MACKEY FUNCTORS OVER THE GROUP $\mathbb{Z}/2$ AND COMPUTATIONS IN HOMOLOGICAL ALGEBRA

by

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DISSERTATION ABSTRACT

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Mackey functors over the group $\mathbb{Z}/2$ are useful in the study of $\mathbb{Z}/2$-equivariant cohomology. In this dissertation we establish results which are useful for homological algebra computations for certain Mackey rings over $\mathbb{Z}/2$. We also provide some Ext computations for Mackey modules over Mackey rings. Additionally, we study the bigraded ring $\mathbb{M}_2$ (which is the Bredon cohomology of a point) and its Mackey ring analog. This includes a computation of $\operatorname{Ext}(k, k)$ over $\mathbb{M}_2$ and a computation of $\operatorname{Ext}(M, k)$ for certain $\mathbb{M}_2$-modules M as well as a proof that the Mackey ring analog is self-injective as a bigraded Mackey ring.

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In loving memory of Betty Raies.

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CHAPTER I

INTRODUCTION

In this dissertation we examine Mackey functors. We evolve the theory behind them and look at properties of a particular Mackey functor that is prevalent in equivariant topology.

Classical algebraic topology uses abelian groups (rings, and modules) to measure topological spaces through homotopy, cohomology theories, and other constructions. In equivariant algebraic topology the role of abelian groups is played by objects called Mackey functors. For a finite group G the category of G-Mackey functors has a symmetric monoidal product (called a box product) and so one can construct ring objects and module objects in the usual way. This dissertation concerns some homological computations for certain Mackey rings and modules for the group $\mathbb{Z}/2$.

1.1 Background on Equivariant Topology

Consider the category of topological spaces **Top** and the category of G-equivariant topological spaces $G - \mathbf{Top}$ for a finite group G. Equivariant cohomology provides a Bredon contravariant functor

$$H^i_G : G - \mathbf{Top} \to \mathbf{Ab} \tag{1.1.1}$$

for integers $i > 0$. One also finds that the cup product induces a functor

$$H^*_G : G - \mathbf{Top} \to \mathbf{Rings}. \tag{1.1.2}$$

Definitions and a rigorous treatment of these topics is given by May in [May96].

Consider the group $G = \mathbb{Z}/2$. The transitive G-sets are given by cosets G/H for $H \leq G$. In this case, there is the two-point set $G \cong G/\langle 0 \rangle$ and the singleton set $\langle 0 \rangle \cong G/G$. The non-trivial element of G swaps the elements of G and, of course, acts trivially on the element of $\langle 0 \rangle$. There is a map of G-sets $t : G \to G$ which swaps the elements and the quotient map of G-sets $p : G \to \langle 0 \rangle$. This yields the following diagram of G-sets:

$$\begin{tikzcd} G \arrow[loop left, "t"] \arrow[r, bend right, "p"'] & \langle 0 \rangle. \end{tikzcd}$$

The theory of G-sets tells us that there are no other non-identity maps of G-sets in the previous diagram.

Consider an object $X \in G - \mathbf{Top}$. By considering G and G/G as elements of $G - \mathbf{Top}$ with the discrete topology we can apply the functor $X \times (-)$ to the above diagram to get the following:

$$\begin{tikzcd} X \times G \arrow[loop left, "\mathrm{id} \times t"] \arrow[r, bend right, "\mathrm{id} \times p"'] & X \times \langle 0 \rangle. \end{tikzcd}$$

We can then apply the Bredon cohomology functor H^*_G which yields the following diagram of graded rings:

$$\begin{tikzcd} H^*_G(X \times G) \arrow[loop left, "(\mathrm{id} \times t)^*"] & H^*_G(X \times \langle 0 \rangle). \arrow[l, bend left, "(\mathrm{id} \times p)^*"] \end{tikzcd}$$

Furthermore, the map $\mathrm{id} \times p : X \times_G G \to X \times_G \langle 0 \rangle$ is a two-sheeted cover. It follows that there is a transfer map $(\mathrm{id} \times p)_* : H^*_G(X \times_G G) \to H^*_G(X \times_G \langle 0 \rangle)$ (see [Hat02]). We can add this to the previous diagram to get

$$\begin{tikzcd} H^*_G(X \times G) \arrow[loop left, "(\mathrm{id} \times t)^*"] \arrow[rr, bend left, "(\mathrm{id} \times p)_*"] & & H^*_G(X \times \langle 0 \rangle). \arrow[ll, bend left, "(\mathrm{id} \times p)^*"] \end{tikzcd}$$

Note that $(\mathrm{id} \times p)_*$ is a map of graded abelian groups but it is not a map of rings. Given a space X this previous diagram (of graded abelian groups) will be denoted $\mathcal{H}^*_G(X)$. Based on the properties of the transfer map, one can check that this diagram satisfies

- $(\mathrm{id} \times p)^* \circ (\mathrm{id} \times p)_* = \mathrm{id} + (\mathrm{id} \times t)^*$,
- $(\mathrm{id} \times p)_* \circ (\mathrm{id} \times t)^* = (\mathrm{id} \times p)_*$,
- $(\mathrm{id} \times t)^* \circ (\mathrm{id} \times p)^* = (\mathrm{id} \times p)^*$, and
- $(\mathrm{id} \times t)^* \circ (\mathrm{id} \times t)^* = \mathrm{id}$.

Diagrams of graded abelian groups of this form which satisfy those four axioms will be called *Mackey functors* over the group $G = \mathbb{Z}/2$ (see section 2.1). These form the objects of a category $\underline{\mathrm{Mack}}$ and the morphisms are natural transformations between the diagrams. Since all of the constructions above were functorial we have functors

$$\mathcal{H}^*_G : G - \mathbf{Top} \to \underline{\mathrm{Mack}}. \tag{1.1.3}$$

There is also a forgetful functor $U : \underline{\mathrm{Mack}} \to \mathbf{Ab}$ which maps a Mackey functor $\mathcal{M}$ to the rightmost group in the diagram. (In the notation of the rest of the paper, $U(\mathcal{M}) = M_\bullet$.) It is clear from the construction of $\mathcal{H}^*_G$ that the following diagram commutes:

$$\begin{tikzcd} & \underline{\mathrm{Mack}} \arrow[d, "U"] \\ G-\mathbf{Top} \arrow[ur, "\mathcal{H}^*_G"] \arrow[r, "H^*_G"'] & \mathbf{Ab} \end{tikzcd}$$

In this way, the cohomology with Mackey functor coefficients $\mathcal{H}^*_G(X)$ provides additional structure that one can study. Furthermore, it turns out that $\underline{\mathrm{Mack}}$ is a symmetric monoidal category (see section 2.2) and that $\mathcal{H}^*_G(X)$ is a ring object in $\underline{\mathrm{Mack}}$. We can then replace **Ab** and $\underline{\mathrm{Mack}}$ with **Rings** and the category of Mackey rings in the previous diagram.

Let $P = \{*\} \in \mathbb{Z}/2 - \mathbf{Top}$ be a single point. The cohomologies of this space are the ring $\mathbb{M}_2 = H^*_{\mathbb{Z}/2}(P)$ and the Mackey ring $\mathcal{M}_2 = \mathcal{H}^*_{\mathbb{Z}/2}(P)$. These rings are stated explicitly and explored in Chapter IV. In particular, we will prove a structure theorem for $\mathrm{Ext}^*_{\mathbb{M}_2}(M, \mathbb{Z}/2)$ (for certain $\mathbb{M}_2$-modules, M) and we will prove that $\mathcal{M}_2$ is self-injective.

If G is an arbitrary finite group then there is still a Mackey ring valued cohomology functor $\mathcal{H}^i_G$. Mackey functors over other groups form different diagrams but the construction is similar.

1.2 Background on Mackey functors

Full treatments of Mackey functors can be found in [May96] and [Web00]. In this paper we are only concerned with the case where $G = \mathbb{Z}/2$ and we give a rigorous treatment of that case in Chapter II. In this section, however, we will give an overview of the general case. There are several different equivalent definitions of Mackey functors; we provide just one of them here.

Definition 1.2.1. Let G be a finite group. A *Mackey functor* over G is an additive functor $\mathcal{M} : \mathscr{B}G^{\mathrm{op}} \to \mathbf{Ab}$ where $\mathscr{B}G$ is the category defined in Definition 1.2.2. The category $G - \underline{\text{Mack}}$ is the resulting functor category whose morphisms are natural transformations. (When we write $\underline{\text{Mack}}$ in future sections we mean $\mathbb{Z}/2 - \underline{\text{Mack}}$.)

Definition 1.2.2. Let G be a finite group. The *Burnside Category* $\mathscr{B}G$ is defined as follows:

- The objects are finite (left) G-sets.
- The set $\mathrm{Hom}_{\mathscr{B}G}(A, B)$ is the set of equivalence classes of diagrams of (left) G-sets

  $$\begin{tikzcd}[column sep=small] & X \arrow[dl] \arrow[dr] & \\ B & & A. \end{tikzcd}$$

  These diagrams are called *spans* and two spans are equivalent if there exists a commutative diagram of the form

  $$\begin{tikzcd}[column sep=small] & X \arrow[dl] \arrow[dr] \arrow[dd, "\cong"] & \\ B & & A. \\ & X' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

  Composition of spans is obtained from the pullback

  $$\begin{tikzcd}[column sep=small] & Y \arrow[dl] \arrow[dr] & \\ C & & B \end{tikzcd} \circ \begin{tikzcd}[column sep=small] & X \arrow[dl] \arrow[dr] & \\ B & & A \end{tikzcd} = \begin{tikzcd}[column sep=small] & & P \arrow[dl] \arrow[dr] & & \\ & Y \arrow[dl] \arrow[dr] & & X \arrow[dl] \arrow[dr] & \\ C & & B & & A \end{tikzcd}$$

  where $P = Y \times_B X$.

In $G - \underline{\text{Mack}}$ there is an obvious direct sum given by $(\mathcal{M} \oplus \mathcal{N})(X) = \mathcal{M}(X) \oplus \mathcal{N}(X)$. In [Shu10] Shulman describes a box product $- \square - : \underline{\text{Mack}} \times \underline{\text{Mack}} \to \underline{\text{Mack}}$. This acts as the tensor product under which $\underline{\text{Mack}}$ is a symmetric monoidal category. (In section 2.2 we describe the box product for $G = \mathbb{Z}/2$; see Theorem 2.1.14 for a description of the unit and the maps which define that structure.) As a result we can form a category of Mackey rings and a category of $\mathcal{R}$-modules for any Mackey ring $\mathcal{R}$ in the usual way.

Note that a Mackey functor $\mathcal{M} : \mathscr{B}G^{\mathrm{op}} \to \mathbf{Ab}$ is determined by its values on the sets G/H for subgroups $H \leq G$ because every G-set is isomorphic to a coproduct of those sets. This allows us to represent Mackey functors as finite diagrams of abelian groups where the shape of the diagram depends on G. In the case where $G = \mathbb{Z}/2$ there are only two subgroups, namely $\langle 0 \rangle$ and $\mathbb{Z}/2$, and so the resulting diagram only has two objects. In [Shu10], Shulman explains how in the case when $G = \mathbb{Z}/2$, Mackey functors can be represented by diagrams of the form

$$\begin{tikzcd} M_\theta \arrow[loop left, "t"] \arrow[r, bend left, "p_*"] & M_\bullet \arrow[l, bend left, "p^*"] \end{tikzcd}$$

(subject to some axioms). In section 2.1 we adopt these diagrams as our definition of Mackey functors. One finds that a Mackey ring is a diagram where both objects are rings (along with some conditions on the maps) and that if $\mathcal{R}$ is a Mackey ring then an $\mathcal{R}$-module is a diagram where the objects are modules over the corresponding rings (again with some conditions on the maps). This is discussed further in section 2.2.

1.3 Summary of Results

We conclude this chapter with a brief summary of the new results found in the rest of the dissertation.

Free Mackey Modules

In **Ab** free modules are all isomorphic to $\bigoplus_{i\in I}\mathbb{Z}$ for some set I. Let $\mathcal{A}$ be the unit in the symmetric monoidal structure on $\underline{\text{Mack}}$. Mackey functors $\bigoplus_{i\in I}\mathcal{A}$ are all free in $\underline{\text{Mack}}$ but there are not enough of those modules (in the categorical sense). There is another Mackey functor $\mathcal{F}_\theta(\mathbb{Z})$ which is free but which is not isomorphic to $\bigoplus_{i\in I}\mathcal{A}$ for some set I. Mackey functors of the form

$$\left(\bigoplus_{i\in I}\mathcal{A}\right)\oplus\left(\bigoplus_{j\in J}\mathcal{F}_\theta(\mathbb{Z})\right) \tag{1.3.1}$$

are all free and, moreover, there are enough free Mackey functors of that form.

We investigate the free modules of $\mathcal{R}-\underline{\text{Mod}}$. Similarly, the regular module $\mathcal{R}$ is free and there is a second module $\mathcal{F}_\theta(\mathbb{Z})\,\square\,\mathcal{R}$ such that there are enough free modules of the form

$$\left(\bigoplus_{i\in I}\mathcal{R}\right)\oplus\left(\bigoplus_{j\in J}\mathcal{F}_\theta(\mathbb{Z})\,\square\,\mathcal{R}\right). \tag{1.3.2}$$

We provide a working model for $\mathcal{F}_\theta(\mathbb{Z})\,\square\,\mathcal{R}$ and explore some of its properties.

Injective Mackey Modules

A module Q is defined to be injective if for any map $f : X \to Q$ and any injection $\iota : X \to Y$ there is an extension $\bar{f} : Y \to Q$ such that $\bar{f}\iota = f$. Baer's Criterion (see [Wei94]) states that in the case of traditional modules over a ring R it is enough to check the cases where $Y = R$ and X is an ideal of R.

We develop an analog to Baer's Criterion for Mackey modules over a Mackey ring $\mathcal{R}$. It turns out that one needs to check both the case when $Y = \mathcal{R}$ as well as the case when Y is the free module $\mathcal{F}_\theta(\mathbb{Z})\,\square\,\mathcal{R}$ discussed above.

Homological Algebra Calculations

Suppose that $\mathcal{C}$ is a tensor category. If R is a ring object and M is an R-bimodule one can form the square-zero extension of R by M which is a ring object that we denote $\mathcal{E}_R(M)$. As an example, if $\mathcal{C} = \mathbf{Ab}$ and R is a ring then $\mathcal{E}_R(R) \cong R[x]/\langle x^2 \rangle$. A standard computation in homological algebra shows that

$$\operatorname{Ext}^*_{\mathcal{E}_R(R)}(R, R) \cong R[x]. \tag{1.3.3}$$

In the case where $\mathcal{C} = \underline{\text{Mack}}$ and $\mathcal{R}$ is a Mackey ring we compute the ring

$$\operatorname{Ext}^*_{\mathcal{E}_\mathcal{R}(\mathcal{M})}(\mathcal{R}, \mathcal{R}) \tag{1.3.4}$$

when $\mathcal{M} = \mathcal{R}$ and when $\mathcal{M} = \mathcal{F}_\theta(\mathbb{Z}) \,\square\, \mathcal{R}$. In particular,

$$\operatorname{Ext}^*_{\mathcal{E}_\mathcal{R}(\mathcal{R})}(\mathcal{R}, \mathcal{R}) \cong R_\bullet[x] \tag{1.3.5}$$

When $\mathcal{M} = \mathcal{F}_\theta(\mathbb{Z}) \,\square\, \mathcal{R}$ the general result is perhaps too complicated to present here. However, if $\mathcal{R}$ is the specific Mackey ring where $R_\bullet = R_\theta = R$ (with $\operatorname{char}(R) = 2$), $p^* = \operatorname{id}$, $p_* = 0$, and $t = \operatorname{id}$ then

$$\operatorname{Ext}^*_{\mathcal{E}_\mathcal{R}(\mathcal{F}_\theta(\mathbb{Z}) \,\square\, \mathcal{R})}(\mathcal{R}, \mathcal{R}) \cong R\langle x_1, x_2, x_3, \ldots \rangle \tag{1.3.6}$$

where the algebra on the right is the graded non-commutative algebra generated by the x_i with $\deg(x_i) = i$.

Homological algebra over the ring $\mathbb{M}_2$

A description of the (traditional) ring $\mathbb{M}_2$ is given in Chapter IV. It is a $k = \mathbb{Z}/2$-algebra and we find that there is a ring homomorphism $k[\tau, \rho] \hookrightarrow \mathbb{M}_2$ which yields a split short exact sequence of $k[\tau, \rho]$-modules,

$$0 \to k[\tau, \rho] \to \mathbb{M}_2 \to J \to 0. \tag{1.3.7}$$

Suppose M is an $\mathbb{M}_2$-module which is finite length as a module over $k[\tau, \rho]$ (i.e. which is finite dimensional over $k = \mathbb{Z}/2$). We prove that

$$\mathrm{Ext}^*_{\mathbb{M}_2}(M, k) \cong \mathrm{Ext}^*_{k[\tau,\rho]}(M, k) \otimes k[\alpha] \tag{1.3.8}$$

where α is in degree 3. We further prove that the action of the ring

$$\mathrm{Ext}^*_{\mathbb{M}_2}(k, k) \cong \mathrm{Ext}^*_{k[\tau,\rho]}(k, k) \otimes k[\alpha] \tag{1.3.9}$$

on the module in Equation 1.3.8 is the obvious natural action.

Furthermore, it is known that the traditional ring $\mathbb{M}_2$ is self-injective (see [May18]). We use this analog to Baer's Criterion to show that the Mackey ring $\mathcal{M}_2$ is also self-injective.

CHAPTER II

MACKEY FUNCTORS, MACKEY RINGS, AND MACKEY MODULES

In this chapter we review some terminology and background results surrounding Mackey functors as well as provide some new results that will be useful in the computations found in later chapters. Most of the results found in this chapter are attributed to Schulman in [Shu10]; we aim to build on her work.

2.1 A review of Mackey functors

A *Mackey functor* over the group G is a functor from the Burnside category $\mathscr{B}G$ into the category of abelian groups as discussed in [Shu10]. In this paper we will be concerned only with the case where $G = \mathbb{Z}/2$. In this case a Mackey functor $\mathcal{M}$ is represented by a diagram in **Ab** of the form

$$\begin{tikzcd} M_\theta \arrow[loop left, "t"] \arrow[r, bend left, "p_*"] & M_\bullet \arrow[l, bend left, "p^*"] \end{tikzcd}$$

which satisfies the following conditions:

- $p^* p_* = 1 + t$
- $p_* t = p_*$
- $t p^* = p^*$
- $t^2 = 1$

The map p^* is called *restriction*, the map p_* is called *transfer*, and the map t is called *twist*. For the remainder of the paper the term *Mackey functor* will refer to a Mackey functor over the group $\mathbb{Z}/2$. A morphism of Mackey functors is a natural transformation of functors or, in our case, maps $f_\bullet$ and f_θ in the diagram below

$$\begin{tikzcd}
M_\theta \arrow[loop left] \arrow[r, bend left] \arrow[d, "f_\theta"] & M_\bullet \arrow[l, bend left] \arrow[d, "f_\bullet"] \\
N_\theta \arrow[loop left] \arrow[r, bend left] & N_\bullet \arrow[l, bend left]
\end{tikzcd}$$

which commute with the restriction, transfer, and twist maps whenever possible. An isomorphism of Mackey functors is a morphism $f : \mathcal{M} \to \mathcal{N}$ where $f_\bullet : M_\bullet \to N_\bullet$ and $f_\theta : M_\theta \to N_\theta$ are both isomorphisms. The category of Mackey functors will be called <u>Mack</u>.

Remark 2.1.1. Whenever possible we will use calligraphic letters to represent Mackey functors and roman letters to represent the corresponding abelian groups. For example, if $\mathcal{M}$ is a Mackey functor then we will use $M_\bullet$ and M_θ for the two abelian groups in the diagram without explicitly specifying. We will try to remain consistent with this convention but it should be clear from context when it is necessary to break it.

Example 2.1.2. There are two Mackey functors which will be important for the remainder of the paper. If $A \in \mathbf{Ab}$ then the Mackey functors $\mathcal{F}_\bullet(A)$ and $\mathcal{F}_\theta(A)$ are shown below:

$$\mathcal{F}_\bullet(A) : \begin{tikzcd}
A \arrow[loop left, "1"] \arrow[r, bend left, "\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]"] & A^2 \arrow[l, bend left, "\left[\begin{smallmatrix} 1 & 2 \end{smallmatrix}\right]"]
\end{tikzcd}
\qquad
\mathcal{F}_\theta(A) : \begin{tikzcd}
A^2 \arrow[loop left, "\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]"] \arrow[r, bend left, "\left[\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right]"] & A \arrow[l, bend left, "\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]"]
\end{tikzcd}$$

For reasons that are made clear in Theorem 2.1.3 we will make the following definitions:

$$\mathbb{I}_\bullet = (1,0) \in (\mathcal{F}_\bullet(\mathbb{Z}))_\bullet \qquad\qquad \mathbb{I}_\theta = (1,0) \in (\mathcal{F}_\theta(\mathbb{Z}))_\theta$$

Theorem 2.1.3. *Let $\mathcal{M}$ be a Mackey functor.*

- *The function $\alpha : \mathrm{Hom}_{\underline{\mathrm{Mack}}}(\mathcal{F}_\bullet(\mathbb{Z}), \mathcal{M}) \to M_\bullet$ given by $\alpha(f) = f_\bullet(\mathbb{I}_\bullet)$ is an isomorphism of Abelian groups.*

- *The function $\beta : \mathrm{Hom}_{\underline{\mathrm{Mack}}}(\mathcal{F}_\theta(\mathbb{Z}), \mathcal{M}) \to M_\theta$ given by $\beta(f) = f_\theta(\mathbb{I}_\theta)$ is an isomorphism of Abelian groups.*

Proof. Consider a map $f : \mathcal{F}_\bullet(\mathbb{Z}) \to \mathcal{M}$ as follows:

$$\begin{tikzcd}
\mathbb{Z} \arrow[loop left, "1"] \arrow[r, bend left, "\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]"] \arrow[d, "f_\theta"'] & \mathbb{Z}^2 \arrow[l, bend left, "\left[\begin{smallmatrix}1 & 2\end{smallmatrix}\right]"] \arrow[d, "f_\bullet"] \\
M_\theta \arrow[loop left, "t"] \arrow[r, bend left, "p_*"] & M_\bullet \arrow[l, bend left, "p^*"]
\end{tikzcd}$$

Since f is a map of Mackey functors, $f_\theta\left[\begin{smallmatrix}1 & 2\end{smallmatrix}\right] = p^* f_\bullet$ and hence

$$f_\theta(1) = f_\theta\left(\left[\begin{smallmatrix}1 & 2\end{smallmatrix}\right]\mathbb{I}_\bullet\right) = p^*(f_\bullet(\mathbb{I}_\bullet)). \tag{2.1.4}$$

Similarly, $f_\bullet\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] = p_* f_\theta$ and hence

$$f_\bullet(0,1) = f_\bullet\left(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]1\right) = p_*(p^*(f_\bullet(\mathbb{I}_\bullet))). \tag{2.1.5}$$

We see that the map f is determined uniquely by $f_\bullet(\mathbb{I}_\bullet)$ and hence α is injective. Since $(\mathcal{F}_\bullet(\mathbb{Z}))_\bullet = \mathbb{Z}^2$ and $(\mathcal{F}_\bullet(\mathbb{Z}))_\theta = \mathbb{Z}$ are free abelian groups we see that any choice of $f_\bullet(\mathbb{I}_\bullet) \in M_\bullet$ induces such a map f and hence α is surjective. The proof that β is a bijection is similar. $\square$

There is a direct sum in Mack which is induced by taking the abelian group sum at each spot in the diagram. Given $\mathcal{M}, \mathcal{N} \in$ Mack there is also a *box product* $\mathcal{M} \square \mathcal{N}$. This is discussed further in [Shu10] where it is shown that for our purposes $\mathcal{M} \square \mathcal{N}$ can be defined as follows.

- $(\mathcal{M} \square \mathcal{N})_\theta = M_\theta \otimes N_\theta$
- $(\mathcal{M} \square \mathcal{N})_\bullet$ is the quotient of $(M_\theta \otimes N_\theta) \oplus (M_\bullet \otimes N_\bullet)$ generated by the following relations:

$$\begin{array}{ll} m_\bullet \otimes p_* n_\theta \sim p^* m_\bullet \otimes n_\theta & \text{for } m_\bullet \in M_\bullet \text{ and } n_\theta \in N_\theta \\ p_* m_\theta \otimes n_\bullet \sim m_\theta \otimes p^* n_\bullet & \text{for } m_\theta \in M_\theta \text{ and } n_\bullet \in N_\bullet \\ t m_\theta \otimes t n_\theta \sim m_\theta \otimes n_\theta & \text{for } m_\theta \in M_\theta \text{ and } n_\theta \in N_\theta. \end{array}$$

- The twist map on $(\mathcal{M} \square \mathcal{N})_\theta$ is induced by the diagonal action, i.e.

$$m_\theta \otimes n_\theta \mapsto t m_\theta \otimes t n_\theta \tag{2.1.6}$$

  for $m_\theta \in M_\theta$ and $n_\theta \in N_\theta$.

- The restriction map $(\mathcal{M} \square \mathcal{N})_\bullet \mapsto (\mathcal{M} \square \mathcal{N})_\theta$ is the map induced by

$$(m_\bullet \otimes n_\bullet) + (m_\theta \otimes n_\theta) \mapsto (p^* m_\bullet \otimes p^* n_\bullet) + (m_\theta \otimes n_\theta) + (t m_\theta \otimes t n_\theta) \tag{2.1.7}$$

  for $m_\bullet \in M_\bullet$, $n_\bullet \in N_\bullet$, $m_\theta \in M_\theta$, and $n_\theta \in N_\theta$.

- The transfer map $(\mathcal{M} \square \mathcal{N})_\theta \mapsto (\mathcal{M} \square \mathcal{N})_\bullet$ is induced by the inclusion

$$M_\theta \otimes N_\theta \to (M_\theta \otimes N_\theta) \oplus (M_\bullet \otimes N_\bullet) \tag{2.1.8}$$

Note that the elements $m_\theta \otimes n_\theta$ in $(\mathcal{M} \square \mathcal{N})_\bullet$ are the image of those same elements from $(\mathcal{M} \square \mathcal{N})_\theta$ under transfer. The element $m_\theta \otimes n_\theta \in (\mathcal{M} \square \mathcal{N})_\bullet$ is equal to $p_*(m_\theta \otimes n_\theta)$ where $m_\theta \otimes n_\theta$ is considered to be an element of $M_\theta \otimes N_\theta = (\mathcal{M} \square \mathcal{N})_\theta$. This is because the transfer map p_* is induced by the inclusion in Equation 2.1.8.

Theorem 2.1.9. *If $\mathcal{M}$ is any Mackey functor we have $\mathcal{M} \square \mathcal{F}_\theta(\mathbb{Z}) \cong \mathcal{F}_\theta(M_\theta)$.*

Proof. We start by computing $\mathcal{M} \square \mathcal{F}_\theta(\mathbb{Z})$. For convenience, call this Mackey functor $\mathcal{Q}$.

First we have

$$Q_\theta = M_\theta \otimes (\mathcal{F}_\theta(\mathbb{Z}))_\theta = M_\theta \otimes \mathbb{Z}^2 \cong M_\theta^2 \tag{2.1.10}$$

and

$$\begin{aligned} Q_\bullet &= [(M_\bullet \otimes (\mathcal{F}_\theta(\mathbb{Z}))_\bullet) \oplus (M_\theta \otimes (\mathcal{F}_\theta(\mathbb{Z}))_\theta)] \, / \sim \\ &= \left[(M_\bullet \otimes \mathbb{Z}) \oplus (M_\theta \otimes \mathbb{Z}^2)\right] / \sim \\ &\cong [M_\bullet \oplus M_\theta \oplus M_\theta] \, / \sim \end{aligned} \tag{2.1.11}$$

where the relations are as follows:

$$\begin{aligned} &(0, 0, p^* m_\bullet) \sim (m_\bullet, 0, 0) \sim (0, p^* m_\bullet, 0) && \text{for } m_\bullet \in M_\bullet \\ &(p_* m_\theta, 0, 0) \sim (0, m_\theta, m_\theta) && \text{for } m_\theta \in M_\theta \\ &(0, m_\theta, 0) \sim (0, 0, t m_\theta) && \text{for } m_\theta \in M_\theta. \end{aligned}$$

One can then compute that the map

$$\phi : M_\theta \to [M_\bullet \oplus M_\theta \oplus M_\theta] \, / \sim \qquad \text{where} \qquad \phi(m_\theta) = [(0, m_\theta, 0)] \tag{2.1.12}$$

is an isomorphism. Hence $Q_\bullet \cong M_\theta$. One can also compute the twist, transfer, and restriction maps on $\mathcal{Q}$ as follows:

- The twist map $M_\theta^2 \to M_\theta^2$ is given by $(p,q) \mapsto (tq, tp)$.

- The transfer map $M_\theta^2 \to M_\theta$ is given by $(p,q) \mapsto p + tq$.

- The restriction map $M_\theta \to M_\theta^2$ is given by $r \mapsto (r, tr)$.

Note that in the maps above, t is the twist map from $\mathcal{M}$. This concludes the computation of $\mathcal{Q} = \mathcal{M} \,\Box\, \mathcal{F}_\theta(\mathbb{Z})$.

Now define $f : \mathcal{Q} \to \mathcal{F}_\theta(M_\theta)$ as follows:

- $f_\theta : M_\theta^2 \to M_\theta^2$ is given by $f_\theta(p,q) = (p, tq)$.

- $f_\bullet : M_\theta \to M_\theta$ is given by $f_\bullet(r) = r$.

