^{b_1,b_2} = 1$ and $a_{b_1,b_2}^{b_1',b_2'} \in q^{-\frac{1}{2d}}\mathbb{Q}[q^{-\frac{1}{2d}}]$ if $(b_1',b_2') \neq (b_1,b_2)$. We note that in the sum, $f(b_1') \neq f(b_1)$ unless both are zero since $|f(b_1')| = |b_1'| < |b_1| = |f(b_1)|$ and similarly $g(b_2') \neq g(b_2)$ unless both are zero. Now

$$\overline{(f \otimes g)(b_1 \diamond b_2)} = (f \otimes g)(\overline{b_1 \diamond b_2}) = (f \otimes g)(b_1 \diamond b_2).$$

If $f(b_1)$ and $g(b_2)$ are both nonzero, then we see that $(f \otimes g)(b_1 \diamond b_2)$ is a bar-invariant element such that

$$(f \otimes g)(b_1 \diamond b_2) - f(b_1) * g(b_2) \in \sum_{(b_1',b_2') \neq (f(b_1),g(b_2))} q^{-\frac{1}{2d}}\mathbb{Q}[q^{-\frac{1}{2d}}]b_1' * b_2'.$$

By Theorem 4.13, we see that $(f \otimes g)(b_1 \diamond b_2) = f(b_1) \diamond g(b_2)$.

If, on the other hand, at least one of $f(b_1)$ and $g(b_2)$ is zero, then we have

$$(f \otimes g)(b_1 \diamond b_2) = \sum_{(b_1',b_2') < (b_1,b_2)} a_{b_1,b_2}^{b_1',b_2'} q^{(|f(b_1')|,|g(b_2')|)/2} f(b_1') \otimes g(b_2').$$

Assume for the sake of contradiction that $(f \otimes g)(b_1 \diamond b_2) \neq 0$. Since only finitely many $a_{b_1,b_2}^{b_1',b_2'}$ are nonzero, we may choose $(b_1^\circ, b_2^\circ) \in \mathcal{B}_{M_1} \times \mathcal{B}_{M_2}$ to be maximal in the partial order such that $(b_1^\circ, b_2^\circ) < (b_1, b_2)$, $f(b_1^\circ) \neq 0$, $g(b_2^\circ) \neq 0$, and $a_{b_1,b_2}^{b_1^\circ,b_2^\circ} \neq 0$. Then the coefficient of $f(b_1^\circ) * g(b_2^\circ)$ in the expansion of $\overline{(f \otimes g)(b_1 \diamond b_2)}$ in terms of $\mathcal{B}_{N_1} * \mathcal{B}_{N_2}$ is $\overline{a_{b_1,b_2}^{b_1^\circ,b_2^\circ}} \in q^{\frac{1}{2d}}\mathbb{Q}[q^{\frac{1}{2d}}]$. However, we showed that $\overline{(f \otimes g)(b_1 \diamond b_2)} = (f \otimes g)(b_1 \diamond b_2)$ and the coefficient of $f(b_1^\circ) * g(b_2^\circ)$ in the expansion of $(f \otimes g)(b_1 \diamond b_2)$ in terms of $\mathcal{B}_{N_1} * \mathcal{B}_{N_2}$ is $a_{b_1,b_2}^{b_1^\circ,b_2^\circ} \in q^{-\frac{1}{2d}}\mathbb{Q}[q^{-\frac{1}{2d}}]$. Hence we have achieved a contradiction, so if at least one of $f(b_1)$ and $g(b_2)$ is zero, then $(f \otimes g)(b_1 \diamond b_2) = 0$. We have shown that $(f \otimes g)(\mathcal{B}_{M_1} \diamond \mathcal{B}_{M_2}) \subset (\mathcal{B}_{N_1} \diamond \mathcal{B}_{N_2}) \cup \{0\}$.

We now show that the associativity isomorphism of $U_q(\mathfrak{g})$-**WMod** is indeed a morphism in $U_q(\mathfrak{g})$-**BaseMod**. In light of our work with barred modules, this is equivalent to showing that given based modules $(M_1, \mathcal{B}_1)$, $(M_2, \mathcal{B}_2)$, and $(M_3, \mathcal{B}_3)$, $(b_1 \diamond b_2) \diamond b_3 = b_1 \diamond (b_2 \diamond b_3)$ in $M_1 \otimes M_2 \otimes M_3$ for $b_1 \in M_1$, $b_2 \in M_2$, $b_3 \in M_3$. However, it is easily seen from their defining properties that both are the unique bar-invariant element $b_1 \diamond b_2 \diamond b_3$ such that

$$b_1 \diamond b_2 \diamond b_3 - b_1 * b_2 * b_3 \in \sum_{(b_1',b_2',b_3') \neq (b_1,b_2,b_3)} \tilde{a}_{b_1,b_2,b_3}^{b_1',b_2',b_3'} b_1' * b_2' * b_3'$$

for some $\tilde{a}_{b_1,b_2,b_3}^{b_1',b_2',b_3'} \in q^{-\frac{1}{2d}}\mathbb{Q}[q^{-\frac{1}{2d}}]$, where we write

$$b_1 * b_2 * b_3 = q^{[(|b_1|,|b_2|)+(|b_1|,|b_3|)+(|b_2|,|b_3|)]/2} b_1 \otimes b_2 \otimes b_3.$$

So the associativity isomorphism of $U_q(\mathfrak{g})$-**WMod** is a morphism in $U_q(\mathfrak{g})$-**BaseMod**.

Finally, we perform the trivial check that the left unit isomorphism of $U_q(\mathfrak{g})$-**WMod** is a morphism of $U_q(\mathfrak{g})$-**BaseMod** and omit the proof for the right unit isomorphism. As usual, our work with barred modules reduces the work necessary and we simply check that, given a based module $(M, \mathcal{B})$ and $b \in \mathcal{B}$, the left unit isomorphism sends $1 \diamond b$ to b. This follows from the trivial fact that $1 \diamond b = 1 \otimes b$. $\square$

Example 4.17. Consider type A_{m-1} for $m \geq 2$. Retaining Convention 2.13, except using the bilinear form $(\varepsilon_i, \varepsilon_j) = \delta_{i,j} - 1$ in place of the usual $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$, the multiplication map

$$\mathcal{A}_q[Mat_{m,1}] \otimes \mathcal{A}_q[Mat_{m,1}] \to \mathcal{A}_q[Mat_{m,1}]$$

is a morphism of based modules

$$(\mathcal{A}_q[Mat_{m,1}] \otimes \mathcal{A}_q[Mat_{m,1}], \mathcal{B}_{m,1} \diamond \mathcal{B}_{m,1}) \to (\mathcal{A}_q[Mat_{m,1}] \otimes \mathcal{A}_q[Mat_{m,1}], \mathcal{B}_{m,1}).$$

Another way to think about the preceding example is as a twist of multiplication by a bicharacter. Namely, we put a $\mathbb{Z}$-grading on $\mathcal{A}_q[Mat_{m,1}]$ via $|x_{i,1}|_{\mathbb{Z}} = 1$ and declare $x \cdot y = q^{-\frac{1}{2}|x|_{\mathbb{Z}}|y|_{\mathbb{Z}}}xy$ for homogeneous $x, y \in \mathcal{A}_q[Mat_{m,1}]$. Using the usual bilinear form, this new multiplication gives a morphism of based modules

$$(\mathcal{A}_q[Mat_{m,1}] \otimes \mathcal{A}_q[Mat_{m,1}], \mathcal{B}_{m,1} \diamond \mathcal{B}_{m,1}) \to (\mathcal{A}_q[Mat_{m,1}] \otimes \mathcal{A}_q[Mat_{m,1}], \mathcal{B}_{m,1}).$$

We conjecture that $\mathcal{A}_q[G/U]$ behaves similarly.

Conjecture 4.18. For each Cartan matrix C and choice of bilinear form, the twisted multiplication

$$xv_\lambda \otimes yv_\mu \mapsto q^{-(\lambda,\mu)/2}(xv_\lambda)(yv_\mu) = q^{(\lambda,|y|)-(\lambda,\mu)/2}xyv_{\lambda+\mu}$$

is a morphism of based modules

$$(\mathcal{A}_q[G/U] \otimes \mathcal{A}_q[G/U], \mathcal{B} \diamond \mathcal{B}) \to (\mathcal{A}_q[G/U], \mathcal{B}),$$

where $\mathcal{B}$ is the dual canonical basis.

If the preceding conjecture is accurate, iterating this twisted multiplication would give us an infinite family of morphisms in $U_q(\mathfrak{g})$-**BaseMod**. Namely, for each $n \geq 1$, we would have a morphism

$$(\mathcal{A}_q[G/U]^{\otimes n}, \mathcal{B}^{\diamond n}) \to (\mathcal{A}_q[G/U], \mathcal{B}).$$

When $m, n \geq 2$, the multiplication in $\mathcal{A}_q[Mat_{m,n}]$ is not nearly so well-behaved. For example, in each such case, we have the element $q^{(|x_{1,1}|,|x_{2,2}|)/2} x_{2,2} \otimes x_{1,1}$ in $\mathcal{B}_{m,n} \diamond \mathcal{B}_{m,n}$. We see that

$$q^{(|x_{1,1}|,|x_{2,2}|)/2} x_{2,2} \otimes x_{1,1} \mapsto q^{(|x_{1,1}|,|x_{2,2}|)/2}(x_{1,1}x_{2,2} + (q - q^{-1})x_{1,2}x_{2,1}).$$

The right-hand side is not bar-invariant or a multiple of a bar-invariant element, let alone a basis element. As such, we conclude that there is no bilinear form (or bicharacter) making the (twisted) multiplication of $\mathcal{A}_q[Mat_{m,n}]$ into a map of based modules if $m, n \geq 2$.

This observation seems to cause doubt about the veracity of the preceding conjecture, since $\mathcal{A}_q[SL_m/U]$ embeds into $\mathcal{A}_q[Mat_{m,m-1}]$ as a $U_q(\mathfrak{g})$-module algebra. However, we recall that if $j \geq 2$, then for any i, $x_{i,j}$ is not in the image of the embedding. Therefore this particular phenomenon is not possible and the conjecture remains intact. In fact, the following conjecture would provide supporting evidence for the preceding conjecture, if true.

Conjecture 4.19. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are upper global crystal bases for $U_q(\mathfrak{g})$-modules M_1 and M_2, respectively, then $\mathcal{B}_1 \diamond \mathcal{B}_2$ is an upper global crystal basis for $M_1 \otimes M_2$.

As with the barred setting, there is a natural analogue of based modules for module algebras.

Definition 4.20. A *based module algebra* is a based module $(A, \mathcal{B})$ such that, if $^-$ is the associated bar, then $(A, ^-)$ is a barred module algebra.

Example 4.21. For any $m \geq 2$ and $n \geq 1$, let $\mathcal{B}_{m,n}$ be the dual canonical basis for $\mathcal{A}_q[Mat_{m,n}]$, obtained via the identification

$$\mathcal{A}_q[Mat_{m,n}] \cong U_q(\mathbf{i}_{m,n}) \subset U_q(\mathfrak{sl}_{m+n}),$$

$$x_{k,\ell} \leftrightarrow X_{\mathbf{i}_{m,n},(k-1)n+\ell}$$

where $\mathbf{i}_{m,n} = (n, n-1, \ldots, 1, n+1, n, \ldots, 2, \ldots, m+n-1, m+n-2, \ldots, m)$ and $U_q(\mathbf{i}_{m,n})$ is the corresponding quantum Schubert cell as in [3], with generators $X_{\mathbf{i}_{m,n},j}$ for $j \in [1, mn]$. Then $(\mathcal{A}_q[Mat_{m,n}], \mathcal{B}_{m,n})$ is a based module algebra.

Definition 4.22. $U_q(\mathfrak{g})$-**BaseModAlg** is the category whose objects are based module algebras. Given based module algebras $(A_1, \mathcal{B}_1)$ and $(A_2, \mathcal{B}_2)$, a morphism $(A_1, \mathcal{B}_1) \to (A_2, \mathcal{B}_2)$ is a morphism φ of the associated barred module algebras such that $\varphi(\mathcal{B}_1) \subset \mathcal{B}_2 \cup \{0\}$.

Combining the theory of barred module algebras and based modules, the following theorem is now proven.

Theorem 4.23. $U_q(\mathfrak{g})$-**BaseModAlg** is monoidal with

$$(A_1, \mathcal{B}_1) \otimes (A_2, \mathcal{B}_2) := (A_1 \underline{\otimes} A_2, \mathcal{B}_1 \diamond \mathcal{B}_2)$$

and unit object $(\mathbb{Q}(q^{\frac{1}{2d}}), \{1\})$, where $A_1 \underline{\otimes} A_2$ is the braided tensor product, considered a $U_q(\mathfrak{g})$-module in the usual way, and $\mathbb{Q}(q^{\frac{1}{2d}})$ is the trivial module algebra. The associativity and left and right unit isomorphisms are the same as those for $U_q(\mathfrak{g})$-**WModAlg**.

