SOME EXTENSION ALGEBRAS OF STANDARD MODULES OVER KHOVANOV-LAUDA-ROUQUIER ALGEBRAS OF TYPE $\mathtt{A}$, INCLUDING A-INFINITY STRUCTURE

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DISSERTATION ABSTRACT

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We give an explicit description of the category of Yoneda extensions of standard modules over KLR algebras for two special cases in Lie type $\mathtt{A}$. In these two special cases, the A_∞-category structure of the Yoneda category is formal. We give an example to show that, in general, the A_∞-category structure of the Yoneda category is non-formal.

This dissertation includes unpublished co-authored material

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For Paige and Anna

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CHAPTER I

INTRODUCTION

This chapter contains unpublished co-authored material. The summary of results in section 1.2 appears (also as a summary of results) in [2, §1] and was written in close collaboration by the author and Alexander Kleshchev.

This dissertation contains unpublished co-authored material. Chapter II contains material which appears in [1, §2]. The purpose of this material both in this dissertation and in [1] is to provide background for the remaining work and was written by the author, David J. Steinberg, and Alexander Kleshchev. Chapter III contains material which appears in [1, §3]. The author and David J. Steinberg independently performed the relevant computations under the supervision of and with assistance from Alexander Kleshchev. The results were written initially by David J. Steinberg and were revised by the author and Alexander Kleshchev. Chapter IV contains material which appears in [1, §4]. The author performed the relevant computations under the supervision of and with assistance from Alexander Kleshchev. The results were written initially by the author with revisions by the author and Alexander Kleshchev. Chapter V contains material which appears in [1, §5]. The author performed the relevant computations under the supervision of Alexander Kleshchev. The results were written by the author.

1.1. The big picture

Khovanov-Lauda-Rouquier (KLR) algebras are connected to fundamental objects in the representation theory of Lie groups and Lie algebras. This connection can be summarized by the following diagram, which is explained below. Horizontal arrows in the diagram represent relationships which carry more or less equivalent amounts of data and structure, upward arrows represent an enrichment of data and structure, and downward arrows represent a specialization or collapsing of data and structure.

$$\begin{tikzcd}
 & & & \boxed{R\text{–proj and } R\text{–fmod}} \arrow[d, shift left, "\text{Grothendieck group}"] \\
 & & \boxed{\mathcal{U}_q} \arrow[r, "\text{positive part of integral form}"'] \arrow[d, shift left, "\lim_{q\to 1}"] & \boxed{\mathbf{f} \text{ and } \mathbf{f}^*} \arrow[u, shift left, "\text{categorification}"] \\
\boxed{G} \arrow[r, shift left, "\text{differentiation}"] & \boxed{\mathfrak{g}} \arrow[l, shift left, "\text{exponential map}"] \arrow[r, "\text{universal enveloping algebra}"'] & \boxed{\mathcal{U}} \arrow[u, shift left, "q\text{-deformation}"] &
\end{tikzcd}$$

Beginning with a compact, simply connected Lie group G, one associates to it a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$. The representation theories of G and $\mathfrak{g}$ are the same in the sense that there is an equivalence of categories $\mathrm{rep}(G) \cong \mathrm{rep}(\mathfrak{g})$. One then passes to the (associative) universal enveloping algebra $\mathcal{U}$ for which there is an equivalence of categories $\mathcal{U}$–Mod $\cong \mathrm{Rep}(\mathfrak{g})$. The *quantized universal enveloping algebra* $\mathcal{U}_q$ of Drinfeld and Jimbo [5, 6] is a q-deformation which specializes to $\mathcal{U}$ in the limit as $q \to 1$. Much of the structure of $\mathcal{U}_q$ can be understood from the *positive part* $\mathcal{U}_q^+ \subseteq \mathcal{U}_q$. Moreover, Lusztig [7] defines dual *integral forms* $\mathbf{f}, \mathbf{f}^* \subseteq \mathcal{U}_q^+$ such that $\mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbf{f} \cong \mathcal{U}_q^+ \cong \mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbf{f}^*$. Finally, Khovanov and Lauda [8, 9] and, independently, Rouquier [10] define the family of *KLR algebras* $\{R_\theta \mid \theta \in Q_+\}$ (where Q_+ is the positive part of the root lattice for the root system corresponding to G or $\mathfrak{g}$), together with induction functors $\mathrm{Ind}_{\theta,\eta} : (R_\theta \otimes R_\eta)$–mod $\to R_{\theta+\eta}$–mod giving a monoidal structure on the category R–mod $:= \bigoplus_{\theta \in Q_+} R_\theta$–mod. There are dual isomorphisms between $\mathbf{f}$ and the split

Grothendieck group [R–proj] of finitely-generated projective modules, and between $\mathbf{f}^*$ and the Grothendieck group [R–fmod] of finite-dimensional modules.

Let us now restrict our attention to simply laced Lie types ($\mathtt{A}$, $\mathtt{D}$, and $\mathtt{E}$). Under certain conditions, the isomorphisms $\mathbf{f} \xrightarrow{\sim} [R\text{–proj}]$ and $[R\text{–fmod}] \xrightarrow{\sim} \mathbf{f}^*$ identify various important bases of $\mathbf{f}$ and $\mathbf{f}^*$ with certain families of modules:

$$
\begin{array}{ccccccc}
[R\text{–proj}] & \xleftarrow{\quad\sim\quad} & \mathbf{f} & & \mathbf{f}^* & \xleftarrow{\quad\sim\quad} & [R\text{–fmod}] \\
\left\{\begin{matrix}\text{Indecomposable} \\ \text{projectives } P(\pi)\end{matrix}\right\} & \leftarrow\!\shortmid & \begin{matrix}\text{Canonical} \\ \text{basis}\end{matrix} & & \begin{matrix}\text{Dual canonical} \\ \text{basis}\end{matrix} & \longleftarrow\!\shortmid & \left\{\begin{matrix}\text{Irreducible} \\ \text{modules } L(\pi)\end{matrix}\right\} \qquad (1.1) \\
\left\{\begin{matrix}\text{“Standard”} \\ \text{modules } \Delta(\pi)\end{matrix}\right\} & \longleftarrow\!\shortmid & \begin{matrix}\text{PBW} \\ \text{basis}\end{matrix} & & \begin{matrix}\text{Dual PBW} \\ \text{basis}\end{matrix} & \leftarrow\!\shortmid & \left\{\begin{matrix}\text{“Proper standard”} \\ \text{modules } \bar{\Delta}(\pi)\end{matrix}\right\}. \qquad (1.2)
\end{array}
$$

The above families are each indexed by the set of *Kostant partitions*. A Kostant partition is a tuple $\pi = (\beta_1^{m_1}, \dots, \beta_t^{m_t})$ such that $m_i \in \mathbb{Z}_{\geq 0}$ and $\beta_1 > \cdots > \beta_t$ are positive roots, ordered according to some fixed convex total order. If, in addition, $m_1\beta_1 + \cdots + m_t\beta_t = \theta$, we say that π is a Kostant partition of θ and write $\pi \in \Pi(\theta)$. Note that the standard modules are not, in general, projective. Rather, they have finite length projective resolutions and therefore represent classes in the Grothendieck group [R–proj].

Crucially, the algebras R_θ can be defined over a field $\mathbb{F}$ of arbitrary characteristic p (in fact, we typically work over the integers), and the Grothendieck groups for varying p are all isomorphic. The identification (1.2) holds in arbitrary characteristic and is due to Kato [11] (when $p = 0$) and Brundan, Kleshchev, and McNamara [12] (for arbitrary p). The identification (1.1) is due to Rouquier [13] and Varagnolo and Vasserot [14], and holds only when $p = 0$: counterexamples of Williamson [15] (cf. [12, §2.6]) show that (1.1) fails when $p > 0$. Nevertheless, in characteristic p, the families $\{P(\pi)\}$ and $\{L(\pi)\}$ categorify *some* pair of dual bases of $\mathbf{f}$ and $\mathbf{f}^*$, respectively, which we shall refer to as the *p-canonical basis* and *dual p-canonical basis*. One is then lead to inquire about the change of basis between the (dual) p-canonical basis and the (dual) PBW basis.

For $\theta \in Q_+$, the category R_θ–mod is an *affine highest weight category* [12, 16, 17, 18] (see section 2.1 for details). In particular, the projective indecomposables admit *standard filtrations*, i.e., finite filtrations with subquotients among the standard modules. The duality of the two change of bases between the (dual) p-canonical basis and the (dual) PBW basis is then categorified by *affine BGG reciprocity*:

$$(P(\tau) : \Delta(\pi))_q = [\bar{\Delta}(\pi) : L(\tau)]_q \tag{1.3}$$

where the left-hand side stands for the graded multiplicity of $\Delta(\pi)$ in a standard filtration for $P(\tau)$, and the right-hand side stands for the graded multiplicity of $L(\tau)$ as a composition factor in $\bar{\Delta}(\pi)$.

Since the projective indecomposables admit standard filtrations, the left-hand side of (1.3) can be computed in the full subcategory $\mathcal{F}(\Delta) \subseteq R_\theta$–mod of modules which admit a standard filtration. Keller and Lefévre-Hasegawa [19, 20, 21] (see also [22], [12, Corollary 3.14] and [16, Theorem 4.28]) provide machinery for describing $\mathcal{F}(\Delta)$ using only information from the Yoneda algebra $\mathrm{Ext}^\bullet_{R_\theta}(\Delta, \Delta)$, where $\Delta = \bigoplus_{\pi \in \Pi(\theta)}$, see section 2.2 for details. The author, Kleshchev, and Steinberg give explicit projective resolutions for the standard modules in [2], and in this dissertation, we use those resolutions to explicitly describe the Yoneda algebra in certain special cases as outlined in the next section.

1.2. Summary of results

Throughout, we work over an arbitrary principal ideal domain $\Bbbk$ (since everything is defined over $\mathbb{Z}$, one could just consider the case $\Bbbk = \mathbb{Z}$). We also use $\mathbb{F}$ to denote a field with characteristic p. Let $R_{\theta,\mathbb{F}}$ be a Khovanov-Lauda-Rouquier (KLR) algebra of finite Lie type over $\mathbb{F}$ corresponding to $\theta \in Q_+$ [8, 9, 10]. It is known that $R_{\theta,\mathbb{F}}$ is affine quasihereditary [12, 11, 23, 17], and in particular it comes with a family of *standard modules* $\{\Delta(\pi)_{\mathbb{F}} \mid \pi \in \mathrm{KP}(\theta)\}$, where $\mathrm{KP}(\theta)$ is the set of Kostant partitions of θ. KLR algebras are defined over $\Bbbk$, so we have a $\Bbbk$-algebra R_θ with $R_{\theta,\mathbb{F}} \cong R_\theta \otimes_\Bbbk \mathbb{F}$. The standard modules have natural $\Bbbk$-forms $\Delta(\pi)$ with $\Delta(\pi)_{\mathbb{F}} \cong \Delta(\pi) \otimes_\Bbbk \mathbb{F}$. All modules and algebras are explicitly graded, and we refer to these gradings as KLR gradings.

We now assume that the Lie type is $\mathtt{A}_\infty$ with simple roots $\{\alpha_i \mid i \in \mathbb{Z}\}$ so that the set of positive roots Φ_+ is $\{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j \mid i \leq j\}$. There is a natural lexicographic total order $>$ on Φ_+. Let Q_+ be the positive root lattice, and fix $\theta \in Q_+$. If $\theta = \sum k_i\alpha_i$, we define the *height* of θ as $\mathrm{ht}(\theta) := \sum k_i$. A *Kostant partition* of θ is a sequence $\pi = (\beta_1^{m_1}, \ldots, \beta_t^{m_t})$ where $m_1, \ldots, m_t \in \mathbb{Z}_{>0}$, $\beta_1 > \cdots > \beta_t$ are positive roots, and $m_1\beta_1 + \cdots + m_t\beta_t = \theta$.

We consider the Yoneda algebra $\mathcal{E}_\theta = \mathrm{Ext}^\bullet_{R_\theta}(\Delta, \Delta)$ (where $\Delta = \bigoplus_{\pi\in\mathrm{KP}(\theta)} \Delta(\pi)$) as the $\Bbbk$-linear category whose objects are $\mathrm{KP}(\theta)$, and the set of morphisms from $\rho \in \mathrm{KP}(\theta)$ to $\sigma \in \mathrm{KP}(\theta)$ is

$$\mathcal{E}^\bullet_\theta(\rho, \sigma) := \mathrm{Ext}^\bullet_{R_\theta}(\Delta(\rho), \Delta(\sigma)).$$

The composition gf of $g \in \mathcal{E}^\bullet_\theta(\sigma, \tau)$ and $f \in \mathcal{E}^\bullet_\theta(\rho, \sigma)$ is obtained using the composition of lifts of g in $\mathrm{Hom}_{R_\theta}(P^\sigma_\bullet, P^\tau_\bullet)$ and f in $\mathrm{Hom}_{R_\theta}(P^\rho_\bullet, P^\sigma_\bullet)$, where $P^\pi_\bullet$ is a projective resolution of $\Delta(\pi)$ for $\pi \in \mathrm{KP}(\theta)$. The category $\mathcal{E}_\theta$ has a *homological grading* for which the homogeneous components are $\mathcal{E}_\theta^m(\rho,\sigma) := \mathrm{Ext}^m_{R_\theta}(\Delta(\rho),\Delta(\sigma))$, and a *KLR grading* which is inherited from the KLR grading on the standard modules. We use q to denote the KLR degree shift functor. Theorems A and B describe the category $\mathcal{E}_\theta$ (as a bigraded category) in two special cases: (1) when θ is an arbitrary positive root, and (2) when θ is of type $\mathtt{A}_2$, i.e. θ is of the form $c_1\alpha_1 + c_2\alpha_2$.

<u>The case where θ is a positive root</u>

Let $\theta = \alpha_a + \alpha_{a+1} + \cdots + \alpha_{b+1} \in \Phi_+$. Set $l := b+2-a = \mathrm{ht}(\theta)$ and consider the polynomial algebra $\mathcal{X} := \Bbbk[x_1,\dots,x_l]$. We consider $\mathcal{X}$ to be graded with $\deg x_r = 2$. Note that $\mathrm{KP}(\theta)$ is in bijection with the set of subsets of $[1, l-1]$: the subset associated to $\rho = (\beta_1,\dots,\beta_u) \in \mathrm{KP}(\theta)$ is $D_\rho := \{d_1,\dots,d_{u-1}\}$ where $d_t := \mathrm{ht}(\beta_1) + \cdots + \mathrm{ht}(\beta_t)$. For such D_ρ, set $d_0 := 0$ and $d_u := l$, and let J^ρ be the ideal of $\mathcal{X}$ generated by all $x_r - x_s$ such that there is $1 \le t \le u$ with $d_{t-1} < r, s \le d_t$. Define $\mathcal{X}^\rho := \mathcal{X}/J^\rho$. If $D_\rho \subseteq D_\sigma$, then $J^\sigma \subseteq J^\rho$ so we have a natural projection $\mathtt{p}^\sigma_\rho : \mathcal{X}^\sigma \twoheadrightarrow \mathcal{X}^\rho$. We use the notation $C \subseteq_m D$ to indicate that $C \subseteq D$ with $|D \setminus C| = m$.

Theorem A. *Let $\theta = \alpha_a + \alpha_{a+1} + \cdots + \alpha_{b+1} \in \Phi_+$ be a positive root. We have*

$$\mathcal{E}_\theta^m(\rho,\sigma) \cong \begin{cases} q^{-m}\mathcal{X}^\rho & \text{if } D_\rho \subseteq_m D_\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

If $D_\rho \subseteq_m D_\sigma \subseteq_n D_\tau$ with $f \in q^{-m}\mathcal{X}^\rho \cong \mathcal{E}_\theta(\rho,\sigma)$ and $g \in q^{-n}\mathcal{X}^\sigma \cong \mathcal{E}_\theta(\sigma,\tau)$, then the composition of g with f is given by $\mathtt{p}^\sigma_\rho(g)f \in q^{-(m+n)}\mathcal{X}^\rho \cong \mathcal{E}_\theta(\rho,\tau)$.

The $\mathtt{A}_2$ case

For a nonnegative integer k, let Λ_k be the algebra of symmetric polynomials in k variables. We impose a grading on Λ_k where linear symmetric polynomials have degree 2. The space Λ_k is a free $\Bbbk$-module with basis $\{s_\lambda \mid \lambda \in \mathscr{P}(k)\}$, where $\mathscr{P}(k)$ is the set of partitions with at most k parts, and s_λ is the *Schur polynomial* corresponding to λ [24, §I.3]. Letting V be the free graded $\Bbbk$-module with basis $\{v_0, v_1, v_2, \ldots\}$ such that $\deg v_i := 2i$, there is an isomorphism of graded $\Bbbk$-modules

$$\gamma_k : \Lambda_k \xrightarrow{\sim} q^{-k(k-1)}\textstyle\bigwedge^k V,\ \ s_{(\lambda_1,\ldots,\lambda_k)} \mapsto v_{\lambda_k} \wedge v_{\lambda_{k-1}+1} \wedge \cdots \wedge v_{\lambda_1+k-1} \tag{1.4}$$

where $\bigwedge^k V$ is the kth exterior power of V. Define

$$- \star - : \Lambda_a \otimes \Lambda_b \to q^{2ab}\Lambda_{a+b},\ \ f \otimes g \mapsto \gamma_{a+b}^{-1}(\gamma_a(f) \wedge \gamma_b(g)). \tag{1.5}$$

Considering Λ_{a+b} to be a subalgebra of $\Lambda_a \otimes \Lambda_b$ in the obvious way, we have that $\Lambda_a \otimes \Lambda_b$ is free as a Λ_{a+b}-module with basis $\{s_\lambda \otimes 1 \mid \lambda \in \mathscr{P}(a,b)\}$, where $\mathscr{P}(a,b)$ is the set of partitions with at most a nonzero parts, the first part being at most b, see [25, PARTL.1.5] and [26, Proposition 2.6.8]. Moreover, [25, SCHUB.1.7] provides an explicit algorithm for writing any element of $\Lambda_a \otimes \Lambda_b$ as a Λ_{a+b}-linear combination of the basis elements $s_\lambda \otimes 1$.

Let $c_1, c_2 \in \mathbb{Z}_{\geq 0}$ and $\theta = c_1\alpha_1 + c_2\alpha_2$. Note that there is a bijection

$$[0, \min\{c_1, c_2\}] \xleftrightarrow{\sim} \mathrm{KP}(\theta),\ \ r \mapsto (\alpha_2^{c_2-r}, (\alpha_1+\alpha_2)^r, \alpha_1^{c_1-r}),\ \ r_\rho \mapsfrom \rho.$$

For $\rho, \sigma \in \mathrm{KP}(\theta)$ with $r_\rho \geq r_\sigma$, let

$$\omega(\rho,\sigma) := -(r_\rho - r_\sigma)(1 + (c_1 - r_\rho) + (c_2 - r_\rho)),$$
$$\Lambda(\rho,\sigma) := q^{\omega(\rho,\sigma)}\Lambda_{c_2-r_\rho} \otimes \Lambda_{r_\rho - r_\sigma} \otimes \Lambda_{r_\sigma} \otimes \Lambda_{c_1 - r_\rho},$$
$$\mathscr{P}_{\rho,\sigma} := \mathscr{P}(r_\rho - r_\sigma, r_\sigma).$$

If $f \in \Lambda_{r_\rho - r_\sigma}$, we write

$$f^{\rho,\sigma} := 1_{\Lambda_{c_2-r_\rho}} \otimes f \otimes 1_{\Lambda_{r_\sigma}} \otimes 1_{\Lambda_{c_1-r_\rho}} \in \Lambda(\rho,\sigma).$$

Then note that $\Lambda(\rho,\sigma)$ is a free right $\Lambda(\rho,\rho)$-module with basis $\{s_\lambda^{\rho,\sigma} \mid \lambda \in \mathscr{P}_{\rho,\sigma}\}$. We make $\Lambda(\rho,\sigma)$ into a left $\Lambda(\sigma,\sigma)$-module via the composition of algebra homomorphisms:

$$\xi : \Lambda(\sigma,\sigma) \hookrightarrow \Lambda_{c_2-r_\rho} \otimes \Lambda_{r_\rho-r_\sigma} \otimes \Lambda_{r_\sigma} \otimes \Lambda_{r_\rho-r_\sigma} \otimes \Lambda_{c_1-r_\rho} \twoheadrightarrow q^{-\omega(\rho,\sigma)}\Lambda(\rho,\sigma);$$

the first map uses the embeddings $\Lambda_{c_2-r_\sigma} \hookrightarrow \Lambda_{c_2-r_\rho} \otimes \Lambda_{r_\rho-r_\sigma}$ and $\Lambda_{c_1-r_\sigma} \hookrightarrow \Lambda_{r_\rho-r_\sigma} \otimes \Lambda_{c_1-r_\rho}$, and the second map is $a \otimes b \otimes c \otimes d \otimes e \mapsto a \otimes bd \otimes c \otimes e$ (which we think of as identifying the two factors of $\Lambda_{r_\rho-r_\sigma}$).

If $\rho, \sigma, \tau \in \mathrm{KP}(\theta)$ with $r_\rho \geq r_\sigma \geq r_\tau$, the tensor product $\Lambda(\sigma,\tau) \otimes_{\Lambda(\sigma,\sigma)} \Lambda(\rho,\sigma)$ is now a free right $\Lambda(\rho,\rho)$-module with basis

$$\{s_\mu^{\sigma,\tau} \otimes s_\lambda^{\rho,\sigma} \mid \mu \in \mathscr{P}_{\sigma,\tau}, \lambda \in \mathscr{P}_{\rho,\sigma}\}$$

and we define a map of right $\Lambda(\rho,\rho)$-modules

$$\Theta : \Lambda(\sigma,\tau) \otimes_{\Lambda(\sigma,\sigma)} \Lambda(\rho,\sigma) \to \Lambda(\rho,\tau),\ s_\mu^{\sigma,\tau} \otimes s_\lambda^{\rho,\sigma} \mapsto (s_\mu \star s_\lambda)^{\rho,\tau}.$$

Let

$$- \diamond - : \Lambda(\sigma,\tau) \otimes_{\Bbbk} \Lambda(\rho,\sigma) \to \Lambda(\rho,\tau),\ g \otimes f \mapsto \Theta(g \otimes f).$$

Thus, to compute $g \diamond f$ for some $g \in \Lambda(\sigma,\tau)$ and $f \in \Lambda(\rho,\sigma)$, the following steps must be performed: (1) write $g = \sum_{\mu \in \mathscr{P}_{\sigma,\tau}} s_\mu^{\sigma,\tau} g_\mu$ with $g_\mu \in \Lambda(\sigma,\sigma)$, (2) for each $\mu \in \mathscr{P}_{\sigma,\tau}$, write $\xi(g_\mu) f = \sum_{\lambda \in \mathscr{P}_{\rho,\sigma}} s_\lambda^{\rho,\sigma} h_{\mu,\lambda}$ with $h_{\mu,\lambda} \in \Lambda(\rho,\rho)$, (3) we have

$$g \diamond f = \sum_{\mu \in \mathscr{P}_{\sigma,\tau},\, \lambda \in \mathscr{P}_{\rho,\sigma}} (s_\mu \star s_\lambda)^{\rho,\tau} h_{\mu,\lambda}.$$

Theorem B. *Let $c_1, c_2 \in \mathbb{Z}_{\geq 0}$ and $\theta = c_1\alpha_1 + c_2\alpha_2$. We have*

$$\mathcal{E}_\theta^m(\rho,\sigma) = \begin{cases} \Lambda(\rho,\sigma) & \text{if } m = r_\rho - r_\sigma \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

If $r_\rho \geq r_\sigma \geq r_\tau$ with $f \in \Lambda(\rho,\sigma) \cong \mathcal{E}_\theta(\rho,\sigma)$ and $g \in \Lambda(\sigma,\tau) \cong \mathcal{E}_\theta(\sigma,\tau)$, then the composition of g with f is given by $g \diamond f \in \Lambda(\rho,\tau) \cong \mathcal{E}_\theta(\rho,\tau)$.

Formality

For θ as in Theorem A or B, note that $\mathrm{Ext}_{R_\theta}(\Delta(\rho),\Delta(\sigma))$ is torsion-free as a $\Bbbk$-module. We do not know if this is true in general.

Also note that because $\mathrm{Ext}_{R_\theta}(\Delta(\rho),\Delta(\sigma))$ is concentrated in homological degree $|X_\sigma \setminus X_\rho|$ (in the case of Theorem A) or $r_\rho - r_\sigma$ (in the case of Theorem B), the A_∞-category structure of $\mathcal{E}_\theta$ must have $m_n = 0$ unless $n = 2$, so that $\mathcal{E}_\theta$ is *intrinsically formal*, see [20, §3.3]. In chapter V, we show that intrinsic formality and even formality does not occur in general:

Example C. *If $\theta = \alpha_1 + 2\alpha_2 + \alpha_3$, then the A_∞-category $\mathcal{E}_\theta$ is non-formal.*

1.3. The structure of the dissertation

The proofs of Theorems A and B occupy chapters III and IV, respectively. In the preliminary chapter II, we review affine highest weight categories (section 2.1), A_∞-categories and the homological algebra machinery (section 2.2, see the appendix for an application of this machinery in the context of truncated polynomial rings $\Bbbk[x]/(x^n)$), the definition of the KLR algebra R_θ and the standard modules $\Delta(\pi)$, and the resolutions of the standard modules from [2] (section 2.3).

In sections 3.1 and 4.1, we record the relevant special cases of the projective resolution $P^\rho_\bullet$ of $\Delta(\rho)$ constructed in [2]. This resolution is finite and has the form $P^\rho_\bullet = \cdots \xrightarrow{d_1} P^\rho_1 \xrightarrow{d_0} P^\rho_0 \xrightarrow{\varepsilon_\rho} \Delta(\rho)$ with $P^\rho_n = \bigoplus_{x \in X_n} q^{s_x} R_\theta 1_x$ for some explicit index set X_n, integers s_x, and idempotents 1_x. The map $d_n : P^\rho_{n+1} \to P^\rho_n$ can be described as right multiplication by an $X_{n+1} \times X_n$ matrix $(d_n^{y,x})$ for some $d_n^{y,x} \in 1_y R_\theta 1_x$.

In sections 3.2 and 4.3, we use the isomorphism $\mathrm{Hom}_{R_\theta}(R_\theta 1_x, \Delta(\sigma)) \cong 1_x \Delta(\sigma)$ to describe the complex $\mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, \Delta(\sigma))$ in terms of familiar objects from commutative algebra; in the case of Theorem A, these objects are polynomial rings and in the case of Theorem B, they are rings of symmetric polynomials. It turns out that in both cases, the complex $\mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, \Delta(\sigma))$ is isomorphic to a Koszul complex corresponding to an explicit regular sequence, and we can therefore compute its homology $H(\mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, \Delta(\sigma))) =: \mathcal{E}_\theta(\rho, \sigma)$ as a bigraded $\Bbbk$-module.

It remains to describe the composition in the category $\mathcal{E}_\theta$. This is done in sections 3.3 and 4.4, where we explicitly lift elements of $\mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, \Delta(\sigma))$ to

$\mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, P^\sigma_\bullet)$. The function composition map $\mathrm{Hom}^\bullet_{R_\theta}(P^\sigma_\bullet, P^\tau_\bullet) \otimes \mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, P^\sigma_\bullet) \to \mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, P^\tau_\bullet)$ induces a map on homology $\mathcal{E}_\theta(\sigma, \tau) \otimes \mathcal{E}_\theta(\rho, \sigma) \to \mathcal{E}_\theta(\rho, \tau)$ which is the composition in the category $\mathcal{E}_\theta$.

In chapter V, we provide details for Example C.

CHAPTER II

BACKGROUND

This chapter, specifically section 2.3, contains material which appears in [1, §2]. The purpose of this material both in this dissertation and in [1] is to provide background for the remaining work and was written by the author, David J. Steinberg, and Alexander Kleshchev.

2.1. Affine highest weight categories

Highest weight categories were introduced by Cline, Parshall and Scott [27] to axiomatize the study of many fundamental categories appearing in representation theory, including the BGG category $\mathcal{O}$. Let R be a finite-dimensional $\mathbb{F}$-algebra, let Π be a set indexing the simple modules $L(\pi)$, and choose a partial order $\prec$ on Π. We require $\mathrm{End}_R(L(\pi)) \cong \mathbb{F}$ for each $\pi \in \Pi$. For a finite-dimensional R-module V, denote by $[V : L(\pi)]$ the multiplicity with which $L(\pi)$ appears in a composition series for V. Let $P(\pi)$ be a projective cover of $L(\pi)$ and let $\Delta(\pi)$ be the (unique) maximal quotient of $P(\pi)$ such that $[\Delta(\pi) : L(\tau)] = 0$ unless $\tau \preceq \pi$, and $[\Delta(\pi) : L(\pi)] = 1$. Thus, the matrix $[\Delta(\pi) : L(\tau)]_{\pi,\tau\in\Pi}$ of composition multiplicities is unitriangular with respect to $\prec$. We call the modules $\Delta(\pi)$ the *standard modules.*

We say that an R-module V admits a *standard filtration* if there is a chain of submodules $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ such that each of the subquotients V_n/V_{n-1} (for $n = 1, 2, \dots, n$) are isomorphic to a standard module. If V admits a standard filtration, we denote by $(V : \Delta(\pi))$ the multiplicity with which $\Delta(\pi)$ appears as a subquotient. We say that R–mod is a highest weight category with respect to $\prec$ if each $P(\tau)$ admits a standard filtration such that $(P(\tau) : \Delta(\pi)) = 0$

unless $\tau \preceq \pi$ and $(P(\tau) : \Delta(\pi)) = 1$. Thus, the matrix $(P(\tau) : \Delta(\pi))_{\pi,\tau\in\Pi}$ of filtration multiplicities is unitriangular with respect to $\prec$.

An important consequence is *BGG reciprocity*: for $\pi, \tau \in \Pi$, we have

$$(P(\tau) : \Delta(\pi)) = [\Delta(\pi) : L(\tau)]. \tag{2.1}$$

The Grothendieck classes of simple modules and projective indecomposable modules form bases, respectively, of the Grothendieck group $[R\text{–mod}]$ of finitely-generated R-modules, and the split Grothendieck group $[R\text{–proj}]$ of finitely-generated projective R-modules. In fact, these bases are dual with respect to the non-degenerate pairing $[R\text{–proj}] \times [R\text{–mod}] \to \mathbb{Z}$ which is defined, for finitely-generated projective P and finitely-generated V by

$$([P], [V]) = \dim \mathrm{Hom}_R(P, V). \tag{2.2}$$

Under the additional assumption that the standard modules have finite length projective resolutions (for example if the algebra R has finite global dimension), the standard modules represent classes in both Grothendieck groups, and since the two matrices $[\Delta(\pi) : L(\tau)]_{\pi,\tau\in\Pi}$ and $(P(\tau) : \Delta(\pi))_{\pi,\tau\in\Pi}$ are unitriangular, the classes of the standard modules form bases in both Grothendieck groups. Moreover, BGG reciprocity shows that these bases are dual to each other with respect to the pairing (2.2).

Motivated from the fact that KLR algebras (which are infinite-dimensional) have many properties reminiscent of highest weight categories [12, 17], affine highest weight categories were axiomatized by Kleshchev [18]. Let R be a (possibly infinite-dimensional) Noetherian Laurentian graded $\mathbb{F}$-algebra, let Π be a set indexing the simple modules $L(\pi)$ up to degree shift, and choose a partial order $\prec$ on Π. For a finitely-generated R-module V, denote by $[V : L(\pi)]_q$ the graded multiplicity with which $L(\pi)$ appears as a composition factor of V. Since R is Laurentian, this is a Laurent series in q with non-negative coefficients. Let $P(\pi)$ be a projective cover of $L(\pi)$ and let $\Delta(\pi)$ be the (unique) maximal quotient of $P(\pi)$ such that $[\Delta(\pi) : L(\tau)]_q = 0$ unless $\tau \leq \pi$. We call the modules $\Delta(\pi)$ the *standard modules*. We also define the *proper standard module* $\bar{\Delta}(\pi)$ to be the (unique) maximal quotient of $P(\pi)$ such that $[\bar{\Delta}(\pi) : L(\tau)]_q = 0$ unless $\tau \leq \pi$ and $[\bar{\Delta}(\pi) : L(\pi)]_q = 1$. Note that it is the proper standard modules whose definition mirrors the standard modules in the finite-dimensional case. Also note that $\bar{\Delta}(\pi)$ is a quotient of $\Delta(\pi)$.

