THE DISTRIBUTION OF THE CUSPED HYPOCYCLOIDAL MAHLER MEASURE

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DISSERTATION ABSTRACT

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We explore *generalized Mahler measures* associated to regions in the complex plane. These generalized Mahler measures describe the complexity of polynomials in $\mathbb{C}[x]$ by comparing the geometry of their roots to compact subsets of $\mathbb{C}$. Citing past work connecting the Mahler measure to the unit disk and the reciprocal Mahler measure to the interval $[-2, 2]$, we explore a family of *cusped hypocycloidal Mahler measures* $\mu^{(N)}$ associated to the $(N + 1)$-cusped hypocycloids, using potential theory to show how a generalized Mahler measure may be constructed from Jensen's formula.

Let s be a complex variable, and d a positive integer. To every generalized Mahler measure Φ we define the *complex moment function* $H_d(\Phi; s)$ which provides information about the range of values Φ takes on degree d polynomials in $\mathbb{C}[x]$. These functions are analytic in the half-plane $\Re(s) > d$. We will show how $H_d(s)$ may be represented as the determinant of a Gram matrix in a Hilbert space determined by Φ and s. We thus discover properties of $H_d(\mu^{(N)}; s)$ as a rational function of s.

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CHAPTER I

INTRODUCTION

Mahler Measure

Definition 1. *The* Mahler measure $\mu(p)$ *of a polynomial* p *with leading coefficient* a *and zeroes* $\alpha_1, \alpha_2, ..., \alpha_n$ *is defined by* $\mu(p) = |a| \prod_{i=1}^{n} \max\{1, |\alpha_i|\}$.

As a simple example, the Mahler measure of any cyclotomic polynomial is 1. The Mahler measure is often used as a measure of complexity of polynomials.

Jensen's Formula

For any $\alpha \in \mathbb{C}$ and $r > 0$,

$$\int_0^1 \log |re^{2\pi i\theta} - \alpha| \, d\theta = \log \max\{r, |\alpha|\}.$$

This statement is called Jensen's formula. By Jensen's formula, the Mahler measure corresponds to the geometric mean around the unit circle:

$$\mu(p) = \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \log |p(e^{i\theta})| \, d\theta \right).$$

These two representations highlight the Mahler measure's role as a measure of complexity of polynomials; we have a contrast between the roots or the coefficients of the polynomial.

Lehmer's Problem

The Mahler measure has been the subject of significant study, especially regarding the unsolved problem known as Lehmer's conjecture [1].

Unsolved problem 1. (Lehmer's problem, 1933) *Does there exist an $\epsilon > 0$ such that if $f(x)$ is an irreducible, non-cyclotomic polynomial in $\mathbb{Z}[x]$, then $\mu(f) > 1 + \epsilon$?*

The best known lower bound for a nontrivial Mahler measure is $\mu(p) = 1.17628...$, with one polynomial satisfying this being Lehmer's polynomial $P(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$; however, it remains open whether this is a true lower bound.

C. Smyth [2] provided a partial solution in the case of non-reciprocal polynomials. A polynomial of degree n is *reciprocal* if $f(x) = x^n p(1/x)$ for some polynomial $p(x) \neq x - 1$. Smyth showed that if f is an irreducible, non-reciprocal polynomial then $\mu(f) \geq \mu(x^3 - x - 1) = 1.32....$ Lehmer's problem can thus be considered in terms of reciprocal polynomials.

This thesis does not focus on Lehmer's problem; this conjecture is referenced here because it gives context to much of past exploration of the Mahler measure. Rather, we focus here on distributions of Mahler measures, which was first explored by S-J. Chern and J. Vaaler in [3]; C. Sinclair continued this line of work in [4], with a particular focus on the *reciprocal Mahler measure* μ_1 [5] defined by $\mu_1(f) = \mu(f(x + 1/x))$.

Volume Calculations

For positive integer d, we may consider the Mahler measure restricted to the set of polynomials of degree at most d; specifically, one may view a polynomial of degree at most d as a vector in $\mathbb{C}^{d+1}$ based on its coefficients, and thus view μ as a function on $\mathbb{C}^{d+1}$. The Mahler measure satisfies most axioms of a vector norm, all except the triangle inequality, so we can construct a set akin to the unit ball. We shall call this set the degree d *complex star body* of μ, and denote it

$$\mathcal{V}_d = \{\mathbf{a} \in \mathbb{C}^{d+1} : \mu(\mathbf{a}) \leq 1\}.$$

We may similarly define the degree d *real star body* of μ as

$$\mathcal{U}_d = \{\mathbf{a} \in \mathbb{R}^{d+1} : \mu(\mathbf{a}) \leq 1\};$$

however, this thesis will focus primarily on the former.

Chern and Vaaler [3] cleverly utilized a Mellin transform to determine the volumes of $\mathcal{U}_d$ and $\mathcal{V}_d$; they considered a *monic* Mahler measure $\tilde{\mu} : \mathbb{C}^d \to [1, \infty)$, where $\tilde{\mu}(\mathbf{b})$ is defined as the Mahler measure of the monic polynomial whose non-leading coefficients are the entries of $\mathbf{b}$. Writing λ_d and λ_{2d} for the Lebesgue measures on $\mathbb{R}^d$ and $\mathbb{C}^d$, respectively, we may define *distribution functions*:

$$f_d(\xi) := \lambda_d\{\mathbf{b} \in \mathbb{R}^d : \tilde{\mu}(\mathbf{b}) \leq \xi\};$$
$$h_d(\xi) := \lambda_{2d}\{\mathbf{b} \in \mathbb{C}^d : \tilde{\mu}(\mathbf{b}) \leq \xi\}.$$

These functions contain information on the range of values of the Mahler measure on the set of monic polynomials of degree d, from $\mathbb{R}[x]$ and $\mathbb{C}[x]$

respectively. The *Mellin transforms* of these are given by

$$\widehat{f_d}(s) = \int_0^\infty \xi^{-s-1} f_d(\xi)\, d\xi,$$
$$\widehat{h_d}(s) = \int_0^\infty \xi^{-s-1} h_d(\xi)\, d\xi,$$

where s is a complex variable. For $\Re(s) > d$ these integrals converge to an analytic function, thus encoding information on the range of values of the Mahler measure of monic polynomials of degree d into analytic functions. As it turns out,

$$\mathrm{vol}(\mathcal{U}_d) = \lambda_{d+1}(\mathcal{U}_d) = 2\widehat{f_d}(d+1);$$
$$\mathrm{vol}(\mathcal{V}_d) = \lambda_{2d+2}(\mathcal{V}_d) = 2\pi\widehat{h_d}(2d+2).$$

These values can then be computed using a change of variables:

$$\widehat{f_d}(s) = \frac{1}{s}\int_{\mathbb{R}^d} \tilde{\mu}(\mathbf{b})^{-s}\, d\lambda_d(\mathbf{b});$$
$$\widehat{h_d}(s) = \frac{1}{2s}\int_{\mathbb{R}^d} \tilde{\mu}(\mathbf{b})^{-2s}\, d\lambda_{2d}(\mathbf{b}).$$

We will define functions

$$F_d(s) := \int_{\mathbb{R}^d} \tilde{\mu}(\mathbf{b})^{-s} d\lambda_d(\mathbf{b});$$
$$H_d(s) := \int_{\mathbb{C}^d} \tilde{\mu}(\mathbf{b})^{-2s} d\lambda_{2d}(\mathbf{b}),$$

respectively, the real and complex *moment functions* of μ, which similarly to f_d and h_d converge on $\Re(s) > d$. A second change of variables allows for integration over root vectors of the polynomials instead of coefficient vectors. Chern and Vaaler then showed that both $F_d(s)$ and $H_d(s)$ analytically continue to rational functions with simple poles at positive integers and high multiplicity roots at the origin:

Theorem 1. (Chern, Vaaler) *Let J be the integer part of $d/2$; then*

$$H_d(s) = \frac{\pi^d}{d!}\prod_{n=1}^{d}\frac{s}{s-n};$$
$$F_d(s) = \left(2^d\prod_{j=1}^{J}\left(\frac{2j}{2j+1}\right)^{d-2j}\right)\prod_{j=0}^{J-1}\frac{s}{s-(d-2j)}.$$

Notably, and perhaps surprisingly, this means that the volume of $\mathcal{U}_d$ is a rational number, and the volume of $\mathcal{V}_d$ is a rational number times π^d. Further, since H_d and F_d are built from Mellin transforms of h_d and f_d, we can recover explicit formulae for f_d and h_d from the Mellin inversion formula. Thus, Chern and Vaaler showed that f_d and h_d are polynomials of degree d, with respectively rational coefficients and rational coefficients times π^d.

Sinclair [4] showed that these volumes as special values of Mellin transforms can be applied to *generalized Mahler measures* Φ. Given a function Φ on polynomials, we may consider star bodies $\mathcal{U}_d(\Phi)$ and $\mathcal{V}_d(\Phi)$, distribution functions $f_d(\Phi;\xi)$ and $h_d(\Phi;\xi)$, and moment functions $F_d(\Phi;s)$ and $H_d(\Phi;s)$. Fascinatingly, many of the properties of the volume calculations above continued on to other cases Sinclair considered. To start, taking μ_1 as the reciprocal Mahler measure, Sinclair found that $H_d(\mu_1;s)$ has an analytic continuation to a rational function of s:

Theorem 2. (Sinclair,[5])

$$H_d(\mu_1;s) = (2\pi)^d\prod_{n=1}^{d}\frac{s}{s^2-n^2}.$$

Much like $H_d(\mu; s)$, $H_d(\mu_1; s)$ has poles at integers, and the origin as a root of multiplicity d. Similarly to above, a corollary of this is that $h_d(\mu_1; \xi)$ is a reciprocal Laurent polynomial of degree d.

Sinclair then considered a family of generalized Mahler measures μ_q defined by $\mu_q(f) = \mu(f(x + q/x))$, for $q \in [0, 1]$, as well as discussing the possiblility of exploring other generalized Mahler measures of the form $\Phi(f) = \mu(f \circ F)$ for some Laurent polynomial F [4].

Theorem 3. (Sinclair) *If $q \in [0, 1]$, then $H_d(\mu_q; s)$ analytically continues to the rational function of s given by*

$$H_d(\mu_q; s) = \frac{\pi^d s^d}{d!} \prod_{n=1}^{d} \frac{(1 - q^{2n})s + (1 + q^{2n})n}{s^2 - n^2}.$$

Note that μ_0 is simply the Mahler measure, and μ_1 is the reciprocal Mahler measure; thus, for $q \in [0, 1]$, $H_d(\mu_q; s)$ serves as a "path" of moment functions between $H_d(\mu; s)$ and $H_d(\mu_1; s)$. Notable for the purposes of this computation, there is a link between these μ_q and a family of ellipses deforming the unit circle to the interval $[-2, 2]$.

The key to finding $H_d(\Phi; s)$ to be a rational function of s was the realization that the moment function can be written as the determinant of a matrix in a Hilbert space associated to Φ. This determinant arises from a Gram matrix, allowing $H_d(\Phi; s)$ to be considered as the volume of a parallelepiped in the associated Hilbert space.

To this end, define the complex measure $\nu = \nu(\Phi)$ on $\mathbb{C}$ by $d\nu(\alpha) = \varphi(\alpha)^{-s}\varphi(\bar{\alpha})^{-s} d\lambda_2(\alpha)$, where $\varphi : \mathbb{C} \to (0, \infty)$ is a *root function* associated with

Φ. Then $L^2(\nu)$ is a Hilbert space equipped with inner product

$$\langle f, g\rangle = \int_{\mathbb{C}} \varphi(\alpha)^{-2s} f(\alpha)\overline{g(\alpha)} d\lambda_2(\alpha)$$

for $f, g \in L^2(\nu)$, along with a norm $\mathfrak{N}(f)^2 = \mathfrak{N}(f; s)^2 := \langle f, f\rangle$. For $\Re(s) > d$, we can see that any polynomial with degree less than d is in $L^2(\nu)$.

Now, let $Q = \{Q_n(\alpha) : n = 1, 2, ..., d\}$ be a set of monic polynomials in $\mathbb{C}[x]$ with $\deg(Q_n) = n - 1$; we call such a set a *complete family of polynomials*. Each Q_n is in $L^2(\nu)$, and Q spans a parallelepiped in the Hilbert space. The Gram matrix of Q is then the $d \times d$ matrix whose ℓ, m entry is $\langle Q_\ell, Q_m\rangle$. This is then a symmetric matrix depending on Q, Φ, and s.

