Macroeconomics of Structural Change and Population Ageing

by

Mingyu “Kevin” Song

A dissertation accepted and approved in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
in Economics

Dissertation Committee:

Shankha Chakraborty, Chair
Alfredo Burlando, Core Member
David Evans, Core Member
Youchang Wu, Institutional Representative

University of Oregon

Winter 2025

DISSERTATION ABSTRACT

Mingyu “Kevin” Song

Doctor of Philosophy in Economics

Title: Macroeconomics of Structural Change and Population Ageing

This dissertation studies the macroeconomics of structural change and population ageing.

Chapter 1 investigates the influence of international trade on structural transformation (ST) in emerging economies. Analyzing rapidly growing, trade-dependent countries such as South Korea, Singapore, and Malaysia, we observe deviations from standard ST patterns: these economies maintain unusually high manufacturing shares in both production and trade, while the volume of service trade exceeds that of agriculture. To explain these anomalies, we develop a general equilibrium model for open economies that incorporates both tradable and non-tradable services. Our findings suggest that partial tradability of services accelerates the shift of resources toward the service sector, reducing the manufacturing share. However, in economies with strong manufacturing productivity, the decline in the manufacturing share is mitigated by sustained competitiveness in manufacturing. These results underscore the importance of service tradability and sector-specific productivity growth in understanding ST in rapid-growing economies.

Chapters 2 and 3 address the challenges of population aging in South Korea. Chapter 2 develops a three-period overlapping generations (OLG) model with endogenous fertility to evaluate the effectiveness of government policies in raising the Total Fertility Rate (TFR). The model incorporates government interventions through income taxation, parental leave payments, child allowances, and public education. Calibrated to South Korean data — where the TFR declined to a world-low of 0.72 in 2023 — our simulations indicate that increasing child allowances is the most effective policy to boost fertility rates. However, this policy also leads to reduced human capital growth and lower output and consumption per worker, due to the standard trade-off between the quantity and quality of children. Additionally, our findings reveal that the high cost of private education significantly deters higher fertility.

Chapter 3 extends previous analyses by incorporating pension benefits that are also affected by population ageing. Calibrating the model to South Korean data, we evaluate the effectiveness of various government policies — child allowances, parental leave payments,

public education, pension benefits and income tax rate — on fertility rates and key macroeconomic variables. Although child allowances remain the most effective policy for raising fertility, it also entails reduced pension benefits per recipient. While a higher tax rate encourages fertility and initially raises pension benefits, it also weakens long-term human capital accumulation and wage growth, eventually eroding per-recipient pension benefits and per-worker economic well-being.

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my committee chair, Shankha Charkraborty, whose endless support and invaluable guidance have been the cornerstone of my graduate studies. I am also deeply indebted to Alfredo Burlando, David Evans, and Youchang Wu for their sharp and insightful feedback on my dissertation, which has greatly refined my work and opened my eyes to perspectives I had not previously considered.

I am grateful to my family and friends, both within and beyond the University of Oregon, for their constant encouragement. I would like to specially acknowledge my wife, Yerim Lee, for her unwavering support and encouragement.

To my parents, who have supported me unconditionally.

Table of Contents

List of Figures

List of Tables

Chapter 1 - Structural Transformation in an Open Economy

1 Introduction

Structural transformation (ST), defined as the reallocation of economic activity across broad sectors, plays a critical role in shaping a nation's growth trajectory and societal dynamics. This reallocation, often driven by technological advancements — as witnessed during the agricultural and industrial revolutions — not only alters the dominant sectors of an economy, but also significantly affects how people work and consume within a country. Thus, analyzing these sectoral shifts can enhance our understanding of current economic patterns and planning for potential future changes.

Numerous theoretical studies have sought to pinpoint the driving forces behind this economic reallocation. Kongsamut, Rebelo, and Xie (2001) emphasize income effects, suggesting that changes in income and the non-homothetic demand for different goods can lead to shifts in value-added shares across sectors. Highlighting the importance of relative price effects, Ngai and PIssarides (2007) argue that sectors with accelerated productivity growth witness a decline in their relative prices, affecting the reallocation of resources. Furthermore, Acemoglu and Guerrieri (2008) suggest that differences in the capital-to-labor ratio across sectors, or capital deepening, trigger ST. Herrendorf, Herrington, and Valentinyi (2015) evaluate the influence of sectoral technology on ST and argue that Cobb-Douglas production functions better capture the postwar U.S. ST than constant elasticity of substitution functions. Buera and Kaboski (2012b) highlight the significance of scale technology, while Buera and Kaboski (2012a) focus on the rise of the service sector driven by high-skilled labor and the shifting balance between home and market production of services.

Research on ST has expanded beyond closed economy models, recognizing the impact of international trade on the pace of development. Matsuyama (1992) finds a positive relationship between agricultural productivity and economic growth, although this relationship turns negative in small open economies. Matsuyama (2009) emphasize the importance of adopting an open economy model, especially incorporating interdependence across countries, to understand the global decline in the manufacturing sector. They propose that asymmetric productivity growth across sectors and countries leads to changes in comparative advantage in the open economy. Additionally, Uy, Yi, and Zhang (2013) show that changes in comparative advantage combined with the Baumol effect can generate the hump-shaped pattern observed in the manufacturing sector.

Empirical data indicate that most developed economies exhibit similar trends as they grow: a declining agricultural share, a hump-shaped pattern in the manufacturing share and a rising service share. However, South Korea - the primary focus of this study - has exhibited distinct ST patterns. Given that South Korea's economy is strongly trade-dependent, several studies have explored the influence of trade policies, tariffs, and subsidies on its ST ((Betts, Giri, and Verma, 2016); (Sposi, 2012); (Teignier, 2018)).

A non-decreasing manufacturing share is essential for understanding ST dynamics and long-term economic growth. A stable or increasing share suggests unique growth trajectories driven by sustained productivity gains, comparative advantage, or strategic industrial policies. This is particularly relevant for economies where manufacturing is pivotal for export performance, job creation, and technological diffusion, challenging the conventional narrative of inevitable de-industrialization and highlighting the sector's enduring contribution to economic development.

This paper has three main objectives: first, to examine how well these ST patterns apply across different countries using the Groningen Growth and Development Centre 10-Sector Database; second, to propose a new, albeit simple, classification of service trade based on available, limited data from the World Bank; and third, to compare the long-run equilibria of open economies with partially tradable services and those with non-tradable services, evaluating how the tradability of service affects equilibrium.

Utilizing Korean sector-specific data on capital shares, export volumes, and import volumes, this paper argues for more accurate estimation of certain exogenous variables in the model. Although the paper does not explicitly investigate South Korea's atypical ST patterns, it may serve as a crucial stepping-stone for studying ST behaviors in other emerging and developing countries. Methodologically, it follows the approach of Teignier (2018), using a three-sector neoclassical growth model with two distinct production functions.

The rest of the paper is organized as follows: Section Section 2 examines the stylized facts of ST across diverse measurements and countries, along with changes in trade shares per sector. Section Section 3 elaborates on the structural model and the theoretical basis of ST. Section Section 4 discusses the analysis of equilibrium and steady state of the model. Section Section 5 discusses the current limitations of the study and outlines future steps for simulations. Section Section 6 concludes.

2 Some Evidence on Structural Transformation

Investigating the structural transformation (ST) of countries requires establishing two critical measurements: development level and sectoral shares. Real GDP per capita is typically used to gauge a country's level of development. For assessing shifts in sectoral shares, three widely accepted measures are employment, value added, and consumption expenditure. Employment is calculated based on the number of workers or hours worked, while value added and consumption expenditure are measured in terms of prices, either nominal or real. Despite concerns about potential discrepancies between nominal and real value added, studies by Kuznets and Murphy (1966) and Herrendorf, Rogerson, and Valentinyi (2014) have shown that both methods yield similar patterns over time. Furthermore, these studies demonstrate that all three measures provide comparable outcomes for shifts in sectoral shares.

As economies develop over time, most advanced countries have witnessed three common trends: a decrease in agricultural sector's share of both GDP and employment, a humped-shaped pattern in manufacturing share and an increase in service share. In particular, when a country's GDP level surpasses a certain threshold, the share of the manufacturing sector begins to decline while the service sector's share rises sharply. Herrendorf, Rogerson, and Valentinyi (2014) demonstrated that this pivot point, for developed countries, was when the log GDP per capita was around 9. Interestingly, they also observed that South Korea's manufacturing sector shares did not decline even after reaching this turning point. This unique trend inspires us to consider the ST patterns in other emerging and developing countries.

This section presents a detailed examination of various countries' ST patterns relative to their income levels, investigating whether the stylized facts of ST apply broadly across nations. Unlike previous studies that primarily focused on developed countries, this analysis utilizes data from the Groningen Growth and Development Centre 10-Sector Database (GGDC) and the Penn World Table 9.1 (PWT) to include a wider range of countries. Given that a country's GDP growth is not sustained continuously due to economic downturns or shocks, sectoral changes are evaluated based on both GDP per capita and year-to-year changes. Evaluating annual data provides insights into whether a nation has experienced increasing or decreasing sectoral share patterns over time.[1]

[1]Using GDP per capita allows examination of ST patterns with economic growth. However, in periods when GDP per capita remains unchanged, plots of sectoral shares can cluster, making it challenging to examine ST patterns.

2.1 Data Description

The data used in this study are sourced from the GGDC and the PWT. Three specific measurements from the available economic activity measures within the three sectors are adopted: employment, nominal value added, and real value added.[2] For detailed information on data sources and sector assignments, please refer to Section A.2. Variations in the agricultural, manufacturing, and service sectors are computed relative to changes in a country's logged GDP per capita.[3]

Countries are classified according to the World Bank's criteria, categorizing income levels into four groups based on Gross National Income per capita.[4] The PWT data used in this study include 14 high-income, 7 upper-middle-income, and 3 lower-middle-income countries.[5]

To ensure clarity, every figure follows a uniform format. The x-axis represents the development level of countries, measured by log GDP per capita, while the y-axis denotes sectoral shares within each measurement. High-income countries are represented on the left side of the figures, while the right side presents data for both upper- and lower-middle-income countries.

2.2 Sectoral Shares by Income Level

2.2.1 Employment

Figure 1 illustrates the sectoral shares of employment relative to GDP per capita for both high-income and middle-income countries. The graphs on the left side demonstrate that most high-income countries generally adhere to the stylized patterns, with a few exceptions found in the manufacturing and service sectors.[6] In particular, in the manufacturing sector, Spain (ESP), Japan (JPN), South Korea (KOR) and Singapore (SGP) do not exhibit the expected declining pattern after the turning point identified by Herrendorf, Rogerson, and Valentinyi (2014).

The graphs on the right side of Figure 1 illustrate that many middle-income countries are still approaching their respective turning points[7] in terms of GDP per capita, following

[2] The hours-worked measure was excluded due to limited data, which accounts for eight OECD countries.

[3] Figures by years, instead of GDP per capita, are provided in Section A.3.

[4] High-income countries (\$12,376 or more); upper-middle-income (\$3,996–\$12,375); lower-middle-income (\$1,026–\$3,995); low-income (\$1,025 or less) countries.

[5] High-income countries (14): CHL, DNK, ESP, FRA, GBR, ITA, JPN, KOR, NLD, SWE, TWN, USA, HKG, SGP; upper-middle-income countries (7): BRA, COL, MEX, MYS, PER, THA, VEN; lower-middle-income countries (3): IDN, IND, PHL.

[6] For instance, Chile exhibits a clustered pattern within these sectors. A further examination on year basis (see Section A.3) reveals that these patterns align with the stylized facts over time.

[7] According to Herrendorf, Rogerson, and Valentinyi (2014), developed countries typically reach this pivot

patterns observed in previously developed nations. However, it remains uncertain whether these middle-income nations will follow the same development trajectories. Notably, countries like Mexico (MEX) and Malaysia (MYS) have surpassed this benchmark but do not conform to the expected patterns in the manufacturing sector.[8]

2.2.2 Nominal Value Added

Figure 2 presents the results derived from nominal value-added data, largely consistent with those from the employment data. Most high-income countries generally align with the stylized patterns, except for certain discrepancies in the manufacturing and service sectors. Notably, Chile's patterns, although clustered in the manufacturing and service sectors, do not accord with the stylized facts, as evident from the annual data in Section A.3. In the manufacturing sector, neither South Korea (KOR) nor Singapore (SGP) demonstrate a declining trend, despite their GDP per capita surpassing the turning point. Similarly, in the service sector, the sectoral shares of KOR and SGP do not consistently increase after the turning point.

Among middle-income countries whose GDP per capita exceeds the turning point, Brazil (BRA) and Venezuela (VEN) display sectoral shares clustered around the turning point. A detailed examination of the annual data in Section A.3 reveals that VEN largely adheres to the stylized facts across all sectors, while BRA deviates. Specifically, in BRA, the agricultural share remains unchanged, the manufacturing sector share increases, and the service share decreases over time. Malaysia (MYS) exhibits similar irregularities, with the manufacturing share rising and the service share either remaining constant or decreasing around the turning point.

2.2.3 Real Value Added

Figure 3 shows the analysis of real value added data, which reveals that some nations deviate substantially from the stylized facts. Most high-income countries' patterns align with previous findings. However, Sweden (SWE) demonstrates an increasing trend in its manufacturing share after an initial decline, whereas the opposite is observed in the service share. Notably, South Korea (KOR) and Singapore (SGP) display distinct patterns compared to other high-income nations. In the manufacturing sector, neither country demonstrates a declining trend after the turning point; in fact, KOR shows an increasing

when the log GDP per capita is around 9.

[8] While Venezuela exhibits patterns clustered around the turning point across all sectors, the year-based data presented in Section A.3 confirm adherence to the stylized facts over time.

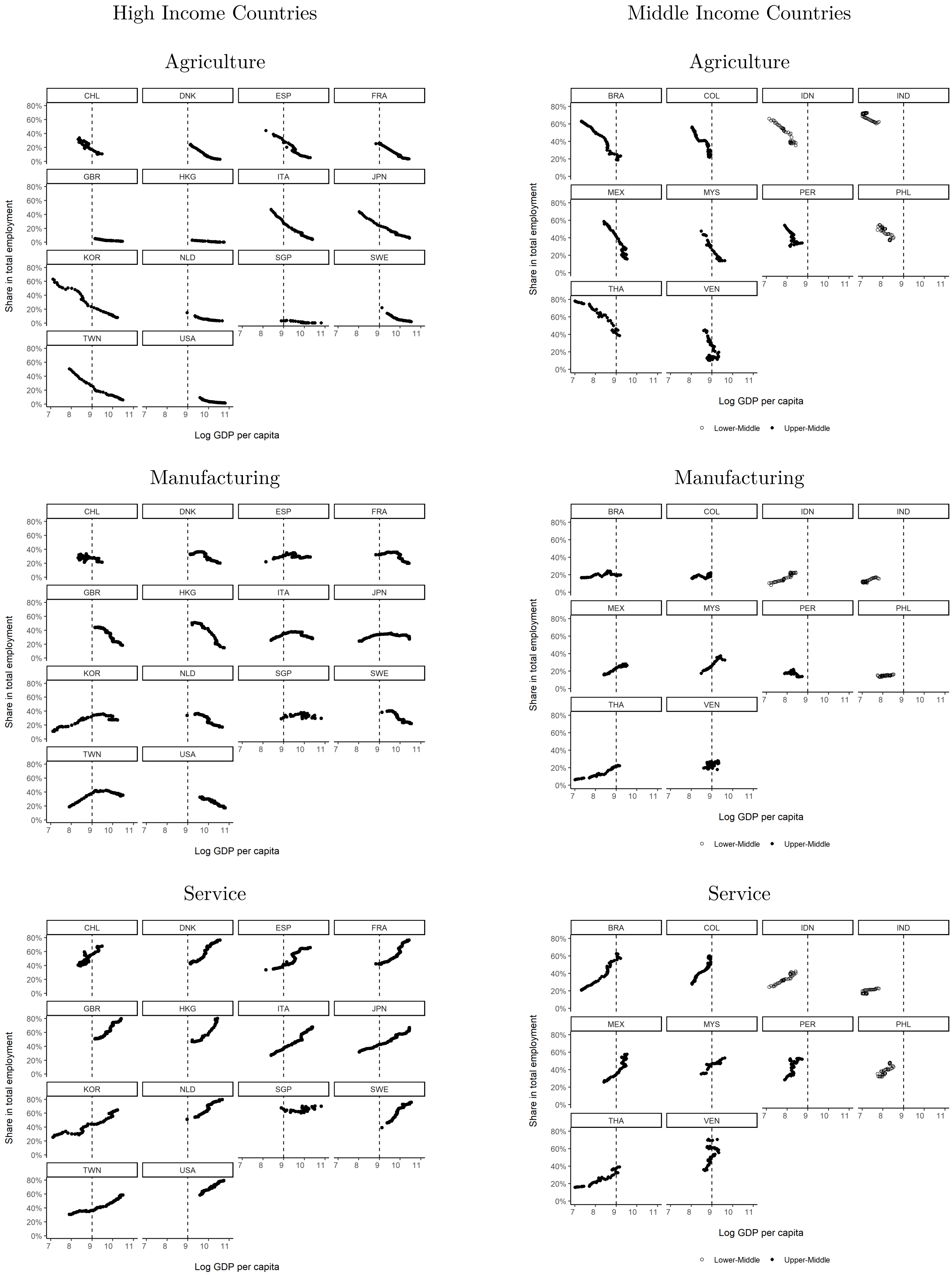


Figure 1: Employment share Sectors and Income (GDPC)

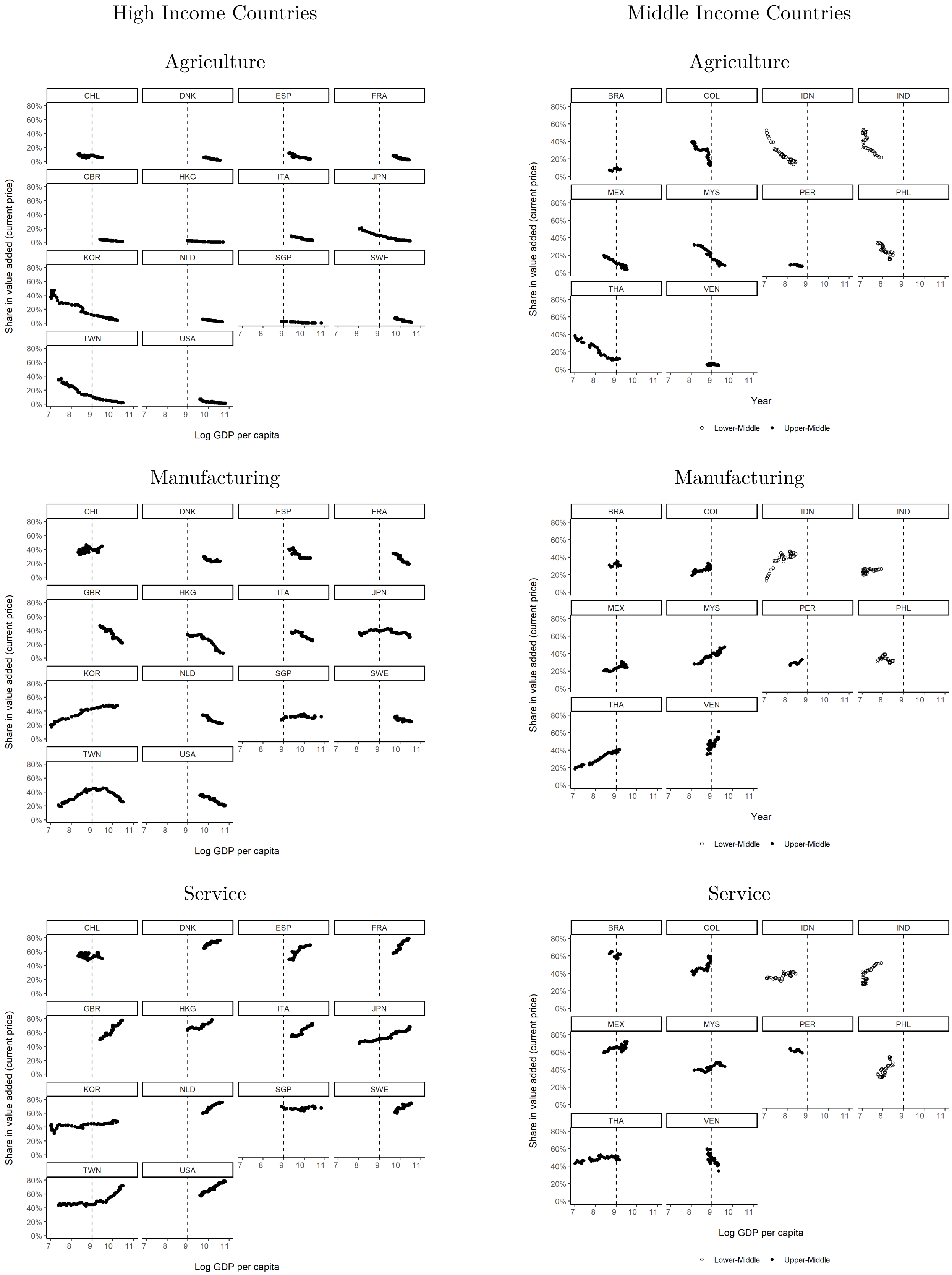


Figure 2: Nominal Value-added share by Sectors and Income (GDPC)

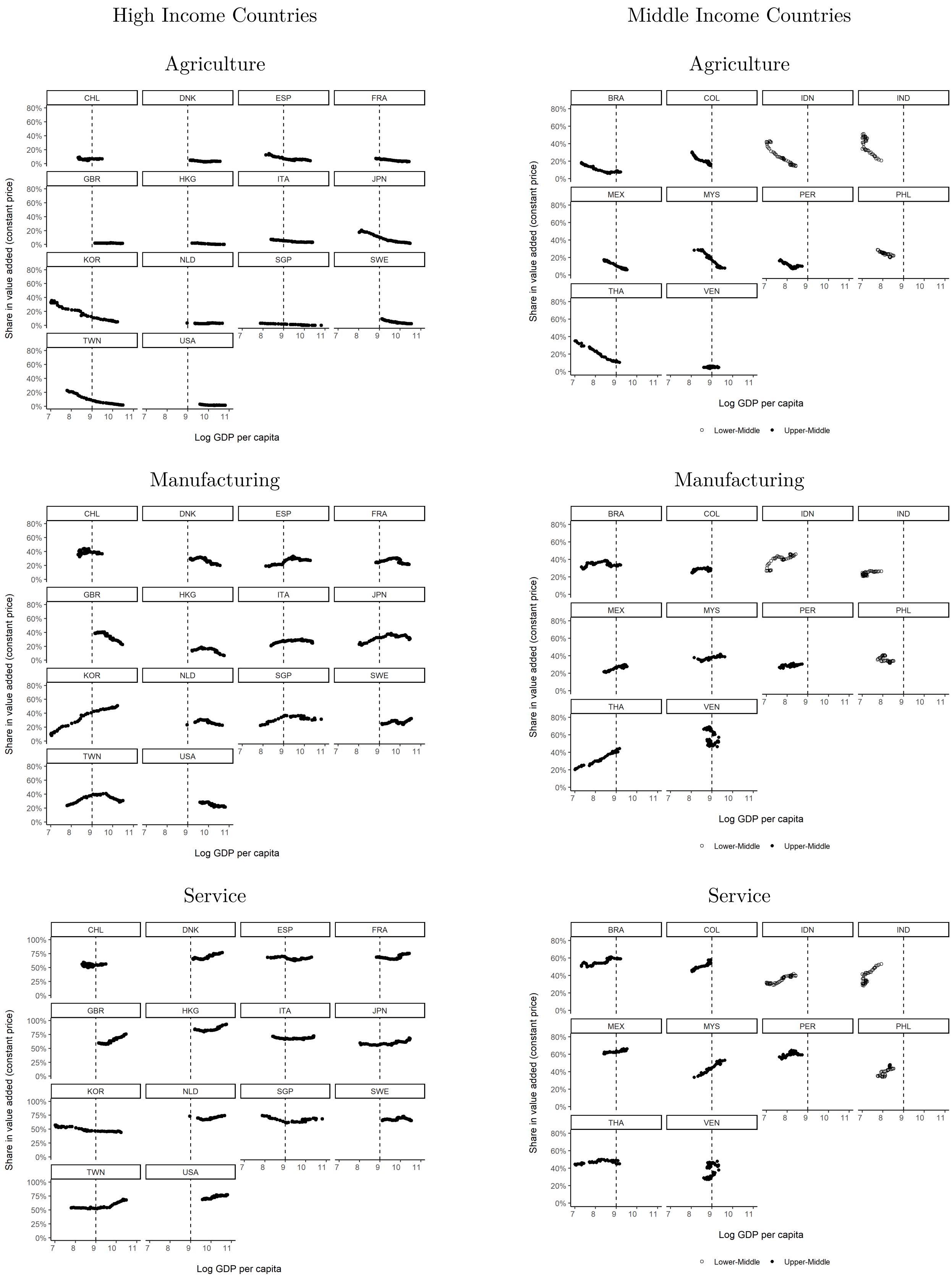


Figure 3: Real Value-added share by Sectors and Income (GDPC)

pattern. In the service sector, both countries exhibit a descending trend, contrary to expectations. These deviations in the manufacturing and service sectors for KOR and SGP warrant further exploration.