Clearly f_θ and $f_\bullet$ are isomorphisms of abelian groups since the twist is an isomorphism. It follows that f is an isomorphism and hence $\mathcal{M} \,\Box\, \mathcal{F}_\theta(\mathbb{Z}) \cong \mathcal{F}_\theta(M_\theta)$. $\square$

The Mackey functor $\mathcal{F}_\theta(\mathbb{Z})$ is meant to act as free in the θ-spot. There is a generator g in $(\mathcal{F}_\theta(\mathbb{Z}))_\theta$. There is also the twist of the generator $tg \in (\mathcal{F}_\theta(\mathbb{Z}))_\theta$ as well as the transfer of the generator $p_* g \in (\mathcal{F}_\bullet(\mathbb{Z}))_\bullet$. We can then picture $\mathcal{F}_\theta(\mathbb{Z})$ as follows:

$$\begin{tikzcd} \mathbb{Z}\langle g\rangle \oplus \mathbb{Z}\langle tg\rangle \arrow[loop left] \arrow[rr, bend left] & & \mathbb{Z}\langle p_* g\rangle . \arrow[ll, bend left] \end{tikzcd}$$

The Mackey functor relations allow us to do the rest of the computations in this particular Mackey functor.

- $t(tg) = g$ because $t^2 = 1$.

- $p_*(tg) = p_* g$ because $p_* t = p_*$.

- $p^*(p_* g) = g + tg$ beacuse $p^* p_* = 1 + t$.

Note that in relating this picture to the definition of $\mathcal{F}_\theta(\mathbb{Z})$ in Example 2.1.2 we can choose the generator g to be either of $(\pm 1, 0)$ or $(0, \pm 1)$ in $(\mathcal{F}_\theta(\mathbb{Z}))_\theta = \mathbb{Z}^2$. However, because of the isomorphism presented in the proof of Theorem 2.1.9, it is convenient to think of g as $(1, 0) = \mathbb{I}_\theta$.

Theorem 2.1.9 explains that $\mathcal{M} \,\square\, \mathcal{F}_\theta(\mathbb{Z})$ is isomorphic to $\mathcal{F}_\theta(M_\theta)$. It is helpful to frame that isomorphism in terms of the previous diagram. The elements of $(\mathcal{M} \,\square\, \mathcal{F}_\theta(\mathbb{Z}))_\theta$ are of the form $m_\theta \otimes g + t(m'_\theta \otimes g)$ and elements of $(\mathcal{M} \,\square\, \mathcal{F}_\theta(\mathbb{Z}))_\bullet$ are of the form $p_*(m_\theta \otimes g)$. The diagram we can draw for $\mathcal{M} \,\square\, \mathcal{F}_\theta(\mathbb{Z})$ is as follows:

$$(\mathcal{M} \,\square\, \mathcal{F}_\theta(\mathbb{Z}))_\theta \leftrightarrows (\mathcal{M} \,\square\, \mathcal{F}_\theta(\mathbb{Z}))_\bullet$$

$$\begin{tikzcd}
\boxed{t(m_\theta \otimes g)} \arrow[dr, dashed] & \\
\boxed{m_\theta \otimes g} \arrow[r, dashed] & \boxed{p_*(m_\theta \otimes g)}
\end{tikzcd}$$

We can, again, use the Mackey functor relations to do all of the desired computations. Here is an example of one such computation:

$$p^*(p_*(m_\theta \otimes g)) = (1 + t)(m_\theta \otimes g) = m_\theta \otimes g + t(m_\theta \otimes g). \tag{2.1.13}$$

Schulman showed in [Shu10] that $\square$ acts as a tensor in a symmetric monoidal category as stated in the following result.

Theorem 2.1.14. *The category* Mack *forms a symmetric monoidal category where the box product is the tensor and* $\mathcal{A} = \mathcal{F}_\bullet(\mathbb{Z})$ *acts as the unit. The associativity isomorphism* $(\mathcal{M} \,\square\, \mathcal{N}) \,\square\, \mathcal{P} \cong \mathcal{M} \,\square\, (\mathcal{N} \,\square\, \mathcal{P})$ *is clear and the twist map* $\tau : \mathcal{M} \,\square\, \mathcal{N} \to \mathcal{N} \,\square\, \mathcal{M}$ *is induced by the obvious isomorphisms*

$$\begin{gathered} M_\theta \otimes N_\theta \cong N_\theta \otimes M_\theta \qquad \textit{and} \\ [(M_\bullet \otimes N_\bullet) \oplus (M_\theta \otimes N_\theta)]/\sim \;\cong [(N_\bullet \otimes M_\bullet) \oplus (N_\theta \otimes M_\theta)]/\sim \end{gathered} \tag{2.1.15}$$

For the remainder of the paper, we will use $\mathcal{A}$ to refer to the unit in the symmetric monoidal structure. Theorem 2.1.16 is proven in [Shu10] and will be useful for the rest of the paper when discussing maps out of box products.

Theorem 2.1.16. *If $\mathcal{M}, \mathcal{N}, \mathcal{P} \in \underline{\text{Mack}}$ then a map $f : \mathcal{M} \square \mathcal{N} \to \mathcal{P}$ is determined uniquely by abelian group maps $f_\bullet : M_\bullet \otimes N_\bullet \to P_\bullet$ and $f_\theta : M_\theta \otimes N_\theta \to P_\theta$ subject to the following relations (called Frobenius relations):*

$$\begin{aligned}
&f_\theta(p^* m_\bullet \otimes p^* n_\bullet) = p^* f_\bullet(m_\bullet \otimes n_\bullet) && \textit{for } m_\bullet \in M_\bullet \textit{ and } n_\bullet \in N_\bullet \\
&f_\bullet(p_* m_\theta \otimes n_\bullet) = p_* f_\theta(m_\theta \otimes p^* n_\bullet) && \textit{for } m_\theta \in M_\theta \textit{ and } n_\bullet \in N_\bullet \\
&f_\bullet(m_\bullet \otimes p_* n_\theta) = p_* f_\theta(p^* m_\bullet \otimes n_\theta) && \textit{for } m_\bullet \in M_\bullet \textit{ and } n_\theta \in N_\theta \\
&f_\theta(t m_\theta \otimes t n_\theta) = t f_\theta(m_\theta \otimes n_\theta) && \textit{for } m_\theta \in M_\theta \textit{ and } n_\theta \in N_\theta
\end{aligned}$$

For Mackey functors $\mathcal{M}$ and $\mathcal{N}$ the set $\text{Hom}_{\underline{\text{Mack}}}(\mathcal{M}, \mathcal{N})$ has the structure of an abelian group. One can also construct an internal Hom object $\underline{\text{Hom}}(\mathcal{M}, \mathcal{N}) \in \underline{\text{Mack}}$ which, when used in homologocial algebra, allows us to realize $\underline{\text{Ext}}^*_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$ as a Mackey functor. We now provide that construction.

Consider the following three maps of Mackey functors:

- $g : \mathcal{F}_\bullet(\mathbb{Z}) \to \mathcal{F}_\theta(\mathbb{Z})$ defined by $g_\bullet(\mathbb{I}_\bullet) = p_*(\mathbb{I}_\theta)$
- $f : \mathcal{F}_\theta(\mathbb{Z}) \to \mathcal{F}_\bullet(\mathbb{Z})$ defined by $f_\theta(\mathbb{I}_\theta) = p^*(\mathbb{I}_\bullet)$
- $\tau : \mathcal{F}_\theta(\mathbb{Z}) \to \mathcal{F}_\theta(\mathbb{Z})$ defined by $\tau_\theta(\mathbb{I}_\theta) = t(\mathbb{I}_\theta)$

Together these create the diagram of Mackey functors shown below:

$$\begin{tikzcd}
\mathcal{F}_\theta(\mathbb{Z}) \arrow[loop left, "\tau"] \arrow[rr, bend left, "f"] && \mathcal{F}_\bullet(\mathbb{Z}). \arrow[ll, bend left, "g"]
\end{tikzcd}$$

Now, the objects in the diagram above are not abelian groups but one can check that the maps τ, f, and g satisfy the Mackey functor axioms. That is, $gf = \mathrm{id}_{\mathcal{F}_\theta(\mathbb{Z})} + \tau$, $f\tau = f$, $\tau g = g$, and $\tau^2 = \mathrm{id}_{\mathcal{F}_\theta(\mathbb{Z})}$.

We can then apply the functor $(-)\,\Box\,\mathcal{M}$ to this diagram. This results in the following:

$$\begin{tikzcd} \mathcal{F}_\theta(\mathbb{Z})\,\Box\,\mathcal{M} \arrow[loop left] \arrow[rr, bend left] & & \mathcal{F}_\bullet(\mathbb{Z})\,\Box\,\mathcal{M}. \arrow[ll, bend left] \end{tikzcd}$$

Finally, we can apply the functor $\mathrm{Hom}_{\underline{\mathrm{Mack}}}(-, \mathcal{N})$ which results in the diagram of abelian groups shown below:

$$\begin{tikzcd} \mathrm{Hom}_{\underline{\mathrm{Mack}}}(\mathcal{F}_\theta(\mathbb{Z})\,\Box\,\mathcal{M}, \mathcal{N}) \arrow[loop left] \arrow[rr, bend left] & & \mathrm{Hom}_{\underline{\mathrm{Mack}}}(\mathcal{F}_\bullet(\mathbb{Z})\,\Box\,\mathcal{M}, \mathcal{N}). \arrow[ll, bend left] \end{tikzcd}$$

Since the Mackey functor axioms held in the original diagram and we have only applied additive functors, they still hold in this final diagram of abelian groups. This Mackey functor will be called an *internal Hom* or a *Hom object* and is denoted $\underline{\mathrm{Hom}}(\mathcal{M}, \mathcal{N})$. Note that $(\underline{\mathrm{Hom}}(\mathcal{M}, \mathcal{N}))_\bullet = \mathrm{Hom}_{\underline{\mathrm{Mack}}}(\mathcal{M}, \mathcal{N})$.

It is possible to simplify the last diagram by observing that $\mathcal{F}_\bullet(\mathbb{Z})\,\Box\,\mathcal{M} \cong \mathcal{M}$ and by choosing an isomorphism $\mathcal{F}_\theta(\mathbb{Z})\,\Box\,\mathcal{M} \cong \mathcal{F}_\theta(M_\theta)$ as demonstrated in Theorem 2.1.9.

At this point we prove that the functors $-\,\Box\,\mathcal{X}$ and $\underline{\mathrm{Hom}}(\mathcal{X}, -)$ are adjoint as one might expect.

Theorem 2.1.17. *If $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{X}$ are Mackey functors then we have*

$$\mathrm{Hom}_{\underline{\mathrm{Mack}}}(\mathcal{M}\,\Box\,\mathcal{X}, \mathcal{N}) \cong \mathrm{Hom}_{\underline{\mathrm{Mack}}}(\mathcal{M}, \underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{N})). \tag{2.1.18}$$

That is, the functor $-\,\Box\,\mathcal{X} : \underline{\mathrm{Mack}} \to \underline{\mathrm{Mack}}$ *is left-adjoint to the functor* $\underline{\mathrm{Hom}}(\mathcal{X}, -) : \underline{\mathrm{Mack}} \to \underline{\mathrm{Mack}}$.

Proof. This theorem is similar to the usual proof that $-\otimes X : \mathbf{Ab} \to \mathbf{Ab}$ is left adoint to $\mathrm{Hom}(X, -) : \mathbf{Ab} \to \mathbf{Ab}$ for abelian groups, X. We will provide the bijection

$$\Theta : \mathrm{Hom}_{\underline{\mathrm{Mack}}}(\mathcal{M}\,\Box\,\mathcal{X}, \mathcal{N}) \to \mathrm{Hom}_{\underline{\mathrm{Mack}}}(\mathcal{M}, \underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{N})). \tag{2.1.19}$$

Choose a map $f \in \mathrm{Hom}_{\underline{\mathrm{Mack}}}(\mathcal{M}\,\Box\,\mathcal{X}, \mathcal{N})$. By Theorem 2.1.16 f is determined by maps $f_\theta : M_\theta \otimes X_\theta \to N_\theta$ and $f_\bullet : M_\bullet \otimes X_\bullet \to N_\bullet$.

We will now define an element $\phi \in \mathrm{Hom}_{\underline{\mathrm{Mack}}}(\mathcal{M}, \underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{N}))$ which will become $\Theta(f)$. Based on the definition of $\underline{\mathrm{Hom}}$ we need to construct maps ϕ_θ and $\phi_\bullet$ in the diagram below.

$$\begin{tikzcd}
M_\theta \arrow[loop left] \arrow[r, bend left] \arrow[d, "\phi_\theta"'] & M_\bullet \arrow[l, bend left] \arrow[d, "\phi_\bullet"'] \\
\mathrm{Hom}_{\underline{\mathrm{Mack}}}(\mathcal{F}_\theta(\mathbb{Z})\,\Box\,\mathcal{X}, \mathcal{N}) \arrow[loop left] \arrow[r, bend left] & \mathrm{Hom}_{\underline{\mathrm{Mack}}}(\mathcal{F}_\bullet(\mathbb{Z})\,\Box\,\mathcal{X}, \mathcal{N}). \arrow[l, bend left]
\end{tikzcd}$$

Choose an element $m_\theta \in M_\theta$. Recall that in order to specify a map $\mathcal{F}_\theta(\mathbb{Z})\,\Box\,\mathcal{X} \to \mathcal{N}$ it is sufficient to define it on elements of the form $g_\theta \otimes x_\theta \in (\mathcal{F}_\theta(\mathbb{Z})\,\Box\,\mathcal{X})_\theta$ where g_θ is a chosen generator for $\mathcal{F}_\theta(\mathbb{Z})$. Define

$$\big[\phi_\theta(m_\theta)\big](g_\theta \otimes x_\theta) = f_\theta(m_\theta \otimes x_\theta). \tag{2.1.20}$$

Now choose an element $m_\bullet \in M_\bullet$. Similarly, in order to specify a map $\mathcal{F}_\bullet(\mathbb{Z})\,\Box\,\mathcal{X} \to \mathcal{N}$ it is sufficient to define it on elements of the form $g_\bullet \otimes x_\bullet \in (\mathcal{F}_\theta(\mathbb{Z})\,\Box\,\mathcal{X})_\bullet$ where

$g_\bullet$ is a chosen generator for $\mathcal{F}_\bullet(\mathbb{Z})$. Define

$$\big[\phi_\bullet(m_\bullet)\big](g_\bullet \otimes x_\bullet) = f_\theta(m_\bullet \otimes x_\bullet). \tag{2.1.21}$$

We leave it to the reader to justify that ϕ_θ and $\phi_\bullet$ define a map $\phi : \mathcal{M} \to \underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{N})$ of Mackey functors. The only requirement is that there must be some compatibility between the generators g_θ and $g_\bullet$.

Define $\Theta(f)$ to be the map ϕ constructed above. We also leave it to the reader to justify that Θ is a bijection. □

Corollary 2.1.22. *For any given Mackey functor $\mathcal{X}$ the functor $-\,\Box\,\mathcal{X}$ is right-exact and the functor $\underline{\mathrm{Hom}}(\mathcal{X}, -)$ is left-exact.*

Proof. This is a direct corollary of Theorem 2.1.17. □

2.2 Mackey rings and modules

One can now define the notion of Mackey rings and Mackey modules just as in any symmetric monoidal category.

Definition 2.2.1. A *Mackey ring* is an object $\mathcal{R} \in \underline{\mathrm{Mack}}$ combined with a unit map $\mathcal{A} \to \mathcal{R}$ and a multiplication map $\mathcal{R}\,\Box\,\mathcal{R} \to \mathcal{R}$ which satisfy the appropriate associativity and unital axioms.

If $\mathcal{R}$ is a Mackey ring then a left Mackey module over $\mathcal{R}$ is an object $\mathcal{M} \in \underline{\mathrm{Mack}}$ combined with a map $\mu_{\mathcal{M}} : \mathcal{R}\,\Box\,\mathcal{M} \to \mathcal{M}$ that satisfies the appropriate associativity and unital axioms. A right Mackey module is defined in the analagous way.

A morphism $f : \mathcal{M} \to \mathcal{N}$ of Mackey modules over $\mathcal{R}$ is a morphism in $\underline{\mathrm{Mack}}$ such that $f \circ \mu_{\mathcal{M}} = \mu_{\mathcal{N}} \circ (\mathrm{id}\,\Box\,f)$. The category of left Mackey modules over $\mathcal{R}$ will be called $\mathcal{R} - \underline{\text{Mod}}$ and the category of right Mackey modules over $\mathcal{R}$ will be called $\underline{\text{Mod}} - \mathcal{R}$.

Theorem 2.2.2. *The structure of a Mackey ring $\mathcal{R}$ is determined by operations $\odot : R_\bullet \otimes R_\bullet \to R_\bullet$ and $\circledcirc : R_\theta \otimes R_\theta \to R_\theta$ which make $(R_\bullet, +, \odot)$ and $(R_\theta, +, \circledcirc)$ into rings and which satisfy the following relations:*

$$\begin{aligned}
&p^* x_\bullet \circledcirc p^* y_\bullet = p^*(x_\bullet \odot y_\bullet) && \text{for } x_\bullet, y_\bullet \in R_\bullet \\
&p_* x_\theta \odot y_\bullet = p_*(x_\theta \circledcirc p^* y_\bullet) && \text{for } x_\theta \in R_\theta \text{ and } y_\bullet \in R_\bullet \\
&x_\bullet \odot p_* y_\theta = p_*(p^* x_\bullet \circledcirc y_\theta) && \text{for } x_\bullet \in R_\bullet \text{ and } y_\theta \in R_\theta \\
&t x_\theta \circledcirc t y_\theta = t(x_\theta \circledcirc y_\theta) && \text{for } x_\theta, y_\theta \in R_\theta.
\end{aligned}$$

Proof. This is a simple matter of chasing through the properties in Definition 2.2.1. See [Shu10] for a complete proof. □

The symbols $\circledcirc$ and $\odot$ are used in Theorem 2.2.2 for clarity but those symbols will be suppressed for the sake of readability in the remainder of the paper.

It can be helpful to understand conditions equivalent to those in Theorem 2.2.2.

- It is clear that the first condition in Theorem 2.2.2 is equivalent to the condition that p^* is a ring map.
- It is also clear that the fourth condition in Theorem 2.2.2 is equivalent to the condition that t is a ring map.
- Since $p^* : R_\bullet \to R_\theta$ is a ring map we have induced functors $R_\theta - \mathbf{Mod} \to R_\bullet - \mathbf{Mod}$ and $\mathbf{Mod} - R_\theta \to \mathbf{Mod} - R_\bullet$. Hence we can realize R_θ as a left or right $R_\bullet$-module. The condition

$$p_* x_\theta \odot y_\bullet = p_*(x_\theta \circledcirc p^* y_\bullet) \qquad \text{for } x_\theta \in R_\theta \text{ and } y_\bullet \in R_\bullet$$

is equivalent to the condition that $p_* : R_\theta \to R_\bullet$ is a map of right $R_\bullet$-modules and the condition

$$x_\bullet \odot p_* y_\theta = p_*(p^* x_\bullet \circledcirc y_\theta) \qquad \text{for } x_\bullet \in R_\bullet \text{ and } y_\theta \in R_\theta$$

is equivalent to the condition that p_* is a map of left $R_\bullet$-modules.

Theorem 2.2.3. *If $\mathcal{R}$ is a Mackey ring then the structure of a module $\mathcal{M} \in \mathcal{R} - \underline{\mathrm{Mod}}$ is determined by the module structures $M_\bullet \in R_\bullet - \mathbf{Mod}$ and $M_\theta \in R_\theta - \mathbf{Mod}$ which satisfy the following relations:*

$$\begin{aligned}
&(p^* r_\bullet)(p^* m_\bullet) = p^*(r_\bullet m_\bullet) && \text{for } r_\bullet \in R_\bullet \text{ and } m_\bullet \in M_\bullet \\
&(p_* r_\theta) m_\bullet = p_*(r_\theta(p^* m_\bullet)) && \text{for } r_\theta \in R_\theta \text{ and } m_\bullet \in M_\bullet \\
&r_\bullet(p_* m_\theta) = p_*((p^* r_\bullet) m_\theta) && \text{for } r_\bullet \in R_\bullet \text{ and } m_\theta \in M_\theta \\
&(t r_\theta)(t m_\theta) = t(r_\theta m_\theta) && \text{for } r_\theta \in R_\theta \text{ and } m_\theta \in M_\theta.
\end{aligned}$$

Proof. This theorem is also a straightforward consequence of chasing through the properties in Definition 2.2.1. □

Definition 2.2.4. Let $\mathcal{R}$ be a Mackey ring and let $\mathcal{M} \in \mathcal{R} - \underline{\mathrm{Mod}}$. A submodule of $\mathcal{M}$ is a Mackey functor $\mathcal{N}$ as shown below where $N_\theta \leq M_\theta$ and $N_\bullet \leq M_\bullet$ are submodules:

$$\begin{tikzcd}
N_\theta \arrow[loop left, "t|_{N_\theta}"] \arrow[r, bend left, "p_*|_{N_\theta}"] & N_\bullet \arrow[l, bend left, "p^*|_{N_\bullet}"]
\end{tikzcd}$$

Each map in the diagram is meant to be the restriction of the corresponding map in $\mathcal{M}$.

Lemma 2.2.5. *Submodules $N_\bullet \leq M_\bullet$ and $N_\theta \leq M_\theta$ determine a submodule $\mathcal{N} \leq \mathcal{M}$ if and only if $p_*(N_\theta) \subseteq N_\bullet$, $p^*(N_\bullet) \subseteq N_\theta$, and $t(N_\theta) \subseteq N_\theta$.*

Proof. The "only if" direction is clear from Definition 2.2.4. For the "if" direction, observe that as long as the maps $p_*|_{N_\theta}$, $p^*|_{N_\bullet}$, and $t|_{N_\theta}$ are well-defined, the Mackey functor axioms are inherited from $\mathcal{M}$. $\square$

Example 2.2.6. If $\mathcal{R}$ is a Mackey ring and $f : \mathcal{M} \to \mathcal{N}$ is a morphism in $\mathcal{R} - \underline{\text{Mod}}$ then the submodules $\ker f_\bullet \leq M_\bullet$ and $\ker f_\theta \leq M_\theta$ form a submodule of $\mathcal{M}$.

- If $m_\bullet \in \ker f_\bullet$ then $f_\theta p^* m_\bullet = p^* f_\bullet m_\bullet = 0$ and hence $p^* m_\bullet \in \ker f_\theta$. Thus $p^*(\ker f_\bullet) \subseteq \ker f_\theta$.
- If $m_\theta \in \ker f_\theta$ then $f_\bullet p_* m_\theta = p_* f_\theta m_\theta = 0$ and hence $p_* m_\theta \in \ker f_\bullet$. Thus $p_*(\ker f_\theta) \subseteq \ker f_\bullet$.
- If $m_\theta \in \ker f_\theta$ then $f_\theta t m_\theta = t f_\theta m_\theta = t0 = 0$ and hence $t m_\theta \in \ker f_\theta$. Thus $t(\ker f_\theta) \subseteq \ker f_\theta$.

This submodule is naturally called $\ker f$.

Definition 2.2.7. A left (resp. right, resp. two-sided) ideal of a Mackey ring $\mathcal{R}$ is a left (resp. right, resp. two-sided) $\mathcal{R}$-submodule of $\mathcal{R}$.

2.3 Free modules

In $\underline{\text{Mack}}$ the *free* objects are those Mackey functors $\mathcal{F}$ where

$$\mathcal{F} \cong \left(\bigoplus_{\lambda \in \Lambda} \mathcal{F}_\bullet(\mathbb{Z}) \right) \oplus \left(\bigoplus_{\omega \in \Omega} \mathcal{F}_\theta(\mathbb{Z}) \right) \tag{2.3.1}$$

for some aribitrary sets Λ and Ω. Theorem 2.1.3 yields the following isomorphisms:

$$\begin{aligned}
&\mathrm{Hom}_{\underline{\mathrm{Mack}}}(\mathcal{F},\mathcal{M}) \\
&\quad\cong \left(\prod_{\lambda\in\Lambda} \mathrm{Hom}_{\underline{\mathrm{Mack}}}(\mathcal{F}_{\bullet}(\mathbb{Z}),\mathcal{M})\right) \times \left(\prod_{\omega\in\Omega} \mathrm{Hom}_{\underline{\mathrm{Mack}}}(\mathcal{F}_{\theta}(\mathbb{Z}),\mathcal{M})\right) \\
&\quad\cong \left(\prod_{\lambda\in\Lambda} M_{\bullet}\right) \times \left(\prod_{\omega\in\Omega} M_{\theta}\right).
\end{aligned} \tag{2.3.2}$$

The composite isomorphism

$$\mathrm{Hom}_{\underline{\mathrm{Mack}}}(\mathcal{F},\mathcal{M}) \to \left(\prod_{\lambda\in\Lambda} M_{\bullet}\right) \times \left(\prod_{\omega\in\Omega} M_{\theta}\right) \tag{2.3.3}$$

is the map

$$f \mapsto \left((f_{\bullet}(\mathbb{I}_{\bullet}^{\lambda}))_{\lambda\in\Lambda}, (f_{\theta}(\mathbb{I}_{\theta}^{\omega}))_{\omega\in\Omega}\right). \tag{2.3.4}$$

Free modules in $\mathcal{R}-\underline{\mathrm{Mod}}$ look like $\mathcal{R}\,\square\,\mathcal{F}$ where $\mathcal{F}$ is a free object in $\underline{\mathrm{Mack}}$. These take the form

$$\left(\bigoplus_{\lambda\in\Lambda} \mathcal{R}\,\square\,\mathcal{F}_{\bullet}(\mathbb{Z})\right) \oplus \left(\bigoplus_{\omega\in\Omega} \mathcal{R}\,\square\,\mathcal{F}_{\theta}(\mathbb{Z})\right). \tag{2.3.5}$$

Since $\mathcal{A} = \mathcal{F}_{\bullet}(\mathbb{Z})$ is the unit in the symmetric monoidal structure on $\underline{\mathrm{Mack}}$ we have $\mathcal{R}\,\square\,\mathcal{F}_{\bullet}(\mathbb{Z}) \cong \mathcal{R}$ and by Theorem 2.1.9 we have that $\mathcal{R}\,\square\,\mathcal{F}_{\theta}(\mathbb{Z}) \cong \mathcal{F}_{\theta}(R_{\theta})$. It follows that free modules in $\mathcal{R}-\underline{\mathrm{Mod}}$ are those of the form

$$\left(\bigoplus_{\lambda\in\Lambda} \mathcal{R}\right) \oplus \left(\bigoplus_{\omega\in\Omega} \mathcal{F}_{\theta}(R_{\theta})\right). \tag{2.3.6}$$

In order to make useful resolutions out of these free modules we need to know that they are projective and that there are enough of them.

Theorem 2.3.7. *Let $\mathcal{R}$ be a Mackey ring. Then the free $\mathcal{R}$-modules are projective in $\mathcal{R} - \underline{\text{Mod}}$.*

Proof. There are obvious forgetful functors $U_\theta : \underline{\text{Mack}} \to \mathbf{Ab}$ and $U_\bullet : \underline{\text{Mack}} \to \mathbf{Ab}$ where $U_\theta(\mathcal{M}) = M_\theta$ and $U_\bullet(\mathcal{M}) = M_\bullet$. In [Shu10], Schulman gives a left adjoint to these functors. With this adjoint one can use the usual categorical argument to show that $\mathcal{F}_\theta(\mathbb{Z})$ and $\mathcal{F}_\bullet(\mathbb{Z})$ (and hence direct sums thereof) are projective objects in $\underline{\text{Mack}}$. An application of Theorem 2.1.17 then shows that $\mathcal{R} \,\square\, \mathcal{F}_\theta(\mathbb{Z})$ and $\mathcal{R} \,\square\, \mathcal{F}_\bullet(\mathbb{Z})$ (and hence direct sums thereof) are projective objects in $\mathcal{R} - \underline{\text{Mod}}$. $\square$

Theorem 2.3.8. *There are enough free modules in $\mathcal{R} - \underline{\text{Mod}}$. That is, for every $\mathcal{M} \in \mathcal{R} - \underline{\text{Mod}}$ there exists a free module $\mathcal{F} \in \mathcal{R} - \underline{\text{Mod}}$ and an epimorphism $f : \mathcal{F} \to \mathcal{M}$.*

Proof. Let

$$\mathcal{F} = \left(\bigoplus_{m_\bullet \in M_\bullet} \mathcal{F}_\bullet(\mathbb{Z}) \right) \oplus \left(\bigoplus_{m_\theta \in M_\theta} \mathcal{F}_\theta(\mathbb{Z}) \right) \tag{2.3.9}$$

and define $f : \mathcal{F} \to \mathcal{M}$ such that

$$f_\bullet(\mathbb{I}_\bullet^{m_\bullet}) = m_\bullet \qquad \text{and} \qquad f_\theta(\mathbb{I}_\theta^{m_\theta}) = m_\theta. \tag{2.3.10}$$

Note that $\mathcal{F}$ is a free object in $\underline{\text{Mack}}$ and f is a morphism of Mackey functors but not a morphism in $\mathcal{R} - \underline{\text{Mod}}$. Clearly this is an epimorphism since $f_\bullet$ and f_θ are surjective.