Theorem 4. (Sinclair, [4]) *Let Q be a complete family of monic polynomials. Then $H_n(s) = \det(W_Q)$.*

N-Cusped Hypocycloids

This thesis will consider generalized Mahler measures for the family of shapes known as the *(n-cusped) hypocycloids*. For $n \in \mathbb{Z}$, the n-cusped hypocycloid is most commonly defined in terms of rolling circles: suppose a circle is inscribed inside another circle with radius n-times that of the inner circle. Choose a point on the boundary of the inner circle, then allow the inner circle to roll along the boundary from inside the outer circle; the curve traced by the chosen point forms a simple closed loop, called the n-cusped hypocycloid. See Figure 1 for a 3-cusped hypocycloid and Figure 2 for a 5-cusped hypocycloid.

Hypocycloids also have a description in terms of conformal and quasi-conformal maps on the complex plane. The map $z \mapsto z + \dfrac{z^{-N}}{N}$ is a conformal

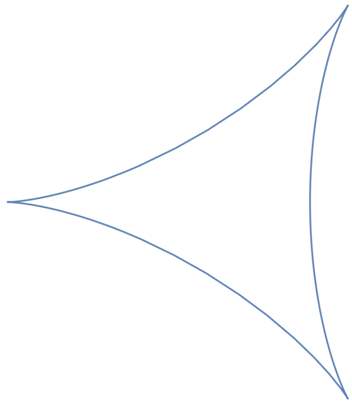

FIGURE 1 A rendition of the 3-cusped hypocycloid.

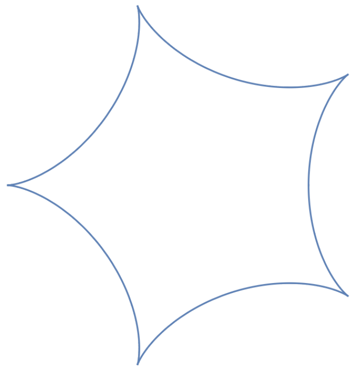

FIGURE 2 A rendition of the 5-cusped hypocycloid.

map sending points on the exterior of the closed unit disk to the exterior of the region enclosed by the $(N+1)$-cusped hypocycloid. Likewise, the map $z \mapsto \dfrac{\bar{z}^N}{N}$ is a quasiconformal map between the closed unit disk and the interior of the hypocycloid [8].

Of interesting note, the 2-cusped hypocycloid corresponds to the real interval $[-2, 2]$. This has important connections to reciprocal polynomials, and thus by Smyth [2] relates to potential solutions to Lehmer's conjecture; this prompted Sinclair's study of the reciprocal Mahler measure [5]. Under the rolling circles definition, the unit circle may similarly be considered the 1-cusped hypocycloid.

We can thus see that for the first two levels of cusps, $H_d(s)$ has integer poles $\leq d$ (with the 2-cusped case adding negative poles that the 1-cusped did not have) and the origin as a root of high multiplicity.

The question arose, then, what happens with the $H_d(s)$ corresponding to higher cusped hypocycloids? Does it maintain the high multiplicity roots at the origin, and where are the poles?

This thesis explores generalized Mahler measures corresponding to the cusped hypocycloids: we define the *Nth cusped hypocycloidal Mahler measure* by $\mu^{(N)}(f) := \mu(f(x + \frac{x^{-N}}{N}))$, for $N > 1$.

Theorem 5. (Main Theorem) *For $N > 1$ an integer, $H_d(\mu^{(N)}; s)$ analytically continues to a rational function of s, which is π^d times a rational function with rational coefficients, has nonzero integer poles, the origin as a root of multiplicity d, and all other roots are negative. Further, the numerator and denominator have matching degrees, bounded by $(d-1)(N+1)$.*

One can see that $H_d(\mu^{(N)}; s)$ indeed continues many of the properties of $H_d(\mu_q; s)$, suggesting that these might be continued for other generalized Mahler measures explored in the future. In addition, it is not hard to calculate $H_d(\mu^{(N)}; s)$ for specific choices of N and d, and thus we can calculate $\mathrm{vol}(\mathcal{V}_d)$, as well as other volumes we will discuss later; see Table 1 for some particular results.

Counting Points of Bounded Height

This thesis has applications of the asymptotics to $H_d(\mu^{(N)}; s)$. Schanuel [**?**] pioneered a study of counting algebraic numbers of bounded height through the asymptotics of a height function; Masser and Vaaler [**?**] explored how this concept may be applied to counting polynomials using the Mahler measure as a height on

TABLE 1 Sample calculations of $\mathrm{vol}(\mathcal{V}_d)$ for various values of $N+1$ cusps and polynomial degrees d

$(N+1)$ vs. d	2	3	4	5
3 cusps	$\frac{3}{28}\pi^3$	$\frac{262}{10395}\pi^4$	$\frac{18125}{1437696}\pi^5$	$\frac{311481}{2896974080}\pi^6$
4 cusps	$\frac{7}{60}\pi^3$	$\frac{5104}{184275}\pi^4$	$\frac{18063359375}{3341114297136}\pi^5$	$\frac{2272135257604}{3295289986463475}\pi^6$
5 cusps	$\frac{1}{8}\pi^3$	$\frac{67}{2112}\pi^4$	$\frac{41758354375}{6744887525376}\pi^5$	$\frac{63792691434842763}{61482997256500019200}\pi^6$

polynomials. These counting problems may be extended to our generalized Mahler measures.

First, we may calculate the volume of monic polynomials as the limit of $H_d(s)$ as $s \to \infty$; see 2 for some examples of the asymptotics of $H_d(\mu^{(N)}; s)$.

Further, the volumes of star bodies approximately gives the number of lattice points in the region, which corresponds to polynomials with Gaussian integer coefficients. Consider $\eta_d(T) = \#\{f \in \mathbb{Z}[i][x] : \mu^{(N)}(f) \leq T, \deg(f) \leq d\}$. Then, by the geometry of numbers (under some assumptions regarding the boundary of the star body),

$$\eta_d(T) \sim \mathrm{vol}(\mathcal{V}_d)T^{d+1} + O(T^d)$$

as $T \to \infty$. Similarly, if $\tilde{\eta}_d(T)$ is the monic version of $\eta_d(T)$, then $\eta_d(T) \approx h_d(T)$ as $T \to \infty$. In this way h_d, $\mathrm{vol}(\mathcal{V}_d)$, and H_d capture information for the counting of Gaussian integer polynomials of bounded degree and bounded height for our generalized Mahler measures.

TABLE 2 Asymptotics of $H_d(\mu^{(N)}; s)$ for various degree d and cusps $N+1$

$(N+1)$ vs. d	2	3	4	5
3 cusps	$\frac{1}{4}\pi^2$	$\frac{7}{192}\pi^3$	$\frac{49}{12288}\pi^4$	$\frac{1617}{5242880}\pi^5$
4 cusps	$\frac{5}{18}\pi^2$	$\frac{65}{1458}\pi^3$	$\frac{24245}{4251528}\pi^4$	$\frac{935857}{13947137604}\pi^5$
5 cusps	$\frac{5}{16}\pi^2$	$\frac{695}{12288}\pi^3$	$\frac{375995}{50331648}\pi^4$	$\frac{2756870539}{3298534883328}\pi^5$

Structure of this thesis

In chapter 2, we cover key theorems of potential theory that enable our discussion on generalized Mahler measures. In chapter 3, we discuss in depth the results regarding $H_d(s)$ found first by Chern and Vaaler, then by Sinclair. In chapter 4, we discuss results pertaining to the hypocycloidal Mahler measure, along with lines of further study stemming from this work.

CHAPTER II

GENERALIZED MAHLER MEASURES

We return to the representation of the Mahler measure provided by Jensen's formula:

$$\mu(f) = \exp\left(\int_0^1 \log|f(e^{2\pi i\theta})|d\theta\right).$$

While the root definition of the Mahler measure is defined on polynomials, this form may be more generically applied to Laurent polynomials, since it recovers the Mahler measure from the coefficients rather than the roots. Further, we can use this form to consider *generalized Mahler measures*: for particular choices of measure ν on $\mathbb{C}$, we will explore

$$\Phi(f) = \exp\left(\int_{\mathbb{C}} \log|f(z)|\, d\nu(z)\right).$$

Using Jensen's formula, we will likewise develop *root functions* $\varphi : \mathbb{C} \to (0, \infty)$ such that

$$\Phi(f) = |a| \prod_{n=1}^{d} \varphi(\alpha_n),$$

where f is a degree d polynomial with leading coefficient a.

Developing these choices of ν and φ will come from an exploration of potential theory. Everything in the following section is background, and already well established; we will focus here on the definitions and theorems directly important to our discussion, though we will mention some well established theorems when necessary. For a more thorough discussion of potential theory, Ransford

[6] gives an excellent discussion on the subject. We will end this chapter with a discussion on how potential theory impacts this thesis.

Potential Theory

Subharmonic and Harmonic Functions

Before discussing potentials, we must first introduce subharmonic functions.

Definition 2. *Let $U \subset \mathbb{C}$ be an open set, and let $u : U \to [-\infty, \infty)$ be a function which is not identically $-\infty$; u is* upper semicontinuous *at $\alpha \in U$ if*

$$\limsup_{z \to \alpha} u(z) \leq u(\alpha).$$

If u is upper semicontinuous at every $\alpha \in U$, we say it is upper semicontinuous on U.

Importantly, if u is upper semicontinuous on U and K is a compact subset of U, then u is bounded above and attains it maximum on K.

If there exists $\rho = \rho(\alpha) > 0$ such that, for $0 \leq r < \rho$,

$$u(\alpha) \leq \frac{1}{2\pi} \int_0^{2\pi} u(\alpha + re^{i\theta})\, d\theta,$$

then u satisfies the *submean inequality* at α. The submean inequality gives that $f(\alpha)$ is smaller than the average value taken by f along the boundary of a disk of sufficiently small radius around α. To understand how to think of it, the submean inequality is analagous to the property of locally convex functions: if $f : (a, b) \to [-\infty, \infty)$ is locally convex at $x \in (a, b)$, then there exists $r > 0$ such that for

$0 \le \epsilon < r$,

$$f(x) \le \frac{f(x-\epsilon)+f(x+\epsilon)}{2},$$

and f is smaller than the average value of f taken on an interval of sufficiently small length around x.

We come now to subharmonic functions:

Definition 3. *If u is upper semicontinuous and satisfies the submean inequality at every $\alpha \in U$, then u is* subharmonic *on U. If both u and $-u$ are subharmonic on U, we may call u* harmonic *on U.*

Notably, harmonic functions are continuous and have equality for the submean inequality.

Potentials and Logarithmic Potentials

Definition 4. *If ν is a finite Borel measure on $\mathbb{C}$ with compact support K, its* potential *is the function $p_\nu : \mathbb{C} \to [-\infty, \infty)$ with*

$$p_\nu(z) := \exp \int_K \log |z-w| \, d\nu(w),$$

for $z \in \mathbb{C}$.

Importantly, the potential p_ν is subharmonic on $\mathbb{C}$. As we will explain later, this name arises as an analog of the potential energy of a physical system. At times, it will be easier to work instead with the *logarithmic potential* of ν, given by $\log p_\nu$; in fact, some texts refer to the logarithmic potential simply as the potential. We shall start with a proof that the logarithmic potential is subharmonic.

Theorem 6. *Let ν be a finite Borel measure on $\mathbb{C}$ with compact support K; then* $\log p_\nu$ *is subharmonic on $\mathbb{C}$ and harmonic on $\mathbb{C}\backslash K$.*

Proof. First, note that subharmonicity is a local condition, so it suffices to show that $\log p_\nu$ is subharmonic on every relatively compact open set $U \subset \mathbb{C}$.

Define $v : \mathbb{C} \times \mathbb{C} \to [-\infty, \infty)$ by $v(\alpha, w) = \log|w - \alpha|$. It may be seen that v is subharmonic in each variable, and is thus also upper semicontinuous in each variable. Thus there exists some c such that $v(\alpha, w) < c$ on $\bar{U} \times K$. Now, since $v(\alpha, w) - c$ is negative on $\bar{U} \times K$, by Fatou's lemma it follows that

$$\begin{aligned}\limsup_{z\to\alpha} \log p_\nu(z) - c &= \limsup_{z\to\alpha} \int_K v(z,w) - c\, d\nu(w) \\ &\leq \int_K \limsup_{z\to\alpha} v(z,w) - c\, d\nu(w) \\ &= \log p_\nu(\alpha) - c.\end{aligned}$$

Thus, $\log p_\nu$ is upper semicontinuous.

