 

# Improving Math Education in Elementary Schools: A Short Book for Teachers

Dave Moursund, moursund@uoregon.edu

Teacher Education, College of Education

University of Oregon, Eugene, Oregon 97403

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http://darkwing.uoregon.edu/~moursund/Math/El Math Book.pdf

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## Contents

 

# Preface

This short book addresses the problem that our elementary school math education system is not as successful as many people would like it to be, and it is not as successful as it could be. It is designed as supplementary material for use in a Math Methods course for preservice elementary school teachers. However, it can also be used by inservice elementary school teachers and for students enrolled in Math for Elementary Teachers courses.

## Procedures and Procedural Thinking

One of the big and unifying ideas in this book is *procedures and procedural thinking.* From the point of view of the elementary school math teachers, a major goal is to help students learn some math procedures and learn how to think in terms of using these procedures to solve problems. The same idea lies at the core of the field of computer and information science. However, there is a difference between how math people and computer people approach the big idea of procedure. They both think about two kinds of procedures:

1. Algorithms. These are step-by-step procedures that can be proved to solve a certain type of problem or accomplish a certain type of task in a finite number of steps. You know algorithms for addition, subtraction, multiplication, and division of integers. You know many other algorithms, such as an algorithm for alphabetizing a list of words and an algorithm for looking up a word in a dictionary, and determining whether it in or is not in the dictionary you are using).
2. Heuristics. These are step-by-step procedures that are designed to solve or help solve a certain type of problem or accomplish a certain type of task, but are not guaranteed to actually do so. As you work to solve a challenging math problem, you likely use heuristic procedures such as draw a picture, look up information in a book, ask a friend, attempt to break the problem into a set of smaller problems, and guess and check.

Computer people think specifically about creating and using procedures that can be carried out by a computer, while math people tend to focus their attention on procedures that can be carried out by people. Of course, many people are both math and computer oriented, and the disciplines of math and computers strongly overlap. People and computers working together can outdo people alone or computers alone in a very wide range of problem-solving situations.

## Some Big Ideas

The math content level of this book is very modest, and the math prerequisite is also very modest. However, there are a number of ideas that require use of higher-order thinking and the ability to quest deeply for meaning and understanding.

For example, think about problem solving. Because this is a math-oriented book for elementary teachers, your first thoughts might be about the types of math problems students learn to solve while in elementary school. But, expand your thinking. Problem solving is an important aspect of every academic discipline. Of course, the nature of problems in the language arts, science, and social sciences is different than the types of problems students learn about in math.

You know that math provides tools useful in problem solving in every academic discipline. This may raise some **Big Ideas** questions in your mind, such as:

1. How do I teach math in a manner that will help my students learn to make use of math as an aid to solving problems in math and in all the other academic disciplines, as well as in other parts of their everyday lives?
2. What is the same and what is different in solving math problems versus solving problems in other academic disciplines?

When solving math problems—or problems in any other discipline—a person makes use of their brain as well as a wide range of tools. Pencil and paper can be thought of as technology-based tools designed to help in representing and solving math problems as well as problems in other disciplines. Here are two Big Ideas related to tools and brains:

3. Computers—more generally, Information and Communication Technology—provide a wide variety of aids to representing and solving problems in all academic disciplines.
4. There is a huge and steadily growing collection of information about the human brain and mind. Research in the areas of brain and mind is making a significant contribution to the science of teaching and learning. [Some good, short articles on brain science and education are available at Brain Connection (n.d.)]

This book provides a brief introduction to math education aspects of the craft and science of teaching and learning. The four Big Ideas help to unify this book. Each chapter ends with a few applications that can be used in teaching math at the elementary school, and then a set of activities targeted at readers of this book.

As with most of my current writing efforts, this book is a "work in progress." It is regularly being added to and revised. Your input and suggestions are welcome.

## One of My Pet Peeves

Finally, I close this Preface with one of my pet peeves. Elementary school teachers often talk to students about getting "the" right answer to a problem. Students grow up with the idea that each math problem has exactly one and only one right answer. This is a wrong concept. Read the math problem examples given below. In the future, I hope you will no longer talk about getting "the" right answer in math.

1. Find two integers that are greater than 1 and less than 10. [There are lots of correct answers.]
2. Find two odd integers that add up to an even integer. [There are lots of correct answers.]
3. Find two even integers that add up to an odd integer. [There are no such integers.]

 

4. Find an integer that lies between 0 and 1. [There is no such integer.]
5. Find a fraction that lies between 0 and 1. [There are lots of correct answers.]

Dave Moursund

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# Chapter 1: Introduction

This book is about the craft and science of teaching and learning math at the elementary school level. The goal of this book is to help improve math educational at the elementary school level. The Preface contains four unifying Big Ideas. It also contains a few math problems designed to convince you to never ever again talk about math being a subject in which the goal is to get "the" right answer to a problem. If you skipped over the Preface, I recommend that you read it now.

At the current time, reading and math are the two most emphasized components of the elementary school curriculum. Although reading and math are taught as two separate and distinct subjects, it is clear that they are related. For example, most teachers are familiar with the idea of "reading in the content areas." And, mathematicians know that math is a language. And, of course, the math curriculum makes extensive use of "word" problems that require students to extract (and then solve) a math problem from a situation described in words.

Throughout our country, there is a top down movement to establish high standards for student achievement in these reading and math, and to improve our educational system so that these high standards are met. In recent years, education has become a political issue, and many politicians want to be considered as leaders of educational reform. They tend to approach educational reform by trying to mandate higher standards through the use of a punishment (and perhaps some possible rewards) type of system.

In educational circles, both reading instruction and math instruction tend to evoke considerable controversy. In essence, the issues are what the standards should be (what students should learn, how this learning should be assessed), and how students should be taught. In reading, there is considerable agreement about the goal of having students achieve an adequate level of reading fluency (speed, accuracy, comprehension) by the end of the third grade so that they can begin to make effective use of reading as an aid to learning throughout the curriculum. The controversy tends to lie in teaching methods, such as phonics versus whole language, and in the content of the materials that students read. Controversy also lies in whom or what to blame because a large number of students do not achieve the reading level fluency goal by the end of the third grade.

In math, both the content and the pedagogy issues remain unresolved. However, there is considerable agreement that the results being produced by our current math education system, whether the approach is "back to basics" or "new-new math," are not nearly as successful as many people would like. Michael Battista provides an excellent summary of the situation in a 1999 article.

> For most students, school mathematics is an endless sequence of memorizing and forgetting facts and procedures that make little sense to them. Though the same topics are taught and retaught year after year, the students do not learn them. Numerous scientific studies have shown that traditional methods of teaching mathematics not only are ineffective but also seriously stunt the growth of students' mathematical reasoning and problem-solving skills. Traditional methods ignore recommendations by professional organizations in mathematics education, and they ignore modern scientific research on how children learn mathematics (Battista, 1999).

Think about the quote from Michael Battista. Is it a good description of your personal math learning experiences? Does the description fit some of the children and adults that you know? Many math education leaders agree that Battista is correct. There is much less agreement about how to make progress in solving this educational problem.

## Math Expertise: Content and Maturity

You have a level of math expertise that you have developed over years of informal and formal study and use of math. Likely you know some people who have greater math expertise than you, and you know some people who have less math expertise than you. You may have an opinion about yourself, such as "I am good at math." or "I am not very good at math." As a perspective elementary school teacher, you need to be concerned about whether your level of math expertise is sufficient to help your future students make satisfactory progress in building their own math expertise.

Math expertise can be divided into two major components: math content and math maturity (see figures 1.1 and 1.2). Much of the math coursework you have taken focused on math content—for example, learning many different arithmetic, algebraic, and geometric procedures and how to use these procedures to solve a wide range of math problems. (Note: The term *procedure* is discussed in the Preface.)

Math maturity focuses on areas such as understanding, solving problems you have not previously encountered, theorem proving, precise mathematical communication, mathematical logic and reasoning, knowing how to learn math, problem posing, transfer of learning (being able to use one's math knowledge and make math connections over a wide range of disciplines and in novel settings), and interest (including intrinsic motivation) in math.

The idea that a math problem may have no solutions, one solution, or more than one solution is part of math maturity. The idea that a solution or a solution process may be more or less clever, beautiful, or elegant is also part of math maturity. Math maturity is an idea that is not specific to any particular content area in math. To a large extent, math maturity does not depend on knowing some specific part of the content of math. A person may have a high level of math content knowledge and a low level of math maturity, or vice versa. I will discuss math maturity more in a later chapter.

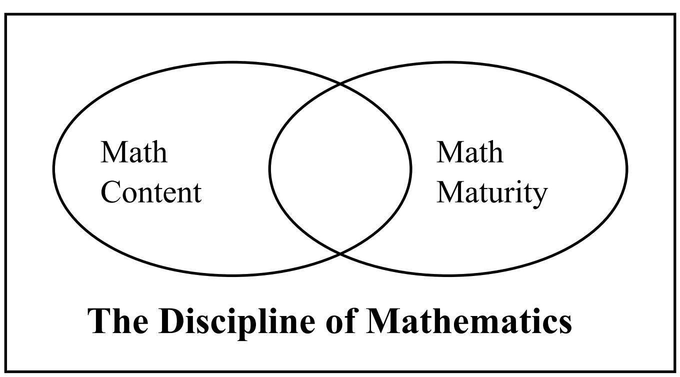



Figure 1.1 Venn diagram showing overlap of math content and math maturity.

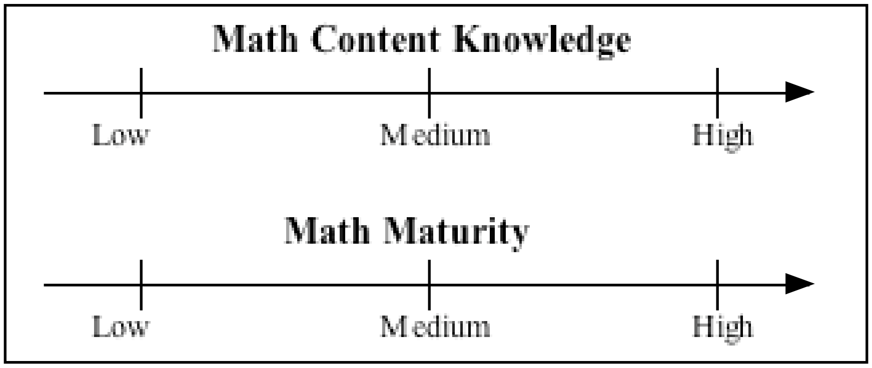



Figure 1.2 Separate expertise scales for math content and math maturity.

## A Good Math Teacher

A good teacher of math has an appropriate level of expertise both in the discipline of math and in the discipline of teaching. Lee Shulman coined the phrase *content pedagogical knowledge* in order to emphasize the importance of a teachers having specific pedagogical knowledge and skills within the disciplines that they teach (Shulman, 1987). Lee Shulman is President of the Carnegie Foundation for the Improvement of Education (Carnegie, n.d.) Figure 1.3 expands on Shulman's work to emphasize *discipline pedagogical knowledge* as one of the keys to good teaching in any discipline.

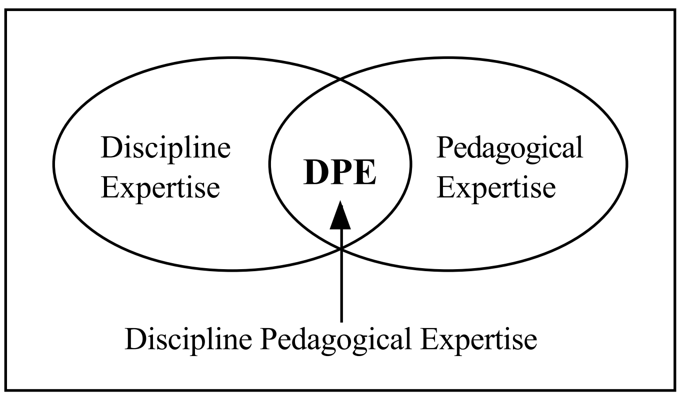



Figure 1.3 Discipline Pedagogical Expertise

To be an effective teacher of math, you need both math content knowledge and math maturity. In addition, you need to know how to teach math—that is, you need math pedagogical knowledge designed to help your students learn math content and gain in their math maturity. Research by Liping Ma (1999) and others suggests that the majority of elementary school teachers in the United States are relatively ill prepared in math pedagogy.

Note: It is evident that not all people agree with the statements in the previous paragraph. Many states have alternative routes to teacher certification that are based mainly on content knowledge. Many people seem to feel that content knowledge is the "be all, end all" to the qualifications needed to be a good teacher. A summary of current research on this issue is available in Emerick et al. (2004).

Elementary school teachers tend to teach math in the way that they were taught. That is, much of what you know about being a teacher of elementary school mathematics you learned while you were in elementary school. This creates a cycle in which the next generation of students is taught in much the same manner as the previous generation. This cycle can and must be broken if the quality of math education that our students receive is to be significantly improved. You, personally, can make a significant difference for your students. The ideas presented in this document will help you.

## Invention of Reading and Writing

Reading, writing, and mathematics are closely connected. The Sumerians (who lived in the area that is now Iraq) developed writing about 5,200 years ago (Acosta, n.d.). This soon led to the development of schools and formal schooling to teaching reading, writing, and arithmetic. While schools have made considerable progress over the years, there is still a considerable similarity between schools 5,000 years ago and schools today.

You are undoubtedly familiar with the curriculum ideas of “reading across the curriculum” and “writing across the curriculum.” Reading and writing are important components of each discipline, and we want students to learn to read and write within each discipline they study. Marilyn Burns is well known for her many math education books (Burns, 1995). The following quotation is from Burns (2004), an article that contains a number of examples of having children write during their elementary school math instruction.

> … and for my first 20 years as a middle school and elementary school teacher, writing played no role in my math teaching.
>
> Today, my view has changed completely. I can no longer imagine teaching math without making writing an integral aspect of student learning.

Later in the article Marilyn Burns explains some of the roles of writing in math instruction:

> Writing in math class supports learning because it requires students to organize, clarify, and reflect on their ideas—all useful processes for making sense of mathematics. In addition, when students write, their papers provide a window into their understandings, their misconceptions, and their feelings about the content there’re learning.

Marilyn Burns then goes on to describe general categories of writing in math, including keeping journals, solving math problems, explaining math ideas, and writing about learning processes. She argues that such writing is an important component of a modern math education.

In my opinion, reading, writing, speaking, and listening in math all fall into the general area of math maturity. By now, in your reading of this chapter, you should have a good start on construction your own working understanding of the term *math maturity*. To self-assess your progress, imagine that a parent of one of your students asks you what it means for a student to be gaining math maturity. What would you say?

## Math, a Human Endeavor

From a historical point of view, writing has facilitated a steady accumulation of human knowledge, including math.

> God created the integers; all the rest is the work of man. (Leopold Kronecker, 1923-1891)

The quotation from Kronecker captures the idea that math is a steadily growing discipline. The invention of writing has made possible more than 5,000 years of growth of and accumulation of mathematical knowledge that can be shared with others. You know lots of things about math that people did not know 5,000 years ago. For example, you know about and make use fractions, the number zero, the decimal point, and decimal notation. You also make routine use of applications that are strongly based on math. For example, you tell time using a digital or analog watch. You use money. You understand the concept of distance and you know how to make use of instruments such as a ruler to measure distance.

Math has become so important in our society and so routinely used in our society that children begin to learn math well before they enter kindergarten, and math is a required part of the school curriculum well into high school. Most colleges require students to take some math, and most likely you have taken a sequence of courses titled Math for Elementary Teachers. Your informal and formal studies and use of math have led to your current level of math expertise.

> Mathematics is one of humanity's great achievements. By enhancing the capabilities of the human mind, mathematics has facilitated the development of science, technology, engineering, business, and government. (Kilpatrick, Swafford, and Findell, 2002)

The idea that people created math and that math is a human endeavor are thoroughly embedded in many books about math. Indeed, Harold Jacobs wrote a secondary school math book titled *Mathematics, a Human Endeavor* that has been widely used (Jacobs, 1994). There is a tremendous amount of materials about the history of math available on the Web (History Topics Index, n.d.).

## Elementary School Applications

Each chapter except the final one contains a few ideas for classroom applications at the elementary school level. These are meant to be suggestive, and are by no means comprehensive. Each teacher will need to build their own pieces of curriculum, instruction, and assessment to appropriately implement the ideas. An underlying goal in the use of such classroom applications is for you, the teacher, to learn more about how the minds of your students work.

1.1 Ask your students, "What is math?" Younger students can provide oral answers, while older students can both talk about and write on this topic. Look for responses that seem to focus on math content and other responses that seem to focus on math maturity. Use responses to carry on class discussion designed to broaden student insights into the discipline of math. If your students mention the idea of getting "the" right answer, use that as a teachable moment to increase their mathematical maturity.

1.2 Ask your students, "How do you learn math?" Use responses to help students gain insights into the fact that there are a variety of ways to learn math, and that different students may learn math in different ways. As a variation on this question, explore student insights into how one knows that they have learned a math topic (or, indeed, a topic in any discipline) well enough. You might get an answer such as, "When I get a good grade on the test." If that answer comes up, you have a teachable moment. I am assuming that you agree that there is much more to learning and understanding than getting high scores on tests.

## Activities for Self-Assessment, Assignments, and Group Discussions

The last section of each chapter contains activities that can be used in teaching from and/or learning from this book.

1.1 Think about your own elementary school math education experiences and what you have observed in visits to elementary school since then. What seems to be working well and what does not seem to be working well? Be as specific as possible.

1.2 Think about your knowledge and experience in the areas of reading math (reading in the content areas) and writing math (writing in the content areas). How is such reading and writing the same as and different from just plain reading and writing?

1.3 This book is specifically designed for preservice teachers who are currently taking a math pedagogy course. Prior to this, such students have had years and years of instruction in mathematics. When I think about this, I conclude that most of what they know about math pedagogy will have come from what they happened to pick up through their years and years of math coursework. Share your thinking about this situation. What might you be learning in your teacher education program of study that will help to break the model of teachers teaching math in the way that they were taught?

# Chapter 2: Academic Disciplines

Elementary school teachers are responsible for teaching a wide range of disciplines such as art, language arts, math, music, science, and social science. Although the focus of this book is on math, let's begin by taking a more general approach. What is a discipline, and how does one distinguish between disciplines?

Each discipline can be defined by its unique combination of:

- The types of problems, tasks, and activities it addresses.
- Its tools, methodologies, and types of evidence and arguments used in solving problems, accomplishing tasks, and recording and sharing accumulated results.
- Its accumulated accomplishments such as results, achievements, products, performances, scope, power, uses, impact on the societies of the world, and so on.
- Its history, culture, language (including notation and special vocabulary), and methods of teaching, learning, and assessment.

When you read this list, did you just "bleep" over the details, or did you pause at each bulleted item and reflect on its meaning to you and to our educational system? Did you select a discipline that you know well and check on your insights into how each of the listed items fits or fails to fit your knowledge of the discipline? Did you think about what is the same and what is different between human natural language and the language of mathematics or the language of music? Did you think about what items you think should be added to the list, and what might be deleted?

For example, what is the same and what is different between a math problem and a problem in art, language arts, health, math, music, or social science? What is the same and what is different when using math to solve a math problem versus using math to help solve a problem in art, language arts, etc.?

Research in brain science is beginning to give us important insights about one of the things that is the same across various disciplines. The brain learns by storing patterns. When a person encounters a problem situation, his or her brain attempts to match the perceived pattern of the problem situation with one or more stored patterns. If an appropriately closely similar stored pattern is recognized, this becomes a starting point for dealing with the problem situation. Thus, learning and problem solving in all disciplines have to do with developing and storing brain patterns, in pattern matching or pattern recognition, and in making use of stored patterns (Goldberg, 2005).

## Reading in Various Disciplines

You know how to read, and you have had experience in reading in many different disciplines. That does not automatically mean that you are skilled in reading in each discipline, or that you are skilled in reading to learn within a specific discipline. The bulleted discipline-definition list is full of relatively complex words and ideas. It is easy to read such a list and gain almost no understanding of the information it is attempting to convey. Reading for deep understanding and learning is a lot different than reading for entertainment. Students need to learn to read in a reflective manner that leads to learning and understanding. They need specific help and practice in learning to read in different disciplines.

Reflect on your learning experiences in learning to read math! For example, when you were in the fifth grade, did you have access to the math books used while you were in the third grade, so that you could look up and perhaps relearn some of the material covered in the third grade? Do you currently have easy access to the math books you used in middle school and secondary school, so that you can use them as needed?

The purpose of these questions is to get you to begin thinking about how the Web is changing education. The Web can be thought of as a global library. Part of a good education is learning how to make use of libraries. You surely want your students to learn to read math well enough so that they can make use of the Web to find and read math information that they happen to need at some particular time.

What follows are a few examples of sample questions from third grade math assessment in various states in the United States (Brainchild, n.d.). Examine them from the point of view of the math-oriented reading level that these questions assume. Some of the problems seemed poorly (indeed, ambiguously) worded to me.

### Example 1: Math, 3rd Grade

Julia made a chart showing the length of each of the six longest rivers in the world. Which river listed on the chart is the third longest?

| River | Length |
|---|---|
| Amazon | 3,900 miles |
| Danube | 1,770 miles |
| Mississippi | 2,350 miles |
| Nile | 4,150 miles |
| Volga | 2,300 miles |
| Yangtze | 3,450 miles |

A. Volga

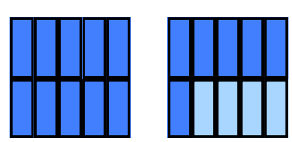
B. Mississippi

C. Danube

D. Yangtze

### Example 2: Math, 3rd Grade

Which decimal tells how much is shaded?



A. 1.06

B. 1.4

C. 1.6

D. 10.6

### Example 3: Math, 3rd Grade

Jackie's dad baked 36 chocolate chip cookies and 24 peanut butter cookies on Monday. On Tuesday, he baked 12 cherry chip, and 15 mixed nut cookies. Jackie reaches into the cookie jar and pulls out a cookie. Which kind of cookie is she least likely to pull out?