Since $\mathcal{R} \,\square\, (-)$ is a right-exact functor the map $\mathcal{R} \,\square\, \mathcal{F} \to \mathcal{R} \,\square\, \mathcal{M}$ obtained by applying $\mathcal{R} \,\square\, (-)$ to $f : \mathcal{F} \to \mathcal{M}$ is an epimorphism. Finally, the map $\mathcal{R} \,\square\, \mathcal{M} \to \mathcal{M}$ which defines the module structure on $\mathcal{M}$ is an epimorphism and the composition

$$\mathcal{R} \,\square\, \mathcal{F} \to \mathcal{R} \,\square\, \mathcal{M} \to \mathcal{M} \tag{2.3.11}$$

is the desired epimorphism of $\mathcal{R}$-modules. $\square$

Theorem 2.1.9 provides an isomorphism $\mathcal{R} \,\square\, \mathcal{F}_\theta(\mathbb{Z}) \cong \mathcal{F}_\theta(R_\theta)$. It is clear how $\mathcal{R}$ acts on $\mathcal{R} \,\square\, \mathcal{F}_\theta(\mathbb{Z})$ and Theorem 2.3.12 describes the action of $\mathcal{R}$ on $\mathcal{F}_\theta(R_\theta)$ under that isomorphism:

Theorem 2.3.12. *Let $\mathcal{R}$ be a Mackey ring and recall that $\mathcal{F}_\theta(R_\theta)$ is the following $\mathcal{R}$-module:*

$$\begin{tikzcd} R_\theta^2 \arrow[loop left, "\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}"] \arrow[r, bend left, "\begin{bmatrix} 1 & 1 \end{bmatrix}"] & R_\theta \arrow[l, bend left, "\begin{bmatrix} 1 \\ 1 \end{bmatrix}"] \end{tikzcd}$$

From Theorem 2.1.9 we have that $\mathcal{R} \,\square\, \mathcal{F}_\theta(\mathbb{Z}) \cong \mathcal{F}_\theta(R_\theta)$ and under this isomorphism the action of $\mathcal{R}$ on $\mathcal{F}_\theta(R_\theta)$ is as follows:

- *The map $R_\theta \otimes R_\theta^2 \to R_\theta^2$ is given by $r_\theta \otimes (x_\theta, y_\theta) \mapsto (r_\theta x_\theta, (tr_\theta) y_\theta)$ for $r_\theta \in R_\theta$ and $(x_\theta, y_\theta) \in R_\theta^2$.*

- *The map $R_\bullet \otimes R_\theta \to R_\theta$ is given by $r_\bullet \otimes x_\theta \mapsto (p^* r_\bullet) x_\theta$ for $r_\bullet \in R_\bullet$ and $x_\theta \in R_\theta$.*

Before continuing to the proof it can be helpful to recall that $\mathcal{R} \,\square\, \mathcal{F}_\theta(\mathbb{Z})$ can be realized as the following diagram:

$$\begin{tikzcd} (\mathcal{R} \,\square\, \mathcal{F}_\theta(\mathbb{Z}))_\theta \arrow[rr, shift left] & & (\mathcal{R} \,\square\, \mathcal{F}_\theta(\mathbb{Z}))_\bullet \arrow[ll, shift left] \\ \boxed{t(r_\theta \otimes g)} \arrow[drr, dashed] & & \\ \boxed{r_\theta \otimes g} \arrow[rr, dashed] & & \boxed{p_*(r_\theta \otimes g)} \end{tikzcd}$$

This notation can help us understand Theorem 2.3.12 more clearly.

- If $r_\theta \in R_\theta$ and $s_\theta \otimes g \in (\mathcal{R} \,\square\, \mathcal{F}_\theta(\mathbb{Z}))_\theta$ then $r_\theta \cdot (s_\theta \otimes g) = (r_\theta s_\theta) \otimes g$.

- If $r_\theta \in R_\theta$ and $t(s_\theta \otimes g) \in (\mathcal{R} \,\Box\, \mathcal{F}_\theta(\mathbb{Z}))_\theta$ then

$$r_\theta\big[t(s_\theta \otimes g)\big] = t^2(r_\theta)t(s_\theta \otimes g) = t\big((tr_\theta)(s_\theta \otimes g)\big) = t\Big(\big((tr_\theta)s_\theta\big) \otimes g\Big). \quad (2.3.13)$$

- If $r_\bullet \in R_\bullet$ and $p_*(s_\theta \otimes g) \in (\mathcal{R} \,\Box\, \mathcal{F}_\theta(\mathbb{Z}))_\bullet$ then

$$r_\bullet\big[p_*(s_\theta \otimes g)\big] = p_*\big((p^* r_\bullet)(s_\theta \otimes g)\big) = p_*\Big(\big((p^* r_\bullet)s_\theta\big) \otimes g\Big) \qquad (2.3.14)$$

We can compare these equations with the results in the statement of Theorem 2.3.12 and see that they are analogous. The proof is shown below but these techniques can help us remember the module structure.

Proof of Theorem 2.3.12. The proof of Theorem 2.1.9 provides an isomorphism

$$\phi : \mathcal{F}_\theta(R_\theta) \to \mathcal{R} \,\Box\, \mathcal{F}_\theta(\mathbb{Z}). \qquad (2.3.15)$$

Let

$$\mu : \mathcal{R} \,\Box\, \mathcal{R} \to \mathcal{R} \qquad (2.3.16)$$

be the map which defines the ring structure on $\mathcal{R}$ and let

$$\alpha : \mathcal{R} \,\Box (\mathcal{R} \,\Box\, \mathcal{F}_\theta(\mathbb{Z})) \to (\mathcal{R} \,\Box\, \mathcal{R}) \,\Box\, \mathcal{F}_\theta(\mathbb{Z}) \qquad (2.3.17)$$

be the associativity isomorphism. Then the map η which makes the diagram below commute defines the $\mathcal{R}$-module structure of $\mathcal{F}_\theta(R_\theta)$.

$$\begin{tikzcd}
\mathcal{R} \,\Box\, \mathcal{F}_\theta(R_\theta) \arrow[rrr, dashed, "\eta"] \arrow[d, "\mathrm{id} \,\Box\, \phi^{-1}"'] & & & \mathcal{F}_\theta(R_\theta) \\
\mathcal{R} \,\Box (\mathcal{R} \,\Box\, \mathcal{F}_\theta(\mathbb{Z})) \arrow[r, "\alpha"'] & (\mathcal{R} \,\Box\, \mathcal{R}) \,\Box\, \mathcal{F}_\theta(\mathbb{Z}) \arrow[rr, "\mu \,\Box\, \mathrm{id}"'] & & \mathcal{R} \,\Box\, \mathcal{F}_\theta(\mathbb{Z}) \arrow[u, "\phi"']
\end{tikzcd}$$

All that remains is to compute the maps η_θ and $\eta_\bullet$ by evaluating the above composition on elements of the appropriate sets.

First, choose $r \in R_\theta$ and $(x, y) \in (\mathcal{F}_\theta(R_\theta))_\theta = R_\theta^2$. After referencing Theorem 2.1.9 we see that

$$\phi_\theta^{-1}(x, y) = x \otimes (1, 0) + ty \otimes (0, 1) \tag{2.3.18}$$

and hence

$$(\mathrm{id} \,\square\, \phi^{-1})_\theta(r \otimes (x, y)) = r \otimes (x \otimes (1, 0)) + r \otimes (ty \otimes (0, 1)). \tag{2.3.19}$$

It is clear that

$$\begin{aligned}(\mu \,\square\, \mathrm{id})_\theta\Big(\alpha_\theta\big(r \otimes (x \otimes (1,0)) + r \otimes (ty \otimes (0,1))\big)\Big) \\ = (\mu \,\square\, \mathrm{id})_\theta\big((r \otimes x) \otimes (1,0) + (r \otimes ty) \otimes (0,1)\big) \\ = rx \otimes (1,0) + r(ty) \otimes (0,1).\end{aligned} \tag{2.3.20}$$

Another reference to Theorem 2.1.9 then gives us that

$$\phi_\theta(rx \otimes (1, 0) + r(ty) \otimes (0, 1)) = (rx, 0) + (0, (tr)y) = (rx, (tr)y). \tag{2.3.21}$$

Finally, this shows that the $\mathcal{R}$-module structure of $\mathcal{F}_\theta(R_\theta)$ is given by $r \otimes (x, y) \mapsto (rx, (tr)y)$ in the θ-component, as desired.

Now, choose $r \in R_\bullet$ and $x \in (\mathcal{F}_\theta(R_\theta))_\theta = R_\theta$. A similar computation shows that $r \otimes x \mapsto (p^* r)x$. $\square$

2.4 Injective Modules

Baer's Criterion (see [Wei94]) gives a condition for checking that a module $M \in R - \mathbf{Mod}$ is injective for traditional rings R. Theorem 2.4.1 below is a generalization of that theorem for Mackey functors.

Theorem 2.4.1. *A module $\mathcal{M} \in \mathcal{R} - \underline{\text{Mod}}$ is injective if and only if both of the following conditions are satisfied:*

1. *For every monomorphism of $\mathcal{R}$-modules $\iota : \mathcal{I} \to \mathcal{R} \square \mathcal{F}_\bullet(\mathbb{Z})$ and every morphism $f : \mathcal{I} \to \mathcal{M}$ there exists a morphism $g : \mathcal{R} \square \mathcal{F}_\bullet(\mathbb{Z}) \to \mathcal{M}$ such that $g\iota = f$ as in the diagram below:*

$$\begin{tikzcd} \mathcal{I} \arrow[r, tail, "\iota"] \arrow[d, "f"'] & \mathcal{R} \square \mathcal{F}_\bullet(\mathbb{Z}) \arrow[dl, dashed, bend left, "g"] \\ \mathcal{M} & \end{tikzcd}$$

2. *For every monomorphism $\tau : \mathcal{J} \to \mathcal{R} \square \mathcal{F}_\theta(\mathbb{Z})$ and every morphism $f : \mathcal{J} \to \mathcal{M}$ there exists a morphism $g : \mathcal{R} \square \mathcal{F}_\theta(\mathbb{Z}) \to \mathcal{M}$ such that $g\tau = f$ as in the diagram below:*

$$\begin{tikzcd} \mathcal{J} \arrow[r, tail, "\tau"] \arrow[d, "f"'] & \mathcal{R} \square \mathcal{F}_\theta(\mathbb{Z}) \arrow[dl, dashed, bend left, "g"] \\ \mathcal{M} & \end{tikzcd}$$

All morphisms and objects are taken to be in $\mathcal{R} - \underline{\text{Mod}}$.

Proof. If $\mathcal{M}$ is injective then it is clear that both conditions hold. We will prove the other implication. Assume that the two conditions hold and consider the following diagram of $\mathcal{R}$-modules:

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, tail, "i"] \arrow[d, "f"'] & \mathcal{B} \\
\mathcal{M} &
\end{tikzcd}
$$

In order to prove that $\mathcal{M}$ is injective we will produce a map $g : \mathcal{B} \to \mathcal{M}$ such that $gi = f$.

Consider the following set:

$$\Lambda = \left\{ (\mathcal{K}, g) \;\middle|\; \mathcal{A} \leq \mathcal{K} \leq \mathcal{B},\ g : \mathcal{K} \to \mathcal{M},\ g|_{\mathcal{A}} = f \right\} \tag{2.4.2}$$

The set Λ has a partial order where $(\mathcal{K}, g) \leq (\mathcal{K}', g')$ whenever $\mathcal{K} \leq \mathcal{K}'$ and $g'|_{\mathcal{K}} = g$. If $S \subseteq \Lambda$ is a totally ordered subset then it is clear that S has a maximal element given by the union of the elements in S. Since S is also nonempty (because $(\mathcal{A}, f) \in S$) then, by Zorn's Lemma, Λ has a maximal element $(\mathcal{K}, g)$. We wish to show that $\mathcal{K} = \mathcal{B}$ and hence that g is the required map. Assume, to the contrary, that $\mathcal{K}$ is a proper submodule of $\mathcal{B}$.

It follows that either $K_\bullet$ is a proper traditional submodule of $B_\bullet$ or K_θ is a proper traditional submodule of B_θ. Instead of treating each case seperately we will choose an element x_* so that either $x_* \in B_\bullet \setminus K_\bullet$ or $x_* \in B_\theta \setminus K_\theta$. Note that the star is meant to represent an index; x_* is either $x_\bullet \in B_\bullet$ or $x_\theta \in B_\theta$ depending on the case. Mimicking this notation, let $\mathcal{F}_*$ be the Mackey module $\mathcal{R} \,\square\, \mathcal{F}_*(\mathbb{Z})$ and note that morphisms out of F_* are determined by the image of the element $\mathbb{I}_*$ (i.e. either $\mathbb{I}_\bullet$ or $\mathbb{I}_\theta$).

We can now form the Mackey submodule $\langle x_* \rangle \leq \mathcal{B}$ which is generated by x_*. Specifically, $\langle x_* \rangle$ is the intersection of all submodules which contain x_*. We can also form the module $\langle x_* \rangle + \mathcal{K}$ as the following pushout:

$$\begin{tikzcd}
\langle x_* \rangle \cap \mathcal{K} \arrow[r, hook] \arrow[d, hook] & \mathcal{K} \arrow[d, hook, dashed] \\
\langle x_* \rangle \arrow[r, hook, dashed] & \langle x_* \rangle + \mathcal{K}
\end{tikzcd}$$

Define a map $u : \mathcal{F}_* \to \langle x_* \rangle$ such that $\mathbb{I}_* \mapsto x_*$ and consider the following diagram:

$$\begin{tikzcd}
u^{-1}(\langle x_* \rangle \cap \mathcal{K}) \arrow[r] \arrow[d, hook] & \langle x_* \rangle \cap \mathcal{K} \arrow[r, hook] \arrow[d, hook] & \mathcal{K} \arrow[d, hook] \arrow[ddr, hook, bend left] \arrow[dddrr, bend left, "g"] & & \\
\mathcal{F}_* \arrow[r, two heads, "u"] & \langle x_* \rangle \arrow[r, hook] \arrow[drr, hook, bend right] & \langle x_* \rangle + \mathcal{K} & & \\
& & & \mathcal{B} & \\
& & & & \mathcal{M}
\end{tikzcd}$$

Note that the map $u^{-1}(\langle x_* \rangle \cap \mathcal{K}) \to \langle x_* \rangle \cap \mathcal{K}$ is the restriction of u.

The inclusions $\mathcal{K} \hookrightarrow \mathcal{B}$ and $\langle x_* \rangle \hookrightarrow \mathcal{B}$ induce an inclusion $\langle x_* \rangle + \mathcal{K} \hookrightarrow \mathcal{B}$ by the universal property of pushout. This means that $\langle x_* \rangle + \mathcal{K}$ can be realized as a submodule of $\mathcal{B}$ and, in particular, that $\mathcal{A} \leq \langle x_* \rangle + \mathcal{K} \leq \mathcal{B}$.

Now consider the map $u^{-1}(\langle x_* \rangle \cap \mathcal{K}) \to \mathcal{M}$ given by the composition along the top of the previous diagram. Since $u^{-1}(\langle x_* \rangle \cap \mathcal{K})$ is a submodule of $\mathcal{F}_* = \mathcal{R} \,\square\, \mathcal{F}_*(\mathbb{Z})$ this map can be extended to a map $h : \mathcal{F}_* \to \mathcal{M}$ by our initial assumption (i.e. by the appropriate condition in the statement of the theorem). We now have the following diagram:

$$
\begin{tikzcd}
u^{-1}(\langle x_* \rangle \cap \mathcal{K}) \arrow[r] \arrow[d, hook] & \langle x_* \rangle \cap \mathcal{K} \arrow[r, hook] \arrow[d, hook] & \mathcal{K} \arrow[d, hook] \arrow[ddr, hook, bend left] \arrow[dddrr, hook, bend left, "g"] & & \\
\mathcal{F}_* \arrow[r, two heads, "u"] \arrow[ddrrrr, bend right, "h"'] & \langle x_* \rangle \arrow[r, hook] \arrow[drr, hook, bend right] & \langle x_* \rangle + \mathcal{K} \arrow[dr, hook] & & \\
& & & \mathcal{B} & \\
& & & & \mathcal{M}
\end{tikzcd}
$$

At this point, if we can show that the left-most square is a pushout diagram then we reach the desired contradiction. Indeed, if both squares are pushouts then, by basic category theory, the larger rectangle is a pushout diagram as well. The maps $h : \mathcal{F}_* \to \mathcal{M}$ and $g : \mathcal{K} \to \mathcal{M}$ then induce a map $\hat{g} : \langle x_* \rangle + \mathcal{K} \to \mathcal{M}$ by the universal property of pushouts. It is then clear that $\hat{g}|_{\mathcal{K}} = g$ and hence $\hat{g}|_{\mathcal{A}} = (\hat{g}|_{\mathcal{K}})|_{\mathcal{A}} = g|_{\mathcal{A}} = f$. Thus the pair $(\langle x_* \rangle + \mathcal{K}, \hat{g})$ is an element of Λ and, since $x_* \notin K_*$, we have $(\mathcal{K}, g) < (\langle x_* \rangle + \mathcal{K}, \hat{g})$ which violates the maximality of $(\mathcal{K}, g)$.

To complete the proof it remains to show that the left-most square in the previous diagram is a pushout square, which is a direct result of Lemma 2.4.3. $\square$

Lemma 2.4.3. *Fix a Mackey ring $\mathcal{R}$. Let $u : \mathcal{X} \twoheadrightarrow \mathcal{Y}$ be a surjective map of $\mathcal{R}$-modules, let $\mathcal{J}$ be a submodule of $\mathcal{Y}$, and let $\tau : \mathcal{J} \hookrightarrow \mathcal{Y}$ be the inclusion. Define $t = u|_{u^{-1}(\mathcal{J})}$, define $\iota : u^{-1}(\mathcal{J}) \hookrightarrow \mathcal{X}$ to be the inclusion, and define $\mathcal{P}$ to be the pushout in the following diagram:*

$$
\begin{tikzcd}
u^{-1}(\mathcal{J}) \arrow[r, "t"] \arrow[d, hook, "\iota"'] & \mathcal{J} \arrow[d, dashed, "\beta"] \\
\mathcal{X} \arrow[r, dashed, "\alpha"'] & \mathcal{P}
\end{tikzcd}
$$

Then $\mathcal{Y}$ is isomorphic to $\mathcal{P}$. In other words, the following is a pushout diagram:

$$\begin{tikzcd}
u^{-1}(\mathcal{J}) \arrow[r, "t"] \arrow[d, hook, "\iota"'] & \mathcal{J} \arrow[d, hook, "\tau"] \\
\mathcal{X} \arrow[r, two heads, "u"'] & \mathcal{Y}
\end{tikzcd}$$

Proof. First we make a reduction. Note that pushouts of Mackey functors are calculated by computing the pushouts of the module diagrams in each of the two positions. More specifically, there are forgetful functors $\mathcal{U}_\bullet : \mathcal{R} - \underline{\text{Mod}} \to R_\bullet - \mathbf{Mod}$ and $\mathcal{U}_\theta : \mathcal{R} - \underline{\text{Mod}} \to R_\theta - \mathbf{Mod}$ where $\mathcal{U}_\bullet(\mathcal{M}) = M_\bullet$ and $\mathcal{U}_\theta(\mathcal{M}) = M_\theta$. Computing the pushout of a diagram D in $\mathcal{R} - \underline{\text{Mod}}$ is equivalent to computing the pushouts of $\mathcal{U}_\bullet(D)$ in $R_\bullet - \mathbf{Mod}$ and $\mathcal{U}_\theta(D)$ in $R_\theta - \mathbf{Mod}$.

As a result, it is sufficient to prove Lemma 2.4.3 when the objects and morphisms in question lie in $R - \mathbf{Mod}$ for an ordinary ring R (i.e. not a Mackey ring). Consider the solid-line diagram of R-modules shown below where all objects are given definitions analogous to those in the statement of the lemma.

$$\begin{tikzcd}
u^{-1}(J) \arrow[r, "t"] \arrow[d, hook, "\iota"'] & J \arrow[d, "\beta"] \arrow[dr, hook, bend left, "\tau"] & \\
X \arrow[r, "\alpha"'] \arrow[drr, two heads, bend right, "u"'] & P \arrow[dr, dashed, "\phi"] & \\
 & & Y
\end{tikzcd}$$

It is sufficient to prove that P is isomorphic to Y.

The maps $u : X \to Y$ and $\tau : J \to Y$ induce a map $\phi : P \to Y$ by the universal property of pushout which is represented by the dashed arrow in the previous diagram. We will show that ϕ is an isomorphism.

Since u is a surjection and $\phi\alpha = u$ it follows that ϕ must also be a surjection. It remains to show that ϕ is inective. To that end, suppose that there is some $s \in P$ such that $\phi(s) = 0$. By definition there exists some $x \in X$ and some $j \in J$ such that

$s = \alpha(x) + \beta(j)$. Then

$$0 = \phi(s) = \phi(\alpha(x) + \beta(j)) = u(x) + \tau(j) \tag{2.4.4}$$

and hence $u(x) = \tau(-j)$. Since $u(x)$ is in the image of τ (i.e. since $u(x) \in J$) it follows that $x \in u^{-1}(J)$ (i.e. that x is in the image of ι.). Observe that $\tau(t(x)) = u(\iota(x)) = u(x) = \tau(-j)$ which forces $t(x) = -j$ since τ is injective. Finally, it follows that

$$s = \alpha(x) + \beta(j) = \alpha(\iota(x)) - \beta(t(x)) = \alpha(\iota(x)) - \alpha(\iota(x)) = 0 \tag{2.4.5}$$

and hence ϕ is injective, as desired. $\square$

2.5 Homological Algebra

If $\mathcal{M} \in \mathcal{R} - \underline{\mathrm{Mod}}$ then we can construct a free resolution $\mathcal{F}_* \twoheadrightarrow \mathcal{M}$ since $\mathcal{R} - \underline{\mathrm{Mod}}$ has enough free modules. In order to compute $\underline{\mathrm{Tor}}_i^{\mathcal{R}}(\mathcal{M}, \mathcal{N})$ we need to be able to perform box product "over $\mathcal{R}$" which is provided by Definition 2.5.1 below.

Definition 2.5.1. Let $\mathcal{M}$ be a right $\mathcal{R}$-module and let $\mathcal{N}$ be a left $\mathcal{R}$-module. We define

$$\mathcal{M} \underset{\mathcal{R}}{\Box} \mathcal{N} = \mathrm{coeq}\,(\mathcal{M} \Box \mathcal{R} \Box \mathcal{N} \rightrightarrows \mathcal{M} \Box \mathcal{N}) \tag{2.5.2}$$

where the two maps in the coequalizer on the right are induced from the maps $\mathcal{M} \Box \mathcal{R} \to \mathcal{M}$ and $\mathcal{R} \Box \mathcal{N} \to \mathcal{N}$.

In homological algebra we often need only compute box products of the form $\mathcal{F} \Box_{\mathcal{R}} \mathcal{N}$ when $\mathcal{F}$ is a free module in $\mathcal{R} - \underline{\mathrm{Mod}}$. It is clear that $\mathcal{R} \Box_{\mathcal{R}} \mathcal{M} \cong \mathcal{M}$. To compute $\mathcal{F}_\theta(R_\theta) \square_{\mathcal{R}} \mathcal{M}$ observe the following:

$$\begin{aligned}
\mathcal{F}_\theta(R_\theta) \underset{\mathcal{R}}{\square} \mathcal{M} &\cong (\mathcal{F}_\theta(\mathbb{Z}) \square \mathcal{R}) \underset{\mathcal{R}}{\square} \mathcal{M} \\
&\cong \mathcal{F}_\theta(\mathbb{Z}) \square \left(\mathcal{R} \underset{\mathcal{R}}{\square} \mathcal{M}\right) \\
&\cong \mathcal{F}_\theta(\mathbb{Z}) \square \mathcal{M} \cong \mathcal{F}_\theta(M_\theta)
\end{aligned} \tag{2.5.3}$$

Definition 2.5.4. Let $\mathcal{M}$ be a right $\mathcal{R}$-module, let $\mathcal{N}$ be a left $\mathcal{R}$-module, and let $\mathcal{F}_*$ be a free resolution of $\mathcal{M}$. If we apply the functor $(-) \square_{\mathcal{R}} \mathcal{N}$ and compute homology of the resulting complex we get

$$\underline{\mathrm{Tor}}_i^{\mathcal{R}}(\mathcal{M}, \mathcal{N}) = H_i\left(\mathcal{F}_* \underset{\mathcal{R}}{\square} \mathcal{N}\right). \tag{2.5.5}$$

This functor is called *internal Tor*.

In section 2.1 we discussed the two different versions of Hom. During that discussion we constructed an internal Hom for $\underline{\mathrm{Mack}}$ but we need a similar construction of an object $\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$ in $\underline{\mathrm{Mack}}$. Let $\mathcal{M}$ and $\mathcal{N}$ be left $\mathcal{R}$-modules. In that construction we developed the diagram of Mackey functors below.

$$\begin{tikzcd}
\mathcal{M} \square \mathcal{F}_\theta(\mathbb{Z}) \arrow[loop left, "\tau"] \arrow[rr, bend left, "f"] & & \mathcal{M} \square \mathcal{F}_\bullet(\mathbb{Z}) \arrow[ll, bend left, "g"]
\end{tikzcd}$$

Since this is a diagram of $\mathcal{R}$-modules we can apply $\mathrm{Hom}_{\mathcal{R}}(-, \mathcal{N})$. The resulting diagram is then an object in $\underline{\mathrm{Mack}}$ that we call $\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$.

There is also a definition of $\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$ using an equalizer that is given in Definition 2.5.6 below and which has some symmetry with the coequalizer definition of $\mathcal{M}\,\Box_{\mathcal{R}}\,\mathcal{N}$ in Definition 2.5.1. It is a simple exercise to prove that the construction above is equivalent to the definition below.

Definition 2.5.6. Given left $\mathcal{R}$-modules $\mathcal{M}$ and $\mathcal{N}$ and a map $f : \mathcal{M} \to \mathcal{N}$ of Mackey functors (i.e. not $\mathcal{R}$-modules) we can make two maps $\mathcal{R}\,\Box\,\mathcal{M} \to \mathcal{N}$ out of the two paths around the following diagram:

$$\begin{tikzcd} \mathcal{R}\,\Box\,\mathcal{M} \arrow[r, "\mu_{\mathcal{M}}"] \arrow[d, "\mathrm{id}_{\mathcal{R}}\,\Box\, f"'] & \mathcal{M} \arrow[d, "f"] \\ \mathcal{R}\,\Box\,\mathcal{N} \arrow[r, "\mu_{\mathcal{N}}"'] & \mathcal{N} \end{tikzcd}$$

This induces two maps $\underline{\mathrm{Hom}}(\mathcal{M},\mathcal{N}) \to \underline{\mathrm{Hom}}(\mathcal{R}\,\Box\,\mathcal{M},\mathcal{N})$. One can also realize $\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{M},\mathcal{N})$ as the equalizer

$$\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{M},\mathcal{N}) = \mathrm{eq}\Big(\underline{\mathrm{Hom}}(\mathcal{M},\mathcal{N}) \rightrightarrows \underline{\mathrm{Hom}}(\mathcal{R}\,\Box\,\mathcal{M},\mathcal{N})\Big). \tag{2.5.7}$$

The functors Hom and $\underline{\mathrm{Hom}}$ induce functors Ext and $\underline{\mathrm{Ext}}$ in the usual way.

Definition 2.5.8. Let $\mathcal{M}$ and $\mathcal{N}$ be left $\mathcal{R}$-modules and let $\mathcal{F}_*$ be a free resolution of $\mathcal{M}$.

- If we apply the functor $\mathrm{Hom}_{\mathcal{R}-\underline{\mathrm{Mod}}}(-,\mathcal{N}) : \mathcal{R}-\underline{\mathrm{Mod}} \to \mathbf{Ab}$ to $\mathcal{F}_*$ and compute homology of the resulting complex we get

$$\mathrm{Ext}^i_{\mathcal{R}}(\mathcal{M},\mathcal{N}) = H^i(\mathrm{Hom}_{\mathcal{R}-\underline{\mathrm{Mod}}}(\mathcal{F}_*,\mathcal{N})). \tag{2.5.9}$$

- If we apply the functor $\underline{\mathrm{Hom}}_{\mathcal{R}}(-,\mathcal{N}) : \mathcal{R}-\underline{\mathrm{Mod}} \to \mathcal{R}-\underline{\mathrm{Mod}}$ to $\mathcal{F}_*$ and compute homology of the resulting complex we get

$$\underline{\mathrm{Ext}}^i_{\mathcal{R}}(\mathcal{M},\mathcal{N}) = H^i(\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{F}_*,\mathcal{N})). \tag{2.5.10}$$

The functor $\underline{\mathrm{Ext}}_{\mathcal{R}}(-,-)$ is called *internal Ext*.

CHAPTER III

EXT-ALGEBRA COMPUTATIONS OVER MACKEY ALGEBRAS

Suppose $\mathcal{C}$ is a tensor[1] category, R is a ring object in $\mathcal{C}$, and M is an R-bimodule object. One can then construct a ring object $\mathcal{E}_R(M)$ as follows:

- As an object $\mathcal{E}_R(M)$ is defined to be $R \oplus M$.
- Observe that

$$(R \oplus M) \otimes (R \oplus M) \cong (R \otimes R) \oplus (R \otimes M) \oplus (M \otimes R) \oplus (M \otimes M). \quad (3.0.1)$$

  The structure map $\mathcal{E}_R(M) \otimes \mathcal{E}_R(M) \to \mathcal{E}_R(M)$ is the map induced by the multiplication map $R \otimes R \to R$, the module maps $R \otimes M \to M$ and $M \otimes R \to M$, and the zero map $M \otimes M \to M$.

One might call $\mathcal{E}_R(M)$ the *square-zero extension* of R by M. In the examples below, we will consider examples in which $\mathcal{C}$ is $\underline{\text{Mack}}$. We will also assume that $\mathcal{R}$ is a commutative Mackey ring and that the left and right $\mathcal{R}$-module structures on objects $\mathcal{M} \in \mathcal{R} - \underline{\text{Mod}}$ are the same.