For each $\alpha \in U$, there exists $\rho > 0$ such that for $0 \leq r < \rho$,

$$\log|w - \alpha| \leq \frac{1}{2\pi} \int_0^{2\pi} \log|w - \alpha + re^{i\theta}| d\theta,$$

with equality when $w \neq \alpha$. Thus, for $0 \leq r < \rho$,

$$\begin{aligned}\log p_\nu(\alpha) &= \int_{\mathbb{C}} \log|w - \alpha| d\nu(w) \\ &\leq \frac{1}{2\pi} \int \mathbb{C} \int_0^{2\pi} \log|w - \alpha + re^{i\theta}| d\theta\, d\nu(w) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{C}} \log|w - \alpha + re^{i\theta}| d\nu(w)\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log p_\nu(\alpha + re^{i\theta}) d\theta\end{aligned}$$

so that $\log p_\nu$ satisfies the submean inequality. Note that equality here holds if $\alpha \notin K$. Thus, $\log p_\nu$ is subharmonic on $\mathbb{C}$ and harmonic on $\mathbb{C}\backslash K$. $\square$

Lemma 1. *Let ν be a Borel probability measure on $\mathbb{C}$ with compact support K; then $p_\nu \sim |\alpha|$ as $|\alpha| \to \infty$.*

Proof. First,

$$\begin{aligned} p_\nu(\alpha) &= \exp\left\{\int_K \log|\alpha - w|\, d\nu(w)\right\} \\ &= \exp\left\{\log|\alpha| + \int_K \log|1 - w/\alpha|\, d\nu(w)\right\} \\ &= |\alpha|\, \exp\left\{\log \int_K \log|1 - w/\alpha|\, d\nu(w)\right\}. \end{aligned}$$

Thus,

$$\lim_{|\alpha|\to\infty} \frac{p_\nu(\alpha)}{|\alpha|} = \lim_{|\alpha|\to\infty} \exp\left\{\log|\alpha| + \int_K \log|1 - w/\alpha|\, d\nu(w)\right\}.$$

Setting $c = \sup_{w\in K}|w|$, we see that for all $w \in K$

$$\log|1 - w/\alpha| \leq \log(1 + c/|\alpha|).$$

Now, if $|\alpha| > c$, then $\log|1 - w/\alpha| \leq 2$, so by dominated convergence theorem,

$$\lim_{|\alpha|\to\infty} \exp\left\{\int_K \log|1 - w/\alpha|\, d\nu(w)\right\} = 1.$$

Thus, $p_\nu \sim |\alpha|$ as $|\alpha| \to \infty$. $\square$

We continue with a value which proves critical for finding our ideal choice of measures.

Definition 5. *If ν is a finite Borel measure on $\mathbb{C}$ with compact support K, its* energy $I(\nu)$ *is*

$$I(\nu) := \int_K \log p_\nu(z)\, d\nu(z) = \int_K \int_K \log |z-w|\, d\nu(w)\, d\nu(z).$$

To understand the origin of the term "energy", consider ν as a distribution of electric charges in $\mathbb{C}$; then $p_\nu(z)$ is the potential energy felt by a particle at point z, making $I(\nu)$ the total energy of the system of charges.

Much as it helps to consider $\log p_\nu$ in place of p_ν at times, we will likewise consider a parallel value to the energy.

Definition 6. *Let $\mathcal{P}(K)$ be the set of Borel probability measures supported on K. The* capacity *of K is the quantity*

$$c(K) = \exp\left(\sup_{\nu\in\mathcal{P}(K)} I(\nu)\right).$$

Capacity zero sets fulfill a similar role of "negligible" sets in potential theory as sets of measure zero do in measure theory; in fact, capacity zero sets have measure zero.

We arrive now at the way to choose probability measures to produce generalized Mahler measures.

Definition 7. *Let $K \subset \mathbb{C}$ be a compact set, and consider $\mathcal{P}(K)$ the collection of all Borel probability measures on K. If there exists $\nu_K \in \mathcal{P}(K)$ with*

$$I(\nu_K) = \sup_{\nu \in \mathcal{P}(K)} I(\nu)$$

then ν_K is called an equilibrium measure *for K. The corresponding potential $p_{\nu_K} = p_K$ is called an* equilibrium potential *for K.*

As a first example, we can quickly see that if ν is an equilibrium measure corresponding to the unit disc, then Jensen's formula gives that the Mahler measure may be written:

$$\mu(f) = \exp\left(\int_{\mathbb{C}} \log|f(z)|\, d\nu(z)\right).$$

We now may give a rigorous definition for these generalized Mahler measures:

Definition 8. *Let $K \subset \mathbb{C}$ be a compact set with equilibrium measure ν; the* generalized Mahler measure over K *is*

$$\Phi(f) := \exp\left(\int_{\mathbb{C}} \log|f(z)|\, d\nu(z)\right).$$

We continue with additional properties of potential theory to understand how to work with these generalized Mahler measures.

Theorem 7. (*Frostman's Theorem) Let K be a compact set in $\mathbb{C}$ with $c(K) > 0$, and suppose the equilibrium potential p_K is continuous; then $p_K \geq c(K)$ on $\mathbb{C}$, and $p_K = c(K)$ on K.*

We shall make great use of Frostman's theorem for simplifying certain calculations in chapter 4.

Proof. We will work with the logarithmic potential to show that $\log p_K \geq I(\nu_K)$ on $\mathbb{C}$ and $\log p_K - I(\nu_K)$ on K. Since we assumed $c(K) > 0$, we have $I(\nu_K) > -\infty$.

For $n \geq 1$, set $U_n = \{z \in K : \log p_\nu(z) > I(\nu) + 1/n\}$, and call $K_n = \overline{U_n}$. We will show that U_n is empty. Assume there is some $n \geq 1$ such that U_n is nonempty; notice that the corresponding K_n must have positive Lebesgue measure, so $c(K_n) > 0$ as well. We thus may find $\nu \in \mathcal{P}(K_n)$ such that $I(\nu) > -\infty$. By definition, $I(\nu_K) = \int_K \log p_K(z) d\nu_K(z)$, so there exists some $z_1 \in \operatorname{supp} \nu_K$ such that $\log p_K(z_1) \leq I(\nu_K)$. By upper semicontinuity there exists $r_1 > 0$ such that for $z \in \bar{\Delta}(z_1, r_1)$,

$$\log p_K(z) < I(\nu_K) + \frac{1}{2n}.$$

Now, $\bar{\Delta}(z_1, r_1) \cap K_n$ is empty, and since $z_1 \in \operatorname{supp} \nu_K$, we must have $\nu_K(\bar{\Delta}(z_1, r_1)) > 0$. Call $a = \nu_K(\bar{\Delta}(z_1, r_1))$ and define a signed measure on K:

$$\sigma = \begin{cases} \nu & \text{on } K \\ -\nu_K/a & \text{on } \bar{\Delta}(z_1, r_1) \\ 0 & \text{elsewhere} \end{cases} \cdot$$

For each $t \in (0, a)$, define a measure on K by $\nu_t = \nu_K + t\sigma$. It is easily verifiable that ν_t satisfies the criteria to be a measure, and since $\sigma(K) = 0$ we can also see that ν_t is a probability measure on K. Now,

$$\begin{aligned} I(\nu_t) - I(\nu_K) &= \int_K \int_K \log |z - w| d\nu_t(w)\, d\nu_t(z) - \int_K \int_K \log |z - w| d\nu_K(w)\, d\nu_K(z) \\ &= 2t \int_K \int_K \log |z - w| d\nu_K(z)\, d\sigma(w) + t^2 \int_K \int_K \log |z - w| d\sigma(z)\, d\sigma(w). \end{aligned}$$

The second integral in the last line is finite, since $I(\nu) > -\infty$ implies that $I(|\sigma|) > -\infty$. Thus the second integral is a constant depending on σ, which we shall call η, so

$$\begin{aligned}
I(\nu_t) - I(\nu_K) &= 2t \int_K \log p_K(w) d\sigma(w) + t^2\eta \\
&= 2t \int_{K_n} \log p_K(w) d\nu(w) - \frac{2t}{a} \int_{\bar{\Delta}(z_1,r_1)} \log p_K(w) d\nu_K(w) + t^2\eta \\
&\geq 2t \left(\left(I(\nu) + \frac{1}{n} \right) - \left(I(\nu) + \frac{1}{2n} \right) \right) + t^2\eta \\
&= t \left(\frac{1}{n} + t\eta \right).
\end{aligned}$$

For sufficiently small t, we have $I(\nu_t) > I(\nu_K)$, contradicting the status of ν_K as an equilibrium measure of K. Thus U_n must be empty for all $n \geq 1$, so that $\log p_K \leq I(\nu_K)$ on K.

We now show that $\log p_K \geq I(\nu_K)$ on the support of ν_K; by a theorem of potential theory called the minimum principle, this implies that $\log p_K \geq I(\nu_K)$ over all of $\mathbb{C}$. Thus we will conclude that $\log p_K \geq I(\nu_K)$ on $\mathbb{C}$, and $\log p_K = I(\nu_K)$ on K.

For each $n \geq 1$, we define

$$V_n = \{z \in \text{supp } \nu_K : \log p_K(z) < I(\nu_K)\}.$$

We will again show that V_n is empty for each $n \geq 1$ and thus that $\log p_K(z) \geq I(\nu_K)$ on the support of ν_K. By the minimum principle, $\log p_K \geq I(\nu_K)$ on all $\mathbb{C}$, providing part 1 of the theorem.

Assume that V_n is nonempty for some $n \geq 1$, and choose $z_2 \in V_n$. By upper semicontinuity, there exists $r_2 > 0$ such that $\log p_K < I(\nu_K) - 1/n$ on $\bar{\Delta}(z_2, r_2)$.

Since $z_2 \in \operatorname{supp} \nu_K$, we have $\nu_K(\bar{\Delta}(z_2, r_2)) > 0$; call $b = \nu_K(\bar{\Delta}(z_2, r_2))$. We previously showed that $\log p_K \leq I(\nu_K)$ on K, so

$$\begin{aligned}
I(\nu_K) &= \int_K \log p_K(z) d\nu_K(z) \\
&= \int_{\bar{\Delta}(z_2,r_2)} \log p_K(z) d\nu_K(z) + \int_{K \setminus \bar{\Delta}(z_2,r_2)} \log p_K(z) d\nu_K(z) \\
&\leq b\left(I(\nu_K) - \frac{1}{n}\right) + (1-b) I(\nu_K) \\
&< I(\nu_K),
\end{aligned}$$

providing a clear contradiction. Thus V_n must be empty for each $n \geq 1$, completing the proof. $\square$

Green's Functions

We turn now to a method of finding explicit formulae for equilibrium potentials; Green's functions will enable us to determine Jensen's formulae for generalized Mahler measures formed from equilibrium measures. Here we consider $\mathbb{C}_\infty$ the extended complex plane.

Definition 9. *If D is a proper subdomain on $\mathbb{C}_\infty$, a* Green's funciton *for D is a map $g_D : D \times D \to (-\infty, \infty]$ such that, for any $w \in D$, $g_D(\cdot, w)$ is harmonic on $D \setminus \{w\}$ and bounded outside each neighborhood of w; $g_D(w, w) = \infty$, and as $z \to w$,*

$$g_D(z, w) = \begin{cases} \log|z| + O(1) & w = \infty \\ -\log|z - w| + O(1) & w \neq \infty; \end{cases}$$

and $g_D(z, w) \to 0$ as $z \to \zeta$ for each $\zeta \in \partial D$ outside a capacity 0 subset of ∂D.

Our next theorem gives some insight into the utility of Green's functions for our study of generalized Mahler measures.

Theorem 8. (Subordination Principle) *Let D_1 and D_2 be domains in $\mathbb{C}_\infty$ with nonzero-capacity boundaries, and let $f : D_1 \to D_2$ be a meromorphic function. Then $g_{D_2}(f(z), f(w)) \geq g_{D_1}(z, w)$, with equality if f is a conformal mapping of D_1 onto D_2.*

We will primarily utilize the case of equality when f is conformal. First, we will require a lemma providing the positivity of Green's functions.

Lemma 2. *Let D be a domain with Green's function g_D; then $g_D(z, w) > 0$ for all $z, w \in D$.*

For a proof of this lemma, see chapter 4.4 of Ransford [6]. Returning to a proof of the subordination principle:

Proof. We first consider the case where $w \neq \infty$ and $f(w) \neq \infty$; for $z \in D_1 \backslash \{w\}$, define $u(z) = g_{D_1}(z, w) - g_{D_2}(f(z), f(w))$. One can see that u is subharmonic on $D_1 \backslash \{w\}$, bounded above outside every neighborhood of w, and

$$\lim_{z \to w} u(z) = \log \left| \frac{f(z) - f(w)}{z - w} \right| + O(1) = \log |f'(w)| + O(1)$$

so that u is bounded above on all of $D_1 \backslash \{w\}$. The preceding lemma gives that $g_{D_2} > 0$ so that

$$\limsup_{z \to \zeta} u(z) \leq \lim_{z \to \zeta} g_{D_1}(z, w) = 0$$

which means that by the maximum principle, a theorem of potential theory, $u \leq 0$ on $D_1 \backslash \{w\}$. Thus, for $w \neq \infty$ and $f(w) \neq \infty$, $g_{D_1}(z, w) \leq g_{D_2}(f(z), f(w))$. For the case where f is a conformal map, we may take the same argument with $f^{-1} : D_2 \to D_1$ to obtain equality.

