A CATEGORIFICATION OF THE POSITIVE HALF OF QUANTUM $\mathfrak{sl}_3$ AT A PRIME ROOT OF UNITY

by

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DISSERTATION ABSTRACT

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We place a differential on $\dot{\mathcal{U}}^+_{\mathfrak{sl}_3}$ and show that $\dot{\mathcal{U}}^+_{\mathfrak{sl}_3}$ is Fc-filtered. This gives a categorification of the positive half of quantum $\mathfrak{sl}_3$ at a prime root of unity.

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For Alison.

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CHAPTER I

INTRODUCTION

In [1], Crane and Frenkel conjectured that when q is a root of unity, a categorification of $\mathbf{U}_q(\mathfrak{sl}_2)$ should exist. Over the last 10 years, much progress has been made towards this end. In [9] Aaron Lauda categorifies $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ for a generic q. Lauda's categorification is extended to $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ in [6]. Similarly, the positive half $\mathbf{U}_q(\mathfrak{g})$ of an arbitrary quantum group is categorified at a generic q in [5].

The general procedure in these categorifications has been to define a diagrammatic 2-category $\mathcal{U}$ and its Karoubi envelope $\dot{\mathcal{U}}$ where the indecomposable 1-morphisms correspond to basis elements in $\dot{\mathbf{U}}$. The nilHecke algebra, and more generally the KLR-algebras, govern how 1-morphisms in $\dot{\mathcal{U}}$ decompose. Using the diagrammatic description of these categories, one shows that the relations in $\dot{\mathbf{U}}$ are lifted to the categorical level by giving direct sum decompositions of 1-morphisms. It is then shown that the split Grothendieck group of $\dot{\mathcal{U}}$ is isomorphic to an integral form of $\dot{\mathbf{U}}$

For q a prime root of unity, a categorification of $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ was achieved in [2, 3]. Categorification at a prime root of unity involves taking the previously defined category $\dot{\mathcal{U}}$ and giving it the structure of a p-DG category. A *differential* ∂ is placed on the 2-morphisms of $\dot{\mathcal{U}}$ which satisfies a certain Leibniz rule and for which $\partial^p = 0$. The p-DG Grothendieck group of $\dot{\mathcal{U}}$ is a module over $\mathbb{O}_p := \mathbb{Z}[q]/(1 + q^2 + \cdots + q^{2(p-1)})$. For categorification at a generic q the relations in $\dot{\mathbf{U}}$ lift to direct sum decompositions. This is not a strong enough condition to ensure compatibility with the p-DG structure and to guarantee a relation on the Grothendieck group. Instead one needs to check that these direct sum decompositions are actually *Fc filtrations* (see Definition 2.7.5).

There are currently no techniques which make this easy to show in the abstract. Instead one needs to check by hand that explicit idempotent decompositions satisfy the conditions of an Fc-filtration. For $\mathfrak{sl}_2$, Lauda gave idempotent decompositions for $\mathcal{EF}$ and $\mathcal{FE}$. What was still needed was a decomposition for the divided powers which was given by the Stošić formula in [7]. Elias and Qi showed that these decompositions, as well as two others, were indeed Fc-filtrations. In [12], Stošić also gave idempotent decompositions for $\dot{\mathcal{U}}^+_{\mathfrak{sl}_3}$. In the last chapter we show that this decomposition is an Fc-filtration.

At this stage it is not clear how to proceed any further since there are few other idempotent decompositions which are currently known. Indeed Lusztig's canonical basis is much more complicated for $\mathbf{U}^+_{\mathfrak{sl}_4}$ consisting of 14 different types of monomials [13].

There are a few complications to be aware of when checking that an idempotent decomposition is an Fc-filtration. The first is that there may be multiple p-DG structures that one can place on $\dot{\mathcal{U}}$. Additionally, there may be many different idempotent decompositions that one could use. It is possible for a decomposition to be an Fc-filtration with respect to one differential but not with respect to another. This is one issue we encountered working with $\dot{\mathcal{U}}^+_{\mathfrak{sl}_3}$. In [2, 3] much of the work, including many of the formulas, had been derived for one particular differential ∂_1. Unfortunately the idempotents in [12] are not an Fc-filtration with respect to this differential. They are instead compatible with another differential, ∂_{-1}. Because of the complexity of Stošić's computation, we have chosen to use his original idempotents and use ∂_{-1}. In Remark 3.3.1, we explain how to adapt these idempotents to obtain the idempotents which are an Fc-filtration with respect to ∂_1.

Another complication we encountered was in deriving the needed formulas for ∂_{-1}. The formulas can be derived in two ways. One way uses the symmetries of $\dot{\mathcal{U}}$ along with the differential ∂_1. Another way to derive the formulas is by placing a differential on $\mathcal{U}$ which induces a differential on the *partial idempotent completion* see (2.94). When checking that both methods produced the same results, it was discovered that some of the formulas in [3] were incorrectly justified. The correct justification for their formulas involves using a different idempotent to define the divided powers $\mathcal{F}^a$. We mention this because we will also use a different idempotent to define the divided powers.

We now sketch the organization of this paper. The second chapter is intended to provide the relevant background material which we will use in the following chapter. We have chosen to motivate much of this material though the example of quantum $\mathfrak{sl}_2$. In doing so this chapter is also intended to serve as an analogy for our later work with $\mathbf{U}_q^+(\mathfrak{sl}_3)$. We begin by recalling Lusztig's idempotented quantum group $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$, which will be categorified. We define Lauda's category $\dot{\mathcal{U}}$ and explain how the thick calculus of [7] can be used to perform computation directly inside of $\dot{\mathcal{U}}$. Crucially, this thick calculus has been used to provide explicit direct sum decompositions of arbitrary 1-morphisms in $\dot{\mathcal{U}}$ in terms of indecomposable 1-morphisms. Next, we recall notions of p-DG algebras and p-DG categories and the notion of a fantastic filtration which allows one to describe the p-DG Grothendieck of a p-DG category in terms of the Grothendieck group of the underlying category. We recall how a p-DG structure has been placed on $\dot{\mathcal{U}}$ and explain how these ideas lead to a categorification of quantum $\mathfrak{sl}_2$ at a prime root of unity.

The last chapter accomplishes a similar result except for the positive half of quantum $\mathfrak{sl}_3$. We include relevant definitions and explain how the thick calculus works in this new setting. Resembling the case for $\mathfrak{sl}_2$, we use the explicit idempotent decomposition from [12]. This is sufficient to decompose any 1-morphism as a direct sum of indecomposable 1-morphisms. We place a p-DG structure on $\dot{\mathcal{U}}^+_{\mathfrak{sl}_3}$ which extends the one of the differentials given in [8, 2, 3] and derive some formulas for this differential. Our main theorem is a computation of the p-DG Grothendieck group of $\dot{\mathcal{U}}^+_{\mathfrak{sl}_3}$. We achieve this by showing that the direct sum decomposition of arbitrary 1-morphisms in terms of indecomposable 1-morphisms is actually an Fc-filtration.

CHAPTER II

PRELIMARIES

Quantum $\mathfrak{sl}_2$

In this section we define quantum $\mathfrak{sl}_2$, which we will denote $\mathbf{U}$. We will also describe the idempotented form of $\mathbf{U}$, denoted $\dot{\mathbf{U}}$, and recall some of its properties.

Definition 2.1.1. Quantum $\mathfrak{sl}_2$ is the $\mathbb{Q}(q)$-algebra generated by $E, F, K^{\pm 1}$ subject to the following relations

$$KK^{-1} = K^{-1}K = 1 \tag{2.1}$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \tag{2.2}$$

$$KE = q^2 EK \tag{2.3}$$

$$KF = q^{-2} FK. \tag{2.4}$$

$\mathbf{U}$ is a Hopf algebra with coproduct, $\Delta : \mathbf{U} \to \mathbf{U} \otimes_{\mathbb{Q}(q)} \mathbf{U}$, given by

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, \quad \Delta(E) = E \otimes 1 + K \otimes E,$$

$$\Delta(F) = 1 \otimes F + F \otimes K^{-1}.$$

The counit is given by

$$\varepsilon(E) = \varepsilon(F) = 0, \quad \varepsilon(K^{\pm 1}) = 1,$$

while the antipode is given by

$$S(E) = -K^{-1}E, \quad S(F) = -FK, \quad S(K) = K^{-1}.$$

For $a \in \mathbb{Z}$ we define *quantum* a to be $[a] := \dfrac{q^a - q^{-a}}{q - q^{-1}}$. A quick calculation shows that for $a \in \mathbb{N}$, $[a] = q^{a-1} + q^{a-3} + \ldots q^{1-a} = \sum_{i=0}^{a-1} q^{a-1-2i}$. The *quantum factorials* are defined by $[a]! := [a][a-1]\ldots[1]$ and *quantum binomial coefficients* by $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = \dfrac{[a]!}{[b]![a-b]!}$. We also define the *divided powers* of E, F to be

$$E^{(a)} := \frac{E^a}{[a]!}, \quad F^{(a)} := \frac{F^a}{[a]!}.$$

A *weight vector* with *weight* n in a $\mathbf{U}$-module is a vector v such that $Kv = q^n v$. By (2.3), E takes a weight vector with weight n to a weight vector of weight $n+2$. Similarly F takes a weight n vector to a weight $n-2$ vector.

Next we define the Lusztig's idempotented quantum group $\dot{\mathbf{U}}$ which formally adds idempotents projecting to weight spaces. To do this we first define ${}_m\dot{\mathbf{U}}_n$ as follows

$${}_m\dot{\mathbf{U}}_n := \mathbf{U}/\left((K - q^m)\mathbf{U} + \mathbf{U}(q^n - K)\right), \tag{2.5}$$

and set

$$\dot{\mathbf{U}} := \bigoplus_{n,m\in\mathbb{Z}} {}_m\dot{\mathbf{U}}_n. \tag{2.6}$$

We then define 1_n as the identity element inside ${}_n\mathbf{U}_n$.

The idempotented quantum group, $\dot{\mathbf{U}}$, does not have a unit (nor does it contain E, nor F) since the sums

$$\sum_{n\in\mathbb{Z}} 1_n, \quad \sum_{n\in\mathbb{Z}} E1_n, \quad \sum_{n\in\mathbb{Z}} F1_n$$

are not elements of $\dot{\mathbf{U}}$. On any weight representation of $\mathbf{U}$, the idempotented form, $\dot{\mathbf{U}}$, has elements which act as 1, E, and F since the sums above act locally-finitely on any element. The category of $\dot{\mathbf{U}}$-modules is equivalent to the category of $\mathbf{U}$-modules which are weight modules.

Remark 2.1.2. Roughly speaking, we have replaced the unit in $\mathbf{U}$ with an infinite collection of mutually orthogonal idempotents, 1_n indexed by $\mathbb{Z}$ (the weight lattice of $\mathfrak{sl}_2$). The idempotent 1_n should be thought of as an operator which projects a vector in any weight modules of $\mathbf{U}$ onto the n'th weight space. Very informally, one thinks of the following 'identities' as holding in $\dot{\mathbf{U}}$:

$$1 = \sum_{n\in\mathbb{Z}} 1_n, \quad E = \sum_{n\in\mathbb{Z}} E1_n, \quad F = \sum_{n\in\mathbb{Z}} F1_n.$$

Remark 2.1.3. It is possible to define $\dot{\mathbf{U}}$ without having defined $\mathbf{U}$ first. As generators, one has $E1_n, F1_n, 1_n$ for $n \in \mathbb{Z}$ with the following relations:

i. $1_n 1_m = \delta_{n,m} 1_n$

ii. $EF1_n - FE1_n = [n]1_n$ (c.f. (2.2))

iii. $E1_n = 1_{n+2}E$, and $F1_n = 1_{n-2}F$ (c.f. (2.3, 2.4)).

A particularly important reason to consider $\dot{\mathbf{U}}$ instead of $\mathbf{U}$ is the existence of Lusztig's canonical basis $\dot{\mathbb{B}}$, which has positive structure constants. This will prove crucial later.

$$\dot{\mathbb{B}} = \{E^{(a)}F^{(b)}1_n \mid a, b \in \mathbb{N}, n \in \mathbb{Z}, n \leq b-a\} \cup \{F^{(b)}E^{(a)}1_n \mid a, b \in \mathbb{N}, n \in \mathbb{Z}, n \geq b-a\}. \tag{2.7}$$

$\mathbf{U}$ has an integral form, $\mathbf{U}_{\mathcal{A}}$, which is the $\mathbb{Z}[q, q^{-1}]$-subalgebra generated by the divided powers and $K^{\pm 1}$. The integral form $\mathbf{U}_{\mathcal{A}}$ gives rise to an integral form $\dot{\mathbf{U}}_{\mathcal{A}}$ which is spanned by $1_m E^{(a)} F^{(b)} 1_n, 1_m F^{(b)} E^{(a)} 1_n$.

The following relations define $\dot{\mathbf{U}}_{\mathcal{A}}$:

$$E^{(a)} E^{(b)} 1_n = \begin{bmatrix} a+b \\ a \end{bmatrix} E^{(a+b)} 1_n, \tag{2.8}$$

$$F^{(a)} F^{(b)} 1_n = \begin{bmatrix} a+b \\ a \end{bmatrix} F^{(a+b)} 1_n, \tag{2.9}$$

$$E^{(a)} F^{(b)} 1_n = \sum_{j=0}^{\min(a,b)} \begin{bmatrix} a-b+n \\ j \end{bmatrix} F^{(b-j)} E^{(a-j)} 1_n, \tag{2.10}$$

$$F^{(b)} E^{(a)} 1_n = \sum_{j=0}^{\min(a,b)} \begin{bmatrix} b-a-n \\ j \end{bmatrix} E^{(a-j)} F^{(b-j)} 1_n. \tag{2.11}$$

There are numerous automorphisms and antiautomorphisms of $\dot{\mathbf{U}}$ [9]. In particular, two of these will be important for our purposes.

– We denote by $\psi : \dot{\mathbf{U}}_{\mathcal{A}} \to \dot{\mathbf{U}}_{\mathcal{A}}$ the $\mathbb{Q}$-linear isomorphism which is defined by

$$\psi(E^{(a)} 1_n) = E^{(a)} 1_n, \quad \psi(F^{(a)} 1_n) = F^{(a)} 1_n, \quad \psi(q) = q^{-1}.$$

– We denote by $\tau : \dot{\mathbf{U}}_{\mathcal{A}} \to \dot{\mathbf{U}}_{\mathcal{A}}$ the $\mathbb{Q}$-linear antiautomorphism defined by

$$\tau(q) = q^{-1}, \quad \tau(E^{(a)} 1_n) = q^{-a(a+n)} 1_n F^{(a)}, \quad \tau(F^{(a)} 1_n) = q^{a(n-a)} 1_n E^{(a)}.$$

The 2-category $\dot{\mathcal{U}}$

In [9] Lauda categorifies $\dot{\mathbf{U}}$. He constructs a 2-category $\mathcal{U}$, whose Karoubi envelope is denoted $\dot{\mathcal{U}}$, and proves that the Grothendieck group of $\dot{\mathcal{U}}$ is isomorphic to $\dot{\mathbf{U}}_{\mathcal{A}}$. The 2-category $\mathcal{U}$ has a natural description using the notation of string diagrams. See [9, Section 4] for more about string diagrams.

Definition 2.2.1. $\mathcal{U}$ is the $\Bbbk$-linear 2-category with:

- Objects: There is one object of $\mathcal{U}$ for each $n \in \mathbb{Z}$.
- 1-morphisms: The 1-morphisms of $\mathcal{U}$ are formal direct sums of grading shifts of composites of the morphisms

$$1_n : n \to n$$

$$1_{n+2}\mathcal{E}1_n : n \to n+2$$

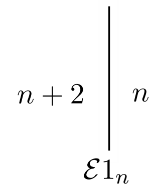
$$1_n\mathcal{F}1_{n+2} : n+2 \to n. \tag{2.12}$$

  Where there can be no confusion, instead of writing $1_{n+2}\mathcal{E}1_n$ we will simply write $\mathcal{E}$ or possibly $\mathcal{E}1_n$. In this way, we write $\mathcal{E}^a1_n$ for the a-fold composition $1_{n+2a}\mathcal{E}1_{n+2a-2}\dots1_{n+2}\mathcal{E}1_n$. We use similar notation for $1_n\mathcal{F}1_{n+2}$. We denote shifts of morphisms as $\mathcal{E}_n\{s\}, \mathcal{F}_n\{s\}, 1_n\{s\}$ for $s \in \mathbb{Z}$.

- 2-morphisms:

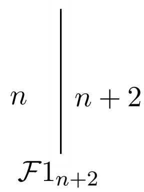
  i) There are degree zero identity 2-morphisms for each 1-morphism which are depicted in the usual way for string diagrams. For $\mathcal{E}1_n$, the 2-

morphism $\mathrm{Id}_{\mathcal{E}1_n}$ is depicted as



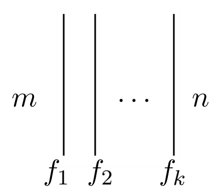
, (2.13)

while for $\mathcal{F}1_{n+2}$, the 2-morphism $\mathrm{Id}_{\mathcal{F}1_{n+2}}$ is depicted as



. (2.14)

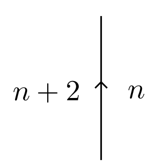
The 2-morphism Id_{1_n} is depicted as a region labeled by n. We use similar notation for composite 1-morphisms. The identity $\mathrm{Id}_{f_1 \dots f_k}$ where each $f_i \in \{\mathcal{E}, \mathcal{F}\}$ is depicted as



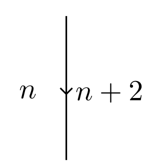
. (2.15)

We will often omit the labels $\mathcal{E}, \mathcal{F}$ by drawing arrows on the vertical lines. An upward arrow will denote $\mathcal{E}$ and a downward arrow will denote $\mathcal{F}$.

$\mathrm{Id}_{\mathcal{E}1_n}$:

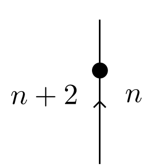


$\mathrm{Id}_{\mathcal{F}1_{n+2}}$:

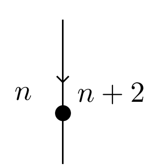


(2.16)

ii) For each integer n, there are the following 2-morphisms of degree 2.

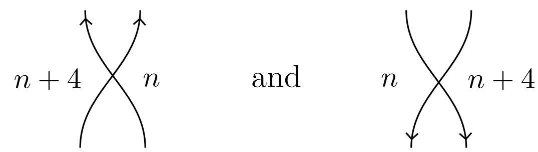


and

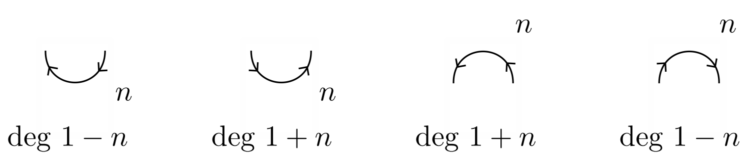


We will allow dots to carry labels of $a \in \mathbb{Z}_+$, by which we mean the a fold composition of the dot with itself. For example,

iii) For each integer n, there are the following 2-morphisms of degree -2.