If $\mathcal{M}$ is a module in $\mathcal{R} - \underline{\text{Mod}}$ the module $\mathcal{E}_{\mathcal{R}}(\mathcal{M}) = \mathcal{R} \oplus \mathcal{M}$ can be equipped with a the structure of an $\mathbb{N}$-graded module which will be useful in the following results. We consider the $\mathcal{R}$ summand to be in degree 0 and the $\mathcal{M}$ summand to be in degree 1. (Note that $\mathcal{R}$ is given the trivial graded ring structure where everything is in degree 0.) It can be helpful to visualize $\mathcal{E}_{\mathcal{R}}(\mathcal{M})$ as follows:

---

[1]We need $\mathcal{C}$ to be a symmetric monoidal category in which the tensor product distributes over direct sums.

$$
\begin{tikzcd}
 & \mathcal{M} \arrow[d, dash] & & M_\theta \arrow[loop left] \arrow[r, bend left] \arrow[d, dash] & M_\bullet \arrow[l, bend left] \arrow[d, dash] & \text{(degree 1)} \\
\mathcal{E}_{\mathcal{R}}(\mathcal{M}): & \mathcal{R} & = & R_\theta \arrow[loop left] \arrow[r, bend left] & R_\bullet \arrow[l, bend left] & \text{(degree 0)}
\end{tikzcd}
$$

The vertical lines indicate direct sum between the pieces in various degrees. According to Theorem 2.2.2, the Mackey ring structure on $\mathcal{E}_{\mathcal{R}}(\mathcal{M})$ is determined by the ring structure of the two components of the Mackey functor. Explicitly, the multiplicative structure on $\mathcal{E}_{\mathcal{R}}(\mathcal{M})_\bullet = R_\bullet \oplus M_\bullet$ is as follows:

- If $r_\bullet, s_\bullet \in R_\bullet$ then $r_\bullet s_\bullet$ is computed using the ring structure on $R_\bullet$.
- If $m_\bullet, n_\bullet \in M_\bullet$ then $m_\bullet n_\bullet = 0$.
- If $r_\bullet \in R_\bullet$ and $m_\bullet \in M_\bullet$ then $r_\bullet m_\bullet$ and $m_\bullet r_\bullet$ are computed using the left and right $R_\bullet$-module structure on $M_\bullet$.

Analogous rules are used to compute products in $\mathcal{E}_{\mathcal{R}}(\mathcal{M})_\theta = R_\theta \oplus M_\theta$.

3.1 Computing $\mathrm{Ext}_{\mathcal{E}}(\mathcal{R}, \mathcal{R})$ when $\mathcal{E} = \mathcal{E}_{\mathcal{R}}(\mathcal{R})$

We consider the special case of $\mathcal{E} = \mathcal{E}_{\mathcal{R}}(\mathcal{R})$ for any Mackey ring, $\mathcal{R}$. Below is a symbolic diagram of $\mathcal{E}$.

$$
\begin{tikzcd}
 & \mathcal{R} \arrow[d, dash] & & R_\theta\langle p^*\tau\rangle \arrow[r, bend left] \arrow[d, dash] & R_\bullet\langle \tau\rangle \arrow[l, bend left] \arrow[d, dash] & \text{(degree 1)} \\
\mathcal{E}_{\mathcal{R}}(\mathcal{R}): & \mathcal{R} & = & R_\theta\langle 1_\theta\rangle \arrow[r, bend left] & R_\bullet\langle 1_\bullet\rangle \arrow[l, bend left] & \text{(degree 0)}
\end{tikzcd}
$$

The $\bullet$-component is $R_\bullet[\tau]/\tau^2$ and the θ-component is $R_\theta[p^*\tau]/(p^*\tau)^2$.

Theorem 3.1.1. *Let $\mathcal{R}$ be a Mackey ring and let $\mathcal{E} = \mathcal{E}_{\mathcal{R}}(\mathcal{R})$. Then there is an augmented free resolution of $\mathcal{R} \in \mathcal{E} - \underline{\mathrm{Mod}}$ of the following form:*

$$\ldots \to \mathcal{E} \to \mathcal{E} \to \mathcal{E} \twoheadrightarrow \mathcal{R}. \tag{3.1.2}$$

Furthermore, as Abelian groups, $\mathrm{Ext}^n_{\mathcal{E}}(\mathcal{R}, \mathcal{R}) \cong R_\bullet$ for all $n \geq 0$ and, as a ring, $\mathrm{Ext}^*_{\mathcal{E}}(\mathcal{R}, \mathcal{R}) \cong R_\bullet[x]$ where x is one of the following:

- *The cocycle $[\pi] \in \mathrm{Ext}^1_{\mathcal{E}}(\mathcal{R}, \mathcal{R})$ where $\pi : \mathcal{E} \to \mathcal{R}$ is shown below:*

$$\begin{tikzcd} \mathcal{R} \arrow[d, dash] & \\ \mathcal{R} \arrow[r, "\mathrm{id}"] & \mathcal{R} \end{tikzcd}$$

- *The Yoneda extension $\mathcal{R} \to \mathcal{E} \to \mathcal{R}$ in $\mathrm{Ext}^1_{\mathcal{E}}(\mathcal{R}, \mathcal{R})$ shown below:*

$$\begin{tikzcd} \mathcal{R} \arrow[r, "\mathrm{id}"] & \mathcal{R} \arrow[d, dash] & \\ & \mathcal{R} \arrow[r, "\mathrm{id}"] & \mathcal{R} \end{tikzcd}$$

Proof. The ring $E_\bullet$ is $R_\bullet$ in degree 0 and $R_\bullet$ in degree 1, as well. We use $\mathbb{I}_\bullet$ to denote the element $1_\bullet \in R_\bullet$ in degree 0 and, for the purposes of this proof, we will define τ to be the element $1_\bullet \in R_\bullet$ in degree 1.

It is easy to verify that the map $\pi : \mathcal{E} \to \mathcal{R}$ is a map of Mackey rings and hence a map of $\mathcal{E}$-modules. It is clear that $\ker(\pi) = \sum \mathcal{R}$. (By $\sum \mathcal{M}$ we mean the graded module $\mathcal{M}$ shifted upward by one degree. In general, $(\sum^k \mathcal{M})_i = \mathcal{M}_{i-k}$.) Next, define the map $\phi : \sum \mathcal{E} \to \mathcal{E}$ where $\phi(\mathbb{I}_\bullet) = \tau$. It is similarly easy to verify that ϕ is the identity map $\mathcal{R} \to \mathcal{R}$ in degree 1, the trivial map $\mathcal{R} \to 0$ in degree 2, and the trivial map $0 \to \mathcal{R}$ in degree 0. It is then clear that the following augmented chain complex is a free resolution of $\mathcal{R}$:

$$\xrightarrow{d_3=\sum^3 \phi} \sum^3 \mathcal{E} \xrightarrow{d_2=\sum^2 \phi} \sum^2 \mathcal{E} \xrightarrow{d_1=\sum \phi} \sum \mathcal{E} \xrightarrow{d_0=\phi} \mathcal{E} \xrightarrow{\pi} \mathcal{R}$$

It becomes more clear when the resolution is stratified by degree as shown below:

$$\begin{tikzcd}
{} \arrow[r, "\mathrm{id}"] & \mathcal{R} \arrow[d, dash] & & & & \\
& \mathcal{R} \arrow[r, "\mathrm{id}"] & \mathcal{R} \arrow[d, dash] & & & \\
& & \mathcal{R} \arrow[r, "\mathrm{id}"] & \mathcal{R} \arrow[d, dash] & & \\
& & & \mathcal{R} \arrow[r, "\mathrm{id}"] & \mathcal{R} \arrow[d, dash] & \\
& & & & \mathcal{R} \arrow[r, two heads, "\mathrm{id}"] & \mathcal{R}
\end{tikzcd}$$

Recall that $\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, \mathcal{R}) = R_\bullet$. Then, by inspecting the grading structure, we see that applying the functor $\mathrm{Hom}_{\mathcal{E}}(-, \mathcal{R})$ to our resolution yields the following:

$$\ldots \xleftarrow{0} R_\bullet \xleftarrow{0} R_\bullet \xleftarrow{0} R_\bullet \xleftarrow{0} R_\bullet \tag{3.1.3}$$

This shows that $\mathrm{Ext}^i_{\mathcal{E}}(\mathcal{R}, \mathcal{R}) \cong R_\bullet$ for all $i \in \mathbb{Z}_{\geq 0}$.

All that remains is to compute the ring structure on $\mathrm{Ext}^*_{\mathcal{E}}(\mathcal{R}, \mathcal{R})$. Choose an element $\mu \in R_\bullet \cong \mathrm{Ext}^p_{\mathcal{E}}(\mathcal{R}, \mathcal{R})$ and an element $\lambda \in R_\bullet \cong \mathrm{Ext}^q_{\mathcal{E}}(\mathcal{R}, \mathcal{R})$. The element μ can be represented by a map $g : \mathcal{E} \to \mathcal{R}$ where $g(\mathbb{I}_\bullet) = \mu$ and the element λ can be represented by a map $f : \mathcal{E} \to \mathcal{R}$ where $f(\mathbb{I}_\bullet) = \lambda$.

Now define maps $f_i : \mathcal{E} \to \mathcal{E}$ where $f_i(\mathbb{I}_\bullet) = \lambda$ for all $i \in \mathbb{Z}_{\geq 0}$. Observe that

$$\pi(f_0(\mathbb{I}_\bullet)) = \pi(\lambda) = \lambda = f(\mathbb{I}_\bullet) \tag{3.1.4}$$

and so $\pi \circ f_0 = f$. Next, for $i \in \mathbb{Z}_{\geq 0}$, observe that

$$f_i(d_{q+i}(\mathbb{I}_\bullet)) = f_i(\tau) = \tau f_i(\mathbb{I}_\bullet) = \tau\lambda \tag{3.1.5}$$

and

$$d_i(f_{i+1}(\mathbb{I}_\bullet)) = d_i(\lambda) = \lambda d_i(\mathbb{I}_\bullet) = \lambda\tau. \tag{3.1.6}$$

Since $\mathcal{R}$ was assumed to be a commutative ring we have that $f_i \circ d_{q+i} = d_i \circ f_{i+1}$. This shows that the diagram below is commutative.

$$\begin{tikzcd}
{} \arrow[r, "d_{q+2}"] & \mathcal{E} \arrow[r, "d_{q+1}"] \arrow[d, "f_2"'] & \mathcal{E} \arrow[r, "d_q"] \arrow[d, "f_1"'] & \mathcal{E} \arrow[d, "f_0"'] \arrow[dr, "f"] & \\
{} \arrow[r, "d_2"'] & \mathcal{E} \arrow[r, "d_1"'] & \mathcal{E} \arrow[r, "d_0"'] & \mathcal{E} \arrow[r, two heads, "\pi"'] & \mathcal{R}
\end{tikzcd}$$

By definition, the product $\mu * \lambda \in \mathrm{Ext}_{\mathcal{E}}^{p+q}(\mathcal{R}, \mathcal{R})$ is the element represented by the composition

$$\mathcal{E} \xrightarrow{f_p} \mathcal{E} \xrightarrow{g} \mathcal{R}.$$

We then have

$$g(f_p(\mathbb{I}_\bullet)) = g(\lambda) = \lambda\mu. \tag{3.1.7}$$

This shows that the product $\mu * \lambda$ in $\mathrm{Ext}_{\mathcal{E}}^*(\mathcal{R}, \mathcal{R})$ is the product $\mu\lambda \in R_\bullet \cong \mathrm{Ext}_{\mathcal{E}}^{p+q}(\mathcal{R}, \mathcal{R})$.

Finally, define a map $\Theta : \mathrm{Ext}_{\mathcal{E}}^*(\mathcal{R}, \mathcal{R}) \to R_\bullet[x]$ such that

$$\Theta_i : \mathrm{Ext}_{\mathcal{E}}^i(\mathcal{R}, \mathcal{R}) \to R_\bullet[x] \text{ is given by } \lambda \mapsto \lambda x^i. \tag{3.1.8}$$

The argument above shows that Θ is an isomorphism of rings, as desired.

It is clear that $\Theta([\pi]) = x$. It is then straightforward to verify that $[\pi]$ is the Yoneda extension described in the problem statement. $\square$

3.2 Computing $\mathrm{Ext}_{\mathcal{E}}(\mathcal{R},\mathcal{R})$ when $\mathcal{E} = \mathcal{E}_{\mathcal{R}}(\mathcal{F}_\theta(R_\theta))$

We now consider the special case of $\mathcal{E} = \mathcal{E}_{\mathcal{R}}(\mathcal{F}_\theta(\mathbb{Z})\,\Box\,\mathcal{R})$ for any Mackey ring, $\mathcal{R}$. Our goal is to compute $\mathrm{Ext}^*_{\mathcal{E}}(\mathcal{R},\mathcal{R})$ but first we try to understand some of the objects involved.

We start with any arbitrary Mackey ring $\mathcal{R}$. The module $\widehat{\mathcal{R}} = \mathcal{F}_\theta(\mathbb{Z})\,\Box\,\mathcal{R}$ is isomorphic to $\mathcal{F}_\theta(R_\theta)$ by Theorem 2.2.2. Then, based on the discussion at the beginning of this section, the ring $\mathcal{E} = \mathcal{E}_{\mathcal{R}}(\widehat{\mathcal{R}})$ can be visualized as follows:

$$
\begin{tikzcd}
 & \widehat{\mathcal{R}} \arrow[d, dash] & = & R_\theta^2 \arrow[loop left, "{\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]}"] \arrow[r, bend left, "{[1\ 1]}"] \arrow[d, dash] & R_\theta \arrow[l, bend left, "{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}"] \arrow[d, dash] \\
\mathcal{E}: & \mathcal{R} & = & R_\theta \arrow[loop left, "t"] \arrow[r, bend left, "p_*"] & R_\bullet \arrow[l, bend left, "p^*"]
\end{tikzcd}
$$

The ring structure on $\mathcal{E}$ is not immediately clear. However, after a careful application of the discussion earlier in the section and Theorem 2.3.12 we find the following:

($\bullet$) Products in $E_\bullet = R_\bullet \oplus R_\theta$:

- $*$ In degree 0, if $r_\bullet, s_\bullet \in R_\bullet$ then $r_\bullet s_\bullet$ is computed according to the ring structure on $R_\bullet$.
- $*$ In degree 1, if $r_\theta, s_\theta \in R_\theta$ then $r_\theta s_\theta = 0$.
- $*$ If $r_\bullet \in R_\bullet$ (in degree 0) and $s_\theta \in R_\theta$ (in degree 1) then

$$r_\bullet s_\theta = (p^* r_\bullet) s_\theta \tag{3.2.1}$$

where p^* is the restriction map in $\mathcal{R}$ and the product $(p^* r_\bullet)s_\theta$ is computed according to the ring structure of R_θ.

(θ) Products in $E_\theta = R_\theta \oplus R_\theta^2$:

* In degree 0, if $r_\theta, s_\theta \in R_\theta$ then $r_\theta s_\theta$ is computed according to the ring structure on R_θ.
* In degree 1, if $(r_\theta, s_\theta), (x_\theta, y_\theta) \in R_\theta^2$ then $(r_\theta, s_\theta)(x_\theta, y_\theta) = 0$.
* If $r_\theta \in R_\theta$ (in degree 0) and $(x_\theta, y_\theta) \in R_\theta^2$ (in degree 1) then

$$r_\theta(x_\theta, y_\theta) = (r_\theta x_\theta, (tr_\theta)y_\theta) \tag{3.2.2}$$

where t is the twist map in $\mathcal{R}$ and both products $r_\theta x_\theta$ and $(tr_\theta)y_\theta$ are computed according to the ring structure on R_θ.

Symbolically we can realize the Mackey functor $\mathcal{E}$ as shown below.

$$\begin{tikzcd}
& \widehat{\mathcal{R}} \arrow[d, dash] & = & R_\theta\langle\eta\rangle \oplus R_\theta\langle t\eta\rangle \arrow[r, bend left] \arrow[d, dash] & R_\theta\langle p_*\eta\rangle \arrow[l, bend left] \arrow[d, dash] \\
\mathcal{E}: & \mathcal{R} & = & R_\theta\langle 1_\theta\rangle \arrow[r, bend left, "p_*"] & R_\bullet\langle 1_\bullet\rangle \arrow[l, bend left, "p^*"]
\end{tikzcd}$$

The element $\eta \in E_\theta$ in degree 1 is meant to represent a generator of $\mathcal{F}_\theta(R_\theta)$. (It is the same as the η use in the proof of Theorem 3.2.7 below.) Thinking about $\mathcal{E}$ in this way can be helpful in keeping track of the products. Note that an element $(x_\theta, y_\theta) \in (\mathcal{F}_\theta(R_\theta))_\theta$ is meant to correspond to $x_\theta\eta + t(y_\theta\eta)$ or $x_\theta\eta + (ty_\theta)(t\eta)$.

We next examine $\widehat{\mathcal{E}} = \mathcal{F}_\theta(\mathbb{Z}) \,\square\, \mathcal{E} \cong \mathcal{F}_\theta(\widehat{E}_\theta)$. After a careful application of Theorem 2.2.2 one finds that $\mathcal{F}_\theta(\widehat{E}_\theta)$ takes the following form:

$$\begin{tikzcd}
 & \widehat{\mathcal{R}}^2 \arrow[d, no head] & & (R_\theta^2)^2 \arrow[r, bend left] \arrow[d, no head] & R_\theta^2 \arrow[l, bend left] \arrow[d, no head] \\
\widehat{\mathcal{E}}: & \widehat{\mathcal{R}} & = & R_\theta^2 \arrow[r, bend left] & R_\theta \arrow[l, bend left]
\end{tikzcd}$$

The action of $\mathcal{E}$ on $\widehat{\mathcal{E}}$ can be understood through an application of Theorem 2.3.12, but there is one special case that will be helpful in the future. Suppose that $(a, b) \in R_\theta^2 \subseteq E_\theta$ is in degree 1 and that $(p, q) \in R_\theta^2 \subseteq \widehat{E}_\theta$ is in degree 0. Then the product $(a, b) * (p, q) \in \widehat{E}_\theta$ is computed as follows:

$$\begin{aligned}
(a, b) * (p, q) &= ((a, b) * p, t(a, b) * q) \\
&= (p * (a, b), q * (b, a)) \\
&= ((pa, (tp)b), (qb, (tq)a))
\end{aligned} \tag{3.2.3}$$

The map t is the twist map on R_θ and multiplication in the last line of this equation takes place in R_θ.

The symbolic visualization of $\widehat{\mathcal{E}}$ is shown below.

$$\begin{tikzcd}
\widehat{E}_\theta \arrow[r, shift right] & \widehat{E}_\bullet \arrow[l, shift right]
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{\eta\mathbb{I}_\theta} \arrow[dr, dashed] & & \\
\boxed{(t\eta)(t\mathbb{I}_\theta)} \arrow[r, dashed] & \boxed{p_*(\eta\mathbb{I}_\theta) = p_*((t\eta)(t\mathbb{I}_\theta))} & \\
\boxed{(t\eta)\mathbb{I}_\theta} \arrow[dr, dashed] & & \\
\boxed{\eta(t\mathbb{I}_\theta)} \arrow[r, dashed] & \boxed{p_*((t\eta)\mathbb{I}_\theta) = p_*(\eta(t\mathbb{I}_\theta))} & \text{(deg 1)}
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{t\mathbb{I}_\theta} \arrow[dr, dashed] & & \\
\boxed{\mathbb{I}_\theta} \arrow[r, dashed] & \boxed{p_*\mathbb{I}_\theta} & \text{(deg 0)}
\end{tikzcd}$$

As usual, the Mackey functor axioms are enough to work out all of the necessary calculations in the previous diagram. For example, the product of $p_*\eta \in E_\bullet$ and $p_*\mathbb{I}_\theta \in \widehat{E}_\theta$ is calculated as follows:

$$\begin{aligned}
(p_*\eta)(p_*\mathbb{I}_\theta) &= p_*(p^*(p_*\eta) \cdot \mathbb{I}_\theta) \\
&= p_*((\eta + t\eta) \cdot \mathbb{I}_\theta) \\
&= p_*(\eta\mathbb{I}_\theta) + p_*((t\eta)\mathbb{I}_\theta).
\end{aligned} \tag{3.2.4}$$

As a remark, it may be helpful to point out that in the proof of Theorem 3.2.7 below we define $\eta \in R^2_\theta \subseteq E_\theta$ to be the degree 1 element $(1, 0)$ and $\mathbb{I}_\theta \in R^2_\theta \subseteq \widehat{E}_\theta$ is the degree 0 element $(1, 0)$. An application of Equation 3.2.3 shows the following:

$$\begin{aligned}
\eta\mathbb{I}_\theta &= (1, 0) * (1, 0) = ((1, 0), (0, 0)), \\
(t\eta)(t\mathbb{I}_\theta) &= (0, 1) * (0, 1) = ((0, 0), (1, 0)), \\
(t\eta)\mathbb{I}_\theta &= (0, 1) * (1, 0) = ((0, 1), (0, 0)), \text{ and} \\
\eta(t\mathbb{I}_\theta) &= (1, 0) * (0, 1) = ((0, 0), (0, 1)).
\end{aligned} \tag{3.2.5}$$

It is clear that these are the four generators of $(R_\theta^2)^2$.

Finally, for the sake of comparisson, we can repeat the calculation done in Equation 3.2.3 using the previous diagram. Recall that an element $(a, b) \in R_\theta^2 \subseteq E_\theta$ in degree 1 is represented as $a\eta + t(b\eta)$ and that an element $(p, q) \in R_\theta^2 \subseteq \widehat{E}_\theta$ in degree 0 is represented as $p\mathbb{I}_\theta + t(q\mathbb{I}_\theta)$. We then have

$$
\begin{aligned}
&\big(a\eta + t(b\eta)\big) * \big(p\mathbb{I}_\theta + t(q\mathbb{I}_\theta)\big) \\
&\qquad = (a\eta)(p\mathbb{I}_\theta) + (a\eta)\big(t(q\mathbb{I}_\theta)\big) + \big(t(b\eta)\big)(p\mathbb{I}_\theta) + \big(t(b\eta)\big)\big(t(q\mathbb{I}_\theta)\big) \\
&\qquad = (ap)(\eta\mathbb{I}_\theta) + \big(a(tq)\big)\big(\eta(t\mathbb{I}_\theta)\big) + \big((tb)p\big)\big((t\eta)\mathbb{I}_\theta\big) + \big(t(bq)\big)\big(t(\eta\mathbb{I}_\theta)\big).
\end{aligned}
\tag{3.2.6}
$$

Note that our rings were assumed to be commutative so $ap = pa$ and $bq = qb$.

Before finally continuing to the computation below, recall that maps $f : \mathcal{E} \to \mathcal{M}$ of $\mathcal{E}$-modules are determined uniquely by $f_\bullet(\mathbb{I}_\bullet) \in M_\bullet$ and, similarly, maps $g : \widehat{\mathcal{E}} \to \mathcal{M}$ of $\mathcal{E}$-modules are determined uniquely by $g_\theta(\mathbb{I}_\theta) \in M_\theta$. It follows that maps $\widehat{\mathcal{E}}^s \to \mathcal{M}$ are determined by the images of the s generators. Furthermore, recall that

- $\mathbb{I}_\bullet$ is the identity element in the $R_\bullet$ summand of $E_\bullet = R_\bullet \oplus R_\theta$ and
- $\mathbb{I}_\theta$ is the element $(1, 0) \in R_\theta^2$ in the degree 0 summand of $\widehat{E}_\theta = R_\theta^2 \oplus R_\theta^4$.

For clarity we will distinguish the identity elements of $R_\bullet$ and R_θ by denoting them $1_\bullet$ and 1_θ, respectively.

Theorem 3.2.7. *Let $\mathcal{R}$ be a Mackey ring, let $\widehat{\mathcal{R}} = \mathcal{F}_\theta(\mathbb{Z}) \,\square\, \mathcal{R} \cong \mathcal{F}_\theta(R_\theta)$, and let $\mathcal{E} = \mathcal{E}_{\mathcal{R}}(\widehat{\mathcal{R}})$. Then there is an augmented free resolution of $\mathcal{R} \in \mathcal{E} - \underline{\mathrm{Mod}}$ of the form*

$$\ldots \to \mathcal{P}_3 \to \mathcal{P}_2 \to \mathcal{P}_1 \to \mathcal{P}_0 \twoheadrightarrow \mathcal{R} \tag{3.2.8}$$

where

$$\mathcal{P}_k = \begin{cases} \mathcal{E} & \textit{if } k = 0, \textit{ and} \\ \widehat{\mathcal{E}}^{(2^{k-1})} & \textit{if } k > 0. \end{cases} \tag{3.2.9}$$

and $\widehat{\mathcal{E}} = \mathcal{F}_\theta(\mathbb{Z}) \,\square\, \mathcal{E} \cong \mathcal{F}_\theta(E_\theta)$. *Furthermore,*

$$\operatorname{Ext}^k_{\mathcal{E}}(\mathcal{R}, \mathcal{R}) \cong \begin{cases} R_\bullet & \textit{if } k = 0, \textit{ and} \\ R_\theta^{(2^{k-1})} & \textit{if } k > 0. \end{cases} \tag{3.2.10}$$

Proof. We start by constructing the free resolution. The objects of our resolution $\mathcal{P}_* \twoheadrightarrow \mathcal{R}$ are given in the statement of the theorm. It remains to define the $\mathcal{E}$-module maps $d_k : \mathcal{P}_{k+1} \to \mathcal{P}_k$ for $k \geq 0$ and a map $\pi : \mathcal{P}_0 \twoheadrightarrow \mathcal{R}$.

Define $\pi : \mathcal{E} \to \mathcal{R}$ such that $\pi(\mathbb{I}_\bullet) = 1_\bullet \in R_\bullet$. It is clear that this map is the identity map $\mathcal{R} \to \mathcal{R}$ in degree 0 and the zero map $\widehat{\mathcal{R}} \to 0$ in degree 1. It follows that π is surjective and $\ker \pi = \sum \widehat{\mathcal{R}}$.

Recall that the degree 1 component of E_θ is R_θ^2. Define $\eta \in E_\theta$ to be the degree 1 element $\eta = (1_\theta, 0)$. Define $d_0 : \sum \widehat{\mathcal{E}} \to \mathcal{E}$ such that $d_0(\mathbb{I}_\theta) = \eta$. This map the takes the form shown below.

$$\begin{tikzcd} \widehat{\mathcal{R}}^2 \arrow[d, dash] & \\ \widehat{\mathcal{R}} \arrow[r, "\mathbb{I}_\theta \mapsto \eta"] & \widehat{\mathcal{R}} \arrow[d, dash] \\ & \mathcal{R} \end{tikzcd}$$

It is clear that d_0 is the trivial map $0 \to \mathcal{R}$ in degree 0 and the zero map $\widehat{\mathcal{R}}^2 \to 0$ in degree 2. Since e and η are both the element $(1_\theta, 0) \in \widehat{R}_\theta = R_\theta^2$ it follows that d_0 is the identity map $\widehat{\mathcal{R}} \to \widehat{\mathcal{R}}$ in degree 1. We see that $\ker d_0 = \sum^2 \widehat{\mathcal{R}}^2$ and that $\operatorname{im} d_0 = \sum \widehat{\mathcal{R}}$.

For the remainder of the proof we will omit the suspensions. All maps are assumed to be maps of graded $\mathcal{E}$-modules and the degrees of the maps and the required suspensions can be deduced by examination if necessary.

The module $\widehat{\mathcal{E}}^2$ has two generators, which we will denote x and y. Define a map $\phi : \widehat{\mathcal{E}}^2 \to \widehat{\mathcal{E}}$ where $\phi(x) = \eta\mathbb{I}_\theta$ and $\phi(y) = (t\eta)\mathbb{I}_\theta$. This map takes the form shown below.

$$\begin{tikzcd}
(\widehat{\mathcal{R}}^2)^2 \arrow[d, dash] & \\
\widehat{\mathcal{R}}^2 \arrow[r, "\substack{x\mapsto\eta\mathbb{I}_\theta \\ y\mapsto(t\eta)\mathbb{I}_\theta}"] & \widehat{\mathcal{R}}^2 \arrow[d, dash] \\
& \widehat{\mathcal{R}}
\end{tikzcd}$$

In degree 1 the θ-component of this map is $R^4_\theta \to R^4_\theta$. In the domain the four generators are $\{x, tx, y, ty\}$ and in the codomain the four generators are $\{\eta\mathbb{I}_\theta, (t\eta)(t\mathbb{I}_\theta), (t\eta)\mathbb{I}_\theta, \eta(t\mathbb{I}_\theta)\}$.

- Since $\phi(x) = \eta\mathbb{I}_\theta$ we have $\phi(tx) = t(\eta\mathbb{I}_\theta) = (t\eta)(t\mathbb{I}_\theta)$.
- Since $\phi(y) = (t\eta)\mathbb{I}_\theta$ we have $\phi(ty) = t((t\eta)\mathbb{I}_\theta) = \eta(t\mathbb{I}_\theta)$.

It follows that in degree 1 the θ-component of ϕ is an isomorphism. Since $\widehat{\mathcal{E}}^2$ is free in the θ-component it can be shown that ϕ is an isomorphism of Mackey functors in degree 1. Note that the kernel of ϕ is $\widehat{\mathcal{R}}^4$ and the image of ϕ is $\widehat{\mathcal{R}}^2$.