To cover the case when $w = \infty$, let $\mathcal{F}$ be the conformal map $z \mapsto 1/z$, and let D_1' be the image of D_1 under this map. Notice now that if $f \circ \mathcal{F}(w) \neq \infty$, then

$$g_{D_1}(1/z, 1/w) = g_{D_1'}(z, w) = g_{D_2}(f \circ \mathcal{F}(z), f \circ \mathcal{F}(w)$$

so that

$$g_{D_1}(z, \infty) = g_{D_1'}(1/z, 0) = g_{D_2}(f(z), f(\infty)).$$

The case where $f(w) = \infty$ follows from a similar inversion. $\square$

We finish our discussion on Green's functions with a corollary to the subordination principle, which will be helpful in our construction of generalized Mahler measures:

Corollary 1. *Let K be a simply connected compact subset of $\mathbb{C}$ with positive capacity and continuous potential. Let $D_1 = \mathbb{C}_\infty \backslash \bar{\Delta}$ and $D_2 = \mathbb{C}_\infty \backslash K$, and let $f : D_1 \to D_2$ be a conformal map with $f(\infty) = \infty$; then*

$$p_K(\alpha) = \begin{cases} c(K) & \alpha \in K \\ c(K)|f^{-1}(\alpha)| & \alpha \notin K \end{cases}.$$

We thus have a construction for the equilibrium potential p_K in terms of the capacity of K and a conformal map from the complement of the unit disk to the complement of K. Specifically, the potential is identically the capacity within K (as stated by Frostman's Theorem), and outside K it is the capacity times the absolute inverse of the conformal map. This may seem unwieldly for some conformal maps, but our conformal maps will prove easy to work with, particularly after an appropriate change of coordinates we will come to later.

Generalized Jensen's Formula

We come now to an application of Jensen's formula to generalized Mahler measures Φ for compact set K, along with a discussion on how to construct the associated root function φ. Recall that if ν is a Borel probability measure on $\mathbb{C}$ with compact support, then $p_\nu(\alpha) \sim |\alpha|$ as $|\alpha| \to \infty$. By Jensen's formula, if $f(x) = a\prod_{n=1}^{d}(x - \alpha_n)$, then

$$\Phi(f) = \exp\left(\int_{\mathbb{C}} \log|f(z)|\, d\nu_K(z)\right) = |a|\prod_{n=1}^{d} p_K(\alpha)$$

showing that $\varphi = p_K$; the construction provided by the corollary above reveals why we refer to φ as a root function for K.

Sinclair [4] studied the generalized Mahler measures for the family of ellipses

$$E_q = \left\{x + iy \in \mathbb{C} : \frac{x^2}{(1+q)^2} + \frac{y^2}{(1-q)^2} \leq 1\right\}$$

for $q \in [0, 1]$, where E_0 is the closed unit disk, and E_1 is the degenerate ellipse $[-2, 2]$. The family of conformal maps $z \mapsto z + q/z$ send the exterior of the unit disk to the exterior of these regions E_q. Using the technology covered above, one may verify that these ellipses have capacity $c(E_q) = 1$.

We will similarly consider a family of closed regions with capacity 1. We shall define the closed $(N + 1)$-*cusped hypocycloid* $\mathcal{H}_N$ as the complement of the image of the exterior of the unit circle under the conformal map $\mathcal{C}_N : z \mapsto z + \dfrac{z^{-N}}{N}$.

Importantly, this family of hypocycloids each has capacity 1. Thus, by the corollary above,

$$p_{\mathcal{H}_N}(\alpha) = \begin{cases} 1 & \alpha \in \mathcal{H}_N \\ |\mathcal{C}_N^{-1}(\alpha)| & \alpha \notin \mathcal{H}_N \end{cases}.$$

We thus have a simple expression for our root function φ of $\mu^{(N)}$.

CHAPTER III

VOLUMES OF POLYNOMIALS

We come now to a discussion of the distributions of generalized Mahler measures. Our goal will be to discuss the volume of star bodies of generalized Mahler measures:

$$\mathrm{vol}(\mathcal{V}_d(\Phi)) = \lambda_{2d+2}\{\mathbf{a} \in \mathbb{C}^{d+1} : \Phi(\mathbf{a}) \le 1\}$$

by use of the cumulative distance function of $\tilde{\Phi}$

$$h_d(\xi) = h_d(\Phi; \xi) = \lambda_{2d}\{\mathbf{b} \in \mathbb{C}^d : \tilde{\Phi}(\mathbf{b}) \le \xi\}$$

and the complex moment function of Φ,

$$H_d(s) = H_d(\Phi; s) = \int_{\mathbb{C}^d} \tilde{\Phi}(\mathbf{b})^{-2s} d\lambda_{2d}(\mathbf{b})$$

These objects each describe the range of values of Φ on polynomials in $\mathbb{C}[x]$ of degree d. One particular connection we will cover is that the volume of $\mathcal{V}_d(\Phi)$ is a special value of the Mellin transform of $h_d(\Phi; \xi)$. Similarly, the complex moment function of Φ is closely related to the Mellin transform of $h_d(\Phi; \xi)$.

The determination of $H_d(\Phi; s)$ will be the primary focus of this and the final chapters. In particular, $H_d(s)$ may be expressed as the determinant of a particular matrix, which allows us to describe the volume $\mathcal{V}_d(\Phi)$ as the volume of a parallelepiped in a Hilbert space determined by Φ. Utilizing this Hilbert space will allow us to refine $H_d(\Phi; s)$ and the volume of $\mathcal{V}_d(\Phi)$ in terms of a family of orthogonal polynomials produced by Φ. Given an expression for $H_d(s)$, we can recover $h_d(\xi)$ from the Mellin inversion theorem.

In this chapter, we will explore the formulae for $H_d(\Phi; s)$ and $h_d(\Phi; \xi)$ for cases of Φ covered in previous research. As a point of interest, all examples of $H_d(s)$ computed here and in the final chapter have a meromorphic continuation to all of $\mathbb{C}$. Information on the distribution of the generalized Mahler measures can be recovered from the values of the poles and roots of these meromorphic functions; in particular, $H_d(s)$ is a rational function of s with poles at nonzero integers, and the origin is a root of multiplicity d. Under certain criteria, the coefficients of this rational function will be rational numbers times π^d. These results will lead to finding that $h_d(\xi)$ is a Laurent polynomial on $[1, \infty)$, and in the special cases the coefficients will be rational numbers times π^d.

Mellin Transformation

We start by discussing the method covered by Chern and Vaaler [3] to express the volume of $\mathcal{V}_d$ as a special value of a Mellin transform of a function found from Φ.

Definition 10. *The* Mellin transform *of a function* $f : [0, \infty) \to \mathbb{R}$ *is*

$$\widehat{f}(s) := \int_0^\infty x^{-s-1} f(x)\, dx$$

where s is a complex variable.

If this integral converges, it does so in the complex strip defined by $a < \Re(s) < b$, where a and b are the extended real numbers defined by the asymptotic behavior of $f(x)$ as $x \to \infty$ and $x \to 0$, respectively. Where the integral converges,

$\widehat{f}$ is an analytic function: consider a triangle Δ in the region $a < \Re(s) < b$; then

$$\begin{aligned}\int_\Delta \widehat{f}(s)\,ds &= \int_\Delta \int_0^\infty x^{-s-1} f(x)\,dx\,ds \\ &= \int_{0^+}^\infty f(x) \int_\Delta x^{-s-1}\,ds\,dx = 0,\end{aligned}$$

where the last is due to x^{-s} being analytic. Thus, Morera's theorem gives that $\widehat{f}$ is analytic in the region of convergence.

Theorem 9. (Chern and Vaaler [3]) *Let Φ be a generalized Mahler measure, then the volume of the degree d star body of Φ is*

$$\lambda_{2d+2}(\mathcal{V}_d) = 2\pi \widehat{h_d}(2d+2).$$

Proof. The volume of $\mathcal{V}_d$ may be written as

$$\lambda_{2d+2}(\mathcal{V}_d) = \int_{\mathbb{C}} \lambda_{2d}\{\mathbf{b} \in \mathbb{C}^d : \Phi(\mathbf{b}, z) \leq 1\}\,d\lambda_2(z).$$

By homogeneity, we can rewrite

$$\begin{aligned}\lambda_{2d}\{\mathbf{b} \in \mathbb{C}^d : \Phi(\mathbf{b}, z) \leq 1\} &= \lambda_{2d}\{z\mathbf{c} \in \mathbb{C}^d : (z\mathbf{c}, z) \leq 1\} \\ &= |z|^{2d} \lambda_{2d} \left\{ \mathbf{c} \in \mathbb{C}^d : \Phi(\mathbf{c}, 1) \leq \frac{1}{|z|} \right\}.\end{aligned}$$

By switching to polar coordinates and setting $\xi = 1/r$, it follows

$$
\begin{aligned}
\lambda_{2d+2}(\mathcal{V}_d) &= \int_{\mathbb{C}} |z|^{2d} \lambda_{2d} \left\{ \mathbf{c} \in \mathbb{C}^d : \Phi(\mathbf{c}, 1) \leq \frac{1}{|z|} \right\} d\lambda_2(z) \\
&= 2\pi \int_0^\infty r^{2d+1} \lambda_{2d} \left\{ \mathbf{c} \in \mathbb{C}^d : \Phi(\mathbf{c}, 1) \leq \frac{1}{r} \right\} dr \\
&= 2\pi \int_0^\infty \xi^{-2d-3} \left\{ \mathbf{c} \in \mathbb{C}^d : \Phi(\mathbf{c}, 1) \leq \xi \right\} d\xi \\
&= 2\pi \int_0^\infty \xi^{-2d-3} \left\{ \mathbf{c} \in \mathbb{C}^d : \tilde{\Phi}(\mathbf{c}) \leq \xi \right\} d\xi \\
&= 2\pi \widehat{h_d}(2d+2).
\end{aligned}
$$

□

To better understand h_d, it helps to consider the set-up geometrically. The set of coefficient vectors of monic polynomials of degree d forms a d-dimensional hyperplane in $\mathbb{C}^{d+1}$; for T sufficiently large, the dilated star body $T\mathcal{V}_d$ intersects this hyperplane. We can consider $h_d(T)$ as the d dimensional Lebesgue measure of the intersection of this hyperplane with $T\mathcal{V}_d$. We can now analyze the asymptotic behavior of $h_d(T)$ as $T \to \infty$ or $T \to 0$.

Lemma 3. (Chern and Vaaler) *Let Φ be a generalized Mahler measure, and $h_d : [0, \infty) \to [0, \infty)$ be defined as above. Then there exists $\epsilon > 0$ such that h_d is identically zero on $[0, \epsilon)$; and $h_d(T) = O(T^{2d})$ as $T \to \infty$, specifically*

$$\lim_{T \to \infty} \frac{h_d(T)}{T^{2d}} = \lambda_{2d}(\mathcal{V}_{d-1}).$$

Proof. Let Δ^{d+1} be the $d+1$ dimensional unit polydisk centered at the origin. Since $\mathcal{V}_d$ is bounded, there exists $\eta > 0$ such that $\mathcal{V}_d \subset \eta\Delta^{d+1}$, so $T\mathcal{V}_d \subset T\eta\Delta^{d+1}$. Now we consider the hyperplane $B \subset \mathbb{C}^{d+1}$ of coefficient vectors of monic polynomials of degree d,

$$B = \{(\mathbf{b}, 1) : \mathbf{b} \in \mathbb{C}^d\}.$$

It follows that $(B \cap T\mathcal{V}_d) \subset (B \cap T\eta\Delta^{d+1})$. Now, the set $(B \cap T\eta\Delta^{d+1})$ is a d-dimensional polydisk with radius $T\eta$ when $T\eta \geq 1$, and is empty otherwise. Thus

$$h_d(T) \leq \lambda_{2d}(B \cap T\eta\Delta^{d+1}).$$

With regards to the first claim, let $\epsilon = 1/\eta$; if $T < \epsilon$ then from the above observation, $(B \cap T\epsilon\Delta^{d+1})$ is empty. Thus $h_d(T) = 0$ if $T < \epsilon$.