A. mixed nuts

B. cherry chip

C. peanut butter

D. chocolate chip

### Example 4: Math, 3rd Grade

In which pair is the second number 10 greater than the first number?

A. 147 and 157

B. 156 and 146

C. 324 and 1324

D. 234 and 334

### Example 5: Math, 3rd Grade

The second grade class ordered five pizzas with mushrooms and pepperoni. The third grade class ordered eight pizzas with pineapple, ham, and mushrooms. The fourth grade class ordered four pizzas with olives, mushrooms, and onions. What is the most popular topping on these pizzas?

| Grade | Pizzas Ordered | Topping |
|---|---|---|
| 2 | 5 | Mushroom Pepperoni |
| 3 | 8 | Pineapple, Ham Mushrooms |
| 4 | 4 | Olives, Onions Mushrooms |

A. pepperoni

B. pineapple

C. olives

D. mushrooms

The reading skills of third graders vary considerably. In a typical third grade class one might to expect reading skills to vary from about 1st grade to 5th grade or higher. I assume that the reading level of these questions is at the 3rd grade level.

But each question requires far more than just reading the words. The student must gain some understanding of the problem. And, solving the problem is a multi-step process. The level of arithmetic computational skill required to solve the various 3rd grade problems is quite modest.

These examples illustrate only one aspect of reading in math. Math reading gets still more complex as the text includes a wider range of math symbols, math vocabulary, geometric shapes and arguments based on the precise vocabulary and logic of math, and so on. Think about the increasing reading complexity as you read the questions given below.

**Example 6: Math, 5th Grade**

Two congruent triangles are plotted on a coordinate plane. Which of the following is **not** true?

A. Their angles are equal.

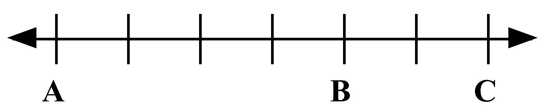
B. Their sides are equal.

C. They are the same type of triangle.

D. Their coordinates are the same.

**Example 7: Math, 7th Grade**

What percent of AC is AB?




A. 60%

B. 40%

C. 50%

D. 20%

**Example 8: Math, 9th Grade**

Which of the following are **not** equal to each other?

A. 0.75 and 3/4

B. $\sqrt{25}$ and 5

C $2^4$ and $4^2$

D. 1/5 and 0.5

## Your Knowledge of Medicine vs. Your Knowledge of Math

The bulleted discipline-definition list given earlier in this chapter includes the idea of accumulated accomplishments within a discipline. Just for the fun of it, think about your knowledge of the accumulated accomplishments in medicine. You know quite a bit about a wide range of diseases, germs, bacteria, viruses, a wide range of drugs and vaccines, various types of surgery, and so on. You know some things about DNA, cloning, and genetic engineering. Perhaps you know your blood type, and that there are different blood types. Your accumulated knowledge in medicine is well beyond that of the best physicians and medical researchers of a few hundred years ago.

Now, contrast that with your knowledge of the accumulated accomplishments in math. Can you name some of the accumulated accomplishments of math, and how does your list compare to your knowledge of medicine? (Remember, Isaac Newton and others developed calculus about 350 years ago, and its mathematical foundations go back a long time before then.)

What might you conclude from this activity? Medicine is an important and ongoing part of your life. You have learned a lot about the discipline of medicine through your informal efforts and the efforts of our schools. This is because medicine is relevant to your everyday life. Think about what aspects of math are relevant to your everyday life. Think about what aspects of math are relevant to the everyday lives of elementary school students. What might you and other elementary school teachers do to make math more relevant across the entire curriculum and in the lives of your students?

## Big Ideas

This book contains a relatively high density of Big Ideas. If you read this book in the same manner and at the same rate as you read a short story or a novel, you will gain very little from it. To gain appreciable benefit from reading this book, you will need to read in a reflective manner, pausing frequently to think about what you already know and how it fits in with what you are reading. You will need to construct meaning that integrates into and adds to your current knowledge and understanding. That is, you will need to practice constructivist learning (Ryder , n.d.).

In essence, that is what the learning theory called constructivism is all about. Constructivism is a learning theory applicable to learning in each discipline. It is a theory about developing patterns in one's brain, and then building on these patterns. It is important in the teaching and learning of math, as well as all other disciplines. Thus, you might want to spend a little time thinking about your preparation to help your future students learn math (and other disciplines) in a constructivist manner (Math Forum, n.d.).

The activities in this document are intended to encourage you to think, and to think about your thinking. Thinking about your thinking is called metacognition. It is an important component of formal and informal education at all grade levels.

More specifically, the ideas in this document are intended to encourage you to think about what you, personally, can do to improve our educational system. If this document does not lead to you, personally, making changes designed to improve upon the "traditional" curriculum and methods of instruction, then this document will have failed as an aid to improving your education.

Teaching is a very challenging and demanding profession. Good teachers are always learning and growing professionally. You may find it useful to make a copy of the discipline-defining bulleted list so you can refer back to it as you develop lesson plans and as you engage in your everyday activities as a (constructivist) teacher.

Moreover, you should structure your professional career as a teacher to allow significant time for learning. There is a huge amount of research and practitioner knowledge on the craft and science of teaching and learning. Bransford (2000) provides an excellent overview of this field, and the book can be read free from the Website listed in the reference.

On September 30, 2004 the National Science Foundation announced it had committed $36.5 million to fund three major research centers in the area of Learning About Learning (NSF, 2004). Quoting from this announcement:

> How do we learn? This most fundamental ability comes about through the complex interplay of genes, brain-based neural mechanisms, developmental trajectories, and social and physical environments. These processes of learning are just beginning to be understood. A deeper

> understanding of learning will allow scientists and educators to devise methods for improving how humans learn and develop machines that can perform tasks intelligently and independently.
>
> NSF has launched the new Science of Learning Centers to meet the challenge of learning about learning. Their goal is to make new discoveries about the foundations of learning across a wide range of learning situations—from processes at the cellular level to complex processes engaging different brain areas, to behaviors of individuals, to interactions in the classroom, to learning in informal settings, to learning performed by computer algorithms.

Learning about learning is an important research topic. However, it is also a core component of learning throughout all formal and informal education. One of your jobs as a teacher of mathematics is to help your students gain increasing knowledge and skills about how to learn math—that is, help them to increase this aspect of their math maturity.

## Elementary School Applications

2.1 Carry on a discussion with your class about two or three of the subjects (disciplines, with math being one of the disciplines) they are learning about. Students are to talk about how the subjects are the same and how they are different. They are to talk about how one shows knowledge and skill in each of the subjects, and how that is the same and different. They are to talk about which subject is the most fun and which is the least fun, and why. As you listen to and participate in this conversation, listen for comments about problems and problem solving. If a student talks about problem solving in a non-math discipline, or if no student mentions this idea, use this as a teachable moment to expand on the fact that problem solving is part of every discipline.

2.2 Carry on a discussion with your class about uses they have made of things learned in school. Help them to explore what it means to make use of things they are learning. A use might be just bringing a topic up in a conversation with parents, siblings, or others. Or, it might be to answer a question, help solve a problem, or help accomplish a task. Make sure the discussion includes a focus on uses they have made of math learned in school. As you listen to their comments about use of math, pay particular attention to whether the applications are based on what they are learning in school, or whether they can be learned and used without going to school. Children throughout the world who grow up in environments that do not include formal schooling still manage to learn a lot of math that they use in an everyday basis.

## Activities for Self-Assessment, Assignments, and Group Discussions

2.1 Spend some time thinking about the bulleted discipline-defining list from the point of view of your preparation to teach the various subjects you will teach in elementary school. Select the discipline that you feel you know best, and summarize your discipline-specific knowledge and skills from the point of view of the four bulleted items. Then do a compare and contrast with a second discipline. Share some of your insights and feelings from doing this activity.

2.2 Think about the elementary school math curriculum that was in place when you were in school and/or that you have observed in more recent visits to schools. Discuss some of the aspects of the discipline of mathematics that are in the curriculum, some aspects that you feel should be added to the curriculum, and some aspects that you feel should be deleted from the curriculum.

2.3 Suppose that you decide to stop reading this book right after reading Activity 2.3. Name some things that you have learned that can make a significant contribution to improving the quality of education that your future students will receive. Then think about variations of this question that you might use with your students. After your students complete a lesson or a unit, ask them to talk about what they have just learned that makes a significant difference to them.

# Chapter 3: The Discipline of Mathematics

It is not easy to give a useful and simple answer to the question: What is mathematics? Many mathematicians and math educators have attempted to answer this question. Here are two examples.

> Mathematics is an inherently social activity, in which a community of trained practitioners (mathematical scientists) engages in the science of patterns—systematic attempts, based on observation, study, and experimentation, to determine the nature or principles of regularities in systems … The tools of mathematics are abstraction, symbolic representation, and symbolic manipulation. However, being trained in the use of these tools no more means that one thinks mathematically than knowing how to use shop tools makes one a craftsman. Learning to think mathematically means (a) developing a mathematical point of view—valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade, and using those tools in the service of the goal of understanding structure—mathematical sense-making (Schoenfeld, 1992).

> … Hence, mathematical education should be centered on encouraging students to think for themselves: to conjecture, to analyze, to argue, to critique, to prove or disprove, and to know when an argument is valid or invalid. Perhaps the unique component of mathematics which sets it apart from other disciplines in the academy is proof—the demand for succinct argument that from a logical foundation for the veracity of a claim (Padraig & McLoughlin, 2002).

## Proof

The word *proof* comes up in most attempts to define mathematics. Of course, the idea of proof or proving something is not restricted just to mathematics. A trial lawyer attempts to prove his or her case. A person attempts to prove that another person is wrong in a particular situation. Alternatively, the person attempts to prove that he or she is right.

Each discipline has its own ideas and standards about what constitutes a proof. Math proofs are designed to answer, once and for all, the correctness or incorrectness of a "mathematical" assertion. Suppose, for example, that I am exploring the sum of three consecutive integers. I see that 6 + 7 + 8 = 21, and 11 + 12 + 13 = 26. After looking at a lot of examples, I conjecture that if the first integer is odd, then the sum is an even integer; if the first integer is even, then the sum is an odd integer. Looking at lots of example, and not finding any counter examples, may increase my confidence that my conjectures are correct. But, my failure to find a counter example does not constitute a proof. Think about definitions of odd and even integers. See if you can construct a convincing proof that my conjectures are correct.

Then think about whether elementary school students, once they have encountered definitions of odd and even integers, might be able to develop convincing proofs. If the conjecture given above is too difficult for students at a particular age, how about considering the simpler conjecture that the sum of two even integers is even, or that the sum of two odd integers is even. A young child attacking these tasks might make use of small cubes, physically lining up rows of cubes to represent integers, and then arguing from the patterns that result.

Finally, be aware that there are lots of simple proof-type situations that can be constructed for use in the elementary school setting. To give one more example, suppose that students have learned the mathematical word *mean.* You might then have them computer the mean of various sets of three consecutive integers, looking for a pattern. Quite likely some of the students will note that the answers they obtain are always the middle of the three consecutive integers. Can they construct a convincing argument that this is always the case? What if one wants to find the mean of five consecutive integers?

### Fluency and Proficiency

The terms fluency and proficiency are often used in talking about goals and expertise in mathematics. The following definition of math proficiency is quoted from *Adding It Up* (2001), a report written for the National Academy of Sciences.

> Mathematical proficiency, as we see it, has five components, or *strands:*
>
> - *conceptual understanding*—comprehension of mathematical concepts, operations, and relations
> - *procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
> - *strategic competence*—ability to formulate, represent, and solve mathematical problems
> - *adaptive reasoning*—capacity for logical thought, reflection, explanation, and justification
> - *productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

Warning! The mathematical proficiency bulleted list reflects many hundreds of hours of thinking by some of the world's leading math educators. Did you read it in a reflective manner? Did you work to construct your own meaning? What aspects of the presented ideas will you remember five minutes from now, a day from now, or a year from now?

For the most part, answers to the "what is math" question do not depend on specific areas of math content. The question and answers are part of math maturity. As you think about the mathematical proficiency bulleted list, you are working to increase an aspect of your math maturity that is very important to being a good teacher of math. As you construct and/or make use of a math lesson plan, you can think about how it fits in with and contributes to increasing your students' mathematical proficiency.

## Weaknesses in Our Math Education System

Many people argue that our math education system is not as good as it could be. They argue that students are not acquiring a sufficient level of math proficiency. Deborah Ball was the chair of a group of people studying the development of proficiency in math.

> Developing proficiency in mathematics is important for all students. However, when considered in light of current standards, or compared with performance in other countries, evidence on student achievement in mathematics makes clear the need for substantial improvement. U.S. students do not, as a group, achieve high levels of mathematical proficiency. The nation must seek to narrow the achievement gaps between white students and students of color, between middle-class students and students living in poverty; gaps that have persisted over the past decade (Ball, 2002).

Over the years, there have been several important international studies that help us understand math education in the United States versus math education in other countries.

> If you look at state and national assessments of math and science competence among our country's elementary and secondary schools today, you'll discover small pockets of excellence amid a broad swath of mediocrity. In fact, only a minority of U.S. students are meeting math and science proficiency benchmarks.
>
> International assessments from the Trends in International Mathematics and Science Study (TIMSS ) show U.S. students are at or below the international average and significantly behind their peers in Japan and Canada. TIMSS compared our most advanced students with those from 15 other nations, and the brightest U.S. students scored dead last against international competitors in advanced math and physics assessments (Ruetters, 2002).

## Math Education Reform

There is a significant battle going on between the “back to basics” math education reformers, and the “new math (new-new math) reformers. The Mathematically Correct (n.d.) Website presents arguments against the ideas of the new-new math education reformers. Quoting from their Website:

> Mathematics achievement in America is far below what we would like it to be. Recent "*reform*" efforts only aggravate the problem. As a result, our children have less and less exposure to rigorous, content-rich mathematics.
>
> The advocates of the new, fuzzy math have practiced their rhetoric well. They speak of higher-order thinking, conceptual understanding and solving problems, but they neglect the systematic mastery of the fundamental building blocks necessary for success in any of these areas. Their focus is on things like calculators, blocks, guesswork, and group activities and they shun things like algorithms and repeated practice. The new programs are shy on fundamentals and they also lack the mathematical depth and rigor that promotes greater achievement.

The Standards produced by the National Council of Teachers of Mathematics (NCTM, n.d.) represent the sense of direction of new-new math reform. The NCTM Standards are divided into five content standards and five process standards. None of the ten standards say anything about computation in their titles. The ten NCTM Standards contain a total of 33 goals. Exactly one of the 33 goals talks about computation—the traditional focus of much of the elementary school math curriculum! This particular goal statement is the Numbers and Operations standard, and it says “compute fluently and make reasonable estimates.”

More general information about proposed math reforms is available in Mathematically Sane (n.d.).

Perhaps the most important thing to understand about math education reform is that it is complex and controversial. Progress in brain science and in the field of computers and information science contribute to this complexity. Lots of people feel that our math education system needs to be changed. Different stakeholder groups have widely varying opinions on the types of changes that will produce an increased level of mathematical proficiency in our students.

## Elementary School Applications

3.1 Pick a simple math exercise that is appropriate to the math level of your students. For example, the exercise might we, “What is two plus three?” at a first grade level. After you students agree on an answer, carry on a discussion using questions such as: A) How do you know that this is a correct answer?; B) Is there more than one correct answer?; and C) How would you go about changing the mind of someone who thinks that this is a wrong answer?

3.2 Take a careful look at a math unit that you have taught or are preparing to teach. Think about what you want your students to gain in *conceptual understanding, procedural fluency, strategic competence, strategic competence, adaptive reasoning, and productive disposition.* Analyze the math unit from the point of view of how it contributes in these five different areas.

3.3 Take a careful look at a math unit that you have taught or are preparing to teach. Analyze the math unit from the point of view of how it contributes in these five different areas.

- comprehension of mathematical concepts, operations, and relations
- skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- ability to formulate, represent, and solve mathematical problems
- capacity for logical thought, reflection, explanation, and justification
- habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

3.4 Ask your students if they can think of a math problem that has more than one right answer. The goal is to lead the class to find examples that are appropriate to their current level of understanding of math. You may need to provide a first or second example before the class is able to generate additional example. First graders can deal with, "Find two counting numbers that add up to six. Somewhat older students can deal with, "Find two counting numbers that multiply together to give 12." and "Using the unit squares, make a rectangular pattern whose area is 24."

## Activities for Self-Assessment, Assignments, and Group Discussions

3.1 As an elementary school teacher, you will likely encounter both the back to basics approach to math education and the new-new math approach to math education. Think about how easy it is to fall back into the mode of teaching the way you were taught (thus, revert to a focus on computation and the other basics), versus learning and teaching a new-new math curriculum. Spend some time making a list of topics and ideas that you feel are new-new math, and a list of topics and ideas that you feel are stressed by the back to basics movement.

3.2 Review the "what is math" quotations given in this section. Think about which (if any) of the ideas in these quotations can be integrated into elementary school math. Think about this from the point of view of, "The way the twig is bent is the way the tree will grow." Argue for and against the idea that elementary school math should place much less emphasis on paper and pencil computation and much more emphasis on topics that lay a different type of foundation for students as they continue to study math in middle school, high school, and beyond.

3.3 Name one big and important idea from this chapter that you are apt to remember and make use of as a teacher of math. What is it about this idea that resonates with you and is likely to stay with you?

# Chapter 4: Mathematical Maturity

One of your goals as a teacher is to help your students increase their levels of expertise within the various disciplines you teach. To be an effective math teacher, for example, you need an appropriate balance of math content knowledge and math maturity as you help your students to gain both increasing math content knowledge and skills, and increasing math maturity. You also need both general pedagogical expertise and math-specific pedagogical expertise.

Figure 4.1 contains two expertise scales. A teacher or a student may be at substantially different levels on these two scales. An appropriate balance (between the two scales) for one person may not be appropriate for another person, since it depends on interests, abilities, goals, and so on. My personal opinion is that our math education system places much more emphasis on math content than on math maturity. I conjecture that for most students, this leads to an inappropriate balance between these two aspects of math expertise. However, I am not aware of specific research that either supports or argues against my conjecture.

> You may wonder what research-oriented math educators do. One answer is that they formulate hard math education research questions that have not been previously answered, and they attempt to answer them. Consider the challenge of doing research in the area of "balance" between a student's math content knowledge and math maturity. Do we have a good definition of math content knowledge and good measures of a student's math content knowledge? Do we have a good definition of math maturity and good measures of a student's math maturity? What might we mean by saying that for a particular student, the student's math content knowledge is appropriately in balance with the student's math maturity? What types of instructional interventions do we have available that lead to relatively precisely measurable increases in math content knowledge or in math maturity? As you can see, this is a complex and challenging area of research.
>
> What I find particularly interesting is that ordinary, everyday math teachers are expected to take appropriate classroom action in this content vs. maturity area, even though the needed research has not been done.

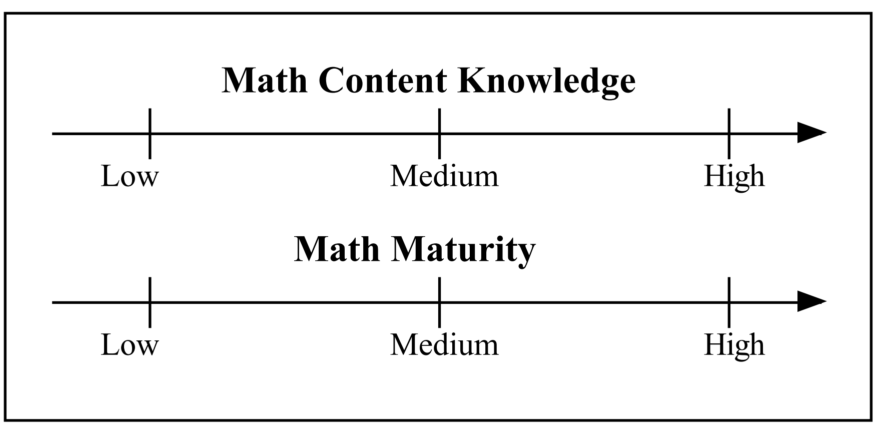



Figure 4.1. Scales for math content knowledge and math maturity.

## Math Content

There is considerable agreement about the scope and sequence of PK-12 math education content in the US. At the elementary school level, for example, a modest number of textbook series capture most of the market. This also holds true at the secondary school level and on into higher education. Clearly, one measure of a person's progress toward increasing Math Content Expertise is the level of coursework that has been completed, the grades received in these courses, and the quality and rigor of the coursework.

However, math can be learned through other ways than just taking courses. Moreover, there is a large amount of math that is not included in the commonly available coursework. And, although a modest number of textbook producers tend to dominate the market, there are many other materials available. Moreover, many teachers do not rigorously follow the textbooks adopted by their school districts.

Finally, (long pause, drum roll), we need to remember that many students forget most of the math they have "covered" in their math courses. Through introspection, you can decide to what extent you have forgotten much of what was covered in some of your high school math courses, such as the geometry course that most likely had.

Teachers of math tend to be driven by the need to "cover" the curriculum, to "get through" the book and the planned lessons. They do this even though they know that students will forget most of what is covered. As I reflect about this situation, I tend to feel embarrassed about much of the teaching that I have done over the years.

## Components of Math Maturity

The term math maturity is widely used by mathematicians and math educators. For example, a middle school teacher may say, "I don't think Pat has the necessary math maturity to take an algebra course right now."

Math maturity is **not** primarily knowledge of specific math content areas or skill in memorizing and accurately using arithmetic and other math procedures. Perhaps Pat is weak in math reasoning, tends to learn math by rote memorization, has little interest in math, and shows little persistence in working on challenging problems.

Here is a list of some possible components of math maturity. Note that one can argue that each is "merely" a component of Math Content Knowledge. However, when people use the term math maturity they tend to be interested in those aspects of the topics listed below that are not dependent on specific Math Content Knowledge.