Since $\mathcal{P}_2 = \widehat{\mathcal{E}}^2$ and $\mathcal{P}_1 = \widehat{\mathcal{E}}$ we can define $d_1 = \phi$. Then $\ker d_0 = \widehat{\mathcal{R}}^2 = \operatorname{im} d_1$ which ensures exactness at $\mathcal{P}_1$. Furthermore, since $\mathcal{P}_{k+1} = \widehat{\mathcal{E}}^{(2^k)}$ and $\mathcal{P}_k = \widehat{\mathcal{E}}^{(2^{k-1})}$ we can define $d_k : (\widehat{\mathcal{E}}^2)^{(2^{k-1})} \to \widehat{\mathcal{E}}^{(2^{k-1})}$ so that $d_k = \phi^{(2^{k-1})}$. Based on the discussion above we see that

$$\begin{aligned} \ker d_{k-1} &= (\ker \phi)^{(2^{k-2})} = (\widehat{\mathcal{R}}^4)^{(2^{k-2})} = \widehat{\mathcal{R}}^{(2^k)} \text{ and} \\ \operatorname{im} d_k &= (\operatorname{im} \phi)^{(2^{k-1})} = (\widehat{\mathcal{R}}^2)^{(2^{k-1})} = \widehat{\mathcal{R}}^{(2^k)} \end{aligned} \tag{3.2.11}$$

for $k > 1$. This ensures exactness at each $\mathcal{P}_k$. The diagram below shows the graded structure of the resulting chain complex $\mathcal{P}_* \twoheadrightarrow \mathcal{R}$ for clarity.

$$\begin{tikzcd}
{} \arrow[r, "\cong"] & \widehat{\mathcal{R}}^8 \arrow[d, dash] & & & & \\
& \widehat{\mathcal{R}}^4 \arrow[r, "\cong"] & \widehat{\mathcal{R}}^4 \arrow[d, dash] & & & \\
& & \widehat{\mathcal{R}}^2 \arrow[r, "\cong"] & \widehat{\mathcal{R}}^2 \arrow[d, dash] & & \\
& & & \widehat{\mathcal{R}} \arrow[r, "\cong"] & \widehat{\mathcal{R}} \arrow[d, dash] & \\
& & & & \mathcal{R} \arrow[r, two heads, "\cong"] & \mathcal{R}
\end{tikzcd}$$

Next, we'd like to argue that the map $d_k^* : \operatorname{Hom}_{\mathcal{E}}(\mathcal{P}_k, \mathcal{R}) \to \operatorname{Hom}_{\mathcal{E}}(\mathcal{P}_{k+1}, \mathcal{R})$ is the zero map. If $\psi \in \operatorname{Hom}_{\mathcal{E}}(\mathcal{P}_k, \mathcal{R})$ is non-zero then, by inspecting degrees, one finds that $\psi \circ d_k$ is the composition shown below.

$$\begin{tikzcd}
\widehat{\mathcal{R}}^{(2^{k+1})} \arrow[d, dash] & & \\
\widehat{\mathcal{R}}^{(2^k)} \arrow[r, "d_k"] & \widehat{\mathcal{R}}^{(2^k)} \arrow[d, dash] & \\
& \widehat{\mathcal{R}}^{(2^{k-1})} \arrow[r, "\psi"] & \mathcal{R}
\end{tikzcd}$$

Clearly this composition is zero for all $\psi \in \operatorname{Hom}_{\mathcal{E}}(\mathcal{P}_{k-1}, \mathcal{R})$ so d_k^* is also zero.

Finally, for free modules we have

$$\operatorname{Hom}_{\mathcal{E}}(\widehat{\mathcal{E}}^s, \mathcal{R}) \cong R_\theta^s \qquad \text{and} \qquad \operatorname{Hom}_{\mathcal{E}}(\mathcal{E}^s, \mathcal{R}) \cong R_\bullet^s. \tag{3.2.12}$$

Since the modules in the resolution $\mathcal{P}_*$ are free it follows that applying the functor $\mathrm{Hom}_{\mathcal{E}}(-,\widehat{\mathcal{R}})$ to the resolution yields

$$\ldots \xleftarrow{0} R_\theta^4 \xleftarrow{0} R_\theta^2 \xleftarrow{0} R_\theta \xleftarrow{0} R_\bullet. \tag{3.2.13}$$

The result then follows. $\square$

We now wish to compute the ring structure on $\mathrm{Ext}^*_{\mathcal{E}}(\mathcal{R},\mathcal{R})$. First, we introduce some notation. In Theorem 3.2.7 we constructed an $\mathcal{E}$-module resolution for $\widehat{\mathcal{E}}$ of the form

$$\ldots \to \mathcal{P}_3 \to \mathcal{P}_2 \to \mathcal{P}_1 \to \mathcal{P}_0 \twoheadrightarrow \mathcal{R} \tag{3.2.14}$$

where

$$\mathcal{P}_k = \begin{cases} \mathcal{E} & \text{if } k=0, \text{ and} \\ \widehat{\mathcal{E}}^{(2^{k-1})} & \text{if } k>0 \end{cases} \tag{3.2.15}$$

and $\widehat{\mathcal{E}} = \mathcal{F}_\theta(\mathbb{Z}) \,\square\, \mathcal{E} \cong \mathcal{F}_\theta(E_\theta)$. For $k>0$ each of the $\widehat{\mathcal{E}}$-summands of $\mathcal{P}_k$ contains a generator $\mathbb{I}_\theta \in \mathcal{E}_\theta$. We will denote these 2^{k-1} generators by e_I for indices I in a set Λ_{k-1} described below.

Definition 3.2.16. We define a *sign sequence* to be a finite sequence of the symbols $+$ and $-$. If I is a sign sequence, denote its length by $\ell(I)$. Define Λ to be the set of all sign sequences and define $\Lambda_n = \{I \in \Lambda \mid \ell(I) = n\}$. As an example, $\Lambda_2 = \{++,+-,-+,--\}$.

Finally, if $J \in \Lambda$, define J° to be the sign sequence where all of the symbols $+$ and $-$ are swapped. For example, if $J = -+--$ then $J^\circ = +-++$.

It is clear that Λ_{k-1} contains 2^{k-1} sign sequences and hence is an appropriate index set for the generators of $\mathcal{P}_k = \widehat{\mathcal{E}}^{(2^{k-1})}$. Note that when $k=1$ there is one sign sequence in Λ_0, namely the empty sequence, and the corresponding generator will simply be written as e. We will now express the differentials in the resolution $\mathcal{P}_*$ in terms of these generators.

Recall that the degree 1 component of E_θ is R_θ^2 and $\eta \in E_\theta$ is the degree 1 element $(1_\theta, 0) \in R_\theta^2$. The map $d_0 : \widehat{\mathcal{E}} \to \mathcal{E}$ was defined to send the generator of $\widehat{\mathcal{E}}$ to η. According to this new notation, $d_0(e) = \eta$.

The map $d_1 : \widehat{\mathcal{E}}^2 \to \widehat{\mathcal{E}}$ sent the two generators of $\widehat{\mathcal{E}}^2$ to $\eta\mathbb{I}_\theta$ and $(t\eta)\mathbb{I}_\theta$. The generator $\mathbb{I}_\theta \in \widehat{\mathcal{E}} = \mathcal{P}_1$ is now called e and the two generators in $\mathcal{P}_2 = \widehat{\mathcal{E}}^2$ are e_+ and e_-. Assign these generators so that

$$d_1(e_+) = \eta e \qquad \text{and} \qquad d_1(e_-) = (t\eta)e. \tag{3.2.17}$$

Later differentials $d_k : \mathcal{P}_{k+1} \to \mathcal{P}_k$ are sums of d_1 (which we called ϕ in the previous proof). It follows that for each generator e_I of $\mathcal{P}_k$ (for $I \in \Lambda_{k-1}$) we must choose two generators in $\mathcal{P}_{k+1}$ to get sent to ηe_I and $(t\eta)e_I$. Obviously there are many such choices. We choose these generators so that

$$d_k(e_{+I}) = \eta e_I \qquad \text{and} \qquad d_k(e_{-I}) = (t\eta)e_I \tag{3.2.18}$$

for all $k > 1$ and for all $I \in \Lambda_{k-2}$. Note that "$+I$" and "$-I$" in the above equation refer to concatenation. For example, if $I = +-$ then $+I = ++-$ and $-I = -+-$.

From this point forward we will omit the subscripts on the differentials when doing so does not cause confusion. In summary,

$$\begin{aligned} d(e) &= \eta, \\ d(e_{+I}) &= \eta e_I \text{ for all } I \in \Lambda, \text{ and} \\ d(e_{-I}) &= (t\eta)e_I \text{ for all } I \in \Lambda. \end{aligned} \tag{3.2.19}$$

The important piece of indexing to remember going forward is that the generators of $\mathcal{P}_k$ are indexed by sign sequences of length $k-1$. That is, if $e_I \in \mathcal{P}_k$ then $\ell(I) = k-1$.

This notation gives us a way to express the elements of $\mathrm{Ext}^*_{\mathcal{E}}(\mathcal{R}, \mathcal{R})$ as explained below. After Definition 3.2.20 we can now proceed to compute the ring structure on $\mathrm{Ext}^*_{\mathcal{E}}(\mathcal{R}, \mathcal{R})$.

Definition 3.2.20. Choose $k > 0$ and $I \in \Lambda_{k-1}$. Define $\widehat{e}_I \in \mathrm{Hom}_{\mathcal{E}}(\mathcal{P}_k, \mathcal{R})$ to be the map which is dual to e_I. That is, $\widehat{e}_I(e_I) = 1_\theta$ and $\widehat{e}_I(e_J) = 0$ for $J \in \Lambda_{k-1} \setminus \{I\}$. Then $\widehat{e}_I$ is a cocycle in $\mathcal{P}_*$ and hence represents an element of $\mathrm{Ext}^*_{\mathcal{E}}(\mathcal{R}, \mathcal{R})$ which will also be called $\widehat{e}_I$. It follows as a result of Theorem 3.2.7 that

$$\mathrm{Ext}^k_{\mathcal{E}}(\mathcal{R}, \mathcal{R}) = \left\{ \sum_{I \in \Lambda_{k-1}} \lambda_I \widehat{e}_I \mid \lambda_I \in R_\theta \right\} \tag{3.2.21}$$

for $k > 0$.

Choose $I \in \Lambda_{k-1}$. By standard arguments in homological algebra, the map $\widehat{e}_I : \mathcal{P}_k \to \mathcal{R}$ lifts to maps $f_i : \mathcal{P}_{k+i} \to \mathcal{P}_i$ for $i \geq 0$ which make the diagram below commute.

$$\begin{tikzcd} \cdots \arrow[r] & \mathcal{P}_{k+3} \arrow[r, "d"] \arrow[d, "f_3"'] & \mathcal{P}_{k+2} \arrow[r, "d"] \arrow[d, "f_2"'] & \mathcal{P}_{k+1} \arrow[r, "d"] \arrow[d, "f_1"'] & \mathcal{P}_k \arrow[d, "f_0"'] \arrow[dr, "\widehat{e}_I"] & \\ \cdots \arrow[r] & \mathcal{P}_3 \arrow[r, "d"'] & \mathcal{P}_2 \arrow[r, "d"'] & \mathcal{P}_1 \arrow[r, "d"'] & \mathcal{P}_0 \arrow[r, two heads, "\pi"'] & \mathcal{R} \end{tikzcd}$$

When the objects in this diagram are replaced by their definitions in Theorem 3.2.7 we get the following new diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & \widehat{\mathcal{E}}^{(2^{n+2})} \arrow[r, "d"] \arrow[d, "f_3"'] & \widehat{\mathcal{E}}^{(2^{n+1})} \arrow[r, "d"] \arrow[d, "f_2"'] & \widehat{\mathcal{E}}^{(2^{n})} \arrow[r, "d"] \arrow[d, "f_1"'] & \widehat{\mathcal{E}}^{(2^{n-1})} \arrow[d, "f_0"'] \arrow[dr, "\widehat{e}_I"] & \\
\cdots \arrow[r] & \widehat{\mathcal{E}}^4 \arrow[r, "d"'] & \widehat{\mathcal{E}}^2 \arrow[r, "d"'] & \widehat{\mathcal{E}} \arrow[r, "d"'] & \mathcal{E} \arrow[r, two heads, "\pi"'] & \mathcal{R}
\end{tikzcd}$$

Lemma 3.2.23 below gives an example of one such lift. In the statement we omit the subscripts on the f_i. When we write $f(e_{J+I}) = e_J$, one can deduce the subscript on f from the length of I and J if needed. We chose $I \in \Lambda_{k-1}$ so $\ell(I) = k - 1$. Thus

$$\ell(J + I) = \ell(J) + \ell(+) + \ell(I) = \ell(J) + 1 + (k - 1) = k + \ell(J) \tag{3.2.22}$$

and it follows that $J + I \in \Lambda_{k+\ell(J)}$ and so $e_{J+I} \in \mathcal{P}_{k+\ell(J)+1}$. Hence when we write $f(e_{J+I}) = e_J$ we mean $f_{\ell(J)+1}(e_{J+I}) = e_J$. However, these subscripts are rarely useful.

Lemma 3.2.23. *Choose $I \in \Lambda_{k-1}$. The map $\widehat{e}_I : \mathcal{P}_k \to \mathcal{R}$ lifts to maps $f_i : \mathcal{P}_{k+i} \to \mathcal{P}_i$ for $i \geq 0$ which make the diagram below commute.*

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{P}_{k+3} \arrow[r, "d"] \arrow[d, "f_3"'] & \mathcal{P}_{k+2} \arrow[r, "d"] \arrow[d, "f_2"'] & \mathcal{P}_{k+1} \arrow[r, "d"] \arrow[d, "f_1"'] & \mathcal{P}_k \arrow[d, "f_0"'] \arrow[dr, "\widehat{e}_I"] & \\
\cdots \arrow[r] & \mathcal{P}_3 \arrow[r, "d"'] & \mathcal{P}_2 \arrow[r, "d"'] & \mathcal{P}_1 \arrow[r, "d"'] & \mathcal{P}_0 \arrow[r, two heads, "\pi"'] & \mathcal{R}
\end{tikzcd}$$

One such lift is as follows:

- *$f(e_I) = 1_\theta$ and $f(e_K) = 0$ whenever $K \in \Lambda_{k-1} \setminus \{I\}$.*
- *$f(e_{J \pm K}) = 0$ if $K \in \Lambda_{k-1} \setminus \{I\}$ and $J \in \Lambda$.*
- *$f(e_{J+I}) = e_J$ for any $J \in \Lambda$.*
- *$f(e_{J-I}) = t(e_{J^\circ})$ for any $J \in \Lambda$. (See Definition 3.2.16 for the definition of J°.)*

Note that the sequences above are being concatenated so that $J + I$ is the symbol $+$ concatenated between J and I. For example, if $J = +$ and $I = -+$ then $J + I = + + -+$ and $J - I = + - -+$.

Proof. We start by explaining the definition of f. The domain of the maps f_i are the modules $\mathcal{P}_{k+i}$ for $i \geq 0$. The generators of those modules are e_S where $\ell(S) \geq k - 1$ and since the chain complexes involved are free it is sufficient to define f on those generatores. The generators e_S then fall into two categories.

- If $\ell(S) = k - 1$ then the generator in question must either be e_I or e_K where $K \in \Lambda_{k-1} \setminus \{I\}$. ($S'$ is just the right-most $k - 1$ signs.) The definition of $f(e_S)$ in this case is given by the first bullet point in the statement.

- If $\ell(S) > k - 1$ then we can write S as $J \pm S'$ where $J \in \Lambda$ is any sign sequence and $\ell(S') = k - 1$. In this case the generator must either be one of e_{J+I} for some $J \in \Lambda$, e_{J-I} for some $J \in \Lambda$, or $e_{J\pm K}$ for some $J \in \Lambda$ and some $K \in \Lambda_{k-1} \setminus \{I\}$. The definition of $f(e_S)$ for these generators is given by the last three bullet points in the statement.

Given this definition of f, the only thing to check is that the diagram commutes. We first check the right-most triangle. It is clear that

$$\begin{aligned} &\pi(f_0(e_I)) = 1_\theta = \widehat{e}_I(e_I) \text{ and} \\ &\pi(f_0(e_K)) = 0 = \widehat{e}_I(e_K) \text{ when } K \neq I \end{aligned} \tag{3.2.24}$$

so $\pi \circ f_0 = \widehat{e}_I$. To verify that the squares commute we will show that $f(d(e_S)) = d(f(e_S))$ for generators e_S with $\ell(S) > k - 1$. This comes down to examining several cases.

First, consider a generator e_S where $\ell(S) = k$. Then S must be one of $+I$, $-I$, or $\pm K$ for some $K \in \Lambda_{k-1} \setminus \{I\}$. In the case when $S = +I$ we have

$$f(d(e_{+I})) = f(\eta e_I) = \eta f(e_I) = \eta 1_\theta = \eta = d(e) = d(f(e_{+I})). \tag{3.2.25}$$

Nearly identical calculations show

$$\begin{aligned} f(d(e_{-I})) = t\eta = d(f(e_{-I})) \quad &\text{and} \\ f(d(e_{\pm K})) = 0 = d(f(e_{\pm K})) \quad &\text{for } K \in \Lambda_{k-1} \setminus \{I\}. \end{aligned} \tag{3.2.26}$$

This verifies that $f(d(e_S)) = d(f(e_S))$ when $\ell(S) = k$.

Now consider a generator e_S where $\ell(S) > k$. Then we can find some $J \in \Lambda$ and some $K \in \Lambda_{k-1}$ such that $S = \pm J \pm K$. Note that K is the right-most $k-1$ signs of S and it is possible that $K = I$. Note, also, that J might be empty. This leads to eight cases depending on the two signs and whether $K = I$ or $K \neq I$.

There are four cases when $K = I$. In the particular case when $S = +J + I$ for some $J \in \Lambda$ we have

$$\begin{aligned} d(f(e_{+J+I})) &= d(e_{+J}) = \eta e_J \quad \text{and} \\ f(d(e_{+J+I})) &= f(\eta e_{J+I}) = \eta f(e_{J+I}) = \eta e_J. \end{aligned} \tag{3.2.27}$$

The other three cases when $K = I$ are similar and the reader can verify the following:

$$\begin{aligned} &d(f(e_{+J-I})) = \eta t(e_{J^\circ}) = f(d(e_{+J-I})), \\ &d(f(e_{-J+I})) = (t\eta)e_J = f(d(e_{-J+I})), \quad \text{and} \\ &d(f(e_{-J-I})) = t(\eta(e_{J^\circ})) = f(d(e_{-J-I})). \end{aligned} \tag{3.2.28}$$

In all four cases where $K \neq I$ the computation is simple and the reader can verify that

$$f(d(e_{\pm J \pm K})) = 0 = d(f(e_{\pm J \pm K})). \tag{3.2.29}$$

We've now verified that $d(f(e_S)) = f(d(e_S))$ whenever $\ell(S) > k$. This completes the proof. $\square$

In Corollary 3.2.31 we provide a similar lift for maps $\lambda\widehat{e}_I : \mathcal{P}_k \to \mathcal{R}$ where $I \in \Lambda_{k-1}$ and $\lambda \in R_\theta$. In case it is unclear, this is the map defined so that

$$\begin{aligned} (\lambda\widehat{e}_I)(e_I) &= \lambda \quad \text{and} \\ (\lambda\widehat{e}_I)(e_K) &= 0 \quad \text{for all } K \in \Lambda_{k-1} \setminus \{I\}. \end{aligned} \tag{3.2.30}$$

Corollary 3.2.31. *Choose $I \in \Lambda_{k-1}$ and choose some $\lambda \in R_\theta$. Then the map $\lambda\widehat{e}_I : \mathcal{P}_k \to \mathcal{R}$ lifts to maps $g_i : \mathcal{P}_{k+i} \to \mathcal{P}_k$ for $i \geq 0$ which make the diagram below commute.*

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{P}_{k+3} \arrow[r, "d"] \arrow[d, "g_3"'] & \mathcal{P}_{k+2} \arrow[r, "d"] \arrow[d, "g_2"'] & \mathcal{P}_{k+1} \arrow[r, "d"] \arrow[d, "g_1"'] & \mathcal{P}_k \arrow[d, "g_0"'] \arrow[dr, "\lambda\widehat{e}_I"] & \\
\cdots \arrow[r] & \mathcal{P}_3 \arrow[r, "d"'] & \mathcal{P}_2 \arrow[r, "d"'] & \mathcal{P}_1 \arrow[r, "d"'] & \mathcal{P}_0 \arrow[r, two heads, "\pi"'] & \mathcal{R}
\end{tikzcd}$$

One such lift is given by $g(e_J) = \lambda f(e_J)$ for all $J \in \Lambda$ where the maps f_i are those provided by Lemma 3.2.23.

Proof. To check the commutivity of the right-most triangle, choose a generator e_J where $\ell(J) = k - 1$. Since $\pi f_0 = \widehat{e}_I$ by Lemma 3.2.23 observe that

$$\pi(g_0(e_J)) = \pi(\lambda f_0(e_J)) = \lambda\pi(f_0(e_J)) = \lambda(\widehat{e}_I(e_J)) = (\lambda\widehat{e}_I)(e_J). \tag{3.2.32}$$

Hence $\pi g_0 = \lambda\widehat{e}_I$.

Next consider a generator e_J where $\ell(J) > k-1$. Since $fd = df$ we have

$$d(g(e_J)) = d(\lambda f(e_J)) = \lambda d(f(e_J)) = \lambda f(d(e_J)) = g(d(e_J)) \tag{3.2.33}$$

and so $gd = dg$. $\square$

Theorem 3.2.34 below describes the product structure on $\mathrm{Ext}^*_{\mathcal{E}}(\mathcal{R},\mathcal{R})$. In light of Definition 3.2.20 it suffices to compute the product of elements of the form $\lambda\widehat{e}_I$ and $\mu\widehat{e}_J$.

Theorem 3.2.34. *Choose $I \in \Lambda_{q-1}$, choose $J \in \Lambda_{p-1}$, and choose $\lambda, \mu \in R_\theta$. The map*

$$\mathrm{Ext}^p_{\mathcal{E}}(\mathcal{R},\mathcal{R}) \otimes \mathrm{Ext}^q_{\mathcal{E}}(\mathcal{R},\mathcal{R}) \to \mathrm{Ext}^{p+q}_{\mathcal{E}}(\mathcal{R},\mathcal{R}) \tag{3.2.35}$$

which defines the product structure on $\mathrm{Ext}^*_{\mathcal{E}}(\mathcal{R},\mathcal{R})$ is given by

$$(\mu\widehat{e}_J) \otimes (\lambda\widehat{e}_I) \mapsto (\mu\lambda)\widehat{e}_{J+I} + ((t\mu)\lambda)\widehat{e}_{J^\circ - I} \tag{3.2.36}$$

Proof. Corollary 3.2.31 provides the lift g shown in the diagram below.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{P}_{p+q} \arrow[r] \arrow[d, "g_p"'] & \cdots \arrow[r] & \mathcal{P}_{q-1} \arrow[r] \arrow[d, "g_1"'] & \mathcal{P}_q \arrow[d, "g_0"'] \arrow[dr, "\lambda\widehat{e}_I"] & \\
\cdots \arrow[r] & \mathcal{P}_p \arrow[r] \arrow[d, "\mu\widehat{e}_J"'] & \cdots \arrow[r] & \mathcal{P}_1 \arrow[r] & \mathcal{P}_0 \arrow[r, two heads] & \mathcal{R} \\
 & \mathcal{R} & & & &
\end{tikzcd}$$

By the definition of the product structure on $\mathrm{Ext}^*_{\mathcal{E}}(\mathcal{R},\mathcal{R})$, the product $(\mu\widehat{e}_J) * (\lambda\widehat{e}_I)$ is the element represented by the composition

$$\mathcal{P}_{p+q} \xrightarrow{g_p} \mathcal{P}_p \xrightarrow{\mu\widehat{e}_J} \mathcal{R}.$$

We will compute this composition by evaluating it on the generators of $\mathcal{P}_{p+q}$. Elements of Λ_{p+q-1} can be represented as $S \pm T$ where $S \in \Lambda_{p-1}$ and $T \in \Lambda_{q-1}$ because

$$\ell(S \pm T) = \ell(S) + \ell(\pm) + \ell(T) = (p-1) + 1 + (q-1) = p + q - 1. \tag{3.2.37}$$

It remains to compute $\mu\widehat{e}_J(g_p(e_{S\pm T}))$ for all choices of S and T. By Corollary 3.2.31 we have the following:

- $g_p(e_{S+I}) = \lambda e_S$
- $g_p(e_{S-I}) = \lambda t(e_{S^\circ})$
- $g_p(e_{S\pm T}) = 0$ if $T \neq I$

Furthermore we have that $\mu\widehat{e}_J(e_J) = \mu$ and $\mu\widehat{e}_J(e_S) = 0$ whenever $S \neq J$. Combining these facts we have the following:

$$\mu\widehat{e}_J(g_p(e_K)) = \begin{cases} \mu\lambda & \text{if } K = J + I \\ (t\mu)\lambda & \text{if } K = J^\circ - I \\ 0 & \text{otherwise} \end{cases} . \tag{3.2.38}$$

It follows that the resulting element of $\mathrm{Ext}_{\mathcal{E}}^{p+q-1}(\mathcal{R}, \mathcal{R})$ is

$$(\mu\lambda)\widehat{e}_{J+I} + ((t\mu)\lambda)\widehat{e}_{J^\circ - I}, \tag{3.2.39}$$

as desired. $\square$

The ring structure on $\mathrm{Ext}_{\mathcal{E}}^{*}(\mathcal{R}, \mathcal{R})$ is a recognizable one at least in some specific cases. Theorem 3.2.40 below gives one such instance.

Theorem 3.2.40. *Let R be a (traditional) ring of characteristic 2 and let $\mathcal{R}$ be*

$$\begin{tikzcd} R \arrow[loop left, "\mathrm{id}"] \arrow[r, bend left, "0"] & R. \arrow[l, bend left, "\mathrm{id}"] \end{tikzcd}$$

It is easy to verify that $\mathcal{R}$ is a Mackey ring. Let $\widehat{\mathcal{R}} = \mathcal{F}_\theta(\mathbb{Z}) \Box \mathcal{R}$ and $\mathcal{E} = \mathcal{E}_{\mathcal{R}}(\widehat{\mathcal{R}})$. Then there is a graded ring isomorphism

$$\mathrm{Ext}^*_{\mathcal{E}}(\mathcal{R}, \mathcal{R}) \cong R\langle x_1, x_2, x_3, \ldots \rangle \tag{3.2.41}$$

where x_i is in degree i and the ring on the right denotes noncommutative polynomials in the x_i.

Proof. Define a graded R-algebra $\mathscr{A}_R$ as follows:

- The generators of $\mathscr{A}_R$ (as an R-algebra) are $\widehat{e}_I$ for $I \in \Lambda$ with $\deg(\widehat{e}_I) = \ell(I)+1$.
- Multiplication on $\mathscr{A}_R$ is induced by $\widehat{e}_J \widehat{e}_I = \widehat{e}_{J+I} + \widehat{e}_{J^\circ - I}$ for all $I \in \Lambda$.

Since the twist map in $\mathcal{R}$ is the identity and since $R_\bullet = R_\theta = R$ it is then clear that

$$\mathrm{Ext}^*_{\mathcal{E}}(\mathcal{R}, \mathcal{R}) \cong \mathscr{A}_R \tag{3.2.42}$$

by the previous results in this section. It is also clear that

$$\mathscr{A}_R = R \otimes \mathscr{A}_{\mathbb{Z}/2} \tag{3.2.43}$$

since the characteristic of R was assumed to be 2.

Define a graded R-algebra $\mathscr{B}_R = R\langle x_1, x_2, \ldots\rangle$ where x_i is in degree i. It is clear that $\mathscr{B}_R \cong R \otimes \mathscr{B}_{\mathbb{Z}/2}$. By Lemma 3.2.45 below, $\mathscr{A}_{\mathbb{Z}/2} \cong \mathscr{B}_{\mathbb{Z}/2}$ so

$$\operatorname{Ext}^*_{\mathcal{E}_R}(\mathcal{R}, \mathcal{R}) \cong \mathscr{A}_R \cong R \otimes \mathscr{A}_{\mathbb{Z}/2} \cong R \otimes \mathscr{B}_{\mathbb{Z}/2} \cong \mathscr{B}_R \tag{3.2.44}$$

and the result follows. □

It remains only to prove Theorem 3.2.40 in the special case when $R = \mathbb{Z}/2$.

Lemma 3.2.45. *If $R = \mathbb{Z}/2$ and $\mathcal{R}$ and $\mathcal{E}$ are the corresponding Mackey rings discussed in Theorem 3.2.40 then*

$$\operatorname{Ext}^*_{\mathcal{E}}(\mathcal{R}, \mathcal{R}) \cong R\langle x_1, x_2, x_3, \ldots\rangle. \tag{3.2.46}$$

Proof. The structure of $\operatorname{Ext}^*_{\mathcal{E}}(\mathcal{R}, \mathcal{R})$ is described previously in this section (see Theorem 3.2.34 for the culmination of that discussion). For $k > 0$ define I_{k-1} to be the sign sequence of length $k-1$ consisting only of the symbol $+$. That is,

$$I_{k-1} = \underbrace{+ + \cdots +}_{k-1 \text{ times}}. \tag{3.2.47}$$

Define $\widehat{e}_k = \widehat{e}_{I_{k-1}}$ and observe that $\widehat{e}_k \in \operatorname{Ext}^k_{\mathcal{E}}(\mathcal{R}, \mathcal{R})$. (Following the previously-established convention, assume I_0 is the empty sequence and that $e_1 = e$.)

Define a graded ring homomorphism

$$\Theta : R\langle x_1, x_2, x_3, \ldots\rangle \to \operatorname{Ext}^*_{\mathcal{E}}(\mathcal{R}, \mathcal{R}) \tag{3.2.48}$$

such that $\Theta(x_k) = \widehat{e}_k$ for all $k > 0$. We wish to prove that Θ is a bijection.

Define A to be the ideal consisting of all the elements in the graded ring $\mathrm{Ext}^*_{\mathcal{E}}(\mathcal{R},\mathcal{R})$ with positive degree. By Lemma 3.2.50 below, $A/A^2 = \langle [\widehat{e}_1], [\widehat{e}_2], [\widehat{e}_3], \ldots \rangle$. It is clear that Θ surjects onto A/A^2 and hence Θ is surjective.