As for the second claim, let $B_{1/T} = \{(\mathbf{b}, 1/T) : \mathbf{b} \in \mathbb{C}^d\}$; the set of polynomials with leading coefficient $1/T$ and distance 1 is $B_{1/T} \cap \mathcal{V}_d$. Now, $B_{1/T} = (1/T)B_1$, so

$$B_{1/T} \cap \mathcal{V}_d = \frac{1}{T}(B \cap T\mathcal{V}_d).$$

Notice that $(B_{1/T} \cap \mathcal{V}_d) \to \mathcal{V}_{d-1}$ as $T \to \infty$, since a leading coefficient of 0 just makes the polynomial one degree lower. It follows that

$$\begin{aligned}
\lambda_{2d}(\mathcal{V}_{d-1}) &= \lim_{T\to\infty} \lambda_{2d}(B_{1/T} \cap \mathcal{V}_d) \\
&= \lim_{T\to\infty} \lambda_{2d}\left(\frac{1}{T}(B \cap T\mathcal{V}_d\right) \\
&= \lim_{T\to\infty} \frac{h_d(T)}{T^{2d}}
\end{aligned}$$

as desired. $\square$

Notice from this that the volume of the degree $d-1$ star body is the leading coefficient of the leading term of $h_d(T)$:

$$h_d(T) = \lambda_{2d}(\mathcal{V}_d)T^{2d} + o(T^{2d}).$$

To understand why this should be expected, consider how h_d is defined by taking the volume of slices of $\mathcal{V}_d$, while $\mathcal{V}_{d-1}$ embeds into $\mathcal{V}_d$ as a slice.

Going forward we shall consider $\widehat{h_d}(2s)$ instead of $\widehat{h_d}(s)$. Note that the integral composing $\widehat{h_d}(2s)$ is convergent as $s = d+1$, since $\lambda_{2d+2}(\mathcal{V}_d)$ is finite. In particular, $\widehat{h_d}(2s)$ is convergent and analytic in the region $\Re(s) > d$. Considering $\widehat{h_d}(2s)$ as a Lebesgue-Stieltjies integral, we can use integration by parts:

$$\widehat{h_d}(2s) = -\frac{\xi^{-2s}h_d(\xi)}{2s}\bigg|_0^\infty + \frac{1}{2s}\int_0^\infty \xi^{-2s} dh_d(\xi).$$

From the preceding lemma, $h_d(0) = 0$ and $h_d(\xi)$ is dominated by $C\xi^{2d}$, for some constant C. Note that the first term from the integration by parts vanishes for $\Re(s) > d$. With a change of variables we write

$$\widehat{h_d}(2s) = \frac{1}{2s}\int_{\mathbb{C}^d} \tilde{\Phi}(\mathbf{a})^{-2s}\, d\lambda_{2d}(\mathbf{a})$$

revealing the connection between the Mellin transform of h_d and H_d. By the preceding theorem, the volume of $\mathcal{V}_d$ is

$$\lambda_{2d+2}(\mathcal{V}_d) = \frac{\pi H_d(d+1)}{d+1}.$$

Moreover, if $H_d(s)$ has a meromorphic continuation to a neighborhood of $s = d$, then the lemma gives that the volume of $\mathcal{V}_{d-1}$ is the residue of the pole at $s = d$ [3].

Though we do not use it further in this thesis, this fact proves useful in verifying computational results.

We now introduce a change of variables from root vectors to coefficient vectors. Let $n \leq d$ be a positive integer, and let $e_n : \mathbb{C}^d \to \mathbb{C}$ be the nth elementary symmetric function:

$$e_n(\boldsymbol{\alpha}) = (-1)^n \sum_{t \in P_n^d} \prod_{\ell=1}^{n} \alpha_{t(\ell)},$$

where $P_n^d = \{t : \{1, 2, ..., n\} \to \{1, 2, ..., d\} | t(1) < t(2) < ... < t(n)\}$. Notice

$$\prod_{n=1}^{d} (x - \alpha_n) = x^d + \sum_{n=1}^{d} e_n(\boldsymbol{\alpha}) x^{d-n}.$$

Now, let $E_d : \mathbb{C}^d \to \mathbb{C}^d$ be the map whose nth coordinate function is e_n; then E_d sends root vectors to (monic) coefficient vectors. Importantly, each monic polynomial is uniquely determined by its roots, and every permutation of roots leaves $E_d(\alpha)$, so the degree of E_d is $d!$. It is well known that the Jacobian of E_d is $|V(\boldsymbol{\alpha})|^2$ where

$$V(\boldsymbol{\alpha}) = \prod_{1 \leq m < n \leq d} (\alpha_n - \alpha_m).$$

We provide an outline of the proof: first, it is easy to take the Jacobian with partial derivatives. The determinant of this matrix is a symmetric polynomial, and its degree may be verified to match that of the Vandermonde. Further, these polynomials have the same zeroes, so they must be equal up to a constant multiple, which turns out to be 1.

Notice from this formula that $|V(\boldsymbol{\alpha})| \neq 0$ for almost all points in $\mathbb{C}^d$. Finaly, change of variables $\mathbf{a} = E_d(\boldsymbol{\alpha})$ provides

$$H_d(s) = \frac{1}{d!} \int_{\mathbb{C}^d} \left(\prod_{n=1}^{d} \varphi(\alpha_n)^{-2s} \right) |V(\boldsymbol{\alpha})|^2 \, d\lambda_{2d}(\boldsymbol{\alpha})$$

At a glance, this may seem like a more complicated formulation for $H_d(\Phi; s)$; however we will capitalize on how $V(\boldsymbol{\alpha})$ may be expressed as the Vandermonde determinant. This fact along with some combinatorics and Fubini's Theorem allows for $H_d(\Phi; s)$ to be further rewritten as a determinant whose entries are integrals over $\mathbb{C}$; we will interpret these entries as values of inner products of polynomials in a Hilbert space determined by Φ.

Recall, if ν is the measure supported on the complex plane given by $d\nu(\alpha) = \varphi^{-2s}(\alpha) d\lambda_2(\alpha)$, where s is a complex parameter, then $L^2(\nu)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int_{\mathbb{C}} \varphi(\alpha)^{-2s} f(\alpha) \overline{g(\alpha)} d\lambda_2(\alpha).$$

This inner product induces a norm $\mathfrak{N}(f)^2 = \mathfrak{N}(f; s)^2 := \langle f, f \rangle$. For $\Re(s) > d$, any polynomial in $\mathbb{C}[x]$ with degree less than d is in $L^2(\nu)$.

Let $Q = \{Q_n(\alpha) : n = 1, 2, ..., d\}$ be a set of monic polynomials in $\mathbb{C}[x]$ with $\deg(Q_n) = n - 1$; we will call such a set a *complete* family of polynomials. Each Q_n is in $L^2(\nu)$, and Q spans a parallelepiped in the Hilbert space. The Gram matrix of Q is a $d \times d$ matrix, whose ℓ, k entry is $\langle Q_\ell, Q_k \rangle$; this is a symmetric matrix whose ℓ, k entry depends on Q_ℓ, Q_k, φ, and s. Importantly, the determinant of this matrix may be interpreted as the volume of the parallelepiped spanned by Q in the Hilbert space. As it turns out, the determinant of W_Q is equivalent to $H_d(s)$

Theorem 10. (Sinclair [4]) *Let Q be any complete family of monic polynomials; then*

$$H_d(s) = \det(W_Q).$$

Since this theorem holds for any complete family of polynomials, we may choose a family Q that makes $\det(W_Q)$ simple to evaluate. Note also that while $\Re(s) > d$, we may leave s as a parameter to be chosen later, so we may consider the inner product as independent of d.

A powerful corollary arises from choosing an orthogonal set of polynomials for the complete family Q.

Corollary 2. *Let $\Re(s) > d$, and let Q be a complete family of monic polynomials with*

$$\langle Q_\ell, Q_k \rangle = \delta_{\ell k} \mathfrak{N}(Q_k; s)^2$$

where $\delta_{\ell k} = 1$ if $\ell = k$, and 0 otherwise; then

$$H_d(s) = \prod_{n=1}^{d} \mathfrak{N}(Q_n; s)^2.$$

We now introduce two lemmas to prove this important theorem.

Lemma 4. *Let $I = I(\ell, k)$ be a $d \times d$ matrix; then*

$$\det(I) = \frac{1}{d!} \sum_{\tau \in S_d} \sum_{\sigma \in S_d} \operatorname{sgn}(\tau) \operatorname{sgn}(\sigma) \prod_{n=1}^{d} I(\tau(n), \sigma(n)).$$

Proof. First note

$$\prod_{n=1}^{d} I(\tau(n), \sigma(n)) = \prod_{n=1}^{d} I(n, \sigma \circ \tau^{-1}(n)),$$

so we may write

$$\begin{aligned}
&\frac{1}{d!}\sum_{\tau\in S_d}\sum_{\sigma\in S_d}\operatorname{sgn}(\tau)\operatorname{sgn}(\sigma)\prod_{n=1}^{d} I(\tau(n),\sigma(n))\\
=&\frac{1}{d!}\sum_{\tau\in S_d}\sum_{\sigma\in S_d}\operatorname{sgn}(\sigma\circ\tau^{-1})\prod_{n=1}^{d} I(n,\sigma\circ\tau^{-1}(n))\\
=&\frac{1}{d!}\sum_{\tau\in S_d}\sum_{\sigma\in S_d}\operatorname{sgn}(\sigma)\prod_{n=1}^{d} I(n,\sigma(n))\\
=&\sum_{\sigma\in S_d}\operatorname{sgn}(\sigma)\prod_{n=1}^{d} I(n,\sigma(n))\\
=&\det(I).
\end{aligned}$$

□

Lemma 5. *Let Q be a complete family of monic polynomials; then*

$$V(\boldsymbol{\alpha}) = \det(V_Q)$$

where V_Q is the $d\times d$ matrix whose ℓ, k entry is $V_Q(\ell,k) = Q_\ell(\alpha_k)$.

Proof. Notice that we may write

$$V_Q = \begin{pmatrix} 1 & 0 & \cdots & 0\\ * & 1 & \cdots & 0\\ * & * & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ * & * & \cdots & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & \cdots & 1\\ \alpha_1 & \alpha_2 & \cdots & \alpha_d\\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_d^2\\ \vdots & \vdots & \ddots & \vdots\\ \alpha_1^{d-1} & \alpha_2^{d-1} & \cdots & \alpha_d^{d-1} \end{pmatrix}$$

where $*$ represent entries that are not necessarily zero. The second matrix is easily recognized as the Vandermonde matrix; it is well known that the determinant of the Vandermonde is $V(\boldsymbol{\alpha})$, which leads to the desired result. □

We may now proceed to our proof that $H_d(s) = \det(W_Q)$:

Proof. We have previously shown that

$$H_d(s) = \frac{1}{d!} \int_{\mathbb{C}^d} \left(\prod_{n=1}^{d} \varphi(\alpha_n)^{-2s} \right) |V(\boldsymbol{\alpha})|^2 \, d\lambda_{2d}(\boldsymbol{\alpha}).$$

By expanding $\det(V_Q)$ as a sum over permutations from S_d, we write

$$|V(\boldsymbol{\alpha})|^2 = \sum_{\tau \in S_d} \sum_{\sigma \in S_d} \operatorname{sgn}(\tau) \operatorname{sgn}(\sigma) \prod_{n=1}^{d} Q_{\tau(n)}(\alpha_n) \overline{Q_{\sigma(n)}(\alpha_n)}.$$

We thus have

$$H_d(s) = \frac{1}{d!} \sum_{\tau \in S_d} \sum_{\sigma \in S_d} \int_{\mathbb{C}^d} \prod_{n=1}^{d} \varphi(\alpha_n)^{-2s} Q_{\tau(n)}(\alpha_n) \overline{Q_{\sigma(n)}(\alpha_n)} \, d\lambda_{2d}(\boldsymbol{\alpha}).$$

Now $H_d(s)$ is convergent for $\Re(s) > d$, so Fubini's Theorem gives

$$H_d(s) = \frac{1}{d!} Q_{\tau(n)}(\alpha_n) \overline{Q_{\sigma(n)}(\alpha_n)} \prod_{n=1}^{d} \langle Q_{\tau(n)}, Q_{\sigma(n)} \rangle,$$

which gives the result by the first lemma above. $\square$

Mahler Measure Cases

Chern and Vaaler [3] explored the unit circle case, corresponding to the Mahler measure. In this case, the inner product on monomials follows

$$\begin{aligned} \langle \alpha^\ell, \alpha^k \rangle &= \int_{\mathbb{C}} \varphi(\alpha)^{-2s} \alpha^\ell \bar{\alpha}^k \, d\lambda_2(\alpha) \\ &= \int_0^\infty \varphi(r)^{-2s} r^{\ell+k+1} dr \int_0^{2\pi} e^{(\ell-k)i\theta} d\theta. \end{aligned}$$

By Parseval's formula,

$$\int_0^{2\pi} e^{(\ell-k)i\theta} d\theta = \begin{cases} 0 & \ell \neq k \\ 1 & \ell = k \end{cases}$$

so the set $\{1, \alpha, \alpha^2, ..., \alpha^{d-1}\}$ is a complete family of orthogonal polynomials in $L^2(\nu)$. Thus W_Q a diagonal matrix, making the computation of $H_d(s)$ comparatively easy; they found

$$H_d(\mu; s) = \frac{\pi^d}{d!} \prod_{n=1}^{d} \frac{s}{s-n}.$$

We can see here that the poles of $H_d(s)$ are all positive integers, and the origin is a root with multiplicity d.