Among middle-income countries, the findings largely correspond with those discovered in the previous data. An intriguing observation is that the sectoral shares of Brazil (BRA) and Malaysia (MYS) largely conform to the stylized facts, with the exception of MYS's manufacturing sector. Even after MYS's GDP per capita surpasses the turning point, the real value added share of the manufacturing sector does not follow the anticipated declining trend.

Overall Interpretation

The data indicate that several nations, specifically South Korea (KOR), Singapore (SGP) and Malaysia (MYS), deviate from the stylized ST patterns across multiple measures. These findings suggest that the so-called "stylized" facts do not universally apply to all nations regardless of their income levels. Notably, these exceptional countries share common characteristics: they have experienced rapid economic growth over the past several decades and are recognized as economies heavily reliant on trade.[9] These findings imply that the sectoral changes of some rapidly growing economies may not necessarily align with the historical experiences of previously developed countries, emphasizing the potential uniqueness and divergent development pathways of such economies.

2.3 Trades within Sectors

To understand why certain countries display divergent ST patterns, incorporating service trade considerations into economic models could provide valuable insights. To implement this perspective, suitable data that describes the trade values of the three key economic sectors are required. Despite potential variances in specific sub-industries, the World Bank and the Bank of Korea offer relevant data on service trade. In this section, each sector's trade share is evaluated — calculated as the aggregate of export and import shares represented as a percentage of GDP. This analysis uses World Bank data for the same set of countries discussed in the previous section. For detailed information on data sources and sector assignments, please refer to Section A.2. It is important to note that Hong Kong (HKG), Singapore (SGP), and Venezuela (VEN) are treated separately due to their exceptionally

[9]However, data from the GGDC and the World Bank reveal that Hong Kong (HKG) and Taiwan (TWN) — two economies renowned for their rapid growth and trade dependence — have experienced sectoral shifts that align with the stylized facts.

high manufacturing trade shares, which exceed 150% of their respective GDPs, thus causing visualization challenges.

Figure 4 shows that there is an increasing pattern in the manufacturing trade share among high-income countries. This trend is particularly noticeable in South Korea (KOR) and the Netherlands (NLD), where the manufacturing trade share surpasses 75% of their respective GDPs. Furthermore, most of these high-income countries also exhibit an upward trend in their service sector's trade shares, corroborating the importance of acknowledging the service sector as tradable. Among middle-income countries, the manufacturing trade shares of Mexico (MEX), the Philippines (PHL), and Thailand (THA) distinctly surpass those of their counterparts.

Considering KOR, MYS and SGP, whose patterns diverge from the stylized facts, their manufacturing trade shares, as shown in Figures 4 and 5, constitute a greater portion of GDP than in other countries, except for HKG. Additionally, these three countries share the characteristic that their service trade shares exceed those of agriculture, further emphasizing the importance of treating services as tradable.[10] These findings suggest that the distinctive ST patterns of these countries might be largely affected by their reliance on trade, providing a plausible explanation for their divergence from the stylized facts.

3 Model

In this section, we develop a structural model for small open economies, drawing on the frameworks established by Greenwood, Hercowitz, and Krusell (1997), Hayashi and Prescott (2008), and Teignier (2018). The model has four sectors: agriculture, manufacturing, tradable services and non-tradable services. Agricultural and all service goods are used exclusively for consumption, while manufacturing goods serve both consumption and investment purposes. This sectoral breakdown allows us to capture the nuances of ST, especially the role of trade in shaping sectoral dynamics.

3.1 Households

We consider a representative household with a population size of N_t. The household owns capital and land, which they rent to firms, in addition to supplying labor. Utility is derived

[10] Tradable services include industries such as financial intermediation, information and communication technology (ICT) and high-skilled professional services like legal, consulting, and engineering. Globalization has expanded cross-border financial services, while digital advancements have driven the international exchange of ICT services. Additionally, the integration of global value chains has increased demand for professional services and boosted trade in travel and tourism.

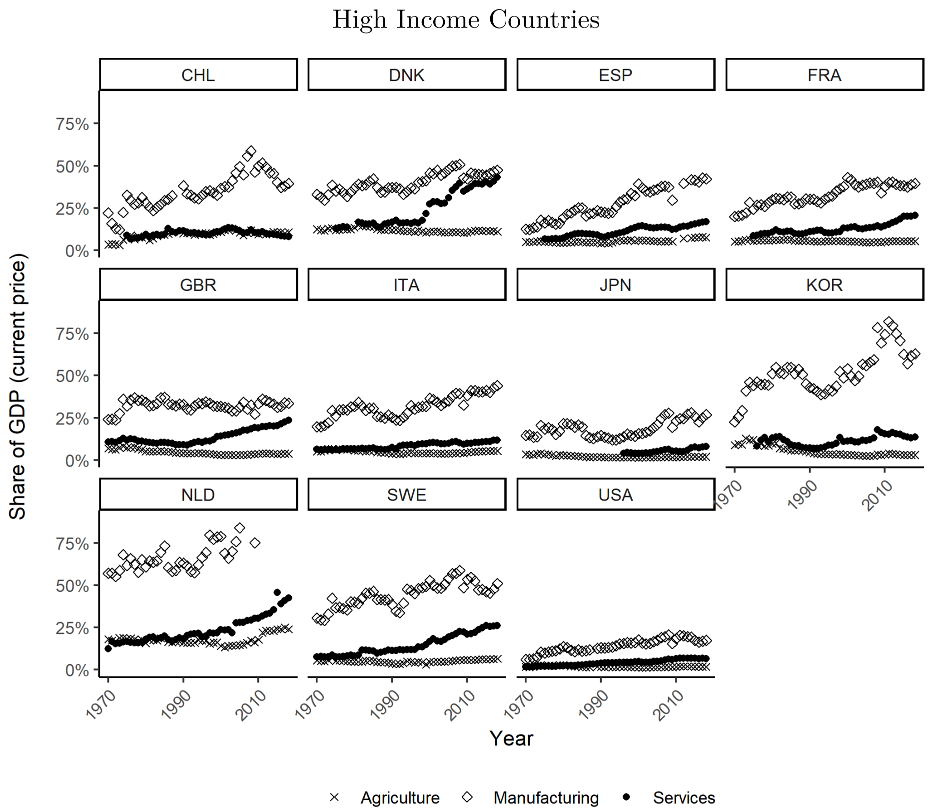


Figure 4: Trade share by Sectors and Income (Year)

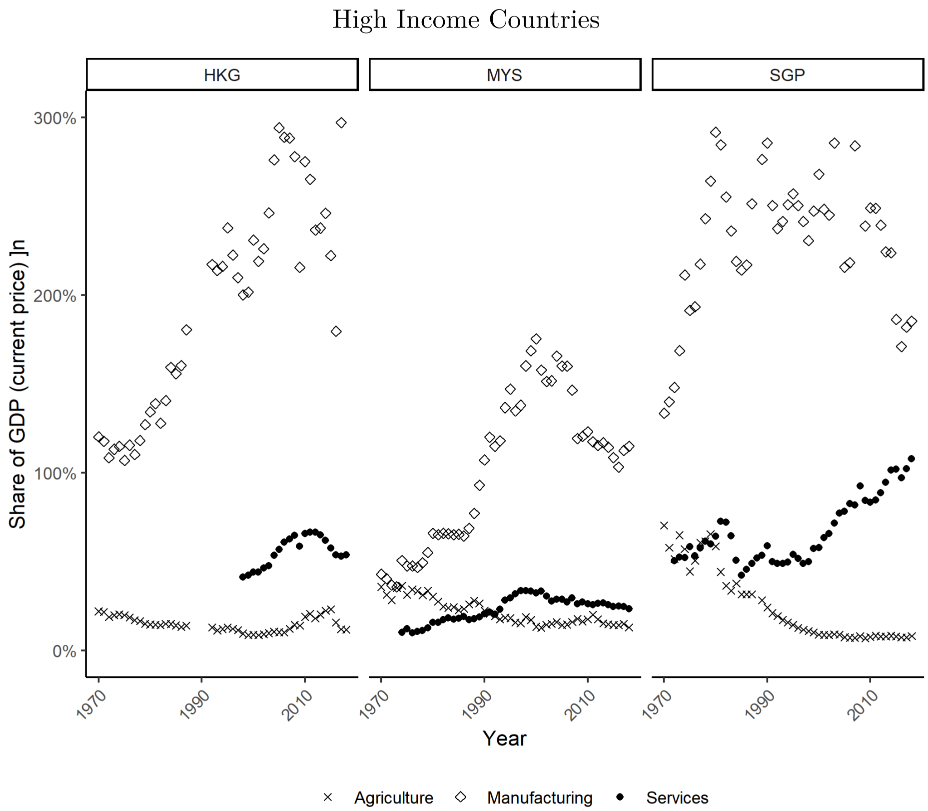


Figure 5: Trade share by Sectors and Income (Year) - HKG, MYS and SGP

from consuming four types of goods: agricultural goods (c_{at}), manufacturing goods (c_{mt}), tradable services (c^T_{st}, only if $\theta > 0$) and non-tradable services (c^N_{st}).

In the model, agricultural and manufacturing goods are fully tradable, while the tradability of the service sector is governed by the parameter $\theta \in [0, 1)$. Specifically, when $\theta = 0$, services are entirely non-tradable, resulting in three sectors: agriculture, manufacturing, and non-tradable services. Conversely, if $\theta > 0$, services become partially tradable, thereby introducing a distinct tradable services sector while maintaining non-tradable services as a separate sector. The higher values of θ indicates a greater degree of tradability in services.

Each period, households allocate their income between consumption and saving, aiming to maximize their lifetime utility. The household's optimization problem is formulated as:

$$\max_{\{c_{at}, c_{mt}, c_{st}, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \log C_t$$

$$\text{where} \quad C_t \equiv \left[\omega_a^{\frac{1}{\varepsilon}} \left(c_{at} - \bar{c}_a\right)^{\frac{\varepsilon-1}{\varepsilon}} + \omega_m^{\frac{1}{\varepsilon}} \left(c_{mt}\right)^{\frac{\varepsilon-1}{\varepsilon}} + \omega_s^{\frac{1}{\varepsilon}} \left(c_{st} + \bar{c}_s\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$c_{st} \equiv \left[\theta \left(c_{st}^T\right)^{\frac{\sigma-1}{\sigma}} + (1-\theta)\left(c_{st}^N\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \qquad \theta \in [0,1)$$

$$\text{s.t} \quad p_{at}c_{at} + c_{mt} + p_{st}^T c_{st}^T + p_{st}^N c_{st}^N + k_{t+1} = (1-\delta-\nu+R_t)k_t + w_t + q_t\left(\frac{L}{N_t}\right).$$

The variable C_t is the composite consumption, where c_{st} is the composite service consumption. The parameter ω_i is preference weight for a good in sector i. $\bar{c}_a$ and $\bar{c}_s$ indicate the subsistence consumption level of agricultural goods and the initial endowment of service goods, respectively, introducing non-homothetic preferences into the model.[11] ε and σ represent the elasticity of substitution between the consumption goods and between tradable and non-tradable services respectively.

In the budget constraint, p_{at}, p_{st}^T and p_{st}^N represent the relative prices of agricultural, tradable and non-tradable service goods respectively, with the price of manufacturing goods normalized to one. The household earns income from renting capital at rate R_t, supplying labor at wage w_t and renting out land at rate q_t. The term $\frac{L}{N_t}$ represent the per capita stock of land, with total land supply L assumed constant over time. The parameters ρ, δ, and ν represent the discount rate, capital depreciation rate and population growth rate, respectively.

First-Order Conditions From the household's optimization problem, we derive the first-order conditions (FOCs) with respect to each consumption good and capital accumulation. These conditions equate the marginal utility per unit of expenditure across different goods, adjusted by their relative prices and the non-homothetic nature of preferences. First, using the Lagrangian, we can write FOCs for each sector as follows:

$$\frac{1}{C_t}C_t^{\frac{1}{\varepsilon}}\omega_a^{\frac{1}{\varepsilon}}\left(c_{at}-\bar{c}_a\right)^{-\frac{1}{\varepsilon}} = \lambda_t p_{at} \tag{1}$$

$$\frac{1}{C_t}C_t^{\frac{1}{\varepsilon}}\omega_m^{\frac{1}{\varepsilon}}\left(c_{mt}\right)^{-\frac{1}{\varepsilon}} = \lambda_t \tag{2}$$

$$\frac{1}{C_t}C_t^{\frac{1}{\varepsilon}}\omega_s^{\frac{1}{\varepsilon}}\left(c_{st}+\bar{c}_s\right)^{-\frac{1}{\varepsilon}}\left(c_{st}\right)^{\frac{1}{\sigma}}\theta\left(c_{st}^T\right)^{-\frac{1}{\sigma}} = \lambda_t p_{st}^T \tag{3}$$

$$\frac{1}{C_t}C_t^{\frac{1}{\varepsilon}}\omega_s^{\frac{1}{\varepsilon}}\left(c_{st}+\bar{c}_s\right)^{-\frac{1}{\varepsilon}}\left(c_{st}\right)^{\frac{1}{\sigma}}(1-\theta)\left(c_{st}^N\right)^{-\frac{1}{\sigma}} = \lambda_t p_{st}^N, \tag{4}$$

where λ_t denotes the Lagrange multiplier on the budget constraint in period t. By substituting λ_t from the equation (2) into the other three conditions, we can express the relative consumption of each sector as below:

$$c_{at} - \bar{c}_a = \frac{\omega_a}{\omega_m}p_{at}^{-\varepsilon}c_{mt}$$

[11] With $\bar{c}_a > 0$ and $\bar{c}_s > 0$, an increase in income leads to a decrease in the expenditure share of agricultural goods and an increase in the share allocated to services.

$$
\begin{aligned}
c_{st}^T &= \left(\frac{\theta}{p_{st}^T}\right)^{\sigma}\left[\frac{\omega_s}{\omega_m}\left(\frac{c_{mt}}{c_{st}+\bar{c}_s}\right)\right]^{\frac{\sigma}{\varepsilon}} c_{st} \\
c_{st}^N &= \left(\frac{1-\theta}{p_{st}^N}\right)^{\sigma}\left[\frac{\omega_s}{\omega_m}\left(\frac{c_{mt}}{c_{st}+\bar{c}_s}\right)\right]^{\frac{\sigma}{\varepsilon}} c_{st} \\
\text{or } c_{st}^T &= \left(\frac{\theta}{1-\theta}\right)^{\sigma}\left(\frac{p_{st}^N}{p_{st}^T}\right)^{\sigma} c_{st}^N
\end{aligned}
$$

These equations illustrate how the consumption of each good responds to changes in relative prices, preference weights and the degree of tradability.

Euler Equation Intertemporal optimization leads to the Euler equation, capturing the trade-off between current and future consumption:

$$
\begin{aligned}
\frac{1}{C_t} &= \lambda_t \left(\omega_a p_{at}^{1-\varepsilon} + \omega_m + \omega_s p_{st}^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}} \\
&\equiv \lambda_t P_t \\
\Rightarrow \ (1+\rho)\frac{P_{t+1}C_{t+1}}{P_t C_t} &= 1-\delta-\nu+R_{t+1},
\end{aligned}
$$

where P_t is the composite price index and p_{st} is the average price of services weighted by consumption shares. This Euler equation reflects how the household adjusts consumption over time in response to changes in prices, income and returns on savings.[12]

3.2 Firms

Production in each sector is carried out by a infinite numbers of identical firms, operating under perfect competition and constant returns to scale (CRS) technologies. The production functions for each sector are specified as:

$$
\begin{aligned}
y_{at} &= (k_{at})^{\eta}(A_{at}n_{at})^{\beta}\left(\frac{L}{N_t}\right)^{1-\eta-\beta} \\
y_{mt} &= (k_{mt})^{\alpha}(A_{mt}n_{mt})^{1-\alpha} \\
y_{st}^N &= (k_{st}^T)^{\alpha}(A_{st}^N n_{st}^N)^{1-\alpha} \\
y_{st}^T &= (k_{st}^T)^{\chi}(A_{st}^T n_{st}^T)^{1-\chi}, \quad \text{where } 0<\alpha<\chi<1.
\end{aligned}
$$

The variables y_{it}, k_{it} and n_{it} denote output per capita, capital stock per capita and employment share in each sector i, respectively. The parameters α, η and χ represent capital share where β represent labor share. With $0 < \alpha < \chi < 1$, we assume that the tradable service sector is more capital-intensive than the non-tradable service sector. The

[12]Details of the derivation of Euler equation can be found in Section A.1.1

total factor productivity (TFP) in each sector, A_{it}, grows at a constant rate $\gamma_i > 0$. The land input, $\frac{L}{N_t}$, is specific to the agricultural sector and decreases over time due to growing population at the rate of ν.

Given the assumption of perfect competition and mobility across sectors, all firms face the same input prices: wage rate w_t, rental rate of capital R_t and rental rate of land q_t. Firms maximize profits by equating marginal products to input prices, leading to equilibrium conditions that determine factor allocations and sectoral outputs. Using the firms' first-order conditions and equilibrium in factor markets, we derive expressions for the relative prices of agricultural and service goods:

$$
\begin{aligned}
p_{at} &= \left(\frac{\alpha}{\eta}\right)^{\alpha}\left(\frac{1-\alpha}{\beta}\right)^{1-\alpha}(k_{at})^{\alpha-\eta}\, n_{at}^{1-\alpha-\beta} A_{mt}^{1-\alpha} A_{at}^{-\beta}\left(\frac{N_t}{L}\right)^{1-\eta-\beta} \\
p_{st}^{T} &= \left(\frac{\alpha}{\chi}\right)^{\alpha}\left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha}\left(k_{st}^{T}\right)^{\alpha-\chi}\left(n_{st}^{T}\right)^{\chi-\alpha} A_{mt}^{1-\alpha}\left(A_{st}^{T}\right)^{\chi-1} \\
p_{st}^{N} &= \left(\frac{A_{mt}}{A_{st}^{N}}\right)^{1-\alpha}
\end{aligned}
$$

These equations show how relative prices are affected by factor intensities, productivity differences and sectoral allocations of capital and labor.

3.3 Market Clearing Conditions

To ensure general equilibrium, market clearing conditions must hold in both factor and goods markets. To ensure all available labor and capital are employed in production, following equation must hold:

$$
\begin{aligned}
n_{at} + n_{mt} + n_{st}^{T} + n_{st}^{N} &= 1 \\
k_{at} + k_{mt} + k_{st}^{T} + k_{st}^{N} &= k_t \\
\Leftrightarrow \ \kappa_{at} + \kappa_{mt} + \kappa_{st}^{T} + \kappa_{st}^{N} &= 1 \quad \text{where } \kappa_{jt} \equiv \frac{k_{jt}}{k_t}
\end{aligned}
$$

The goods market clearing conditions for each sector are:

$$
\begin{aligned}
c_{at} + x_{at} &= y_{at} \\
c_{mt} + x_{mt} + k_{t+1} - (1-\delta-\nu)\,k_t &= y_{mt} \\
c_{st}^{T} + x_{st} &= y_{st}^{T} \\
c_{st}^{N} &= y_{st}^{N},
\end{aligned}
$$

where x_{it} represents net exports in sector i. The first three equations state that domestic production in the agricultural, manufacturing and tradable service sectors meets the combined demand from domestic consumption and net exports. The fourth equation indicates that consumption equals production in the non-tradable service sector, reflecting its closed nature. Finally, in an open economy, the trade balance condition requires that the value of exports equals the value of imports:

$$p_{at}x_{at} + x_{mt} + p^T_{st}x_{st} = 0,$$

which assumes no borrowing or lending with the rest of the world.

4 Equilibrium and Steady State Analysis

This section analyzes the equilibrium and steady-state outcomes of the model in two open economy scenarios: one where services are non-tradable and one where services are partially tradable. These analyses build upon common underlying assumptions regarding sectoral structures and productivity growth rates. Specifically, we assume:

1. Complementarity - The elasticity of substitution across three sectors - agriculture, manufacturing and service - is less than one ($0 < \varepsilon < 1$). Similarly, tradable and non-tradable service goods are assumed to be complements, with $0 < \sigma < 1$.