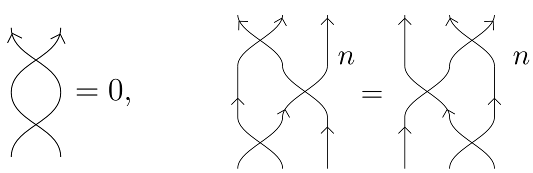


iv) For each integer n, there are the following 2-morphisms with given degree.

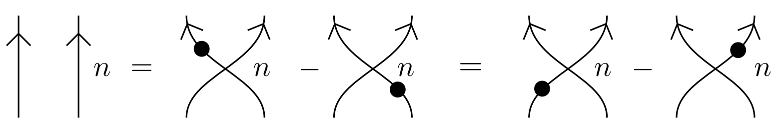


The 2-morphisms are required to satisfy the following relations.

I) NilHecke relations



(2.17)



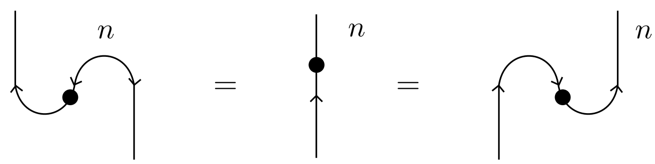
(2.18)

II) Isotopy relations: cups and caps are biadjoint morphisms. Diagrammatically this is equivalent to the equality of the following diagrams.

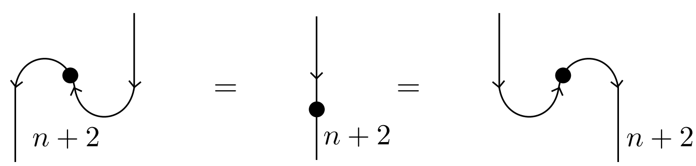
(2.19)

(2.20)

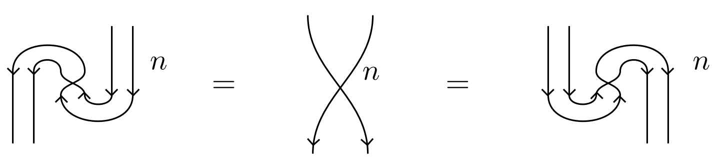
Further, all 2-morphisms are required to be cyclic with respect to the biadjoint structure. That is,



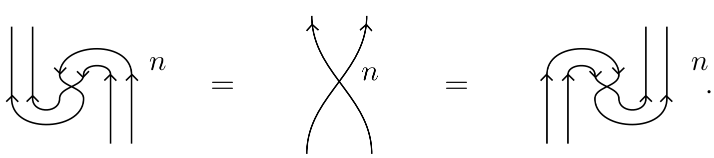
(2.21)



(2.22)



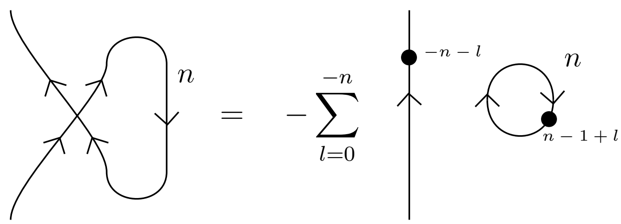
(2.23)



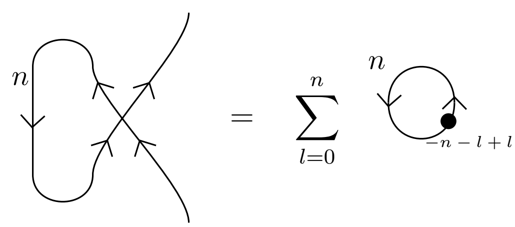
(2.24)

These relations mean that planar diagrams up to isotopy unambiguously represent 2-morphisms.

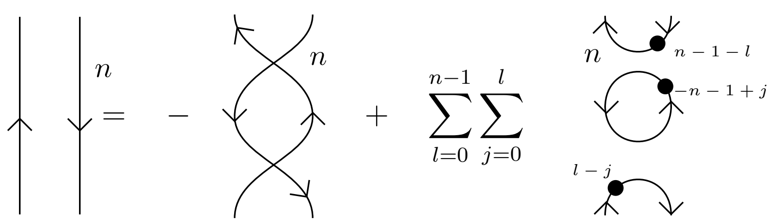
III) $\mathfrak{sl}_2$ relations: The following relations are in place to ensure that the quantum $\mathfrak{sl}_2$ relations of E, F lift.



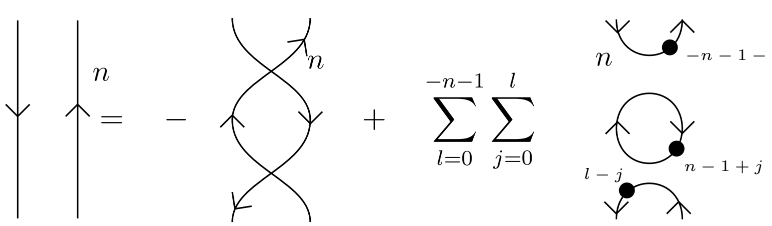
(2.25)



(2.26)



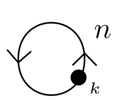
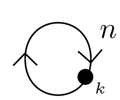
(2.27)



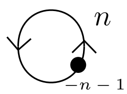
(2.28)

Remark 2.2.2. Notice in the above equations that it is possible for a sum to have a decreasing index (for example $\sum_3^2$). Any time this happens, the sum is taken to be zero.

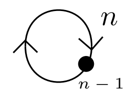
IV) Negative degree bubbles are zero. For $k \in \mathbb{Z}$,

 $= 0 \quad$ if $k < -n-1$,  $= 0 \quad$ if $k < n-1$. (2.29)

Degree 0 dotted bubbles are equal to 1.

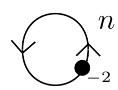


$= 1 \quad \text{for } n \leq -1,$

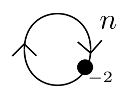


$= 1 \quad \text{for } n \geq 1. \qquad (2.30)$

Remark 2.2.3. Formal symbols, called *fake bubbles*, are defined in [9]. Fake bubbles are dotted bubbles where the dot is allowed to carry a negative label. For example

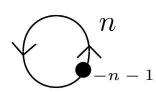


or



.

Of course it doesn't make any sense to compose the dot with itself -2 times. Instead, fake bubbles are formal symbols which are introduced to stand for specific 2-morphisms in $\mathcal{U}$. The benefit of these symbols is that they give a convenient way to write formulas. We define the only fake bubble which we will use below.



$= 1 \qquad (2.31)$

Remark 2.2.4. In [5], Khovanov and Lauda categorify the positive half of an arbitrary quantum group in a similar manner. The presentation of $\mathcal{U}$ is simplified since one only needs upward pointing strands–one 'color' of upward pointing strand for each generator. Caps, cups, and downward pointing strands are no longer needed as generators. From the relations above, only I and V are needed as well as new relations involving the crossings of strands of different colors. In the next chapter we will be concerned with categorification of the positive half of quantum $\mathfrak{sl}_3$.

We also introduce some helpful notation. For any Laurent polynomial, $f = \sum_{i\in\mathbb{Z}} a_i x^i$, and any 1-morphism M, we define

$$\bigoplus_f M := \sum_{i\in\mathbb{Z}} a_i M\{i\}. \tag{2.32}$$

In particular, we have $\bigoplus_{[n]} 1_n = 1_n\{n-1\} + 1_n\{n-3\} + \cdots + 1_n\{1-n\}$.

$\mathcal{U}$ has many properties one would expect from a categorification of $\dot{\mathbf{U}}_{\mathcal{A}}$.

- There are symmetries of diagrams in $\mathcal{U}$ giving lifts of certain algebra maps in $\dot{\mathbf{U}}_{\mathcal{A}}$. The lift of τ from Section 2.1 is given by $\tilde{\tau}$ which corresponds to rotating a diagram by 180 degrees. Similarly there is a lift of ψ, denoted $\tilde{\psi}$, which corresponds to reflecting across a horizontal axis and then inverting the orientation of the strands.

- The $\dot{\mathbf{U}}$ relations between E and F lift to direct sum decompositions in $\mathcal{U}$. More specifically we have [9, Theorem 5.10],

$$\mathcal{E}\mathcal{F}1_n \cong \mathcal{F}\mathcal{E}1_n \oplus_{[n]} 1_n, \tag{2.33}$$

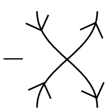
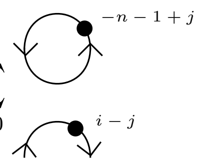
$$\mathcal{F}\mathcal{E}1_n \cong \mathcal{E}\mathcal{F}1_n \oplus_{[n]} 1_n. \tag{2.34}$$

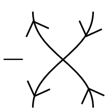
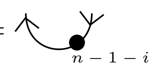
We outline the proof of (2.33) following the arguments given in [9]. The proof provides a useful illustration of how string diagrams can be used to demonstrate a direct sum decomposition. Similar arguments will play an important role later. The general procedure is to find a pair of 2-morphisms for each potential direct summand, corresponding to the projection to and the inclusion from this summand. One then can use previously developed diagrammatic identities to show that the 2-morphisms give a direct sum decomposition. Explicitly, if λ_i is the inclusion from the i-th summand, and σ_i is the projection, then one needs to show that the compositions $e_i := \lambda_i\sigma_i$ form a collection of mutually orthogonal idempotents with

$$\sum_i e_i = \mathrm{Id}_{\mathcal{E}\mathcal{F}1_n}. \tag{2.35}$$

A difficulty which we do not address is how to arrive at the maps in the first place.

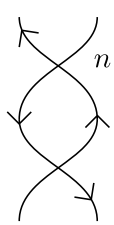
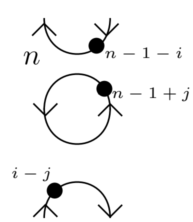
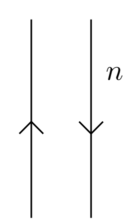
To prove (2.33), define

$\sigma_n := -$  , $\sigma_i := \sum_{j=0}^{s}$  for $0 \le i < n$, (2.36)

$\lambda_n := -$  , $\lambda_i :=$  for $0 \le i < n$, (2.37)

and set $e_i = \lambda_i\sigma_i$.

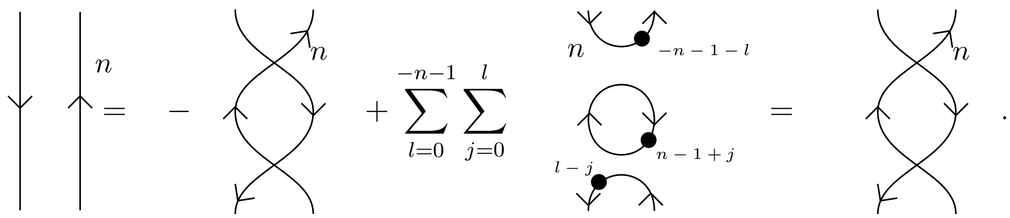
We first show (2.35). Diagrammatically, this corresponds to showing the equality of the following diagrams.

$-$  $+ \sum_{i=0}^{n-1}\sum_{j=0}^{l}$  $=$  (2.38)

But the equality above is precisely relation (2.27). Hence, (2.35) holds.

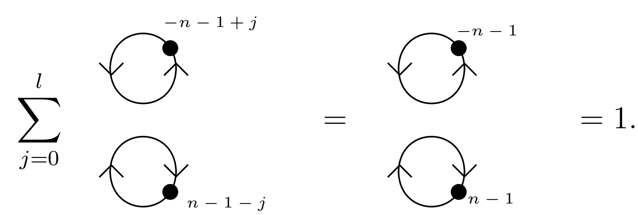
To show that the e_i are mutually orthogonal idempotents, it suffices to show the following two statements. First, that $\sigma_n\lambda_n = \mathrm{Id}_{\mathcal{F}\mathcal{E}1_n}$. Second, that for i, j not both equal to n, $\sigma_i\lambda_j = \delta_{i,j}\mathrm{Id}_{1_n}$. Notice that for $n \geq 0$ the summation in relation

(2.28) contains a decreasing index, which by convention is 0 in $\mathcal{U}$. That means that


(2.39)

Thus, we have proved the first statement, that $\sigma_n \lambda_n = \mathrm{Id}_{\mathcal{F}\mathcal{E}1_n}$.

To prove the second statement, we start by showing that the composition $\sigma_i \lambda_i = \mathrm{Id}_{1_n}$ for $i < n$. Composing, we get


(2.40)

The first equality holds because for $j > 0$ the bottom bubbles in the summands have negative degree and so are zero. This leaves a single non-zero summand. The second equality holds because the top bubble is a fake bubble defined in (2.31) to be 1, while the bottom bubble is also 1 by relation (2.30). The proof that $\sigma_i \lambda_j = 0$ for $i \neq j$ is similar.

Lusztig's canonical basis for $\dot{\mathbf{U}}$ is given by products of the divided powers, which are defined to be $E^{(a)} := \frac{E^a}{[a]!}$. In $\mathcal{U}$ there are no 1-morphisms to represent the divided powers. We want there to be 1-morphisms $\mathcal{E}^{(a)}1_n$ with

$$\mathcal{E}^a 1_n \cong \bigoplus_{[a]!} \mathcal{E}^{(a)} 1_n. \tag{2.41}$$

The hope is to look for a collection of idempotent 2-morphisms in $\mathrm{Hom}(\mathcal{E}^a 1_n, \mathcal{E}^a 1_n)$ whose images are all isomorphic, giving the decomposition in (2.41). We will denote a particular one of these idempotents as e_a (see Section 2.4). Since idempotents need not split in $\mathcal{U}$, we first need to pass to the Karoubi envelope to ensure that idempotents split.

In light of some issues that will be discussed in Remark 2.7.4, we choose a slightly different notion than that of the usual Karoubi envelope. To contrast the two, we will define both the usual Karoubi envelope below as well as the notion we will use, the partial idempotent completion. This is to ensure that we are able to extend a p-DG structure from $\mathcal{U}$ to $\dot{\mathcal{U}}$.

Definition 2.2.5. For an additive category $\mathcal{C}$, the *Karoubi envelope* $\mathrm{Kar}(\mathcal{C})$ is the category which has as objects pairs (X, ε) where X is an object of $\mathcal{C}$ and ε is an idempotent in $\mathrm{Hom}_{\mathcal{C}}(X, X)$. Given two objects $(X, \varepsilon), (Y, \varepsilon')$ a morphism is a triple $(\varepsilon', f, \varepsilon) : (X, \varepsilon) \to (Y, \varepsilon')$ where $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a morphism which is unchanged under precomposition with ε and postcomposition with ε' on the left. That is, $f = \varepsilon' f \varepsilon$, and $\mathrm{Hom}_{\mathrm{Kar}(\mathcal{C})}((X, \varepsilon), (Y, \varepsilon')) = \varepsilon' \mathrm{Hom}_{\mathcal{C}}(X, Y)\varepsilon$.

The definition of $\mathrm{Kar}(\mathcal{C})$ ensures that every idempotent $\varepsilon \in \mathrm{End}_{\mathcal{C}}(X)$ will factor through $(X, \varepsilon) \in \mathrm{Kar}(\mathcal{C})$. A category where every idempotent splits is called *Karoubian*. There is a functor from $\mathcal{C}$ to $\mathrm{Kar}(\mathcal{C})$ sending X to (X, Id_X) which is an equivalence of categories when $\mathcal{C}$ is Karoubian.

Definition 2.2.6. Fix an additive category $\mathcal{C}$ and a collection $\mathbb{X} = \{(X_i, \varepsilon_i)\}$ of idempotents $\varepsilon_i \in \mathrm{End}_{\mathcal{C}}(X_i)$. The *partial idempotent completion* $\mathcal{C}(\mathbb{X})$ is the category with objects $\mathrm{Ob}(\mathcal{C}) \cup \mathbb{X}$ and morphisms given by

$$\mathrm{Hom}_{\mathcal{C}(\mathbb{X})}((X_i, \varepsilon_i), (X_j, \varepsilon_j)) = \varepsilon_j \mathrm{Hom}_{\mathcal{C}}(X_i, X_j)\varepsilon_i$$

$$\mathrm{Hom}_{\mathcal{C}(\mathbb{X})}(X, (X_i, \varepsilon_i)) = \varepsilon_i \mathrm{Hom}_{\mathcal{C}}(X, X_i)$$

$$\mathrm{Hom}_{\mathcal{C}(\mathbb{X})}((X_i,\varepsilon_i),X) = \mathrm{Hom}_{\mathcal{C}}(X_i,X)\varepsilon_i$$

$$\mathrm{Hom}_{\mathcal{C}(\mathbb{X})}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y).$$

That is, $\mathcal{C}(\mathbb{X})$ is the full subcategory of $\mathrm{Kar}(\mathcal{C})$ whose objects are (X,ε_i) and (X,Id_X) for $X \in \mathrm{Ob}(\mathcal{C})$ and $\varepsilon_i \in \mathbb{X}$.

We denote by $\dot{\mathcal{U}}$ the partial idempotent completion of $\mathcal{U}$ with respect to the idempotents (E^a, e_a). That is, $\dot{\mathcal{U}} := \mathcal{U}(\mathbb{X})$ for $\mathbb{X} = \{(E^a,e_a) \mid a \in \mathbb{N}\}$. The definition of $\dot{\mathcal{U}}$ given in [9] is as the Karoubi envelope. It turns out that that $\mathcal{U}(\mathbb{X})$ is Karoubian and that $\mathrm{Kar}(\mathcal{U}) \cong \mathcal{U}(\mathbb{X})$. Still, in light of Remark 2.7.4, we choose to define $\dot{\mathcal{U}}$ as the partial idempotent completion.

We define the lifts of the divided powers as $\mathcal{E}^{(a)}1_n := \left(\mathcal{E}^a 1_n\left\{\frac{a(1-a)}{2}\right\}, e_a\right)$. The direct sum decomposition in (2.41) is [9, Proposition 9.4].

The relations in $\dot{\mathbf{U}}$ lift to the decompositions

$$\mathcal{E}^{(a)}\mathcal{E}^{(b)}1_n \cong \bigoplus_{\left[\begin{smallmatrix}a+b\\ a\end{smallmatrix}\right]} \mathcal{E}^{(a+b)}1_n, \tag{2.42}$$

$$\mathcal{F}^{(a)}\mathcal{F}^{(b)}1_n \cong \bigoplus_{\left[\begin{smallmatrix}a+b\\ a\end{smallmatrix}\right]} \mathcal{F}^{(a+b)}1_n, \tag{2.43}$$

$$\mathcal{F}^{(b)}\mathcal{E}^{(a)}1_n \cong \bigoplus_{j=0}^{\min(a,b)} \bigoplus_{\left[\begin{smallmatrix}b-a-n\\ j\end{smallmatrix}\right]} \mathcal{E}^{(a-j)}\mathcal{F}^{(b-j)}1_n \quad \text{if } n < -2a+2, \tag{2.44}$$

$$\mathcal{E}^{(a)}\mathcal{F}^{(b)}1_n \cong \bigoplus_{j=0}^{\min(a,b)} \bigoplus_{\left[\begin{smallmatrix}a-b+n\\ j\end{smallmatrix}\right]} \mathcal{F}^{(b-j)}\mathcal{E}^{(a-j)}1_n \quad \text{if } n > 2b-2. \tag{2.45}$$

It is also shown in [9, Proposition 9.9] that the 1-morphisms

$$\mathcal{E}^{(a)}\mathcal{F}^{(b)}1_n\{s\} \quad \text{for } a,b \in \mathbb{N}, \quad s \in \mathbb{Z}, \quad n \leq b-a,$$

$$\mathcal{F}^{(b)}\mathcal{E}^{(a)}1_n\{s\} \quad \text{for } a,b \in \mathbb{N}, \;\; s \in \mathbb{Z}, \;\; n \geq b-a,$$

are indecomposable and that no two of them are isomorphic with the exception that $\mathcal{F}^{(b)}\mathcal{E}^{(a)}1_n\{s\} \cong \mathcal{E}^{(a)}\mathcal{F}^{(b)}1_n\{s\}$ when $n = b-a$. Up to isomorphism, these are the only indecomposable 1-morphisms and every 1-morphism can be written as the direct sum of such indecomposable 1-morphisms [9, Proposition 9.10].