Elliptical Case

Sinclair [4] considered a family of ellipses of capacity 1. In particular, the conformal map $z \mapsto z + \frac{q}{z}$ (for $q \in [0, 1)$) maps the exterior of the unit disc to the exterior of an ellipse; this leads to a set $\{H_d(\mu_q; s) : q \in [0, 1]\}$ which forms a curve of rational functions between $H_d(\mu; s)$ and $H_d(\mu_1; s)$.

Theorem 11. (Sinclair) *Let d be a positive integer. If $q \in [0, 1]$, then $H_d(\mu_q; s)$ analytically continues to a rational function of s given by*

$$H_d(\mu_q; s) = \frac{\pi^d s^d}{d!} \prod_{n=1}^{d} \frac{(1-q^{2n})s + (1+q^{2n})n}{s^2 - n^2}.$$

For $q > 0$ the poles of $H_d(\mu_q; s)$ are at a mix of positive and negative integers, in contrast to the strictly positive integer poles of $H_d(\mu; s)$. However, $H_d(\mu_q; s)$ maintains the origin as a root of multiplicity d. For $q \in (0, 1)$, there are an additional d roots in the half plane $\Re(s) < 0$. Of interest, in addition to being rational in s, $H_d(\mu_q; s)$ is polynomial in q.

We will not provide a full proof of Sinclair's result, though we will outline the start of a key lemma providing $\langle f, g\rangle$ for monomials:

Lemma 6. *Let $J, K < d$ be postive integers, and let $q \in [0,1]$; then $\langle \alpha^{J-1}, \alpha^{K-1}\rangle$ analytically continues to a a rational function of s:*

$$2\pi \sum_{n=1}^{d} \left\{ q^{J/2} \left(\binom{J-1}{\frac{J+n}{2}} - \binom{J-1}{\frac{J+n}{2}-1} \right) \right\} \left\{ q^{K/2} \left(\binom{K-1}{\frac{K+n}{2}} - \binom{K-1}{\frac{K+n}{2}-1} \right) \right\} \times \frac{s}{2n} \left(\frac{s(q^{-n} - q^n) + n(q^{-n} + q^n)}{s^2 - n^2} \right)$$

where the binomial terms are zero for non-integer or negative entries.

Sinclair starts by setting up the integral

$$\langle \alpha^{J-1}, \alpha^{K-1}\rangle = \int_{\mathbb{C}} \varphi(\alpha)^{-2s} \alpha^{J-1} \bar{\alpha}^{K-1} d\lambda_2(\alpha)$$

then splits the integral into two regions: one inside the ellipse E_q, and one outside the ellipse. By Frostman's theorem, φ is identically $c(E_q) = 1$ inside the ellipse. By taking a change of variables along the conformal map $\alpha \mapsto z + q/z$ mapping the interior (respectively exterior) of the circle to the interior (resp. exterior) of the ellipse, we may integrate instead the regions inside and outside the unit disk. This substitution takes $\varphi(\alpha)$ to $|z|$ outside the ellipse, so that we may bypass φ entirely. From here, Sinclair expands the powers of z according to the binomial theorem (introducing the binomial coefficients seen above). After a switch to polar coordinates, rearranging and identifying cases where the integral would be zero leads to the expression above.

Interval Case

As a special case, for $q = 1$ the elliptical case reduces to the interval $[-2, 2]$, with

$$H_d(\mu_1; s) = (2\pi)^d \prod_{n=1}^{d} \frac{s}{s^2 - n^2}.$$

Of interesting note, this interval serves both as a degenerate ellipse, and is considered the 2-cusped hypocycloid (what we would denote $\mathcal{H}_1$ in the notation introduced in the preceding chapter). We note however that the methods used for the calculation of $H_d(\mu^{(N)}; s)$ next chapter cannot be applied to this 2-cusped hypocycloid, as we will assume the presence of both an interior and exterior region.

We note now some similarities on H_d among all three cases explored above; in all cases, H_d is π^d times a rational function from $\mathbb{Q}(s)$, with integer poles $\leq d$, the origin as a root of multiplicity d, and matching degree of numerator and denominator for $q \neq 1$. A notable difference is that while the Mahler measure case has only positive poles, the elliptical and interval cases have equal numbers of positive and negative poles. We will see a similar extension of poles when we come to the cusped hypocycloids, with additional negative poles beyond $-d$.

CHAPTER IV

THE CUSPED HYPOCYCLOIDAL MAHLER MEASURE

Before coming to the main results of this thesis, let us summarize our journey so far. Our goal has been to discuss generalized Mahler measures for the region K

$$\Phi(f) = \exp\left(\int_{\mathbb{C}} \log|f(z)| d\nu_K(z)\right)$$

and in particular the volumes of the star body $\mathcal{V}_d$ corresponding to the generalized Mahler measure. From Green's functions, we saw that the equilibrium potential may be realized in terms of the capacity of the region K and a conformal map between the exteriors of the unit disk and K. From Chern and Vaaler, we may realize the volume of $\mathcal{V}_d$ in terms of $H_d(s)$, which in turn from Sinclair may be written as the determinant of the matrix whose entries are inner products $\langle \alpha^M, \alpha^L \rangle = \int_{\mathbb{C}} \varphi(\alpha)^{-2s} f(\alpha)\overline{g(\alpha)} d\lambda_2(\alpha)$.

The Distribution of the Cusped Hypocycloidal Mahler Measure

We come now to a discussion of the *cusped hypocycloidal Mahler measure.* We'll start by acknowledging the differences compared to the elliptical cases.

In the elliptical case, the integral $\langle f, g\rangle$ was broken into two regions: inside and outside the ellipse. Sinclair used the change of coordinates $\alpha = z + q/z$ to transform this to inside and outside the circle, capitalizing on how this is a conformal map to utilize the subordination principle. This technique breaks down for the cusped hypocycloids; while the conformal map $\alpha = z + \dfrac{z^{-N}}{N}$ may be applied to the outside of the hypocycloid, the non-smoothness of the cusps along the boundary means we cannot apply the same conformal map to the inner integral. We instead capitalize on two facts; first, by Frostman's theorem φ is identically 1 within the hypocycloid. Second, we may instead apply the quasi-conformal map $\alpha = z + \dfrac{\bar{z}^N}{N}$ from the interior of the circle to the interior of the hypocycloid; this is still an integrable expression, and so we may bypass the cusps.

We now introduce the inner product for the the Hilbert space associated to $\mu^{(N)}$.

Lemma 7. (Main Lemma) *Let L, M, N, d be nonnegative integers such that $M, L < d$ and $N \geq 2$; then $\langle \alpha^M, \alpha^L \rangle$ equals*

$$\pi \sum_{n=\frac{-N}{N+1}M}^{\frac{1}{N+1}M} \binom{M}{n + \frac{N}{N+1}M} \binom{L}{n + \frac{N}{N+1}L} \left(\frac{1}{N}\right)^{\frac{M+L}{N+1} - 2n} \frac{n(N+1)+1}{(n + \frac{N}{N+1}M + 1)(n + \frac{N}{N+1}L + 1)}$$
$$\times \frac{s}{s - n(N+1) - 1}$$

for $L \equiv M \mod (N+1)$; and 0 otherwise. This sum is taken with integer increments (so the elements of the sum are integers plus $\frac{M}{N+1}$). Further, the binomial coefficients are taken as zero for non-integer entries, or bottom entries less than zero or greater than the top entry.

As an example, taking $N = 2$, $M = 1$, and $L = 4$, we have:

$$\begin{aligned}
\langle \alpha^1, \alpha^4 \rangle &= \pi \sum_{n=-2/3}^{1/3} \binom{1}{n + \frac{2}{3}} \binom{4}{n + \frac{8}{3}} \left(\frac{1}{2}\right)^{\frac{5}{3} - 2n} \frac{3n+1}{(n + \frac{2}{3} + 1)(n + \frac{8}{3} + 1)} \left(\frac{s}{s - 3n - 1}\right) \\
&= \pi \left(1 \binom{4}{3} \left(\frac{1}{2}\right) \left(\frac{2}{8}\right) \left(\frac{s}{s-2}\right) + 1 \binom{4}{2} \left(\frac{1}{2}\right)^3 \left(\frac{-1}{3}\right) \left(\frac{s}{s+1}\right) \right) \\
&= \left(\frac{\pi}{2}\right) \frac{s}{s-2} - \left(\frac{\pi}{4}\right) \frac{s}{s+1}.
\end{aligned}$$

TABLE 3 Sample calculations of vol($\mathcal{V}_d$) for various values of N+1 cusps and polynomial degrees d

cusps $(N+1)$	d: 2	3	4	5
3 cusps	$\frac{3}{28}\pi^3$	$\frac{262}{10395}\pi^4$	$\frac{18125}{1437696}\pi^5$	$\frac{311481}{2896974080}\pi^6$
4 cusps	$\frac{7}{60}\pi^3$	$\frac{5104}{184275}\pi^4$	$\frac{18063359375}{3341114297136}\pi^5$	$\frac{2272135257604}{3295289986463475}\pi^6$
5 cusps	$\frac{1}{8}\pi^3$	$\frac{67}{2112}\pi^4$	$\frac{41758354375}{6744887525376}\pi^5$	$\frac{63792691434842763}{61482997256500019200}\pi^6$

We leave the proof of this identity to its own section at the end of this chapter.

Volume of complex polynomials

Recall from chapter 3 that

$$\lambda_{2d+2}(\mathcal{V}_d) = \frac{\pi H_d(d+1)}{d+1}.$$

Since $H_d(s)$ is a determinant of the Gram matrix of terms $\langle \alpha^M, \alpha^L \rangle$, our main lemma allows us to compute volumes of star bodies for any choice of degree d or cusps $N+1$; see Table 3 for some examples.

Of interesting note, Schanuel [**?**] showed that the volume of a region gives the approximate number of lattice points of the region. In the case of our volumes of polynomials, lattice points correspond to polynomials with Gaussian integer coefficients. Because our generalized Mahler measures are homogeneous, this means we can easily compute the volume of "balls" of hypocycloidal Mahler measure $\leq T$

TABLE 4 Asymptotics of $H_d(\mu^{(N)}; s)$ for various degree d and cusps $N+1$

cusps $(N+1)$	d: 2	3	4	5
3 cusps	$\frac{1}{4}\pi^2$	$\frac{7}{192}\pi^3$	$\frac{49}{12288}\pi^4$	$\frac{1617}{5242880}\pi^5$
4 cusps	$\frac{5}{18}\pi^2$	$\frac{65}{1458}\pi^3$	$\frac{24245}{4251528}\pi^4$	$\frac{935857}{13947137604}\pi^5$
5 cusps	$\frac{5}{16}\pi^2$	$\frac{695}{12288}\pi^3$	$\frac{375995}{50331648}\pi^4$	$\frac{2756870539}{3298534883328}\pi^5$

from the volume of $\mathcal{V}_d$. This gives the asymptotic (main term) for the number of Gaussian integer polynomials of degree $\leq d$ and "height" $\leq T$ as $T \to \infty$.

Building off of Schanuel, Masser and Vaaler [**?**] showed that $H_d(\mu; s)$ asymptotically approaches the volume of monic polynomials with bounded Mahler measure. This similarly extends to the generalized Mahler measures; see Table 4 for examples of the volume of monic polynomials of bounded Mahler measure.

Analysis of poles and roots of rational function

We note for the following theorems that the inner product $\langle \alpha^M, \alpha^L \rangle$ is a sum of terms $\frac{s}{s-k}$; this enables a characterization of the rational function $H_d(\mu^{(N)}; s)$.

Theorem 12. *For $(N+1)$-cusps and degree d, the negative poles of $H_d(\mu^{(N)}; s)$ are a subset of*

$$\{N+1-Nd, N+2-Nd, ..., -2, -1\}$$

and the positive poles are $\{1, 2, ..., d\}$.

Proof. We start by acknowledging why $s = 0$ is not a pole; notice that each term in the sum includes

$$(n(N+1)+1)\,\frac{s}{s-n(N+1)-1}.$$

Thus, the term $\dfrac{s}{s-0}$ is attached to a coefficient 0, and may be disregarded.

We will now simplify the coefficients of $\langle \alpha^M, \alpha^L \rangle$ as a single term C_n, and consider the sum as

$$\sum_{n=\frac{-N}{N+1}M}^{\frac{1}{N+1}M} \frac{C_n s}{s-n(N+1)-1}$$

The poles are provided by the expression $s - n(N+1) - 1$. For a fixed M, the poles range from $1 - NM$ to $M + 1$, with increments of $N + 1$. It follows that for M ranging from 0 to $d - 1$, the poles range from $N + 1 - Nd$ to d.