As in the finite-dimensional case, we say that an R-module V admits a *standard filtration* if there is a chain of submodules $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ such that each of the subquotients V_n/V_{n-1} (for $n = 1, 2, \ldots, n$) are isomorphic to a standard module, up to degree shift. If V admits a standard filtration, we denote by $(V : \Delta(\pi))_q$ the graded multiplicity with which $\Delta(\pi)$ appears as a subquotient. We say that R–mod is an affine highest weight category with respect to $\prec$ if the following conditions hold:

- for each $\pi \in \Pi$, the algebra $B_\pi := \mathrm{End}_R(\Delta(\pi))^{\mathrm{op}}$ is an *affine algebra*, that is, a quotient of a polynomial algebra by a homogeneous ideal,
- for each $\tau, \pi \in \Pi$, the right B_π-module $\mathrm{Hom}_R(P(\tau), \Delta(\pi))$ is free of finite rank,
- for each $\tau \in \Pi$, $P(\tau)$ admits a standard filtration such that $(P(\tau) : \Delta(\pi))_q = 0$ unless $\tau \preceq \pi$ and $(P(\tau) : \Delta(\tau))_q = 1$.

Analogous to the finite-dimensional case, we have *affine BGG reciprocity*: for $\pi, \tau \in \Pi$, we have

$$(P(\tau) : \Delta(\pi))_q = [\bar{\Delta}(\pi) : L(\tau)]_{q^{-1}}.$$

In our main application to KLR algebras, there is an additional duality present, in which case affine BGG reciprocity reads

$$(P(\tau) : \Delta(\pi))_q = [\bar{\Delta}(\pi) : L(\tau)]_q. \tag{2.3}$$

We again have the Grothendieck groups [R–mod] and [R–proj], for which the classes of simple modules and projective indecomposable modules form dual bases with respect to the non-degenerate pairing (2.2). The proper standard modules represent classes in [R–mod], and under the additional assumption that the standard modules have finite length projective resolutions (for example if the algebra R has finite global dimension which is the case for KLR algebras), the standard modules represent classes in [R–proj]. Since the two matrices $([\bar{\Delta}(\pi) : L(\tau)]_q)_{\pi,\tau\in\Pi}$ and $((P(\tau) : \Delta(\pi))_q)_{\pi,\tau\in\Pi}$ are unitriangular, the classes of the proper standard modules form a basis for [R–mod] and the classes of the standard modules form a basis for [R–proj]. Moreover, affine BGG reciprocity shows that these bases are dual to each other with respect to the pairing (2.2).

2.2. Homological algebra

A_∞-categories

The theory of A_∞-spaces, A_∞-algebras, and A_∞-categories has topological origin due to Stasheff [28, 29]. The prototypical example of an A_∞-space is the loop space ΩX of a pointed topological space (X, x_0). The loop space ΩX

possesses a multiplication, namely, concatenation of loops. The multiplication is not associative, rather it is associative up to homotopy—for loops $\gamma_1, \gamma_2, \gamma_3$, there is a homotopy $(\gamma_1\gamma_2)\gamma_3 \to \gamma_1(\gamma_2\gamma_3)$. Similarly, for an additional loop γ_4, the homotopies

$$((\gamma_1\gamma_2)\gamma_3)\gamma_4 \to (\gamma_1(\gamma_2\gamma_3))\gamma_4 \to \gamma_1((\gamma_2\gamma_3)\gamma_4) \to \gamma_1(\gamma_2(\gamma_3\gamma_4)) \quad \text{and}$$
$$((\gamma_1\gamma_2)\gamma_3)\gamma_4 \to (\gamma_1\gamma_2)(\gamma_3\gamma_4) \to \gamma_1(\gamma_2(\gamma_3\gamma_4))$$

are not equal, rather there is a higher homotopy between them. Continuing in this way, we obtain a sequence of higher homotopies $[0,1]^{n-2} \times (\Omega X)^n \to \Omega X$. The singular chain complex $C^\bullet\Omega X$ is therefore endowed with maps $m_n : (C^\bullet\Omega X)^{\otimes n} \to C^\bullet\Omega X$ which turn out to satisfy the axioms of an A_∞-algebra.

In what follows, we follow the sign conventions of Keller [20] (see also [19, 21]). Note that these conventions include the Koszul-Quillen sign convention: For $\mathbb{Z}$-graded $\Bbbk$-modules A, B, C, and D, homogeneous elements $a \in A$ and $b \in B$, and graded maps $f : A \to C$, $g : B \to D$, we have

$$(f \otimes g)(a \otimes b) = \begin{cases} f(a) \otimes g(b) & \text{if } g \text{ is an even map or } a \text{ is of even degree} \\ -f(a) \otimes g(b) & \text{if } g \text{ is an odd map and } a \text{ is of odd degree.} \end{cases}$$

A (small, strictly unital) A_∞-*category* $\mathcal{A}$ over $\Bbbk$ consists of the following data:

- a set of objects $\mathrm{ob}(\mathcal{A})$,
- a $\mathbb{Z}$-graded $\Bbbk$-module of morphisms $\mathcal{A}^\bullet = \bigoplus_{\pi,\rho\in\mathrm{ob}(\mathcal{A})} \mathcal{A}^\bullet(\pi,\rho)$,
- for each $n \in \mathbb{Z}_{>0}$, a degree $2-n$ $\Bbbk$-linear map $m_n : (\mathcal{A}^\bullet)^{\otimes n} \to \mathcal{A}^\bullet$,

subject to the following conditions:

- for each $n \in \mathbb{Z}_{>0}$ and objects $\pi_0, \ldots, \pi_n, \pi'_1, \ldots, \pi'_{n-1}$, the restriction of the map m_n to $\mathcal{A}^\bullet(\pi'_{n-1}, \pi_n) \otimes \mathcal{A}^\bullet(\pi'_{n-2}, \pi_{n-1}) \otimes \cdots \otimes \mathcal{A}^\bullet(\pi'_1, \pi_2) \otimes \mathcal{A}^\bullet(\pi_0, \pi_1)$ is zero unless $\pi_i = \pi'_i$ (for $1 \leq i \leq n-1$), and in that case factors through $\mathcal{A}^\bullet(\pi_0, \pi_n)$,

- for each $n \in \mathbb{Z}_{>0}$, we have

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(\mathrm{id}_{\mathcal{A}^\bullet}^{\otimes r} \otimes m_s \otimes \mathrm{id}_{\mathcal{A}^\bullet}^{\otimes s}) = 0, \tag{2.4}$$

- for each object $\pi \in \mathrm{ob}(\mathcal{A})$, there exists a *strict identity* $1_\pi \in \mathcal{A}^0(\pi, \pi)$ such that $m_2(1_\pi \otimes x) = m_2(x \otimes 1_\pi) = x$ for any $x \in \mathcal{A}^\bullet$, and for $n \neq 2$ and elements $x_1, \ldots, x_n \in \mathcal{A}^\bullet$, if any of the x_i are equal to 1_π, we have

$$m_n(x_1 \otimes \cdots \otimes x_n) = 0.$$

The strict identity axiom is omitted in typical treatments, but its inclusion is convenient for our purposes. If $\mathcal{A}$ has only one object, it is called an A_∞*-algebra*. If m_1 is zero, $\mathcal{A}$ is called *minimal*. An A_∞-category with $m_n = 0$ for $n > 2$ is a differential graded category. The axioms show that the data of the map m_n can be expressed as a collection of maps

$$m_n^{\pi_0, \ldots, \pi_n} : \mathcal{A}^\bullet(\pi_{n-1}, \pi_n) \otimes \cdots \otimes \mathcal{A}^\bullet(\pi_0, \pi_1) \to \mathcal{A}^\bullet(\pi_0, \pi_n)$$

for all choices of objects $\pi_0, \ldots, \pi_n \in \mathrm{ob}(\mathcal{A})$.

A (strictly unital) A_∞*-functor* F from $\mathcal{A} = (\mathcal{A}^\bullet, m_1, m_2, \ldots)$ to $\mathcal{B} = (\mathcal{B}^\bullet, m_1, m_2, \ldots)$ consists of the following data:

- a function $F : \mathrm{ob}(\mathcal{A}) \to \mathrm{ob}(\mathcal{B})$,

- for each $n \in \mathbb{Z}_{\geq 0}$, a degree $1-n$ $\Bbbk$-linear map $F_n : (\mathcal{A}^\bullet)^{\otimes n} \to \mathcal{B}^\bullet$,

subject to the following conditions:

- for each $n \in \mathbb{Z}_{>0}$, and objects $\pi_0, \ldots, \pi_n, \pi'_1, \ldots, \pi'_{n-1} \in \mathrm{ob}(\mathcal{A})$, the restriction of the the map F_n to $\mathcal{A}^\bullet(\pi'_{n-1}, \pi_n) \otimes \mathcal{A}^\bullet(\pi'_{n-2}, \pi_{n-1}) \otimes \cdots \otimes \mathcal{A}^\bullet(\pi'_1, \pi_2) \otimes \mathcal{A}^\bullet(\pi_0, \pi_1)$ is zero unless $\pi_i = \pi'_i$ (for $1 \leq i \leq n-1$), and in that case factors through $\mathcal{B}^\bullet(F(\pi_0), F(\pi_n))$,

- for each $n \in \mathbb{Z}_{>0}$, we have

$$\sum_{r+s+t=n} (-1)^{r+st} F_{r+1+t}(\mathrm{id}_{\mathcal{A}^\bullet}^{\otimes r} \otimes m_s \otimes \mathrm{id}_{\mathcal{A}^\bullet}^{\otimes s}) = \sum_{i_1+\cdots+i_k=n} (-1)^{\mathsf{s}} m_k(F_{i_1} \otimes \cdots \otimes F_{i_k}) \tag{2.5}$$

  where $\mathsf{s} := \sum_{j=1}^{k} (k-j)(i_j - 1)$.

- for each object $\pi \in \mathrm{ob}(\mathcal{A})$, we have $F_1(1_\pi) = 1_{F(\pi)}$, and for $n > 1$ and elements $x_1, \ldots, x_n \in \mathcal{A}^\bullet$, if any of the x_i are equal to 1_π, we have

$$F_n(x_1 \otimes \cdots \otimes x_n) = 0.$$

The axioms show that the data of the map F_n can be expressed as a collection of maps

$$F_n^{\pi_0,\ldots,\pi_n} : \mathcal{A}^\bullet(\pi_{n-1}, \pi_n) \otimes \cdots \otimes \mathcal{A}^\bullet(\pi_0, \pi_1) \to \mathcal{B}^\bullet(\pi_0, \pi_n)$$

for all choices of objects $\pi_0, \ldots, \pi_n \in \mathrm{ob}(\mathcal{A})$.

Some special cases of relations (2.4) and (2.5) have familiar interpretations and nice consequences. When $n = 1$, the relations read

$$m_1^2 = 0, \quad F_1 m_1 = m_1 F_1$$

so m_1 is a differential and F_1 is a chain map. In particular, we may speak of the *homology* of $\mathcal{A}^\bullet$, which we denote by $H\mathcal{A}^\bullet$, and F_1 induces a map $[F_1] : H\mathcal{A}^\bullet \to H\mathcal{B}^\bullet$. When $n = 2$, the relations read

$$m_1 m_2 = m_2(m_1 \otimes 1 + 1 \otimes m_1),$$
$$F_1 m_2 = m_2(F_1 \otimes F_1) + m_1 F_2 + F_2(m_1 \otimes 1 + 1 \otimes m_1),$$

so that m_1 is a derivation with respect to m_2, and we may speak of the map $[m_2] : H\mathcal{A}^\bullet \otimes H\mathcal{A}^\bullet \to H\mathcal{A}^\bullet$ induced by homology, which then commutes with $[F_1]$. When $n = 3$, the relation (2.4) reads

$$m_2(1 \otimes m_2 - m_2 \otimes 1) = m_1 m_3 + m_3(m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1),$$

so that if either of m_1 or m_3 is zero, then m_2 is associative. In particular, $[m_2]$ is associative, so $H\mathcal{A}$ is a $\mathbb{Z}$-graded $\Bbbk$-linear category with the same object set as $\mathcal{A}$, and $[F_1]$ is a functor from $H\mathcal{A}$ to $H\mathcal{B}$. If $[F_1]$ is an isomorphism of categories, then F is called a *quasi-isomorphism.*

The *composition* of two A_∞-functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ is given on objects by $(GF)(\pi) = G(F(\pi))$ for $\pi \in \mathrm{ob}(\mathcal{A})$, and on morphisms by

$$(GF)_n := \sum_{i_1 + \cdots + i_k = n} (-1)^{\mathsf{s}} G_k(F_{i_1} \otimes \cdots \otimes F_{i_k})$$

where s is as in (2.5) The *identity* functor on $\mathcal{A}$ is the A_∞-functor $1_\mathcal{A} : \mathcal{A} \to \mathcal{A}$ with $1_\mathcal{A}(\pi) = \pi$ for each $\pi \in \mathrm{ob}(\mathcal{A})$, $(1_\mathcal{A})_1 = \mathrm{id}_{\mathcal{A}^\bullet}$, and $(1_\mathcal{A})_n = 0$ for $n > 1$. An *isomorphism* of A_∞-categories is an A_∞-functor $F : \mathcal{A} \to \mathcal{B}$ such that there exists an A_∞-functor $G : \mathcal{B} \to \mathcal{A}$ such that $GF = 1_\mathcal{A}$ and $FG = 1_\mathcal{B}$.

We have seen that the homology $H\mathcal{A}$ of $\mathcal{A}$ is a $\mathbb{Z}$-graded $\Bbbk$-linear category. In fact, $H\mathcal{A}$ also carries (uniquely) the structure of an A_∞-category which contains, in the following sense, the same information as $\mathcal{A}$:

Theorem 2.6. *[30] Let $\mathcal{A} = (\mathcal{A}^\bullet, m_1, m_2, \ldots)$ be an A_∞-category and $H\mathcal{A}$ its homology. If each morphism space in $H\mathcal{A}$ is a free $\mathbb{Z}$-graded $\Bbbk$-module, then $H\mathcal{A}$ carries the structure of an A_∞-category $H\mathcal{A} = (H\mathcal{A}^\bullet, M_1, M_2, \ldots)$ such that*

1. *$M_1 = 0$ and $M_2 = [m_2]$,*
2. *there exists a quasi-isomorphism $F : H\mathcal{A} \to \mathcal{A}$ such that $F(\pi) = \pi$ for each $\pi \in \mathrm{ob}(\mathcal{A})$ and $[F_1] = \mathrm{id}_{H\mathcal{A}^\bullet}$.*

Moreover, the A_∞-category structure on $H\mathcal{A}$ satisfying 1 and 2 is unique up to (non-unique) isomorphism of A_∞-categories.

Such an A_∞-category structure on $H\mathcal{A}$ is called a *minimal model* of $\mathcal{A}$. The A_∞-category $\mathcal{A}$ is called *formal* if its minimal model can be chosen so that $M_n = 0$ for $n \neq 2$. A (small) $\mathbb{Z}$-graded $\Bbbk$-linear category $\mathcal{B}$ is *intrinsically formal* if every A_∞-category $\mathcal{A}$ whose homology is isomorphic to $\mathcal{B}$ as a $\mathbb{Z}$-graded $\Bbbk$-linear category is formal.

Kadeishvili's original proof is constructive and yields an inductive algorithm for producing a minimal model in the special case where $\mathcal{A}$ is a differential-graded algebra, i.e., $m_n = 0$ for $n > 2$:

Algorithm 2.7. [30, Proof of Theorem 1] Let $\mathcal{A} = (\mathcal{A}^\bullet, m_1, m_2, \ldots)$ be an A_∞-category with $m_n = 0$ for $n > 2$ and $H\mathcal{A}$ its homology (as a $\mathbb{Z}$-graded $\Bbbk$-linear category). The following algorithm produces an A_∞-category structure $(H\mathcal{A}^\bullet, M_1, M_2, \ldots)$ and an A_∞-functor $F : H\mathcal{A} \to \mathcal{A}$ which satisfies the conditions in Theorem 2.6.

Step 1: Let $M_1 = 0$ and take $F_1 : H\mathcal{A}^\bullet \to \mathcal{A}^\bullet$ to be a cycle-choosing homomorphism of $\mathbb{Z}$-graded $\Bbbk$-modules. Set $n := 2$.

Step 2: Since $m_k = 0$ for $k > 2$, we may rewrite (2.5) as

$$m_1 F_n = F_1 M_n - U_n, \tag{2.8}$$

where

$$\begin{aligned} U_n := m_2 \sum_{i=1}^{n-1} (-1)^{i-1} (F_i \otimes F_{n-i}) \\ - \sum_{s=2}^{n-1} \sum_{t=0}^{n-s} (-1)^{n-s-t+st} F_{n-s+1} (\mathrm{id}_{H\mathcal{A}^\bullet}^{\otimes n-s-t} \otimes M_s \otimes \mathrm{id}_{H\mathcal{A}^\bullet}^{\otimes t}). \end{aligned} \tag{2.9}$$

The factorizations

$$U_n^{\pi_0,\ldots,\pi_n} : H\mathcal{A}^\bullet(\pi_{n-1},\pi_n) \otimes \cdots \otimes H\mathcal{A}^\bullet(\pi_0,\pi_1) \to \mathcal{A}^\bullet(\pi_0,\pi_n) \tag{2.10}$$

will also be useful in applications of this algorithm. One can check that $m_1 U_n = 0$. Thus, since M_k, F_k have been defined for $k < n$, we take M_n to be the (well-defined) homology class $[U_n]$ of U_n.

Step 3: Note that $[F_1 M_n - U_n] = [F_1 M_n] - [U_n] = M_n - M_n = 0$, so $F_1 M_n - U_n$ is a boundary, and choose F_n such that $m_1 F_n = F_1 M_n - U_n$. Increment n and return to Step 2.

The machine

Let R be a $\Bbbk$-algebra and let $\Delta = \{\Delta_\pi \mid \pi \in \Pi\}$ be a collection of R-modules, and suppose we are interested in the full subcategory $\mathcal{F} = \mathrm{filt}(\Delta)$ of

R–Mod consisting of modules which admit a finite filtration whose subquotients are among the Δ_π. In appendix A, we discuss an example where Δ is the collection of simple R-modules, in which case $\mathcal{F}$ is the category of finite-length R-modules. In our main application, Δ is the collection of standard modules over a KLR algebra (or more generally, the standard modules in an affine highest weight category). In this case, since the projective covers P_π have filtrations by the standard modules Δ_τ, the multiplicities $(P_\pi : \Delta_\tau)_q$ involved in affine BGG reciprocity 2.3 are present in the category $\mathcal{F}$. Keller and Lefèvre-Hasegawa [19, 20, 21] provide a procedure for reconstructing $\mathcal{F}$ using the A_∞-category structure on $\mathcal{E} = \bigoplus_{\pi,\rho\in\Pi} \mathcal{E}^\bullet(\pi,\rho)$ where $\mathcal{E}^\bullet(\pi,\rho) := \mathrm{Ext}^\bullet_R(\Delta_\pi, \Delta_\rho)$.

For each $\pi \in \Pi$, we fix a projective resolution $P^\bullet_\pi \xrightarrow{\varepsilon_\pi} \Delta_\pi$ with differential d. We then have the chain complex $\mathrm{Hom}^\bullet_R(P_\pi, \Delta_\rho)$ whose homology, by definition, is the $\mathbb{Z}$-graded $\Bbbk$-module

$$\mathcal{E}^\bullet(\pi,\rho) = H(\mathrm{Hom}^\bullet_R(P^\bullet_\pi, \Delta_\rho)). \tag{2.11}$$

We also have the chain complex $\mathrm{Hom}^\bullet_R(P^\bullet_\pi, P^\bullet_\rho)$ with differential δ given by

$$\delta(\varphi) := d\varphi - (-1)^m \varphi d \tag{2.12}$$

for $\varphi \in \mathrm{Hom}^m_R(P^\bullet_\pi, P^\bullet_\rho)$. There is an isomorphism

$$H\,\mathrm{Hom}^\bullet_R(P^\bullet_\pi, P^\bullet_\rho) \xrightarrow{\sim} \mathcal{E}^\bullet(\pi,\rho) \tag{2.13}$$

induced by the maps

$$\mathrm{Hom}^m_R(P^\bullet_\pi, P^\bullet_\rho) \to \mathrm{Hom}_R(P^m_\pi, \Delta_\rho),\ \varphi \mapsto (-1)^{\frac{m(m+1)}{2}} \varepsilon_\rho(\varphi|_{P^m_\pi}). \tag{2.14}$$

Let $\mathcal{H}$ be the differential graded category with object set Π, morphism spaces $\mathcal{H}^\bullet(\pi, \rho) := \mathrm{Hom}^\bullet_R(P^\bullet_\pi, P^\bullet_\rho)$, differential δ, and composition $\mathcal{H}^\bullet(\rho, \sigma) \otimes \mathcal{H}^\bullet(\pi, \rho) \to \mathcal{H}^\bullet(\pi, \sigma)$ given by composition of homomorphisms $\mathrm{Hom}^\bullet_R(P^\bullet_\rho, P^\bullet_\sigma) \otimes \mathrm{Hom}^\bullet_R(P^\bullet_\pi, P^\bullet_\rho) \to \mathrm{Hom}^\bullet_R(P^\bullet_\pi, P^\bullet_\sigma)$. We can therefore consider $\mathcal{E}$ as a $\mathbb{Z}$-graded $\Bbbk$-linear category, and in light of Theorem 2.6, as an A_∞-category. Moreover, any of the higher multiplications M_n on the A_∞-category $\mathcal{E}$ can be computed using Algorithm 2.7.

Following [19], we now describe how to reconstruct the category $\mathcal{F}$ using the A_∞-category structure on $\mathcal{E}$. We define the *category of twisted stalks* $\mathrm{twst}\,\mathcal{E}$ over $\mathcal{E}$. A *twisted stalk* M over $\mathcal{E}$ is the following data:

- a sequence $\pi_1, \ldots, \pi_n \in \mathrm{ob}(\mathcal{E})$ of objects of $\mathcal{E}$,
- an $n \times n$ matrix $\delta = (\delta_{ij})$ with $\delta_{ij} \in \mathcal{E}^1(\pi_j, \pi_i)$

satisfying the conditions:

- δ is strictly upper-triangular, i.e., $\delta_{ij} = 0$ for $i \geq j$,
- we have
$$\sum_{t=1}^{\infty} (-1)^{\frac{t(t-1)}{2}} M_t(\delta^{\otimes t}) = 0, \tag{2.15}$$
where M_k denotes the natural extension of M_k to matrices with coefficients in the morphism spaces of $\mathcal{E}$ (note that since δ is strictly upper-triangular, only finitely many terms of the sum are nonzero).

A *morphism of twisted stalks* $M = (\pi_1, \ldots, \pi_n, \delta) \to M' = (\pi'_1, \ldots, \pi'_{n'}, \delta')$ is the datum of an $n' \times n$ matrix $f = (f_{ij})$ with $f_{ij} \in \mathcal{E}^0(\pi_j, \pi'_i)$, satisfying the condition

$$\sum_{s,t \geq 0} (-1)^{\frac{(s+t)(s+t-1)}{2}+s} M_{s+t+1}((\delta')^{\otimes s} \otimes f \otimes \delta^{\otimes t}) = 0. \tag{2.16}$$

The *composition* of two morphisms $f : M \to M'$ and $g : M' \to M''$ is given by

$$gf = \sum_{r,s,t \geq 0} (-1)^{\frac{(r+s+t)(r+s+t-1)}{2}+s} M_{r+s+t+2}((\delta'')^{\otimes r} \otimes g \otimes (\delta')^{\otimes s} \otimes f \otimes \delta^{\otimes t}). \tag{2.17}$$

Note that the identity morphism id_M on $M = (\pi_1, \ldots, \pi_n, \delta)$ is given by the $n \times n$ diagonal matrix with $(\mathrm{id}_M)_{ii} = 1_{\pi_i}$, the strict identity on π_i.

Theorem 2.18. *[19] There is an equivalence of categories* $\mathrm{twst}\, \mathcal{E} \xrightarrow{\sim} \mathcal{F}$ *which takes the twisted stalk* $(\pi_1, \ldots, \pi_n, \delta)$ *to an R-module which admits a filtration with subquotients given by* $\Delta_{\pi_1}, \ldots, \Delta_{\pi_n}$*. In particular, the equivalence takes the twisted stalk* $(\pi, \delta = 0)$ *to* Δ_π*.*

2.3. KLR algebras and their modules

Basic notation

For $r, s \in \mathbb{Z}$, we use the segment notation $[r, s] := \{t \in \mathbb{Z} \mid r \leq t \leq s\}$, $[r, s) := \{t \in \mathbb{Z} \mid r \leq t < s\}$, etc.

Let q be a variable, and $\mathbb{Z}((q))$ be the ring of Laurent series. For $n \in \mathbb{Z}_{\geq 0}$, we define

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_\pm := q^{\pm(n-1)}[n], \quad [n]^!_{(\pm)} := [1]_{(\pm)}[2]_{(\pm)} \cdots [n]_{(\pm)},$$

and if $0 \leq m \leq n$,

$$\begin{bmatrix} n \\ m \end{bmatrix}_{(\pm)} := \frac{[n]^!_{(\pm)}}{[m]^!_{(\pm)}[n-m]^!_{(\pm)}}.$$

We denote by $\mathfrak{S}_d$ the symmetric group on d letters considered as a Coxeter group with generators $\{s_r := (r, r+1) \mid 1 \leq r < d\}$ and the corresponding length function ℓ. The longest element of $\mathfrak{S}_d$ is denoted w_0 or $w_{0,d}$. By definition, $\mathfrak{S}_d$ acts on $[1, d]$ on the left. For a set I the d-tuples from I^d are written as words $\boldsymbol{i} = i_1 \cdots i_d$. The group $\mathfrak{S}_d$ acts on I^d via place permutations: $w \cdot \boldsymbol{i} = i_{w^{-1}(1)} \cdots i_{w^{-1}(d)}$.

Given a composition $\mu = (\mu_1, \ldots, \mu_k)$ of d, we have the corresponding standard parabolic subgroup $\mathfrak{S}_\mu := \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_k} \leq \mathfrak{S}_d$. We denote by $\mathscr{D}^\mu$ the set of the shortest coset representatives for $\mathfrak{S}_d/\mathfrak{S}_\mu$.

Let R be a $\mathbb{Z}$-graded $\Bbbk$-algebra. We denote by R–Mod the category of graded left R-modules. The morphisms in this category are all homogeneous degree zero R-homomorphisms, which we denote $\hom_R(-,-)$. For $V \in R$–Mod, let $q^d V$ denote its grading shift by d, so if V_m is the degree m component of V, then $(q^d V)_m = V_{m-d}$. More generally, for a Laurent series $a = a(q) = \sum_d a_d q^d \in \mathbb{Z}((q))$ with non-negative coefficients, we set $aV := \bigoplus_d (q^d V)^{\oplus a_d}$. For $U, V \in R$–Mod, we set $\mathrm{Hom}_R(U, V) := \bigoplus_{d \in \mathbb{Z}} \mathrm{Hom}_R(U, V)_d$, where

$$\mathrm{Hom}_R(U, V)_d := \hom_R(q^d U, V) = \hom_R(U, q^{-d} V).$$

We define $\mathrm{Ext}^m_R(U, V)$ and $\mathrm{End}_R(U)$ similarly from $\mathrm{ext}^m_R(U, V)$ and $\mathrm{end}_R(U)$.

For a free $\Bbbk$ module V of finite rank, we denote the rank of V by $\dim V$. A graded $\Bbbk$-module $V = \bigoplus_{m \in \mathbb{Z}} V_m$ is called *Laurentian* if the graded components V_m are free of finite rank for all $m \in \mathbb{Z}$ and $V_m = 0$ for $m \ll 0$. The *graded rank* of a

Laurentian $\Bbbk$-module V is

$$\dim_q V := \sum_{m\in\mathbb{Z}} (\dim V_m) q^m \in \mathbb{Z}((q)).$$

We say that an R-module V is *Laurentian* if it is so as a $\Bbbk$-module.

Recall that the ground ring $\Bbbk$ is assumed to be a PID. The following standard result often allows us to reduce to the case where $\Bbbk$ is a field.

Lemma 2.19. *If $\varphi : V \to W$ is a degree 0 homomorphism of Laurentian $\Bbbk$-modules such that the induced map $\overline{\varphi} : V/JV \to W/JW$ is an isomorphism of $\Bbbk/J$-vector spaces for every maximal ideal J, then φ is an isomorphism.*

Symmetric polynomials and the nil-Hecke algebra

We impose a grading on the polynomial algebra $\mathcal{X}_d := \Bbbk[x_1, \dots, x_d]$ such that $\deg(x_r) = 2$. The symmetric group $\mathfrak{S}_d$ acts by automorphisms on $\mathcal{X}_d$ via $(w \cdot f)(x_1, \dots, x_d) := f(x_{w(1)}, \dots, x_{w(d)})$. The symmetric polynomial algebra $\Lambda_d := \mathcal{X}_d^{\mathfrak{S}_d}$ has a basis consisting of Schur polynomials

$$\{s_\lambda \mid \lambda \in \mathscr{P}(d)\},$$

where $\mathscr{P}(d)$ is the set of partitions with at most d nonzero parts, see [24, §I.3].

For a composition $\mu = (\mu_1, \dots, \mu_k)$ of d, the algebra of μ-partially symmetric polynomials is $\Lambda_\mu := \mathcal{X}_d^{\mathfrak{S}_\mu}$. We often write $\Lambda_{\mu_1,\dots,\mu_k}$ for Λ_μ and identify it with $\Lambda_{\mu_1} \otimes \cdots \otimes \Lambda_{\mu_k}$. For $a, b \in \mathbb{Z}_{\geq 0}$, let $\mathscr{P}(a, b)$ be the set of partitions with at most a nonzero parts, each part being at most b. The following is known:

Proposition 2.20. *The algebra Λ_μ is free as a Λ_d-module, and in the case where $\mu = (a, b)$ with $a + b = d$, a basis is given by*

$$\{s_\lambda \otimes 1 \mid \lambda \in \mathscr{P}(a, b)\}.$$

Proof. The freeness assertion is [25, PARTL.1.5], which, when combined with [26, Proposition 2.6.8], gives the basis assertion. □

For an integer r with $1 \leq r < d$ and a reduced decomposition $w = s_{r_1} \cdots s_{r_k} \in \mathfrak{S}_d$, the Demazure operators on $\mathcal{X}_d$ are defined as follows:

$$\partial_r := \frac{\mathrm{id}_{\mathcal{X}_d} - s_r}{x_{r+1} - x_r} \quad \text{and} \quad \partial_w := \partial_{r_1} \cdots \partial_{r_k}.$$

Note that ∂_w does not depend on the choice of reduced decomposition and is a degree $-2\ell(w)$ element of $\mathrm{End}_{\Bbbk} \mathcal{X}_d$.

For integers $r, i, j \geq 0$ with $r + i + j \leq d$, define

$$U_{r;i,j} \in \mathfrak{S}_d \tag{2.21}$$

to be the permutation which maps the interval $[r+1, r+i]$ increasingly onto $[r+1+j, r+i+j]$, and the interval $[r+i+1, r+i+j]$ increasingly onto $[r+1, r+j]$, and fixes all other elements of $[1, d]$. For example, we have $U_{r;1,1} = s_{r+1} = (r+1\ r+2)$. Recalling (1.5), we have:

Proposition 2.22. *[31, Proposition 2.9] Let $a, b \in \mathbb{Z}_{\geq 0}$ with $f \in \Lambda_a$ and $g \in \Lambda_b$. Then $\partial_{U_{0;a,b}}(f \otimes g) = f \star g$.*

Proof. Since we use different conventions from [31], we provide a translation for the reader's convenience. For $w \in \mathfrak{S}_{a+b}$, let $\partial'_w := (-1)^{\ell(w)} \partial_w$. If $k \in \mathbb{Z}_{\geq 0}$, recalling

(1.4), let $\gamma'_k := (-1)^{\binom{k}{2}}\gamma_k$ and note that $\gamma'_k(s_\lambda) = v_{\lambda_1+k-1} \wedge v_{\lambda_2+k-2} \wedge \cdots \wedge v_{\lambda_k} \in \bigwedge^k V$ for any $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathscr{P}(k)$. By [31, Proposition 2.9], we have $\partial'_{U_{0;a,b}}(f \otimes g) = (\gamma'_{a+b})^{-1} m(\gamma'_a \otimes \gamma'_b)(f \otimes g)$, so

$$\begin{aligned}\partial_{U_{0;a,b}}(f \otimes g) &= (-1)^{\ell(w)} \partial'_{U_{0;a,b}}(f \otimes g)\\ &= (-1)^{ab}(\gamma'_{a+b})^{-1} m(\gamma'_a \otimes \gamma'_b)(f \otimes g)\\ &= (-1)^{ab+\binom{a+b}{2}+\binom{a}{2}+\binom{b}{2}}(\gamma_{a+b})^{-1} m(\gamma_a \otimes \gamma_b)(f \otimes g)\\ &= f \star g. \qquad \square\end{aligned}$$

Define

$$x_0 = x_{0,d} := \prod_{r=1}^{d} x_r^{r-1} \in \mathcal{X}_d.$$

The following is well-known and easy to check:

Lemma 2.23. *We have $\partial_{w_0}(f x_0) = f$ for all $f \in \Lambda_d$.*

The *nil-Hecke algebra* $\mathcal{NH}_d$ is given by generators $\tau_1, \dots, \tau_{d-1}$, $x_1, \dots, x_d$ subject only to the relations

$$x_r x_t = x_t x_r,\ \tau_r^2 = 0,\ \tau_r \tau_{r+1} \tau_r = \tau_{r+1} \tau_r \tau_{r+1},\ \tau_r \tau_s = \tau_s \tau_r\ (|r-s| > 1),$$

$$\tau_r x_r = x_{r+1} \tau_r - 1,\ \tau_r x_{r+1} = x_r \tau_r + 1,\ \tau_r x_s = x_s \tau_r\ (s \neq r, r+1).$$

For a reduced decomposition $w = s_{r_1} \cdots s_{r_k} \in \mathfrak{S}_d$, we have a well-defined element $\tau_w := \tau_{r_1} \cdots \tau_{r_k}$. It is well-known that $\{\tau_w x_1^{k_1} \dots x_d^{k_d} \mid w \in \mathfrak{S}_d,\ k_1, \dots, k_d \in \mathbb{Z}_{\geq 0}\}$ is a basis of $\mathcal{NH}_d$. In particular, we identify $\mathcal{X}_d$ as a subalgebra of $\mathcal{NH}_d$.