The graded ring homomorphism Θ is a map of finite-rank free R-modules in each degree. By Theorem 3.2.7 we have that

$$\mathrm{rank}_R\big(\mathrm{Ext}^k_{\mathcal{E}}(\mathcal{R},\mathcal{R})\big) = \begin{cases} 1 & \text{if } k = 0, \text{ and} \\ 2^{k-1} & \text{if } k > 0. \end{cases} \tag{3.2.49}$$

It is a simple exercise in combinatorics to check that the ranks of $R\langle x_1, x_2, x_3, \ldots \rangle$ in each degree agree with the ranks in Equation 3.2.49. Since Θ is a surjective map of finite-rank free R-modules of the same rank in each degree it follows that Θ is also bijective in each degree. Hence Θ is a bijection, as desired. $\square$

Lemma 3.2.50. *Let $R = \mathbb{Z}/2$ and let $\mathcal{R}$ and $\mathcal{E}$ be defined as in Theorem 3.2.40. Define A to be the ideal consisting of all the elements in the graded ring $\mathrm{Ext}^*_{\mathcal{E}}(\mathcal{R},\mathcal{R})$ with positive degree. Then $A/A^2 = \langle [\widehat{e}_1], [\widehat{e}_2], [\widehat{e}_3], \ldots \rangle$ where the elements $\widehat{e}_k \in \mathrm{Ext}^k_{\mathcal{E}}(\mathcal{R},\mathcal{R})$ are those described in the proof of Theorem 3.2.40.*

Proof. We first wish to justify that if $\ell(J) = \ell(I)$ and $J \neq I$ then $\widehat{e}_J + \widehat{e}_I \in A^2$. Let $n = \ell(J) = \ell(I)$. There exist sequences $V, U, K \in \Lambda$ such that

$$\widehat{e}_J = \widehat{e}_{VK} \qquad \text{and} \qquad \widehat{e}_I = \widehat{e}_{UK}. \tag{3.2.51}$$

Not all of the signs in J and I can match since $J \neq I$ but some of them might. The sequence K is a sign sequence that matches the right-most signs of both J and I. Note that K could be the empty sequence (which is forced to happen when the right-most sign of J and the right-most sign of I are different) but V and U cannot be empty (because then both J and I would equal K). This means that $0 \leq \ell(K) < n$. Furthermore, we can assume that K is chosen to be the longest possible sign sequence (and hence V and U are the shortest possible) for which Equation 3.2.51 holds. Under this assumption the choices of V, U, and K are unique and the right-most sign of V and U must be different.

We wish to proceed by reverse induction on $k = \ell(K)$. In the base case, suppose $k = n-1$. Then V and U have length 1 so $J = \pm K$ and $I = \mp K$. It follows that

$$\widehat{e} \cdot \widehat{e}_K = \widehat{e}_{+K} + \widehat{e}_{-K} = \widehat{e}_J + \widehat{e}_I. \tag{3.2.52}$$

Hence $\widehat{e}_J + \widehat{e}_I \in A^2$.

Now suppose that $k < n-1$. The induction hypothesis is that $\widehat{e}_X + \widehat{e}_Y \in A^2$ whenever the right-most $k+1$ (or more) signs of X and Y match. Since $R = \mathbb{Z}/2$ we have

$$\begin{aligned} \widehat{e}_J + \widehat{e}_I &= \widehat{e}_{VK} + \widehat{e}_{UK} \\ &= \widehat{e}_{VK} + \widehat{e}_{U^\circ K} + \widehat{e}_{U^\circ K} + \widehat{e}_{UK}. \end{aligned} \tag{3.2.53}$$

Our choice of $V, U, K \in \Lambda$ guaranteed that the right-most signs of V and U were different. Hence the right-most signs of V and U° match. It follows that the right-most $k+1$ signs of the sequences VK and $U^\circ K$ match so, by the induction hypothesis, $\widehat{e}_{VK} + \widehat{e}_{U^\circ K} \in A^2$.

Now, since $\ell(U) > 0$ we can write $U = \overline{U}S$ where $\ell(S) = 1$. By Theorem 3.2.34,

$$\widehat{e}_{\overline{U}} \cdot \widehat{e}_K = \widehat{e}_{\overline{U}+K} + \widehat{e}_{\overline{U}^\circ-K} \qquad \text{and} \qquad \widehat{e}_{\overline{U}^\circ} \cdot \widehat{e}_K = \widehat{e}_{\overline{U}^\circ+K} + \widehat{e}_{\overline{U}-K}. \tag{3.2.54}$$

In the case when $S = +$ we have that $UK = \overline{U} + K$ and $U^\circ K = \overline{U}^\circ - K$. In the case when $S = -$ we have that $UK = \overline{U} - K$ and $U^\circ K = \overline{U}^\circ + K$. In either case, Equation 3.2.54 shows that $\widehat{e}_{U^\circ K} + \widehat{e}_{UK} \in A^2$.

We have now shown that $\widehat{e}_{VK} + \widehat{e}_{U^\circ K} \in A^2$ and $\widehat{e}_{U^\circ K} + \widehat{e}_{UK} \in A^2$. Since A^2 is closed under addition it follows that $\widehat{e}_J + \widehat{e}_I \in A^2$ by Equation 3.2.53. Hence, by induction, if $\ell(J) = \ell(I)$ and $J \neq I$ then $\widehat{e}_J + \widehat{e}_I \in A^2$.

We now wish to argue that $\widehat{e}_K \notin A^2$ for all $K \in \Lambda$. By Theorem 3.2.34 we know that $\widehat{e}_J \cdot \widehat{e}_I = \widehat{e}_{J+I} + \widehat{e}_{J^\circ - I}$. Now consider a sum

$$\widehat{e}_J \cdot \widehat{e}_I + \widehat{e}_{J'} \cdot \widehat{e}_{I'} = \widehat{e}_{J+I} + \widehat{e}_{J^\circ - I} + \widehat{e}_{J'+I'} + \widehat{e}_{(J')^\circ - I'}. \tag{3.2.55}$$

Since R has characteristic 2 the terms on the right side of that sum can cancel pairwise but after cancellation an even number of terms must remain. Now consider an arbitrary sum $x = \sum_\alpha \widehat{e}_{J_\alpha} \cdot \widehat{e}_{I_\alpha}$, i.e. an arbitrary element of A^2. When those products are expanded and we write $x = \sum_\beta \widehat{e}_{L_\beta}$, pairwise cancellation means that this new sum must also contain an even number of terms. In particular, the resulting sum cannot contain a single term so if $K \in \Lambda$ then $\widehat{e}_K \notin A^2$.

We have shown that if $\widehat{e}_K \in A$ then $[\widehat{e}_K]$ is non-zero in A/A^2. We have also shown that for $k > 0$ elements $[\widehat{e}_K]$ with $\ell(K) = k - 1$ are all equivalent in A^2 (since $\widehat{e}_K + \widehat{e}_{K'} \in A^2$ if $\ell(K) = \ell(K')$) and hence we can choose one such element, namely $[\widehat{e}_k] = [\widehat{e}_{I_{k-1}}]$, to represent them. Since it is clear that $[\widehat{e}_k] \neq [\widehat{e}_j]$ when $k \neq j$, the result follows. $\square$

3.3 A Conjecture about the Tensor Algebra

Recall the situation from the beginning of the chapter where $\mathcal{C}$ is a tensor category, R is a ring object, and M is an R-bimodule object. We constructed the square-zero extention of R by M, called $\mathcal{E}_R(M)$. We now wish to construct the tensor algebra, $T_R(M)$.

As an $\mathbb{N}$-graded object in $\mathcal{C}$,

$$T_R(M) = R \oplus M \oplus (M \otimes_R M) \oplus (M \otimes_R M \otimes_R M) \oplus \dots \tag{3.3.1}$$

where R is in degree 0 and $M^{\otimes k}$ is in degree k. An R-algebra structure is given by the bimodule maps

$$\begin{aligned} R \otimes M^{\otimes k} &\to M^{\otimes k} \\ M^{\otimes k} \otimes R &\to M^{\otimes k} \end{aligned} \tag{3.3.2}$$

and the natural map

$$M^{\otimes p} \otimes M^{\otimes q} \to M^{\otimes (p+q)}. \tag{3.3.3}$$

Consider the case when $\mathcal{C} = \mathbf{Ab}$. It is straightforward to show that

$$\mathrm{Ext}_{\mathcal{E}_R(M)}(R, R) \cong T_R(M). \tag{3.3.4}$$

As an example, consider $M = R$. First, we have

$$\mathcal{E}_R(R) \cong \frac{R[x]}{\langle x^2 \rangle} \quad \text{and so} \quad \mathrm{Ext}_{\mathcal{E}_R(R)}(R, R) \cong R[x] \tag{3.3.5}$$

Furthermore, since $R^{\otimes k} \cong R$ we have that

$$T_R(R) = R \oplus R \oplus R \oplus R \oplus \dots \quad \text{and so} \quad T_R(R) \cong R[x]. \tag{3.3.6}$$

Hence $\operatorname{Ext}_{\mathcal{E}_R(R)}(R, R) \cong T_R(R)$.

One might suspect a similar isomorphism in the case when $\mathcal{C} = \underline{\text{Mack}}$. Note that $\operatorname{Ext}_{\mathcal{E}_{\mathcal{R}}(\mathcal{M})}(\mathcal{R}, \mathcal{R})$ and $T_{\mathcal{R}}(\mathcal{M})$ do not live in the same category; the former is a traditional ring and the latter is a Mackey functor. Instead, Conjecture 3.3.7 uses the internal Ext construction.

Conjecture 3.3.7. $\underline{\operatorname{Ext}}_{\mathcal{E}_{\mathcal{R}}(\mathcal{M})}(\mathcal{R}, \mathcal{R}) \cong T_{\mathcal{R}}(\mathcal{M})$ *for any Mackey ring* $\mathcal{R}$ *and* $\mathcal{R}$-*bimodule* $\mathcal{M}$.

We wish to support this conjecture by showing that

$$(\underline{\operatorname{Ext}}_{\mathcal{E}_{\mathcal{R}}(\mathcal{M})}(\mathcal{R}, \mathcal{R}))_\bullet \cong (T_{\mathcal{R}}(\mathcal{M}))_\bullet. \tag{3.3.8}$$

in the case where $\mathcal{M} = \mathcal{R}$ and when $\mathcal{M} = \mathcal{F}_\theta(\mathbb{Z}) \,\Box\, \mathcal{R}$. To do this, first recall that

$$(\underline{\operatorname{Ext}}_{\mathcal{E}_{\mathcal{R}}(\mathcal{M})}(\mathcal{R}, \mathcal{R}))_\bullet \cong \operatorname{Ext}_{\mathcal{E}_{\mathcal{R}}(\mathcal{M})}(\mathcal{R}, \mathcal{R}). \tag{3.3.9}$$

Consider $\mathcal{M} = \mathcal{R}$. Since $\mathcal{R}^{\Box k} \cong \mathcal{R}$ we have

$$T_{\mathcal{R}}(\mathcal{R}) = \mathcal{R} \oplus \mathcal{R} \oplus \mathcal{R} \oplus \dots \tag{3.3.10}$$

and, in particular, $(T_{\mathcal{R}}(\mathcal{R}))_\bullet \cong R_\bullet[x]$. Furthermore, by Theorem 3.1.1, we have

$$(\underline{\operatorname{Ext}}_{\mathcal{E}_{\mathcal{R}}(\mathcal{R})}(\mathcal{R}, \mathcal{R}))_\bullet \cong \operatorname{Ext}_{\mathcal{E}_{\mathcal{R}}(\mathcal{R})}(\mathcal{R}, \mathcal{R}) \cong R_\bullet[x] \tag{3.3.11}$$

and hence

$$(\underline{\mathrm{Ext}}_{\mathcal{E}_{\mathcal{R}}(\mathcal{R})}(\mathcal{R},\mathcal{R}))_\bullet \cong (T_{\mathcal{R}}(\mathcal{R}))_\bullet. \tag{3.3.12}$$

Now consider $\mathcal{M} = \widehat{\mathcal{R}} = \mathcal{F}_\theta(\mathbb{Z}) \,\square\, \mathcal{R}$. In section 3.2 we computed $\mathrm{Ext}_{\mathcal{E}_{\mathcal{R}}(\widehat{\mathcal{R}})}(\mathcal{R},\mathcal{R})$. We now proceed to compute $(T_{\mathcal{R}}(\widehat{\mathcal{R}}))_\bullet$ as well.

Recall the structure of $\widehat{\mathcal{R}} = \mathcal{F}_\theta(\mathbb{Z}) \,\square\, \mathcal{R}$ described in Chapter II:

$$(\mathcal{R} \,\square\, \mathcal{F}_\theta(\mathbb{Z}))_\theta \;\leftrightarrows\; (\mathcal{R} \,\square\, \mathcal{F}_\theta(\mathbb{Z}))_\bullet$$

$$\begin{tikzcd}
\boxed{tg} \arrow[dr, dashed] & \\
\boxed{g} \arrow[r, dashed] & \boxed{p_*(g) = p_*(tg)}
\end{tikzcd}$$

The element g in this diagram is a chosen generator for $\widehat{\mathcal{R}}$. Now recall that for any Mackey functors $\mathcal{X}$ and $\mathcal{Y}$, $(\mathcal{X} \,\square\, \mathcal{Y})_\theta = X_\theta \otimes Y_\theta$. This means that

$$(T_{\mathcal{R}}(\widehat{\mathcal{R}}))_\theta \cong R_\theta \oplus R_\theta^2 \oplus \left(R_\theta^2\right)^{\otimes 2} \oplus \left(R_\theta^2\right)^{\otimes 3} \oplus \dots \tag{3.3.13}$$

with $(R_\theta^2)^{\otimes k}$ in degree k. Furthermore, the R_θ-linear generators of $(T_{\mathcal{R}}(\widehat{\mathcal{R}}))_\theta$ are

$$\begin{array}{ll}
\text{Degree 1:} & g,\ tg \\
\text{Degree 2:} & g\otimes g,\ g\otimes tg,\ tg\otimes g,\ tg\otimes tg \\
\text{Degree 3:} & g\otimes g\otimes g,\ g\otimes g\otimes tg,\ g\otimes tg\otimes g,\ g\otimes tg\otimes tg, \\
 & tg\otimes g\otimes g,\ tg\otimes g\otimes tg,\ tg\otimes tg\otimes g,\ tg\otimes tg\otimes tg.
\end{array} \tag{3.3.14}$$

Let β_k be the set of generators in degree $k > 0$ and let $\beta = \bigcup_{k>0} \beta_k$. Then, in general,

$$\beta_k = \{\phi \otimes g \mid \phi \in \beta_{k-1}\} \cup \{\phi \otimes tg \mid \phi \in \beta_{k-1}\}. \tag{3.3.15}$$

Note that $t(\phi \otimes tg) = t\phi \otimes g$ which means that the twist map on $(T_{\mathcal{R}}(\widehat{\mathcal{R}}))_\theta$ induces a bijection between $\{\phi \otimes g \mid \phi \in \beta_{k-1}\}$ and $\{\phi \otimes tg \mid \phi \in \beta_{k-1}\}$. Note also that in degree k the rank of $(T_{\mathcal{R}}(\widehat{\mathcal{R}}))_\theta$ is 2^k as an R_θ-module.

It is easy to compute that $\widehat{\mathcal{R}} \square \widehat{\mathcal{R}} \cong \widehat{\mathcal{R}} \oplus \widehat{\mathcal{R}}$. Inductively then we have

$$\widehat{\mathcal{R}}^{\otimes k} \cong \widehat{\mathcal{R}}^{\oplus 2^{k-1}}. \tag{3.3.16}$$

In particular, since $(\widehat{\mathcal{R}})_\bullet \cong R_\theta$, this tells us that in degree k, $(T_{\mathcal{R}}(\widehat{\mathcal{R}}))_\bullet$ is a free R_θ-module of rank 2^{k-1}.

We now wish to produce generators for $(T_{\mathcal{R}}(\widehat{\mathcal{R}}))_\bullet$. In degree k, $(T_{\mathcal{R}}(\widehat{\mathcal{R}}))_\bullet$ contains elements of the form $p_*(\phi)$ for $\phi \in \beta_k$ subject to the relation that $p_*(\phi) = p_*(t\phi)$. Note that since $|\beta_k| = 2^k$, there are 2^{k-1} of these elements. Based on the properties of $\widehat{\mathcal{R}}$ it follows that these 2^{k-1} elements generate $(T_{\mathcal{R}}(\widehat{\mathcal{R}}))_\bullet$. In summary, we have developed the following picture for $T_{\mathcal{R}}(\widehat{\mathcal{R}})$:

$$\begin{array}{ccccl}
\vdots & & & \vdots & \\
\hline
\boxed{\phi \otimes g} \quad \boxed{t\phi \otimes tg} & \dashrightarrow & & \boxed{p_*(\phi \otimes g)} & \text{(deg k)} \\
\hline
\vdots & & & \vdots & \\
\hline
\boxed{g \otimes g} \quad \boxed{tg \otimes tg} & \dashrightarrow & & \boxed{p_*(g \otimes g) = p_*(tg \otimes tg)} & \\
\boxed{g \otimes tg} \quad \boxed{tg \otimes g} & \dashrightarrow & & \boxed{p_*(g \otimes tg) = p_*(tg \otimes g)} & \text{(deg 2)} \\
\hline
\boxed{g} \quad \boxed{tg} & \dashrightarrow & & \boxed{p_*(g) = p_*(tg)} & \text{(deg 1)} \\
\hline
\boxed{p^* 1_\bullet} & \dashleftarrow & & \boxed{1_\bullet} & \text{(deg 0)}
\end{array}$$

Let $1_\theta = p^*(1_\bullet)$ and note that 1_θ is the unit in the algebra $(T_{\mathcal{R}}(\widehat{\mathcal{R}}))_\theta$ (since p^* is a ring map). For the remainder of the section we will use the set

$$\gamma = \{p_*(\phi \otimes g) \mid \phi \in \beta \cup \{1_\theta\}\} \tag{3.3.17}$$

as our R_θ-linear generators[2] for $(T_{\mathcal{R}}(\widehat{\mathcal{R}}))_\bullet$ in positive degree.

Lastly, we need to understand multiplication on $(T_{\mathcal{R}}(\widehat{\mathcal{R}}))_\bullet$. It suffices to understand multiplication of the generators so choose generators $p_*(\phi \otimes g), p_*(\psi \otimes g) \in \gamma$. First, recall the fact that $p^*p_* = 1 + t$ and the Frobenius relations in Theorem 2.2.2. Also, recall that $(T_{\mathcal{R}}(\widehat{\mathcal{R}}))_\bullet$ is a (traditional) tensor algebra and so the product of $\phi \otimes g$ and $\psi \otimes g$ in $(T_{\mathcal{R}}(\widehat{\mathcal{R}}))_\bullet$ is simply $\phi \otimes g \otimes \psi \otimes g$. We then have

$$\begin{aligned} p_*(\phi \otimes g) \cdot p_*(\psi \otimes g) &= p_*((\phi \otimes g) \otimes p^*p_*(\psi \otimes g)) \\ &= p_*((\phi \otimes g) \otimes (1+t)(\psi \otimes g)) \\ &= p_*((\phi \otimes g) \otimes (\psi \otimes g + t\psi \otimes tg)) \\ &= p_*(\phi \otimes g \otimes \psi \otimes g) + p_*(\phi \otimes g \otimes t\psi \otimes tg) \\ &= p_*(\phi \otimes g \otimes \psi \otimes g) + p_*(t\phi \otimes tg \otimes \psi \otimes g). \end{aligned} \tag{3.3.18}$$

We can now describe the isomorphism $\mathrm{Ext}_{\mathcal{E}_{\mathcal{R}}(\widehat{\mathcal{R}})}(\mathcal{R}, \mathcal{R}) \cong (T_{\mathcal{R}}(\widehat{\mathcal{R}}))_\bullet$. Consider the sets β and Λ. There is a map

$$[-] : \beta \cup \{1_\theta\} \to \Lambda \tag{3.3.19}$$

[2] We use $\beta \cup \{1_\theta\}$ instead of just β so that γ includes $p_*(1_\theta \otimes g) = p_*(g)$.

where $[1_\theta]$ is the empty sequence, $[g] = +$, $[tg] = -$, and we let $[-]$ commute with the operations of tensor in $\beta \cup \{1\}$ and concatenation in Λ. As examples,

$$[g \otimes tg \otimes g] = + - + \quad \text{and} \quad [tg \otimes tg \otimes g \otimes tg] = - - + - \,. \tag{3.3.20}$$

It is clear that Θ is a bijection. Furthermore, it is clear that for any $\phi \in \beta \cup \{1_\theta\}$ we have $[t\phi] = [\phi]^\circ$.

We can define a map of $R_\bullet$-modules

$$\Theta : (T_{\mathcal{R}}(\widehat{\mathcal{R}}))_\bullet \to \operatorname{Ext}_{\mathcal{E}_{\mathcal{R}}(\widehat{\mathcal{R}})}(\mathcal{R}, \mathcal{R}) \tag{3.3.21}$$

such that $\Theta(1_\bullet) = 1_\bullet$ and

$$\Theta(p_*(\phi \otimes g)) = \widehat{e}_{[\phi]} \tag{3.3.22}$$

for generators $p_*(\phi \otimes g) \in \beta \cup \{1_\theta\}$. To show that this is a map of $R_\bullet$-algebras we need only show that it is a homomorphism.

Choose $p_*(\phi \otimes g), p_*(\psi \otimes g) \in \gamma$. On one hand,

$$\Theta(p_*(\phi \otimes g)) \cdot \Theta(p_*(\psi \otimes g)) = \widehat{e}_{[\phi]} \cdot \widehat{e}_{[\psi]} = \widehat{e}_{[\phi]+[\psi]} + \widehat{e}_{[\phi]^\circ - [\psi]}. \tag{3.3.23}$$

On the other hand,

$$\begin{aligned}
&\Theta(p_*(\phi \otimes g) \cdot p_*(\psi \otimes g)) \\
&\qquad = \Theta(p_*(\phi \otimes g \otimes \psi \otimes g) + p_*(t\phi \otimes tg \otimes \psi \otimes g)) \\
&\qquad = \widehat{e}_{[\phi \otimes g \otimes \psi]} + \widehat{e}_{[t\phi \otimes tg \otimes \psi]} \\
&\qquad = \widehat{e}_{[\phi]+[\psi]} + \widehat{e}_{[\phi]^\circ - [\psi]}.
\end{aligned} \tag{3.3.24}$$

Hence Θ is a homomorphism. Since Θ is an abelian group isomorphism by inspection it is an isomorphism of $R_\bullet$-algebras.

CHAPTER IV

HOMOLOGICAL PROPERTIES OF THE RINGS $\mathbb{M}_2$ AND $\mathcal{M}_2$

In this section we investigate certain homological questions about a ring $\mathbb{M}_2$ and a Mackey ring $\mathcal{M}_2$ that show up in $\mathbb{Z}/2$-equivariant algebraic topology.

Definition 4.0.1. Let k be a field. Define a k-algebra $\mathbb{M}_2$ as follows: elements $\tau^i\rho^j$ and $\frac{\theta}{\tau^i\rho^j}$ for $i, j \in \mathbb{Z}_{\geq 0}$ form a k-basis for $\mathbb{M}_2$ and multiplication on this basis is given by

- $(\tau^i\rho^j)(\tau^p\rho^q) = \tau^{i+p}\rho^{j+q}$,
- $(\frac{\theta}{\tau^i\rho^j})(\frac{\theta}{\tau^p\rho^q}) = 0$,
- $(\tau^i\rho^j)(\frac{\theta}{\tau^p\rho^q}) = 0$ if $i > p$ or $j > q$, and
- $(\tau^i\rho^j)(\frac{\theta}{\tau^p\rho^q}) = \frac{\theta}{\tau^{p-i}\rho^{q-j}}$ if $i \leq p$ and $j \leq q$.

We will consider $\mathbb{M}_2$ to be a $\mathbb{Z}^2$-graded module where $\tau^i\rho^j$ is in degree $(j, j+i)$ and $\frac{\theta}{\tau^i\rho^j}$ is in degree $(-j, -i-j-2)$.

For the rest of this chapter we will write $R = k[\tau, \rho]$ and $S = \mathbb{M}_2$ for convenience. It is clear that R is a $\mathbb{Z}^2$-graded ring but we will differ from normal convention and give $\tau^i\rho^j$ a degree of $(j, j+i)$. For the remainder of this chapter we will assume that all R-modules and S-modules are $\mathbb{Z}^2$-graded.

There is a ring map $S \twoheadrightarrow R$ under which $\tau^i\rho^j \mapsto \tau^i\rho^j$ and $\frac{\theta}{\tau^i\rho^j} \mapsto 0$ and a ring inclusion $R \hookrightarrow S$. The former map allows us to regard an R-module M as an S-module which we will call ${}_SM$ and the latter map allows us to regard an S-module N as an R-module, which we will call ${}_RN$. Note that the regular S-module splits over R as ${}_RS \cong R \oplus J$ where J is the R-submodule of ${}_RS$ generated by all elements $\frac{\theta}{\tau^i\rho^j}$.

Note, also, that the kernel of the map $S \twoheadrightarrow k$ sending 1 to 1 is $\langle \tau, \rho \rangle$; this is because $\frac{\theta}{\tau^i \rho^j} = (\frac{\theta}{\tau^{i+1}\rho^j})\tau \in \langle \tau, \rho \rangle$.

The goal of this chapter is to relate $\mathrm{Ext}^*_S({}_R M, k)$ to $\mathrm{Ext}^*_R(M, k)$ for finite length R-modules M. In particular, we compute $\mathrm{Ext}^*_S(k, k)$ as a ring.

4.1 A Resolution of J

First, we will produce an R-module resolution for J. For $i \in \mathbb{N}$ consider the submodule $\langle \frac{\theta}{\tau^i \rho^i} \rangle \leq J$. The following is a free resolution of this module where $d_2(t) = (\rho^{i+1}t, -\tau^{i+1}t)$ and $d_1(a, b) = \tau^{i+1}a - \rho^{i+1}b$:

$$0 \longrightarrow R \xrightarrow{d_2} R^2 \xrightarrow{d_1} R \longrightarrow\!\!\!\!\rightarrow \langle \tfrac{\theta}{\tau^i \rho^i} \rangle$$

Of course, the map $R \twoheadrightarrow \langle \frac{\theta}{\tau^i \rho^i} \rangle$ is given by $1 \mapsto \frac{\theta}{\tau^i \rho^i}$. These chain complexes form a directed system in $\mathrm{Ch}(R - \mathbf{Mod})$ via chain maps of the form

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d, hook] \\
0 \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "f_2"'] & R^2 \arrow[r] \arrow[d, "f_1"'] & R \arrow[r, two heads] \arrow[d, "f_0"'] & \langle \frac{\theta}{\tau^i \rho^i} \rangle \arrow[d, hook] \\
0 \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & R^2 \arrow[r] \arrow[d] & R \arrow[r, two heads] \arrow[d] & \langle \frac{\theta}{\tau^{i+1} \rho^{i+1}} \rangle \arrow[d, hook] \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

where f_2 is the identity map, $f_1(a, b) = (\rho a, \tau b)$, and $f_0(t) = \tau \rho t$. On the homology of the individual chain complexes, the induced maps become

$$\begin{tikzcd}
0 \arrow[d, "\hat{f}_2"'] & 0 \arrow[d, "\hat{f}_1"'] & \langle \frac{\theta}{\tau^i \rho^i} \rangle \arrow[d, "\hat{f}_0"'] \\
0 & 0 & \langle \frac{\theta}{\tau^{i+1} \rho^{i+1}} \rangle
\end{tikzcd}$$

and $\hat{f}_0(t) = \tau\rho t$. By computing the colimits of the columns in the previous two diagrams we have that the directed limit of the original directed system is

$$0 \longrightarrow R \longrightarrow \tau^{-1}R \oplus \rho^{-1}R \longrightarrow (\tau\rho)^{-1}R$$

and its homology is zero except at the right-most spot, where the homology is J. Hence this resulting complex is a flat R-module resolution of J.

4.2 Calculating the Groups $\mathrm{Ext}^i_{\mathbb{M}_2}(k,k)$

Now suppose that M is a finite length R-module. Since ${}_RS \cong J \oplus R$ we have that a resolution of ${}_RS$ is given by

$$R \longrightarrow \tau^{-1}R \oplus \rho^{-1}R \longrightarrow (\tau\rho)^{-1}R \oplus R$$

and tensoring this complex with M (over R) yields

$$M \longrightarrow \cancelto{0}{\tau^{-1}M} \oplus \cancelto{0}{\rho^{-1}M} \longrightarrow \cancelto{0}{(\tau\rho)^{-1}M} \oplus M.$$

Note that $\tau^{-1}M$, $\rho^{-1}M$, and $(\tau\rho)^{-1}M$ are all zero because M is finite length and hence killed by a sufficiently large power of τ, ρ, and $\tau\rho$. Hence we have the following isomorphisms of R-modules:

$$\mathrm{Tor}_2^R(M, {}_RS) \cong M, \qquad \mathrm{Tor}_1^R(M, {}_RS) \cong 0, \quad \text{and} \qquad \mathrm{Tor}_0^R(M, {}_RS) \cong M \tag{4.2.1}$$

Choose a projective resolution $0 \to P_2 \to P_1 \to P_0 \twoheadrightarrow M$ of M as an R-module and consider the chain complex $P_\bullet \otimes_R S$. We wish to compute the homology of this complex. The R-module structure of $H_*(P_\bullet \otimes_R S)$ is given by $\mathrm{Tor}_*^R(M, {}_RS)$ and hence the following isomorphisms hold as R-modules:

$$H_2(P_\bullet \otimes_R S) \cong M, \qquad H_1(P_\bullet \otimes_R S) \cong 0, \quad \text{and} \qquad H_0(P_\bullet \otimes_R S) \cong M \tag{4.2.2}$$

By definition, the element $\theta \in S$ acts as zero on ${}_SM$ and, by Corollary 4.2.4 below, θ acts as zero on $H_*(P_\bullet \otimes_R S)$. Hence the isomorphisms above also hold as S-modules.