The 2-category $\dot{\mathcal{U}}$ gives a categorification of $\dot{\mathbf{U}}$ in the following sense. The split Grothendieck group of $\dot{\mathcal{U}}$ is isomorphic to $\dot{\mathbf{U}}_{\mathcal{A}}$ as $\mathbb{Z}[q,q^{-1}]$-modules where the grading shift $\{1\}$ corresponds to multiplication by q [9, Theorem 9.13]. Moreover, any Grothendieck group is equipped with a natural $\mathcal{A}$-basis given by the symbols of the indecomposable 1-morphisms (up to isomorphism and grading shift). This basis matches with Lusztig's canonical basis.

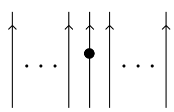
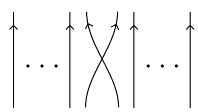
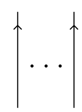
While adding idempotents is an essential step, it means that the diagrammatic description of $\mathcal{U}$ in [9, Section] can not be used directly for computations in $\dot{\mathcal{U}}$. In [7], the authors develop diagrammatic notation for performing computations in $\dot{\mathcal{U}}$ itself, which they call thick calculus, and which we explain in section 2.4. To lay the groundwork for this thick calculus, we first define the nilHecke ring, which we do in the next section.

The nilHecke ring

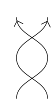
For a fixed n, if one considers the $\mathbb{Z}$-span of diagrams in $\mathrm{Hom}_{\mathcal{U}}(\mathcal{E}^a 1_n, \mathcal{E}^a 1_n)$ which involve only dots and crossings (but not bubbles), and imposes the relations in $\mathcal{U}$, one gets the *nilHecke ring* on a strands, denoted $\mathcal{NH}_a$. We present a more common definition below.

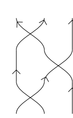
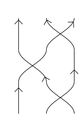
Definition 2.3.1. The *nilHecke ring* on a strands, $\mathcal{NH}_a$, is a unital subring of the ring of endomorphisms of the group $\mathbb{Z}[x_1,\ldots,x_a]$ generated by two families of generators: $x_1, \ldots, x_a$ which act by multiplication by x_i, and $\partial_1, \ldots, \partial_{a-1}$, called *divided difference* operators. The ∂_i act via $\partial_i(f) = \dfrac{f - s_i f}{x_i - x_{i+1}}$ where $s_i f$ swaps the variables x_i and x_{i+1}. We give $\mathcal{NH}_a$ a grading where the x_i are of degree 2 and the ∂_i of degree -2. In later sections we will consider the generators above over a field $\Bbbk$ instead of $\mathbb{Z}$. When we do so we will refer to it as the nilHecke algebra.

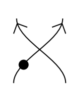
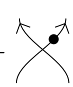
Remark 2.3.2. To interpret $\mathcal{NH}_a$ diagrammatically, x_i corresponds to placing a dot on the i'th strand (from the left) while ∂_i corresponds to a crossing of the i'th and $i+1$'st strands. The unit in $\mathcal{NH}_a$ is depicted by the 2-morphism $\mathrm{Id}_{\mathcal{E}^a}$. That is,

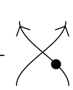
 $:= x_i \quad , \quad$  $:= \partial_i \quad , \quad$  $:= 1.$ (2.46)

From the definition of the operators above, one can deduce the defining relations of $\mathcal{NH}_a$. The column on the left below gives the defining relations of $\mathcal{NH}_a$ while the column on the right shows the diagrammatic interpretation of the relations. Notice that the first four relations correspond to the nilHecke relations in $\mathcal{U}$. The last three relations hold in $\mathcal{U}$ because of rectilinear isotopy of diagrams, something which is automatically implied using string diagram notation for 2-categories.

$\partial_i^2 = 0 \qquad \longleftrightarrow \qquad$  $= 0$ (2.47)

$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \qquad \longleftrightarrow \qquad$  $=$  (2.48)

$\partial_i x_i - x_{i+1} \partial_i = 1 \qquad \longleftrightarrow \qquad$  $-$  $= \uparrow\ \uparrow$ (2.49)

$x_i \partial_i - \partial_i x_{i+1} = 1 \qquad \longleftrightarrow \qquad$  $-$  $= \uparrow\ \uparrow$ (2.50)

$$\left.\begin{aligned} &\partial_i\partial_j = \partial_j\partial_i \ \text{ if } |i-j|>1 \\ &x_i x_j = x_j x_i \\ &\partial_i x_j = x_j \partial_i \ \text{ if } |i-j|>1 \end{aligned}\right\} \quad \longleftrightarrow \qquad \text{rectilinear isotopy of diagrams} \qquad (2.51)$$

The braid relations in (2.48) imply that for $w \in S^a$ and $w = s_{i_1} \dots s_{i_k}$ a reduced expression of w, the operator $\partial_w := \partial_{s_{i_1}} \dots \partial_{s_{i_k}}$ is well defined. That is, ∂_w does not depend on the choice of reduced expression. Denote the longest element of S^a by w_0. The *Demazure operator*, ∂_{w_0}, is a linear map from $\mathbb{Z}[x_1, \dots, x_a] \to \mathbb{Z}[x_1, \dots, x_a]^{S_a}$. This operator will be used in Section 2.4 to define Schur polynomials.

We record some properties of the nilHecke ring below:

- The center of $\mathcal{NH}_a$ is isomorphic to symmetric polynomials in a variables.
- $\mathcal{NH}_a$ is isomorphic to $a! \times a!$ matrices with coefficients in symmetric polynomials in a variables (see [9, Proposition 3.5]). It is this isomorphism which enables the definition of the divided powers $\mathcal{E}^{(a)}$ in $\dot{\mathcal{U}}$.

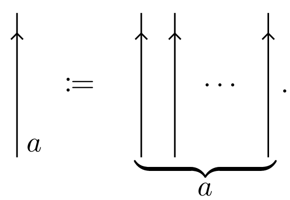
Thick calculus for $\dot{\mathcal{U}}$

In [7] the authors develop a diagrammatic calculus which can be used to represent and decompose certain 2-morphisms between products of divided powers in $\dot{\mathcal{U}}$. They call this *thick calculus*. Using thick calculus, the authors are able to prove the direct sum decompositions in (2.45) and (2.44) using the same techniques as we used in the proof of (2.33). That is, they give explicit projection and inclusion maps which decompose the products of the divided powers. Having this idempotent decomposition is an essential component of the categorification of

$\dot{\mathbf{U}}$ at a prime root of unity given in [3], which we will review in Section 2.9. We recall the relevant parts of thick calculus from [7] below.

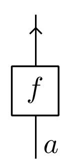
More notation for diagrams in $\mathcal{U}$

To begin, we introduce some new notation which is meant to simplify certain diagrams in $\mathcal{U}$. We will draw Id_{E^a} as a single upward strand with label a. That is,

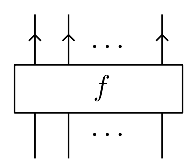


(2.52)

Next we will define what the authors of [7] call box notation. For any element in $f \in NH_a$, we will use the following notation to depict f as a 2-morphism in $\mathcal{U}$.

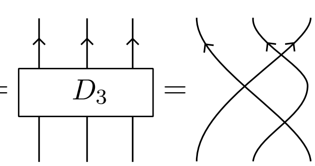


or alternatively



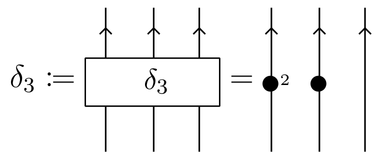
We give names to certain 2-morphisms in $\mathrm{End}(E^a)$. D_a will denote the Demazure operator ∂_{w_0} in $\mathcal{NH}_a$ viewed as a 2-morphism in $\mathcal{U}$. For example, D_3 is shown below.

$D_3 :=$

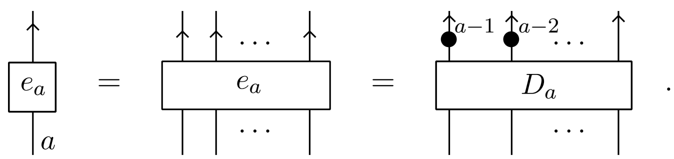


We define the 2-morphisms $\delta_a : E^a \to E^a$ in the following way. Let $\delta_a = x_1^{a-1}x_2^{a-2}\dots x_{a-1} \in \mathcal{NH}_a$. As above, we now consider δ_a as a 2-morphism in $\mathcal{U}$.

For example, δ_3 is shown below.



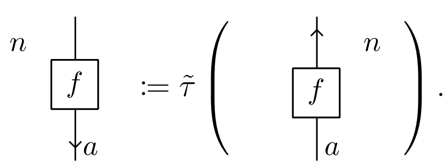
Next, we define e_a to be the composition $\delta_a D_a$. Diagrammatically, this is



(2.53)

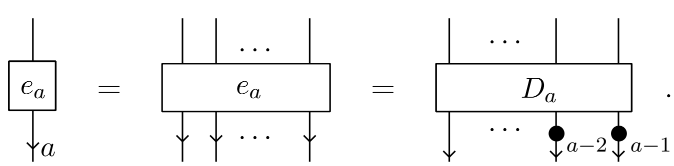
In [5, Lemma 5] the authors show that $D_a e_a = D_a$. This implies that e_a is an idempotent since $\delta_a D_a e_a = \delta_a D_a = e_a$. It is this idempotent which we used in Section 2.2 to define the divided powers $\mathcal{E}^{(a)}1_n$ in $\dot{\mathcal{U}}$. Recall that we defined the divided powers as $\mathcal{E}^{(a)} := \left(\mathcal{E}^a\{\frac{a(1-a)}{2}\}, e_a\right)$.

Finally, we can apply our functor $\tilde{\tau}$ to any of the above diagrams to obtain 2-morphisms involving downward pointing strands. We define box notation for downward strands in the following way:



(2.54)

In particular,



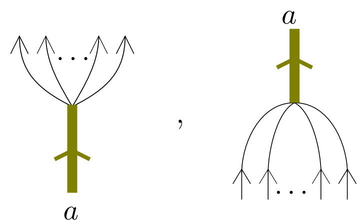
(2.55)

Diagrammatics for $\dot{\mathcal{U}}$

With this notation in hand, we are ready to extend the diagrammatics of $\mathcal{U}$ to $\dot{\mathcal{U}}$. By naturally considering $\mathcal{U}$ as a full subcategory of $\dot{\mathcal{U}}$, we think of $\dot{\mathcal{U}}$ as being obtained from $\mathcal{U}$ by adding some additional 1-morphisms. Each new 1-morphism is a direct summand of a 1-morphism in $\mathcal{U}$. This means that to extend our diagrammatics to $\dot{\mathcal{U}}$ we will need new notation for each added 1-morphism. We will also need new notation for 2-morphisms giving the projection and inclusion to and from the added summand (see the remark below). Instead of defining new symbols for *every* new 1-morphism, we will only define symbols for the divided powers. After all, the morphisms between the divided powers are of particular importance since they govern how the 1-morphisms $\mathcal{E}^{(a)}\mathcal{F}^{(b)}$ decompose.

Remark 2.4.1. In light of the above comments, at first it seems necessary to add projection and inclusion 2-morphisms for *each* summand appearing in the decomposition $E^a 1_n = \bigoplus_{[a]!} E^{(a)} 1_n$–that is, that we need to add $[a]!$ worth of projection and inclusion 2-morphisms. It turns out that this is unnecessary. It suffices to add a single symbol for projection to the summand with the highest grading shift and a single symbol for inclusion of the summand with the lowest grading shift. All other projection and inclusion morphisms can be obtained by composing these with existing 2-morphisms in $\mathcal{U}$.

We will represent $\mathcal{E}^{(a)}$ using a *thick strand* of thickness a. In this way $\mathcal{E}^{(1)} = \mathcal{E}$ is represented by our previously used thin strands. The inclusion and projection of the divided powers in the decomposition of E^a given in (2.41) are depicted as



$$(2.56)$$

where the inclusion is from the lowest degree copy of $E^{(a)}$ and projection is onto the highest degree copy. We call these 2-morphisms *complete splitters*. Recall that a 2-morphism in $\dot{\mathcal{U}}$ is a triple (e, f, e') where f is a 2-morphism in $\mathcal{U}$ and e, e' are idempotent 2-morphisms in $\mathcal{U}$ such that $e'fe = f$. The complete splitters above represent the triples $(e_a, D_a, \mathrm{Id}_{\mathcal{E}^a})$ and $(\mathrm{Id}_{\mathcal{E}^a}, e_a, e_a)$ respectively.

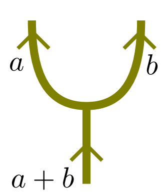
Remark 2.4.2. The use of color in the diagrams above currently has no significance. In the next chapter the colors of strands will take on meaning.

The identity 2-morphism for $\mathcal{E}^{(a)}$ is the triple (e_a, e_a, e_a), which will draw as an oriented strand of thickness a. Similarly, a downward arrow of thickness a will denote the 2-morphism $(\tilde{\tau}(e_a), \tilde{\tau}(e_a), \tilde{\tau}(e_a))$.


$:= (e_a, e_a, e_a) = \mathrm{Id}_{\mathcal{E}^{(a)}}$,

$:= (\tilde{\tau}(e_a), \tilde{\tau}(e_a), \tilde{\tau}(e_a)) = \mathrm{Id}_{\mathcal{F}^{(a)}}$ (2.57)

The authors also develop notation for certain other 2-morphisms in $\dot{\mathcal{U}}$. We include some of these, referred to as *splitter diagrams*, below (see [7, Sections 4.2-4.5] for a complete list).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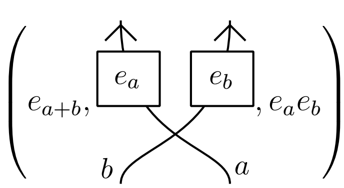
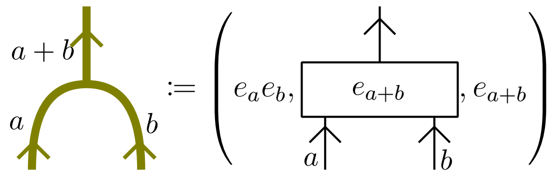
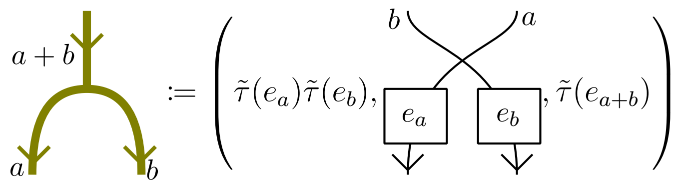


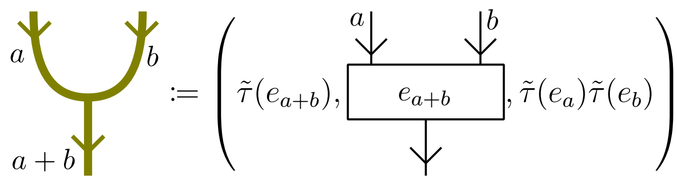
(2.58)



(2.59)

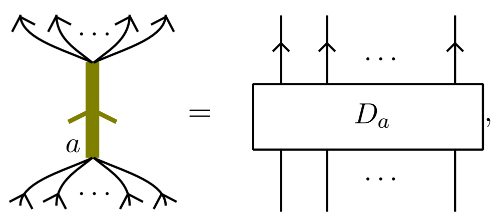


(2.60)

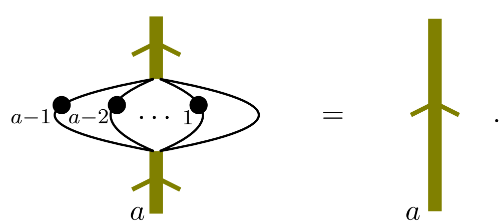


(2.61)

Remark 2.4.3. In [3, Proposition 5.2], the authors explain how all of the new diagrams for $\dot{\mathbf{U}}$ can be obtained by adding to $\mathcal{U}$ the complete splitters in (2.56) along with the relations



(2.62)



(2.63)

Schur polynomials

We want to develop notation for this thick calculus which is similar to the box notation defined in 2.4. As before, we continue to label boxes on thin strands with any element of $\mathcal{NH}_a$. For thick strands, we will only label boxes with elements in the center of $\mathcal{NH}_a$. That is, we will only decorate thick strands with symmetric polynomials. We are particularly interested in Schur polynomials. Schur polynomials form a linear basis for symmetric polynomials. This basis has certain nice properties with respect to $\dot{\mathcal{U}}$ which we will exploit later.

We say that an a-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_a)$ is a *partition with at most a rows* if $\lambda_i \geq \lambda_{i+1} \geq 0$ for all i. We will also set $|\lambda| := \sum_i \lambda_i$. Let $P(a)$ denote the set of partitions with at most a parts. The set of Schur polynomials in a variables are indexed by partitions with at most a parts.