Theorem 4.24. The isomorphism

$$\mathcal{A}_q[Mat_{m,n_1}]\underline{\otimes}\mathcal{A}_q[Mat_{m,n_2}] \cong \mathcal{A}_q[Mat_{m,n_1+n_2}],$$

induces an isomorphism of based module algebras

$$(\mathcal{A}_q[Mat_{m,n_1}], \mathcal{B}_{m,n_1}) \otimes (\mathcal{A}_q[Mat_{m,n_2}], \mathcal{B}_{m,n_2}) \cong (\mathcal{A}_q[Mat_{m,n_1+n_2}], \mathcal{B}_{m,n_1+n_2}).$$

Proof. We begin by recalling from [3] that, for any $m \geq 2$ and $n \geq 1$, the quantum Schubert cell $U_q(\mathbf{i}_{m,n})$ has generators $X_{\mathbf{i}_{m,n},j}$ for $j \in [1, mn]$ and $X^{\mathbf{a}}_{\mathbf{i}_{m,n}}$ is defined to be the product of $X^{a_1}_{\mathbf{i}_{m,n},1}X^{a_2}_{\mathbf{i}_{m,n},2}\cdots X^{a_{mn}}_{\mathbf{i}_{m,n},mn}$ and the unique power of $q^{\frac{1}{2d}}$ such that if we expand $\overline{X^{\mathbf{a}}_{\mathbf{i}_{m,n}}} - X^{\mathbf{a}}_{\mathbf{i}_{m,n}}$ in the basis $\{X^{a'_1}_{\mathbf{i}_{m,n},1}X^{a'_2}_{\mathbf{i}_{m,n},2}\cdots X^{a'_{mn}}_{\mathbf{i}_{m,n},mn} \mid \mathbf{a}' \in \mathbb{Z}^{mn}_{\geq 0}\}$, the coefficient of $X^{a_1}_{\mathbf{i}_{m,n},1}X^{a_2}_{\mathbf{i}_{m,n},2}\cdots X^{a_{mn}}_{\mathbf{i},mn}$ is zero. In this setting, this power is easily seen to be

$$\frac{1}{2}\left(\sum_{k=1}^{m}\sum_{1\leq \ell<\ell'\leq n} a_{(k-1)n+\ell}a_{(k-1)n+\ell'} + \sum_{\ell=1}^{n}\sum_{1\leq k<k'\leq m} a_{(k-1)n+\ell}a_{(k'-1)n+\ell}\right).$$

When $n = n_1 + n_2$ with $n_1, n_2 \geq 1$, twice this can be split into three summands, two of which are similar:

$$\begin{aligned}
&\sum_{k=1}^{m}\sum_{1\leq \ell<\ell'\leq n} a_{(k-1)n+\ell}a_{(k-1)n+\ell'} + \sum_{\ell=1}^{n}\sum_{1\leq k<k'\leq m} a_{(k-1)n+\ell}a_{(k'-1)n+\ell}\\
&\quad = \left(\sum_{k=1}^{m}\sum_{1\leq \ell<\ell'\leq n_1} a_{(k-1)n+\ell}a_{(k-1)n+\ell'} + \sum_{\ell=1}^{n_1}\sum_{1\leq k<k'\leq m} a_{(k-1)n+\ell}a_{(k'-1)n+\ell}\right)\\
&\qquad + \left(\sum_{k=1}^{m}\sum_{n_1+1\leq \ell<\ell'\leq n} a_{(k-1)n+\ell}a_{(k-1)n+\ell'} + \sum_{\ell=n_1+1}^{n}\sum_{1\leq k<k'\leq m} a_{(k-1)n+\ell}a_{(k'-1)n+\ell}\right)\\
&\qquad + \sum_{k=1}^{m}\sum_{\ell=1}^{n_1}\sum_{\ell'=n_1+1}^{n} a_{(k-1)n+\ell}a_{(k-1)n+\ell'}
\end{aligned}$$

Recall that if $k < k'$ and $\ell < \ell'$, then $x_{k,\ell'}x_{k',\ell} = x_{k',\ell}x_{k,\ell'}$. Furthermore, we have

$$|X^{\mathbf{a}}_{\mathbf{i}_{m,n}}| = \sum_{k=1}^{m} \left(\sum_{\ell=1}^{n} a_{(k-1)n+\ell} \right) \varepsilon_k.$$

Altogether, we see that $X^{\mathbf{a}}_{\mathbf{i}_{m,n}}$ factors nicely:

$$X^{\mathbf{a}}_{\mathbf{i}_{m,n}} = q^{(|X^{s(\mathbf{a})}_{\mathbf{i}_{m,n}}|,|X^{t(\mathbf{a})}_{\mathbf{i}_{m,n}}|)/2} X^{s(\mathbf{a})}_{\mathbf{i}_{m,n}} X^{t(\mathbf{a})}_{\mathbf{i}_{m,n}}$$

where $s(\mathbf{a})$ and $t(\mathbf{a})$ denote the mn-tuples $\mathbf{a}'$ and $\mathbf{a}''$, respectively, with

$$a'_{(k-1)n+\ell} := \begin{cases} a_{(k-1)n+\ell} & \text{if } 1 \leq \ell \leq n_1 \\ 0 & \text{if } \ell > n_1 \end{cases}$$

and

$$a''_{(k-1)n+\ell} := \begin{cases} a_{(k-1)n+\ell} & \text{if } n_1+ \leq \ell \leq n \\ 0 & \text{if } \ell \leq n_1 \end{cases}.$$

Letting

$$\Psi : U_q(\mathbf{i}_{m,n}) \to U_q(\mathbf{i}_{m,n_1}) \underline{\otimes} U_q(\mathbf{i}_{m,n_2})$$

be the composition of the standard isomorphism

$$\mathcal{A}_q[Mat_{m,n}] \tilde{\to} \mathcal{A}_q[Mat_{m,n_1}] \underline{\otimes} \mathcal{A}_q[Mat_{m,n_2}]$$

with the isomorphisms $U_q(\mathbf{i}_{m,n}) \tilde{\to} \mathcal{A}_q[Mat_{m,n}]$ and

$$\mathcal{A}_q[Mat_{m,n_1}] \underline{\otimes} \mathcal{A}_q[Mat_{m,n_2}] \tilde{\to} U_q(\mathbf{i}_{m,n_1}) \underline{\otimes} U_q(\mathbf{i}_{m,n_2}),$$

we see that it suffices to show that $\Psi(b_{\mathbf{i}_{m,n},\mathbf{a}}) = b_{\mathbf{i}_{m,n_1},s'(\mathbf{a})} \diamond b_{\mathbf{i}_{m,n_2},t'(\mathbf{a})}$ for any $\mathbf{a} \in \mathbb{Z}_{\geq 0}^{mn}$, where we write $s'(\mathbf{a})_{(k-1)n_1+\ell} = a_{(k-1)n+\ell}$ for $1 \leq k \leq m$, $1 \leq \ell \leq n_1$ and $t'(\mathbf{a})_{(k-1)n_2+\ell} = a_{(k-1)n+(n_1+\ell)}$ for $1 \leq k \leq m$, $1 \leq \ell \leq n_2$. Using the definition of the diamond basis, this is equivalent to showing that

$$\Psi(b_{\mathbf{i}_{m,n},\mathbf{a}}) - b_{\mathbf{i}_{m,n_1},s'(\mathbf{a})} * b_{\mathbf{i}_{m,n_2},t'(\mathbf{a})} \in \sum_{\substack{(b_{\mathbf{i}_{m,n_1},\mathbf{a}_1}, b_{\mathbf{i}_{m,n_2},\mathbf{a}_2}) \\ \neq (b_{\mathbf{i}_{m,n_1},s'(\mathbf{a})}, b_{\mathbf{i}_{m,n_2},t'(\mathbf{a})})}} q^{-\frac{1}{2d}}\mathbb{Q}[q^{-\frac{1}{2d}}] b_{\mathbf{i}_{m,n_1},\mathbf{a}_1} * b_{\mathbf{i}_{m,n_2},\mathbf{a}_2}.$$

According to the above, we have

$$\Psi(X_{\mathbf{i}_{m,n}}^{\mathbf{a}'}) = \Psi(q^{(|X_{\mathbf{i}_{m,n}}^{s(\mathbf{a}')}|,|X_{\mathbf{i}_{m,n}}^{t(\mathbf{a}')}|)/2} X_{\mathbf{i}_{m,n}}^{s(\mathbf{a}')} X_{\mathbf{i}_{m,n}}^{t(\mathbf{a}')}) = q^{(|X_{\mathbf{i}_{m,n_1}}^{s'(\mathbf{a}')}|,|X_{\mathbf{i}_{m,n_2}}^{t'(\mathbf{a}')}|)/2} X_{\mathbf{i}_{m,n_1}}^{s'(\mathbf{a}')} \otimes X_{\mathbf{i}_{m,n_2}}^{t'(\mathbf{a}')}$$

for any $\mathbf{a}' \in \mathbb{Z}_{\geq 0}^{mn}$. Then

$$\begin{aligned}
\Psi(b_{\mathbf{i}_{m,n},\mathbf{a}}) &= \Psi\left(\sum_{\mathbf{a}' \preceq \mathbf{a}} c_{\mathbf{i}_{m,n},\mathbf{a}}^{\mathbf{a}'} X_{\mathbf{i}_{m,n}}^{\mathbf{a}'}\right) \\
&= \sum_{\mathbf{a}' \preceq \mathbf{a}} c_{\mathbf{i}_{m,n},\mathbf{a}}^{\mathbf{a}'} q^{(|X_{\mathbf{i}_{m,n_1}}^{s'(\mathbf{a}')}|,|X_{\mathbf{i}_{m,n_2}}^{t'(\mathbf{a}')}|)/2} X_{\mathbf{i}_{m,n_1}}^{s'(\mathbf{a}')} \otimes X_{\mathbf{i}_{m,n_2}}^{t'(\mathbf{a}')} \\
&= \sum_{\substack{\mathbf{a}' \preceq \mathbf{a} \\ \mathbf{a}'' \preceq s'(\mathbf{a}') \\ \mathbf{a}''' \preceq t'(\mathbf{a}')}} c_{\mathbf{i}_{m,n},\mathbf{a}}^{\mathbf{a}'} (c_{\mathbf{i}_{m,n_1}}^{-1})_{s'(\mathbf{a}')}^{\mathbf{a}''} (c_{\mathbf{i}_{m,n_2}}^{-1})_{t'(\mathbf{a}')}^{\mathbf{a}'''} q^{(|X_{\mathbf{i}_{m,n_1}}^{s'(\mathbf{a}')}|,|X_{\mathbf{i}_{m,n_2}}^{t'(\mathbf{a}')}|)/2} b_{\mathbf{i}_{m,n_1},\mathbf{a}''} \otimes b_{\mathbf{i}_{m,n_2},\mathbf{a}'''} \\
&= \sum_{\substack{\mathbf{a}' \preceq \mathbf{a} \\ \mathbf{a}'' \preceq s'(\mathbf{a}') \\ \mathbf{a}''' \preceq t'(\mathbf{a}')}} c_{\mathbf{i}_{m,n},\mathbf{a}}^{\mathbf{a}'} (c_{\mathbf{i}_{m,n_1}}^{-1})_{s'(\mathbf{a}')}^{\mathbf{a}''} (c_{\mathbf{i}_{m,n_2}}^{-1})_{t'(\mathbf{a}')}^{\mathbf{a}'''} b_{\mathbf{i}_{m,n_1},\mathbf{a}''} * b_{\mathbf{i}_{m,n_2},\mathbf{a}'''}
\end{aligned}$$

where we write $b_{\mathbf{i},\mathbf{a}} = \sum_{\mathbf{a}' \preceq \mathbf{a}} c_{\mathbf{i},\mathbf{a}}^{\mathbf{a}'} X_{\mathbf{i}}^{\mathbf{a}'}$ and $X_{\mathbf{i}}^{\mathbf{a}} = \sum_{\mathbf{a}' \preceq \mathbf{a}} (c_{\mathbf{i}}^{-1})_{\mathbf{a}}^{\mathbf{a}'} b_{\mathbf{i},\mathbf{a}'}$ with $c_{\mathbf{i},\mathbf{a}}^{\mathbf{a}'}, (c_{\mathbf{i}}^{-1})_{\mathbf{a}}^{\mathbf{a}'} \in q^{-1}\mathbb{Z}[q^{-1}] \subset q^{-\frac{1}{2d}}\mathbb{Q}[q^{-\frac{1}{2d}}]$ for $\mathbf{a}' \preceq \mathbf{a}$ and $c_{\mathbf{i},\mathbf{a}}^{\mathbf{a}} = 1 = (c_{\mathbf{i}}^{-1})_{\mathbf{a}}^{\mathbf{a}}$. The result follows. $\square$

CHAPTER V

QUANTUM CLUSTER ALGEBRAS

Throughout this chapter, Γ is an Abelian group and $\mathbb{K}$ is a fixed field.

5.1. Graded Quantum Cluster Algebras

For this section, fix $N \in \mathbb{Z}_{\geq 0}$, and $\mathbf{ex} \subset [1, N]$.

Definition 5.1. Let $\tilde{B}$ be an $N \times \mathbf{ex}$ integer matrix, $\mathbf{q}$ an $N \times N$ multiplicatively skew-symmetric matrix with entries in $\mathbb{K}$, and $\mathbf{g}$ an element of Γ^N, considered as a column vector and called a *grading vector*. We say the triple $(\mathbf{q}, \tilde{B}, \mathbf{g})$ is *compatible* if $(\mathbf{q}, \tilde{B})$ is a compatible pair and $\tilde{B}^T \mathbf{g} = \mathbf{0}$.

We extend the mutation of compatible pairs to mutation of compatible triples. Namely, for fixed $k \in \mathbf{ex}$ and $\epsilon \in \{+, -\}$, we set

$$\mu_k(\mathbf{q}, \tilde{B}, \mathbf{g}) = (\mu_k(\mathbf{q}), \mu_k(\tilde{B}), \mu_k(\mathbf{g})) := \left({}^{E_\epsilon^T}\mathbf{q}^{E_\epsilon}, E_\epsilon \tilde{B} F_\epsilon, E_\epsilon^T \mathbf{g} \right) \qquad \text{(Equation 5.1.)}$$

where E_ϵ and F_ϵ are as in (Equation 2.1.) and (Equation 2.2.).

Proposition 5.2. Let $(\mathbf{q}, \tilde{B}, \mathbf{g})$ be a compatible triple. Then with $\mu_k(\mathbf{q}, \tilde{B}, \mathbf{g})$ as defined in (Equation 5.1.), the following hold.

1. $\mu_k(\mathbf{q}, \tilde{B}, \mathbf{g})$ is independent of the choice of ϵ.

2. $\mu_k(\mathbf{q}, \tilde{B}, \mathbf{g})$ is a compatible triple.

3. μ_k is involutive on compatible triples.

Proof. We note that $\mu_k(\mathbf{g})_i = g_i$ for $i \neq k$ and

$$\mu_k(\mathbf{g})_k = \left(\sum_{i \in [1,N] \setminus \{k\}} \max(-\epsilon b_{i,k}, 0) g_i \right) - g_k.$$

But we have

$$0 = \sum_{i \in [1,N]} b_{i,k} g_i = \epsilon \left(\sum_{\substack{i \in [1,N] \setminus \{k\} \\ \epsilon b_{i,k} > 0}} \max(\epsilon b_{i,k}, 0) g_i - \sum_{\substack{i \in [1,N] \setminus \{k\} \\ -\epsilon b_{i,k} > 0}} \max(-\epsilon b_{i,k}, 0) g_i \right),$$

showing that

$$\sum_{\substack{i \in [1,N] \setminus \{k\} \\ -(-\epsilon) b_{i,k} > 0}} \max(-(-\epsilon) b_{i,k}, 0) g_i = \sum_{\substack{i \in [1,N] \setminus \{k\} \\ -\epsilon b_{i,k} > 0}} \max(-\epsilon b_{i,k}, 0) g_i$$

and so we see that $\mu_k(\mathbf{g})$ is independent of ϵ. The first claim now follows from previously established theory.

The second claim follows from previously established theory and the easy observation that

$$\begin{aligned}
(\mu_k(\tilde{B}))^T \mu_k(\mathbf{g}) &= (E_\epsilon \tilde{B} F_\epsilon)^T (E_\epsilon^T \mathbf{g}) \\
&= F_\epsilon^T \tilde{B}^T (E_\epsilon^2)^T \mathbf{g} \\
&= F_\epsilon^T \tilde{B}^T \mathbf{g} \\
&= \mathbf{0}.
\end{aligned}$$

Finally, observe that if E'_ϵ is obtained using $\mu_k(\tilde{B})$ instead of $\tilde{B}$, then $E'_\epsilon = E_{-\epsilon}$. Then the fact that $\mu_k(\mathbf{g})$ was independent of the choice of ϵ implies that

$$\mu_k^2(\mathbf{g}) = (E_{-\epsilon}^2)^T \mathbf{g} = \mathbf{g}.$$

The third claim now follows from previously established theory. $\square$

Let $\mathcal{F}$ be a division ring over $\mathbb{K}$ and $M : \mathbb{Z}^N \to \mathcal{F}$ a toric frame. Then the assignment $X_i \mapsto M(e_i)$ yields a well-defined embedding of $\mathbb{K}$-algebras $\varphi_M : \mathcal{T}_{\mathbf{q}(M)^{\cdot 2}} \hookrightarrow \mathcal{F}$ and $\mathcal{F} = Frac(\varphi_M(\mathcal{T}_{\mathbf{q}(M)^{\cdot 2}}))$. Given $\mathbf{g} \in \Gamma^N$, $\varphi_M(\mathcal{T}_{\mathbf{q}(M)^{\cdot 2}})$ is Γ-graded by setting $|\varphi_M(X_i)| = g_i$ for each $i \in [1, N]$.