Theorem 2.24. *[8, Theorem 2.9] The center of $\mathcal{NH}_d$ is equal to Λ_d.*

We consider $\mathcal{X}_d$ as an $\mathcal{NH}_d$-module with τ_w acting by ∂_w and x_r acting by multiplication with x_r. Then we have the following easy to check and well-known properties:

Lemma 2.25. *Let $d \in \mathbb{Z}_{\geq 0}$, $f \in \mathcal{X}_d$, and $w \in \mathfrak{S}_d$. Then in $\mathcal{NH}_d$, we have*

$$\tau_{w_0} f \tau_w = \tau_{w_0} \partial_{w^{-1}}(f) \quad \textit{and} \quad \tau_w f \tau_{w_0} = \partial_w(f) \tau_{w_0}.$$

Define the following elements of $\mathcal{NH}_d$:

$$e_d := x_{0,d} \tau_{w_{0,d}} \quad \text{and} \quad e'_d := \tau_{w_{0,d}} x_{0,d}. \tag{2.26}$$

Lemmas 2.23 and 2.25 yield:

Lemma 2.27. *In $\mathcal{NH}_d$, the elements e_d and e'_d are idempotents. Moreover,*

1. $\tau_{w_0} f \tau_{w_0} = 0$ *for any $f \in \mathcal{X}_d$ with* $\deg f < d(d-1)$, *and*

2. $\tau_{w_0} x_0 \tau_{w_0} = \tau_{w_0}$.

KLR algebras

From now on, we set $I := \mathbb{Z}$. If $i, j \in I$ with $|i-j| = 1$ we set $\varepsilon_{i,j} := j - i \in \{1, -1\}$. We identify I with the set of vertices of the Dynkin diagram of type $\mathtt{A}_\infty$ and denote by $(\mathsf{c}_{i,j})_{i,j \in I}$ the corresponding Cartan matrix so that $\mathsf{c}_{i,j} = 2$ if $i = j$, $\mathsf{c}_{i,j} = -1$ if $|i-j| = 1$, and $\mathsf{c}_{i,j} = 0$ otherwise. We use the notation Q_+, Φ_+, ht, etc. introduced in chapter I.

For $\theta \in Q_+$ of height d, we define $I^\theta := \{\boldsymbol{i} = i_1 \cdots i_d \in I^d \mid \alpha_{i_1} + \cdots + \alpha_{i_d} = \theta\}$. Let $\mathbb{Z}((q)) \cdot I^\theta := \bigoplus_{\boldsymbol{i} \in I^\theta} \mathbb{Z}((q)) \cdot \boldsymbol{i}$. For $1 \leq k \leq t$, suppose $\theta_k \in Q_+$ are such that $\theta_1 + \cdots + \theta_t = \theta$, and set $d_k := \mathrm{ht}(\theta_k)$. If $\boldsymbol{i}^k \in I^{\theta_k}$, then the concatenation $\boldsymbol{i}^1 \cdots \boldsymbol{i}^t$

is considered as an element of I^θ. Set $\boldsymbol{i}^1 \cdots \boldsymbol{i}^t =: i_1 \cdots i_d$. Then the *quantum shuffle product* is

$$\boldsymbol{i}^1 \circ \cdots \circ \boldsymbol{i}^t := \sum_{w \in \mathscr{D}^{(d_1,\ldots,d_t)}} q^{-e(w)} w \cdot (\boldsymbol{i}^1 \cdots \boldsymbol{i}^t) \in \mathbb{Z}((q)) \cdot I^\theta, \tag{2.28}$$

where $e(w) := \sum_{n<m,\, w(n)>w(m)} \mathsf{c}_{i_n,i_m}$. If $a_k \in \mathbb{Z}((q)) \cdot I^{\theta_k}$, we define $a_1 \circ \cdots \circ a_t \in \mathbb{Z}((q)) \cdot I^\theta$ extending (2.28) by linearity.

The *KLR algebra* [8, 9, 10] corresponding to θ as above is the unital $\Bbbk$-algebra R_θ (with identity denoted 1_θ) with generators

$$\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^\theta\} \cup \{y_1, \ldots, y_d\} \cup \{\psi_1, \ldots, \psi_{d-1}\}$$

and defining relations

$$y_r y_s = y_s y_r; \tag{R1}$$

$$1_{\boldsymbol{i}} 1_{\boldsymbol{j}} = \delta_{\boldsymbol{i},\boldsymbol{j}} 1_{\boldsymbol{i}} \quad \text{and} \quad \sum_{\boldsymbol{i} \in I^\theta} 1_{\boldsymbol{i}} = 1_\theta; \tag{R2}$$

$$y_r 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}} y_r \quad \text{and} \quad \psi_r 1_{\boldsymbol{i}} = 1_{s_r \cdot \boldsymbol{i}} \psi_r; \tag{R3}$$

$$(\psi_r y_t - y_{s_r(t)} \psi_r) 1_{\boldsymbol{i}} = \delta_{i_r,i_{r+1}} (\delta_{t,r+1} - \delta_{t,r}) 1_{\boldsymbol{i}}; \tag{R4}$$

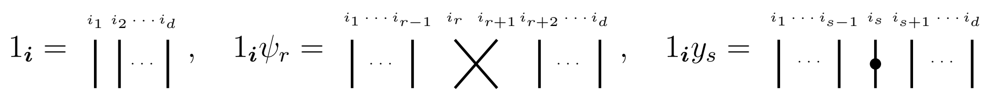
$$\psi_r^2 1_{\boldsymbol{i}} = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ \varepsilon_{i_r,i_{r+1}} (y_r - y_{r+1}) 1_{\boldsymbol{i}} & \text{if } |i_r - i_{r+1}| = 1, \\ 1_{\boldsymbol{i}} & \text{otherwise;} \end{cases} \tag{R5}$$

$$\psi_r \psi_s = \psi_s \psi_r \text{ if } |r - s| > 1; \tag{R6}$$

$$(\psi_{r+1} \psi_r \psi_{r+1} - \psi_r \psi_{r+1} \psi_r) 1_{\boldsymbol{i}} = \begin{cases} \varepsilon_{i_r,i_{r+1}} 1_{\boldsymbol{i}} & \text{if } |i_r - i_{r+1}| = 1 \text{ and } i_r = i_{r+2}, \\ 0 & \text{otherwise.} \end{cases} \tag{R7}$$

The right-hand sides of relations (R4) and (R7), when they are nonzero, will be referred to as *error terms*. The algebra R_θ is graded with $\deg 1_{\boldsymbol{i}} = 0$; $\deg(y_s) = 2$; $\deg(\psi_r 1_{\boldsymbol{i}}) = -\mathsf{c}_{i_r, i_{r+1}}$.

We will use the Khovanov-Lauda [8] diagrammatic notation for elements of R_θ. In particular, for $\boldsymbol{i} = i_1 \cdots i_d \in I^\theta$, $1 \le r < d$ and $1 \le s \le d$, we denote



For each element $w \in \mathfrak{S}_n$, fix a reduced expression $w = s_{r_1} \cdots s_{r_l}$ which determines an element $\psi_w = \psi_{r_1} \cdots \psi_{r_l}$. This element depends on the reduced expression of w.

Theorem 2.29. *[8, Theorem 2.5],[10, Theorem 3.7] Let $\theta \in Q_+$ and $d = \mathrm{ht}(\theta)$. Then the following sets are $\Bbbk$-bases of R_θ:*

$$\{\psi_w y_1^{k_1} \dots y_d^{k_d} 1_{\boldsymbol{i}} \mid w \in \mathfrak{S}_d,\ k_1, \dots, k_d \in \mathbb{Z}_{\ge 0},\ \boldsymbol{i} \in I^\theta\},$$
$$\{y_1^{k_1} \dots y_d^{k_d} \psi_w 1_{\boldsymbol{i}} \mid w \in \mathfrak{S}_d,\ k_1, \dots, k_d \in \mathbb{Z}_{\ge 0},\ \boldsymbol{i} \in I^\theta\}.$$

We identify the polynomial algebra

$$\mathcal{Y}_d := \Bbbk[y_1, \dots, y_d] \tag{2.30}$$

with the subalgebra of R_θ generated by $\{y_1, \dots, y_d\}$ according to Theorem 2.29.

The following lemma often simplifies calculations in R_θ.

Lemma 2.31. *Let $i_1, \dots, i_l \in I$ be distinct, $\theta := \alpha_{i_1} + \cdots + \alpha_{i_l}$, $\boldsymbol{i} = i_1 \cdots i_l \in I^\theta$, and $w, w' \in \mathfrak{S}_l$. In R_θ, we have $\psi_{w'}\psi_w 1_{\boldsymbol{i}} = \psi_{w'w} 1_{\boldsymbol{i}}$ unless there are $r, s \in [1, l]$ such that $|i_r - i_s| = 1$, $r < s$, $w(r) > w(s)$, and $w'w(r) < w'w(s)$.*

Proof. As $i_1, \dots, i_l$ are distinct, the braid relation (R7) holds without error term in R_θ. Moreover, so long as there is no pair r, s as in the statement, the only quadratic relations we need to use are of the form $\psi_t^2 1_{\boldsymbol{j}} = 1_{\boldsymbol{j}}$. Therefore $\psi_w \psi_{w'} 1_{\boldsymbol{i}}$ simplifies directly to $\psi_{ww'} 1_{\boldsymbol{i}}$ as claimed. □

Parabolic subalgebras and divided power idempotents

Let $\theta_1, \dots, \theta_t \in Q_+$ and set $\theta := \theta_1 + \dots + \theta_t$. Let

$$1_{\theta_1,\dots,\theta_t} := \sum_{\boldsymbol{i}^1 \in I^{\theta_1},\dots,\boldsymbol{i}^t \in I^{\theta_t}} 1_{\boldsymbol{i}^1 \dots \boldsymbol{i}^t} \in R_\theta.$$

Then we have an algebra embedding

$$\iota_{\theta_1,\dots,\theta_t} : R_{\theta_1} \otimes \dots \otimes R_{\theta_t} \hookrightarrow 1_{\theta_1,\dots,\theta_t} R_{\theta_1+\dots+\theta_t} 1_{\theta_1,\dots,\theta_t} \tag{2.32}$$

obtained by horizontal concatenation of the Khovanov-Lauda diagrams. For $r_1 \in R_{\theta_1}, \dots, r_t \in R_{\theta_t}$ we often write

$$r_1 \circ \dots \circ r_t := \iota_{\theta_1,\dots,\theta_t}(r_1 \otimes \dots \otimes r_t).$$

For example,

$$1_{\boldsymbol{i}^1} \circ \dots \circ 1_{\boldsymbol{i}^t} = 1_{\boldsymbol{i}^1 \dots \boldsymbol{i}^t} \qquad (\boldsymbol{i}^1 \in I^{\theta_1}, \dots, \boldsymbol{i}^t \in I^{\theta_t}). \tag{2.33}$$

We fix for the moment $i \in I$, $d \in \mathbb{Z}_{\geq 0}$ and take $\theta = d\alpha_i$. Then we have an isomorphism $\varphi : \mathcal{NH}_d \xrightarrow{\sim} R_{d\alpha_i}$, $x_r \mapsto y_r$, $\tau_s \mapsto \psi_s$. Recalling (2.26) and Lemma 2.27, the following element is an idempotent in $R_{d\alpha_i}$:

$$1_{i^{(d)}} := \varphi(e'_d).$$

Now let $\theta \in Q_+$ be arbitrary. We define I^θ_{div} to be the set of all expressions of the form $i_1^{(d_1)} \cdots i_r^{(d_r)}$ with $d_1, \dots, d_r \in \mathbb{Z}_{\geq 0}$, $i_1, \dots, i_r \in I$ and $d_1\alpha_{i_1} + \cdots + d_r\alpha_{i_r} = \theta$. We refer to such expressions as *divided power words*. We identify I^θ with the subset of I^θ_{div} which consists of all divided power words as above with all $d_k = 1$. We use the same notation for concatenation of divided power words as for concatenation of words. For $\boldsymbol{i} = i_1^{(d_1)} \cdots i_r^{(d_r)} \in I^\theta_{\mathrm{div}}$, we define

$$\boldsymbol{i}^!_{(\pm)} := [d_1]^!_{(\pm)} \cdots [d_r]^!_{(\pm)}, \quad \text{and} \quad \hat{\boldsymbol{i}} := i_1^{d_1} \cdots i_r^{d_r} \in I^\theta, \tag{2.34}$$

and the corresponding *divided power idempotent* is

$$1_{\boldsymbol{i}} = 1_{i_1^{(d_1)} \cdots i_r^{(d_r)}} := 1_{i_1^{(d_1)}} \circ \cdots \circ 1_{i_r^{(d_r)}} \in R_\theta.$$

We have the following generalization of (2.33):

$$1_{\boldsymbol{i}^1} \circ \cdots \circ 1_{\boldsymbol{i}^t} = 1_{\boldsymbol{i}^1 \dots \boldsymbol{i}^t} \qquad (\boldsymbol{i}^1 \in I^{\theta_1}_{\mathrm{div}}, \dots, \boldsymbol{i}^t \in I^{\theta_t}_{\mathrm{div}}).$$

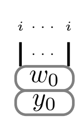
Lemma 2.35. *In the algebra $R_{d\alpha_i}$, if $r_1 + \cdots + r_t = d$ then $1_{i^{(r_1)} \dots i^{(r_t)}} \psi_{w_{0,d}} = \psi_{w_{0,d}}$ and $1_{i^{(r_1)} \dots i^{(r_t)}} 1_{i^{(d)}} = 1_{i^{(d)}}$.*

Proof. Write $\psi_{w_{0,d}} = (\psi_{w_{0,r_1}} \circ \cdots \circ \psi_{w_{0,r_t}})\psi_u$ for some $u \in \mathfrak{S}_d$ and use Lemma 2.27.

□

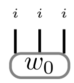
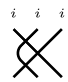
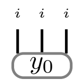
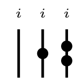
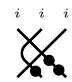
To be used as part of the Khovanov-Lauda diagrammatics, we denote

$$\psi_{w_{0,d}} =: \boxed{w_0}, \quad y_{0,d} =: \boxed{y_0}, \quad 1_{i^{(d)}} =$$



$$=: \boxed{i^d}.$$

For example, if $d = 3$, we have

$1_{i^3}\psi_{w_0} =$  $=$  $,\quad 1_{i^3}y_0 =$  $=$  $,\quad 1_{i^{(3)}} =$  $.$

More generally, we denote

$$1_{i_1^{(d_1)}\cdots i_r^{(d_r)}} =: \boxed{i_1^{d_1}} \cdots \boxed{i_r^{d_r}}.$$

<u>Modules over R_θ</u>

Let $\theta \in Q_+$. Recalling (2.34), for a Laurentian R_θ-module V and $\boldsymbol{i} \in I^\theta_{\mathrm{div}}$, by [8, §2.5], we have

$$\dim_q(1_{\boldsymbol{i}}V) = \frac{1}{\boldsymbol{i}^!_+}\dim_q(1_{\hat{\boldsymbol{i}}}V), \tag{2.36}$$

which explains the usage of the term "divided power word" for $\boldsymbol{i} \in I^\theta_{\mathrm{div}}$. If V is a Laurentian R_θ-module then each $1_{\boldsymbol{i}}V$ is a Laurentian $\Bbbk$-module, and so we can define the *formal character* of V as follows:

$$\mathrm{ch}_q V := \sum_{\boldsymbol{i}\in I^\theta}(\dim_q 1_{\boldsymbol{i}}V)\cdot \boldsymbol{i} \in \mathbb{Z}((q))\cdot I^\theta.$$

Note that $\mathrm{ch}_q(q^dV) = q^d\,\mathrm{ch}_q(V)$.

For $\theta_1,\dots,\theta_t \in Q_+$ and $\theta := \theta_1+\cdots+\theta_t$, recalling (2.32), we have a functor

$$\mathrm{Ind}_{\theta_1,\dots,\theta_t} = R_\theta 1_{\theta_1,\dots,\theta_t}\otimes_{R_{\theta_1}\otimes\cdots\otimes R_{\theta_t}} - : (R_{\theta_1}\otimes\cdots\otimes R_{\theta_t})\text{–Mod} \to R_\theta\text{–Mod}\,. \tag{2.37}$$

For $V_1 \in R_{\theta_1}$–Mod, $\dots, V_t \in R_{\theta_t}$–Mod, we denote by $V_1 \boxtimes \cdots \boxtimes V_t$ the $\Bbbk$-module $V_1 \otimes \cdots \otimes V_t$, considered naturally as an $(R_{\theta_1} \otimes \cdots \otimes R_{\theta_t})$-module, and set

$$V_1 \circ \cdots \circ V_t := \mathrm{Ind}_{\theta_1,\dots,\theta_t} V_1 \boxtimes \cdots \boxtimes V_t.$$

By Theorem 2.29, setting $d_k := \mathrm{ht}(\theta_k)$, we have

$$V_1 \circ \cdots \circ V_t = \bigoplus_{w \in \mathscr{D}^{(d_1,\dots,d_t)}} \psi_w \otimes V_1 \otimes \cdots \otimes V_t. \tag{2.38}$$

If $V_1, \dots, V_t$ are Laurentian, then by [8, Lemma 2.20], recalling (2.28), we have

$$\mathrm{ch}_q(V_1 \circ \cdots \circ V_t) = \mathrm{ch}_q(V_1) \circ \cdots \circ \mathrm{ch}_q(V_t). \tag{2.39}$$

For $v_1 \in V_1, \dots, v_t \in V_t$, we denote

$$v_1 \circ \cdots \circ v_t := 1_{\theta_1,\dots,\theta_t} \otimes v_1 \otimes \cdots \otimes v_t \in V_1 \circ \cdots \circ V_t.$$

If $\boldsymbol{i}^1 \in I^{\theta_1}_{\mathrm{div}}, \dots, \boldsymbol{i}^t \in I^{\theta_t}_{\mathrm{div}}$, it is easy to check that

$$R_{\theta_1} 1_{\boldsymbol{i}^1} \circ \cdots \circ R_{\theta_t} 1_{\boldsymbol{i}^t} \cong R_\theta 1_{\boldsymbol{i}^1 \cdots \boldsymbol{i}^t}. \tag{2.40}$$

Since $R_\theta 1_{\theta_1,\dots,\theta_t}$ is a free right $R_{\theta_1} \otimes \cdots \otimes R_{\theta_t}$-module of finite rank by Theorem 2.29, we get the following well-known properties:

Proposition 2.41. *The functor* $\mathrm{Ind}_{\theta_1,\dots,\theta_t}$ *is exact and sends finitely generated projectives to finitely generated projectives.*

Let again $\theta_1, \dots, \theta_t \in Q_+$ and $\theta = \theta_1 + \cdots + \theta_t$. Suppose $(C^k_\bullet, d_k)$ is a chain complexes of R_{θ_k}-modules for $1 \le k \le t$. Let $(C^1_\bullet \circ \cdots \circ C^t_\bullet)_n := \bigoplus_{p_1+\cdots+p_t=n} C^1_{p_1} \circ \cdots \circ C^t_{p_t}$, and for $x_1 \in C^1_{p_1}, \dots, x_t \in C^t_{p_t}$, define

$$d(x_1 \circ \cdots \circ x_t) := \sum_{k=1}^{t} (-1)^{p_{k+1}+\cdots+p_t} x_1 \circ \cdots \circ x_{k-1} \circ d_k(x_k) \circ x_{k+1} \circ \cdots \circ x_t.$$

Then $(C^1_\bullet \circ \cdots \circ C^t_\bullet, d)$ is a chain complex of R_θ-modules. Proposition 2.41 and [32, Lemma 2.7.3] immediately imply the following.

Lemma 2.42. *If $C^k_\bullet$ is a projective resolution of $M_k \in R_{\theta_k}$–Mod for $1 \le k \le t$, then $C^1_\bullet \circ \cdots \circ C^t_\bullet$ is a projective resolution of $M_1 \circ \cdots \circ M_t \in R_\theta$–Mod.*

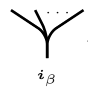
Standard modules

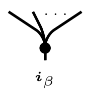
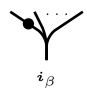
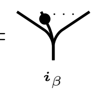
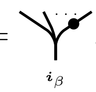
The algebra R_θ is affine quasihereditary in the sense of [23]. In particular, it comes with an important class of *standard modules*, which we now describe explicitly following [12].

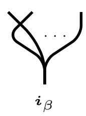
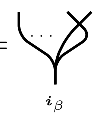
Fix $\beta = \alpha_i + \cdots + \alpha_j \in \Phi_+$ of height $l := j-i+1$, and set $\boldsymbol{i}_\beta := i\,(i+1) \cdots j \in I^\beta$. We define the R_β-module $\Delta(\beta)$ to be a cyclic R_β-module generated by a vector v_β of degree 0 with defining relations

- $1_{\boldsymbol{i}} v_\beta = \delta_{\boldsymbol{i}, \boldsymbol{i}_\beta} v_\beta$ for all $\boldsymbol{i} \in I^\beta$;
- $\psi_r v_\beta = 0$ for all $1 \le r < l$;
- $y_r v_\beta = y_s v_\beta$ for all $1 \le r, s \le l$.

The module $\Delta(\beta)$ can be considered as an $(R_\beta, \Bbbk[x])$-bimodule with the right action given by $v_\beta x := y_1 v_\beta$. Diagrammatically, we represent

$v_\beta =$  ,

$v_\beta x =$  $= y_1 v_\beta =$  $= y_2 v_\beta =$  $= \cdots = y_l v_\beta =$  ,

$\psi_1 v_\beta =$  $= \cdots = \psi_{l-1} v_\beta =$  $= 0.$

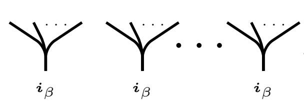
The following lemma is easy to check.

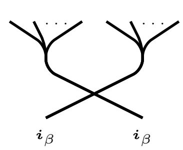
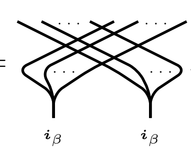
Lemma 2.43. *Let $\beta \in \Phi_+$. Then there is an isomorphism of right $\Bbbk[x]$-modules $\Bbbk[x] \to \Delta(\beta),\ 1 \mapsto v_\beta$.*

For $m \in \mathbb{Z}_{\geq 0}$ and $\beta \in \Phi_+$, the $R_{m\beta}$-module $\Delta(\beta)^{\circ m}$ is cyclicly generated by $v_\beta^{\circ m}$. As explained in [12, §3.2], $\mathcal{NH}_m$ acts on $\Delta(\beta)^{\circ m}$ on the right so that

$$v_\beta^{\circ m} x_r = v_\beta^{\circ(r-1)} \circ (v_\beta x) \circ v_\beta^{\circ(m-r)}, \quad v_\beta^{\circ m} \tau_s = v_\beta^{\circ(s-1)} \circ (\psi_{w_{l,l}}(v_\beta \circ v_\beta)) \circ v_\beta^{\circ(m-s-1)},$$

where $w_{l,l}$ is the longest element of $\mathscr{D}^{(l,l)}$. Diagrammatically, we represent

$v_\beta^{\circ m} =$  ,

$(v_\beta \circ v_\beta)\tau_1 =$  $= \psi_{w_{l,l}}(v_\beta \circ v_\beta) =$  .

Let $\leq$ be the lexicographic total order on Φ_+, i.e. for $\beta = \alpha_i + \cdots + \alpha_j \in \Phi_+$ and $\beta' = \alpha_{i'} + \cdots + \alpha_{j'} \in \Phi_+$, we have $\beta < \beta'$ if and only if either $i < i'$ or

$i = i'$ and $j < j'$. Given $\theta \in Q_+$, a *Kostant partition* of θ is a sequence $\pi = (\beta_1^{m_1}, \dots, \beta_t^{m_t})$ such that $m_1, \dots, m_t \in \mathbb{Z}_{>0}$, $\beta_1 > \cdots > \beta_t$ are positive roots, and $m_1\beta_1 + \cdots + m_t\beta_t = \theta$. We denote by $\mathrm{KP}(\theta)$ the set of all Kostant partitions of θ. For $\pi = (\beta_1^{m_1}, \dots, \beta_t^{m_t}) \in \mathrm{KP}(\theta)$,

$$\hat{\Delta}(\pi) := q^{\binom{m_1}{2}+\cdots+\binom{m_t}{2}} \Delta(\beta_1)^{\circ m_1} \circ \cdots \circ \Delta(\beta_t)^{\circ m_t} \tag{2.44}$$

can now be considered as an $(R_\theta, \mathcal{NH}_{m_1} \otimes \cdots \otimes \mathcal{NH}_{m_t})$-bimodule. Recalling (2.26), we define the corresponding *standard module* as

$$\Delta(\pi) := \hat{\Delta}(\pi)(e_{m_1} \otimes \cdots \otimes e_{m_t}).$$

Setting

$$\Lambda_\pi := \Lambda_{m_1,\dots,m_t}, \tag{2.45}$$

by Theorem 2.24, $\Delta(\pi)$ is naturally an (R_θ, Λ_π)-bimodule. In fact, by [16, Theorem 2.17], the bimodule structure yields the isomorphism

$$\mathrm{End}_{R_\theta}(\Delta(\pi))^{\mathrm{op}} \cong \Lambda_\pi. \tag{2.46}$$

The module $\Delta(\pi)$ is cyclic as a left R_θ-module with *standard generator*

$$v_\pi := (v_{\beta_1}^{\circ m_1} \circ \cdots \circ v_{\beta_t}^{\circ m_t})(e_{m_1} \otimes \cdots \otimes e_{m_t}). \tag{2.47}$$

Noting that $\Delta(\pi) = \Delta(\beta_1^{m_1}) \circ \cdots \circ \Delta(\beta_1^{m_t})$, by [12, Lemma 3.10], we have an isomorphism of R_θ-modules

$$\hat{\Delta}(\pi) \cong [m_1]^!_+ \cdots [m_t]^!_+ \Delta(\pi). \tag{2.48}$$

If $\Bbbk$ is a field, the modules $\{\Delta(\pi) \mid \pi \in \mathrm{KP}(\theta)\}$ are the standard modules for an affine quasihereditary structure on the algebra R_θ, see [12, 23]. If $\Bbbk = \mathbb{Z}$ or $\mathbb{Z}_p$, they can be thought of as integral forms of the standard modules, see [16, §4].

Standard resolutions

Some of the main ingredients in the machinery outlined in section 2.2 are projective resolutions of the standard modules. Let $\theta \in Q_+$. If $\rho = (\beta_1^{m_1}, \ldots, \beta_t^{m_t}) \in \mathrm{KP}(\theta)$, then by construction, we have $\Delta(\rho) = \Delta(\beta_1^{m_1}) \circ \cdots \circ \Delta(\beta_t^{m_t})$. Since the induction product (2.37) is exact and sends projectives to projectives, a projective resolution for $\Delta(\rho)$ given by

$$P^\rho_\bullet = P^{\beta_1^{m_1}}_\bullet \circ \cdots \circ P^{\beta_t^{m_t}}_\bullet,$$

where $P^{(\beta_i^{m_i})}_\bullet$ are projective resolutions for $\Delta(\beta_i^{m_i})$. We may therefore consider the *semicuspidal* case where ρ is of the form $\rho = (\beta^m)$. When $m = 1$, this was achieved in [12], and was generalized to arbitrary $m \geq 1$ in [2]. We record the form of the resolution here.

Let $\beta = \alpha_1 + \cdots + \alpha_l$ and $\theta = m\beta$. We consider the set of compositions

$$\Lambda := \{\lambda = (\lambda_1, \ldots, \lambda_{l-1}) \mid 0 \leq \lambda_i \leq m\}.$$

Let $\Lambda(n) := \{\lambda \in \Lambda \mid \lambda_1 + \cdots + \lambda_{l-1} = n\}$. For $\lambda \in \Lambda$, define

$$\begin{aligned}
s_\lambda &:= -l\frac{m(m-1)}{2} + (m+1)n - \sum_{i=1}^{l-1} \lambda_i^2, \\
\boldsymbol{i}^\lambda &:= 1^{(m-\lambda_1)}2^{(m-\lambda_2)} \cdots (l-1)^{(m-\lambda_{l-1})} l^{(m)} (l-1)^{(\lambda_{l-1})} \cdots 2^{(\lambda_2)} 1^{(\lambda_1)} \in I_{\text{div}}^{m\beta}, \\
e_\lambda &:= 1_{\boldsymbol{i}^\lambda} \in R_{m\beta}, \\
P_\lambda &:= q^{s_\lambda} R_{m\beta} e_\lambda,
\end{aligned}$$

so that P_λ are projective. The terms of the resolution are now defined by

$$P_n^{(\beta^m)} := \bigoplus_{\lambda \in \Lambda(n)} P_\lambda.$$

Given $\mu \in \Lambda(n+1)$ and $\lambda \in \Lambda(n)$, explicit elements $d_n^{\mu,\lambda} \in e_\mu R_\theta e_\lambda$ are defined in [2], so that right multiplication with $d_n^{\mu,\lambda}$ is a map $P_\mu \to P_\lambda$. The matrix $d_n = (d_n^{\mu,\lambda})_{\mu \in \Lambda(n+1), \lambda \in \Lambda(n)}$ then represents a map $P_{n+1}^{(\beta^m)} \to P_n^{(\beta^m)}$. We also define a natural augmentation map $\mathsf{p} : P_0^{(\beta^m)} \to \Delta(\beta^m)$. Note that $\Lambda(n)$ is empty for $n > m(l-1)$, so that $P_n^{(\beta^m)} = 0$.

Theorem 2.49. *[2] The sequence*

$$0 \longrightarrow P_{m(l-1)}^{(\beta^m)} \longrightarrow \cdots \longrightarrow P_{n+1}^{(\beta^m)} \xrightarrow{d_n} P_n^{(\beta^m)} \longrightarrow \cdots \longrightarrow P_0^{(\beta^m)} \xrightarrow{\mathsf{p}} \Delta(\beta^m) \longrightarrow 0$$

is a projective resolution of the standard module $\Delta(\beta^m)$.

For $\rho \in \mathrm{KP}(\theta)$, suppose we have a projective resolution of $\Delta(\rho)$ of the form $P_\bullet^\rho = \cdots \xrightarrow{d_1} P_1^\rho \xrightarrow{d_0} P_0^\rho \xrightarrow{\varepsilon_\rho} \Delta(\rho)$ with $P_n^\rho = \bigoplus_{x \in X_n} q^{s_x} R_\theta 1_x$ for some index set X_n, integers s_x, and idempotents 1_x. The map $d_n : P_{n+1}^\rho \to P_n^\rho$ can be described as right multiplication by an $X_{n+1} \times X_n$ matrix $D_n = (d_n^{y,x})$ for some $d_n^{y,x} \in 1_y R_\theta 1_x$.

Using the isomorphism $\mathrm{Hom}_{R_\theta}(q^n R_\theta 1_x, \Delta(\sigma)) \xrightarrow{\sim} q^{-n} 1_x \Delta(\sigma)$ and recalling (2.46), we obtain:

Lemma 2.50. *There is an isomorphism of complexes of (right) Λ_σ-modules*

$$\mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, \Delta(\sigma)) \cong T^{P^\rho_\bullet}_\bullet(\Delta(\sigma)),$$

where $T^{P^\rho_\bullet}_\bullet(\Delta(\sigma)) = \cdots \overset{d^1}{\leftarrow} T_1 \overset{d^0}{\leftarrow} T_0$ with $T_n = \bigoplus_{x \in X_n} q^{-s_x} 1_x \Delta(\sigma)$ and d^n given by left multiplication with the $X_{n+1} \times X_n$ matrix D_n.