Lemma 4.2.3. *Let N be a multigraded S-module and let $n \in N$. If $\theta n \neq 0$ then $\langle n \rangle \cong S$ and hence $S \leq N$.*

Proof. Choose a monomial $v = \tau^i \rho^j \in S$. Observe that $\frac{\theta}{v}(vn) = \theta n$. Since $\theta n \neq 0$ it follows that $vn \neq 0$. Similarly, $v\left(\frac{\theta}{v} n\right) = \theta n$ and hence $\frac{\theta}{v} n \neq 0$. It follows that the (multigraded) map of S modules $S \to N$ defined by $t \mapsto tn$ is injective and clearly its image is $\langle n \rangle$. This proves the result. $\square$

Corollary 4.2.4. *If N is an S-module such that ${}_RN$ is finite length then θ acts as zero on N.*

Proof. Recall that $\dim_k(N) < \infty$ since ${}_RN$ is finite length. Suppose, to the contrary, that there exists some $n \in N$ such that $\theta n \neq 0$. Then, by Lemma 4.2.3, $S \leq N$. Hence $R \leq {}_RS \leq {}_RN$ and so ${}_RN$ does not have finite length. $\square$

We now wish to find a projective S-module resolution of ${}_SM$. At this point we have developed the following exact complex of S-modules:

$$ {}_SM \hookrightarrow P_2 \otimes_R S \xrightarrow{d_2} P_1 \otimes_R S \xrightarrow{d_1} P_0 \otimes_R S \twoheadrightarrow {}_SM $$

Now consider a new complex

$$\begin{tikzcd}
\cdots \arrow[r, "d_1"] & P_0 \otimes_R S \arrow[r, "d_3"] & P_2 \otimes_R S \arrow[r, "d_2"] & P_1 \otimes_R S \arrow[dll, "d_1"] & \\
 & P_0 \otimes_R S \arrow[r, "d_3"] & P_2 \otimes_R S \arrow[r, "d_2"] & P_1 \otimes_R S \arrow[r, "d_1"] & P_0 \otimes_R S
\end{tikzcd}$$

where $d_3 : P_0 \otimes_R S \to P_2 \otimes_R S$ is the following composition:

$$
\begin{tikzcd}
P_0 \otimes_R S \arrow[r, two heads] \arrow[rr, bend right, "d_3"'] & \operatorname{coker}(d_1) \cong {}_S M \cong \ker(d_2) \arrow[r, hook] & P_2 \otimes_R S
\end{tikzcd}
$$

This is an acyclic complex of free S-modules and hence resolves $\operatorname{coker}(d_1) \cong {}_S M$. This resolution is used to prove Theorem 4.2.5 below.

Theorem 4.2.5. *Let M be a finite length R-module. Then*

1. $\operatorname{Ext}^i_S({}_S M, k) \cong \operatorname{Ext}^i_R(M, k)$ *for* $i \in \{0, 1, 2\}$ *and*

2. $\operatorname{Ext}^i_S({}_S M, k) \cong \operatorname{Ext}^{i-3}_S({}_S M, k)$ *for all* $i \geq 3$.

Proof. Recall that $0 \to P_2 \to P_1 \to P_0 \twoheadrightarrow M$ is a projective resolution for the R-module M. Write $d_2 = \bar{d}_2 \otimes \mathrm{id}_S$ and $d_1 = \bar{d}_1 \otimes \mathrm{id}_S$ where $\bar{d}_2 : P_2 \to P_1$ and $\bar{d}_1 : P_1 \to P_0$ are the maps in this resolution.

Apply the functor $\operatorname{Hom}_S(-, k)$ to the resolution for ${}_S M$ constructed previously. Lemma 4.2.9 below shows that when the functor $\operatorname{Hom}_S(-, k)$ is applied to d_3 the result is the zero map. This fact, combined with the repitition in the resolution, shows that the resulting complex breaks up into exact sequences of the following form:

$$
0 \leftarrow \operatorname{Hom}_S(P_2 \otimes_R S, k) \leftarrow \operatorname{Hom}_S(P_1 \otimes_R S, k) \leftarrow \operatorname{Hom}_S(P_0 \otimes_R S, k) \leftarrow 0. \tag{4.2.6}
$$

This proves part 2 of the theorm.

By adjunction we have

$$
\operatorname{Hom}_S(P_i \otimes_R S, k) \cong \operatorname{Hom}_R(P_i, \operatorname{Hom}_S(S, k)) \cong \operatorname{Hom}_R(P_i, k). \tag{4.2.7}
$$

The naturality of the adjunction then tells us that these exact sequences become

$$0 \leftarrow \mathrm{Hom}_R(P_2, k) \xleftarrow{\bar{d}_2^*} \mathrm{Hom}_R(P_1, k) \xleftarrow{\bar{d}_1^*} \mathrm{Hom}_R(P_0, k) \leftarrow 0. \tag{4.2.8}$$

The homology of this complex is $\mathrm{Ext}^i_R(M, k)$ which proves part 1 of the theorem. □

Lemma 4.2.9. *Let $d_3 : P_0 \otimes_R S \to P_2 \otimes_R S$ be the map described prior to Theorem 4.2.5. The induced map*

$$d_3^* : \mathrm{Hom}_S(P_2 \otimes_R S, k) \to \mathrm{Hom}_S(P_0 \otimes_R S, k) \tag{4.2.10}$$

is the zero map.

Proof. We repeat the notation used for $\bar{d}_1$ and $\bar{d}_2$ in the proof of Theorem 4.2.5. Choose a map $\phi : P_2 \otimes_R S \to k$. Then $d_3^*(\phi)$ is the composition $\phi \circ d_3$, or

$$P_0 \otimes_R S \twoheadrightarrow \mathrm{coker}(d_1) \cong \ker(d_2) \hookrightarrow P_2 \otimes_R S \xrightarrow{\phi} k. \tag{4.2.11}$$

Our goal is to prove that this composition is the zero map. For the rest of this proof assume that all tensors are taken over R.

We first claim that $\ker d_2 \subseteq P_2 \otimes J$. (Recall that $R \subseteq S$ and $S = R \oplus J$. In particular, J is spanned by elements of the form $\frac{\theta}{m}$ for monomials $m \in R$.) Choose an element $x \in \ker d_2$. Then we can write

$$x = \sum_{i=0}^{N} v_i \otimes m_i + \sum_{i=0}^{N'} v_i' \otimes \tfrac{\theta}{m_i'} \tag{4.2.12}$$

for integers $0 \le N, N' < \infty$, elements $v_i, v_i' \in P_2$, and monomials $m_i, m_i' \in R$. Note that every element $\frac{\theta}{m} \in J$ is annihilated by some power of ρ. Choose k large enough so that ρ^k annihilates each $\frac{\theta}{m'_i}$ (which is possible because there are only $N' < \infty$ such elements). Note that $\rho^k \in R$ also annihilates elements $v'_i \otimes \frac{\theta}{m'_i}$. Since $\ker d_2$ is a submodule of $P_2 \otimes S$ we have that $\rho^k x \in \ker d_2$. Furthermore,

$$\begin{aligned} \rho^k x &= \sum_{i=0}^{N} \rho^k (v_i \otimes m_i) + \sum_{i=0}^{N'} \rho^k (v'_i \otimes \tfrac{\theta}{m'_i}) \\ &= \sum_{i=0}^{N} \rho^k (v_i \otimes m_i) \\ &= \sum_{i=0}^{N} (\rho^k v_i m_i) \otimes 1. \end{aligned} \tag{4.2.13}$$

Thus we have

$$\begin{aligned} 0 = d_2(\rho^k x) &= d_2 \left(\sum_{i=0}^{N} (\rho^k v_i m_i) \otimes 1 \right) \\ &= (\bar{d}_2 \otimes \mathrm{id}) \left(\sum_{i=0}^{N} (\rho^k v_i m_i) \otimes 1 \right) = \bar{d}_2 \left(\sum_{i=0}^{N} \rho^k v_i m_i \right) \otimes 1. \end{aligned} \tag{4.2.14}$$

Recall that $\bar{d}_2 : P_2 \to P_1$ is the map from the resolution $0 \to P_2 \to P_1 \to P_0 \twoheadrightarrow M$ and hence is injective. We can now conclude that

$$\begin{aligned} \bar{d}_2 \left(\sum_{i=0}^{N} \rho^k v_i m_i \right) &= 0, \\ \rho^k \bar{d}_2 \left(\sum_{i=0}^{N} v_i m_i \right) &= 0, \\ \bar{d}_2 \left(\sum_{i=0}^{N} v_i m_i \right) &= 0, \\ \text{and finally } \sum_{i=0}^{N} v_i m_i &= 0. \end{aligned} \tag{4.2.15}$$

Finally, we have that

$$0 = \left(\sum_{i=0}^{N} v_i m_i\right) \otimes 1 = \sum_{i=0}^{N}(v_i m_i \otimes 1) = \sum_{i=0}^{N}(v_i \otimes m_i) \tag{4.2.16}$$

and hence, recalling Equation 4.2.12, we have

$$x = \sum_{i=0}^{N} v_i \otimes m_i + \sum_{i=0}^{N'} v_i' \otimes \tfrac{\theta}{m_i'} = \sum_{i=0}^{N'} v_i' \otimes \tfrac{\theta}{m_i'}. \tag{4.2.17}$$

Thus $x \in P_2 \otimes J$. This proves that $\ker d_2 \subseteq P_2 \otimes J$ as we claimed.

We now claim that $\phi(P_2 \otimes J) = 0$. Choose $x \in P_2 \otimes J$. Then

$$x = \sum_{j=0}^{M} u_j \otimes \tfrac{\theta}{m_j} \tag{4.2.18}$$

for an integer $0 \leq M < \infty$, elements $u_j \in P_2$, and monomials $m_j \in R$. Since each $\frac{\theta}{m_j}$ is divisible by ρ we have that

$$x = \rho\left(\sum_{j=0}^{M} u_j \otimes \tfrac{\theta}{\rho m_j}\right) \quad \text{and so} \quad x = \rho x' \text{ for } x' = \sum_{j=0}^{M} u_j \otimes \tfrac{\theta}{\rho m_j} \in P_2 \otimes J. \tag{4.2.19}$$

Note that since $x' \in P_2 \otimes J$ we have that $d_2(x') \in k$. Then, since everything in k is annihilated by ρ, it then follows that

$$\phi(x) = \phi(\rho x') = \rho\phi(x') = 0. \tag{4.2.20}$$

This proves the claim that $\phi(P_2 \otimes J) = 0$.

We are now ready to prove the lemma. We have shown that $\ker d_2 \subseteq P_2 \otimes J$ and that $\phi(P_2 \otimes J) = 0$. It then follows that the composition

$$\ker d_2 \hookrightarrow P_2 \otimes S \xrightarrow{\phi} k \tag{4.2.21}$$

is zero. It is then clear that the composition shown in Equation 4.2.11 is zero and hence d_3^* is zero, as desired. $\square$

In summary, Theorem 4.2.5 tells us that as a vector space $\mathrm{Ext}_S^*({}_SM, k)$ is isomorphic to $\mathrm{Ext}_R^*({}_SM, k) \otimes_k k[\alpha]$ where α has degree 3. For the remainder of the paper we will omit subscripts; it should be clear from context whether M is being considered an R-module or an S-module.

4.3 Calculating the Ring Structure on $\mathrm{Ext}^i_{\mathbb{M}_2}(k, k)$

Consider the case when $M = k$. This module is resolved over R by the Koszul complex $0 \to R \to R^2 \to R \twoheadrightarrow k$ and we have shown that it is resolved over S by a resolution of the form

$$\cdots \xrightarrow{d_3} S \xrightarrow{d_2} S^2 \xrightarrow{d_1} S \xrightarrow{d_3} S \xrightarrow{d_2} S^2 \xrightarrow{d_1} S. \tag{4.3.1}$$

Recall that $d_1(a, b) = \tau a - \rho b$ and $d_2(t) = (\rho t, -\tau t)$. Also, $d_3(x) = x\theta$ for all $x \in k$ and $d_3(x) = 0$ for all $x \in S \setminus k$.

We now wish to compute the Yoneda extensions corresponding to basis elements of $\mathrm{Ext}_S^*(k, k)$. Two linearly independent Yoneda extensions in $\mathrm{Ext}_S^1(k, k)$ are given by

$$k \xrightarrow{1 \mapsto \tau} S/\langle \tau^2, \rho\rangle \xrightarrow{1 \mapsto 1} k \qquad \text{and} \qquad k \xrightarrow{1 \mapsto \rho} S/\langle \tau, \rho^2\rangle \xrightarrow{1 \mapsto 1} k. \tag{4.3.2}$$

Call these extensions h_τ and h_ρ, respectively.

There is a map of algebras $\mathrm{Ext}^*_R(k,k) \to \mathrm{Ext}^*_S(k,k)$. This is most easily seen on Yoneda extensions as the map $X_* \mapsto {}_S(X_*)$. There is also an algebra map $\mathrm{Ext}^*_S(k,k) \to \mathrm{Ext}^*_R(k,k)$ given by $X_* \mapsto {}_R(X_*)$ and it is clear that the composition

$$\mathrm{Ext}^*_R(k,k) \to \mathrm{Ext}^*_S(k,k) \to \mathrm{Ext}^*_R(k,k) \tag{4.3.3}$$

is the identity. Hence the map $\mathrm{Ext}^*_R(k,k) \to \mathrm{Ext}^*_S(k,k)$ is injective. We recall that

$$\mathrm{Ext}^*_R(k,k) = \frac{k[\tau,\rho]}{\langle \tau^2, \rho^2 \rangle} \tag{4.3.4}$$

and it is easy to verify that h_τ and h_ρ are the image of τ and ρ. This shows that $h_\tau h_\rho = h_\rho h_\tau$ and that this element is non-zero in $\mathrm{Ext}^2_S(k,k)$.

Define the extension $\alpha \in \mathrm{Ext}^3_S(k,k)$ as

$$k \xrightarrow{1 \mapsto \theta} S \xrightarrow{d_2} S^2 \xrightarrow{d_1} S \xrightarrow{1 \mapsto 1} k \tag{4.3.5}$$

We first wish to verify that $\alpha h_\tau = h_\tau \alpha$ and that this element is non-zero in $\mathrm{Ext}^4_S(k,k)$. Define $\phi : S^2 \to k$ to be $\phi(a,b) = [a]$. We will show that the cocycle ϕ represents both αh_τ and $h_\tau \alpha$.

In the diagram below, the top row is the resolution of k and the bottom row is the Yoneda extension corresponding to $h_\tau \alpha$.

$$\begin{tikzcd}
\cdots \arrow[r] & S^2 \arrow[r, "d_1"] \arrow[d, "\phi"] & S \arrow[r, "d_3"] \arrow[d, "\pi"] & S \arrow[r, "d_2"] \arrow[d, "\mathrm{id}"] & S^2 \arrow[r, "d_1"] \arrow[d, "\mathrm{id}"] & S \arrow[r, two heads] \arrow[d, "\mathrm{id}"] & k \arrow[d, "\mathrm{id}"] \\
& k \arrow[r, "1 \mapsto \tau"'] & S/\langle \tau^2, \rho \rangle \arrow[r, "1 \mapsto \theta"'] & S \arrow[r, "d_2"'] & S^2 \arrow[r, "d_1"'] & S \arrow[r, "1 \mapsto 1"'] & k
\end{tikzcd}$$

It is clear (since $d_1(1,0) = \tau$) that the diagram above commutes. This shows that ϕ represents $h_\tau \alpha$.

In the diagram below the top row is again the resolution of k while the bottom row is the Yoneda extension corresponding to αh_τ.

$$\begin{tikzcd}
\cdots \arrow[r] & S^2 \arrow[r, "d_1"] \arrow[d, "\phi"] & S \arrow[r, "d_3"] \arrow[d, "f_4"] & S \arrow[r, "d_2"] \arrow[d, "f_3"] & S^2 \arrow[r, "d_1"] \arrow[d, "f_2"] & S \arrow[r, two heads] \arrow[d, "f_1"] & k \arrow[d, "\mathrm{id}"] \\
& k \arrow[r, "1 \mapsto \theta"'] & S \arrow[r, "d_2"'] & S^2 \arrow[r, "d_1"'] & S \arrow[r, "1 \mapsto \tau"'] & S/\langle \tau^2, \rho \rangle \arrow[r, "1 \mapsto 1"'] & k
\end{tikzcd}$$

The f_i are given by

$$f_1(t) = [t], \quad f_2(a,b) = a, \quad f_3(t) = (0, -t), \quad \text{and} \quad f_4(t) = \left(\tfrac{\theta}{\tau}\right)t. \tag{4.3.6}$$

The reader can verify that this diagram commutes which shows that ϕ represents αh_τ. We have thus verified that $\alpha h_\tau = h_\tau \alpha$ is non-zero. A similar argument shows that $\alpha h_\rho = h_\rho \alpha$ is non-zero in $\mathrm{Ext}^4_S(k,k)$ (represented by the cocycle $S^2 \to k$ where $(a,b) \mapsto [b]$). It is also straightforward to show that $h_\tau h_\rho \alpha$ is non-zero in $\mathrm{Ext}^5_S(k,k)$ (represented by the cocycle $S \to k$ given by the quotient map).

We now wish to verify that the element $\alpha^i \in \mathrm{Ext}^{3i}_S(k,k)$ is non-zero for all $i > 1$. In the diagram below the top row is the resolution of k and the bottom row is the Yoneda extension for α^i.

$$\begin{tikzcd}
\cdots \arrow[r] & S \arrow[r, "d_3"] \arrow[d, "\psi"] & S \arrow[r, "d_2"] \arrow[d, "\mathrm{id}"] & S^2 \arrow[r, "d_1"] \arrow[d, "\mathrm{id}"] & \cdots \arrow[r, "d_2"] & S^2 \arrow[r, "d_2"] \arrow[d, "\mathrm{id}"] & S \arrow[r, two heads] \arrow[d, "\mathrm{id}"] & k \arrow[d, "\mathrm{id}"] \\
& k \arrow[r, "1 \mapsto \theta"'] & S \arrow[r, "d_2"'] & S^2 \arrow[r, "d_1"'] & \cdots \arrow[r, "d_2"'] & S^2 \arrow[r, "d_1"'] & S \arrow[r, two heads] & k
\end{tikzcd}$$

It is clear that this diagram commutes when the map $\psi : S \to k$ is the quotient map. This shows that $\alpha^i \neq 0$.

These facts together justify that

$$\begin{aligned}\operatorname{Ext}_S^{(3i)}(k,k) &= \langle \alpha^i \rangle,\\ \operatorname{Ext}_S^{(3i+1)}(k,k) &= \langle h_\tau \alpha^i, h_\rho \alpha^i \rangle, \text{and}\\ \operatorname{Ext}_S^{(3i+2)}(k,k) &= \langle h_\tau h_\rho \alpha^i \rangle.\end{aligned} \tag{4.3.7}$$

Furthermore, the products computed justify Theorem 4.3.8 below.

Theorem 4.3.8. *The ring* $\operatorname{Ext}_S^*(k,k)$ *is isomorphic to* $\operatorname{Ext}_R^*(k,k) \otimes_k k[\alpha]$ *as a* k*-algebra.*

Note that $\operatorname{Ext}_R^*(k,k)$ is isomorphic to $k[\tau,\rho]/\langle \tau^2, \rho^2 \rangle$. The isomorphism $\Phi : k[\tau,\rho]/\langle \tau^2,\rho^2\rangle \otimes_k k[\alpha] \to \operatorname{Ext}_S^*(k,k)$ is given by $\Phi(\tau) = h_\tau$, $\Phi(\rho) = h_\rho$, and $\Phi(\alpha^i) = \alpha^i$.

4.4 Calculating the Module Structure on $\operatorname{Ext}_{\mathbb{M}_2}^*(M,k)$

We now wish to understand the action of $\operatorname{Ext}_S^*(k,k)$ on $\operatorname{Ext}_S^*(M,k)$. That action follows directly from Theorem 4.4.1 below.

Theorem 4.4.1. *Suppose* M *is a finite length* R*-module. If* $u \in \operatorname{Ext}_S^i(M,k)$ *is non-zero then* αu *is also non-zero. Additionally, the map* $\operatorname{Ext}_S^i(M,k) \to \operatorname{Ext}_S^{i+3}(M,k)$ *given by* $u \mapsto \alpha u$ *is an isomorphism of vector spaces.*

Proof. We will proceed by induction on the length of M. If $\ell(M) = 1$ then $M = k$ and this is verified above. If $\ell(M) > 1$ then, since M has finite length, there exists a short exact sequence of the form

$$0 \to k \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0 \tag{4.4.2}$$

where $1 \leq \ell(N) < \ell(M)$. Consider the corresponding long exact sequence under $\mathrm{Ext}^*_S(-, k)$:

$$\cdots \leftarrow \mathrm{Ext}^i_S(k,k) \xleftarrow{\iota^*} \mathrm{Ext}^i_S(M,k) \xleftarrow{\pi^*} \mathrm{Ext}^i_S(N,k) \xleftarrow{\partial} \mathrm{Ext}^{i-1}_S(k,k) \leftarrow \cdots \tag{4.4.3}$$

It is clear that the maps ι^* and π^* are maps of left $\mathrm{Ext}^*_S(k,k)$-modules. We claim that the boundary map ∂ is, as well. This can be shown by considering the corresponding maps of Yoneda extensions. First, recall that $\partial : \mathrm{Ext}^*_S(k,k) \to \mathrm{Ext}^{*+1}_S(N,k)$ is given by $\partial(V) = V \cdot \mathscr{E}$ where $\mathscr{E}$ is the short exact sequence in Equation 4.4.2 (and $V \cdot \mathscr{E}$ indicates concatenation). Choose Yoneda extensions $U, V \in \mathrm{Ext}^*_S(k,k)$. Then

$$U \cdot \partial(V) = U \cdot (V \cdot \mathscr{E}) = (U \cdot V) \cdot \mathscr{E} = \partial(U \cdot V) \tag{4.4.4}$$

and so ∂ is a map of $\mathrm{Ext}^*_S(k,k)$-modules.

It follows that the diagram (of vector spaces) shown below is commutative when the vertical maps are all given by the (left) action of $\alpha \in \mathrm{Ext}^*_S(k,k)$.

$$\begin{tikzcd}
\mathrm{Ext}^{i+1}_S(N,k) \arrow[d, "\alpha\cdot"] & \mathrm{Ext}^i_S(k,k) \arrow[l] \arrow[d, "\alpha\cdot"] & \mathrm{Ext}^i_S(M,k) \arrow[l] \arrow[d, "\alpha\cdot"] & \mathrm{Ext}^i_S(N,k) \arrow[l] \arrow[d, "\alpha\cdot"] & \mathrm{Ext}^{i-1}_S(k,k) \arrow[l] \arrow[d, "\alpha\cdot"] \\
\mathrm{Ext}^{i+4}_S(N,k) & \mathrm{Ext}^{i+3}_S(k,k) \arrow[l] & \mathrm{Ext}^{i+3}_S(M,k) \arrow[l] & \mathrm{Ext}^{i+3}_S(N,k) \arrow[l] & \mathrm{Ext}^{i+2}_S(k,k) \arrow[l]
\end{tikzcd}$$

Note that $\ell(k) \leq \ell(N) < \ell(M)$ so, by induction, the four outermost maps are isomorphisms. Therefore, by the Five Lemma (see [Wei94]), it follows that the map $\mathrm{Ext}^*_S(M,k) \to \mathrm{Ext}^{*+3}_S(M,k)$ given by the (left) action of α is an isomorphism of vector spaces. This completes the proof. $\square$

We have already verified for $0 \leq i < 3$ that $\mathrm{Ext}^i_S(M,k)$ is isomorphic to $\mathrm{Ext}^i_R(M,k)$ and for $i \geq 3$ that $\mathrm{Ext}^i_S(M,k) \cong \mathrm{Ext}^{i-3}_S(M,k)$. We have also verified that multiplication on the left by α induces an isomorphism $\mathrm{Ext}^i_S(M,k) \to \mathrm{Ext}^{i+3}_S(M,k)$. It follows that $\mathrm{Ext}^*_S(k,k) \cong \mathrm{Ext}^*_R(k,k) \otimes_k k[\alpha]$ acts on the module $\mathrm{Ext}^*_S(M,k) \cong \mathrm{Ext}^*_R(M,k) \otimes_k k[\alpha]$ in the natural way.

4.5 Generalization to Higher Dimensions

The ring S is the two-dimensional case of a broader concept. Let $R_n = k[x_1, x_2, \ldots, x_n]$ and define S_n as the n-dimensional analog of S. That is, S_n is generated as a vector space by elements of the form m and $\frac{\theta}{m}$ where m is a monomial in R_n. Multiplication on S_n is done in the obvious way (i.e. $\theta^2 = x_i\theta = 0$). There are analogous results to the above work in these higher-dimensional cases. Suppose that M is a finite-length R_n-module and that $P_\bullet$ is a free resolution of M. There is a resolution of ${}_{S_n}M$ given by

$$\cdots \to P_0 \otimes S_n \to P_n \otimes S_n \to \cdots \to P_0 \otimes S_n \to P_n \otimes S_n \to \cdots \to P_0 \otimes S_n \quad (4.5.1)$$

One can compute this resolution in the same way; we can write ${}_{R_n}S_n \cong R_n \oplus J_n$ and compute the following flat resolution of J_n:

$$0 \to R \to \bigoplus_{1 \le i_1 \le n} x_{i_1}^{-1} R \to \bigoplus_{1 \le i_1 < i_2 \le n} (x_{i_1} x_{i_2})^{-1} R \to \cdots \to (x_1 x_2 \cdots x_n)^{-1} R \to 0 \quad (4.5.2)$$

It then follows that $\mathrm{Ext}^*_{S_n}(k,k) \cong \mathrm{Ext}^*_{R_n}(k,k) \otimes_k k[\alpha]$ (where α has degree $n+1$) and that $\mathrm{Ext}^*_{S_n}(M,k) \cong \mathrm{Ext}^*_{R_n}(M,k) \otimes_k k[\alpha]$ with the expected action.

4.6 Properties of the Mackey Ring $\mathcal{M}_2$

In the previous sections of this chapter we analyzed the ring $\mathbb{M}_2$. We called it S for convenience but we will now abandon that notation. Additionally, we will assume that $k = \mathbb{Z}/2$. Recall that there is a $k[\tau, \rho]$-module isomorphism $\mathbb{M}_2 \cong k[\tau, \rho] \oplus J$ where J consists of k-linear combinations of elements of the form $\frac{\theta}{m}$ for monomials $m \in k[\tau, \rho]$.

Recall that $\mathbb{M}_2$ is $\mathbb{Z}^2$-graded and we consider all $\mathbb{M}_2$-modules (and hence ideals of $\mathbb{M}_2$) to be $\mathbb{Z}^2$-graded, as well. We first classify the ideals of $\mathbb{M}_2$.

Lemma 4.6.1. *If $I \trianglelefteq \mathbb{M}_2$ and $I \cap k[\tau, \rho]$ is non-empty then $J \subset I$.*

Proof. Suppose that $I \cap k[\tau, \rho]$ is non-empty. Since I is a $\mathbb{Z}^2$-graded ideal we can assume that there is a monomial $m \in I \cap k[\tau, \rho]$. Consider an element $\frac{\theta}{n} \in J$. Observe that $\frac{\theta}{mn} \in \mathbb{M}_2$ and I is an ideal of $\mathbb{M}_2$ so

$$\frac{\theta}{n} = \frac{\theta}{mn} \cdot m \in I. \tag{4.6.2}$$

It follows that $J \subset I$. $\square$

Corollary 4.6.3. *If $I \trianglelefteq \mathbb{M}_2$ then either $I \subseteq J$ or there exists an ideal $I' \trianglelefteq k[\tau, \rho]$ and a $k[\tau, \rho]$-module isomorphism $I \cong I' \oplus J$.*

The Mackey functor in Definition 4.6.4 has useful applications in equivariant homotopy theory.

Definition 4.6.4. The Mackey functor $\mathcal{M}_2$ is defined as follows:

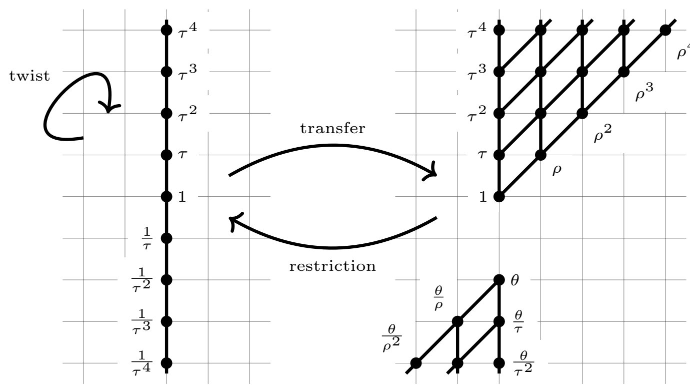
- $(\mathcal{M}_2)_\bullet = k[\tau, \tau^{-1}]$
- $(\mathcal{M}_2)_\theta = \mathbb{M}_2$
- The twist map $t : k[\tau, \tau^{-1}] \to k[\tau, \tau^{-1}]$ is the identity map.
- The restriction map $p^* : \mathbb{M}_2 \to k[\tau, \tau^{-1}]$ is defined so that $p^*(\tau^i) = \tau^i$ for $i \geq 0$, and $p^*(m) = 0$ for all other monomials $m \in k[\tau, \rho] \subseteq \mathbb{M}_2$, and $p^*(\frac{\theta}{m}) = 0$ for all $m \in k[\tau, \rho] \subseteq \mathbb{M}_2$.

- The transfer map $p_* : k[\tau, \tau^{-1}] \to \mathbb{M}_2$ is defined so that

$$p_*(\tau^i) = \begin{cases} \frac{\theta}{\tau^{(-2-i)}} & \text{if } i \leq 2, \text{ and} \\ 0 & \text{if } i > 2. \end{cases} \tag{4.6.5}$$

Both $k[\tau, \tau^{-1}]$ and $\mathbb{M}_2$ are rings and it is straightforward to check that the Mackey functor $\mathcal{M}_2$ has the structure of a Mackey ring.