Definition 5.3. A *graded quantum seed of* $\mathcal{F}$ is a triple $(M, \tilde{B}, \mathbf{g})$, where $(M, \tilde{B})$ is a quantum seed of $\mathcal{F}$ and $\mathbf{g} \in \Gamma^N$ is such that $(\mathbf{q}(M), \tilde{B}, \mathbf{g})$ is a compatible triple.

Akin to mutation of compatible triples, we also have mutation of graded quantum seeds. Namely, mutations of a toric frame and a grading vector depend only on the exchange matrix and not on each other. Additionally, this mutation is involutive and yields an equivalence relation on the set of quantum seeds. Namely, $(M, \tilde{B}, \mathbf{g}) \sim (M', \tilde{B}', \mathbf{g})$ if and only if there exists a sequence of mutations which can be successively applied to $(M, \tilde{B}, \mathbf{g})$ to yield $(M', \tilde{B}', \mathbf{g}')$ (or, equivalently, vice versa).

Proposition 5.4. Let $(M, \tilde{B}, \mathbf{g})$ be a graded quantum seed of $\mathcal{F}$. Then $\mu_k(M)(e_i)$ is a homogeneous element of $\varphi_M(\mathcal{T}_{\mathbf{q}(M)^{\cdot 2}})$ for all $i \in [1, N]$ and $|\mu_k(M)(e_i)| = \mu_k(\mathbf{g})_i$.

Proof. By definition,

$$\mu_k(M)(e_i) = \begin{cases} M([b^k]_+ - e_k) + M([b^k]_- - e_k) & \text{if } i = k \\ M(e_i) & \text{otherwise} \end{cases}.$$

Now, since we know that $M(e_i)$ is homogeneous, it suffices to show that $M([b^k]_+ - e_k)$ and $M([b^k]_- - e_k)$ are homogeneous of the same graded degree. But we have

$$M([b^k]_+ - e_k) = \mathcal{S}_{\mathbf{q}(M)}([b^k]_+ - e_k)\varphi_M(X_1^{\max(b_{1,k},0)} \cdots X_k^{-1} \cdots X_N^{\max(b_{N,k},0)}) \text{ and}$$

$$M([b^k]_- - e_k) = \mathcal{S}_{\mathbf{q}(M)}([b^k]_- - e_k)\varphi_M(X_1^{\max(-b_{1,k},0)} \cdots X_k^{-1} \cdots X_N^{\max(-b_{N,k},0)}),$$

which are of respective degree

$$\left(\sum_{i=1}^{N} \max(b_{i,k}, 0)g_i\right) - g_k \quad \text{and} \quad \left(\sum_{i=1}^{N} \max(-b_{i,k}, 0)g_i\right) - g_k.$$

As previously observed, these are equal. $\square$

The following corollary is then immediate.

Corollary 5.5. Let $(M, \tilde{B}, \mathbf{g})$ be a graded quantum seed and $k \in \mathbf{ex}$. Then the gradings induced on $\varphi_M(\mathcal{T}_{\mathbf{q}(M)^{\cdot 2}})$ and $\varphi_{\mu_k(M)}(\mathcal{T}_{\mathbf{q}(\mu_k(M))^{\cdot 2}})$ by $\mathbf{g}$ and $\mathbf{g}'$, respectively, agree on the intersection

$$\varphi_M(\mathcal{T}_{\mathbf{q}(M)^{\cdot 2}}) \cap \varphi_{\mu_k(M)}(\mathcal{T}_{\mathbf{q}(\mu_k(M))^{\cdot 2}}) \subset \mathcal{F}.$$

Corollary 5.6. Let $(M, \tilde{B}, \mathbf{g})$ be a graded quantum seed of $\mathcal{F}$. Then, for any $\mathbf{inv} \subset [1, N] \setminus \mathbf{ex}$ and unital subring $\mathbb{k}$ of $\mathbb{K}$ containing $q_{i,j}(M)$ for all $i, j \in [1, N]$, $\mathbb{k}(M, \tilde{B}, \mathbf{inv})$ is a graded subalgebra of $\varphi_M(\mathcal{T}^{\mathbb{k}}_{\mathbf{q}(M)^{\cdot 2}})$.

Proof. We already know from the quantum Laurent phenomenon that $\mathbb{k}(M, \tilde{B}, \mathbf{inv})$ is a subalgebra of $\varphi_M(\mathcal{T}^{\mathbb{k}}_{\mathbf{q}(M)^{\cdot 2}})$. It only remains to use Corollary 5.5 to observe that all cluster variables are homogeneous. Indeed, if $k_1, \ldots, k_m \in \mathbf{ex}$ are chosen so that $(\mu_{k_m} \cdots \mu_{k_1})(M, \tilde{B}, \mathbf{g}) = (M', \tilde{B}', \mathbf{g}')$, then by the quantum Laurent phenomenon, we have

$$(\mu_{k_m}\cdots\mu_{k_1})(M)(e_i)\in\bigcap_{j=1}^{m}\varphi_{(\mu_{k_j}\cdots\mu_{k_1})(M)}\left(\mathcal{T}_{\mathbf{q}((\mu_{k_j}\cdots\mu_{k_1})(M))^{\cdot 2}}\right)$$

for all $i\in[1,N]$. Since the grading of each $\varphi_{(\mu_{k_j}\cdots\mu_{k_1})(M)}\left(\mathcal{T}_{\mathbf{q}((\mu_{k_j}\cdots\mu_{k_1})(M))^{\cdot 2}}\right)$ agrees with its "neighbors" on their pairwise intersections and $(\mu_{k_m}\cdots\mu_{k_1})(M)(e_i)$ is homogeneous in $\varphi_{(\mu_{k_m}\cdots\mu_{k_1})(M)}\left(\mathcal{T}_{\mathbf{q}((\mu_{k_m}\cdots\mu_{k_1})(M))^{\cdot 2}}\right)$, it must also be homogeneous in $\varphi_M\left(\mathcal{T}_{\mathbf{q}(M)^{\cdot 2}}\right)$. □

Definition 5.7. Given a graded quantum seed $(M,\tilde{B},\mathbf{g})$, $\mathbf{inv}\subset[1,N]\setminus\mathbf{ex}$, and a unital subring $\Bbbk$ of $\mathbb{K}$ containing $q_{i,j}(M)$ for $i,j\in[1,N]$, the *graded quantum cluster algebra* $\Bbbk(M,\tilde{B},\mathbf{g},\mathbf{inv})$ is simply $\Bbbk(M,\tilde{B},\mathbf{inv})$ as an algebra, equipped with the Γ-grading of Corollary 5.6.

We will often denote a graded quantum cluster algebra $\Bbbk(M,\tilde{B},\mathbf{g},\mathbf{inv})$ instead by $\Bbbk(\mathbf{q}(M),\tilde{B},\mathbf{g},\mathbf{inv})$ or by $\Bbbk(\mathbf{q},\tilde{B},\mathbf{g},\mathbf{inv})$. We now turn our attention to graded tensor products of graded quantum cluster algebras.

5.2. Graded Tensor Products

In this section, we assume $\langle,\rangle:\Gamma\times\Gamma\to\mathbb{K}^{\times}$ is a bicharacter. Namely, for all $m\in\mathbb{Z}$ and $\gamma,\gamma',\gamma''\in\Gamma$, the following hold.

1. $\langle m\gamma,\gamma'\rangle=\langle\gamma,\gamma'\rangle^m=\langle\gamma,m\gamma'\rangle$
2. $\langle\gamma+\gamma',\gamma''\rangle=\langle\gamma,\gamma''\rangle\langle\gamma',\gamma''\rangle$
3. $\langle\gamma,\gamma'+\gamma''\rangle=\langle\gamma,\gamma'\rangle\langle\gamma,\gamma''\rangle$

Note that the set of bicharacters is a group under point-wise multiplication and we will sometimes consider the bicharacter $\langle,\rangle^m$ for $m\in\mathbb{Z}$.

The following lemma is very well-known.

Lemma 5.8. Let $\Bbbk$ be a unital subring of $\mathbb{K}$ and suppose A and B are Γ-graded $\Bbbk$-algebras so that $\langle |a|, |b| \rangle \in \Bbbk$ for all homogeneous $a \in A$ and $b \in B$. Then the Γ-graded $\Bbbk$-module $A \otimes_{\Bbbk} B$ naturally has the structure of a Γ-graded $\Bbbk$-algebra with multiplication given on pure tensors by $(a \otimes b)(a' \otimes b') = \langle |a'|, |b| \rangle aa' \otimes bb'$ for homogeneous $a, a' \in A$ and $b, b' \in B$.

The Γ-graded algebra of the preceding lemma is called a *graded tensor product* and denoted by $A \otimes_{\Bbbk}^{\langle,\rangle} B$. The main goal of this section is to define graded quantum cluster algebra structures on graded tensor products of graded quantum cluster algebras. The following theorem is a first step in that direction. However, we first need a bit of notation. Namely, for any $S \subset \mathbb{Z}$ and $m \in \mathbb{Z}$, set

$$S[m] := \{s + m \mid s \in S\}.$$

Theorem 5.9. Let $(\mathbf{q}', \tilde{B}', \mathbf{g}')$ and $(\mathbf{q}'', \tilde{B}'', \mathbf{g}'')$ be compatible triples with corresponding indexing sets $[1, N']$, $\mathbf{ex}'$, $[1, N'']$, and $\mathbf{ex}''$. Then the following assignments define a compatible triple $(\mathbf{q}, \tilde{B}, \mathbf{g})$ with indexing sets $[1, N]$ and $\mathbf{ex} := \mathbf{ex}' \cup \mathbf{ex}''[N']$, where $N = N' + N''$.

$$q_{i,j} := \begin{cases} q'_{i,j} & \text{if } i, j \in [1, N'] \\ \langle g'_j, g''_{i-N_1} \rangle & \text{if } i \in [N'+1, N] \text{ and } j \in [1, N'] \\ \langle g'_i, g''_{j-N_1} \rangle^{-1} & \text{if } i \in [1, N'] \text{ and } j \in [N'+1, N] \\ q''_{i-N', j-N'} & \text{if } i, j \in [N'+1, N] \end{cases}$$

$$b_{i,j} := \begin{cases} b'_{i,j} & \text{if } i \in [1, N'] \text{ and } j \in \mathbf{ex}' \\ b''_{i-N',j-N'} & \text{if } i \in [N'+1, N] \text{ and } j \in \mathbf{ex}''[N'] \\ 0 & \text{else} \end{cases}$$

$$g_i := \begin{cases} g'_i & \text{if } i \in [1, N'] \\ g''_{i-N'} & \text{if } i \in [N'+1, N] \end{cases}$$

Ignoring indexing difficulties, the assignments in Theorem 5.9 can be viewed as the assignments

$$\mathbf{q} := \begin{bmatrix} \mathbf{q}' & (\mathbf{x}^{-I_{N'}})^T \\ \mathbf{x} & \mathbf{q}'' \end{bmatrix}, \ \tilde{B} := \begin{bmatrix} \tilde{B}' & 0 \\ 0 & \tilde{B}'' \end{bmatrix}, \text{ and } \mathbf{g} := \begin{bmatrix} \mathbf{g}' \\ \mathbf{g}'' \end{bmatrix}, \qquad \text{(Equation 5.2.)}$$

where $\mathbf{x}$ is the $N'' \times N'$ matrix with entries given by $x_{i,j} := \langle g'_j, g''_i \rangle$ for all $i \in [1, N'']$ and $j \in [1, N']$.

Proof. First, note that $\mathbf{q}$ is obviously multiplicatively skew-symmetric. Now a simple computation shows that for $i \in [1, N]$ and $j \in \mathbf{ex}$,

$$\prod_{k=1}^{N} q_{k,i}^{b_{k,j}} = \begin{cases} 1 & \text{if } i \neq j \\ \prod_{k=1}^{N'} (q'_{k,j})^{b'_{k,j}} & \text{if } i = j \in \mathbf{ex}' \\ \prod_{k=1}^{N''} (q''_{k,j-N'})^{b''_{k,j-N'}} & \text{if } i = j \in \mathbf{ex}''[N'] \end{cases}.$$

Here we use the assumption that $(\tilde{B}')^T \mathbf{g}' = \mathbf{0}$ and $(\tilde{B}'')^T \mathbf{g}'' = \mathbf{0}$. Now, since $(\mathbf{q}', \tilde{B}', \mathbf{g}')$ and $(\mathbf{q}'', \tilde{B}'', \mathbf{g}'')$ are compatible triples, it follows that $\prod_{k=1}^{N} q_{k,i}^{b_{k,j}} = 1$ for all $i \in [1, N]$ and $j \in \mathbf{ex} \setminus \{i\}$, as well as that $\prod_{k=1}^{N} q_{k,j}^{b_{k,j}}$ is not a root of unity for any $j \in \mathbf{ex}$. Finally, it is clear that $\tilde{B}^T \mathbf{g} = \mathbf{0}$. □

Henceforth, we will denote by $(\mathbf{q}', \tilde{B}', \mathbf{g}') * (\mathbf{q}'', \tilde{B}'', \mathbf{g}'')$ the compatible triple $(\mathbf{q}, \tilde{B}, \mathbf{g})$ constructed in Theorem 5.9 from two compatible triples.