□

It should be noted that for d not a nonzero multiple of $N + 1$, the poles will be a strict subset of the possible poles; as noted above, for a fixed M the spacing of poles is by $N+1$, thus skipping over some of the possible poles between $N+1-Nd$ and -1. The positive poles are captured by the term $n = \frac{M}{N+1}$ in the sum, so all poles from 1 to d are included.

Theorem 13. *For $(N + 1)$-cusps and degree d, $s = 0$ is a root of $H_d(\mu^{(N)}; s)$ with multiplicity d.*

Proof. We note that $s = 0$ is certainly a root since $\langle \alpha^M, \alpha^L \rangle$ is a sum of terms all with $s = 0$ as a root. Now, $H_d(\mu^{(N)}; s)$ is a determinant of a $d \times d$ matrix whose entries all have $s = 0$ as a root, and thus has 0 as a root with multiplicity d. □

Theorem 14. *The remaining roots of $H_d(\mu^{(N)}; s)$ are all nonpositve.*

Note from Descartes' Rule of Signs that if all coefficients of a polynomial are positive, then none of the roots are positive. It thus suffices to show that the numerator polynomial of the simplified $\langle \alpha^M, \alpha^L \rangle$ has no negative coefficients. This follows from the term $\left(\frac{1}{N}\right)^{\frac{L+M}{N+1}-2n}$ in our expression for the inner product; terms in the sum with negative coefficients are more strongly bounded by powers $\left(\frac{1}{N}\right)^k$, so that positive terms in the combined rational function outweight the negative terms.

Analysis of diagonal block matrices

While we have an construction for evaluating $H_d(\mu^{(N)}; s)$, we do not currently have a closed form expression for this function. In this section, we explore avenues for finding a closed form expression by utilizing the structure of the Gram matrix whose determinant gave us $H_d(s)$.

In the following analysis, we'll refer to our Gram matrix as $\mathbf{A}$, and focus on block matrices $\mathbf{A}_{ij}$ of size $N+1$ (operating under the assumption that $d = K(N+1)$ for some $K \in \mathbb{Z}$. Importantly, since $\langle \alpha^M, \alpha^L \rangle = 0$ unless $M \equiv L \mod (N+1)$, the block matrices are diagonal.

<u>Cholesky Decomposition</u>

As a Hermitian matrix, we may rewrite the matrix using the Cholesky decomposition $\mathbf{A} = \mathbf{LDL}^*$, where $\mathbf{D}$ is a diagonal matrix, $\mathbf{L}$ is a lower triangular matrix with entries of 1 along the diagonal, and $\mathbf{L}^*$ is the conjugate transpose of $\mathbf{L}$. H. Fang [7] worked out a block representation of this decomposition; if $\mathbf{A} = (\mathbf{A}_{ij})$, then we can represent $\mathbf{L} = (\mathbf{L}_{ij})$ and $\mathbf{D} = (\mathbf{D}_j)$, where

$$\begin{aligned}
\mathbf{D}_j &= \mathbf{A}_{jj} - \sum_{k=1}^{j-1} \mathbf{L}_{jk}\mathbf{D}_k\mathbf{L}_{jk}^T \\
\mathbf{L}_{ij} &= \left(\mathbf{A}_{ij} - \sum_{k=1}^{j-1} \mathbf{L}_{jk}\mathbf{D}_k\mathbf{L}_{jk}^T\right)\mathbf{D}_j^{-1} \qquad i > j \\
\mathbf{L}_{ii} &= \mathbf{I} \qquad \mathbf{L}_{ij} = \mathbf{L}_{ji}^T.
\end{aligned}$$

Since the block matrices $\mathbf{A}_{ij}$ are diagonal, $\mathbf{L}_{ij} = \mathbf{L}_{ij}^T$, and matrix arithmetic may be performed entry-wise, so we may write

$$\begin{aligned}
\mathbf{D}_j &= \mathbf{A}_{jj} - \sum_{k=1}^{j-1} \mathbf{L}_{jk}^2\mathbf{D}_k \\
\mathbf{L}_{ij} &= \mathbf{A}_{ij}\mathbf{D}_j^{-1} - \sum_{k=1}^{j-1} \mathbf{L}_{jk}^2\mathbf{D}_k\mathbf{D}_j^{-1}.
\end{aligned}$$

The term $\mathbf{D}_j^{-1}$ serves as a telescoping role, eliminating poles present in blocks prior to j. Since the main diagonal entries of $\mathbf{L}$ are all 1, it follows that $\det(\mathbf{A}) = \det(\mathbf{D})$.

Block Matrix Determinant

An alternative consideration comes from capitalizing only on the commuting nature of the diagonal block matrices $\mathbf{A}_{ij}$; from Kovacs, Silver, and Williams [**?**], the determinants of commuting-block matrices may be computed as:

$$\det(\mathbf{A}) = \det\left(\sum_{\sigma\in S_k} (\operatorname{sgn}(\sigma)) \prod_{i=1}^{k} \mathbf{A}_{i\sigma(i)}\right).$$

In essence, this formula treats $\mathbf{A}$ not as a $d \times d$ matrix with complex entries, but as a $K \times K$ matrix (with $d = K(N+1)$) over the ring of commuting matrices.

It shall be convenient to index the blocks $\mathbf{A}_{ij}$ by $0 \leq i, j \leq K$. Let us define $\mathcal{D}(\mathbf{A}) = \sum_{\sigma \in S_k} (\mathrm{sgn}(\sigma)) \prod_{i=1}^{k} \mathbf{A}_{i\sigma(i)}$, and consider the entries of a particular $\mathbf{A}_{ij}$ to determine the behavior of $\mathcal{D}(\mathbf{A})$. Recall again how $\langle \alpha^M, \alpha^L \rangle = 0$ unless $M \equiv L \mod (N+1)$; thus the powers of the nonzero entries of $\mathbf{A}_{ij}$ are equivalent mod $N+1$. For example,

$$\mathbf{A}_{0,1} = \begin{pmatrix} \langle \alpha^0, \alpha^{N+1} \rangle & 0 & 0 & \cdots & 0 \\ 0 & \langle \alpha^1, \alpha^{N+2} \rangle & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \langle \alpha^N, \alpha^{2N+1} \rangle \end{pmatrix}$$

and in general,

$$\mathbf{A}_{ij} = \begin{pmatrix} \langle \alpha^{i(N+1)}, \alpha^{j(N+1)} \rangle & 0 & \cdots & 0 \\ 0 & \langle \alpha^{i(N+1)+1}, \alpha^{j(N+1)+1} \rangle & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \langle \alpha^{i(N+1)+N}, \alpha^{j(N+1)+N} \rangle \end{pmatrix}.$$

Remember again that algebra with diagonal matrices may be performed entry-wise; taking $\mathcal{D}(\mathbf{A})$ as a “determinant” of the blocks $\mathbf{A}_{ij}$, notice that $\mathcal{D}(\mathbf{A})$ is itself diagonal, and its entries may be thought of as determinants of $K \times K$

submatrices constructed from characteristic classes modulo $N+1$:

$$\mathcal{D}(\mathbf{A}) = \begin{pmatrix} \det([\langle \alpha^i, \alpha^j \rangle]_{i,j\equiv 0}) & 0 & \cdots & 0 \\ 0 & \det([\langle \alpha^i, \alpha^j \rangle]_{i,j\equiv 1}) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \det([\langle \alpha^i, \alpha^j \rangle]_{i,j\equiv N}) \end{pmatrix}$$

leading to

$$\det(\mathbf{A}) = \prod_{k=0}^{N} \det([\langle \alpha^i, \alpha^j \rangle]_{i,j\equiv k \mod (N+1)}).$$

This serves as a fascinating parallel to checkerboard matrices, such as the Gram matrix Sinclair worked with in dealing with the reciprocal Mahler measure [4]. A known method for taking the determinant of a checkerboard (alternating nonzero and zero) matrix is to break the nonzero entries into two matrices and multiply their determinants. The checkerboard matrix is simply the case $N+1=2$ for this result.

Proof of Main Lemma of Inner Product Identity

For polynomials f and g, $\langle f, g \rangle = \int_{\mathbb{C}} \varphi(\alpha)^{-2s} f(\alpha)\overline{g(\alpha)} d\lambda_2(\alpha)$. Using polynomials z^M and z^L, then integrating over the $(N+1)$-cusped hypocycloid $\mathcal{H}_N$ and its complement ${\mathcal{H}_N}^C$, we obtain

$$= \int_{\mathcal{H}_N} \alpha^M \bar{\alpha}^L d\lambda_2(\alpha) + \int_{{\mathcal{H}_N}^C} \varphi(\alpha)^{-2s} \alpha^M \bar{\alpha}^L d\lambda_2(\alpha)$$

since within the hypocycloid, $\varphi = 1$. Now, within the hypocycloid we set $\alpha = z + \frac{\bar{z}^N}{N}$, while outside we set $\alpha = z + \frac{z^{-N}}{N}$ to shift the integration to the unit disc D and its complement:

$$\int_D \left(z + \frac{\bar{z}^N}{N}\right)^M \left(\bar{z} + \frac{z^N}{N}\right)^L (1 - |z|^{2(N-1)}) d\lambda_2(z)$$
$$+ \int_{D^C} |z|^{-2s} \left(z + \frac{z^{-N}}{N}\right)^M \left(\bar{z} + \frac{\bar{z}^{-N}}{N}\right)^L |1 - z^{-N-1}|^2 d\lambda_2(z).$$

We then expand the polynomial terms as sums with binomial coefficients and switch to polar coordinates, obtaining:

$$\int_0^1 \int_0^{2\pi} \sum_{m=0}^{M} \sum_{\ell=0}^{L} \binom{M}{m} \binom{L}{\ell} r^{m+N(M-m)+\ell+N(L-\ell)+1} \left(\frac{1}{N}\right)^{M-m+L-\ell}$$
$$\times e^{i\theta(m-N(M-m)-\ell+N(L-\ell))}(1 - r^{2(N-1)}) d\theta \, dr$$
$$+ \int_1^\infty \int_0^{2\pi} \sum_{m=0}^{M} \sum_{\ell=0}^{L} \binom{M}{m} \binom{L}{\ell} r^{-2s+m-N(M-m)+\ell-N(L-\ell)+1} \left(\frac{1}{N}\right)^{M-m+L-\ell}$$
$$\times e^{i\theta(m-N(M-m)-\ell+N(L-\ell))}(1 - r^{-(N+1)} e^{-i\theta(N+1)} - r^{-(N+1)} e^{i\theta(N+1)} + r^{-2(N+1)}) d\theta \, dr.$$

By Parseval's, the integral with respect to θ is 0 for all terms in the double sum except those that give $e^{i\theta \times 0}$; we thus eliminate the sum over ℓ by taking $\ell = m + \frac{N}{N+1}(L - M)$ in the left integral, and

$$\ell = \begin{cases} m + \frac{N}{N+1}(L-M) \\ m + \frac{N}{N+1}(L-M) + 1 \\ m + \frac{N}{N+1}(L-M) - 1 \\ m + \frac{N}{N+1}(L-M) \end{cases}$$

for the four terms at the end of the right integral, giving

$$
\begin{aligned}
&2\pi \int_0^1 \sum_{m=0}^{M} \binom{M}{m} \binom{L}{m+\frac{N}{N+1}(L-M)} r^{2m(1-N)+\frac{2N}{N+1}L+\frac{2N^2}{N+1}M+1} \\
&\qquad \times \left(\frac{1}{N}\right)^{\frac{2n+1}{N+1}M+\frac{1}{N+1}L-2m} (1-r^{2(N-1)})dr \\
&+ 2\pi \int_1^\infty \sum_{m=0}^{M} \binom{M}{m} \Bigg[\binom{L}{m+\frac{N}{N+1}(L-M)} r^{-2s+2m(N+1)-2NM+1} \\
&\qquad\quad \times \left(\frac{1}{N}\right)^{\frac{2N+1}{N+1}M+\frac{1}{N+1}L-2m} (1+r^{-2(N+1)}) \\
&\qquad - \binom{L}{m+\frac{N}{N+1}(L-M)+1} r^{-2s+2m(N+1)-2NM+1} \left(\frac{1}{N}\right)^{\frac{2N+1}{N+1}M+\frac{1}{N+1}L-2m-1} \\
&\qquad - \binom{L}{m+\frac{N}{N+1}(L-M)-1} r^{-2s+2m(N+1)-2NM-2N-1} \left(\frac{1}{N}\right)^{\frac{2N+1}{N+1}M+\frac{1}{N+1}L-2m+1} \Bigg] dr.
\end{aligned}
$$

Note that the binomial terms previously had only integer entries; we thus note that $\langle \alpha^M, \alpha^L \rangle = 0$ if $M \not\equiv L \mod (N+1)$; going forward, we thus assume $M \equiv L \mod (N+1)$. To see the symmetry of $\langle \alpha^M, \alpha^L \rangle$, we substitute $m = n + \frac{N}{N+1}M$.

