A SPECIAL ENDOMORPHISM OF THE STANDARD GAITSGORY CENTRAL OBJECT OF THE AFFINE HECKE CATEGORY

by

JAY HATHAWAY

A DISSERTATION

Presented to the Department of Mathematics
and the Division of Graduate Studies of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

December 2023

DISSERTATION APPROVAL PAGE

Student: Jay Hathaway

Title: A Special Endomorphism of the Standard Gaitsgory Central Object of the Affine Hecke Category

This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Ben Elias	Chair
Nicolas Addington	Core Member
Victor Ostrik	Core Member
Nicholas Proudfoot	Core Member
Anca Cristea	Institutional Representative

and

Krista Chronister	Vice Provost for Graduate Studies

Original approval signatures are on file with the University of Oregon Division of Graduate Studies.

Degree awarded December 2023

DISSERTATION ABSTRACT

Jay Hathaway

Doctor of Philosophy

Department of Mathematics

December 2023

Title: A Special Endomorphism of the Standard Gaitsgory Central Object of the Affine Hecke Category

Using the combinatorial description of the standard Gaitsgory central object of the (extended, graded) affine type A Hecke category due to Elias, we show the existence of and explicitly describe the unique endomorphism that lifts right multiplication by the i-th fundamental weight on the i-th component of the associated graded of its Wakimoto filtration. We give work in progress on describing a conjectural program to categorify the Vershik-Okounkov approach to the representation theory of the affine Hecke algebra. Here this endomorphism will play a role. This is the affine version of the program described by Gorsky, Negut, and Rasmussen in finite type A.

CURRICULUM VITAE

NAME OF AUTHOR: Jay Hathaway

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR, USA
University of Texas, Austin, TX, USA

DEGREES AWARDED:

Doctor of Philosophy, Mathematics, 2023, University of Oregon
Bachelor of Sciences, Mathematics, 2017, University of Texas

AREAS OF SPECIAL INTEREST:

Statistics
Machine Learning
Representation Theory

PROFESSIONAL EXPERIENCE:

Graduate Employee, Department of Mathematics, University of Oregon, 2017-2023

ACKNOWLEDGEMENTS

Thank you to Ben Elias, my advisor, for all his help with the mathematical content of this thesis, particularly for his expertise on all things Weyl groups, braid groups, Hecke algebras, and Soergel bimodules. Also I must mention his support and help during graduate school allowing me to have many amazing experiences in math. Lastly I must thank him for his helpful editorial comments.

Thank you especially to Eugene Gorsky for originally suggesting to search for the endomorphism χ. I would like to thank him again and also Andrei Negut for explaining how to think of flag commuting stacks as iterated graded affine bundles.

Thank you to Nick Proudfoot for teaching me about GIT and symplectic quotients, and for his support during my time in graduate school as both Graduate Studies Director and later as Department Head.

Thank you to Elijah Bodish for being a huge presence in the positive experiences I've had in graduate school, and also for his helpful advice on thesis writing which helped me break through writers block.

Thank you to several other mathematical friends for keeping school interesting with their conversations: Surya Raghavendran, Daniel Chupin, Holt Bodish, Max Vargas, and Corey Brooke, and many others I fail to mention.

Thank you James Murray and David Levin for supporting my transition toward an interest in statistics and machine learning.

I was supported by Ben Elias's NSF CAREER grant DMS-1553032 and the NSF FRG grant DMS-1800498.

TABLE OF CONTENTS

Chapter | Page

Chapter Page

Chapter Page

Chapter	Page

CHAPTER I

INTRODUCTION

1.1 Spectral Theory

Given an associative algebra A, one method of studying its representation theory is to study the simultaneous eigenvalues of its center $Z(A)$. The simultaneous eigenspaces of $Z(A)$ are A-submodules, so simultaneously diagonalizing $Z(A)$ is a way to find isotypic components. Sometimes the simultaneous eigenvalues of $Z(A)$ are difficult to understand, but there is a larger commutative subalgebra S of A, with $Z(A) \subset S$, whose simultaneous eigenvalues are easier to understand. While S-eigenspaces are not A-submodules, studying them can nevertheless help to understand the structure of A-modules. A classic example of this is in Lie theory when one studies the simultaneous eigenvalues of the Cartan subalgebra of a Lie algebra, i.e. the weights.

The notion of simultaneous eigenvalues can be reinterpreted in the language of algebraic geometry. A simultaneous eigenvalue of S is a closed point in $\mathrm{Spec} S$, and studying the simultaneous S-eigenspaces of an A-module amounts to studying its support as a quasi-coherent sheaf on $\mathrm{Spec} S$. One satisfying feature of this picture is that the symmetrizer onto the simultaneous eigenspace corresponding to a point $\lambda \in \mathrm{Spec} S$ is given by the Dirac delta function δ_λ.

1.2 Vershik-Okounkov Approach

Let's now expand on an important example of spectral theory. In [22], Vershik and Okounkov reformuate the representation theory of the symmetric group by studying the spectrum of the subalgebra of its group algebra generated by the Young-Jucys-Murphy Operators j_i 2.5.15. Remarkably, these operators commute, and the center $Z(\mathbb{C}[S_n])$ is shown to be generated by symmetric polynomials of the j_i.

As explained in [44], this setup deforms to the Hecke algebra $\mathbf{H}^n_{\mathrm{fin}}$ of S_n. Here too there are Jucys-Murphys operators j_i, they all commute, and symmetric polynomials in the j_i generate $Z(\mathbf{H}^n_{\mathrm{fin}})$.

1.3 Categorified Spectral Theory

The categorification of an algebra is a monoidal category $\mathcal{C}$. The Drinfel'd center of $\mathcal{C}$ consists of objects A in $\mathcal{C}$ equipped with natural isomoprhisms $A \otimes (-) \xrightarrow{\sim} (-) \otimes A$, which forms a braided monoidal category. Forgetting these natural isomorphisms, we obtain a full subcategory of $\mathcal{C}$ which we refer to as the *naive* Drinfel'd center. Sometimes it is a symmetric monoidal full subcategory. A large source of symmetric monoidal categories is $\mathrm{Coh}(Y)$ where Y is a (derived) scheme or stack. The categorical analogue of the discussion in section 1.1 is the following.

Given a monoidal category $\mathcal{C}$ can we find a symmetric monoidal full subcategory $\mathcal{S}$, containing the naive Drinfel'd center of $\mathcal{C}$, and a (derived) scheme or stack Y so that $\mathcal{S}$ is realized, by concretely defined functors, as the category of coherent sheaves on Y? If we can, then the skyscraper sheaves at closed points in Y give categorical symmetrizers. In the setting of triangulated and dg-categories, the categorical symmetrizer is given by a "normalized" skyscraper sheaf, to account for the fact that while skycraper sheaves are idempotent under the usual tensor product, they are not idempotent under the derived tensor product.

In [6], Elias and Hogancamp introduce the theory of categorical diagonalization. Given a (graded) dg-pretriangulated monoidal category $\mathcal{C}$ and an invertible object F, then F is said to be categorically prediagonalizable if there exist 'eigencones' $[\mathbf{1}(\lambda_i) \stackrel{\alpha_i}{\to} F]$ such that

$$\bigotimes_i \left(\mathbf{1}(\lambda_i) \stackrel{\alpha_i}{\to} F \right) = 0.$$

Here $\mathbf{1}(\lambda_i)$ is a grading shift of the monoidal unit. Thus λ_i is the eigenvalue and α_i is the eigenmap. Given a prediagonalizable functor whose eigenvalues are sufficiently distict in a category $\mathcal{C}$ which is suitably complete, Elias and Hogancamp construct objects P_λ categorifing the Lagrange interpolators:

$$P_{\lambda_i} := \frac{\prod_{j \neq i}(F - \lambda_j I)}{\prod_{j \neq i}(\lambda_i - \lambda_j)},$$

thus giving categorical projectors onto eigencategories. Categorifying the denominator in the above formula requires working with semi-infinite complexes hence the necessity of a completeness assumption.

In [3, Chapter 4], it is explained how the work of Elias-Hogancamp can be interpreted via algebraic geometry. The space Y is a projective space with a certain torus action prescribed by the eigenvalues of F, so the fixed points correspond to eigenvalues. It has an affine stratification via the torus action given by the Bialynicki-Birula decomposition. Let U_λ denote the ascending set, an affine chart, and let p_λ denote the corresponding torus fixed point. The categorical symmetrizers are recovered by pushing forward $S^\bullet \nu_\lambda^\vee$ where ν_λ is the normal bundle of U_λ. This sheaf is equal to a multiple of the skyscraper sheaf $\mathcal{O}_{p_\lambda}$ after descending to K-theory and is idempotent under the derived tensor product of $\mathcal{O}_Y$-modules.

1.4 Hecke category

There is a category, the finite type A Hecke category $\mathcal{H}^n_{\text{fin}}$, which categorifies $\mathbf{H}^n_{\text{fin}}$. It has several constructions: representation-theoretic (BGG category $\mathcal{O}$), algebraic (Soergel bimodules), combinatorial (Elias-Williamson), and geometric (sheaves/D-modules on flag varieties). There has been much progress in recent years to categorify the Vershik-Okounkov approach by understanding the categorified spectrum of the full subcategory $\mathcal{J}$ of $\mathcal{H}^n_{\text{fin}}$ generated (under direct sums, tensor products, grading shifts, and cones/cocones) by the Jucys-Murphy objects J_i.

The Jucys-Murphy subalgebra J of $\mathbf{H}^n_{\text{fin}}$ generated by the Jucys-Murphy braids j_i can also be generated by the full twists FT_i. Thus to understand its spectrum, one could have simultaneously diagonalized the FT_i rather than the j_i as in the original paper of Vershik-Okounkov. To this end, in [7], Elias and Hogancamp apply their theory of categorical diagonalization to the objects of $\mathcal{H}^n_{\text{fin}}$ corresponding to the full twists, which we also denote FT_i. Since the FT_i also generate (in the stable dg/triangulated monoidal sense) the category $\mathcal{J}$, this allows them to construct categorical projectors onto simulataneous eigencategories for $\mathcal{J}$.

1.5 Flag Hilbert Schemes and Categorified Projectors

Now, wouldn't it be great if the results of Elias and Hogancamp diagonalizing categorified full twists could be recast in terms of algebraic geometry? This is exactly what Gorsky, Negut, and Rasmussen (GNR) sought to do in [3]. They propose coherent sheaves on the (derived) flag Hilbert scheme of points in the plane supported on the x-axis (FHilb_n) as the spectral incarnation of the Jucys-Murphy category $\mathcal{J}$.

Now, the inclusion $\iota : J \to \mathbf{H}^n_{\text{fin}}$ has an adjoint, $\iota_* : \mathbf{H}^n_{\text{fin}} \to J$. Likewise, the inclusion of the center $\iota : Z(\mathbf{H}^n_{\text{fin}}) \hookrightarrow \mathbf{H}^n_{\text{fin}}$ also has an adjoint $\iota_* : \mathbf{H}^n_{\text{fin}} \to Z(\mathbf{H}^n_{\text{fin}})$. This is because the standard form 2.5.12 restricts to non-degenerate forms on J and $Z(\mathbf{H}^n_{\text{fin}})$. They conjecture this can be categorified by adjoint functors

$$\mathcal{H}^n_{\text{fin}} \underset{\iota^*}{\overset{\iota_*}{\rightleftarrows}} \text{Coh}_{\mathbb{C}^* \times \mathbb{C}^*}(\text{FHilb}_n), \qquad \iota^* \dashv \iota_*. \tag{1.5.1}$$

The torus equivariance comes from the torus action on the plane. These functors make $\mathcal{H}^n_{\text{fin}}$ into a *category over* (see 6.2.1) the Hilbert scheme. These functors exchange the Jucys-Murphy objects J_i with the i-th tautological line bundle $\mathcal{L}_i$ on FHilb_n.

Similarly, we let $\mathcal{Z}$ the naive Drinfel'd center of $\mathcal{H}^n_{\text{fin}}$. Let Hilb_n denote the (derived) Hilbert scheme of n points in the plane supported on the x-axis. They also conjecture that $\mathcal{H}^n_{\text{fin}}$ has adjoint functors

$$\mathcal{H}^n_{\text{fin}} \underset{\iota^*}{\overset{\iota_*}{\rightleftarrows}} \text{Coh}_{\mathbb{C}^* \times \mathbb{C}^*}(\text{Hilb}_n), \tag{1.5.2}$$

and that the essential image of ι^* should be related to the center $\mathcal{Z}$. These functors can be constructed from the ones to FHilb_n by using the map p : $\text{FHilb}_n \to \text{Hilb}_n$.

Remarkably, GNR show how you can recover the categorified projectors of Elias-Hogancamp by considering skyscrapers sheaves at torus fixed-points in FHilb_n.

1.5.1 Connection to Knot Theory. Let $\text{Tr} : \mathbf{H}_{\text{fin}} \to \mathbb{Z}[a^{\pm 1}](v)$ denote the Jones-Ocneanu trace as in [12, Theorem 5.2.2]. For a planar braid β, we have $\text{Tr}(\beta)$ is the HOMFLY invariant ([40]) of the braid closure $\overline{\beta}$ of β which is a link in

S^3. Let $\iota_* : \mathbf{H}^n_{\text{fin}} \to J$ denote the adjoint of the inclusion. Then

$$\mathrm{Tr}(\beta) = \mathrm{Tr}\mid_J (\iota_*(\beta)). \tag{1.5.3}$$

Thus HOMFLY invariants may be computed just by considering traces of elements of J. The HOMFLY invariant is categorified by the derived categorical trace of the object corresponding to β in the Hecke category, or more concretely by the total Hochshild cohomology of its Rouquier complex of Soergel bimodules, which explicitly gives the *HOMFLY homology* of $\overline{\beta}$. Let $\mathrm{HHH}(\overline{\beta})$ denote the HOMFLY homology of $\overline{\beta}$. Ignoring issues of matching gradings, GNR conjecture that equation 1.5.3 is categorified by:

$$\mathrm{HHH}^0(\overline{\beta}) = R\Gamma_{\mathrm{FHilb}}(\iota_* F_\beta), \tag{1.5.4}$$

with similar interpretations of HHH^i for all i.

1.5.2 Inductive Approach. Using the fact that FHilb_n can be realized as an iterated (dg) projective bundle, GNR give an inductive approach to constructing the functors in 1.5.1. The tautological bundle $\mathcal{T}_n$ on FHilb_n is filtered by the tautological line bundles $\mathcal{L}_i$. An important piece of the inductive construction is the need for the incarnation of $\mathcal{T}_n$ in the Hecke category, which should be $\iota^*\mathcal{T}_n$, along with incarnations of its endomorphisms X and Y. We denote this object as T_n It should carry a filtration by the Jucys-Murphy objects J_i. The construction then involves checking some concrete properties of T_n, X and Y purely in the Hecke category [3, Conjecture 3.9].

While the object T_n and the endomorphisms X, Y have not yet been constructed, a version of T_n and Y have been constructed in the setting of affine

Hecke categories. This leads us to consider an affine version of the whole GNR story.

1.6 Flag Commuting Stack and Affine Hecke Category

Let $\mathbf{H}^n_{\text{ext}}$ denote the extended affine Hecke algebra of type A_n (see 2.5.2). It contains a (non-unique) copy of the weight lattice Λ_{wt} of $\mathfrak{gl}_n$, generated by the Wakimoto braids y_i 2.4.14. The subalgebra $\mathbb{Z}[v^{\pm 1}][\Lambda_{\text{wt}}] \subset \mathbf{H}^n_{\text{ext}}$ plays the role of J. It is known that symmetric polynomials in the y_i generate $Z(\mathbf{H}^n_{\text{ext}})$. Thus one could understand the representation theory of $\mathbf{H}^n_{\text{ext}}$ through the simulaneous eigenvalues of $\mathbb{Z}[v^{\pm 1}][\Lambda_{\text{wt}}]$. The inclusions $\iota_* : \mathbb{Z}[v^{\pm 1}][\Lambda_{\text{wt}}] \to \mathbf{H}^n_{\text{ext}}$ and $\iota : Z(\mathbf{H}^n_{\text{ext}}) \to \mathbf{H}^n_{\text{ext}}$ have adjoints $\iota_* : \mathbf{H}^n_{\text{ext}} \to \mathbb{Z}[v^{\pm 1}][\Lambda_{\text{wt}}]$ and $\iota_* : \mathbf{H}^n_{\text{ext}} \to Z(\mathbf{H}^n_{\text{ext}})$ because the standard form of $\mathbf{H}^n_{\text{ext}}$ remains non-degenerate after restricting to these subalgebras.

There is a categorification of $\mathbf{H}^n_{\text{ext}}$, the extended affine Hecke category $\mathcal{H}^n_{\text{ext}}$ (see 3.3.19). It has Wakimoto objects W_i (see 4.1.12) categorifying the y_i. They generate a full subcategory (in the stable dg/triangulated monoidal sense) which we denote Wak. Due to motivation coming from the geometric Langlands program that we will discuss later, it is expected that the flag commuting stack FComm_n, which is the derived stack parametrizing commuting pairs of upper-triangular $n \times n$ matrices X and Y with Y nilpotent, plays the role in the affine setting that FHilb_n played in the finite setting of GNR.

This yields an 'affine version' of the GNR conjecture categorifying the inclusion $\mathbb{Z}[v^{\pm 1}][\lambda_{\text{wt}}] \to \mathbf{H}^n_{\text{ext}}$ and its adjoint.

Conjecture 1.6.1. *(Conjecture 7.3.3) There exists functors:*

$$\mathcal{H}^n_{\text{ext}} \underset{\iota^*}{\overset{\iota_*}{\rightleftarrows}} \mathrm{Coh}_{\mathbb{C}^* \times \mathbb{C}^*}(\mathrm{FComm}_n), \qquad \iota^* \dashv \iota_*, \tag{1.6.1}$$

making $\mathcal{H}^n_{\mathrm{ext}}$ into a category over FComm_n. These functors exchange the i-th tautological line bundle of FComm_n with the i-th Wakimoto object W_i.

As was communicated to us by Gorsky and Negut [39], the underlying classical stack $\mathrm{FComm}^{\mathrm{cl}}_n$ can be realized as an iterated graded affine bundle (see 7.1.3). We conjecture (Conjecture 7.1.4) that this lifts to derived enhancements.

In section 7.3 we describe an inductive procedure to construct the functors in 1.6.1. Let $\mathcal{T}_n$ denote the tautological vector bundle of FComm_n (see 7.1.1). Similarly to the program described by GNR, ours involves checking some properties (see 7.3.2) of the object $\mathcal{V}_n$ corresponding to $\iota^*(\mathcal{T}_n)$ along with its endomorphisms X and Y.

1.6.1 Relating Finite and Affine GNR Conjectures. We assume the existence of the yet undefined flattening functor $\flat : \mathcal{H}_{\mathrm{ext}} \to \mathcal{H}_{\mathrm{fin}}$ from the extended affine Hecke category to the finite Hecke category [27, 1.6], which categorifies the map described in 2.4.4. Like the bifunctor in 1.9.1, this functor is well-defined on the additive category of Soergel bimodules $\mathbb{S}\mathrm{Bim}$ (see [17]), but extending it to the homotopy category remains open.

We expect the GNR conjecture and the affine GNR conjecture to be compatble in the following way. Note that there is map $\pi : \mathrm{FHilb}_n \to \mathrm{FComm}_n$ given by forgetting the cyclic vector (the stability condition). We conjecture that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{H}^n_{\mathrm{ext}} \arrow[r, shift left, "\iota_*"] \arrow[d, "\flat"'] & \mathrm{Coh}_{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{FComm}_n) \arrow[l, shift left, "\iota^*"] \arrow[d, "\pi^*"'] \\
\mathcal{H}^n_{\mathrm{fin}} \arrow[r, shift left, "\iota_*"] & \mathrm{Coh}_{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{FHilb}_n) \arrow[l, shift left, "\iota^*"]
\end{tikzcd} \quad . \tag{1.6.2}$$

Here, the flattening functor sends the Rouquier complex for the i-th Wakimoto braid to the Rouquier complex for the i-th Jucys-Murphy braid.

1.6.2 Connection to Knots in the Thickened 2-Torus. Given a cylindrical braid β (see 2.4.6), one can associate an object F_β of $\mathcal{H}_{\text{ext}}$, its Rouquier complex 4.1. The closure of a cylindrical braid is naturally a link in the thickened 2-torus $T^2 \times I$. Using total Hochschild cohomology of F_β, we define the *toroidal HOMFLY-PT homology* of such a link in 4.1.16. We hope that in further work one could exploit the geometry of FComm to establish similar idenitities as (GNR) do for ordinary HOMFLY homology in the case of toroidal HOMFLY homology.

1.7 Standard Gaitsgory Central Object

The object $\mathcal{V}_n$ should carry an n-layer filtration, where the i-th graded component of the associated graded is the i-th Wakimoto object W_i. Such an object of $\mathcal{H}_{\text{ext}}$ is implicit in the work of Gaitsgory (see [10]) in constructing central objects of affine Hecke categories. We won't go in to the details of that construction, which involves nearby cycles of sheaves. We now summarize some features of Gaitsgory's construction in type A. To a representation V of GL_n, Gaitsgory constructs an object $\mathcal{Z}(V)$ of $\mathcal{H}_{\text{ext}}$ satisfying:

1. $\mathcal{Z}(V)$ can be upgraded to an object of the Drinfel'd center.

2. $\mathcal{Z}(V)$ carries a filtration by Wakimoto objects corresponding to the weight filtration of V.

3. $\mathcal{Z}(V)$ carries a nilpotent endomorphism μ, the log monodromy, corresponding to the principal nilpotent operator on V, i.e. the sum $\sum_{\alpha_i \in \Delta} e_i$ of the raising operators e_i for each of the simple roots.

4. $\mathcal{Z}(V)$ is in the heart of the perverse t-structure on $\mathcal{H}_{\text{ext}}$.

Thus if V is the standard representation of GL_n, the object $\mathcal{Z}(V)$, along with its nilpotent log monodromy endomorphism μ, account for $\mathcal{V}_n$ and Y.

In the setting of the diagrammatic Hecke category, Elias (see [27]) gives a purely combinatorial construction of $\mathcal{V}_n$ and Y. Because that category is defined over $\mathbb{Z}$, this construction also applies to the setting of parity sheaves by the work of [23] relating the diagrammatic Hecke category to parity sheaves. We must also mention the work of Achar and Rider [14] constructing this object directly in the setting of parity sheaves. Their construction agrees with Elias's up to homotopy equivalence.

1.8 Main result and The Map χ

While $\mathcal{V}_n$ and Y are accounted for in the works mentioned above, the map X is still missing. The main technical result of this work is proving the existence and uniquenes of X, and identifying an explicit formula for it in terms of the combinatorial description of $\mathcal{V}_n$ due to Elias.

Theorem 1.8.1. *(Main Theorem 5.1.6) There exists a unique chain-map $\chi : \mathcal{V}_n \to \mathcal{V}_n(2)$, upper-triangular with respect to the Wakimoto filtration of $\mathcal{F}$, which lifts right multiplication by the fundamental weight x_i on the i-th graded component of the associated graded of the filtration.*

With our result, the necessary ingredients to pursue an affine version of the GNR program are all accounted for.

1.9 Related Work and Motivation

1.9.1 Elliptic Hall Algebra and HOMFLY-PT Skein of the Torus.

Motivated by work of Morton and Samuelson [31] giving an isomorphism between the HOMFLY-PT skein algebra of the 2-torus and the elliptic Hall algebra, Gorsky and Negut [30] explore the relation between the derived trace of the extended affine Hecke category $\mathcal{H}_{\mathrm{ext}}$, i.e. its cocenter, and the K-theoretic elliptic Hall algebra. The latter category is built from Hecke correspondences between commuting varieties, given by 2-step flag commuting varieties.

Let $\mathcal{H}^n_{\mathrm{ext}}$ denote the extended affine Hecke category of type A_{n-1}. Let $\mathrm{Tr}(\mathcal{H}^n_{\mathrm{ext}})$ denote its derived trace. There is a yet undefined bifunctor

$$\mathcal{H}^n_{\mathrm{ext}} \otimes \mathcal{H}^m_{\mathrm{ext}} \to \mathcal{H}^{n+m}_{\mathrm{ext}} \tag{1.9.1}$$

categorifying the inclusion in 2.6.3. They work under the assumption that induced functor on the trace categories $\mathrm{Tr}(\mathcal{H}_{\mathrm{ext}})$ should categorify concatenation in the HOMFLY-PT Skein algebra of the 2-torus. They show that the categorical commutators between certain objects in the trace categories $\mathrm{Tr}(\mathcal{H}_{\mathrm{ext}})$ and the K-theoretic Hall algebra agree.

1.9.2 Betti Geometric Langlands. Bezrukanikov's equivalance [2] identifies the extended affine Hecke category with $\mathrm{Coh}_{\mathrm{GL}_n}(Z)$ where Z is the (derived) Steinberg variety. Using this equivalence, Ben-Zvi, Nadler, and Preygel show that the cocenter of the extended affine Hecke category is naturally $\mathrm{Coh}(\mathrm{Comm})$. Their models for these categories, derived cocenters, and derived spaces all use the language of modern derived algebraic geometry and ∞-category theory.

In subsequent work, Nadler, Li, and Yun [38] exploit this connection between the (derived) commuting variety and the extended affine Hecke category to compute its dg-coordinate ring.

In remarkable recent work of Li and Ho [43], the authors upgrade the results of Ben-Zvi, Nadler, and Preygel, and also those of Bezrukavnikov, Finkelberg, and Ostrik (see [1]) on centers and traces of Hecke categories to the setting of graded Hecke categories (e.g. mixed or parity sheaves, Soergel Bimodules, diagrammatic category). In finite type A, they establish a proof of GNR's conjecture computing HOMFLY homology via sheaves on Hilb_n.

1.9.3 Matrix Factorizations and Hilbert Schemes. In [21], and [20], Oblomkov and Rozansky describe categories of matrix factorizations MF, and AMF, along with strict braid group actions from the ordinary braid group $\mathrm{Br}_{\mathrm{fin}} \to MF$ and the cylindrical braid group $\mathrm{Br}_{\mathrm{ext}} \to AMF$. They also give functors

$$MF \to \mathrm{Coh}(\mathrm{Hilb}), \qquad AMF \to \mathrm{Coh}(\mathrm{Comm}). \tag{1.9.2}$$

For ordinary braids, their images in $\mathrm{Coh}(\mathrm{Hilb})$ appear to agree with the complexes conjectured by (GNR) to compute HOMFLY-PT homology. It is not known whether MF or AMF are equivalent to $\mathcal{H}_{\mathrm{fin}}$ and $\mathcal{H}_{\mathrm{ext}}$ respectively. It also not yet known whether the functor $AMF \to \mathrm{Coh}(\mathrm{Comm})$ is trace-like in the sense of [34].

CHAPTER II

WEYL GROUPS, BRAID GROUPS, AND HECKE ALGEBRAS

2.1 Weyl Groups

Definition 2.1.1. (Finite Weyl Group). Let $(W_{\text{fin}}, S_{\text{fin}})$ denote the Coxeter system of type A_{n-1}. Here $S_{\text{fin}} = \{1, ..., n-1\}$ and $\{s_i\}_{i \in S_{\text{fin}}}$ is a set of generators for W_{fin}, with the relations

$$s_i^2 = 1, \tag{2.1.1}$$

$$(s_i s_j)^{m_{ij}} = \text{id}, \tag{2.1.2}$$

where

$$m_{ij} = \begin{cases} 1 & \text{if } i = j. \\ 3 & \text{if } i = j \pm 1 \cdot \\ 2 & \text{else.} \end{cases} \tag{2.1.3}$$

Note that W_{fin} is isomorphic to the symmetric group S_n, and the generator s_i corresponds to the transposition $((i)(i+1))$ in cycle notation.

Definition 2.1.2. Let $\Bbbk$ be a commutative integral domain. A *realization* of a Coxeter system (W, S) over $\Bbbk$ is a triple $(\mathfrak{h}, \{\alpha_s\}_{s \in S} \subset \mathfrak{h}^*, \{\alpha_s^\vee\}_{s \in S} \subset \mathfrak{h})$, where $\mathfrak{h}$ is a free, finite rank $\Bbbk$-module and the following hold

1. For all $s \in S$, $\alpha_s(\alpha_s^\vee) = 2$.

2. The assignment

$$S \times \mathfrak{h} \to \mathfrak{h}$$

$$(s, v) \mapsto v - \alpha_s(v)\alpha_s^\vee$$

extends to a W-action on $\mathfrak{h}$.

3. If $m_{st} = 2$ then $\alpha_s(\alpha_t^\vee) = \alpha_t(\alpha_s^\vee) = 0$.

4. If $\alpha_s(\alpha_t^\vee) = \alpha_t(\alpha_s^\vee) = 0$, then m_{st} is even.

The α_s are called the *simple roots* and the $\alpha_s^\vee$ are called the *simple coroots*. In the following we will consider special examples where $\Bbbk = \mathbb{C}$ and $\mathfrak{h}$ is a Euclidean space with Euclidean form $(\cdot, \cdot)$ and we may identify $\mathfrak{h} \cong \mathfrak{h}^*$.

Example 2.1.3. (The $\mathfrak{gl}_n$ realization of W_{fin}). We set the following notation.

$$H = \mathbb{C}\{x_1, ..., x_n\} \cong \mathbb{C}^n. \tag{2.1.4}$$

$$\alpha_i = \alpha_i^\vee = x_i - x_{i+1}. \tag{2.1.5}$$

$$\Delta = \{\alpha_i \mid 1 \leq i \leq n-1\} \subset \mathbb{C}^n. \tag{2.1.6}$$

$$\Lambda_{\text{rt}} = \mathbb{Z}\Delta \tag{2.1.7}$$

$$\Lambda_{\text{wt}} = \mathbb{Z}\{x_1, ..., x_n\} \tag{2.1.8}$$

Identifying $H \cong H^*$ via standard Euclidean form, the data (H, Δ, Δ) gives a realization of W_{fin} over $\mathbb{C}$. We define a faithful action of W_{fin} on H where, for $i \in S_{\text{fin}}$, s_i acts as the transposition $((i)(i+1))$ on the set $\{x_i\}$ of basis vectors. This is reflection across the hyperplane orthogonal to $\alpha_i \in H$.

We note that H may be identified with a maximal toral subalgebra for the reductive Lie algebra $\mathfrak{gl}_n$, Δ may be identified as a choice of simple roots for the corresponding root system, and W_{fin} as the corresponding Weyl group. In this setup Λ_{rt} and Λ_{wt} are the corresponding root and weight lattices respectively of $\mathfrak{gl}_n$. From this we obtain the usual sl_n realization of W_{fin} by taking the quotient of H by the vector space spanned by $\sum_{i=1}^n x_i$.

Definition 2.1.4. (Affine Weyl Group) Let $(W_{\text{aff}}, S_{\text{aff}})$ denote the Coxeter system of type $\widetilde{A}_{n-1}$. Here $S_{\text{aff}} = \{0, 1, ..., n-1\}$ and $\{s_i\}_{i \in S_{\text{aff}}}$ is a set of generators for W_{aff} with the relations

$$s_i^2 = 1, \tag{2.1.9}$$

$$(s_i s_j)^{m_{ij}} = \text{id}, \tag{2.1.10}$$

where

$$m_{ij} = \begin{cases} 1 & \text{if } i = j \\ 3 & \text{if } i = j \pm 1 \text{ mod } n \\ 2 & \text{else} \end{cases} \cdot \tag{2.1.11}$$

We identify $S_{\text{aff}} \cong \mathbb{Z}/n\mathbb{Z}$ as a set, as suggested by these relations. Note that for $i = 1, ..., n-1$, the relations among the s_i are the same as in 2.1.3, therefore W_{fin} is a subgroup of W_{aff}. The Coxeter system $(W_{\text{aff}}, S_{\text{aff}})$ has an automorphism

$$\tau : W_{\text{aff}} \to W_{\text{aff}} \tag{2.1.12}$$

defined by

$$\tau(s_i) = s_{i+1}.$$

Remark 2.1.5. The group W_{aff} does not act linearly on the vector space H in 2.1.3 however we can define an action by affine linear transformations. The s_i for $i = 1, ..., n$ act as before, while s_0 acts by reflection across an affine hyperplane orthogonal to

$$\alpha_{\text{long}} = \sum_{i=1}^{n} \alpha_i$$

and translated from the origin by x_1. This action can be linearized by adding a new parameter.

Example 2.1.6. (The $\mathfrak{gl}_n$ realization of W_{aff}). We reuse the notation from 2.1.3 and we set the following notation.

$$\hat{H} = H \oplus \mathbb{C}\{\delta\} \cong \mathbb{C}^{n+1}. \tag{2.1.13}$$

$$\alpha_{\text{long}} = \sum_{\alpha \in \Delta} \alpha. \tag{2.1.14}$$

$$\alpha_0 = -\alpha_{\text{long}} + \delta. \tag{2.1.15}$$

$$\alpha_0^\vee = -\alpha_{\text{long}}. \tag{2.1.16}$$

$$\hat{\Delta} = \Delta \cup \{\alpha_0\}. \tag{2.1.17}$$

$$\hat{\Delta}^\vee = \delta \cup \{\alpha_0^\vee\} \tag{2.1.18}$$

We endow $\hat{H}$ with the standard Euclidean form. We note that $\delta = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \alpha_i$. We define a linear action of W_{aff} on $\hat{H}$ as follows. We set δ to be invariant under W_{aff}. We set s_i for $i = 1, ..., n$ to act by permuting the basis vectors $\{x_j\}$ as the transposition $(i(i+1))$. We define the action of s_0 on the basis vectors as follows.

$$s_0(x_1) = x_n + \delta, \qquad s_0(x_n) = x_1 - \delta, \qquad s_0(x_i) = x_i \text{ for } i \neq 1, n. \tag{2.1.19}$$

Equivalently, for $i \in \mathbb{Z}/n\mathbb{Z}$, s_i acts by reflection across the linear hyperplane orthogonal to α_i. This makes the data $(\hat{H}, \hat{\Delta}, \hat{\Delta}^\vee)$ into a realization of $(W_{\text{aff}}, S_{\text{aff}})$.

This realization of W_{aff} is compatible with the automorphism τ in the the following way. We define a map $\tau : \hat{H} \to \hat{H}$ by

$$\tau(\delta) = \delta, \qquad \tau(x_n) = x_1 - \delta, \qquad \tau(x_i) = x_{i+1} \text{ for } 1 \leq i \leq n-1. \tag{2.1.20}$$

It satisfies the the following compatibility.

$$\tau(s) \cdot \tau(x) = \tau(s \cdot x), \qquad \tau(\alpha_i) = \alpha_{(i+1 \mod n)}, \qquad \text{for } s \in W_{\text{aff}} \text{ and } x \in \hat{H}. \tag{2.1.21}$$

We recover the affine action of W_{aff} by projecting this linear action to the affine hyperplane $H + \delta$.

We recall some fundamental notions about Coxeter groups. Let (W, S) be an abstract Coxeter system.

Definition 2.1.7. Let $w \in W$. An *expression* $\underline{w}$ for w is a tuple, or word, $(s_{i_1}, ..., s_{i_k})$ in S such that the product $s_{i_1} \cdots s_{i_k}$ in W is equal to w.

Definition 2.1.8. The *length* of w, denoted by $l(w)$, is the minimal k for which w has an expression of length k. Such an expression with minimal length is called a *reduced expression*.

Definition 2.1.9. Let $R \subset W$ denote the set of conjugates of elements of S. These are called the *reflections* in W. The *Bruhat order* on W is the transitive completion of the relation defined by $x < y$ if $sx = y$ or $xs = y$ for some $s \in R$ and with $l(x) < l(y)$.

2.2 Loop vs Coxeter

The presentation of W_{aff} discussed in 2.1.4 is called the *Coxeter presentation* of W_{aff}. Recall the action of W_{fin} on $\Lambda_{\text{rt}} \subset H$. The following isomorphism is called the *loop presentation* of W_{aff}.

Proposition 2.2.1. *(Loop Presentation).*

$$W_{\text{aff}} \cong W_{\text{fin}} \ltimes \Lambda_{\text{rt}}. \tag{2.2.1}$$

Recall the faithful action of W_{aff} by affine linear transformations on H, where s_0 acts by reflection across an affine hyperplane. Let $t_{\alpha_{\text{long}}}$ indicate translation by α_{long}. Let s_{long} indicate reflection across the hyperplane orthogonal to α_{long} in H. Then

$$t_{\alpha_{\text{long}}} = s_0 s_{\text{long}}. \tag{2.2.2}$$

Taking conjugates of $t_{\alpha_{\text{long}}}$ under W_{fin}, we get the translations t_α for $\alpha \in \Delta$, which generate a normal copy of Λ_{rt} inside W_{aff}. The presentation of W_{aff} with generators s_i for $i \in S_{\text{fin}}$ and t_α for $\alpha \in \Delta$ is the loop presentation.

2.3 Extended Affine Weyl Group

Recall the automorphism τ of the Coxeter system $(W_{\text{aff}}, S_{\text{aff}})$. We define a new group, the *extended affine Weyl group*, by making τ an inner automorphism.

Definition 2.3.1. The *extended affine Weyl group* is defined as

$$W_{\text{ext}} := W_{\text{aff}} \ltimes \mathbb{Z}. \tag{2.3.1}$$

We set ω to be the generator of $\mathbb{Z}$, and we set $\omega x \omega^{-1} = \tau(x)$ for $x \in W_{\text{aff}}$.

Definition 2.3.2. We call the presentation of W_{ext} given by s_i for $i \in S_{\text{aff}}$, and ω, the *Coxeter presentation* of W_{ext}.

Definition 2.3.3. We call $(l, s_{i_1}, s_{i_2}, ..., s_{i_k})$, for $i_m \in S_{\text{aff}}$, an expression for $w \in W_{\text{ext}}$ if $\omega^l s_{i_1} \cdots s_{i_k} = w$. Here l is a (possibly negative) integer, and is uniquely determined by w. The expression is reduced if k is minimal.

We now describe a geometric interpretation of W_{ext} in terms of a faithful action by affine linear transformations on H. Recall the action of W_{fin} on $\Lambda_{\text{wt}} \subset H$.

Let $\Lambda_{\mathrm{wt}} \subset \mathrm{Aut}(H)$ consist of the translations along the lattice Λ_{wt}. The following decomposition is called the *loop presentation* of W_{ext}.