Proposition 5.10. Let $(M, \tilde{B}, \mathbf{g})$, $(M', \tilde{B}', \mathbf{g}')$, and $(M'', \tilde{B}'', \mathbf{g}'')$ be graded quantum seeds in division algebras $\mathcal{F}$, $\mathcal{F}'$, and $\mathcal{F}''$ such that $\mathcal{F}'$ and $\mathcal{F}''$ are subalgebras of $\mathcal{F}$ and $(\mathbf{q}(M), \tilde{B}, \mathbf{g}) = (\mathbf{q}(M'), \tilde{B}', \mathbf{g}') * (\mathbf{q}(M''), \tilde{B}'', \mathbf{g}'')$. Suppose, furthermore, that

$$M(e_i) = \begin{cases} M'(e_i) & \text{if } i \in [1, N'] \\ M''(e_{i-N'}) & \text{if } i \in [N'+1, N] \end{cases}.$$

Then, for $k \in \mathbf{ex}'$,

$$(\mathbf{q}(\mu_k(M)), \mu_k(\tilde{B}), \mu_k(\mathbf{g})) = (\mathbf{q}(\mu_k(M')), \mu_k(\tilde{B}'), \mu_k(\mathbf{g}')) * (\mathbf{q}(M''), \tilde{B}'', \mathbf{g}'')$$

and

$$\mu_k(M)(e_i) = \begin{cases} \mu_k(M')(e_i) & \text{if } i \in [1, N'] \\ M''(e_{i-N'}) & \text{if } i \in [N'+1, N] \end{cases}.$$

Similarly, for $k \in \mathbf{ex}''[N']$,

$$(\mathbf{q}(\mu_k(M)), \mu_k(\tilde{B}), \mu_k(\mathbf{g})) = (\mathbf{q}(M'), \tilde{B}', \mathbf{g}') * (\mathbf{q}(\mu_{k-N}(M'')), \mu_{k-N}(\tilde{B}''), \mu_{k-N}(\mathbf{g}''))$$

and

$$\mu_k(M)(e_i) = \begin{cases} M'(e_i) & \text{if } i \in [1, N'] \\ \mu_k(M'')(e_{i-N'}) & \text{if } i \in [N'+1, N] \end{cases}.$$

Proof. We prove the statement only for $k \in \mathbf{ex}'$, since the case where $k \in \mathbf{ex}''[N']$ is nearly identical. We begin by observing that E_ϵ and F_ϵ are of the form

$$E_\epsilon = \begin{bmatrix} E'_\epsilon & 0 \\ 0 & I_{N''} \end{bmatrix} \quad \text{and} \quad F_\epsilon = \begin{bmatrix} F'_\epsilon & 0 \\ 0 & I_{N''} \end{bmatrix}.$$

We compute, using the formulas of (Equation 5.2.). First, we observe that the (i,k)-th entry of $\mathbf{x}^{E'_\epsilon}$ is given by

$$\begin{aligned} \prod_{\ell=1}^{N'} \langle g_\ell, g_i \rangle^{\max(-\epsilon b_{\ell,k},0)} &= \left\langle \sum_{\ell=1}^{N'} \max(-\epsilon b_{\ell,k},0) g'_\ell, g''_i \right\rangle \\ &= \langle ((E'_\epsilon)^T \mathbf{g}')_k, g''_i \rangle \\ &= \langle \mu_k(\mathbf{g}')_k, g''_i \rangle, \end{aligned}$$

while the (i,j)-th entry is given by $\langle g'_j, g''_i \rangle = \langle \mu_k(\mathbf{g}')_j, g''_i \rangle$ for $j \neq k$. Hence, if we write $\mu_k(\mathbf{x}) := (\langle \mu_k(\mathbf{g}')_j, g''_i \rangle)$, then we have shown that $\mathbf{x}^{E'_\epsilon} = \mu_k(\mathbf{x})$. Now,

$$\begin{aligned} \mathbf{q}(\mu_k(M)) &= \mu_k(\mathbf{q}(M) \\ &= {}^{E_\epsilon^T}\mathbf{q}(M)^{E_\epsilon} \\ &= \begin{bmatrix} (E'_\epsilon)^T & 0 \\ 0 & I_{N''} \end{bmatrix} \begin{bmatrix} \mathbf{q}(M') & (\mathbf{x}^{-I_{N''}})^T \\ \mathbf{x} & \mathbf{q}(M'') \end{bmatrix} \begin{bmatrix} E'_\epsilon & 0 \\ 0 & I_{N''} \end{bmatrix} \\ &= \begin{bmatrix} {}^{(E'_\epsilon)^T}\mathbf{q}(M')^{E'_\epsilon} & {}^{(E'_\epsilon)^T}((\mathbf{x}^{-I_{N''}})^T) \\ \mathbf{x}^{E'_\epsilon} & \mathbf{q}(M'') \end{bmatrix} \\ &= \begin{bmatrix} \mu_k(\mathbf{q}(M')) & ((\mathbf{x}^{E'_\epsilon})^{-I_{N''}})^T \\ \mathbf{x}^{E'_\epsilon} & \mathbf{q}(M'') \end{bmatrix} \\ &= \begin{bmatrix} \mu_k(\mathbf{q}(M')) & (\mu_k(\mathbf{x})^{-I_{N''}})^T \\ \mu_k(\mathbf{x}) & \mathbf{q}(M'') \end{bmatrix}, \end{aligned}$$

$$
\begin{aligned}
\mu_k(\tilde{B}) &= E_\epsilon \tilde{B} F_\epsilon \\
&= \begin{bmatrix} E'_\epsilon & 0 \\ 0 & I_{N''} \end{bmatrix} \begin{bmatrix} \tilde{B}' & 0 \\ 0 & \tilde{B}'' \end{bmatrix} \begin{bmatrix} F'_\epsilon & 0 \\ 0 & I_{N''} \end{bmatrix} \\
&= \begin{bmatrix} E'_\epsilon \tilde{B}' F'_\epsilon & 0 \\ 0 & \tilde{B}'' \end{bmatrix} \\
&= \begin{bmatrix} \mu_k(\tilde{B}') & 0 \\ 0 & \tilde{B}'' \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\mu_k(\mathbf{g}) &= E_\epsilon^T \mathbf{g} \\
&= \begin{bmatrix} E'_\epsilon & 0 \\ 0 & I_{N''} \end{bmatrix} \begin{bmatrix} \mathbf{g}' \\ \mathbf{g}'' \end{bmatrix} \\
&= \begin{bmatrix} E'_\epsilon \mathbf{g}' \\ \mathbf{g}'' \end{bmatrix} \\
&= \begin{bmatrix} \mu_k(\mathbf{g}') \\ \mathbf{g}'' \end{bmatrix}.
\end{aligned}
$$

It is now clear that

$$(\mathbf{q}(\mu_k(M)), \mu_k(\tilde{B}), \mu_k(\mathbf{g})) = (\mathbf{q}(\mu_k(M')), \mu_k(\tilde{B}'), \mu_k(\mathbf{g}')) * (\mathbf{q}(M''), \tilde{B}'', \mathbf{g}'').$$

The proposition now follows once we observe that

$$\mu_k(M)(e_i) = M(e_i) = M'(e_i) = \mu_k(M')(e_i)$$

if $i \in [1, N'] \setminus \{k\}$, while

$$\begin{aligned}\mu_k(M)(e_k) &= M([b^k]_+ - e_k) + M([b^k]_- - e_k) \\ &= M([(b')^k]_+ - e_k) + M([(b')^k]_- - e_k) \\ &= M'([(b')^k]_+ - e_k) + M'([(b')^k]_- - e_k) \\ &= \mu_k(M')(e_k).\end{aligned}$$

Here the second equality is easily deduced from the assumptions that $M(e_i) = M'(e_i)$ and $q_{i,j} = q'_{i,j}$ for $i, j \in [1, N']$. □

Theorem 5.11. Let $(\mathbf{q}', \tilde{B}', \mathbf{g}')$ and $(\mathbf{q}'', \tilde{B}'', \mathbf{g}'')$ be compatible triples, as well as $\mathbf{inv}' \subset [1, N'] \setminus \mathbf{ex}'$ and $\mathbf{inv}'' \subset [1, N''] \setminus \mathbf{ex}''$. Set $(\mathbf{q}, \tilde{B}, \mathbf{g}) := (\mathbf{q}', \tilde{B}', \mathbf{g}') * (\mathbf{q}'', \tilde{B}'', \mathbf{g}'')$ and $\mathbf{inv} := \mathbf{inv}' \sqcup \mathbf{inv}''[N']$. Then, if $\mathbb{k}$ is any unital subring of $\mathbb{K}$ containing $q_{i,j}$ for all $i, j \in [1, N]$ (and so, in particular, $\mathbb{k}$ also contains $q'_{i,j}$ for $i, j \in [1, N']$ and $q''_{i,j}$ for $i, j \in [1, N'']$), we have an isomorphism of $\mathbb{k}$-algebras

$$\mathbb{k}(\mathbf{q}', \tilde{B}', \mathbf{g}', \mathbf{inv}') \otimes_{\mathbb{k}}^{\langle,\rangle^2} \mathbb{k}(\mathbf{q}'', \tilde{B}'', \mathbf{g}'', \mathbf{inv}'') \cong \mathbb{k}(\mathbf{q}, \tilde{B}, \mathbf{g}, \mathbf{inv}).$$

Proof. We first note an obvious isomorphism of Γ-graded $\mathbb{k}$-algebras.

$$\begin{gathered}\mathcal{T}^{\mathbb{k}}_{\mathbf{q}^{\cdot 2}} \cong \mathcal{T}^{\mathbb{k}}_{(\mathbf{q}')^{\cdot 2}} \otimes_{\mathbb{k}}^{\langle,\rangle^2} \mathcal{T}^{\mathbb{k}}_{(\mathbf{q}'')^{\cdot 2}} \\ X_i^{\pm 1} \mapsto \begin{cases} X_i^{\pm 1} \otimes 1 & \text{if } i \in [1, N'] \\ 1 \otimes X_{i-N'}^{\pm 1} & \text{if } i \in [N'+1, N] \end{cases}\end{gathered}$$

Now, the quantum Laurent phenomenon gives embeddings (again of Γ-graded $\mathbb{k}$-algebras)

$$\Bbbk(\mathbf{q}, \tilde{B}, \mathbf{g}, \mathbf{inv}) \hookrightarrow \mathcal{T}^{\Bbbk}_{\mathbf{q}^{\cdot 2}} \quad \text{and}$$

$$\Bbbk(\mathbf{q}', \tilde{B}', \mathbf{g}', \mathbf{inv}') \otimes_{\Bbbk}^{\langle,\rangle^2} \Bbbk(\mathbf{q}'', \tilde{B}'', \mathbf{g}'', \mathbf{inv}'') \hookrightarrow \mathcal{T}^{\Bbbk}_{(\mathbf{q}')^{\cdot 2}} \otimes_{\Bbbk}^{\langle,\rangle^2} \mathcal{T}^{\Bbbk}_{(\mathbf{q}'')^{\cdot 2}}.$$

In light of the above isomorphism, we may consider both $\Bbbk(\mathbf{q}, \tilde{B}, \mathbf{g}, \mathbf{inv})$ and $\Bbbk(\mathbf{q}', \tilde{B}', \mathbf{g}', \mathbf{inv}') \otimes_{\Bbbk}^{\langle,\rangle^2} \Bbbk(\mathbf{q}'', \tilde{B}'', \mathbf{g}'', \mathbf{inv}'')$ as subalgebras of a common division ring, namely $\mathcal{F} := Frac(\mathcal{T}_{\mathbf{q}^{\cdot 2}})$. The claim of the theorem then, is that these subalgebras coincide. We consider the induced algebra embeddings

$$\iota' : \mathcal{T}_{(\mathbf{q}')^{\cdot 2}} \hookrightarrow \mathcal{T}_{\mathbf{q}^{\cdot 2}} \hookrightarrow \mathcal{F} \quad \text{and} \quad \iota'' : \mathcal{T}_{(\mathbf{q}'')^{\cdot 2}} \hookrightarrow \mathcal{T}_{\mathbf{q}^{\cdot 2}} \hookrightarrow \mathcal{F}.$$

Let $\mathcal{F}' := Frac(\iota'(\mathcal{T}_{(\mathbf{q}')^{\cdot 2}})) \subset \mathcal{F}$ and $\mathcal{F}'' := Frac(\iota''(\mathcal{T}_{(\mathbf{q}'')^{\cdot 2}})) \subset \mathcal{F}$.

We now show that the generators of the two algebras in question coincide in $\mathcal{F}$. Choose toric frames $M : \mathbb{Z}^N \to \mathcal{F}$, $M' : \mathbb{Z}^{N'} \to \mathcal{F}'$, and $M'' : \mathbb{Z}^{N''} \to \mathcal{F}''$ satisfying $M(e_i) = X_i \in \mathcal{T}_{\mathbf{q}^{\cdot 2}} \subset \mathcal{F}$ for $i \in [1, N]$, $M'(e_i) = M(e_i)$ for $i \in [1, N']$, and $M''(e_i) = M(e_{i+N'})$ for $i \in [1, N'']$, yielding $\mathbf{q}(M) = \mathbf{q}$, $\mathbf{q}(M') = \mathbf{q}'$, and $\mathbf{q}(M'') = \mathbf{q}''$. This is possible precisely because $q_{i,j} = q'_{i,j}$ for $i, j \in [1, N']$ and $q_{i,j} = q''_{i-N',j-N'}$ for $i, j \in [N' + 1, N]$.

In light of Proposition 5.16, it is now clear that, given arbitrary $k_1, \ldots, k_m \in [1, N]$, we may find $i_1, \ldots, i_{m'}, j_1, \ldots, j_{m''} \in [1, m]$ so that $m' + m'' = m$, $i_1 \leq \cdots \leq i_{m'}$, $j_1 \leq \cdots \leq j_{m''}$, $k_{i_1}, \ldots, k_{i_{m'}} \in [1, N']$, $k_{j_1}, \ldots, k_{j_{m''}} \in [N' + 1, N]$, and

$$\begin{aligned} \mu_{k_1} \cdots \mu_{k_m}(M, \tilde{B}, \mathbf{g}) &= \mu_{k_{i_1}} \cdots \mu_{k_{i_{m'}}} \mu_{k_{j_1}} \cdots \mu_{k_{j_{m''}}}(M, \tilde{B}, \mathbf{g}) \\ &= \mu_{k_{j_1}} \cdots \mu_{k_{j_{m''}}} \mu_{k_{i_1}} \cdots \mu_{k_{i_{m'}}}(M, \tilde{B}, \mathbf{g}). \end{aligned}$$

We therefore conclude that

$$\mu_{k_1}\cdots\mu_{k_m}(M)(e_i)=\begin{cases}\mu_{k_{i_1}}\cdots\mu_{k_{i_{m'}}}(M')(e_i) & \text{if } i\in[1,N']\\ \mu_{k_{j_1}-N'}\cdots\mu_{k_{j_{m''}}-N'}(M'')(e_i) & \text{if } i\in[N'+1,N]\end{cases}.$$

Hence the generators of $\Bbbk(\mathbf{q},\tilde{B},\mathbf{g},\mathbf{inv})$ and $\Bbbk(\mathbf{q}',\tilde{B}',\mathbf{g}',\mathbf{inv}')\otimes_{\Bbbk}^{\langle,\rangle^2}\Bbbk(\mathbf{q}'',\tilde{B}'',\mathbf{g}'',\mathbf{inv}'')$ coincide in $\mathcal{F}$. □

5.3. The Single-Parameter Case

In the case of single-parameter quantum cluster algebras (i.e. where $q\in\mathbb{K}^\times$ is not a root of unity and has a specified $2d$-th root $q^{\frac{1}{2d}}$), we take $(,):\Gamma\times\Gamma\to\frac{1}{d}\mathbb{Z}$ to be a bilinear pairing and define a bicharacter $\langle,\rangle:\Gamma\times\Gamma\to\mathbb{K}^\times$ by $\langle\gamma,\gamma'\rangle=q^{\frac{1}{2}(\gamma,\gamma')}$.