This yields an isomorphism $\mathcal{E}_\theta(\rho, \sigma) \cong H(T^{P^\rho_\bullet}_\bullet(\sigma))$ of Λ_σ-modules. One can also use the resolutions $P^\rho_\bullet$, $P^\sigma_\bullet$, and $P^\tau_\bullet$ to describe the composition map $\mathcal{E}_\theta(\sigma, \tau) \otimes \mathcal{E}_\theta(\rho, \sigma) \to \mathcal{E}_\theta(\rho, \tau)$. Indeed, let $\mathrm{Hom}^m_{R_\theta}(P^\rho_\bullet, P^\sigma_\bullet)$ denote the homological degree m homomorphisms. Then $\mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, P^\sigma_\bullet)$ is a complex with respect to the differential (2.12). Recall the isomorphism (2.13). The composition map $\mathcal{E}^n_\theta(\sigma, \tau) \otimes \mathcal{E}^m_\theta(\rho, \sigma) \to \mathcal{E}^{n+m}_\theta(\rho, \tau)$ is induced from the composition of homomorphisms $\mathrm{Hom}^n_{R_\theta}(P^\sigma_\bullet, P^\tau_\bullet) \otimes \mathrm{Hom}^m_{R_\theta}(P^\rho_\bullet, P^\sigma_\bullet) \to \mathrm{Hom}^{n+m}_{R_\theta}(P^\rho_\bullet, P^\tau_\bullet)$.

CHAPTER III

THE CASE WHERE θ IS A ROOT

This chapter contains material which appears in [1, §3]. The author and David J. Steinberg independently performed the relevant computations under the supervision of and with assistance from Alexander Kleshchev. The results were written initially by David J. Steinberg and were revised by the author and Alexander Kleshchev.

Throughout this chapter, $\theta = \alpha_a + \cdots + \alpha_{b+1}$ (with $a \leq b+1$) is a positive root of height $l = b + 2 - a$. Note that for $\pi = (\pi_1, \dots, \pi_u) \in \mathrm{KP}(\theta)$, the positive roots $\pi_1, \dots, \pi_u$ are all distinct. There is a bijection from $\mathrm{KP}(\theta)$ to the set of all subsets of $[a, b]$

$$\pi = (\pi_1, \dots, \pi_u) \mapsto C_\pi := \{\max(\operatorname{supp} \pi_2), \max(\operatorname{supp} \pi_3), \dots, \max(\operatorname{supp} \pi_u)\},$$

where, for a root $\alpha = \alpha_i + \cdots + \alpha_j$, we let $\operatorname{supp} \alpha := [i, j]$. For $\pi, \tau \in \mathrm{KP}(\theta)$, if $C_\tau \supseteq C_\pi$, we say that τ is a *refinement* of π and write $\tau \supseteq \pi$. If, in addition, $|C_\tau \setminus C_\pi| = n$, we write $\tau \supseteq_n \pi$ and say that τ is an *n-refinement* of π. If $C_\tau \setminus C_\pi = \{i\}$ for some $i \in [a, b]$, we write $\mathrm{ref}^i(\pi) := \tau$. For example, we have $\mathrm{ref}^i((\theta)) = (\alpha_{i+1} + \cdots + \alpha_{b+1}, \alpha_a + \cdots + \alpha_i)$.

If $\tau = (\tau_1, \dots, \tau_t) \in \mathrm{KP}(\theta)$, the elements of

$$\mathscr{D}^\tau := \mathscr{D}^{(\mathrm{ht}(\tau_1), \dots, \mathrm{ht}(\tau_t))} \tag{3.1}$$

are called τ-*shuffles*. Set

$$d_v := \mathrm{ht}(\tau_1) + \cdots + \mathrm{ht}(\tau_v) \qquad (0 \le v \le t).$$

We say that integers $r, s \in [1, l]$ are τ-*equivalent* if there is some $v \in [1, t]$ with $d_{v-1} < r, s \le d_v$. Recalling (2.45) and (2.46), we have

$$\Lambda_\tau = \Bbbk[x_1, \dots, x_t] \cong \mathrm{End}_{R_\theta}(\Delta(\tau))^{\mathrm{op}}.$$

We have a surjection

$$\mathtt{p}_\tau : \mathcal{Y}_l \twoheadrightarrow \Lambda_\tau,\ y_r \mapsto x_v \text{ if } d_{v-1} < r \le d_v, \tag{3.2}$$

so that $\mathtt{p}_\tau(y_r) = \mathtt{p}_\tau(y_s)$ if and only if r and s are τ-equivalent. If, in addition, $\tau \supseteq \pi = (\pi_1, \dots, \pi_u)$, then $\mathtt{p}_\pi$ factors as $\mathtt{p}_\pi = \mathtt{p}_\pi^\tau \mathtt{p}_\tau$, where the surjection $\mathtt{p}_\pi^\tau$ is defined as follows: for each $m \in [0, u]$, we have $\mathrm{ht}(\pi_1) + \cdots + \mathrm{ht}(\pi_m) = d_{v_m}$ for some $v_m \in [0, t]$; now

$$\mathtt{p}_\pi^\tau : \Lambda_\tau \twoheadrightarrow \Lambda_\pi,\ x_r \mapsto x_m \text{ if } v_{m-1} < r \le v_m. \tag{3.3}$$

3.1. The resolution $S_\bullet^\rho$

In this section, we fix $\rho = (\rho_1, \dots, \rho_t) \in \mathrm{KP}(\theta)$. For $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j \in \Phi_+$, let

$$\boldsymbol{j}_\alpha := i(i+1)\cdots j \in I^\alpha, \quad e_\alpha := 1_{\boldsymbol{j}_\alpha} \in R_\alpha.$$

Then set

$$\boldsymbol{j}_\rho := \boldsymbol{j}_{\rho_1} \cdots \boldsymbol{j}_{\rho_t} \in I^\theta, \quad e_\rho := 1_{\boldsymbol{j}_\rho} \in R_\theta. \tag{3.4}$$

For $\pi, \tau \in \mathrm{KP}(\theta)$, let $w(\tau,\pi) \in \mathfrak{S}_l$ be the unique permutation with

$$w(\tau,\pi) \cdot \boldsymbol{j}_\pi = \boldsymbol{j}_\tau, \tag{3.5}$$

so that $e_\tau \psi_{w(\tau,\pi)} = \psi_{w(\tau,\pi)} e_\pi$. If $\tau = \mathrm{ref}^i(\pi)$, we set

$$s(\tau,\pi) := (-1)^{|C_\pi \cap [a,i)|}.$$

For $n \in \mathbb{Z}_{\geq 0}$, we set

$$S_n^\rho := \bigoplus_{\pi \supseteq_n \rho} q^n R_\theta e_\pi.$$

The boundary map $S_{n+1}^\rho \to S_n^\rho$ is defined to be right multiplication with the matrix

$$d_n := (d_n^{\tau,\pi})_{\substack{\tau \supseteq_{n+1} \rho \\ \pi \supseteq_n \rho}}, \text{ where } d_n^{\tau,\pi} := \begin{cases} s(\tau,\pi) e_\tau \psi_{w(\tau,\pi)} e_\pi & \text{if } \tau \supseteq \pi, \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

We define the augmentation map by

$$\varepsilon_\rho : S_0^\rho = R_\theta e_\rho \to \Delta(\rho), \;\; h e_\rho \mapsto h v_\rho,$$

where v_ρ is the standard generator for $\Delta(\rho)$, see (2.47).

Lemma 3.7. *The following is a projective resolution of $\Delta(\rho)$:*

$$0 \longrightarrow S_{l-t}^\rho \longrightarrow \cdots \longrightarrow S_{n+1}^\rho \xrightarrow{d_n} S_n^\rho \longrightarrow \cdots \longrightarrow S_0^\rho \xrightarrow{\varepsilon_\rho} \Delta(\rho) \longrightarrow 0.$$

Proof. Given a complex $C_\bullet$, we denote by $\overline{C}_\bullet$ the same complex but with all the boundary maps negated. If $v \in [1,t]$, we let

$$\widehat{S}_\bullet^{(\rho_v)} = \begin{cases} \overline{S}_\bullet^{(\rho_v)} & \text{if } t-v \text{ is odd,} \\ S_\bullet^{(\rho_v)} & \text{if } t-v \text{ is even.} \end{cases}$$

Using (2.40) and the fact that the resolutions $\overline{S}_\bullet^{(\rho_v)}$ and $S_\bullet^{(\rho_v)}$ are isomorphic, it is easy to note that

$$S_\bullet^\rho = \widehat{S}_\bullet^{(\rho_1)} \circ \cdots \circ \widehat{S}_\bullet^{(\rho_t)} \cong S_\bullet^{(\rho_1)} \circ \cdots \circ S_\bullet^{(\rho_t)},$$

so by Lemma 2.42, we have reduced to the case $t = 1$, i.e. $\rho = (\rho_1) = (\theta)$.

To complete the proof, we show that $S_\bullet^{(\theta)} \cong P_\bullet$, where $P_\bullet$ is a resolution of $\Delta(\theta)$ constructed in [12, §4.5] (see also [2]) and which we now recall. For $\pi \in \mathrm{KP}(\theta)$, put

$$\boldsymbol{i}^\pi := a^{\delta_{a \notin C_\pi}} (a+1)^{\delta_{a+1 \notin C_\pi}} \cdots b^{\delta_{b \notin C_\pi}} (b+1) b^{\delta_{b \in C_\pi}} \cdots (a+1)^{\delta_{a+1 \in C_\pi}} a^{\delta_{a \in C_\pi}} \in I^\theta,$$

where, for a statement s, $\delta_s = 1$ if s is true, and $\delta_s = 0$ if s if false. For $n \in \mathbb{Z}_{\geq 0}$, set $P_n := \bigoplus_{\pi \supseteq_n (\theta)} q^n R_\theta 1_{\boldsymbol{i}^\pi}$. If $\pi, \tau \in \mathrm{KP}(\theta)$ with $\tau = \mathrm{ref}^i(\pi)$, let $u(\tau,\pi) \in \mathfrak{S}_l$ be determined from $u(\tau,\pi) \cdot \boldsymbol{i}^\pi = \boldsymbol{i}^\tau$, and define the matrix

$$\partial_n := (\partial_n^{\tau,\pi})_{\substack{\tau \supseteq_{n+1} (\theta) \\ \pi \supseteq_n (\theta)}}, \text{ where } \partial_n^{\tau,\pi} := \begin{cases} s(\tau,\pi) 1_{\boldsymbol{i}^\tau} \psi_{u(\tau,\pi)} 1_{\boldsymbol{i}^\pi} & \text{if } \tau \supseteq \pi, \\ 0 & \text{otherwise.} \end{cases}$$

Right multiplication with ∂_n defines a map $P_{n+1} \to P_n$. By [12, Theorem 4.12] (see also [2, Theorem A]), noting that $P_0 = S_0^{(\theta)}$, we have that

$$0 \longrightarrow P_{b+1-a} \longrightarrow \cdots \longrightarrow P_{n+1} \xrightarrow{\partial_n} P_n \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\varepsilon(\theta)} \Delta(\theta) \longrightarrow 0$$

is a projective resolution of $\Delta(\theta)$.

For $\pi \in \mathrm{KP}(\theta)$, let $w(\pi) \in \mathfrak{S}_l$ be the unique permutation with $w(\pi) \cdot \boldsymbol{i}^\pi = \boldsymbol{j}_\pi$ so that $e_\pi \psi_{w(\pi)} = \psi_{w(\pi)} 1_{\boldsymbol{i}^\pi}$. We have the map

$$S_n^{(\theta)} = \bigoplus_{\pi \supseteq_n (\theta)} q^n R_\theta e_\pi \to \bigoplus_{\pi \supseteq_n (\theta)} q^n R_\theta 1_{\boldsymbol{i}^\pi} = P_n,$$
$$(h_\pi e_\pi)_{\pi \supseteq_n (\theta)} \mapsto (h_\pi e_\pi \psi_{w(\pi)} 1_{\boldsymbol{i}^\pi})_{\pi \supseteq_n (\theta)}.$$

To check that this yields an isomorphism of complexes, let $\tau := \mathrm{ref}^i(\pi)$ for some $i \in [a, b] \setminus C_\pi$, and check the following using Lemma 2.31:

- $e_\tau \psi_{w(\tau)} \psi_{u(\tau,\pi)} = e_\tau \psi_{w(\tau,\pi)} \psi_{w(\pi)}$,
- $e_\pi \psi_{w(\pi)} \psi_{w(\pi)^{-1}} = e_\pi$,
- $1_{\boldsymbol{i}^\pi} \psi_{w(\pi)^{-1}} \psi_{w(\pi)} = 1_{\boldsymbol{i}^\pi}$. □

3.2. The $\Bbbk$-module $\mathcal{E}_\theta(\rho, \sigma)$

Throughout this section, we fix $\rho, \sigma \in \mathrm{KP}(\theta)$. Recall the word $\boldsymbol{j}_\sigma \in I^\theta$ and the idempotent e_σ from (3.4). Writing $\boldsymbol{j}_\sigma = j_1 j_2 \cdots j_l$, for $i \in [a, b+1]$, there exists a unique $r \in [1, l]$ such that $j_r = i$, and we denote $r_\sigma(i) := r$.

We have the standard module $\Delta(\sigma)$ with generator $v_\sigma \in e_\sigma \Delta(\sigma)$, see (2.47). As in section 2.3, we consider $\Delta(\sigma)$ as an $(R_\theta, \Lambda_\sigma)$-bimodule. Recalling

Lemma 2.50, we write $T^\rho_\bullet(\sigma) := T^{S^\rho_\bullet}_\bullet(\Delta(\sigma))$, so that $T^\rho_n(\sigma) = \bigoplus_{\pi \supseteq_n \rho} q^{-n} e_\pi \Delta(\sigma)$. Recall (3.2) and (3.1).

Lemma 3.8. *In $\Delta(\sigma)$, we have:*

1. $y_r v_\sigma = v_\sigma \mathtt{p}_\sigma(y_r)$*; in particular,* $y_r v_\sigma = y_s v_\sigma$ *if r and s are σ-equivalent;*

2. $\psi_w v_\sigma = 0$ *whenever $w \in \mathfrak{S}_l$ is not a σ-shuffle.*

Moreover, $\Delta(\sigma)$ is free as a right Λ_σ-module with basis $\{\psi_w v_\sigma \mid w \in \mathscr{D}^\sigma\}$.

Proof. Use Lemma 2.43 and (2.38). □

Recalling (3.5), we now get:

Lemma 3.9. *Let $\pi, \sigma \in \mathrm{KP}(\theta)$. If $\sigma \not\supseteq \pi$, then $e_\pi \Delta(\sigma) = 0$. If $\sigma \supseteq_n \pi$, then there is an isomorphism of right Λ_σ-modules*

$$q^n \Lambda_\sigma \xrightarrow{\sim} e_\pi \Delta(\sigma),\ f \mapsto \psi_{w(\pi,\sigma)} v_\sigma f.$$

Proof. If σ is a refinement of π, then every pair of σ-equivalent integers is also π-equivalent, so $w(\pi, \sigma)$ is a σ-shuffle and the result follows from Lemma 3.8 since the map takes the basis element $1 \in q^n \Lambda_\sigma$ to the basis element $\psi_{w(\pi,\sigma)} v_\sigma \in e_\pi \Delta(\sigma)$, and in this case we have $\deg(\psi_{w(\pi,\sigma)} e_\sigma) = n$.

On the other hand, if $\sigma \not\supseteq \pi$, then there is some $i \in C_\pi$ with $i \notin C_\sigma$. It follows that $r_\sigma(i)$ and $r_\sigma(i) + 1$ are σ-equivalent but $w(\pi, \sigma)(r_\sigma(i)) > w(\pi, \sigma)(r_\sigma(i) + 1)$, so $w(\pi, \sigma)$ is not a σ-shuffle, and $e_\pi \Delta(\sigma) = 0$ by Lemma 3.8(2). □

Corollary 3.10. *We have:*

1. *If $\sigma \not\supseteq \rho$, then $T^\rho_\bullet(\sigma) = 0$.*

2. If $\sigma \supseteq_m \rho$, then as right Λ_σ-modules,

$$T_n^\rho(\sigma) = \bigoplus_{\sigma \supseteq \pi \supseteq_n \rho} q^{-n} \psi_{w(\pi,\sigma)} v_\sigma \cdot \Lambda_\sigma \cong \bigoplus_{\sigma \supseteq \pi \supseteq_n \rho} q^{m-2n} \Lambda_\sigma.$$

In particular, $T_n^\rho(\sigma) = 0$ for $n > m$ and

$$T_m^\rho(\sigma) = q^{-m} e_\sigma \Delta(\sigma) = v_\sigma \cdot \Lambda_\sigma \cong q^{-m} \Lambda_\sigma.$$

Proof. If σ is not a refinement of ρ, then it cannot be a refinement of any $\pi \supseteq \rho$, which implies (1). Let $\sigma \supseteq_m \rho$. Recall that $T_n^\rho(\sigma) = \bigoplus_{\pi \supseteq_n \rho} q^{-n} e_\pi \Delta(\sigma)$. Let $\pi \supseteq_n \rho$. If $\sigma \not\supseteq \pi$, then $e_\pi \Delta(\sigma) = 0$. Otherwise $e_\pi \Delta(\sigma) = \psi_{w(\pi,\sigma)} v_\sigma \cdot \Lambda_\sigma$. □

Lemma 3.11. *Let $\pi \in \mathrm{KP}(\theta)$. Suppose that $\sigma \supseteq \mathrm{ref}^i(\pi)$ for some $i \in [a,b]$. Then*

$$\psi_{w(\mathrm{ref}^i(\pi),\pi)} \psi_{w(\pi,\sigma)} e_\sigma = \psi_{w(\mathrm{ref}^i(\pi),\sigma)} (y_{r_\sigma(i)} - y_{r_\sigma(i+1)}) e_\sigma.$$

Proof. Let $\tau = \mathrm{ref}^i(\pi)$. We have that i and $i+1$ are the only adjacent elements of $[a, b+1]$ with $r_\tau(i) > r_\tau(i+1)$ and $r_\pi(i) < r_\pi(i+1)$. Since $\sigma \supseteq \tau = \mathrm{ref}^i(\pi)$, we have $r_\sigma(i) > r_\sigma(i+1)$. Now we compute:

$$\begin{aligned}
\psi_{w(\tau,\pi)} \psi_{w(\pi,\sigma)} e_\sigma &= \psi_{w(\tau,\pi) s_{r_\pi(i)}} \psi^2_{r_\pi(i)} \psi_{s_{r_\pi(i)} w(\pi,\sigma)} e_\sigma \\
&= \psi_{w(\tau,\pi) s_{r_\pi(i)}} (y_{r_\pi(i+1)} - y_{r_\pi(i)}) \psi_{s_{r_\pi(i)} w(\pi,\sigma)} e_\sigma \\
&= \psi_{w(\tau,\pi) s_{r_\pi(i)}} \psi_{s_{r_\pi(i)} w(\pi,\sigma)} (y_{r_\sigma(i)} - y_{r_\sigma(i+1)}) e_\sigma \\
&= \psi_{w(\tau,\sigma)} (y_{r_\sigma(i)} - y_{r_\sigma(i+1)}) e_\sigma,
\end{aligned}$$

where the first equality is obtained using the fact that the braid relations (R7) hold without error term in R_θ, the second comes by applying a non-trivial quadratic relation (R5) on strands colored i and $i+1$, the third is obtained using the relation (R4), and the last comes from Lemma 2.31. $\square$

The proof of Theorem 3.12 amounts to showing that $T^\rho_\bullet(\sigma)$ is isomorphic to a certain Koszul complex (see [32, §4.5]) which we now define. Suppose $\sigma \supseteq_m \rho$, and write $C_\sigma \setminus C_\rho = \{i_1, \dots, i_m\}$ with $i_1 < \dots < i_m$. Let N be the free right Λ_σ-module of graded rank mq^{-2}, that is, $N := q^{-2}\Lambda_\sigma^{\oplus m}$. For $k \in [1, m]$, we denote $\epsilon_k := (0, \dots, 0, 1, 0, \dots, 0) \in N$ (with "1" in the kth entry). Recalling (3.2), define

$$z_k := s(\mathrm{ref}^{i_k}(\rho), \rho)\mathrm{p}_\sigma(y_{r_\sigma(i_k)} - y_{r_\sigma(i_k+1)}) \in \Lambda_\sigma \qquad (k = 1, \dots, m),$$

$$Z := (z_1, \dots, z_m) = \epsilon_1 z_1 + \dots + \epsilon_m z_m \in N.$$

Note that Z is a homogeneous degree 0 element of N. We consider the Koszul complex $q^m \bigwedge^\bullet N$ associated to the regular sequence Z for the ring Λ_σ:

$$0 \longleftarrow q^m \textstyle\bigwedge^m N \longleftarrow \cdots \longleftarrow q^m \bigwedge^{n+1} N \longleftarrow q^m \bigwedge^n N \longleftarrow \cdots \longleftarrow q^m \bigwedge^0 N \longleftarrow 0$$

$$Z \wedge a \longmapsfrom a$$

where $\bigwedge^n N$ is the nth exterior power of the free Λ_σ-module N. Note that $\bigwedge^n N$ has a Λ_σ-basis $\{\epsilon_{k_1} \wedge \dots \wedge \epsilon_{k_n} \mid 1 \le k_1 < \dots < k_n \le m\}$.

Let $\sigma \supseteq_m \rho$. By Corollary 3.10(2), the mth component of $\mathrm{Hom}^\bullet_{R_\theta}(S^\rho_\bullet, \Delta(\sigma))$ is the last nonzero component and it can be identified with $\mathrm{Hom}_{R_\theta}(q^m R_\theta e_\sigma, \Delta(\sigma))$. Thus, every element of the mth component is a cocycle, so there is a surjective map

$$[-] : \mathrm{Hom}_{R_\theta}(q^m R_\theta e_\sigma, \Delta(\sigma)) \twoheadrightarrow \mathcal{E}^m_\theta(\rho, \sigma) = H^m(\mathrm{Hom}^\bullet_{R_\theta}(S^\rho_\bullet, \Delta(\sigma))), \ \varphi \mapsto [\varphi],$$

where $[\varphi]$ is the cohomology class of φ. Moreover, by Lemma 3.9, we have an isomorphism

$$\xi : q^{-m}\Lambda_\sigma \xrightarrow{\sim} \mathrm{Hom}_{R_\theta}(q^m R_\theta e_\sigma, \Delta(\sigma)),\ f \mapsto (e_\sigma \mapsto v_\sigma f).$$

We consider Λ_ρ to be a Λ_σ-module via the homomorphism $\mathtt{p}^\sigma_\rho : \Lambda_\sigma \twoheadrightarrow \Lambda_\rho$, see (3.3).

Theorem 3.12. *Let $\rho, \sigma \in \mathrm{KP}(\theta)$. If $\sigma \not\supseteq \rho$, then $\mathcal{E}_\theta(\rho,\sigma) = 0$. If $\sigma \supseteq_m \rho$, then $\mathcal{E}_\theta(\rho,\sigma) = \mathcal{E}^m_\theta(\rho,\sigma)$ and there is an isomorphism of Λ_σ-modules $\mathcal{E}^m_\theta(\rho,\sigma) \xrightarrow{\sim} q^{-m}\Lambda_\rho$ which makes the following diagram of Λ_σ-modules commute:*

$$\begin{tikzcd} q^{-m}\Lambda_\sigma \arrow[r, "\sim", "\xi"'] \arrow[d, "\mathtt{p}^\sigma_\rho"] & \mathrm{Hom}_{R_\theta}(q^m R_\theta e_\sigma, \Delta(\sigma)) \arrow[d, "{[-]}"] \\ q^{-m}\Lambda_\rho & \mathcal{E}^m_\theta(\rho,\sigma). \arrow[l, dashed, "\sim"] \end{tikzcd}$$

Proof. If $\sigma \not\supseteq \rho$, then $\mathcal{E}_\theta(\rho,\sigma) = 0$ by Lemma 2.50 and Corollary 3.10(1).

Assume now that $\sigma \supseteq_m \rho$ and write $C_\sigma \setminus C_\rho = \{i_1 < \cdots < i_m\}$. If $\sigma \supseteq \pi \supseteq_n \rho$, we set $B(\pi) := \{k \in [1,m] \mid i_k \in C_\pi\}$. Note that $|B(\pi)| = n$. We define a map $\Theta_n : T^\rho_n(\sigma) \to q^m \bigwedge^n N$ of Λ_σ-modules by defining it on the Λ_σ-basis $\{\psi_{w(\pi,\sigma)} v_\sigma \mid \sigma \supseteq \pi \supseteq_n \rho\}$ of $T^\rho_n(\sigma)$, see Corollary 3.10. If $B(\pi) = \{k_1 < \cdots < k_n\}$, then define

$$\Theta_n(\psi_{w(\pi,\sigma)} v_\sigma) := \epsilon_{k_1} \wedge \cdots \wedge \epsilon_{k_n}.$$

It is easy to see that Θ_n is an isomorphism of (graded) Λ_σ-modules. To show that Θ_n defines an isomorphism of complexes $T^\rho_\bullet(\sigma) \to q^m \bigwedge^\bullet N$, we must verify that the following square commutes:

$$\begin{CD} T^\rho_{n+1}(\sigma) @<{d_n\cdot -}<< T^\rho_n(\sigma) \\ @V{\Theta_{n+1}}VV @VV{\Theta_n}V \\ q^m{\bigwedge}^{n+1}N @<{Z\wedge -}<< q^m{\bigwedge}^n N, \end{CD}$$

where d_n is the matrix defined in (3.6). We check this using an arbitrary basis element $\psi_{w(\pi,\sigma)}v_\sigma \in T^\rho_n(\sigma)$. We have

$$\begin{aligned} \Theta_{n+1}(d_n\psi_{w(\pi,\sigma)}v_\sigma) &= \sum_{\sigma\supseteq\tau\supseteq_1\pi} s(\tau,\pi)\Theta_{n+1}(\psi_{w(\tau,\pi)}\psi_{w(\pi,\sigma)}v_\sigma) \\ &= \sum_{k\in[1,m]\setminus B(\pi)} s(\mathrm{ref}^{i_k}(\pi),\pi)\Theta_{n+1}(\psi_{w(\mathrm{ref}^{i_k}(\pi),\pi)}\psi_{w(\pi,\sigma)}v_\sigma) \\ &= \sum_{k\in[1,m]\setminus B(\pi)} (-1)^{B(\pi)\cap[1,k)}\Theta_{n+1}(\psi_{w(\mathrm{ref}^{i_k}(\pi),\sigma)}v_\sigma)z_k \\ &= \sum_{k\in[1,m]\setminus B(\pi)} (-1)^{B(\pi)\cap[1,k)}(\epsilon_{k_1}\wedge\cdots\wedge\epsilon_k\wedge\cdots\wedge\epsilon_{k_n})z_k \\ &= Z\wedge(\epsilon_{k_1}\wedge\cdots\wedge\epsilon_{k_n}) \\ &= Z\wedge\Theta_n(\psi_{w(\pi,\sigma)}v_\sigma) \end{aligned}$$

where the second equality follows by noting that $k \mapsto \mathrm{ref}^{i_k}(\pi)$ defines a bijection from $[1,m]\setminus B(\pi)$ to the set of 1-refinements of π which are refined by σ, the third by Lemma 3.11 and the observation that $s(\mathrm{ref}^{i_k}(\pi),\pi) = (-1)^{B(\pi)\cap[1,k)}s(\mathrm{ref}^{i_k}(\rho),\rho)$, and the remaining equalities follow from the definitions.

Since $q^m{\bigwedge}^\bullet N$ is a Koszul complex corresponding to a regular sequence, we now have that $\mathcal{E}^n_\theta(\rho,\sigma) \cong H^n(q^m{\bigwedge}^\bullet N) = 0$ unless $n = m$. The proof is complete in view of Lemma 2.50 upon noting that the kernel of $\mathtt{p}^\sigma_\rho : \Lambda_\sigma \twoheadrightarrow \Lambda_\rho$ is the ideal generated by $(z_1, \ldots, z_m)$, so $\mathtt{p}_\rho^\sigma$ induces an isomorphism

$$H^m(q^m {\textstyle\bigwedge}^\bullet N) = \frac{q^{-m}\Lambda_\sigma}{(z_1, \ldots, z_m)} \xrightarrow{\sim} q^{-m}\Lambda_\rho. \qquad \square$$

3.3. The category $\mathcal{E}_\theta$

Throughout this section, we use Theorem 3.12 to identify $\mathcal{E}_\theta(\rho, \sigma) = \mathcal{E}_\theta^m(\rho, \sigma)$ with $q^{-m}\Lambda_\rho$ whenever $\sigma \supseteq_m \rho \in \mathrm{KP}(\theta)$.

Let $\sigma \supseteq_m \rho \in \mathrm{KP}(\theta)$. For any $\hat{f} \in q^{-m}\mathcal{Y}_l$ and $\pi \supseteq_k \sigma$, set

$$\hat{f}^\pi_{\rho,\sigma} := (-1)^{\frac{m(m+1)}{2} + mk} w(\pi, \sigma) \cdot \hat{f}.$$

We define an element of $\mathrm{Hom}^m_{R_\theta}(S^\rho_\bullet, S^\sigma_\bullet)$ by

$$\begin{aligned} \varphi^{\hat{f}}_{\rho,\sigma} : S^\rho_{m+k} = \bigoplus_{\pi \supseteq_{m+k} \rho} q^{m+k} R_\theta e_\pi &\to \bigoplus_{\tau \supseteq_k \sigma} q^k R_\theta e_\tau = S^\sigma_k \\ (h_\pi e_\pi)_{\pi \supseteq_{m+k} \rho} &\mapsto (h_\tau \hat{f}^\tau_{\rho,\sigma} e_\tau)_{\tau \supseteq_k \sigma}. \end{aligned} \tag{3.13}$$

Recalling the differential (2.12) on $\mathrm{Hom}^\bullet_{R_\theta}(S^\rho_\bullet, S^\sigma_\bullet)$, we have:

Lemma 3.14. *Let $\sigma \supseteq_m \rho \in \mathrm{KP}(\theta)$. If $f \in q^{-m}\Lambda_\rho = \mathcal{E}_\theta^m(\rho, \sigma)$ and $\hat{f} \in q^{-m}\mathcal{Y}_l$ are such that $\mathtt{p}_\rho(\hat{f}) = f$, then*

1. *$\delta(\varphi^{\hat{f}}_{\rho,\sigma}) = 0$, and*
2. *the isomorphism (2.13) sends the cohomology class of $\varphi^{\hat{f}}_{\rho,\sigma}$ to f.*

Proof. We prove (1) by checking that the following diagram either commutes (if m is even) or anticommutes (if m is odd) whenever $\tau \supseteq_1 \pi \supseteq_k \sigma$:

$$\begin{tikzcd}
q^{k+m+1}R_\theta e_\tau \arrow[r, "-\cdot d^{\tau,\pi}_{k+m}"] \arrow[d, "-\cdot \hat{f}^{\tau}_{\rho,\sigma}"'] & q^{k+m}R_\theta e_\pi \arrow[d, "-\cdot \hat{f}^{\pi}_{\rho,\sigma}"] \\
q^{k+1}R_\theta e_\tau \arrow[r, "-\cdot d^{\tau,\pi}_{k}"] & q^{k}R_\theta e_\pi.
\end{tikzcd}$$

Modulo the signs, this is checked by the computation:

$$\psi_{w(\tau,\pi)}(w(\pi,\sigma)\cdot \hat{f})e_\pi = (w(\tau,\pi)w(\pi,\sigma)\cdot \hat{f})\psi_{w(\tau,\pi)}e_\pi = (w(\tau,\sigma)\cdot \hat{f})\psi_{w(\tau,\pi)}e_\pi,$$

and the signs are taken care of by

$$(-1)^{\frac{m(m+1)}{2}+mk}s(\tau,\pi) = (-1)^m(-1)^{\frac{m(m+1)}{2}+m(k+1)}s(\tau,\pi).$$

To prove (2), first note that the restriction of $\varphi^{\hat{f}}_{\rho,\sigma}$ to S^ρ_m has image in S^σ_0 and can therefore be realized as

$$\begin{aligned}
\varphi^{\hat{f}}_{\rho,\sigma}|_{S^\rho_m} : S^\rho_m = \bigoplus_{\pi\supseteq_m\rho} q^m R_\theta e_\pi &\to R_\theta e_\sigma = S^\sigma_0 \\
(h_\pi e_\pi)_{\pi\supseteq_m\rho} &\mapsto h_\sigma \hat{f}^{\sigma}_{\rho,\sigma} e_\sigma = (-1)^{\frac{m(m+1)}{2}} h_\sigma \hat{f} e_\sigma.
\end{aligned}$$

By Corollary 3.10(2), we identify $\mathrm{Hom}_{R_\theta}(S^\rho_m, \Delta(\sigma))$ with $\mathrm{Hom}_{R_\theta}(q^m R_\theta e_\sigma, \Delta(\sigma))$. Then the image of $\varphi^{\hat{f}}_{\rho,\sigma}$ under the map (2.14) is

$$(-1)^{\frac{m(m+1)}{2}}\varepsilon_\sigma(\varphi^{\hat{f}}_{\rho,\sigma}|_{S^\rho_m}) = (e_\sigma \mapsto v_\sigma f) \in \mathrm{Hom}_{R_\theta}(q^m R_\theta e_\sigma, \Delta(\sigma)).$$

An application of Theorem 3.12 completes the proof. $\square$

Lemma 3.15. *Let* $\tau \supseteq \sigma \supseteq_m \rho \in \mathrm{KP}(\theta)$*. If* $f \in q^{-m}\Lambda_\rho$ *and* $\hat{f} \in q^{-m}\mathcal{Y}_l$ *are such that* $\mathtt{p}_\rho(\hat{f}) = f$*, then we also have* $\mathtt{p}_\rho(w(\tau,\sigma)\cdot\hat{f}) = f$*.*

Proof. Since $\tau \supseteq \sigma \supseteq \rho$, we have that r and $w(\tau,\sigma)(r)$ are ρ-equivalent for any $r \in [1,l]$, so $\mathtt{p}_\rho(w(\tau,\sigma)\cdot y_r) = \mathtt{p}_\rho(y_r)$. This implies the result. □

Lemma 3.16. *Let* $\tau \supseteq_n \sigma \supseteq_m \rho \in \mathrm{KP}(\theta)$*. If* $f \in q^{-m}\Lambda_\rho$*, and* $g \in q^{-n}\Lambda_\sigma$*, then there exist* $\hat{f} \in q^{-m}\mathcal{Y}_l$*,* $\hat{g} \in q^{-n}\mathcal{Y}_l$*, and* $\widehat{\mathtt{p}_\rho^\sigma(g)f} \in q^{-(m+n)}\mathcal{Y}_l$ *with* $\mathtt{p}_\rho(\hat{f}) = f$*,* $\mathtt{p}_\sigma(\hat{g}) = g$*, and* $\mathtt{p}_\rho(\widehat{\mathtt{p}_\rho^\sigma(g)f}) = \mathtt{p}_\rho^\sigma(g)f$*, such that*

$$\varphi_{\sigma,\tau}^{\hat{g}}\varphi_{\rho,\sigma}^{\hat{f}} = \varphi_{\rho,\tau}^{\widehat{\mathtt{p}_\rho^\sigma(g)f}}.$$

Proof. Choose any two lifts $\hat{f} \in q^{-m}\mathcal{Y}_l$ and $\hat{g} \in q^{-n}\mathcal{Y}_l$ of f and g, respectively. By Lemma 3.15, since $\tau \supseteq \sigma$, we have that $w(\tau,\sigma)\cdot\hat{f}$ is also a lift of f, and so $\widehat{\mathtt{p}_\rho^\sigma(g)f} := \hat{g}(w(\tau,\sigma)\cdot\hat{f}) \in q^{-(m+n)}\mathcal{Y}_l$ is a lift of $\mathtt{p}_\rho^\sigma(g)f$. By (3.13), it suffices to show that for any $\pi \supseteq \tau$ we have $(\widehat{\mathtt{p}_\rho^\sigma(g)f})_{\rho,\tau}^\pi = \hat{g}_{\sigma,\tau}^\pi\hat{f}_{\rho,\sigma}^\pi$. Modulo the signs, this holds by the following computation:

$$\begin{aligned} w(\pi,\tau)\cdot(\hat{g}(w(\tau,\sigma)\cdot\hat{f})) &= (w(\pi,\tau)\cdot\hat{g})(w(\pi,\tau)w(\tau,\sigma)\cdot\hat{f}) \\ &= (w(\pi,\tau)\cdot\hat{g})(w(\pi,\sigma)\cdot\hat{f}), \end{aligned}$$

and if $\pi \supseteq_k \tau$, the signs are taken care of by

$$(-1)^{\frac{(m+n)(m+n+1)}{2}+(m+n)k} = (-1)^{\frac{m(m+1)}{2}+m(k+n)}(-1)^{\frac{n(n+1)}{2}+nk}.$$ □

We combine Lemmas 3.16 and 3.14 to obtain the following theorem.