2. TFP Growth - TFP grows exogenously at positive rates across all sectors ($\gamma_i > 0$), with TFP growth in the manufacturing sector exceeding that in the non-tradable service sector($\gamma_m > \gamma^N_s$).

4.1 Open Economy with Non-Tradable Service

Given the initial capital stock k_0 and exogenous variables $\{A_{at}, A_{mt}, A_{st}, N_t, p_{at}\}^{\infty}_{t=0}$, the equilibrium conditions can be summarized by the following: a set of prices $\{p_{st}, w_t, R_t, q_t\}^{\infty}_{t=0}$, sectoral shares of employment and capital $\{n_{at}, n_{mt}, n_{st}, \kappa_{at}, \kappa_{mt}, \kappa_{st}\}^{\infty}_{t=0}$ and quantities $\{k_t, y_{at}, y_{mt}, y_{st}, c_{at}, c_{mt}, c_{st}, x_{at}, x_{mt}\}^{\infty}_{t=0}$ which satisfy: (i) Household's optimization, (ii) Firm's optimization and (iii) Market clearing conditions.[13] In this case, only agricultural and manufacturing goods are tradable and the relative price of agricultural goods grows exogenously at the rate γ^a_p.

To ensure convergence to a steady state, the following condition must hold:

$$\gamma^a_p < (1-\eta)\gamma_m + (1-\eta-\beta)\nu - \beta\gamma_a, \tag{5}$$

[13] Section A.1.2 contains the detail derivation and descriptions for the competitive equilibrium.

The equation (5) ensures that the decline in agricultural prices is sufficiently rapid to drive a continuous reduction in the agricultural sector's share, leading to a reallocation of resources toward manufacturing and services. In the steady state, the following outcomes are observed:

1. Agricultural input shares converge to zero: $\kappa_a^{ss} = n_a^{ss} = 0$.
2. Manufacturing sector retains a positive share: $\kappa_m^{ss} = n_m^{ss} = \frac{\alpha(\gamma_m+\delta+\nu)}{\rho+\delta+\nu}$.
3. Service input shares are given by: $\kappa_s^{ss} = n_s^{ss} = 1 - \kappa_m^{ss}$.
4. Net exports of manufacturing converge to zero: $\lim_{t\to\infty} \hat{x}_{mt} = 0$.
5. Consumption of manufacturing converges to zero: $\lim_{t\to\infty} \hat{c}_{mt} = 0$.

Here, $\hat{c}_{mt} \equiv \frac{c_{mt}}{A_{mt}}$ and $\hat{x}_{it} \equiv \frac{x_{it}}{A_{mt}}$, representing detrended variables for simplicity. In this scenario, ST is primarily driven by declining agricultural prices and relative productivity differences between sectors. Over time, resources are reallocated from agriculture to manufacturing and services, facilitating the economy's shift toward a more diversified structure. Since land is specific to agriculture, it is never optimal to specialize entirely in manufacturing or services. Consequently, countries with low agricultural productivity or limited land endowments will shift resources away from agriculture more rapidly.

4.2 Open Economy with Partially Tradable Service

In the second scenario, we extend the model to include partially tradable services. Given the initial capital stock k_0 and exogenous variables $\{A_{at}, A_{mt}, A_{st}^T, A_{st}^N, N_t, p_{at}, p_{st}^T\}_{t=0}^{\infty}$, the equilibrium conditions can be summarized by the following: a set of prices $\{p_{st}^N, w_t, R_t, q_t\}_{t=0}^{\infty}$, sectoral shares of employment and capital $\{n_{at}, n_{mt}, n_{st}^T, n_{st}^N, \kappa_{at}, \kappa_{mt}, \kappa_{st}^T, \kappa_{st}^N\}_{t=0}^{\infty}$ and quantities $\{k_t, y_{at}, y_{mt}, y_{st}^T, y_{st}^N c_{at}, c_{mt}, c_{st}, c_{st}^T, c_{st}^N, x_{at}, x_{mt}, x_{st}\}_{t=0}^{\infty}$ which satisfy: (i) Household's optimization, (ii) Firm's optimization and (iii) Market clearing conditions.[14] In this case, relative prices of both agricultural and tradable service goods are determined in the global market.

The steady-state outcomes are governed by the following conditions:

1. The relative price decline in agriculture is sufficiently rapid:

$$\gamma_p^a < (1-\eta)\gamma_m + (1-\eta-\beta)\nu - \beta\gamma_a.$$

[14] Section A.1.1 contains the detail derivation and descriptions for the competitive equilibrium.

2. The relative price of agriculture declines, while that of tradable services increases:

$$\gamma_p^a < 0 < \gamma_p^s. \tag{6}$$

3. Price growth in tradable services is slower than in non-tradable services:[15]

$$\gamma_p^s < (1-\alpha)(\gamma_m - \gamma_s^N). \tag{7}$$

Under these conditions, the steady-state outcomes are:

1. Agricultural input shares converge to zero:
   $\kappa_a^{ss} = n_a^{ss} = 0.$

2. Manufacturing input shares are given by:
   $(\kappa_m^{ss})^\alpha \, (n_m^{ss})^{1-\alpha} = \frac{\alpha(\gamma_m+\delta+\nu)}{\rho+\delta+\nu} - \left(\frac{\alpha}{\chi}\right)^\alpha \left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha} (\kappa_{ss}^T)^\alpha (n_{ss}^T)^{1-\alpha}.$

3. Tradable service has positive input shares:
   $n_{ss}^T = \frac{\alpha(1-\chi)\kappa_{ss}^T}{\chi(1-\alpha)-(\chi-\alpha)\kappa_{ss}^T}$ and $\kappa_{ss}^T = \frac{\chi(1-\alpha)n_{ss}^T}{\alpha(1-\chi)-(\alpha-\chi)n_{ss}^T}.$

4. Non-tradable service maintains positive input shares:
   $n_{ss}^N = \frac{\chi(1-\alpha)\kappa_{ss}^N}{\chi(1-\alpha)-(\chi-\alpha)\kappa_{ss}^T}$ and $\kappa_{ss}^N = \frac{\chi(1-\alpha)n_{ss}^N}{\alpha(1-\chi)-(\alpha-\chi)n_{ss}^T}.$

5. Net exports of manufacturing goods turn negative in the long run:
   $\lim_{t\to\infty} \hat{x}_{mt} = -\lim_{t\to\infty} p_{st}^T \hat{x}_{st} = -\left(\frac{\alpha}{\chi}\right)^\alpha \left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha} (\kappa_{ss}^T)^\alpha (n_{ss}^T)^{1-\alpha}.$

6. Consumption of manufacturing and tradable services converge to zero, while consumption of non-tradable services remains positive:
   $\lim_{t\to\infty} \hat{c}_{mt} = \lim_{t\to\infty} \hat{c}_{st}^T = 0$ and $\lim_{t\to\infty} \hat{c}_{st}^N \neq 0.$

In this scenario, ST is accelerated by slower price growth in tradable services, driven by higher productivity and exposure to international competition. As a result, resources are reallocated from agriculture and manufacturing into the service sector, facilitating a transition toward a service-oriented economy.

[15] In equilibrium, $p_{st}^N = \left(\frac{A_{mt}}{A_{st}^N}\right)^{1-\alpha}$. Tradable services experience slower price growth due to higher productivity and exposure to international competition, while non-tradable services face faster price increases, driven by higher labor intensity and limited competition.

4.3 Structural Transformation Dynamics

In both open economy scenarios, resources are gradually reallocated away from the agricultural sector as the economy approaches its steady state. This reallocation is primarily driven by the declining relative international price of agricultural goods. Over time, the steady-state shares of agricultural capital and labor converge to zero. However, the dynamics differ between the two cases due to the tradability of services.

Non-Tradable Services Scenario When services are non-tradable ($\theta = 0$), the manufacturing sector retains a positive share of resources in the steady state: $\kappa_m^{ss} = n_m^{ss} = \frac{\alpha(\gamma_m+\delta+\nu)}{\rho+\delta+\nu}$. The service sector grows, but its expansion is limited by its non-tradability: $\kappa_s^{ss} = n_s^{ss} = 1 - \kappa_m^{ss}$. Net exports of manufacturing goods converge to zero in the long run, as the economy balances domestic production and consumption without significant trade in services.

Partially Tradable Services Scenario When services become partially tradable ($0 < \theta < 1$), the manufacturing sector's steady-state share declines: $\kappa_m^{ss} = n_m^{ss} = \frac{\alpha(\gamma_m+\delta+\nu)}{\rho+\delta+\nu} - \left(\frac{\alpha}{\chi}\right)^{\alpha} \left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha} (\kappa_{ss}^T)^{\alpha} (n_{ss}^T)^{1-\alpha}$. The service sector's share expands correspondingly, reflecting a reallocation of resources toward the production and export of tradable services.

In this scenario, net exports of manufacturing goods turn negative, indicating that the economy imports more manufacturing goods than it exports. Resources shift toward tradable services, which are exported, while manufacturing imports rise to meet domestic consumption and investment needs. Although the manufacturing sector's relative importance diminishes, it still retains a portion of resources. These adjustments illustrate a global trend toward service-oriented economies, especially in countries with higher degrees of service tradability.

4.4 Transition Dynamics

During the transition to the steady state, resource allocation across sectors is affected by sector-specific TFP growth rates and the relationship between domestic and international prices. In agricultural sector, countries with low agricultural productivity or limited land resources experience a faster decline in their agricultural sector, increasingly relying on imports to meet domestic demand. In economies with strong manufacturing productivity growth (high γ_m), the reallocation of resources from manufacturing to services is slower. Robust productivity supports the manufacturing sector's share, preventing a sharp decline.

In contrast, countries with lower γ_m, they encounter declining manufacturing sectoral shares in labor and capital input, while their shares in service sector rises (stylized facts).

In addition, the model identifies two distinct phases of ST. Initially, when service goods are not tradable (θ is zero), resources are reallocated from agriculture to both manufacturing and services, leading to simultaneous growth in both sectors. After the initial phase, as service goods become tradable ($\theta > 0$), global demand for tradable services rises, prompting a larger reallocation of resources toward this sector. Consequently, the agricultural sector continues to decline, while the service sector expands significantly, and the manufacturing sector's share either stagnates or decreases. These dynamics highlight the importance of trade openness and sectoral productivity growth in determining the pace and direction of ST.

4.5 Implications for South Korea

South Korea presents a notable case in the context of ST due to its relatively high manufacturing share compared to other developed countries. Unlike the typical pattern where manufacturing shares decline and service sector shares rise sharply, South Korea's manufacturing sector has remained resilient.

This deviation can be explained by the model's outcomes, highlighting the crucial role of manufacturing TFP growth in determining the long-run share of manufacturing. Specifically, the steady-state share of resources allocated to manufacturing is:

$$(\kappa_m^{ss})^{\alpha}\,(n_m^{ss})^{1-\alpha} = \frac{\alpha(\gamma_m + \delta + \nu)}{\rho + \delta + \nu} - \left(\frac{\alpha}{\chi}\right)^{\alpha}\left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha}(\kappa_{ss}^{T})^{\alpha}(n_{ss}^{T})^{1-\alpha}.$$

South Korea's higher TFP growth rate in manufacturing (γ_m) enables it to sustain a larger allocation of resources in this sector[16], even as global trends have shifted other economies toward more service-oriented structures. Consequently, South Korea's manufacturing sector maintains competitiveness and resource allocation, diverging from the common pattern observed in other developed nations.

4.6 Summary of Results

The equilibrium and steady-state analysis demonstrates how sector-specific productivity growth rates, preferences, and the tradability of services influence an economy's ST. By

[16] According to Kim (2004), during the 1980s, South Korea's manufacturing sector had the relatively higher growth contribution rate at approximately 40%, compared to other OECD countries such as Belgium (30.6%), Japan (27.1%), France (7.7%) and Norway (-2.3%). By the 1990s, South Korea maintained a strong position with a manufacturing growth contribution of 43.7%, following Sweden (58.2%) and Finland (57.5%)

incorporating partially tradable services into the model, we gain a nuanced understanding of the mechanisms driving deviations from the stylized facts observed in rapidly growing, trade-dependent economies like South Korea. Our findings suggest that economies with strong manufacturing productivity may retain larger manufacturing sectors, while limiting the rise of service sectoral shares.

5 Discussion

This section addresses the current limitations of the study and outlines future steps to enhance the robustness of the findings and contribute meaningfully to the literature.

Data Limitations

A primary limitation of this study is the unavailability of detailed trade data by industry within the service sector. Currently, trade data for services are aggregated, hindering a nuanced analysis of the tradability of different service industries. To refine the model and its implications, access to disaggregated trade data in services is essential. Addressing this data gap requires either awaiting the availability of such detailed data or employing estimation methodologies to infer trade in services at the industry level.

Estimating Trade in Services

One potential approach to mitigate the data limitation is to estimate the tradability of services using the methodology proposed by Gervais and Jensen (2019). Their approach estimates the tradability of services based on thresholds established for the manufacturing sector. For example, they found that approximately 75% of the value added in manufacturing is tradable, compared to only 28% in the service sector. Implementing this methodology would involve converting data classified under the Standard Industrial Classification (SIC) or the North American Industry Classification System (NAICS) to align with the Standard International Trade Classification (SITC) used in this study. This conversion would allow for a more precise estimation of tradable and non-tradable components within the service sector.

6 Concluding Remarks

This study examines the extent to which the stylized facts of structural transformation (ST), as highlighted by Herrendorf, Rogerson, and Valentinyi (2014), apply across different countries using data from the Penn World Table and the Groningen Growth and Development Centre. Our analysis confirms that these stylized patterns — a declining agricultural share, a hump-shaped manufacturing share and a continuously increasing service share — generally hold for the majority of high-income and upper-middle-income countries.

However, when analyzing the ST patterns of South Korea, Singapore and Malaysia across three different measures— employment, nominal value added and real value added — we observe deviations from the stylized facts. These countries, characterized by rapid economic growth over recent decades, maintain higher manufacturing shares than expected and exhibit unique ST patterns. Data indicate that their manufacturing trade shares are significantly higher than those of other countries and their service trade shares exceed their agricultural trade shares.

To account for these observations, we developed a general equilibrium model for open economies that includes both tradable and non-tradable services. Our model demonstrates that when services are tradable, resources reallocate toward the service sectors, accelerating ST and reducing the manufacturing sector's share. However, in economies with strong manufacturing productivity, like South Korea, the decline in the manufacturing sector's share is less pronounced. The high productivity in manufacturing sustains its competitiveness, mitigating the reallocation of resources toward services even in the presence of tradable services.

Our findings suggest that sector-specific productivity growth rates play a crucial role in shaping the path of ST. In economies with high manufacturing productivity, the manufacturing sector retains a larger share of resources despite the tradability of services. This highlights the importance of accounting for productivity differentials and the tradability of services when analyzing ST in rapidly growing economies.

Moving forward, considering the rising trend of service trade in rapidly growing countries, we plan to extend our structural model to more explicitly account for tradable service goods. Given the relatively limited exploration of such models, we anticipate that further development and simulation will yield significant insights into the mechanisms driving deviations from traditional ST patterns. While the complete role of the open economy in explaining the ST of emerging countries has not been fully demonstrated, this paper provides a foundation for future research endeavors aimed at understanding and managing economic development in an increasingly globalized world.

Chapter 2 - The Effect of Child Allowances and Public Spending on Fertility

1 Introduction

The demographic landscape of developed countries is undergoing significant changes due to declining fertility rates and increasing life expectancy. These shifts have led to concerns about the rising dependency ratio — the proportion of non-working-age individuals to the working-age population — which poses substantial challenges for healthcare systems, social security financing, and overall economic productivity. A higher dependency ratio implies that fewer workers are available to support a growing number of dependents, including retirees, which can strain public finances and hinder economic growth (Barr and Diamond, 2006; Heer and Irmen, 2014).

Two primary factors contribute to the escalating dependency ratio. First, advancements in medical technology and improved living standards have resulted in a steady increase in life expectancy. To mitigate this issue, various policies have been proposed, such as extending working years by raising the retirement age (Gruber and Wise, 2002), reducing pension benefits per individual (Feldstein and Samwick, 1998; Diamond, 2004), increasing the eligible age for pension benefits (Feldstein and Liebman, 2002; Coile and Gruber, 2007), or raising taxes to offset the increased number of beneficiaries (Holzmann, 2005). However, implementing such policies presents political and practical challenges (Pierson, 1996; Mulligan and Sala-i Martin, 1999; Wise, 2012).

The second factor is the persistently low fertility rate, which is the focus of this paper. Fertility rates have consistently declined alongside economic growth in developed countries, with several potential reasons such as higher female workforce participation (Goldin, 2006), increased child-rearing costs (Laroque and Salanié, 2014), and lower marriage rates (Baudin, De La Croix, and Gobbi, 2015). In response to low fertility, many countries have introduced economic incentives designed to encourage higher birth rates. These measures include direct child subsidies, expansions of public childcare infrastructure, and other policies aimed at alleviating the financial burden of raising children (Milligan, 2005).[17]

While traditional neoclassical growth theory suggests that a lower population growth

[17] An alternative strategy involves increasing immigration to augment the working-age population (Delogu, Docquier, and Machado, 2018). Although this approach has its own implications, the present paper concentrates on policies related to the economic incentives.

rate may increase per-capita output and consumption, this perspective often overlooks the complexity of real-world fiscal systems. In practice, declining fertility intensifies population ageing, raises the dependency ratio, and ultimately places substantial pressures on public finances. With fewer workers contributing to the tax base, government revenues fall, undermining the sustainability of welfare and social insurance programs.

Previous research incorporating endogenous fertility choices has used different modeling approaches, primarily Endogenous Growth Models and Overlapping Generations (OLG) Models. Endogenous Growth Models often focus on the long-term implications of demographic changes on economic growth, considering factors like human capital accumulation and technological progress. For example, Jones (2022) constructs a model that predicts two stable states — Expanding Cosmos and Empty Planet — based on decreasing population growth rates. The study concludes that if the economy adopts optimal allocation promptly, it converges to the Expanding Cosmos; otherwise, it risks getting trapped in the Empty Planet outcome.

OLG Models are well-suited for analyzing individual fertility decisions and the intergenerational transfer of resources, allowing for the examination of how policies influence household decisions regarding the number of children, labor supply, savings, and investment in human capital. Fougère and Mérette (1999) compare 15-period OLG and endogenous growth models to investigate the effects of population ageing on economic growth in seven OECD countries, finding that ageing could increase incentives for future generations to invest more in human capital, thereby enhancing economic growth. Hock and Weil (2012) use a continuous-time OLG model and state that increasing fertility is the consumption-maximizing response to greater longevity but point out that larger transfer payments will decrease fertility, leading to further population ageing. van Groezen and Meijdam (2008) find that it is optimal to introduce child allowances that increase with longevity, and if child-rearing costs depend on wages, savings should be taxed. Aaronson, Lange, and Mazumder (2014) analyze the intensive (number of children) and extensive (decision to have children) margins of fertility in the U.S. Their results show that higher wages are positively associated with fertility on both margins, while higher taxes and childcare costs have a negative impact.

In recent years, several studies have analyzed the effects of South Korea's fertility policies. For instance, Hong, Kim, Lim, and Yeo (2016) found a 1,000 USD increase in cash grant amounts raises the crude birth rate by 4.4%. Similarly, Choi, Yellow Horse, and Yang (2018) finds that family and career assisting policies—such as childcare leave, family allowance, and workplace daycare facilities—interact with age to shape fertility intentions among working

Korean women, with particularly pronounced effects for those under 35.[18]

However, in the existing literature, a comprehensive comparison of the efficacy of various fertility policies has not been extensively explored. Moreover, existing studies often focus on specific policies or contexts, lacking a broader evaluation of multiple policy instruments and their interactions. This gap is particularly pronounced in the case of South Korea, a country with the lowest fertility rate in the world. To address this gap, this paper develops a three-period OLG model to evaluate the impacts of multiple policies — income taxation, parental leave payments, child allowances, and public education - on fertility rate and other key economic variables. By calibrating the model to South Korean data, we assess the individual effects of each policy and investigate whether increases in fertility come at the expense of per-worker output and consumption compared to a scenario without these policy interventions.

Our analysis reveals several findings. First, enhancing child allowances is the most effective policy for increasing the fertility rate, leading to a 31% rise in the TFR when increased by 10 percentage points (pp). Second, increasing parental leave payments and income tax rates by 10pp enhance the TFR by 22% and 24%, respectively. Third, we find that when the TFR decreases, households can enjoy a better standard of living with higher output per capital and consumption per adult due to increased human capital accumulation shared among fewer individuals. Lastly, we identify the high cost of private education as a critical factor deterring higher fertility. Policies aimed at reducing private education expenses could have a pronounced positive effect on fertility rates.

Our contributions to the literature are threefold. First, we provide a comprehensive analysis of different fertility policies within a unified OLG framework, allowing for direct comparisons of their effectiveness. Second, by incorporating endogenous labor supply and education decisions, our model captures the trade-offs households face between the quantity and quality of children, as well as between labor market participation and child-rearing. Third, focusing on South Korea, a country with one of the lowest fertility rates globally, we offer policy insights that are particularly relevant for countries facing similar demographic challenges.

The remainder of the paper is organized as follows. Section 2 provides some factual

[18] Housing prices are frequently cited as a major factor contributing to low fertility rates in Korea, given their substantial impact on household finances. Seo (2013) demonstrates that rising housing prices disproportionately deter fertility among renters relative to homeowners, with a 10% price increase correlating with a 0.5% to 1.06% reduction in renters' fertility rates. Kim and Choi (2023) shows that regions characterized by economic activity and elevated housing prices tend to exhibit lower rates of marriage and childbirth.

background on demographic changes and government policies in developed countries. Section 3 presents the baseline model and identifies the equilibrium and steady states. Section 4 examines the quantitative analysis, including the dynamic analysis and the simulation of the model. Section 5 evaluates the impact of different types of government policies. Section 6 discusses the role of private education costs along with the limitations of the model. Finally, Section 7 concludes with a summary.

2 Some facts

2.1 Demographic Changes

Demographic changes have significant implications for a country's economic development, social policies and overall societal well-being. Numerous studies have focused on shifts in total population, age composition, and Total Fertility Rates (TFR)[19]. Notably, South Korea's TFR was recorded at 0.72 in 2023, much lower than the OECD average of 1.5 in 2022 and well below the replacement rate of 2.1 children per woman (OECD, 2023b).

Population decline can result from a decrease in TFR or an increase in the death rate. However, given advancements in medical treatments, the primary cause of population decrease is likely the declining TFR. A decreasing population implies a reduction in both the labor supply and aggregate output, which can lead to economic slowdown and eventual recession (Bloom, Canning, and Sevilla, 2001). Moreover, with increasing life expectancy, the number of retirees will grow, while the working-age population shrinks, resulting in a higher ratio of beneficiaries to contributors within the pension system.