In the single-parameter setting, the definition of a compatible triple is altered as follows. A triple $(\Lambda,\tilde{B},\mathbf{g})$ is compatible if $(\Lambda,\tilde{B})$ is a single-parameter compatible pair and $\tilde{B}^T\mathbf{g}=0$. This is easily seen to imply the compatibility of the triple $((q^{\frac{1}{2}}I_N)^\Lambda,\tilde{B},\mathbf{g})$ in the sense of Section 5.1..

For convenience and later use, we now record the single-parameter analogues of the definitions and results (without proof) of Sections 5.1. and 5.2..

Proposition 5.12. Let $(\Lambda,\tilde{B},\mathbf{g})$ be a single-parameter compatible triple. Then setting $\mu_k(\Lambda,\tilde{B},\mathbf{g})=(E_\epsilon^T\Lambda E_\epsilon,E_\epsilon\tilde{B}F_\epsilon,E_\epsilon^T\mathbf{g})$, the following hold.

1. $\mu_k(\Lambda,\tilde{B},\mathbf{g})$ is independent of the choice of ϵ.

2. $\mu_k(\Lambda,\tilde{B},\mathbf{g})$ is a single-parameter compatible triple.

3. μ_k is involutive on single-parameter compatible triples.

Definition 5.13. A *single-parameter graded quantum seed of $\mathcal{F}$* is a triple $(M, \tilde{B}, \mathbf{g})$, where $(M, \tilde{B})$ is a single-parameter quantum seed of $\mathcal{F}$ and $\mathbf{g} \in \Gamma^N$ is such that $(\Lambda(M), \tilde{B}, \mathbf{g})$ is a single-parameter compatible triple.

As with ordinary single-parameter quantum cluster algebras, every single-parameter graded quantum seed is automatically a graded quantum seed. Therefore, we won't define a single-parameter graded quantum cluster algebra, except to say that is a graded quantum cluster algebra coming from a single-parameter graded quantum seed.

Lemma 5.14. Let $\Bbbk$ be a unital subring of $\mathbb{K}$ and suppose A and B are Γ-graded $\Bbbk$-algebras so that $q^{\pm(|a|,|b|)} \in \Bbbk$ for all homogeneous $a \in A$ and $b \in B$. Then the Γ-graded $\Bbbk$-module $A \otimes_{\Bbbk} B$ naturally has the structure of a Γ-graded $\Bbbk$-algebra with multiplication given on pure tensors by $(a \otimes b)(a' \otimes b') = q^{(|a'|,|b|)} aa' \otimes bb'$ for homogeneous $a, a' \in A$ and $b, b' \in B$.

The Γ-graded algebra of the preceding lemma is called a *graded tensor product* and denoted by $A \overline{\otimes}_{\Bbbk}^{(,)} B$.

Theorem 5.15. Let $(\Lambda', \tilde{B}', \mathbf{g}')$ and $(\Lambda'', \tilde{B}'', \mathbf{g}'')$ be single-parameter compatible triples with corresponding indexing sets $[1, N']$, $\mathbf{ex}'$, $[1, N'']$, and $\mathbf{ex}''$. Then the following assignments define a single-parameter compatible triple $(\Lambda, \tilde{B}, \mathbf{g})$ with indexing sets $[1, N]$ and $\mathbf{ex} := \mathbf{ex}' \cup \mathbf{ex}''[N']$, where $N = N' + N''$.

$$\lambda_{i,j} := \begin{cases} \lambda'_{i,j} & \text{if } i, j \in [1, N'] \\ (g'_j, g''_{i-N_1}) & \text{if } i \in [N'+1, N] \text{ and } j \in [1, N'] \\ (g'_i, g''_{j-N_1})^{-1} & \text{if } i \in [1, N'] \text{ and } j \in [N'+1, N] \\ \lambda''_{i-N', j-N'} & \text{if } i, j \in [N'+1, N] \end{cases}$$

$$b_{i,j} := \begin{cases} b'_{i,j} & \text{if } i \in [1, N'] \text{ and } j \in \mathbf{ex}' \\ b''_{i-N',j-N'} & \text{if } i \in [N'+1, N] \text{ and } j \in \mathbf{ex}''[N'] \\ 0 & \text{else} \end{cases}$$

$$g_i := \begin{cases} g'_i & \text{if } i \in [1, N'] \\ g''_{i-N'} & \text{if } i \in [N'+1, N] \end{cases}$$

Ignoring indexing difficulties, the assignments in Theorem 5.15 can be viewed as the assignments

$$\Lambda := \begin{bmatrix} \Lambda' & -\mathbf{x}^T \\ \mathbf{x} & \Lambda'' \end{bmatrix}, \ \tilde{B} := \begin{bmatrix} \tilde{B}' & 0 \\ 0 & \tilde{B}'' \end{bmatrix}, \text{ and } \mathbf{g} := \begin{bmatrix} \mathbf{g}' \\ \mathbf{g}'' \end{bmatrix}, \qquad \text{(Equation 5.3.)}$$

where $\mathbf{x}$ is the $N'' \times N'$ matrix with entries given by $x_{i,j} := (g'_j, g''_i)$ for all $i \in [1, N'']$ and $j \in [1, N']$. Henceforth, we will denote by $(\Lambda', \tilde{B}', \mathbf{g}') * (\Lambda'', \tilde{B}'', \mathbf{g}'')$ the single-parameter compatible triple $(\Lambda, \tilde{B}, \mathbf{g})$ constructed in Theorem 5.15 from two single-parameter compatible triples.

Proposition 5.16. Let $(M, \tilde{B}, \mathbf{g})$, $(M', \tilde{B}', \mathbf{g}')$, and $(M'', \tilde{B}'', \mathbf{g}'')$ be single-parameter graded quantum seeds in division algebras $\mathcal{F}$, $\mathcal{F}'$, and $\mathcal{F}''$ such that $\mathcal{F}'$ and $\mathcal{F}''$ are subalgebras of $\mathcal{F}$ and $(\Lambda(M), \tilde{B}, \mathbf{g}) = (\Lambda(M'), \tilde{B}', \mathbf{g}') * (\Lambda(M''), \tilde{B}'', \mathbf{g}'')$. Suppose, furthermore, that

$$M(e_i) = \begin{cases} M'(e_i) & \text{if } i \in [1, N'] \\ M''(e_{i-N'}) & \text{if } i \in [N'+1, N] \end{cases}.$$

Then, for $k \in \mathbf{ex}'$,

$$(\Lambda(\mu_k(M)), \mu_k(\tilde{B}), \mu_k(\mathbf{g})) = (\Lambda(\mu_k(M')), \mu_k(\tilde{B}'), \mu_k(\mathbf{g}')) * (\Lambda(M''), \tilde{B}'', \mathbf{g}'')$$

and

$$\mu_k(M)(e_i) = \begin{cases} \mu_k(M')(e_i) & \text{if } i \in [1, N'] \\ M''(e_{i-N'}) & \text{if } i \in [N'+1, N] \end{cases}.$$

Similarly, for $k \in \mathbf{ex}''[N']$,

$$(\Lambda(\mu_k(M)), \mu_k(\tilde{B}), \mu_k(\mathbf{g})) = (\Lambda(M'), \tilde{B}', \mathbf{g}') * (\Lambda(\mu_{k-N}(M'')), \mu_{k-N}(\tilde{B}''), \mu_{k-N}(\mathbf{g}''))$$

and

$$\mu_k(M)(e_i) = \begin{cases} M'(e_i) & \text{if } i \in [1, N'] \\ \mu_k(M'')(e_{i-N'}) & \text{if } i \in [N'+1, N] \end{cases}.$$

Theorem 5.17. Let $(\Lambda', \tilde{B}', \mathbf{g}')$ and $(\Lambda'', \tilde{B}'', \mathbf{g}'')$ be single-parameter compatible triples, as well as $\mathbf{inv}' \subset [1, N'] \setminus \mathbf{ex}'$ and $\mathbf{inv}'' \subset [1, N''] \setminus \mathbf{ex}''$. Set $(\Lambda, \tilde{B}, \mathbf{g}) := (\Lambda', \tilde{B}', \mathbf{g}') * (\Lambda'', \tilde{B}'', \mathbf{g}'')$ and $\mathbf{inv} := \mathbf{inv}' \sqcup \mathbf{inv}''[N']$. Then, if $\Bbbk$ is any unital subring of $\mathbb{K}$ containing $q^{\pm\frac{1}{2}\lambda_{i,j}}$ for all $i, j \in [1, N]$ (and so, in particular, $q^{\pm\frac{1}{2}\lambda'_{i,j}}$ for $i, j \in [1, N']$ and $q^{\pm\frac{1}{2}\lambda''_{i,j}}$ for $i, j \in [1, N'']$), we have an isomorphism of $\Bbbk$-algebras

$$\Bbbk_q(\Lambda', \tilde{B}', \mathbf{g}', \mathbf{inv}')\overline{\otimes}_{\Bbbk}^{(,)}\Bbbk_q(\Lambda'', \tilde{B}'', \mathbf{g}'', \mathbf{inv}'') \cong \Bbbk_q(\Lambda, \tilde{B}, \mathbf{g}, \mathbf{inv}).$$

Recalling that

$$\mathcal{A}_q[U]^{\underline{\otimes} n} \cong \mathcal{A}_q[U]^{\overline{\otimes} n}, \quad \mathcal{A}_q[B]^{\underline{\otimes} n} \cong \mathcal{A}_q[B]^{\overline{\otimes} n},$$

the following corollary is then immediate.

Corollary 5.18. For any $n \geq 1$, $\mathcal{A}_q[U]^{\underline{\otimes} n}$ and $\mathcal{A}_q[B]^{\underline{\otimes} n}$ are (single-parameter) graded quantum cluster algebras.

5.4. The Search for a Quantum Seed

Set $\mathbb{K} = \mathbb{Q}(q^{\frac{1}{2d}})$ for the rest of the chapter. All quantum cluster algebras appearing in this section are single-parameter, although we will stop saying so.

Corollary 5.18 gives rise to a graded quantum seed $(M, \tilde{B}, \mathbf{g}, \mathbf{inv})$ so that there is an isomorphism of graded $\mathbb{K}$-algebras $\Psi : \mathbb{K}(M, \tilde{B}, \mathbf{g}, \mathbf{inv}) \to \mathcal{A}_q[B]^{\underline{\otimes} n}$. For each $\lambda \in \mathcal{P}^+$ and $i \in [1, n]$, abbreviate by $v_{\lambda,i}$ the element $1 \otimes \cdots \otimes v_\lambda \otimes \cdots \otimes 1 \in \mathcal{A}_q[B]^{\underline{\otimes} n}$, where v_λ appears in the i-th place. Then for each $i \in I$ and $j \in [1, n]$, there is some $k_{i,j} \in \mathbf{inv}$ such that $\Psi(M(e_{k_{i,j}})) = v_{\omega_i,j}$. In fact, this yields a bijection $I \times [1, n] \tilde{\to} \mathbf{inv}$, $(i, j) \mapsto k_{i,j}$. Now, as $\mathcal{A}_q[B]$ is a localization of $\mathcal{A}_q[G/U]$ by the multiplicative set $\{v_\lambda \mid \lambda \in \mathcal{P}^+\}$, it follows that for each $x \in \mathcal{A}_q[B]$ there exists some $\lambda \in \mathcal{P}^+$ such that $x v_\lambda \in \mathcal{A}_q[G/U]$. Then for each $i \in [1, N] \setminus \mathbf{inv}$, there exist nonnegative integers $(s_{i,j})_{j \in \mathbf{inv}}$ such that

$$\Psi\left(M\left(e_i + \sum_{j \in \mathbf{inv}} s_{i,j} e_j\right)\right) \in \mathcal{A}_q[G/U]^{\underline{\otimes} n}.$$

Setting $s_{i,j} = \delta_{i,j}$ for all other $(i, j) \in [1, N] \times [1, N]$, we obtain a matrix

$$S = (s_{i,j})_{i,j \in [1,N]} \in GL_N(\mathbb{Z}),$$

which we also view as an invertible additive map $S : \mathbb{Z}^N \to \mathbb{Z}^N$. Set $M_S := M \circ S$, $\tilde{B}_S = S^{-1}\tilde{B}$, and $\mathbf{g}_S := S^T \mathbf{g}$. One easily checks that $(M_S, \tilde{B}_S, \mathbf{g}_S)$ is a graded quantum seed. Let $\mathbf{inv}_S$ be any subset of $\mathbf{inv}$. The following lemma is obvious.

Lemma 5.19. $\mathbb{K}(M_S, \tilde{B}_S, \mathbf{g}_S, \mathbf{inv}_S)$ is a subalgebra of $\mathbb{K}(M, \tilde{B}, \mathbf{g}, \mathbf{inv})$. In particular, $\mathbb{K}(M_S, \tilde{B}_S, \mathbf{g}_S, \mathbf{inv}) = \mathbb{K}(M, \tilde{B}, \mathbf{g}, \mathbf{inv})$.

Conjecture 5.20. There is some choice of $(s_{i,j})_{(i,j)\in([1,N]\setminus\mathbf{inv})\times\mathbf{inv}}$ such that, if $(M', \tilde{B}', \mathbf{g}')$ is mutation equivalent to $(M_S, \tilde{B}_S, \mathbf{g}_S)$, then $\Psi(M'(e_i)) \in \mathcal{A}_q[G/U]^{\underline{\otimes} n}$ for all $i \in [1, N]$.

Conjecture 5.21. Set $\mathbf{ex}_{thaw} = \mathbf{ex} \cup \{k_{i,j} \mid i \in I,\ j \in [1, n-1]\}$. Given $(M_S, \tilde{B}_S, \mathbf{g}_S)$ as in the preceding conjecture, there is an $N \times \mathbf{ex}_{thaw}$ integer matrix $\tilde{B}_{S,thaw}$ such that the $N \times \mathbf{ex}$ submatrix of $\tilde{B}_{S,thaw}$ is $\tilde{B}_S$, $(M_S, \tilde{B}_{S,thaw}, \mathbf{g}_S)$ is a graded quantum seed, and $\mathbb{K}(M_S, \tilde{B}_{S,thaw}, \mathbf{g}_S, \varnothing) \cong \mathcal{A}_q[G/U]^{\underline{\otimes} n}$.

In the case of $G = SL_m$, there is another natural place to search for the quantum cluster structures that appear in the preceding conjectures. Namely, for any integer $m \geq 2$, there is a natural $U_q(\mathfrak{sl}_m)$-module subalgebra of $\mathcal{A}_q[Mat_{m,m-1}]$ isomorphic to $\mathcal{A}_q[SL_m/U]$. In fact, it is a quantum cluster subalgebra in an appropriate sense, which we do not address here.

Conjecture 5.22. For all integers $m \geq 2$ and $n \geq 1$, $\mathcal{A}_q[SL_m/U]^{\underline{\otimes} n}$ is a quantum cluster subalgebra of $\mathcal{A}_q[Mat_{m,(m-1)n}]$.