Theorem 3.17. *Let $\tau \supseteq_n \sigma \supseteq_m \rho \in \mathrm{KP}(\theta)$. The composition in the category $\mathcal{E}_\theta$ is given by*

$$\begin{array}{ccc}
\mathcal{E}_\theta^n(\sigma,\tau) \otimes \mathcal{E}_\theta^m(\rho,\sigma) & \longrightarrow & \mathcal{E}_\theta^{m+n}(\rho,\tau) \\
\wr\| & & \wr\| \\
q^{-n}\Lambda_\sigma \otimes q^{-m}\Lambda_\rho & \longrightarrow & q^{-(m+n)}\Lambda_\rho \\
g \otimes f & \longmapsto & \mathtt{p}_\rho^\sigma(g) f.
\end{array}$$

CHAPTER IV

THE $\mathtt{A}_2$ CASE

This chapter contains material which appears in [1, §4]. The author performed the relevant computations under the supervision of and with assistance from Alexander Kleshchev. The results were written initially by the author with revisions by the author and Alexander Kleshchev.

Throughout this chapter, we use a special notation

$$\alpha := \alpha_1,\ \beta := \alpha_2,\ \gamma := \alpha_1 + \alpha_2,$$

so that α, β, γ are now the positive roots of the root system of type $\mathtt{A}_2$. We fix

$$\theta := a\alpha + b\beta$$

with $a, b \in \mathbb{Z}_{\geq 0}$. There is a bijection

$$\sigma : [0, \min\{a, b\}] \xrightarrow{\sim} \mathrm{KP}(\theta),\ s \mapsto \sigma(s) := (\beta^{b-s}, \gamma^s, \alpha^{a-s}).$$

The standard R_θ-modules are

$$\Delta(s) := \Delta(\sigma(s)) \qquad (0 \leq s \leq \min\{a, b\}).$$

We denote the standard generator of $\Delta(s)$ by $v_s := v_{\sigma(s)}$, see (2.47). Recall that

$$\mathrm{End}_{R_\theta}(\Delta(s))^{\mathrm{op}} \cong \Lambda_{\sigma(s)} = \Lambda_{b-s,s,a-s} = \Lambda_{b-s} \otimes \Lambda_s \otimes \Lambda_{a-s},$$

see (2.46).

4.1. The resolution $P^r_\bullet$

Let $0 \le r \le \min\{a, b\}$. We recall the resolution of $\Delta(r)$ defined in [2]. For $n \in [0, r]$, recalling (2.21), we define:

$$
\begin{aligned}
\boldsymbol{j}_{r,n} &:= 2^{b-r}1^{r-n}2^r1^{n+a-r} \in I^\theta, \\
\boldsymbol{i}_{r,n} &:= 2^{(b-r)}1^{(r-n)}2^{(r)}1^{(n)}1^{(a-r)} \in I^\theta_{\mathrm{div}}, \\
e_{r,n} &:= 1_{\boldsymbol{i}_{r,n}} \in R_\theta, \\
\mathtt{s}_{r,n} &:= n(r-n+1) - \binom{a-r}{2} - \binom{b-r}{2} - r(r-1), \\
x_{r,n} &:= U_{b-n-1;1,r+n} = (b+r, b+r-1, \dots, b-n) \in \mathfrak{S}_{a+b}, \\
d_{r,n} &:= e_{r,n+1}\psi_{x_{r,n}}e_{r,n} \in R_\theta, \\
P^r_n &:= q^{\mathtt{s}_{r,n}} R_\theta e_{r,n}.
\end{aligned}
$$

Note that right multiplication with $d_{r,n}$ yields the degree zero R_θ-homomorphism $-\cdot d_{r,n} : P^r_{n+1} \to P^r_n$. Define $u_r \in \mathfrak{S}_{2r}$ by

$$
u_r(i) = \begin{cases} r - \frac{i-1}{2} & \text{if } i \text{ is odd,} \\ 2r - \frac{i-2}{2} & \text{if } i \text{ is even.} \end{cases} \tag{4.1}
$$

Note that $(\psi_{w_{0,b-r}} \circ \psi_{u_r} \circ \psi_{w_{0,a-r}})v_r$ does not depend on the reduced expressions for $w_{0,b-r}$, u_r, or $w_{0,a-r}$, so we have a well-defined map

$$
\varepsilon_r : P^r_0 \twoheadrightarrow \Delta(r),\ xe_{r,0} \mapsto x(\psi_{w_{0,b-r}} \circ \psi_{u_r} \circ \psi_{w_{0,a-r}})v_r. \tag{4.2}
$$

By [2, Theorem A] and Lemma 2.42, we have:

Lemma 4.3. *The following sequence is a projective resolution of* $\Delta(r)$*:*

$$0 \longrightarrow P_r^r \longrightarrow \cdots \longrightarrow P_{n+1}^r \xrightarrow{-\cdot d_{r,n}} P_n^r \longrightarrow \cdots \longrightarrow P_0^r \xrightarrow{\varepsilon_r} \Delta(r) \longrightarrow 0.$$

4.2. Weight spaces of standard modules

The following lemmas are useful for finding bases for certain weight spaces of the standard modules $\Delta(s)$. The first of them concerns the nil-Hecke algebra and is well-known and easy to check. Recall the notation from section 2.3 and the $(R_{c\alpha_i}, \mathcal{NH}_c)$-bimodule structure on $\hat{\Delta}(\alpha_i^c)$ from section 2.3.

Lemma 4.4. *Let* $i \in I$ *and* $c \in \mathbb{Z}_{\geq 0}$*. The map*

$$q^{-\binom{c}{2}}\mathcal{X}_c \to \hat{\Delta}(\alpha_i^c), \;\; f \mapsto v_{\alpha_i}^{\circ c} f \tau_{w_0}$$

is an injective map of right Λ_c*-modules with image* $\Delta(\alpha_i^c)$*.*

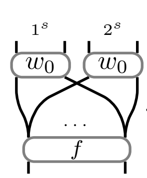
Recalling (4.1), we have:

Lemma 4.5. *The map*

$$q^{-2\binom{s}{2}}\Lambda_s \to \hat{\Delta}(\gamma^s), \;\; f \mapsto \psi_{u_s} v_\gamma^{\circ s} f$$

is an injective map of right Λ_s*-modules with image* $1_{1^{(s)}2^{(s)}}\Delta(\gamma^s)$*.*

Diagrammatically, the map in the lemma is given by

$f \mapsto$ 

.

Proof. In view of Lemma 2.19, we may assume that $\Bbbk$ is a field. Let M be the free graded $\Bbbk$-module with basis $\mathscr{D}^{(2^s)}$, where the degree of the basis element $w \in \mathscr{D}^{(2^s)}$ is set to equal to the degree of $\psi_w 1_{(12)^s}$ in $R_{s\alpha+s\beta}$. Lemma 2.43 and (2.38) show that the map $\xi : q^{\binom{s}{2}} M \otimes_\Bbbk \mathcal{X}_s \to \hat{\Delta}(\gamma^s),\ w \otimes f \mapsto \psi_w v_\gamma^{\circ s} f$ is an isomorphism of right $\mathcal{X}_s$-modules.

Let φ be the map in the statement. Then φ is the composition of the inclusion $q^{-2\binom{s}{2}}\Lambda_s \hookrightarrow q^{\binom{s}{2}} M \otimes_\Bbbk \mathcal{X}_s,\ f \mapsto u_s \otimes f$ with ξ, so φ is injective. It is now enough to show that $\varphi(f) \in 1_{1^{(s)}2^{(s)}}\Delta(\gamma^s)$ and that $\dim_q(1_{1^{(s)}2^{(s)}}\Delta(\gamma^s)) = q^{-2\binom{s}{2}} \dim_q(\Lambda_s)$. For the first claim, note that $\psi_{u_s} v_\gamma^{\circ s}$ is a nonzero element of smallest possible degree in $1_{1^s 2^s}\hat{\Delta}(\gamma^s)$, so recalling (2.26), we have

$$
\begin{aligned}
1_{1^{(s)}2^{(s)}}\psi_{u_s} v_\gamma^{\circ s} f e_s &= \psi_{u_s} v_\gamma^{\circ s} f x_0 \tau_{w_0} \\
&= \psi_{u_s} v_\gamma^{\circ s} f x_0 \tau_1 \tau_{s_1^{-1} w_0} \\
&= \psi_{u_s} v_\gamma^{\circ s} \tau_1 f x_0 \tau_{s_1^{-1} w_0} + \psi_{u_s} v_\gamma^{\circ s} \partial_1(f x_0) \tau_{s_1^{-1} w_0} \\
&= \psi_{u_s} v_\gamma^{\circ s} \partial_1(f x_0) \tau_{s_1^{-1} w_0} \\
&\ \,\vdots \\
&= \psi_{u_s} v_\gamma^{\circ s} \partial_{w_0}(f x_0) \\
&= \psi_{u_s} v_\gamma^{\circ s} f,
\end{aligned}
$$

where the first equality comes from Lemma 2.35 and (2.26), the third comes from the relations in $\mathcal{NH}_s$, the fourth follows because $\deg(\psi_{u_s} v_\gamma^{\circ s} \tau_1) = \deg(\psi_{u_s} v_\gamma^{\circ s}) - 2 < \deg(\psi_{u_s} v_\gamma^{\circ s})$, and the last holds by Lemma 2.23.

As for graded dimension, we have

$$\begin{aligned}\dim_q(1_{1^{(s)}2^{(s)}}\Delta(\gamma^s)) &= \frac{1}{\left([s]^!_+\right)^3}\dim_q(1_{1^s2^s}\hat{\Delta}(\gamma^s))\\ &= \frac{q^{2\binom{s}{2}}\left([s]^!_-\right)^2}{\left([s]^!_+\right)^3}(\dim_q \mathcal{X}_1)^s\\ &= q^{-2\binom{s}{2}}\dim_q \Lambda_s.\end{aligned}$$

where the first equality follows from (2.48) and (2.36), the second from (2.44), (2.39), and Lemma 2.43, and the last from an elementary computation. □

4.3. The $\Bbbk$-module $\mathcal{E}_\theta(r,s)$

In this section we fix $r, s \in [0, \min\{a, b\}]$. Recalling Lemma 2.50, we write $T^r_\bullet(s) := T^{Pr}_\bullet(\Delta(s))$. The terms of this complex are of the form

$$T^r_n(s) = q^{-\mathbf{s}_{r,n}} e_{r,n}\Delta(s) \qquad (n = 0, \dots, r).$$

If $r \geq s$ and $0 \leq n \leq r - s$, we define

$$\begin{aligned}\omega_n(r,s) &:= -(r-s)(1+(a-r)+(b-r)) + (r-s-n+1)(r-s-n),\\ K_n &:= q^{\omega_n(r,s)}\Lambda_{b-r,r-s,s,r-s-n,n,a-r},\\ w_n &:= U_{b-r;s,r-s-n}U_{b-n;r-s,s}U_{b-r+s;r-s,r-s-n}U_{b-r;r-s,s}U_{b;s,r-s-n}.\end{aligned}$$

The diagram for $1_{\boldsymbol{j}_{r,n}}\psi_{w_n}1_{\boldsymbol{j}_{s,0}}$ is the top part of the diagram below. Observe that $\Lambda_{\sigma(s)} = \Lambda_{b-s,s,a-s} \subseteq K_n$ in a natural way, so we may consider K_n as a $\Lambda_{\sigma(s)}$-module. Recalling (4.1), we have

Lemma 4.6. *Suppose* $0 \leq n \leq r$.

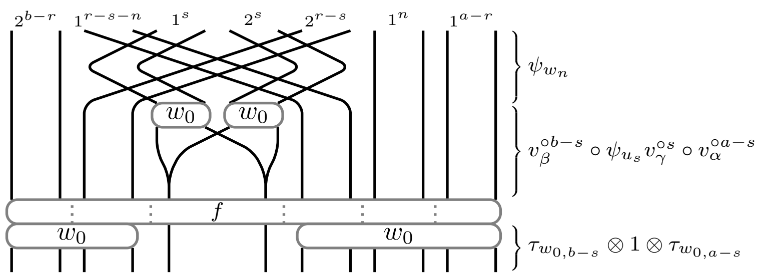
1. *If* $n > r - s$, *then* $T_n^r(s) = 0$.

2. *If* $n \le r - s$, *then the map*

$$\Xi_n : K_n \to q^{-\mathbf{s}_{r,n}}\hat{\Delta}(s),$$
$$f \mapsto \psi_{w_n}(v_\beta^{\circ b-s} \circ \psi_{u_s} v_\gamma^{\circ s} \circ v_\alpha^{\circ a-s})\, f\,(\tau_{w_{0,b-s}} \otimes 1 \otimes \tau_{w_{0,a-s}})$$

is an injective degree zero map of $\Lambda_{\sigma(s)}$*-modules with image* $T_n^r(s)$.

Diagrammatically, Ξ_n is given by

$f \mapsto$ 

Proof. (1) The condition $n > r - s$ is equivalent to the condition $a - r + n > a - s$, which easily implies, by (2.39), that $1_{\boldsymbol{j}_{r,n}}\Delta(s) = 0$ hence $e_{r,n}\Delta(s) = 0$.

(2) It is straightforward to check that Ξ_n is homogeneous and $\Lambda_{\sigma(s)}$-equivariant. Define

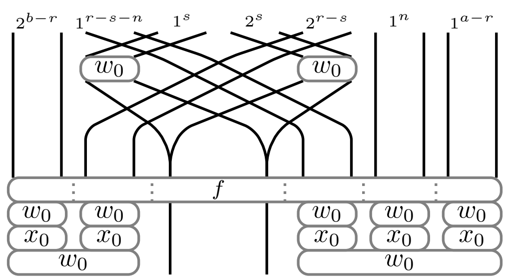
$$\xi_\alpha : q^{-\binom{a-s}{2}}\Lambda_{r-s-n,n,a-r} \hookrightarrow q^{-\binom{a-s}{2}}\mathcal{X}_{a-s} \xrightarrow{\sim} \Delta(\alpha^{a-s})$$
$$\xi_\beta : q^{-\binom{b-s}{2}}\Lambda_{b-r,r-s} \hookrightarrow q^{-\binom{b-s}{2}}\mathcal{X}_{b-s} \xrightarrow{\sim} \Delta(\beta^{b-s})$$
$$\xi_\gamma : q^{-2\binom{s}{2}}\Lambda_s \xrightarrow{\sim} 1_{1^{(s)}2^{(s)}}\Delta(\gamma^s)$$

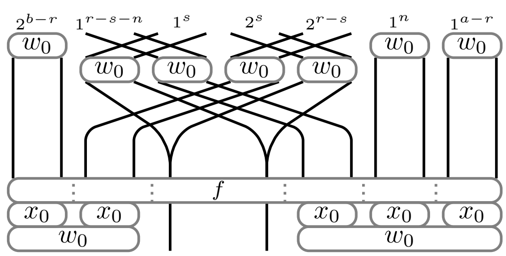
where the isomorphisms are from Lemmas 4.4 and 4.5. Let $d = (r-s)(r-s-n) - s(r-s-n) - s(r-s)$ and define

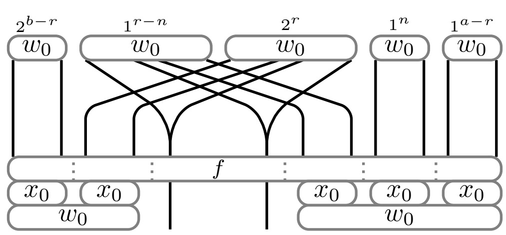
$$\xi : \Delta(\beta^{b-s}) \otimes 1_{1^{(s)}2^{(s)}}\Delta(\gamma^s) \otimes \Delta(\alpha^{a-s}) \to q^{-d}1_{\boldsymbol{j}_{r,n}}\Delta(s),\ x \otimes y \otimes z \mapsto \psi_{w_n}(x \circ y \circ z).$$

Since $w_n \in \mathscr{D}^{(b-s,2s,a-s)}$, (2.38) shows that ξ is injective. Observing that $\Xi_n = \xi(\xi_\beta \otimes \xi_\gamma \otimes \xi_\alpha)$, we see that Ξ_n is injective with image in $1_{\boldsymbol{j}_{r,n}}\Delta(s)$.

Now we prove that $\operatorname{im}\Xi_n \subseteq e_{r,n}\Delta(s)$ by showing that $e_{r,n}\Xi_n(f) = \Xi_n(f)$. We have, using the relations in R_θ,

$\Xi_n(f) =$ 

$=$ 

$=$ 

where the first equality follows from Lemma 2.35, the second follows because f is symmetric in the variables as indicated by the vertical dotted lines, hence, by Theorem 2.24, commutes with the parabolic subalgebra $\mathcal{NH}_{b-r} \otimes \mathcal{NH}_{r-s} \otimes \mathcal{NH}_s \otimes$

$\mathcal{NH}_{r-s-n} \otimes \mathcal{NH}_n \otimes \mathcal{NH}_{a-r}$, and the last equality is straightforward. Now we have $e_{r,n}\Xi_n(f) = \Xi_n(f)$ by Lemma 2.35.

To complete the proof, in view of Lemma 2.19, we assume that $\Bbbk$ is a field and check that $\dim_q K_n = \dim_q q^{-\mathtt{s}_{r,n}} e_{r,n}\Delta(s)$. For brevity, we denote $[M] := \dim_q M$. Let $z = (r-s)s + (r-s-n)s + (r-s)(r-s-n)$. We have

$$\begin{aligned}
\left[1_{\boldsymbol{j}_{r,n}}\Delta(s)\right] &= q^z \begin{bmatrix} r-n \\ s \end{bmatrix}_- \begin{bmatrix} r \\ s \end{bmatrix}_- \left[\Delta(\beta^{b-s})\right] \left[1_{1^s 2^s}\Delta(\gamma^s)\right] \left[\Delta(\alpha^{a-s})\right] \\
&= q^z \begin{bmatrix} r-n \\ s \end{bmatrix}_- \begin{bmatrix} r \\ s \end{bmatrix}_- ([s]^!_+)^2 \left[\Delta(\alpha^{a-s})\right] \left[1_{1^{(s)}2^{(s)}}\Delta(\gamma^s)\right] \left[\Delta(\alpha^{a-s})\right] \\
&= q^{\omega_n(r,s)+\mathtt{s}_{r,n}} \frac{[r-n]^!_+[r]^!_+}{[r-n-s]^!_+[r-s]^!_+}[\mathcal{X}_{b-s}][\Lambda_s][\mathcal{X}_{a-s}],
\end{aligned}$$

where the first equality follows from (2.39), then second from (2.36), and the last from Lemmas 4.4 and 4.5. Thus,

$$\begin{aligned}
q^{-\mathtt{s}_{r,n}}\left[e_{r,n}\Delta(s)\right] &= \frac{q^{-\mathtt{s}_{r,n}}}{[b-r]^!_+[r-n]^!_+[r]^!_+[n]^!_+[a-r]^!_+}\left[1_{\boldsymbol{j}_{r,n}}\Delta(s)\right] \\
&= q^{\omega_n(r,s)} \frac{[\mathcal{X}_{b-s}]}{[b-r]^!_+[r-s]^!_+}[\Lambda_s]\frac{[\mathcal{X}_{a-s}]}{[r-s-n]^!_+[n]^!_+[a-r]^!_+} \\
&= q^{\omega_n(r,s)}[\Lambda_{b-r}][\Lambda_{r-s}][\Lambda_s][\Lambda_{r-s-n}][\Lambda_n][\Lambda_{a-r}],
\end{aligned}$$

where the first equality is (2.36), the second is by the above computation, and the last is by an elementary computation. $\square$

Because of Lemma 4.6(1), we assume for the rest of the section that $r \geq s$. We use Lemma 4.6(2) to understand the complex $T^r_\bullet(s)$ and compute its cohomology. First, we re-express the coboundary map of $T^r_\bullet(s)$. For $0 \leq n < r-s$, set

$$g_n := \prod_{k=1}^{r-s}(x_{b+r-s-n} - x_{b-r+k}) \in \mathcal{X}_{a+b-s}.$$

If $f \in K_n$, observe that $fg_n \in \Lambda_{b-r,r-s,s,r-s-n-1,1,n,a-r}$ so by Proposition 2.22, we have a map

$$\delta_n : K_n \to K_{n+1},\ \ f \mapsto \partial_{U_{b+r-s-n-1;1,n}}(fg_n).$$

Recalling the isomorphisms $\Xi_n : K_n \xrightarrow{\sim} T_n^r(s)$ from Lemma 4.6(2), we have:

Lemma 4.7. *If $0 \leq n < r-s$, then the following diagram commutes:*

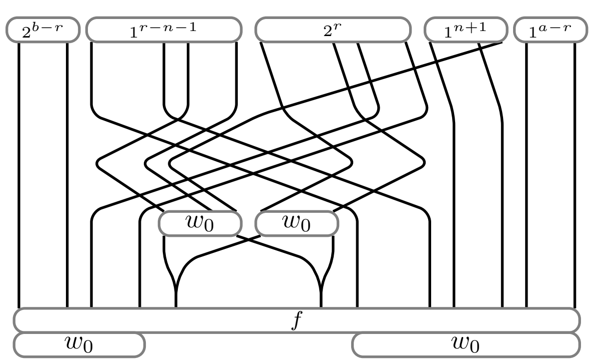
$$\begin{tikzcd}
K_{n+1} \arrow[d, "\Xi_{n+1}"] & K_n \arrow[l, "\delta_n"'] \arrow[d, "\Xi_n"] \\
T_{n+1}^r(s) & T_n^r(s). \arrow[l, "d_{r,n}\cdot -"']
\end{tikzcd}$$

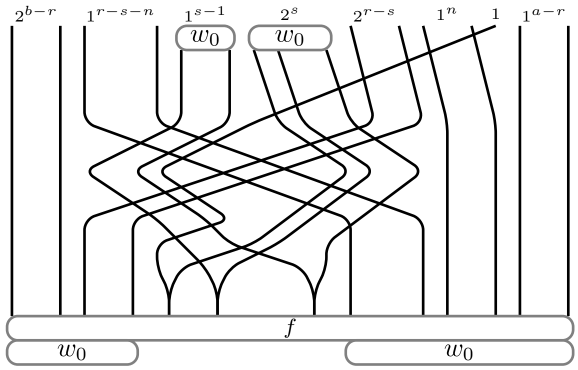
In particular, the maps δ_n make $K_\bullet$ into a complex isomorphic to $T_\bullet^r(s)$.

Proof. There exist polynomials $h_j \in \Bbbk[y_{b-r+1}, \ldots, y_{b-s}]$ and $k_j \in \Bbbk[y_{b+r-n}]$ such that $\prod_{i=1}^{r-s}(y_{b+r-n} - y_{b-r+i}) = \sum_j h_j k_j$. Let $f \in K_n$ and note that

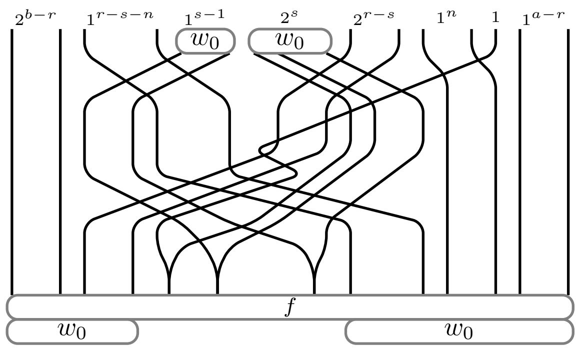
$$d_{r,n}\Xi_n(f) = e_{r,n+1}\psi_{x_{r,n}}e_{r,n}\Xi_n(f) = e_{r,n+1}\psi_{x_{r,n}}\Xi_n(f)$$

since $\operatorname{im}\Xi_n \subseteq T_n^r(s) = q^{-\mathsf{s}_{r,n}}e_{r,n}\Delta(s)$ by Lemma 4.6. We compute $e_{r,n+1}\psi_{x_{r,n}}\Xi_n(f)$:

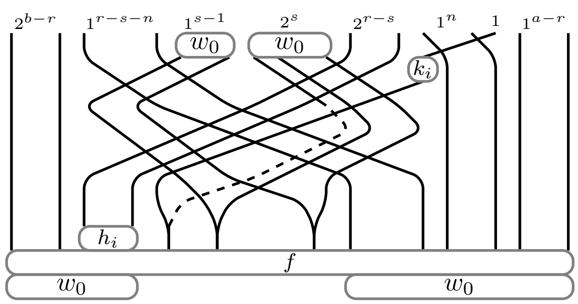


$=$ 

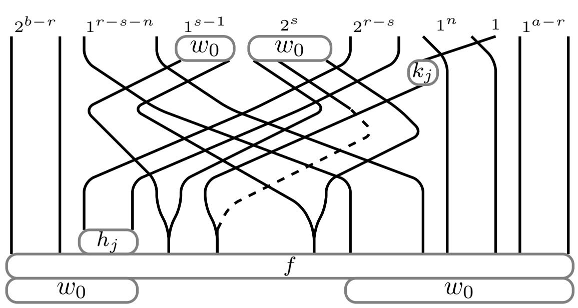
by the relation (R7) and Lemma 2.35

$=$ 

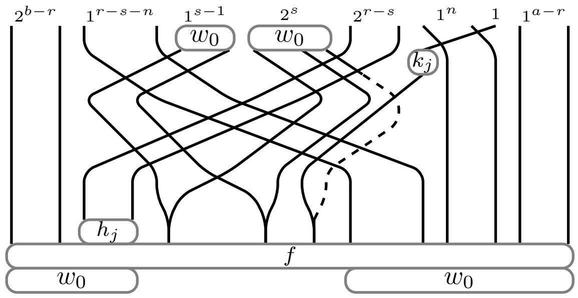
by the relation (R7)

$= \sum_j$ 

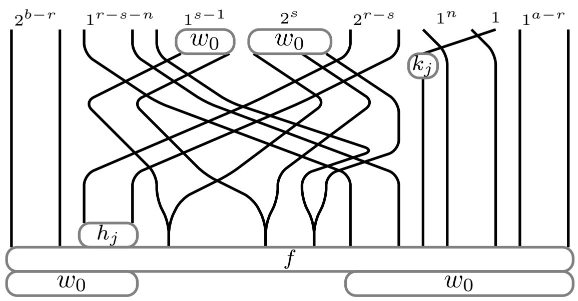
by several applications of the relation (R5)

$= \sum_j$ 

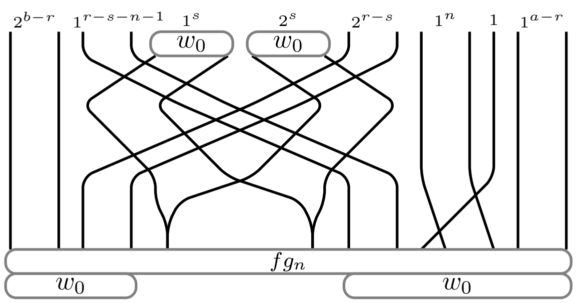
(see ($*$) below)

$= \sum_j$ 

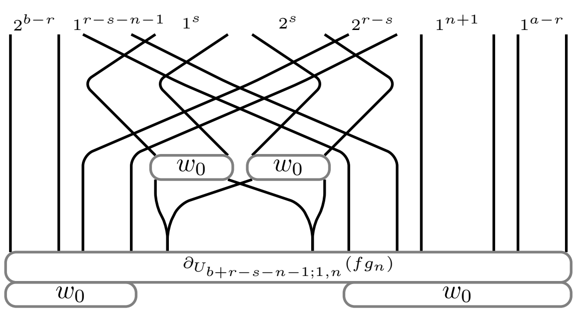
repeat the argument from the previous step several times

$= \sum_j$ 

(see (∗∗) below)

$=$ 

by the relation (R7)

$=$ 

by Lemma 2.25 and the relation (R7).

(∗) To obtain this equality, we attempt to move the portion of dashed strand in the previous diagram to the left by applying a special case of relation (R7):

$$\psi_{t+1}\psi_t\psi_{t+1}1_{\boldsymbol{i}} = \psi_t\psi_{t+1}\psi_t 1_{\boldsymbol{i}} + 1_{\boldsymbol{i}} \text{ (if } i_{t+1} = i_t + 1 \text{ and } i_t = i_{t+2}) \tag{4.8}$$

several times. In all except the last application, the error term $1_{\boldsymbol{i}}$ causes the rest of the diagram to become 0, so we only keep the term $\psi_t\psi_{t+1}\psi_t 1_{\boldsymbol{i}}$. In the last application, because of the defining relations in $\Delta(s)$, the term $\psi_t\psi_{t+1}\psi_t 1_{\boldsymbol{i}}$ causes the rest of the diagram to become 0, so we only keep the error term $1_{\boldsymbol{i}}$, which yields the desired diagram.