To address this issue, possible solutions include increasing the TFR as a medium or long-term strategy, adopting more capital-intensive production methods at the firm level, or increasing the labor supply through foreign workers or immigrants (Lee and Mason, 2009). In practice, countries often focus on raising fertility rates and allowing more foreign workers or immigrants to alleviate labor supply shortages. Consequently, the primary strategies can be divided into two categories: increasing fertility rates or permitting more immigrants to address labor supply constraints.

[19] There are three general ways of calculating fertility rates: (1) Crude Birth Rate $= \frac{\text{total live births}}{\text{total population}} \times 1,000$; (2) General Fertility Rate $= \frac{\text{total live births}}{\text{female population aged 15-44}} \times 1,000$; (3) Total Fertility Rate $= \sum_{10-49}$ Age-Specific Fertility Rates $\times\, 5$, where age-specific fertility rates are calculated for 5-year age groups from 10-14 up to 45-49. The TFR represents the average number of children a woman would have over her lifetime if current age-specific fertility rates remained constant.

2.2 Current Outlook

In traditional growth models, such as the Solow and Ramsey models, a decreasing population or TFR is not necessarily problematic since lower population growth can lead to capital deepening, increasing output per worker and enhancing productivity (Solow, 1956). However, this perspective primarily focuses on per capita outcomes and may overlook potential challenges such as the social security implications of an ageing population (Cutler, Poterba, Sheiner, Summers, and Akerlof, 1990).

This paper begins by examining how the TFR of developed countries has changed, with a focus on South Korea. According to the World Bank (2023), the top three countries with the highest TFR were located in Africa, while the lowest three were Hong Kong (0.7), South Korea (0.78)[20], and Puerto Rico (0.9). Considering their population sizes, the data highlight the severe situation of low fertility rates in South Korea, which had a population of over 51 million in 2020, compared to Hong Kong's 7.5 million and Puerto Rico's 3.3 million.

Developed Countries

Figure 6 illustrates the trends in TFR across a range of developed countries. Consistent with long-term demographic trends, OECD countries show a persistent decline in TFR. Within Europe, France and Sweden exhibit relatively higher TFRs at 1.79 and 1.52, respectively, largely supported by comprehensive family-oriented policies and financial incentives (Thévenon, 2011). In contrast, countries such as Spain and Italy continue to experience consistently low TFRs at 1.16 and 1.24 (Billari and Kohler, 2004). The United States, while following a declining trend, maintains a comparatively higher TFR due to factors such as immigration and cultural diversity. In Asia, Japan faces similar fertility challenges, with a TFR of 1.26, driven by an ageing population and economic constraints (Matsukura, Shimizutani, Mitsuyama, Lee, and Ogawa, 2018). However, South Korea faces a more severe demographic issue, with an exceptionally low TFR of 0.72, showing the steepest decline over the observed period.

South Korea

South Korea's low fertility rate is a complex issue. One significant reason is the cultural and societal changes over the past few decades. There has been a substantial increase in women's workforce participation, leading to a decline in the traditional family structure where women were mainly responsible for child-rearing (Ma, 2014). Women are now delaying marriage

[20] South Korea reported 0.72 of TFR in 2023.

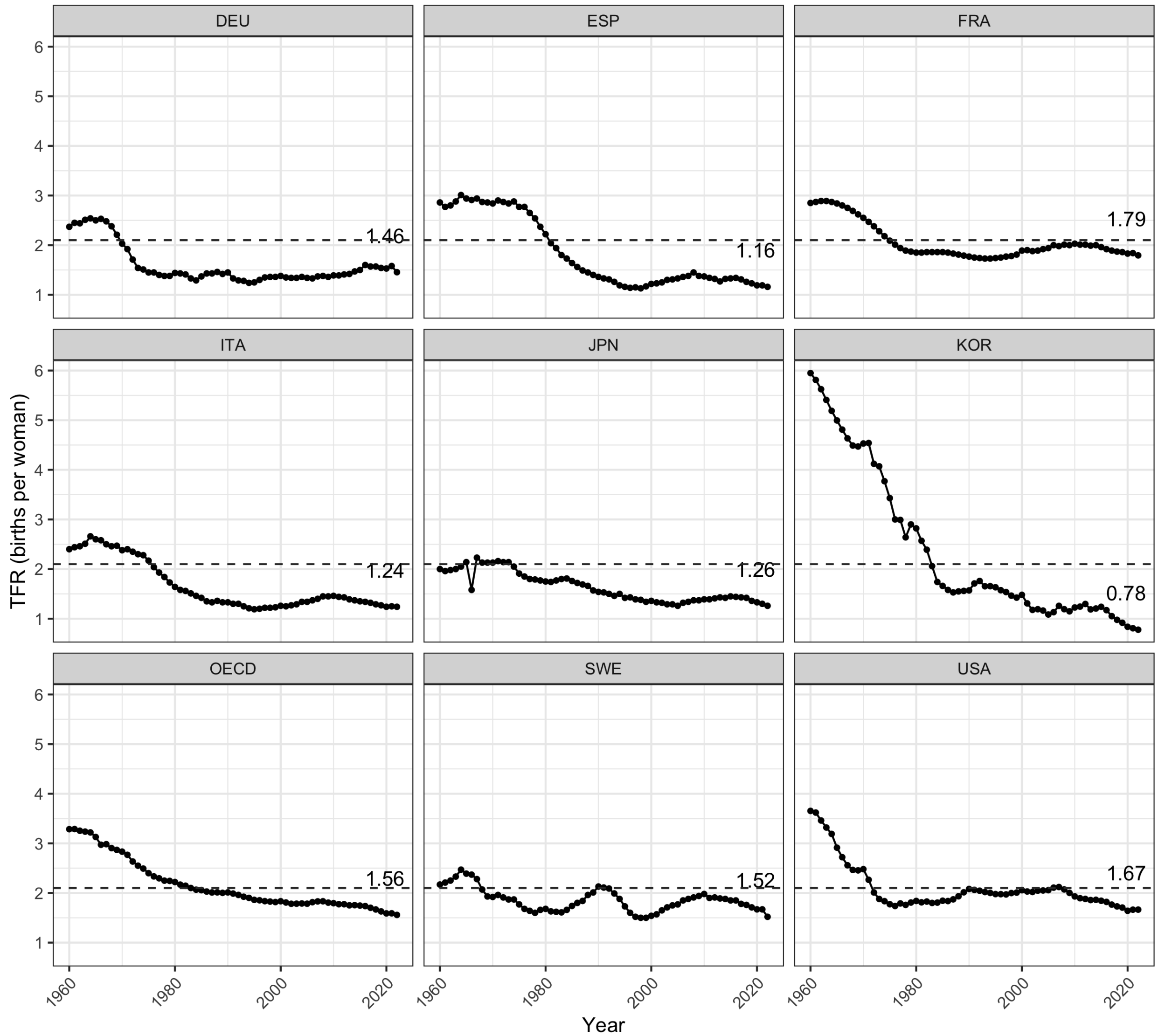


Figure 6: TFR of Developed Countries

Note: The dotted line in the figure represents the replacement rate (2.1). The numbers displayed in the figure represent the latest TFR for each country.

and childbirth, opting instead to focus on their careers and personal growth (Yoo, 2016). Furthermore, the high cost of living and the pressure to succeed in a competitive society make it difficult for young couples to afford raising children (Lee, 2009).

In a nationwide analysis, Figure 7 highlights notable trends among four age groups. The data indicate that fertility rates for the 15–19 and 40–49 age groups have exhibited minimal variation over time, consistently remaining at low levels. In contrast, significant shifts have been observed in the 20–29 and 30–39 age groups. Between 1993 and 2021, the TFR for the

30–39 age group increased from 76 to 120, while the 20–29 age group experienced a drastic decline from 248 to 32. This suggests that the significant decrease in South Korea's TFR is predominantly driven by the 20–29 age group.

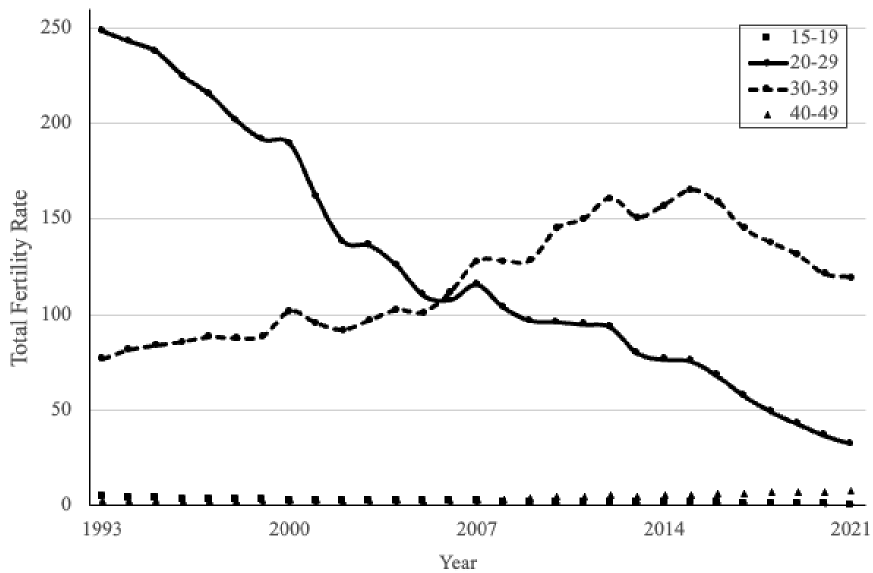


Figure 7: TFR of Korea by Age Group (KOSIS, 2022b)

The low TFR in South Korea is also affected by specific circumstances in metropolitan areas. According to KOSIS (2022b), while the national TFR is 0.72, it drops to 0.59 in Seoul[21]. Additionally, BAIK (2021) shows that over the past 20 years, the decline in TFR in Seoul has been around 44%, while the decline in provincial areas was approximately 34%. This discrepancy can be attributed to factors such as the delay in marriage among the younger generation in metropolitan areas.

Table 1 illustrates that regardless of gender and location, the marriage rates within the 25-29 age group have declined significantly. Specifically, between 2010 and 2019, there has been a decrease of over 20pp for males and around 30pp for females, aligning with the data presented in Figure 7. These two data indicate that the decline of TFR in Korea is predominantly driven by the lower marriage rate and lower TFR within 20-29 age group, especially who live in metropolitan areas. To address this issue, the South Korean government has implemented several policies, including financial incentives for families who have children, increased support for working mothers, and improved childcare services. However, changing cultural and societal attitudes towards family and child-rearing may take time and require more

[21]Seoul, the capital of South Korea, is home to approximately 17% of the country's total population.

comprehensive measures to address the root causes of the low fertility rate (Jones, 2019; Cheng, 2020).

		2010		2013		2015		2017		2019	
		Male	Female	Male	Female	Male	Female	Male	Female	Male	Female
25-29	Nationwide	49.6	79.1	47.1	79.9	41.2	72.9	33.5	60.6	27.8	50.4
	Seoul	41.7	68.8	39.0	68.0	33.9	62.6	27.2	49.5	22.2	39.9
30-34	Nationwide	58.5	52.0	64.2	51.7	62.4	51.8	56.4	48.4	51.1	46.9
	Seoul	62.4	46.9	68.3	56.9	67.6	57.7	59.6	52.4	54.3	49.5

Table 1: Marriage Rate in South Korea (Age 25-34)

Note: The marriage rate is measured as the number of marriages per 1,000 people.

2.3 Government Policies

Developed Countries

Many developed countries are making efforts to increase TFR through various policies focusing on financial incentives and support for families. Figure 8 illustrates the TFR trends in Australia, Hungary, and Sweden.

First, Australia has provided financial incentives such as the "Baby Bonus" and family tax benefits, as well as paid parental leave for both mothers and fathers (Drago, Sawyer, Shreffler, Warren, and Wooden, 2011). The government supports affordable childcare and early education programs, making it easier for parents to balance work and family life. Despite these efforts, Australia's TFR has remained relatively stable, with some fluctuations over the years.

Second, Hungary has introduced a comprehensive family support system, including generous tax benefits for families with multiple children, housing subsidies, and low-interest loans. The government has also focused on promoting traditional family values and encouraging women to have more children. While these measures have led to a slight increase in TFR, challenges remain due to persistent economic issues and emigration of young people.

Finally, Sweden, known for its progressive family policies, offers extensive parental leave, with both parents entitled to share up to 480 days of paid leave per child. Additionally, Sweden provides subsidized, high-quality childcare and preschool programs, enabling parents to better balance work and family life (Thévenon, 2011). These policies have contributed to a relatively high TFR compared to other European countries, but the rate has still experienced some fluctuations over time.

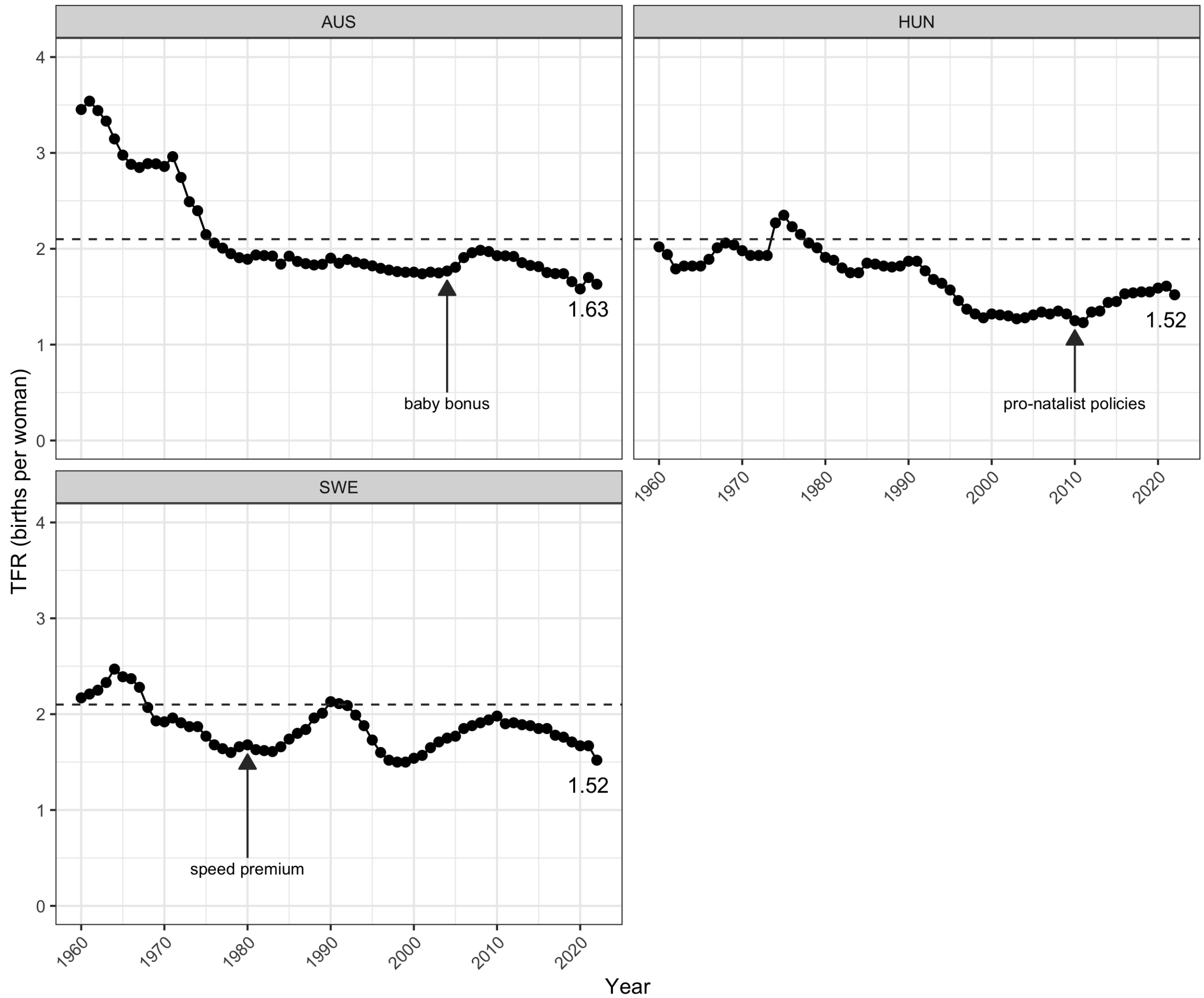


Figure 8: TFR of AUS, HUN and SWE

Note: The arrows in the figure represent the starting years of each country's fertility policy.

South Korea

The South Korean government has implemented various policies to address the issue of low TFR, such as financial incentives, generous parental leave, childcare support, housing support, and education support. These policies aim to reduce the financial and social barriers to having children and promote a better work-life balance. Furthermore, South Korea has established key committees to address demographic changes, including the Presidential Committee on Ageing Society and Population Policy and the Presidential Committee for

Balanced National Development.[22] In addition, South Korea is planning to establish the Administration of Overseas Koreans to manage immigration-related matters (Lee and Noh, 2021). Despite these efforts, a persistent decline in TFR has been observed. Therefore, this paper aims to investigate the effects of parental leave support, child allowances, and public spending on fertility rates.

3 The Model

Building upon the declining endogenous fertility rates, this model investigates the effectiveness of government fertility support policies within a three-period overlapping generations (OLG) framework, following the foundational work of Diamond (1965) and Samuelson (1958). Each agent progresses through three life stages — Child, Young, and Old — but makes decisions only during the latter two. Agents cares about their own consumption and the number and human capital of their children. During the Young stage, they allocate their time between working and raising children and distribute their wage income among consumption, savings, and educational expenditures for their children. In the Old stage, individuals only consume based on their savings.

3.1 Human Capital

Parents can enhance their child's future human capital h_{t+1} by investing on education e_t according to

$$h_{t+1} = a(e_0 + e_t)^{\delta} g_t^{1-\delta} \quad \text{where } \delta \in (0,1). \tag{8}$$

A child is born with a basic level of human capital e_0 and does not make any independent decision. The parameter a represents the learning ability of a child and g_t denotes the public education per child provided by the government. The parameter δ reflects the relative shares of education investments between parents and government.

3.2 Households and Preferences

A parent maximizes her expected lifetime utility:

$$U_t = \log c_t^y + \gamma \log(n_t^{\theta} h_{t+1}^{1-\theta}) + \beta \log c_{t+1}^o \tag{9}$$

[22] The Presidential Committee on Ageing Society and Population Policy develops strategies to address the ageing population and low fertility rates. The Presidential Committee for Balanced National Development works to promote balanced regional development by reducing disparities between urban and rural areas and fostering regional competitiveness throughout South Korea.

by choosing $\{c_t^y, c_{t+1}^o, s_t, n_t, e_t\}$ subject to:

$$
\begin{aligned}
c_t^y + s_t + (e_t + \varepsilon - \alpha_t^c)n_t &= (1-\tau)(1-(\nu-\alpha_t^p)n_t)w_t h_t \\
c_{t+1}^o &= (1+r_{t+1})s_t \\
h_{t+1} &= a\left(e_0 + e_t\right)^{\delta} g_t^{1-\delta},
\end{aligned}
$$

where input prices (w_t, r_t) and the government policies - parental leave payments (α_t^p), child allowances (α_t^c) and public education (g_t) - are taken as given. The parameter $\beta \in (0,1)$ is the subjective discount rate; γ represents the weight placed on offspring utility; and $\theta \in (0,1)$ captures the trade-off between the quantity and quality of children.

In this model, raising a child incurs fixed time costs $\nu \in (0,1)$ and goods costs $\varepsilon > 0$. These costs can be partially offset by parental leave payments $\alpha_t^p \in [0,\nu)$ and child allowances $\alpha_t^c \in [0,\varepsilon)$ respectively. Specifically, a parent first determines the number of children n_t. Within one unit of time, the fixed time cost νn_t reduces the available work hours. Then the government provides parental leave payments α_t^p, compensating part of the fixed time cost of child rearing as an income transfer.

The household optimization problem can yield either interior or corner solutions, depending on the level of wage income. When the wage income $w_t h_t$ is below a certain threshold $\hat{E}_t$ [23], parents choose not to invest in their children's education due to low marginal gains from quality investment.

Interior Solutions

Substituting constraints and human capital into equation (9), the household problem is reduced to choosing education expenditure e_t, fertility rate n_t and savings s_t. When $n_t > 0$ and their wage income exceeds the threshold $\hat{E}_t$, the interior equilibrium is characterized by:

$$
e_t = \frac{\delta(1-\theta)(1-\tau)(\nu-\alpha_t^p)w_t h_t}{\theta+\theta\delta-\delta} + \frac{\delta(1-\theta)(\varepsilon-\alpha_t^c)-\theta e_0}{\theta+\theta\delta-\delta} \tag{10}
$$

$$
n_t = \frac{\gamma(\theta+\theta\delta-\delta)(1-\tau)w_t h_t}{(1+\beta+\gamma\theta)\Bigg((1-\tau)(\nu-\alpha_t^p)w_t h_t + \varepsilon - \alpha_t^c - e_0\Bigg)} \tag{11}
$$

$$
s_t = \frac{\beta(1-\tau)}{(1+\gamma\theta+\beta)} w_t h_t. \tag{12}
$$

We assume $\theta + \theta\delta - \delta > 0$ to ensure feasible values for e_t and n_t.

Equation (10) shows that parental investment in education is positively related to their wage income $w_t h_t$. As human capital h_t increases, parents earn higher income, enabling

[23] $\hat{E}_t \equiv \frac{\delta(1-\theta)(\varepsilon-\alpha_t^c)-\theta e_0}{\delta(1-\theta)(1-\tau)(\nu-\alpha_t^p)}$, which will be described more in detail in the next part.

them to invest more in each child's education. Parents who prioritize child quality (lower θ) invest more in education. Higher fixed costs of time (ν) and goods (ε) increases education expenditure, highlighting the quantity-quality trade-off in child-rearing decisions.

Regarding government interventions, equation (10) suggests that reducing the child-rearing costs of time ($\nu - \alpha_t^p$) or goods ($\varepsilon - \alpha_t^c$) lowers parental investment in education.[24] In contrast, an increase in public education g_t enhances the marginal returns to educational investment, thereby raising education expenditures following the equation (8). A higher tax rate τ reduces disposable income, leading parents to spend less on education.