Proposition 2.3.4. *(Loop presentation). The following decomposition holds.*

$$W_{\mathrm{ext}} \cong W_{\mathrm{fin}} \ltimes \Lambda_{\mathrm{wt}}. \tag{2.3.2}$$

From it we infer a faithful action of W_{ext} on H by affine linear transformations.

The generator $(1, 0, ..., 0)$ of Λ_{wt} is given by $\omega s_{n-1} s_{n-1} \cdots s_1$. The rest of the generators are given by conjugation by ω. These are related to the Wakimoto braids discussed in Definition 2.4.14.

We note that $W_{\mathrm{ext}}/W_{\mathrm{aff}} \cong \Lambda_{\mathrm{wt}}/\Lambda_{\mathrm{rt}} \cong \mathbb{Z}$.

Recall that $\hat{H}$ was compatible with the automorphsim τ as a realization of $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$. We define a faithful action of W_{ext} by linear transformations on $\hat{H}$ by setting

$$\omega \cdot x = \tau(x), \qquad \text{for } x \in \hat{H}.$$

2.4 Braid Groups

We recall the definition of the braid group of a Coxeter group.

Definition 2.4.1. The *Braid Group* Br_W of a Coxeter system (W, S) is given by the presentation with generators

$$f_i, \qquad \text{for } i \in S \tag{2.4.1}$$

and the relations

$$\underbrace{f_i f_j \cdots}_{m_{ij} \text{ times}} = \underbrace{f_j f_i \cdots}_{m_{ij} \text{ times}}. \tag{2.4.2}$$

We note there is always a homomorphism $h : \mathrm{Br}_W \to W$.

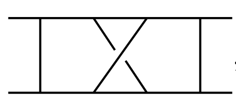
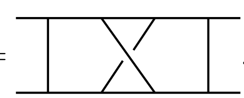
Definition 2.4.2. For $w \in W$, we set f_w to be the braid diagram such that $h(f_w) = w$, with f_w having the minimal number of crossings among such diagrams and only having positive crossings according to the convention we set in 2.4.3. We call f_w the positive lift of w. Equivalently, if $(s_{i_1}, ..., s_{i_k})$ is a reduced expression for w, we have

$$f_w = f_{i_1} \cdots f_{i_k}.$$

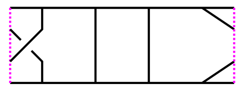
Notation 2.4.3. We set Br_{fin} and Br_{aff} to be the braid groups of W_{fin} and W_{aff} respectively.

Proposition 2.4.4. *There is an inclusion* $\mathrm{Br}_{\text{fin}} \hookrightarrow \mathrm{Br}_{\text{aff}}$ *as in the case of the corresponding Weyl groups.*

2.4.1 String Diagrams and Braid Diagrams. We introduce braid diagrams in the planar strip to understand elements of Br_{fin}. For example, we draw the simple crossings f_2 and f_2^{-1} when $n = 4$.

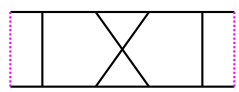
$f_2 =$  , $\quad f_2^{-1} =$  . (2.4.3)

This sets our convention for a *positive crossing* and *negative* crossing respectively. We interpret the elements of Br_{aff} as braid diagrams on a cylinder, or a planar strip with the left and right ends identified. The generators f_i for $i \neq 0$ are the same simple crossings as in the finite case, while f_0 wraps behind the cylinder. We draw it below in the case $n = 4$.

$f_0 =$  (2.4.4)

Diagrammatically, the elements of Br_{fin} correspond to braid diagrams that don't wrap behind the cylinder.

We view elements of W_{fin} and W_{aff} diagrammatically in terms of string diagrams on a cylinder. Here, we do not keep track of whether strands cross over or under. We draw the string diagram for s_2 in the case $n = 4$ below.

$s_2 = s_2^{-1} =$  (2.4.5)

We think of W_{fin} as corresponding to string diagrams that do not go behind the cylinder.

Proposition 2.4.5. *The vertical stacking or concatenation of braid diagrams corresponds to multiplication in* Br_{fin} *and* Br_{aff}*. Likewise, vertical stacking of string diagrams is multiplication in* W_{fin} *and* W_{aff}*.*

2.4.2 Cylindrical Braid Group.

Definition 2.4.6. (Cylindrical Braid Group). We define the braid group of W_{ext}, which we denote Br_{ext}, to be the group generated by

$$\omega, \qquad f_i, \qquad \text{for } i \in S_{\text{aff}} \tag{2.4.6}$$

with the relations

$$\underbrace{f_i f_j \cdots}_{m_{ij}\text{ times}} = \underbrace{f_j f_i \cdots}_{m_{ij}\text{ times}}, \qquad \omega f_i \omega^{-1} = f_{(i+1 \mod n)}. \tag{2.4.7}$$

From this presentation, we have

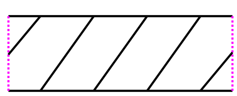
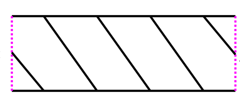
$$\mathrm{Br}_{\mathrm{ext}} \cong \mathrm{Br}_{\mathrm{aff}} \ltimes \mathbb{Z} \tag{2.4.8}$$

We also call this group the *cylindrical braid group*.

Remark 2.4.7. We have a homomorphism $\mathrm{Br}_{\mathrm{ext}} \to W_{\mathrm{ext}}$.

Let us expand on why this group is the cylindrical braid group, and not $\mathrm{Br}_{\mathrm{aff}}$. Indeed, not all cylindrical braid diagrams are obtained from those generated by f_i for $i \in S_{\mathrm{aff}}$ under vertical concatenation. Such braid diagrams can only have zero *winding number*.

The following is the cylindrical braid diagram for ω and ω^{-1}, which have winding numbers 1 and -1 respectively.

$\omega =$ , $\qquad \omega^{-1} =$ . (2.4.9)

All braid diagrams can be generated by ω and f_i for $i \in S_{\mathrm{aff}}$ under vertical stacking.

Proposition 2.4.8. *The group* $\mathrm{Br}_{\mathrm{ext}}$ *is isomorphic, via mapping the generators* f_i *and* ω *as described above, to the group of braid diagrams, considered up to isotopy, on the cylinder.*

Definition 2.4.9. We define the *winding number* of a cylindrical braid to be the number of times it wraps around the cylinder, with counterclockwise rotation being set as positive winding. In fact, winding number is the group homomorphism

$$\mathrm{wind} : \mathrm{Br}_{\mathrm{ext}} \to \mathrm{Br}_{\mathrm{ext}} \,/\, \mathrm{Br}_{\mathrm{aff}} \cong \mathbf{Z}. \tag{2.4.10}$$

We note that $\mathrm{wind}(\omega^k) = k$.

Definition 2.4.10. We define the evaluation homomorphism

$$\mathrm{ev} : \mathrm{Br}_{\mathrm{ext}} \to W_{\mathrm{fin}} \cong S_n, \tag{2.4.11}$$

in terms of cylindrical braid diagrams, by mapping a braid diagram to its underlying permutation on the n-marked points on the boundary circles of the cylinder.

Definition 2.4.11. We define a *pure cylindrical braid* to be a braid in the kernel of $ev : \mathrm{Br}_{\mathrm{ext}} \to S_n$. These form a subgroup called $\mathrm{PBr}_{\mathrm{ext}}$.

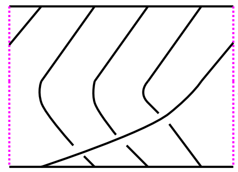
Because the underlying permuation of a pure cylindrical braid is the identity, we can keep track of the winding of each individual strand of the braid. This allows us to define the following.

Definition 2.4.12. We define the homomorphism

$$\underline{\mathrm{wind}} : \mathrm{PBr}_{\mathrm{ext}} \to \mathbf{Z}^n \cong \Lambda_{\mathrm{wt}}, \tag{2.4.12}$$

in terms of cylindrical braid diagrams, by mapping a diagram on n strands to the n-vector of integers recording the winding number of each strand. We identify $\mathbb{Z}^n$ with Λ_{wt} in the standard way.

Example 2.4.13. Consider the following cylindrical braid diagram.


$$. \tag{2.4.13}$$

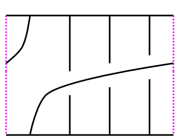
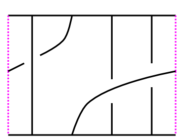
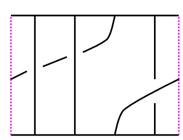
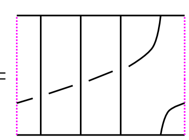
It is the diagram corresponding to $\omega f_3 f_2 f_1$ for $n = 4$. Here, we have

$$\underline{\mathrm{wind}}(\omega f_3 f_2 f_1) = (1, 0, 0, 0).$$

2.4.3 Translation Lattices.

Let us introduce a particular lift of Λ_{wt} under $\underline{\mathrm{wind}}$.

Definition 2.4.14. (Wakimoto Braids). For $i \leq y \leq n$, let y_i denote the pure cylindrical braid where the i-th strand wraps to the right around the cylinder, passing over the j-th strand for $j > i$ and under it for $j < i$. We draw the y_i for $n = 4$.

$y_1 =$  $y_2 =$  $y_3 =$  $y_4 =$  . (2.4.14)

We refer to y_i as the i-th Wakimoto braid. In terms of the generators ω and f_i, we have

$$y_i = f_{i-1}^{-1} \cdots f_2^{-1} \omega f_{n-1} \cdots f_{i+1} f_i \tag{2.4.15}$$

Proposition 2.4.15. *The elements y_i commute with each each other and* $\underline{\mathrm{wind}}$ *gives an isomorphism between the group they generate and Λ_{wt}. Let $\epsilon_i = (0, \cdots, 1, \cdots, 0) \in \Lambda_{\mathrm{wt}}$.*

One has $\underline{\mathrm{wind}}(y_i) = \epsilon_i$ *and*

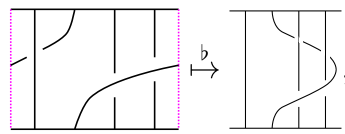
$$\prod_{i=1}^{n} y_i = \omega^n \tag{2.4.16}$$

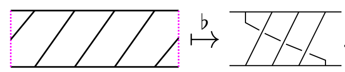
Remark 2.4.16. This lift of Λ_{wt} is not unique. Let us describe a way to obtain other interesting lifts. Fix an ordering σ of $1, \cdots n$. Let y_i^σ denote the pure cylindrical braid which wraps around the cylinder from the right, crossing over the j-th strand if $j < i$ and crossing under if $j > i$ under the ordering specified by σ.

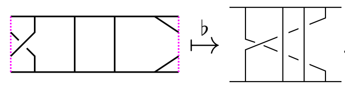
Orderings of n points are an S_n-torsor and may be identified with S_n by identifying the ordering $n > n-1 > \cdots > 2 > 1$ with $\mathrm{id} \in S_n$. Then the y_i defined earlier correspond to y_i^{id}.

In addition to these lifts of Λ_{wt} we can get other copies of Λ_{wt} by conjugating these by an arbitrary braid. However, the lattices given by orderings described above are useful because the y_i^σ have no repeated crossings. This is useful for categorification, as the Rouquier complexes for braids with no repeated crossings are minimal complexes.

2.4.4 Flattening. By drawing the cylinder as a strip with the left and right edges identified as in the images before, we've fixed a projection $\pi : S^1 \times I \times (-\epsilon, \epsilon) \to I \times I \times (-\epsilon, \epsilon)$, a map from the thickened cylinder to the thickened planar strip. This induces a map from cylindrical braids, which live in the thickened cylinder, to planar braids which live in the thickened strip. Indeed, this projection also induces a group homomorphism $\flat : \mathrm{Br}_{\mathrm{ext}} \to \mathrm{Br}_{\mathrm{fin}}$. We draw some examples of flattening in $n = 4$ below.

$$y_2 = $$  $$, \tag{2.4.17}$$

$$\omega = $$  $$. \tag{2.4.18}$$

$$f_0 = $$  $$. \tag{2.4.19}$$

We can explicitly define $\flat$ in terms of generators.

Definition 2.4.17. Let $\flat : \mathrm{Br}_{\mathrm{ext}} \to \mathrm{Br}_{\mathrm{fin}}$ be the homomorphism defined on generators by

$$\flat(f_i) = f_i \text{ for } 1 \le i \le n-1, \tag{2.4.20}$$

$$\flat(\omega) = f_1 f_2 \cdots f_{n-1}, \tag{2.4.21}$$

$$\flat(f_0) = f_{n-1}^{-1} \cdots f_2^{-1} f_1 f_2 \cdots f_{n-1}. \tag{2.4.22}$$

2.4.5 Bar Involution.

Definition 2.4.18. Let β be cylindrical braid. We set $\overline{\beta}$ to be the cylindrical braid diagram where all the positive crossings are changed to negative crossings and vice versa, according to the convention we set in 2.4.3. This gives an involution on $\mathrm{Br}_{\mathrm{fin}}$, $\mathrm{Br}_{\mathrm{aff}}$, and $\mathrm{Br}_{\mathrm{ext}}$ which we denote as the *bar involution*.

Let f_w be the positive lift of $w \in W$. We note that

$$\overline{f_w} = f_{w^{-1}}^{-1}. \tag{2.4.23}$$

We note that $\overline{\omega} = \omega$, hence the bar involution preserves winding number and also $\underline{\mathrm{wind}}$ of pure braids.

2.5 Hecke Algebras

Now that we've discussed the Braid groups of $W_{\text{fin}}, W_{\text{aff}}$, and W_{ext} we may introduce their Hecke algebras.

Definition 2.5.1. (Hecke Algebra). Let (W, S) be a Coxeter system and Br its braid group. The Hecke algebra $\mathbf{H}$ of (W, S) is obtained as a quotient of the $\mathbf{Z}[v, v^{-1}]$-group algebra of Br by the relations

$$(f_i + v)(f_i - v^{-1}) = 0. \tag{2.5.1}$$

We note that this definition applies to Br_{ext} giving its Hecke algebra $\mathbf{H}_{\text{ext}}$.

Definition 2.5.2. We set $\mathbf{H}_{\text{fin}}, \mathbf{H}_{\text{aff}}$, and $\mathbf{H}_{\text{ext}}$ to be the Hecke algebras of $W_{\text{fin}}, W_{\text{aff}}$, and W_{ext} respectively.

Notation 2.5.3. In the following we will set T_{s_i} to be the image of the quotient of f_i. When the choice of generator is not relevant, we may drop the index i and refer to these generators as T_s.

Remark 2.5.4. The flattening map $\flat : \mathrm{Br}_{\text{ext}} \to \mathrm{Br}_{\text{fin}}$ extends linearly over $\mathbb{Z}[v, v^{-1}]$ and descends to a map $\flat : \mathbf{H}_{\text{ext}} \to \mathbf{H}_{\text{fin}}$.

Proposition 2.5.5. *The following decompositions hold.*

$$\mathbf{H}_{\text{aff}} \cong \mathbf{H}_{\text{fin}} \ltimes \Lambda_{\text{rt}}, \tag{2.5.2}$$

$$\mathbf{H}_{\text{ext}} \cong \mathbf{H}_{\text{fin}} \ltimes \Lambda_{\text{wt}}, \tag{2.5.3}$$

$$\mathbf{H}_{\text{ext}} \cong \mathbf{H}_{\text{aff}} \rtimes \mathbf{Z}. \tag{2.5.4}$$

We set ω to be the generator of $\mathbb{Z}$ in 2.5.4.

2.5.1 Bases and Pairings.

Let (W, S) be a Coxeter system (e.g $(W_{\text{fin}}, S_{\text{fin}})$ or $(W_{\text{aff}}, S_{\text{aff}})$), Br its braid group and $\mathbf{H}$ its Hecke algebra. We may also consider $(W, \mathrm{Br}, \mathbf{H})$ to be $(W_{\text{ext}}, \mathrm{Br}_{\text{ext}}, \mathbf{H}_{\text{ext}})$ with appropriate modification.

Proposition 2.5.6. *(Standard Basis). Let $w \in W$ and let f_w be a positive lift of w in* Br. *Let T_w denote the image of f_w in $\mathbf{H}$. Equivalently, for a reduced expression $(s_{i_1}, ..., s_{i_k})$ of w, we have*

$$T_w = T_{s_{i_1}} \cdots T_{s_{i_k}}.$$

The set $\{T_w \mid w \in W\}$ are a basis of $\mathbf{H}$ over $\mathbb{Z}[v, v^{-1}]$.

Proposition 2.5.7. *(Standard Basis of $\mathbf{H}_{\text{ext}}$). The set $\{\omega^k T_w \mid w \in W_{\text{aff}}, k \in \mathbb{Z}\}$ are a basis of $\mathbf{H}_{\text{ext}}$ over $\mathbb{Z}[v, v^{-1}]$.*

To prove this one must use the multiplication rules in this basis which are expressed in terms of the *Bruhat order* on W. We will not do so.

Definition 2.5.8. (Bar Involution) Recall the *bar involution* $\beta \mapsto \overline{\beta}$ defined on braids β. We extend it to and involution of $\mathbf{H}$ as follows.

$$\overline{T_s} = T_s^{-1}, \qquad \overline{v} = v, \qquad \overline{(ab)} = \overline{a} \cdot \overline{b}. \tag{2.5.5}$$

In the case of $\mathbf{H}_{\text{ext}}$, we also set

$$\overline{\omega} = \omega.$$

Definition 2.5.9. (Kazhdan-Lusztig anti-involution). We define an anti-involution $\alpha : \mathbf{H} \to \mathbf{H}$ as follows.

$$\alpha(T_s) = T_s^{-1}, \qquad \alpha(v) = v^{-1}, \qquad \alpha(ab) = \alpha(b)\alpha(a). \tag{2.5.6}$$

In the case of $\mathbf{H}_{\mathrm{ext}}$, we also set

$$\alpha(\omega) = \omega.$$

Definition 2.5.10. (Standard Trace) The standard trace $\epsilon : \mathbf{H} \to \mathbb{Z}[v, v^{-1}]$ is defined on the standard basis by

$$\epsilon(T_w) = \delta_{w,\mathrm{id}}.$$

It extracts the coefficient of T_{id}.

Proposition 2.5.11. *The following hold for ϵ.*

1.
$$\epsilon(T_x T_y) = \begin{cases} 1 & \text{if } x = y^{-1} \\ 0 & \text{else} \end{cases}$$

2. $\epsilon(ab) = \epsilon(ba)$ *for all* $a, b \in \mathbf{H}$.

This allows us to define a form on $\mathbf{H}$ which is $\mathbb{Z}[v, v^{-1}]$-sequilinear with respect to the anti-involution on $\mathbb{Z}[v, v^{-1}]$ given by $v \mapsto v^{-1}$.

Definition 2.5.12. The *standard form* $(\cdot, \cdot) : \mathbf{H} \times \mathbf{H} \to \mathbb{Z}[v, v^{-1}]$ is the $\mathbb{Z}[v, v^{-1}]$-sequilinear form defined by $(a, b) := \epsilon(\alpha(a)b)$.

Proposition 2.5.13. *The standard form is non-degenerate.*

Remark 2.5.14. As we will mention again in the following, the standard form is categorified by the graded-Hom spaces between Soergel bimodules.

Definition 2.5.15. We define the *Jucys-Murphy subalgebra* $J = \flat(\mathbb{Z}[v^{\pm}][\Lambda_{\mathrm{wt}}]) \subset \mathbf{H}_{\mathrm{fin}}$. The braids $j_i = \flat(y_i)$ are called the *multiplicative Jucys-Murphy braids*.

Proposition 2.5.16. *1. The standard form on $\mathbf{H}_{\mathrm{fin}}$ restricts to a nondegenerate form on the Jucys-Murphy subalgebra J.*

2. *The standard form on* $\mathbf{H}_{\text{ext}}$ *restricts to a nondegenerate form on* $\mathbb{Z}[v, v^{-1}][\Lambda_{\text{wt}}]$.

Theorem 2.5.17. *(Kazhdan-Lusztig Basis/Positivity Theorem)* $\mathbf{H}$ *has a unique basis* $\{b_w \mid w \in W\}$ *such that*

1. *(self-duality):* $\overline{b_x} = b_x$

2. b_x *has the form*

$$b_x = T_x + \sum_{y<x} h_{y,x} T_y \text{ for some } h_{y,x} \in v\mathbb{Z}[v]$$

3. $b_s b_t = \sum_u C_{st}^u b_u$, *where* $C \in \mathbb{N}[v, v^{-1}]$

The polynomials $h_{y,x}$ are the famous *Kazhdan-Lusztig polynomials* and their positivity, along with the positivity of the C_{st}^u, is the result of the famous Kazhdan-Lusztig positivity theorem.

Example 2.5.18. For all $s \in S$, $b_s = T_s + v$.

The following relates to part (3) of Theorem 2.5.17.

Corollary 2.5.19. *Let* $\underline{w} = (s_{i_1}, ..., s_{i_k})$ *be a reduced expression for* $w \in W$. *Then*

$$b_{s_{i_1}} \cdots b_{s_{i_k}} = b_w + \sum_{x<w} C_{xw} b_x \tag{2.5.7}$$

where $C_{xw} \in \mathbb{N}[v, v^{-1}]$. *The element on the left-hand side of the equation is called the Bott-Samelson element for the expression* $\underline{w}$.

Definition 2.5.20. (KL Basis of $\mathbf{H}_{\text{ext}}$). We define the set $\{\omega^k b_w \mid w \in W_{\text{aff}}, k \in \mathbb{Z}\}$ to be the *Kazhdan-Lusztig basis* of $\mathbf{H}_{\text{ext}}$.

Proposition 2.5.21. *The Kazhdan-Lusztig basis is asymptotically orthonormal with respect to* $(\cdot,\cdot)$*. That is, for all* $x, y \in W$*,*

$$(b_x, b_y) \in \begin{cases} 1 + v\mathbb{Z}[v] & \textit{if } x = y \\ v\mathbb{Z}[v] & \textit{else} \end{cases} \tag{2.5.8}$$

If we set $v = 0$, they are orthonormal.

Remark 2.5.22. Left multiplication by an element fixed under α is self-biadjoint with respect to $(\cdot,\cdot)$. This occurs in particular for the elements b_s where $s \in S$.

2.6 Traces, Inclusions, and Link Invariants.

2.6.1 Inclusions. In the following we will be working with Weyl groups, braid groups, and Hecke algebras of different ranks simultaneously, so we introduce the following notation.

Notation 2.6.1. Let $\mathbf{H}^n_{\mathrm{fin}}$ be the Hecke algebra of type A_{n-1} let W^n_{fin} be the Coxeter group (i.e. the symmetric group S_n), and let $\mathrm{Br}^n_{\mathrm{fin}}$ be its braid group. Let W^n_{aff} be the Coxeter group of type $\widetilde{A}_{n-1}$, let $\mathrm{Br}^n_{\mathrm{aff}}$ its braid group and $\mathbf{H}^n_{\mathrm{aff}}$ its Hecke algebra. Let W^n_{ext}, $\mathrm{Br}^n_{\mathrm{ext}}$, and $\mathbf{H}^n_{\mathrm{ext}}$ be the extended incarnations of those objects with the same rank.

Proposition 2.6.2. *There are inclusion maps*

$$\iota : W^n_{\mathrm{fin}} \times W^m_{\mathrm{fin}} \hookrightarrow W^{n+m}_{\mathrm{fin}} \tag{2.6.1}$$

$$\iota : \mathrm{Br}^n_{\mathrm{fin}} \times \mathrm{Br}^m_{\mathrm{fin}} \hookrightarrow \mathrm{Br}^{n+m}_{\mathrm{fin}} \tag{2.6.2}$$

$$\iota : \mathbf{H}^n_{\mathrm{fin}} \otimes_{\mathbb{Z}[v^{\pm 1}]} \mathbf{H}^m_{\mathrm{fin}} \hookrightarrow \mathbf{H}^{n+m}_{\mathrm{fin}} \tag{2.6.3}$$

We make sense of the case $m = 1$ *by setting* $W^1_{\mathrm{fin}} = \mathrm{Br}^1_{\mathrm{fin}} = \{\mathrm{id}\}$ *and* $\mathbf{H}^1_{\mathrm{fin}} = \mathbb{Z}[v, v^{-1}]$.

Diagrammatically, one may think of these as horizontally composing braid or string diagrams. We will be especially interested in the inclusion $\iota : \mathbf{H}^n_{\text{fin}} \hookrightarrow \mathbf{H}^{n+1}_{\text{fin}}$.

There are analogs of these inclusion in the affine setting i.e. with braids and strings on a cylinder.

Proposition 2.6.3. *There are inclusion maps*

$$\iota : W^n_{\text{ext}} \times W^m_{\text{ext}} \hookrightarrow W^{n+m}_{\text{ext}} \tag{2.6.4}$$

$$\iota : \mathrm{Br}^n_{\text{ext}} \times \mathrm{Br}^m_{\text{ext}} \hookrightarrow \mathrm{Br}^{n+m}_{\text{ext}} \tag{2.6.5}$$

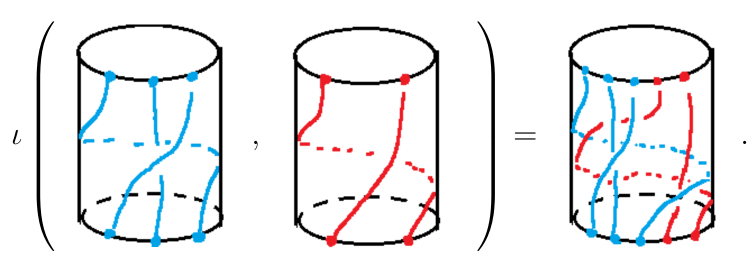
$$\iota : \mathbf{H}^n_{\text{ext}} \otimes_{\mathbb{Z}[v^{\pm 1}]} \mathbf{H}^m_{\text{ext}} \hookrightarrow \mathbf{H}^{n+m}_{\text{ext}} \tag{2.6.6}$$

We make sense of the case $m = 1$ by setting $W^1_{\text{ext}} = \mathrm{Br}^1_{\text{ext}} = \mathbb{Z}$ and $\mathbf{H}^1_{\text{ext}} = \mathbb{Z}[v, v^{-1}][\mathbb{Z}]$.

We do not have horizontal stacking for cylindrical braids. What we can do is place one cylinder in the interior of another, which we'll call *internal composition* or *internal stacking* of cylindrical braids. Diagrammatically, the above inclusions come from internal composition of a cylindrical braid on m strands with a cylindrical braid on n strands.

We draw an example of ι in the case $n = 3$, $m = 2$.

Example 2.6.4.



(2.6.7)

In the above, the red strands pass behind the blue strands.

We'll mainly be interested in the inclusions $\iota : \mathrm{Br}^n_{\mathrm{ext}} \hookrightarrow \mathrm{Br}^{n+1}_{\mathrm{ext}}$, and $\iota : \mathbf{H}^n_{\mathrm{ext}} \hookrightarrow \mathbf{H}^{n+1}_{\mathrm{ext}}$, which diagrammatically correspond to inserting one strand inside the cylinder.

Proposition 2.6.5. *The inclusion $\iota : \mathrm{Br}^n_{\mathrm{ext}} \hookrightarrow \mathrm{Br}^{n+1}_{\mathrm{ext}}$ sends the i-th Wakimoto braid in $\mathrm{Br}^n_{\mathrm{ext}}$ to the i-th Wakimoto braid in $\mathrm{Br}^{n+1}_{\mathrm{ext}}$ where $i = 1, ..., n$.*

Proposition 2.6.6. *The inclusions of finite and extended affine Hecke algebras are isometric with respect to the standard forms.*

2.6.2 Traces. Let $\mathbf{H} = \mathbf{H}_{\mathrm{fin}}, \mathbf{H}_{\mathrm{aff}}$, or $\mathbf{H}_{\mathrm{ext}}$. Let us define some trace maps on $\mathbf{H}$ that we will later discuss categorifications of, and that allow one to define link invariants. The first is the standard trace we have already defined $\epsilon : \mathbf{H} \to \mathbb{Z}[v, v^{-1}]$. We observe that it is equivalent to $(1, \cdot) : \mathbf{H} \to \mathbb{Z}[v, v^{-1}]$, so it will later be categorified in the setting of Hecke categories by taking graded-Hom with monoidal unit.

The standard trace on $\mathbf{H}_{\mathrm{fin}}$ is the specialization (at $(a = 0)$) of a trace valued in $\mathbb{Z}[a^{\pm 1}](v)$ originally defined by Ocneanu via a Skein relation. We state the relevant theorem.

Theorem 2.6.7. *(Ocneanu) There is a unique family of $\mathbb{Z}[v, v^{-1}]$-linear maps*

$$\mathrm{Tr}_n : \mathbf{H}^n_{\mathrm{fin}} \to \mathbb{Z}[a^{\pm 1}](v)$$

satisfying

1. $\mathrm{Tr}_n(xy) = \mathrm{Tr}_n(yx)$,

2. $\mathrm{Tr}_{n+1}(\iota(x)) = \{0\}\, \mathrm{Tr}_n(X)$,

3. $\mathrm{Tr}_{n+1}(\iota(x))b_{s_n} = \{1\}\,\mathrm{Tr}_n(x)$,

4. $\mathrm{Tr}_0(1) = 1$,

where

$$\{n\} = \frac{av^{-n} - a^{-1}v^n}{v - v^{-1}}.$$

Definition 2.6.8. Let β be a planar braid diagram, viewed as an element of $\mathbf{H}^n_{\mathrm{fin}}$. Let L denote its *braid closure*. The HOMFLY-PT invariant of L is defined to be $\mathrm{Tr}_n(\beta)$.

Proposition 2.6.9. *The standard trace* $(1, \cdot) : \mathbf{H}_{\mathrm{fin}} \to \mathbb{Z}[v^{\pm 1}]$ *factors through the adjoint of the inclusion of the Jucys-Murphy algebra* $\iota : J \hookrightarrow \mathbf{H}_{\mathrm{fin}}$.

Proof. We note that $(1, x)_{\mathbf{H}_{\mathrm{fin}}} = (\iota(1), x)_{\mathbf{H}_{\mathrm{fin}}} = (1, \iota_*(x))_J$. □

Remark 2.6.10. As far as we know, it is an open problem to realize the standard trace of $\mathbf{H}^n_{\mathrm{ext}}$ as the specialization at $a = 0$ of a Jones-Ocneanu trace. While such a trace hasn't been defined, a related object, the HOMFLY-PT skein algebra of the 2-torus T^2, has been studied in [31]. Such a trace would give a yet undefined *toroidal HOMFLY-PT invariant* for a link in the thickened 2-torus. The categorification of such an invariant is implicitly studied in [30].

2.6.3 Partial Traces.

The inclusion $\iota : \mathbf{H}^n_{\mathrm{fin}} \hookrightarrow \mathbf{H}^{n+1}_{\mathrm{fin}}$ has an adjoint with respect to the standard form $p_n := \iota_* : \mathbf{H}^{n+1}_{\mathrm{fin}} \to \mathbf{H}^n_{\mathrm{fin}}$. Both ι and p_n have been categorified by Hogancamp in [32]. Likewise $\iota : \mathbf{H}^n_{\mathrm{ext}} \hookrightarrow \mathbf{H}^{n+1}_{\mathrm{ext}}$ has an adjoint $p_n := \iota_* : \mathbf{H}^{n+1}_{\mathrm{ext}} \to \mathbf{H}^n_{\mathrm{ext}}$. As we will elaborate on later, categorifying ι and its adjoint in the extended affine setting remains an open problem.

Proposition 2.6.11. *The standard trace* $(1, \cdot) : \mathbf{H}^n_{\mathrm{fin}} \to \mathbb{Z}[v^{\pm 1}]$ *is equal to the repeated partial trace* $p_2 p_3 \ldots p_n$.

Exercise 2.6.12. Prove proposition 2.6.11 using induction and the fact that p_2 is the adjoint with respect to the standard form $(\cdot, \cdot)$. The base case $n = 2$ is the definition of the standard trace.

2.6.4 Link Invariants.

Definition 2.6.13. Let β be a planar braid diagram, viewed as an element of $\mathbf{H}^n_{\mathrm{fin}}$. Let L denote its *braid closure*, which is a link in the 3-sphere S^3. The HOMFLY-PT invariant of L is defined to be $\mathrm{Tr}_n(\beta)$.

Theorem 2.6.14. *(HOMFLY [40]). The HOMFLY-PT invariant is an invariant of L (up to scalar, we ignore normalization). It is invariant under the Markov moves, because Tr_n is.*

Definition 2.6.15. Let β be a cylindrical braid, and let L be its closure which is a link in $T^2 \times I$. We define the toroidal HOMFLY-PT (at $a = 0$) invariant of L to be $(1, \beta) \in \mathbb{Z}[v^{\pm 1}]$.

Conjecture 2.6.16. *The toroidal HOMFLY-PT invariant is an invariant of L as a link in $T^2 \times I$.*

CHAPTER III

HECKE CATEGORIES

3.1 Soergel Bimodules

We follow the book [28] closely in the following.

Let (W, S) be a Coxeter system and let $\mathfrak{h}$ be a realization of it over $\mathbb{C}$. Let $R = \operatorname{Sym} \mathfrak{h}^*$, viewed as a graded ring with $\mathfrak{h}^*$ in degree 2. Note that R has an action of W. We define the Demazure operators.

Definition 3.1.1. For each element $s \in S$, we have the Demazure operator $\partial_s : R \to R^s(-2)$ where R^s is the s-invariant subring of R. It is defined as

$$\partial_s(f) = \frac{f - s(f)}{\alpha_s}. \tag{3.1.1}$$

Note that $f - s(f)$ is divisible by α_s since it is s-anti-invariant and α_s generates the s-anti-invariants as an R^s-module.

Lemma 3.1.2. 1. *The map ∂_s is an R^s-bimodule map*

2. *We have $\partial_s^2 = 0, s \circ \partial_s = \partial_s,$ and $\partial_s \circ s = -\partial_s$.*

3. *(Twisted Leibniz), For $f, g \in R$, the following equation holds:*

$$\partial_s(fg) = \partial_s(f)g + s(f)\partial_s(g). \tag{3.1.2}$$

Proposition 3.1.3. *The Demazure operator $\partial_s : R \to R^s$ makes R into a Frobenius ring extension of R^s.*

We will not give details on Frobenius extensions and Frobenius objects, but will refer to the relevant section on them and related diagrammatics in [28, Chapter 7].

Notation 3.1.4. Let M be a graded R-bimodule. Let $M(n)$ denote the grading shift by n. Here, $M(n)^i = M^{i+n}$. Let R-grbimod denote the category of graded R-bimodules.

Notation 3.1.5. For $s \in S$, we consider the following graded R-bimodules.

$$B_s := R \otimes_{R^s} R(1). \tag{3.1.3}$$

Definition 3.1.6. Let $\underline{w} = (s_{i_1}, ..., s_{i_k})$ be a word in S. The *Bott-Samelson bimodule* associated to $\underline{w}$ is

$$\mathrm{BS}(\underline{w}) := B_{s_{i_1}} \otimes \cdots \otimes B_{s_{i_k}}. \tag{3.1.4}$$

Definition 3.1.7. The category $\mathbb{BS}\mathrm{Bim}$ of Bott-Samelson bimodules associated to (W, S) and $\mathfrak{h}$ is the smallest full subcategory of R-grbimod containing all the Bott-Samelson bimodules and direct sums of them.

The category $\mathbb{BS}\mathrm{Bim}$ is closed under tensor product, because the tensor product of two Bott-Samelson bimodules is a Bott-Samelson bimodule. The category $\mathbb{BS}\mathrm{Bim}$ is generated, as an additive monoidal category, by the objects R, and B_s for $s \in S$, in R-grbimod. The following definition is due to Soergel [26].

Definition 3.1.8. The category $\mathbb{S}\mathrm{Bim}$ of Soergel bimodules associated to (W, S) and $\mathfrak{h}$ is the smallest full subcategory of $R\text{-grbimod}$ containing all the Bott-Samelson bimodules, and closed under taking direct sums and direct summands.

The category $\mathbb{S}\mathrm{Bim}$ is a graded Krull-Schmidt category, i.e. it has unique decompositions into indecomposables, with graded local endomorphism rings. We note that $\mathbb{S}\mathrm{Bim}$ is the Karoubi envelope of $\mathbb{BS}\mathrm{Bim}$.

Example 3.1.9. Let $(W, S) = (W_{\text{fin}}, S_{\text{fin}})$, the Coxeter system of type A_{n-1}. Let $\mathfrak{h} = H$. Here $R = \mathbb{C}[x_1, ..., x_n]$. We set $\mathbb{S}\mathrm{Bim}_n$ to be the corresponding category of Soergel bimodules.

Example 3.1.10. Let $(W, S) = (W_{\text{aff}}, S_{\text{aff}})$, the Coxeter system of type $\widetilde{A}_{n-1}$. Let $\mathfrak{h} = \hat{H}$. Here $R = \mathbb{C}[x_1, ..., x_n, \delta]$. We set $A\mathbb{S}\mathrm{Bim}_n$ to be the corresponding category of Soergel bimodules.

Let $\mathbf{H}$ be the Hecke algebra of (W, S). Let $[\mathbb{S}\mathrm{Bim}]_{\oplus}$ indicate the additive Grothendieck ring of Soergel bimodules, an additive monoidal category, viewed as a $\mathbb{Z}[v^{\pm 1}]$-algebra where v corresponds to the internal grading.

Theorem 3.1.11. *(Soergel Categorification Theorem [26]).*

1. *There is a $\mathbb{Z}[v^{\pm 1}]$-algebra isomorphism $c : \mathbf{H} \to [\mathbb{S}\mathrm{Bim}]_{\oplus}$ defined on the Kahzdan-Lusztig generators by*

$$b_s \mapsto [B_s]$$

for all $s \in S$.

2. *There is a bijection between W and the indecomposables of $\mathbb{S}\mathrm{Bim}$. Let $\underline{w}$ be any reduced expression of $w \in W$. The indecomposable B_w is a summand of $\mathrm{BS}(\underline{w})$ with multiplicity 1. All other summands are $B_y(k)$ with $y < w$ and $k \in \mathbb{Z}$.*

Notation 3.1.12. Let $A, B \in R$-grbimod. We set the notation for the graded hom.

$$\mathrm{HOM}(A, B) = \bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}(A, B(m)).$$

The bar involution is categorified by the following duality functor.