($**$) To obtain this equality, we again attempt to move the portion of dashed strand in the previous diagram to the left by applying the relation (4.8) several times. In the first application, the error term $1_{\boldsymbol{i}}$ yields the desired diagram, so we wish to show that the term $\psi_t\psi_{t+1}\psi_t 1_{\boldsymbol{i}}$ causes the rest of the diagram to become 0. Since f is symmetric in the variables $x_{b+1}, \ldots, x_{b+r-s-n}$, the error term in any application of the relation (4.8) other than the first causes the rest of the diagram to become 0. However, after the last application of the relation, the term $\psi_t\psi_{t+1}\psi_t 1_{\boldsymbol{i}}$ also causes the rest of the diagram to become 0 because of the defining relations in $\Delta(s)$.

The expression represented by the last diagram above is $\Xi_{n+1}(\delta_n(f))$, which completes the proof of the lemma. $\square$

Given an interval $(c, d]$ and a polynomial in $d - c$ variables, we denote

$$f(\underline{x}_{(c,d]}) := f(x_{c+1}, \ldots, x_d).$$

For example, if $0 \leq m \leq d - c$ then we have the mth elementary symmetric function

$$E_m(\underline{x}_{(c,d]}) = \sum_{c<i_1<\cdots<i_m\leq d} x_{i_1} \cdots x_{i_m}.$$

Now, for $0 \le k < r-s$, we define

$$z_k := (-1)^{r-s-k}(E_{r-s-k}(\underline{x}_{(b-r,b-s]}) - E_{r-s-k}(\underline{x}_{(b,b+r-s]})).$$

These elements are considered as elements of the algebra

$$\Lambda^{r,s} := \Lambda_{b-r,r-s,s,r-s,a-r}. \tag{4.9}$$

Note that since $\Lambda^{r,s} \subseteq \Lambda_{b-r,r-s,s,r-s-n,n,a-r}$, each K_n is naturally a (right) $\Lambda^{r,s}$-module. We use this to interpret the right-hand side of the lemma below as an element of K_{n+1}.

Lemma 4.10. *For $0 \le n < r-s$ and a symmetric polynomial f in n variables, we have*

$$\delta_n(f(\underline{x}_{(b+r-s-n,b+r-s]})) = \sum_{k=0}^{r-s-1} \left(x_{b+r-s-n}^k \star f(\underline{x}_{(b+r-s-n,b+r-s]})\right) z_k.$$

Proof. For brevity, write $f = f(\underline{x}_{(b+r-s-n,b+r-s]})$. We observe

$$g_n = \prod_{k=1}^{r-s}(x_{b+r-s-n} - x_{b-r+k}) = \sum_{k=0}^{r-s}(-1)^{r-s-k}x_{b+r-s-n}^k E_{r-s-k}(\underline{x}_{(b-r,b-s]}),$$

so that

$$\delta_n(f) = \sum_{k=0}^{r-s}(-1)^{r-s-k}\partial_{U_{b+r-s-n-1;1,n}}(x_{b+r-s-n}^k f)E_{r-s-k}(\underline{x}_{(b-r,b-s]}). \tag{4.11}$$

Next, using the identity

$$x_{b+r-s-n}^{r-s} = -\sum_{k=0}^{r-s-1}(-1)^{r-s-k}x_{b+r-s-n}^k E_{r-s-k}(\underline{x}_{(b,b+r-s]}),$$

(4.11) becomes

$$\delta_n(f) = \sum_{k=0}^{r-s-1} \partial_{U_{b+r-s-n-1;1,n}}(x_{b+r-s-n}^k f) z_k,$$

and the result follows from Proposition 2.22. $\square$

The proof of Theorem 4.13 below amounts to showing that $T^r_\bullet(s) \cong K_\bullet$ is isomorphic to a certain Koszul complex which we now define. Let N be the free right $\Lambda^{r,s}$-module of graded rank $\sum_{k=0}^{r-s-1} q^{2k-2(r-s)}$. For $k = 0, \dots, r-s-1$, we have the basis element $\epsilon_k := 1 \in q^{2k-2(r-s)}\Lambda^{r,s} \subseteq N$. We set

$$Z := (z_0, \dots, z_{r-s-1}) = \epsilon_0 z_0 + \cdots + \epsilon_{r-s-1} z_{r-s-1} \in N.$$

Note that Z is a homogeneous degree 0 element of N. We consider the Koszul complex $q^{\omega_0(r,s)} \bigwedge^\bullet N$ associated to the regular sequence Z for the algebra $\Lambda^{r,s}$ (see [32, §4.5]), which has the form

$$\cdots \longleftarrow q^{\omega_0(r,s)} \bigwedge\nolimits^{n+1} N \longleftarrow q^{\omega_0(r,s)} \bigwedge\nolimits^{n} N \longleftarrow \cdots$$
$$Z \wedge a \longmapsfrom a$$

where $\bigwedge^n N$ is the nth exterior power of the free $\Lambda^{r,s}$-module N. Note that $\bigwedge^n N$ has basis $\{\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_n} \mid 0 \le i_1 < \cdots < i_n < r-s\}$. Recall (1.5).

By Lemmas 2.50 and 4.6(1), the complex $\mathrm{Hom}^\bullet_{R_\theta}(P^r_\bullet, \Delta(s))$ is zero in degrees larger than $r-s$, so every element of the $r-s$ component is a cocycle and there is a surjective map

$$[-] : \mathrm{Hom}_{R_\theta}(q^{\mathsf{s}_{r,r-s}} R_\theta e_{r,r-s}, \Delta(s)) \twoheadrightarrow \mathcal{E}^{r-s}_\theta(r,s) = H^{r-s}(\mathrm{Hom}^\bullet_{R_\theta}(P^r_\bullet, \Delta(s))),\ \varphi \mapsto [\varphi],$$

where $[\varphi]$ is the cohomology class of φ. Moreover, by Lemma 4.6(2), we have an isomorphism

$$\xi : K_{r-s} \xrightarrow{\sim} \mathrm{Hom}_{R_\theta}(q^{\mathtt{s}_{r,r-s}} R_\theta e_{r,r-s}, \Delta(s)),\ f \mapsto (e_{r-s,s} \mapsto v_s f).$$

For $r, s \in \mathbb{Z}_{\geq 0}$ with $\min\{a,b\} \geq r \geq s$, we define

$$\Lambda(r,s) := q^{\omega_{r-s}(r,s)} \Lambda_{b-r,r-s,s,a-r}.$$

Note that there is a surjection

$$\mathtt{p}_{r,s} : q^{\omega_{r-s}(r,s)} \Lambda^{r,s} \twoheadrightarrow \Lambda(r,s),\ f_1 \otimes f_2 \otimes f_3 \otimes f_4 \otimes f_5 \mapsto f_1 \otimes f_2 f_4 \otimes f_3 \otimes f_5, \tag{4.12}$$

obtained by identifying the two Λ_{r-s} components. Since $\Lambda_{\sigma(s)} = \Lambda_{b-s,s,a-s} \subseteq \Lambda^{r,s}$, we consider $\Lambda^{r,s}$ to be a right $\Lambda_{\sigma(s)}$-module, and we consider $\Lambda(r,s)$ to be a right $\Lambda_{\sigma(s)}$-module via the composition of algebra homomorphisms

$$\Lambda_{\sigma(s)} \hookrightarrow \Lambda^{r,s} \overset{\mathtt{p}_{r,s}}{\twoheadrightarrow} \Lambda_{b-r,r-s,s,a-r} = q^{-\omega_{r-s}(r,s)} \Lambda(r,s)$$

and then degree shift. Note that $\mathtt{p}_{r,s}$ is a $\Lambda_{\sigma(s)}$-homomorphism.

Theorem 4.13. *Let $0 \leq r, s \leq \min\{a,b\}$. If $r < s$, then $\mathcal{E}_\theta(r,s) = 0$. If $r \geq s$, then $\mathcal{E}_\theta(r,s) = \mathcal{E}_\theta^{r-s}(r,s)$ and there is an isomorphism of right $\Lambda_{\sigma(s)}$-modules $\mathcal{E}_\theta^{r-s}(r,s) \xrightarrow{\sim} \Lambda(r,s)$ such that the following diagram of right $\Lambda_{\sigma(s)}$-modules commutes:*

$$\begin{tikzcd}
q^{\omega_{r,s}(r-s)} \Lambda^{r,s} \arrow[r, "\sim", "\xi"'] \arrow[d, "\mathtt{p}_{r,s}"] & \mathrm{Hom}_{R_\theta}(q^{\mathtt{s}_{r,r-s}} R_\theta e_{r,r-s}, \Delta(s)) \arrow[d, "{[-]}"] \\
\Lambda(r,s) & \mathcal{E}_\theta^{r-s}(r,s). \arrow[l, dashed, "\sim"']
\end{tikzcd}$$

Proof. If $r < s$, then $\mathcal{E}_\theta(r,s) = 0$ by Lemmas 2.50 and 4.6(1), so assume $r \geq s$. For $0 \leq n \leq r-s$ and $\lambda \in \mathscr{P}(n, r-s-n)$, let $s_\lambda := s_\lambda(\underline{x}_{(b+r-s-n,b+r-s]}) \in K_n$. By Proposition 2.20, $\{s_\lambda \mid \lambda \in \mathscr{P}(n, r-s-n)\}$ is a basis of K_n as an $\Lambda^{r,s}$-module, so there exists an isomorphism $\Theta_n : K_n \to q^{\omega_0(r,s)} \bigwedge^n N$ of $\Lambda^{r,s}$-modules such that

$$\Theta_n(s_\lambda) = \epsilon_{\lambda_n} \wedge \epsilon_{\lambda_{n-1}+1} \wedge \cdots \wedge \epsilon_{\lambda_1+n-1}.$$

We claim that the maps Θ_n define an isomorphism of complexes between $K_\bullet$ and $q^{\omega_0(r,s)} \bigwedge^\bullet N$. We must verify that the following square commutes:

$$\begin{tikzcd} K_{n+1} \arrow[d, "\Theta_{n+1}"] & & K_n \arrow[ll, "\delta_n"'] \arrow[d, "\Theta_n"] \\ q^{\omega_0(r,s)} \bigwedge^{n+1} N & & q^{\omega_0(r,s)} \bigwedge^n N. \arrow[ll, "Z\wedge -"'] \end{tikzcd}$$

Fix some $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathscr{P}(n, r-s-n)$ and set $X := \{\lambda_n, \lambda_{n-1}+1, \ldots, \lambda_1 + n-1\}$. We then have by Lemma 4.10 and (1.5):

$$\begin{aligned} \Theta_{n+1}(\delta_n(s_\lambda)) &= \sum_{k \in [0,r-s) \setminus X} (-1)^{|X \cap [0,k)|} (\epsilon_{\lambda_n} \wedge \cdots \wedge \epsilon_k \wedge \cdots \wedge \epsilon_{\lambda_1+n-1}) z_k \\ &= \sum_{k \in [0,r-s) \setminus X} (\epsilon_k \wedge \epsilon_{\lambda_n} \wedge \cdots \wedge \epsilon_{\lambda_1+n-1}) z_k \\ &= Z \wedge \Theta_n(s_\lambda). \end{aligned}$$

Since Z is a regular sequence, we have $\mathcal{E}_\theta^n(r,s) \cong H^n(q^{\omega_0(r,s)} \bigwedge^\bullet N) = 0$ unless $n = r-s$. We complete the proof using Lemmas 2.50 and 4.7 and the observation that by the fundamental theorem of elementary symmetric polynomials, the kernel of $\mathtt{p}_{r,s} \; : \; q^{\omega_{r-s}(r,s)} \Lambda^{r,s} \twoheadrightarrow \Lambda(r,s)$ is the ideal generated by $(z_0, \ldots, z_{r-s-1})$, so $\mathtt{p}_{r,s}$

induces an isomorphism of $\Lambda^{r,s}$-modules (and therefore of $\Lambda_{\sigma(s)}$-modules)

$$H^{r-s}(q^{\omega_0(r,s)}{\textstyle\bigwedge}^{\bullet}N) = q^{\omega_{r-s}(r,s)}\Lambda^{r,s}/(z_0,\ldots,z_{r-s-1}) \xrightarrow{\sim} \Lambda(r,s). \qquad \square$$

4.4. The category $\mathcal{E}_\theta$

Throughout this section, we use Theorem 4.13 to identify $\mathcal{E}_\theta^{r-s}(r,s)$ with $\Lambda(r,s) = q^{\omega_{r-s}(r,s)}\Lambda_{b-r,r-s,s,a-r}$ whenever $\min\{a,b\} \geq r \geq s \geq 0$.

In this section, we will need to consider not only partially symmetric polynomials in the variables x but also partially symmetric polynomials in the variables y. This will be important since elements of $\mathcal{Y}_{a+b}$ will be considered as elements of R_θ, cf. (2.30). For any d we have an isomorphism

$$\iota_{y\to x} : \mathcal{Y}_d \xrightarrow{\sim} \mathcal{X}_d,\ y_r \mapsto x_r.$$

We will use the notation $\Lambda_m^{\mathcal{Y}} := \mathcal{Y}_m^{\mathfrak{S}_m}$ for the symmetric polynomials in $y_1,\ldots,y_m$. More generally, given a composition $\mu = (\mu_1,\ldots,\mu_k)$ of d, we have the algebra of μ-partially symmetric polynomials $\Lambda_\mu^{\mathcal{Y}} := \mathcal{Y}_d^{\mathfrak{S}_\mu}$. We often write $\Lambda^{\mathcal{Y}}_{\mu_1,\ldots,\mu_k}$ for $\Lambda_\mu^{\mathcal{Y}}$ and identify it with $\Lambda^{\mathcal{Y}}_{\mu_1}\otimes\cdots\otimes\Lambda^{\mathcal{Y}}_{\mu_k}$. The isomorphism $\iota_{y\to x}$ restricts to the isomorphism $\iota_{y\to x} : \Lambda_\mu^{\mathcal{Y}} \xrightarrow{\sim} \Lambda_\mu$.

For integers r,s,t with $\min\{a,b\} \geq r \geq s \geq t \geq 0$, define the following:

$$\hat{\Lambda}(r,s) := q^{\omega_{r-s}(r,s)}\Lambda^{\mathcal{Y}}_{b-r,r-s,s,s,r-s,a-r} \subseteq \mathcal{Y}_{a+b},$$
$$\hat{\Lambda}(r,s,t) := q^{\omega_{r-t}(r,t)-4(r-s)(s-t)}\Lambda^{\mathcal{Y}}_{b-r,r-s,s-t,t,t,s-t,r-s,a-r} \subseteq \mathcal{Y}_{a+b}.$$

Recalling (4.9), there is a surjection

$$\hat{\mathsf{p}}_{r,s} : \hat{\Lambda}(r,s) \twoheadrightarrow q^{\omega_{r-s}(r,s)}\Lambda^{r,s}, \;\; f_1 \otimes f_2 \otimes f_3 \otimes f_4 \otimes f_5 \otimes f_6 \mapsto \iota_{y\to x}\big(f_1 \otimes f_2 \otimes f_3 f_4 \otimes f_5 \otimes f_6\big).$$

Recalling (4.12), let

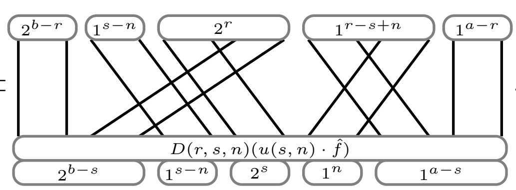
$$\mathsf{q}_{r,s} := \mathsf{p}_{r,s}\hat{\mathsf{p}}_{r,s} : \hat{\Lambda}(r,s) \to \Lambda(r,s).$$

Let again $\min\{a,b\} \geq r \geq s \geq t \geq 0$, and $0 \leq n \leq r-s$. Set

$$\begin{aligned}
D(r,s,n) &:= \prod_{\substack{i\in(b+s-n,b+s] \\ j\in(b-r,b-s]}} (y_i - y_j) \in \mathcal{Y}_{a+b}, \\
u(s,n) &:= U_{b-n;n,s} \in \mathfrak{S}_{a+b}, \\
v(r,s,n) &:= U_{b-r;r-s,2s-n}U_{b+s-n;n,r-s} \in \mathfrak{S}_{a+b}, \\
w(s,t) &:= U_{b-s;t,s-t}U_{b;s-t,t}U_{b-s+t;s-t,s-t} \in \mathfrak{S}_{a+b}, \\
x(r,s,t) &:= U_{b-r;r-s,s-t}U_{b+t;s-t,r-s} \in \mathfrak{S}_{a+b}.
\end{aligned}$$

Let $\min\{a,b\} \geq r \geq s \geq n \geq 0$. For any $\hat{f} \in \hat{\Lambda}(r,s)$, set

$$\hat{f}^{n}_{r,s} := (-1)^{\binom{r-s+1}{2}+n(r-s)} e_{r,r-s+n}\psi_{v(r,s,n)}D(r,s,n)(u(s,n)\cdot\hat{f})e_{s,n}$$

$= \pm$ 

.

Define

$$\varphi^{\hat{f}}_{r,s} : P^r := \bigoplus_{m=0}^{r} q^{\mathsf{s}_{r,m}} R_\theta e_{r,m} \to \bigoplus_{n=0}^{s} q^{\mathsf{s}_{s,n}} R_\theta e_{s,n} =: P^s,$$

$$(h_m e_{r,m})_{m=0}^{r} \mapsto (h_{r-s+n} \hat{f}^n_{r,s} e_{s,n})_{n=0}^{s}.$$

We think of $\varphi^{\hat{f}}_{r,s}$ as an element of $\mathrm{Hom}^{r-s}_{R_\theta}(P^r_\bullet, P^s_\bullet)$. Recalling the differential (2.12) on $\mathrm{Hom}^\bullet_{R_\theta}(P^r_\bullet, P^s_\bullet)$, we have:

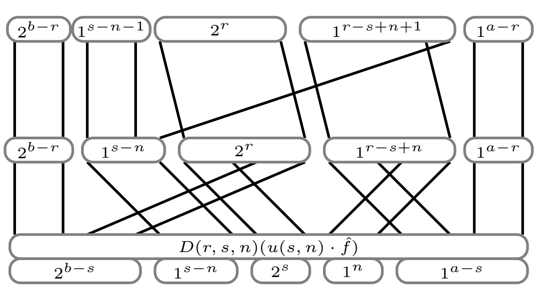
Lemma 4.14. *Suppose* $0 \leq s \leq r \leq \min\{a, b\}$ *and let* $f \in \Lambda(r, s) = \mathcal{E}_\theta(r, s)$*. If* $\hat{f} \in \hat{\Lambda}(r, s)$ *is such that* $\mathsf{q}_{r,s}(\hat{f}) = f$*, then*

1. $\delta(\varphi^{\hat{f}}_{r,s}) = 0$*, and*
2. *the isomorphism (2.13) sends the cohomology class of* $\varphi^{\hat{f}}_{r,s}$ *to* f*.*

Proof. We prove (1) by checking that the following diagram either commutes (if $r - s$ is even) or anticommutes (if $r - s$ is odd) whenever $0 \leq n < r - s$.

$$\begin{tikzcd}
q^{\mathsf{s}_{r,n+1+r-s}} R_\theta e_{r,n+1+r-s} \arrow[r, "-\cdot d_{r,r-s+n}"] \arrow[d, "-\cdot \hat{f}_{n+1}"] & q^{\mathsf{s}_{r,r-s+n}} R_\theta e_{r,r-s+n} \arrow[d, "-\cdot \hat{f}_n"] \\
q^{\mathsf{s}_{s,n+1}} R_\theta e_{s,n+1} \arrow[r, "-\cdot d_{s,n}"] & q^{\mathsf{s}_{s,n}} R_\theta e_{s,n}.
\end{tikzcd}$$

We compute $\pm d_{r,r-s+n} \hat{f}_n$:



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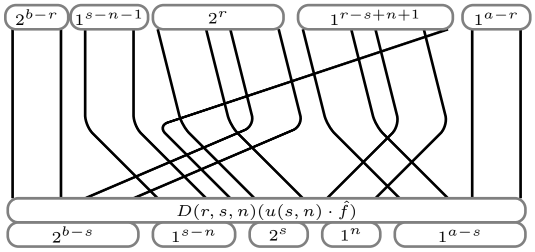
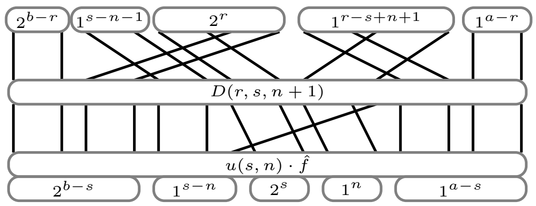


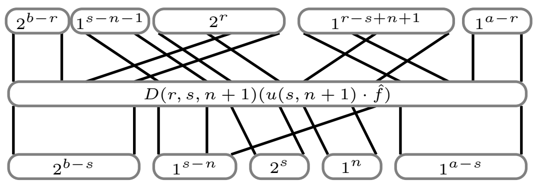
by Lemma 2.35

=



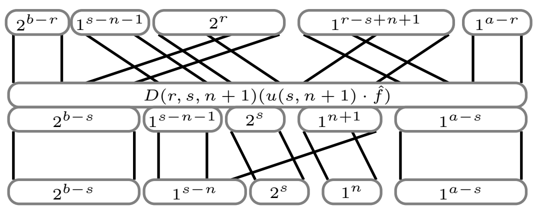
by several applications of the relations (R5) and (R4), and by Lemma 2.24

=



by Lemma 2.24 and the relation (R4)

=

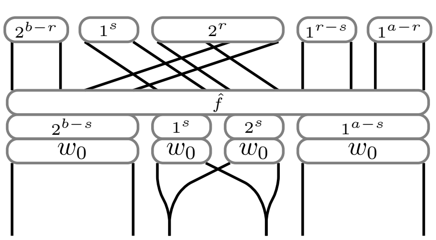


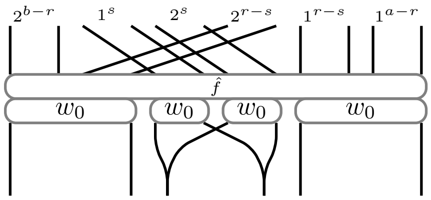
by Lemma 2.35.

The last diagram represents $\pm\hat{f}_{n+1}d_{s,n}$. The signs are taken care of by

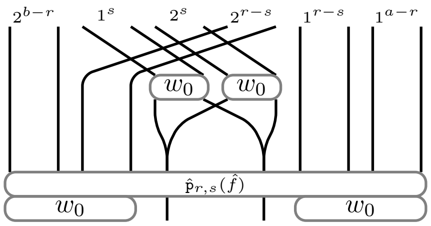
$$(-1)^{\binom{r-s+1}{2}+(n+1)(r-s)} = (-1)^{r-s}(-1)^{\binom{r-s+1}{2}+n(r-s)}.$$

To prove (2), recalling (4.2), we compute $(-1)^{\frac{(r-s)(r-s+1)}{2}}\varepsilon_s(e_{r,r-s}\hat{f}^0_{r,s})$:



$=$ 

by Lemma 2.35

$=$ 

by Lemma 2.24, the relation (R4), and the defining relations of $\Delta(s)$.

The last diagram represents $\Xi_{r-s}(\hat{\mathsf{p}}_{r,s}(\hat{f}))$, where Ξ_{r-s} is as in Lemma 4.6(2). Thus, the image of $\varphi_{r,s}^{\hat{f}}$ under the map (2.14) is

$$(e_{r,r-s} \mapsto \Xi_{r-s}(\hat{\mathsf{p}}_{r,s}(\hat{f}))) \in \mathrm{Hom}_{R_\theta}(q^{\mathsf{s}_{r,r-s}} R_\theta e_{r,r-s}, \Delta(s)).$$

The proof is complete upon an application of Theorem 4.13. $\square$

For $\hat{f} \in \hat{\Lambda}(r,s)$ and $\hat{g} \in \hat{\Lambda}(s,t)$, define

$$-\hat{\diamond}- : \hat{\Lambda}(s,t) \otimes \hat{\Lambda}(r,s) \to \hat{\Lambda}(r,t),\ \hat{g} \otimes \hat{f} \mapsto \partial_{x(r,s,t)}\big(D(r,s,s-t)\hat{g}(w(s,t)\cdot \hat{f})\big). \tag{4.15}$$

Lemma 4.16. *Let $0 \le t \le s \le r \le \min\{a,b\}$. If $\hat{f} \in \hat{\Lambda}(r,s)$ and $\hat{g} \in \hat{\Lambda}(s,t)$, then $\varphi_{s,t}^{\hat{g}} \varphi_{r,s}^{\hat{f}} = \varphi_{r,t}^{\hat{g}\hat{\diamond}\hat{f}}$.*

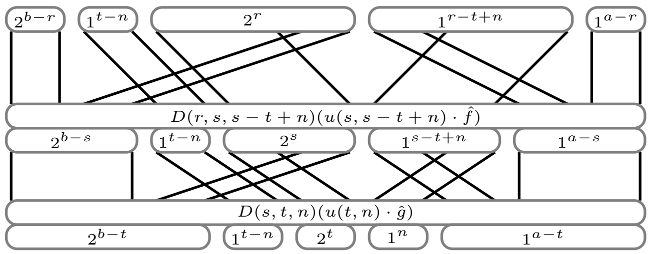
Proof. Let $y(s,t,n) := U_{b-s;t-n,s-t} \in \mathfrak{S}_d$ and note that

$$u(t,n)^{-1} y(s,t,n) u(s,s-t+n) = w(s,t) \tag{4.17}$$

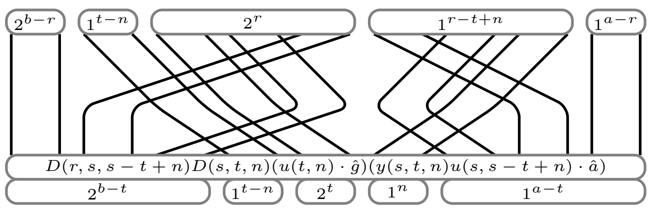
and

$$D(r,s,s-t+n)D(s,t,n) = D(r,t,n)D(r,s,s-t). \tag{4.18}$$

We compute $\hat{f}_{r,s}^{s-t+n}\hat{g}_{s,t}^{n}$:

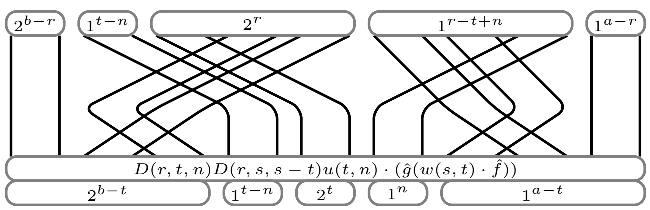


$=$



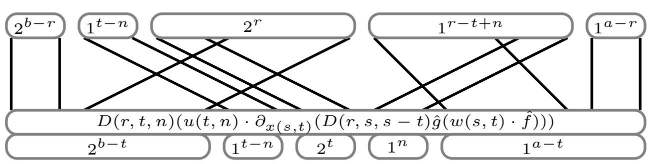
by (R4), Lemma 2.35, and Theorem 2.24

$=$



by (4.17), (4.18), and the relation (R7)

$=$



by Lemma 2.25.

The last diagram represents $(\hat{g}\,\hat{\diamond}\,\hat{f})^{n}$. $\square$

Lemma 4.19. *Let* $0 \leq t \leq s \leq r \leq \min\{a,b\}$, $\lambda \in \mathscr{P}(r-s,s)$, *and* $\mu \in \mathscr{P}(s-t,t)$, *and set* $\hat{f} := 1 \otimes s_\lambda \otimes 1 \otimes 1 \otimes 1 \otimes 1 \in \hat{\Lambda}(r,s)$ *and* $\hat{g} := 1 \otimes s_\mu \otimes 1 \otimes 1 \otimes 1 \otimes 1 \in \hat{\Lambda}(s,t)$. *Then*

$$\hat{g}\,\hat{\diamond}\,\hat{f} = 1 \otimes (s_\lambda \star s_\mu) \otimes 1 \otimes 1 \otimes 1 \otimes 1 \in \hat{\Lambda}(r,t).$$

Proof. We use the notation λ^c for the conjugate of a partition λ. For a partition $\kappa = (\kappa_1, \ldots, \kappa_{s-t}) \in \mathscr{P}(s-t, r-s)$, let $\hat{\kappa} := (r-s-\kappa_{s-t}, \ldots, r-s-\kappa_1)^c \in$

$\mathscr{P}(r-s,s-t)$. We have $\hat{f} = w(s,t)\hat{f}$ and

$$D(r,s,s-t) = \sum_{\kappa \in \mathscr{P}(s-t,r-s)} (-1)^{|\hat{\kappa}|} 1 \otimes s_{\hat{\kappa}} \otimes 1 \otimes 1 \otimes 1 \otimes s_\kappa \otimes 1 \otimes 1 \in \Lambda(r,s,t)$$

by [24, I.4 Example 5], so that

$$\begin{aligned}
\hat{g} \diamond \hat{f} &= \sum_{\kappa \in \mathscr{P}(s-t,r-s)} (-1)^{|\hat{\kappa}|} \partial_{x(r,s,t)}(1 \otimes (s_{\hat{\kappa}} s_\lambda) \otimes s_\mu \otimes 1 \otimes 1 \otimes s_\kappa \otimes 1 \otimes 1) \\
&= \sum_{\kappa \in \mathscr{P}(s-t,r-s)} (-1)^{|\hat{\kappa}|} 1 \otimes ((s_{\hat{\kappa}} s_\lambda) \star s_\mu) \otimes 1 \otimes 1 \otimes (s_\kappa \star 1) \otimes 1 \\
&= 1 \otimes (s_\lambda \star s_\mu) \otimes 1 \otimes 1 \otimes (s_{(r-s)^{(s-t)}} \star 1) \otimes 1 \\
&= 1 \otimes (s_\lambda \star s_\mu) \otimes 1 \otimes 1 \otimes 1 \otimes 1. \qquad \square
\end{aligned}$$

We consider $\Lambda(r,s)$ as a right $\Lambda(r,r)$ module via the natural algebra embedding

$$\Lambda(r,r) = \Lambda_{b-r,r,a-r} \hookrightarrow \Lambda_{b-r,r-s,s,a-r} = q^{-\omega_{r-s}(r,s)} \Lambda(r,s).$$

If $f \in \Lambda_{r-s}$, we write

$$f^{r,s} := 1_{\Lambda_{b-r}} \otimes f \otimes 1_{\Lambda_r} \otimes 1_{\Lambda_{a-r}} \in \Lambda(r,s).$$

By Proposition 2.20, $\Lambda(r,s)$ is a free right $\Lambda(r,r)$-module with basis

$$\{s_\lambda^{r,s} \mid \lambda \in \mathscr{P}(r-s,s)\}.$$

We also make $\Lambda(r,s)$ into a left $\Lambda(s,s)$-module via the composition of algebra homomorphisms:

$$\Lambda(s,s) = \Lambda_{b-s,s,a-s} \hookrightarrow \Lambda_{b-r,r-s,s,r-s,a-r} = \Lambda^{r,s} \xrightarrow{\mathrm{p}_{r,s}} q^{-\omega_{r-s}(r,s)}\Lambda(r,s), \tag{4.20}$$

the first map being the natural embedding. (This is similar to the definition of the *right* $\Lambda_{\sigma(s)}$-module structure on $\Lambda(r,s)$ used in the previous section; in fact $\Lambda(s,s)$ can be identified with $\mathrm{End}_{R_\theta}(\Delta(s)) = \Lambda_{\sigma(s)}^{\mathrm{op}}$, see the proof of Theorem 4.22.)

If $0 \le t \le s \le r \le \min\{a,b\}$, then the tensor product $\Lambda(s,t) \otimes_{\Lambda(s,s)} \Lambda(r,s)$ is now a free right $\Lambda(r,r)$-module with basis

$$\{s_\mu^{s,t} \otimes s_\lambda^{r,s} \mid \mu \in \mathscr{P}(s-t,t), \lambda \in \mathscr{P}(r-s,s)\} \tag{4.21}$$

and we define a map of right $\Lambda(r,r)$-modules

$$\Theta : \Lambda(s,t) \otimes_{\Lambda(s,s)} \Lambda(r,s) \to \Lambda(r,t),\ s_\mu^{s,t} \otimes s_\lambda^{r,s} \mapsto (s_\mu \star s_\lambda)^{r,t}.$$

Let

$$- \diamond - : \Lambda(s,t) \otimes_{\Bbbk} \Lambda(r,s) \to \Lambda(r,t),\ g \otimes f \mapsto \Theta(g \otimes f).$$

Theorem 4.22. *Let $0 \le t \le s \le r \le \min\{a,b\}$. The composition in the category $\mathcal{E}_\theta$ is given by*

$$\begin{array}{ccc} \mathcal{E}_\theta(s,t) \otimes \mathcal{E}_\theta(r,s) & \longrightarrow & \mathcal{E}_\theta(r,t) \\ \wr\Vert & & \wr\Vert \\ \Lambda(s,t) \otimes \Lambda(r,s) & \longrightarrow & \Lambda(r,t) \\ g \otimes f & \longmapsto & g \diamond f. \end{array}$$

Proof. According to Lemmas 4.14 and 4.16, the composition of $g \in \Lambda(s,t)$ with $f \in \Lambda(r,s)$ is given by $\mathsf{q}_{r,t}(\hat{g} \diamond \hat{f})$, where $\hat{g} \in \hat{\Lambda}(s,t)$ and $\hat{f} \in \hat{\Lambda}(r,s)$ are such that $\mathsf{q}_{s,t}(\hat{g}) = g$ and $\mathsf{q}_{r,s}(\hat{f}) = f$.