A diagram of $\mathcal{M}_2$ is shown below.



Note that a Mackey functor $\mathcal{M}$ is considered graded over a monoid Q provided that M_θ and $M_\bullet$ are both Q-graded and that the restriction, transfer, and twist maps are all Q-graded as well. We will consider $k[\tau, \tau^{-1}]$ to be a $\mathbb{Z}^2$-graded ring where the degree of τ^i is $(0, i)$. Then the Mackey ring $\mathcal{M}_2$ is a $\mathbb{Z}^2$-graded Mackey functor and this graded structure makes it easier to keep track of the restriction and transer maps in Definition 4.6.4. The previous diagram of $\mathcal{M}_2$ honors this graded structure; the element 1 on either side is in degree $(0, 0)$.

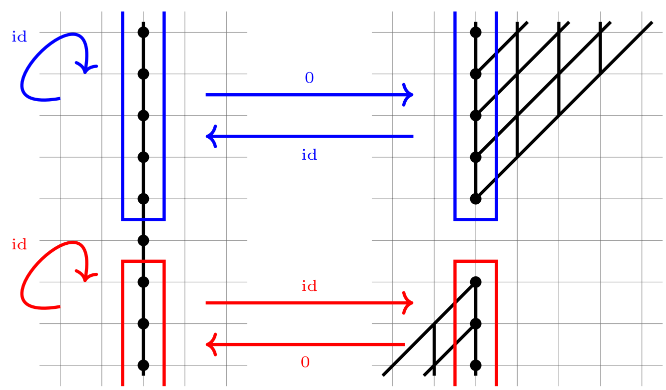
Note that the only $\mathbb{Z}^2$-degrees in which both of $k[\tau, \tau^{-1}]$ and $\mathbb{M}_2$ are non-trivial are $(0, n)$ for $n \neq -1$. Furthermore, the corresponding components of both rings are $k \cong \mathbb{Z}/2$ in those degrees. Let $D = \{(0, n) \in \mathbb{Z}^2 \mid n \neq -1\}$. We can then make the following observations about the restriction and transfer maps:

- In every degree $d \in \mathbb{Z}^2 \setminus D$ the restriction and transfer maps are maps of the form $0 \to k$, $k \to 0$, or $0 \to 0$ and hence are forced to be zero.

- For degrees $(0, n) \in D$ where $n \geq 0$, restriction is the identity map $k \to k$ and transfer is the zero map $k \to k$.

- For degrees $(0, n) \in D$ where $n \leq -2$, restriction is the zero map $k \to k$ and transfer is the identity map $k \to k$.

The diagram below summarizes these observations. Only the maps in highlighted degrees can be non-zero and on these gradings the map $k \to k$ can only be 0 or id, as indicated.



May proved that the (traditional) ring $\mathbb{M}_2$ is self-injective (see [May18]) and here we prove that $\mathcal{M}_2$ is a self-injective Mackey ring. Recall from Theorem 2.4.1 that our criterion for verifying the injectivity of a Mackey module requires us to understand the ideals of the corresponding Mackey ring. Lemma 4.6.6 below provides a categorization of the ideals of $\mathcal{M}_2$.

Lemma 4.6.6. *If $\mathcal{I}$ is an ideal of $\mathcal{M}_2$ then it must be of one of the following two forms:*

1. *$I_\theta = 0$ and $\tau^i \notin I_\bullet$ for all $i \geq 0$.*

2. *$I_\theta = k[\tau, \tau^{-1}]$ and $\frac{\theta}{\tau^i} \in I_\bullet$ for all $i \geq 0$*

Additionally, any choice of ideals $I_\theta \trianglelefteq k[\tau, \tau^{-1}]$ and $I_\bullet \trianglelefteq \mathbb{M}_2$ which fall into one of those two categories will form an ideal $\mathcal{I} \trianglelefteq \mathcal{M}_2$.

Proof. Recall from Lemma 2.2.5 that an ideal $\mathcal{I} \trianglelefteq \mathcal{M}_2$ will consist of ideals $I_\bullet \trianglelefteq (\mathcal{M}_2)_\bullet = \mathbb{M}_2$ and $I_\theta \trianglelefteq (\mathcal{M}_2)_\theta = k[\tau, \tau^{-1}]$ such that the restrictions of all the maps in $\mathcal{M}_2$ to the appropriate ideals are well-defined. Also note that $k[\tau, \tau^{-1}]$ is a graded field and hence only has two ideals.

- Suppose that $I_\theta = 0$. The only non-zero outputs of p^* are linear combinations of $\tau^i \in \mathbb{M}_2$ for $i \geq 0$. Hence, in order for $p^*|_{I_\bullet}$ to be well-defined, we must have $\tau^i \notin I_\bullet$ for all $i \geq 0$. Clearly $p_*|_{I_\theta}$ and $t|_{I_\theta}$ will both be well-defined and hence any such ideal $I_\bullet$ will yield an ideal $\mathcal{I} \trianglelefteq \mathcal{M}_2$ in this case.

- Suppose that $I_\theta = k[\tau, \tau^{-1}]$. The image of p_* is all linear combinations of elements $\frac{\theta}{\tau^i} \in \mathbb{M}_2$ for $i \geq 0$. In order for $p_*|_{I_\theta}$ to be well-defined, $I_\bullet$ must contain all of those elements. It is clear that $p^*|_{I_\bullet}$ and t will both be well-defined and hence any such ideal $I_\bullet$ will yield and ideal $\mathcal{I} \trianglelefteq \mathcal{M}_2$ in this case.

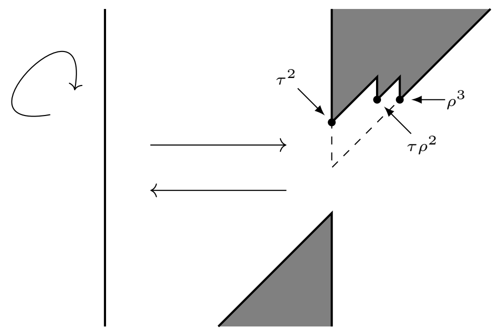
Since the only ideals of $k[\tau, \tau^{-1}]$ are 0 and $k[\tau, \tau^{-1}]$ we have now covered all possible cases. $\square$

Example 4.6.7. We'd now like to consider an example of some of the ideals of $\mathcal{M}_2$.

- Consider the ideal $I_\bullet = \langle \tau^2, \tau\rho^2, \rho^3 \rangle \trianglelefteq \mathbb{M}_2$. Note that $J \subseteq I_\bullet$ (this is proven in Lemma 4.6.1). If we let $I_\theta = k[\tau, \tau^{-1}]$ then the resulting Mackey functor

$$\begin{tikzcd} k[\tau,\tau^{-1}] \arrow[loop left, "\mathrm{id}"] \arrow[r, bend left, "p_*"] & I_\bullet \arrow[l, bend left, "p^*|_{I_\bullet}"] \end{tikzcd}$$

is an ideal of $\mathcal{M}_2$ based on Lemma 4.6.6. Shown below is a diagram of this ideal:

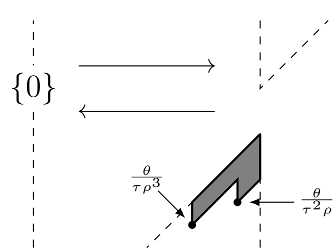


This ideal is of the second type described by Lemma 4.6.6.

- Consider the ideal $I_\bullet = \langle \frac{\theta}{\tau\rho^3}, \frac{\theta}{\tau^2\rho} \rangle \trianglelefteq \mathbb{M}_2$. It is easy to check that $I_\bullet \subseteq J$. Based on Lemma 4.6.6 the Mackey functor

$$\begin{tikzcd} \{0\} \arrow[loop left, "0"] \arrow[r, bend left, "0"] & I_\bullet \arrow[l, bend left, "0"] \end{tikzcd}$$

is an ideal of $\mathcal{M}_2$. Shown below is a diagram of this ideal:

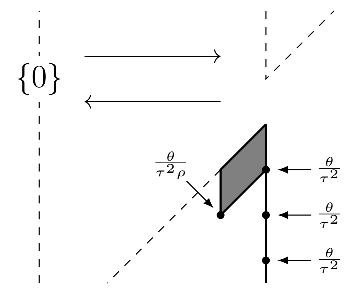


This ideal is of the first type described by Lemma 4.6.6.

- Ideals $I_\bullet \trianglelefteq \mathbb{M}_2$ need not be finitely generated. Consider

$$I_\bullet = \left\langle \left\{ \frac{\theta}{\tau^2\rho^2} \right\} \cup \left\{ \frac{\theta}{\tau^2}, \frac{\theta}{\tau^4}, \frac{\theta}{\tau^6}, \dots \right\} \right\rangle. \tag{4.6.8}$$

If we let $I_\theta = \{0\}$ the diagram of the resulting Mackey ideal is as follows:



As a brief aside, now that we understand the ideals of $\mathcal{M}_2$ it becomes clear that it is not a Noetherian ring.

Theorem 4.6.9. *The Mackey ring $\mathcal{M}_2$ is not Noetherian.*

Proof. First note that $\mathbb{M}_2$ is not Noetherian and this theorem is essentially a direct corollary of that fact.

For $i \in \mathbb{N}$ let $v_i = \frac{\theta}{\tau^i\rho^i} \in \mathbb{M}_2$. Consider ideals $J_i \trianglelefteq \mathbb{M}_2$ where $J_i = \langle v_i \rangle$. Note that J_i does not contain J since $v_{i+1} \in J \setminus J_i$ and hence $J_i \neq \mathbb{M}_2$ for all $i \in \mathbb{N}$. Additionally, it is clear that J_i is a proper subset of J_{i+1} since $v_{i+1} \in J_{i+1} \setminus J_i$. It follows that the collection $\{J_i\}_{i\in\mathbb{N}}$ is an ascending chain of ideals in $\mathbb{M}_2$.

As a result of Corollary 4.6.3 we see that $J_i \subseteq J$. Furthermore, by Lemma 4.6.6, there exist ideals $\mathcal{J}_i \trianglelefteq \mathcal{M}_2$ of the following form:

$$\mathcal{J}_i : \quad \begin{tikzcd} \{0\} \arrow[loop left, "0"] \arrow[r, bend left, "0"] & J_i \arrow[l, bend left, "0"] \end{tikzcd}$$

It is clear that $\{\mathcal{J}_i\}_{i\in\mathbb{N}}$ is an ascending chain of ideals and that $\mathcal{J}_i \neq \mathcal{J}_{i+1}$ for all $i \in \mathbb{N}$. Hence $\mathbb{M}_2$ is non-Noetherian. $\square$

We are now ready to show that $\mathcal{M}_2$ is a self-injective ring.

Theorem 4.6.10. *The Mackey functor $\mathcal{M}_2$ is an injective object in the category of graded $\mathcal{M}_2$-modules.*

Proof. This proof will use the criteron described in Theorem 2.4.1. We need to prove two things:

1. For every inclusion of $\mathcal{M}_2$-modules $\iota : \mathcal{I} \to \mathcal{M}_2$ and every morphism $f : \mathcal{I} \to \mathcal{M}_2$ there exists a morphism $g : \mathcal{M}_2 \to \mathcal{M}_2$ such that $g\iota = f$ as in the diagram below:

$$\begin{tikzcd} \mathcal{I} \arrow[r, tail, "\iota"] \arrow[d, "f"'] & \mathcal{M}_2 \arrow[dl, dashed, bend left, "g"] \\ \mathcal{M}_2 & \end{tikzcd}$$

2. For every inclusion $\iota : \mathcal{J} \to \mathcal{M}_2 \,\square\, \mathcal{F}_\theta(\mathbb{Z})$ and every morphism $f : \mathcal{J} \to \mathcal{M}_2$ there exists a morphism $g : \mathcal{M}_2 \,\square\, \mathcal{F}_\theta(\mathbb{Z}) \to \mathcal{M}_2$ such that $g\iota = f$ as in the diagram below:

$$\begin{tikzcd} \mathcal{J} \arrow[r, tail, "\iota"] \arrow[d, "f"'] & \mathcal{M}_2 \,\square\, \mathcal{F}_\theta(\mathbb{Z}) \arrow[dl, dashed, bend left, "g"] \\ \mathcal{M}_2 & \end{tikzcd}$$

In this context we are dealing with graded modules and maps of graded modules. The only adjustment that needs to be made is to understand that the codomains of the maps f and g in the diagrams above may contain a shift in the $\mathbb{Z}^2$-grading.

We start by addressing the first of these two conditions. Choose an ideal $\mathcal{I} \trianglelefteq \mathcal{M}_2$ and a $\mathcal{M}_2$-map $f : \mathcal{I} \to \mathcal{M}_2$. Since the choice of g is clear when $f = 0$, assume $f \neq 0$. Since $\mathbb{M}_2$ is an injective $\mathbb{M}_2$-module there exists a map $g_\bullet : I_\bullet \to \mathcal{M}_2$ of $\mathcal{M}_2$-modules such that $g_\bullet \iota_\bullet = f_\bullet$. However, $\mathcal{M}_2 \cong \mathcal{M}_2 \square \mathcal{F}_\bullet(\mathbb{Z})$ is free. Hence there is a map of $\mathcal{M}_2$-modules $g : \mathcal{M}_2 \to \mathcal{M}_2$ determined by $\mathbb{I}_\bullet \mapsto g_\bullet(1_\bullet)$. We claim that for this choice of g it follows that $g_\theta \iota_\theta = f_\theta$.

We first make some reductions. If I_θ is trivial then it is clear that $g_\theta \iota_\theta = f_\theta$ so assume $I_\theta = k[\tau, \tau^{-1}]$. Now, recall that f, g, and ι are all graded maps wih the understanding that the codomain of f and g undergoes a shift; write this shift as (a, b). That is, if an element $x \in \mathcal{I}$ has degree (a', b') then the degree of $f(x)$ is $(a' + a, b' + b)$. The same must also hold for elements $x \in \mathcal{M}_2$ and $g(x)$. Note that the non-zero elements in the θ-components of $\mathcal{M}_2$ and $\mathcal{I}$ all have degree $(0, n)$. This means that if $a \neq 0$ then the maps $f_\bullet$ and $g_\bullet$ must both be zero and it is clear that $g_\theta \iota_\theta = f_\theta$ in that case. Hence we can assume that $a = 0$.

Since g is a graded map and $1_\bullet$ has degree $(0, 0)$ we have that the degree of $g_\bullet(\mathbb{I}_\bullet)$ is $(0, b)$. If $b \geq 0$ then $g_\bullet(\mathbb{I}_\bullet) = \tau^b$ and if $b \leq -2$ then $g_\bullet(\mathbb{I}_\bullet) = \frac{\theta}{\tau^{b-2}}$.

- Suppose $b \geq 0$. First we have

$$g_\theta(\iota_\theta \tau^i) = g_\theta(\tau^i) = g_\theta(\tau^i p^* \mathbb{I}_\bullet) = \tau^i g_\theta(p^* \mathbb{I}_\bullet) = \tau^i p^*(g_\bullet \mathbb{I}_\bullet) = \tau^i \tau^b = \tau^{i+b}. \tag{4.6.11}$$

  Next, observe that

$$\begin{aligned} p_*(f_\theta(\tau^{-b-2})) = f_\bullet(p_*(\tau^{-b-2})) = f_\bullet\left(\tfrac{\theta}{\tau^b}\right) = g_\bullet\left(\iota_\bullet\left(\tfrac{\theta}{\tau^b}\right)\right) \\ = g_\bullet\left(\tfrac{\theta}{\tau^b}\right) = \tfrac{\theta}{\tau^b} g_\bullet(1_\bullet) = \tfrac{\theta}{\tau^b}\tau^b = \theta. \end{aligned} \tag{4.6.12}$$

The degree of τ^{-b-2} is $(0, -b-2)$ and so it follows that the degree of $f_\theta(\tau^{-b-2})$ is $(0, -2)$. The component of $k[\tau, \tau^{-1}]$ that lies in degree $(0, -2)$ is $\{0, \tau^{-2}\}$. Since $p_*(f_\theta(\tau^{-b-2})) \neq 0$ it must be that $f_\theta(\tau^{-b-2}) \neq 0$. This forces $f_\theta(\tau^{-b-2}) = \tau^{-2}$. Finally,

$$f_\theta(\tau^i) = f_\theta(\tau^{(i+b+2)+(-b-2)}) = \tau^{i+b+2} f_\theta(\tau^{-b-2}) = \tau^{i+b+2}\tau^{-2} = \tau^{i+b}. \quad (4.6.13)$$

Thus $g_\theta \iota_\theta = f_\theta$ in this case.

- Suppose $b \leq -2$. An argument nearly identical to the previous one shows that in this case we must have that $f_\theta = 0$ and $g_\theta = 0$. Hence $g_\theta \iota_\theta = f_\theta$ in this case, as well.

This completes the proof justifying the first of the two conditions.

We now verify the second condition. For convenience, let $\mathbb{F} = k[\tau, \tau^{-1}]$. (Recall that here we are assuming $k = \mathbb{Z}/2$.) As a consequence of Theorem 2.1.9 we have that $\mathcal{M}_2 \square \mathcal{F}_\theta(\mathbb{Z}) \cong \mathcal{F}_\theta(\mathbb{F})$ and

$$\mathcal{F}_\theta(\mathbb{F}) = \begin{tikzcd} \mathbb{F}^2 \arrow[loop left, "t = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]"] \arrow[rr, bend left, "p_* = \left[\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right]"] & & \mathbb{F}. \arrow[ll, bend left, "p^* = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]"] \end{tikzcd}$$

Choose a submodule $\mathcal{J} \leq \mathcal{F}_\theta(\mathbb{F})$ and choose a map $f : \mathcal{J} \to \mathcal{M}_2$. Observe that (since $k = \mathbb{Z}/2$, $\mathbb{F}$ is a field, and $\mathcal{J}$ is a homogeneous ideal) there are only the following five choices of $J_\theta \leq \mathbb{F}^2$:

$$0 \oplus 0, \qquad \mathbb{F} \oplus 0, \qquad 0 \oplus \mathbb{F}, \qquad \Delta, \qquad \mathbb{F}^2 \qquad (4.6.14)$$

where $\Delta = \langle(1,1)\rangle \leq \mathbb{F}^2$. We now consider five cases based on the choice of J_θ and in each case we construct a map $g : \mathcal{F}_\theta(\mathbb{F}) \to \mathcal{M}_2$ such that $g\iota = f$ (where $\iota : \mathcal{J} \to \mathcal{F}_\theta(\mathbb{F})$ is the inclusion).

First we show that two of those five cases aren't possible. Suppose that $J_\theta = \mathbb{F}\oplus 0$. Lemma 2.2.5 states that $J_\bullet$ must contain $p_*(J_\theta) = \mathbb{F}$ which forces $J_\bullet = \mathbb{F}$. Lemma 2.2.5 also states that J_θ must contain $p^*(J_\bullet) = \Delta$ but it is clear that $\Delta \not\subseteq \mathbb{F} \oplus 0$. Hence there is no submodule $\mathcal{J} \leq \mathcal{F}_\theta(\mathbb{F})$ such that $J_\theta = \mathbb{F}\oplus 0$. An identical argument shows that there is no submodule $\mathcal{J} \leq \mathcal{F}_\theta(\mathbb{F})$ such that $J_\theta = 0 \oplus \mathbb{F}$.

Next we show that two of those cases are trivial. If $J_\theta = 0$ then Lemma 2.2.5 forces $J_\bullet = 0$ and hence $\mathcal{J} = 0$ which forces $f = 0$. In this case, we can choose $g = 0$ and then clearly $g\iota = f$. Similarly, if $J_\theta = \mathbb{F}^2$ then Lemma 2.2.5 forces $J_\bullet = \mathbb{F}$ and hence $\mathcal{J} = \mathcal{F}_\theta(\mathbb{F})$ which forces $\iota = \mathrm{id}$. In this case, we can choose $g = f$ and then clearly $g\iota = f$.

Finally, suppose $J_\theta = \Delta$ and suppose $f : \mathcal{J} \to \mathcal{M}_2$ is a map of the following form:

$$\begin{tikzcd}
\Delta \arrow[loop left, "\mathrm{id}"] \arrow[r, bend left, "p_* = 0"] \arrow[d, "f_\theta"] & J_\bullet \arrow[l, bend left, "p^*"] \arrow[d, "f_\bullet"] \\
\mathbb{F} \arrow[loop left, "\mathrm{id}"] \arrow[r, bend left, "p_*"] & \mathbb{M}_2 \arrow[l, bend left, "p^*"]
\end{tikzcd}$$

Note that $p_* = 0$ in the above diagram because for $(\tau^j, \tau^j) \in \Delta$ we have

$$p_*(\tau^j, \tau^j) = \tau^j + \tau^j = 2\tau^j = 0. \tag{4.6.15}$$

The rest of this proof is dedicated to showing that there exists a map $g : \mathcal{F}_\theta(\mathbb{F}) \to \mathcal{M}_2$ such that $g\iota = f$ in this specific case.

We first wish to argue that f_θ is the zero map. Suppose, to the contrary, that f_θ is non-zero. Then, since $f_\theta : \Delta \to \mathbb{F}$ is a map of graded $\mathbb{F}$-modules this forces $f_\theta(\tau^i, \tau^i) = \tau^{i+j}$ for some fixed $j \in \mathbb{Z}$. If $i = -j-2$ then we have

$$\begin{aligned} p_*(f_\theta(\tau^i, \tau^i)) = p_*(\tau^{-2}) = \theta \quad \text{and} \\ f_\bullet(p_*(\tau^i, \tau^i)) = f_\bullet(0) = 0. \end{aligned} \tag{4.6.16}$$

Since f is a map of Mackey functors we have $p_* f_\theta = f_\bullet p_*$ and so $\theta = 0$ which is a contradiction. This shows that f_θ must be the zero map.

If $f_\bullet$ is also the zero map then $f : \Delta \to \mathcal{M}_2$ must be zero, in which case we can choose $g = 0$ and have $g\iota = f$. We are left with the case where $f : \Delta \to \mathcal{M}_2$ is a graded map of $\mathcal{M}_2$-modules and $f_\bullet$ is non-zero. An application of Lemma 2.2.5 shows that $J_\bullet$ can either be $\mathbb{F} \trianglelefteq \mathbb{F}$ or $0 \trianglelefteq \mathbb{F}$ so, since $f_\bullet \neq 0$, we must have $J_\bullet = \mathbb{F}$.

We are now considering a map $f : \Delta \to \mathcal{M}_2$ of the form

$$\begin{tikzcd}
\Delta \arrow[loop left, "\mathrm{id}"] \arrow[r, bend left, "p_* = 0"] \arrow[d, "f_\theta"] & \mathbb{F} \arrow[l, bend left, "p^* = \mathrm{id}"] \arrow[d, "f_\bullet"] \\
\mathbb{F} \arrow[loop left, "\mathrm{id}"] \arrow[r, bend left, "p_*"] & \mathbb{M}_2 \arrow[l, bend left, "p^*"]
\end{tikzcd}$$

where $f_\bullet : \mathbb{F} \to \mathbb{M}_2$ is non-zero. (In the above diagram when we write $p^* = \mathrm{id}$ we mean the map $x \mapsto (x, x)$.) We first wish to make some reductions regarding the image of $f_\bullet$. In what follows we use the action of $\mathcal{M}_2$ on $\mathcal{M}_2 \square \mathcal{F}_\theta(\mathbb{Z}) \cong \mathcal{F}_\theta(\mathbb{F})$; a description of this action can be found in Theorem 2.3.12.[1]

- Suppose $\tau^a\rho^b$ is in the image of $f_\bullet$. Then there is some $\tau^i \in \mathbb{F}$ where $f_\bullet(\tau^i) = \tau^a\rho^b$. By checking the gradings we then find that $f_\bullet(\tau^{i-a-1})$ is forced to equal 0. Since $f_\bullet$ is a map of $\mathbb{M}_2$-modules we have

$$\tau^a\rho^b = f_\bullet(\tau^i) = f_\bullet(\tau^{a+1}\tau^{i-a-1}) = \tau^{a+1}f_\bullet(\tau^{i-a-1}) = 0, \tag{4.6.17}$$

  which is a contradiction. Thus the image of $f_\bullet$ does not contain an element of the form $\tau^a\rho^b$.

- Suppose $\frac{\theta}{\tau^a\rho^b}$ with $b > 0$ is in the image of $f_\bullet$. Then there is some $\tau^i \in \mathbb{F}$ such that $f_\bullet(\tau^i) = \frac{\theta}{\tau^a\rho^b}$. After noting that $\rho\tau^i \in \mathbb{F}$ is zero we have

$$0 = f_\bullet(0) = f_\bullet\rho\tau^i = \rho f_\bullet(\tau^i) = \rho\left(\frac{\theta}{\tau^a\rho^b}\right) = \frac{\theta}{\tau^a\rho^{b-1}}. \tag{4.6.18}$$

  This is a contradiction since $b > 0$ and so $\frac{\theta}{\tau^a\rho^{b-1}} \in \mathbb{M}_2$ is non-zero. It follows that the image of $f_\bullet$ does not contain an element of the form $\frac{\theta}{\tau^a\rho^b}$ where $b > 0$.

- Suppose $\frac{\theta}{\tau^a}$ is in the image of $f_\bullet$. (Note that in this case there is no contradiction to be found.) Then there is some $\tau^i \in \mathbb{F}$ such that $f_\bullet(\tau^i) = \frac{\theta}{\tau^a}$. If $j > i$ then

$$f_\bullet(\tau^j) = \tau^{j-i}f_\bullet(\tau^i) = \tau^{j-i}\left(\frac{\theta}{\tau^a}\right) = \frac{\theta}{\tau^{a+i-j}} \tag{4.6.19}$$

[1] In particular, we use that elements $\tau^i \in \mathbb{M}_2$ where $i \geq 0$ act on $(\mathcal{F}_\theta(\mathbb{F})) = \mathbb{F}_\bullet$ as expected and all other elements of $\mathbb{M}_2$ act as zero.

where, of course, we mean that $f_\bullet(\tau^j)$ is zero if $a+i-j<0$. Furthermore, if $j<i$ then

$$\tau^{i-j} f_\bullet(\tau^j) = f_\bullet(\tau^{i-j}\tau^j) = f_\bullet(\tau^i) = \frac{\theta}{\tau^a}. \tag{4.6.20}$$

It follows that $f_\bullet(\tau^j)$ is non-zero and, in particular, it must be that $f_\bullet(\tau^j) = \frac{\theta}{\tau^{a+i-j}}$ (this is clear by examining the graded structure of $\mathbb{M}_2$).

As a result of the above reductions we have shown that a non-zero graded map of $\mathbb{M}_2$-modules $\mathbb{F} \to \mathbb{M}_2$ must be of the form $\tau^j \mapsto \frac{\theta}{\tau^{k-j}}$ for some $k \in \mathbb{Z}$. Assume that the map $f_\bullet$ takes this form. For the rest of this proof when we write an element $\frac{\theta}{\tau^n} \in \mathbb{M}_2$ where $n \in \mathbb{Z}$ it will be under the assumption that we mean this element to be zero in cases where $n<0$.

Recall that maps $\mathcal{F}_\theta(\mathbb{F}) \to \mathcal{M}_2$ are uniquely determined by the image of $\mathbb{I}_\theta = (1,0) \in (\mathcal{F}_\theta(\mathbb{F}))_\theta = \mathbb{F}^2$. Define $g : \mathcal{F}_\theta(\mathbb{F}) \to \mathcal{M}_2$ such that $g_\theta(1,0) = \tau^{-2-k}$. We wish to prove that $g\iota = f$. We will do this by showing that $g|_{\mathcal{J}} = f$ or, equivalently, by showing that $g_\theta(x) = f_\theta(x)$ for all $x \in J_\theta = \Delta$ and $g_\bullet(x) = f_\bullet(x)$ for all $x \in J_\bullet = \mathbb{F}$.

Choose an element $(\tau^j, \tau^j) \in \Delta$. Observe that

$$\begin{aligned} g(\tau^j, \tau^j) &= \tau^j g(1,1) = \tau^j g\big((1,0) + t(1,0)\big) \\ &= \tau^j\big(\tau^{-2-k} + t(\tau^{-2-k})\big) = \tau^j\big(\tau^{-2-k} + \tau^{-2-k}\big) = 0. \end{aligned} \tag{4.6.21}$$

Hence $g_\theta(\tau^j, \tau^j) = 0 = f_\theta(\tau^j, \tau^j)$, as desired.

Choose an element $\tau^j \in \mathbb{F}$. Before computing $g_\bullet(\tau^j)$ it helps to recall the transfer maps[2] in $\mathcal{F}_\theta(\mathbb{F})$ and $\mathcal{M}_2$: for $(a,b) \in (\mathcal{F}_\theta(\mathbb{F}))_\theta = \mathbb{F}^2$ we have $p_*(a,b) = a+b$ and for

[2]Note that we do not give the usual description of the transfer map in $\mathcal{M}_2$ but rather a description adapted to the notation in this proof.

$\tau^n \in (\mathcal{M}_2)_\theta = \mathbb{F}$ we have $p_*(\tau^n) = \frac{\theta}{\tau^{-n-2}}$. Now observe that

$$g_\bullet(\tau^j) = g_\bullet\big(p_*(\tau^j, 0)\big) = p_*\big(g_\theta(\tau^j, 0)\big) = p_*\big(\tau^j g_\theta(1, 0)\big) = p_*(\tau^{-2-k+j}) = \frac{\theta}{\tau^{k-j}} \tag{4.6.22}$$

and hence $g_\bullet(\tau^j) = f_\bullet(\tau^j)$, as desired.

Finally, we have satisfied the conditions laid out in Theorem 2.4.1 and hence $\mathcal{M}_2$ is an injective $\mathcal{M}_2$-module. $\square$

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