1. An understanding of the math that one has had adequate opportunities to learn. A good way to think about this is in terms of lower-order versus higher-order knowledge and skills. An increasing level of understanding is reflected by movement toward the higher-order end of the scale. Bloom's Taxonomy is a six-level scale: knowledge, comprehension, application, analysis, synthesis, and evaluation. Although developed about 50 years ago, is still a useful aid in understanding lower-order and higher-order (Bloom's Taxonomy, n.d.).
2. Considering mathematics as a language suggests three related components of Mathematical Maturity:
    - A. Mathematical speaking and listening fluency.
    - B. Mathematical reading and writing fluency.

C. Thinking and reasoning in the language of mathematics. Gary Marcus (2004, p. 124) indicates that thought and language are only loosely connected. Many mathematicians and other people clearly develop and make use of mental representations (mental images, mental pictures) that are not words. For example, Albert Einstein, when describing his discovery of special relativity said: "Words and sentences, whether written or spoken, do not seem to play any part in my thought processes. The psychological entities that serve as building blocks for my thoughts are certain signs or images, more or less clear, that I can reproduce and recombine at will." (Marcus, 2004, p219.)

3. Ability to pose and represent math problems, and to ask insightful mathematical questions. This includes the ability to recognize math aspects of a problem situation in a wide range of disciplines and represent them mathematically.
4. Ability to effectively use one's Math Content Knowledge to solve or help solve the types of math problems that arise in (3) above. Making connections within mathematics, and transfer of one's math learning to other disciplines.
5. Ability to learn mathematics, and to build upon one's current mathematical knowledge. In the field of reading, people talk about learning to read and then reading to learn. In our current education system approximately 70% of students learn to read well enough by the end of the third grade so that they can use their reading knowledge as a significant aid to learning in other disciplines. In a parallel to this, we can think about "learning to math and them mathing to learn." One important goal in math education is for students to gain enough knowledge and skills in math so that they can make effective use of their knowledge and skills to learn more math and to use their math across the disciplines.
6. Other factors affecting Math Maturity include attitude, interest, motivation, focused attention, perseverance, and acceptance of and fitting into the "culture" of mathematics.

Lets look at a few examples to help illustrate some of the components and concepts of Math Maturity. Consider your content knowledge of multi-digit multiplication and long division. You have memorized and extensively practiced paper and pencil algorithms for these two mathematical operations. How do these two algorithms relate to your understanding of arithmetic, math in general, connections within math, transfer outside of math, and so on?

When you think about multiplication of two numbers, perhaps some sort of mental model (pattern, picture, procedure) pops from your long term memory into your conscious, working memory. Think about what "pops into your head." Is it applicable to multiplying decimal fractions or other fractions? Is it applicable to multiplying irrational numbers? Is it applicable to multiplying algebraic expressions? Is it applicable to multiplying functions? The point is, multiplication is a mathematical concept that is an important component of many different parts of math. When you are teaching students about multiplication of integers, you are helping them to build a mental pattern (a "chunk of knowledge that they label "multiplication") that they will construct additional knowledge and understanding upon in the future.

How might one go about teaching multiplication and division so that students "really understand" these two topics? Or, perhaps you believe that learning math is mainly an activity of memorizing without understanding, and that the math education you received while in elementary school was just fine? How might handheld calculators fit into this discussion?

As a second example, consider the idea of dividing by fractions. Probably you have memorized an algorithm that is summarized by "invert and multiply." Think about your understanding of this procedure. Can you explain it, use it, and justify or prove it? Is it applicable in algebra, or does it just work in arithmetic? Can you give practical examples that are meaningful to you and to the students you will teach of when and why one might want to divide by a fraction?

Next, consider what you know about plane geometry. The chances are that you had a year length course on this topic while in high school, and that you had additional instruction on this topic while taking a Math for Elementary Teachers course. Think about what you remember, and attempt to divide it into the "content" and "maturity." This is not an easy task. But, for example, perhaps you remember the general concept that there are theorems and that you studied proofs of many different theorems. Perhaps you remember that there are paper and pencil straight edge and compass constructions. There is a good chance that you remember some of the vocabulary, but that you may not remember how to prove very many theorems or do very many of the constructions. What are your current thoughts on why one might want to learn to state and prove some theorems in plane geometry and to be able to do straight edge and compass constructions?

## Problem Posing and Question Asking

Posing math problems and asking math questions constitute one of the most important topics in the math maturity list, and this topic is often overlooked in the math curriculum. In January 2004 the NCTM issued the following call for papers for an October 2005 Focus issue of Teaching Children Mathematics:

> The Editorial Panel of *Teaching Children Mathematics (TCM)* is seeking manuscripts that discuss or exemplify the role of problem posing and problem solving in the pre-K–6 mathematics classroom. The importance of this focus topic is reflected in NCTM's *Principles and Standards for School Mathematics*, which calls for teachers to regularly ask students to pose and solve interesting problems based on a wide variety of situations. By highlighting problem posing and problem solving, the Editorial Panel aims to provide teachers and teacher educators with resources to assist in their efforts to integrate problem posing and problem solving in the pre-K–6 mathematics classroom. Although problem posing and problem solving go hand in hand, manuscripts that specifically address problem posing are welcome. Accessed 11/2/03: http://my.nctm.org/eresources/article_summary.asp?URI=TCM2004-01-253a&from=B.

Of course, posing problems and asking questions are an essential component of every discipline.

> In any discipline, it is essential to help students understand our ignorance. They should come to appreciate the range of questions that remain open and, most importantly, the fact that countless interesting questions have yet to be thought of. Such an understanding is an invitation to join in the discussion. **When teachers present mathematics as a predetermined set of facts to be transmitted, the implicit message is that students are separate from those who created the mathematic** (Problem Posing, n.d.; bold added for emphasis.).

Learning to pose and/or recognize math problems and math questions in "real world" and school settings contributes to understanding of math and transfer of learning of one's math knowledge. There is a substantial amount of literature on math problem posing that can be accessed from the Web. Using the search engine Google and the search term *problem posing mathematics*, I got 122,000 hits on 12/28/04. The literature indicates that math problem posing has been extensively studied, can be used at all grade levels and in college, can be an important component in a Math Methods course, and is a challenge to teachers.

As an example of this challenge, many discussions about what is mathematics include a statement about finding patterns. To expand on this a bit, think about the mathematics involved in finding a possible pattern, describing the pattern, testing if the description seems to be accurate, conjecturing that the description is accurate, and proving that the description is correct. Perhaps I am a student at an early level in grade school. I have just learned about odd and even integers. In playing around, I add some pairs of odd integers and see a pattern that each time the sum is an even integer. I conjecture that the sum of two odd integers is always an even integer. But, I may lack the where withal to create a convincing proof of this.

Now, think about my teacher. Does my teacher know enough math to facilitate my exploration of this topic, to provide feedback on the correctness of incorrectness of steps I am taking to "prove" my conjecture, or to actually construct a proof that will be convincing to students in my classroom?

Much of the literature on math problem posing focuses on students developing word problems that are suggested by a particular environment or by a particular math calculation. Liping Ma (1999), for example, based part of her doctoral research on asking elementary school teachers in the US and China to create a word problem that is solved by the calculation 1 3/4 ÷ 1/2. See if you can do this calculation and if you think of a "real world" problem in which it is appropriate to carry out this calculation. This type of problem-posing activity can be used with any computational procedure students are studying.

## Math Use in a Typical Day

Think about uses that you make of math in a typical day. Your list might include telling time, estimating or measuring distances, counting a variety of things (such as calories or carbohydrates), doing exact or approximate arithmetic calculations, spending and keeping track of money, using your mental map of a town in order to drive from one locations to another, telling a friend how to drive to where you live, and so on. Here are a couple of interesting ways to think about your list:

1. Which of the uses on your list were learned in school, and how did you become skilled at transferring this school learning to settings in your everyday life?
2. Which of these uses did you learn outside of school (perhaps from other people, by discovery, by reading), and what does this tell you about your ability to learn math-types of things outside of formal schooling?
3. How is your list similar to and different from the lists your colleagues would likely create, and how do such differences get taken into consideration in the math curriculum, instruction, and assessment in the elementary school?

The heart of math maturity and math content is being able to use your math knowledge and skills to deal with the types of math-related problems and tasks that you encounter. If your life and career depend heavily on "school math," you will build a working knowledge of this school math and it will become a part of your everyday life. You will develop the math-related knowledge, skills, and habits of mind that are important to you in this type of everyday life.

On the other hand, if much of the math that you studied in school has little use in your everyday life, then you will likely forget most of that content. Your math-type knowledge, skills, and habits of mind will grow in the areas and types of uses that are useful to you in your everyday life.

## Elementary School Applications

4.1 Once a week, at the beginning of the math instruction period, ask your students to give examples of any use they have made outside of math period of the math studied in the past week. Younger students can do this orally, in a whole class discussion. Older students might write about this in their math journals.

4.2 This is for students near the end of the first grade, and older. Ask your students, "Which are you better at—reading, or math? Explain why you gave the answer you did." Then ask your students to talk about their thoughts and feelings concerning word problems in math. Look for insights that you feel represent increasing understanding and maturity.

## Activities for Self-Assessment, Assignments, and Group Discussions

4.1 Think about the mathematics instruction you received before you started college and while in college. Focus specifically on those aspects of your math education that seemed to be designed to increase your math maturity. Name some of these activities and analyze their effectiveness. For example, have you received specific instruction on how to read math, how to learn math, and how to retrieve math information from reference books and the Web?

4.2 Two of the Big Ideas in math are *variable* and *function.* What do these two words (concepts) mean to you? What sort of mental model, picture, or idea pops into your conscious working memory when you think about the term *variable* or the term *function*? In what sense is each a part of your Math Content Knowledge and in what sense is each a part of your Math Maturity? It might help you in your thinking if you make a list of times or situations in your everyday life where you make use of these two concepts in a math-related manner. For example, you might say, "I've got so many balls in the air, I don't know what is most apt to happen." Roughly speaking, this is a statement about dealing with a lot of variables and how they relate to each other.

4.3 Make up some questions that you feel are appropriate to use with students at a particular grade level, and that are designed to help assess the math maturity of such students. Try your instrument with some students (probably in a one-on-one setting) and report on the results.

# Chapter 5: Problem Solving

Problem solving lies at the heart of each discipline. However, the nature of the problems being addressed and the methodologies being used varies considerably from discipline to discipline. This chapter provides a brief introduction to *problem* and *problem solving.* As you read this chapter, keep in mind that math is both a discipline in its own right and is also a powerful aid to representing and solving problems in many other disciplines.

## Definition

Here is a definition of the word *problem* that I have found useful in my teaching of preservice and inservice teachers at all grade levels and in a variety of subject areas:

> You (personally) have a problem if the following four conditions are satisfied:
>
> 1. You have a clearly defined given initial situation.
> 2. You have a clearly defined goal (a desired end situation). Some writers talk about having multiple goals in a problem. However, such a multiple goal situation can be broken down into a number of single goal problems.
> 3. You have a clearly defined set of resources that may be applicable in helping you move from the given initial situation to the desired goal situation. These typically include some of your time, knowledge, and skills. Resources might include money, the Web, and the telephone system. There may be specified limitations on resources, such as rules, regulations, guidelines, and timelines for what you are allowed to do in attempting to solve a particular problem.
> 4. You have some ownership—you are committed to using some of your own resources, such as your knowledge, skills, time, and energy, to achieve the desired final goal.

The fourth component of this definition is particularly important. Unless a student has ownership—an appropriate combination of intrinsic and extrinsic motivation—the student does not have a problem. Motivation, especially intrinsic motivation, is a huge topic in its own right, and I will not attempt to explore it in detail in this book. You certainly know that many teachers are not very successful in helping their students to develop intrinsic motivation in their math studies. As students progress through elementary school and into secondary school, the math problem solving that they study seems to have less and less meaning and intrinsic motivation for many students.

As noted at the start of this chapter, problem solving lies at the core of each discipline. Perhaps you have heard people ask questions such as "Why do I need to study math?" or "Why do I need to study xxxx (where xxxx is some other discipline that is a required part of the curriculum)?"

While there are many possible answers to such questions, a unifying answer is that by doing so you will be able to solve a variety of problems that you cannot currently solve. You will learn about some of the important accomplishments within the discipline, some of its history, and some of its language. As you learn the language and notation, you will get better in making use of and building on the accumulated knowledge of the discipline. You will learn to precisely represent problems to be solved and tasks to be accomplished so that you can communicate your needs and interests to other people and to Information and Communication Technology (ICT) systems.

ICT provides powerful information retrieval systems (an aid to building on the previous work of others) as well as tools that can solve or greatly aid in solving a wide range of problems. A later chapter of this book is devoted to ICT and math education.

## George Polya

George Polya was one of the leading mathematicians of the 20th century, and he wrote extensively about problem solving. His 1945 book, *How to Solve It: A New Aspect of Mathematical Method,* is well known in math education circles (Polya, 1957).

*The Goals of Mathematical Education* (Polya, 1969) is a talk that he gave to a group of elementary school teachers.

> To understand mathematics means to be able to do mathematics. And what does it mean doing mathematics? In the first place it means to be able to solve mathematical problems. For the higher aims about which I am now talking are some general tactics of problems—to have the right attitude for problems and to be able to attack all kinds of problems, not only very simple problems, which can be solved with the skills of the primary school, but more complicated problems of engineering, physics and so on, which will be further developed in the high school. But the foundations should be started in the primary school. And so I think an essential point in the primary school is to introduce the children to the tactics of problem solving. Not to solve this or that kind of problem, not to make just long divisions or some such thing, but to develop a general attitude for the solution of problems.

In this statement, Polya is talking both about problem solving throughout the field of math, and also about use of math in solving problems in other disciplines. He is also talking about "the right attitude and to be able to attack all kinds of problems." This statement is about math maturity, rather than about knowledge of any specific math content.

As the following quotation from the same talk indicates, Polya was particularly concerned with helping students learn to think mathematically when working on problems.

> We wish to develop all the resources of the growing child. And the part that mathematics plays is mostly about thinking. Mathematics is a good school of thinking. But what is thinking? The thinking that you can learn in mathematics is, for instance, to handle abstractions. Mathematics is about numbers. Numbers are an abstraction. When we solve a practical problem, then from this practical problem we must first make an abstract problem. Mathematics applies directly to abstractions. Some mathematics should enable a child at least to handle abstractions, to handle abstract structures.

Notice the emphasis on representing problems in the abstract words and symbols of math. Later in this book I will present some ideas from Piaget and others on cognitive developmental theory. Problem solving and abstraction lie at the Formal Operations end of the Piagetian scale for cognitive development. As we teach math, we are attempting to help students move up this cognitive development scale.

## Building on Previous Work

One of the most important ideas in problem solving is to build on the previous work of yourself and others. That is, one way to solve a problem is to retrieve from your own memory either a solution to the problem or a method for solving the problem. Another way is to retrieve this information from another person, from a book, from a machine such as a cash register, or from a calculator or a computer. If you are repeatedly faced by a particular problem or type of problem, it is very useful to memorize one or more solutions to the problem, or a general method for solving the problem in a timely fashion.

Mathematics is a very large discipline because a large number of people have been working throughout recorded history to build and accumulate knowledge in this field. A research mathematician may spend years working on a single problem or a small group of related problems. If the mathematician is successful, then information about solving the problem or group of problems is published and becomes part of the accumulated knowledge of the field.

The human race's accumulated knowledge in mathematics is stored in hundreds of thousands of books, monographs, journals, Web publications, and other forms of publication. Much of this accumulated knowledge is only accessible to those who have studied math at a graduate school level. While it is easy to talk about the importance of building on the accumulated knowledge of oneself and others, it can take many years of hard work to develop the knowledge needed to read and understand the accumulated research knowledge in a discipline.

Moreover, currently most of the accumulated knowledge in a field such as math is not readily available. It is scattered throughout the libraries of the world, and it is written in many different languages. Over time, such difficulties of accessing materials will decrease as the materials are digitized and become accessible through the Web. Progress in the computer translation of languages will also help.

To summarize, one goal in math education needs to be that students learn to access the accumulated math knowledge that is appropriate to their educational level and needs, and to learn to make use of this accumulated knowledge to solve problems and accomplish tasks. One aspect of this is having students learn to read math well enough so that they can "look up" and read the math that they have studied in their previous years of studying math in school. A somewhat different way to think about this is that when a student is learning a math topic, the student should be learning enough to "relearn" the topic in the future, after a substantial amount of forgetting has occurred.

## To Memorize or not to Memorize: That is the Question

You carry quite a bit of accumulated math knowledge in your head. There, it is available for use in pattern matching, recall, and use. In math, as in each other discipline, one approach to learning to solve a problem or type of problem is via rote memorization. Just memorize the problem and a solution or a solution method. Practice until a desired level of speed and accuracy has been achieved. Continue to practice from time to time in order to retain the desired level of speed and accuracy.

This is the approach our educational system uses for the one-digit addition and multiplication facts. While some students can master this limited number of memorization tasks relatively easily, others struggle throughout school and even on in their adulthood. Still, there is general agreement of both the "back top basics" and the "now math" groups that such memorization is desirable.

It is interesting to think about this memorization difficulty. Many of the people who have trouble memorizing math facts and procedures have little trouble in learning to spell thousands of different words, memorizing lots of different songs, and so on. Perhaps some of the difference in these types of memorization is that "rote memory" is closely associated with meaning and understanding. If each word to be spelled were just a random collection of letters (a different random collection for each word), then spelling would be very difficult. (For many students, memorizing math facts is like memorizing words made up of random letters. ) But, if one is learning to spell words that one knows and uses in everyday life, and if the spelling is relatively closely related to the sound of the word, then spelling is relatively easy. Indeed, if one's written language is completely phonetic (each word is spelled the way it sounds), then spelling is indeed easy.

To close this section, here are my opinions on the specific topic of calculation. In my opinion, it makes no sense to think of memorizing answers to every possible addition, subtraction, multiplication, and division of integers problem. Instead, people have developed a variety of algorithms (detailed step by step sets of instructions) that can be memorized and applied. From the point of view of difficulty in memorization, difficulty in building speed and accuracy, and difficulty in maintaining speed and accuracy over time, this approach is only marginally successful. For many students, a huge amount of their math education time and effort is spent learning to effectively deal with the problems of arithmetic computation of integers.

Here are three rather obvious observations about this situation:

1. The rote memorization approach, along with practice to build and maintain speed and accuracy, is not closely related to understanding nor to transfer of learning in which one is expected to use arithmetic to represent and solve problems.
2. The rote memory approach takes a lot of learning time, does little to increase math maturity, and does little to lay a good foundation for future learning in math.
3. Via this rote memory approach, students are spending a lot of time learning to do stuff that a calculator can do more rapidly.

## Polya's 6-Step (Heuristic) Strategy

The research literature on problem solving is quite large, and math education includes a number of heuristic strategies for attaching math problems. Examples of heuristic strategies include: draw a picture; break a big problem into smaller pieces; trial and error; develop a somewhat similar but simpler problem; and do library research. Each of these examples is a heuristic—a plan of action that may help, but is not guaranteed to help. This is in contrast with an algorithm, which is guaranteed to solve a particular category of problem or accomplish a particular task in a finite number of steps.

Thinking mathematically and solving math problems are large topics and are important components of any math or math education course. While these two topics are beyond the scope of this short, all readers should be interested in Polya's (1957) general heuristic strategy for attempting to solve any math problem. I have reworded his strategy so that it is applicable to a wide range of problems in a wide range of disciplines—not just in math. This six-step strategy can be called the Polya Strategy or the Six Step strategy. Note that there is no guarantee that use of the Six Step strategy will lead to success in solving a particular problem. You may lack the knowledge, skills, time, and other resources needed to solve a particular problem, or the problem might not be solvable.

1. Understand the problem. Among other things, this includes working toward having a well-defined (clearly defined) problem. You need an initial

understanding of the Givens, Resources, and Goal. This requires knowledge of the domain(s) of the problem, which could well be interdisciplinary. You need to make a personal commitment (Ownership) to solving the problem.

2. Determine a plan of action. This is a thinking activity. What strategies will you apply? What resources will you use, how will you use them, in what order will you use them? Are the resources adequate to the task? On hard problems, it is often difficult to develop a plan of action. Research into this situation suggests that many good problem solvers "sleep on the problem." That is, after working on a problem for quite awhile with little or no success, they put the problem out of mind and do something else for days or even weeks. What may well happen is that a subconscious level the mind continues to work on the problem. Eventually, an "ah-ha" occurs.
3. Think carefully about possible consequences of carrying out your plan of action. Focus major emphasis on trying to anticipate undesirable outcomes. What new problems will be created? You may decide to stop working on the problem or return to step 1 as a consequence of this thinking.
4. Carry out your plan of action. Do so in a thoughtful manner. This thinking may lead you to the conclusion that you need to return to one of the earlier steps. Note that this reflective thinking leads to increased expertise.
5. Check to see if the desired goal has been achieved by carrying out your plan of action. Then do one of the following:
    A. If the problem has been solved, go to step 6.
    B. If the problem has not been solved and you are willing to devote more time and energy to it, make use of the knowledge and experience you have gained as you return to step 1 or step 2.
    C. Make a decision to stop working on the problem. This might be a temporary or a permanent decision. Keep in mind that the problem you are working on may not be solvable, or it may be beyond your current capabilities and resources.
6. Do a careful analysis of the steps you have carried out and the results you have achieved to see if you have created new, additional problems that need to be addressed. Reflect on what you have learned by solving the problem. Think about how your increased knowledge and skills can be used in other problem-solving situations. (Work to increase your reflective intelligence!)

Many of the steps in this six-step strategy require careful thinking. However, there are a steadily growing number of situations in which much of the work of step 4 can be carried out by a computer. The person who is skilled at using a computer for this purpose may gain a significant advantage in problem solving, as compared to a person who lacks computer knowledge and skill.

## Computers and Math Problem Solving

I find the diagram given in figure 5.1 to be particularly useful when I talk about computers and math problem solving at the precollege level. With some effort, this diagram can be modified to fit problem solving in other disciplines.