Conjecture 5.22 is already known in the case where $n = 1$. In the case where $m = 2$, it is obvious because $\mathcal{A}_q[SL_2/U]^{\underline{\otimes} n}$ and $\mathcal{A}_q[Mat_{2,n}]$ are actually isomorphic. The case where $m = 3$ is already much more complicated. However, even here, we can say the conjecture is true if $n = 2$.

Conjecture 5.22 also has a very desirable consequence. Namely, if it is true, then all quantum cluster monomials of $\mathcal{A}_q[SL_m/U]^{\underline{\otimes} n}$ are contained in $\mathcal{B}^{\diamond n}$, where $\mathcal{B}$ is the dual canonical basis of $\mathcal{A}_q[SL_m/U]$. This may appear to be a happy coincidence, but we conjecture that this phenomenon is prevalent.

Conjecture 5.23. Suppose $(A, \mathcal{B})$ and $(A', \mathcal{B}')$ are locally finite based module algebras such that $\mathcal{B}$ and $\mathcal{B}'$ are upper global crystal bases and A and A' are quantum cluster algebras such that their respective quantum cluster monomials are contained in $\mathcal{B}$ and $\mathcal{B}'$. Then $A \underline{\otimes} A'$ has the structure of a quantum cluster algebra such that the corresponding quantum cluster monomials are contained in $\mathcal{B} \diamond \mathcal{B}'$.

5.5. Example: $G = SL_3(\mathbb{C})$ and $n = 2$

In this section, we explicitly run through the algorithm proposed by the conjectures of the preceding section, noting that $\mathcal{A}_q[SL_3/U]$ is the $U_q(\mathfrak{g})$-module subalgebra of $\mathcal{A}_q[B]$ generated by $v_1 := v_{\omega_1}$ and $v_2 := v_{\omega_2}$. Then it is easy to see that $\mathcal{A}_q[G/U]$ is generated as an algebra by the following elements:

$$v_1,\ v_2,\ v_1x_1,\ v_2x_2,\ v_1\frac{q^{1/2}x_2x_1 - q^{-1/2}x_1x_2}{q-q^{-1}},\ \text{and } v_2\frac{q^{1/2}x_1x_2 - q^{-1/2}x_2x_1}{q-q^{-1}}.$$

Now $\mathcal{A}_q[B] \cong \mathbb{K}_q(\Lambda, \tilde{B}, \mathbf{g}, \{4,5\})$, where

$$\Lambda = \begin{bmatrix} 0 & -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \end{bmatrix},\ \tilde{B} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix},\ \text{and } \mathbf{g} = \begin{bmatrix} \varepsilon_2 - \varepsilon_1 \\ \varepsilon_3 - \varepsilon_1 \\ \varepsilon_3 - \varepsilon_1 \\ \varepsilon_1 \\ \varepsilon_1 + \varepsilon_2 \end{bmatrix}$$

with the column of $\tilde{B}$ labeled by 1. The inverse of this isomorphism is given on the initial variables by

$$
\begin{aligned}
&X_1 \mapsto x_1, \\
&X_2 \mapsto \frac{q^{1/2}x_2x_1 - q^{-1/2}x_1x_2}{q-q^{-1}}, \\
&X_3 \mapsto \frac{q^{1/2}x_1x_2 - q^{-1/2}x_2x_1}{q-q^{-1}}, \\
&X_4 \mapsto v_1, \text{ and} \\
&X_5 \mapsto v_2,
\end{aligned}
$$

where we write $X_i = M(e_i)$ for $i \in [1,5]$. Using the fact that $\mathcal{A}_q[B]\underline{\otimes}\mathcal{A}_q[B] \cong \mathcal{A}_q[B]\overline{\otimes}\mathcal{A}_q[B]$, we have isomorphisms

$$
\begin{aligned}
\mathcal{A}_q[B]\underline{\otimes}\mathcal{A}_q[B] &\cong \mathbb{K}_q(\Lambda, \tilde{B}, \mathbf{g}, \{4,5\})\overline{\otimes}\mathbb{K}_q(\Lambda, \tilde{B}, \mathbf{g}, \{4,5\})) \\
&\cong \mathbb{K}_q(\Lambda', \tilde{B}', \mathbf{g}', \{4,5,9,10\}),
\end{aligned}
$$

where

$$
\Lambda' = \begin{bmatrix}
0 & -1 & 1 & 1 & 0 & -2 & -1 & -1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & -1 & -2 & -2 & 1 & 1 \\
-1 & 0 & 0 & 1 & 1 & -1 & -2 & -2 & 1 & 1 \\
-1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 & -1 & -1 \\
0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & -1 & -2 \\
2 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & 0 \\
1 & 2 & 2 & -1 & -1 & 1 & 0 & 0 & 1 & 1 \\
1 & 2 & 2 & -1 & -1 & -1 & 0 & 0 & 1 & 1 \\
-1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 \\
0 & -1 & -1 & 1 & 2 & 0 & -1 & -1 & 0 & 0
\end{bmatrix},
$$

$$\tilde{B}' = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } \mathbf{g}' = \begin{bmatrix} \varepsilon_2 - \varepsilon_1 \\ \varepsilon_3 - \varepsilon_1 \\ \varepsilon_3 - \varepsilon_1 \\ \varepsilon_1 \\ \varepsilon_1 + \varepsilon_2 \\ \varepsilon_2 - \varepsilon_1 \\ \varepsilon_3 - \varepsilon_1 \\ \varepsilon_3 - \varepsilon_1 \\ \varepsilon_1 \\ \varepsilon_1 + \varepsilon_2 \end{bmatrix},$$

with the columns of $\tilde{B}'$ labeled by $\{1, 6\}$. The inverse of this isomorphism is Ψ from the previous section and is given by

$$\begin{aligned}
\Psi(X_1) &= x_1 \otimes 1, \\
\Psi(X_2) &= \frac{q^{1/2}x_2x_1 - q^{-1/2}x_1x_2}{q - q^{-1}} \otimes 1, \\
\Psi(X_3) &= \frac{q^{1/2}x_1x_2 - q^{-1/2}x_2x_1}{q - q^{-1}} \otimes 1, \\
\Psi(X_4) &= v_1 \otimes 1, \\
\Psi(X_5) &= v_2 \otimes 1, \\
\Psi(X_6) &= 1 \otimes x_1 - x_1 \otimes 1, \\
\Psi(X_7) &= 1 \otimes \frac{q^{1/2}x_2x_1 - q^{-1/2}x_1x_2}{q - q^{-1}} - q^{-1/2}x_2 \otimes x_1 \\
&\quad + q^{-1}\frac{q^{1/2}x_1x_2 - q^{-1/2}x_2x_1}{q - q^{-1}} \otimes 1,
\end{aligned}$$

$$\Psi(X_8) = 1 \otimes \frac{q^{1/2}x_1x_2 - q^{-1/2}x_2x_1}{q-q^{-1}} - q^{-1/2}x_1 \otimes x_2 + q^{-1}\frac{q^{1/2}x_2x_1 - q^{-1/2}x_1x_2}{q-q^{-1}} \otimes 1,$$

$$\Psi(X_9) = 1 \otimes v_1, \text{ and}$$

$$\Psi(X_{10}) = 1 \otimes v_2.$$

Precomposing our toric frame with the element of $GL_{10}(\mathbb{Z})$:

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

we obtain $\mathcal{A}_q[B]\underline{\otimes}\mathcal{A}_q[B] \cong \mathbb{K}_q(\Lambda'', \tilde{B}'', \mathbf{g}'', \{4,5,9,10\})$ where

$$\Lambda'' = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & -1 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & -2 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{B}'' = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ -1 & 1 \\ 1 & -1 \\ 0 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, \text{ and } \mathbf{g}'' = \begin{bmatrix} \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_2 + \varepsilon_3 \\ \varepsilon_1 \\ \varepsilon_1 + \varepsilon_2 \\ \varepsilon_1 + \varepsilon_2 \\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \\ \varepsilon_1 \\ \varepsilon_1 + \varepsilon_2 \end{bmatrix},$$

with the columns of $\tilde{B}''$ labeled by $\{1, 6\}$. The inverse of the isomorphism is given by

$$
\begin{aligned}
X_1 &\mapsto q^{1/2} v_1 x_1 \otimes 1, \\
X_2 &\mapsto q^{1/2} v_1 \frac{q^{1/2} x_2 x_1 - q^{-1/2} x_1 x_2}{q - q^{-1}} \otimes 1, \\
X_3 &\mapsto q^{1/2} v_2 \frac{q^{1/2} x_1 x_2 - q^{-1/2} x_2 x_1}{q - q^{-1}} \otimes 1, \\
X_4 &\mapsto v_1 \otimes 1, \\
X_5 &\mapsto v_2 \otimes 1, \\
X_6 &\mapsto q^{1/2} v_1 \otimes v_1 x_1 - q^{-1/2} v_1 x_1 \otimes v_1, \\
X_7 &\mapsto q^{1/2} v_2 \otimes v_1 \frac{q^{1/2} x_2 x_1 - q^{-1/2} x_1 x_2}{q - q^{-1}} - v_2 x_2 \otimes v_1 x_1 \\
&\quad + q^{-1/2} v_2 \frac{q^{1/2} x_1 x_2 - q^{-1/2} x_2 x_1}{q - q^{-1}} \otimes v_1, \\
X_8 &\mapsto q^{1/2} v_1 \otimes v_2 \frac{q^{1/2} x_1 x_2 - q^{-1/2} x_2 x_1}{q - q^{-1}} - v_1 x_1 \otimes v_2 x_2 \\
&\quad + q^{-1/2} v_1 \frac{q^{1/2} x_2 x_1 - q^{-1/2} x_1 x_2}{q - q^{-1}} \otimes v_2, \\
X_9 &\mapsto 1 \otimes v_1, \text{ and} \\
X_{10} &\mapsto 1 \otimes v_2.
\end{aligned}
$$

These are clearly elements of $\mathcal{A}_q[G/U]\underline{\otimes}\mathcal{A}_q[G/U]$. As the given quantum cluster algebra is of type $A_1 \times A_1$, we readily verify that the images of all other quantum cluster variables are elements of $\mathcal{A}_q[G/U]\underline{\otimes}\mathcal{A}_q[G/U]$:

$$
\begin{aligned}
\mu_1(X_1) &\mapsto q^{1/2} v_2 x_2 \otimes 1 \text{ and} \\
\mu_6(X_6) &\mapsto q v_2 \otimes v_2 x_2 - v_2 x_2 \otimes v_2,
\end{aligned}
$$

where we write $\mu_k(X_k)$ as a short-hand for $\mu_k(M)(e_k)$. Hence we see that $\mathbb{K}_q(\Lambda'', \tilde{B}'', \mathbf{g}'', \varnothing)$ may be considered as a subalgebra of $\mathcal{A}_q[G/U]\underline{\otimes}\mathcal{A}_q[G/U]$. Let $\mathcal{F}$ be the skew-field of fractions of $\mathcal{A}_q[G/U]\underline{\otimes}\mathcal{A}_q[G/U]$ (or equivalently, $\mathcal{A}_q[B]\underline{\otimes}\mathcal{A}_q[B]$).

We observe that the triple $(\Lambda'', \tilde{B}''', \mathbf{g}'')$ with

$$\tilde{B}''' = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is also compatible, where the columns of $\tilde{B}'''$ are labeled by $\{1, 4, 5, 6\}$. Hence we have an inclusion of algebras

$$\mathbb{K}_q(\Lambda'', \tilde{B}''', \mathbf{g}'', \varnothing) \hookrightarrow \mathcal{F}.$$

As $\mathbb{K}_q(\Lambda'', \tilde{B}''', \mathbf{g}'', \varnothing)$ is of type D_4 and hence has finitely many cluster variables, we may explicitly compute all of their images (where we don't repeat those already given previously):

$$\begin{aligned} \mu_4(X_4) &\mapsto 1 \otimes q^{1/2} v_1 x_1, \\ \mu_1\mu_6\mu_5(X_1) &\mapsto 1 \otimes q^{1/2} v_2 x_2, \\ \mu_1\mu_5\mu_4(X_1) &\mapsto 1 \otimes q^{1/2} v_1 \frac{q^{1/2} x_2 x_1 - q^{-1/2} x_1 x_2}{q - q^{-1}}, \end{aligned}$$

$$\begin{aligned}
\mu_4\mu_1\mu_6\mu_5(X_4) &\mapsto 1 \otimes q^{1/2} v_2 \frac{q^{1/2}x_1x_2 - q^{-1/2}x_2x_1}{q-q^{-1}},\\
\mu_1\mu_5(X_1) &\mapsto q^{1/2}v_1 \otimes v_1 \frac{q^{1/2}x_2x_1 - q^{-1/2}x_1x_2}{q-q^{-1}}\\
&\quad - q^{-1/2}v_1 \frac{q^{1/2}x_2x_1 - q^{-1/2}x_1x_2}{q-q^{-1}} \otimes v_1,\\
\mu_5\mu_4(X_5) &\mapsto qv_2 \otimes v_2 \frac{q^{1/2}x_1x_2 - q^{-1/2}x_2x_1}{q-q^{-1}}\\
&\quad - v_2 \frac{q^{1/2}x_1x_2 - q^{-1/2}x_2x_1}{q-q^{-1}} \otimes v_2,\\
\mu_6\mu_5(X_6) &\mapsto qv_1x_1 \otimes v_2x_2 - q^{-1/2}v_1 \frac{q^{1/2}x_2x_1 - q^{-1/2}x_1x_2}{q-q^{-1}} \otimes v_2,\\
\mu_5\mu_1\mu_4\mu_1\mu_5(X_5) &\mapsto qv_2x_2 \otimes v_1x_1 - q^{-1/2}v_2 \frac{q^{1/2}x_1x_2 - q^{-1/2}x_2x_1}{q-q^{-1}} \otimes v_1,\\
\mu_5(X_5) &\mapsto qv_1x_1 \otimes v_1 \frac{q^{1/2}x_2x_1 - q^{-1/2}x_1x_2}{q-q^{-1}}\\
&\quad - v_1 \frac{q^{1/2}x_2x_1 - q^{-1/2}x_1x_2}{q-q^{-1}} \otimes v_1x_1,\\
\mu_4\mu_6\mu_1(X_4) &\mapsto q^{3/2}v_2x_2 \otimes v_2 \frac{q^{1/2}x_1x_2 - q^{-1/2}x_2x_1}{q-q^{-1}}\\
&\quad - q^{1/2}v_2 \frac{q^{1/2}x_1x_2 - q^{-1/2}x_2x_1}{q-q^{-1}} \otimes v_2x_2.
\end{aligned}$$

We see that all quantum cluster variables are contained in $\mathcal{A}_q[G/U]\underline{\otimes}\mathcal{A}_q[G/U] \subset \mathcal{F}$ and the set of all quantum cluster variables clearly contains a generating set for $\mathcal{A}_q[G/U]\underline{\otimes}\mathcal{A}_q[G/U]$. Hence we see that we have

$$\mathcal{A}_q[G/U]\underline{\otimes}\mathcal{A}_q[G/U] \cong \mathbb{K}_q(\Lambda'', \tilde{B}''', \mathbf{g}'', \varnothing).$$

CHAPTER VI

BIALGEBROIDS

Since bialgebroids are not so widely known as bialgebras (which are a special case), we begin with some examples of bialgebroids which are not bialgebras.