Equation (11) presents the fertility choice of a household. It implies that parents will have more children if the fixed costs (ν, ε) are lower. To ensure that fertility n_t decreases as wage income $w_t h_t$ increases, we impose the condition $\varepsilon < e_0$, which ensures that the substitution effect dominates the income effect of higher disposable income.[25]

In terms of policy implications for fertility, equation (11) indicates that reducing the costs of raising a child, both time and goods, leads parents to choose larger families. This outcome contrasts with the effect observed for education expenditure, highlighting the trade-off between child quantity and quality. Furthermore, a higher tax rate τ prompts households to have more children. Facing reduced disposable income, parents find additional labor hours less rewarding and instead allocate more time to raise children.

Equation (12) indicates that household savings are positively related to the discount rate β and negatively related to the tax rate τ. Parents who value future consumption more (higher β) save more, while higher taxes reduce disposable income and savings. Additionally, a higher preference for children (higher γ or θ) reduces savings due to increased time allocation on child-rearing or increased expenditure on education.

[24] In the government section, it will be discussed that if one policy parameter increases, another must decrease to maintain budget balance. Here, we consider the individual effects of each policy instrument in isolation.

[25] The substitution effect arises from higher opportunity costs of time and decreased marginal benefits of investing in education due to higher e_0. Parents may feel that having fewer children is sufficient to achieve their desired utility, as each child inherently possesses higher capabilities. Even if the financial cost per child decreases (lower ε), the overall cost of raising additional children — including the high value of parents' time — leads parents to prioritize allocating resources (time and money) to other areas, such as labor market participation, consumption and saving. This reinforces the preference for a smaller family size despite increased earnings and lower child-rearing costs.

Corner Solutions

If a parent decides to have children, but their wage income is below the threshold $w_t h_t < \hat{E}_t$, they choose not to invest in education ($e_t = 0$). Then the fertility rate and savings become:

$$n_t = \frac{\gamma\theta(1-\tau)w_t h_t}{(1+\beta+\gamma\theta)\bigg((1-\tau)(\nu-\alpha_t^p)w_t h_t + \varepsilon - \alpha_t^c\bigg)}$$

$$s_t = \frac{\beta(1-\tau)}{(1+\gamma\theta+\beta)} w_t h_t.$$

If parents choose not to have a child ($n_t = 0$), they incur no child-rearing or education costs, and their savings simplify to $s_t = (1-\tau)w_t h_t$.

3.3 Aggregate Production and Factor Prices

The economy consists of numerous competitive firms with constant returns to scale production technology:

$$Y_t = AK_t^\alpha L_t^{1-\alpha}, \quad \text{where } A > 0 \text{ and } \alpha \in (0,1).$$

Here, Y_t is aggregate output, K_t is physical capital, and L_t is *effective* labor input. Effective labor L_t is defined as:

$$L_t = (1-\nu n_t)h_t N_t^y,$$

where N_t^y is the young population (worker) at time t, h_t is the human capital of current worker, and $1-\nu n_t$ represents the the fraction of time allocated to work.[26] Defining capital per effective labor as $k_t = \frac{K_t}{L_t}$, the firm's profit maximization leads to the price functions of effective labor and capital:

$$w_t = (1-\alpha)Ak_t^\alpha \tag{13}$$

$$r_t = \alpha Ak_t^{\alpha-1}. \tag{14}$$

3.4 Government

The government's revenue comes solely from wage taxation at rate τ and is allocated to child allowances α_t^c, parental leaves supports α_t^p and public education g_t. Assuming a balanced budget each period, the government's budget constraint at period t is:

$$\tau(1-(\nu-\alpha_t^p)n_t)w_t h_t N_t^y = \alpha_t^c n_t N_t^y + \alpha_t^p n_t w_t h_t N_t^y + g_t N_{t+1}^y.$$

[26]It is noted that parental leave payments α_t^p do not affect the working time, as discussed in Section 3.2.

Each area of government spending is a fixed proportion σ^i of its revenue, where $i \in c, p, e$ and $\sigma^c + \sigma^p + \sigma^e = 1$. Thus, the government policies are:

$$\alpha_t^p = \sigma^p \tau(1 - (\nu - \alpha_t^p)n_t)n_t^{-1} = \frac{\sigma^p \tau(1 - \nu n_t)}{(1 + \sigma^p \tau)n_t} \tag{15}$$

$$\alpha_t^c = \sigma^c \tau(1 - (\nu - \alpha_t^p)n_t)w_t h_t n_t^{-1} = \frac{\sigma^c \tau(1 - \nu n_t)w_t h_t}{(1 + \sigma^p \tau)n_t} \tag{16}$$

$$g_t = \sigma^e \tau(1 - (\nu - \alpha_t^p)n_t)w_t h_t n_t^{-1} = \frac{\sigma^e \tau(1 - \nu n_t)w_t h_t}{(1 + \sigma^p \tau)n_t}. \tag{17}$$

3.5 Market Clearing Condition

Capital market clearing requires:

$$K_{t+1} = s_t N_t^y. \tag{18}$$

This equation states that the physical capital available in the next period is determined by the total savings of the current young population. It is assumed that physical capital fully depreciates after one period.

3.6 Dynamic Equilibrium

Given the initial values of capital stock K_0, population N_0^y and human capital h_0, the dynamic competitive equilibrium consists of sequences of prices $\{w_t, r_{t+1}\}_{t=0}^{\infty}$, quantities $\{k_t, h_t, y_t\}_{t=0}^{\infty}$, household decisions $\{c_t^y, c_{t+1}^o, s_t, n_t, e_t\}_{t=0}^{\infty}$, and government policies $\{\alpha_t^p, \alpha_t^c, g_t\}_{t=0}^{\infty}$ that satisfy: (i) Household's optimization, (ii) Firm's optimization, (iii) Market clearing conditions and (iv) Government budget constraint. These equilibrium conditions can be simplified to involve primarily $\frac{h_{t+1}}{h_t}$, n_t, and k_{t+1}:[27]

$$\frac{h_{t+1}}{h_t} = \frac{H_1(H_2 n_t - H_3)^{\delta}(1 - \nu n_t)^{1-\delta} w_t}{n_t} \tag{19}$$

$$n_t = \frac{C_1 w_t h_t}{A_1 + B_1 w_t h_t} \tag{20}$$

$$k_{t+1} = \frac{D_1 w_t}{(1 - \nu n_{t+1})n_t} \times \frac{h_t}{h_{t+1}}, \tag{21}$$

where w_t is given by the equation (13) and H_1, H_2, H_3, B_1, C_1, and D_1 are positive constants determined by the model's parameters.[28] It is noted that A_1 is negative, given the condition $\varepsilon - e_0 < 0$.

[27] Derivations of equilibrium conditions and simplification are provided in Section B.2.

[28]

Equation (19) indicates that the growth rate of human capital depends on fertility, wage, and the parameters governing education investment and government policies. It is noted that $H_2 n_t - H_3 > 0$ must hold for equilibrium.[29] Equation (20) shows that fertility choices depend on current human capital and wage. To ensure a declining fertility rate, we maintain the assumption $\varepsilon - e_0 < 0$, as this affects the sign of $A_1 < 0$ and the dynamics of n_t. Equation (21) demonstrates that capital per effective labor in the next period depends on current wage, the growth rate of human capital, and fertility choices of current and next generation.

3.7 Steady States

Assuming interior solutions, the steady state is the fixed points of the equation (19), (21) and (20) where $n_t = n_{t+1} = \bar{n}$, $k_t = k_{t+1} = \bar{k}$, and the growth rate of human capital is constant, $g_{t+1} \equiv \frac{h_{t+1}}{h_t} = \bar{g}_h$. Since h_t grows steadily, we can determine:

$$
\begin{aligned}
\bar{g}_h &= \frac{H_1(H_2\bar{n} - H_3)^{\delta}(1-\nu\bar{n})^{1-\delta}(1-\alpha)A\bar{k}^{\alpha}}{\bar{n}} \\
\bar{n} &= \frac{C_1}{B_1} \\
\bar{k} &= \frac{D_1}{H_1(H_2\bar{n} - H_3)^{\delta}(1-\nu\bar{n})^{1-\delta}(1-\nu\bar{n})}
\end{aligned}
$$

This implies that the economy achieves a balanced growth path where variables such as fertility rate n_t, output y_t, wage w_t, and interest rate r_{t+1} remain constant, while human capital h_t, education expenditure e_t, and savings s_t grow at steady rates.

To assess local stability, we construct a dynamic system in Julia, incorporating equations (19), (20), and (21). We evaluate the Jacobian matrix at the steady state to check whether eigenvalues are less than one. These eigenvalues provide information regarding the interdependencies and reaction mechanisms within the system, albeit without constraining the analysis to a linear approximation.

$$A_1 = (1+\beta+\gamma\theta)(1+\sigma^p\tau)(\varepsilon - e_0) \qquad H_1 = \frac{a(\sigma^e\tau)^{1-\delta}(1+2\sigma^p\tau)}{1+\sigma^p\tau}\left[\frac{\delta(1-\theta)}{(\theta+\theta\delta-\delta)(1+2\sigma^p\tau)}\right]^{\delta}$$

$$B_1 = (1+\beta+\gamma\theta)\left[(1-\tau)+2(1-\tau)\sigma^p\tau+\sigma^c\tau\right]\nu \qquad H_2 = ((1-\tau)(1+2\sigma^p\tau)+\sigma^c\tau)\nu$$

$$D_1 = \frac{\beta(1-\tau)}{1+\beta+\gamma\theta} \qquad H_3 = \tau((1-\tau)\sigma^p+\sigma^c))$$

$$C_1 = \gamma(\theta+\theta\delta-\delta)(1-\tau)(1+\sigma^p\tau) + (1+\beta+\gamma\theta)\left[(1-\tau)\sigma^p\tau+\sigma^c\tau\right]$$

[29] In the subsequent chapter, we calibrate the parameter values to ensure that $n_t > \frac{H_3}{H_2} = 0.22$, which corresponds to an empirical TFR of 0.462 when accounting for the replacement rate of 2.1.

4 Quantitative Analysis

In this section, we calibrate the model using South Korean data to assess if it aligns with observed economic and demographic trends, particularly the declining fertility rate. We also examine the stability of the steady state and the economy's response to perturbations. Detailed information on parameters is provided in Appendix B.1.

4.1 Parameters

Parameters	Description	Values	Source
Human capital			
a	Learning ability	1.0	Calibration
e_0	Innate skills	1.0	Calibration
δ	Share of private educ	0.423	OECD & KOSIS
$1-\delta$	Share of public educ	0.577	OECD & KOSIS
Preference			
β	Discount factor	0.36	$(0.96)^{25}$
θ	Utility share of fertility	0.38	Calibration
γ	Child preference	0.7	Calibration
ν	Fixed cost of *time* to raise a child	0.17	KIHSA (2021)
ε	Fixed cost of *goods* to raise a child	0.75	Calibration
Production			
α	Capital intensity	0.3	Literature
A	Total Factor Productivity	23.59	Calibration
Government			
τ	Income tax rate	0.216	NABO (2023)
σ_p	Parental leave payments	0.017	NABO (2021)
σ_c	Child allowances	0.111	NABO (2021)
σ_e	Public education	0.873	NABO (2021)

Table 2: Parameters

Human Capital Parameters

In the human capital production function, we normalize the learning ability parameter to $a = 1$ and set the innate skill level to $e_0 = 1$. The shares of education expenditure by parents (δ) and the government ($1-\delta$) are calculated using data from KOSIS (2023) and OECD's "Education at a Glance" reports for between 2007 and 2020. We take the average over these years to obtain $\delta = 0.422$, indicating that approximately 42.2% of child education expenditure is sourced from parents, while the remaining 57.8% comes from government funding.

Preference Parameters

Given that each generation spans 25 years in the model, we compute the discount rate $\beta = 0.96^{25} \approx 0.36$. The utility share of fertility θ and child preference γ are calibrated to 0.38 and 0.7, respectively, to ensure the model's declining TFR pattern for the first two periods and to maintain the feasibility conditions of equilibrium.[30] The fixed time cost ν is derived from Korea Institute for Health and Social Affairs survey data (KIHSA, 2021)[31] on child care time by employed married couples. Assuming 16 waking hours per day (960 minutes), households allocate their time between work and child-rearing activities. According to the data, the combined daily childcare time is approximately 329 minutes. Therefore, we calculate the average time spent on childcare as $\nu = \frac{329}{1920} \approx 0.171$.[32]

Production Parameters

The capital intensity parameter α is set to 0.3, a conventional value in macroeconomic models. The total factor productivity A is calibrated to meet an initial equilibrium condition, as will be explained below.

Government Policy Parameters

We consider four key parameters for government policies: the tax rate τ, parental leave payments σ_p, child allowances σ_c, and public education σ_e. The tax rate τ is set to 0.216, based on the average tax revenue as a percentage of GDP in South Korea from 2000 to 2022 from National Assembly Budget Office (NABO, 2023). The policy parameters σ_p, σ_c, and σ_e are set to 0.017, 0.111, and 0.873, respectively, based on data from NABO (2021). The value of σ_e is sourced from government expenditures on public education data from OECD's "Education at a Glance", consistent with the estimation of δ.

4.2 Simulation

Using the calibrated parameters, we simulate the model over ten periods (equivalent to 250 years) to examine the behavior of key economic variables and assess alignment with observed data. We normalize the initial capital stock K_0 and adult population N_0^y to 1.0. The initial human capital level h_0 is calibrated to match the growth rate of GDP per worker in South Korea between 1995 and 2020, resulting in $h_0 = 2.43$. Given the absence of significant family

[30] One of the feasibility conditions for equilibrium is $\theta + \theta\delta - \delta > 0$.

[31] KOSIS has restructured this data and it can be accessed via KOSIS - Household Survey.

[32] According to BLS (2024), Table 10, adults spent an average of 247.8 minutes per day caring for household children (under age 13) on weekdays, with men spending 198.6 minutes and women spending 292.8 minutes.

policies in 1995, we set the policy parameters for that period as $\sigma_p = \sigma_c = 0$ and $\sigma_e = 1.0$, indicating that all government spending was allocated to public education. From the second period onward, we adjust the policy parameters to reflect the data from 2020, as described in the previous subsection.

We begin by matching South Korea's TFR in 1995, which was 1.63. Adjusting for the replacement rate of 2[33], we obtain $n_0 = \frac{1.63}{2} \approx 0.815$. Using the equations (20) and (18), we solve for the initial wage w_0 and the capital per effective worker k_0 given h_0:

$$w_0 = \frac{n_0 A_1}{C_1 h_0 - n_0 B_1 h_0} \text{ and } k_0 = \frac{K_0}{(1-\nu n_0) h_0 N_0^y}.$$

Finally, we calibrate the total factor productivity parameter A using the equation (13) to ensure consistency between the calculated w_0 and k_0, where $A = \frac{w_0}{(1-\alpha)k_0^{\alpha}} =\approx 23.59$.

Alignment with Data

Fertility rate: In the initial period, as we aimed, the fertility rate n_0 matches the observed TFR of South Korea in 1995. In the second period, the model predicts $n_1 \approx 0.517$, corresponding to a TFR of approximately 1.03 when adjusted (TFR $= n_1 \times 2$). The actual TFR in 2020 was 0.84. Thus, while the model captures the declining trend, it slightly overestimates the fertility rate. We impose the condition $\varepsilon < e_0$ to ensure a declining fertility rate consistent with empirical observations. However, the steady-state values of the TFR and other variables are not specifically targeted in our calibration. Rather, they are endogenously determined by the model's dynamics, given the calibrated parameters and initial conditions set to match empirical data.

Return on Capital: The initial rate of return on capital r_0 is 11.87. Adjusted for the 25-year, this translates to an annual rate of approximately 10.4% ($11.87^{\frac{1}{25}} - 1 \approx 0.104$). The data from Feenstra, Inklaar, and Timmer (2015) reports that South Korea's real internal rate of return was 13.35% in 1995 and 8.38% in 2019, with an average of around 10.92%. Thus, the model's rate of return aligns reasonably well with the data.

GDP Growth: The model predicts a growth rate of GDP per worker of approximately 344% between the initial and second periods, which aligns with the data from Feenstra, Inklaar, and Timmer (2015) that we targeted earlier.[34] Furthermore, data shows that the

[33] Unlike the real-world replacement rate of 2.1, which accounts for child mortality, we use 2.0 since our model does not account for child mortality risk.

[34] From Feenstra, Inklaar, and Timmer (2015), we take rgdpna (output-side real GDP) and emp (number of persons engaged) to measure GDP per worker by $\frac{\text{rgdpna}t}{\text{emp}t}$.

growth rate of real GDP per hours worked between 1995 to 2020 was around 390%, whereas the model yields 317%.[35]

Figure 9 presents the simulation results for key variables over time. We observe that after the first periods, n_t converges to the steady state value with a difference less than 1%. For the other variables, after two periods, each approaches its steady state value, differing by less than 1%. Specifically, variables such as n_t, g_{t+1}, w_t and y_t converge to their steady-state values by period T=5. However, both output per worker $\frac{Y_t}{N_t^y}$[36] and consumption per adult[37] continue to grow indefinitely due to ongoing human capital accumulation.

The simulation results suggest that even with higher income levels, households maintain a low fertility rate. They choose to allocate additional income toward consumption, savings, and education. This behavior underscores the quantity-quality trade-off in child-rearing decisions, where the substitution effect dominates the income effect of higher income. As discussed in Section 3, the increased wage income raises the opportunity cost of time spent on child-rearing, leading parents to opt for fewer children while enhancing the human capital of each child.

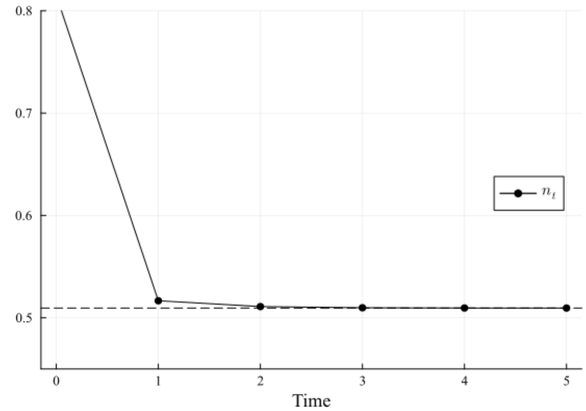
4.3 Dynamic Analysis

We examine the stability of the steady state and the economy's response to perturbations. The non-linear dynamic system, represented by equations (19), (20), and (21) has the steady-state values: $\bar{g}_h = 4.84$, $\bar{n} = 0.51$ and $\bar{k} = 1.34$. The eigenvalues of the Jacobian matrix evaluated at the steady state are 0.004, $0.148+0.522i$, and $0.148-0.522i$. Since all eigenvalues have moduli less than 1, the steady state is locally stable. The complex eigenvalues indicate that the system exhibits a spiral sink toward the steady state, while the small eigenvalue associated with n_t suggests rapid convergence of the fertility rate. Although h_0 and K_0 are initial values, h_1 and k_0 are determined endogenously through the choice variable n_0. Given that all eigenvalues have moduli less than one and there is one choice variable, the steady state is indeterminate. Detailed graphs illustrating the dynamic paths and further interpretations are provided in Section B.3.

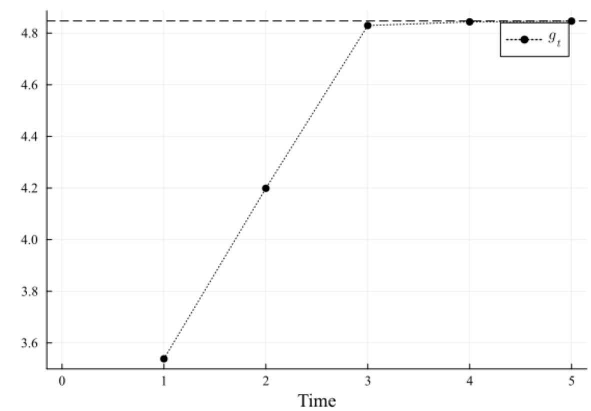
[35] We calculated total hours worked by multiplying emp and avh (Average annual hours worked by persons engaged). In the model, the effective unit of labor measures total hours worked adjusted by human capital. By multiplying human capital into GDP per worker, we were able to find the value.

[36] It is to be noted that $L_t = (1 - \nu n_t)\, h_t N_t^y$ represents the effective labor input, incorporating both human capital and labor hours, and is different from N_t^y.

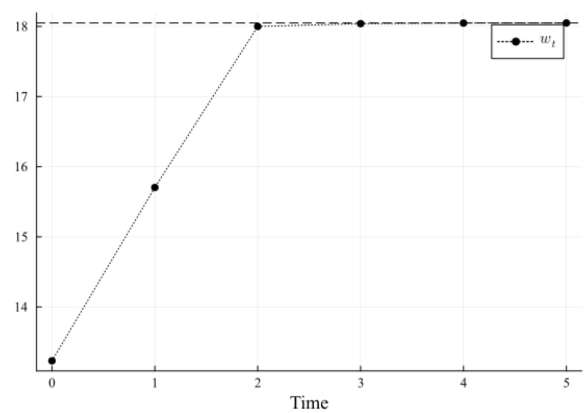
[37] Consumption per adult at time t is calculated as: $\frac{c_t^y N_t^y + c_t^o N_t^o}{N_t^y + N_t^o}$. At $t = 0$, the old population and consumption are set to $N_0^o = \frac{N_0^y}{n_0}$ and $c_0^o = c_0^y$, respectively.



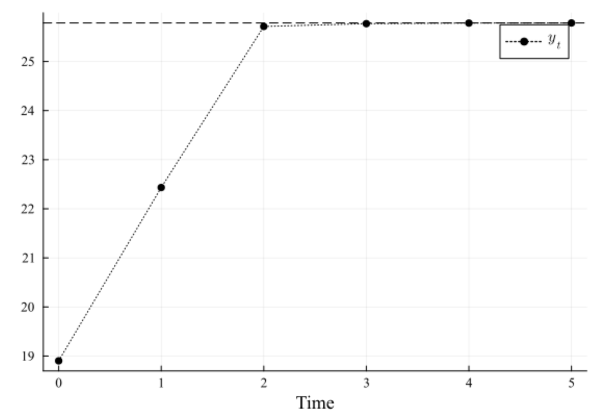
TFR



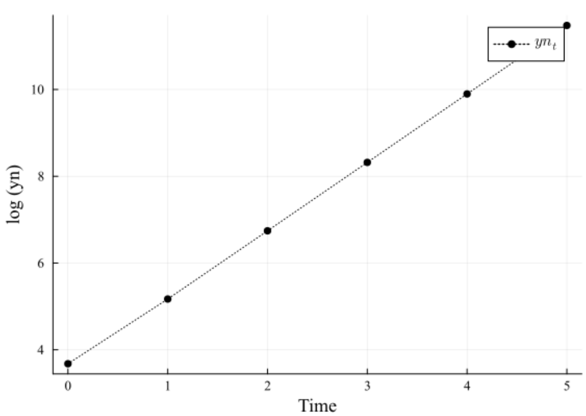
Growth rate of Human Capital



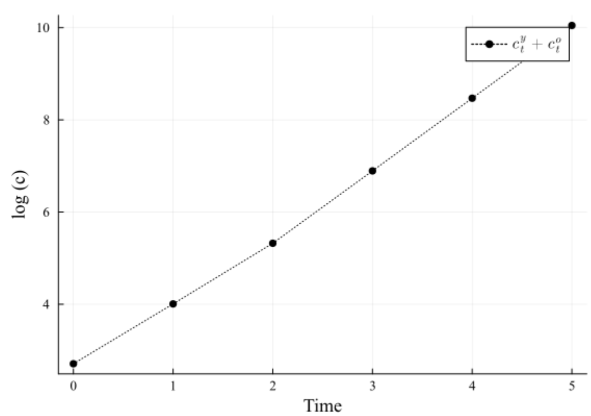
Wage rate



Output per effective worker



Output per worker



Consumption per adult

Figure 9: Simulation

5 Policy Experiments

This section examines the effects of government policy interventions on the fertility rate (n_t) and key economic variables within our model. Our objective is to identify the most effective policy instruments for increasing the fertility rate in South Korea, given its persistent decline. We also assess how these policy changes influence household welfare and macroeconomic indicators such as total population, human capital growth (g_t), wage rates (w_t), output per worker (yn_t), and consumption per worker (c_t).