Definition 3.1.13. The (right) *duality functor* is the contravariant autoequivalence $\mathbb{D}$ of $\mathbb{R}$-grbimod given by

$$\mathbb{D}(-) := \mathrm{Hom}^{\bullet}_{-R}(-, R). \tag{3.1.5}$$

To a bimodule B it assigns the space of right R-module maps to R. We view it as a graded R-bimodule via $(r \cdot f \cdot r')(b) = rf(b)r'$ for $r, r' \in R$, $f \in \mathbb{D}(B)$, and $b \in B$. We note how $\mathbb{D}$ intertwines grading shifts, that $\mathbb{D}(B(k)) \cong \mathbb{D}(B)(-k)$.

Proposition 3.1.14. *([28, Proposition 18.1]) For all $w \in W$, we have $\mathbb{D}(B_w) \cong B_w$.*

Theorem 3.1.15. *(Soergel Hom Formula [26, Theorem 5.15]). Let A and B be Soergel bimodules. Then $\mathrm{HOM}(A, B)$ is free as a left or right graded R-module, of graded rank:*

$$\mathrm{grdrk}\,\mathrm{HOM}(A, B) = ([A]_{\oplus}, [B]_{\oplus}). \tag{3.1.6}$$

So the graded hom categorifies the standard form on $\mathbf{H}$.

Example 3.1.16. $\mathrm{grdrk}\,\mathrm{HOM}(R, B_s) = v$ because $\mathrm{HOM}(R, B_s)$ is generated, as a right R-module, by the 'start-dot' (see 3.2.2). We recall that $b_s = T_s + v$ and thus $(1, b_s) = v$.

Remark 3.1.17. If $I \subset S$, then the restriction of the standard form on the Hecke algebra $\mathbf{H}_S$ associated to S restricts to the standard form on the Hecke algebra $\mathbf{H}_I$ associated to I. There is a full subcategory of $\mathbb{S}\mathrm{Bim}_{(W,S)}$, generated by R, B_s for $s \in I$. This differs from the category $\mathbb{S}\mathrm{Bim}_{(W_I,I)}$, Soergel bimodules for the corresponding parabolic subgroup, in that the graded Hom spaces between corresponding objects are still free of the same graded ranks but over a potentially different polynomial ring.

For example, The subcategory of $A\mathbb{S}\mathrm{Bim}_n$ generated by R and B_s for $s = 1, ...n$ is equivalent to $\mathbb{S}\mathrm{Bim}_n$ with Hom spaces extended from $R = \mathbb{C}[x_1, ..., x_n]$ to $R' = \mathbb{C}[x_1, ..., x_n, \delta]$. Simply apply $(-) \otimes_R R'$ on all Hom spaces.

3.1.1 Extended affine Soergel bimodules. Consider $(W, S) = (W_{\text{aff}}, S_{\text{aff}})$ and $\mathfrak{h} = \hat{H}$. Recall that R has an action of W_{ext} via $\tau : R \to R$. Let R_τ denote the bimodule defined in the following way. It is free of rank 1 as either a left or right R-module. The left action is the ordinary one. The right action by $x \in R$ is multiplication by $\tau(x)$. Let $R_{\tau^{-1}}$ defined in the same way except the right action by $x \in R$ is multiplication by $\tau^{-1}(x)$. Following the notation of [27], we'll set $\Omega := R_\tau$ and $\Omega^{-1} := R_{\tau^{-1}}$. Note, for $k \in \mathbb{Z}$, that $\Omega^k \cong R_{\tau^k}$, the R-bimodule where the right action by $x \in R$ is given by $\tau^k(x)$.

Definition 3.1.18. The category $EA\mathbb{S}\mathrm{Bim}_n$ of extended affine Soergel bimodules is the smallest full subcategory of R-grbimod containing Ω, Ω^{-1}, R, and B_s for $s \in S_{\text{aff}} = \mathbb{Z}/n\mathbb{Z}$ and closed under tensor product, direct sum, and direct summand.

Remark 3.1.19. The category $A\mathbb{S}\mathrm{Bim}_n$ is a full subcategory of $EA\mathbb{S}\mathrm{Bim}_n$.

Lemma 3.1.20. *There are bimodule isomorphisms:*

$$\Omega \otimes \Omega^{-1} \cong R \cong \Omega^{-1} \otimes \Omega, \tag{3.1.7}$$

$$\Omega \otimes B_s \cong B_{\tau(s)} \otimes \Omega. \tag{3.1.8}$$

Let $\mathcal{F} \in EA\mathbb{S}\mathrm{Bim}$. The lemma above implies that, for some $k \in \mathbb{Z}$, we have $\mathcal{F} \cong \Omega^k \mathcal{T}$ where $\mathcal{T} \in A\mathbb{S}\mathrm{Bim}$. Moreover, Elias proves the following in [27].

Proposition 3.1.21. *Let* $A, B \in A\mathbb{S}\mathrm{Bim}$.

$$\mathrm{Hom}(\Omega^k A, \Omega^l B) \cong \begin{cases} 0 & k \neq l, \\ R_\tau^k \otimes_R \mathrm{Hom}(B, C) & k = l. \end{cases} \tag{3.1.9}$$

Theorem 3.1.22. *(Soergel Categorification theorem for* $\mathbf{H}_{\mathrm{ext}}$ *[16, Theorem 2.5]).*

1. *There is a* $\mathbb{Z}[v^{\pm 1}]$*-algebra isomorphism* $c : \mathbf{H}_{\mathrm{ext}} \to [EA\mathbb{S}\mathrm{Bim}]_{\oplus}$ *defined on the Kahzdan-Lusztig generators by*

   $$\omega^k b_s \mapsto [\Omega^k B_s]$$

   for all $s \in S_{\mathrm{aff}}$ *and* $k \in \mathbb{Z}$.

2. *There is a bijection between* W_{ext} *and the indecomposables of* $EA\mathbb{S}\mathrm{Bim}$. *They are of the form* $\Omega^k B_w$ *where* $w \in W_{\mathrm{aff}}$ *and* $k \in \mathbb{Z}$.

3.2 Diagrammatics

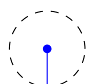
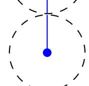
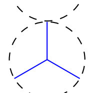
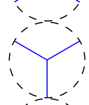
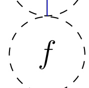
Notation 3.2.1. In [27] Elias defines a diagrammatic category $\mathcal{ED}$ and a functor $\varphi : \mathcal{ED} \to EA\mathbb{S}\mathrm{Bim}$. This functor is an equivalence after considering *Karoubi envelopes*. The category $\mathcal{ED}$ is an extension of the diagrammatic category of the Elias-Williamson $\mathcal{D}$ associated to $(W_{\mathrm{aff}}, S_{\mathrm{aff}}, H)$ described in [9], whose Karoubi envelope was shown to be equivalent to $A\mathbb{S}\mathrm{Bim}$ via a functor we also call $\varphi : \mathcal{D} \to A\mathbb{S}\mathrm{Bim}$.

Each of these categories is described as a unital additive monoidal category, generated by Frobenius algebra objects. A general discussion of this framework for defining diagrammatic categories can be found in [28, Chapter 8]. Here, isotopy and cyclicity are enforced as relations.

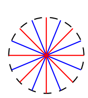
Our utilization of the diagrammatic categories will be simply to efficiently describe morphisms between Soergel bimodules themselves.

We abusively set R to be the unit of $\mathcal{D}$ and the generating objects to be B_s for $s \in S_{\text{aff}}$, as these are the Soergel bimodules they are mapped to under φ. While we will not aim to list all of the generating morphisms and relations of $\mathcal{D}$, we list a few, along with the corresponding morphism of Soergel bimodules. The following diagrams are from [9].

Notation 3.2.2. (One Color Generating Morphisms).

end-dot		deg 1	$B_s \to R$	$f \otimes g \mapsto fg$
start-dot		deg 1	$R \to B_s$	$1 \mapsto \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$
tri-valent vertex		deg -1	$B_sB_s \to B_s$	$1 \otimes g \otimes 1 \mapsto \partial_s g \otimes 1$
		deg -1	$B_s \to B_sB_s$	$1 \otimes 1 \mapsto 1 \otimes 1 \otimes 1$
polynomial		deg f	$R \to R$	$1 \mapsto f$

Notation 3.2.3. (The $2m_{st}$-valent vertex). For $s, t \in S$, we have a diagram for the isomorphism.

 deg 0 $\underbrace{B_sB_t\ldots}_{m_{st}} \xrightarrow{\sim} \underbrace{B_tB_s\ldots}_{m_{st}}$

For S_{aff} recall that all m_{st} are equal to 2 or 3, so there are 4 and 6 valent vertices as generating morphisms.

Let $i \in S$ be represented by the color blue. Consider the vertical composition of the i-colored start-dot, with the end-dot, which is the 'barbell':

$$: R \to B_i(1) \to R(2) \tag{3.2.1}$$

We also have multiplication by α_i. The following equality holds.

$\alpha_i =$  $: R \to R(2).$ (3.2.2)

We make use of the following relation extensively.

Lemma 3.2.4. *(Polynomial forcing). Here $s \in S$ is represented by the color red and $f \in R$.*

f  $=$  $s(f) +$  $\partial_s(f).$ (3.2.3)

The category $\mathcal{ED}$ is extended from $\mathcal{D}$ by adding new generators which we abusively call Ω and Ω^{-1}. There are new generating morphisms and relations added to ensure that 3.1.20 holds in $\mathcal{ED}$. We will not list these as we do not use them. We refer the reader to [27, Section 3.2] for details.

3.3 Chain complexes and Pseudocomplexes

The additive category $\mathbb{S}\mathrm{Bim}$ is enough to categorify the Kazhdan-Lusztig basis of **H**. We want to categorify $\mathbb{Z}[v^{\pm 1}]$-linear combinations of the Kazhdan-Lusztig and Bott-Samelson elements of the Hecke algebra. To do so we must work with chain complexes over $\mathbb{S}\mathrm{Bim}$.

Definition 3.3.1. We define the *Hecke Category*, $\mathcal{H}_W$, associated to $(W, S, \mathfrak{h})$ to be the bounded chain complex category $\mathrm{Ch}^b(\mathbb{S}\mathrm{Bim})$, viewed as a pre-triangulated dg-category.

Notation 3.3.2. Given a chain complex $M = (M^i, d^i)$, we denote by $M[k]$ the homological shift of M by k. Here $M[k]^i = M^{i+k}$ and $d[k]^i = d^{i+k}$.

Notation 3.3.3. We set notation for the *Hom complex*

$$\underline{\mathrm{Hom}}(A, B) = \bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}(A, B[m]).$$

We view it as a chain complex with the usual differential. It carries the structure of a left dg-module over $\underline{\mathrm{Hom}}(B, B)$ and a right dg-module over $\underline{\mathrm{Hom}}(A, A)$.

Remark 3.3.4. The setting of pre-triangulated dg-categories is the one we work with in following sections. These are sometimes referred to as *dg-enhanced* triangulated categories. They are an abstraction of triangulated categories arising as the homotopy category of chain complexes over an additive category. We refer the reader to [33] for more. For a pre-triangulated dg-category $\mathcal{C}$, its homotopy category $\mathrm{Ho}(\mathcal{C})$ is a triangulated category.

Notation 3.3.5. In the following, to align ourselves with the conventions in the literature, we will obscure whether we are working in the dg-category $\mathrm{Ch}^b(\mathbb{S}\mathrm{Bim})$ or its triangulated homotopy category $K^b(\mathbb{S}\mathrm{Bim}) := \mathrm{Ho}(\mathrm{Ch}^b(\mathbb{S}\mathrm{Bim}))$. We will simply refer to the category $K^b(\mathbb{S}\mathrm{Bim})$, or to $\mathcal{H}_W$. When the dg-structure is relevant, we will mean the category $\mathrm{Ch}^b(\mathbb{S}\mathrm{Bim})$. When the triangulated structure is relevant, for example when considering a triangulated Grothendieck ring, we will mean its homotopy category.

Definition 3.3.6. We set $\mathcal{H}_{\mathrm{fin}}$ to be the category $\mathcal{H}_W$ for $(W_{\mathrm{fin}}, S_{\mathrm{fin}}, H)$.

Definition 3.3.7. The duality functor $\mathbb{D}$ extends to complexes in $\mathcal{H}_{\mathrm{fin}}$. Because it is contravariant, it reverses the differentials in the complex, hence reverses the homological grading. In terms of Soergel diagrams, it corresponds to flipping diagrams upside-down.

Proposition 3.3.8. *Let $\mathcal{C}$ be an additive monoidal category, and let $\mathcal{K}^b(\mathcal{C})$ be its bounded homotopy category, viewed as a triangulated monoidal category. Let $[\mathcal{C}]_\oplus$ be the additive Grothendieck ring of $\mathcal{C}$, and let $[\mathcal{K}^b(\mathcal{C})]_T$ be the triangulated Grothendieck ring of $\mathcal{K}^b(\mathcal{C})$. Then we have*

$$[\mathcal{C}]_\oplus = [\mathcal{K}^b(\mathcal{C})]_T. \tag{3.3.1}$$

3.3.1 Pseudocomplexes.

Let us describe the setup that will be relevant in the rest of this work. We will later need to define important objects which are not chain complexes, but *pseudocomplexes*. We refer the reader to [18, Chapter 4] for more on the general setup for pseudocomplexes. We follow [27, Section 5.1] for a restricted setting of this theory.

We set $\mathcal{C} = EA\mathbb{S}\mathrm{Bim}$ and we recall the element $\delta \in \mathrm{End}_\mathcal{C}(R) \cong \mathbb{C}[x_1, ..., x_n, \delta]$.

Definition 3.3.9. (Pseudocomplexes). A *pseudocomplex* over $\mathcal{C}$ is the data $X = (X^i, d^i)_{i\in\mathbb{Z}}$ of objects $X^i \in \mathcal{C}$ and morphisms $d^i : X^i \to X^{i+1}$ of degree zero, such that $d^{i+1} \circ d^i \in \mathrm{HOM}(X^i, X^{i+2}) \cdot \delta$.

These objects are similar to, and include, ordinary chain complexes except where the differential is not required to square to zero. Instead we only require it to square to a multiple of δ.

Definition 3.3.10. Let X and Y be pseudocomplexes. A *pseudochain map* $f : X \to Y$ is a collection $f^i : X^i \to Y^i$ such that $d^i_Y f^i - f^{i+1} d^i_X \in \mathrm{Hom}(A, B)\delta$.

Definition 3.3.11. Each pseudocomplex X is equipped with a *monodromy map*

$$\mu_X \in \mathrm{Hom}(X, X(-2)[2])$$

given by $d_X^2 = \mu_X \cdot \delta$. In fact, μ_X is a pseudochain-map. For homologically bounded pseudocomplexes, μ_X is nilpotent.

Definition 3.3.12. We set $\mathrm{Ch}_\delta(\mathcal{C})$ to be the category of pseudocomplexes, twisted by δ, over $\mathcal{C}$. The morphisms in this category are the pseudo chain maps.

Proposition 3.3.13. *Let X and Y be psudocomplexes. The category $\mathrm{Ch}_\delta(\mathcal{C})$ inherits a monoidal structure from $\mathcal{C}$, where complex $X \otimes Y$ and its differential are defined in the usual way as with ordinary complexes.*

Remark 3.3.14. In order for $X \otimes Y$ to be a pseudocomplex, it is needed that left and right multiplication on morphisms by δ are the same, which holds because δ is invariant under W_{aff} and τ. Indeed, $d^2_{X \otimes Y} = d^2_X \otimes 1 + 1 \otimes d^2_Y$. Now $1 \otimes d^2_Y$ is in the ideal generated by δ acting on the right, and $d^2_X \otimes 1$ is in the ideal generated by δ acting in the middle (right of X, left of Y). The left, right, and middle multiplications by δ all agree so $d^2_{X \otimes Y}$ is right multiple of δ.

Proposition 3.3.15. *Let X and Y be pseudocomplexes. Then*

$$\mu_{X \otimes Y} = \mu_X \otimes 1 + 1 \otimes \mu_Y. \tag{3.3.2}$$

Definition 3.3.16. A pseudohomotopy of pseudocomplexes is a morphism $h \in \mathrm{Hom}(X, Y[-1])$. The corresponding nulhomotopic map is $dh + hd$.

Definition 3.3.17. Let $\mathcal{I}$ denote the ideal of $\mathrm{Ch}_\delta(\mathcal{C})$ generated by the nulhomotopic maps, and by the map of right mulitplication by δ. We define the homotopy category $K_\delta(\mathcal{C})$ to be the quotient of $\mathrm{Ch}_\delta(\mathcal{C})$ by $\mathcal{I}$.

The usual concept of mapping cones of morphisms of chain complexes applies in the psuedocomplex setting to pseudochain maps.

Proposition 3.3.18. *$K^b_\delta(\mathcal{C})$ is triangulated.*

See [18] for details.

If we take the quotient of $\mathrm{Ch}^b_\delta(\mathcal{C})$ by the ideal generated by δ, we get ordinary chain complexes over $\mathcal{C}$ modulo its ideal generated by δ. We will need to consider this later as it is not clear to us how to enhance the category of pseudo-chain complexes into a dg-category. In the following, the dg-structure will be relevant for our conjecture, so we must pass back to ordinary chain complexes which may be dg-enhanced.

Definition 3.3.19. (Extended affine Hecke category). We define the *extended affine Hecke category*, $\mathcal{H}_{\mathrm{ext}}$, to be the quotient of $\mathrm{Ch}^\delta(\mathcal{C})$ by the ideal generated by δ.

This is the same category as if we had defined $EA\mathbb{S}\mathrm{Bim}$ by working with the realization $R = \mathbb{C}[x_1, ..., x_n]$ in the first place, by setting δ to zero, and then taking chain-complexes over it. Thus it may be dg-enhanced and we view it as a dg-category.

It is still helpful to work with pseudocomplexes, rather than their images in the quotient, as it makes their monodromy endomorphisms, μ_X, self-evident. Practically we will work with these complexes without killing δ. When we say two morphisms in this category are homotopic, we will mean up to homotopy and the ideal of δ. However, when the dg-category structure is important to us, we will mean ordinary complexes over $\mathcal{C}/\delta$, having killed δ.

CHAPTER IV

CATEGORIFIED BRAIDS AND STANDARD GAITSGORY COMPLEX

4.1 Rouquier Complexes

In this section we discuss the complexes in $\mathcal{H}_{\text{ext}}$ associated to braids in Br_{ext}. Let ω, and f_i for $i \in S_{\text{aff}}$ denote the generators of Br_{ext} as in 2.4.6.

Claim 4.1.1. *The generators ω and f_i of Br_{ext}, and their inverses, are categorified by the following objects of $\mathcal{H}_{\text{ext}}$.*

$$\omega \rightsquigarrow \underline{\Omega} \tag{4.1.1}$$

$$\omega^{-1} \rightsquigarrow \underline{\Omega^{-1}} \tag{4.1.2}$$

$$f_i \rightsquigarrow F_{s_i} := \left(\underline{B}_{s_i} \longrightarrow R(1) \right) \tag{4.1.3}$$

$$f_i^{-1} \rightsquigarrow F_{s_i}^{-1} := \left(R(-1) \longrightarrow \underline{B}_{s_i} \right) \tag{4.1.4}$$

The underline in the chain complexes above indicates homological degree zero.

Proposition 4.1.2. *(Rouquier [24]). There are homotopy equivalences*

$$F_s \otimes F_s^{-1} \simeq R \simeq F_s^{-1} \otimes F_s, \tag{4.1.5}$$

$$\underbrace{F_s \otimes F_t \otimes \cdots}_{m_{st} \text{ times}} \simeq \underbrace{F_t \otimes F_s \otimes \cdots}_{m_{st} \text{ times}} \tag{4.1.6}$$

for all $s, t \in S_{\text{aff}}$.

A consequence of Lemma 3.1.20 is the following.

Proposition 4.1.3. *There is an isomorphism:*

$$\Omega \otimes F_s \cong F_{\tau(s)} \otimes \Omega. \tag{4.1.7}$$

Definition 4.1.4. Let $\beta \in \mathrm{Br}_{\mathrm{ext}}$ denote a cylindrical braid. Given a braid word $\mathbf{b}$ for β, we may write $\beta = \omega^l f_{i_1} f_{i_2} \cdots f_{i_k}$. We define the *Rouquier complex* $F_{\mathbf{b}}$ associated to $\mathbf{b}$ to be

$$F_{\mathbf{b}} := \Omega^l \otimes F_{s_{i_1}} \otimes \cdots \otimes F_{s_{i_k}}. \tag{4.1.8}$$

The following lemma follows from 3.1.21.

Lemma 4.1.5. *Let $x = \omega^k \beta$ and $y = \omega^\ell \gamma$ be in $\mathrm{Br}_{\mathrm{ext}}$. There is an isomorphism as R-bimodules:*

$$\mathrm{Hom}_{K^b(EA\mathbb{S}\mathrm{Bim})}(F_x, F_y) \cong \begin{cases} 0 & \text{if } k \neq \ell, \\ R_{\tau^k} \otimes_R \mathrm{Hom}_{K^b(A\mathbb{S}\mathrm{Bim})}(F_\beta, F_\gamma) & \text{if } k = \ell. \end{cases} \tag{4.1.9}$$

In [27, 4.4], Elias proves the following.

Theorem 4.1.6. *(Rouquier Canonicity for $\mathrm{Br}_{\mathrm{ext}}$). Let $\mathbf{b}_1$, $\mathbf{b}_2$ be braid words representing the same braid $\beta \in \mathrm{Br}_{\mathrm{ext}}$. There exists a homotopy equivalence*

$$\psi_{\mathbf{b}_1,\mathbf{b}_2} \colon F_{\mathbf{b}_1} \to F_{\mathbf{b}_2}.$$

These homotopy equivalences are compatible:

$$\psi_{\mathbf{b}_1,\mathbf{b}_3} = \psi_{\mathbf{b}_2,\mathbf{b}_3} \circ \psi_{\mathbf{b}_1,\mathbf{b}_2}. \tag{4.1.10a}$$

For any simple reflection $s \in S_{\mathrm{aff}}$, we have

$$\psi_{\mathbf{b}_1 f_s,\mathbf{b}_2 f_s} = \psi_{\mathbf{b}_1,\mathbf{b}_2} \otimes \mathrm{id}_{F_s}, \tag{4.1.10b}$$

$$\psi_{\mathbf{b}_1 f_s^{-1},\mathbf{b}_2 f_s^{-1}} = \psi_{\mathbf{b}_1,\mathbf{b}_2} \otimes \mathrm{id}_{F_s^{-1}}. \tag{4.1.10c}$$

The above are equalities in the homotopy category.

Proof. Let $\mathfrak{M}$ denote the monoidal ideal of $EA\mathbb{S}\mathrm{Bim}$ generated by id_{B_x} for $x \in W_{\text{aff}}$ with $x \neq 1$, and by positive degree polynomials $f \in \mathrm{END}(R)$. Let $v : \mathcal{H}_{\text{ext}} \to \mathcal{H}_{\text{ext}}/\mathfrak{M}$ denote the functor which is the quotient by this ideal. Taking the quotient by this ideal, one recovers the category $\mathbf{Vect}_{\text{gr}}^{\mathbb{C}} \boxtimes \mathbb{Z}$.

Here $\mathbf{Vect}_{\text{gr}}^{\mathbb{C}}$ denotes graded vector spaces and $\mathbb{Z}$ denotes the category with objects $k \in \mathbb{Z}$ and morphisms

$$\mathrm{Hom}(k, l) = \mathbb{C}\delta_{kl}.$$

Extending to complexes, we get a functor $v : \mathcal{H}_{\text{ext}} \to \mathrm{Ch}^b(\mathbf{Vect}_{\text{gr}}^{\mathbb{C}} \boxtimes \mathbb{Z})$.

Note that $v(R) = \mathbb{C} \boxtimes 0$ and $v(\Omega^k) = \mathbb{C} \boxtimes k$, and $v(B_x) = 0 \boxtimes 0$. The Because $F_{\mathbf{b}_1}$ and $F_{\mathbf{b}_2}$ are invertible objects in $K^b(EA\mathbb{S}\mathrm{Bim})$ by 4.1.5, a homotopy equivalence between them is already determined up to scalar. The issue is in choosing compatible scalars. The complexes $F_{\mathbf{b}_1}$ and $F_{\mathbf{b}_2}$ have a unique copy of $R(k) \otimes \Omega^l[-k]$ where l is the winding number and k is the braid exponent: the number of positive crossings - the number of negative crossings. An isomorphism between them must descend to an isomorphism after taking the quotient by $\mathfrak{M}$. An isomorphism exists by applying 4.1.5 and 4.1.6. We choose the isomorphism which descends to the identity on the copy of $\mathbb{C}(k)[-k] \boxtimes l$ in $v(F_{\mathbf{b}_1})$ and $v(F_{\mathbf{b}_2})$ S. $\square$

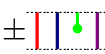
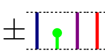
The theorem defines what we call a *strict* action of Br_{ext} on $\mathcal{H}_{\text{ext}}$.

Notation 4.1.7. We will keep the notation $v : \mathcal{H}_{\text{ext}} \to \mathbf{Vect}_{\text{gr}}^{\mathbb{C}} \boxtimes \mathbb{Z}$. This functor will be referred to briefly again in the following.

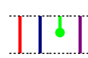
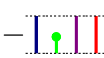
4.1.1 Sign convention for Rouquier complexes.

We introduce a helpful way of understanding the sign of the differentials in Rouquier complexes from the diagrammatics. Let P and Q be chain complexes. The typical convention for the sign of the differential on $P^i \otimes Q^j$ is $d_P \otimes 1 + (-1)^i \otimes d_Q$. We'll use a different, yet isomorphic, convention for the differentials on Rouquier complexes.

Given a Rouquier complex $F_{\mathbf{b}}$, in terms of Soergel diagrammatics, all of its differentials are, up to sign, either a start-dot (see 3.2.2) tensored with identity maps id_{B_i} on the left and right, or an end-dot, also tensored with identity maps id_{B_i} on the left or right. For example the following diagrams may be matrix entries of the differential.

$\pm$ , $\pm$  (4.1.11)

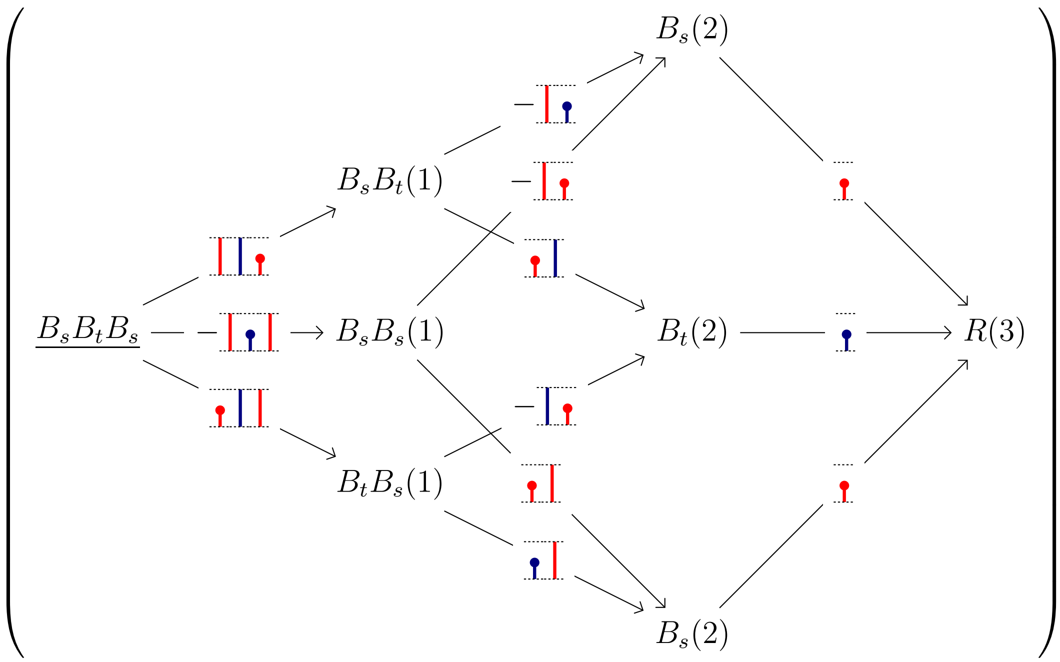
We will use the convention where the sign on this entry of the differential is $(-1)^l$, where l is the number of strands to the left of the start-dot or end-dot. So, in our example, the entries would have the following signs:

, $-$ . (4.1.12)

We show the entries of the differential, with this sign rule, for $F_{\mathbf{b}} = F_s F_t F_s$, with s being represented by red and t being represented by blue. The underline indicates homological degree zero.

Example 4.1.8.

$F_sF_tF_s =$



4.1.2 Wakimoto Complexes.

Definition 4.1.9. Let $\lambda \in \Lambda_{\mathrm{wt}}$. Recall the corresponding Wakimoto braid $w(\lambda) = y_1^{\lambda_1} \cdots y_n^{\lambda_n}$. We set $W(\lambda) := F_{w(\lambda)}$, the Rouquier complex for this braid.

Remark 4.1.10. By Rouquier canonicity, the Wakimoto complexes give a strict action of Λ_{wt} on $\mathcal{H}_{\mathrm{ext}}$. In fact, the homotopy equivalences picked out by Rouquier canonicity ensure that the Wakimoto complexes commute in the categorical sense.

Proposition 4.1.11. *Let $\lambda, \nu \in \Lambda_{\mathrm{wt}}$. Then we have canonical isomorphisms:*

$$W(\lambda) \otimes W(\nu) \cong W(\lambda + \nu) \cong W(\nu) \otimes W(\lambda). \tag{4.1.13}$$

Notation 4.1.12. We set $W_i = W(\epsilon_i)$, the Rouquier complex categorifiying the Wakimoto braid $y_i \in \mathrm{Br}_{\mathrm{ext}}$.

4.1.3 Toroidal HOMFLY Homology.

Consider the total Hochschild cohomology functor $\mathrm{HH}^* := \mathrm{R}^\bullet \mathrm{HOM}(R, -) : R\text{-grbimod} \to D^b(R\text{-grmod})$, where the target category is the derived category. Extending to complexes, it yields a functor:

$$\mathrm{HH}^* : K^b(R\text{-grbimod}) \to K^b(D^b(\mathbf{Vect}^{\mathbb{C}}_{\mathrm{gr}})). \tag{4.1.14}$$

Here, we are implicitly applying the forgetful functor

$$K^b(D^b(R\text{-grmod})) \stackrel{\text{for}}{\to} K^b(D^b(\mathbf{Vect}^{\mathbb{C}}_{\mathrm{gr}}).$$

Now, the target category of this functor carries 3 gradings! The homological grading coming from D^b, which we call the Hochschild grading, the homological grading coming from K^b, and the internal grading on $\mathbf{Vect}^{\mathbb{C}}_{\mathrm{gr}}$ which is the v-grading.

Definition 4.1.13. (HOMFLY Homology). Given an ordinary planar braid $\beta \in \mathrm{Br}_{\mathrm{fin}}$, the HOMFLY homology complex of its closure $L = \overline{\beta}$, a link in S^3, is defined to be

$$\mathrm{HHH}(L) = \mathrm{HH}^*(F_\beta). \tag{4.1.15}$$

The total Hochschild homology categorifies the Jones-Ocneanu trace in the following sense:

Theorem 4.1.14. *(Khovanov [41]).*

1. *The homology modules of the bicomplex* $\mathrm{HHH}(L)$, *up to grading shift, give an invariant of L as a link in* S^3.

2. *The complex* $\mathrm{HHH}(L)$ *categorifies the HOMFLY-PT invariant. Let* $P(L) \in \mathbb{Z}[a^{\pm 1}](v)$ *denote the HOMFLY-PT invariant of* L*. Then*

$$P(L) = \sum_{i,j} (-1)^i (-a^2 v^2)^j \operatorname{grdrk}_v \mathrm{HHH}_{i,j}(L). \tag{4.1.16}$$

We can also apply the total Hochschild cohomology functor in the extended affine setting. This ultimately gives us a functor

$$\mathrm{HH}^* : K^b(EA\mathbb{S}\mathrm{Bim}_n) \to K^b(D^b(\mathbf{Vect}^{\mathbb{C}}_{\mathrm{gr}})). \tag{4.1.17}$$

Conjecture 4.1.15. *The functor above is 'trace-like' in the sense of [34].*

Now we define the following.

Definition 4.1.16. Given a cylindrical braid β and its braid closure L, a link in $T^{\times} I$, we define the *toroidal HOMFLY-PT complex* of L:

$$\mathrm{HHH}(L) := \mathrm{HH}^*(F_\beta) \in K^b(D^b(\mathbf{Vect}^{\mathbb{C}}_{\mathrm{gr}})). \tag{4.1.18}$$

Conjecture 4.1.17. *Up to grading shifts and isomorphism in* $K^b(D^b(\mathbf{Vect}^{\mathbb{C}}_{gr}))$*, we conjecture that* $\mathrm{HHH}(L)$ *is an invariant of* L *as a link in* $T^2 \times I$*.*

Remark 4.1.18. It would be interesting to demonstrate a relation between toroidal HOMFLY homology of $\overline{\beta}$ and the ordinary HOMFLY homology of $\overline{\flat(\beta)}$.

Conjecture 4.1.19. *Let* $\beta = f_{i_1} \cdots f_{i_k}$ *where* $i_1, ..., i_k \subset S_{\text{fin}} \subset S_{\text{aff}}$*, for one of the* n *copies of* S_{fin} *in* S_{aff} *obtained by omitting* j *of* S_{aff}*. We conjecture that the toroidal HOMFLY homology complex of* $\overline{\beta}$ *and the ordinary HOMFLY homology complex of* $\overline{\flat(\beta)}$ *are quasi-isomorphic.*

4.2 Standard Gaitsgory Complex

In [27, Section 8.1], Elias defines a pseudocomplex $\mathcal{F}$ over $A\mathbb{S}\mathrm{Bim}$. The complex $\mathcal{V} = \Omega\mathcal{F}$ is then shown to have the desired properties of the standard Gaitsgory central object of $\mathcal{H}_{\mathrm{ext}}$.

Proposition 4.2.1. *There is a natural isomorphism of functors*

$$\mathcal{F} \otimes (-) \cong \tau(-) \otimes \mathcal{F} : EA\mathbb{S}\mathrm{Bim} \to \mathcal{H}_{\mathrm{ext}}, \tag{4.2.1}$$

where τ is the endofunctor of $A\mathbb{S}\mathrm{Bim}_n$ sending the generator B_s to $B_{\tau(s)}$. We have $\tau(-) = \Omega \otimes (-) \otimes \Omega^{-1}$.

Therefore, the complex $\mathcal{V} = \Omega\mathcal{F}$ is proven to commute, in the categorical sense, with the additive category $EA\mathbb{S}\mathrm{Bim}$. This is a first step to showing it is central in $\mathcal{H}_{\mathrm{ext}}$. Elias also shows the following.

Proposition 4.2.2. *(Wakimoto Filtration). The complex $\mathcal{V}$ has a filtration*

$$0 = \mathcal{V}^0 \hookrightarrow \mathcal{V}^1 \hookrightarrow \cdots \hookrightarrow \mathcal{V}^n = \mathcal{V}.$$

The i-subquotient $\mathcal{V}^i/\mathcal{V}^{i-1}$ is isomorphic to the i-th Wakimoto complex W_i. This filtration comes from a filtration of $\mathcal{F}$:

$$0 = \mathcal{F}^0 \hookrightarrow \mathcal{F}^1 \hookrightarrow \cdots \hookrightarrow \mathcal{F}^n = \mathcal{F}.$$

The i-th subquotient $\mathcal{F}^i/\mathcal{F}^{i-1}$ is isomorphic to $\Omega^{-1}W_i$.

Because $\mathcal{V}$ is a bounded pseudocomplex, it is equipped with a nilpotent 'monodromy' operator $\mu : \mathcal{V} \to \mathcal{V}[2]$.

Recall the functor $v : \mathcal{H}_{\text{ext}} \to \mathrm{Ch}^b(\mathbf{Vect}^{\mathbb{C}}_{\text{gr}} \boxtimes \mathbb{Z})$. Under this functor, we have $v(\mathcal{V}) = V \boxtimes 1$, where V is the standard representation of $\mathfrak{gl}_n$, viewed as a chain complex with a copy of $\mathbb{C}$, for each weight space, in homological degrees -(n-1) to n-1 with zero differential. Here, μ becomes the principal nilpotent operator $\sum_i e_i$ where i ranges over the simple roots of $\mathfrak{gl}_n$ and the e_i are the raising operators. The Wakimoto filtration becomes the weight filtration of V, and the i-th Wakimoto W_i is sent to the weight vector for ϵ_i.

In the following we'll develop the notation necessary to define $\mathcal{F}$.

4.2.1 Cyclic Orders and Signs. The objects in each homological degree of $\mathcal{F}$ will be indecomposable Soergel bimodules B_X associated to a subset $X \subset S_{\text{aff}}$.

Definition 4.2.3. Let $X \subsetneq S$. Choose $\aleph \in S \setminus X$. This choice determines a total order on S, where $\aleph < \aleph + 1 < \aleph + 2 < \ldots < \aleph - 1$. Restricting this order to the subset X, we write $X = \{x_1 < x_2 < \ldots < x_d\}$. We call such an order on X a *cyclic order*. Let $h_X \in W_{\text{aff}}$ denote the element $s_{x_d} \cdots s_{x_2} s_{x_1}$.

Definition 4.2.4. We set ρ_k to be the cyclic order on S_{aff} which starts at k.

Proposition 4.2.5. *The element h_X does not depend on the cyclic order on X (i.e. the choice of $\aleph$). Any two reduced expressions coming from a cyclic order on X will be related by braid relations of the form $s_k s_j = s_j s_k$ for $|j - k| > 1$.*

Definition 4.2.6. Let $X \subsetneq S_{\text{aff}}$. Suppose we have a cyclic ordering on X as in 4.2.3. We set define B_X to be the Bott-Samelson bimodule $\mathrm{BS}(s_{x_d} \cdots s_{x_1})$. It is indecomposable, and isomorphic to the bimodule B_{h_X}.