6.1. Examples of Bialgebroids

Example 6.1. [25, Example in Section 7] Let $\Bbbk$ be a subring of $\mathbb{K}$ and let H be a cocommutative bialgebra over $\Bbbk$ such that $\mathbb{K}$ is an H-module algebra. Then the $\Bbbk$-algebra $\mathbb{K} \rtimes H$ with base abelian group $\mathbb{K} \otimes_{\Bbbk} H$ may be given the structure of a bialgebroid over $\mathbb{K}$ of Sweedler type via the assignments

- $(k' \bullet h)(k \bullet h') = k'(h_{(1)} \triangleright k) \bullet h_{(2)}h'$
- $\eta(k) = k \bullet 1$
- $\Delta(k \bullet h) = (k \bullet h_{(1)}) \otimes (1 \bullet h_{(2)})$
- $\varepsilon(k \bullet h) = k\varepsilon_H(h)$.

Example 6.1 is an immediate source of many examples of bialgebroids over $\mathbb{K}$ of Sweedler type which are not bialgebras over $\mathbb{K}$. Galois extensions yield our first concrete example.

Example 6.2. If $\mathbb{K}$ is a Galois extension of a field $\Bbbk$ with Galois group $G := \mathrm{Gal}(\mathbb{K}/\Bbbk)$, then $\mathbb{K}$ is a $\Bbbk G$-module algebra and hence the cross product $\mathbb{K} \rtimes \Bbbk G$ is a bialgebroid over $\mathbb{K}$ of Sweedler type.

In [5], Berenstein and Richmond constructed $\mathbb{K}$-coalgebra structures on "twisted group algebras" which match the construction in Example 6.1. Our next example addresses this and the associated generalized nil Hecke algebra.

Example 6.3. Fix a field $\Bbbk$ and an index set I. Following [5, Section 4], let V be the $\Bbbk$-vector space with basis $\{\alpha_i \mid i \in I\}$ and $W = \langle s_i \mid i \in I \rangle$ a Coxeter semigroup (by assumption, actually a monoid) with a compatible quasi-Cartan matrix $A = (a_{ij})$. The assignment

$$s_i(\alpha_j) = \alpha_i - a_{ij}\alpha_j$$

defines an action of W on V. This induces the structure of a $\Bbbk W$-module algebra on the symmetric algebra $S(V)$ and its field of fractions $Frac(S(V)) =: Q$. As in Example 6.1, this makes $Q \rtimes \Bbbk W =: Q_W$ into a bialgebroid over Q of Sweedler type.

The generalized nil Hecke algebra $\mathcal{H}_A(W)$ is the $\Bbbk$-subalgebra of Q_W generated by $S(V) \bullet 1$ and the Demazure elements

$$x_i := \frac{1}{\alpha_i} \bullet (s_i - 1).$$

$\mathcal{H}_A(W)$ is automatically an $S(V)$-ring via the map $S(V) \to \mathcal{H}_A(W)$, $v \mapsto v \bullet 1$. However, just like Q_W, $\mathcal{H}_A(W)$ is not a bialgebra over $S(V)$ since the image of $S(V)$ is not central (and hence $\mathcal{H}_A(W)$ is not an $S(V)$-algebra). It is, however, a bialgebroid over $S(V)$ of Sweedler type in the following manner.

It was proved in [5] that $\{x_w \mid w \in W\}$ is a basis for $\mathcal{H}_A(W)$ over $S(V)$ and for Q_W over Q, satisfying

$$\Delta_{Q_W}(x_w) = \sum_{u,v \in W} p^w_{uv} x_u \otimes x_v$$

for some $p^w_{uv} \in S(V)$. Since we also have $\varepsilon_{Q_W}(x_w) = \delta_{w,1} \in S(V)$, the Q-coalgebra structure on Q_W actually induces an $S(V)$-coalgebra structure of $\mathcal{H}_A(W)$. Δ and ε are given on the generating Demazure elements by

$$\Delta(x_i) = \alpha_i x_i \otimes x_i + 1 \otimes x_i + x_i \otimes 1, \qquad \varepsilon(x_i) = 0.$$

It is then easily argued that these induced maps make $\mathcal{H}_A(W)$ into a bialgebroid over $S(V)$ of Sweedler type.

In [21], Kostant and Kumar introduce a similar construction, but in place of $S(V)$, they have $R(T)$, which we explain in the following example.

Example 6.4. Suppose G is a Kac-Moody group over $\mathbb{C}$, B the standard Borel subgroup, and T the compact maximal torus. Let $R(T)$ be the group ring of the character group of T and let Q be the field of fractions of $R(T)$. W is the corresponding Weyl group, generated by simple reflections s_i. We denote by e^λ the character corresponding to the integral weight λ.

W acts on Q by field automorphisms, making Q a $\mathbb{Z}W$-module algebra. As above, this makes $Q \rtimes \mathbb{Z}W =: Q_W$ into a bialgebroid over Q of Sweedler type. (Note that our Q_W is ring anti-isomorphic to the Q_W of [21] via $q \bullet w \mapsto \delta_{w^{-1}} q$.)

Let Y be the subring of Q_W generated by $R(T) \bullet 1$ and the idempotent elements

$$y_i := \frac{1}{1 - e^{-\alpha_i}} \bullet (s_i + 1)$$

(as in [21]). Y is automatically an $R(T)$-ring via the map $R(T) \to Y$, $q \mapsto q \bullet 1$, but unfortunately

$$\varepsilon_{Q_W}(y_i) = \frac{2}{1 - e^{-\alpha_i}} \notin R(T).$$

This means that Y is not (so easily, at least) a bialgebroid over $R(T)$ of Sweedler type as $\mathcal{H}_A(W)$ was over $S(V)$ in Example 6.3. Instead, we consider Y^t, the subring of Q_W generated by $R(T) \bullet 1$ and the indempotent elements

$$\overline{y}_i := \frac{1}{1-e^{\alpha_i}} \bullet s_i + \frac{1}{1-e^{-\alpha_i}} \bullet 1 = (1 \bullet (s_i + 1)) \left(\frac{1}{1-e^{-\alpha_i}} \bullet 1 \right).$$

We have

$$\Delta_{Q_W}(\overline{y}_i) = (1-e^{\alpha_i})\overline{y}_i \otimes \overline{y}_i + e^{\alpha_i}(\overline{y}_i \otimes 1 + 1 \otimes \overline{y}_i + 1 \otimes 1), \quad \varepsilon_{Q_W}(\overline{y}_i) = 1.$$

As in Example 6.3, we conclude that Y^t is a bialgebroid over $R(T)$ of Sweedler type.

6.2. The Construction and Applications

Theorem 6.5. Let B be a cocommutative bialgebroid over $\mathbb{K}$ of Sweedler type and let A be a B-module and a $\mathbb{K}$-bialgebra, such that m_A, η_A, and ε_A are B-module homomorphisms. Suppose further that $R : B \to A \otimes_{\mathbb{K}} A$ is a (left) $\mathbb{K}$-linear map such that

1. $R(1) = 1 \otimes 1 \qquad (\in A \otimes_{\mathbb{K}} A)$
2. $R(bb') = R(b_{(1)})(b_{(2)} \triangleright R(b')) \qquad (\in A \otimes_{\mathbb{K}} A)$
3. $R(b_{(1)})(b_{(2)} \triangleright \Delta_A(a)) = \Delta_A(b_{(1)} \triangleright a)R(b_{(2)}) \qquad (\in A \otimes_{\mathbb{K}} A)$
4. $(\Delta_A \otimes \mathrm{id}_A)(R(b_{(1)}))R_{12}(b_{(2)}) = (\mathrm{id}_A \otimes \Delta_A)(R(b_{(1)}))R_{23}(b_{(2)}) \qquad (\in A \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} A)$
5. $m_A \circ ((\eta_A \circ \varepsilon_A) \otimes \mathrm{id}_A) \circ R = \eta_A \circ \varepsilon_B = m_A \circ (\mathrm{id}_A \otimes (\eta_A \circ \varepsilon_A)) \circ R \qquad (\in A)$

where we use the notation $R_{ij}(b)$ $(1 \le i < j \le 3)$ to denote that $R(b)$ appears in the i^{th} and j^{th} places of the three-fold tensor. Then the $\mathbb{K}$-ring $A \rtimes B$ may be given the structure of a bialgebroid over $\mathbb{K}$ of Sweedler type via the additional assignments

$$\Delta(a \bullet b) = (a_{(1)}(b_{(1)})^{(1)} \bullet b_{(2)}) \otimes (a_{(2)}(b_{(1)})^{(2)} \bullet b_{(3)}), \qquad \varepsilon(a \bullet b) = \varepsilon_A(a)\varepsilon_B(b)$$

where we use Sweedler-like notation to write $R(b) = b^{(1)} \otimes b^{(2)}$.

Proof. We check the axioms of Definition 2.7 one-by-one, skipping (1) in light of Lemma 2.12.

(2) **Coassociativity of Δ:**

By definition and since Δ_A is multiplicative,

$$((\Delta \otimes \mathrm{id}) \circ \Delta)(a \bullet b)$$
$$= (\Delta \otimes \mathrm{id})((a_{(1)}(b_{(1)})^{(1)} \bullet b_{(2)}) \otimes (a_{(2)}(b_{(1)})^{(2)} \bullet b_{(3)}))$$
$$= (a_{(1)}((b_{(1)})^{(1)})_{(1)}(b_{(2)})^{(1)} \bullet b_{(3)}) \otimes (a_{(2)}((b_{(1)})^{(1)})_{(2)}(b_{(2)})^{(2)} \bullet b_{(4)}) \otimes (a_{(3)}(b_{(1)})^{(2)} \bullet b_{(5)}).$$

Using property (4) for R,

$$(a_{(1)}((b_{(1)})^{(1)})_{(1)}(b_{(2)})^{(1)} \bullet b_{(3)}) \otimes (a_{(2)}((b_{(1)})^{(1)})_{(2)}(b_{(2)})^{(2)} \bullet b_{(4)}) \otimes (a_{(3)}(b_{(1)})^{(2)} \bullet b_{(5)})$$
$$= (a_{(1)}(b_{(1)})^{(1)} \bullet b_{(3)}) \otimes (a_{(2)}((b_{(1)})^{(2)})_{(1)}(b_{(2)})^{(1)} \bullet b_{(4)}) \otimes (a_{(3)}((b_{(1)})^{(2)})_{(2)}(b_{(2)})^{(2)} \bullet b_{(5)}).$$

Since B is cocommutative, then by definition,

$$(a_{(1)}(b_{(1)})^{(1)} \bullet b_{(3)}) \otimes (a_{(2)}((b_{(1)})^{(2)})_{(1)}(b_{(2)})^{(1)} \bullet b_{(4)}) \otimes (a_{(3)}((b_{(1)})^{(2)})_{(2)}(b_{(2)})^{(2)} \bullet b_{(5)})$$

$$= (a_{(1)}(b_{(1)})^{(1)} \bullet b_{(2)}) \otimes (a_{(2)}((b_{(1)})^{(2)})_{(1)}(b_{(3)})^{(1)} \bullet b_{(4)}) \otimes (a_{(3)}((b_{(1)})^{(2)})_{(2)}(b_{(3)})^{(2)} \bullet b_{(5)})$$

$$= (\mathrm{id} \otimes \Delta)((a_{(1)}(b_{(1)})^{(1)} \bullet b_{(2)}) \otimes (a_{(2)}(b_{(1)})^{(2)} \bullet b_{(3)}))$$

$$= ((\mathrm{id} \otimes \Delta) \circ \Delta)(a \bullet b).$$

So Δ is coassociative.

Compatibility of Δ and ε:

By definition,

$$\begin{aligned}\varepsilon((a \bullet b)_{(1)})(a \bullet b)_{(2)} &= \varepsilon(a_{(1)}(b_{(1)})^{(1)} \bullet b_{(2)})(a_{(2)}(b_{(1)})^{(2)} \bullet b_{(3)}) \\ &= \varepsilon_A(a_{(1)}(b_{(1)})^{(1)})\varepsilon_B(b_{(2)})a_{(2)}(b_{(1)})^{(2)} \bullet b_{(3)}.\end{aligned}$$

Since ε_A is multiplicative,

$$\begin{aligned}&\varepsilon_A(a_{(1)}(b_{(1)})^{(1)})\varepsilon_B(b_{(2)})a_{(2)}(b_{(1)})^{(2)} \bullet b_{(3)} \\ &\quad = \varepsilon_A(a_{(1)})a_{(2)}\varepsilon_A((b_{(1)})^{(1)})(b_{(1)})^{(2)} \bullet \varepsilon_B(b_{(2)})b_{(3)}.\end{aligned}$$

Using also property (5) for R and the compatibilities of Δ_A with ε_A and Δ_B with ε_B,

$$\begin{aligned}\varepsilon_A(a_{(1)})a_{(2)}\varepsilon_A((b_{(1)})^{(1)})(b_{(1)})^{(2)} \bullet \varepsilon_B(b_{(2)})b_{(3)} &= a\varepsilon_B(b_{(1)}) \bullet b_{(2)} \\ &= a \bullet \varepsilon_B(b_{(1)})b_{(2)} \\ &= a \bullet b.\end{aligned}$$

One can similarly show that $\varepsilon((a \bullet b)_{(2)})(a \bullet b)_{(1)} = a \bullet b$ and hence Δ and ε are compatible.