5.1 Policy Instruments and Methodology

Drawing on economic theory and empirical evidence, we select policy parameters that significantly influence fertility decisions. Theoretical models (Barro and Becker, 1989; Galor and Weil, 1996; Doepke, 2004) suggest that fertility choices depend on the costs and benefits of child-rearing, including financial expenses, time commitments, and investments in child quality. Many developed countries have implemented family-friendly and pronatalist policies to address low fertility rates, and empirical studies demonstrate that such policies effectively influence fertility rates (Gauthier and Hatzius, 1997; D'Addio and Mira d'Ercole, 2005; Milligan, 2005; Thévenon, 2011; Luci-Greulich and Thévenon, 2014).

We focus on two main policy instruments: family support policies, represented by government expenditure shares on parental leave payments (σ^p), child allowances (σ^c), and public education (σ^e); and the income tax rate (τ). Adjusting these parameters allows us to analyze how changes in government expenditure allocation and taxation affect both fertility rate and key economic variables.

Our baseline simulation is consistent with the parameters established in the previous section, where the government maintains the current levels of policy shares: $\sigma^p = 0.017$, $\sigma^c = 0.111$, and $\sigma^e = 0.873$ from the second period onward. This setup captures the effects of ongoing government support for family policies and projects how the TFR and economic growth may evolve based on existing policies. We conduct a series of policy experiments by changing one policy instrument at a time while keeping others constant. For changes in government expenditure shares, we ensure the government's budget constraint is satisfied by adjusting other expenditure shares proportionally.

5.2 Family Support Policies

We explore four scenarios:

1. No Family Support Policies: Government expenditure is exclusively allocated to public education ($\sigma^p = \sigma^c = 0$ and $\sigma^e = 1$) in all periods. This scenario assesses the baseline dynamics without any family-related government intervention.

2. Parental Leave Payments: Increasing σ^p by 10 percentage points (pp) to 0.117, with a proportional decrease in σ^e to maintain the budget balance.

3. Child Allowances: Increasing σ^c by 10pp to 0.211, offset by a proportional reduction in σ^e.

4. Public Education Funding: Increasing σ^e by 10pp to 0.973, with σ^p and σ^c reduced equally to balance the budget.

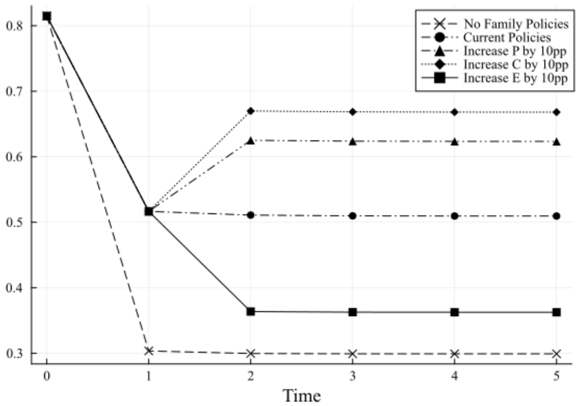
Figure 10 presents the simulation results for these scenarios. First, in terms of the TFR, it shows that increasing child allowances by 10pp is the most effective way to boost fertility. Specifically, when σ^c increases from 0.111 to 0.211 at T=2, n_2 rises from 0.51 (equivalent to 1.02 in actual TFR) to 0.67 (1.34). Enhancing parental leave payments by 10pp increases n_2 0.63 (1.26). Conversely, increasing public education funding by 10pp reduces n_2 to 0.36 (0.72). Without any family support policies, n_t decreases further to 0.303 (0.606) at T=1 and 0.299 (0.598) at T=2.

Second, total population[38] declines throughout the simulation, as none of the policies achieve the replacement fertility rate. The results indicate that policy effects on total population are consistent with those on fertility. For instance, between T=1 and T=2, expanding child allowances reduces total population by 32.1%, whereas increasing parental leave payments or maintaining current policies results in declines of 33% and 35%, respectively. By T=5, relative to the initial population, child allowances preserve 40% of the population, while current policies retain only 19.4% and the absence of family support measures leaves just 8.3%.

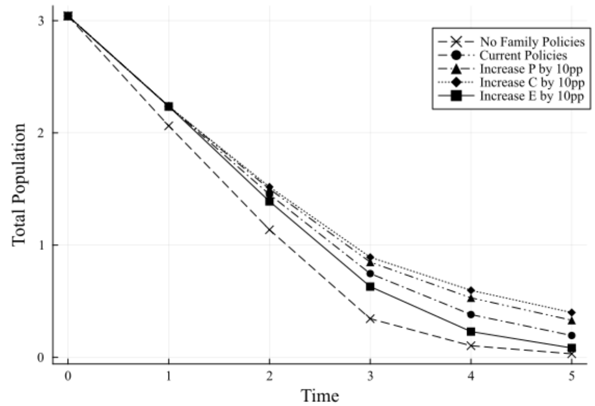
Third, the growth rate of human capital g_t has an inverse relationship with fertility rate n_t. Scenarios that result in higher fertility rate tend to have lower human capital growth rates due to the quantity-quality trade-off in child-rearing decisions. Fourth, the wage rate is positively related with n_t. Policies that leads to higher fertility rate lead to increased wage rates, reflecting the expanded labor force which affects the marginal productivity of labor.

Finally, with respect to output per worker and consumption per adult (young and old), the simulations indicate that scenarios featuring lower family support — and consequently

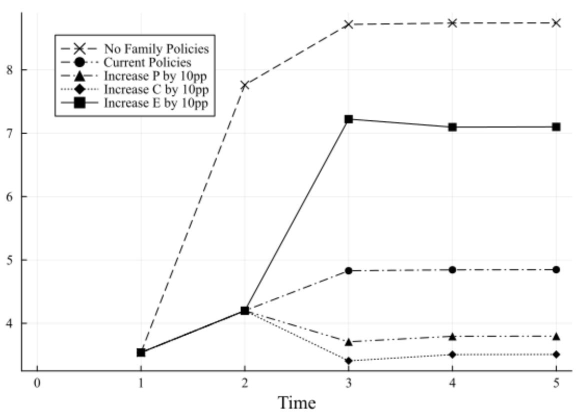
[38]Total population at time t is calculated as the sum of the old, young (worker) and child populations. At $t = 0$, the old population is set to $\frac{N_0^y}{n_0}$.



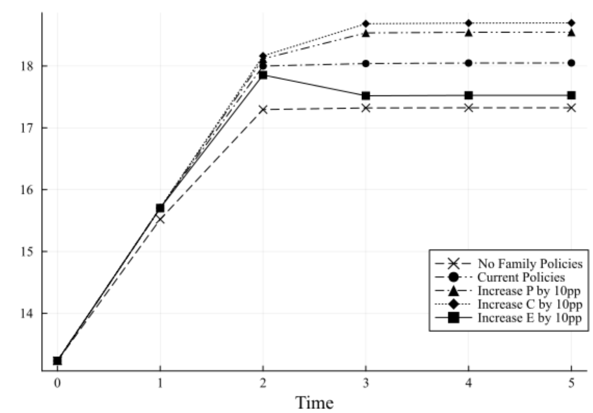
TFR



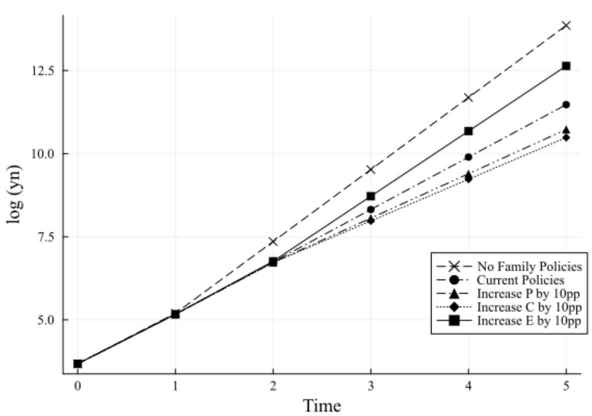
Total Population



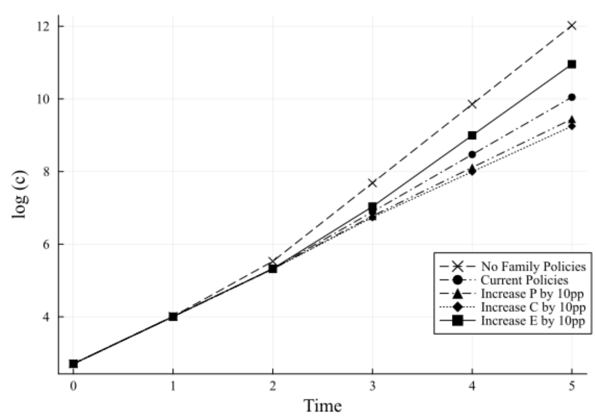
Growth rate of Human Capital



Wage rate



Output per worker



Consumption per adult

Figure 10: Policy Experiment 1 - Family Supports

lower TFRs — produce higher levels of both variables. This outcome arises from increased human capital being distributed among a smaller labor force, thereby enhancing per-person productivity and consumption.

These results imply that increasing child allowances is the most effective family support policy for raising the fertility rate. However, this comes at the cost of reduced human capital growth and lower output and consumption per worker, emphasizing the quantity-quality trade-off in child-rearing decisions. To achieve the replacement fertility rate (TFR of 2.0), it would require substantial policy adjustments that may not be economically feasible. Specifically, the government would need to increase child allowances by 33pp ($\sigma^c = 0.441$) or parental leave payments by 51pp ($\sigma^p = 0.527$).

5.3 Tax Adjustments

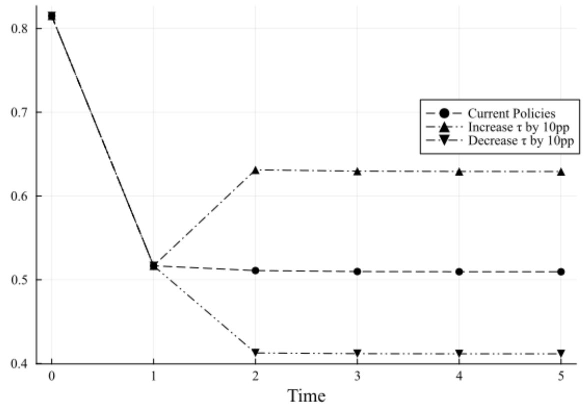
We examine the impact of changing the income tax rate τ by increasing and decreasing by 10pp from the baseline value, adjusting it to 0.316 and 0.116 respectively, from period T=2 onward. Figure 11 depicts the simulation results for this policy experiment. In terms of TFR, it shows that increasing the tax rate at T=2 raises the n_2 from 0.51 (1.02) to 0.63 (1.26), while decreasing the tax rate lowers n_2 to 0.41 (0.82). This indicates that a higher tax rate reduces disposable income, prompting households to substitute labor market participation with child-rearing activities, thereby increasing the fertility rate.

Moreover, higher tax rates lead to lower output per worker and consumption per adult due to reduced labor supply and potential distortions in economic incentives. While adjusting the tax rate influences fertility decisions, achieving the replacement rate solely through tax policy would require a high tax rate of approximately 0.551 (55.1%), which seems impractical.

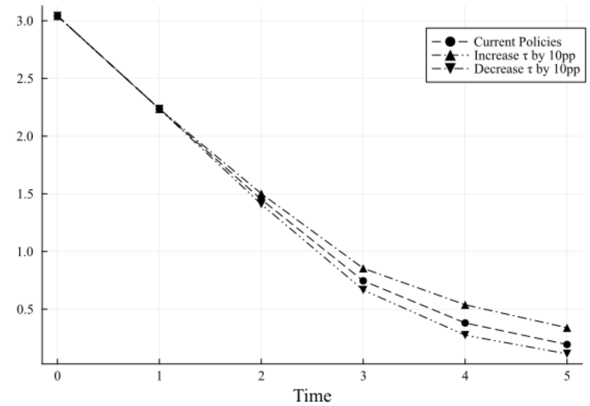
Summary of Policy Experiment Results

Our policy experiments reveal that increasing child allowances is the most effective family support policy for raising the fertility rate. However, achieving the replacement rate solely through this measure would require substantial adjustments — specifically, increasing child allowances by 33 percentage points, which may not be economically or politically feasible. Adjusting the tax rate also significantly impacts fertility decisions, but reaching the replacement rate would necessitate an impractically high tax rate of approximately 55.1%. There are also trade-offs between higher fertility rates and per capita economic indicators such as consumption and output, emphasizing the need to balance policy objectives.

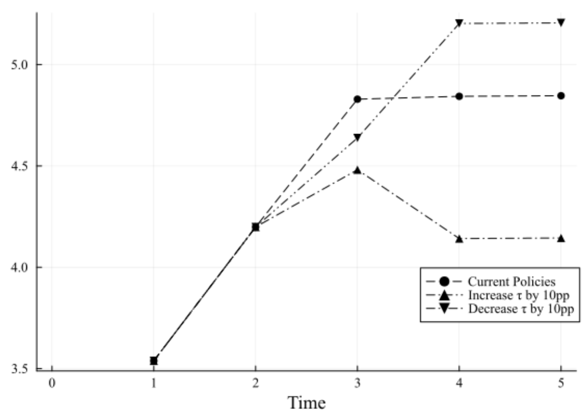
These results align with the literature indicating that family policies can improve fertility



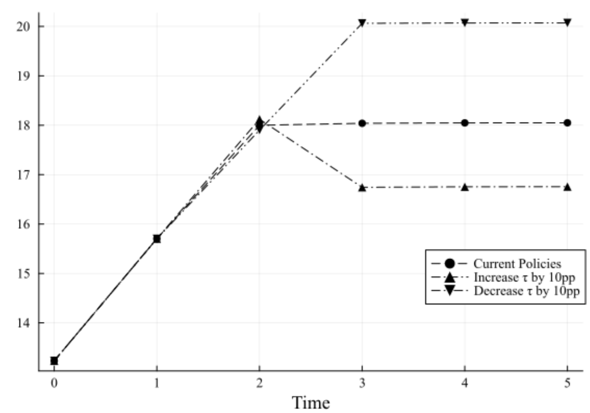
TFR



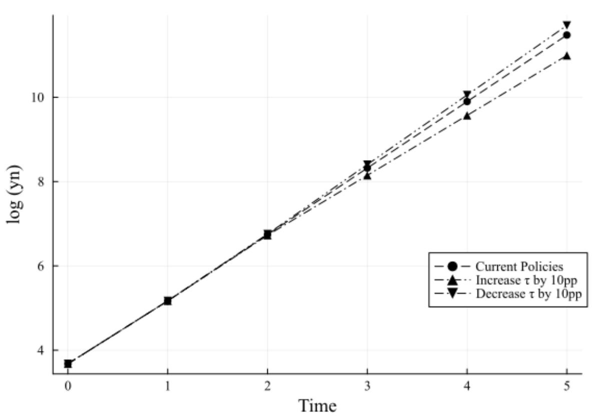
Total Population



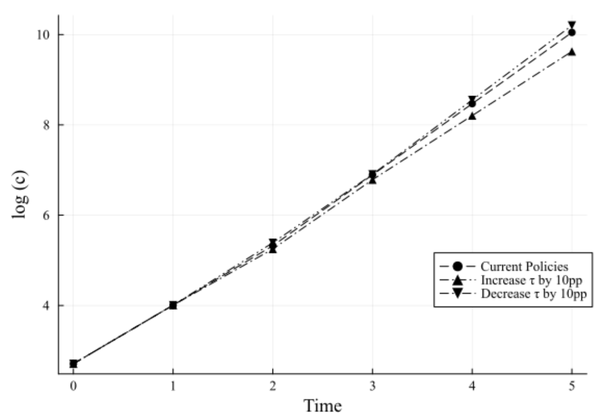
Growth rate of Human Capital



Wage rate



Output per worker



Consumption per adult

Figure 11: Policy Experiment 2 - Tax Rate

rates when they effectively reduce the opportunity costs of child-rearing and support women's participation in the labor force. Additionally, studies have shown that cash transfers and parental leave policies can have positive effects on fertility, though their effectiveness varies by country and context.

6 Discussion

Despite extensive efforts by the South Korean government, including significant financial investments, the fertility rate has continued to decline, reaching a historic low of 0.72 in 2023. Similar trends are observed in other developed countries that have implemented family support policies. Although countries such as Australia and France have seen stabilization in fertility rates, they remain below the replacement level. Hungary has recently exhibited an increasing trend in TFR, though still below the replacement rate; however, considering factors such as significant financial investment (amounting to 4.6% of their GDP on average), rising housing prices and emigration of young people, it remains to be seen whether this trend is sustainable.

Since the enactment of the Low Fertility and Aging Society Framework Act in 2005, the South Korean government has implemented measures to counteract declining fertility rates. Their financial investment in low fertility initiatives surged from approximately $769 million in 2006 to $33 billion in 2021, totalling $215 billion over 16 years.[39] Despite these efforts, the TFR has declined annually, attracting criticism regarding the effectiveness of the expenditure.

Our findings suggest that while family support policies can positively influence the fertility rate, achieving the replacement rate requires policy adjustments that may not be economically or politically feasible. For instance, increasing child allowances by 33pp or parental leave payments by 51pp represents a substantial fiscal commitment. Alternatively, a mixed policy approach may offer a more practical solution. For example, increasing the tax rate by 8.4pp (0.4) and enhancing child allowances by 10pp could collectively achieve the replacement fertility rate by 2045. Given that South Korea's tax rate increased from 22% in 2003 to approximately 32% in recent years, a further increase to 40% over the next two decades may be plausible.

[39]Budget figures are from NABO (2021); currency conversion based on an exchange rate of $1 = 1,300 KRW.

The Role of Private Education

One significant factor of South Korea's low fertility rate is the high cost of private education. Many studies have highlighted that the competitive education environment and societal pressures in South Korea lead parents to invest heavily in their children's education, discouraging them from having more children. According to Kim, Tertilt, and Yum (2024), status externalities in education[40] contribute significantly to low birth rates in Korea. The competitive education environment and societal pressures lead parents to invest heavily in their children's education, discouraging them from having more children. In our simulation, the private education expenditure is embedded in $\delta = 0.422$, which is not a a policy parameter. However, when we adjust δ, our model simulations show that it significantly affects the fertility rate. Figure 12 presents the simulation results for different values of δ.

In terms of TFR, it shows that decreasing δ by 10pp (0.322) at T=2 increases the n_2 from 0.51 (1.02) to 0.66 (1.32). Conversely, increasing δ by 10pp (0.522) reduces n_2 to 0.36 (0.72). A higher δ increases the cost burden of education on parents, intensifying the quantity-quality trade-off and leading to lower fertility rates. The simulation results suggest that managing δ through policies that reduce private education costs could significantly affect fertility rates. In fact, reducing δ by approximately 33pp ($\delta \simeq 0.1$) could enable achieving the replacement rate with current policy parameters unchanged.

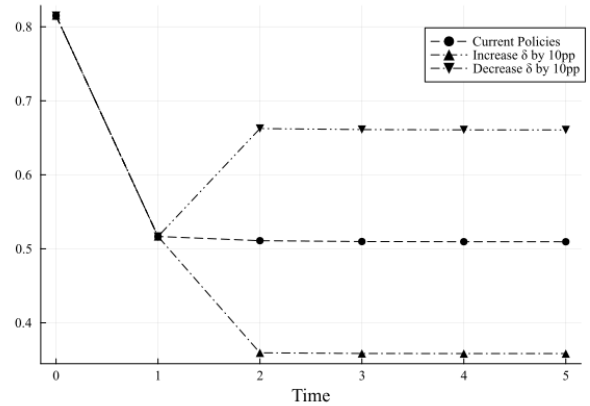
However, the challenge remains significant. Private education expenditure in Korea increased by 45.8% between 2013 and 2023, from \$14.3 to \$20.9 billion, and is projected to rise further.[41] In addition, under-reporting of private education expenditures, especially for early childhood and supplementary education, may lead to an underestimation of δ. According to survey data from KIHSA (2021), households with children under age 7 spend an average of \$68.5 per month on education, whereas expenditures increase to \$328.5 for elementary school students and \$329.23 for middle and high school students. A considerable proportion of students (31.7% in 2024) also retake the College Scholastic Ability Test (CSAT), further increasing private education costs.[42]

Although the Korean government has announced plans to develop a comprehensive dataset on private education expenditures for these particular groups, policies focusing on improving the quality of public education and mitigating the societal phenomenon of "education fever" will alleviate financial pressures on parents. Such measures could be more effective in increasing the TFR than solely expanding family support or tax rates.

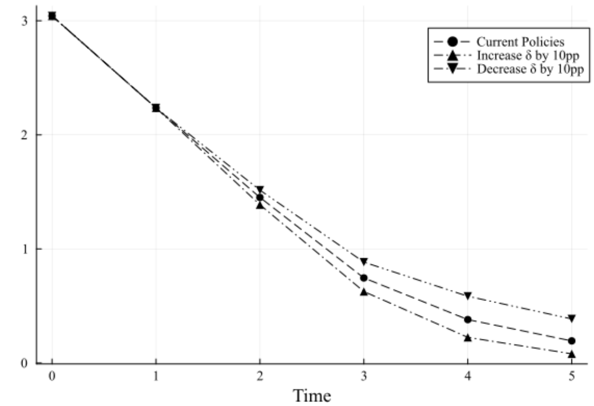
[40] The value parents place on their children's educational achievements relative to others.