The definition of B_X above depends on the choice of cyclic order. Given two cyclic orders on X, we have two reduced expressions, $\underline{h_1}$, and $\underline{h_2}$, for h_X, hence two Bott-Samelson bimodules $\mathrm{BS}(\underline{h_1})$ and $\mathrm{BS}(\underline{h_2})$. They both deserve to be called B_X. We now fix a family of isomorphisms $\mathrm{rex}(\underline{h_1}, \underline{h_2}) : \mathrm{BS}(h_1) \to \mathrm{BS}(h_2)$ which are compatible like in 4.1.10a. We will use these different constructions to canonically identify these different constructions of B_X.

Until these issues are settled, we use the following notation to disambiguate different constructions of B_X. For $k \notin X$, if we use the cyclic order ρ_k to obtain a reduced expression $\underline{h}_X$ for h_X, then we write $B_X^{(k)}$ for $\mathrm{BS}(\underline{h}_X)$.

Because these two reduced expressions are related only by braid moves of the form $s_i s_j = s_j s_i$, the Soergel diagram making them isomorphic will involve only involve 4-valent vertices from 3.2.2.

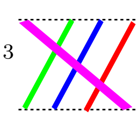
Definition 4.2.7. (Signed Rex Move). Let $X \subsetneq S_{\text{aff}}$. Given two cyclic orders of X, we have two isomorphic Bott-Samelson bimodules $\mathrm{BS}(h_1)$ and $\mathrm{BS}(h_2)$. Let $d : \mathrm{BS}(h_1) \to \mathrm{BS}(h_2)$ be any diagram built from the 4-valent vertices, making them isomorphic. Let m be the number of vertices in the Soergel diagram for this isomorphism. We defined the *signed rex move* to be:

$$\mathrm{rex} := (-1)^m d : \mathrm{BS}(h_1) \to \mathrm{BS}(h_2).$$

It is independent of the choice of diagram d.

Example 4.2.8. Let $n = 8$ and let $X = 2, 3, 4, 6 \subset S_{\text{aff}} = \mathbb{Z}/8\mathbb{Z}$. There are two distinct cyclic orders 2346 and 6234. Below we draw the signed rex move: $\mathrm{rex} : \mathrm{BS}(2345) \to \mathrm{BS}(6234)$. Here 2 is represented by green, 3 by blue, 4 by red, and 6 by pink.

$$\mathrm{rex} = (-1)^3 \tag{4.2.2}$$



Now that we are able to switch reduced expressions for h_X at will, we can pick our favorite to induce sign conventions on morphisms.

Definition 4.2.9. Let $X, Y \subsetneq S_{\text{aff}}$ with $Y = X \cup \{i\}$. Choose $k \notin Y$ and give both X and Y the cyclic order induced from ρ_k. We set $\forall_X^Y \colon B_X^{(k)} \to B_Y^{(k)}(1)$ and $A_Y^X \colon B_Y^{(k)} \to B_X^{(k)}(1)$ to be the signed dot maps discussed in 4.1.1.

Lemma 4.2.10. *For $k, l \notin Y$, let $\forall^{(k)}$ and $A^{(k)}$ (resp. $\forall^{(l)}$ and $A^{(l)}$) be the maps defined above using the cyclic order ρ_k (resp. ρ_l). Then the following diagrams commute.*

$$\begin{tikzcd} B_Y^{(k)} \arrow[r, "\text{rex}"] \arrow[d, "A^{(k)}"'] & B_Y^{(l)} \arrow[d, "A^{(l)}"'] \\ B_X^{(k)} \arrow[r, "\text{rex}"] & B_X^{(l)} \end{tikzcd}, \qquad \begin{tikzcd} B_Y^{(k)} \arrow[r, "\text{rex}"] & B_Y^{(l)} \\ B_X^{(k)} \arrow[r, "\text{rex}"] \arrow[u, "\forall^{(k)}"] & B_X^{(l)} \arrow[u, "\forall^{(l)}"] \end{tikzcd}. \tag{4.2.3}$$

Checking this is an easy computation. Given the above lemma, Definition 4.2.9 defines a map $B_X \to B_Y$ which is independent of the choice of cyclic orders, up to canonical identifications by the map in Definition 4.2.7.

4.2.2 The Pseudocomplex $\mathcal{F}$.

Definition 4.2.11. For $k \in \mathbb{Z}$ let P_k denote the following set of subsets of S_{aff}:

$$P_k = \{X \subsetneq S_{\text{aff}} \mid |X| = m \text{ where } 0 \leq m \leq n-1-|k| \text{ and } n-1-k-m \text{ is even}\}. \tag{4.2.4}$$

So, for example, P_0 consists of all proper subsets of S whose parity agrees with $n-1$, while $P_{n-1} = P_{1-n} = \{\emptyset\}$, and P_k is empty for $|k| \geq n$. Note that $P_k = P_{-k}$.

Definition 4.2.12. Let $\mathcal{F} = (\mathcal{F}^i, d^i)$ denote the precomplex with

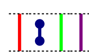
$$\mathcal{F}^k = \bigoplus_{X \in P_k} B_X(k). \tag{4.2.5}$$

Thus P_k indexes the summands in homological degree k. Let X in P_i, and let $d^i : \mathcal{F}^i \to \mathcal{F}^{i+1}$. Let $d^i_{B_X}$ denote the column of d^i when restricted to the direct summand $B_X(i) \overset{\oplus}{\subset} \mathcal{F}^i$. Then

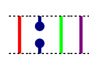
$$d^i_{B_X} = \bigoplus_{\substack{i \notin X \\ Xi \in P_{k+1}}} \forall_X^{Xi} \oplus \bigoplus_{\substack{j \in X \\ X \setminus j \in P_{k+1}}} A_X^{X \setminus j}. \tag{4.2.6}$$

Remark 4.2.13. We often refer to a summand of $\mathcal{F}$ by the index X, rather than specifying a pair (X, k) where $X \in P_k$. Implicitly, whenever we refer to X as indexing a summand of $\mathcal{F}$, there is implicitly some k such that $X \in P_k$. We treat as distinct summands $X \in P_k$ and $X \in P_l$ for $k \neq l$.

It is not obvious that $\mathcal{F}$ is a pseudocomplex, i.e. that $d^{i+1} \circ d^i \in \mathrm{HOM}(\mathcal{F}^i, \mathcal{F}^{i+1})\delta$. A priori, the entries of $d^{i+1} \circ d^i$ will involve middle multiplication by simple roots α_j like

,

and middle 'broken-dot-maps' like:

.

Elias proves that after you exploit (3.2.4) to move all the polynomials to the right, that the broken-dots cancel out, and that $d^{i+1} \circ d^i \in \mathrm{HOM}(\mathcal{F}^i, \mathcal{F}^{i+1})\delta$.

Proposition 4.2.14. *[27, Proposition 8.25] The complex $\mathcal{F}$ is a well-defined pseudocomplex, and a true complex after setting $\delta = 0$. It is concentrated in homological degrees $-(n-1)$ through $(n-1)$.*

Elias also proves that $\mathcal{F}$ has the desired properties of the standard Gaitsgory complex, which include the following.

Proposition 4.2.15. *The pseudocomplex $\mathcal{F}$ is self-dual, i.e. $\mathbb{D}(\mathcal{F}) = \mathcal{F}$. It is also perverse [28, Definition 19.5]. Hence, $\mathcal{V}$ has these properties.*

4.2.3 Examples.

4.2.3.1 n=1. We haven't discussed the category $\mathcal{H}_{\text{ext}}$ for $n = 1$. We define it as follows.

Definition 4.2.16. We define the extended affine Hecke category, for n=1, to be the category $\mathbf{Vect}_{\text{gr}}^{\mathbb{C}} \boxtimes (\mathbb{Z})$. Here the category $\mathbb{Z}$ is the additive monoidal category generated by an invertible object Ω, with grading shifts (i), and with graded hom spaces

$$\mathrm{Hom}_{\mathbb{Z}}(\Omega^k, \Omega^k) = \begin{cases} 0 & \text{if } k \neq l, \\ \mathbb{C}[x_1, \delta] & \text{else,} \end{cases}$$

where x is in degree 2.

Let $R = \mathbb{C}[x_1, \delta]$. We can think of Ω as being the twisted bimodule R_τ where $\tau(x_1) = x_1 - \delta$. Then, as an R-bimodule, we have $\mathrm{Hom}(\Omega^k, \Omega^k) = R_{\tau^k}$.

We should think of this category as categorifying cylindrical braids on a single strand. When there is one strand, there can be no crossings, only winding.

In this setting $\mathcal{V} = \Omega$. The nilpotent monodromy operator μ is the zero map, as $\mathcal{V}$ is concentrated in homological degree 0. We also have $W_1 = \Omega$. The map $\chi : \mathcal{V} \to \mathcal{V}(2)$ of chapter V is simply right multiplication by x_1.

4.2.3.2 n=2. For $n = 2$, we have

$$\mathcal{V} = \Omega\left(\begin{tikzcd} & B_1 \arrow[dr] & \\ R(-1) \arrow[ur] \arrow[dr] & & R(1) \\ & B_0 \arrow[ur] & \end{tikzcd} \right). \tag{4.2.7}$$

Now we observe its Wakimoto filtration. It has

$$W_1 = \Omega\left(\begin{tikzcd} & B_1 \arrow[dr] & \\ & & R(1) \end{tikzcd} \right) \tag{4.2.8}$$

as a subcomplex. This is the Wakimoto complex $W((1,0))$, which is the Rouquier complex for the braid $y_1 = \omega f_1$. The quotient by this complex is

$$W_2 = \Omega\left(\begin{tikzcd} R(-1) \arrow[dr] & & \\ & B_0 & \end{tikzcd} \right) \tag{4.2.9}$$

which is the Wakimoto complex $W((0,1))$, the Rouquier complex for the braid $y_2 = \omega f_0^{-1}$. The nilpotent monodromy operator $\mu : \mathcal{V} \to \mathcal{V}[2](-2)$ is

$$\mu = \mathrm{id}_\Omega \otimes \left(\begin{tikzcd} & & & B_1 \arrow[dr] & \\ & & R(-1) \arrow[ur] \arrow[dr] \arrow[ddd, "\mathrm{id}_R"] & & R(1) \\ & & & B_0 \arrow[ur] & \\ & B_1(-2) \arrow[dr] & & & \\ R(-3) \arrow[ur] \arrow[dr] & & R(-1) & & \\ & B_0(-2) \arrow[ur] & & & \end{tikzcd} \right). \tag{4.2.10}$$

Oberve that $d^2 = \delta \cdot \mu$.

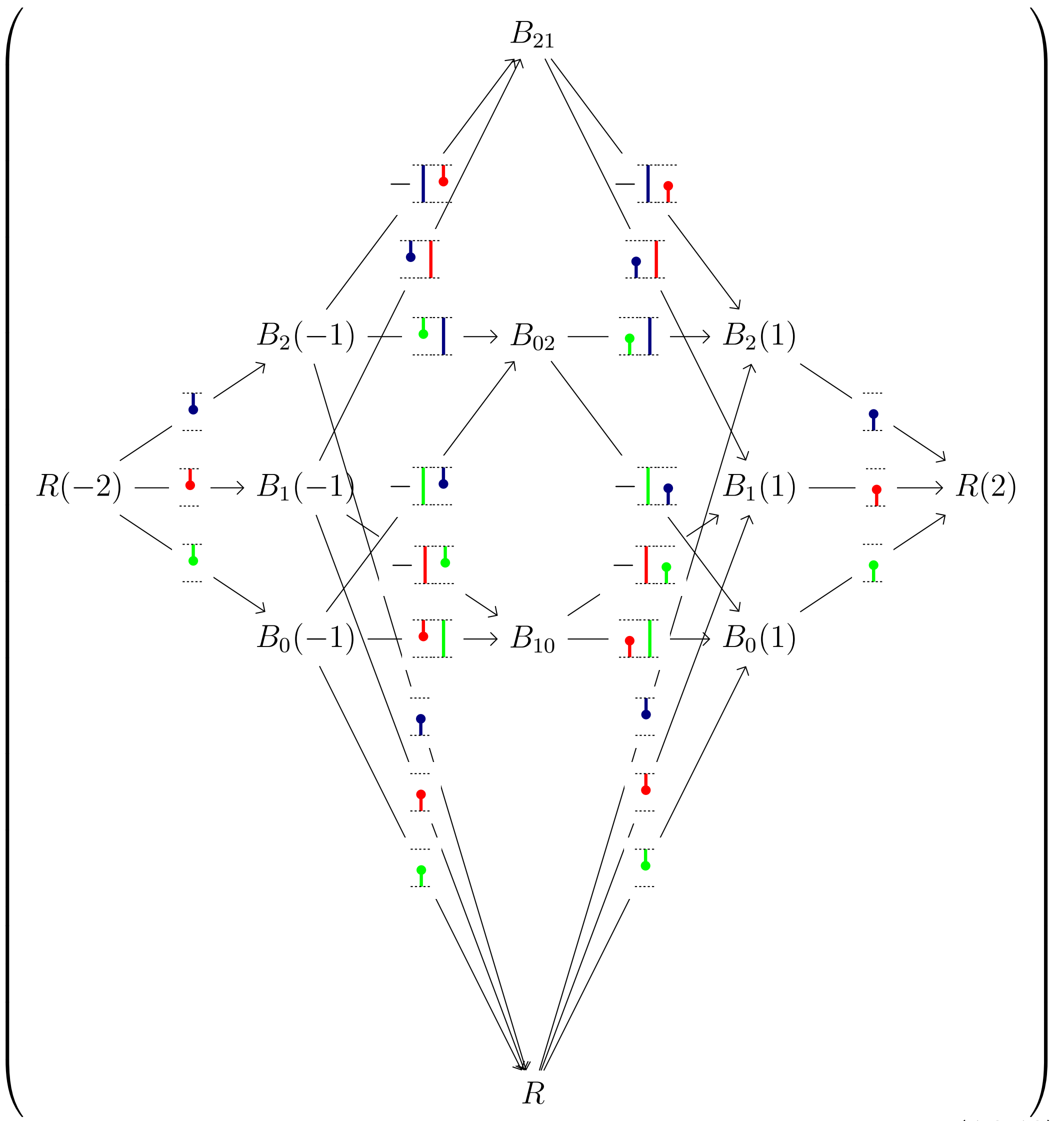
Now we give the operator $\chi : \mathcal{V} \to \mathcal{V}(2)$. It is given in homological degree -1 by $\chi^{-1} = x_2$, i.e. multiplication by x_2, and in homological degree 1 by $\chi^1 = x_1$. In homological degree 0 it is given by

$$\chi^0 = \mathrm{id}_\Omega \otimes \begin{pmatrix} x_1 & \\ 0 & x_2 \end{pmatrix}. \tag{4.2.11}$$

It is a short-exercise in polynomial forcing (3.2.4) to check this is a pseudochain map.

_4.2.3.3 n=3._ For $n = 3$, we have

$\mathcal{V} = \Omega$



.

(4.2.12)

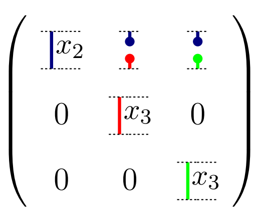
Here is the Wakimoto filtration:

$$W_1 = \Omega F_2 F_1 = \left(\begin{tikzcd} B_{21} \arrow[r] \arrow[dr] & B_2(1) \arrow[dr] & \\ & B_1(1) \arrow[r] & R(2) \end{tikzcd}\right) \tag{4.2.13}$$

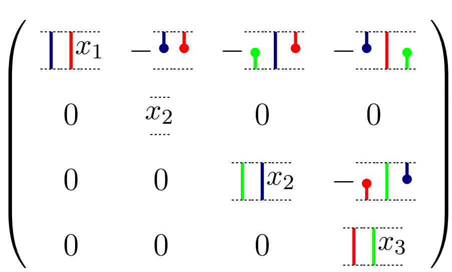
$$W_2 = \Omega F_0^{-1} F_2 = \left(\begin{tikzcd} B_2(-1) \arrow[dr] \arrow[ddr] & & \\ & B_{02} \arrow[dr] & \\ & R \arrow[r] & B_0(1) \end{tikzcd}\right) \tag{4.2.14}$$

$$W_3 = \Omega F_1^{-1} F_0^{-1} = \left(\begin{tikzcd} & & B_{10} \\ R(-2) \arrow[r] \arrow[dr] & B_1(-1) \arrow[ur] & \\ & B_0(-1) \arrow[uur] & \end{tikzcd}\right) \tag{4.2.15}$$

Now we describe $\chi : \mathcal{V} \to \mathcal{V}(2)$ Here $\chi^{-2} = x_3$. In homological degree -1 it is given by

$$\chi^{-1} = \mathrm{id}_\Omega \otimes$$  $.$ (4.2.16)

In homological degree 0 it is given by

$$\chi^{0} = \mathrm{id}_\Omega \otimes$$  $.$ (4.2.17)

In homological degree $+1$ it is given by

$$\chi^{+1} = \mathrm{id}_\Omega \otimes \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_1 & \\ 0 & 0 & x_2 \end{pmatrix}. \tag{4.2.18}$$

Lastly in homological degree 2, $\chi^2 = x_1$.

4.2.4 Wakimoto filtration. We now discuss some features of Elias's proof of Proposition 4.2.2. We observe that the Wakimoto braid $y_i = \omega\widetilde{h}_{X_i}$, where $X_i = S \setminus \{i-1\}$ and $\widetilde{h}_{X_i}$ is a negative-positive lift of the element $h_{X_i} \in W_{\mathrm{aff}}$. For any subset $X \subset X_i$, the subexpression of h_{X_i} corresponding to this subset is h_X.

Let's ignore the copies of Ω in $\mathcal{V}$ and the Wakimoto complexes W_i. Then we are dealing with the filtration of the complex $\mathcal{F}$ and Rouquier complexes for the negative-positive lifts of Coxeter elements h_{X_i}. We call this Rouquier complex Y_i. Note that $Y_i = \Omega^{-1}W_i$. Observe that Y_i is a cube complex with vertices B_X for $X \subset X_i$. Recall our sign convention for the differential on Rouquier complexes. By this convention, the differential of a summand in Y_i agrees with the corresponding differential of that summand of $\mathcal{F}$. Now all that remains to prove that $\mathcal{F}$ is filtered with subquotients Y_i is to prove the combinatorial statement the summands $\coprod P_k$ of $\mathcal{F}$ can be paritioned into cubes Y_i, and that there are no nonzero differentials from the cube Y_i to the cube Y_j when $j > i$.

Let P_k as in 4.2.11. We set

$$U_m = \bigsqcup_k \{X \in P_k \mid k = n - 1 - |X| - 2m\}. \tag{4.2.19}$$

Note that $\bigsqcup_m U_m$ gives a partition of P, where m ranges from 0 to $n-1$. As an example, note that U_0 consists of all summands B_X that sit in homological degree $n-1-|X|$.

Notation 4.2.17. By abuse of notation, we also let Y_k denote the set of subsets $X \subset S_{\text{aff}}$ (or more precisely pairs (X, k) with $X \in P_k$ but we shorten to just X, see Remark 4.2.13) for which B_X is a summand of Y_k. In each of these subsets, $k-1 \notin X$. We define it by setting $Y_k \cap U_m$ to consists of $X \in U_m$ for which $k-1 \notin X$ and $\{0, 1, ..., k-2\} \cap X$ has size $k-1-m$.

We argue these sets $Y_k \cap U_m$ are disjoint. Suppose $X \in Y_k \cap Y_l \cap U_m$ with $k < l$. Then $X \cap \{0, 1, ..., l-2\}$ must have size $l-1-m$, and $X \cap \{0, 1, ..., k-2\}$ has size $k-1-m$. Thus we must have $X \cap \{k-1, k, ..., l-2\}$ has size $l-k$. But then $k-1 \in X$, a contradiction.

We refer the reader to [27, Theorem 8.40] for the rest of the proof concerning the differentials.

We make the following remark.

Remark 4.2.18. Earlier (see Remark 4.2.13) we mentioned that given $X \subset S_{\text{aff}}$, one can specify which summand of $\mathcal{F}$ we are speaking of by indicating the homological degree k, i.e. $X \in P_k$. We note now that we may also specify the summand by indicating which layer l of the Wakimoto filtration it lies in. In other words, saying $X \in Y_l$ uniquely determines its homological degree k.

CHAPTER V

THE MAP χ

5.1 Main goal and theorem

Notation 5.1.1. We set $\chi : \mathcal{F} \to \mathcal{F}(2)$ to be a linear map, and $\chi^i_{X,Y} : B_X \to B_Y(2)$ where B_X and B_Y are both summands of $\mathcal{F}$ in homological degree i. These are the matrix entries of $\chi^i : \mathcal{F}^i \to \mathcal{F}^i(2)$.

We eventually want χ to have these three properties:

1. $\chi^i_{X,X}$ =right multiplication by x_l when $B_X \in Y_l$.

2. $\chi^i_{X,Y} = 0$ when $Y \neq X$, $B_X \in Y_l$, $B_Y \in Y_{l'}$, and $l' \geq l$.

3. χ is a pseudochain map.

We reiterate Remarks 4.2.13 and 4.2.18 from the previous chapter.

In this chapter, we will first state our theorem, providing χ satisfying $1-3$. When we prove the theorem, we will derive conditions equivalent to 3 given 1 and 2.

Notation 5.1.2.

- B_X - The Bott-Samelson bimodule associated to an ordering of a subset $X \subset S$. It is indecomposable.

- $X_k := X \cup \{k\}$

- $X_{k,j} := X \cup \{k, j\}$

- $X_{k,j,i} := X \cup \{k, j, i\}$

- $X_{\setminus i} := X \setminus \{i\}$

- $X_{\setminus i \setminus j} := X \setminus \{i, j\}$

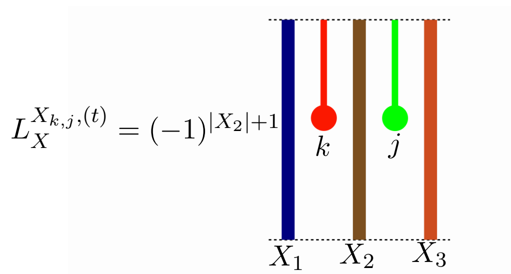
- $X_{\backslash i \backslash j \backslash k} := X \setminus \{i, j, k\}$
- $X_{k \backslash j} := (X \cup \{k\}) \setminus \{j\}$
- $X_{k,j \backslash i} := (X \cup \{k, j\}) \setminus \{i\}$
- $X_{k \backslash j \backslash i} := (X \cup \{k\}) \setminus \{i, j\}$
- $\forall_X^{X_k}$ The signed start dot map $B_X \to B_{X_k}(2)$. These are some of the matrix entries of the differential of $\mathcal{F}$.
- $A_X^{X_{\backslash k}}$ The signed final dot map $B_X \to B_{X_{\backslash k}}$. These are the rest of the matrix entries of the differential of $\mathcal{F}$.
- Y_l - the l-th layer of the Wakimoto filtration, combinatorially defined as in Notation 4.2.17.

5.1.1 Double-dot maps. The matrix entries $\chi^i_{X,Y} : B_X \to B_Y(2)$ are degree two maps between indecomposable Soergel bimodules. The summands of $\mathcal{F}^i$ are Bott-Samelson bimodules B_X for a Coxeter element of the parabolic subgroup of W_{aff} given by $X \subset S$. The only degree two maps between these, up to scalar, are *double-dot maps*. These are compositions of a start/end dot, (possibly) a (signed) rex move (see 4.2.7), and another start/end dot. We define these double-dot maps, and give a sign convention for them so that they will play well with the signs on the differential in $\mathcal{F}$.

Definition 5.1.3. (Signed double-dot maps). Let $X \subset S$ and B_X the indecomposable Soergel bimodule given by a cyclic ordering of X.

1. (Add two). Let $k, j \in S - X$ and assume $X_{k,j} \neq S$. Choose $t \notin X_{k,j}$, and use the cyclic order starting at t, ρ_t, for both X and $X_{k,j}$. Assume without

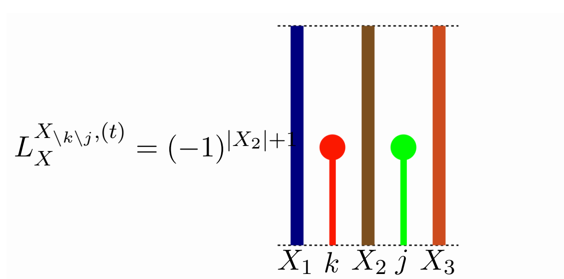
loss of generality that $k > j$ in that cyclic order. In this cyclic order we write $X_{k,j} = X_1 k X_2 j X_3$ and $X = X_1 X_2 X_3$. Such a cyclic order exists up to possibly reversing the roles of j and k. With these cyclic orders, the signed double-dot map $B_X \to B_{X_{k,j}}(2)$ is



.

Note it depends on t.

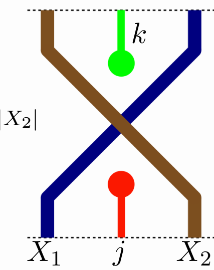
2. (Remove two). Let $k, j \in X$. Choose $t \notin X$, and use the cyclic order ρ_t. Assume without loss of generality that $k > j$ in this cyclic order. We write $X = X_1 k X_2 j X_3$ and $X_{\backslash k \backslash j} = X_1 X_2 X_3$ according to this cyclic order. . With these cyclic orders, the signed double dot map $B_X \to B_{X_{\backslash k \backslash j}}(2)$ is



.

3. (Add and Remove). Let $k \in S - X$ and $j \in X$. Let X have the cyclic order $X_1 j X_2$ which starts at k, and let $X_{k \backslash j}$ have the cyclic order $X_2 k X_1$, which

starts at j. The signed double dot map $B_X \to B_{X_{k\backslash j}}(2)$ is

$$L_X^{X_{k\backslash j}} = (-1)^{|X_1|+|X_2|+|X_1|\cdot|X_2|}$$ 

.

We note that $|X_1| + |X_2| + |X_1| \cdot |X_2|$ is even if and only if $|X_1|$ and $|X_2|$ are both even.

Lemma 5.1.4.

1. *Let $t, t' \notin X_{k,j}$, and $t < j < t' < k < t$ in the circle representing S_{aff}. Then the double-start-dot $L_X^{X_{k,j},(t)}$ is equal to $-L_X^{X_{k,j},(t')}$ up to canonical isomorphisms of the source and target via 4.2.7.*

2. *Similarly, let $t, t' \notin X$. Let $k, j \in X$ $t < j < t' < k < t$ in the circle representing S_{aff}. Then the double-end-dot $L_X^{X_{\backslash k \backslash j},(t)}$ is equal to $-L_X^{X_{\backslash k \backslash j},(t')}$ up to canonical isomorphisms of the source and target via 4.2.7.*

Proof. This is left the reader, but we do a very similar calculation below in Example 5.4.10. □

5.1.2 Criterion for χ and main theorem.

Recall that our simple reflections are parametrized by $S_{\text{aff}} = \mathbb{Z}/n\mathbb{Z}$. However our choice of Wakimoto filtration breaks the rotational symmetry of S_{aff}. In the rest of this chapter, we will parametrize the simple reflections by $\{0, ..., n-1\}$.

Definition 5.1.5. Let B_X be a summand of $\mathcal{F}^i$ which lies in the l-th layer of the Wakimoto filtration, Y_l. We define *the block of X over $l-1$*, $A(X, l)$, to be the largest contiguous subset $\{l, l+1, ..., l+m\} \subset X$ starting at l. We reiterate that the sequence $l, l+1, ..., l+m$ does not cross from $n-1$ to 0. If $l \notin X$, then $A(X, l) = \emptyset$. We set $\xi(X, l) = |A(X, l)| + 1$.

We now state the main theorem of this chapter.

Theorem 5.1.6. *There exists a unique pseudochain-map $\chi : \mathcal{F} \to \mathcal{F}(2)$, upper-triangular with respect to the Wakimoto filtration of $\mathcal{F}$, which lifts right multiplication by x_i on the i-th graded component of the associated graded. The matrix entries $\chi^i_{Z,X} : B_Z \to B_X$ of the signed double-dot maps are determined by the following criterion.*

Let B_X be a summand of $\mathcal{F}$ in Y_l, the l-th layer of the Wakimoto filtration

- *(C1). If both $j, k \geq l$ and at least one of j, k are in $A(X, l)$, then we have*

$$\chi^i_{X_{\backslash j \backslash k}, X} = L^{X,(l-1)}_{X_{\backslash j \backslash k}}.$$

Note we are using the cyclic order ρ_{l-1}. Without loss of generality we assume $j < k$. Then, by applying Lemmas 5.2.2 and 5.2.4 we have that $B_{X_{\backslash j \backslash k}}$ is in Y_{j+1}, so this matrix entry of χ^i goes from the $j+1$-th layer Y_{j+1} to the l-th layer Y_l.

In all other cases, we have

$$\chi^i_{X_{\backslash j \backslash k}, X} = 0.$$

- *(C2). If $j \in A(X, l)$ and $k \leq l-1$, then*

$$\chi^i_{X_{k \backslash j}, X} = L^X_{X_{k \backslash j}}.$$

By applying Lemmas 5.2.2 and 5.2.4 we have that $B_{X_{k \backslash j}}$ is in Y_{j+1}, so this matrix entry of χ^i goes from the $j+1$-th layer Y_{j+1} to the l-th layer Y_l.

In all other cases, we have

$$\chi^i_{X_{k\setminus j},X} = 0.$$

– *(C3). For all $j, k \in S \setminus X$, we have*

$$\chi^i_{X_{j,k},X} = 0.$$

Notation 5.1.7. Given the signed double-dot maps, from now on we abusively set $\chi^i_{X,Y}$ to be the *coefficient* of the signed double-dot map in the map $B_X \to B_Y(2)$, rather than the map itself. We also remove the superscripts (t) from the signed double-start-dot $L_X^{X_{k,j},(t)}$ and the signed double-end-dot $L_X^{X_{\setminus k \setminus j},(t)}$. We instead take t to be $l-1$ where l is the Wakimoto layer of the target and source respectively.

Remark 5.1.8. In the proof that χ is unique, we'll be studying a general linear map $\chi : \mathcal{F} \to \mathcal{F}(2)$ defined as above but with different coefficients. To make sense of the coefficient $\chi^i_{X,Y}$ for a double-end-dot, we will need to specify a cyclic order. We use $l-1$ where l is the Wakimoto layer of the source. However, we'll show this coefficient must be zero anyway, so the cyclic order, and hence the sign of the coefficient (see lemma 5.1.4), is irrelevant.

5.2 More on the Wakimoto filtration of $\mathcal{F}$.

We expand on properties of the Wakimoto filtration of $\mathcal{F}$.

Proposition 5.2.1. *The summand B_X in homological degree i is in Y_l if and only if:*

1. $l - 1 \notin X$.
2. $|X \cap \{0, ..., l-2\}| = l - 1 + \frac{i-n+1+|X|}{2}$.

Proof. Recall the set U_m (see 4.2.19) where $U_m = \bigsqcup_i \{X \in P^i \mid i = n-1-|X|-2m\}$. We defined the Wakimoto filtration layer Y_l such that $Y_l \cap U_m$ consists of $X \in U_m$ such that $|X \cap \{0, ..., l-2\}| = l-1-m$. If $X \in P^i$ then $m = -(\frac{i-n+1+|X|}{2})$. $\square$

Lemma 5.2.2. *(Movement in the Wakimoto filtration 1). Let B_X be a summand of $\mathcal{F}$ in Y_l in homological degree i. For $Z \subset S$ which differs from X by one element, the following accounts for the layer of the Wakimoto filtration that the summand B_Z in homological degree $i-1$ sits in.*

1. *Let $j \in X$, and $B_{X\setminus j} \in \mathcal{F}^{i-1}$. Then*

$$B_{X_{\setminus j}} \in \begin{cases} Y_l & \text{if } j < l-1 \\ Y_{j+1} & \text{if } j \in A(X,l) \\ Y_{l+\xi(X,l)} & \text{if } j > l+\xi(X,l)-2 \end{cases}$$

2. *Let $j \in S - X$, and $B_{X_j} \in \mathcal{F}^{i-1}$. Then*

$$B_{X_j} \in \begin{cases} Y_l & \text{if } j > l-1 \\ Y_{l+\xi(X,l)} & \text{if } j \leq l-1 \end{cases}$$

Proof. We use proposition 5.2.1 extensively in this proof.

Proof of 1: If $j < l-1$, then

$$\begin{aligned} |X_{\setminus j} \cap \{0, ..., l-2\}| &= |X \cap \{0, ..., l-2\}| - 1 && (5.2.1) \\ &= l-1+\frac{i-n+1+|X|}{2} - 1 && (5.2.2) \\ &= l-1+\frac{(i-1)-n+1+|X_{\setminus j}|}{2}. && (5.2.3) \end{aligned}$$

Hence $B_{X_{\backslash j}}$ is in Y_l. If $j \in A(X,l)$, then

$$\begin{aligned}
|X_{\backslash j} \cap \{0, ..., (j+1)-2\}| &= |X \cap \{0, ..., l-2\}| + (j-l) && (5.2.4)\\
&= l - 1 + \frac{i-n+1+|X|}{2} + (j-l) && (5.2.5)\\
&= j - 1 + \frac{i-n+1+|X|}{2} + 1 - 1 && (5.2.6)\\
&= (j+1) - 1 + \frac{i-n+1+|X|-2}{2} && (5.2.7)\\
&= (j+1) - 1 + \frac{(i-1)-n+1+|X_{\backslash j}|}{2}. && (5.2.8)
\end{aligned}$$

Hence $B_{X_{\backslash j}}$ is in Y_{j+1}. Finally, if $j > l + \xi(X,l) - 2$, then

$$\begin{aligned}
|X_{\backslash j} \cap \{0, ..., l+\xi(X,l)-2\}| &= |X \cap \{0, ..., l-2\}| + (\xi(X,l)-1) && (5.2.9)\\
&= (l+\xi(X,l)) - 1 + \frac{i-n+1+|X|}{2} - 1 && (5.2.10)\\
&= (l+\xi(X,l)) - 1 + \frac{(i-1)-n+1+|X_{\backslash j}|}{2}. && (5.2.11)
\end{aligned}$$

Hence $B_{X_{\backslash j}}$ is in $Y_{l+\xi(X,l)}$.

Proof of 2: If $j > l - 1$, then

$$\begin{aligned}
|X_j \cap \{0, ..., l-2\}| &= |X \cap \{0, ..., l-2\}| && (5.2.12)\\
&= l - 1 + \frac{i-n+1+|X|+(1-1)}{2} && (5.2.13)\\
&= l - 1 + \frac{(i-1)-n+1+|X_j|}{2}. && (5.2.14)
\end{aligned}$$

Hence B_{X_j} is in Y_l. If $j \leq l-1$, then

$$|X_j \cap \{0, ..., l+\xi(X,l)-2\}| = |X \cap \{0, ..., l-2\}| + 1 + (\xi(X,l)-1) \tag{5.2.15}$$
$$= (l+\xi(X,l)) - 1 + \frac{i-n+1+|X|}{2} \tag{5.2.16}$$
$$= (l+\xi(X,l)) - 1 + \frac{i-n+1+|X|+(1-1)}{2} \tag{5.2.17}$$
$$= (l+\xi(X,l)) - 1 + \frac{(i-1)-n+1+|X_j|}{2}. \tag{5.2.18}$$

Hence B_{X_j} is in $Y_{l+\xi(X,l)}$. $\square$

We can apply a similar analysis to determine the layer of B_Z in degree $i+1$, but to do se we need the block of X below $l-1$ instead.

Definition 5.2.3. Let $B_X \in \mathcal{F}^i$ and in Y_l. We define *the block of X under $l-1$*, $C(X,l)$, to be the largest contiguous subset $\{l-m, l-m+1, ..., l-2\} \subset X$ ending at $l-2$. We reiterate that this subset does not cross from $n-1$ to 0. We set $\gamma(X,l) = |C(X,l)| - 1$. Note that $l \geq 1$. When $l = 1$, then $C(X,l)$ must be the empty set.

Lemma 5.2.4. *(Movement in the Wakimoto filtration 2). Let B_X be a summand of $\mathcal{F}$ in Y_l in homological degree i. For $Z \subset S$ which differs from X by one element, the following accounts for the layer of the Wakimoto filtration that the summand B_Z in homological degree $i+1$ sits in.*

1. *Let* $j \in X$*, and* $B_{X_{\setminus j}} \in \mathcal{F}^{i+1}$*. Then*

$$B_{X\setminus j} \in \begin{cases} Y_l & \textit{if } j > l-2 \\ Y_{j+1} & \textit{if } j \in C(X,l) \\ Y_{l-2-\gamma(X,l)} & \textit{if } j < l-2-\gamma(X,l) \end{cases}$$

2. *Let* $j \in S - X$*, and* $B_{X_j} \in \mathcal{F}^{i+1}$*. Then*

$$B_{X_j} \in \begin{cases} Y_l & \textit{if } j < l-1 \\ Y_{l-2-\gamma(X,l)} & \textit{else} \end{cases}$$

Proof. Again, we use proposition 5.2.1 extensively in this proof.

Proof of 1: If $j > l-2$, then

$$|X_{\setminus j} \cap \{0, ..., l-2\}| = |X \cap \{0, ..., l-2\}| \tag{5.2.19}$$
$$= l - 1 + \frac{i - n + 1 + |X|}{2} \tag{5.2.20}$$
$$= l - 1 + \frac{i - n + 1 + |X| + (1-1)}{2} \tag{5.2.21}$$
$$= l - 1 + \frac{(i+1) - n + 1 + |X_{\setminus j}|}{2}. \tag{5.2.22}$$

Hence $B_{X_{\setminus j}}$ is in Y_l. If $j \in C(X,l)$, then

$$|X_{\setminus j} \cap \{0,...,(j+1)-2\} = |X \cap \{0,...,l-2\}| - (l-1-j) \tag{5.2.23}$$

$$= l-1+\frac{i-n+1+|X|}{2} - (l-1-j) \tag{5.2.24}$$

$$= (j+1)-1+\frac{i-n+1+|X|+(1-1)}{2} \tag{5.2.25}$$

$$= (j+1)-1+\frac{(i+1)-n+1+|X_{\setminus j}|}{2}. \tag{5.2.26}$$

Hence $B_{X_{\setminus j}}$ is in Y_{j+1}. If $j < l-2-\gamma(X,l)$, then

$$|X_{\setminus j} \cap \{0,...,(l-2-\gamma(X,l))-2\} = |X \cap \{0,...,l-2\}| - 1 - (\gamma(X,l)+1) \tag{5.2.27}$$

$$= l-1+\frac{i-n+1+|X|}{2} - 1 - (\gamma(X,l)+1) \tag{5.2.28}$$

$$= (l-2-\gamma(X,l))-1+\frac{i-n+1+|X|+(1-1)}{2} \tag{5.2.29}$$

$$= (l-2-\gamma(X,l))-1+\frac{(i+1)-n+1+|X_{\setminus j}|}{2}. \tag{5.2.30}$$

Hence $B_{X_{\setminus j}}$ is in $Y_{l-2-\gamma(X,l)}$.