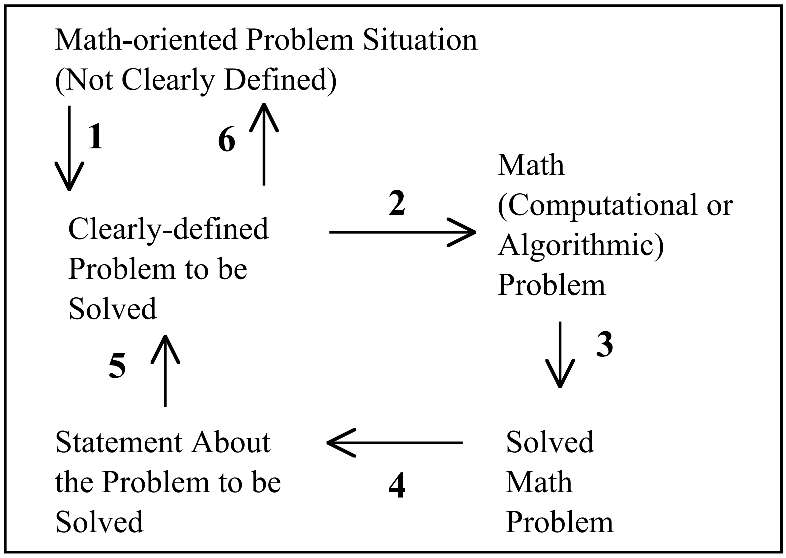



Figure 5.1 Math problem solving.

The six steps illustrated are 1) Problem posing and problem recognition; 2) mathematical modeling; 3) Using a computational or algorithmic procedure to solve a computational or algorithmic math problem; 4) Mathematical "unmodeling"; 5) Thinking about the results to see if the Clearly-defined Problem has been solved; and 6) Thinking about whether the original Problem Situation has been resolved. Steps 5 and 6 also involve thinking about related problems and problem situations that one might want to address or that are created by the process or attempting to solve the original Clearly-defined Problem or resolve the original Problem Situation.

In some sense, all of the steps except (3) involve higher-order knowledge and skills. They require a significant level of math maturity. Step (3) lends itself to a rote memory approach. It is highly desirable that students develop speed and accuracy in certain types of mathematical operations. However, the human mind is not good at memorizing math procedures and then carrying them out rapidly and accurately with the assistance of pencil and paper. On the other hand, calculators and computers are really good at carrying out math procedures.

PK-12 teachers who teach math tend to estimate that about 75% of the math education curriculum focuses on (3). [Note: This is an estimate I have made based upon working with a very large number of teachers. I don't know of any published research that backs up my assertion.] This leaves about 25% of the learning time and effort focusing on the remaining five steps. Appropriate use of calculators and computers as tools, and Computer-Assisted Learning, could easily decrease the time spent on (3) to 50% or less of the total math education time. This would allow a doubling of the time (from 25% to 50%) devoted to instruction and practice on the higher-order knowledge and skill areas.

## Correctness of a Solution

Suppose that you were given the task of writing a persuasive paper about some aspect of our national election system. You ask the teacher, "How long does it need to be?" The teacher says that it needs to be sufficiently long to accomplish the task, and that grading will be based on the quality of the paper. The question for you is, how can you tell when you have accomplished the task?

Remember, problem solving is part of every discipline. With the broad definition of *problem* that we are using, your writing task is a (writing) problem to be solved. It is certainly different than a math problem! Think about doing a compare and contrast with a math assignment. Here are some of my thoughts as I pretend to be a student:

1. If the teacher had just said how long the paper was to be, I would know I was close to done when I had achieved the required length. That is a little bit like an assignment in math where I am supposed to do all of the odd numbered problems at the end of a chapter. I know I am done when I have completed all of the odd numbered problems. But, I may have made mistakes in solving some of the problems. I guess that is a little like having errors in the writing and in the logic of the persuasive arguments.
2. The teacher didn't tell me if I needed to have a bibliography. I suppose I do, because this seems like the type of writing problem that requires research. When I am doing a math assignment I sometimes need to look back in the book to see how to solve a particular type of problem. Occasionally I can't find an example in the book, perhaps because it is a problem from last year or several years ago. I guess it is easier to do library research in non-math areas.
3. In my writing, I will have a goal of convincing the reader of "something," through my careful logic and using information from the literature. First I need to get a clear idea of my goal—what I want to convince my reader about. I suppose this is a little bit like solving a math problem. In solving a math problem, I usually have a clear goal, and I carry out a sequence of steps. Each step is sort of like a piece of an argument, moving me in a logical fashion towards my goal.
4. I know that writing is a process, and that I will be doing "revise, revise, revise" to produce as good a product as possible in the time that I am willing to devote to the writing task. I know that my paper will not be perfect—that will more time, I could make it better. This seems different than solving a math problem. When I solve a math problem and get an answer, I am done. That assumes, of course, that I have some way of telling that I have gotten a correct answer. Of course, my math problem (task) might be to make a proof. That is sort of like making a persuasive argument. But, in math it is possible to make a persuasive argument that is really convincing. I guess that is what a math theorem is all about.
5. Etcetera, etcetera, etcetera.

The $4^{th}$ point in the list is especially important in math. In some math problem-solving situations it is possible to check an answer. For example, addition can be checked by subtraction, division by multiplication, and so on. In some math problems one can check an answer by testing to see if it meets the conditions specified in the problem. For example, suppose I am supposed to find three consecutive positive integers whose sum is a perfect square of an integer. If I find an answer, I can easily check to see if it is correct. If I can't find an answer, I can always try to prove that there is no answer. But that requires me to develop a carefully constructed chain of logical argument that will be convincing to my readers.

This whole section tends to fall into the area of mathematical maturity. As a person increases in mathematical maturity, they are both more able to and more confident in being able to judge the correctness of their mathematical work.

## Elementary School Applications

5.1 Having a person "think out loud" as they attempt to solve a problem is a standard research tool. It can also be useful both as an aid to learning and as a vehicle through which a teacher can gain insight into a student's learning and problem-solving difficulties. Select some math problems (as distinguished from math exercises) of a difficulty level appropriate to your students. Train your students in carrying out this thinking out loud activity through use of volunteers who role model it. In this training process, you are role modeling how to interact with the out loud-thinker, and how to provide appropriate feedback. Gradually work toward the situation in which students can work in pairs or small teams, with a student thinking out loud in the team, explaining his or her thinking processes when attempting to solve a problem. The listeners or listeners practice interaction with the talker, gaining skill in listening and providing appropriate feedback.

5.2 This chapter contains a 4-part definition of the term "problem." Since problem and problem solving are key components of each discipline you teach, it seems reasonable that your students should be learning definitions of these terms that are appropriate to their developmental level and the disciplines they are studying. Set yourself a teaching goal of having your students understand meanings for math problem and math problem solving that are appropriate to the level at which you teach. You might begin such a lesson by first asking students to say what they think a math problem is, and what they think math problem solving is. You might then continue by looking at some examples of problems and problem situations that may or may not be math problems, and carrying on a discussion with your students about these examples and non-examples. You might continue by asking your students what it means to solve a math problem. For example, in this discussion you might hear a student say, "Do things to get the right answer." You might use that response to explore situations in which a math problem has no solution, only one solution, or more than one solution. You might raise the question, how can one tell if a proposed answer is right? This is a big and important topic in its own right.

## Activities for Self-Assessment, Assignments, and Group Discussions

5.1 People teaching math often try to distinguish between an exercise and a problem. An exercise is practice in carrying out a procedure, or applying and carrying out a procedure, that the students have recently encountered. A problem is more challenging, requiring higher-order cognition. The diagram in figure 5.1 shows that a number of steps are required in working from a typical math-related problem situation to a solved problem. What are your personal insights into the amount of math education time in elementary school spent on exercises versus time spent on problems?

5.2 You know that there are 50 states in the United States, that each has a geographical location, Governor, state capital, two Senators, a number of Representatives, and so on. Think about what data for each state is worthwhile for most students to memorize. As you do this, think about the concepts such as geographical location, state capital, government and governmental officials, and so on. If a person learns the concepts, then information about specific details can be retrieved relatively quickly from the Web or other resources. What are your current thoughts on what to memorize and what to "understand" and be able to look up? What would it take to change your current position?

5.2 Think about some "real world" math problems that you have encountered recently. How did you go about solving these problems? For example, which did you solve by quick recall of memorized information, on which did you seek help on, on which did you make use of calculators or computers, and what other approaches did you use?

# Chapter 6: Intelligence

Historically, the study of the human brain (one of a person's organs) and the study of the human mind (think of the mind as a product of the brain) have been distinct disciplines. Computer-oriented people tend to think of the brain as hardware (they call it wetware) and the mind as software.

The study of the mind is currently part of the field of psychology, while the study of the brain is part of the discipline of neuroscience. However, in recent years, the mind and brain disciplines have begun to merge.

Jacques Hadamard (1865-1963) was a prolific and well-respected research mathematician and teacher. In one of his books, he explored the working of the mathematical mind (Hadamard, 1945). In the first chapter, while talking about the difficulty of this task, he notes:

> That difficulty is not only an intrinsic one, but one which, in an increasing number of instances, hampers the progress of our knowledge: I mean the fact that the subjects involves two disciplines, psychology and mathematics, and would require, in order to be treated adequately, that one be both a psychologist and a mathematician.

If Hadamard were alive today, he would likely be impressed by the progress that is occurring in brain and mind science, and in applications of computers to the teaching, learning, and doing math. However, he would likely argue that we still have a long way to go before we have a thorough understanding of the psychology of invention in the mathematical field. This is, indeed, a challenging area of research and development.

## What is Intelligence?

Intelligence is the ability to learn and to take actions that make use of one's learning. Clearly, intelligence is not limited just to humans (NSF Press Release, 10/27/04). However, the ability of an ordinary person to learn a natural language such as English demonstrates a very high level of intelligence on the intelligence scale of all life on earth. Indeed, although students in our elementary schools vary in intelligence, all are highly intelligent on the scale of all intelligent creatures on earth.

For many years, psychologists studying the human brain/mind have tried to measure its capabilities. Quite a bit of this work has focused on defining intelligence and measuring a person's intelligence.

> The concept that intelligence could be or should be tested began with a nineteenth-century British scientist, Sir Francis Galton. Galton was known as a dabbler in many different fields, including biology and early forms of psychology. After the shake-up from the 1859 publishing of Charles Darwin's "The Origin of Species," Galton spent the majority of his time trying to discover the relationship between heredity and human ability (History of I.Q., n.d.).

Howard Gardner (1993), David Perkins (1995), and Robert Sternberg (1988) are researchers who have written widely sold books about intelligence. Of these three, Howard Gardner is probably the best known by PK-12 educators, because his theory of Multiple Intelligences has proven quite popular in PK-12 education (Mckenzie). However, there are many researchers who have contributed to the extensive and continually growing collection of research papers on the intelligence (Yekovich 1994). The following definition of human intelligence is a composite from various authors, especially Gardner, Perkins, and Sternberg. Intelligence is a combination of the abilities to:

1. Learn. This includes all kinds of informal and formal learning via any combination of experience, education, and training.
2. Pose problems. This includes recognizing problem situations and transforming them into more clearly defined problems.
3. Solve problems. This includes solving problems, accomplishing tasks, and fashioning products.

Ways to measure intelligence were first developed more than 120 years ago, and this continues to be an active field of research and development. A very simplified summary of the current situation consists of:

1. There are a variety of IQ tests that produce one number or a small collection of numbers as measures of a person's intelligence. Most of these tests place a high emphasis on the linguistic and mathematical/logical aspects of intelligence. Increases in Math Content Knowledge and in Math Maturity tend to contribute to scoring higher on IQ tests.
2. The "one number" approach (the general intelligence, or "g" factor) was developed by Charles Spearman in 1904, and it still has considerable prominence.
3. Many people have proposed and discussed the idea of multiple intelligences. In the past two decades, the work of Howard Gardner has helped to publicize this idea. Logical/mathematical, spatial, and linguistic are three of the eight Multiple Intelligences identified by Gardner (n.d.), and they all relate to learning and using mathematics
4. Significant decreases in the intelligence of children result from starvation, lack of needed vitamins and minerals, and exposure to various poisons such as lead and mercury (Nutrition, n.d.). Significant differences also result from other aspects of a child's home environment, such as education level of the adults in the environment and socioeconomic status (ASCD, 2004).

## Fluid and Crystallized Intelligence

While Howard Gardner and Robert Sternberg have garnered a lot of publicity during the past couple of decades for their work on intelligence, many really important ideas have been developed by other people. One of these is the idea that "g" can be divided into two major components: fluid intelligence (biologically-based) (gF) and crystallized intelligence (acquired knowledge base) (gC).

> The theory of fluid and crystallized intelligence … proposes that primary abilities are structured into two principal dimensions, namely, fluid (***Gf***) and crystallized (***Gc***) intelligence. The first common factor, ***Gf***, represents a measurable outcome of the influence of biological factors on intellectual development (i.e., heredity, injury to the central nervous system), whereas the second common factor, ***Gc***, is considered the main manifestation of influence from education, experience, and acculturation. ***Gf-Gc*** theory disputes the notion of a unitary structure, or general intelligence,

> as well as, especially in the origins of the theory, the idea of a structure comprising many restricted, slightly different abilities (McArdle , et al.,2002).

In casual conversations about intelligence and IQ, people tend to forget about the meaning of the "Q" in IQ. The human brain grows considerably during a person's childhood, with full maturity being reached in the early to mid 20s for most people. Both gF and gC increase during this time. Recent research suggests that gF then begins a slow decline. However, with appropriate education and cognitive experiences, gC continues to grow well into a person's 50s (McArdle et al.; 2002).

## Street Smarts and Folk Math

Robert Sternberg is well known for his triarchic model of intelligence. Very roughly speaking, he divides intelligence into the three parts: creativity, street smarts, and school smarts. Here is a somewhat different way of explaining his theory. Think of creativity as being gF, while street smarts and school smarts are two broad categories in which one develops gC. If a person is raised in a preliterate hunter-gather community living in a jungle, the person will develop a high level of "hunter-gather living in a jungle" street smarts. Since the person will not be exposed to reading, writing, and books, the person will not develop an appreciable level of school smarts.

The following is quoted from Cianciolo and Sternberg (2004, p20).

> **School's eye views of intelligence**
>
> Shirley Brice Heath (Heath, 1983), an ethnographer, studied mismatches between notions of intelligence held in the home and those held in the school environment, and observed the effects of these mismatches on the development of language in children. In three communities, Heath discovered that as home socialization practices diverged from those valued by school environments, performance in school suffered. For example, in one community, verbal interaction typically involved highly fanciful storytelling and clever put-downs. Students from this community experienced difficulty in school, where fanciful stories were perceived as lies, and putdowns were not a valued part of the school's social environment. In another community, parents modeled their verbal exchanges after modes of knowledge transmission in the church, which discouraged dialogue and fantasy. Students from this community excelled in verbatim recall, but experienced great difficulty when novel storytelling was required.

Research suggests similar findings in math. Quoting from Sternberg (2002) in which he argues that there is more to intelligence than just IQ:

> For example, Carraher, Carraher, and Schliemann (1985) studied a group of children that is especially relevant for assessing intelligence as adaptation to the environment. The group was of Brazilian street children. Brazilian street children are under great contextual pressure to form a successful street business. If they do not, they risk death at the hands of so-called "death squads," which may murder children who, unable to earn money, resort to robbing stores (or who are suspected of resorting to robbing stores). The researchers found that the same children who are able to do the mathematics needed to run their street business are often little able or unable to do school mathematics. In fact, the more abstract and removed from real-world contexts the problems are in their form of presentation, the worse the children do on the problems. These results suggest that differences in context can have a powerful effect on performance.
>
> Such differences are not limited to Brazilian street children. Lave (1988) showed that Berkeley housewives who successfully could do the mathematics needed for comparison shopping in the supermarket were unable to do the same mathematics when they were placed in a classroom and given isomorphic problems presented in an abstract form. In other words, their problem was not at the level of mental processes but at the level of applying the processes in specific environmental contexts.

Gene Maier (n.d.) was one of the founders of the Math Learning Center that has offices in Salem and Portland, Oregon, and he served as its President for many years. One of his areas of interest is "folk math" versus school math. He notes that many people (including cabinet makers, carpenters, mill wrights, street urchins throughout the world, and lots of other people with little or no formal education) make routine use of math to help solve the types of problems they encounter on the job and in their day-to-day lives. By and large they make use of folk math (their math-oriented street smarts) rather than school math.

Of course, many other people have thought about the ideas underlying street smarts and school smarts. For example, Jerome Bruner has had a significant impact on our educational system. Quoting from Bruner (n.d.):

> It is surely the case that schooling is only one small part of how a culture inducts the young into its canonical ways. Indeed, schooling may even be at odds with a culture's other ways of inducting the young into the requirements of communal living.... What has become increasingly clear... is that education is not *just* about conventional school matters like curriculum or standards or testing. What we resolve to do in school only makes sense when considered in the broader context of what the society intends to accomplish through its educational investment in the young. How one conceives of education, we have finally come to recognize, is a function of how one conceives of culture and its aims, professed and otherwise. (Jerome S. Bruner 1996: ix-x)

The street smarts versus school smarts analysis helps to explain why children raised in poverty (low socioeconomic environments) tend to be a year behind average in school smarts by the time they begin school. Their early childhood learning focuses on gaining street smarts knowledge and skills that help them survive and prosper in a poverty environment. Here is a brief summary of recent research in this area (ASCD, 2004):

> In general, as socioeconomic status increased, the degree of environmental influence on measured IQ scores decreased. For the most impoverished families, almost 60 percent of the variability in scores was explained by environmental differences, whereas the percentage of variation in scores attributable to genetic difference was essentially zero. For the high-SES grouping, almost 90 percent of the variance in scores was explained by genetic differences.
>
> The effect of environment on the IQ of young children can be significant, particularly for children living in poverty. As the influence of poverty decreases, the importance of environmental conditions as a limiting factor on intelligence also decreases. By addressing the environmental issues created by poverty, it may be possible to weaken the link between low socioeconomic status and poor student performance on IQ (and other) tests.

It is interesting to carry this line of thought a little further. Some children grow up in an environment that is school smarts mathematically "rich." I am an example of such a person, since both my father and mother were on the faculty in the Department of Mathematics at the University of Oregon. I grew up in a culture that placed high value on knowing and using math. This environment helped to "grow" my math oriented gF and gC.

My conclusion is that one of the reasons for the relatively poor success of our formal, school smarts math education system is that the math environment many of our children grow up in before they start school and the math environment they encounter both at home and in school during the early years of their formal education is not particularly "rich" in its support of school mathematical development. This idea illustrated in the following quote from an American Association for the Advancement of Science report (New, 1998). The article by Rebecca New is one of many related articles available at AAAS (1998).

> Teacher attitudes and knowledge may also account for much of the inequitable treatment of preschool mathematics, science, and technology. The field of early childhood education has struggled for much of the second half of this century to establish a reputation of professionalism. However, the knowledge base deemed essential for teachers' scientific and professional status derives almost exclusively from the child study movement and the field of developmental psychology. Few states require early childhood educators to have formal professional knowledge in the content areas as a condition of certification. Consequently, the experiences in science, mathematics, and technology that many early childhood educators bring with them to the classroom are limited by their personal histories as learners in those domains.

I also conclude that many people grow up rather weak in their folk math development, because they are not raised and taught in environments that are explicitly designed to foster cognitive growth of street smarts mathematics (folk math). They find that much of the school math they learn is not particularly to their outside of school needs.

## Elementary School Applications

6.1 Quite a bit of a young student's attitude toward math comes from math-related attitudes in the home environment. As you work with individual students in the elementary school, it is helpful to have insights into the home math environment and attitudes that your students have grown up in. You can garner some of this information by engaging your students in whole class discussions about the interests in and attitudes towards math that they encounter at home. You might ask, for example, if there is someone in their home situation who particularly likes math, or someone who thinks that math is really hard, or that boys are better at math than girls (or, vice versa).

6.2 Many people find that math is fun. Indeed, most children find that math is fun while they are at the primary level, but many then find it to be less fun is they move into the upper elementary grades. As a teacher, you need to learn what aspects of math are fun (hence, perhaps intrinsically motivating) to your students. You can do this by observing and talking to your students as you try out a wide range of different math activities that other teachers have found to be fun. Many fine examples, along with videos of teachers using the ideas and materials, can be found at the PBS Teacher Source (n.d.). There are a large number of other Websites that contain free math materials for use in the elementary school, and many of these are "fun" oriented. For example, see Elementary School Math Center (n.d.).

## Activities for Self-Assessment, Assignments, and Group Discussions

6.1 What are your personal thoughts on nature versus nurture as determiners of intelligence? What personal knowledge and experience do you have that supports your position? How does your position fit into the way you plan to work with young students?

6.2 Think about the math that you routinely use in your day-to-day life. Give examples of the folk math aspects that you see in this use of math. Give some ideas about what schools might do to increase the folk math knowledge and skills of students.

6.3 What are your personal attitudes towards math and the learning of math, and what seems to have led to these attitudes?

# Chapter 7: Cognitive Development

This chapter is about cognitive development of the mind. The word *mind* has a number of different definitions. Quoting from Encarta® World English Dictionary © 1999 Microsoft Corporation:

1. the center of consciousness that generates thoughts, feelings, ideas, and perceptions and stores knowledge and memories
2. the capacity to think, understand, and reason (often used in combination)
3. …

Most definitions of mind include the term consciousness, which is a very complex idea. Many people consider the neurobiology of consciousness to be the last major unsolved problem in biology.

> Since school activities focus principally on conscious learning and behavior, the biology of consciousness will thus help to formulate credible 21st century theories of teaching and learning. But since consciousness is also integral to religious belief and cultural behavior, its relationship to educational theory will certainly be controversial. Educational leaders will obviously have to understand consciousness in order to deal intelligently with the complex issues it will raise. (Sylwester, 2004).