[41] Data from KOSIS (2023); currency conversion based on an exchange rate of \$1 = 1,300 KRW.

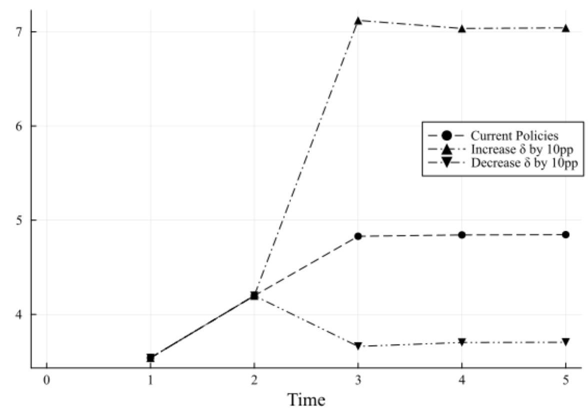
[42] Source: Korea Institute for Curriculum and Evaluation (KICE)



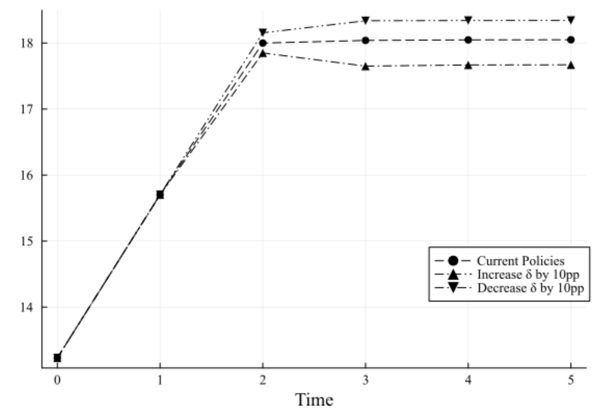
TFR



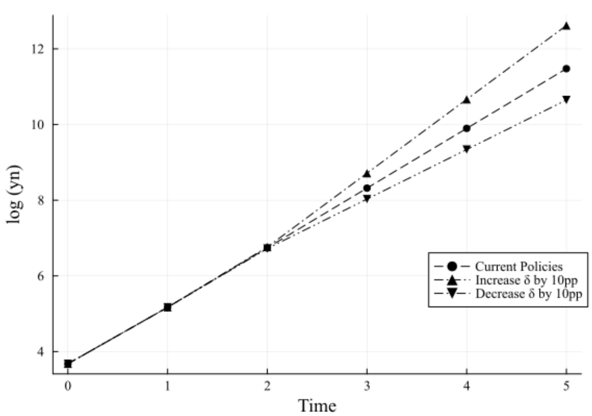
Total Population



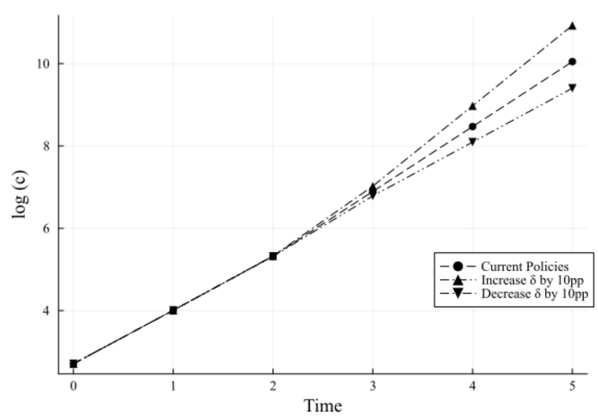
Growth rate of Human Capital



Wage rate



Output per worker



Consumption per adult

Figure 12: Policy Experiment 3 - Private Education Expenditure Share

Limitations

It is essential to acknowledge the limitations of our analysis. First, we assume that government revenues are allocated exclusively for family support policies; however, in reality, government budgets are distributed across various sectors, including healthcare, infrastructure, defense, and other public services. Particularly with respect to parental leave benefits (σ^p), we assume unrestricted access for all households, yet data from KOSIS (2022a) indicate that only 65.5% of eligible females and 4.1% of eligible males actually utilized parental leave in 2021. Consequently, the model may overstate the effect of parental leave benefits, given the relatively low actual uptake.

Second, the model assumes homogeneous households and does not consider the dynamics of marriage and partnership formation. Introducing heterogeneity and incorporating household formation dynamics could offer a more detailed understanding of fertility behavior. Third, our model does not account for societal factors that affect fertility decisions, such as housing prices, labor market conditions, gender equality, and cultural expectations. These factors can significantly influence individuals' decisions to marry and have children.

7 Conclusion

This study addresses the persistent decline in South Korea's total fertility rate (TFR), which poses significant demographic and economic challenges. Despite substantial government interventions, their TFR has continued to decrease, reaching a historic low of 0.72 in 2023.

We developed a three-period overlapping generations model incorporating endogenous fertility choices, human capital accumulation and four types of government policies: income tax, parental leave payments, child allowances and public education funding. Calibrated to South Korean data, the dynamic analysis revealed that the economy converges to a locally stable steady state through a spiral sink path, where the interplay between capital accumulation and human capital growth causes oscillations.

Through simulations of various policy experiments, we aimed to identify the most effective instruments for increasing the fertility rate and to assess their broader economic implications. The results indicate that increasing child allowances is the most effective family support policy for boosting the fertility rate. Specifically, enhancing child allowances by 10 percentage points (pp) leads to a 31% rise in the TFR, while increasing parental leave payments by 10pp results in a 22% increase. However, achieving the replacement rate solely through these measures would require an increase in child allowances by 33pp and parental leave payments by 51pp. Increasing the income tax rate by 10pp would also raise the TFR by 24%, but reaching the replacement level would necessitate an impractically high tax rate of approximately 55.1%.

Another insight from our analysis is the significant role of private education in fertility decisions. The share of education expenditure by parents (δ) substantially affects the fertility rate. An increase in δ enhances the quantity-quality trade-off faced by households, leading to lower fertility rates, while reducing δ could have a pronounced positive effect on fertility. Given the rising and potentially under-reported private education expenditures in South Korea, addressing the high cost of private education emerges as a crucial factor.

These findings highlight the importance of comprehensive strategies that combine financial support with measures addressing underlying societal factors, particularly the high expenditure on private education by parents. Policies aimed at improving the quality of public education and mitigating the "education fever" could alleviate the financial burden on parents and encourage higher fertility.

In conclusion, reversing South Korea's declining fertility trend is a complex challenge requiring multifaceted policy interventions. While financial incentives can contribute to increasing fertility rates, it is essential to address underlying factors such as the intense academic competitiveness among students and high expenditures on private education.

Chapter 3 - The Effect of Government Policies on Fertility under Pensions

1 Introduction

The sustainability of public pension systems in developed countries has become a matter of concern due to demographic shifts characterized by declining fertility rates and increasing life expectancy. These trends have led to a rising dependency ratio, where a shrinking working-age population must support an expanding elderly population.

The sustainability of public pension systems in developed countries has become a matter of concern due to demographic shifts characterized by declining fertility rates and increasing life expectancy. These trends have led to a rising dependency ratio, in which a shrinking working-age population must support an expanding elderly population. As fertility rates remain persistently low, fewer future workers will be available to contribute to pension schemes, putting long-term pressure on the fiscal balance. This dynamic not only threatens the solvency of pension programs but also raises fundamental questions about intergenerational equity and social stability.

Advancements in healthcare and medical technology have extended life expectancies, intensifying the strain on pension systems. Without policy interventions, the proportion of non-working elderly individuals will continue to grow, exacerbating the dependency ratio. Proposed solutions include extending working years by raising the retirement age, reducing individual pension benefits, increasing the eligibility age for pension entitlements, and raising taxes to finance the expanding beneficiary pool. However, these measures often face political resistance and implementation challenges (Pierson, 1996; Mulligan and Sala-i Martin, 1999).

Building upon the model developed in Chapter 2, we incorporate pension benefits into the government's policy framework and assess their influence on household fertility decisions. In addition, we investigate how each policy affects pension benefits alongside other key economic variables.

A declining TFR reduces the future workforce, raising concerns about the long-run sustainability of pay-as-you-go (PAYG) pension systems that depend on current workers to fund current retirees. Nevertheless, in our framework, pension benefits are determined as a fixed share of total tax revenues, and we do not explicitly model rising life expectancy. Under these assumptions, while fewer workers decrease overall tax revenues and thus the total pension budget, higher human capital accumulation (and consequently higher wages) can partially offset the negative impact, potentially stabilizing or even increasing per-recipient pension benefits.

The relationship between pension systems and fertility rates has been extensively studied in the economic literature. Wigger (1999) shows that introducing PAYG pension schemes can lead to reduced fertility rates, which impedes economic growth. They suggest that partial privatization of the pension system may encourage higher fertility and promote economic expansion. Similarly, Fougère and Mérette (1999) use overlapping generations (OLG) and endogenous growth models to explore how population ageing affects economic growth in OECD countries, finding that ageing populations may incentivize future generations to invest more in human capital, potentially enhancing economic performance.

Further research utilizing OLG models has explored various aspects of this demographic challenge. In Tabata (2005), pension reforms mitigate the adverse growth effects associated with population ageing in economies with high old-age dependency ratios and substantial defined-benefit pension schemes. Heer, Polito, and Wickens (2020) identify a threshold dependency ratio beyond which pension reforms become necessary, focusing on the United States and EU-15 countries. Hock and Weil (2012) argue that increasing fertility rates is a consumption-maximizing response to greater longevity but large transfer payments can suppress fertility, exacerbating population ageing. Van Groezen, Leers, and Meijdam (2003) show that Pareto improvements require introducing child allowance schemes alongside reductions in PAYG taxes. Building on this, van Groezen and Meijdam (2008) find that optimal policy involves implementing child allowances that increase with longevity and taxing savings if child-rearing costs depend on wages. Cipriani (2014) demonstrates that ageing populations, due to falling fertility and rising longevity, can adversely affect pension systems in the long run, especially when fertility is endogenously determined.

Researchers have also investigated the implications of retirement age policies. Lacomba and Lagos (2006) suggest that under a defined-contribution pension scheme, population ageing may lead to a preference for a lower legal retirement age, while increased life expectancy could delay the desired retirement age. Heijdra and Romp (2009) show that ageing populations tend to raise the desired retirement age and stimulate human capital investments, which can partially offset the negative impact of ageing on economic growth. Their findings indicate that well-designed pension policies can help mitigate the detrimental effects of demographic shifts on economic performance.

Despite considerable attention given to the interaction between pension systems and fertility rates, comprehensive analysis of the effectiveness of various fertility-enhancing policies are relatively scarce. This paper aims to fill this gap by comparing the effect of different fertility policies. The model includes pension benefits into a dynamic model of household fertility decisions, allowing for a thorough examination of how government policies affect both fertility rates and other economic variables of interests. By analyzing these measures within the constraints of government budgets, we seek to identify "optimal" strategies for promoting higher fertility rates.

The remainder of the paper is organized as follows: Section Section 2 briefly discusses the challenges of an ageing population in South Korea. Section Section 3 outlines the extended model that incorporates pension benefits into household decision-making. Section Section 4 presents the calibration and dynamic analysis of the model. Section Section 5 describes the policy experiments and provides an analysis of various policy interventions. Section Section 6 explores future research directions aimed at addressing the ageing population issue. Finally, Section Section 7 concludes the paper.

2 The case of South Korea

South Korea presents a compelling case study due to its rapidly aging population and persistently low fertility rates. The country's Total Fertility Rate (TFR) has been on a continuous decline, reaching the lowest level globally in 2023. This demographic trend has resulted in a U-shaped trajectory of the age dependency ratio over past decades. In the 1960s, South Korea had a high dependency ratio due to a large youth population relative to the working-age cohort, stemming from high birth rates. As the nation underwent rapid economic development, fertility rates plummeted, leading to a reduced number of dependents per working-age individual and a declining dependency ratio.

In recent years, however, the dependency ratio has begun to rise again, primarily due to an increasing proportion of elderly citizens. The combination of low fertility and extended life expectancy has amplified the old-age dependency ratio, intensifying the pressure on the working-age population to support a growing number of retirees. Figures 13 and 14 illustrate these demographic shifts, highlighting the challenges faced by South Korea.

To address these challenges, the South Korean government has implemented various policies, including financial incentives for families with children, enhanced support for working mothers and improved childcare services. Despite these efforts, shifts in cultural and societal attitudes toward family and child-rearing have been gradual, suggesting that more comprehensive and multifaceted approaches may be necessary to effectively tackle the underlying causes of the low fertility rate.

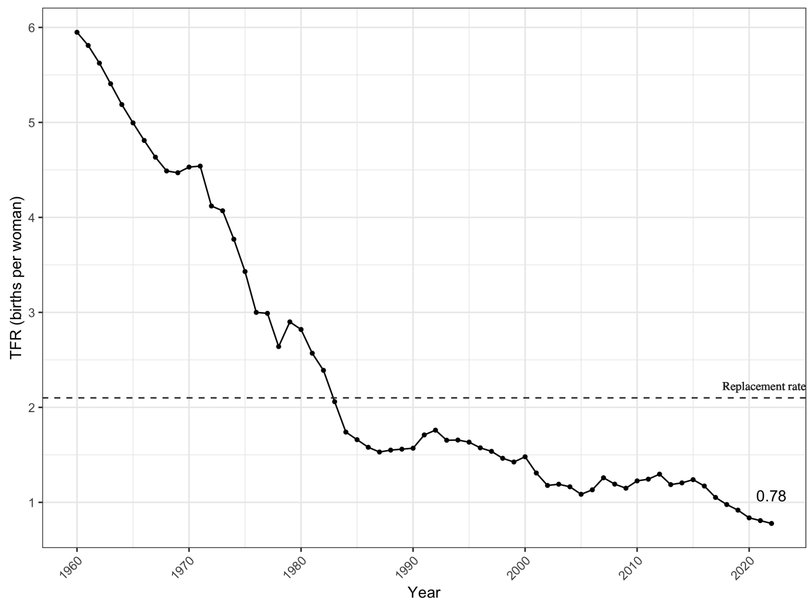


Figure 13: Total Fertility Rate of South Korea

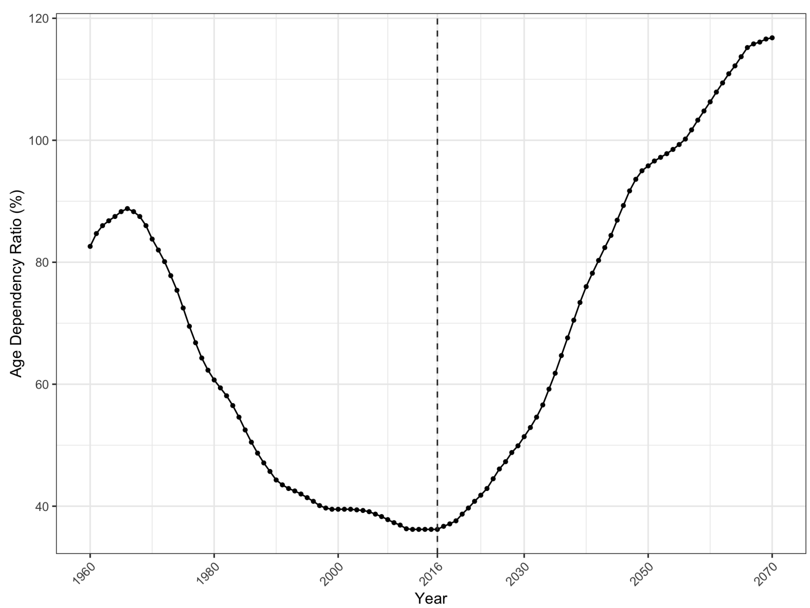


Figure 14: Age Dependency Ratio of South Korea

Note: The "Age Dependency Ratio" figure was created using data from 1960–2019. For 2020–2070 projections, KOSIS employed the Cohort Component Method, which separately projects fertility, mortality and net migration for each birth cohort. The base population advances yearly using projected survival rates and net international migration, with new birth cohorts added annually by applying projected fertility rates to the female population.

3 The Model

In this section, we extend the model from Chapter 2 by incorporating pension benefits into the government's policy framework. Households now receive pension benefits η_{t+1} from the government during their old age. While the fundamental structure remains similar to the baseline model, the inclusion of pension benefits introduces some changes to household decisions and government policies.

Households and Preferences

Households make decisions to maximize their expected lifetime utility, considering consumption during their working years, the utility derived from their children, and consumption during retirement. The utility function is specified as:

$$U_t = \log c_t^y + \gamma \log(n_t^\theta h_{t+1}^{1-\theta}) + \beta \log c_{t+1}^o$$

by choosing $\{c_t^y, c_{t+1}^o, s_t, n_t, e_t\}$ subject to constraints

$$\begin{aligned} c_t^y + s_t + (e_t + \varepsilon - \alpha_t^c)n_t &= (1-\tau)(1-(\nu - \alpha_t^p)n_t)w_t h_t \\ c_{t+1}^o &= (1+r_{t+1})s_t + \eta_{t+1} \\ h_{t+1} &= a\,(e_0 + e_t)^\delta\, g_t^{1-\delta} \end{aligned} \tag{22}$$

and taking input prices (w_t, r_{t+1}) and the government policies $(g_t, \alpha_t^c, \alpha_t^p, \eta_{t+1})$ as given. In this model, the household's consumption during seniorhood c_{t+1}^o depends on both the savings accumulated during the working years and the pension benefits η_{t+1} provided by the government.

Interior Solution

Assuming that wage income $w_t h_t$ is sufficiently high (exceeding a threshold $\hat{E}_t$) to ensure an interior solution, households optimize by choosing e_t, n_t and s_t. The first-order conditions yield:

$$e_t = \frac{\delta(1-\theta)(1-\tau)(\nu-\alpha_p)w_t h_t}{\theta+\theta\delta-\delta} + \frac{\delta(1-\theta)(\varepsilon-\alpha_t^c)-\theta e_0}{\theta+\theta\delta-\delta}$$

$$n_t = \frac{\gamma(\theta+\theta\delta-\delta)\left((1-\tau)w_t h_t + \dfrac{\eta_{t+1}}{1+r_{t+1}}\right)}{(1+\beta+\gamma\theta)\left((1-\tau)(\nu-\alpha_t^p)w_t h_t + \varepsilon - \alpha_t^c - e_0\right)}$$

$$s_t = \frac{\beta(1-\tau)w_t h_t}{1+\beta+\gamma\theta} - \left(\frac{1+\gamma\theta}{1+\beta+\gamma\theta}\right)\frac{\eta_{t+1}}{1+r_{t+1}}$$

Note that the expression for e_t remains the same as in the baseline model, while n_t and s_t are now affected by the pension benefits η_{t+1}. Anticipated old-age pension benefits affect current decisions regarding fertility and savings. Specifically, higher expected pension benefits reduce the need for savings and, as a result, increase fertility in partial equilibrium.

Government

The government collects revenue exclusively through a proportional income tax τ on wage income. It maintains a balanced budget each period, allocating its revenue to four expenditure categories: child allowances α_t^c, parental leave payments α_t^p, public education g_t and pension benefits η_t. The government's budget constraint in period t is given by:

$$\tau(1-(\nu-\alpha_t^p)n_t)w_t h_t N_t^y = \alpha_t^c n_t N_t^y + \alpha_t^p n_t w_t h_t N_t^y + g_t N_{t+1}^y + \eta_t N_{t-1}^y,$$

where N_{t-1}^y is the number of old in period t. Each government spending component is a fixed proportion σ^i,of total tax revenue, where $i \in \{c, p, e, \eta\}$. The proportions satisfy $\sigma^c + \sigma^p + \sigma^e + \sigma^\eta = 1$ and $\sigma^i \in [0,1]$. Under these assumptions, the policy variables are determined as:

$$\begin{aligned}
\alpha_t^p &= \sigma^p\tau(1-(\nu-\alpha_t^p)n_t)n_t^{-1} = \frac{\sigma^p\tau(1-\nu n_t)}{(1+\sigma^p\tau)n_t} \\
\alpha_t^c &= \sigma^c\tau(1-(\nu-\alpha_t^p)n_t)w_t h_t n_t^{-1} = \frac{\sigma^c\tau(1-\nu n_t)w_t h_t}{(1+\sigma^p\tau)n_t} \\
g_t &= \sigma^e\tau(1-(\nu-\alpha_t^p)n_t)w_t h_t n_t^{-1} = \frac{\sigma^e\tau(1-\nu n_t)w_t h_t}{(1+\sigma^p\tau)n_t} \\
\eta_t &= \sigma^\eta\tau(1-(\nu-\alpha_t^p)n_t)w_t h_t n_{t-1} = \frac{\sigma^\eta\tau(1-\nu n_t)w_t h_t n_{t-1}}{1+\sigma^p\tau}.
\end{aligned} \tag{23}$$

The pension benefit η_t depends on the previous generation's fertility n_{t-1}, reflecting a Pay-As-You-Go (PAYG) pension system where current workers fund the pensions of current retirees.

3.1 Dynamic Equilibrium

Given the initial values for physical capital K_0, population N_0^y, human capital h_0 and pension benefits $\eta_0 = 0$, the dynamic competitive equilibrium consists of sequences of prices $\{w_t, r_{t+1}\}_{t=0}^\infty$, quantities $\{k_t, h_t, y_t\}_{t=0}^\infty$, household decisions $\{c_t^y, c_{t+1}^o, s_t, n_t, e_t\}_{t=0}^\infty$, and government policies $\{\alpha_t^p, \alpha_t^c, g_t, \eta_{t+1}\}_{t=0}^\infty$ that satisfy: (i) Household's optimization, (ii) Firm's optimization, (iii) Market clearing conditions and (iv) Government budget constraint. These equilibrium conditions can be simplified to involve primarily $\frac{h_{t+1}}{h_t}$, n_t, and k_{t+1}:

$$
\begin{aligned}
\frac{h_{t+1}}{h_t} &= \frac{H_1(H_2 n_t - H_3)^{\delta}(1-\nu n_t)^{1-\delta} w_t}{n_t} \\
&\equiv F\left(n_t, k_t\right) && (24) \\
n_t &= \frac{C_1 w_t h_t + \gamma(\theta + \theta\delta - \delta)(1+\sigma^p\tau)\dfrac{\eta_{t+1}}{1+r_{t+1}}}{A_1 + B_1 w_t h_t} \\
&\equiv G\left(\frac{h_{t+1}}{h_t}, n_t, n_{t+1}, k_t, k_{t+1}\right) && (25) \\
k_{t+1} &= \frac{D_1 w_t h_t - (1+\gamma\theta)(1+\sigma^p\tau)\dfrac{\eta_{t+1}}{1+r_{t+1}}}{(1-\nu n_{t+1}) n_t h_{t+1}} \\
&\equiv H\left(\frac{h_{t+1}}{h_t}, k_t, k_{t+1}, n_t, n_{t+1}\right), && (26)
\end{aligned}
$$

where w_t is a function k_t and H_1, H_2, H_3, B_1, C_1, and D_1 are positive constants same as outlined in Chapter 2. Importantly, the inclusion of pension benefits η_{t+1} introduces additional terms in the equations for n_t and k_{t+1}. The pension benefits, as outline in (23), depend on the fertility choices and human capital of both the current and next generations, creating an inter-generational linkage. With higher expected pension benefits, households are incentivized to have more children (increasing n_t) and reduce their savings s_t, leading to lower capital accumulation k_{t+1}. This dynamic reflects the PAYG pension system's impact on individual decisions and overall economic outcomes.