Proof of 2: If $j < l-1$, then

$$|X_j \cap \{0, ..., l-2\}| = |X \cap \{0, ..., l-2\}| + 1 \tag{5.2.31}$$
$$= l - 1 + \frac{i-n+1+|X|}{2} + 1 \tag{5.2.32}$$
$$= l - 1 + \frac{i-n+1+|X|+2}{2} \tag{5.2.33}$$
$$= l - 1 + \frac{(i+1)-n+1+|X_j|}{2}. \tag{5.2.34}$$

Hence B_{X_j} is in Y_l. If $j \geq l-1$, then

$$|X_j \cap \{0, ..., (l-2-\gamma(X,l))-2\}| = |X \cap \{0, ..., l-2\}| - (\gamma(X,l)+1) \tag{5.2.35}$$
$$= l - 1 + \frac{i-n+1+|X|}{2} - (\gamma(X,l)+1) + (1-1) \tag{5.2.36}$$
$$= (l-2-\gamma(X,l)) - 1 + \frac{i-n+1+|X|}{2} + 1 \tag{5.2.37}$$
$$= (l-2-\gamma(X,l)) - 1 + \frac{i-n+1+|X|+2}{2} \tag{5.2.38}$$
$$= (l-2-\gamma(X,l)) - 1 + \frac{(i+1)-n+1+|X_j|}{2}. \tag{5.2.39}$$

Hence B_{X_j} is in $Y_{l-2-\gamma(X,l)}$.

$\square$

5.3 Commutative squares.

In order for χ to be a pseudochain map, χ must commute with the differential of $\mathcal{F}$ modulo δ. Thinking in terms of the matrix entries of χ, this reduces to the observation that the following types of squares must commute modulo δ.

$$\begin{tikzcd}
B_{X_{\setminus j}} \oplus \left(\bigoplus_{k\in X_{\setminus j}} B_{X_{\setminus k}}\right) \oplus \left(\bigoplus_{k\in S-X} B_{X_k}\right) \arrow[r, "d^{i-1}"] & B_X \\
B_{X_{\setminus j}} \arrow[u, "\chi^{i-1}"] \arrow[r, "d^{i-1}"] & B_X \oplus \left(\bigoplus_{k\in X_{\setminus j}} B_{X_{\setminus j\setminus k}}\right) \oplus \left(\bigoplus_{k\in S-X} B_{X_{k\setminus j}}\right) \arrow[u, "\chi^i"']
\end{tikzcd} \tag{5.3.1}$$

$$\begin{tikzcd}
B_{X_j} \oplus \left(\bigoplus_{k\in X} B_{X_{\setminus k}}\right) \oplus \left(\bigoplus_{k\in S-X_j} B_{X_k}\right) \arrow[r, "d^{i-1}"] & B_X \\
B_{X_j} \arrow[u, "\chi^{i-1}"] \arrow[r, "d^{i-1}"] & B_X \oplus \left(\bigoplus_{k\in X} B_{X_{j\setminus k}}\right) \oplus \left(\bigoplus_{k\in S-X_j} B_{X_{j,k}}\right) \arrow[u, "\chi^i"']
\end{tikzcd} \tag{5.3.2}$$

$$\begin{tikzcd}
B_{X_{\setminus i}} \oplus B_{X_{\setminus j}} \oplus B_{X_{\setminus k}} \arrow[r, "d^{i-1}"] & B_X \\
B_{X_{\setminus i\setminus j\setminus k}} \arrow[u, "\chi^{i-1}"] \arrow[r, "d^{i-1}"] & B_{X_{\setminus j\setminus k}} \oplus B_{X_{\setminus i\setminus k}} \oplus B_{X_{\setminus i\setminus j}} \arrow[u, "\chi^i"']
\end{tikzcd} \tag{5.3.3}$$

$$\begin{tikzcd}
B_{X_{\setminus i}} \oplus B_{X_{\setminus j}} \oplus B_{X_k} \arrow[r, "d^{i-1}"] & B_X \\
B_{X_{k\setminus i\setminus j}} \arrow[u, "\chi^{i-1}"] \arrow[r, "d^{i-1}"] & B_{X_{k\setminus i}} \oplus B_{X_{k\setminus j}} \oplus B_{X_{\setminus i\setminus j}} \arrow[u, "\chi^i"']
\end{tikzcd} \tag{5.3.4}$$

$$\begin{tikzcd}
B_{X_i} \oplus B_{X_j} \oplus B_{X_{\setminus k}} \arrow[r, "d^{i-1}"] & B_X \\
B_{X_{i,j\setminus k}} \arrow[u, "\chi^{i-1}"] \arrow[r, "d^{i-1}"] & B_{X_{j\setminus k}} \oplus B_{X_{i\setminus k}} \oplus B_{X_{i,j}} \arrow[u, "\chi^i"']
\end{tikzcd} \tag{5.3.5}$$

$$\begin{tikzcd}
B_{X_i} \oplus B_{X_j} \oplus B_{X_k} \arrow[rr, "d^{i-1}"] & & B_X \\
B_{X_{i,j,k}} \arrow[u, "\chi^{i-1}"] \arrow[r, "d^{i-1}"] & B_{X_{j,k}} \oplus B_{X_{i,k}} \oplus B_{X_{i,j}} \arrow[ur, "\chi^i"'] &
\end{tikzcd} \tag{5.3.6}$$

The commutativity of these six squares give several equations which must hold modulo δ.

Now, let $B_X \in Y_l$ and set $\xi = \xi(X,l)$. We also set $A = A(X,l)$.

The square (5.3.1) gives the following equations in $\mathrm{Hom}(B_{X_{\setminus j}}, B_X(3))$. If $j \in \{0,1,...,l-2\}$, then $B_{X_{\setminus j}} \in Y_l$ by Lemma 5.2.2, so we get the equation

$$\begin{aligned}
&\forall^X_{X_{\setminus j}} x_l + \sum_{k\in X_{\setminus j}} \chi^{i-1}_{X_{\setminus j}, X_{\setminus k}} \forall^X_{X_{\setminus k}} L^{X_{\setminus k}}_{X_{\setminus j}} + \sum_{k\in S-X} \chi^{i-1}_{X_{\setminus j}, X_k} A^X_{X_k} L^{X_k}_{X_{\setminus j}} \\
=&\forall^X_{X_{\setminus j}} x_l + \sum_{k\in X_{\setminus j}} \chi^{i}_{X_{\setminus j\setminus k}, X} L^X_{X_{\setminus j\setminus k}} A^{X\setminus j\setminus k}_{X_{\setminus j}} + \sum_{k\in S-X} \chi^{i}_{X_{k\setminus j}, X} L^X_{X_{k\setminus j}} \forall^{X_{k\setminus j}}_{X_{\setminus j}}.
\end{aligned} \tag{5.3.7a}$$

If $j \in A$, then $B_{X_{\setminus j}} \in Y_{j+1}$ so we get the equation

$$\begin{aligned}
&\forall^X_{X_{\setminus j}} x_{j+1} + \sum_{k\in X_{\setminus j}} \chi^{i-1}_{X_{\setminus j}, X_{\setminus k}} \forall^X_{X_{\setminus k}} L^{X_{\setminus k}}_{X_{\setminus j}} + \sum_{k\in S-X} \chi^{i-1}_{X_{\setminus j}, X_k} A^X_{X_k} L^{X_k}_{X_{\setminus j}} \\
=&\forall^X_{X_{\setminus j}} x_l + \sum_{k\in X_{\setminus j}} \chi^{i}_{X_{\setminus j\setminus k}, X} L^X_{X_{\setminus j\setminus k}} A^{X\setminus j\setminus k}_{X_{\setminus j}} + \sum_{k\in S-X} \chi^{i}_{X_{k\setminus j}, X} L^X_{X_{k\setminus j}} \forall^{X_{k\setminus j}}_{X_{\setminus j}}.
\end{aligned} \tag{5.3.7b}$$

If $j > l + \xi - 2$, then $B_{X_{\setminus j}} \in Y_{l+\xi}$ so we get the equation

$$\begin{aligned}
&\forall^X_{X_{\setminus j}} x_{l+\xi} + \sum_{k\in X_{\setminus j}} \chi^{i-1}_{X_{\setminus j}, X_{\setminus k}} \forall^X_{X_{\setminus k}} L^{X_{\setminus k}}_{X_{\setminus j}} + \sum_{k\in S-X} \chi^{i-1}_{X_{\setminus j}, X_k} A^X_{X_k} L^{X_k}_{X_{\setminus j}} \\
=&\forall^X_{X_{\setminus j}} x_l + \sum_{k\in X_{\setminus j}} \chi^{i}_{X_{\setminus j\setminus k}, X} L^X_{X_{\setminus j\setminus k}} A^{X\setminus j\setminus k}_{X_{\setminus j}} + \sum_{k\in S-X} \chi^{i}_{X_{k\setminus j}, X} L^X_{X_{k\setminus j}} \forall^{X_{k\setminus j}}_{X_{\setminus j}}.
\end{aligned} \tag{5.3.7c}$$

Note that these three equations only differ by the first term, which involves a diagonal entry of χ^{i-1}, and thus depends on where $B_{X_{\backslash j}}$ sits in the Wakimoto filtration.

The square (5.3.2) gives the following equations in $\mathrm{Hom}(B_{X_j}, B_X(3))$. If $j \notin \{0, 1, ..., l-1\}$, then $B_{X_j} \in Y_l$, so we get the equation

$$\begin{aligned}
&A^X_{X_j} x_l + \sum_{k \in X} \chi^{i-1}_{X_j, X_{\backslash k}} \forall^X_{X_{\backslash k}} L^{X_{\backslash k}}_{X_j} + \sum_{k \in S - X_j} \chi^{i-1} A^X_{X_k} L^{X_k}_{X_j} \\
=&A^X_{X_j} x_l + \sum_{k \in X} \chi^i_{X_{j \backslash k}, X} L^X_{X_{j \backslash k}} A^{X_{j \backslash k}}_{X_j} + \sum_{k \in S - X_j} \chi^i_{X_{j,k}, X} L^X_{X_{j,k}} \forall^{X_{j,k}}_{X_j}.
\end{aligned} \tag{5.3.8a}$$

If $j \in \{0, 1, ..., l-1\}$, then $B_{X_j} \in Y_{l+\xi}$, so we get the equation

$$\begin{aligned}
&A^X_{X_j} x_{l+\xi} + \sum_{k \in X} \chi^{i-1}_{X_j, X_{\backslash k}} \forall^X_{X_{\backslash k}} L^{X_{\backslash k}}_{X_j} + \sum_{k \in S - X_j} \chi^{i-1} A^X_{X_k} L^{X_k}_{X_j} \\
=&A^X_{X_j} x_l + \sum_{k \in X} \chi^i_{X_{j \backslash k}, X} L^X_{X_{j \backslash k}} A^{X_{j \backslash k}}_{X_j} + \sum_{k \in S - X_j} \chi^i_{X_{j,k}, X} L^X_{X_{j,k}} \forall^{X_{j,k}}_{X_j}.
\end{aligned} \tag{5.3.8b}$$

Again, these equations only differ by the first term.

The next four squares only involve off-diagonal entries of χ, so we do not have to break them into cases like we did with the squares above.

The square (5.3.3) gives the equation

$$\begin{aligned}
&\chi^{i-1}_{X_{\backslash i \backslash j \backslash k}, X_{\backslash i}} \forall^X_{X_{\backslash i}} L^{X_{\backslash i}}_{X_{\backslash i \backslash j \backslash k}} + \chi^{i-1}_{X_{\backslash i \backslash j \backslash k}, X_{\backslash j}} \forall^X_{X_{\backslash j}} L^{X_{\backslash j}}_{X_{\backslash i \backslash j \backslash k}} + \chi^{i-1}_{X_{\backslash i \backslash j \backslash k}, X_{\backslash k}} \forall^X_{X_{\backslash k}} L^{X_{\backslash k}}_{X_{\backslash i \backslash j \backslash k}} \\
=&\chi^i_{X_{\backslash j \backslash k}, X} L^X_{X_{\backslash j \backslash k}} \forall^{X_{\backslash j \backslash k}}_{X_{\backslash i \backslash j \backslash k}} + \chi^i_{X_{\backslash i \backslash k}, X} L^X_{X_{\backslash i \backslash k}} \forall^{X_{\backslash i \backslash k}}_{X_{\backslash i \backslash j \backslash k}} + \chi^i_{X_{\backslash i \backslash j}, X} L^X_{X_{\backslash i \backslash j}} \forall^{X_{\backslash i \backslash j}}_{X_{\backslash i \backslash j \backslash k}}.
\end{aligned} \tag{5.3.9}$$

The square (5.3.4) gives the equation

$$\begin{aligned}&\chi^{i-1}_{X_{k\backslash i\backslash j},X_{\backslash i}}\forall^{X}_{X_{\backslash i}}L^{X_{\backslash i}}_{X_{k\backslash i\backslash j}}+\chi^{i-1}_{X_{k\backslash i\backslash j},X_{\backslash j}}\forall^{X}_{X_{\backslash j}}L^{X_{\backslash j}}_{X_{k\backslash i\backslash j}}+\chi^{i-1}_{X_{k\backslash i\backslash j},X_{k}}A^{X}_{X_{k}}L^{X_{k}}_{X_{k\backslash i\backslash j}}\\=&\chi^{i}_{X_{k\backslash i},X}L^{X}_{X_{k\backslash i}}\forall^{X_{k\backslash i}}_{X_{k\backslash i\backslash j}}+\chi^{i}_{X_{k\backslash j},X}L^{X}_{X_{k\backslash j}}\forall^{X_{k\backslash j}}_{X_{k\backslash i\backslash j}}+\chi^{i}_{X_{\backslash i\backslash j},X}L^{X}_{X_{\backslash i\backslash j}}A^{X_{\backslash i\backslash j}}_{X_{k\backslash i\backslash j}}.\end{aligned}\tag{5.3.10}$$

The square (5.3.5) gives the equation

$$\begin{aligned}&\chi^{i-1}_{X_{i,j\backslash k},X_{i}}A^{X}_{X_{i}}L^{X_{i}}_{X_{i,j\backslash k}}+\chi^{i-1}_{X_{i,j\backslash k},X_{j}}A^{X}_{X_{j}}L^{X_{j}}_{X_{i,j\backslash k}}+\chi^{i-1}_{X_{i,j\backslash k},X_{\backslash k}}\forall^{X}_{X_{\backslash k}}L^{X_{\backslash k}}_{X_{i,j\backslash k}}\\=&\chi^{i}_{X_{j\backslash k},X}L^{X}_{X_{j\backslash k}}A^{X_{j\backslash k}}_{X_{i,j\backslash k}}+\chi^{i}_{X_{i\backslash k},X}L^{X}_{X_{i\backslash k}}A^{X_{i\backslash k}}_{X_{i,j\backslash k}}+\chi^{i}_{X_{i,j},X}L^{X}_{X_{i,j}}\forall^{X_{i,j}}_{X_{i,j\backslash k}}.\end{aligned}\tag{5.3.11}$$

The square (5.3.6) gives the equation

$$\begin{aligned}&\chi^{i-1}_{X_{i,j,k},X_{i}}A^{X}_{X_{i}}L^{X_{i}}_{X_{i,j,k}}+\chi^{i-1}_{X_{i,j,k},X_{j}}A^{X}_{X_{j}}L^{X_{j}}_{X_{i,j,k}}+\chi^{i-1}_{X_{i,j,k},X_{k}}A^{X}_{X_{k}}L^{X_{k}}_{X_{i,j,k}}\\=&\chi^{i}_{X_{j,k},X}L^{X}_{X_{j,k}}A^{X_{j,k}}_{X_{i,j,k}}+\chi^{i}_{X_{i,k},X}L^{X}_{X_{i,k}}A^{X_{i,k}}_{X_{i,j,k}}+\chi^{i}_{X_{i,j},X}L^{X}_{X_{i,j}}A^{X_{i,j}}_{X_{i,j,k}}.\end{aligned}\tag{5.3.12}$$

5.4 Solving for the coefficients of χ.

In order to solve the equations of the previous section to get recursive formulas for the coefficients $\chi^{i}_{X,Y}$ we will need the following lemma.

Lemma 5.4.1. *(Pushing Polynomials). Let $l-1\notin X$, set $\xi=\xi(X,l)$, and set $A=A(X,l)$. For any cyclic order on X, let x^{L}_{i} and x^{R}_{i} denote left and right multiplication by x_i on B_X respectively. The following equations hold modulo δ.*

$$x^{R}_{l+\xi}-x^{R}_{l}=\left(\sum_{k\in S-X,k\neq l+\xi-1}A^{X}_{X_k}\forall^{X_k}_{X}+\sum_{k\in X,k\notin A}\forall^{X}_{X_{\backslash k}}A^{X_{\backslash k}}_{X}\right).\tag{5.4.1a}$$

$$x^{R}_{l}-x^{R}_{l+\xi}=\left(A^{X}_{X^{X}_{l+\xi-1}}\forall^{X_{l+\xi-1}}_{X}+\sum_{k=l}^{l+\xi-2}\forall^{X}_{X_{\backslash k}}A^{X_{\backslash k}}_{X}\right).\tag{5.4.1b}$$

Proof. Since $l-1 \notin X$, we give X the cyclic order starting at $l-1$, and consider B_X to be the corresponding Bott-Samelson bimodule. By the proof of [27, Theorem 14.7], we have that, modulo δ,

$$x^L_{l+\xi-1} - x^R_{l+\xi} = -\left(\sum_{k \in S-X, k \neq l+\xi-1} A^X_{X_k} \forall^{X_k}_X + \sum_{k \in X, k \neq l+\xi-1} \forall^X_{X_{\setminus k}} A^{X_{\setminus k}}_X \right). \tag{5.4.2}$$

Using the polynomial forcing relation (3.2.4) repeatedly to move the left multiplication by $x_{l+\xi-1}$ competely to the right by introducing broken lines, we get the equation

$$x^L_{l+\xi-1} = -\left(\sum_{k=l}^{l+\xi-2} \forall^X_{X_{\setminus k}} A^{X_{\setminus k}}_X \right) + x^R_l. \tag{5.4.3}$$

Substituting this into equation 5.4.2 gives the equation:

$$x^R_l - x^R_{l+\xi} = -\left(\sum_{k \in S-X, k \neq l+\xi-1} A^X_{X_k} \forall^{X_k}_X + \sum_{k \in X, k \notin A} \forall^X_{X_{\setminus k}} A^{X_{\setminus k}}_X \right). \tag{5.4.4}$$

Negating the whole equation gives us equation 5.4.1a. Note that,

$$\sum_{k \in S \setminus X} A^X_{X_k} \forall^{X_k}_X + \sum_{k \in X} \forall^X_{X_{\setminus k}} A^{X_{\setminus k}}_X = 0 \bmod \delta \tag{5.4.5}$$

This follows from Elias's proof (cf. [27, Proposition 8.25]) that $\mathcal{F}$ is a pseudocomplex. Subtracting this from 5.4.1a gives the equation:

$$x^R_{l+\xi} - x^R_l = -\left(A^X_{X^X_{l+\xi-1}} \forall^{X_{l+\xi-1}}_X + \sum_{k=l}^{l+\xi-2} \forall^X_{X_{\setminus k}} A^{X_{\setminus k}}_X \right). \tag{5.4.6}$$

Negation of the whole equation gives 5.4.1b. $\square$

Keeping track of sign differences will be a hassle in the following discussion. For instance, in (5.3.7a), we can match the terms $\forall^{X}_{X_{\setminus k}} L^{X_{\setminus k}}_{X_{\setminus j}}$ on the left-hand side to the terms $L^{X}_{X_{\setminus j \setminus k}} A^{X_{\setminus j \setminus k}}_{X_{\setminus j}}$ on the right-hand side. These two terms differ by a sign depending on how j and k are situated in the cyclic ordering of X. We introduce the following notation to deal with such sign differences.

Definition 5.4.2. (Sign Rule). Fix a cyclic order ρ on S. Then

$$\operatorname{sgn}_{\rho}(j,k) := \begin{cases} 1 & \text{if } j > k \text{ in } \rho, \\ -1 & \text{if } j < k \text{ in } \rho. \end{cases} \tag{5.4.7}$$

Notation 5.4.3. Given a subset $X \subset S$, we abusively set $\rho(X)$ to indicate *any* cyclic ordering ρ_t of S starting at some $t \in S - X$. We only use this convention for the purpose of using the notation of Definition 5.4.2 when the choice of cyclic order is irrelevant.

Again,we let $B_X \in Y_l$ in homological degree i, and set ξ equal to $\xi(X,l)$.

Lemma 5.4.4. *Equation* (5.3.7a) *holds if and only if the following equations,* (5.4.8) *and* (5.4.9)*, hold.*

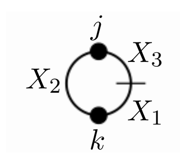
Since $j < l-1$ we have $B_{X_{\setminus j}} \in Y_l$.

$$\chi^{i}_{X_{k \setminus j}, X} = \chi^{i-1}_{X_{\setminus j}, X_k} \operatorname{sgn}_{\rho(X_k)}(j,k) \quad \text{for } k \in S - X. \tag{5.4.8}$$

$$\chi^{i}_{X_{\setminus k \setminus j}, X} = \chi^{i-1}_{X_{\setminus j}, X_{\setminus k}} \operatorname{sgn}_{\rho(X)}(j,k) \quad \text{for } k \in X_{\setminus j}. \tag{5.4.9}$$

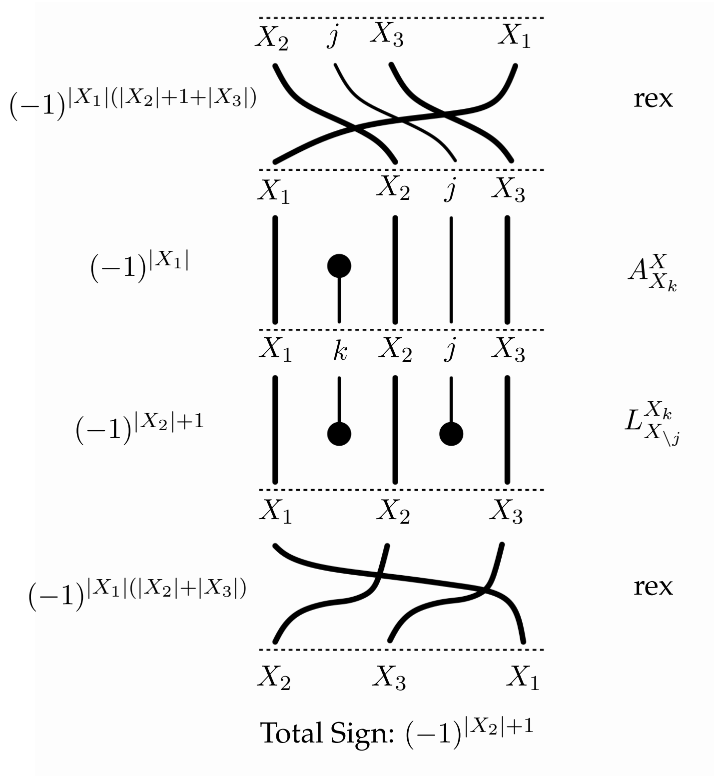
Proof. First we argue that each term on the left-hand side of (5.3.7a) matches a term on the right-hand side up to sign, and we verify that equations (5.4.8) and (5.4.9) are equivalent to matching of signs. Then we argue that the terms were linearly independent, so that the equality in equation (5.3.7a) holds if and only if the signs match.

We work out the sign difference to get equation 5.4.8. This equation comes from matching the terms $\chi^{i}_{X_{k\backslash j},X} L^{X}_{X_{k\backslash j}} \forall^{X_{k\backslash j}}_{X_{\backslash j}}$ with the terms $\chi^{i-1}_{X_{\backslash j},X_k} A^{X}_{X_k} L^{X_k}_{X_{\backslash j}}$ in equation 5.3.7a. Note that $L^{X}_{X_{k\backslash j}} \forall^{X_{k\backslash j}}_{X_{\backslash j}}$ and $A^{X}_{X_k} L^{X_k}_{X_{\backslash j}}$ are equal up to a sign. To determine this sign, we must determine the total signs on $L^{X}_{X_{k\backslash j}} \forall^{X_{k\backslash j}}_{X_{\backslash j}}$ and $A^{X}_{X_k} L^{X_k}_{X_{\backslash j}}$ coming from signed (double) dot maps, and rex moves, then compare these signs. Let X_k have the cyclic order $X_1 k X_2 j X_3$. Give X the cyclic order $X_2 j X_3 X_1$ induced by k, give $X_{k\backslash j}$ the cyclic order $X_3 X_1 k X_2$ induced by j, and give $X_{\backslash j}$ the cyclic order $X_2 X_3 X_1$ inherited from X. The following picture represents where the sets X_1, X_2, X_3, and j, k are situated in S.



(5.4.10)

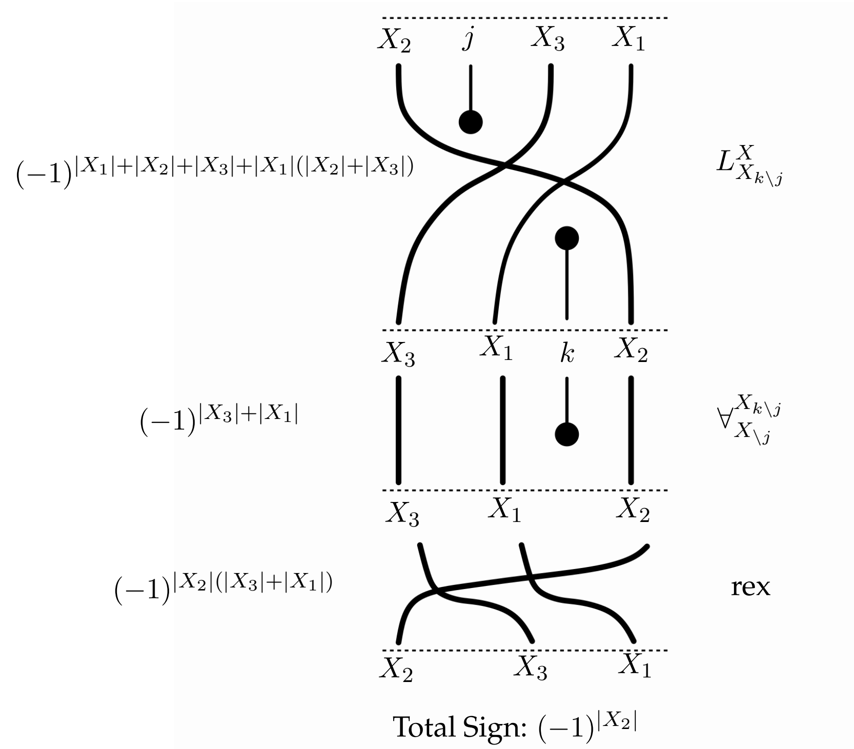
The tick mark indicates where the cyclic order on X_k begins. Below we decompose $A^{X}_{X_k} L^{X_k}_{X_{\backslash j}}$ into dot maps and signed rex moves, and calculate the total sign



(5.4.11)

The labels on the right indicate what the component is, while on the left the sign on this component is indicated.

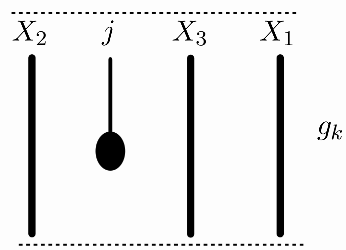
Now we decompose $L^X_{X_{k\setminus j}} \forall^{X_{k\setminus j}}_{X_{\setminus j}}$ into signed double-dot maps and signed rex moves, and calculate the total sign.



(5.4.12)

Note that the diagrams (5.4.11) and (5.4.12) are equal, ignoring coefficients, because X_3 is distant from $X_1 k X_2$ and can be slid over the rest of the diagram. Overall, the sign difference between these two diagrams is $\mathrm{sgn}_{\rho(X_k)}(j,k)$ as in (5.4.8).

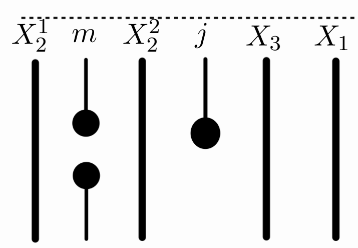
Now we argue that the diagrams in (5.4.12) are linearly independent as k ranges over $S \setminus X$. Using Lemma 3.2.4, we can force the k-colored barbell to the right, past all the strands in X_2. The result will be a linear combination of terms, some which break strands in X_2 and one *pure sliding term* where no strands are broken. The pure sliding term is equal to



(5.4.13)

for some polynomial g_k. The polynomials g_k are equal to $\alpha_k + \sum_{k<m<j} c_m \alpha_m$ for some coefficient c_m. The polynomials g_k, hence the pure sliding terms, are linearly independent as k varies.

Meanwhile, the term with a broken strand looks like



(5.4.14)

where $X_2 = X_2^1 m X_2^2$. All terms with a broken strand lie in an ideal of the category of morphisms factoring through objects of shorter length. Modulo this ideal, the original diagrams of (5.4.11) are equal to their pure sliding terms, hence are linearly independent as k varies.

One must also argue independence with the other kinds of diagrams appearing in equation (5.3.7a), indexed by $k \in X \setminus j$. These diagrams look like diagram (5.4.14) except with broken strand labeled by k instead of m, and they lie in the ideal of shorter terms, thus they are linearly independent from the terms indexed by $k \in S \setminus X$. If equation (5.3.7a) holds, then all signs must match for terms indexed by $k \in X \setminus j$, and these terms cancel from both sides of (5.3.7a). It remains to show that the terms indexed by $k \in S \setminus X$ are linearly independent from each other. This is straightforward using the light leaves basis of [29], as these diagrams are distinct basis elements. $\square$

Remark 5.4.5. The proofs of Lemmas 5.4.6 through 5.4.14 below will follow a very similar argument. One matches terms in sums on both sides of the equations, and argues their coefficients match if and only if certain formulas hold. Then one argues linear independence of the terms in each sum. The proof of linear independence in each case is similar to the proof given above, and we omit it. We focus on the problem of matching the signs.

Lemma 5.4.6. *The equation* (5.3.7b) *holds if and only if the sign equations* (5.4.15)-(5.4.17) *below hold.*

We set $\xi = \xi(X, l)$ and $A = A(X, l)$. This is the first nontrivial case. As in 5.4.4, we can match terms on the left-hand side to terms on the right-hand side, but we need some additional terms terms on the right to cancel out with $\forall^X_{X_{\setminus j}} x_l$ to give $\forall^X_{X_{\setminus j}} x_{j+1}$. Here we apply equation 5.4.1a, post-composed with $\forall^X_{X_{\setminus j}}$, applied to $B_{X_{\setminus j}}$ where $l + \xi - 2 = j - 1$. We note that we must use 5.4.1a as opposed to 5.4.6 because there aren't any terms involving a j-colored barbell. This distinction will become clearer in the next case. We then have

$$\forall^X_{X_{\setminus j}}(x_{j+1} - x_l) = \sum_{k \neq j, k \in S - X_{\setminus j}} \forall^X_{X_{\setminus j}} A^{X_{\setminus j}}_{X_{k \setminus j}} \forall^{X_{k \setminus j}}_{X_{\setminus j}} + \sum_{k \neq l, \ldots, j-1, k \in X_{\setminus j}} \forall^X_{X_{\setminus j}} \forall^{X_{\setminus j}}_{X_{\setminus k \setminus j}} A^{X_{\setminus k \setminus j}}_{X_{\setminus j}}$$

Accounting for sign differences, we get the following equations.

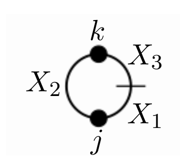
Since $j \in A$ we have $B_{X_{\backslash j}} \in Y_{j+1}$.

$$\begin{aligned} \chi^i_{X_{k \backslash j}, X} &= \chi^{i-1}_{X_{\backslash j}, X_k} \mathrm{sgn}_{\rho(X_k)}(j,k) + 1 \\ &\text{for } k \in S - X. \end{aligned} \tag{5.4.15}$$

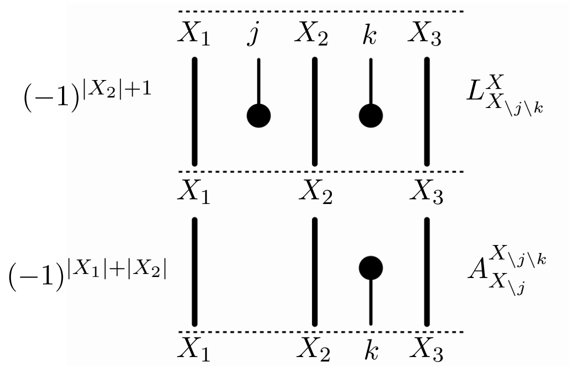
$$\begin{aligned} \chi^i_{X_{\backslash k \backslash j}, X} &= \left(\chi^{i-1}_{X_{\backslash j}, X_{\backslash k}} - 1 \right) \mathrm{sgn}_{\rho(X)}(j,k) \\ &\text{for } k \in X_{\backslash j}, \qquad k \neq l, ..., j-1. \end{aligned} \tag{5.4.16}$$

$$\begin{aligned} \chi^i_{X_{\backslash k \backslash j}, X} &= \chi^{i-1}_{X_{\backslash j}, X_{\backslash k}} \mathrm{sgn}_{\rho(X)}(j,k) \\ &\text{for } k \in X_{\backslash j}, \qquad k = l, ..., j-1. \end{aligned} \tag{5.4.17}$$

Example 5.4.7. We explain the signs in equation 5.4.16. In this equation, as in equation 5.4.9, we match the terms $\chi^i_{X_{\backslash j \backslash k}, X} L^X_{X_{\backslash j \backslash k}} A^{X \backslash j \backslash k}_{X_{\backslash j}}$ with the terms $\chi^{i-1}_{X_{\backslash j}, X_{\backslash k}} \forall^X_{X_{\backslash k}} L^{X_{\backslash k}}_{X_{\backslash j}}$. So $\chi^i_{X_{\backslash j \backslash k}, X} = \chi^{i-1}_{X_{\backslash j}, X_{\backslash k}} \mathrm{sgn}_{\rho(X)}(j,k) \pm 1$, where the ± 1 comes from the sign difference between $L^X_{X_{\backslash j \backslash k}} A^{X \backslash j \backslash k}_{X_{\backslash j}}$ and $\forall^X_{X_{\backslash j}} \forall^{X_{\backslash j}}_{X_{\backslash k \backslash j}} A^{X_{\backslash k \backslash j}}_{X_{\backslash j}}$. We now analyze this sign difference. Let X have the cyclic order $X = X_1 j X_2 k X_3$, let $X_{\backslash j}$ and $X_{\backslash k \backslash j}$ have the cyclic orders inherited from X. The following picture depicts where j, k and the subsets $X_1, X_2,$ and X_3 are situated in S. The tick mark is where the cyclic order on X starts.

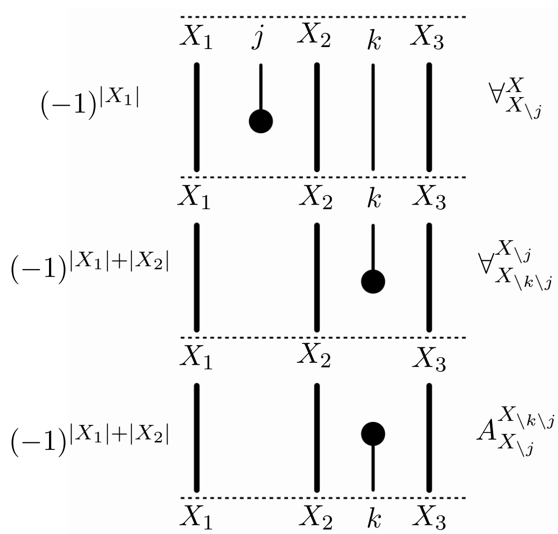


Below we analyze the total sign on $L^{X}_{X_{\backslash j \backslash k}} A^{X \backslash j \backslash k}_{X_{\backslash j}}$.



Total Sign: $(-1)^{|X_1|+1}$

Now we analyze the total sign on $\forall^{X}_{X_{\backslash j}} \forall^{X_{\backslash j}}_{X_{\backslash k \backslash j}} A^{X_{\backslash k \backslash j}}_{X_{\backslash j}}$



Total Sign: $(-1)^{|X_1|}$

Observe the sign difference is -1. A similar analysis would observe no sign difference when $k > j$ in the cyclic order on X. Thus $\chi^{i}_{X_{\backslash j \backslash k}, X} = \chi^{i-1}_{X_{\backslash j}, X_{\backslash k}} \mathrm{sgn}_{\rho(X)}(j,k) + \mathrm{sgn}_{\rho(X)}(k,j)$, which gives equation $5.4.16$.

Lemma 5.4.8. *Equation* (5.3.7c) *holds if and only if the sign equations* (5.4.18)-(5.4.21) *below hold.*

This case is similar to 5.4.6 - we now need some additional terms terms on the right to cancel out with $\forall^X_{X_{\setminus j}} x_l$ to give $\forall^X_{X_{\setminus j}} x_{l+\xi}$. The difference is that here we apply equation 5.4.6, precomposed with $\forall^X_{X_{\setminus j}}$. This is because we do see an $(l+\xi-1)$-colored barbell on the right-hand side of equation 5.3.7c, so 5.4.1a, which excludes this barbell, does not apply. Accounting for sign differences between terms on the right-hand side of 5.3.7c and corresponding terms in 5.4.6 precomposed with $\forall^X_{X_{\setminus j}}$, we get the following.