(3) By definition,

$$
\begin{aligned}
(a \bullet b)_{(1)} k \otimes (a \bullet b)_{(2)} &= (a_{(1)}(b_{(1)})^{(1)} \bullet b_{(2)})k \otimes (a_{(2)}(b_{(1)})^{(2)} \bullet b_{(3)}) \\
&= (a_{(1)}(b_{(1)})^{(1)} \bullet b_{(2)})(k \bullet 1) \otimes (a_{(2)}(b_{(1)})^{(2)} \bullet b_{(3)}) \\
&= (a_{(1)}(b_{(1)})^{(1)}(b_{(2)} \triangleright k) \bullet b_{(3)}) \otimes (a_{(2)}(b_{(1)})^{(2)} \bullet b_{(4)}) \\
&= (a_{(1)}(b_{(1)})^{(1)} \bullet b_{(3)}) \otimes (a_{(2)}(b_{(1)})^{(2)}(b_{(2)} \triangleright k) \bullet b_{(4)}).
\end{aligned}
$$

Using also the assumption that B is cocommutative, then again by definition,

$$
\begin{aligned}
&(a_{(1)}(b_{(1)})^{(1)} \bullet b_{(3)}) \otimes (a_{(2)}(b_{(1)})^{(2)}(b_{(2)} \triangleright k) \bullet b_{(4)}) \\
&\quad = (a_{(1)}(b_{(1)})^{(1)} \bullet b_{(2)}) \otimes (a_{(2)}(b_{(1)})^{(2)}(b_{(3)} \triangleright k) \bullet b_{(4)}) \\
&\quad = (a_{(1)}(b_{(1)})^{(1)} \bullet b_{(2)}) \otimes (a_{(2)}(b_{(1)})^{(2)} \bullet b_{(3)})(k \bullet 1) \\
&\quad = (a_{(1)}(b_{(1)})^{(1)} \bullet b_{(2)}) \otimes (a_{(2)}(b_{(1)})^{(2)} \bullet b_{(3)})k \\
&\quad = (a \bullet b)_{(1)} \otimes (a \bullet b)_{(2)} k.
\end{aligned}
$$

(4) **Unit:**

We perform the straightforward computation, using property (1) for R.

$$
\begin{aligned}
\Delta(1) &= (1^{(1)} \bullet 1) \otimes (1^{(2)} \bullet 1) \\
&= (1 \bullet 1) \otimes (1 \bullet 1)
\end{aligned}
$$

Multiplicative:

By definition and since Δ_A and Δ_B (or the corresponding corestriction) are multiplicative,

$$\begin{aligned}
&\Delta((a \bullet b)(a' \bullet b')) \\
&\quad = \Delta(a(b_{(1)} \triangleright a') \bullet b_{(2)}b') \\
&\quad = ((a(b_{(1)} \triangleright a'))_{(1)}((b_{(2)}b')_{(1)})^{(1)} \bullet (b_{(2)}b')_{(2)}) \\
&\qquad \otimes ((a(b_{(1)} \triangleright a'))_{(2)}((b_{(2)}b')_{(1)})^{(2)} \bullet (b_{(2)}b')_{(3)}) \\
&\quad = (a_{(1)}(b_{(1)} \triangleright a')_{(1)}(b_{(2)}b'_{(1)})^{(1)} \bullet b_{(3)}b'_{(2)}) \\
&\qquad \otimes (a_{(2)}(b_{(1)} \triangleright a')_{(2)}(b_{(2)}b'_{(1)})^{(2)} \bullet b_{(4)}b'_{(3)}).
\end{aligned}$$

Using property (2) for R,

$$\begin{aligned}
&(a_{(1)}(b_{(1)} \triangleright a')_{(1)}(b_{(2)}b'_{(1)})^{(1)} \bullet b_{(3)}b'_{(2)}) \otimes (a_{(2)}(b_{(1)} \triangleright a')_{(2)}(b_{(2)}b'_{(1)})^{(2)} \bullet b_{(4)}b'_{(3)}) \\
&\quad = (a_{(1)}(b_{(1)} \triangleright a')_{(1)}(b_{(2)})^{(1)}(b_{(3)} \triangleright (b'_{(1)})^{(1)}) \bullet b_{(5)}b'_{(2)}) \\
&\qquad \otimes (a_{(2)}(b_{(1)} \triangleright a')_{(2)}(b_{(2)})^{(2)}(b_{(4)} \triangleright (b'_{(1)})^{(2)}) \bullet b_{(6)}b'_{(3)}).
\end{aligned}$$

By property (3) for R,

$$\begin{aligned}
&(a_{(1)}(b_{(1)} \triangleright a')_{(1)}(b_{(2)})^{(1)}(b_{(3)} \triangleright (b'_{(1)})^{(1)}) \bullet b_{(5)}b'_{(2)}) \\
&\quad \otimes (a_{(2)}(b_{(1)} \triangleright a')_{(2)}(b_{(2)})^{(2)}(b_{(4)} \triangleright (b'_{(1)})^{(2)}) \bullet b_{(6)}b'_{(3)}) \\
&= (a_{(1)}(b_{(1)})^{(1)}(b_{(2)} \triangleright a'_{(1)})(b_{(4)} \triangleright (b'_{(1)})^{(1)}) \bullet b_{(6)}b'_{(2)}) \\
&\quad \otimes (a_{(2)}(b_{(1)})^{(2)}(b_{(3)} \triangleright a'_{(2)})(b_{(5)} \triangleright (b'_{(1)})^{(2)}) \bullet b_{(7)}b'_{(3)}).
\end{aligned}$$

Since B is cocommutative,

$$
\begin{aligned}
&(a_{(1)}(b_{(1)})^{(1)}(b_{(2)} \triangleright a'_{(1)})(b_{(4)} \triangleright (b'_{(1)})^{(1)}) \bullet b_{(6)}b'_{(2)}) \\
&\quad \otimes (a_{(2)}(b_{(1)})^{(2)}(b_{(3)} \triangleright a'_{(2)})(b_{(5)} \triangleright (b'_{(1)})^{(2)}) \bullet b_{(7)}b'_{(3)}) \\
&= (a_{(1)}(b_{(1)})^{(1)}(b_{(2)} \triangleright a'_{(1)})(b_{(3)} \triangleright (b'_{(1)})^{(1)}) \bullet b_{(6)}b'_{(2)}) \\
&\quad \otimes (a_{(2)}(b_{(1)})^{(2)}(b_{(4)} \triangleright a'_{(2)})(b_{(5)} \triangleright (b'_{(1)})^{(2)}) \bullet b_{(7)}b'_{(3)}).
\end{aligned}
$$

Using the assumption that m_A is a B-module homomorphism,

$$
\begin{aligned}
&(a_{(1)}(b_{(1)})^{(1)}(b_{(2)} \triangleright a'_{(1)})(b_{(3)} \triangleright (b'_{(1)})^{(1)}) \bullet b_{(6)}b'_{(2)}) \\
&\quad \otimes (a_{(2)}(b_{(1)})^{(2)}(b_{(4)} \triangleright a'_{(2)})(b_{(5)} \triangleright (b'_{(1)})^{(2)}) \bullet b_{(7)}b'_{(3)}) \\
&= (a_{(1)}(b_{(1)})^{(1)}(b_{(2)} \triangleright (a'_{(1)}(b'_{(1)})^{(1)})) \bullet b_{(4)}b'_{(2)}) \\
&\quad \otimes (a_{(2)}(b_{(1)})^{(2)}(b_{(3)} \triangleright (a'_{(2)}(b'_{(1)})^{(2)})) \bullet b_{(5)}b'_{(3)}).
\end{aligned}
$$

Again by the assumption that B is cocommutative, then by definition,

$$
\begin{aligned}
&(a_{(1)}(b_{(1)})^{(1)}(b_{(2)} \triangleright (a'_{(1)}(b'_{(1)})^{(1)})) \bullet b_{(4)}b'_{(2)}) \\
&\quad \otimes (a_{(2)}(b_{(1)})^{(2)}(b_{(3)} \triangleright (a'_{(2)}(b'_{(1)})^{(2)})) \bullet b_{(5)}b'_{(3)}) \\
&= (a_{(1)}(b_{(1)})^{(1)}(b_{(2)} \triangleright (a'_{(1)}(b'_{(1)})^{(1)})) \bullet b_{(3)}b'_{(2)}) \\
&\quad \otimes (a_{(2)}(b_{(1)})^{(2)}(b_{(4)} \triangleright (a'_{(2)}(b'_{(1)})^{(2)})) \bullet b_{(5)}b'_{(3)}) \\
&= [(a_{(1)}(b_{(1)})^{(1)} \bullet b_{(2)}) \otimes (a_{(2)}(b_{(1)})^{(2)} \bullet b_{(3)})] \\
&\quad [(a'_{(1)}(b'_{(1)})^{(1)} \bullet b'_{(2)}) \otimes (a'_{(2)}(b'_{(1)})^{(2)} \bullet b'_{(3)})] \\
&= \Delta(a \bullet b)\Delta(a' \bullet b').
\end{aligned}
$$

So Δ is multiplicative.

(5) We perform the straight-forward computation.

$$\varepsilon(1 \otimes 1) = \varepsilon_A(1)\varepsilon_B(1) = (1)(1) = 1$$

(6) By definition,

$$\begin{aligned} \varepsilon((a \bullet b)(a' \bullet b')) &= \varepsilon(a(b_{(1)} \triangleright a') \bullet b_{(2)}b') \\ &= \varepsilon_A(a(b_{(1)} \triangleright a'))\varepsilon_B(b_{(2)}b'). \end{aligned}$$

Since ε_A is multiplicative and $\varepsilon_B(bb') = \varepsilon_B(b\varepsilon_B(b'))$,

$$\varepsilon_A(a(b_{(1)} \triangleright a'))\varepsilon_B(b_{(2)}b') = \varepsilon_A(a)\varepsilon_A(b_{(1)} \triangleright a')\varepsilon_B(b_{(2)}\varepsilon_B(b')).$$

Using the assumption that ε_A is a B-module homomorphism, then definitions,

$$\begin{aligned} \varepsilon_A(a)\varepsilon_A(b_{(1)} \triangleright a')\varepsilon_B(b_{(2)}\varepsilon_B(b')) &= \varepsilon_A(a)\varepsilon_B(b_{(1)}\varepsilon_A(a'))\varepsilon_B(b_{(2)}\varepsilon_B(b')) \\ &= \varepsilon_A(a)\varepsilon_B(\varepsilon_B(b_{(1)}\varepsilon_A(a'))b_{(2)}\varepsilon_B(b')). \end{aligned}$$

By the assumption that $\Delta_B(B) \subset B \times_{\mathbb{K}} B$,

$$\varepsilon_A(a)\varepsilon_B(\varepsilon_B(b_{(1)}\varepsilon_A(a'))b_{(2)}\varepsilon_B(b')) = \varepsilon_A(a)\varepsilon_B(\varepsilon_B(b_{(1)})b_{(2)}\varepsilon_A(a')\varepsilon_B(b')).$$

Since Δ_B and ε_B are compatible and by definition,

$$\begin{aligned}\varepsilon_A(a)\varepsilon_B(\varepsilon_B(b_{(1)})b_{(2)}\varepsilon_A(a')\varepsilon_B(b')) &= \varepsilon_A(a)\varepsilon_B(b\varepsilon_A(a')\varepsilon_B(b')) \\ &= \varepsilon_A(a)\varepsilon_B(b\varepsilon(a' \bullet b')) \\ &= \varepsilon(a \bullet (b\varepsilon(a \bullet b))) \\ &= \varepsilon((a \bullet b)\varepsilon(a' \bullet b')). \end{aligned}$$ □

Example 6.6. Let B be a cocommutative bialgebroid over $\mathbb{K}$ of Sweedler type and let A be a bialgebra in the category of B-modules. Then $R = \Delta_A \circ \eta_A \circ \varepsilon_B$ satisfies the hypotheses of Theorem 6.5, making $A \rtimes B$ into a bialgebroid over $\mathbb{K}$ of Sweedler type with $\Delta(a \bullet b) = (a_{(1)} \bullet b_{(1)}) \otimes (a_{(2)} \bullet b_{(2)})$.

Example 6.7. (cf. [24, Example 3.4.3]) Let $q \in \mathbb{C}$ be an ℓ^{th} root of unity for any odd $\ell > 1$ and consider $\mathbb{K} = \mathbb{Q}(q)$. If we denote by $C_2 = \{1, z\}$ the cyclic group of order 2, then $\mathbb{Q}(q)$ is a $\mathbb{Q}C_2$-module algebra (with $z \triangleright q = q^{-1}$), meaning that $\mathbb{Q}(q) \rtimes \mathbb{Q}C_2$ is a (cocommutative) bialgebroid over $\mathbb{Q}(q)$ of Sweedler type.

Let $\mathfrak{u}_q(\mathfrak{sl}_2)$ denote the $\mathbb{Q}(q)$-Hopf algebra generated (as an algebra) by symbols $K^{\pm 1}$, E, and F, with relations and structure given by

$$KK^{-1} = K^{-1}K = 1, \qquad K^\ell = 1, \qquad E^\ell = F^\ell = 0,$$

$$KE = q^2EK, \qquad KF = q^{-2}FK, \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, \quad ccc$$

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, \qquad \Delta(E) = E \otimes K + 1 \otimes E, \qquad \Delta(F) = F \otimes 1 + K^{-1} \otimes F,$$

$$\varepsilon(K^{\pm 1}) = 1, \qquad \varepsilon(E) = 0, \qquad \varepsilon(F) = 0,$$

$$S(K^{\pm 1}) = K^{\mp 1}, \quad S(E) = -EK^{-1}, \quad S(F) = -KF.$$

$\mathfrak{u}_q(\mathfrak{sl}_2)$ is quasitriangular with R-matrix

$$\mathcal{R} = \frac{1}{\ell}\left(\sum_{a,b=0}^{\ell-1} q^{-2ab} K^a \otimes K^b\right)\left(\sum_{n=0}^{\ell-1} \frac{(q-q^{-1})^n}{[n;q^{-2}]!} E^n \otimes F^n\right),$$

where $[m;q^{-2}] = \dfrac{1-q^{-2m}}{1-q^{-2}}$ and $[n;q^{-2}]! = \displaystyle\prod_{m=1}^{n} [m;q^{-2}]$.

The following assignments make $\mathfrak{u}_q(\mathfrak{sl}_2)$ into a $\mathbb{Q}(q) \rtimes \mathbb{Q}C_2$-module such that $m_{\mathfrak{u}_q(\mathfrak{sl}_2)}$, $\eta_{\mathfrak{u}_q(\mathfrak{sl}_2)}$, and $\varepsilon_{\mathfrak{u}_q(\mathfrak{sl}_2)}$ are homomorphisms of $\mathbb{Q}(q) \rtimes \mathbb{Q}C_2$-modules.

$$(1 \bullet z) \triangleright K^{\pm 1} = K^{\mp 1} \qquad (1 \bullet z) \triangleright E = K^{-1}E \qquad (1 \bullet z) \triangleright F = FK$$

Furthermore, if we use Sweedler-like notation to write $\mathcal{R} = \mathcal{R}_{(1)} \otimes \mathcal{R}_{(2)}$, then setting $R(1 \bullet 1) = 1 \otimes 1$ and $R(1 \bullet z) = \mathcal{R}_{(2)} \otimes \mathcal{R}_{(1)}$ defines a $\mathbb{Q}(q)$-linear map $R : \mathbb{Q}(q) \rtimes \mathbb{Q}C_2 \to \mathfrak{u}_q(\mathfrak{sl}_2) \otimes_{\mathbb{Q}(q)} \mathfrak{u}_q(\mathfrak{sl}_2)$. As defined, R satisfies the hypotheses of Theorem 6.5, making $\mathfrak{u}_q(\mathfrak{sl}_2) \rtimes (\mathbb{Q}(q) \rtimes \mathbb{Q}C_2)$ into a bialgebroid over $\mathbb{Q}(q)$ of Sweedler type.

Remark 6.8. Example 6.7 can be generalized to $\mathfrak{u}_q(\mathfrak{g})$ for any semisimple Lie algebra $\mathfrak{g}$ (with appropriate restrictions on ℓ). We refer the reader to [13, Section 9.3 B] for the definitions of $\mathfrak{u}_q(\mathfrak{g})$ and the corresponding R-matrix. The action of $\mathbb{Q}(q) \rtimes \mathbb{Q}C_2$ on $\mathfrak{u}_q(\mathfrak{g})$ is completely analogous, e.g. $(1 \bullet z) \triangleright E_i = K_i^{-1}E_i$.

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