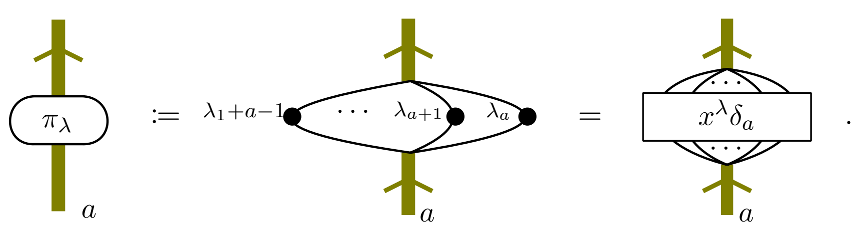
Let ∂_{w_0} be the Demazure operator, and $\lambda = (\lambda_1, \ldots, \lambda_a) \in P(a)$. The Schur polynomial corresponding to λ, which we call π_λ, is given by

$$\pi_\lambda := \partial_{w_0} \left(x_1^{\lambda_1+a-1} x_2^{\lambda_2+a-2} \ldots x_{a-1}^{\lambda_{a-1}+1} x_a^{\lambda_a} \right). \tag{2.64}$$

Notice that if we define x^λ as $x_1^{\lambda_1} \ldots x_a^{\lambda_a}$, then the Schur polynomial above is

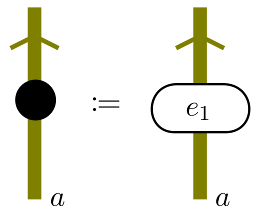
$$\pi_\lambda = \partial_{w_0} \left(\delta_a x^\lambda \right).$$

Diagrammatically we define



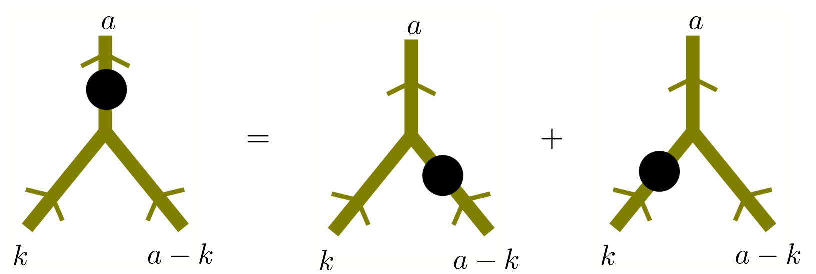
(2.65)

We will use a single dot on a thick strand (of thickness a) to denote the first elementary polynomial in a variables, e_1.

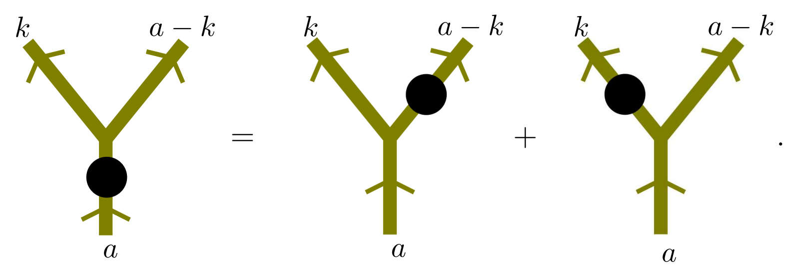


(2.66)

Using this notation, we have the following identites:

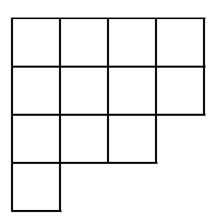


(2.67)



(2.68)

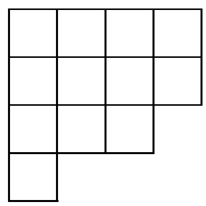
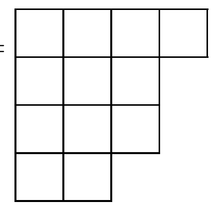
We define $P(a,b)$ to be the set of partitions with at most a rows where each $\lambda_i \leq b$. We identify partitions with Young diagrams where we use the English notation for Young diagrams. That is, the partition $(4,4,3,1)$ will be drawn as



.

We think of $P(a)$ as Young diagrams which have at most a rows. We think of $P(a,b)$ as Young diagrams which have at most a rows and b columns. We will also place a partial order on partitions and say that $\beta \leq \alpha$ if the Young diagram for α can be obtained from the Young diagram for β by iteratively adding boxes.

We adopt much of the notation for various Schur polynomials from [12]. For a partition α, we denote by $\bar{\alpha}$ the partition represented by the transpose of α. For example, for

$\alpha =$  , $\bar{\alpha} =$  .

Given $\alpha = (\alpha_1, \ldots, \alpha_a) \in P(a,b)$, we define $\hat{\alpha}$ by $\hat{\alpha} = \overline{(b-\alpha_a, \ldots, b-\alpha_1)}$. This can be interpreted as rotating 180 degrees and then transposing all the boxes in an a by b rectangle which are not contained in α.

Given 3 partitions α, β, and γ, we will denote the Littlewood-Richardson coefficients $c^{\gamma}_{\alpha,\beta}$, which are defined by the identity

$$\pi_\alpha \pi_\beta = \sum_{\gamma} c^{\gamma}_{\alpha,\beta} \pi_\gamma.$$

Given 2 partitions α and γ, we denote the skew-Schur polynomial $\pi_{\gamma/\alpha}$, which is defined by

$$\pi_{\gamma/\alpha} = \sum_{\beta} c^{\gamma}_{\alpha,\beta} \pi_\beta$$

We denote the partition $(\underbrace{b, b, \ldots, b}_{a})$ by $K_{a,b}$. Finally, for any $\alpha \leq K_{a,b}$, we denote the partition $(b-\alpha_a, \ldots, b-\alpha_1)$ by $K_{a,b} - \alpha$.

For our purposes, one significance of thick calculus is its use in the proof of [7, Theorem 5.9], which states that for $n \geq b-a$ there is a direct sum decomposition

$$\mathcal{E}^{(a)}\mathcal{F}^{(b)} \cong \bigoplus_{j=0}^{\min(a,b)} \bigoplus_{\alpha\in P(j,n+a-b-j)} \mathcal{F}^{(b-j)}\mathcal{E}^{(a-j)}1_n\{2|\alpha| - j(n+a-b)\}. \tag{2.69}$$

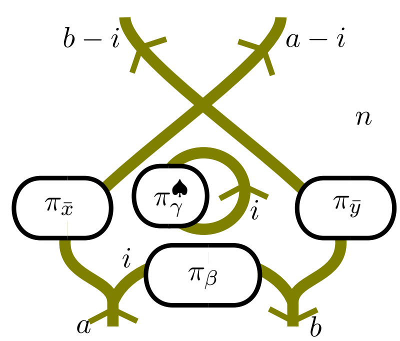
When $n \leq b-a$, there is a decomposition

$$\mathcal{F}^{(b)}\mathcal{E}^{(a)} \cong \bigoplus_{j=0}^{\min(a,b)} \bigoplus_{\alpha\in P(j,-n-a+b-j)} \mathcal{E}^{(a-j)}\mathcal{F}^{(b-j)}1_n\{2|\alpha| - j(b-a-n)\}. \tag{2.70}$$

Remark 2.4.4. After reindexing the second direct sum using the identity $\begin{bmatrix} a+b \\ b \end{bmatrix} = \sum_{\lambda\in P(a,b)} q^{2|\lambda|-ab}$, these are the same isomorphisms given in (2.45) and (2.44).

The isomorphisms above follow from the so-called Stošić formula, which explicitly give idempotent decompositions of the direct sums above. The projection maps are

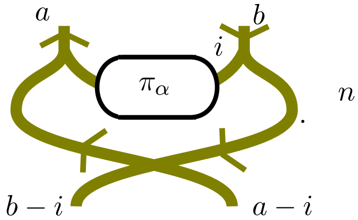
$$\sigma^i_\alpha := (-1)^{ab} \sum_{\beta,\gamma,x,y} (-1)^{\frac{i(i+1)}{2}+|x|+|y|} c^{K_i}_{\alpha,\beta,\gamma,x,y}$$



, (2.71)

while the inclusion maps are

$$\lambda^i_\alpha :=$$



. (2.72)

Remark 2.4.5. The spades which appear in the notation for thick bubbles are defined in [7, Section 4.5]. They are intended to emphasize the degree of the thick bubbles.

Quantum $\mathfrak{sl}_2$ at a prime root of unity

Before defining quantum groups at a root of unity, we fix some notation. Throughout this section we let p be prime. We denote the n'th cyclotomic polynomial in the variable q by $\Psi_n(q)$. We define

$$\mathbb{O}_p \coloneqq \mathbb{Z}[q]/\left(\Psi_p(q^2)\right).$$

Recall that $\Psi_p(q^2) = 1 + q^2 + \cdots + q^{2(p-1)}$. Also notice that

$$[p] = q^{p-1} + q^{p-3} + \cdots + q^{1-p} = q^{1-p}(1 + q^2 + \cdots + q^{2(p-1)}).$$

This means that $[p] = 0$ in $\mathbb{O}_p$.

The quantum groups $\mathbf{U}$ and $\dot{\mathbf{U}}$ were defined in Section 2.1 as algebras over $\mathbb{Q}(q)$. Given a $\mathbb{Z}[q, q^{-1}]$ subalgebra, we can specialize q^2 to be a p'th root of 1. This leads to the idea of a *quantum group at a prime root of unity.* There are different notions of what quantum $\mathfrak{sl}_2$ at a root of unity is, in part because there are different integral forms a quantum group. Below, we recall 3 of these notions. Note that they generalize to arbitrary quantum groups.

In $\mathbf{U}$ we have already defined the integral form of divided powers, $\mathbf{U}_{\mathcal{A}}$. We can also consider the $\mathbb{Z}[q, q^{-1}]$ subalgebra of $\mathbf{U}$ generated by (the non-divided powers) $E, F, K^{\pm 1}$. Taking the integral form of non-divided powers and specializing q^2 to be p'th root of unity via base change leads to the *Kac-De Concini* notion of a quantum group at a root of unity. By adding the relation $E^p = F^p = 0$ to the Kac-De Concini version, we get the *small quantum group*, which we denote $\mathfrak{u}_p$.

A third notion of a quantum group at a root of unity, due to Beilinson, Lusztig, and MacPherson, is what we call the *BLM* form and denote $\mathbf{U}_p$. The BLM form is obtained by base change from Lusztig's integral form $\mathbf{U}_{\mathcal{A}}$. That is

$$\mathbf{U}_p = \mathbf{U}_{\mathcal{A}} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{O}_p. \tag{2.73}$$

Note that $E^p = 0$ in $\mathbf{U}_p$ because

$$E^p \otimes 1 = [p]! E^{(p)} \otimes 1 = 0 \tag{2.74}$$

if q^2 is a p'th root of 1. Similarly $F^p = 0$. In this way we see that the small quantum group is both a sub and quotient of the BLM version.

We will mostly be concerned with the BLM version. As before, instead of working with $\mathbf{U}_p$ we prefer to work with the idempotented form,

$$\dot{\mathbf{U}}_p \coloneqq \dot{\mathbf{U}}_{\mathcal{A}} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{O}_p. \tag{2.75}$$

Remark 2.5.1. When giving a presentation for $\dot{\mathbf{U}}_p$ over $\mathbb{O}_p$ by generators and relations (for instance see [3, Definition 6.1]) it is common to call the generators 'divided powers'. Previously, divided powers had been defined by the formula $E^{(a)} = \dfrac{E^a}{[a]!}$, but that formula can no longer hold for $a \geq p$ since $[p] = 0$. Instead, the divided power generators of $\dot{\mathbf{U}}_p$ are the elements $E^{(a)} \otimes 1$ using the definition in (2.75).

The small quantum group $\mathfrak{u}_p(\mathfrak{sl}_2)$ is categorified in [2] and the BLM form $\dot{\mathbf{U}}_p(\mathfrak{sl}_2)$ is categorified in [3]. In the next section we review *p-DG algebras* and *p-DG categories*. These are the structures used to give the categorifications of quantum $\mathfrak{sl}_2$ at a prime root of unity.

p-DG algebras and categories

Our goal in this section is to define p-DG categories and their Grothendieck groups. We spend much of the section recalling definitions and properties from [2, 3, 8]. A particularly remarkable feature is that under certain assumptions, the Grothendieck group of a p-DG category can be described in terms of the usual (non p-DG) Grothendieck group of the underlying category. Our main theorem involves showing that $\dot{\mathcal{U}}^+_{\mathfrak{sl}_3}$ satisfies these assumptions. We start by defining p-DG algebras and categories.

Definition 2.6.1. Let $\Bbbk$ be a field of characteristic p. We call the tuple (A, ∂) a *p-DG algebra* if A is a $\mathbb{Z}$-graded algebra over $\Bbbk$ and ∂ is a *p-nilpotent* (meaning that $\partial^p = 0$) degree 2 endomorphism of A satisfying the following Leibniz rule: for all $f, g \in A$,

$$\partial(fg) = \partial(f)g + f\partial(g). \tag{2.76}$$

We refer to ∂ as a *differential*. For simplicity we will denote the p-DG algebra (A, ∂) as A_∂ or simply as A when the differential is understood.

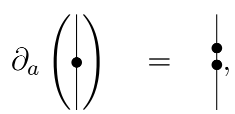
Remark 2.6.2. Notice that the Leibniz rule implies that

$$\partial^n(fg) = \sum_{i=0}^{n} \binom{n}{i} \partial^{n-i}(f)\partial^i(g)$$

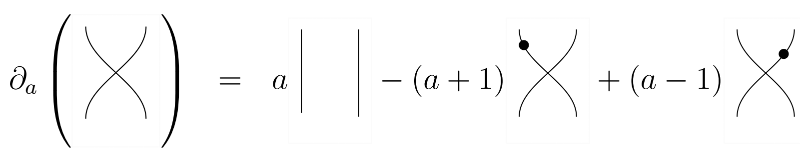
(where $\partial^0(f)$ is taken to be f). In particular, $\partial^p(fg) = \partial^p(f)g + f\partial^p(g)$ since $\Bbbk$ is of characteristic p. Thus it suffices to check p-nilpotence on generators of A.

Example 2.6.3. We can place a differential on the polynomial ring $A = \Bbbk[x_1, \ldots, x_n]$ which is defined on generators by $\partial(x_i) = x_i^2$ and then extended via the Leibniz rule. By the Leibniz rule, $\partial^n(x_i) = n!x_i^{n+1}$ and so ∂ is p-nilpotent on generators and thus everywhere on A. For degree conventions to match, we use the grading where $\deg(x_i) = 2$.

Example 2.6.4. We consider the nilHecke algebra (over $\Bbbk$) on n strands. In [8] a family of differentials, ∂_a, are defined on $\mathcal{NH}_n$. These differentials are defined locally as follows and then extended by the Leibniz rule.



(2.77)



(2.78)

Example 2.6.3 shows that ∂_a are p-nilpotent on dots. The differentials are p-nilpotent on crossings if and only if $a \in \mathbb{F}_p$ ([8, Lemma 3.6]). We will return to this example once we have defined p-DG Grothendieck groups. Of particular interest will be $\partial_{\pm 1}$.

Definition 2.6.5. A *left p-DG module* (M, ∂_M) over a p-DG algebra (A, ∂_A) is a graded left module over A equipped with a $\Bbbk$-linear, p-nilpotent differential ∂_M of degree 2 which satisfies the following Leibniz rule, for all $a \in A, m \in M$,

$$\partial_M(am) = \partial_A(a)m + a\partial_M(m). \tag{2.79}$$

As before, we will often simply write M instead of (M, ∂_M).

Example 2.6.6. Let A_∂ be as in example 2.6.3. Let M be the left regular module A. We explore the differentials we can place on M to give M the structure of an A_∂-module. The Leibniz rule for modules means we can possibly have a differential ∂_M where $\partial_M(1) \neq 0$. We set $\partial_M(1) = \sum_i a_i x_i$ and extend via the Leibniz rule: $\partial_M(f) = \partial_A(f) + \partial_M(1)$ for any polynomial $f \in M$. One can compute that

$$\partial_M^p(1) = \sum_i a_i(a_i+1)(a_i+2)\dots(a_i+p-1)x_i^p,$$

and so $\partial^p(1)$ will be zero if and only if one of $a_i, a_i+1, \dots a_i+p-1$ is zero for each i. Thus ∂_M is p-nilpotent if and only if each $a_i \in \mathbb{F}_p$. We recover the p-DG algebra structure of A by taking each $a_i = 0$. When $\partial_M(1)$ is non-zero, we call the differential on M a *twisted differential* and M a *twisted regular module.*

A morphism between p-DG modules $f : M \to N$ is a degree preserving A-module morphism which commutes with the respective differentials.

Example 2.6.7. Consider $\Bbbk$ itself as a p-DG algebra. The Leibniz rule implies that any differential on $\Bbbk$ is necessarily trivial. The Leibniz rule for $\Bbbk_\partial$-modules means that any differential is actually just a p-nilpotent $\Bbbk$-linear map. We use the term *p-complexes* for $\Bbbk_\partial$-modules. These are analogous to chain complexes but instead of $\partial^2 = 0$, we have $\partial^p = 0$. Except for the degree of the differential, these are the same thing as Mayer's complexes in [11].

Just as for ordinary chain complexes, there is a notion of homotopy between two p-complexes. Given morphisms between $\Bbbk_\partial$-modules, $f, g : M \to N$, we say that

f, g are *homotopic* if there is a $\Bbbk$-linear map h of degree $2-2p$ such that

$$f - g = \sum_{i=0}^{p-1} \partial_N^i \circ h \circ \partial_M^{p-i-1}. \tag{2.80}$$

Example 2.6.8. Let A_∂ be a p-DG algebra and M, N be A_∂-modules. Let $\mathrm{Hom}_A^i(M, N)$ be A-module maps of degree i. One can place a differential on the total space $\mathrm{Hom}_A(M, N) := \bigoplus_i \mathrm{Hom}_A^i(M, N)$ as follows. For $f \in \mathrm{Hom}_A(M, N)$, $\partial(f)(x) = \partial_N(f(x)) - f(\partial_M(x))$. This differential is p-nilpotent which gives the space $\mathrm{Hom}_A(M, N)$ the structure of a p-complex. The p-DG module morphisms in $\mathrm{Hom}_{A_\partial}(M, N)$ are precisely the morphisms in $\mathrm{Hom}_A^0(M, N)$ which are killed by this differential.

Remark 2.6.9. The p-complexes above can be thought of as modules over $\Bbbk[\partial]/(\partial^p)$. Since $\Bbbk[\partial]/(\partial^p)$ is a Hopf algebra, where comultiplication is given by $\Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial$, a consequence is that the tensor product of p-complexes is again a p-complex. In fact, any p-DG algebra A_∂ can be equipped with a $\Bbbk[\partial]/(\partial^p)$ module structure in a way which makes the category of A_∂-modules closed under tensor product with p-complexes (see [8, remark 2.11]).

The notion of a p-DG category extends that of a p-DG algebra.

Definition 2.6.10. A *p-DG category* $(\mathcal{A}, \partial)$ is a graded $\Bbbk$-linear category $\mathcal{A}$ where the Hom spaces $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ are equipped with a p-nilpotent *differential*, ∂, of degree 2 such that for all objects $X, Y, Z \in \mathcal{A}$ and all $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y), g \in \mathrm{Hom}_{\mathcal{A}}(Y, Z)$ the differential satisfies the Leibniz rule

$$\partial(g \circ f) = \partial(g) \circ f + g \circ \partial(f). \tag{2.81}$$

Notice that for $X, Y \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is a $\Bbbk_\partial$-module.

A *left p-DG module* $\mathcal{M}$ over a p-DG category $\mathcal{A}$ is a covariant functor from $\mathcal{A}$ to p-complexes which commutes with the differential,

$$\mathcal{M}(\partial_{\mathcal{A}}(f)) = \partial(\mathcal{M}(f)).$$

A p-DG module $\mathcal{M}$ is said to be *representable* if it is isomorphic to $\mathrm{Hom}_{\mathcal{A}}(M, -)$ for some object $M \in \mathcal{A}$. Representable modules are the analog for categories of free modules for rings. We will see later that for $\dot{\mathcal{U}}$ we can restrict our focus to representable modules. In light of Example 2.6.6 and Example 2.6.11 (which is still to come), we emphasize that a representable module $\mathcal{M}$ carries the standard differential from $\mathcal{A}$.