This chapter focuses specifically on math cognitive development.

## Piaget's Cognitive Developmental Scale

You are probably familiar with the four-stage Piagetian Developmental Scale shown in figure 7.1 (Huitt and Hummel, 1998).

| Approximate Age | Stage | Major Developments |
|---|---|---|
| Level 1.<br>Birth to 2 years | Sensorimotor | Infants use sensory and motor capabilities to explore and gain understanding of their environments. |
| Level 2.<br>2 to 7 years | Preoperational | Children begin to use symbols. They respond to objects and events according to how they appear to be. |
| Level 3.<br>7 to 11 years | Concrete operations | Children begin to think logically. This stage is characterized by 7 types of conservation: number, length, liquid, mass, weight, area, volume. Increasing intelligence is demonstrated through logical and systematic manipulation of symbols related to concrete objects. Operational thinking—mental actions that are reversible—develops. |
| Level 4.<br>11 years and beyond | formal operations | Thought begins to be systematic and abstract. In this stage, intelligence is demonstrated through the logical use of symbols related to abstract concepts, problem solving, thinking logically about abstract propositions, testing hypotheses, and gaining and using higher-order knowledge and skills. |

Figure 7.1 Piaget's Stages of Cognitive Development

Piaget's stages of cognitive development are not specific to any particular discipline. However, a math-oriented reader of figure 7.1 might decide that Concrete Operations and Formal Operations seem to be somewhat math oriented. Piaget was particularly interested in math aspects of cognitive development. You may want to reread the material quoted from George Polya given in Chapter 5. Even at its most elementary levels, school math tends to be rather abstract. Later in this section I explore a still more math-oriented cognitive development scale.

Cognitive development is dependent on both nature and nurture. Roughly speaking, a child's progress though the first two Piagetian Developmental stages is more strongly dependent on nature, while progress in the latter two stages is more strongly dependent on nurture. However, nature versus nurture is not that simple. Marcus (2004) argues that the two are so thoroughly intertwined that is hopeless to attempt to separate them. Moreover, his arguments provide strong support for the value of high quality informal and formal education.

Although the Piagetian scale has only four labeled levels, it is a continuous scale. It is a common mistake to think of a person either being at Formal Operations or not being at Formal Operations. It is much more accurate to think of a person making progress in moving through a stage and gradually moving into the early part of the next stage. The rate of movement strongly depends on formal and informal education and the environment in which one operates. Moreover, a person may be well into Formal Operations in a one discipline such as history, and not yet have reached the beginnings of Formal Operations in another discipline such as math.

There are a variety of instruments used to measure cognitive development, and with such an instrument one can define a specific score as being the minimum score to be labeled "Formal Operations." When that is done, researchers find that only about 35% of children in industrialized societies have achieved Formal Operations by the time they finish high school (MacDonald , n.d.).

The following quotation provide additional information about the attainment of formal operations.

> However, data from similar cross-sectional studies of adolescents do not support the assertion that all individuals will automatically move to the next cognitive stage as they biologically mature. Data from adult populations provides essentially the same result: Between 30 to 35% of adults attain the cognitive development stage of formal operations (Kuhn, Langer, Kohlberg & Haan, 1977). For formal operations, it appears that maturation establishes the basis, but a special environment is required for most adolescents and adults to attain this stage (Huitt & Hummel, 2003).

> Many studies suggest our [college] students' ability to reason with abstractions is strikingly limited, that a majority are not yet "formal operational" (Gardiner, 1998).

These findings suggest that we need to take a careful look at the cognitive expectations in courses in all disciplines and at all grade levels. For example, the study of causality and the generating and testing of hypotheses are key ideas in the discipline of history and in the sciences. A ninth grade history or science course is apt to have a significant emphasis on these ideas. But, these ideas are part of Formal Operations. Unless they are presented and explored in a careful and appropriate Concrete Operations manner, they will be well over the heads of most of the ninth graders. Needless to say, this difficulty grows as one attempts to teach such ideas to still less cognitively developmentally mature students.

## Geometry Cognitive Development Scale

The same sort of analysis is applicable to our math curriculum. About 50 years ago, the Dutch educators Dina and Pierre van Hiele focused some of their research efforts on defining a

Piagetian-type developmental scale for Geometry (van Hiele, n.d.). Their five-level scale is shown in figure 7.2. Notice that the van Hieles, being mathematicians, labeled their first stage Level 0.

| Stage | Description |
| --- | --- |
| Level 0 (Visualization) | Students recognize figures as total entities (triangles, squares), but do not recognize properties of these figures (right angles in a square). |
| Level 1 (Analysis) | Students analyze component parts of the figures (opposite angles of parallelograms are congruent), but interrelationships between figures and properties cannot be explained. |
| Level 2 (Informal Deduction) | Students can establish interrelationships of properties within figures (in a quadrilateral, opposite sides being parallel necessitates opposite angles being congruent) and among figures (a square is a rectangle because it has all the properties of a rectangle). Informal proofs can be followed but students do not see how the logical order could be altered nor do they see how to construct a proof starting from different or unfamiliar premises. |
| Level 3 (Deduction) | At this level the significance of deduction as a way of establishing geometric theory within an axiom system is understood. The interrelationship and role of undefined terms, axioms, definitions, theorems and formal proof is seen. The possibility of developing a proof in more than one way is seen. |
| Level (Rigor) | Students at this level can compare different axiom systems (non-Euclidean geometry can be studied). Geometry is seen in the abstract with a high degree of rigor, even without concrete examples. |

Figure 7.2 Van Hiele five-level developmental scale for geometry.

The van Hieles' scale is mainly a school math (as distinguished from folk math) scale. The van Hieles' work suggested that the typical high school geometry course was being taught at a developmental level considerably above that of the typical students taking such courses. Think carefully about your math experiences as you took algebra and geometry courses in high school. Did some of this coursework seem over your head ("I just don't get it."), forcing you into memorize, regurgitate, and forget mode? The same general question applies to students studying math at all grade levels. When students "just don't seem to get it," the chances are good that the content and the way it is being presented are at an inappropriate cognitive developmental level for the student.

It is evident that moving up the van Hiele geometry cognitive developmental scale requires learning quite a bit of school-math geometry. For most students, this means that progress in moving up this scale is highly dependent on their teachers and the math curriculum. The NCTM Standards list geometry as one of the major content strands, and indicate that geometry is an important part of the elementary school math curriculum (NCTM, n.d.). Thus, elementary school teachers have the opportunity to make a major contribution to helping their students increase their geometry-oriented cognitive development.

## Math Cognitive Development Scale

Figure 7.3 represents my current thinking on a six-level Piagetian-type scale for school mathematics (as distinguished from folk math). It is an amalgamation and extension of ideas of Piaget and the van Hieles. The first three levels are particularly relevant to elementary school students.

| Stage Name | Math Developments |
| --- | --- |

| | |
|---|---|
| Level 1.<br>Piagetian and Math sensorimotor. | Infants use sensory and motor capabilities to explore and gain increasing understanding of their environments. Research on very young infants suggests some innate ability to deal with small quantities such as 1, 2, and 3. As infants gain crawling or walking mobility, they can display innate spatial sense. For example, they can move to a target along a path requiring moving around obstacles, and can find their way back to a parent after having taken a turn into a room where they can no longer see the parent. |
| Level 2.<br>Piagetian and Math preoperational. | During the preoperational stage, children begin to use symbols, such as speech. They respond to objects and events according to how they appear to be. The children are making rapid progress in receptive and generative oral language. They accommodate to the language environments (including math as a language) they spend a lot of time in, so can easily become bilingual or trilingual in such environments.<br>During the preoperational stage, children learn some folk math and begin to develop an understanding of number line. They learn number words and to name the number of objects in a collection and how to count them, with the answer being the last number used in this counting process.<br>A majority of children discover or learn "counting on" and counting on from the larger quantity as a way to speed up counting of two or more sets of objects. Children gain increasing proficiency (speed, correctness, and understanding) in such counting activities.<br>In terms of nature and nurture in mathematical development, both are of considerable importance during the preoperational stage. |
| Level 3.<br>Piagetian and Math concrete operations. | During the concrete operations stage, children begin to think logically. In this stage, which is characterized by 7 types of conservation: number, length, liquid, mass, weight, area, volume, intelligence is demonstrated through logical and systematic manipulation of symbols related to concrete objects. Operational thinking develops (mental actions that are reversible).<br>While concrete objects are an important aspect of learning during this stage, children also begin to learn from words, language, and pictures/video, learning about objects that are not concretely available to them.<br>For the average child, the time span of concrete operations is approximately the time span of elementary school (grades 1-5 or 1-6). During this time, learning math is somewhat linked to having previously developed some knowledge of math words (such as counting numbers) and concepts.<br>However, the level of abstraction in the written and oral math language quickly surpasses a student's previous math experience. That is, math learning tends to proceed in an environment in which the new content materials and ideas are not strongly rooted in verbal, concrete, mental images and understanding of somewhat similar ideas that have already been acquired.<br>There is a substantial difference between developing general ideas and understanding of conservation of number, length, liquid, mass, weight, area, and volume, and learning the mathematics that corresponds to this. These tend to be relatively deep and abstract topics, although they can be taught in very concrete manners. |
| Level 4.<br>Piagetian and Math formal operations.<br>Van Hiele level 2: informal deduction. | Thought begins to be systematic and abstract. In this stage, intelligence is demonstrated through the logical use of symbols related to abstract concepts, problem solving, and gaining and using higher-order knowledge and skills.<br>Math maturity supports the understanding of and proficiency in math at the level of a high school math curriculum. Beginnings of understanding of math-type arguments and proof.<br>Piagetian and Math formal operations includes being able to recognize math aspects of problem situations in both math and non-math disciplines, convert these aspects into math problems (math modeling), and solve the resulting math problems if they are within the range of the math that one has studied. Such transfer of learning is a core aspect of Level 4. |
| Level 5.<br>Abstract mathematical operations.<br>Van Hiele | Mathematical content proficiency and maturity at the level of contemporary math texts used at the senior undergraduate level in strong programs, or first year graduate level in less strong programs. Good ability to learn math through some combination of reading required texts and other math literature, listening to lectures, participating in class discussions, studying on your own, studying in groups, and so on. Solve relatively high level math problems posed by others (such as in the |

| | |
|---|---|
| level 3: deduction. | text books and course assignments). Pose and solve problems at the level of one's math reading skills and knowledge. Follow the logic and arguments in mathematical proofs. Fill in details of proofs when steps are left out in textbooks and other representations of such proofs. |
| Level 6. Mathematician. Van Hiele level 4: rigor. | A very high level of mathematical proficiency and maturity. This includes speed, accuracy, and understanding in reading the research literature, writing research literature, and in oral communication (speak, listen) of research-level mathematics. Pose and solve original math problems at the level of contemporary research frontiers. |

Figure 7.3. Six-stage mathematical cognitive developmental scale.

You, and each of the students you teach, are at some place on this six-level continuous scale. As you teach math, think carefully about what you are doing that will help your students move up this scale. As you study math, think carefully about how this helps you move up the scale.

Here is an example that I found interesting. It contains a sequence of ways in which a student might deal with the task of finding 3 + 4 (Thomas and Tall, 2002).

- **Count-all**: count 3 objects, "1, 2, 3," then 4 objects, "1, 2, 3, 4," then count all the objects, "1, 2, 3, 4, 5, 6, 7," to get 7 objects.
- **Count-both**: count 3 as "1, 2, 3," then count-on 4 as "4, 5, 6, 7."
- **Count-on**: count-on 4 after 3, "4, 5, 6, 7."
- **Count-on from large**r: turn the problem round and count-on 3 after 4 as "5, 6, 7."
- **Derived fact**: "3+4 is one less than 8, so it is 7."
- **Known fact**: "3+4 is 7."

The last of the six bulleted approaches produces the answer most quickly. We merely have students memorize the appropriate number fact. This can be done without any of the understanding that is inherent to the previous five bulleted approaches. Moreover, the memorization does not contribute to a student making some progress in moving through the Preoperational Level of the math cognitive developmental scale.

This example illustrates a major issue in math education. Suppose there is a clearly defined problem or closely related category of problems that we really want students to be able to solve. We can have them memorize (learn by rote) answers or quickly applied algorithms for arriving at answers. Or, we can take a longer route of teaching and learning for understanding. In some cases, rote memorization is "the" right approach. But learning without understanding is quite fragile and provides a very weak framework for further learning. Each math teacher and each math learner is faced by the difficulty of achieving an appropriate balance between the two approached.

## Probability and Math Cognitive Development

The van Hieles examined the secondary school geometry course from the point of view of student cognitive development in geometry. They concluded that there is a significant mismatch between student cognitive development and the typical proof-oriented course being taught at the time they developed their scale.

Other researchers have examined other parts of the math curriculum from a cognitive development point of view. A number of math education researchers have explored the issue of cognitive development and learning probability. A good example of such work is provided in Soen (1997). Quoting from that article:

> Piaget and Inhelder (1975) were the first researchers to study the development of the idea of chance in children. According to them, the concept of probability as a formal set of ideas develops only during the formal operational stage, which occurs about twelve years of age. By that age, children can reason probabilistically about a variety of randomizing devices.
>
> …
>
> Garfield and Ahlgren (1988) contend that before the teaching of probability, students must have an understanding of ratio and proportion. Students must be able to function at the formal operational level. They must have the necessary skills in dealing with abstractions.

The research relating the learning of probability and a student's level of cognitive development suggests that learning for understanding requires students to be at a formal operations level. Remember, even though age 11 or 12 is a biological time for beginning to move into formal operations, only about a third of students have achieved formal operations by the time they finish high school. Thus, research in this area tells us that elementary school students are not ready to develop a formal understanding of probability.

At the current time, the elementary school math curriculum includes a focus on "intuitive" probability. From the point of view of math educators familiar with ideas such as those quoted above, the goal is to have students gain an intuitive (not a formal math-oriented) understanding of some "simple" probability concepts. The realization is that students are simply not developmentally ready for a formal treatment of the topic.

The following example was presented in Chapter 2 of this book. It falls into the category of intuitive probability.

### Example 3: Math, 3rd Grade

> Jackie's dad baked 36 chocolate chip cookies and 24 peanut butter cookies on Monday. On Tuesday, he baked 12 cherry chip, and 15 mixed nut cookies. Jackie reaches into the cookie jar and pulls out a cookie. Which kind of cookie is she least likely to pull out?
>
> A. mixed nuts
>
> B. cherry chip
>
> C. peanut butter
>
> D. chocolate chip

Unfortunately, many elementary school teachers do not have a good intuitive understanding of probability and have not achieved formal operations in math. Thus, the teaching situation is often best described as "the blind leading the blind" and the results are not very good.

I believe that the same analysis holds for the person or team that created this 3rd grade math problem example. Consider, for example, what you know about cookie jars. One typically puts cookies into the jar through an opening in the top, and draws them out in the same manner. Thus, the situation tends to be one of "last in, first out." Cookies placed in a cookie jar do not (magically) arrange themselves in a random order. Thus, a correct answer to this problem depends on whether the Monday's cookies were put into the cookie jar before the Tuesday's cookies. It also depends on whether Tuesday's cookies were put into the jar in the order that they are mentioned in the word problem—thus, the cherry chip cookies going in first and the mixed nut cookies going in last. In that situation, I think that the probability of reaching into the jar and drawing out a mixed nut cookie is 100%.

The problem was written to be somewhat politically correct—the father baking the cookies. But, I am concerned about the small number of cookies baked on Tuesday. That does not fit with my understanding and experience in baking cookies. And, I find my mind is puzzling over what a cherry chip cookie might be. I don't believe I have ever encountered one. All in all I find that this example problem is poorly conceived.

It is easy to talk about an "intuitive understanding," but it is more difficult to state clearly what this might mean. Herbert A. Simon (1916-2001) was a Nobel Prize (in Economics) winning researcher and scholar who made many significant contributions in the areas of problem solving, computers, cognitive psychology, and economics. Many years ago he gave a talk at the University of Oregon, during the celebration of the addition of a new, major building in the Business School complex. In this talk he said, "Intuition is frozen analysis."

What he was saying is that intuition arises from (is based on) careful analysis of lots of examples or cases of the situation in which one eventually has a good intuition. There is quite a bit of literature on *mathematical intuition*. For example a Google search on 12/29/04 of the term produced 430,000 hits. I think that I agree with Simon and that I understand mathematical intuition both according to his insight and in an intuitive manner. My understanding makes me suspicious of the idea that elementary school students can develop a good intuitive understanding of a topic such as probability through the teachings of a teacher who lacks such intuition. Part of my conclusion is rooted in the fact that people had been gambling for thousands of years (presumably some of the people developing a good intuition of the odds in various situations), but that the development of a solid mathematical footing for probability was a major mathematical achievement.

The second of the two quoted paragraphs (the statement by Garfield and Ahlgren) points out another problem in elementary school math. Even at the middle school level, ratio and proportion tend to confound students. They can memorize procedures, but most gain relatively little understanding of what they are doing. Attempts to provide an "intuitive understanding" type of treatment of these topics at still lower grade levels tend to be relatively unsuccessful.

In summary, research on math and cognitive development suggest that attempts to teach these topics at the elementary school and middle school will be fraught with significant difficulties. My conversations with a very large number of teachers suggest that this research result is correct.

## Math Manipulatives

My analysis of Piagetian cognitive development and mathematical cognitive development is that much of the math curriculum students encounter at the precollege level is not being taught in a manner consistent with our understanding of cognitive development. It is being taught at a level of abstraction that is well above the developmental levels of students.

As previously mentioned, this situation tends to force the majority of students into memorize and regurgitate mode, where they develop only a modest understanding of what they are doing. Such mathematical knowledge is fragile and tends to disappear over time. It provides a very weak foundation for a student's future studying of math.

There is general agreement in the math education leadership that math should be taught in a manner that builds understanding, and that a successful math education program can and does help students to achieve understanding. Much of the current reform movement in math focuses on students gaining a higher level of understanding of the content being covered. One approach that is showing good signs of success is to make extensive use of math manipulatives. Math manipulatives fit in well with —help to bridge the gap—of students being at a concrete operations level, and gradually moving toward formal operations.

The ready availability of computers in schools has facilitated the development of computer-based manipulatives (virtual manipulatives), and these are now commonly used in school. Douglas Clement (1999) has written an excellent analysis of physical manipulatives versus virtual manipulatives. Many useful virtual manipulative materials are available free at the Website Virtual Manipulatives (n.d.).

I believe that our math education system is thinking "way too small" as it considers the use of physical and virtual manipulatives. Yes, indeed, such manipulative are useful in developing an understanding of important mathematical concepts, yes, indeed, such manipulatives are quite useful in moving students from the preoperational level to the concrete operations level. But in addition, physical and virtual manipulative lie at the very core of problem solving in many different disciplines.

For example, consider a business person developing a spreadsheet model of a certain aspect of a business, and then using this spreadsheet in posing and answering "What if?" types of questions. The spreadsheet model is a virtual manipulative.

Or, consider researchers developing an appropriate shape for an airplane or a car. They develop physical models that they test in wind tunnels. Nowadays, they develop computer models that they use as they pose and answer "What if?" types of questions. Physical and virtual manipulative are routine tools of these researchers.

Or, think about architects. In the past they developed physical models a\s well as blueprints and other drawings. Now, they develop computer models. They have long recognized the value of various types of physical and virtual models (physical and virtual manipulatives) in representing and solving the problems they face.

In 1998, one of the Chemists who received a Nobel Prize did so on the basis of his work on developing computer models of molecules in chemical reactions. That is, over the previous 15-20 years he had developed virtual manipulatives that proved to be powerful aids to understanding and attacking certain types of problems in chemistry.

I could continue to extend the list, but perhaps the message is becoming clear. Computer models (virtual manipulatives) are now commonly used to help represent and solve problems in many different disciplines. Math educators should take this into consideration as they make use of manipulative to help students learn math. At the same time their students are learning to use manipulative as an aid to learning math, they could be learning about use of manipulative to help represent and solve problems in many other disciplines.

## FOSS, Example from Science Education

Educators in each discipline are aware of the work of Piaget and other research in cognitive psychology. Thus, curriculum developers in each discipline pay attention to how their materials align with the cognitive development of the students who will use the materials.

The Full Option Science System (FOSS) is based on the teaching and learning philosophy of the Lawrence Hall of Science, University of California, Berkley (n.d.). The Lawrence Hall of

Science is one of the world's more successful and well-known hands-on museums of science. Quoting from Foss (n.d.):

> The FOSS program is correlated to human cognitive development. The activities are matched to the way students think at different times in their lives. The research that guides the FOSS developers indicates that humans proceed systematically through predictable, describable years, and that students learn science best from direct experiences in which they describe, sort, and organize observations about objects and organisms. Upper elementary students construct more advanced concepts by classifying, testing, experimenting, and determining cause and effect relationships among objects, organisms, and systems.
>
> FOSS investigations are carefully crafted to guarantee that the cognitive demands placed on students are appropriate for their cognitive abilities. Developmental appropriateness and in-depth exposure to the subject matter with multiple experiences give FOSS its "horizontal curriculum" character (numerous activities that provide a great variety of experiences at a cognitive level) as opposed to a "vertical curriculum" design (activities that attempt to take students to inappropriately complex and abstract levels of understanding). A horizontal curriculum provides challenges for all students and results in a much deeper understanding of the subject.

The FOSS curriculum is based upon a combination of research in science education and years of practical experience in working with young learners. Here are a few of the key quoted from the FOSS Website::

- learning moves from experience to abstractions. FOSS modules begin with hands-on investigations, then move students toward abstract ideas related to those investigations using simulations, models and readings.
- a child's ability to reason changes over time. FOSS designs investigations to enhance their reasoning abilities.
- fewer topics experienced in depth enhance learning better than many topics briefly visited. FOSS provides long-term (8-10 weeks) topical modules for each grade level, and the modules build upon each other within and across each strand, progressively moving students toward the grand ideas of science. The grand ideas of science are never learned in one lesson or in one class year.