First suppose $r = s$. Since $D(s,s,s-t) = 1$ and $x(s,s,t) = 1$, (4.15) shows that $\mathsf{q}_{s,t}(\hat{g} \diamond \hat{f}) = gf \in \Lambda(s,t)$. Thus, the composition map $\Lambda(s,t) \otimes \Lambda(s,s) \to \Lambda(s,s)$ coincides with the right $\Lambda(s,s)$-module structure of $\Lambda(s,t)$.

Now suppose $s = t$. Since $D(r,s,0) = 1$ and $w(s,s) = x(r,s,s) = 1$, (4.15) shows that $\hat{g} \diamond \hat{f} = \hat{g}\hat{f}$, so that $\mathsf{q}_{r,s}(\hat{g} \diamond \hat{f}) = \varphi(g)f \in \Lambda(r,s)$ where φ is the composition in (4.20). Thus, the composition map $\Lambda(s,s) \otimes \Lambda(r,s) \to \Lambda(r,s)$ coincides with the left $\Lambda(s,s)$-module structure of $\Lambda(r,s)$.

Associativity in $\mathcal{E}_\theta$ implies that the composition map $\Lambda(s,t) \otimes \Lambda(r,s) \to \Lambda(r,t)$ is $\Lambda(s,s)$-balanced and $\Lambda(r,r)$-equivariant, and is therefore completely determined by the image of the basis elements in (4.21). Lemma 4.19 now completes the proof.

$\square$

CHAPTER V

NON-FORMALITY OF THE A_∞ STRUCTURE

This chapter contains material which appears in [1, §5]. The author performed the relevant computations under the supervision of Alexander Kleshchev. The results were written by the author.

In the situations of chapters III and IV, note that by homological degree considerations, the A_∞-category structure on $\mathcal{E}_\theta$ is formal. In fact, in the language of section 2.2, the category $\mathcal{E}_\theta$ is intrincically formal. We provide an example to show that the A_∞-category structure on $\mathcal{E}_\theta$ is, in general, non-formal. Recall the definitions, theorems, and machinery from section 2.2.

Let $\theta \in Q_+$. For each $\pi \in \mathrm{KP}(\theta)$, fix a projective resolution $P^\pi_\bullet$ of $\Delta(\pi)$. Consider the differential-graded category $\mathcal{H}_\theta$ whose objects are the Kostant partitions of θ with morphism spaces $\mathcal{H}^\bullet_\theta(\rho,\sigma) := \mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, P^\sigma_\bullet)$. We denote by

$$
\begin{aligned}
m_1^{\rho,\sigma} &: \mathcal{H}^\bullet_\theta(\rho,\sigma) \to \mathcal{H}^\bullet_\theta(\rho,\sigma) \\
m_2^{\rho,\sigma,\tau} &: \mathcal{H}^\bullet_\theta(\sigma,\tau) \otimes \mathcal{H}^\bullet_\theta(\rho,\sigma) \to \mathcal{H}^\bullet_\theta(\rho,\tau)
\end{aligned}
$$

the differential and composition in $\mathcal{H}_\theta$, respectively. Note that $m_1^{\rho,\sigma}$ is precisely δ from (2.12). Being a differential-graded category, $\mathcal{H}_\theta$ is also an A_∞-category, so its homology $\mathcal{E}_\theta$ carries a structure of an A_∞-category according to Theorem 2.6.

For the rest of this chapter, we let $\theta := \alpha_1 + 2\alpha_2 + \alpha_3 \in Q_+$ and set

$$
\pi := (\alpha_2, \alpha_1+\alpha_2+\alpha_3) \qquad \rho := (\alpha_2+\alpha_3, \alpha_1+\alpha_2)
$$
$$
\sigma := (\alpha_3, \alpha_2, \alpha_1+\alpha_2) \qquad \tau := (\alpha_3, (\alpha_2)^2, \alpha_1).
$$

Note that there is one other Kostant partition, $(\alpha_2+\alpha_3, \alpha_2, \alpha_1)$, which will not play a role in our construction. Recall the standard generators (2.47) of the standard modules and the idempotents (2.26) in the nil-Hecke algebra. We define

$$\hat{v}_\tau := v_{\alpha_3} \circ v_{\alpha_2} \circ v_{\alpha_2} \circ v_{\alpha_1} \in \hat{\Delta}(\tau),$$
$$e_\tau := e_1 \otimes e_2 \otimes e_1 \in \mathcal{NH}_1 \otimes \mathcal{NH}_2 \otimes \mathcal{NH}_1,$$

so that $v_\tau = \hat{v}_\tau e_\tau$. We list the resolutions $P^\pi_\bullet$, $P^\rho_\bullet$, $P^\sigma_\bullet$, and $P^\tau_\bullet$ of the corresponding standard modules from [12, Theorem A] and Lemma 2.42 below:

$$0 \to q^2 R_\theta 1_{2321} \xrightarrow{d_1^\pi} qR_\theta 1_{2231} \oplus qR_\theta 1_{2132} \xrightarrow{d_0^\pi} R_\theta 1_{2123} \xrightarrow{\epsilon_\pi} \Delta(\pi) \to 0,$$
$$0 \to q^2 R_\theta 1_{3221} \xrightarrow{d_1^\rho} qR_\theta 1_{2321} \oplus qR_\theta 1_{3212} \xrightarrow{d_0^\rho} R_\theta 1_{2312} \xrightarrow{\epsilon_\rho} \Delta(\rho) \to 0,$$
$$0 \longrightarrow qR_\theta 1_{3221} \xrightarrow{d_0^\sigma} R_\theta 1_{3212} \xrightarrow{\epsilon_\sigma} \Delta(\sigma) \to 0,$$
$$0 \longrightarrow q^{-1} R_\theta 1_{32^{(2)}1} \xrightarrow{\epsilon_\tau} \Delta(\tau) \to 0,$$

where a matrix label stands for right multiplication with that matrix, and

$$d_1^\pi := \begin{bmatrix} -\psi_2 1_{2231} & \psi_3\psi_2 1_{2132} \end{bmatrix}, \quad d_0^\pi := \begin{bmatrix} \psi_3\psi_2 1_{2123} \\ \psi_3 1_{2123} \end{bmatrix}, \quad \epsilon_\pi := \begin{bmatrix} v_\pi \end{bmatrix},$$
$$d_1^\rho := \begin{bmatrix} -\psi_1 1_{2321} & \psi_3 1_{3212} \end{bmatrix}, \quad d_0^\rho := \begin{bmatrix} \psi_3 1_{2312} \\ \psi_1 1_{2312} \end{bmatrix}, \quad \epsilon_\rho := \begin{bmatrix} v_\rho \end{bmatrix},$$
$$d_0^\sigma := \begin{bmatrix} \psi_3 1_{3212} \end{bmatrix}, \quad \epsilon_\sigma := \begin{bmatrix} v_\sigma \end{bmatrix},$$
$$\epsilon_\tau := \begin{bmatrix} \psi_2 v_\tau \end{bmatrix} = \begin{bmatrix} \hat{v}_\tau \tau_2 \end{bmatrix}.$$

One can easily check, using (2.38), that the complexes $T_\bullet^{P^\pi_\bullet}(\Delta(\rho))$, $T_\bullet^{P^\rho_\bullet}(\Delta(\sigma))$, $T_\bullet^{P^\sigma_\bullet}(\Delta(\tau))$, and $T_\bullet^{P^\pi_\bullet}(\Delta(\tau))$ are, respectively, the top complexes in the four diagrams below. Moreover, the diagrams define isomorphisms of complexes.

$$\begin{tikzcd}
0 & q^{-1}1_{2132}\Delta(\rho) \arrow[l] & & 1_{2123}\Delta(\rho) \arrow[ll, "{\left[(d_0^\pi)_{2,1}\right]}"'] & 0 \arrow[l] \\
0 & q^{-1}\mathcal{X}_2 \arrow[l] \arrow[u, "\wr"', "f\mapsto\psi_2 v_\rho f"] & & q\mathcal{X}_2 \arrow[ll, "{[x_2-x_1]}"'] \arrow[u, "\wr"', "f\mapsto\psi_3\psi_2 v_\rho f"] & 0, \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
0 & q^{-1}1_{3212}\Delta(\sigma) \arrow[l] & & 1_{2312}\Delta(\sigma) \arrow[ll, "{\left[(d_0^\rho)_{2,1}\right]}"'] & 0 \arrow[l] \\
0 & q^{-1}\mathcal{X}_3 \arrow[l] \arrow[u, "\wr"', "f\mapsto v_\sigma f"] & & q\mathcal{X}_3 \arrow[ll, "{[x_2-x_1]}"'] \arrow[u, "\wr"', "f\mapsto\psi_1 v_\sigma f"] & 0, \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
0 & 1_{3221}\Delta(\tau) \arrow[l] & & q1_{3212}\Delta(\tau) \arrow[ll, "{\left[(d_0^\sigma)_{1,1}\right]}"'] & 0 \arrow[l] \\
0 & q^{-2}\mathcal{X}_4 \arrow[l] \arrow[u, "\wr"', "f\mapsto\hat{v}_\tau f\tau_2"] & & \mathcal{X}_4 \arrow[ll, "{[x_4-x_3]}"'] \arrow[u, "\wr"', "f\mapsto\psi_3\hat{v}_\tau f\tau_2"] & 0, \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
0 & q^{-1}1_{2321}\Delta(\tau) \arrow[l] & 1_{2231}\Delta(\tau)\oplus 1_{2132}\Delta(\tau) \arrow[l, "d_1^\pi"'] & q1_{2123}\Delta(\tau) \arrow[l, "d_0^\pi"'] & 0 \arrow[l] \\
0 & q^{-2}\mathcal{X}_4 \arrow[l] \arrow[u, "\wr"', "f\mapsto\psi_1\hat{v}_\tau f\tau_2"] & \mathcal{X}_4\oplus\mathcal{X}_4 \arrow[l, "{[\,-(x_3-x_1)\ \ x_4-x_3\,]}"] \arrow[u, "\wr"', "{(f,g)\mapsto(\psi_2\psi_1\hat{v}_\tau f\tau_2,\psi_2\psi_1\psi_3\hat{v}_\tau g\tau_2)}"] & q^2\mathcal{X}_4 \arrow[l, "{\begin{bmatrix} x_4-x_3 \\ x_3-x_1 \end{bmatrix}}"] \arrow[u, "\wr"', "f\mapsto\psi_3\psi_2\psi_3\psi_1\hat{v}_\tau f\tau_2"] & 0 \arrow[l]
\end{tikzcd}$$

Thus, denoting $\mathcal{Z}_k := \Bbbk[z_1,\dots,z_k]$, we have

$$\mathcal{E}_\theta(\pi,\rho) = \mathcal{E}_\theta^1(\pi,\rho) \cong q^{-1}\mathcal{X}_2/(x_1=x_2) \xrightarrow{\sim} q^{-1}\mathcal{Z}_1,\ \bar{x}_1\mapsto z_1$$

$$\mathcal{E}_\theta(\rho,\sigma) = \mathcal{E}_\theta^1(\rho,\sigma) \cong q^{-1}\mathcal{X}_3/(x_1=x_2) \xrightarrow{\sim} q^{-1}\mathcal{Z}_2,\ \bar{x}_1\mapsto z_1, \bar{x}_3\mapsto z_2$$

$$\mathcal{E}_\theta(\sigma,\tau) = \mathcal{E}_\theta^1(\sigma,\tau) \cong q^{-2}\mathcal{X}_4/(x_3=x_4) \xrightarrow{\sim} q^{-2}\mathcal{Z}_3,\ \bar{x}_1\mapsto z_1, \bar{x}_2\mapsto z_2, \bar{x}_3\mapsto z_3$$

$$\mathcal{E}_\theta(\pi,\tau) = \mathcal{E}_\theta^2(\pi,\tau) \cong q^{-2}\mathcal{X}_4/(x_1=x_3=x_4) \xrightarrow{\sim} q^{-2}\mathcal{Z}_2,\ \bar{x}_1\mapsto z_1, \bar{x}_2\mapsto z_2.$$

Example 5.1. There is an A_∞-category structure $\mathcal{E}_\theta = (\mathcal{E}_\theta^\bullet, M_1, M_2, \ldots)$ satisfying the conditions in Theorem 2.6 such that

$$\begin{tikzcd}
\mathcal{E}_\theta^1(\sigma,\tau) \otimes \mathcal{E}_\theta^1(\rho,\sigma) \otimes \mathcal{E}_\theta^1(\pi,\rho) \arrow[r, "M_3^{\pi,\rho,\sigma,\tau}"] \arrow[d, phantom, "\wr\|"] & \mathcal{E}_\theta^2(\pi,\tau) \arrow[d, phantom, "\wr\|"] \\
q^{-2}\mathcal{Z}_3 \otimes q^{-1}\mathcal{Z}_2 \otimes q^{-1}\mathcal{Z}_1 \arrow[r] & q^{-2}\mathcal{Z}_2 \\
z_1^a z_2^b z_3^c \otimes z_1^m z_2^n \otimes z_1^w \arrow[r, mapsto] & z_1^{c+m+w} z_2^{b+n} \dfrac{z_2^a - z_1^a}{z_2 - z_1}.
\end{tikzcd}$$

Moreover, there is no A_∞-category structure on $\mathcal{E}_\theta$ satisfying the conditions in Theorem 2.6 with $M_3 = 0$.

Proof. We apply Algorithm 2.7. We will need to examine the complexes $T_\bullet^{P_\bullet^\pi}(\Delta(\sigma))$ and $T_\bullet^{P_\bullet^\rho}(\Delta(\tau))$. They are, respectively, the top complexes in the two diagrams below, and the diagrams define isomorphisms of complexes.

$$\begin{tikzcd}
0 & q^{-1}1_{2132}\Delta(\sigma) \arrow[l] & 1_{2123}\Delta(\sigma) \arrow[l, "{[(d_0^\pi)_{2,1}]}"'] & 0 \arrow[l] \\
0 & \mathcal{X}_3 \arrow[l] \arrow[u, "\wr"', "f\mapsto \psi_2\psi_1 v_\sigma f"] & q^2\mathcal{X}_3 \arrow[l, "{[x_3-x_1]}"'] \arrow[u, "\wr"', "f\mapsto \psi_3\psi_2\psi_1 v_\sigma f"] & 0 \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
0 & q^{-1}1_{3221}\Delta(\tau) \arrow[l] & 1_{2321}\Delta(\tau) \oplus 1_{3212}\Delta(\tau) \arrow[l, "d_1^\rho"'] & q1_{2312}\Delta(\tau) \arrow[l, "d_0^\rho"'] & 0 \arrow[l] \\
0 & q^{-3}\mathcal{X}_4 \arrow[l] \arrow[u, "\wr"', "f\mapsto \hat{v}_\tau f\tau_2"] & q^{-1}\mathcal{X}_4 \oplus q^{-1}\mathcal{X}_4 \arrow[l, "{[\,-(x_2-x_1)\ \ x_4-x_3\,]}"] \arrow[u, "\wr"', "{(f,g)\mapsto(\psi_1\hat{v}_\tau f\tau_2, \psi_3\hat{v}_\tau g\tau_2)}"] & q\mathcal{X}_4 \arrow[l, "{\begin{bmatrix} x_4-x_3 \\ x_2-x_1 \end{bmatrix}}"] \arrow[u, "\wr"', "f\mapsto \psi_1\psi_3\hat{v}_\tau f\tau_2"] & 0 \arrow[l]
\end{tikzcd}$$

so that

$$\mathcal{E}_\theta^\bullet(\pi,\sigma) = \mathcal{E}_\theta^1(\pi,\sigma) \cong \mathcal{X}_3/(x_1 = x_3) \xrightarrow{\sim} \mathcal{Z}_2,\ \bar{x}_1 \mapsto z_1, \bar{x}_2 \mapsto z_2$$

$$\mathcal{E}_\theta^\bullet(\rho,\tau) = \mathcal{E}_\theta^2(\rho,\tau) \cong q^{-3}\mathcal{X}_4/(x_1 = x_2, x_3 = x_4) \xrightarrow{\sim} q^{-3}\mathcal{Z}_2, \bar{x}_1 \mapsto z_1, \bar{x}_3 \mapsto z_2.$$

Instead of F_1, it will only be relevant to determine the restrictions $F_1^{\pi,\rho}$, $F_1^{\rho,\sigma}$, $F_1^{\sigma,\tau}$, and $F_1^{\rho,\tau}$. According to the algorithm, we are free to take for these any cycle-choosing homomorphisms. We define

$$F_1^{\pi,\rho}(z_1^a) := \begin{tikzcd} 0 \arrow[r] & q^2R_\theta 1_{2321} \arrow[r, "d_1^\pi"] \arrow[dr, "{[y_4^a 1_{2321} \ 0]}"] & qR_\theta 1_{2231} \oplus qR_\theta 1_{2132} \arrow[r, "d_0^\pi"] \arrow[dr, "{\left[\begin{smallmatrix} 0 \\ -\psi_2 y_3^a 1_{2312} \end{smallmatrix}\right]}"] & R_\theta 1_{2123} \arrow[r] & 0, \\ 0 \arrow[r] & q^2R_\theta 1_{3221} \arrow[r, "d_1^\rho"] & qR_\theta 1_{2321} \oplus qR_\theta 1_{3212} \arrow[r, "d_0^\rho"] & R_\theta 1_{2312} \arrow[r] & 0, \end{tikzcd}$$

$$F_1^{\rho,\sigma}(z_1^a z_2^b) := \begin{tikzcd} 0 \arrow[r] & q^2R_\theta 1_{3221} \arrow[r, "d_1^\rho"] \arrow[dr, "{[y_2^a y_4^b 1_{3221}]}"] & qR_\theta 1_{2321} \oplus qR_\theta 1_{3212} \arrow[r, "d_0^\rho"] \arrow[dr, "{\left[\begin{smallmatrix} 0 \\ -y_2^a y_3^b 1_{3212} \end{smallmatrix}\right]}"] & R_\theta 1_{2312} \arrow[r] & 0, \\ & 0 \arrow[r] & qR_\theta 1_{3221} \arrow[r, "d_0^\sigma"] & R_\theta 1_{3212} \arrow[r] & 0, \end{tikzcd}$$

$$F_1^{\sigma,\tau}(z_1^a z_2^b z_3^c) := \begin{tikzcd} 0 \arrow[r] & qR_\theta 1_{3221} \arrow[r, "d_0^\sigma"] \arrow[dr, "{[-y_1^a y_2^b y_4^c 1_{32^{(2)}1}]}"] & R_\theta 1_{3212} \arrow[r] & 0, \\ & 0 \arrow[r] & q^{-1}R_\theta 1_{32^{(2)}1} \arrow[r] & 0, \end{tikzcd}$$

$$F_1^{\rho,\tau}(z_1^a z_2^b) := \begin{tikzcd} 0 \arrow[r] & q^2R_\theta 1_{3221} \arrow[r, "d_1^\rho"] \arrow[drr, "{[-y_2^a y_4^b 1_{32^{(2)}1}]}"] & qR_\theta 1_{2321} \oplus qR_\theta 1_{3212} \arrow[r, "d_0^\rho"] & R_\theta 1_{2312} \arrow[r] & 0, \\ & & 0 \arrow[r] & q^{-1}R_\theta 1_{32^{(2)}1} \arrow[r] & 0. \end{tikzcd}$$

Using the above, we now show that the following choices are in accordance with the algorithm:

$$M_2^{\pi,\rho,\sigma} = 0, \quad F_2^{\pi,\rho,\sigma} = 0, \quad M_2^{\pi,\rho,\tau} = 0, \quad F_2^{\pi,\rho,\tau} = 0 \tag{5.2}$$

By our choices of $F_1^{\pi,\rho}$ and $F_1^{\rho,\sigma}$, we have $U_2^{\pi,\rho,\sigma} = 0$, so $M_2^{\pi,\rho,\sigma} = 0$, and according to (2.8), we may take $F_2^{\pi,\rho,\sigma} = 0$. Since $\mathcal{E}_\theta^\bullet(\pi,\rho)$ is concentrated in homological degree 1 and $\mathcal{E}_\theta^\bullet(\rho,\tau)$ is concentrated in homological degree 2, the image of $U_2^{\pi,\rho,\tau} = m_2^{\pi,\rho,\tau}(F_1^{\rho,\tau} \otimes F_1^{\pi,\rho})$ is in $\mathcal{H}^3_\theta(\pi,\tau)$, which is zero since $P^\pi_\bullet$ has length 2. Thus $M_2^{\pi,\rho,\tau} = 0$ and according to (2.8), we may take $F_2^{\pi,\rho,\tau} = 0$.

We now have

$$\begin{aligned} U_3^{\pi,\rho,\sigma,\tau} &= m_2^{\pi,\sigma,\tau}(F_1^{\sigma,\tau} \otimes F_2^{\pi,\rho,\sigma}) - m_2^{\pi,\rho,\tau}(F_2^{\rho,\sigma,\tau} \otimes F_1^{\pi,\rho}) + F_2^{\pi,\sigma,\tau}(1^{\sigma,\tau} \otimes M_2^{\pi,\rho,\sigma}) \\ &\qquad - F_2^{\pi,\rho,\tau}(M_2^{\rho,\sigma,\tau} \otimes 1^{\pi,\rho}) \\ &= -m_2^{\pi,\rho,\tau}(F_2^{\rho,\sigma,\tau} \otimes F_1^{\pi,\rho}), \end{aligned}$$

so we only need to make a choice for $F_2^{\rho,\sigma,\tau}$. We have $U_2^{\rho,\sigma,\tau} = m_2^{\rho,\sigma,\tau}(F_1^{\sigma,\tau} \otimes F_1^{\rho,\sigma})$ so that $U_2^{\rho,\sigma,\tau}(z_1^a z_2^b z_3^c \otimes z_1^m z_2^n)$ is given by

$$\begin{tikzcd} 0 \arrow[r] & q^2 R_\theta 1_{3221} \arrow[r, "d_1^\rho"] \arrow[drr, "{\left[-y_1^a y_2^{b+m} y_4^{c+n} 1_{32^{(2)}1}\right]}"] & qR_\theta 1_{2321} \oplus qR_\theta 1_{3212} \arrow[r, "d_0^\rho"] & R_\theta 1_{2312} \arrow[r] & 0 \\ & & 0 \arrow[r] & q^{-1} R_\theta 1_{32^{(2)}1} \arrow[r] & 0. \end{tikzcd}$$

Thus

$$M_2^{\rho,\sigma,\tau}(z_1^a z_2^b z_3^c \otimes z_1^m z_2^n) = z_1^{a+b+m} z_2^{c+n}, \tag{5.3}$$

and $(F_1^{\rho,\tau} M_2^{\rho,\sigma,\tau} - U_2^{\rho,\sigma,\tau})(z_1^a z_2^b z_3^c \otimes z_1^m z_2^n)$ is given by the above diagram, except the diagonal arrow is right multiplication with $\left[-y_2^{b+m} y_4^{c+n} (y_2^a - y_1^a) 1_{32^{(2)}1}\right]$. We may now take $F_2^{\rho,\sigma,\tau}(z_1^a z_2^b z_3^c \otimes z_1^m z_2^n)$ to be:

$$\begin{tikzcd} 0 \arrow[r] & q^2 R_\theta 1_{3221} \arrow[r, "d_1^\rho"] & qR_\theta 1_{2321} \oplus qR_\theta 1_{3212} \arrow[r, "d_0^\rho"] \arrow[dr, "{\begin{bmatrix} \psi_1 y_2^{b+m} y_4^{c+n} \frac{y_2^a - y_1^a}{y_2 - y_1} 1_{32^{(2)}1} \\ 0 \end{bmatrix}}"] & R_\theta 1_{2312} \arrow[r] & 0 \\ & & 0 \arrow[r] & q^{-1} R_\theta 1_{32^{(2)}1} \arrow[r] & 0. \end{tikzcd}$$

Now $U_3^{\pi,\rho,\sigma,\tau}(z_1^a z_2^b z_3^c \otimes z_1^m z_2^n \otimes z_1^w)$ is given by:

$$
\begin{tikzcd}
0 \arrow[r] & q^2 R_\theta 1_{2321} \arrow[r, "d_1^\pi"] \arrow[drr, "\left[-\psi_1 y_2^{b+m} y_4^{c+n+w} \frac{y_2^a - y_1^a}{y_2 - y_1} 1_{32^{(2)}1}\right]"] & q R_\theta 1_{2231} \oplus q R_\theta 1_{2132} \arrow[r, "d_0^\pi"] & R_\theta 1_{2123} \arrow[r] & 0, \\
 & & 0 \arrow[r] & q^{-1} R_\theta 1_{32^{(2)}1} \arrow[r] & 0,
\end{tikzcd}
$$

so that $M_3^{\pi,\rho,\sigma,\tau} = [U_3^{\pi,\rho,\sigma,\tau}]$ is as in the theorem statement.

For the second assertion, note that the existence of a second A_∞-category structure $\mathcal{E}_\theta = (\mathcal{E}_\theta^\bullet, N_1, N_2, \ldots)$ satisfying the conditions of Theorem 2.6 implies the existence of an isomorphism of A_∞-categories $G : (\mathcal{E}_\theta^\bullet, M_1, M_2, \ldots) \to (\mathcal{E}_\theta^\bullet, N_1, N_2, \ldots)$ with G_1 being the identity on each morphism space. Assume, toward a contradiction, that such an isomorphism exists, and that $N_3 = 0$.

Recall that we take $M_1 = N_1 = 0$, so when $n = 2$, the relation (2.5) applied to G reads $G_1 M_2 = N_2(G_1 \otimes G_1)$, and since G_1 is the identity, we have $M_2 = N_2$. Now since $N_3 = 0$ by assumption, when $n = 3$, the relation (2.5) reads

$$G_2(M_2 \otimes 1 - 1 \otimes M_2) + M_3 = M_2(1 \otimes G_2 - G_2 \otimes 1). \tag{5.4}$$

The restriction of (5.4) to $\mathcal{E}_\theta^\bullet(\sigma,\tau) \otimes \mathcal{E}_\theta^\bullet(\rho,\sigma) \otimes \mathcal{E}_\theta^\bullet(\pi,\rho)$ is

$$\begin{aligned} &G_2^{\pi,\rho,\tau}(M_2^{\rho,\sigma,\tau} \otimes 1^{\pi,\rho}) - G_2^{\pi,\sigma,\tau}(1^{\sigma,\tau} \otimes M_2^{\pi,\rho,\sigma}) + M_3^{\pi,\rho,\sigma,\tau} \\ &\qquad = M_2^{\pi,\sigma,\tau}(1^{\sigma,\tau} \otimes G_2^{\pi,\rho,\sigma}) - M_2^{\pi,\rho,\tau}(G_2^{\rho,\sigma,\tau} \otimes 1^{\pi,\rho}), \end{aligned} \tag{5.5}$$

so according to (5.2), (5.5) becomes

$$G_2^{\pi,\rho,\tau}(M_2^{\rho,\sigma,\tau} \otimes 1^{\pi,\rho}) + M_3^{\pi,\rho,\sigma,\tau} = M_2^{\pi,\sigma,\tau}(1^{\sigma,\tau} \otimes G_2^{\pi,\rho,\sigma}). \tag{5.6}$$

We have a formula for $M_2^{\rho,\sigma,\tau}$ given by (5.3). We define

$$F_1^{\pi,\sigma}(z_1^m z_2^n) := \begin{tikzcd} 0 \arrow[r] & q^2 R_\theta 1_{2321} \arrow[r, "d_1^\pi"] \arrow[dr, "{[\psi_1 y_2^n y_4^m 1_{3221}]}"] & qR_\theta 1_{2231} \oplus qR_\theta 1_{2132} \arrow[r, "d_0^\pi"] \arrow[dr, "{\left[\begin{smallmatrix} 0 \\ -\psi_2\psi_1 y_2^n y_3^m 1_{3212} \end{smallmatrix}\right]}"] & R_\theta 1_{2123} \arrow[r] & 0, \\ & 0 \arrow[r] & qR_\theta 1_{3221} \arrow[r, "d_0^\sigma"] & R_\theta 1_{3212} \arrow[r] & 0, \end{tikzcd}$$

By our choices of $F_1^{\pi,\sigma}$ and $F_1^{\sigma,\tau}$, we have that $U_2^{\pi,\sigma,\tau}(z_1^a z_2^b z_3^c \otimes z_1^m z_2^n)$ is given by

$$\begin{tikzcd} 0 \arrow[r] & q^2 R_\theta 1_{2321} \arrow[r, "d_1^\pi"] \arrow[drr, "{[-\psi_1 y_1^a y_2^{b+n} y_4^{c+m} 1_{32^{(2)}1}]}"] & qR_\theta 1_{2231} \oplus qR_\theta 1_{2132} \arrow[r, "d_0^\pi"] & R_\theta 1_{2123} \arrow[r] & 0, \\ & & 0 \arrow[r] & q^{-1} R_\theta 1_{32^{(2)}1} \arrow[r] & 0 \end{tikzcd}$$

so that $M_2^{\pi,\sigma,\tau}(z_1^a z_2^b z_3^c \otimes z_1^m z_2^n) = z_1^{a+c+m} z_2^{b+n}$.

Denote the left- and right-hand sides of (5.6) by L and R, respectively. We apply L and R to two elements: $z_1 \otimes 1 \otimes 1$ and $z_2 \otimes 1 \otimes 1$. Note that the map

$$G_2^{\pi,\rho,\sigma} : q^{-1}\mathcal{Z}_2 \otimes q^{-1}\mathcal{Z}_1 \cong \mathcal{E}_\theta^1(\rho,\sigma) \otimes \mathcal{E}_\theta^1(\pi,\rho) \to \mathcal{E}_\theta^1(\pi,\sigma) \cong \mathcal{Z}_2,$$

is a KLR degree 0 map, so we must have $G_2^{\pi,\rho,\sigma}(1 \otimes 1) = 0$. Thus, $R(z_1 \otimes 1 \otimes 1) = R(z_2 \otimes 1 \otimes 1) = 0$. On the other hand, we have $M_3^{\pi,\rho,\sigma,\tau}(z_1 \otimes 1 \otimes 1) = 1$, $M_3^{\pi,\rho,\sigma,\tau}(z_2 \otimes 1 \otimes 1) = 0$, and $M_2^{\rho,\sigma,\tau}(z_1 \otimes 1) = M_2^{\rho,\sigma,\tau}(z_2 \otimes 1) = z_1$, so by (5.6), we have

$$\begin{aligned} 0 = R(z_1 \otimes 1 \otimes 1) = L(z_1 \otimes 1 \otimes 1) = G_2(z_1 \otimes 1) + 1, \\ 0 = R(z_2 \otimes 1 \otimes 1) = L(z_2 \otimes 1 \otimes 1) = G_2(z_1 \otimes 1), \end{aligned}$$

a clear contradiction. $\square$

APPENDIX

A PROOF OF CONCEPT: TRUNCATED POLYNOMIALS

The following exercise is suggested by Keller in [20] to illustrate the phenomenon that the traditional Ext functor results in a loss of information, whereas the Ext functor upgraded to include A_∞ structure via Theorem 2.6 and Algorithm 2.7 preserves information.

For an integer $N > 0$, let $R_N = \Bbbk[x]/(x^N)$ and let $\Delta_N = \Bbbk$ be the unique simple R_N-module (where x acts by zero). Then the category $\mathcal{F}_N := \mathrm{filt}(\Delta_N)$ is the category of finite-dimensional R_N-modules. We will use the machinery from Section 2.2 to explicitly describe the A_∞-algebra $\mathcal{E}_N := \mathrm{Ext}^\bullet_{R_N}(\Delta_N, \Delta_N)$ and, using only information from $\mathcal{E}_N$, reconstruct $\mathcal{F}_N$.

A.1. $\mathcal{E}_N$ as a $\Bbbk$-module

The first ingredient in the machinery is a projective resolution $P_N^\bullet$ of Δ_N. Note that Δ_1 is isomorphic to the left regular R_1-module. It is easy to check that the following gives projective resolutions of Δ_N:

$$\begin{array}{lr} \cdots \longrightarrow 0 \longrightarrow R_1 \longrightarrow 0 \qquad \text{if } N = 1 & \\ \cdots \xrightarrow{x} \underset{4}{R_N} \xrightarrow{x^{N-1}} \underset{3}{R_N} \xrightarrow{x} \underset{2}{R_N} \xrightarrow{x^{N-1}} \underset{1}{R_N} \xrightarrow{x} \underset{0}{R_N} \longrightarrow \underset{-1}{0} \qquad \text{if } N > 1, & \text{(A.1)} \end{array}$$

with augmentation map $\varepsilon_N : R_N \to \Delta_N,\ 1 \mapsto 1$.