The category of left p-DG modules over a p-DG category $\mathcal{A}$ is abelian. We denote this category $\mathcal{A}_\partial$-mod.

Example 2.6.11. One can define twisted representable modules similar to the modules in Example 2.6.6. The Leibniz rule for p-DG categories means that differentials have to act trivially on identity morphisms. However, for a module, this no longer needs to be true.

Consider the representable module $\mathrm{Hom}(M, -)$. For objects $X \in \mathcal{A}$, we view the p-complexes $\mathrm{Hom}(M, X)$ as being $\mathrm{Hom}(M, X) \circ \mathrm{id}_M$. Let ϕ be a degree two morphism in $\mathrm{End}_{\mathcal{A}}(M)$. We define a twisted differential $\tilde{\partial}$ on $\mathrm{Hom}(M, X) \circ \mathrm{id}_M$ by

$$\tilde{\partial}\,(f \circ \mathrm{id}_M) := \partial(f) + f \circ \phi. \tag{2.82}$$

For a general morphism ϕ, this differential may or may not be p-nilpotent. It will be p-nilpotent when $\partial(\phi) = a\phi^2$ for $a \in \mathbb{F}_p$. We call these modules *twisted representable modules.*

Remark 2.6.12. We will use the notation $\mathrm{Hom}_{\mathcal{A}_\partial}(\mathcal{M}, \mathcal{N})$ for morphisms of p-DG modules and $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ for morphisms just as $\mathcal{A}$-modules (where we just view $\mathcal{A}$ as an additive category). Consider the representable p-DG modules $\mathcal{M} = \mathrm{Hom}_{\mathcal{A}}(M, -)$ and $\mathcal{N} = \mathrm{Hom}_{\mathcal{A}}(N, -)$. The natural transformations in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ are given by morphisms $\mathrm{Hom}_{\mathcal{A}}(N, M)$ in the underlying category $\mathcal{A}$. In the same way as before we can place a p-complex structure on these natural transformations, $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$. The degree zero morphisms in $\mathrm{Hom}_{\mathcal{A}}(N, M)$ which are killed by this differential are the p-DG module morphisms.

In the same way as Remark 2.6.9, we can define the notion of homotopy between two morphisms of p-DG modules. For morphisms between two p-DG modules, $f, g : \mathcal{M} \to \mathcal{N}$, we say that f, g are homotopic if there exists a collection of linear maps h_X, one for each $X \in \mathrm{Ob}_{\mathcal{A}}$, where each h_X is a homotopy between $f_{\mathcal{M}(X)}$ and $g_{\mathcal{N}(X)}$. A morphism f between two p-DG modules is called *null-homotopic* if f is homotopic to the zero morphism. We define the *homotopy category* of $\mathcal{A}_\partial$-mod, $\mathcal{H}(\mathcal{A})$, to be the quotient of $\mathcal{A}_\partial$-mod by null-homotopic morphisms. We say that two p-complexes X, Y are *quasi-isomorphic* if there is a map of p-complexes $f : X \to Y$ which yields an isomorphism in $\mathcal{H}(\Bbbk_\partial)$. Similarly, two $\mathcal{A}_\partial$-modules $\mathcal{M}, \mathcal{N}$ are quasi-isomorphic if there is a natural transformation which induces quasi-isomorphism on the underlying p-complexes $f_X : \mathcal{M}(X) \to \mathcal{N}(X)$ for every object $X \in \mathcal{A}$. We formally invert quasi-isomorphisms to obtain the *derived category*, $\mathcal{D}(\mathcal{A})$.

The morphisms in the p-DG derived category are not easy to understand. For instance, a short exact sequence of $\mathcal{A}$-modules

$$0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{O} \to 0,$$

need not be a short exact sequence as p-DG modules and so the relation $[\mathcal{N}] = [\mathcal{M}] + [\mathcal{O}]$ is not guaranteed to hold in the p-DG Grothendieck group. A short exact sequence of $\mathcal{A}$-modules which are also *cofibrant* does lead to a short exact sequence as p-DG modules and to relation in the Grothendieck group. Cofibrant modules should be thought of as analogous to complexes of projective objects in ordinary homological algebra.

Definition 2.6.13. Let $\mathcal{A}$ be a p-DG category and $\mathcal{M}$ a p-DG module over $\mathcal{A}$.

i. $\mathcal{M}$ is said to be *cofibrant* if any surjective quasi-isomorphism of p-DG modules $f : \mathcal{N}_1 \to \mathcal{N}_2$ induces a homotopy equivalence of p-complexes $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}_1) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}_2)$.

ii. $\mathcal{M}$ is a *finite-cell module* if there is a finite filtration on $\mathcal{M}$ by p-DG $\mathcal{A}$-modules $\mathcal{F}^i$,

$$0 = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^n = \mathcal{M},$$

where the subquotients $\mathcal{F}^i/\mathcal{F}^{i-1}$ are all isomorphic to direct sums of representable modules.

iii. Viewing $\mathcal{M}$ as an object in the derived category, we say that $\mathcal{M}$ is *compact* if $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(\mathcal{M}, -)$ commutes with taking direct sums.

At this stage we could try to define the Grothendieck group of a p-DG category to be the Grothendieck group of $\mathcal{D}(\mathcal{A})$. For this to be non-zero we will need to impose some boundedness conditions. We define the *compact derived category* $\mathcal{D}^c(\mathcal{A})$ to be the strictly full subcategory of $\mathcal{D}(\mathcal{A})$ consisting of compact objects. We denote the Grothendieck group of $\mathcal{D}^c(\mathcal{A})$ by $K_0(\mathcal{A}, \partial)$.

Remark 2.6.14. The Grothendieck group of $\mathcal{D}^c(\Bbbk_\partial)$ is $\mathbb{O}_p$ [4]. One remarkable feature about p-DG algebras is that $K_0(A)$ is naturally a module over $K_0(\Bbbk_\partial)$. This is because, following Remark 2.6.9, tensoring A_∂-modules with p-complexes gives rise to exact functors and an action of $\mathcal{H}(\Bbbk_\partial)$ on $\mathcal{H}(A_\partial)$. Since $K_0(\Bbbk_\partial) \cong \mathbb{O}_p$, this gives p-DG Grothendieck groups the structure of an $\mathbb{O}$-module instead of just the usual $\mathbb{Z}[q, q^{-1}]$ structure one would expect. We will soon see that in certain instances there is an isomorphism

$$K_0(\mathcal{A}, \partial) \cong K_0(\mathcal{A}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{O}_p, \tag{2.83}$$

where $K_0(\mathcal{A})$ is the Grothendieck group of $\mathcal{A}$ as an p-DG category.

Example 2.6.15. [**2.6.4 cont**] Set $(\mathcal{NH}, \partial) := \bigoplus_{n\in\mathbb{N}}(\mathcal{NH}_n, \partial)$. For the differentials$\partial_1$ and ∂_{-1} defined in 2.6.4 the p-DG Grothendieck group $K_0(\mathcal{NH}, \partial)$ is isomorphic to $u^+_{\mathfrak{sl}_2}$. But, of course,

$$u^+_{\mathfrak{sl}_2} \cong K_0(\mathcal{NH}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{O}_p. \tag{2.84}$$

Definition 2.6.16. We define a *p-DG 2-category* to be a graded $\Bbbk$-linear 2-category $\mathcal{A}$ where the 1-morphisms are p-DG categories and the differential satisfies the Leibniz rule for both horizontal and vertical composition of 2-morphisms.

The homotopy category is defined as

$$\mathcal{H}(\mathcal{A}) \coloneqq \bigoplus_{n,m \in \mathrm{Ob}(\mathcal{A})} \mathcal{H}(\mathrm{Hom}_{\mathcal{A}}(n,m)). \tag{2.85}$$

Similarly the derived category and Grothendieck group are defined as

$$\mathcal{D}(\mathcal{A}) \coloneqq \bigoplus_{n,m \in \mathrm{Ob}(\mathcal{A})} \mathcal{D}(\mathrm{Hom}_{\mathcal{A}}(n,m)), \qquad K_0(\mathcal{A}) \coloneqq \bigoplus_{n,m \in \mathrm{Ob}(\mathcal{A})} K_0(\mathrm{Hom}_{\mathcal{A}}(n,m)). \tag{2.86}$$

Induced differentials and filtrations

Given a p-DG category $\mathcal{A}$ and a set of idempotents $\{\varepsilon_i\}$, one can consider the partial idempotent completion. In some cases it is possible to extend the differential on $\mathcal{A}$ to the partial idempotent completion. In this section we recall the conditions given in [3] which allow a differential to be extended to an idempotent completion. Of particular interest will be understanding when certain submodules and quotient modules of representable p-DG modules inherit a differential.

Given a p-DG algebra A and an idempotent $\varepsilon \in A$, we can consider when the decomposition $A = A\varepsilon \oplus A(1-\varepsilon)$ holds as p-DG modules. First ,we notice that for any idempotent ε, the Leibniz rule implies that $\varepsilon\partial(\varepsilon)\varepsilon = 2\varepsilon\partial(\varepsilon)\varepsilon$ and so

$$\varepsilon\partial(\varepsilon)\varepsilon = 0. \tag{2.87}$$

This implies, again using the Leibniz rule, that $A\varepsilon$ is preserved by the differential if and only if

$$\varepsilon\partial(\varepsilon) = 0. \tag{2.88}$$

Said another way, if ε is an idempotent with the property that $\varepsilon\partial(\varepsilon) = 0$, then $A\varepsilon$ is a p-DG submodule with differential

$$\partial_{A\varepsilon}(a\varepsilon) := \partial(a\varepsilon) = \partial(a\varepsilon)\varepsilon. \tag{2.89}$$

Now suppose that $A(1-\varepsilon)$ is a p-DG submodule (that $A(1-\varepsilon)$ is preserved by the differential). This is equivalent to $\partial(1-\varepsilon)\varepsilon = 0$. Using the equality

$$0 = \partial((1-\varepsilon)\varepsilon)) = \partial(1-\varepsilon)\varepsilon + (1-\varepsilon)\partial(\varepsilon),$$

this is equivalent to

$$\partial(\varepsilon) = \varepsilon\partial(\varepsilon) \tag{2.90}$$

(and hence to $\partial(\varepsilon)\varepsilon = 0$). Similar to before, we see that $A\varepsilon$ is a p-DG quotient if and only if $\partial(\varepsilon) = \varepsilon\partial(\varepsilon)$. The equation

$$0 = \partial(a\varepsilon(1-\varepsilon)) = a\varepsilon\partial(1-\varepsilon) + \partial(a\varepsilon)(1-\varepsilon), \tag{2.91}$$

implies that in the quotient $A\varepsilon$ we again have the equality $\partial(a\varepsilon) = \partial(a\varepsilon)\varepsilon$. Said again, as a p-DG quotient, $A\varepsilon$ has a differential given by

$$\partial_{A\varepsilon}(ae) := \partial(ae) = \partial(a\varepsilon)\varepsilon. \tag{2.92}$$

Note that by (2.88) and (2.90) $A\varepsilon$ will be both a p-DG submodule and quotient module if and only if $\partial(\varepsilon) = 0$.

Definition 2.7.1. We say that ε is a *subquotient idempotent* if ε is an idempotent in a p-DG submodule $A\varepsilon'$ and $A\varepsilon$ is a quotient of $A\varepsilon'$. In this case we call $A\varepsilon$ a *subquotient summand.*

Lemma 2.7.2. *[3, Lemma 4.4] For a subquotient idempotent ε in a p-DG algebra A, there is an inherited differential on $\varepsilon A\varepsilon$ which is given by*

$$\partial_{\varepsilon A\varepsilon}(\varepsilon a\varepsilon) \coloneqq \varepsilon\partial(\varepsilon a\varepsilon)\varepsilon \tag{2.93}$$

Remark 2.7.3. The discussion above has all been regarding p-DG algebras. The same statements can all be used for representable modules $\mathcal{M}$ over a p-DG category $\mathcal{A}$. See [3, Section 4.5] for one way to do so. In the language of categories, we are interested in subquotient idempotents $\varepsilon \in \mathrm{End}_{\mathcal{A}}(M)$ and $\mathcal{M}\varepsilon = \mathrm{Hom}_{\mathcal{A}}(M, -) \circ \varepsilon$. We refer to these idempotents as *subquotient idempotents* in $\mathcal{A}$.

A consequence of the discussion above is that given a collection of subquotient idempotents in a p-DG category $\mathcal{A}$, the partial idempotent completion $\dot{\mathcal{A}} = \mathcal{A}\left(\{\varepsilon_i\}\right)$ inherits a differential from $\mathcal{A}$. Recall that morphisms in $\dot{\mathcal{A}}$ are triples $\dot{f} = (\varepsilon_i, f, \varepsilon_j)$. The inherited differential is given by

$$\partial_{\dot{\mathcal{A}}}(\dot{f}) \coloneqq \varepsilon_i\partial(\varepsilon_i f\varepsilon_j)\varepsilon_i. \tag{2.94}$$

Remark 2.7.4. It is this reason why we prefer the partial idempotent completion, $\dot{\mathcal{U}}$, over the Karoubi envelope, $\mathrm{Kar}(\mathcal{U})$. The idempotents e_a will be subquotient idempotents and so we will get an induced differential on $\dot{\mathcal{U}}$.

We collect below some propositions and theorems to aid in computing Grothendieck groups of p-DG algebras and categories.

Definition 2.7.5. Let $\mathcal{A}$ be a p-DG category. For an object M, an *Fc-filtration* on M is a finite set of objects $\{N_i \mid i \in I\}$, equipped with inclusion and projection maps σ_i, λ_i for which the following conditions hold

i)

$$\sigma_i \lambda_j = \delta_{i,j} \mathrm{Id}_{N_i}, \tag{2.95}$$

$$\mathrm{Id}_M = \sum_i \lambda_i \sigma_i. \tag{2.96}$$

ii) There exists some total order on I for which $\partial(\sigma_i)\lambda_j = 0$ for $j \geq i$.

Remark 2.7.6. The idea behind an Fc-filtration is that it gives a filtration on the representable module $\mathrm{Hom}_{\mathcal{A}}(M, -)$ where the subquotients are isomorphic to representable modules. The condition involving the differential, $\partial(\sigma_i)\lambda_j = 0$ for $j \geq i$, ensures that the subquotients are actually p-DG subquotients. In the language of Definition 2.7.1, the idempotents $\lambda_i \sigma_i$ are subquotient idempotents. The upshot of this is the following proposition.

Proposition 2.7.7 ([3, Proposition 4.15])**.** *Given an Fc-filtration, the idempotents $\varepsilon_i := \lambda_i \sigma_i$ and the order on I define a filtration on M where the subquotients are p-DG isomorphic to the representable modules N_i.*

Definition 2.7.8. A Karoubian mixed p-DG category with self dual indecomposable objects is *fantastically filtered* if every object has an Fc filtration where the summands are grading shifts of the indecomposable objects.

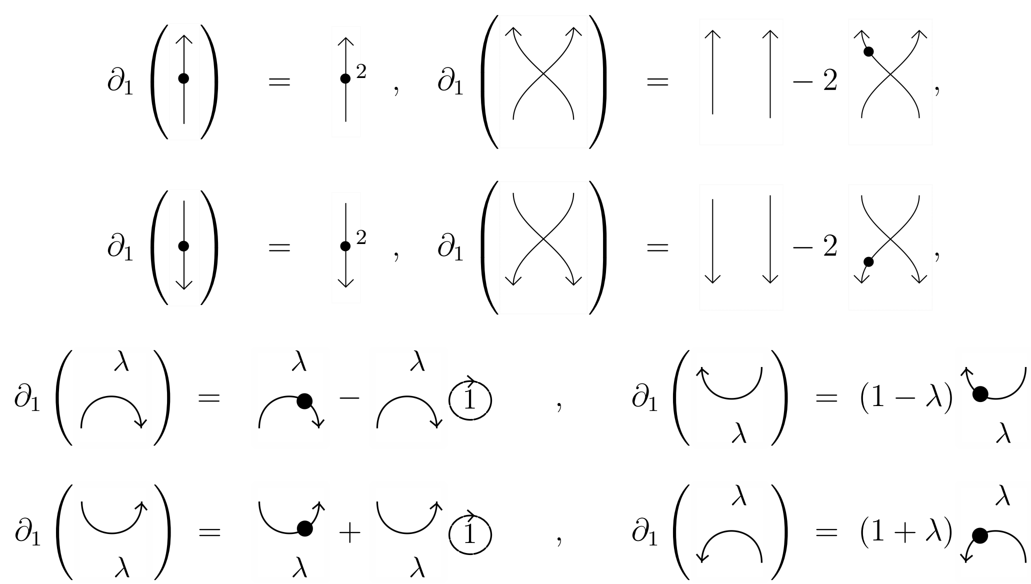
Proposition 2.7.9 ([3, Proposition 4.24])**.** *If a p-DG category is fantastically filtered, then $K_0(\mathcal{A}, \partial) \cong K_0(\mathcal{A}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{O}_p$*

The previous proposition motivates the following: to show that the p-DG Grothendieck group of a category is what you expect it to be, show that the category is fantastically filtered.

$\mathcal{U}$ and $\dot{\mathcal{U}}$ as p-DG categories

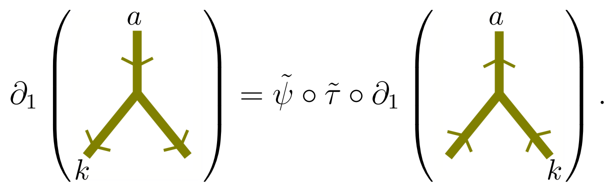
In this section we consider Lauda's $\mathcal{U}$ where $\Bbbk$ is of characteristic p. We sketch the results of [3] which give a categorification of the idempotented form of quantum $\mathfrak{sl}_2$ at a root of unity, $\dot{\mathcal{U}}_p$.

The differential ∂_1 from examples 2.6.4, 2.6.15 is extended to all of $\mathcal{U}$ by the following.



The other differential, ∂_{-1}, can be extended in a similar way. It turns out that the differentials are in a sense dual to each other in the sense that $\partial_{-1} = \tilde{\tau}\partial_1\tilde{\tau}^{-1}$. Similarly $\partial_{-1} = \tilde{\psi}\partial_1\tilde{\psi}$. This means that ∂_1 is left unchanged under conjugation by the composition $\tilde{\tau} \circ \tilde{\psi}$. In order for these differentials to induce a differential on $\dot{\mathcal{U}}$ it must be the case that the idempotents in $\dot{\mathcal{U}}$ are subquotient idempotents (recall that $\dot{\mathcal{U}}$ is the partial idempotent completion with respect to the idempotents e_a.)