Since $j > l+\xi-2$ we have $B_{X_{\setminus j}} \in Y_{l+\xi}$.

$$\begin{aligned} \chi^i_{X_{k\setminus j},X} &= \chi^{i-1}_{X_{\setminus j},X_k} \mathrm{sgn}_{\rho(X_k)}(j,k) \\ &\text{for } k \in S-X, \qquad k \neq l+\xi-1. \end{aligned} \tag{5.4.18}$$

$$\begin{aligned} \chi^i_{X_{k\setminus j},X} &= \chi^{i-1}_{X_{\setminus j},X_k} \mathrm{sgn}_{\rho(X_k)}(j,k) - 1 \\ &\text{for } k \in S-X, \qquad k = l+\xi-1. \end{aligned} \tag{5.4.19}$$

$$\begin{aligned} \chi^i_{X_{\setminus k\setminus j},X} &= \chi^{i-1}_{X_{\setminus j},X_{\setminus k}} \mathrm{sgn}_{\rho(X)}(j,k) \\ &\text{for } k \in X_{\setminus j}, \qquad k \notin A. \end{aligned} \tag{5.4.20}$$

$$\begin{aligned} \chi^i_{X_{\setminus k\setminus j},X} &= \left(\chi^{i-1}_{X_{\setminus j},X_{\setminus k}} + 1\right) \mathrm{sgn}_{\rho(X)}(j,k) \\ &\text{for } k \in X_{\setminus j}, \qquad k \in A. \end{aligned} \tag{5.4.21}$$

Lemma 5.4.9. *Equation* (5.3.8a) *holds if and only if the sign equations* (5.4.22) *and* (5.4.23) *below hold.*

This is another easy case, as we can directly match the terms on the left-hand side to those on the right-hand side, up to a sign difference. We get the following.

$$\text{Since } j > l-1 \text{ we have } B_{X_j} \in Y_l.$$

$$\begin{gathered}\chi^{i}_{X_{j,k},X} = \chi^{i-1}_{X_j,X_k}\operatorname{sgn}_{\rho(X_{j,k})}(j,k) \\ \text{for } k \in S - X_j.\end{gathered} \tag{5.4.22}$$

$$\begin{gathered}\chi^{i}_{X_{j\setminus k},X} = \chi^{i-1}_{X_j,X_{\setminus k}}\operatorname{sgn}_{\rho(X_j)}(j,k) \\ \text{for } k \in X.\end{gathered} \tag{5.4.23}$$

Lemma 5.4.10. *Equation* (5.3.8b) *holds if and only if the sign equations* (5.4.24)-(5.4.27) *hold.*

This is a nontrivial case. We need additional terms terms on the right to cancel out with $A^X_{X_j}x_l$ to give $A^X_{X_j}x_{l+n}$. Since the right-hand side of 5.3.8b does have an $(l+\xi-1)$-colored barbell, we apply 5.4.6 precomposed with $A^X_{X_j}$. Accounting for sign differences between terms on the right-hand side of 5.3.8b and the corresponding terms on the right-hand side of 5.4.6 precomposed with $A^X_{X_j}$, we get the following.

$$\text{Since } j \leq l-1 \text{ we have } B_{X_j} \in Y_{l+\xi}.$$

$$\begin{gathered}\chi^{i}_{X_{j,k},X} = \chi^{i-1}_{X_j,X_k}\operatorname{sgn}_{\rho(X_{j,k})}(j,k) \\ \text{for } k \in S - X_j, \qquad k \neq l+\xi-1.\end{gathered} \tag{5.4.24}$$

$$\begin{gathered}\chi^{i}_{X_{j,k},X} = \left(\chi^{i-1}_{X_j,X_k} - 1\right)\operatorname{sgn}_{\rho(X_{j,k})}(j,k) \\ \text{for } k = l+\xi-1.\end{gathered} \tag{5.4.25}$$

$$\chi^{i}_{X_{j\backslash k},X} = \chi^{i-1}_{X_j,X_{\backslash k}} \mathrm{sgn}_{\rho(X_j)}(j,k) \qquad \text{for } k \in X, \qquad k \notin A. \tag{5.4.26}$$

$$\chi^{i}_{X_{j\backslash k},X} = \chi^{i-1}_{X_j,X_{\backslash k}} \mathrm{sgn}_{\rho(X_j)}(j,k) + 1 \qquad \text{for } k \in A. \tag{5.4.27}$$

For the next four equations, we will not have to apply Lemma 5.4.1 as no diagonal entries of χ appear in them. For each of these equations all the terms appearing are equal up to a sign.

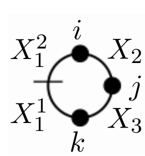
Lemma 5.4.11. *Equation* (5.3.9) *holds if and only if the equation* (5.4.28) *below holds.*

Accounting for sign differences, we get the following.

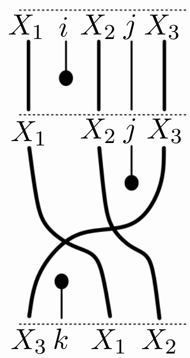
$$\begin{aligned}
&\chi^{i-1}_{X_{\backslash i\backslash j\backslash k},X_{\backslash i}} \mathrm{sgn}_{\rho(X)}(i,j)\mathrm{sgn}_{\rho(X)}(i,k) + \chi^{i-1}_{X_{\backslash i\backslash j\backslash k},X_{\backslash j}} \mathrm{sgn}_{\rho(X)}(j,i)\mathrm{sgn}_{\rho(X)}(j,k)\\
+&\chi^{i-1}_{X_{\backslash i\backslash j\backslash k},X_{\backslash k}} \mathrm{sgn}_{\rho(X)}(k,i)\mathrm{sgn}_{\rho(X)}(k,j)\\
=&\chi^{i}_{X_{\backslash j\backslash k},X} \mathrm{sgn}_{\rho(X)}(i,j)\mathrm{sgn}_{\rho(X)}(i,k) + \chi^{i}_{X_{\backslash i\backslash k},X} \mathrm{sgn}_{\rho(X)}(j,i)\mathrm{sgn}_{\rho(X)}(j,k)\\
+&\chi^{i}_{X_{\backslash i\backslash j},X} \mathrm{sgn}_{\rho(X)}(k,i)\mathrm{sgn}_{\rho(X)}(k,j)
\end{aligned} \tag{5.4.28}$$

Lemma 5.4.12. *Equation* (5.3.10) *holds if and only if the equation* (5.4.29) *below holds.*

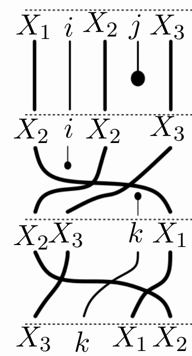
We work out the signs for this equation. Let X have the cyclic order $X_1iX_2jX_3$, induced by k. Let $X_{k\backslash i\backslash j}$ have the cyclic order $X_3kX_1X_2$, induced by j. Let X_k have the cyclic order $X_1^2iX_2jX_3kX_1^1$ where $X_1 = X_1^1 \cup X_1^2$. The following picture depicts where i, j, k and X_1, X_2, X_3 are situated in S. The tick mark indicates where the cyclic order on X_k starts.



Below we work out the total signs of the terms in equation 5.3.10.

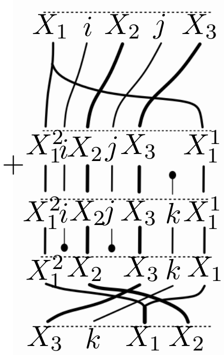


$+$

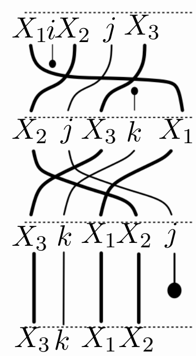


$\chi^{i-1}_{X_{k\setminus i\setminus j}, X_{\setminus i}}(-1)^{|X_3||X_1|+|X_3||X_2|+|X_3|+|X_2|}$ $\quad$ $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_{\setminus j}}(-1)^{|X_3||X_1|+|X_3||X_2|+|X_3|+|X_2|+1}$

$+$

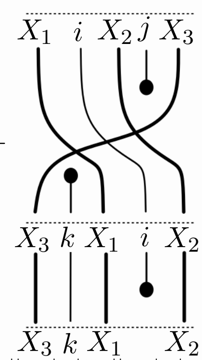


$=$

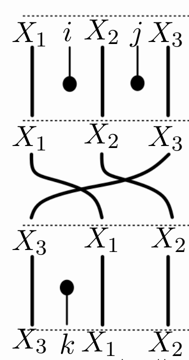


$\chi^{i-1}_{X_{k\setminus i\setminus j}, X_k}(-1)^{|X_3||X_1|+|X_3||X_2|+|X_3|+|X_2|+1}$ $\quad$ $\chi^{i}_{X_{k\setminus i}, X}(-1)^{|X_3||X_1|+|X_3||X_2|+|X_3|+|X_2|+1}$

$+$



$+$



$\chi^{i}_{X_{k\setminus j}, X}(-1)^{|X_3||X_1|+|X_3||X_2|+|X_3|+|X_2|}$ $\quad$ $\chi^{i}_{X_{\setminus i\setminus j}, X}(-1)^{|X_3||X_1|+|X_3||X_2|+|X_3|+|X_2|+1}$

We note the symmetry between i and j in this equation. Thus, the terms involving add-two double dot maps have equal sign regardless of the cyclic order on X. For this equation to hold, it is essential that we give X the cyclic order ρ_k

$$\begin{aligned}&\chi^{i-1}_{X_{k\backslash i\backslash j},X_{\backslash i}}\mathrm{sgn}_{\rho(X)}(j,i)+\chi^{i-1}_{X_{k\backslash i\backslash j},X_{\backslash j}}\mathrm{sgn}_{\rho(X)}(i,j)+\chi^{i-1}_{X_{k\backslash i\backslash j},X_k}\\ =&\chi^{i}_{X_{k\backslash j},X}\mathrm{sgn}_{\rho(X)}(j,i)+\chi^{i}_{X_{k\backslash i},X}\mathrm{sgn}_{\rho(X)}(i,j)+\chi^{i}_{X_{\backslash i\backslash j},X}.\end{aligned}\tag{5.4.29}$$

Lemma 5.4.13. *Equation* (5.3.11) *holds if and only if the equation* (5.4.30) *below holds.*

For this equation to hold, it is essential that we give $X_{i,j\backslash k}$ the cyclic order ρ_k.

$$\begin{aligned}&\chi^{i-1}_{X_{i,j\backslash k},X_i}\mathrm{sgn}_{\rho(X_{i,j\backslash k})}(j,i)+\chi^{i-1}_{X_{i,j\backslash k},X_j}\mathrm{sgn}_{\rho(X_{i,j\backslash k})}(i,j)+\chi^{i-1}_{X_{i,j\backslash k}X_{\backslash k}}\\ =&\chi^{i}_{X_{j\backslash k},X}\mathrm{sgn}_{\rho(X_{i,j\backslash k})}(j,i)+\chi^{i}_{X_{i\backslash k},X}\mathrm{sgn}_{\rho(X_{i,j\backslash k})}(i,j)+\chi^{i}_{X_{i,j},X}\end{aligned}\tag{5.4.30}$$

Lemma 5.4.14. *Equation* (5.3.12) *holds if and only if equation* (5.4.31) *below holds.*

Accounting for sign differences, we get the following.

$$\begin{aligned}&\chi^{i-1}_{X_{i,j,k},X_i}\mathrm{sgn}_{\rho(X_{i,j,k})}(i,j)\mathrm{sgn}_{\rho(X_{i,j,k})}(i,k)+\chi^{i-1}_{X_{i,j,k},X_j}\mathrm{sgn}_{\rho(X_{i,j,k})}(j,i)\mathrm{sgn}_{\rho(X_{i,j,k})}(j,k)\\ +&\chi^{i-1}_{X_{i,j,k},X_k}\mathrm{sgn}_{\rho(X_{i,j,k})}(k,i)\mathrm{sgn}_{\rho(X_{i,j,k})}(k,j)\\ =&\chi^{i}_{X_{j,k},X}\mathrm{sgn}_{\rho(X_{i,j,k})}(i,j)\mathrm{sgn}_{\rho(X_{i,j,k})}(i,k)+\chi^{i}_{X_{i,k},X}\mathrm{sgn}_{\rho(X_{i,j,k})}(j,i)\mathrm{sgn}_{\rho(X_{i,j,k})}(j,k)\\ +&\chi^{i}_{X_{i,j},X}\mathrm{sgn}_{\rho(X_{i,j,k})}(k,i)\mathrm{sgn}_{\rho(X_{i,j,k})}(k,j)\end{aligned}\tag{5.4.31}$$

5.5 Proof of main theorem.

5.5.1 Existence. In order for χ to be a pseudo-chain map, the criterion (C1-C3) in 5.1.6 must be compatible with the equations 5.4.8-5.4.31. We check this.

Showing 5.4.8 holds. For this equation, we have $j < l-1$, thus $\chi^i_{X_{k\backslash j},X} = 0$, and $B_{X_{\backslash j},X} \in Y_l$. If $k \leq l-1$, then $B_{X_k} \in Y_{l+\xi}$ by Lemma 5.2.2. Then $j \leq l-1 \leq (l+\xi-1)$, so 5.1.6 (C1) says that $\chi^{i-1}_{X_{\backslash j},X_k} = 0$. So the equation holds in this case. If

$k > l-1$, then $B_{X_k} \in Y_l$. Since $j < l-1$, the criterion still says $\chi^{i-1}_{X_{\backslash j}, X_k} = 0$, and the equation still holds.

Showing 5.4.9 holds. For this equation we have $j < l-1$ so the criterion says that $\chi^{i}_{X_{\backslash j \backslash k}, X} = 0$. If, no matter what k is, we have then $B_{X_{\backslash k}} \in Y_m$, where $m \geq l$. Since $j < l$, the criterion says that $\chi^{i-1}_{X_{\backslash j}, X_{\backslash k}} = 0$ so the equation holds.

Showing 5.4.15 holds. For this equation $j \in A$. If $k \in \{0, \ldots, l-1\}$ then the criterion says that $\chi^{i}_{X_{k \backslash j}, X} = 1$. In this case, we also have $B_{X_k} \in Y_{l+\xi}$ by Lemma 5.2.2 part (2). Since $j \in \{0, \ldots, (l+\xi)-2\}$, 5.1.6 (C1) says that $\chi^{i-1}_{X_{\backslash j}, X_k} = 0$. So the equation holds. If $k > l-1$, 5.1.6 (C2) says that $\chi^{i}_{X_{k \backslash j}, X} = 0$. We also have that $B_{X_k} \in Y_l$. Hence $\xi(X_k, l)$ is at most $\xi + 1$, which happens in the case $k = l + \xi - 1$. Then j will still be between l and $l + \xi(X_k, l) - 2$. Thus, 5.1.6 (C1) says that $\chi^{i-1}_{X_{\backslash j}, X_k} = 1$. We also know that $k > l + \xi - 2 \geq j$, so we have that $\mathrm{sgn}_{\rho(X_k)}(j,k) = -1$ in the cyclic order of X_k induced by $l-1$. Thus the right-hand side of the equation is 0, so the equation holds.

Showing 5.4.16 holds. For this equation $j \in A$. If $k \leq l-1$, then $\chi^{i}_{X_{\backslash k \backslash j}, X} = 0$ by 5.1.6 (C1). By Lemma 5.2.2 part (1), $B_{X_k} \in Y_l$. By 5.1.6 (C2), $\chi^{i-1}_{X_{\backslash j}, X_{\backslash k}} = 1$, so the right-hand side of the equation is 0 and the equation holds. If $k > l-1$, then $k > j$ in the cyclic order of X induced by $l-1$. Thus $\mathrm{sgn}_{\rho(X)}(j,k) = -1$. Here 5.1.6 (C1) says that $\chi^{i-1}_{X_{\backslash j \backslash k}, X} = 1$. By Lemma 5.2.2 part (1), $B_{X_{\backslash k}} \in Y_m$ where $m \geq k+1 > j+1$. Since $B_{X_{\backslash j}} \in Y_{j+1}$, then $\chi^{i-1}_{X_{\backslash j}, X_{\backslash k}} = 0$ by upper-triangularity. So the right-hand side of the equation is 1, and the equation holds.

Showing 5.4.17 holds. For this equation $j \in A$ and $k < j$ in the cyclic order of X induced by $l-1$. Thus $\mathrm{sgn}_{\rho(X)}(j,k) = 1$. 5.1.6 (C1) says that $\chi^{i}_{X_{\backslash j \backslash k}, X} = 1$. By Lemma 5.2.2 part (1), $B_{X_{\backslash k}} \in Y_{k+1}$. Since $j \in A(X_{\backslash k}, k+1) = \{k+1, k+2, \ldots, l+$

$\xi - 2\}$, and $k \leq (k+1)-1$, 5.1.6 (C2) says that $\chi^{i-1}_{X_{\setminus j}, X_{\setminus k}} = 1$. Thus the equation holds.

Showing 5.4.18 holds. Since $j > l+\xi-2$, 5.1.6 (C2) says that $\chi^{i}_{X_{k \setminus j}, X} = 0$. If $k > l-1$, then $B_{X_k} \in Y_l$ by Lemma 5.2.2 part (1). Since $l, l+1, \ldots, l+\xi-2 \in X$, and $k \in S \setminus X$,we must have $k > l+n-2$. In this equation we also have $k \neq l+n-1$, so $k > l+n-1$. Then $\xi(X_k, l) = \xi$. Since both j and k are greater than $l+n-2$, 5.1.6 (C1) says that $\chi^{i-1}_{X_{\setminus j}, X_k} = 0$, and the equation holds. If $k \leq l-1$ then $B_{X_k} \in Y_{l+\xi}$. Since $k \leq l-1 \leq l+\xi-2$, 5.1.6 (C1) says that $\chi^{i-1}_{X_{\setminus j}, X_k} = 0$, and the equation holds.

Showing 5.4.19 holds. Since $j > l+\xi-2$, 5.1.6 (C2) says that $\chi^{i}_{X_{k \setminus j}, X} = 0$. Since $k = l+\xi-1$, we have $B_{X_k} \in Y_l$ by Lemma 5.2.2 part (2). Then $\xi(X_k, l) = \xi+1$. Since $k \in A(X_k, l)$, 5.1.6 (C1) says that $\chi^{i-1}_{X_{\setminus j}, X_k} = 1$. In the cyclic order of X_k induced by $l-1$, we have $j > k$. Thus $\text{sgn}_{\rho(X_k)}(j,k) = 1$. Thus the right-hand side of the equation is 0, and the equation holds.

Showing 5.4.20 holds. Since $k \notin A$, either $k < l-1$ or $k > l+\xi-2$. Either way, since $j > l+\xi-2$, 5.1.6 (C1) says that $\chi^{i}_{X_{\setminus k \setminus j}, X} = 0$. If $k < l-1$, then $B_{X_{\setminus k}} \in Y_l$ by Lemma 5.2.2 part (1). Since $k \notin A$, we have $\xi(X_{\setminus k}, l) = \xi$. Since $j > l+\xi-2$, 5.1.6 (C2) says that $\chi^{i-1}_{X_{\setminus j}, X_{\setminus k}} = 0$, and the equation holds.

Showing 5.4.21 holds. Since $k \in A$, 5.1.6 (C1) says that $\chi^{i}_{X_{\setminus j \setminus k}, X} = 1$. By Lemma 5.2.2 part (1), $B_{X_{\setminus k}} \in Y_{k+1}$. Since $j > l+n-2$, we have $j \notin A(X_{\setminus k}, k+1) = \{k+1, k+2, ..., l+n-2\}$. Then 5.1.6 (C2) says that $\chi^{i-1}_{X_{\setminus j}, X_{\setminus k}} = 0$. Note $j > k$ in the cyclic order of X induced by $l-1$, so $\text{sgn}_{\rho(X)}(j,k) = 1$. Thus the equation holds.

Showing 5.4.22 holds. 5.1.6 (C3) says that $\chi^{i}_{X_{j,k}, X} = 0$. If $k > l-1$, then $B_{X_k} \in Y_l$ by Lemma 5.2.2 part (2). Since $j > l-1$, 5.1.6 (C2) says that $\chi^{i-1}_{X_j, X_k} = 0$ and the equation holds. If $k \leq l-1$, then $B_{X_k} \in Y_{l+\xi}$. Now since $k < l+\xi$, 5.1.6 (C2) says that $\chi^{i-1}_{X_j, X_k} = 0$ and the equation still holds.

Showing 5.4.23 holds. Since $j > l-1$ 5.1.6 (C2) says that $\chi^i_{X_{j\setminus k},X} = 0$. 5.1.6 (C3) says that $\chi^{i-1}_{X_j,X_{\setminus k}} = 0$, and the equation holds.

Showing 5.4.24 holds. By 5.1.6 (C3), $\chi^i_{X_{i,j},X} = 0$. If $k > l-1$, then $B_{X_k} \in Y_l$ by Lemma 5.2.2 part (2). Since $k \neq l+\xi-1$, we have $\xi(X_k, l) = \xi$. Then since $k \notin X$, $k \notin A$. Then by 5.1.6 (C2), $\chi^{i-1}_{X_j,X_k} = 0$, and the equation holds. If $k \leq l-1$, then $B_{X_k} \in Y_{l+\xi}$. Since $k < l+\xi$, 5.1.6 (C2) says $\chi^{i-1}_{X_j,X_k} = 0$, and the equation holds.

Showing 5.4.25 holds. By 5.1.6 (C3), $\chi^i_{X_{i,j},X} = 0$. Since $k = l+\xi-1$, $B_{X_k} \in Y_l$ by Lemma 5.2.2 part (2). Thus $\xi(X_k, l) = \xi+1$. Since $k = l+\xi-1 \in A(X_k, l)$, and $j \leq l-1$, 5.1.6 (C2) says that $\chi^{i-1}_{X_j,X_k} = 1$. Thus the right-hand side of the equation is 0, and the equation holds.

Showing 5.4.26 holds. Since $k \notin A$, 5.1.6 (C2) says that $\chi^i_{X_{j\setminus k},X} = 0$. 5.1.6 (C3) says that $\chi^{i-1}_{X_j,X_{\setminus k}} = 0$, so the equation holds.

Showing 5.4.27 holds. Since $k \in A$ and $j \leq l-1$, 5.1.6 (C2) says that $\chi^i_{X_{j\setminus k},X} = 1$. 5.1.6 (C3) says that $\chi^{i-1}_{X_j,X_{\setminus k}} = 0$, so the right-hand side of the equation is 1, and the equation holds.

For the next four equations, there are no conditions on what i, j, k are, so we must consider several cases for each.

Showing 5.4.28 holds. We break this into several cases conditioned on i, j, k.

$\underline{i, j, k < l-1:}$ Under these conditions $B_{X_{\setminus i}}, B_{X_{\setminus j}}, B_{X_{\setminus k}} \in Y_l$. The reader may verify that all the coefficients appearing are 0, and the equation holds.

$\underline{i \in A, j, k < l-1:}$ Under these conditions, $B_{X_{\setminus i}} \in Y_{i+1}, B_{X_{\setminus j}} \in Y_l$, and $B_{X_{\setminus i}} \in Y_l$. The reader may verify that all the coefficients are 0, and the equation holds.

$\underline{i > l + \xi - 2, j, k < l - 1:}$ Under these conditions, $B_{X_{\setminus i}} \in Y_{l+n}$, $B_{X_{\setminus j}} \in Y_l$, and $B_{X_{\setminus k}} \in Y_l$. The reader may verify that all the coefficients are 0, and the equation holds.

$\underline{i, j \in A, k < l - 1:}$ Under these conditions $B_{X_{\setminus i}} \in Y_{i+1}$, $B_{X_{\setminus j}} \in Y_{j+1}$, and $B_{X_{\setminus k}} \in Y_l$. By 5.1.6 (C1), all the coefficients are 0, except $\chi^{i-1}_{X_{\setminus i \setminus j \setminus k}, X_{\setminus k}}$, and $\chi^{i}_{X_{\setminus i \setminus j}, X}$, which are both equal to 1. The equation holds.

$\underline{i > l + n - 2, j \in A, k < l - 1:}$ Under these conditions $B_{X_{\setminus i}} \in Y_{l+\xi}$, $B_{X_{\setminus j}} \in Y_{j+1}$, and $B_{X_{\setminus k}} \in Y_l$. By 5.1.6 (C1), all the coefficients are 0, except $\chi^{i-1}_{X_{\setminus i \setminus j \setminus k}, X_{\setminus k}}$, and $\chi^{i}_{X_{\setminus i \setminus j}, X}$, which are both equal to 1. The equation holds.

$\underline{i, j > l + \xi - 2, k < l - 1:}$ Under these conditions $B_{X_{\setminus i}} \in Y_{l+\xi}$, $B_{X_{\setminus j}} \in Y_{l+\xi}$, and $B_{X_{\setminus k}} \in Y_l$. The reader may verify that all the coefficients are 0, and the equation holds.

$\underline{i, j, k \in A:}$ Under these conditions $B_{X_{\setminus i}} \in Y_{i+1}$, $B_{X_{\setminus j}} \in Y_{j+1}$, and $B_{X_{\setminus k}} \in Y_{k+1}$. Assume, without loss of generality, that $i > j > k$. The reader may verify that all the coefficients appearing on the right-hand side of the equation are equal to 1. On the left-hand side, the only non-zero coefficient is $\chi^{i-1}_{X_{\setminus i \setminus j \setminus k}, X_{\setminus k}}$, which is 1 by 5.1.6 (C1). In the cyclic order of X induced by $l - 1$, we have $\mathrm{sgn}_{\rho(X)}(i, j) = \mathrm{sgn}_X(i, k) = \mathrm{sgn}_{\rho(X)}(j, k) = 1$ and $\mathrm{sgn}_{\rho(X)}(j, i) = \mathrm{sgn}_{\rho(X)}(k, i) = \mathrm{sgn}_{\rho(X)}(k, j) = -1$. Both sides of the equation total to 1, and the equation holds.

$\underline{i > l + \xi - 2, j, k \in A:}$ Under these conditions $B_{X_{\setminus i}} \in Y_{l+\xi}$, $B_{X_{\setminus j}} \in Y_{j+1}$, and $B_{X_{\setminus k}} \in Y_{k+1}$. The reader may verify that all the coefficients appearing in the right-hand side of the equation are equal to 1. Assume, without loss of generality, that $j > k$. By 5.1.6 (C1), $\chi^{i-1}_{X_{\setminus i \setminus j \setminus k}, X_{\setminus i}} = 0$, $\chi^{i-1}_{X_{\setminus i \setminus j \setminus k}, X_{\setminus j}} = 0$, and $\chi^{i-1}_{X_{\setminus i \setminus j \setminus k}, X_{\setminus k}} = 1$. The signs work out the same as the previous case too. Both sides of the equation total to 0, and the equation holds.

<u>$i, j > l + \xi - 2, k \in A$:</u> Under these conditions $B_{X_{\setminus i}} \in Y_{l+n}$, $B_{X_{\setminus j}} \in Y_{l+\xi}$, and $B_{X_{\setminus k}} \in Y_{k+1}$. The reader may verify that all the coefficients on the left-hand side are 0. On the right-hand side, by 5.1.6 (C1), we have $\chi^i_{X_{\setminus i \setminus j}, X} = 0$, $\chi^i_{X_{\setminus i \setminus k}, X} = 1$, and $\chi^i_{X_{\setminus j \setminus k}, X} = 1$. Assume, without loss of generality, that $i > j$. In the cyclic order of X induced by $l - 1$, $\mathrm{sgn}_{\rho(X)}(j, i) = \mathrm{sgn}_{\rho(X)}(k, i) = \mathrm{sgn}_{\rho(X)}(k, j) = -1$, and $\mathrm{sgn}_{\rho(X)}(j, k) = 1$. The right-hand side of the equation totals to 0, and the equation holds.

<u>$i, j, k > l + \xi - 2$:</u> Under these conditions $B_{X_{\setminus i}} \in Y_{l+\xi}$, $B_{X_{\setminus j}} \in Y_{l+\xi}$, and $B_{X_{\setminus k}} \in Y_{l+\xi}$. The reader may verify that all the coefficients on the right-hand side of the equation are 0. Assume, without loss of generality, that $i > j > k$. Let $\zeta = \xi(X, l+\xi)$. In any case, $\chi^{i-1}_{X_{\setminus i \setminus j \setminus k}, X_{\setminus k}} = 0$. If $k \notin \{l+\xi, l+\xi+1, \ldots, l+\xi+\zeta-2\}$, then $\chi^{i-1}_{X_{\setminus i \setminus j \setminus k}, X_{\setminus i}} = \chi^{i-1}_{X_{\setminus i \setminus j \setminus k}, X_{\setminus j}} = 0$, and the equation holds. If $k \in \{l + \xi, l + \xi + 1, \ldots, l + \xi + \zeta - 2\}$, then $\chi^{i-1}_{X_{\setminus i \setminus j \setminus k}, X_{\setminus i}} = \chi^{i-1}_{X_{\setminus i \setminus j \setminus k}, X_{\setminus j}} = 1$. In the cyclic order of X induced by $l - 1$, $\mathrm{sgn}_{\rho}(X)(i, j) = \mathrm{sgn}_{\rho(X)}(i, k) = \mathrm{sgn}_{\rho(X)}(j, k) = 1$, and $\mathrm{sgn}_{\rho(X)}(j, i) = -1$. Then the left-hand side of the equation totals to 0, and the equation holds.

Showing 5.4.29 holds. We also break this into several cases conditioned on i, j, k.

<u>$k > l - 1, i, j < l - 1$:</u> Under these conditions on i, j, k we have $B_{X_{\setminus i}}$, $B_{X_{\setminus j}}$, and B_{X_k} are in Y_l. The reader can verify that all the coefficients $\chi^{i-1}_{Z,W}$ and $\chi^i_{Z,W}$ appearing in this equation are 0, so the equation holds.

<u>$k > l - 1, i \in A, j < l - 1$:</u> Under these conditions, B_{X_k} and $B_{X_{\setminus j}}$ are in Y_l, and $B_{X \setminus i} \in Y_{i+1}$. The reader can verify that all the coefficients appearing in this equation are 0, so the equation holds.

$\underline{k > l-1, i > l+\xi-2, j < l-1:}$ Under these conditions, we have $B_{X_{\setminus i}} \in Y_{l+\xi}$, $B_{X_{\setminus j}} \in Y_l$ and $B_{X_k} \in Y_l$. The reader can verify all the coefficients appearing in this equation are 0, so the equation holds.

$\underline{k > l-1, i, j \in A:}$ Under these conditions, $B_{X_k} \in Y_l$, $B_{X_{\setminus i}} \in Y_{i+1}$ and $B_{X_{\setminus j}} \in Y_{j+1}$. Since $k \notin X$, we have $k > l+\xi-2$, so $k > i = (i+1)-1$ and $k > j = (j+1)-1$. Thus 5.1.6 (C2) says then that $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_{\setminus j}}$ and $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_{\setminus i}}$ are 0. 5.1.6 (C1) says that $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_k} = 1$. Since $k \notin \{0, ..., l-1\}$, 5.1.6 (C2) says that $\chi^{i}_{X_{k\setminus i}, X}$ and $\chi^{i}_{X_{k\setminus j}, X}$ are 0. 5.1.6 (C1) says that $\chi^{i}_{X_{\setminus i\setminus j}, X} = 1$. Thus the equation holds.

$\underline{k > l-1, i > l+\xi-2, j \in A:}$ Under these conditions, $B_{X_k} \in Y_l$, $B_{X_{\setminus i}} \in Y_{l+\xi}$, and $B_{X_{\setminus j}} \in Y_{j+1}$. By 5.1.6 (C2), $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_{\setminus i}}$, $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_{\setminus j}}$, $\chi^{i}_{X_{k\setminus i}, X}$ and $\chi^{i}_{X_{k\setminus j}, X}$ are 0. 5.1.6 (C1) says that $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_k}$ and $\chi^{i}_{X_{\setminus i\setminus j}, X}$ are 1, and the equation holds.

$\underline{k > l-1, i, j > l+\xi-2:}$ Under these conditions, $B_{X_k} \in Y_l$, $B_{X_{\setminus i}} \in Y_{l+\xi}$, $B_{X_{\setminus j}} \in Y_{l+\xi}$. The reader can verify that all the coefficients on the right-hand side of the equation are 0. If $k \neq l+\xi-1$, 5.1.6 (C2) says that both $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_{\setminus j}}$ and $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_{\setminus i}}$ are 0. Since $k \neq l+\xi-1$, $B_{X_k} \in Y_l$ and $\xi(X_k, l) = \xi$, so 5.1.6 (C1) says that $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_k} = 0$. Now assume $k = l+\xi-1$. Without loss of generality, assume $i > j$. Let $\zeta = \xi(X_k, l+\xi)$. Note now that $B_{X_k} \in Y_l$ and $\xi(X_k, l) = \xi + \zeta$. If $j \notin A(X_k, l+\xi) = \{l+\xi, (l+\xi)+1, ..., (l+\xi)+\zeta-2\}$, then neither is i since $i > j$. Thus both $i, j > (l+\xi)+\zeta-2$, so the all the coefficients on the left-hand side of the equation are 0. If $j \in A(X_k, l+\xi)$, then 5.1.6 (C2) says that $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_{\setminus i}} = 1$, and $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_{\setminus j}} = 0$. 5.1.6 (C1) says that $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_k} = 1$. Since $i > j > l+\xi-2$, $i > j$ in the cyclic order of X induced by $l-1$. Thus $\text{sgn}_{\rho(X)}(j, i) = -1$. Then the terms on the left-hand side of the equation sum to 0, and the equation holds.

$\underline{k \leq l-1, i, k < l-1:}$ Under these conditions, $B_{X_{\setminus i}} \in Y_l$, $B_{X_{\setminus j}} \in Y_l$, and $B_{X_k} \in Y_{l+\xi}$. The reader may verify that all the coefficients are 0, and the equation holds.

$\underline{k \leq l-1, i \in A, j < l-1:}$ Under these conditions, $B_{X_{\setminus i}} \in Y_{i+1}$, $B_{X_{\setminus j}} \in Y_l$, and $B_{X_k} \in Y_{l+\xi}$. 5.1.6 (C2) says that $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_{\setminus i}} = 0$, $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_{\setminus j}} = 1$, $\chi^{i}_{X_{k\setminus j}, X} = 0$ and $\chi^{i}_{X_{k\setminus i}, X} = 1$. 5.1.6 (C1) says that $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_k} = 0$ and $\chi^{i}_{X_{\setminus i\setminus j}, X} = 0$. The equation holds.

$\underline{k \leq l-1, i > l+\xi-2, j < l-1:}$ Under these conditions $B_{X_{\setminus i}} \in Y_{l+\xi}$, $B_{X_{\setminus j}} \in Y_l$, and $B_{X_k} \in Y_{l+\xi}$. The reader may verify that all the coefficients are 0, and the equation holds.

$\underline{k \leq l-1, i, j \in A:}$ Under these conditions $B_{X_{\setminus i}} \in Y_{i+1}$, $B_{X_{\setminus j}} \in Y_{j+1}$, and $B_{X_k} \in Y_{l+n}$. Assume, without loss of generality, that $i > j$. 5.1.6 (C2) says that $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_{\setminus i}} = 0$, $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_{\setminus j}} = 1$, $\chi^{i}_{X_{k\setminus j}, X} = 1$, and $\chi^{i}_{X_{k\setminus i}, X} = 1$. 5.1.6 (C1) says that $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_k} = 0$ and $\chi^{i}_{X_{\setminus i\setminus j}, X} = 1$. The left-hand side and the right-hand side both total to 1, and the equation holds.

$\underline{k \leq l-1, i > l+\xi-2, j \in A:}$ Under these conditions $B_{X_{\setminus i}} \in Y_{l+\xi}$, $B_{X_{\setminus j}} \in Y_{j+1}$, and $B_{X_k} \in Y_{l+\xi}$. The only nonzero coefficients in the equation are $\chi^{i}_{X_{k\setminus j}, X}$ and $\chi^{i}_{X_{\setminus i\setminus j}, X}$ which are both 1. Under these conditions, $i > j$ in the cyclic order of X induced by k, so $\operatorname{sgn}_X(j, i) = -1$. The right-hand side of the equation totals to 0, and the equation holds.

$\underline{k \leq l-1, i, j > l+\xi-2:}$ Under these conditions $B_{X_{\setminus i}} \in Y_{l+\xi}$, $B_{X_{\setminus j}} \in Y_{l+\xi}$, and $B_{X_k} \in Y_{l+\xi}$. The reader may verify that all the coefficients on the right-hand side of the equation are 0. Let $\zeta = \xi(X_k, l+\xi)$. Assume, without loss of generality, that $i > j$. If $j \notin A(X_k, l+\xi) = \{l+\xi, l+\xi+1, \ldots, l+\xi+\zeta-2\}$, then all the coefficients on the left-hand side of the equation are 0, and the equation holds. If

$j \in A(X_k, l+\xi)$, then 5.1.6 (C2) says that $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_{\setminus i}} = 1$ and $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_{\setminus i}} = 0$. 5.1.6 (C1) says that $\chi^{i-1}_{X_{k\setminus i\setminus j}, X_k} = 1$. Note that $i > j$ in the cyclic order of X induced by k, so $\mathrm{sgn}_{\rho(X)}(j,i) = -1$. Thus the left-hand side of the equation totals to 0, and the equation holds.

Showing 5.4.30 holds. We break this into the following cases.

$\underline{k \in< l-1, i, j > l-1 :}$. Under these conditions $B_{X_i} \in Y_l$, $B_{X_j} \in Y_l$, and $B_{X_{\setminus k}} \in Y_l$. The reader may verify that all the coefficients in the equation are 0.

$\underline{k < l-1, i \leq l-1, j > l-1 :}$ Under these conditions $B_{X_i} \in Y_{l+\xi}$, $B_{X_j} \in Y_l$, and $B_{X_{\setminus k}} \in Y_l$. The reader may verify that all the coefficients in the equation are 0.

$\underline{k < l-1, i, j \leq l-1 :}$ Under these conditions $B_{X_i} \in Y_{l+\xi}$, $B_{X_j} \in Y_{l+\xi}$, and $B_{X_{\setminus k}} \in Y_l$. The reader may verify that all the coefficients in the equation are 0.