Applying the functor $\mathrm{Hom}_{R_N}(-,\Delta_N)$ to the resolution A.1, we obtain the complexes

$$\cdots \longleftarrow 0 \longleftarrow \Bbbk \longleftarrow 0 \qquad \text{if } N=1$$

$$\underset{}{\cdots} \xleftarrow[0]{} \underset{4}{\Bbbk} \xleftarrow[0]{} \underset{3}{\Bbbk} \xleftarrow[0]{} \underset{2}{\Bbbk} \xleftarrow[0]{} \underset{1}{\Bbbk} \xleftarrow[0]{} \underset{0}{\Bbbk} \longleftarrow \underset{-1}{0} \qquad \text{if } N>1.$$

Thus, as graded $\Bbbk$-modules, we see that $\mathcal{E}_1^\bullet$ is isomorphic to $\Bbbk$ concentrated in degree 0, and for $N>1$, $\mathcal{E}_N^\bullet$ is isomorphic to $\bigoplus_{n=0}^{\infty}\Bbbk$. This allows us to distinguish the case $N=1$, but does not distinguish between any other $N>1$. In $\mathcal{E}_N^\bullet$, denote by X_n the element 1 in the degree n copy of $\Bbbk$.

A.2. $\mathcal{E}_N$ as a $\Bbbk$-algebra

We now describe the multiplication on $\mathcal{E}_N$ by lifting homomorphisms $\mathrm{Hom}^\bullet_{R_N}(P_N^\bullet,\Delta_N)$ to $\mathrm{Hom}^\bullet_{R_N}(P_N^\bullet,P_N^\bullet)$ according to 2.14. Since the differentials in the complex $\mathrm{Hom}^\bullet_{R_N}(P_N^\bullet,\Delta_N)$ are all zero, we may identify $\mathrm{Hom}^\bullet_{R_N}(P_N^\bullet,\Delta_N)$ with $\mathcal{E}_N^\bullet$ as graded $\Bbbk$-modules.

In the case $N=1$, the element X_0 lifts to the identity $\mathrm{id}\in\mathrm{Hom}^\bullet_{R_1}(P_1^\bullet,P_1^\bullet)$ and we therefore have $X_0X_0=X_0$. As a graded $\Bbbk$-algebra, we now have $\mathcal{E}_1^\bullet\cong\Bbbk$.

In the case $N=2$, the element $X_n\in\mathrm{Hom}_{R_2}(P_2^n,\Delta_N)$ lifts to the map $\hat{X}_n\in\mathrm{Hom}^n_{R_2}(P_2^\bullet,P_2^\bullet)$ defined below:

$$\begin{tikzcd}
\cdots \arrow[r, "x"] & \overset{n+2}{R_2} \arrow[r, "x"] \arrow[d, "s(n,2)"] & \overset{n+1}{R_2} \arrow[r, "x"] \arrow[d, "s(n,1)"] & \overset{n}{R_2} \arrow[r, "x"] \arrow[d, "s(n,0)"] & \overset{n-1}{R_2} \arrow[r, "x"] \arrow[d] & \cdots \\
\cdots \arrow[r, "x"'] & \underset{2}{R_2} \arrow[r, "x"'] & \underset{1}{R_2} \arrow[r, "x"'] & \underset{0}{R_2} \arrow[r] & \underset{-1,}{0} \arrow[r] & \cdots
\end{tikzcd} \tag{A.2}$$

where

$$s(n,k) = (-1)^{\frac{n(n+1)}{2}+nk}. \tag{A.3}$$

Note that for $n, m, k \in \mathbb{Z}$, we have

$$s(n, m+k)s(m,k) = s(n+m, k). \tag{A.4}$$

It is easy to check that $\hat{X}_m \hat{X}_n = \hat{X}_{m+n}$ and we therefore have $X_m X_n = X_{m+n}$. Setting $X := X_1$, this allows us to identify $\mathcal{E}_2^\bullet$ with $\Bbbk[X]$ as graded algebras (with X in degree 1).

In the case $N > 2$, a lift of the element $X_n \in \mathrm{Hom}_{R_N}(P_N^n, \Delta_N)$ depends on the parity of n. If n is even, we define $\hat{X}_n \in \mathrm{Hom}^n_{R_N}(P_N^\bullet, P_N^\bullet)$ below:

$$\begin{tikzcd}
\cdots \arrow[r, "x"] & \overset{n+2}{R_N} \arrow[r, "x^{N-1}"] \arrow[d, "s(n,2)"] & \overset{n+1}{R_N} \arrow[r, "x"] \arrow[d, "s(n,1)"] & \overset{n}{R_N} \arrow[r, "x^{N-1}"] \arrow[d, "s(n,0)"] & \overset{n-1}{R_N} \arrow[r, "x"] \arrow[d] & \cdots \\
\cdots \arrow[r, "x"'] & \underset{2}{R_N} \arrow[r, "x^{N-1}"'] & \underset{1}{R_N} \arrow[r, "x"'] & \underset{0}{R_N} \arrow[r] & \underset{-1,}{0} \arrow[r] & \cdots
\end{tikzcd} \tag{A.5}$$

and if n is odd, we define $\hat{X}_n$ below:

$$\begin{tikzcd}
\cdots \arrow[r] & \overset{n+3}{R_N} \arrow[r, "x^{N-1}"] \arrow[d, "s(n,3)x^{N-2}"] & \overset{n+2}{R_N} \arrow[r, "x"] \arrow[d, "s(n,2)"] & \overset{n+1}{R_N} \arrow[r, "x^{N-1}"] \arrow[d, "s(n,1)x^{N-2}"] & \overset{n}{R_N} \arrow[r, "x"] \arrow[d, "s(n,0)"] & \overset{n-1}{R_N} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \underset{3}{R_N} \arrow[r, "x"'] & \underset{2}{R_N} \arrow[r, "x^{N-1}"'] & \underset{1}{R_N} \arrow[r, "x"'] & \underset{0}{R_N} \arrow[r] & \underset{-1.}{0} \arrow[r] & \cdots
\end{tikzcd} \tag{A.6}$$

It is easy to check that if at least one of m or n is even, then $\hat{X}_m \hat{X}_n = \hat{X}_{m+n}$ so $X_m X_n = X_{m+n}$. If both m and n are odd, then we have $\varepsilon_N((\hat{X}_m \hat{X}_n)|_{P_N^{n+m}}) = 0$ so $X_m X_n = 0$ according to (2.14). Setting $X := X_1$ and $Y := X_2$, this allows us to identify $\mathcal{E}_N^\bullet$ with $\Bbbk[X, Y]/(X^2)$ (with X in degree 1 and Y in degree 2).

To summarize the situation so far, we have isomorphisms of graded algebras

$$\mathcal{E}_N^\bullet \cong \begin{cases} \Bbbk & \text{if } N = 1 \\ \Bbbk[X] \quad (|X| = 1) & \text{if } N = 2 \\ \Bbbk[X,Y]/(X^2) \quad (|X| = 1, |Y| = 2) & \text{if } N > 2. \end{cases} \tag{A.7}$$

We have therefore distinguished the cases $N = 1$ and $N = 2$, but we see that the algebra structure on $\mathcal{E}_N^\bullet$ is not enough to distinguish between values of $N > 2$.

A.3. $\mathcal{E}_N$ as an A_∞-algebra

We now describe the A_∞-algebra structure on $\mathcal{E}_N^\bullet$, which we shall denote by $(\mathcal{E}_N, M_1, M_2, \ldots)$. According to Theorem 2.6, we have $M_1 = 0$, and M_2 given by the multiplication summarized by (A.7).

For the case $N = 1$, since M_n is of degree $2 - n$, and since $\mathcal{E}_1^\bullet$ is concentrated in degree 0, we see that $M_n = 0$ for $n \neq 2$. For $N > 1$, we use Algorithm 2.7 to recursively compute the maps M_n as well as maps F_n which together form the quasi-isomorphism $\mathcal{E}_N^\bullet \to \mathrm{Hom}^\bullet_{R_N}(P_N^\bullet, P_N^\bullet)$ guaranteed to exist by Theorem 2.6.

For the case $N = 2$, we claim that we may take $M_2(X_i \otimes X_j) = X_{i+j}$, $M_n = 0$ for $n \neq 2$, $F_1(X_i) = \hat{X}_i$, and $F_n = 0$ for $n \neq 1$. Step 1 of the algorithm yields $M_1 = 0$ and $F_1(X_i) = \hat{X}_i$. Recalling (2.10), Step 2 yields $U_2 = m_2(F_1 \otimes F_1)$ so that

$$U_2(X_i \otimes X_j) = \hat{X}_i \hat{X}_j = \hat{X}_{i+j}.$$

We therefore recover $M_2(X_i \otimes X_j) = X_{i+j}$ as desired. For Step 3, note that $(F_1 M_2 - U_2)(X_i \otimes X_j) = \hat{X}_{i+j} - \hat{X}_{i+j} = 0$, so we may take $F_2 = 0$. Returning to Step 2, for

$n > 2$, we have $U_n = 0$ by induction and therefore $M_n = 0$ as claimed. Moreover, $F_1 M_n - U_n = 0$, so according to Step 3, we may take $F_n = 0$ as claimed.

We have proved parts 1 and 2 of the following:

Theorem A.8. *Let $N \geq 1$. There is a minimal model $(\mathcal{E}_N^\bullet, M_1, M_2, \ldots)$ such that*

1. *if $N = 1$, then $\mathcal{E}_1^\bullet \cong \Bbbk$ (concentrated in degree 0) with $M_2(1 \otimes 1) = 1$ and $M_n = 0$ for $n \neq 2$;*
2. *if $N = 2$, then $\mathcal{E}_2^\bullet \cong \bigoplus_{i=0}^{\infty} \Bbbk\{X_i\}$ with $|X_i| = i$, $M_2(X_{i_1} \otimes X_{i_2}) = X_{i_1+i_2}$, and $M_n = 0$ for $n \neq 2$;*
3. *if $N > 2$, then $\mathcal{E}_N^\bullet \cong \bigoplus_{i=0}^{\infty} \Bbbk\{X_i\}$ with $|X_i| = i$,*

$$M_2(X_{i_1} \otimes X_{i_2}) = \begin{cases} X_{i_1+i_2} & \text{if } i_1 i_2 \text{ is even,} \\ 0 & \text{if } i_1 i_2 \text{ is odd,} \end{cases}$$

$$M_N(X_{i_1} \otimes \cdots \otimes X_{i_N}) = \begin{cases} 0 & \text{if } i_1 \cdots i_N \text{ is even,} \\ -s(N,0) X_{i_1+\cdots+i_N+2-N} & \text{if } i_1 \cdots i_N \text{ is odd,} \end{cases}$$

and $M_n = 0$ if $n \neq 2$ and $n \neq N$.

To prove part 3, we define the following elements of $\mathrm{Hom}^\bullet_{R_N}(P^\bullet_N, P^\bullet_N)$ for $n, q \in \mathbb{Z}_{\geq 0}$:

$$B_n^q := \begin{tikzcd} \cdots \arrow[r] & \overset{n+k}{R_N} \arrow[r] \arrow[d, "s(n,k)x^q"] & \cdots \\ \cdots \arrow[r] & \underset{k}{R_N} \arrow[r] & \cdots \end{tikzcd} \qquad \text{for even } n,$$

$$W_n^q := \begin{tikzcd} \cdots \arrow[r] & \overset{n+k}{R_N} \arrow[r] \arrow[d, "\begin{cases} 0 & \text{if } k \text{ is even,} \\ s(n,k)x^q & \text{if } k \text{ is odd,} \end{cases}"] & \cdots \\ \cdots \arrow[r] & \underset{k}{R_N} \arrow[r] & \cdots \end{tikzcd} \qquad \text{for odd } n.$$

We will use the following properties:

Lemma A.9. *Let n be an even nonnegative integer, let m, m' be odd nonnegative integers, and let q, q' be any nonnegative integers. Then*

1. $\delta(B_n^q) = 0$ *so that* B_n^q *is a cycle,*
2. $\delta(W_m^q) = -B_{m+1}^{q+1}$ *so that* B_n^q *is a boundary for* $n, q > 0$,
3. $m_2(W_m^q \otimes W_{m'}^{q'}) = 0$
4. $m_2(\hat{X}_i \otimes W_{m-i}^q) = W_m^q = m_2(W_{m-i}^q \otimes \hat{X}_i)$ *if* $i \leq m$ *is an even integer.*
5. $m_2(\hat{X}_i \otimes W_{n-i}^q + W_{n-j}^q \otimes \hat{X}_j) = B_n^q$ *if* $i, j \leq n$ *are odd nonnegative integers.*

Proof. 1. Since n is even, k and $n+k$ have the same parity so in the square

$$\begin{tikzcd} \overset{n+k+1}{R_N} \arrow[r] \arrow[d, "s(n,k+1)x^q"] & \overset{n+k}{R_N} \arrow[d, "s(n,k)x^q"] \\ \underset{k+1}{R_N} \arrow[r] & \underset{k}{R_N}, \end{tikzcd}$$

the horizontal arrows are multiplication with the same element (either x or x^{N-1}). Moreover, since n is even, we have $s(n,k+1) = (-1)^{\frac{n(n+1)}{2}} = s(n,k)$ so that the square commutes, proving 1.

2. For even and odd k, respectively, the following squares represent W_m^q at homological degree k:

$$
\begin{tikzcd}
\overset{m+k+1}{R_N} \arrow[r, "x^{N-1}"] \arrow[d, "s(m,k+1)x^q"'] \arrow[dr, dashed] & \overset{m+k}{R_N} \arrow[d, "0"] \\
\underset{k+1}{R_N} \arrow[r, "x"'] & \underset{k}{R_N}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overset{m+k+1}{R_N} \arrow[r, "x"] \arrow[d, "0"'] \arrow[dr, dashed] & \overset{m+k}{R_N} \arrow[d, "s(m,k)x^q"] \\
\underset{k+1}{R_N} \arrow[r, "x^{N-1}"'] & \underset{k}{R_N}.
\end{tikzcd}
$$

The map $\delta(W_n^q)$ is the diagonal arrow obtained from doing ↳ + ⌝ in either of the squares. If k is even, since m is odd, we have $s(m,k+1) = -(-1)^{\frac{m(m+1)}{2}}$. If k is odd, since m is also odd, we have $s(m,k) = -(-1)^{\frac{m(m+1)}{2}}$. Now we have

$$-(-1)^{\frac{m(m+1)}{2}} = -(-1)^{\frac{(m+1)(m+2)}{2}} = -s(m+1,k)$$

since $m+1$ is even. Thus, in either case, the diagonal arrow is multiplication with $-s(m+1,k)x^{q+1}$ and is therefore the homological degree k component of $-B_{m+1}^{q+1}$.

3. If k is even, the homological degree k component of W_m^q is zero. If k is odd, since m is also odd, the homological degree $m+k$ component of $W_{m'}^{q'}$ is zero. Therefore, in either case, the homological degree k component of $m_2(W_m^q \otimes W_{m'}^{q'})$ is zero.

4. This is obvious from the definitions and from (A.4).

5. Denote by φ_k the homological degree k component of an element $\varphi \in \mathrm{Hom}^\bullet_{R_N}(P_N^\bullet, P_N^\bullet)$. We examine $m_2(\hat{X}_i \otimes W_{n-i}^q + W_{n-j}^q \otimes \hat{X}_j)_k$. If k is even, then since $(W_{n-j}^q)_k = 0$, we have $m_2(W_{n-j}^q \otimes \hat{X}_j)_k = 0$. Since $(\hat{X}_i)_k$ is multiplication

with $s(i,k)$ and $(W^q_{n-i})_{k+i}$ is multiplication with $s(n-i,k+i)x^q$, we have that $m_2(\hat{X}_i \otimes W^q_{n-i})_k$ is multiplication with $s(n,k)x^q$ using (A.4). If k is odd, then since $(W^q_{n-i})_{k+i} = 0$, we have $m_2(\hat{X}_i \otimes W^q_{n-i})_k = 0$. Since $(\hat{X}_j)_{k+n-j}$ is multiplication with $s(j,k+n-j)$ and $(W_{n-j})_k$ is multiplication with $s(n-j,k)x^q$, we have that $m_2(W^q_{n-j} \otimes \hat{X}_j)$ is multiplication with $s(n,k)x^q$ using (A.4). $\square$

The proofs of the next four Lemmas are applications of Algorithm 2.7, and taken together, the four Lemmas constitute the proof of part 3 of Theorem A.8. Recall the map U_n appearing in Algorithm 2.7.

Lemma A.10. *If $N > 2$, then*

- $U_2(X_{i_1} \otimes X_{i_2}) = \begin{cases} \hat{X}_{i_1+i_2} & \textit{if } i_1 i_2 \textit{ is even,} \\ B^{N-2}_{i_1+i_2} & \textit{if } i_1 i_2 \textit{ is odd,} \end{cases}$

- $M_2(X_{i_1} \otimes X_{i_2}) = \begin{cases} X_{i_1+i_2} & \textit{if } i_1 i_2 \textit{ is even,} \\ 0 & \textit{if } i_1 i_2 \textit{ is odd.} \end{cases}$

Proof. According to (2.9), we have $U_2(X_{i_1} \otimes X_{i_2}) = m_2(\hat{X}_{i_1} \otimes \hat{X}_{i_2})$. The first part now follows from the definitions and (A.4). The second part follows from Lemma A.9.1. $\square$

Lemma A.11. *Let $N > 2$.*

- *If $2 < n \leq N$, then*

$$U_n(X_{i_1} \otimes \cdots \otimes X_{i_n}) = \begin{cases} 0 & \textit{if } i_1 \cdots i_n \textit{ is even,} \\ -s(n,0)B^{N-n}_{i_1+\cdots+i_n+2-n} & \textit{if } i_1 \cdots i_n \textit{ is odd.} \end{cases}$$

- *If $2 < n < N$, then $M_n = 0$.*

- *If $2 \leq n < N$, then*

$$F_n(X_{i_1} \otimes \cdots \otimes X_{i_n}) = \begin{cases} 0 & \text{if } i_1 \cdots i_n \text{ is even,} \\ -s(n,0)W^{N-n-1}_{i_1+\cdots+i_n+1-n} & \text{if } i_1 \cdots i_n \text{ is odd.} \end{cases}$$

Proof. According to (2.8) and Lemma A.10, we must have have

$$m_1 F_2(X_{i_1} \otimes X_{i_2}) = \begin{cases} 0 & \text{if } i_1 i_2 \text{ is even,} \\ -B^{N-2}_{i_1+i_2} & \text{if } i_1 i_2 \text{ is odd,} \end{cases}$$

so by Lemma A.9.2, we may take F_2 as stated since $s(2,0) = -1$. Assume now that $2 < n \leq N$ and that U_k, M_k, and F_k have been defined as stated in Lemmas A.10 and A.11 for $k < n$. Note that by Lemma A.9.3, we have $m_2(F_i \otimes F_{n-i}) = 0$ unless $i = 1$ or $i = n-1$. We also have $M_s = 0$ for $s < n$ unless $s = 2$. Thus,

$$U_n = m_2(F_1 \otimes F_{n-1} + (-1)^n F_{n-1} \otimes F_1) - F_{n-1} \sum_{t=0}^{n-2} (-1)^{n-t} \operatorname{id}^{\otimes n-2-t} \otimes M_2 \otimes \operatorname{id}^{\otimes t}. \quad \text{(A.12)}$$

We now apply U_n to $X := X_{i_1} \otimes \cdots \otimes X_{i_n}$. We handle four cases.

<u>Case:</u> If i_1 is even, then we have $(F_{n-1} \otimes F_1)(X) = 0$ and $F_{n-1}((\operatorname{id}^{\otimes n-2-t} \otimes M_2 \otimes \operatorname{id}^{\otimes t})(X)) = 0$ unless $t = n-2$. Thus, we have

$$U_n(X) = m_2(F_1(X_{i_1}) \otimes F_{n-1}(X_{i_2} \otimes \cdots \otimes X_{i_n})) - F_{n-1}(X_{i_1+i_2} \otimes X_{i_3} \otimes \cdots \otimes X_{i_n}).$$

If any of $i_2, \ldots, i_n$ is even, then both terms above are zero. If all of $i_2, \ldots, i_n$ are odd, then by Lemma A.9.4 we have

$$\begin{aligned}
U_n(X) &= -s(n-1,0)m_2(\hat{X}_{i_1} \otimes W^{N-n}_{i_2+\cdots+i_n+2-n}) + s(n-1,0)W^{N-n}_{i_1+\cdots+i_n+2-n} \\
&= -s(n-1,0)W^{N-n}_{i_1+\cdots+i_n+2-n} + s(n-1,0)W^{N-n}_{i_1+\cdots+i_n+2-n} \\
&= 0.
\end{aligned}$$

Thus, $U_n(X) = 0$.

<u>Case:</u> If i_n is even, then $U_n(X) = 0$ by an argument analogous to the previous case.

<u>Case:</u> If i_k is even for some $1 < k < n$, then $(F_1 \otimes F_{n-1})(X) = (F_{n-1} \otimes F_1)(X) = 0$ and $F_{n-1}((\mathrm{id}^{\otimes n-2-t} \otimes M_2 \otimes \mathrm{id}^{\otimes t})(X)) = 0$ unless $t = n-k$ or $t = n-k-1$. Therefore,

$$\begin{aligned}
U_n(X) = -(-1)^k F_{n-1}(&X_{i_1} \otimes \cdots \otimes X_{i_{k-2}} \otimes X_{i_{k-1}+i_k} \otimes X_{i_{k+1}} \otimes \cdots \otimes X_{i_n} \\
&- X_{i_1} \otimes \cdots \otimes X_{i_{k-1}} \otimes X_{i_k+i_{k+1}} \otimes X_{i_{k+2}} \otimes \cdots \otimes X_{i_n}).
\end{aligned}$$

If any of $i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n$ is even, then both terms above are zero. If all of $i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n$ are odd, then we have

$$U_n(X) = (-1)^k s(n-1,0)(W^{N-n}_{i_1+\cdots+i_n+2-n} - W^{N-n}_{i_1+\cdots+i_n+2-n}) = 0.$$

Thus, $U_n(X) = 0$.

Case: If $i_1, \ldots, i_n$ are all odd, then $i_k + i_{k+1}$ is even for all $1 \leq k < n$ so that $F_{n-1}((\mathrm{id}^{\otimes n-2-t} \otimes M_2 \otimes \mathrm{id}^{\otimes t})(X)) = 0$. Therefore, by Lemma A.9.5, we have

$$\begin{aligned}
U_n(X) &= -(-1)^n s(n-1,0) m_2(\hat{X}_{i_1} \otimes W^{N-n}_{i_2+\cdots+i_n+2-n} + W^{N-n}_{i_1+\cdots+i_{n-1}+2-n} \otimes \hat{X}_{i_n}) \\
&= -(-1)^n s(n-1,0) B^{N-n}_{i_1+\cdots+i_n+2-n} \\
&= -s(n,0) B^{N-n}_{i_1+\cdots+i_n+2-n},
\end{aligned}$$

where the sign computation follows from the Koszul-Quillen sign conventaion and (A.4). Thus, U_n is as claimed.

If $n < N$, we now have $M_n = 0$ by Lemma A.9.2. According to (2.8), we must have

$$m_1 F_n(X_{i_1} \otimes \cdots \otimes X_{i_n}) = \begin{cases} 0 & \text{if } i_1 \cdots i_n \text{ is even,} \\ s(n,0) B^{N-n}_{i_1+\cdots+i_n+2-n} & \text{if } i_1 \cdots i_n \text{ is odd,} \end{cases}$$

so by Lemma A.9.2, we may take F_n as stated. □

Lemma A.13. *If $N > 2$, then*

- $M_N(X_{i_1} \otimes \cdots \otimes X_{i_N}) = \begin{cases} 0 & \textit{if } i_1 \cdots i_N \textit{ is even,} \\ -s(N,0) X_{i_1+\cdots+i_N+2-N} & \textit{if } i_1 \cdots i_N \textit{ is odd,} \end{cases}$
- $F_N = 0$.

Proof. The first part follows from Lemma A.13 because $B^0_{i_1+\cdots+i_n+2-n} = \hat{X}_{i_1+\cdots+i_n+2-n}$ if $i_1 \cdots i_n$ is odd. The second part is immediate from (2.8) because $F_1 M_N = U_N$. □

Lemma A.14. *Let $N > 2$. If $n > N$, then $U_n = 0$, $M_n = 0$, and $F_n = 0$.*

Proof. Assume $n > N$ and that U_k, M_k, and F_k have been defined as stated in Lemmas A.10 through A.14 for $k < n$. Note that by Lemmas A.9.3, and A.13, we have $m_2(F_i \otimes F_{n-i}) = 0$ unless $i = 1$ or $i = n-1$, and in either of those cases, we still have $m_2(F_i \otimes F_{n-i}) = 0$ by induction. Since $F_{n-1} = 0$ by induction, we have $F_{n-1}(\mathrm{id}^{\otimes n-2-t} \otimes M_2 \otimes \mathrm{id}^{\otimes t}) = 0$ for any t. Since $M_k = 0$ unless $k = 2$ or $k = N$, we have

$$U_n = -F_{n-N+1} \sum_{t=0}^{n-N} (-1)^{n-N-t+Nt} \mathrm{id}^{\otimes n-N-t} \otimes M_N \otimes \mathrm{id}^{\otimes t}.$$

We now apply U_n to $X := X_{i_1} \otimes \cdots \otimes X_{i_n}$. If any of $i_1, \ldots, i_n$ is even, then since both F_{n-N+1} and M_N annihilate any tensor of the X_k where some index is even, we have $U_n(X) = 0$. If $i_1, \ldots, i_n$ are all odd, then for any k with $0 \leq k \leq n-N$, we have

$$M_N(X_{i_{k+1}} \otimes \cdots \otimes X_{i_{k+N}}) = -s(N,0)X_{i_{k+1}+\cdots+i_{k+N}+2-N}.$$

Now $i_{k+1} + \cdots + i_{k+N}$ has the same parity as N, so $i_{k+1} + \cdots + i_{k+N} + 2 - N$ has even parity, and therefore

$$F_{n-N+1}((\mathrm{id}^{\otimes k} \otimes M_N \otimes \mathrm{id}^{\otimes n-N-k})(X)) = 0.$$

Thus, $U_n = 0$.

Now we clearly have $M_n = 0$ and since $F_1 M_n = U_n$, we may take $F_n = 0$ by (2.8). $\square$

A.4. Reconstruction of the filtered subcategory

Finally, we apply Keller's reconstruction to explicitly describe the category of twisted stalks $\operatorname{twst}\mathcal{E}_N$ which is equivalent to $\mathcal{F}_N$—in this case, the category of finite-dimensional R_N-modules. We shall handle the cases $N = 1$, $N = 2$, and $N \geq 3$ separately, but they have some features in common. Since the A_∞-category $\mathcal{E}_N$ has only one object, a twisted stalk over $\mathcal{E}_N$ is simply the data of a strictly upper-triangluar square matrix δ with entries in $\mathcal{E}_N^1$ such that

$$\sum_{t=1}^{\infty}(-1)^{\frac{t(t-1)}{2}}M_t(\delta^{\otimes t}) = 0. \tag{A.15}$$

If the matrices δ and δ' have size n and n' a morphism $f : \delta \to \delta'$ is an $n' \times n$ matrix with entries in $\mathcal{E}_N^0$ such that

$$\sum_{s,t\geq 0}(-1)^{\frac{(s+t)(s+t-1)}{2}+s}M_{s+t+1}((\delta')^{\otimes s} \otimes f \otimes \delta^{\otimes t}) = 0. \tag{A.16}$$

The composition of two morphisms $f : \delta \to \delta'$ and $g : \delta' \to \delta''$ is given by

$$gf = \sum_{r,s,t\geq 0}(-1)^{\frac{(r+s+t)(r+s+t-1)}{2}+s}M_{r+s+t+2}((\delta'')^{\otimes r} \otimes g \otimes (\delta')^{\otimes s} \otimes f \otimes \delta^{\otimes t}). \tag{A.17}$$

Since $\mathcal{E}_1$ is concentrated in degree zero, the twisted stalks over $\mathcal{E}_1$ are the square zero matrices of various sizes, the condition (A.15) being trivially satisfied. Denoting by 0_n the square zero matrix of size n (for $n \in \mathbb{Z}_{\geq 0}$), a morphism from 0_n to $0_{n'}$ is therefore an arbitrary $n' \times n$ matrix f with entries in $\mathcal{E}_1^0 \cong \mathbb{k}$ since (A.16) is also trivially satisfied. Finally, if $f : 0_n \to 0_{n'}$ and $g : 0_{n'} \to 0_{n''}$, then their composition is simply given by matrix multiplication since $M_k = 0$ for $k \neq 2$. We therefore recover the category whose object set is (in bijection with) $\mathbb{Z}_{\geq 0}$ and whose morphisms from n to n' are given by the set of $n' \times n$ matrices over $\Bbbk$. This is easily recognizable as equivalent to the category of finite-dimensional vector spaces over $\Bbbk$, which is precisely $\mathcal{F}_1$.

For $N \geq 2$, since $\mathcal{E}_N^1 = \Bbbk\{X_1\}$, we shall denote a twisted stalk over $\mathcal{E}_N$ by δX_1 where δ is a strictly upper-triangular square matrix with entries in $\Bbbk$. Let δX_1 and $\delta' X_1$ be twisted stalks with δ of size n and δ' of size n'. Since $\mathcal{E}_2^0 = \Bbbk\{X_0\}$, we shall denote a morphism from δX_1 to $\delta' X_1$ by $f X_0$, where f is an $n' \times n$ matrix with entries in $\Bbbk$.

We now handle the case $N = 2$. Let δX_1 be a twisted stalk over $\mathcal{E}_2$. Since $M_k = 0$ for $k \neq 2$ and $M_2(X_1 \otimes X_1) = X_2$, the condition (A.15) reduces to $\delta^2 = 0$. Let $f X_0$ be a morphism from δX_1 to $\delta' X_1$. Since $M_2(X_1 \otimes X_0) = M_2(X_0 \otimes X_1) = X_1$, the condition (A.16) reduces to $\delta' f = f \delta$. If $g X_0 : \delta' X_1 \to \delta'' X_1$ is another morphism, then since $M_2(X_0 \otimes X_0) = X_0$, the composition $(g X_0)(f X_0)$ is given from (A.17) as $(gf) X_0$ (where gf denotes usual matrix multiplication). We claim that there is an equivalence F from the category of twisted stalks over $\mathcal{E}_2$ to $\mathcal{F}_2$ (the category of finite-dimensional $\Bbbk[x]/(x^2)$-modules) taking the twisted stalk δX_1 (of size n) to the module which is equal as a vector space to $\Bbbk^n$ where x acts as the matrix δ. The proposed equivalence is full and faithful because the only condition on morphisms, $\delta' f = f \delta$, becomes $x F(f) = F(f) x$. It is essentially surjective because every (square) nilpotent matrix is similar to a strictly upper-triangular one.

We now handle the case $N \geq 3$. Let δX_1 be a twisted stalk over $\mathcal{E}_N$. Since $M_k = 0$ unless $k = 2$ or $k = N$, $M_2(X_1 \otimes X_1) = 0$, and $M_N(X_1^{\otimes N}) = \pm X_2$, the condition (A.15) reduces to $\delta^N = 0$. Let $f X_0$ be a morphism from δX_1 to $\delta' X_1$. Since $M_N(X_1^{\otimes s} \otimes X_0 \otimes X_1^{\otimes t}) = 0$ (for $s + t + 1 = N$) and $M_2(X_1 \otimes X_0) = M_2(X_0 \otimes X_1) = X_1$, the condition (A.16) reduces to $\delta' f = f \delta$. If $g X_0 : \delta' X_1 \to \delta'' X_1$

is another morphism, then since $M_N(X_1^{\otimes r} \otimes X_0 \otimes X_1^{\otimes s} \otimes X_0 \otimes X_1^{\otimes t}) = 0$ (for $r + s + t + 2 = N$) and $M_2(X_0 \otimes X_0) = X_0$, the composition $(gX_0)(fX_0)$ is given from (A.16) as $(gf)X_0$ (where gf denotes usual matrix multiplication). We claim that there is an equivalence F from the category of twisted stalks over $\mathcal{E}_N$ to $\mathcal{F}_N$ (the category of finite-dimensional $\Bbbk[x]/(x^N)$-modules) taking the twisted stalk δX_1 (of size n) to the module which is equal as a vector space to $\Bbbk^n$ where x acts as the matrix δ. The proposed equivalence is full and faithful because the only condition on morphisms, $\delta' f = f\delta$, becomes $xF(f) = F(f)x$. It is essentially surjective because every (square) nilpotent matrix is similar to a strictly upper-triangular one.

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