It is indeed the case that the e_a are subquotient idempotents with respect to the differentials ∂_1, and so the differentials can be extended to $\dot{\mathcal{U}}$. In [3], the authors derive formulas for the induced differential for upward splitters and merges and then use that ∂_1 is fixed under conjugation by $\tilde{\tau} \circ \tilde{\psi}$ to compute the action of the differential on downward splitters and merges. Explicitly,



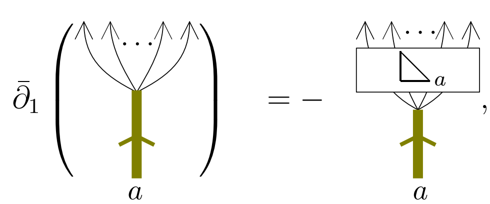
(2.97)

Remark 2.8.1. One subtlety is that this differential no longer agrees with the induced differential on the partial idempotent completion. For the differential to agree, the downward thick strands should be defined using a different idempotent. The idempotent which should be used for the downward strands is $\tilde{\psi}(e_a)$.

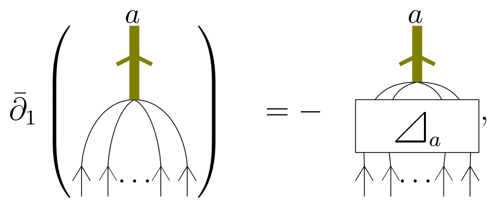
We adopt the notation from [3, (2.2a), (2.2b)] for certain linear polynomials which we also view as elements in $\mathcal{NH}_n$

$$\text{◺}_n \coloneqq \sum_{i=1}^{n}(n-i)x_i \qquad , \qquad \text{◿}_n \coloneqq \sum_{i=1}^{n}(i-1)x_i. \tag{2.98}$$

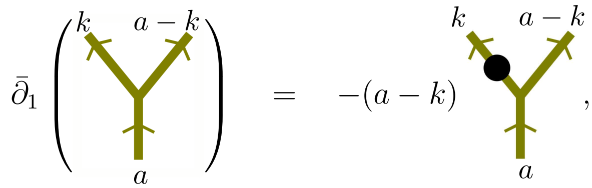
We list some formulas below for the differential on $\dot{\mathcal{U}}$ (see [3] for details). In light of the above remark, if these are viewed as the induced differential $\bar{\partial}_1$, the partial idempotent completion needs to be with respect to different idempotents for the downward pointing strands.



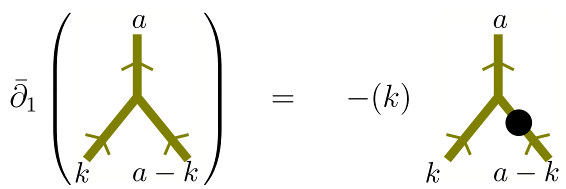
(2.99)



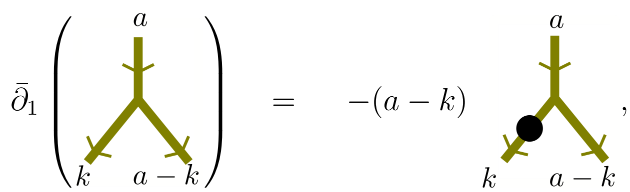
(2.100)



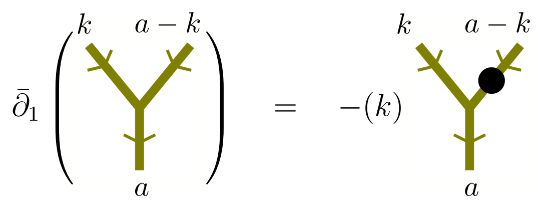
(2.101)



(2.102)



(2.103)



(2.104)

To obtain similar formulas for $\bar{\partial}_{-1}$ one can conjugate by $\tilde{\tau}$.

It is easy to see that the differentials ∂_1 and ∂_{-1} act the same on polynomials. Their action on Schur polynomials has a particularly nice description. We introduce some notation designed to make formulas easier to write. For a partition $\lambda \in P(a,b)$, we denote by $\lambda + \Box$ a new partition which is obtained from λ by adding a single box. We don't allow $\lambda + \Box$ to have $a+1$ rows (we require that it still be in $P(a)$), but we do allow it to have $b+1$ columns. We will index sums by this new symbol. The index in $\sum_{\lambda+\Box}$ is meant to be understood as the sum over all ways to form a new partition by adding a box to λ. The *content* of a box in a Young diagram is the column number minus the row number. The contents of each box have been labeled in the example below.

0	1	2	3
−1	0	1	2
−2	−1	0	
−3			

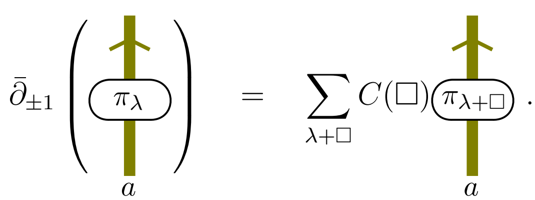
.

For partitions $\lambda + \square$, we denote the content of the added box by $C(\square)$.

Lemma 2.8.2 ([3, Lemma 2.4]). *For any Schur polynomial $\pi_\lambda \in P(a)$ the differentials $\partial_{\pm 1}$ act as*

$$\partial_{\pm 1}(\pi_\lambda) = \sum_{\lambda+\square} C(\square)\pi_{\lambda+\square}. \tag{2.105}$$

When viewing Schur polynomials as elements of $\dot{\mathcal{U}}$, the Lemma above results in the following diagrammatically equality:



(2.106)

A categorification of $\dot{\mathbf{U}}$ at a prime root of unity

The p-DG 2-category $(\dot{\mathcal{U}}, \partial_1)$ categorifies $\dot{\mathbf{U}}_p$. In other words,

$$K_0(\dot{\mathcal{U}}, \partial_1) \cong \dot{\mathbf{U}}_{\mathcal{A}} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{O}_p. \tag{2.107}$$

Relations (2.42)-(2.45) are enough to decompose any 1-morphism in terms of indecomposable 1-morphisms. The task in [3] was to prove that the direct sum decompositions are actually Fc-filtrations. Showing that (2.42) and (2.43) are Fc-filtrations is comparatively straightforward. Showing that (2.44) and (2.45) are Fc-filtrations is much more difficult and crucially relies on the explicit idempotents given in the Stošić formula [7, Theorem 5.9]. The special case $a = b = 1$ is Lauda's original decomposition given in (2.33) and (2.34). It is shown in [7] that $\dot{\mathcal{U}}$ is self-dual and mixed.

This means that $\dot{\mathcal{U}}$ is fantastically filtered and so

$$K_0(\dot{\mathcal{U}}, \partial_1) \cong K_0(\dot{\mathcal{U}}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{O}_p. \tag{2.108}$$

Since $\dot{\mathcal{U}}$ categorifies $\dot{\mathbf{U}}_{\mathcal{A}}$ we have that $K_0(\dot{\mathcal{U}}) \cong \dot{\mathbf{U}}_{\mathcal{A}}$. This gives an isomorphism

$$K_0(\dot{\mathcal{U}}, \partial_1) \cong \dot{\mathbf{U}}_{\mathcal{A}} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{O}_p = \dot{\mathbf{U}}_p. \tag{2.109}$$

CHAPTER III

CATEGORIFICATION OF THE POSITIVE HALF OF QUANTUM SL3 AT A PRIME ROOT OF UNITY

Quantum $\mathfrak{sl}_3$

We begin this chapter by recalling the work that has been done categorifying quantum $\mathfrak{sl}_3^+$. In the first section we define the quantum group $\mathbf{U}_{\mathfrak{sl}_3}^+$ and construct Khovanov and Lauda's diagrammatic category $\mathcal{U}_{\mathfrak{sl}_3}^+$ which categorifies it [5]. We view this category as an extension of $\mathcal{U}_{\mathfrak{sl}_2}^+$ (see Remark 2.2.4).

Similar to the case for quantum $\mathfrak{sl}_2$, a thick calculus has been developed in [12] which enables a diagrammatic representation of 2-morphisms directly inside of $\dot{\mathcal{U}}_{\mathfrak{sl}_3}^+$. Stošić also gives an idempotent decomposition for certain 2-morphisms which is sufficient to decompose any 2-morphism into a direct sum of indecomposable 2-morphisms.

In [8], a family of differentials is placed on the thin category $\mathcal{U}_{\mathfrak{sl}_3}^+$ which gives the structure of a p-DG category. They show that for only two of these differentials, the quantum Serre relations hold. When restricted to $\mathcal{E}_1$ (or $\mathcal{E}_2$) these differentials are precisely the differentials $\partial_{\pm 1}$ in Section 2.8. We then extend these differentials to $\dot{\mathcal{U}}_{\mathfrak{sl}_3}^+$, giving it a p-DG structure. We derive formulas for how the differentials act on thick strands. Our main result (Proposition 3.3.2) is to show that the idempotent decomposition given in [12] together with one of the differentials from [8] is an Fc-filtration. Since, in addition, $\mathcal{U}_{\mathfrak{sl}_3}^+$ is mixed and Karoubian we see that $\mathcal{U}_{\mathfrak{sl}_3}^+$ is Fc-filtered. By Proposition 2.7.9, this gives a categorification of $\dot{\mathcal{U}}_{\mathfrak{sl}_3}^+$ at a prime root of unity.

Remark 3.1.1. It should be noted that the work mentioned above–the categorification of $\mathbf{U}^+_{\mathfrak{sl}_3}$, the development of the thick calculus, and the differential placed on $\mathcal{U}^+_{\mathfrak{sl}_3}$–was all done in the more general setting of quantum $\mathfrak{sl}_n$. The Stošić formula in [12], however, is only sufficient to decompose every 2-morphism in $\mathfrak{sl}_3$. For that reason, we work with $\mathfrak{sl}_3$ specifically. Lusztig's canonical basis for $\mathbf{U}^+_{\mathfrak{sl}_4}$ is considerably more complicated where the basis consists of 14 different types of monomials [13, Theorem 2.2].

In this chapter we drop the subscript $\mathfrak{sl}_3$ letting $\mathbf{U}^+$ denote $\mathbf{U}^+_{\mathfrak{sl}_3}$. As in the previous chapter, where we dropped that subscript $\mathfrak{sl}_2$, in this chapter we drop the subscript $\mathfrak{sl}_3$ letting $\mathbf{U}^+$ denote $\mathbf{U}^+_{\mathfrak{sl}_3}$. We will denote the categorified versions in a similar manner.

Categorified and decategorified quantum $\mathfrak{sl}_3^+$

Definition 3.2.1. The *positive half of quantum* $\mathfrak{sl}_3$, $\mathbf{U}^+$, is the algebra over $\mathbb{Q}(q)$ generated by E_1 and E_2 subject to the relation

$$E_i^2 E_j + E_j E_i^2 = [2] E_i E_j E_i, \qquad \text{for } i \neq j. \tag{3.1}$$

Remark 3.2.2. We will not work with the idempotented version like we did in Section 2.1. This is because we are only dealing with the positive half of quantum $\mathfrak{sl}_3$ and so do not need the projection morphisms.

Similar to the $\mathfrak{sl}_2$ case, we will consider the $\mathbb{Z}[q, q^{-1}]$ subalgebra $\mathbf{U}^+_{\mathcal{A}}$ spanned by the divided powers. Using divided powers, the relation in (3.1) becomes

$$E_i^{(2)} E_j + E_j E_i^{(2)} = E_i E_j E_i, \qquad \text{for } i \neq j. \tag{3.2}$$

We also have the $\mathfrak{sl}_2$ relation

$$E_i^{(a)}E_i^{(b)} = \begin{bmatrix} a+b \\ b \end{bmatrix} E_i^{(a+b)}. \tag{3.3}$$

Lusztig's canonical basis $\mathbb{B}$ of $\mathbf{U}_{\mathcal{A}}^{+}$ is given by

$$\mathbb{B} = \{E_1^{(a)}E_2^{(b)}E_1^{(c)},\ E_2^{(a)}E_1^{(b)}E_2^{(c)} \ \mid\ b \geq a+c, \quad a,b,c, \in \mathbb{N}\}. \tag{3.4}$$

Non-basis elements $E_i^{(a)}E_j^{(b)}E_i^{(c)}$ for $i \neq j$, $b \leq a+c$ can be decomposed in this basis as [10, Lemma 42.1.2]

$$E_i^{(a)}E_j^{(b)}E_i^{(c)} = \sum_{\substack{p+r=b \\ p\leq c \\ r\leq a}} \begin{bmatrix} a+c-b \\ c-p \end{bmatrix} E_i^{(p)}E_j^{(a+c)}E_i^{(r)}. \tag{3.5}$$

In [5], Khovanov and Lauda give a diagrammatic description for a categorification of the positive half of an arbitrary quantum group. The categorification follows a similar construction to that previously discussed for $\mathbf{U}_{\mathfrak{sl}_2}$. For an arbitrary quantum group (thin) strands are labeled with colors, one for each generator of the quantum group. The relations for $\mathfrak{sl}_2$ hold for crossings of strands of the same color and there are new relations regarding the crossing of strands of different colors which depend on $i \cdot j$. For $\mathfrak{sl}_3$, the category is defined diagrammatically as follows.

Definition 3.2.3. The category $\mathcal{U}^+$ is defined to be the $\mathbb{Z}$-linear category which has a single object $*$. The 1-morphisms are formal direct sums of grading shifts of composites of $\mathcal{E}_1, \mathcal{E}_2$. The 2-morphisms are generated by:

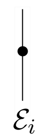
i. Identity 2-morphisms $\mathrm{Id}_{\mathcal{E}_1}$ and $\mathrm{Id}_{\mathcal{E}_2}$, which we depict as

$:= \mathrm{Id}_{\mathcal{E}_1},$


$:= \mathrm{Id}_{\mathcal{E}_2}.$ (3.6)

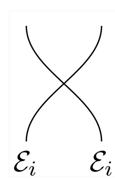
We will frequently omit labeling the strands and instead draw the strands using colors; one color for $\mathcal{E}_1$ and another for $\mathcal{E}_2$.

ii. The following 2-morphisms of degree 2.



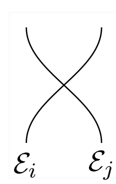
(3.7)

iii. The following 2-morphisms of degree -2.



(3.8)

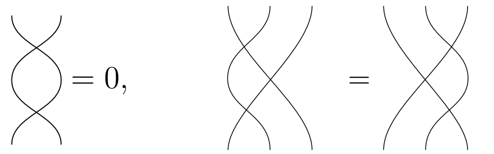
iv. For $i \neq j$, the following 2-morphisms of degree 1.



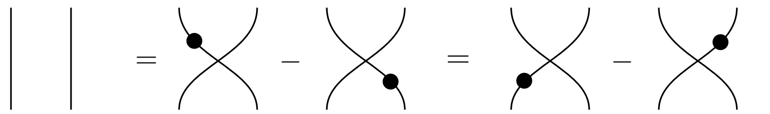
(3.9)

The defining relations are:

1. NilHecke relations hold for strands of a given color. Explicitly, for strands of any single color

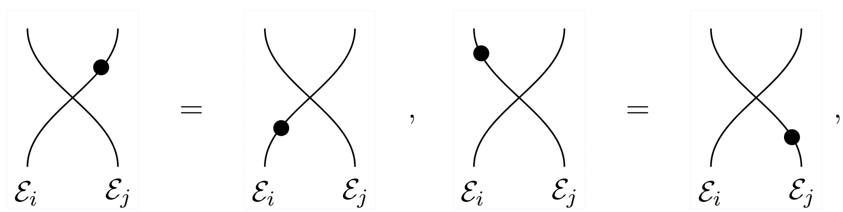


(3.10)

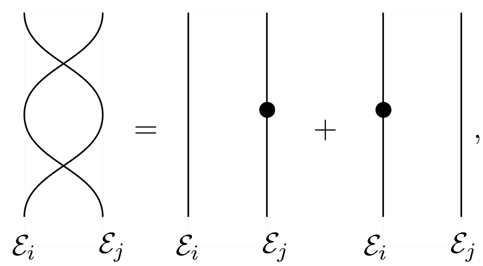


(3.11)

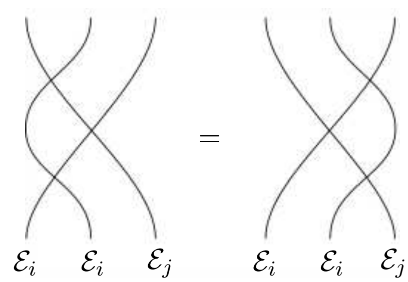
2. Two color relations: for crossings of different colors (for $i \neq j$), we impose the following relations.



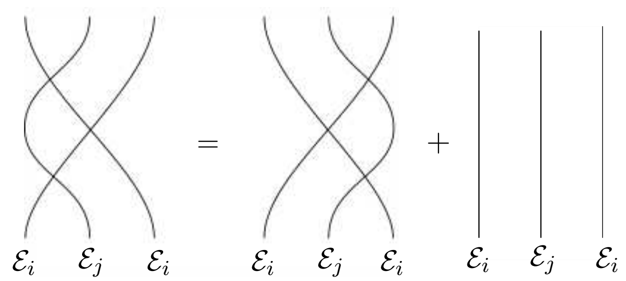
(3.12)



(3.13)



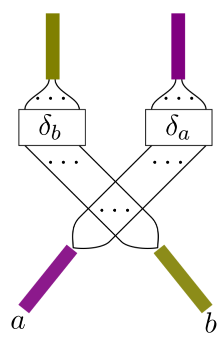
(3.14)



(3.15)

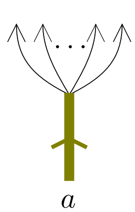
For strands of a single color, we will use the box notation given in Section 2.4. This means that we have idempotents e_a in both $\mathrm{End}(\mathcal{E}_1^a)$ and $\mathrm{End}(\mathcal{E}_2^a)$. We will define $\dot{\mathcal{U}}^+$ to be the partial idempotent completion with respect to the idempotents $\{(\mathcal{E}_1^a, \tilde{\psi}(e_a)),\ (\mathcal{E}_2^a, \tilde{\psi}(e_a))\ \mid a \in \mathbb{N}\}$.

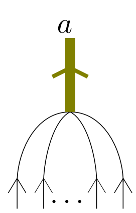
In [12], the thick calculus introduced in [7] is extended to $\dot{\mathcal{U}}^+$ (and more generally to $\dot{\mathcal{U}}^+_{\mathfrak{sl}_n}$). Diagrammatically we can define $\dot{\mathcal{U}}^+$ by adding thick strands for each color to the generators of $\mathcal{U}^+$. We add complete splitters for each color which individually satisfy relations (2.62), (2.63). A crossing of thick strands of different colors is defined by expressing thick edges as thin ones.

$:=$


(3.16)

Remark 3.2.4. We reiterate that our choice of idempotents for the partial idempotent completion is made so that we can work with the differential ∂_{-1}. This is the differential for which the idempotent decomposition given in [12] is an Fc-filtration. This also necessitates defining the splitters as

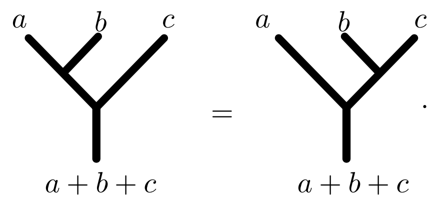


$:= (\tilde{\psi}(e_a), \tilde{\psi}(e_a), \mathrm{Id}_{E_i^a})$,


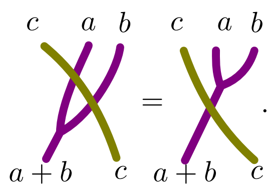
$:= (\mathrm{Id}_{E_i^a}, \tilde{\psi}(e_a), \tilde{\psi}(e_a)).$ (3.17)

Having a diagrammatic description enables us to compute directly in $\dot{\mathcal{U}}^+$. We recall some of the properties from $\dot{\mathcal{U}}^+$ which we will use and which can be deduced from the defining relations.