It is interesting to compare and contrast these ideas and the FOSS approach to education with the various math curricula that are widely used in this country. The first bulleted item notes the challenge of abstraction, and emphasizes the need to move from the concrete to the abstract. The second bulleted item emphasizes working over time to enhance the growing reasoning ability of learners. The third bulleted item addresses the issue of breadth versus depth, indicating that the developers of FOSS favor depth over breadth.

## Elementary School Applications

7.1 You have an understanding of the number line. Probably your understanding is quite a bit different than that of most elementary school students. For example, you have insight into the existence of irrational numbers such as the positive square root of 2, and you can probably make a mark on a number line close to where this number lies. Your broader and deeper understanding developed over the years as you studied math and as your brain continued to develop. Think about the understanding of the number line you expect your average student to have at the begging of the school year and then at the end of the school year for a specific grade level that interests you. Carry on a whole class discussion with your students to gain insight into their current understanding of the number line. Examine the math content you are teaching and how it relates to increasing student understanding of the number line. When appropriate, engage your students in a discussion about how the math content topic fits in with and expands their understanding of the number line.

7.2 Watch your students as they do paper and pencil arithmetic and as they make use of math manipulative to explore various math topics. Likely you will see some students who are better at (more comfortable with) one of these activities as compared to the other. There can be transfer of learning in either direction—from manipulative to abstract symbols, or vice versa. If you see an example of this happening, point it out to the whole class and use the situation to help your students to learn to find and make use of such connections.

7.3 Talk with your students about models, such as toy cars, model airplanes, toy figurines of people and animals, and so on. Move the focus toward the similarities and differences between models and the "real thing." What can one learn from use of models? After this conversation goes on for awhile, move it in the direction of mathematics. In what sense is a mathematical sentence such as "3 + 5 = 8" a model for "If we have one group of three people and a different group of five people, and we combine the two groups, we will have a total of eight people?" Math modeling lies at the very heart of use of math to help represent and solve problems. Work to learn the current level of your students' understanding of this idea, and then to help them expand their understanding.

## Activities for Self-Assessment, Assignments, and Group Discussions

7.1 The chances are good that you are at the Formal Operations level on the four-level Piagetian Cognitive Developmental Scale. Think about where you fall on the six-stage mathematical cognitive developmental scale. Share your insights into your mathematical self that result from this activity.

7.1 Can a teacher be an effective teacher of elementary school mathematics if the teacher is not at Level 4 (Piagetian and Math formal operations) on the six-stage mathematical cognitive developmental scale? Present arguments on each side of this issue, as well as suggestions for an elementary school math teacher who is not at this math cognitive developmental level.

7.3 Explore and share your insights into how math manipulative fit into helping students learn math while at various states in their mathematical development. What do you know about uses of and the effects of using physical manipulatives versus virtual manipulatives (that is, computerized manipulatives)?

7.4 Develop a lesson plan in which students use math manipulatives to help learn some math ideas and, at the same time, increase their understanding of math modeling.

# Chapter 8: Cognitive Neuroscience

Cognitive neuroscience is a relatively new discipline, combining aspects of brain science and mind science that specifically focus on cognition.

> Cognitive neuroscience has emerged in the last decade as an intensely active and influential discipline, forged from interactions among the cognitive sciences, neurology, neuroimaging (including physics and statistics), physiology, neuroscience, psychiatry, and other fields.
>
> …
>
> The cross-disciplinary integration and exploitation of new techniques in cognitive neuroscience has generated a rapid growth in significant scientific advances. Research topics have included sensory processes (including olfaction, thirst, multi-sensory integration), higher perceptual processes (for faces, music, etc.), higher cognitive functions (e.g., decision-making, reasoning, mathematics, mental imagery, awareness), language (e.g., syntax, multi-lingualism, discourse), sleep, affect, social processes, learning, memory, attention, motor, and executive functions (NSF, 2002).

Cognitive neuroscience research using brain imaging is beginning to make significant contributions to our understanding of learning and using math, although this type of research is still in its infancy. For example, by 1999 brain imaging showed different parts of the brain being used in exact calculations than being used in estimations or approximate calculations (Dehaene et al. 1999). This provides scientific evidence to support the idea that teaching students to do exact calculations and teaching students to estimate are distinct topics, and that transfer of learning between these two topics may be a challenge to students and their teachers.

As another example of cognitive neuroscience progress, in an earlier chapter I mentioned that research on gF suggests that this component of g increases into early adulthood. A recently published longitudinal brain imaging study reports results that seem to be consistent with this gF result (Gogtay et al., 2004).

Robert Sylwester is a well-known educator and authority on how better understanding of the brain can shed light on education practices that directly impact the classroom. He writes a monthly column that is available on the Web (Sylwester, 2004). Quoting from his October 2004 article:

> John Dewey, Jean Piaget, and B. F. Skinner helped shape 20th century educational policy and practice by connecting teaching and learning to emerging cultural and scientific developments. Recent dramatic advances in the cognitive neurosciences and computer technology suggest that a similar set of creative educational theorists will soon emerge to help schools connect teaching and learning to 21st century biology and technology.
>
> …
>
> … how is it possible for networks of firing neurons to spark my subjective feelings and thoughts—to transform matter into mind? The search for the meaning and mechanisms of consciousness has historically been the speculative purview of philosophers and theologians (who tended to consider it a disembodied essence beyond the capabilities of biological research). Times change, however, and neuroscientists can now explore the biology of consciousness via the remarkable observational capabilities of brain imaging technology. Although conventional wisdom saw biological explanations of consciousness as emerging in the distant future, the renowned neuroscientist Jean-Pierre Changeux recently wrote, "The day when the autonomy of

> consciousness can be given a neuronal interpretation may not be as far off as was generally supposed"

In brief summary, the discipline of brain and mind science has progressed to a level in which it can and is making significant contributions to teaching and learning.

## Dyscalculia

Brain imaging has identified regions of the brain associated with different types of dyscalculia, a difficulty in learning certain aspects of math (Stanescu-Cosson et al., 2000; Pearson, 2003). Quoting from Geary (1999):

> Over the past several decades important advances have been made in the understanding of the genetic, neural, and cognitive deficits that underlie reading disability (RD), and in the ability to identify and remediate this form of learning disability (LD). Research on learning disabilities in mathematics (MD) has also progressed over the past ten years, but more slowly than the study of RD. One of the difficulties in studying children with MD is the complexity of the field of mathematics. In theory, MD could result from difficulties in the skills that comprise one or many of the domains of mathematics, such as arithmetic, algebra, or geometry. Moreover, each of these domains is very complex, in that each has many subdomains and a learning disability can result from difficulties in understanding or learning basic skills in one or several of these subdomains.
>
> As an example, to master arithmetic, children must understand numbers (e.g., the quantity that each number represents), counting (there are many basic principles of counting that children must come to understand), and the conceptual (e.g., understanding the Base-10 number system) and procedural (e.g., borrowing from one column to the next, as in 43-9) features involved in solving simple and complex arithmetic problems. A learning disability in math can result from difficulties in learning any one, or any combination, of these more basic skills. To complicate matters further, it is possible, and in fact it appears to be the case, that different children with MD have different patterns of strengths and weakness when it comes to understanding and learning these basic skills.

Perhaps 5-7 percent of students have some form of dyscalculia. Early identification of dyscalculia can make a significant contribution to helping students deal with this learning disability. Symptoms of dyscalculia include (Dyscalculia, n.d.):

- Difficulty with numbers;
- Poor understanding of the signs +, -, / and x, or may confuse these mathematical symbols;
- Difficulty with addition, subtraction, multiplication and division or may find it difficult to understand the words “plus,” “add,” “add-together”;
- May reverse or transpose numbers for example 63 for 36, or 785 for 875;
- Difficulty with times tables;
- Poor mental arithmetic skills;
- Difficulty telling the time and following directions.

Our current educational system is not good at early identification of students with dyscalculia. But, early identification and a strong intervention can help a student overcome or more effectively real with this disability. The possible parallel with dyslexia (a serious reading disability) is interesting. Research in this area now strongly suggests that a strong, early intervention can lead to “rewiring” of the brain in a manner that contributes significantly to a person being able to become a fluent reader.

To do a precise diagnosis that a student has dyscalculia requires considerable knowledge and skill. However, an elementary teacher or a parent can easily study the bulleted list given above and do a preliminary screening of students who seem to be having considerable difficulty in learning math. In addition, students identified as dyslexic should also be carefully screened for dyscalculia.

## Attention

Attention is a large and important component of the discipline of neuroscience. A human's five senses bring in an overwhelming amount of data. The brain, at a conscious and subconscious level, pays attention to some of this data; however, it filters out and ignores most of this data.

This presents a challenge both to teachers and to students. In a schoolroom class, a student's brain is processing input from five senses, and it is thinking about lots of other things. For example, it may be sensing that he or she is hungry, bored, will have a lot of fun later in the day, would rather be listening to some good music, is worried about a recent interaction with a friend, and so on. A teacher needs to teach in a manner that catches and holds student attention, and the student needs to learn to focus his or her attention on the learning task at hand.

The following is from Posner & Fan (in press).

> Everyone knows what attention is. It is the taking possession of the mind in clear and vivid form of one out of what seem several simultaneous objects or trains of thought." (James, 1890).
>
> However, this subjective definition does not provide hints that might lead to an understanding of attentional development or pathologies. The theme of our paper is that it is now possible to view attention much more concretely as an organ system.
>
> …
>
> We believe that viewing attention as an organ system aids in answering many perplexing issues raised in cognitive psychology, psychiatry and neurology. … **We can view attention as involving specialized networks to carry out functions such as achieving and maintaining the alert state, orienting to sensory events and controlling thoughts and feelings.** [Bold added for emphasis.]

The study of attention as an organ is now being facilitated by brain imaging technology. Researchers are beginning to understand which parts of the brain are active when a person is paying attention, or focusing attention in a particular manner. This is contributing to increased understanding of Attention Deficit Disorder (ADD) and other attention pathologies.

As a potential or current teacher at the elementary school level, you know that there are many different things that attract student attention away from the topics being addressed in class. One of the reasons that this happens is that some of the school topics are, from a student's point of view, "just plain boring." In math education, for example, this helps to explain why (very roughly speaking) most children find math class time reasonably interesting up through about the 3rd or 4th grade. During those first few years of formal schooling, math tends to contain many new, interesting, empowering, and attention grabbing ideas. After about the 3rd to 4th grade, an increasing number of students find that math class is not particularly interesting and does not hold their attention.

In a later chapter of this book, I discuss computers. One aspect of computer games is that they are attention grabbing and attention holding. At the current time, research indicates that elementary school children are spending more time playing computer games than they are watching television (Science of Mental Health, 2004). Both television and computer games can be viewed as major competitors for a student's attention! Repetitious paper and pencil drill and practice of computational algorithms does not compete well with such media.

## Genetics

The past decade has seen a very high rate of progress in genetics and in decoding the human (and other) genomes. We now have theory and instrumentation that helps us gain increased understanding of the human brain. We have steadily increasing knowledge of the human genome, noninvasive tools for brain imaging, and tools and skills for manipulation of individual genes. This progress has raised the nature versus nurture discussion to an entirely new level. We are gaining increased understanding of nature, and we now have the ability to change nature.

> In the coming decades, we will all collectively as a society need to decide what we think about biotechnology and what applications we are and are not willing to allow. The debates we have now, about cloning and stem cell research, pale in comparison to debates we are likely to encounter as the technology for manipulating genes advances. We are already at the point where it is possible to screen embryos for the predisposition to certain life-threatening illnesses; as we unravel more and more of the genome, we will be able to detect more and more disorders (or predispositions to disorders) well in advance of birth. Ultimately, if we so choose, we may be able to directly manipulate embryonic genomes—add a gene here, delete a gene there. The genes of a child might eventually be more a matter of choice than of chance (Marcus, 2004, p174).

I assume that you are aware of the issue of athletes taking drugs to enhance the development and performance of their physical bodies. And, perhaps you drink beverages that contain caffeine, and you know that caffeine enhances brain alertness and performance. In the coming years there will be a steadily increasing number of "drugs" that can enhance brain development and performance. Thus, as a teacher, you can look forward to having to deal with issues of students who have been genetically enhanced and/or enhanced by a variety of drugs.

## Chunks and Chunking

Here are three different types of human memory:

- Sensory memory stores data from one's senses, and for only a short time. For example, visual sensory memory stores an image for less than a second, and auditory sensory memory stores aural information for less than four seconds.
- Working memory (short term memory) can store and actively process a small number of chunks. It retains these chunks for less than 20 seconds.
- Long-term memory has large capacity and stores information for a long time.

Research on working memory indicates that for most people the size of this memory is about 7 ± 2 chunks (Miller, 1956). This means, for example, that a typical person can read or hear a seven-digit telephone number and remember it long enough to key into a telephone keypad. The word *chunk* is very important. When I was a child, my home phone number was the first two letters of the word diamond, followed by five digits. Thus, to remember the number (which I still do, to this day) I needed to remember only six chunks. But, I had to be able to decipher the first chunk, the word "diamond."

Long-term memory has a very large capacity, but this does not work like computer memory. Input to computer memory can be very rapid (for example, the equivalent of an entire book in a second), and a computer can store such data letter perfect for a long period of time. The human brain can memorize large amount of music, poetry, or other text. But, this is a long and slow process for most people. By dint of hard and sustained effort, an ordinary person can memorize nearly letter perfect the equivalent of a few books. However, the typical person is not very good at this. At the current time, the Web contains the equivalent of tens of millions of books.

On the other hand, the human brain is very good at learning meaningful chunks of information. Think about some of your personal chunks such as constructivism, multiplication, democracy, transfer of learning, and Mozart. Undoubtedly these chunks have different meanings to me than they do for you. As an example, for me, the chunk "multiplication" covers multiplication of positive and negative integers, fractions, decimal fractions, irrational numbers, complex numbers, functions (such as trigonometric and polynomial), matrices, and so on. My breadth and depth of meaning and understanding were developed through years of undergraduate and graduate work in mathematics.

It is useful to think of a chunk as a label or representation (perhaps a word, phrase, visual image, sound, smell, taste, or touch) and a collection of pointers. A chunk has two important characteristics:

1. It can be used by short-term memory in a conscious, thinking, problem-solving process.
2. It can be used to retrieve more detailed information from long-term memory.

Our education system can be substantially improved by taking advantage of our steadily increasing understanding of how the mind/brain learns and then uses its learning in problem solving. Chunking information to be learned and used is a powerful aid to learning and problem solving. However, even if two people receive the same education about a topic, and use the same label for a chunk that they form on that topic, their chunks will be quite different. (This is a key idea in constructivism.)

## Brain Versus Computer

In the early days of computers, people often referred to such machines as *electronic brains*. Even now, more than 50 years later, many people still use this term. Certainly a human brain and a computer have some characteristics in common. However:

- Computers are very good at carrying out tasks in a mechanical, "non-thinking" manner. They are millions of times as fast as humans in tasks such as doing arithmetic calculations or searching through millions of pages of text to find occurrences of a certain set of words. Moreover, they can do such tasks without making any errors.
- Human brains are very good at doing the thinking and orchestrating the processes required in many different very complex tasks such as carrying on a conversation with a person, reading for understanding, posing problems, and solving complex problems. Humans have minds and consciousness. A human's brain/mind capability for "meaningful understanding" is far beyond the capabilities of the most advanced computers we currently have.

There are many things that computers can do much better than human brains, and there are many things that human brains can do much better than computers. Our educational system can be significantly improved by building on the relative strengths of brains and computers, and decreasing the emphasis on attempting to "train" students to compete with computers. We need to increase the focus on students learning to solve problems using the strengths of their brains and the strengths of Information and Communication Technology (ICT).

One of the key issues in studying human brains and computer-as-brains is the human brain-computer brain interface. If we go back to the time of the first computers, the interface was mainly via an electric typewriter device, light displays, and punch paper tape and cards. Later came display screens, touch screens and the mouse. Now we also have voice input and output. We also have wireless connectivity and cell phones.

At the current time, some research projects are working on implanting ICT systems into people's brains. Cochlear implants and retina implants can be considered as part of this overall endeavor, and cochlear implants are now relatively common. Brain implants have been used to help deal with epilepsy. Research is being done on creating a direct connection between a person's brain and a computer located outside the brain. For example, a volunteer in this research program is able to play a simple computer game involving movements of the curser by "thinking" up, down, right, and left.

The point I am making here is that in the past, and continuing into the future, there has been substantial research on improving human-computer interfaces. As a computer user, you likely make routine use of a mouse, video display screen, and a keyboard. In the future, we will see significant progress in still building still more direct brain-computer interfaces. Improvements in the interface will have a significant impact on education. In essence, such improvements contribute to the idea of the compute as an auxiliary brain, or as a brain enhancement. The way I view it is that me (and my human brain/mind) along with my computer system can accomplish a wide range of tasks and solve a wide range of problems that I, all by myself, cannot do.

## Rate of Learning

Howard Gardner has identified eight domains or types of intelligence, including math/logic and spatial. One aspect of having varying levels of intelligence in these various domains is that a person is likely to have some differences in rates of learning and in learning potentials in these various domains.

Differences in rates of learning are very evident for students with special needs. Our school system does not deal very well with varying rates of learning, even though it puts a lot of money into special education. A solid example of this is provided by the math learning of students who are classified as learning disabled.

> The background literature of special education has long shown that students with mild disabilities (a) demonstrate levels of achievement approximating 1 year of academic growth for every 2 or 3 years they are in school (Cawley & Miller, 1989); (b) exit school achieving approximately 5th- to 6th-grade levels (Warner, Alley, Schumaker, Deshler, & Clark, 1980); and (c) demonstrate that on tests of minimum competency at the secondary level, their performance is lower for mathematics than it is for other areas (Grise, 1980). Crawley et al., 2001).

On the other end of the cognitive ability scale, there are quite a few students in school who can learn math much faster than the average student. A typically elementary school class will likely have several students who learn math at least 50% faster than average. That is, these students are capable of making one and a half (or more) years of school math progress per school year. Think about this in terms of "no child left behind."

Rate of learning is also highly dependent on the nature and quality of instruction. There has been quite a bit of research on the value of providing students with individual tutors. Research by Benjamin Bloom (1984), the same person who was responsible for Bloom's Taxonomy, indicates that with individual tutoring a "C" student can become an "A" student. Similar research forms the basis of very small class and individual tutoring approaches to helping students who are making slow progress in learning math and reading.

Students vary tremendously in their rates of learning and their abilities to learn various disciplines. Individualization of instruction and individualization of programs of instruction —such as in an Individualized Education Program (IEP) make a significant contribution to improving the rate of learning of a student. This type of research helps to explain the success of computer-assisted instruction that can provide a type of highly interactive, somewhat individualized type of instruction.

## Augmentation to Brain/Mind

Reading and writing provide an augmentation to short (working) term and long-term memory for personal use and that can be shared with others. Data and information can be stored and retrieved with great fidelity. As Confucius noted about 2,500 years ago, "The strongest memory is not as strong as the weakest ink."

Writing onto paper provides a passive storage of data and information. The "using" of such data and information is done by a human's brain/mind.

Computers add a new dimension to the storage and retrieval of data and information. Computers can process (carry out operations on) data and information. Thus, one can think of a computer as a more powerful augmentation to brain/mind than is provided by static storage on paper or other hardcopy medium. The power, capability, and value of this type of augmentation continue to grow rapidly. Certainly this is one of the most important ideas in education at the current time. At the current time our formal educational system has yet to understand the idea of ICT as an augmentation to the mind/brain.

In thinking about chunks and learning, I see two approaches. In the first approach, a clear framework is provided. Think of the framework as scaffolding for a chunk along with a label for the chunk. One learns the framework and then fits new knowledge and experiences into the framework. In the second approach, one creates their own framework. This is less efficient initially, but perhaps more productive over the long run in the task of helping students learn to learn and to take increasing responsibility for their own learning.

To illustrate, suppose I want to know a modest amount about something that others have carefully studied. Since part of a discipline is how to teach and learn it, I decide to take advantage of this accumulated knowledge. I have the discipline taught to me by an expert teacher.

But now, suppose that I want to extend my knowledge to "my" world and to situations not covered in the standard curriculum. Now, I hope that I have learned to learn on my own. I hope that I have the creativity and skill to discover, invent, find, and so on, and fit my new learning into the old framework. I hope that I can restructure the old framework so that it better fits the new and my needs.

There is one more important piece to this. Suppose that the area that I want to study is one in which computers provides powerful aids to solving its problems. Then I want my chunk to include a link to the capabilities and limitations of computers as an aid to solving the problems. I want to have the knowledge and skills to make use of this computer augmentation to my brain. The next chapter focuses on computers in education.

## Elementary School Applications

8.1 Brainstorming is a very useful strategy in thinking about a problem situation. In some sense, one's brainstorming around a particular idea is like putting together some of the topics that are chunked (in one's brain) with the idea. Present your class with a problem or task related to the math they are studying. Do a whole class brainstorm to illustrate the process of brainstorming. When done, facilitate discussion about how the results relate to the problem or task at hand, and see if the class collectively can solve the problem or accomplish the task. This activity can be used a number of times over a school year. Among other things, it helps to expand an existing chunk that some or perhaps all of the students have.

8.2 Play a short-term memory game with your students. For example, hold up a picture of a geometric figure for the class to view for a few seconds. Then, each class member is to draw the figure from memory. Or, invite a student to come to your desk and view a geometric figure. The student must then go to another student and tell the student how to draw the geometric figure. These activities can be more challenging by adding letters to designate various vertices or edges of the figure, so that these also must be held in short term memory. The activity tends to be easy for students if the figure is one that they know a word for, such as square, rectangle, or triangle. Then the figure is just one chunk. Chunking helps in dealing with the burden of labeled vertices and edges if they are labeled systematically, such as a rectangle with vertices A, B, C, D working counterclockwise.