$\underline{k \in A, i, j > l-1 :}$ Under these conditions $B_{X_i} \in Y_l$, $B_{X_j} \in Y_l$, and $B_{X_{\setminus k}} \in Y_{k+1}$. The reader may verify that all the coefficients in the equation are 0.

$\underline{k \in A, i \leq l-1, j > l-1 :}$ Under these conditions $B_{X_i} \in Y_{l+\xi}$, $B_{X_j} \in Y_l$, and $B_{X_{\setminus k}} \in Y_{k+1}$. The only nonzero coefficients are $\chi^{i-1}_{X_{i,j\setminus k}, X_j}$ and $\chi^{i}_{X_{i\setminus k}, X}$ which are both 1, and the equation holds.

$\underline{k \in A, i, j \leq l-1 :}$ Under these conditions $B_{X_i} \in Y_{l+\xi}$, $B_{X_j} \in Y_{l+\xi}$, and $B_{X_{\setminus k}} \in Y_{k+1}$. The coefficients $\chi^{i-1}_{X_{i,j\setminus k}, X_{\setminus k}}$ and $\chi^{i}_{X_{i,j}, X}$ are 0 and all other coefficients are 1. The equation holds.

$\underline{k > l+\xi-2, i, j > l-1 :}$ Under these conditions $B_{X_i} \in Y_l$, $B_{X_j} \in Y_l$, and $B_{X_{\setminus k}} \in Y_{l+\xi}$. The reader may verify that all the coefficients in the equation are 0.

$\underline{k > l+\xi-2, i \leq l-1, j > l-1 :}$ Under these conditions $B_{X_i} \in Y_{l+\xi}$, $B_{X_j} \in Y_l$, and $B_{X_{\setminus k}} \in Y_{l+\xi}$. The reader may verify that all the coefficients on the right-hand side of the equation are 0. Let $\zeta = \xi(X_i, l+\xi)$. If $k \notin A(X_i, l+\xi) = \{l+\xi, l+\xi+1, \ldots, l+\xi+\zeta-2\}$ then all the coefficients are 0 and the equation holds. If

$k \in A(X_i, l+\xi)$ and $j \neq l+\xi-1$, then all the coefficients on the left-hand side 0 and the equation holds. If $j = l+\xi-1$, then $B_{X_j} \in Y_l$ and $\xi(X_j, l) = (\xi+\zeta)$. Then 5.1.6 (C2) says that $\chi^{i-1}_{X_{i,j\backslash k}, X_i}$ and $\chi^{i}_{X_{i,j\backslash k}, X_j}$ are both equal to 1. The left-hand side totals to 0, and the equation holds.

$\underline{k > l+\xi-2, i, j \leq l-1:}$ Under these conditions $B_{X_i} \in Y_{l+\xi}$, $B_{X_j} \in Y_{l+\xi}$, and $B_{X_{\backslash k}} \in Y_{l+\xi}$. The reader may verify that all the coefficicents on the right-hand side of the equation are 0. Let $\zeta = \xi(X_i, l+\xi)$. If $k \notin A(X_i, l+\xi) = A(X_j, l+\xi) = \{l+\xi, l+\xi+1, \ldots, l+\xi+\zeta-2\}$, then the reader may verify that all the coefficients on the left-hand side are 0, and the equation holds. If $k \in A(X_i, l+\xi) = A(X_j, l+\xi)$, then 5.1.6 (C2) says that $\chi^{i-1}_{X_{i,j\backslash k}, X_i}$ and $\chi^{i-1}_{X_{i,j\backslash k}, X_j}$ are both equal to 1. The left-hand side of the equation totals to 0, and the equation holds.

Showing 5.4.31 holds. 5.1.6 (C3) says all the coefficients appearing in this equation are 0, so the equation holds.

5.5.2 Uniqueness. Now we show the uniqueness of χ. To do this, we show that the equations 5.4.8through 5.4.31 on the coefficients $\chi^i_{X,Y}$ determine the criterion (C1) to (C3) in Theorem 5.1.6.

Deducing Theorem 5.1.6 (C1).

Suppose $j < l-1$. The equation 5.4.9 gives

$$\chi^i_{X_{\backslash k \backslash j}, X} = \chi^{i-1}_{X_{\backslash j}, X_{\backslash k}} \mathrm{sgn}_{\rho(X)}(j,k). \tag{5.5.1}$$

Now, $X_{\backslash j} \in Y_l$ by Lemma 5.2.2. By the same lemma, $X_{\backslash k}$ is in $Y_{l'}$ for $l' \geq l$. Thus $\chi^{i-1}_{X_{\backslash j}, X_{\backslash k}} = 0$ by upper-triangularity. We remind the reader that on Y_l χ should be right multiplication by x_l, so this coefficient is 0 in the case $l' = l$ also. Thus $\chi^i_{X_{\backslash k \backslash j}, X} = 0$.

Suppose both $j, k \geq l$. Assume one of j or k is in $A(X, l)$. Without loss of generality, we take it to be j. If $k > j$, then equation (5.4.16) gives:

$$\chi^i_{X_{\setminus k \setminus j}, X} = \left(\chi^{i-1}_{X_{\setminus j}, X_{\setminus k}} - 1\right) \operatorname{sgn}_{\rho(X)}(j, k). \tag{5.5.2}$$

Now by Lemma 5.2.2, $X_{\setminus j} \in Y_{j+1}$. Under the assumptions on k, we have $X_{\setminus k} \in Y_{l'}$ for $l' \geq j+1$. Thus $\chi^{i-1}_{X_{\setminus j}, X_{\setminus k}} = 0$ by upper-triangularity. Note that under the assumptions we have $\operatorname{sgn}_{\rho_{l-1}}(j, k) = -1$. Hence $\chi^i_{X_{\setminus k \setminus j}, X} = 1$.

If $k < j$, then we may reverse the roles of k and j and arrive at the same conclusion: $\chi^i_{X_{\setminus k \setminus j}, X} = 1$.

Finally suppose both j and k are greater than $l + \xi(X, l) - 2$. Then equation (5.4.20) gives

$$\chi^i_{X_{\setminus k \setminus j}, X} = \chi^{i-1}_{X_{\setminus j}, X_{\setminus k}} \operatorname{sgn}_{\rho(X)}(j, k). \tag{5.5.3}$$

Under these assumptions, we have that both $X_{\setminus j}$ and $X_{\setminus k}$ are in $Y_{l+\xi(X,l)}$ by Lemma 5.2.2. Thus $\chi^{i-1}_{X_{\setminus j}, X_{\setminus k}} = 0$ by upper-triangularity. Hence $\chi^i_{X_{\setminus k \setminus j}, X} = 0$.

Deducing Theorem 5.1.6 (C2).

Suppose $j < l - 1$, then $X_{\setminus j} \in Y_l$ by Lemma 5.2.2. Equation (5.4.8) gives

$$\chi^i_{X_{k \setminus j}, X} = \chi^{i-1}_{X_{\setminus j}, X_k} \operatorname{sgn}_{\rho(X_k)}(j, k). \tag{5.5.4}$$

By Lemma 5.2.2, $X_k \in Y_{l'}$ where $l' \geq l$. Hence $\chi^{i-1}_{X_{\setminus j}, X_k} = 0$ by upper-triangularity. Hence $\chi^i_{X_{k \setminus j}, X} = 0$.

Now suppose $j \in A(X, l)$. Then equation (5.4.15) gives

$$\chi^i_{X_{k \setminus j}, X} = \chi^{i-1}_{X_{\setminus j}, X_k} \operatorname{sgn}_{\rho(X_k)}(j, k) + 1. \tag{5.5.5}$$

By Lemma 5.2.2 we have $X_{\setminus j} \in Y_{j+1}$. If $k \leq l-1$, then $X_k \in Y_{l+\xi(X,l)}$. Then $\chi^{i-1}_{X_{\setminus j},X_k} = 0$ by upper-triangularity. Hence, $\chi^{i}_{X_{k\setminus j},X} = 1$.

If $k > l-1$, then $X_k \in Y_l$. Then (C1), which we have already deduced, says that $\chi^{i-1}_{X_{\setminus j},X_k} = 1$. Since $\mathrm{sgn}_{\rho_{l-1}}(j,k) = -1$, we have $\chi^{i}_{X_{k\setminus j},X} = 0$.

Now suppose $j > l+\xi(X,l)-2$. If $k \neq l+\xi(X,l)-1$, then equation (5.4.18) gives

$$\chi^{i}_{X_{k\setminus j},X} = \chi^{i-1}_{X_{\setminus j},X_k}\mathrm{sgn}_{\rho(X_k)}(j,k). \tag{5.5.6}$$

By Lemma 5.2.2, we have $X_{\setminus j} \in Y_{l+\xi(X,l)}$. If $k \leq l-1$, then $X_k \in Y_{l+\xi(X,l)}$. Then $\chi^{i-1}_{X_{\setminus j},X_k} = 0$ by upper triangularity. Then $\chi^{i}_{X_{k\setminus j},X} = 0$. If $k > l-1$, then $X_k \in Y_l$. Now, neither k or j are in $A(X_k,l)$. Thus $\chi^{i-1}_{X_{\setminus j},X_k} = 0$ by $(C1)$ which we have already deduced. Hence, $\chi^{i}_{X_{k\setminus j},X} = 0$.

If $k = l+\xi(X,l)-1$, then equation (5.4.19) gives

$$\chi^{i}_{X_{k\setminus j},X} = \chi^{i-1}_{X_{\setminus j},X_k}\mathrm{sgn}_{\rho(X_k)}(j,k) - 1. \tag{5.5.7}$$

Now $X_k \in Y_l$. Since $k \in A(X_k,l)$ and $j \geq l$, then $\chi^{i-1}_{X_{\setminus j},X_k} = 1$ by (C1) which we have already deduced. Since $\mathrm{sgn}_{\rho_{l-1}}(j,k) = 1$, we have $\chi^{i}_{X_{k\setminus j},X} = 0$.

Deducing Theorem 5.1.6 (C3).

Assume, without loss of generality, that $j < k$. If $j > l-1$, then equation (5.4.22) gives

$$\chi^{i}_{X_{j,k},X} = \chi^{i-1}_{X_j,X_k}\mathrm{sgn}_{\rho(X_{j,k})}(j,k). \tag{5.5.8}$$

Since both j and k are greater than $l-1$, then X_j and X_k are both in Y_l by Lemma 5.2.2. Thus $\chi^{i-1}_{X_j,X_k} = 0$ by upper-triangularity. Hence $\chi^{i}_{X_{j,k},X} = 0$.

Now suppose $j \leq l-1$. Then $X_j \in Y_{l+\xi(X,l)}$ by Lemma 5.2.2. If $k \neq l+\xi(X,l)-1$, then equation (5.4.24) gives

$$\chi^{i}_{X_{j,k},X} = \chi^{i-1}_{X_j,X_k} \mathrm{sgn}_{\rho(X_{j,k})}(j,k). \tag{5.5.9}$$

If $k \leq l-1$, then $X_k \in Y_{l+\xi(X,l)}$. Then $\chi^{i-1}_{X_j,X_k} = 0$ by upper-triangularity. Hence $\chi^{i}_{X_{j,k},X} = 0$.

If $k > l-1$, then $X_k \in Y_l$. Observe that k is not in $A(X_k, l)$ since $k \neq l+\xi(X,l)-1$. Thus $\chi^{i-1}_{X_j,X_k} = 0$ by (C2) which we have already deduced. Hence $\chi^{i}_{X_{j,k},X} = 0$.

Now suppose $k = l+\xi-1$. Then equation (5.4.25) gives

$$\chi^{i}_{X_{j,k},X} = \left(\chi^{i-1}_{X_j,X_k} - 1\right) \mathrm{sgn}_{\rho(X_{j,k})}(j,k). \tag{5.5.10}$$

Since $k > l-1$, we have $X_k \in Y_l$ by Lemma 5.2.2. Now $k \in A(X_k, l)$. Thus, by (C2) which we have already determined, we have $\chi^{i-1}_{X_j,X_k} = 1$. Hence $\chi^{i}_{X_{j,k},X} = 0$.

CHAPTER VI

CATEGORIES OVER SCHEMES

We review the theory of categories over schemes introduced in [3, 4]. We then make some comments on how [3, Proposition 4.16], which concerns lifting this structure to graded affine and projective bundles, might work in the setting of derived schemes and stacks.

6.1 Classical setting of maps to projective space.

Let X be a projective variety and let $\mathcal{L}$ be a line bundle over X. We say $\mathcal{L}$ is generated by global sections if the map of sheaves

$$\mathcal{O} \otimes \Gamma(X, \mathcal{L}) \to \mathcal{L} \tag{6.1.1}$$

is surjective. If this is the case, then we can choose a basis $s_0, \ldots, s_n$ of $\Gamma(X, \mathcal{L})$ of basepoint-free sections of $\mathcal{L}$ that globally generate. This data gives us a map:

$$X \xrightarrow{\iota} \mathbb{P}^n, \qquad x \mapsto [s_0(x) : \ldots : s_n(x)] \tag{6.1.2}$$

such that $\iota^*\mathcal{O}(k) = \mathcal{L}^{\otimes k}$ for all $k \in \mathbf{Z}$. We then have pushforward and pullback functors

$$\mathrm{Coh}(X) \underset{\iota^*}{\overset{\iota_*}{\rightleftarrows}} \mathrm{Coh}(\mathbb{P}^n) \tag{6.1.3}$$

with adjunction $\iota^* \dashv \iota_*$ between categories of coherent sheaves. The pullback functor ι^* is monoidal with respect to tensor product of $\mathcal{O}$-modules. The functor ι_* is not monoidal but instead satisfies the following *projection formula*:

$$\iota_*(\iota^* M_1 \otimes C \otimes \iota^* M_2) = M_1 \otimes \iota_* C \otimes M_2. \tag{6.1.4}$$

The sections s_i are maps $\mathcal{O}_X \xrightarrow{s_i} \mathcal{L}$. The cone

$$\left[\mathcal{O}_X \xrightarrow{s_i} \mathcal{L}\right] \in D^b \operatorname{Coh}(X) \tag{6.1.5}$$

has nonzero stalk complex only at points $x \in X$ where $s_i(x) = 0$. Thus, if the sections s_i are basepoint-free, the Koszul complex

$$K(s_0, \ldots, s_n) = \bigotimes_i \left[\mathcal{O}_X \xrightarrow{s_i} \mathcal{L}\right] \tag{6.1.6}$$

is quasi-isomorphic to 0.

Given a birational map $X \xrightarrow{\iota} Y$ of schemes, one has $\iota_* \mathcal{O}_X = \mathcal{O}_Y$. In fact this is a suitable definition of birationality. Since $\mathcal{L}$ is very ample, we have

$$X = \operatorname{Proj}\left(\bigoplus_{k=0}^{\infty} \Gamma(X, \mathcal{L}^{\otimes k})\right). \tag{6.1.7}$$

The map ι is induced by the map of graded rings $S^\bullet \mathbb{C}^{n+1} = \mathbb{C}[z_0, ..., z_n] \to \bigoplus_{k=0}^{\infty} \Gamma(X, \mathcal{L}^{\otimes k})$ sending z_i to s_i.

6.2 General Theory

Let X be a scheme or stack. We let $\operatorname{Coh}(X)$ denote its dg-derived category of coherent sheaves, viewed as a dg-pretriangulated category.

Definition 6.2.1. Let $\mathcal{C}$ be a dg-pretriangulated monoidal category. We say having dg functors

$$\mathcal{C} \underset{\iota^*}{\overset{\iota_*}{\rightleftarrows}} \operatorname{Coh}(X) \tag{6.2.1}$$

makes $\mathcal{C}$ a *category over* X if they satisfy

- The functor ι^* is monoidal.

– The adjunction $\iota^* \dashv \iota_*$ holds.

– The *projection formula* holds functorially:

$$\iota_*(\iota^* M_1 \otimes C \otimes \iota^* M_2) = M_1 \otimes \iota_* C \otimes M_2. \tag{6.2.2}$$

We will denote the structure of being a category over X simply by 'the morphism' $\mathcal{C} \xrightarrow{\iota} X$.

Definition 6.2.2. The morphism $\mathcal{C} \xrightarrow{\iota} X$ is called *birational* if $\iota_*(\mathbf{1}) = \mathcal{O}_X$.

Lemma 6.2.3. *If $\mathcal{C} \xrightarrow{\iota} X$ is birational, then $\iota_*\iota^*(\mathcal{F}) = \mathcal{F}$ for all objects $\mathcal{F} \in \mathrm{Coh}(X)$.*

Proof. Observe that

$$\iota_*\iota^*\mathcal{F} = \iota_*(\iota^*\mathcal{F} \otimes \mathbf{1}) = \mathcal{F} \otimes \iota_*\mathbf{1} = \mathcal{F} \otimes \mathcal{O}_X = \mathcal{F}. \tag{6.2.3}$$

□

Remark 6.2.4. When one has a *proper birational* map $X \xrightarrow{\iota} Y$, so pushforward preserves being coherent, then $\iota_*\iota^*\mathcal{F} = \mathcal{F}$ for all $\mathcal{F} \in \mathrm{Coh}(Y)$. This is because, since it is coherent, $\mathcal{F}$ may be presented by copies of $\mathcal{O}_Y$, and one has $\iota_*\iota^*\mathcal{O}_Y = \mathcal{O}_Y$.

Proposition 6.2.5. *Let $\mathcal{C} \xrightarrow{\iota} X$ be birational. Then ι^* is fully faithful, and*

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \iota^*\mathcal{F}) = \Gamma(X, \mathcal{F}) \tag{6.2.4}$$

for all $\mathcal{F} \in \mathrm{Coh}(X)$.

Proof.

$$\mathrm{Hom}_{\mathcal{C}}(\iota^*\mathcal{F}', \iota^*\mathcal{F}) = \mathrm{Hom}_X(\mathcal{F}', \iota_*\iota^*\mathcal{F}) = \mathrm{Hom}_X(\mathcal{F}', \mathcal{F}). \tag{6.2.5}$$

The last equality follows from 6.2.3. □

6.3 Examples

6.3.1 The affine case. Let $\mathcal{C}$ be an additive monoidal category. Let A be a (Noetherian) commutative ring, and suppose we have a finite ring morphism

$$A \xrightarrow{f} \mathrm{End}_{\mathcal{C}}(\mathbf{1}). \tag{6.3.1}$$

Then the chain category $K(\mathcal{C})$, viewed as a dg-category, is a sometimes a category over $\mathrm{Spec}A$. Let us explore the structures present. We have functors

$$K(\mathcal{C}) \underset{\iota^*}{\overset{\iota_*}{\rightleftarrows}} D(A\text{-mod}_{\text{f.g}}). \tag{6.3.2}$$

Here the latter category is the bounded dg-derived category of finitely generated A-modules. The functor $\iota_* : \mathcal{C} \to A$-mod is given by

$$\iota_*(B) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, C), \qquad B \in \mathcal{C} \tag{6.3.3}$$

A priori, this gives a functor $\mathcal{C} \to A^{\mathrm{op}}$. Since A is commutative, $A = A^{\mathrm{op}}$, so we intepret it as a functor to A-mod. By extending to complexes, we get the functor $K(\mathcal{C}) \to D(A\text{-mod}_{\text{f.g.}})$.

Now we describe the functor in the other direction. Since A is Noetherian, every complex of finitely-generated A-modules has a (possibly infinite) resolution by finitely generated free modules. Let FA-mod denote the category of finitely generated free A-modules. The inclusion $K(FA\text{-mod}) \rightarrow D(A\text{-mod}_{\text{f.g.}})$ is an equivalence because A is Noetherian. Therefore, we need only define ι^* on $K(FA\text{-mod})$. We set $\iota^*(A) = \mathbf{1}$ and $\iota^*(a) = f(a)$ for all $a \in A$. This extends to complexes of finitely-generated free A-modules in the obvious way. If $M \in D(A\text{-mod})$, we write $\iota^*(M) = M \otimes_A \mathbf{1}$ to denote this functor, understanding we must take a free resolution to interpret it. Now we ask whether this satisfies the properties of being a *category over*.

- We note that ι^* is monoidal by construction.

- We note that

$$\mathrm{Hom}_{K(\mathcal{C})}(M \otimes_A \mathbf{1}, C) = \mathrm{Hom}_{D(A\text{-mod}_{\text{f.g.}})}(M, \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, C)) \tag{6.3.4}$$

  is seen to hold functorially when M is a complex of free modules, which suffices.

- Likewise, we see that the projection formula

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathcal{F} \otimes (M \otimes_A \mathbf{1}) \otimes \mathcal{G}) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathcal{F}) \otimes_{A\text{-mod}} M \otimes \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathcal{G}) \tag{6.3.5}$$

  reduces to the following for $M = A$:

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathcal{F} \otimes \mathcal{G}) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathcal{F}) \otimes_{A\text{-mod}} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathcal{G}). \tag{6.3.6}$$

  This is not always expected to hold, but does in the example below.

Example 6.3.1. (Example of affine case): Let Y be a scheme. Then $\mathrm{Coh}(Y)$ is a category over its affinization, $\mathrm{Spec}(\Gamma_Y(\mathcal{O}_Y))$.

6.3.2 The projective case. Let $\mathcal{C}$ be a monoidal dg-pretriangulated category, and let F be an object that is invertible in the homotopy category $K(\mathcal{C})$. Assume that the graded algebra:

$$\mathrm{Hom}_{K(\mathcal{C})}(\mathbf{1}, F^\bullet) := \bigoplus_{k=-\infty}^{\infty} \mathrm{Hom}_{K(\mathcal{C})}(1, F^k) \tag{6.3.7}$$

is commutative. The multiplication is given by tensor of morphisms. Suppose $R^\bullet$ is a Noetherian graded commutative $\mathbb{C}$-algebra and we have a finite graded ring homomorphism

$$R^\bullet \xrightarrow{f} \mathrm{Hom}_{K(\mathcal{C})}(\mathbf{1}, F^\bullet). \tag{6.3.8}$$

That is, $\mathrm{Hom}_{K(\mathcal{C})}(\mathbf{1}, F^\bullet)$ is finitely generated over $R^\bullet$. Then $\mathcal{C}$ is sometimes a category over the quotient stack $\mathrm{Spec}R/\mathbb{C}^*$. Let us explore the structures present.

Notation 6.3.2. We will refer to the quotient stack $\mathrm{Spec}R/\mathbb{C}^*$ associated to a graded $\mathbb{C}$-algebra as $\mathrm{grSpec}R^\bullet$.

Recall that:

$$\mathrm{Coh}(\mathrm{grSpec}R^\bullet) = D^b(\{\text{finitely generated graded } R^\bullet\text{-modules}\}). \tag{6.3.9}$$

The functors are given by

$$\iota_*(C) = \mathrm{Hom}_{K(\mathcal{C})}(\mathbf{1}, F^\bullet \otimes C) \tag{6.3.10}$$

and

$$\iota^*(M) = (M^\bullet \otimes_{R^\bullet} F^\bullet)_{\deg 0}. \tag{6.3.11}$$

Here, as in the affine case, we assume M is a complex of free graded modules over $R^\bullet$. Now we ask whether this construction satisfies the properties of being a category over. First, you may be confused as to why the sums in the above constructions are taken over $\mathbb{Z}$ and not $\mathbb{N}$. When you check whether there is an adjunction between the functors ι_* and ι^* in the case $M^\bullet = R^\bullet$, you'll see it to be necessary.

Likewise, if we check the projection formula in the case $M^\bullet = R^\bullet \in R^\bullet\text{-grmod}$, and $B, C \in \mathcal{C}$, we see that we need

$$\mathrm{Hom}_{K(\mathcal{C})}\left(\mathbf{1}, \bigoplus_{k=-\infty}^{\infty} F^k \otimes (A \otimes B)\right) = \tag{6.3.12}$$

$$\mathrm{Hom}_{K(\mathcal{C})}\left(\mathbf{1}, \bigoplus_{k=-\infty}^{\infty} F^k \otimes A\right) \otimes_{R^\bullet} \mathrm{Hom}_{K(\mathcal{C})}\left(\mathbf{1}, \bigoplus_{k=-\infty}^{\infty} F^k \otimes B\right) \tag{6.3.13}$$

to hold. This does not always hold, but does in the examples coming from algebraic geometry like in the case discussed in 6.1 where $\mathcal{C}$ is $\mathrm{Coh}(Y)$ for a scheme Y and F an ample line bundle.

Let's elaborate on the case of this when $R^\bullet = A[z_0, \ldots, z_n]$ for a $\mathbb{C}$-algebra A equipped with a homomorphism to $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$, and the z_i have degree 1. We can thus specify f by giving a morphism

$$\mathbf{1} \overset{\alpha_i}{\to} F. \tag{6.3.14}$$

for each degree 1 generator z_i. This data makes $\mathcal{C}$ into a category over $\mathrm{grSpec} R = \mathbb{A}_A^{n+1}/\mathbb{C}^*$. Like $\mathrm{Proj}(R)$, which we will discuss shortly, $\mathrm{grSpec}(R)$ has line bundles $\mathcal{O}(k)$. These correspond to the graded module $R(k)$. In this setting, we have $\iota_* F = \mathcal{O}(1)$.

We may view $\mathbb{P}^n_A = \mathrm{Proj} R^\bullet$ as an open substack of $\mathbb{A}^{n+1}/\mathbb{C}^*$. The question is when does this structure factor through projective space as follows.

$$\begin{tikzcd} \mathcal{C} \arrow[r, dotted, "\iota'"] \arrow[dr, "\iota'"'] & \mathbb{P}^n_A \arrow[d, hook] \\ & \mathbb{A}^{n+1}_A/\mathbb{C}^* \end{tikzcd} \tag{6.3.15}$$

We recall the following.

Proposition 6.3.3. *(Beilinson's Description of* $\mathrm{Coh}(\mathbb{P}^n_A)$ *[42]). Let* $R = A[z_0, ..., z_n]$*. Each degree 1 generator* z_i *of* $A[z_0, ..., z_n]$ *gives a map* $R^\bullet \xrightarrow{z_i} \mathcal{O}(1)$*. We define the Koszul complex*

$$K(z_0, ..., z_n) := \bigotimes_i \left[R \xrightarrow{s_i} R(1) \right]. \tag{6.3.16}$$

We have

$$\mathrm{Coh}(\mathbb{P}^n_A) = \frac{D^b(\{\textit{finitely generated graded } R\textit{-modules}\})}{K(z_0, ..., z_n) \simeq 0}. \tag{6.3.17}$$

The latter category is a modification of the dg-derived category of finitely generated graded R*-modules, where we add an isomorphism between* $K(z_0, ..., z_n)$ *and* 0*. We note that the line bundle* $\mathcal{O}(k)$ *on* $\mathbb{P}^n_A$ *corresponds to* $R(k)$*.*

Since we think of $\mathbb{P}^n_A$ as an open substack of $\mathbb{A}^n_A/\mathbb{C}^*$ via $\mathbb{P}^n_A = (\mathbb{A}^n_A - \{0\})/\mathbb{C}^*_m \hookrightarrow \mathbb{A}^n_A/\mathbb{C}^*$, we expect sheaves supported at the non-geometric point $0 \in \mathbb{A}^{n+1}_A/\mathbb{C}^*$ to be quasi-isomorphic to 0. The Koszul complex resolves the skyscraper sheaf of the point 0, so it makes sense that we 'set' it to zero.

Beilinson's description of coherent sheaves gives us our answer for when we can 'lift' the structure ι. In $\mathcal{C}$, we must have

$$\bigotimes_i \mathrm{Cone} \left[\mathbf{1} \xrightarrow{\alpha_i} F \right] \simeq 0. \tag{6.3.18}$$

We will have that $\iota^* F = \mathcal{O}(1)$.

Remark 6.3.4. This means that F is categorically pre-diagonalizable with eigencones $\left[\mathbf{1} \xrightarrow{\alpha_i} F\right]$ in the framework of categorical diagonalization given in [6].

6.3.3 Relative Case.

Now we ask more generally, given a map of schemes $Y \xrightarrow{\pi} X$, and $\mathcal{C}$ a category over X via ι, how can we make $\mathcal{C}$ a category over Y so that the following diagram commutes?

$$\begin{tikzcd} \mathcal{C} \arrow[r, dotted, "\iota'"] \arrow[dr, "\iota"'] & Y \arrow[d, "\pi"] \\ & X \end{tikzcd} \tag{6.3.19}$$

6.3.3.1 Projectivization of a Locally Free Sheaf.

We dicuss the case where $Y = \mathbb{P}(V^\vee)$ where V is a locally free sheaf on X. In terms of the relative Proj construction, we have

$$Y = \underline{\mathrm{Proj}}(S^\bullet V). \tag{6.3.20}$$

Here, $S^\bullet V$ is the symmetric algebra of V in $\mathrm{Coh}(X)$. We note that like in the case relative to $\mathrm{Spec} A$ discussed above, we have a relative version of graded Spec, and we can define the *graded affine* bundle $\mathrm{Tot}(V^\vee)/\mathbb{C}^*$, the $\mathbb{C}^*$ quotient stack of the affine bundle $\mathrm{Tot}(V^\vee)$, as

$$\mathrm{Tot}(V^\vee)/\mathbb{C}^* = \underline{\mathrm{grSpec}}(S^\bullet V). \tag{6.3.21}$$

We set $Y' = \underline{\mathrm{grSpec}}(S^\bullet V)$ and note that Y is an open substack of Y'.

We have

$$\mathrm{Coh}(Y') = \{\text{graded } S^\bullet V \text{ modules in } \mathrm{Coh}(X)\}\,. \tag{6.3.22}$$

There is a relative version of Beilinson's description of coherent sheaves on Y, describing $\mathrm{Coh}(Y)$ as a modification of $\mathrm{Coh}(X)$.

Proposition 6.3.5. *[3, Appendix 10.4]*

$$\mathrm{Coh}(Y) = \frac{\{\textit{graded } S^\bullet V \textit{ modules in } \mathrm{Coh}(X)\}}{(S^\bullet V/S^{\bullet>0}V^\vee) \cong 0}. \tag{6.3.23}$$

We note that both Y and Y' come equipped with the tautological line bundle $\mathcal{O}(1)$ corresponding to the object $S^\bullet V(1)$.

Proposition 6.3.6. *Let $Y = \mathbb{P}\mathcal{V}^\vee$ and let $\mathcal{C}$ be a category over X. The data of ι' is equivalent to having $F \in \mathrm{Pic}\,\mathcal{C}$ and a map*

$$\iota^*\mathcal{V} \xrightarrow{\alpha} F \tag{6.3.24}$$

satisfying

$$\mathrm{Tot}\left[\ldots \xrightarrow{\alpha} \iota^*\left(\wedge^k\mathcal{V}\right) \otimes F^{-k} \xrightarrow{\alpha} \ldots\right] \simeq 0 \in \mathcal{C} \tag{6.3.25}$$

The map ι' is birational if ι is birational and satisfies:

$$S^k(V) \cong \iota_*(F^k), \qquad \forall k \geq 0. \tag{6.3.26}$$

Here the map α on the left-hand-side of (6.3.25) is given by contraction by the map α of (6.3.24). We will refer the reader to [3, Prop 4.15] for the full details.

We explain how to define the functors. The functor ι'^* is given as follows. Because ι^* is monoidal, we have morphisms

$$\iota^*(V^{\otimes k}) \to F^k. \tag{6.3.27}$$

Because F is invertible, we factor through

$$\iota^*(S^k(V)) \to F^k. \tag{6.3.28}$$

This gives $\bigoplus_{k=-\infty}^{\infty} F^k$ an action of $\iota^*(S^\bullet V)$. Given M in $\mathrm{Coh}(Y)$ we set

$$\iota'^*(M) := \left(\iota^*(M) \otimes_{\iota^* S^\bullet V} \bigoplus_{k=-\infty}^{\infty} F^k\right)_{\text{deg } 0}. \tag{6.3.29}$$

=

We define the functor ι'_* as follows. Given an object C of $\mathcal{C}$, we set

$$\iota'_*(C) := \iota_*\left(\bigoplus_{k=-\infty}^{\infty} F^k \otimes C\right). \tag{6.3.30}$$

We equip it with an action of $S^\bullet V$ in $\mathrm{Coh}(X)$ via

$$S^\bullet V \otimes_{\mathcal{O}_X} \iota_*\left(\bigoplus_{k=-\infty}^{\infty} F^k \otimes C\right) \longrightarrow \iota_*\left(\bigoplus_{k=-\infty}^{\infty} F^k \otimes C\right). \tag{6.3.31}$$

This morphism is given by the map 6.3.28 and adjunction.

A priori, these maps make $\mathcal{C}$ into a category over $\mathrm{Tot}\, \mathcal{V}^\vee/\mathbb{C}^*$. The condition 6.3.25 combined with 6.3.5 ensure that this descends to the category $\mathrm{Coh}(\mathbb{P}(\mathcal{V}^\vee))$. We note that when α is an isomorphism, then $\iota'^*(O(1)) = F$.

6.3.3.2 *Adapting to Derived Algebraic Geometry.*

Starting with the work of Bezrukavnikov in [2], it has been understood that the spectral (coherent) incarnations of categories of arising in geometric representation theory (e.g. the Hecke category) behave like coherent sheaves on a locally complete intersection X. Generally, when the underlying space X of spectral parameters is not a locally complete intersection, it seems it must be replaced by a derived enhancement, which is a derived stack. For example, the work of Bezrukavnikov realizes the extended affine Hecke category for a reductive group G in terms of coherent sheaves on a derived enhancement of the usual Steinberg variety of the Langlands dual group.

The following definition will be important while we discuss aspects of derived algebraic geometry.

Definition 6.3.7. A connective dg-algebra is a dg-algebra concentrated in non-positive degrees.

We now make some comments on how we expect the theory of categories over classical schemes and stacks to adapt to the setting of derived stacks. We are not going to be very explicit about what framework for derived algebraic geometry we are considering, but we consider derived stacks which are locally modeled on connective (non-positively graded) commutative dg $\mathbb{C}$-algebras, which we denote as $\mathrm{cdgas}^{\leq 0}$. Thus to us a derived stack is an ∞-functor

$$X : (\mathrm{cdgas}^{\leq 0})^{\mathrm{op}} \to \text{gpds} \tag{6.3.32}$$

satisfying relevant descent conditions. The target category is the $(\infty, 1)$-category of infinity groupoids. This is essentially the framework considered in [36] and

[37] and we refer the reader there for actual details. The following discussion should not be taken to be mathematically rigorous.

Given a derived stack X, we denote its underlying classical stack as X^{cl}. We treat $\mathrm{Coh}(X)$ as a dg-pretriangulated monoidal category (aka a stable dg-category).

Because the theory of categories over schemes just requires $\mathrm{Coh}(X)$ to be a monoidal stable dg-category, we define what it means for a category $\mathcal{C}$ to be a category over a derived stack X identically to the definition for classical stacks. The case we will care about in the next chapter involves a version of relative graded Spec, and we are not aware of this in the DAG literature (of which we are unfamiliar). Nevertheless, we propose the following conjecture.

Conjecture 6.3.8. *Let X be a derived stack, and let $\mathcal{F} \in \mathrm{Coh}(X)$ complex of vector bundles on X. Let $S^\bullet\mathcal{F}$ denote the symmetric (dg) algebra of $\mathcal{F}$. Let X^{cl} denote the underlying classical stack of X. Then the derived stack*

$$\mathrm{grSpec}_X S^\bullet\mathcal{F} := \mathrm{Spec}_X(S^\bullet_X\mathcal{F})/\mathbb{C}^* \tag{6.3.33}$$

has the following properties.

1. *There is a map $\pi : \mathrm{grSpec}_X S^\bullet\mathcal{F} \to X$.*

2. *There is a relative description $\mathrm{Coh}(\mathrm{grSpec}_X S^\bullet\mathcal{F}) = \{S^\bullet\mathcal{F}$ graded modules in $\mathrm{Coh}(X)\}$.*

3. *There is a line bundle $\mathcal{O}(k)$ on $\mathrm{grSpec}_X S^\bullet\mathcal{F}$ such that $\pi_*\mathcal{O}(k) \simeq S^k\mathcal{F}$ for all $k \geq 0$, and $\mathcal{O}(k) = S^\bullet(\mathcal{F})(k)$ in the above relative description. In that description, π corresponds to taking the degree 0 part in the $\bullet$-grading. In addition,*

$\mathrm{Coh}(\mathrm{grSpec}_X S^\bullet \mathcal{F})$ *is generated, in the triangulated/stable-dg sense, by the objects* $\mathcal{E}(k) := S^\bullet \mathcal{F}(k) \otimes \mathcal{E}$*, for* $k \in \mathbb{Z}$ *and* $\mathcal{E} \in \mathrm{Coh}(X)$*.*

4. $(\mathrm{grSpec}_X S^\bullet \mathcal{F})_{\mathrm{cl}} = \mathrm{grSpec}_{X^{\mathrm{cl}}} S^\bullet \mathcal{H}^0 \mathcal{F}$.

The first three properties are what we expect from the classical setting, while the fourth property ensures that the construction gives the derived enhancement of the corresponding classical constuction.

A version of this conjecture exists in the setting of dg-schemes and dg-stacks, and requires that $\mathcal{F}$ is (quasi-isomorphic to) a connective complex, so that $S^\bullet \mathcal{F}$ is (quasi-isomorphic to) a connective dg-algebra. This is discussed in [3, Section 10.4].

Given a construction of grSpec in the derived setting, satisfying the properties above, we make the following proposition.

Proposition 6.3.9. *Let* $\mathcal{C}$ *be a dg-pretriangulated category with the structure* ι_*, ι^* *making it a category over a derived stack* X*. Let* $V \in \mathrm{Coh}(X)$ *be a complex of vector bundles on* X *and let* $Y \xrightarrow{\pi} X$ *be given by* $Y = \mathrm{grSpec}_X S^\bullet V$*. Then an invertible object* F *in* $\mathcal{C}$ *and an isomorphism*

$$\iota^* V \xrightarrow{\alpha} F \tag{6.3.34}$$

can be used to make $\mathcal{C}$ *into a category over* Y*. If* α *is an isomorphism, then the map is birational if* ι *satisfies*

$$S^k V \cong \iota_*(F) \tag{6.3.35}$$

Proof. As in Proposition 6.3.6, the map $\iota^*(\mathcal{V}^{\otimes k}) \to F^k$ must factor through $\iota^*(S^k\mathcal{V})$ because F is invertible. Given an object M of $\mathrm{Coh}(Y)$, we define

$$\iota'^*(M) := \left(\iota^*(M) \otimes_{\iota^* S^\bullet V} \bigoplus_{k=-\infty}^{\infty} F^k\right)_{\deg 0}.$$

We note that if α is an isomorphism, then $\iota^*(\mathcal{O}(1)) = F$, since $\pi_*(\mathcal{O}(1)) = F$.