Proposition 3.2.5 (Associativity of splitters [7, Proposition 2.2.4]). *For any color, the following holds:*

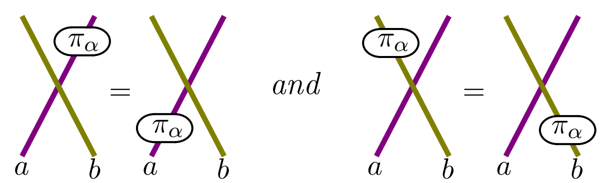


Proposition 3.2.6 (Pitchfork lemma). *For any two colors,*



Proof. After expressing the thick strands as thin strands, repeatedly apply relations (3.12) and (3.14). ∎

Proposition 3.2.7 (Dot slide [12, Proposition 6]). *Thick dots can slide past crossings of different colors.*



In [12, Theorem 2], Stošić shows that the indecomposable 1-morphisms, up to shift, in $\dot{\mathcal{U}}^+$ are $\{\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)},\ \mathcal{E}_2^{(a)}\mathcal{E}_1^{(b)}\mathcal{E}_2^{(c)} \ \mid\ b \geq a+c, \quad a,b,c,\in \mathbb{N}\}$ and that they are pairwise non-isomorphic with the exception that

$$\mathcal{E}_1^{(a)}\mathcal{E}_2^{(a+c)}\mathcal{E}_1^{(c)} \cong \mathcal{E}_2^{(a)}\mathcal{E}_1^{(a+c)}\mathcal{E}_2^{(c)}.$$

These are the self-dual indecomposable 1-morphisms, and they correspond precisely to Lusztig's canonical basis for $\mathbf{U}_{\mathcal{A}}$.

The decomposition of non-basis elements given in (3.5) is categorified by finding an idempotent decomposition with indecomposable summands.

Theorem 3.2.8 ([12, Theorem 3])**.** *For $i \neq j$, $b \leq a + c$ there is a decomposition*

$$\mathcal{E}_i^{(a)} \mathcal{E}_j^{(b)} \mathcal{E}_i^{(c)} = \bigoplus_{\substack{p+r=b \\ p \leq c \\ r \leq a}} \bigoplus_{\alpha \in P(c-p, a-r)} E_j^{(p)} E_i^{(a+c)} E_j^{(r)} \{2|\alpha| - (c-p)(a-r)\}. \tag{3.18}$$

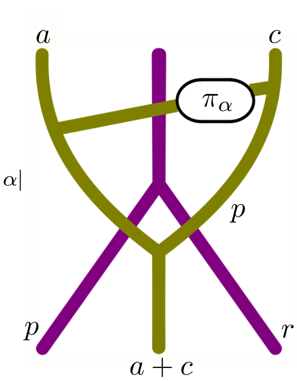
Remark 3.2.9. As in 2.4.4, the right hand side above can be written as

$$\bigoplus_{\substack{p+r=b \\ p \leq c \\ r \leq a}} \begin{bmatrix} a+c-b \\ c-p \end{bmatrix} E_j^{(p)} E_i^{(a+c)} E_j^{(r)},$$

using the equality $\begin{bmatrix} a+c-b \\ c-p \end{bmatrix} = \sum_{\alpha \in P(c-p, a-r)} q^{2|\alpha| - (c-p)(a-r)}$.

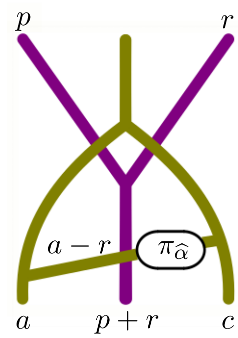
Stošić proves this theorem by explicitly giving projection and inclusion 2-morphisms which decompose the left hand side. The projection and inclusion maps are defined as follows. For every integer p with $\max(0, b-a) \leq p \leq \min(b, c)$, and partition $\alpha \in P(c-p, a-r)$, the inclusion morphisms are

$$\lambda_\alpha^p := (-1)^{r(a+c-r)+|\alpha|}$$



, (3.19)

while the projection morphisms are

$$\sigma_{\alpha}^{p} := $$  $$. \tag{3.20}$$

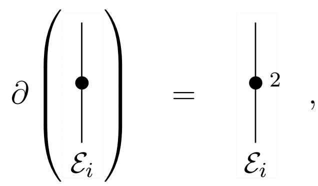
In the next section we place differentials on $\dot{\mathcal{U}}^+$ which extend $\partial_{\pm 1}$. Our main result is to show that this collection of idempotents, together with a partial order, gives an Fc-filtration with respect to one of these differentials. This is enough to decompose any 1-morphism in $\dot{\mathcal{U}}^+$ and so $\dot{\mathcal{U}}^+$ is Fc-filtered. It is also true that $\dot{\mathcal{U}}^+$ is Karoubian and mixed. Hence the p-DG Grothendieck group of $\dot{\mathcal{U}}_3^+$ is isomorphic to $K_0\left(\dot{\mathcal{U}}_3^+\right) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{O}_p$. This gives a categorification of the positive half of quantum $\mathfrak{sl}_3$ at a prime root of unity.

A p-DG structure on $\dot{\mathcal{U}}_{\mathfrak{sl}_3}^+$

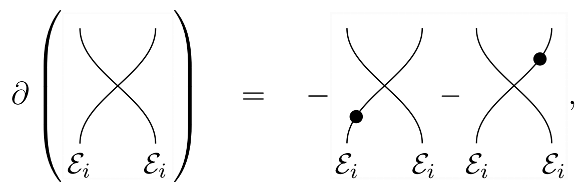
In [8, Lemma 4.3] a family of differentials is placed on KLR algebras giving p-DG structures. For A_2, the Serre relation (3.1) is lifted to $\mathcal{U}^+$ for two of these differentials (see [8, Proposition 4.11]). When restricted to diagrams which only involve strands of a single color, these two differentials act as $\partial_{\pm 1}$ from Section 2.8. We will extend one of their differentials to $\dot{\mathcal{U}}^+$ giving a p-DG structure.

Remark 3.3.1. In what follows we focus on ∂_{-1}, which we will simply denote ∂. With this differential, and a partial order, the idempotent decomposition in Theorem 3.2.8 is an Fc-filtration. An idempotent decomposition that is an Fc-filtration with respect to the differential ∂_1 and where the differential agrees with the induced differential on the partial idempotent completion with respect to the usual idempotents can be obtained by composing the idempotents from [12] with $\tilde{\psi}$. This is the same thing as reflecting the idempotents along a horizontal axis.

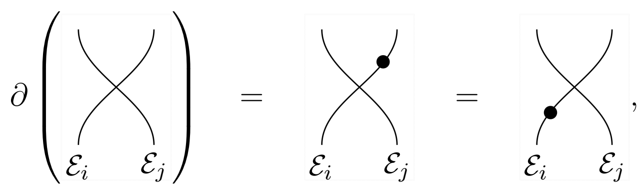
We define ∂ on generators as follows:



(3.21)

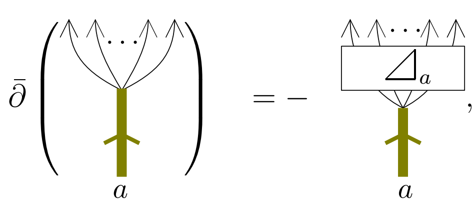


(3.22)

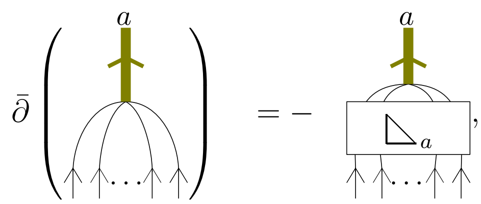


for $i \neq j$. (3.23)

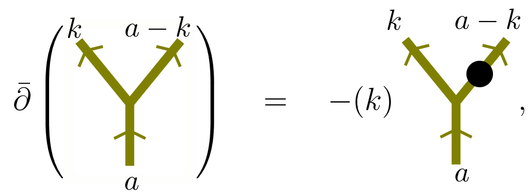
When considering the partial idempotent completion, $\dot{\mathcal{U}}^+$, the idempotents $\{(\mathcal{E}_1^a, \tilde{\psi}(e_a)), (\mathcal{E}_2^a, \tilde{\psi}(e_a))\}$ are all subquotient idempotents and so we get a differential on $\dot{\mathcal{U}}^+$. For single color thick strands and splitters, we can use the identity $\bar{\partial} = \tilde{\tau}\bar{\partial}_1\tilde{\tau}$ and the the formulas in (2.99)-(2.106) to derive the following formulas:



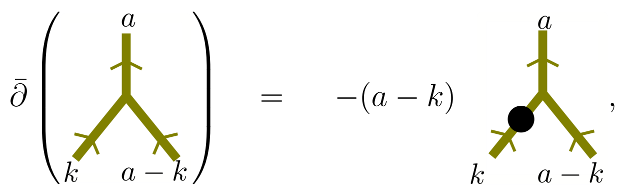
(3.24)



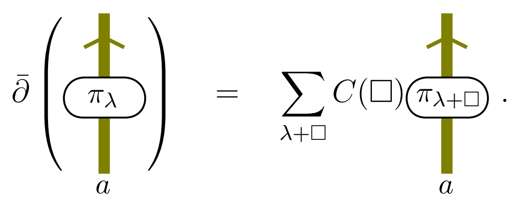
(3.25)



(3.26)

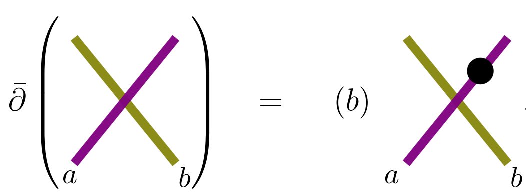


(3.27)



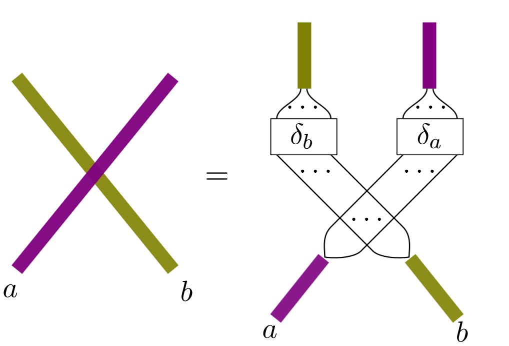
(3.28)

It is also a straightforward computation to show that $\partial(\delta_n) = \triangle_n \delta_n$. On thick 2 color crossings, the differential acts as



(3.29)

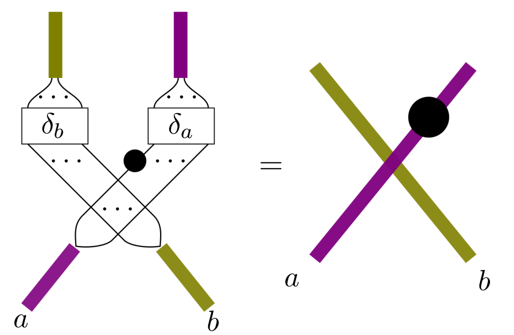
Proof. To prove (3.29), we express the thick crossing by



(3.30)

The Leibniz rule means that ∂ acts via summation over all the ways to apply the differential to various different splitters, crossings, etc.

On the bottommost splitter of each color, the differential acts by multiplication of δ_b (resp. δ_a) by $\triangle_b$ (resp. $\triangle_a$). Each summand has a pair of strands which are symmetric in 2 variables and so is zero. For each color, after applying the differential to the top splitter and box we get the same term with opposite signs. Applying the differential to the 2-color crossings, the only non-zero terms are



(3.31)

and there are precisely b such terms. $\square$

We now proceed to our main result.

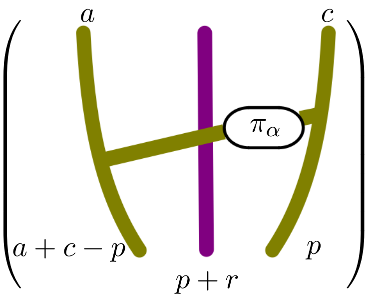
Proposition 3.3.2. *The direct sum decomposition*

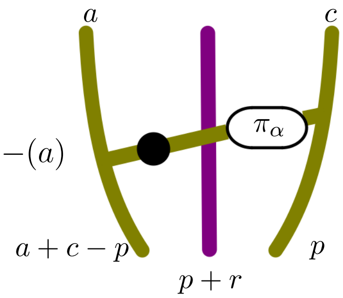
$$\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)} \cong \bigoplus_{p+r=b, p\leq c, r\leq a} \bigoplus_{\alpha\in P(c-p,a-r)} \mathcal{E}_2^{(p)}\mathcal{E}_1^{(a+c)}\mathcal{E}_2^{(r)}\{2|\alpha| - (c-p)(a-r)\}$$

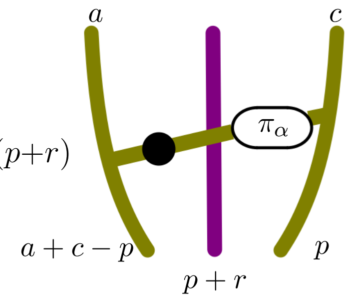
is an Fc-filtration.

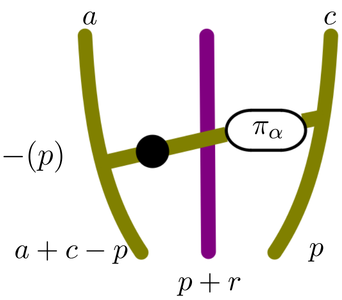
We use the partial order $(\beta, p') \leq (\alpha, p)$ if and only if $p' < p$, or $p' = p$ and $\beta \leq \alpha$. Recall that for two partitions, $\beta \leq \alpha$ means that the Young diagram for α can be obtained from the Young diagram for β by successively adding boxes. We need to check that $\sigma_\beta^{p'}\partial(\lambda_\alpha^p) = 0$ for $(\beta, p') \leq (\alpha, p)$. We start by computing $\partial(\lambda_\alpha^p)$.

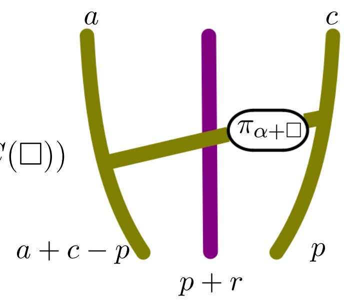
To simplify things we first compute the differential of the top half of λ^p_α.

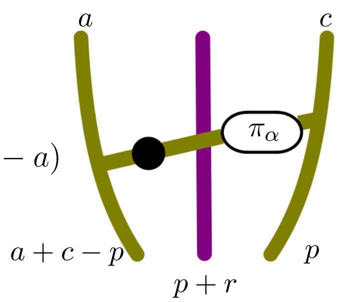
∂ 

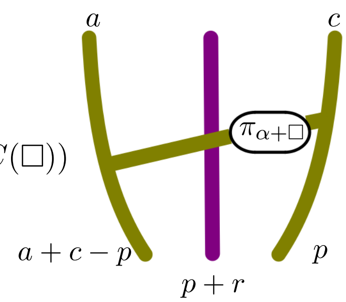
$= -(a)$ 

$+(p+r)$ 

$-(p)$ 

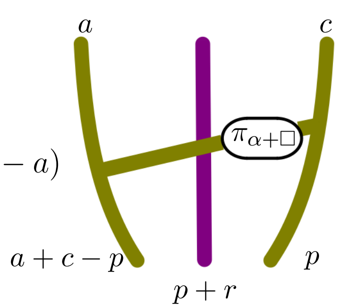
$+ \sum_{\alpha+\square} (C(\square))$ 

$= (r-a)$ 

$+ \sum_{\alpha+\square} (C(\square))$ 

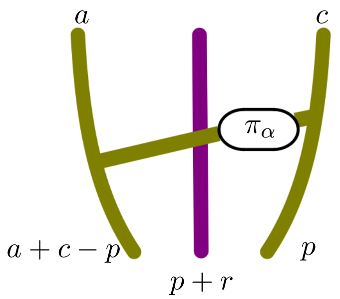
. (3.32)

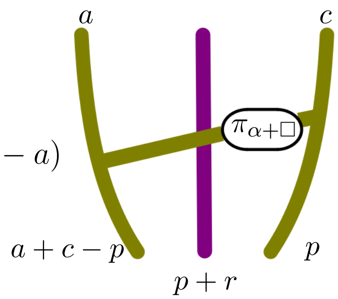
Using the Pieri rule, the first term can be written as

$\sum_{\alpha+\square} (r-a)$ 

. (3.33)

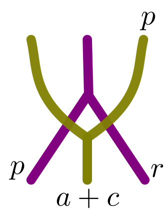
So we have

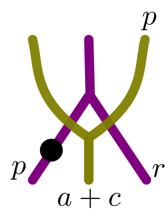
$\partial \Bigg($ 

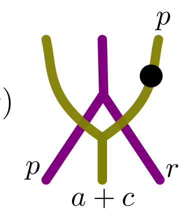
$\Bigg) = \sum_{\alpha+\square} (C(\square) + r - a)$ 

. (3.34)

Let $k = a + c - p - r$. Computing the differential of the bottom half yields

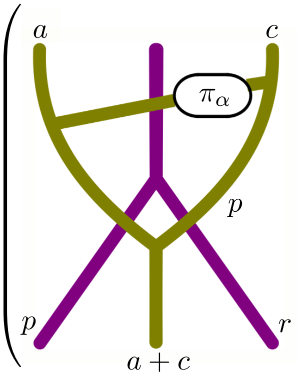
$\partial \Bigg($ 

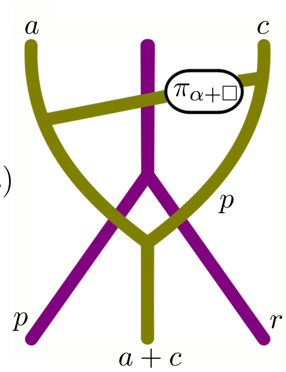
$\Bigg) = (k)$ 

$- (k)$ 

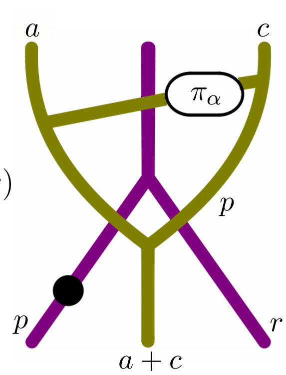
. (3.35)

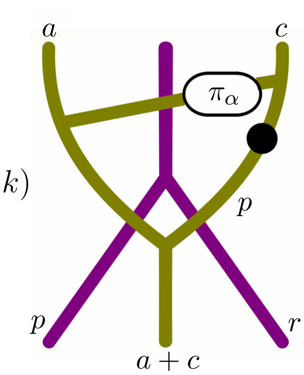
Putting these together, we have computed $\partial(\lambda_\alpha^p)$.