## Activities for Self-Assessment, Assignments, and Group Discussions

8.1 Select a math topic that you feel it is important for an elementary school student to learn. Think about this from the point of view of being a "chunk" that the student will construct in his or her brain/mind. What is a good name for this chunk? What is a good mental image or picture for this chunk? How do you expect this chunk to grow in breadth and depth over time? What are some aspects of this chunk that you expect will serve the student over a lifetime?

8.2 Think carefully about your rate and ease of learning math versus your rate and ease of learning some other discipline that is in the elementary school curriculum. How have you and the school curriculum accommodated these rates of learning during your many years of being a student?

# Chapter 9: Information and Communication Technology (ICT)

Information and Communication Technology (ICT) includes calculators, computers, telecommunications, the Web, digital still and video cameras, digital phones, audio and video recorders and players, and so on. ICT was mentioned a number of times in the previous chapter on cognitive neuroscience. This chapter provides additional ideas on ICT and math education.

ICT and math share much in common. For example, both provide powerful aids to problem solving across a wide range of academic disciplines. Both have tremendous breadth and depth. Much of the theory of the discipline of Computer and Information Science is rooted in mathematics, and much of the notation used in this discipline comes from math. Computer science majors in higher education typically have to take at least two years of college math at the level of calculus, discrete math, or higher.

## ICT Cognitive Development Scale

ICT is a large, vibrant, and rapidly growing field. The International Society for Technology in Education (ISTE) has developed national educational technology standards for students, teachers, and school administrators. These standards have been widely adopted and serve to provide a good sense of direction for the ICT preparation of teachers and their students (ISTE NETS, n.d.).

The discipline of ICT can be divided into ICT Content and ICT Maturity, much like I have done for math earlier in this document. ISTE NETS for Students (ISTE NETS-S) provides recommendations for ICT content the PreK-12 curriculum. ISTE NETS for Teachers provides the recommendation that precollege teachers should meet the ISTE NETS-S and should have a substantial amount of knowledge and skill in educational uses of computers.

Following the same line of reasoning that led to the math cognitive development scale given earlier in this document, I have been working on an ICT cognitive development scale. My current version (very rough draft) is in figure 9.1.

| Stage "Title" | Age and/or Education Levels | Brief Discussion |
|---|---|---|
| Stage 1. Piagetian Sensorimotor. | Age birth to 2 years. Informal education provided by parents, and other caregivers. | Infants use sensory and motor capabilities to explore and gain increasing understanding of their environments.<br>ICT has brought us a wide range of sound and music-producing, talking, moving, walking, interactive, and developmentally appropriate toys for children in Stage 1. These contribute both to general progress in sensory motor growth and also to becoming acquainted with an ICT environment. |
| Stage 2. ICT Preoperational. | Age 2 to 7 years. Includes both informal education and increasingly formal education in preschool, kindergarten, and | During the Piagetian Preoperational stage, children begin to use symbols, such as speech. They respond to objects and events according to how they appear to be. They accommodate to the language environments they spend a lot of time in.<br>ICT provides a type of symbols and symbol sets that are different from the speech, gestures, and other symbol sets that have traditionally been available.<br>TV and interactive ICT-based games and edutainment are a significant |

| | | |
|---|---|---|
| | first grade. | environmental component of many children during Stage 2. During this stage children can develop considerable speed and accuracy in using a mouse, touch pad, and touch screen to interact and problem solve in a 3-dimensional multimedia environment displayed on a 2-dimensional screen. |
| Stage 3. ICT Concrete Operations. | Age 7 to 11 years. Includes informal education and steadily increasing importance of formal education at grades 2-5 in elementary school. | During the Piagetian Concrete Operations stage, children begin to think logically. In this stage intelligence is demonstrated through logical and systematic manipulation of symbols related to concrete objects. Operational thinking (mental actions that are reversible) develops.<br>ISTE has established NETS-Student that includes a statement of what students should be able to do by the end of the fifth grade. During the ICT Concrete Operations stage children:<br>• Learn to use a variety of software tools such as those listed in the 5th grade ISTE NETS-Student, and begin to understand some of the capabilities and limitations of these tools. (They do logical and systematic manipulation of symbols in a computer environment.)<br>• Learn to apply these software tools at a Piagetian Concrete Operations level as an aid to solving a wide range of general curriculum-appropriate problems and tasks. |
| Stage 4. ICT Formal Operations. | Age 11 and beyond. This is an open ended developmental stage, continuing well into adulthood.<br>Requires ICT knowledge, skills, speed, and understanding of topics in ISTE NETS for students finishing the 12th grade. | During the Piagetian Formal Operations stage, thought begins to be systematic and abstract. In this stage, intelligence is demonstrated through the logical use of symbols related to abstract concepts.<br>Formal Operations in ICT includes functioning at a Piagetian Formal Operations level in specific activities such as:<br>1. Communicate accurately, fluently, and with good understanding using the vocabulary, notation, and content of ISTE NETS-S for the 12th grade.<br>2. Given a piece of software and a computer, install and run the software, learn to use the software, explain and demonstrate some of the uses of the software, save a document you have created, and later return to make further use of your saved document.<br>3. Problem solve at the level of detecting and debugging hardware and software problems that occur in routine use of ICT hardware and software.<br>4. Convert (represent, model, pose) real world problems from non-ICT disciplines into ICT problems, and then solve these problems.<br>5. Routinely and comfortably use ICT in the other disciplines you have studied, at a level consistent with and supportive of your cognitive developmental level in these disciplines. |
| Stage 5 and higher. Label or title not yet determined. | As with math, there are ICT developmental levels well above what are typically achieved by well educated high school graduates. | This section remains to be written. |

Figure 9.1 ICT cognitive development and expertise scale.

## Procedures and Procedural Thinking

From a computer programmer point of view, a computer program is being a procedure—a step-by-step set of directions—that can be carried out by a computer. Programmers develop procedures to solve or help solve problems. In doing this, they make use of procedures (both algorithms and heuristics) written by others and by themselves. That is, building on the previous work of oneself and others is a standard approach used by computer programmers.

For an example, consider a team of programmers developing a word processor. It is a relatively simple task to write a program that accepts input from a keyboard, displays the text on a screen, stores it in computer memory, and outputs it when commanded to do so. The task gets a little more difficult when the text contains a variety of fonts and character sizes, bold and italic, and so on. Still, the challenge is not too big.

But, it is sure nice to have an outliner built into a word processor, as well as the ability to insert pictures and other graphics. The programmer's challenge is growing.

Next, consider the idea that a person using a word processor to write a paper may want the software to help in various ways, such as spell checking, grammar checking, formatting for final publication, perhaps generating a table of contents and/or an index, and so on. It is helpful to have provisions for creating tables and lists, alphabetizing a list, or sorting a list into numerical order. The challenge to the programmer continues to grow.

However, a pattern is beginning to emerge. There is the basic word processor. Then, there are a lot of different features (in essence, separate procedures) that can be added to it. Thus, a number of programmers can work on the overall task because it can readily be broken down into a collection of smaller, more manageable tasks.

Moreover, some of the tasks have been done lots of times by other programmers. Sorting a list alphabetically or numerically is a common programming task in a first term computer programming course. Thus, the team of professional programmers working to develop a word processor will make extensive use of these and other procedures that are stored in a library of computer procedures.

Other components may be really challenging. Consider a grammar checker. This task is an area of research, and it is very challenging. The grammar checker in a word processor such as Microsoft Word in gradually getting better due to the efforts of researchers from a variety of different disciplines, including the work of people in artificial intelligence.

In summary, writing a computer program is a particular type of problem solving task. It involves procedural thinking and developing procedures to solve or help solve a problem. The programmer is developing a tool that may be used by problem solvers in many different disciplines, or that may be quite narrow and scope and used only by a few narrow specialists in a particular narrow part of some discipline. Typically, the programming tasks faced by programmers are complex, and a program is typically a large and relatively complex set of instructions.

## Math-Related ICT Topics

Listed below are some math-related ICT topics. The intent of this list is to provide you with a hint of the breadth and depth of this discipline. More detail on a number of these topics is available in my math Website (Moursund, n.d., Math website). If you are interested in still broader aspects of ICT in education, a number of appropriate and free materials are available at Moursund (n.d., Free Materials).

1. The discipline of mathematics is now commonly divided into three major components: pure math, applied math, and computational math.

"Computational" is also a new aspect of many other disciplines. For example, one of the winners of the 1998 Nobel Prize in Chemistry received the award for his past 15 years of work in Computational Chemistry. Computer-based modeling and simulation, based on computational mathematics, is now a common component of each discipline that makes use of mathematics. Such modeling and simulation is very slowly working its way into the math curriculum.

2. Computer algebra systems (CAS) provide very powerful tools to carry out a wide range of mathematical procedures.

   It is now common for students taking high school math courses to learn to use graphing, equation-solving calculators that have some built-in rudiments of CAS. Ideas such as function, equation, and graphing are very important ideas in math. As you work with elementary school students you are laying the foundations for their future learning of these topics. Among other things this means you are helping students to develop chunks in their mind/brain that can grow to include these topics. For more information see Moursund (n.d., Computational Math).

3. Computer-assisted learning (CAL) is gradually improving. We now have Highly Interactive Intelligent Computer-Assisted Learning (HIICAL) systems that are quite good. The meaning of "quite good" can be debated. Research in this area tends to compare test scores of students taught by conventional instructional methods versus test scores of students taught by HIICAL. There is now a significant amount of such software that, on average, leads to better test scores than does conventional instruction (Moursund, 2002).

   HIICAL software can be developed that integrates the power of computer-assisted instruction with the power of CAS systems. That is, we are gradually seeing a merger of powerful computer tools and powerful aids to learning and using the tools. Such software has the potential to lead to major changes in math education. The goal might become to educate students so that they function well mathematically in a world in which such systems are readily available.

4. It is helpful to think about math *training* versus math *education.* Most of what an animal trainer does falls into the category of training, as contrasted with education. Education has a focus on understanding; training has a focus on rote performance.

   Our educational system consists of a mixture of training and education, and it is not easy to draw a clear distinction between the two. Research in computer-assisted learning suggests that this approach to teaching and learning is currently more effective in training than it is in education. Suppose, for example, that we want students to memorize the single digit multiplication facts and to be able to retrieve these facts with great speed and accuracy. This can be considered as a training task, and CAL is quite effective in this teaching/learning situation. Even the simplest of HIICAL designed for such training is able to individualize instruction, detect student weaknesses and address these weaknesses, and assess student speed and accuracy. From those points of view, such a CAL system is definitely more effective than a teacher working with a whole class. As we look to the future of math education, we will see HIICAL becoming a common component.

This document has previously mentioned math manipulatives. Such manipulatives can be used in both training and education modes. However, the current focus on using math manipulatives is in education for understanding and problem solving, rather than on training. For a list of resources on virtual (that is, computer-based) manipulatives for use in math education see Virtual Manipulatives (n.d.).

5. Artificial intelligence (AI) is a branch of the discipline of Computer and Information Science. It focuses on developing hardware and software systems that solve problems and accomplish tasks that—if accomplished by humans—would be considered to be a display of intelligence. As I look toward the future, I see a steady increase in situations where people and AI systems work together to solve problems and accomplish tasks.

   What is artificial intelligence? It is often difficult to construct a definition of a discipline that is satisfying to all of its practitioners. AI research encompasses a spectrum of related topics. Broadly, AI is the computer-based exploration of methods for solving challenging tasks that have traditionally depended on people for solution. Such tasks include complex logical inference, diagnosis, visual recognition, comprehension of natural language, game playing, explanation, and planning (Horvitz, 1990).

   AI is of steadily growing importance in education (Moursund, 2004c). Elementary school students already have a mind/brain chunk in this area, based on the robots and computers they see on television, their electronic toys, and so on. One of your jobs as a teacher is to shape this chunk so that it is more accurate and so that it can better accommodate future learning. For example, a handheld calculator has some intelligence. Think about how this intelligence is similar to and different from human intelligence in math.

6. Distance education is a rapidly growing field. If we use a rather broad definition of distance education, then it is already in common use in elementary schools. When a student uses the Web to retrieve information, this is a form of distance education. When a student uses a help feature in a software package, this is a form of distance education. Much of the CAL that students is accessed through a computer that is remotely located; thus, much of current CAL is a type of distance education.

   Imagine the situation in which HIICAL that covers the entire math curriculum is routinely available to students at home, at school, and wherever else they have access to the Internet. Such a system would also provide access to CAS, large numbers of math resource books, and other aids to learning and using math. While the progress in this direction seems relatively slow, I believe that this situation will be a standard part of many educational systems within the next two decades.

7. In light of goals for students learning math content and gaining in math maturity, how authentic is math assessment? Outside of school testing situations, people who need to make appreciable use of math tend to make use of calculators, computers, and many specialized devices (such as a global

positioning system, computerized laser measuring and surveying systems) as aids to math problem solving. This suggests that authentic assessment in math should be moving in the direction of open book, open notes, open calculator, open computer, and similar forms of assessment. Some progress in this direction has occurred in the use of calculators, but little progress is occurring other aspects of authentic math assessment. See Moursund (n.d., Project-Based learning).

## Elementary School Applications

9.1 Many math educators and others feel it is important for students to develop high accuracy and speed on number facts and simple arithmetic calculators. Thus, they make use of timed tests along with a lot of drill and practice. (Note that many other math educators think that this is not an appropriate way to teach math!). Locate a suitable piece of math education software that provides timed test and/or drill and practice. Have your students use it. Then hold a whole class discussion about what they like and what they don't like about use of the software.

9.2 Provide your students with calculators that have a M+ (that is, a memory that can be added to) key. Carry on a whole class discussion about what is the same and what is different between this calculator and a computer. Make sure that the discussion eventually includes calculator and/or computer memory. Both a calculator and a computer have memory and a central processing unit (CPU). A CPU on a simple calculator can carry out a very limited number of operations such as add, subtract, multiply, and divide. The CPU on a computer may well be able to carry out a hundred or more different operations. Both a calculator and a computer can automatically follow a step-by-step set of instructions.

## Activities for Self-Assessment, Assignments, and Group Discussions

9.1 Many leaders in the field of ICT in education argue that the development of writing, the mass printing and distribution of printed materials made possible by Gutenberg's movable type printing press, and the development of computers are the three most important developments in the history of education. Compare and contrast current and potential roles of ICT in education relative to the contributions made by writing and the printing press.

9.2 Make a list of things that you can do much better than ICT systems, things that ICT systems can do much better than you, and things that you and ICT systems working together can do much better than either can do alone. Analyze your list from the point of view of our current elementary school and teacher education systems.

9.3 Summarize and analyze your thoughts on having most math tests be open book, open calculator, and open computer.

# Chapter 10: Conclusions and Final Thoughts

Math education is a large, complex, and challenging discipline. The formal teaching of math began at the time of the first formal teaching of reading and writing, a little more than 5,000 years ago. During the past 5,000 years, the collected mathematical knowledge of the human race has grown immensely. A number of ideas that challenged the mathematical geniuses of their time have trickled down into the precollege school math curriculum—indeed, even into elementary school.

As the agriculture age has given way to the industrial age and now the information age, the math-related demands placed on people have grown. In information age societies such as the United States, there are now much higher math education expectations than there were in the industrial age or the agricultural age. As our society continues to raise its math education expectations, it is not achieving the math learning gains that it would like.

Because math knowledge and skills are so important in our information age society, you can expect to see continued efforts to "reform" our math education system. This book supports the idea that with appropriate informal and formal teaching and support, students (on average) can gain greater Math Content Knowledge and greater Math Maturity than they are currently obtaining. However, such math education goals leave us with many challenging issues. Here are a few examples:

1. It is likely that well over half of parents and elementary school teachers have not achieved Math Formal Operations. Their levels of School Math Maturity and School Math Content Knowledge are low. Thus, on average, children growing up in our society tend gain their first dozen years (birth through grade school) of informal and formal math education in what I would call relatively poor math education environments. If we want to significantly improve our math education system, we will have to make significant progress toward addressing this problem.

   This means, of course, that significant progress will take decades. As we gradually improve the math education of preservice and inservice elementary school teachers, we will see progress in improving elementary school math education. As we gradually improve elementary school math education, this will eventually lead to parents who will provide a better math education environment for their children. It will also lead to preservice teachers entering teacher education programs with a better preparation in math and in math pedagogy.

2. Our current math education curriculum is often described as being "a mile high and an inch deep" (Ruetters, 20023). I have some trouble understanding what this means, as I don't use linear measure when I am taking about the breadth and depth of a curriculum. However, what I think it means is that many people are concerned about how our curriculum has expanded in breadth, covering more and more topics in a shallower and shallower manner. The curriculum lacks the depth needed for students to gain

understanding and a number of other aspects of increasing math maturity. Our curriculum is not well designed in terms of helping students learn to make connections and to transfer their math knowledge and skills to areas outside of the formal math curriculum.

3. ICT brings new dimensions to both School Math and Folk Math. We have yet to appropriately understand and implement a math education system that adequately takes into consideration the capabilities of ICT as aids to teaching, learning, and using math.

   For example, consider computer tools that are routinely used by graphic artists. They are based on a very large amount of mathematics. However, very few graphic artists feel the need to have studied this underlying mathematics, and few people who teach graphic arts use of computers have appreciable insights into the underlying mathematics. The issue here is somewhat similar to the issue of children using calculators rather than paper and pencil algorithms, or researchers using statistical packages of computer programs without having mastered the underlying mathematics.

   But, the issue is also quite different. The goal of a graphic artist is to solve a graphic artist problem or complete a graphic artist task. The graphic artist has graphic arts knowledge and skills that can provide feedback on progress toward solving the problem or accomplishing the task.

   It turns out that this example identifies a major hole in the overall math curriculum. We are not very successful in helping students understand math at a level where they can detect their own errors. People who routinely use math are able to detect their errors because they have knowledge (intuition, deep insights) into the problems that they are addressing. Even though our math curriculum makes considerable use of "word problems" that provide some context for the problem to be solved, it is rare that a student has a sufficient grasp of the problem setting and meaning so that the student can detect errors in math thinking and in carrying out needed math procedures.

4. There are a variety of math topics that require a student to be at or near math Formal Operations in order to gain a significant understanding of the topic. Examples include probability, ratio and proportion, and algebra. Roughly speaking, if many of the students you are teaching "just don't seem to get it" for certain topics, then there is a good chance that they are not developmentally ready for the topic.

   I think what has happened in the school math curriculum is that it has developed a severe imbalance between the immediate success and long-term success. For a specific category of problems, immediate success is achieved by memorizing (without understanding) how to solve the specific category of problems. Such learning is fragile, does not transfer well to new situations, and tends to be quickly forgotten.

   Long-term success requires learning for understanding and developing significant level of math maturity. To do this, without increasing the amount of time devoted to math instruction, requires decreasing the breadth of the math content covered, and devoting much more time to learning for understanding and increased math maturity.

5. One of the most important ideas in math education is learning to build upon and make effective use of the accumulated knowledge in the discipline of math. An important requirement in this endeavor is that students learn to read (with understanding) math at the levels they have studied. ICT is a powerful aid to learning, a powerful aid to information retrieval, and a powerful aid to carrying out many of the types of procedures that are important in solving math problems. Our current math education system is not doing well in helping students learn to read math and to make effective use of ICT.

In brief summary, our math education system can be a lot better. But, this will require significant improvement in teachers, in appropriate use of ICT, and in our understanding of the human brain and learning processes.

# Appendix A—Chesslandia: A Parable

Moursund, D.G. (March 1987). Chesslandia: A Parable. *The Computing Teacher (Learning and Leading with Technology).* Eugene, OR: ISTE. Accessed 12/29/04: http://darkwing.uoregon.edu/~moursund/dave/LLT-V14-1986-87.html#LLTV14%236.



Chesslandia was aptly named. In Chesslandia, almost everybody played chess. A child's earliest toys were chess pieces, chess boards, and figurines of famous chess masters. Children's bedtime tales focused on historical chess games and on great chess-playing folk heroes. Many of the children's television adventure programs were woven around a theme of chess strategy. Most adults watched chess matches on evening and weekend television.

Language was rich in chess vocabulary and metaphors. "I felt powerless—like a pawn facing a queen." "I sent her flowers as an opening gambit." "His methodical, breadth-first approach to problem solving does not suit him to be a player in our company." "I lacked mobility-— had no choice."

The reason was simple. Citizens of Chesslandia had to cope with the deadly CHESS MONSTER! The CHESS MONSTER, usually just called the CM, was large, strong, and fast. It had a voracious appetite for citizens of Chesslandia, although it could survive on a mixed diet of vegetation and small animals.

The CM was a wild animal in every respect but one. It was born with an ability to play chess and an innate desire to play the game. A CM's highest form of pleasure was to defeat a citizen of Chesslandia at a game of chess, and then to eat the defeated victim. Sometimes a CM would spare a defeated victim if the game was well played, perhaps savoring a future match.

In Chesslandia, young children were always accompanied by adults when they went outside. One could never tell when a CM might appear. The adult carried several portable chessboards. (While CMs usually traveled alone, sometimes a group traveled together. Citizens who were adept at playing several simultaneous chess games had a better chance of survival.)

Formal education for adulthood survival in Chesslandia began in the first grade. Indeed, in kindergarten children learned to draw pictures of chessboards and chess pieces. Many children learned how each piece moves even before entering kindergarten. Nursery rhyme songs and children's games helped this memorization process.

In the first grade, students were expected to master the rudiments of chess. They learned to set up the board, name the pieces, make each of the legal moves, and tell when a game had ended. Students learned chess notation so they could record their moves and begin to read chess books. Reading was taught from the "Dick and Jane Chess Series." Even first graders played important roles in the school play, presented at the end of each year. The play was about a famous chess master and contained the immortal lines: "To castle or not to castle—that is the question."

In the second grade, students began studying chess openings. The goal was to memorize the details of the 1,000 most important openings before finishing high school. A spiral curriculum had been developed over the years. Certain key chess ideas were introduced at each grade level, and then reviewed and studied in more depth each subsequent year.