We define the functor ι'_* as follows. Given an object C of $\mathcal{C}$, we set

$$\iota'_*(C) := \iota_*\left(\bigoplus_{k=-\infty}^{\infty} F^k \otimes C\right). \tag{6.3.36}$$

We equip it with an action of $S^\bullet V$ in $\mathrm{Coh}(X)$ via

$$S^\bullet V \otimes_{\mathcal{O}_X} \iota_*\left(\bigoplus_{k=-\infty}^{\infty} F^k \otimes C\right) \longrightarrow \iota_*\left(\bigoplus_{k=-\infty}^{\infty} F^k \otimes C\right). \tag{6.3.37}$$

This morphism is given by the adjoint of the map $\iota^*(S^k\mathcal{V}) \to F^k$. Assuming Conjecture 6.3.8, the rest of the proof goes like GNR's proof of 6.3.6, except without the need to check the functors descend to the quotient by torsion modules since Y is not the projective bundle. For example, to check $\iota'^* \dashv \iota'_*$, it suffices to check this holds on the generators $\mathcal{E}(i)$. We demonstrate this.

Let $\mathcal{E}(k) = \mathcal{E} \otimes S^{\bullet}V(k) \in \operatorname{Coh}(X)$, viewed as an object of $\operatorname{Coh}(Y)$. Let $M \in \mathcal{C}$. Then,

$$\operatorname{Hom}_{\mathcal{C}}(\iota'^*(\mathcal{E}(k)), M) = \operatorname{Hom}_{\mathcal{C}}(\iota'^*(\mathcal{E} \otimes S^{\bullet}V(k)), M) \tag{6.3.38}$$

$$= \operatorname{Hom}_{\mathcal{C}}\left(\left((\iota^*(S^{\bullet}V(k) \otimes \mathcal{E}) \otimes_{\iota^* S^{\bullet}V} \bigoplus_{l=-\infty}^{\infty} F^l\right)_{\deg 0}, M\right) \tag{6.3.39}$$

$$= \operatorname{Hom}_{\mathcal{C}}\left(\left((\iota^*(S^{\bullet}V(k)) \otimes \iota^*(\mathcal{E}) \otimes_{\iota^* S^{\bullet}V} \bigoplus_{l=-\infty}^{\infty} F^l\right)_{\deg 0}, M\right) \qquad \text{since } \iota^* \text{is monoidal.} \tag{6.3.40}$$

$$= \operatorname{Hom}_{\mathcal{C}}(F^k \otimes \iota^*(\mathcal{E}), M) \qquad \text{since } \alpha \text{ is an iso.} \tag{6.3.41}$$

$$= \operatorname{Hom}_{\mathcal{C}}(\iota^*(\mathcal{E}), M \otimes F^{-k}) \tag{6.3.42}$$

$$= \operatorname{Hom}_{\operatorname{Coh}(X)}(\mathcal{E}, \iota_*(M \otimes F^{-k})) \tag{6.3.43}$$

$$= \operatorname{Hom}_{S^{\bullet}V\text{-mod}}(\mathcal{E} \otimes S^{\bullet}V(k), \iota_*(M \otimes \bigoplus_{l=-\infty}^{\infty} F^l)) \tag{6.3.44}$$

$$= \operatorname{Hom}_{\operatorname{Coh}(Y)}(\mathcal{E}(k), \iota'_*(M)). \tag{6.3.45}$$

□

CHAPTER VII

THE FLAG COMMUTING STACK

7.1 Geometry of FComm

The classical flag commuting variety is the stack quotient

$$\mathrm{FComm}_n^{\mathrm{cl}} = \{X, Y \mid X \in \mathfrak{b}, Y \in [\mathfrak{b}, \mathfrak{b}], [X, Y] = 0\}/B. \tag{7.1.1}$$

Here $\mathfrak{b}$ indicates upper triangular $n \times n$ matrices over $\mathbb{C}$, and B indicates invertible upper triangular matrices acting by the adjoint action on both X and Y. Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. We write

$$\mathrm{FComm}_n^{\mathrm{cl}} = \left((\mathfrak{b} \times \mathfrak{n}) \times_{\mathfrak{g}} \{0\}\right)/B. \tag{7.1.2}$$

It is a classical fiber product, where $\mathfrak{b} \times \mathfrak{n} \overset{[-,-]}{\to} \mathfrak{g}$ is the commutator map.

The dimension of the space of X and Y is $n(n+1)/2 + n(n-1)/2 = n^2$. The affine subscheme of $\mathbb{C}^{n^2}$ cut out by the equation $[X, Y] = 0$ is not a complete intersection. So we replace it by the affine derived scheme with coordinate dg-algebra given by the Koszul complex of the commutator map

$$\mathfrak{b} \times \mathfrak{n} = \mathbb{C}^{n^2} \overset{[-,-]}{\to} \mathbb{C}^{n^2} = \mathfrak{g}, \tag{7.1.3}$$

viewed as a section of trivial $\mathfrak{g}$-bundle over $\mathbb{C}^{n^2}$. This gives a map $\mathfrak{g}^* \otimes \mathcal{O}_{\mathbb{C}^{n^2}} \overset{\mu}{\to} \mathcal{O}_{\mathbb{C}^{n^2}}$. The Koszul complex is

$$\left[\cdots \overset{d_3}{\to} \wedge^2(\mathfrak{g}^* \otimes \mathcal{O}_{\mathbb{C}^{n^2}}) \overset{d_2}{\to} \mathfrak{g}^* \otimes \mathcal{O}_{\mathbb{C}^{n^2}} \overset{d_1 = \mu}{\to} \mathcal{O}_{\mathbb{C}^{n^2}}\right]. \tag{7.1.4}$$

We call this affine derived scheme Z_n.

Definition 7.1.1. The derived version of the flag commuting variety FComm_n we study is the quotient stack

$$\mathrm{FComm}_n = Z_n/B.$$

We equivalently can think of Z_n as a derived fiber product, and hence

$$\mathrm{FComm}_n = ((\mathfrak{b} \times \mathfrak{n}) \overset{L}{\times}_{\mathfrak{g}} \{0\})/B \tag{7.1.5}$$

The space FComm_n inherits a $T = \mathbb{C}^* \times \mathbb{C}^*$-action via the scaling of X and Y respectively, i.e the T action on the factor $\mathfrak{b} \times \mathfrak{n}$.

7.1.1 Tautological Bundles. Since FComm_n is a B-quotient, to any representation V of B there is a vector bundle $\mathcal{V}$ on FComm_n via the associated bundle construction, i.e. $\mathcal{V} = p^*V$ where p is the map $\mathrm{FComm}_n \to \mathrm{pt}/B$. For the n elementary characters of B, we get n tautological line bundles $\mathcal{L}_1, ..., \mathcal{L}_n$. Here, B preserves the full flag

$$0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n.$$

The i-th elementary character of B is given by $B \mid_{F^i/F^{i-1}}$. Likewise, FComm_n has the rank n vector bundle $\mathcal{T}_n$ given by the standard representation $\mathbb{C}^n$ of B. The standard flag fixed by B gives a filtration of vector bundles:

$$0 \hookrightarrow \mathcal{T}_1 \hookrightarrow \mathcal{T}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{T}_n. \tag{7.1.6}$$

Here, the i-th subquotient is equal to $\mathcal{L}_i$. The bundle $\mathcal{T}_n$ is equipped with endomorphisms

$$X \curvearrowright \mathcal{T}_n \curvearrowleft Y. \tag{7.1.7}$$

These endomorphisms preserve the filtration of $\mathcal{T}_n$. Here, Y is nilpotent with respect to the filtration. The endmorphisms X and Y are not T-equivariant and instead have weights t and q respectively with respect to the T-action. We should really write:

$$X : t\mathcal{T}_n \to \mathcal{T}_n. \tag{7.1.8}$$

$$Y : q\mathcal{T}_n \to \mathcal{T}_n. \tag{7.1.9}$$

These give the correct maps in the equivariant category.

7.1.2 Iterated Graded-Affine Bundle. The following idea was communicated to us by Eugene Gorsky and Andrei Negut.

There is a map:

$$\begin{tikzcd} \mathrm{FComm}_n^{\mathrm{cl}} \arrow[d, "\pi"'] \\ \mathrm{FComm}_{n-1}^{\mathrm{cl}} \times \mathbb{C} \end{tikzcd} \tag{7.1.10}$$

given by:

$$(X, Y) \mapsto ((X\mid_{F_{n-1}}, Y\mid_{F_{n-1}}), X\mid_{F_n/F_{n-1}}). \tag{7.1.11}$$

Let's try to understand the fiber of this map. Let (X_{n-1}, Y_{n-1}, x_n) be a point in $\mathrm{FComm}_{n-1}^{\mathrm{cl}} \times \mathbb{C}$. Let (X_n, Y_n) be a point in the fiber. We have to find two $(n-1)$-vectors to fill in the remaining entries of the $n \times n$ matrices X_n and Y_n. We set v

and w to be these vectors as in

$$X_n = \begin{pmatrix} X_{n-1} & v \\ 0 \cdots 0 & x_n \end{pmatrix}, \tag{7.1.12}$$

$$Y_n = \begin{pmatrix} Y_{n-1} & w \\ 0 \cdots 0 & 0 \end{pmatrix}. \tag{7.1.13}$$

Proposition 7.1.2. *Let $V = \mathbb{C}^{n-1}$. We claim that the space of possible pairs of $n-1$ vectors are given, up to scaling, as the dual of the zero-th cohomology of the following complex:*

$$E_{n-1} := V \xrightarrow{(X_{n-1}-x_n\cdot\mathrm{id},-Y_{n-1})} \underline{V \oplus V} \xrightarrow{(Y_{n-1},X_{n-1}-x_n\cdot\mathrm{id})^T} V. \tag{7.1.14}$$

Here, the underline indicates homological degree 0.

Proof. Let v and w be the vectors filling in the remaining entries of X_n and Y_n respectively. The equation $[X_n, Y_n] = 0$ gives $(X_{n-1} - x_n)w + Y_{n-1}v = 0$.

Now, let B_n denote the group of invertible upper-triangular $n \times n$ matrices. Let $B_n = T_n U_n$ where U_n is the unipotent radical of B_n and $T_n = B_n/U_n$ is the torus. Let A_n denote the affine scheme parametrizing the matrices X_n and Y_n. The action of U_n on matrices is a proper free action so the stack quotient $[A_n/B]$ is equivalent to $[(A_n /\!\!/ U_n)/T_n]$ where $(A_n /\!\!/ U_n)$ is the affine (categorical) quotient. We now must consider v and w up to the residual action of $V = \ker[U_n \to U_{n-1}]$ acting via U_n on $n \times n$ matrices. Given $t \in V$, the action by t is precisely:

$$(v, w) \mapsto (v, w) + ((X_{n-1} - x_n) - Y_{n-1})t. \tag{7.1.15}$$

Lastly we must consider residual stacky action of $\mathbb{C}^* = \ker[T_n \to T_{n-1}]$, which acts by scaling. This completes the proof.

$\square$

So we should think of the fiber as a $\mathbb{C}^*$-quotient, because there was no stacky $\mathbb{C}^*$-action on the base factor $\mathbb{C}$. The consequence of the above discussion is the following.

Proposition 7.1.3. *Let x_n denote the coordinate function on $\mathbb{C}$. Consider the following complex of vector bundles on $\mathrm{FComm}^{\mathrm{cl}}_{n-1} \times \mathbb{C}$.*

$$\mathcal{E}_{n-1} := qt\mathcal{T}_{n-1} \xrightarrow{(X-x_n,-Y)} \underline{q\mathcal{T}_{n-1} \oplus t\mathcal{T}_{n-1}} \xrightarrow{(Y,X-x_n)^T} \mathcal{T}_{n-1}. \tag{7.1.16}$$

We have the following diagram.

$$\begin{tikzcd} \mathrm{FComm}^{\mathrm{cl}}_n = \underline{\mathrm{grSpec}}(S^\bullet\mathcal{H}^0(\mathcal{E}_{n-1})) \arrow[d, "\pi"'] \\ \mathrm{FComm}^{\mathrm{cl}}_{n-1} \times \mathbb{C} \end{tikzcd}$$

Thus

$$\mathrm{Coh}(\mathrm{FComm}^{\mathrm{cl}}_n) = \{S^\bullet\mathcal{H}^0(\mathcal{E}_{n-1}) \text{ graded modules in } \mathrm{Coh}(\mathrm{FComm}^{\mathrm{cl}}_{n-1}) \boxtimes \mathbb{C}[x_n]\text{-mod}\}.$$

Recall that graded affine bundles are equipped with tautological bundles $\mathcal{O}(k)$. We can think of $\mathcal{O}(k)$ as the $S^\bullet\mathcal{H}^0(\mathcal{E}_{n-1})$ module given by $S^\bullet\mathcal{H}^0(\mathcal{E}_{n-1})(k)$ where (k) indicates a degree shift by k in the $\bullet$-grading. For $k = 1$, this line bundle is in fact the n-th tautological line bundle $\mathcal{L}_n$ discussed previously. This is because for each k, we can also obtain $\mathcal{O}(k)$ via the associated bundle construction for the weight k character for the residual torus $\mathbb{C}^*$. Restricting the n-th elementary character of B_n to $\mathbb{C}^*$ gives the weight 1 character. For $1 \leq m < n$, we have $\mathcal{L}_m = \pi^*\mathcal{L}_m$, because the first $n-1$ elementary characters of B_n factor through $B_n \to B_{n-1}$. This gives an inductive construction of $\mathrm{FComm}_n^{\mathrm{cl}}$ and its tautological line bundles. We have:

$$p_*(\mathcal{L}_n^k) = S^k(\mathcal{H}^0(\mathcal{E}_{n-1})) \qquad \forall k \geq 0. \tag{7.1.17}$$

From this iterative description of $\mathrm{FComm}_n^{\mathrm{cl}}$ we can see how it is badly behaved. The maps in the complex $\mathcal{E}_{n-1}$ are not always injective or surjective, so $\mathcal{H}^0(\mathcal{E}_{n-1})$ is not a vector bundle.

The map in (7.1.10) certainly lifts to derived enhancements to give a map $\pi : \mathrm{FComm}_n \to \mathrm{FComm}_{n-1} \times \mathbb{C}$. However, it is not obvious that this map is a derived version of a graded affine bundle.

We now make a conjecture about how this iterative description of $\mathrm{FComm}_n^{\mathrm{cl}}$ may exist for its derived enhancement.

Conjecture 7.1.4. *The structure of iterated graded affine bundles for* $\mathrm{FComm}^{\mathrm{cl}}$ *lifts to derived enhancements. There exists a complex of vector bundles* $\mathcal{V}$ *on* $\mathrm{FComm}_{n-1} \times \mathbb{C}$ *such that*

$$\begin{tikzcd} \mathrm{FComm}_n = \mathrm{grSpec}_{\mathrm{FComm}_{n-1} \times \mathbb{C}}(S^\bullet \mathcal{V}) \arrow[d, "\pi"'] \\ \mathrm{FComm}_{n-1} \times \mathbb{C} \end{tikzcd}$$

where $\mathrm{grSpec}_{\mathrm{FComm}_{n-1} \times \mathbb{C}}(S^\bullet \mathcal{V})$ *is the construction in Conjecture 6.3.8, and* $\mathcal{L}_n = \mathcal{O}(1)$.

Conjecture 7.1.5. *In Conjecture 7.1.4, the complex* $\mathcal{V}$ *is quasi-isomorphic to* $\mathcal{E}_{n-1}$.

We make two conjectures rather than one, because our intuition suggests it is more likely that the graded affine bundle structure lifts to derived enhancements *for some* complex of vector bundles $\mathcal{V}$. Nevertheless, in the following we work with the conjecture that $\mathcal{V}$ is equal to $\mathcal{E}_{n-1}$ which we view as less likely. It is important to note that $S^{\bullet}\mathcal{E}_{n-1}$ is not a connective dg-algebra, or obviously quasi-isomorphic to one. This is because $\mathcal{E}_n$ is concentrated in degrees $-1, 0$ and 1, and the first map fails to be always injective while the second fails to be always injective. Hence $S^{\bullet}\mathcal{E}_{n-1}$ is possibly concentrated in all homological degrees. Thus in this particular case, it is conceivable that the correct setting for the above conjecture is one of non-connective derived algebraic geometry or dg-algebraic geometry. Another possibility of course is that $\mathcal{E}_{n-1}$ is the wrong complex of vector bundles after all.

Remark 7.1.6. In [3, Proposition 2.10], it is shown how the classical flag Hilbert scheme can also be described iteratively, as the projectivization of the 0-th homology of a similar complex of vector bundles also concentrated in degrees $-1, 0$ and 1. An important difference though is that the second map in that complex is guaranteed to be surjective, so that complex is actually quasi-isomorphic to a complex supported in degrees -1 and 0. Hence the symmetric algebra of that complex is obviously quasi-isomorphic to a connective dg-algebra. The authors then explain how to construct a family of dg-schemes iteratively from this connective dg-algebra. We initially attempted to work in the setting of dg-schemes and dg-stacks rather than modern DAG, but because $S^{\bullet}\mathcal{E}_{n-1}$ is not connective, or obviously quasi-isomorphic to a connective dg-algebra, we didn't see much hope.

7.2 Categorifying inclusion and trace in the affine setting

In [32], Hogancamp gives categorifications of the inclusion map $\iota : \mathbf{H}^n_{\mathrm{fin}} \to \mathbf{H}^{n+1}_{\mathrm{fin}}$ and its adjoint, the partial trace (2.6.3) $p_n : \mathbf{H}^{n+1}_{\mathrm{fin}} \to \mathbf{H}^n_{\mathrm{fin}}$. He defines adjoint functors:

$$I : \mathcal{H}^n_{\text{fin}} \boxtimes \mathbb{C}[x_{n+1}]\text{-mod} \to \mathcal{H}^{n+1}_{\text{fin}} \qquad \mathrm{Tr} : \mathcal{H}^{n+1}_{\text{fin}} \to \mathcal{H}^n_{\text{fin}} \boxtimes \mathbb{C}[x_{n+1}]\text{-mod}. \tag{7.2.1}$$

He calls the latter functor *partial Hochshild homology*. This functors play a key role in the inductive proof the the GNR conjecture proposed in [3].

Now, we want to explain how one might categorify the inclusion map $\iota : \mathbf{H}^n_{\text{ext}} \to \mathbf{H}^{n+1}_{\text{ext}}$ and its adjoint $p_n : \mathbf{H}^{n+1}_{\text{ext}} \to \mathbf{H}^n_{\text{ext}}$. These are the functors we will need in the inductive proof of the affine GNR conjecture that we propose in the next section.

We state the following conjecture.

Conjecture 7.2.1. *There exist functors:*

$$I : \mathcal{H}^n_{\text{ext}} \boxtimes \mathbb{C}[x_{n+1}]\text{-}mod \to \mathcal{H}^{n+1}_{\text{ext}} \qquad \mathrm{Tr} : \mathcal{H}^{n+1}_{\text{ext}} \to \mathcal{H}^n_{\text{ext}} \boxtimes \mathbb{C}[x_{n+1}]\text{-}mod, \tag{7.2.2}$$

with adjunction $I \dashv \mathrm{Tr}$.

The map I is analogous to p^* and the map Tr is analogous to p_* where p is the map in equation 7.1.10.

Note that ι maps the Kazhdan-Lusztig generators of $\mathbf{H}^n_{\text{ext}}$ as follows:

$$\iota(b_i) = b_i \qquad \text{for } i = 1, ..., n-1, \qquad \iota(b_0) = T_n b_0 T_n^{-1}, \qquad \iota(\omega) = \omega T_n. \tag{7.2.3}$$

We can give a well defined functor on the additive category $I : EA\mathbb{S}\mathrm{Bim}_n \boxtimes \mathbb{C}[x_n]\text{-mod} \to \mathcal{H}^{n+1}_{\text{ext}}$ by describing where the generators go. We have:

$$I(B_0) = F_n B_0 F_n^{-1} \qquad I(B_i) = B_i \qquad \text{for } i = 1, ..., n-1 \qquad I(\Omega) = \Omega F_n. \tag{7.2.4}$$

On generating morphisms, I sends i-colored morphisms in $EA\mathbb{S}\mathrm{Bim}_n$ to their counterparts in $\mathcal{H}^{n+1}_{\mathrm{ext}}$ for $i = 1, ..., n-1$. It sends an n-colored generating morphism φ to $\mathrm{id}_{F_n} \otimes \varphi \otimes \mathrm{id}_{F_n^{-1}}$. It sends an Ω-colored morphism φ (the black diagrams in [27, Definition 3.12]) to $\varphi \otimes \mathrm{id}_{F_n}$. The construction of I, ignoring the tensor factor of $\mathbb{C}[x_n]$-mod, was accomplished in very recent work of Mackaay-Miemietz-Vaz (see [17]). The factor of $\mathbb{C}[x_n]$-mod simply extends scalars on Hom spaces to include multiplication by x_n.

While we conjecture it to be true, it is not guaranteed that I lifts to a functor on the homotopy category of $EA\mathbb{S}\mathrm{Bim}$. This needs to be checked. Let us explain. Let A and B be additive categories. There are two issues that arise:

1. A functor $F : A \to K^b(B)$ doesn't necessarily induce $\hat{F} : K^b(A) \to K^b(B)$.
2. If functors $\hat{F}, \hat{G} : K^b(A) \to K^b(B)$ restrict to functors $F, G : A \to K^b(B)$, and $\varphi : F \to G$ is a natural transformation, then φ doesn't necessarily lift to a natural transformation $\hat{\varphi} : \hat{F} \to \hat{G}$.

Point 2 above is especially relevant to us when extending units and counits of adjunction.

Resolving this issue is the subject of forthcoming work by Elias and Hogancamp [5].

We propose that the adjoint to I is the same as the functor defined by Hogancamp. Given a complex $\mathcal{F}$ in $\mathcal{H}^{n+1}_{\mathrm{ext}}$, we conjecture that

$$\mathcal{F} \overset{\mathrm{Tr}}{\mapsto} \mathrm{Tot}\left[0 \to \mathcal{F} \overset{x_n^L - x_n^R}{\to} \mathcal{F} \to 0 \right] \tag{7.2.5}$$

gives the desired functor. Here, x_n^L and x_n^R denote left and right multiplication by x_n respectively.

7.3 Affine GNR Conjectures

Let $\mathcal{V}_n$ denote the standard Gaitsgory complex in $\mathcal{H}^n_{\text{ext}}$ (see section 4.2), let $\chi : \mathcal{V}_n \to \mathcal{V}_n(2)$ be the map of chapter V, and let $\mathcal{Y} : \mathcal{V} \to \mathcal{V}(-2)[2]$ be the nilpotent monodromy map that $\mathcal{V}$ is equipped with from being a pseudocomplex.

Notation 7.3.1. We set E_n to be the following object of $\mathcal{H}^n_{\text{ext}} \boxtimes \mathbb{C}[x_{n+1}]$-mod:

$$E_{n-1} := \mathrm{Tot}\left[\mathcal{V}_{n-1}[-2] \xrightarrow{(\chi - x_{n+1}, -\mathcal{Y})} \underline{\mathcal{V}_{n-1}(2)[-2] \oplus \mathcal{V}_{n-1}(-2)} \xrightarrow{(\mathcal{Y}, \chi - x_{n+1})^T} \mathcal{V}_{n-1}\right]. \tag{7.3.1}$$

Here, the underline indicates homological degree zero.

For the rest of the section, we assume the functors

$$\mathcal{H}^{n-1}_{\text{ext}} \boxtimes \mathbb{C}[x_n]\text{-mod} \underset{\mathrm{Tr}}{\overset{I}{\rightleftarrows}} \mathcal{H}^n_{\text{ext}}$$

of the last section exist.

We state the following conjecture about objects of $\mathcal{H}_{\text{ext}}$, having drawn comparisons with the corresponding bundles on FComm.

Conjecture 7.3.2. *Let W_i denote the i-th Wakimoto object. We conjecture that:*

1. *We have $I(W_i) = W_i$ for $i = 1, 2, ..., n-1$.*

2. *There exists a homotopy equivalence:*

$$S^k E_{n-1} \simeq \mathrm{Tr}(W_n^{\otimes k}) \qquad \forall k \geq 0. \tag{7.3.2}$$

In particular, when $k = 1$, we conjecture that $E_{n-1} \cong \mathrm{Tr}(W_n)$. Using adjunction, this gives rise to the map $I(E_{n-1}) \rightsquigarrow W_n$.

3. *We have*

$$\mathcal{V}_n = \mathrm{Cone}[I(\mathcal{V}_{n-1}[1]) \to W_n], \tag{7.3.3}$$

for some map $I(\mathcal{V}_{n-1}) \to W_n$. In addition, we propose it is the following map. Composing the inclusion $V_{n-1}[1] \to E_{n-1}$, after applying I, with the map $I(E_{n-1}) \to W_n$ gives this map.

Now we state our main conjecture in full detail, which we called the affine GNR conjecture.

Conjecture 7.3.3. *There exist adjoint functors*

$$\mathcal{H}^n_{\mathrm{ext}} \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathrm{Coh}_{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{FComm}_n) \tag{7.3.4}$$

making $\mathcal{H}_{\mathrm{ext}}$ a category over FComm, *exchanging the q grading with $(2)[-2]$ and the t grading with (-2). Moreover:*

1. *The functor ι_* is birational, i.e. $\iota_*(R) = \mathcal{O}$.*

2. *The functors exchange the i-th Wakimoto with the i-th tautological line bundle $\mathcal{L}_i$.*

3. *The following diagrams of functors commute:*

$$\begin{tikzcd}
\mathcal{H}^n_{\mathrm{ext}} \arrow[r, "\iota_*"] \arrow[d, "\mathrm{Tr}"'] & \mathrm{Coh}_{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{FComm}_n) \arrow[d, "p_*"] \\
\mathcal{H}^{n-1}_{\mathrm{ext}} \boxtimes \mathbb{C}[x_n]\text{-}\mathit{mod} \arrow[r, "\iota_*"] & \mathrm{Coh}_{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{FComm}_{n-1}) \boxtimes \mathbb{C}[x_n]\text{-}\mathit{mod}
\end{tikzcd} \tag{7.3.5}$$

and

$$\begin{tikzcd}
\mathcal{H}^n_{\mathrm{ext}} & \mathrm{Coh}_{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{FComm}_n) \arrow[l, "\iota^*"'] \\
\mathcal{H}^{n-1}_{\mathrm{ext}} \boxtimes \mathbb{C}[x_n]\text{-}\mathit{mod} \arrow[u, "I"] & \mathrm{Coh}_{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{FComm}_{n-1}) \boxtimes \mathbb{C}[x_n]\text{-}\mathit{mod} \arrow[l, "\iota^*"'] \arrow[u, "p^*"']
\end{tikzcd} \tag{7.3.6}$$

In the next section, we explain how to deduce Conjecture 7.3.3 from Conjecture 7.3.2.

A corollary of this conjecture is that the toroidal HOMFLY homology can be computed via coherent sheaves on FComm.

Corollary 7.3.4. *Let β be a cylindrical braid and let L be its closure, a link in the thickened torus $T^2 \times I$. Let F_β be the Rouquier complex. Let $\mathrm{HHH}(L)$ be the toroidal HOMFLY homology of L defined in 4.1.16. Then*

$$\mathrm{HHH}^0(L) = \mathrm{R}\,\Gamma^\bullet(\iota_* F_\beta)$$

up to degree shift.

7.3.1 $n = 1$ Case.

Recall that $\mathcal{H}^1_{\mathrm{ext}} = K^b(\mathbb{Z})$ where $\mathbb{Z}$ is the category defined in 4.2.16. We note that $\mathrm{FComm}_1 = \mathrm{FComm}_1^{\mathrm{cl}} = \mathbb{C} \times (\mathrm{pt}/\mathbb{C}^*)$, so $\mathrm{Coh}_T(\mathrm{FComm}_1) = (t,q)-$graded $\mathbb{C}[x_1]$-mod where x_1 is in degree $(-1,0)$. The functor $\iota^* : \mathrm{Coh}_T(\mathrm{FComm}_1) \to \mathcal{H}^1_{\mathrm{ext}}$ is given by $(-) \otimes \Omega$.

We must consider the additive category $EA\mathbb{S}\mathrm{Bim}_1$ to understand ι_*. Given an object

$$M \otimes \Omega^k \qquad M \in \mathbb{C}[x_1, \delta]\text{-mod}. \tag{7.3.7}$$

the functor ι_* sends it to

$$q^k M \tag{7.3.8}$$

7.3.2 Inductive construction.

Now we show how Conjecture 7.3.3 can be deduced from the earlier conjectures.

Assume the conjecture for all $m < n$.

By the inductive assumption we have functors

$$\mathcal{H}_{\mathrm{ext}}^{n-1} \underset{\iota_{n-1*}}{\overset{\iota_{n-1}^*}{\rightleftarrows}} \mathrm{Coh}_{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{FComm}_n) \tag{7.3.9}$$

satisfying 1 and 2 of Conjecture 7.3.3. Because ι_{n-1}^* exchanges $\mathcal{L}_i$ with W_i, it exchanges $\mathcal{T}_{n-1}$ with $\mathcal{V}_{n-1}$ by part 3 of Conjecture 7.3.2. Since χ and μ are unique, it also exchanges them with X and Y respectively. Hence $\iota_{n-1}^*(\mathcal{E}_{n-1}) = E_{n-1}$. We obtain functors

$$\mathcal{H}_{\mathrm{ext}}^{n} \underset{\iota^*}{\overset{\iota_*}{\rightleftarrows}} \mathrm{Coh}_T(\mathrm{FComm}_n \times \mathbb{C}) \tag{7.3.10}$$

by setting $\iota_* = \iota_{n-1*} \circ \mathrm{Tr}$ and $\iota^* = I \circ \iota_{n-1}^*$. These make $\mathcal{H}^n$ into a category over $\mathrm{FComm}_n \times \mathbb{C}$. Now apply Proposition 6.3.9 using W_n as the invertible object. Take the adjoint of the map $E_{n-1} \xrightarrow{\sim} \mathrm{Tr}(W_n)$ to get a map $I(E_{n-1})$. Note that $I(E_{n-1}) = I(\iota_{n-1}^*(\mathcal{E}_{n-1})) = \iota^*(\mathcal{E}_{n-1})$. Hence we have a map $\iota^*(\mathcal{E}_{n-1}) \to W_n$.

By applying Proposition 6.3.9 we get functors

$$\mathcal{H}_{\mathrm{ext}}^{n} \underset{\iota'^*}{\overset{\iota'_*}{\rightleftarrows}} \mathrm{Coh}_T(\mathrm{FComm}_n). \tag{7.3.11}$$

Since the map $\iota^*(\mathcal{E}_{n-1}) \to W_n$ is an equivalence, we have $\iota^*(\mathcal{L}_n) = \iota^*(\mathcal{O}(1)) = W_n$.

REFERENCES CITED

[1] Roman Bezrukavnikov, Michael Finkelberg, and Victor Ostrik. Character D-modules via Drinfeld center of Harish-Chandra bimodules. *Invent. Math.*, 188(3):589–620, 2012.

[2] Roman Bezrukavnikov. On two geometric realizations of an affine Hecke algebra. Publications math´ematiques de l'IHES´ , 123(1):1–67, 2016.

[3] Andrei Negut, Eugene Gorsky and Jacob Rasmussen. Flag Hilbert schemes, colored projectors and Khovanov-Rozansky homology. Preprint, 2016. arXiv:1608.07308.

[4] Ben Elias. Flattening in type A. In preparation.

[5] Ben Elias and Matthew Hogancamp. Extending isomorphisms of functors to the homotopy category. In preparation.

[6] Ben Elias and Matthew Hogancamp. Categorical diagonalization. Preprint, 2017. arXiv:1707.04349.

[7] Ben Elias and Matthew Hogancamp. Categorical diagonalization of full twists. Preprint, 2017. arXiv:1801.00191.

[8] Ben Elias and Matthew Hogancamp. On the computation of torus link homology. *Compositio Mathematica*, To appear. arXiv:1603.00407.

[9] Ben Elias and Geordie Williamson. Soergel calculus. *Represent. Theory*, 20:295–374, 2016. arXiv:1309.0865.

[10] Dennis Gaitsgory. Construction of central elements in the affine Hecke algebra via nearby cycles. *Invent. Math.*, 144(2):253–280, 2001.

[11] Eugene Gorsky, Alexei Oblomkov, Jacob Rasmussen, and Vivek Shende. Torus knots and the rational DAHA. *Duke Math. J.*, 163(14):2709–2794, 2014.

[12] Jacob Rasmussen. Knots, Polynomials, and categorification. Quantum field theory and manifold invariants, Amer. Math. Soc., Providence, RI, 28, 77-170, 2019.

[13] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[14] Pramod Achar and Laura Rider. Nearby cycles for parity sheaves on a divisor with simple normal crossings. arXiv preprint, 2018. arXiv:1812.07924.

[15] Mikhail Khovanov. Triply-graded link homology and Hochschild homology of Soergel bimodules. *Internat. J. Math.*, 18(8):869–885, 2007.

[16] Marco Mackaay and Anne-Laure Thiel. Categorifications of the extended affine Hecke algebra and the affine q-Schur algebra $\widehat{\mathbf{S}}(n, r)$ for $3 \leq r < n$. *Quantum Topol.*, 8(1):113–203, 2017.

[17] Marco Mackaay, Vanessa Miemietz, Pedro Vaz. Evaluation birepresentations of affine type A Soergel bimodules. arXiv preprint arXiv:2207.02459. 2022 Jul 6.

[18] Shotaro Makisumi. Modular Koszul duality for Soergel bimodules. Preprint, 2017. arXiv:1703.01576.

[19] Alexei Oblomkov and Lev Rozansky. Knot homology and sheaves on the Hilbert scheme of points on the plane. *Selecta Math. (N.S.)*, 24(3):2351–2454, 2018.

[20] Alexei Oblomkov and Lev Rozansky. Affine braid group, JM elements and knot homology. *Selecta Math. (N.S.)*, 24(3):2351–2454, 2018.

[21] Alexei Oblomkov and Lev Rozansky. Soergel bimodules and matrix factorizations. arXiv:2010.14546

[22] Andrei Okounkov and Anatoly Vershik. A new approach to representation theory of symmetric groups. *Selecta Math. (N.S.)*, 2(4):581–605, 1996.

[23] Simon Riche, Pramod Achar, Shotaro Makisumi, and Geordie Williamson. Free-monodromic mixed tilting sheaves on flag varieties. Preprint, 2017. arXiv:1703.05843.

[24] Raphael Rouquier. Categorification of the braid groups. Preprint, 2004. arXiv:math/0409593.

[25] Wolfgang Soergel. Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. *J. Amer. Math. Soc.*, 3(2):421–445, 1990.

[26] Wolfgang Soergel. Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln über Polynomringen. *J. Inst. Math. Jussieu*, 6(3):501–525, 2007.

[27] Ben Elias. Gaitsgory's central sheaves via the diagrammatic Hecke category. arXiv preprint, 2018. arXiv:1811.06188.

[28] Ben Elias, Shotaro Makisumi, Ulrich Thiel, Geordie Williamson. Introduction to Soergel bimodules (Vol. 5). Springer Nature, 2020.

[29] Nicolas Libedinsky. Sur la catégorie des bimodules de Soergel. J. Algebra 320.7 (2008), pp. 2675–2694. url: https://doi.org/10.1016/j.jalgebra.2008.05.027.

[30] Eugene Gorsky, and Andrei Negut. The Trace of the Affine Hecke Category Preprint, 2022. arXiv:2201.07144.

[31] Hugh Morton, Peter Samuelson The HOMFLYPT Skein Algebra of the Torus and The Elliptic Hall Algebra Duke Math. J. 166 (2017), no. 5, 801–854

[32] Matt Hogancamp. Stable homology of torus links via categorified Young symmetrizers I: one row partitions. Preprint, 2015 arXiv:1505.08148

[33] Mikhail Bondal, Mikhael Kapranov. Enhancing triangulated categories. Math Compendium, Vol 181 (1990), No.5, 669–683 (Russian); transl. in USSR Math. USSR Sbornik, vol. 70 (1991), No. 1, pp. 93–107, (MR91g:18010)

[34] Eugene Gorsky, Matthew Hogancamp, and Paul Wedrich. Derived Traces of Soergel Categories. International Mathematics Research Notices 2022, no. 15 (2022): 11304-11400.

[35] David Ben-Zvi, David Nadler, and Anatoly Preygel. A spectral incarnation of affine character sheaves. Compositio Mathematica 153, no. 9 (2017): 1908-1944.

[36] David Ben-Zvi, John Francis, and David Nadler. Integral transforms and Drinfeld centers in derived algebraic geometry. Journal of the American Mathematical Society 23.4 (2010): 909-966.

[37] Dennis Gaitsgory, and Nick Rozenblyum. A study in derived algebraic geometry: Volume I: correspondences and duality. Vol. 221. American Mathematical Society, 2019.

[38] Penghui Li, David Nadler, and Zhiwei Yun. Functions on the commuting stack via Langlands duality. preprint arXiv:2301.02618 (2023).

[39] Eugene Gorsky and Andrei Negut. Personal Communication.

[40] Peter Freyd, David Yetter, Jim Hoste, WB Raymond Lickorish, Kenneth Millett, and Adrian Ocneanu. A new polynomial invariant of knots and links. (1985): 239-246.

[41] Mikhail Khovanov. A functor-valued invariant of tangles, Algebr. Geom. Topol. 2 (2002), 665–741

[42] Alexander Beilinson. Coherent sheaves on Pn and problems of linear algebra. Funkcionalniy analiz i ego pril. 12 (1978), 68–69.

[43] Quoc Ho, and Penghui Li. Graded character sheaves, HOMFLY-PT homology, and Hilbert schemes of points on C2. arXiv preprint, 2023. arXiv:2305.01306.

[44] A.P. Isaev, A. I. Molev, and A. F. Os’ kin. On the idempotents of Hecke algebras. Letters in Mathematical Physics 85 (2008): 79-90