$\partial \Bigg($ 

$\Bigg) = \sum_{\alpha+\square} (C(\square) + r - a)$ 

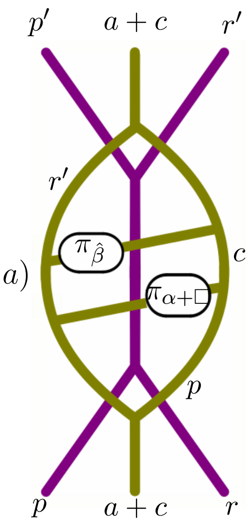
(3.36)

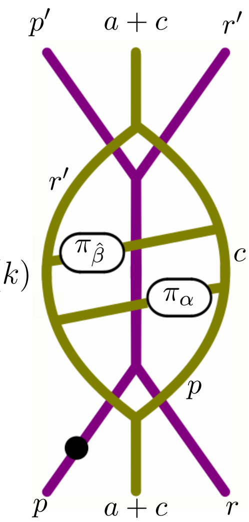
$+(k)$ 

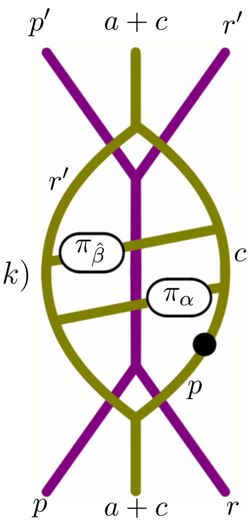
$- (k)$ 

(3.37)

When we compose $\sigma_{\beta}^{p'}\partial(\lambda_{\alpha}^{p})$, we get three terms.

$\sum_{\alpha+\square}(C(\square)+r-a)$ 

$+(k)$ 

$-(k)$ 

(3.38)

We examine the summands of the first term by considering 3 cases exhausting the possibilities of what $\alpha + \square$ can be.

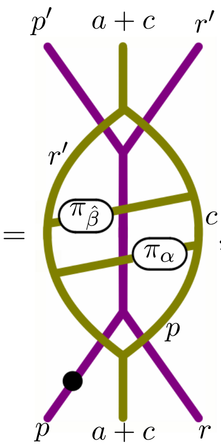
1. If the partition $\alpha + \square$ is in $P(c-p, a-r)$, then we can apply [12, Lemma 4] which implies the summand will be zero for $\beta \leq \alpha$.

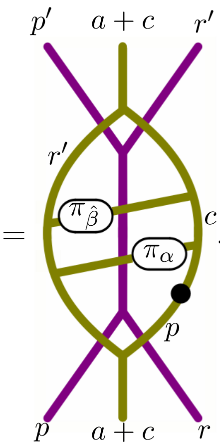
2. If the partition $\alpha + \square$ is not in $P(c-p, a-r)$ and the new box was added to the first row of α, this means the content of the added box was precisely $a - r$ and the coefficient, $C(\square) + r - a$, of that summand is 0.

3. If the partition $\alpha + \square$ is not in $P(c-p, a-r)$ and the new box was not added to the first row of α, then $\alpha + \square$ must have $c - p + 1$ rows and again the summand is 0 (Section 2.8).

Thus, when $\beta \leq \alpha$, the first term of (3.38) vanishes.

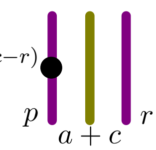
Denote the second and third terms in (3.38) without the coefficients $k, -k$ as Picture 2 and Picture 3 respectively. That is,

Picture 2 = 

, Picture 3 = 

.

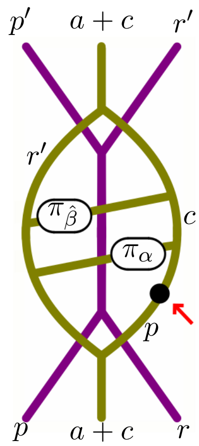
It follows from [12, Theorem 5] that Picture 2 is 0 unless $\beta = \alpha$ and $p = p'$ in which case

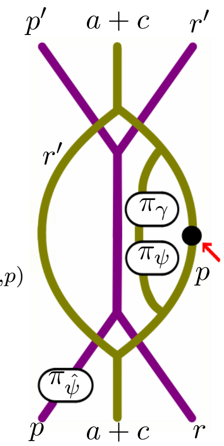
Picture 2 $= \delta_{p,p'}\delta_{\alpha,\beta}(-1)^{|\alpha|+r(a+c-r)}$ 

. (3.39)

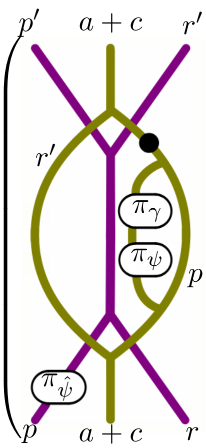
It remains to show that the same is true for Picture 3 since the coefficients show up with opposite signs. This would prove that $\sigma_\beta^{p'}\partial(\lambda_\alpha^p) = 0$ for $(\beta, p') \leq (\alpha, p)$. We resolve Picture 3 following the computation in the proof of Lemma 4 in [12].

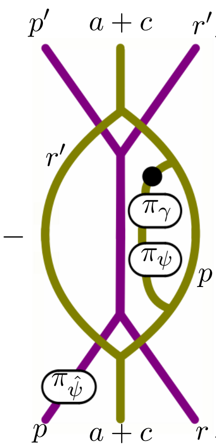
We start by following [12, Lemma 4] verbatim, though with an extra dot which we have highlighted for emphasis, up until before the last equality in the middle of page 269. Now apply (2.67) to send the extra dot upwards through the splitter, resulting in 2 terms. We then apply the Pieri rule to the second term before applying [12, Lemma 3] to both terms. Note that we have drawn our diagrams slightly differently than those in [12]. It is easily seen by the pitchfork lemma and dot slide rule that the two ways of drawing the diagrams are equivalent.



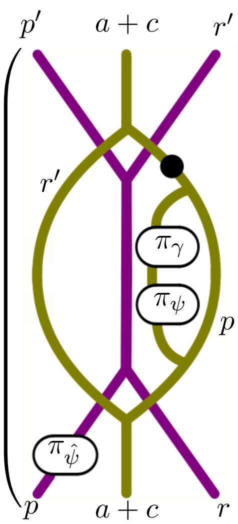
$= \sum_{\psi\in P(a-r'+c-p,p)}$


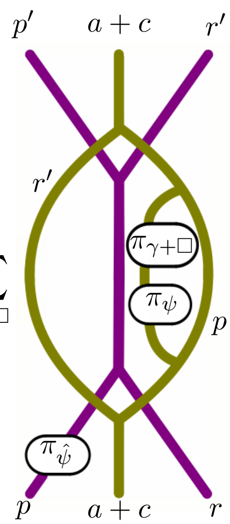
,

$= \sum_{\psi\in P(a-r'+c-p,p)} \Bigg($


$-$


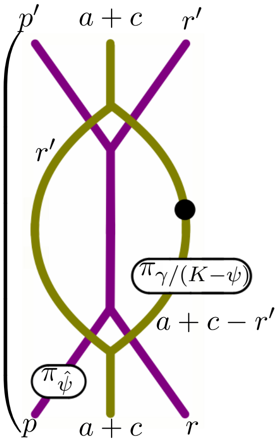
$\Bigg)$,

$\sum_{\psi\in P(a-r'+c-p,p)} \Bigg($


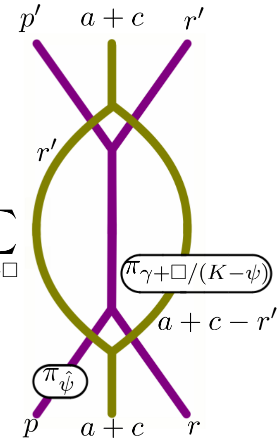
$- \sum_{\gamma+\square}$


$\Bigg)$, (3.40)

$$= \sum_{\psi\in P(a-r'+c-p,p)} \left(\vphantom{\sum} \right.$$



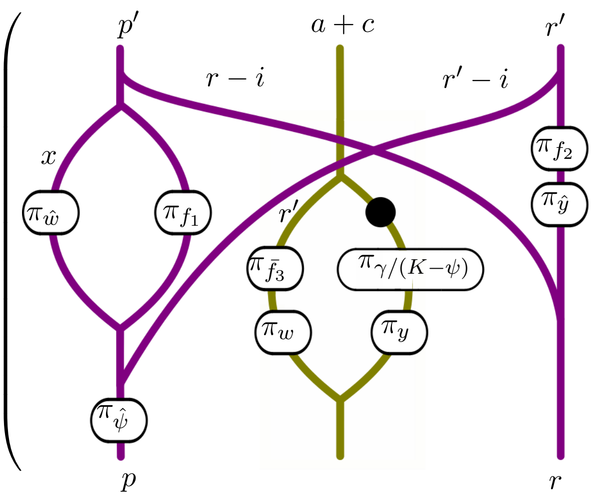
$$- \sum_{\gamma+\Box}$$



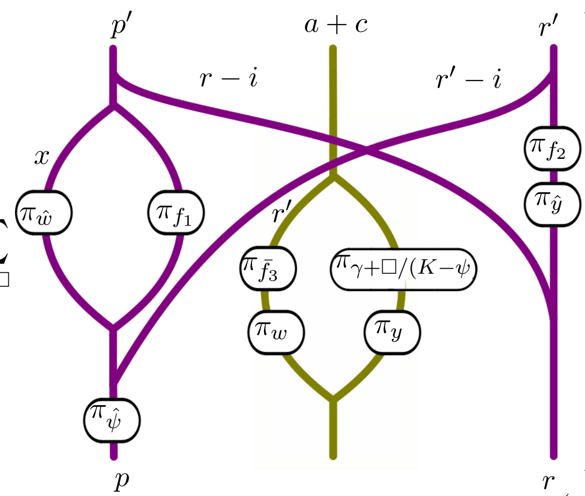
$$\left. \vphantom{\sum} \right) . \tag{3.41}$$

Note that in the last equality in the middle of page 269 in [12], after the change of variables $\phi = \bar{\hat{\psi}}$, the sum was restricted to $\phi \subset \gamma$ since the remaining terms are zero. This is not true for us because of the extra box. Instead we continue to sum over $\phi \in P(a - r' + c - p, p)$. The arguments given in [12] (from the bottom of page 269) resolve diagrams similar to (3.41) though with different labels on the Schur polynomials. His arguments are local away from the polynomials and do not rely on the restricted sum. Thus Picture 3 is equal to (3.42).

$$\sum_{\phi} \sum_{w\in P(r',x)} \sum_{i=0}^{r} \sum_{f_1,f_2,f_3\in P(i,r'-i)} \sum_{y\in P(a+c-r',i)} c^{K_i}_{f_1 f_2 f_3} \left(\vphantom{\sum} \right.$$



$$- \sum_{\gamma+\Box}$$



$$\left. \vphantom{\sum} \right) \tag{3.42}$$

To analyze the digon in the middle of both terms in (3.42) we will need the following lemma.

Lemma 3.3.3. *For partitions $\gamma \in P(a,b)$, $\phi \in P(a)$*

$$\sum_{\nu\in P(a,b)}\sum_{\nu+\square} c^{\gamma}_{\phi,\nu}\,\underset{a}{\boxed{\pi_{\nu+\square}}} - \sum_{\nu'\in P(a,b+1)}\sum_{\gamma+\square} c^{\gamma+\square}_{\phi,\nu'}\,\underset{a}{\boxed{\pi_{\nu'}}} \tag{3.43}$$

$$= \sum_{\nu\in P(a,b)}\sum_{\nu+\square\in P(a,b)} c^{\gamma}_{\phi,\nu}\,\underset{a}{\boxed{\pi_{\nu+\square}}} - \sum_{\nu'\in P(a,b)}\sum_{\gamma+\square} c^{\gamma+\square}_{\phi,\nu'}\,\underset{a}{\boxed{\pi_{\nu'}}} \tag{3.44}$$

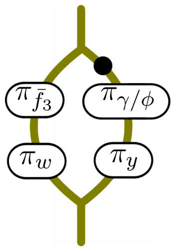
The difference between (3.43) and (3.44) is the restricted sum in (3.44). The lemma is true if and only if (3.45) holds, which we prove below.

$$\sum_{\nu\in P(a,b)}\sum_{(\nu+\square)_1=b+1} c^{\gamma}_{\phi,\nu}\,\underset{a}{\boxed{\pi_{\nu+\square}}} = \sum_{\substack{\nu'\in P(a,b)\\ \nu'_1=b+1}} c^{\gamma+\square}_{\phi,\nu'}\,\underset{a}{\boxed{\pi_{\nu'}}} \quad . \tag{3.45}$$

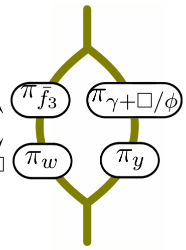
Proof. If $\nu'_1 = b+1$ then $c^{\gamma+\square}_{\phi,\nu'}$ can be non-zero only if $(\gamma+\square)_1 = b+1$ and since $\gamma \in P(a,b)$, $\gamma_1 = b$ and the box was added to γ in the first row. Similarly $c^{\gamma+\square}_{\phi,\nu'}$ can only be non-zero if $\nu'_2 \leq b$. Let $\nu = (b, \nu_2 \ldots, \nu_a)$. The Littlewood-Richardson coefficients $c^{\gamma+\square}_{\phi,\nu'}, c^{\gamma}_{\phi,\nu}$ are equal since the skew tableaux are the same, $(\gamma+\square)\setminus\nu' = \gamma\setminus\nu$.

Similarly, if $(\nu+\square)_1 = b+1$ then $c^{\gamma}_{\phi,\nu}$ can be non-zero only if $\gamma_1 = b$ but then letting $\nu' = (b+1, \nu_2, \ldots \nu_a)$ and $\gamma + \square = (b+1, \gamma_2, \ldots, \gamma_a)$ the corresponding Littlewood-Richardson coefficients again agree. $\square$

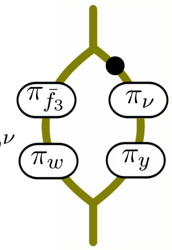
Now we analyze the bigons in the middle part of (3.42). By definition of $\pi_{\gamma/\phi}$,



$$-\sum_{\gamma+\Box}$$

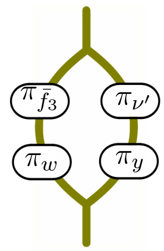


$$= \sum_{\nu\in P(a+c-r',r'-r)} c^{\gamma}_{\phi,\nu}$$



(3.46)

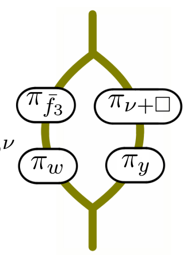
$$-\sum_{\nu'\in P(a+c-r',r'-r+1)} \sum_{\gamma+\Box} c^{\gamma+\Box}_{\phi,\nu'}$$



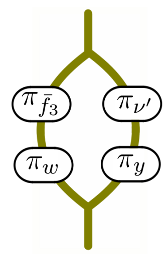
.

Next we apply Lemma 3.3.3 to (3.46). This restricts the size of the partitions we need to sum over and gives the following.

$$\sum_{\nu,\nu+\Box\in P(a+c-r',r'-r)} c^{\gamma}_{\phi,\nu}$$



$$-\sum_{\nu'\in P(a+c-r',r'-r)} \sum_{\gamma+\Box} c^{\gamma+\Box}_{\phi,\nu'}$$



(3.47)

Since $\nu', \nu+\Box \in P(a+c-r', r-r')$, we can make the same argument as in [12] with the only change being that his equation (34) becomes

$$|w| + |f_3| + |y| + |\gamma| + 1 - |\phi| = r'(a+c-r'), \tag{3.48}$$

since our partitions ν have exactly one more box that his. This results in the inequality

$$|\phi| + (a+c-i-p)(r-i) + (r'-i)(r'-r) \le 1. \tag{3.49}$$

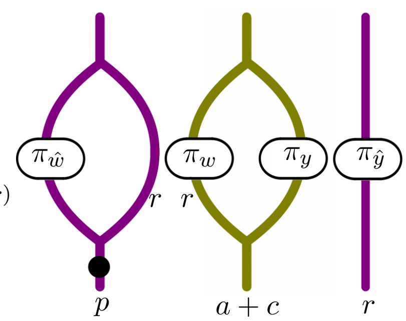
Each parenthetical factor on the left-hand side is non-negative so either $|\phi| = 0$ or $|\phi| = 1$.

If $|\phi| = 0$, then the coefficients $c^{\gamma}_{\phi,\nu}, c^{\gamma+\Box}_{\phi,\nu'}$ in (3.47) are nonzero only when $\nu = \gamma$, $\nu' = \gamma + \Box$, in which case the coefficients are 1 and the terms cancel in (3.47).

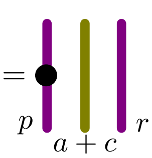
If $|\phi| = 1$, then (3.49) implies that both $(a + c - i - p)(r - i) = 0$ and $(r' - i)(r' - r) = 0$. In addition, since $r' \geq r \geq i$ and $a + c \geq p + r \geq p + i$, it follows that $r' = r = i$. Since $\gamma \in P(a - r' + c - p, r' - r)$, we have that $|\gamma| = 0$. Similarly $|f_i| = 0$. That means that the Littlewood-Richardson coefficients in the first term of (3.47), $c^{\gamma}_{\phi,\nu}$, are all zero. The coefficients in the second term are

$$c^{\gamma+\Box}_{\phi,\nu'} = \begin{cases} 1 & |\nu'| = 0 \\ 0 & \text{otherwise} \end{cases} . \tag{3.50}$$

We have reduced (3.47) to the single term with $|\nu'| = 0, |\gamma| = 0, |\phi| = \Box$. Thus, just as in [12, Lemma 4], (3.42) reduces to

$$\delta_{r,r'}\delta_{\alpha,\alpha'}(-1)^{|\alpha|} \sum_{w\in P(r,x)} \sum_{y\in P(a+c-r,r)}$$  . (3.51)

Note the extra dot. We can follow the final part of the proof of [12, Lemma 4] verbatim to obtain

Picture 3 =  (3.52)

as desired.

Theorem 3.3.4. *The p-DG Grothendieck of the derived category $\mathcal{D}^c(\dot{\mathcal{U}}^+, \partial)$ is isomorphic to $\mathbf{U}_p^+$. In $K_0(\dot{\mathcal{U}}^+, \partial)$ the symbols for the representable in $\{\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)},\ \mathcal{E}_2^{(a)}\mathcal{E}_1^{(b)}\mathcal{E}_2^{(c)} \ \mid \ b \geq a + c, \quad a, b, c, \in \mathbb{N}\}$ are identified with Lusztig's canonical basis.*

Proof. In Proposition 3.3.2 we showed that every 1-morphism in $\mathcal{U}^+$ has an Fc-filtration. It is also true that $\mathcal{U}^+$ is mixed and Karoubian. By Proposition 2.7.9, $K_0(\mathcal{U}^+, \partial) \ \cong \ K_0(\mathcal{U}^+) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{O}_p$. In [12, Theorem 4], it is shown that the indecomposable 1-morphisms are precisely those given in the theorem. $\square$

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