As might be expected, some children had more natural chess talent than others. By the end of the third grade, some students were a full two years behind grade level. Such chess illiteracy caught the eyes of the nation, so soon there were massive, federally funded remediation programs. There were also gifted and talented programs for students who were particularly adept at learning chess. One especially noteworthy program taught fourth grade gifted and talented students to play blindfold chess. (Although CMs were not nocturnal creatures, they were sometimes still out hunting at dusk. Besides, a solar eclipse could lead to darkness during the day.)

Some students just could not learn to play a decent game of chess, remaining chess illiterate no matter how many years they went to school. This necessitated lifelong supervision in institutions or shelter homes. For years there was a major controversy as to whether these students should attend special schools or be integrated into the regular school system. Surprisingly, when this integration was mandated by law, many of these students did quite well in subjects not requiring a deep mastery of chess. However, such subjects were considered to have little academic merit.

The secondary school curriculum allowed for specialization. Students could focus on the world history of chess, or they could study the chess history of their own country. One high school built a course around the chess history of its community, with students digging into historical records and interviewing people in a retirement home.

Students in mathematics courses studied breadth-first versus depth-first algorithms, board evaluation functions, and the underlying mathematical theory of chess. A book titled "A Mathematical Analysis of some Roles of Center Control in Mobility" was often used as a text in the advanced placement course for students intending to go on to college.

Some schools offered a psychology course with a theme on how to psych out an opponent. This course was controversial, because there was little evidence one could psych out a CM. However, proponents of the course claimed it was also applicable to business and other areas.

Students of dance and drama learned to represent chess pieces, their movement, the flow of a game, the interplay of pieces, and the beauty of a well-played match. But such studies were deemed to carry little weight toward getting into the better colleges.

All of this was, course, long long ago. All contact with Chesslandia has been lost for many years.

That is, of course, another story. We know its beginning. The Chesslandia government and industry supported a massive educational research and development program. Of course, the main body of research funds was devoted to facilitating progress in the theory and pedagogy of chess. Eventually, however, quite independently of education, the electronic digital computer was invented.

Quite early on it became evident that a computer could be programmed to play chess. But, it was argued, this would be of little practical value. Computers could never play as well as adult citizens. And besides, computers were very large, expensive, and hard to learn to use. Thus, educational research funds for computer-chess were severely restricted.

However, over a period of years computers got faster, cheaper, smaller, and easier to use. Better and better chess programs were developed. Eventually, portable chess-playing computers were developed, and these machines could play better than most adult citizens. Laboratory experiments were conducted, using CMs from zoos, to see what happened when these machines were pitted against CMs. It soon became evident that portable chess-machines could easily defeat most CMs.

While educators were slow to understand the deeper implications of chess-playing computers, many soon decided that the machines could be used in schools. "Students can practice against the chess-machine. The machine can be set to play at an appropriate level, it can keep detailed records of each game, and it has infinite patience." Parents called for "chess-machine literacy" to be included in the curriculum. Several state legislatures passed requirements that all students in their schools must pass a chess-machine literacy test.

At the same time, a few educational philosophers began to question the merits of the current curricula, even those that included a chess-computer literacy course. Why should the curriculum spend so much time teaching students to play chess? Why not just equip each student with a chess-machine, and revise the curriculum so it focuses on other topics?

There was a call for educational reform, especially from people who had a substantial knowledge of how to use computers to play chess and to help solve other types of problems. Opposition from most educators and parents was strong. "A chess-machine cannot and will never think like an adult citizen. Moreover, there are a few CMs that can defeat the best chess-machine. Besides, one can never tell when the batteries in the chess-machine might wear out." A third grade teacher noted that "I teach students the end game. What will I do if I don't teach students to deal with the end game?" Other leading citizens and educators noted that chess was much more than a game. It was a language, a culture, a value system, a way of deciding who will get into the better colleges or get the better jobs.

Many parents and educators were confused. They wanted the best possible education for their children. Many felt that the discipline of learning to play chess was essential to successful adulthood. "I would never want to become dependent on a machine. I remember having to memorize three different chess openings each week. And I remember the worksheets that we had to do each night, practicing these openings over and over. I feel that this type of homework builds character."

The education riots began soon thereafter.

## Retrospective Comments Added 10/19/04

Some years after writing this editorial, I discovered the following essay:

Peddiwell, J. Abner (1939). The saber-tooth curriculum. NY: McGraw Hill. Accessed 10/19/04: http://aral.cse.msu.edu/CSE103FS04/CSE103Visitor/saber.htm.

This superb essay was written about 50 years before mine. Here are a few paragraphs quoted from that essay.

> Having developed a curriculum, New-Fist took his children with him as he went about his activities. He gave them an opportunity to practice these three subjects [fish-grabbing, horse-clubbing, and tiger-scaring]. The children liked to learn. It was more fun for them to engage in

these purposeful activities than to play with colored stones just for the fun of it. They learned the new activities well, and so the educational system was a success.

As New-Fist's children grew older, it was plain to see that they had an advantage over other children who had never been educated systematically. Some of the more intelligent members of the tribe began to do as New-Fist had done, and the teaching of fish-grabbing, horse-clubbing, and tiger-scaring came more and more to be accepted as the heart of real education.

…

It is to be supposed that all would have gone well forever with this good educational system if conditions of life in that community had remained forever the same. But conditions changed, and life which had once been so safe and happy in the cave-realm valley became insecure and disturbing.

A new ice age was approaching in that part of the world. A great glacier came from the neighboring mountain range to the north. Year after year crept closer and closer to the headwaters of the creek which ran through the valley, until at length it reached the stream and began to melt into the water. Dirt and gravel which the glacier had collected on its long journey were dropped into the creek. The water grew muddy. What had once been a crystal-clear stream in which one could easily see the bottom was now a milky stream which one could not see at all.

At once the life of the community was changed in one very important respect. It was no longer possible to catch fish with the bare hands. The fish could not be seen in the muddy water. For some years, moreover, the fish in this creek had been getting more timid, agile, and intelligent. The stupid, clumsy, brave fish, of which originally there had been a great many, had been caught with the bare hands for fish generation after fish generation, until only fish of superior intelligence and agility were left. These smart fish, hiding in the muddy water under the newly deposited glacial boulders, eluded the hands of the most expertly trained fish-grabbers. Those tribesmen who had studied advanced fish-grabbing in the secondary school could do no better than their less well-educated fellows who had taken only an elementary course in the subject, and even the university graduates with majors in ichthyology were baffled by this problem. No matter how good a man's fish-grabbing education had been, he could not grab fish when he could not find fish to grab.

# Appendix B—Me: A Course of Study

Note: This is a scenario designed to forecast some possible aspects of the future of education. The underlying theme is some possible educational impacts of Information and Communication Technology.

For the "old timers" reading this scenario, you will note that the title is quite similar to "Man: A Course of Study." MACOS was a highly successful set of instructional materials developed for use at the fifth grade level. Jerome Bruner played a major role in this development of this course (Bruner, n.d.). The project was funded by the National Science Foundation and led to a published product that was quite successful. Indeed, the commercial success of the materials led to the NSF changing its approach to the commercialization of materials developed under its funding. For many years after this project, there were strict limitations on such commercialization.

If you want to read some futuristic scenarios written by others, see Visions 2020 (2002).

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# Me: A Course of Study

Assignment # 2: My Informal and Formal Education

Due November 17, 2014

Submitted by Serena Venus Chavez

Student # 2007-3482-3792

I am writing this paper as an assignment in my high school senior year Capstone Projects Course, "Me, A Course of Study." Because this is a required course, I didn't think that I would like it. However, this course is turning out to be one of the best courses I have had in school. It is causing me to think about some things that are important to me. (I'm not saying this just to try to get a good grade. This really is a worthwhile course!)

The assignment suggests that I start by specifying my intended audience and stating the message that I am trying to get across. The assignment also says that I am to do some research, not just make up everything off the top of my head. I have to have a bibliography. I have eight references—I hope that is enough.

My audience is people who want to be teachers and people who are already teachers. My message is that education can be a whole lot better than it currently is, and that I have some ideas the audience should pay attention to. Many of these ideas come from my own educational experiences and from talking to other students throughout the world.

## Grade School

When I started school, I was eager to show off to the teacher and to the other kids. I told the teacher that I knew how to read. So, the teacher gave me a story to read out loud. Disaster! I hadn't heard the story before, and so I could not read it. It turns out that I have a good memory and I could read to my mom because I had memorized the stories she had read to me.

By the end of the second grade, the teacher told my mother that I was reading a year behind grade level, and that she should spend some time helping me during the summer. Finally, in the third grade a reading specialist figured out my problem. He said that my brain works differently than that of people who find it easy to learn to read. He said that I was really smart and good at solving problems, but that I was dyslexic. Of course, I had never heard this word before, but now I know it well (Shaywitz, n.d.).

Three things came out of this. First, I was put into a special reading class where a computer hooked to the Internet did the instruction. The software came from a company called Scientific Learning Corporation, and this company still exists (Scientific Learning Corporation, n.d.). They develop brain-based teaching and learning software. I know that some of this software can teach a lot better than a teacher. This does not dissuade me from wanting to become a teacher. Humans can teach lots of things better than computers.

Second, my mom bought me a laptop computer and the school let me use it whenever I felt it would be helpful. Third, I got to have extra time whenever I was taking a test, and I got to use my computer when I was taking tests that involved doing writing. All of this has helped a lot. I am a good student and a wiz at writing using a word processor and looking up stuff on the Web. But, I am still dyslexic, reading is a painfully slow process for me, my handwriting is terrible, and I am a terrible speller.

When I was doing research for this paper, I spent some time on the Web reading about dyslexia. My research identified that there are now a whole bunch of diagnostic tests that can be given via computer (and immediately scored by the computer) to kindergarteners, and that can identify a whole lot of different learning challenges including dyslexia, dyscalculia, ADHD, poor vision, poor hearing, poor reaction time, and so on. I also learned that many schools are not yet routinely making use of these tests. IMHO, that is inexcusable!

I remember that I really liked math before I went to school, and when I was in the first grade, and I was good at it. But, by the time I got to the third or fourth grade, I wasn't very good at math and I didn't like it very much. My special reading teacher came to my rescue. He said that math is a kind of language, and that learning math is related to learning reading. He said that lots of students who have trouble learning to read also have trouble learning to do arithmetic.

The result of this is that they gave me a calculator and said that I could use it whenever I wanted to. The other kids in the class said that this was unfair, because they didn't get to use calculators very much. But, my special math teacher told me that they were just jealous because I didn't have to spend my time learning to do things that a cheap calculator can do better than a person. I learned that solving math problems and doing arithmetic are really two different things. It turns out that I am pretty good at math problem solving and math types of thinking—I am just not good at paper and pencil arithmetic.

I did some web research on this calculators and math thing. It turns out that the National Council of Teachers of Mathematics has been recommending use of calculators since way back in 1980 (NCTM, n.d.). Still, many schools do not yet fully integrate use of calculators into elementary school. I guess in lots of schools the individual teachers get to decide, and many still decide that students must master paper and pencil arithmetic before they get to regularly use calculators.

That feels very strange to me. This is especially true since many people now wear a wristport that uses voice input, tells time, does arithmetic, connects to the Web, is a telephone, is a video camera, and so on. As part of my research for this project, I talked with some of my nieces and nephews who are now in grade school. They said that they get to use their wristports quite a bit, except when the teacher wants silence in the room.

## Middle School

When I started the sixth grade, I got to go to a really modern middle school. The school gave each student a laptop computer to use at home and school. All of a sudden I found that I was a big brain—I could already do all kinds of things with a computer that the other kids didn't know how to do. I got to help them learn. I think that is when I first decided that it would be fun to be a teacher.

The school had a number of ideas that have turned out to be good for me. For example, each term each student had to take at least one distance-learning course over the Internet. This really fitted my learning style, I had experience from doing some distance learning in the third grade, and I could proceed at my own pace and really go deep into the parts that interested me. I began to learn how to learn on my own and how to be responsible for my own learning. In retrospect, that has turned out to be one of the better parts of my education.

As I was writing this paper, I talked to my mom a lot about it. She said that in the "good old days," students were not given much responsibility for their own learning. If the kids didn't learn, it was the teacher's fault, the school's fault, the parent's fault, the superintendent's fault, the government's fault, and so on.

IMHO, this needs to change. I know that kids can learn to take much more responsibility for their own education. I think that this should be a major focus in all schools, starting in grade school. Of course, we still need teachers. We especially need teachers who are good at helping students to learn on their own and to take responsibility for their own learning.

The distance learning courses I took enrolled students from throughout the country and quite a few from other countries. I began to make friends all over the world. Our computers had little TV cameras so that we could see each other as we were talking to each other and sending instant messages to each other. I had no trouble with talking to students who knew English or Spanish (I was raised in a bilingual home), but I had trouble reading the written Spanish.

Nowadays, when my Internet friends talk to me in languages that I do not know, I have my wristport or computer translate for me. They do a pretty good job—I can usually follow the conversation. My wristport and computer each have a built-in video camera, and I find that seeing who I am talking to also helps.

In my middle school, all students had to take a foreign language starting in the sixth grade. First, I thought I would take Spanish, since it would be easy for me. This is because I learned Spanish at home as I was growing up. But, my mom said that I had to take Arabic, or Chinese, or Japanese. She said that I needed to get an education to prepare me to be a citizen of the world—to understand cultures different than the one I was growing up in. My school offered all of these languages via distance learning.

I am glad she made me do this. I have been studying Japanese for more than six years and am getting good at it. I have a lot of Japanese friends living in Japan, and we talk to each other almost every day. I got to actually meet some of them last summer, when I got to go to Japan for six weeks. This was arranged through Sister Cities, International (Sister Cities, n.d.). I lived in several different Japanese homes during that time, and that certainly helped me learn about Japanese culture.

In doing research for this paper, I looked into what other schools are doing about foreign language instruction. It turns out that most schools are not doing very well. Lots of universities require that students have had two years of a foreign language in to be admitted, but two years (often taken while in high school) isn't enough to do much good. While I was riding the bus to school the other day, I used my wristport to watch my favorite WINN (Worldwide Internet News Network) and they were talking about language instruction in different countries. I think they said that the United States was lowest among 30 developed nations. That really bothered me, because I know that we are a melting pot of people who have come from all over the world. I have a hard time understanding why our schools do so poorly in the language preparation of kids.

## High School

As soon as I met my high school counselor, she asked me about what disciplines I might want to study in high school. I said, you mean "courses?" She said, "no, we want our students here to think more broadly than just courses. A course can help you to make some progress in a discipline, but a discipline is much bigger than a few courses." She then shared with me the following material from an old article written by old professor named David Moursund. I think it was written in 2004, but the copy my advisor gave me did not have an appropriate citation.)

> Each discipline can be defined by its unique combination of:
>
> - The types of problems, tasks, and activities it addresses.
> - Its accumulated accomplishments such as results, achievements, products, performances, scope, power, uses, impact on the societies of the world, and so on.
> - Its history, culture, language (including notation and special vocabulary), and methods of teaching, learning, and assessment.
> - Its tools, methodologies, and types of evidence and arguments used in solving problems, accomplishing tasks, and recording and sharing accumulated results.

My counselor told me that a good mixture of all of these things is needed to have a reasonable level of expertise in a discipline. She said that our high school is special because we focus on "deep" learning of disciplines rather than just taking courses. She said something about the courses in most schools being "A mile wide and an inch deep." I didn't really understand what she meant until I looked this up on the Web. It was a term used to describe math instruction in the United States late in the last century, when the US didn't do very well on international comparisons. (We still don't do very well. One of my goals as I become a teacher is to help fix this situation.) I guess courses in many high schools are "a mile wide and an inch deep."

Another thing that my counselor told me is that our school is really good because it embraced the ideas of a computer as a brain & mind enhancement. The way she was talking made me think of turning people into computers by some sort of implanted computer chips, Scary! I told her I didn't want a computer chip put into my brain (even though I have read that this is now getting to be a common thing).

She said "no, that is not what I mean." She went on to explain that people can do lots of things that computers cannot do, and computers can do lots of things that people cannot do. In each course in our school, students regularly work with computers. In our school we think of a computer as being a brain & mind tool, an enhancement to our brains. Thus, computers are integrated into the courses in a very deep manner.

She also said that many of the tests given in our courses are "authentic." She says that this means that they are hands-on computer and they focus on making use of higher-order thinking to address hard, meaningful problems. What she said reminded me of what she said about gaining increased expertise in various disciplines.

When I started my classes, I was pleased at the type of assignments given. Almost every class had two term projects. One was a team term project and one an individual term project. In several of my classes we have to build interactive, hypermedia Websites. The hypermedia part should be easy for me. I built my first Website while I was in middle school. One of my older brothers helped me and taught me about making Websites that communicate well and are interesting. (I guess this was part of my informal education, although they taught us a little bit about making a Website way back in grade school.) One of my Websites is about all of the dolls that I have collected. Part of this Website is a Blog in which I talk with other people with the same interests in doll collecting and selling dolls on the Web.

Oh, I forgot to tell you. My mom is good at music and we have a piano. When I was really young and we first got a computer at home, she really got into composing and playing music using a computer. I learned how to do this from her. My doll Website has the dolls move to my music and turn so that you can see them from all directions. Also, my mom taught me to sew, and I sew lots of clothes for my dolls. I guess this was all part of my informal education. When I think about it, I realize how lucky I have been to have such a good home environment. Lots of my friends don't know how to sew and to compose music.

Most of the courses in our high school do not have required textbooks. Our math courses are an exception to this. Each year I take a math course, and I usually have a teacher who is determined to "cover" the material and get through the book. These teachers don't seem to realize that just because we cover the material doesn't mean we learn it. I hate it when teachers tell us that we are in hurry up mode so that we can get through the required materials.

The good thing is that my high school math teachers assume that you have access to a computer all of the time. (The teacher said that this was the modern version of having access to a chalkboard or paper and pencil while doing math.) This has been really good for me because I think of math as having a thinking and problem solving part, and a computational part. But in higher math, computation is much more than just doing arithmetic. For example, we learn to "manipulate algebraic expressions." We learned that computer programs to do this have existed for about 40 years (Moursund, n.d.). Their use is thoroughly integrated into our math and science classes. For me, this type of situation is sort of like what I had back in grade school when I was allowed to use a calculator whenever it suited my needs. My computer and I together are a wiz at doing math!

## Thoughts from My Friends

While I was writing this, I spent a lot of time talking to some of my friends from other countries. Most of these people I met through the Internet, so I have never actually "met" them, even though we have talked together, written to each other, and see each other through our telephones.

Several of them said that this is a great paper and I certainly should get high marks. They were particularly impressed by the idea of having a course in which students reflect on their education. They also liked the idea of courses stressing learning to learn and taking responsibility for your own learning. They wished that they had more of this in their schools.

They argued a lot about the idea of getting to use a computer when taking tests. They said that in their schools, this would definitely be cheating. They think it is certainly okay for people to use all of the latest and greatest artificially intelligence on the job, but that the purpose of school is to develop your real intelligence. My response to this was that I think they are terribly out of date. I think schools need to help students learn to live, learn, and work in the real world. My wristport is part of my real world.

## Final Thoughts

In reading back through this paper, I see I have left out some important things. For example, I did not talk about the Virtual Reality Entertainment System that I usually carry with me and I use lots of the time. It includes earphones, special eyeglasses for 3-D, a 3-D mouse, and a rechargeable battery that is good for about 10 hours of use. The VR plugs into my wristport. It has the neatest games, and a bunch of us can all be playing the same game at the same time, interacting with each other in cyberspace.

My VR also ties into the school districts VR education system. In our high school, all students are provided with a free a basic Virtual Reality Educational System. I guess this system has a hardware-based filter, because it can only tie into the district's educational VR system But, most parents buy their kids a system like I have, that is designed for both education and entertainment. This really bugs our teachers, because they can't tell if a student is playing a game or studying.

Another thing I didn't mention is the Brain/Mind course that I took last year. In that course I learned that no two people's brains are the same, and that one's brain is changing all the time. The process of learning is actually a process of changing one's brain. I learned a really scary thing in that course. A brain has various centers or parts, such as an attention part and a pleasure center. The VR games are really good at grabbing your attention and providing pleasure. That explains why some of my friends seem to be addicted to the games. Indeed, I think I am sort of addicted, because sometimes I find it really hard to stop playing.

It also points to another problem that schools have to deal with. I have been reading some of the history of intelligent tutoring systems and use of AI in education (Human Performance Center, n.d.). Even 10 years ago people were producing some highly interactive intelligent computer assisted learning materials produced much better learning gains than human teachers in quite a few different situations. Nowadays, schools want to have students use these aids to learning and they are available on our school districts educational VR system. The makers of the instructional materials want to make them sufficiently entertaining (pleasure giving) and attention grabbing so that students will want to study their materials. Thus, it seems to me that the schools are helping students to become addicted! (I think I had better wait until after I graduate before I ask the principal if he agrees with this.)

In any case, I have been taking some college courses through my VR system, and I already have a bunch of college credits. I guess they used to have "Advance Placement" courses, but now students just take college courses instead.

I still want to be a teacher, and next year I will be enrolling in a university in order to get my teacher's credential. I want to take a bunch of courses in cognitive neuroscience, because it is clear to me that this is an important part of the future of education. However, I am beginning to understand that I already know a lot about how to be a good teacher, because I am a human being. I am reminded of a book that I read many years ago written by Robert Fulghum (2003). I thought the book was so important that I have kept it in my personal library. Quoting from the inside of the front cover of the book:

> Most of what I really need to know about how to live and what to do and how to be I learned in kindergarten. Wisdom was not at the top of the graduate school mountain, but there in the sandpile at Sunday School. These are the things I learned:
>
> Share everything.
>
> Play fair.
>
> Don't hit people.
>
> Put things back where you found them.
>
> Clean up your own mess.
>
> Don't take things that aren't yours.
>
> Say you're sorry when you hurt somebody.
>
> Wash your hands before you eat.
>
> Flush.
>
> Warm cookies and cold milk are good for you.
>
> Live a balanced life.
>
> Take a nap every afternoon.
>
> When you go out into the world, watch out for traffic, hold hands, and stick together.

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