$$
\begin{aligned}
&2\pi \int_0^1 \sum_{n=\frac{-N}{N+1}M}^{\frac{1}{N+1}M} \binom{M}{n+\frac{N}{N+1}M}\binom{L}{n+\frac{N}{N+1}L} \\
&\qquad \times \left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n} r^{2n(1-N)+\frac{2N}{N+1}(L+M)+1}(1-r^{2(N-1)})dr \\
&+ 2\pi \int_1^\infty \sum_{n=\frac{-N}{N+1}M}^{\frac{1}{N+1}M} \binom{M}{n+\frac{N}{N+1}M}\left[\binom{L}{n+\frac{N}{N+1}L}\right. \\
&\qquad\qquad \times \left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n} r^{-2s+2n(N+1)+1}(1+r^{-2(N+1)}) \\
&\qquad - \binom{L}{n+\frac{N}{N+1}L+1}\left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n-1} r^{-2s+2n(N+1)+1} \\
&\qquad \left. -\binom{L}{n+\frac{N}{N+1}L-1}\left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n+1} r^{-2s+2n(N+1)-2N-1}\right] dr.
\end{aligned}
$$

Note that within each binomial term $\binom{K}{k}$, if $k < 0$ or $k > K$, the binomial is identically zero; we may thus reindex some terms by $n \to n+1$ to align powers of r, obtaining

$$2\pi\int_0^1 \sum_{n=\frac{-N}{N+1}M}^{\frac{1}{N+1}M} r^{2n(1-N)+\frac{2N}{N+1}(L+M)+1}\left[\binom{M}{n+\frac{N}{N+1}M}\binom{L}{n+\frac{N}{N+1}L}\left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n}\right.$$
$$\left.-\binom{M}{n+\frac{N}{N+1}M+1}\binom{L}{n+\frac{N}{N+1}L+1}\left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n-2}\right]dr$$
$$+\,2\pi\int_1^\infty \sum_{n=\frac{-N}{N+1}M}^{\frac{1}{N+1}M} r^{-2s+2n(N+1)+1}\left[\binom{M}{n+\frac{N}{N+1}M}\binom{L}{n+\frac{N}{N+1}L}\left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n}\right.$$
$$+\binom{M}{n+\frac{N}{N+1}M+1}\binom{L}{n+\frac{N}{N+1}L+1}\left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n-2}$$
$$-\binom{M}{n+\frac{N}{N+1}M}\binom{L}{n+\frac{N}{N+1}L+1}\left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n-1}$$
$$\left.-\binom{M}{n+\frac{N}{N+1}M+1}\binom{L}{n+\frac{N}{N+1}L}\left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n-1}\right]dr.$$

Integrating gives

$$2\pi\sum_{n=\frac{-N}{N+1}M}^{\frac{1}{N+1}M}\left\{\frac{1}{2n(1-N)+\frac{2N}{N+1}(L+M)+2}\left[\binom{M}{n+\frac{N}{N+1}M}\binom{L}{n+\frac{N}{N+1}L}\left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n}\right.\right.$$
$$\left.-\binom{M}{n+\frac{N}{N+1}M+1}\binom{L}{n+\frac{N}{N+1}L+1}\left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n-2}\right]$$
$$+\frac{1}{2s-2n(N+1)-2}\left[\binom{M}{n+\frac{N}{N+1}M}\binom{L}{n+\frac{N}{N+1}L}\left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n}\right.$$
$$+\binom{M}{n+\frac{N}{N+1}M+1}\binom{L}{n+\frac{N}{N+1}L+1}\left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n-2}$$
$$-\binom{M}{n+\frac{N}{N+1}M}\binom{L}{n+\frac{N}{N+1}L+1}\left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n-1}$$
$$\left.\left.-\binom{M}{n+\frac{N}{N+1}M+1}\binom{L}{n+\frac{N}{N+1}L}\left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n-1}\right]\right\}.$$

Now, by factoring out $\binom{M}{n+\frac{N}{N+1}M}\binom{L}{n+\frac{N}{N+1}L}\left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n}$ from all terms, we arrive at

$$
\begin{aligned}
2\pi \sum_{n=\frac{-N}{N+1}M}^{\frac{1}{N+1}M} & \binom{M}{n+\frac{N}{N+1}M}\binom{L}{n+\frac{N}{N+1}L}\left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n} \left\{ \frac{1}{2n(1-N)+\frac{2N}{N+1}(L+M)+2} \right. \\
& \times \left[1-\left(\frac{\frac{M}{N+1}-n}{n+\frac{N}{N+1}M+1}\right)\left(\frac{\frac{L}{N+1}-n}{n+\frac{N}{N+1}L+1}\right)N^2\right] \\
& \left. +\frac{1}{2s-2n(N+1)-2}\left[1-\frac{\frac{M}{N+1}-n}{n+\frac{N}{N+1}M+1}N\right]\left[1-\frac{\frac{L}{N+1}-n}{n+\frac{N}{N+1}L+1}N\right]\right\}.
\end{aligned}
$$

Expanding the terms and simplifying gives the desired

$$
\begin{aligned}
\pi \sum_{n=\frac{-N}{N+1}M}^{\frac{1}{N+1}M} & \binom{M}{n+\frac{N}{N+1}M}\binom{L}{n+\frac{N}{N+1}L}\left(\frac{1}{N}\right)^{\frac{M+L}{N+1}-2n} \frac{n(N+1)+1}{(n+\frac{N}{N+1}M+1)(n+\frac{N}{N+1}L+1)} \\
& \times \frac{s}{s-n(N+1)-1}.
\end{aligned}
$$

Future work

Real case

We mentioned in the introduction a parallel set of star bodies for a restriction to real polynomials. In chapter 3 we discussed the degree d complex star body $\mathcal{V}_d(\Phi)$ using the cumulative distance function $h_d(\Phi;\xi)$ and the complex moment function $H_d(\Phi;s)$. We may likewise ponder the degree d *real star body* of Φ

$$\mathcal{U}_d(\Phi) := \{\mathbf{u} \in \mathbb{R}^{d+1} : \Phi(\mathbf{u}) \leq 1\}$$

utilizing the cumulative distance function

$$f_d(\Phi; \xi) := \lambda_d\{\mathbf{b} \in \mathbb{R}^d : \tilde{\Phi} \leq \xi\},$$

and *real moment function*

$$F_d(\Phi; s) := \int_{\mathbb{R}^d} \tilde{\Phi}(\mathbf{b})^{-s} d\lambda_d(s).$$

Importantly, this line of study utlizes a skew-symmetric inner product, rather than an inner product as used for studying $H_d(\Phi; s)$. We may define a pair of skew-symmetric inner products $\langle \cdot; \cdot \rangle_{\mathbb{R}}$ and $\langle \cdot; \cdot \rangle_{\mathbb{C}}$ by

$$\langle f; g \rangle_{\mathbb{R}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x)^{-s} \varphi(y)^{-s} f(x) g(y) \operatorname{sgn}(y - x)\, dx\, dy;$$
$$\langle f; g \rangle_{\mathbb{C}} = -2i \int_{\mathbb{C}} \varphi(\beta)^{-s} \varphi(\bar{\beta})^{-s} \overline{f(\beta)} g(\beta) \operatorname{sgn}(\Im(\beta)) d\lambda_2(\beta).$$

Given Q a complete family of monic polynomials in $\mathbb{R}[x]$, we may create $d \times d$ anti-symmetric matrices R_Q and C_Q whose j, k entries are $\langle Q_j; Q_k \rangle_{\mathbb{R}}$ and $\langle Q_j; Q_k \rangle_{\mathbb{C}}$, respectively. Finally, define the antisymmetric matrix $U_Q = R_Q + C_Q$; U_Q serves as an antisymmetric analog to the Gram matrix W_Q explored in the complex case.

As a further difference from the complex case, rather than working with a determinant as in $H_d(s) = \det(W_Q)$, we must instead work with the Pfaffian; for $d = 2J$ and U a $d \times d$ anti-symmetric matrix, the Pfaffian of U is

$$\operatorname{Pf}(U) = \frac{1}{2^J J!} \sum_{\tau \in S_{2J}} \operatorname{sgn}(\tau) \prod_{j=1}^{J} U(\tau(2j-1), \tau(2j)).$$

The Pfaffian is often thought of as the signed square root of the determinant of the matrix. As it turns out, $F_d(s) = \mathrm{Pf}(U_Q)$. Utilizing this, Chern and Vaaler [3] found

Theorem 15. (Chern and Vaaler) *For $d = 2J$,*

$$F_d(\mu; s) = 2^d \prod_{j=1}^{J} \left(\frac{2j}{2j+1}\right)^{d-2j} \prod_{i=0}^{J-1} \frac{s}{s-(d-2i)}.$$

In the case of the reciprocal Mahler measure, Sinclair [4] similarly found

Theorem 16. (Sinclair) *For $d = 2J$,*

$$F_d(\mu_1; s) = \frac{2^d}{d!} \prod_{n=1}^{d} \left(\frac{2n}{2n-1}\right)^{d+1-n} \prod_{j=0}^{J-1} \frac{s^2}{s^2-(d-2j)^2}.$$

From this, one may see that in both cases, $F_d(s) \in \mathbb{Q}(s)$ has numerator and denominator of matching degrees, integer poles $\leq\ d$ (all positive in the Mahler measure case), and the origin as a root of high multiplicity. We thus speculate:

Conjecture 1. *For $N > 1$, $F_d(\mu^{(N)}; s) \in \mathbb{Q}(s)$ has numerator and denominator of matching degrees, integer poles $\leq d$, and the origin as a root of high multiplicity.*

Other hypotrochoids

Much as the ellipses served to smoothly connect the circular and interval cases, we may explore other families of hypotrochoids to smoothly connect the circular and cusped hypocycloids. Much as we had the conformal map $\alpha \mapsto z + q/z$ sending the unit disk to interior of the ellipse E_q for $q \in [0, 1]$, we could consider a conformal map $\alpha \mapsto z + q\dfrac{z^{-N}}{N}$ and quasi-conformal map $\alpha \mapsto z + q\dfrac{\bar{z}^{N}}{N}$ sending the unit disk to the interior of a hypotrochoid with $(N + 1)$ petals; see Figure 3 for an example with 5 cusps. For further discussion, let us call these hypotrochoids $\mathcal{H}_q^{(N)}$,

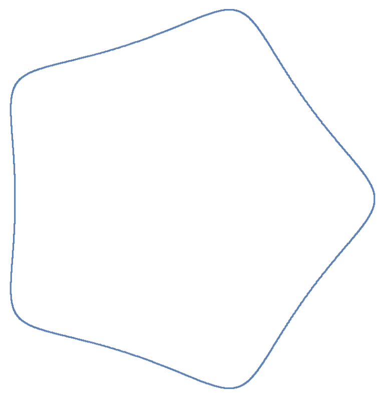

FIGURE 3 A rendition of $\mathcal{H}_{0.4}^{(4)}$, the 5-cusped 0.4-hypotrochoid.

and the associated generalized Mahler measures $\mu_q^{(N)}$. Based on the behaviors of μ_q and $\mu^{(N)}$, we speculate:

Conjecture 2. *For $q \in (0,1)$ and positive integers $N \geq 2$ and d, $H_d(\mu_q^{(N)}; s)$ is rational in s and polynomial in q. As a function of s, it has poles at nonzero integers $\leq d$, the origin as a root of mulitplicity d, and additional roots so that the degress of numerator and denominator are equal.*

We may also wish to explore non-integer cusped hypocycloids. If $N+1$ is a non-integer rational number with reduced form $\frac{a}{b}$, the corresponding hypocycloid has a cusps, but unlike the integer case, the boundary produced by the conformal map intersects itself within the interior of the region; see Figure 4 for an example with $N+1 = 5/3$. Careful use of algebraic geometry could produce a modified conformal map for further exploration.

Conjecture 3. *For $N \in \mathbb{Q}\backslash\mathbb{Z}$, $H_d(\mu^{(N)}; s)$ is rational in s with matching degrees in numerator and denominator. This function has poles at nonzero rational values $\leq d$, the origin as a root of multiplicity d, and additional roots.*

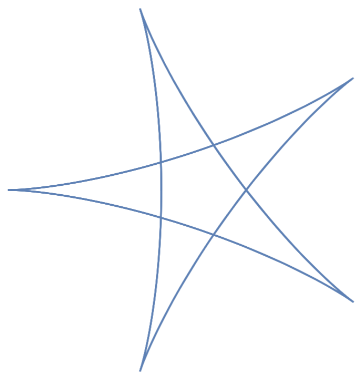

FIGURE 4 A rendition of the $\frac{5}{3}$-cusped hypocycloid.

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