3.2 Steady States

Assuming interior solutions, the steady state is the fixed points of the equation (24), (25) and (26) where $n_t = n_{t+1} = \bar{n}$, $k_t = k_{t+1} = \bar{k}$, and a constant growth rate of human capital $g_{t+1} \equiv \frac{h_{t+1}}{h_t} = \bar{g}_h$. The steady-state conditions are:

$$
\begin{aligned}
\bar{g}_h &= F\left(\bar{n}, \bar{k}\right) && (27) \\
\bar{n} &= G\left(\bar{g}_h, \bar{n}, \bar{k}\right) && (28) \\
\bar{k} &= H\left(\bar{g}_h, \bar{n}, \bar{k}\right). && (29)
\end{aligned}
$$

In the steady state, the economy achieves a balanced growth path where output per effective labor y_t, wage w_t and the return on capital r_{t+1} remain constant, while human capital h_t, (private) education expenditure e_t and savings s_t grow constantly.

To assess local stability, we construct a dynamic system which incorporates equations (24), (25), and (26). We evaluate the Jacobian matrix at the steady state to check whether eigenvalues are less than one. These eigenvalues provide information regarding the interdependencies and reaction mechanisms within the system, albeit without constraining the analysis to a linear approximation.

4 Quantitative Analysis

In this section, we calibrate the extended model using South Korean data, focusing on the year 2020 as the initial period. This choice reflects the most recent data available and allows us to capture the current state of the economy and demographic trends. Our primary objectives are to evaluate the stability of the steady state and to assess the economy's response to policy changes, with a particular focus on fertility rates and key economic variables of interest.

4.1 Parameters

While most of the model parameters remain consistent with those specified in Table 2 from Chapter 2, several adjustments are necessary due to the inclusion of pension benefits and use of 2020 data. Specifically, the government policy parameters — σ^p, σ^c, σ^e and σ^η — are re-calibrated to reflect the updated allocation of government revenue among policy areas, including pension expenditures.

The pension benefits data are sourced from the NPS (2022). Pension benefits per recipient are calculated by dividing the total pension disbursements by the number of recipients. Table 3 presents the adjusted proportions of government revenue allocated to each policy, incorporating pension benefits. The parameter values selected for the year 2020 are highlighted in bold.

Additionally, we update the tax rate (τ) and the share of private education expenditure (δ) based on 2020 data. The tax rate is set at $\tau = 0.277$ (compared to $\tau = 0.216$ in Chapter 2). The parameter δ is adjusted to $\delta = 0.338$ (down from $\delta = 0.425$ in Chapter 2), representing the proportion of private spending in total education expenditure. These adjustments ensure that our model accurately reflects the economic environment of South Korea in 2020.

Parameters	Description	2018	2019	**2020**
σ^p	Parental leave payments	0.013	0.014	**0.014**
σ^c	Child allowances	0.087	0.092	**0.092**
σ^e	Public education funding	0.71	0.701	**0.681**
σ^η	National Pension	0.189	0.194	**0.213**

Table 3: Policy Parameter - with Pension

Note: In Chapter 2, the parameters were $\sigma^p = 0.017$, $\sigma^c = 0.111$ and $\sigma^e = 0.873$, without accounting for pension benefits.

4.2 Dynamic Analysis

Using the calibrated parameters outlined above, we proceed to analyze the dynamic properties of the model. The key equations governing the dynamics are equations (24), (25), and (26), which represent the evolution of human capital growth, fertility choices, and capital accumulation, respectively. The steady-state values are $\bar{g}_h = 6.09$, $\bar{n} = 0.385$, and $\bar{k} = 1.21$. $\bar{n} = 0.385$ corresponds to a Total Fertility Rate (TFR) of 0.77 when adjusted (since $\text{TFR} = 2 \times n_t$), which is below the current South Korea's TFR.

The eigenvalues of the Jacobian matrix evaluated at the steady state are -0.014, $0.14 + 0.536i$, and $0.14 - 0.536i$. Since all eigenvalues have moduli less than 1, the steady state is locally stable. The complex eigenvalues indicate that the system exhibits a spiral sink toward the steady state. Moreover, given that all eigenvalues have moduli less than one and there is one choice variable, the steady state is indeterminate. Detailed graphs illustrating the dynamic paths and further interpretations are provided in Section C.2.

5 Policy Experiments

In this section, we investigate various policy experiments to examine their effects on key variables of interest: the fertility rate n_t, total population, the growth rate of human capital $\frac{h_{t+1}}{h_t}$, wage rate w_t, output per worker yn_t and pension benefits per recipient (old) η_t. Our objective is to identify the most effective policy instruments for increasing the fertility rate in South Korea and to assess how these policy changes influence household welfare and macroeconomic indicators. As in the previous chapter, the government's income is sourced exclusively from wage taxation, but its expenditures are allocated across four key areas: child allowances (α_t^c), parental leave payments (α_t^p), public education (g_t), and pension benefits (η_t). We set the initial fertility rate n_0 equal to the observed data for South Korea in 2020 and then examine the impacts of government policies on each variable of interest.

5.1 Policy Instruments and Methodology

Our baseline simulation employs the parameters established in the previous section, with the government maintaining the current levels of policy expenditure shares: $\sigma^p = 0.014$, $\sigma^c = 0.092$, $\sigma^e = 0.681$, and $\sigma^\eta = 0.213$ for all periods. This configuration reflects the ongoing government support for family policies and allows us to project how the Total Fertility Rate (TFR) and economic growth may evolve under existing policies. We conduct a series of policy experiments by altering one policy instrument at a time while keeping other parameters constant. To ensure the government's budget constraint is satisfied, we adjust other expenditure shares proportionally when changing government expenditure shares. We consider five policy scenarios:

1. No Family Support Policies: Government expenditure is exclusively allocated to public education ($\sigma^e = 1$) in all periods.

2. Parental Leave Payments: Increasing σ^p by 5 percentage points (pp) to 0.064, with a proportional decrease in σ^e to maintain the budget balance.

3. Child Allowances: Increasing σ^c by 5pp to 0.142, offset by a reduction in σ^e.

4. Public Education: Increasing σ^e by 5pp to 0.731, with σ^p and σ^c reduced equally.

5. Pension Benefits: Increasing σ^η by 5pp to 0.263, reducing σ^e equally.

5.2 Family Support Policies

Figure 15 presents the simulation results for these five scenarios. First, with respect the TFR, increasing child allowances or parental leave payments raises fertility, whereas expanding government education funding further reduces it. Adjusting pension benefits alone does not significantly alter the TFR, producing outcomes similar to the baseline scenario at current policy levels. Correspondingly, these patterns extend to total population: child allowances best mitigate population decline, followed by parental leave enhancements and existing policies.

Second, the growth rate of human capital (g_t) remains inversely related to the fertility rate (n_t), illustrating the quantity–quality trade-off in child-rearing. Higher fertility reduces per-child educational investment, diminishing human capital growth. In contrast, policies that boost fertility also raise the wage rate (w_t). Third, scenarios with lower family support (and thus lower TFRs) yield higher output per worker, attributed to higher human capital accumulation shared among fewer workers. In these cases, parents invest more in the education of a smaller number of children, fostering greater human capital per worker and thereby improving productivity.

Finally, turning to pension benefits per old-age recipient (η_t), our simulations indicate that policies increasing the TFR reduce pension benefits on a per-recipient basis. In contrast, lower fertility scenarios lead to fewer retirees and, when accompanied by higher human capital accumulation, it may offset the narrower tax base, resulting in larger benefits per retiree. However, it is crucial to recognize that in our model, pension outlays are set as a fixed proportion σ^η of total government revenue. Real-world PAYG systems, by contrast, typically legislate a constant fraction of pre-retirement or average earnings to ensure stable replacement rates, independent of demographic shifts. Because our framework neither ties pension benefits to prevailing wages nor guarantees a fixed amount, any observed increase in per-recipient pensions under declining fertility should be treated cautiously.

To more accurately capture the guaranteed benefit levels characteristic of real-world PAYG systems, our model could be modified to link pension benefits to the current worker's earning. For instance, setting

$$\eta_t N_{t-1}^y = \tau^\eta(1 - (\nu - \alpha_t^p)n_t)w_t h_t N_t^y$$

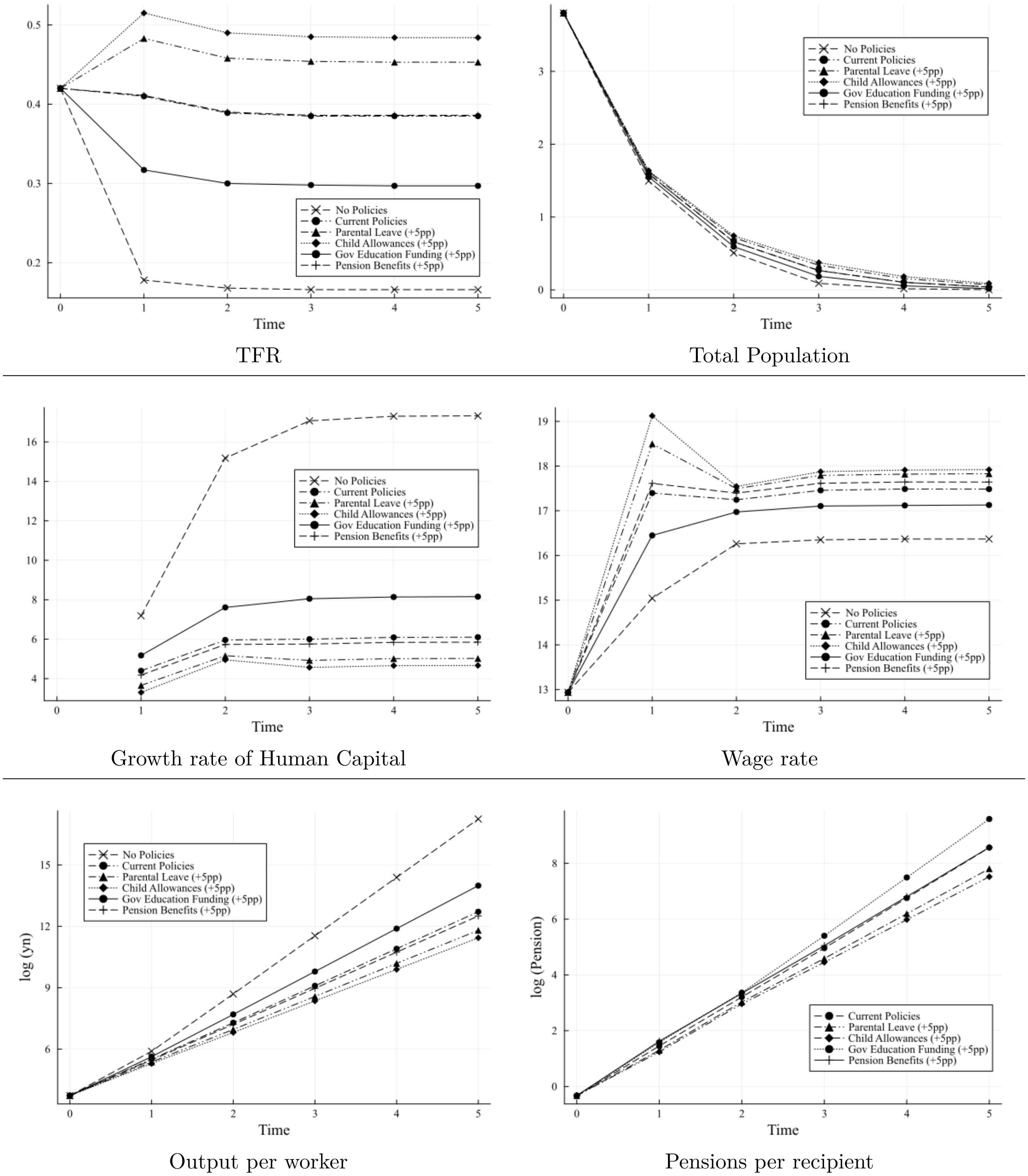
would involve the government collecting a separate pension tax τ^η from household income, directly financing pension benefits for the existing elderly population (Cipriani, 2014; Fanti and Gori, 2013; Kaganovich and Zilcha, 2012). Additionally, to formalize a guaranteed benefit level, one could impose a condition such as

$$\eta_{t+1} > \tau^\eta(1 - (\nu - \alpha_t^p)n_t)w_t h_t.$$

Under this specification, adjustments to fertility or wage levels would necessitate changes in the pension tax rate τ^η to maintain the legislated replacement rate. Incorporating such a wage-linked pension structure into our model would provide a more accurate representation of the intergenerational redistributions in PAYG systems. However, since our current framework employs a fixed budget share for pensions, it may overstate per-recipient pension benefits compared to systems where benefits are legally guaranteed or indexed to wages and demographics.

5.3 Tax rate

We also examine the effects of adjusting the income tax rate τ by $\pm$ 10pp from the baseline, setting $\tau = 0.377$ or $\tau = 0.177$. Figure 16 shows that, consistent with the findings in



TFR | Total Population

Growth rate of Human Capital | Wage rate

Output per worker | Pensions per recipient

Figure 15: Simulation Results of Policy Experiments

Note: The No-Family Support scenario is omitted from the "Pensions per recipient" panel since $\sigma^{\eta} = 0$ under that scenario, leading to zero pension benefits ($\eta_t = 0$) in all periods.

Chapter 2, increasing τ by 10pp results in: (1) a higher TFR, (2) a smaller reduction in total population, (3) a lower growth rate of human capital, (4) a reduced in the wage rate, and (5) lower output per worker.

These outcomes reflect the interplay between income effects and child-rearing decisions. Higher taxes reduce disposable income, encouraging households to increase fertility rather than maintain previous labor market participation or invest heavily in education. While this response raises fertility, it simultaneously suppresses human capital formation and lowers wage growth over time, consequently diminishing per-worker output.

Turning to pension benefits per recipient (η_t), the higher tax scenario ($\tau = 0.377$) initially yields the greatest pension benefits. However, this advantage fades over time (after T=3). As lower wages eventually reduce tax revenues, it allows the baseline and lower-tax scenarios to surpass the high-tax scenario in terms of pension benefits per recipient. In other words, while increasing the tax rate initially bolsters pension revenues and thus benefits, its negative long-run impact on wage growth and human capital accumulation ultimately reduces pension benefits in later periods. Conversely, although lowering the tax rate initially leads to smaller pension benefits, the scenario's higher long-run wages and reduced number of pension recipients improve per-recipient pension benefits after T=4.

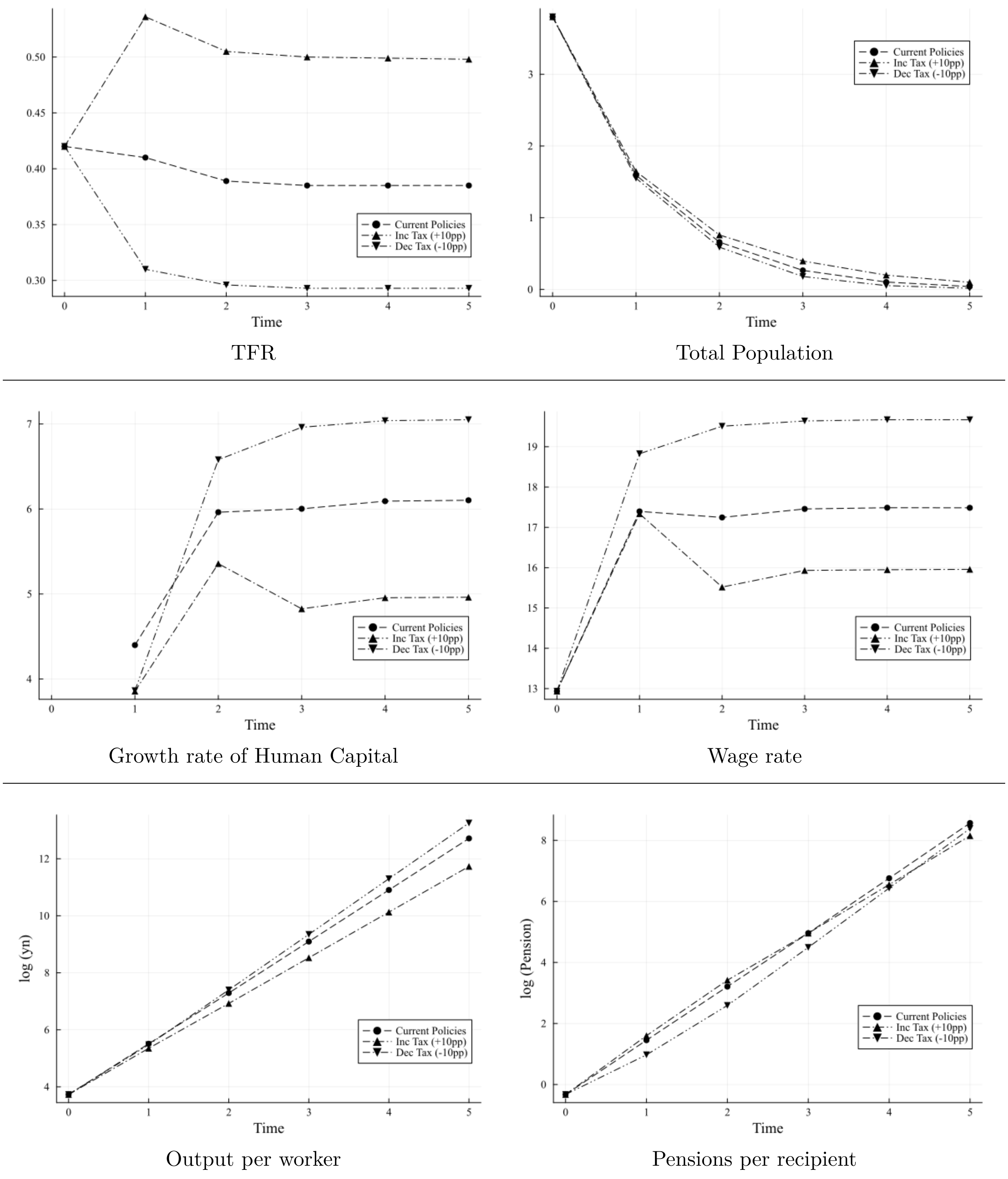
In summary, our policy experiments suggest that increasing child allowances is the most effective approach to raising fertility in the South Korean context. Yet, policymakers must account for the associated trade-offs: higher fertility can dampen human capital growth, reduce output per worker, and diminish pension benefits per old-age recipient in the long run. Adjusting the tax rate similarly involves trade-offs. While a higher tax rate encourages fertility and initially raises pension benefits, it also weakens long-term human capital accumulation and wage growth, eventually eroding per-recipient pension benefits and per-worker economic well-being.

6 Discussion

In this section, we delve into additional factors that may influence fertility rates beyond the scope of our model, consider alternative solutions such as immigration policies and explore the potential impact of mandatory paternity leave on fertility decisions.

Additional Factors Influencing Fertility

While our analysis focuses on the impact of several family support policies and tax rate, it is important to acknowledge that other factors can significantly affect individuals'



TFR | Total Population

Growth rate of Human Capital | Wage rate

Output per worker | Pensions per recipient

Figure 16: Policy Experiment 2 - Tax Rate

decisions to have children. Housing affordability, urbanization, labor market conditions, cultural norms and societal attitudes toward family and child-rearing all play crucial roles in shaping fertility patterns. For instance, high housing prices can deter young couples from starting a family due to financial constraints, and urbanization may lead to lifestyle changes that favor smaller family sizes. Moreover, societal attitudes toward gender roles, work-life balance and the opportunity costs associated with child-rearing can influence fertility decisions. Incorporating these factors into future research could provide a more comprehensive understanding of the determinants of fertility rates.

Alternative Approaches: Immigration Policies

In addition to policies aimed at increasing the fertility rate, many developed countries have turned to immigration as a strategy to sustain their working-age population and alleviate financial pressures on pension systems. Western European nations and the United States, for example, have implemented immigration policies to address labor shortages and demographic imbalances. South Korea is also making efforts to counteract the decline in its working-age population by increasing the inflow of immigrants. According to the Ministry of Employment and Labor, the labor shortage in South Korea rose from approximately 415,000 in the first half of 2021 to 642,000 in the first half of 2022. In response, the government expanded the issuance of E-9 visas to about 110,000 individuals — the highest number to date — allocating foreign workers to sectors such as manufacturing, agriculture, and construction.[43]

The number of foreign residents in South Korea has steadily increased since 2007 and is expected to continue rising. Exploring the role of immigration in sustaining the labor force presents a valuable avenue for future research. Specifically, analyzing the dynamics of immigration policies and labor types could offer insights into the impact of immigrants with varying skill levels on economic output and social welfare. Such a study could help determine the necessary levels of skilled and unskilled immigrant labor to achieve desired economic outcomes.

Related Literature on Immigration and Pension Sustainability

Several studies have explored the impact of immigration on fiscal sustainability and demographic dynamics. Storesletten (2000) examines the potential of sustaining fiscal policy through immigration using a calibrated general equilibrium OLG model. The study finds that selective immigration policies increasing the inflow of working-age high- and medium-skilled

[43] KOSIS data, Statistics on Foreign Residents by Local Government (Province).

immigrants can eliminate the need for future fiscal reforms. Similarly, Leers, Meijdam, and Verbon (2004) investigate the relationship between aging, migration, and endogenous public pensions in a two-period OLG model of a small open economy. Their findings suggest that the beneficial effects of immigration on public pensions may be limited if labor is heterogeneous. They also note that with endogenous public pensions financed by higher taxes, it may take several generations before a steady state is reached.

Berger, Davoine, Schuster, and Strohner (2016) compare cross-country differences in the contribution of future migration to old-age financing using an OLG model with idiosyncratic migration. They estimate that projected immigration flows could offset labor income taxes by significant percentages in countries such as Australia (14.3%), Germany (7.3%), the United Kingdom (6.2%), and Poland (1.7%) by 2060. In the context of Japan, Okamoto (2021) analyze the impact of immigration policies on the economy and welfare of an aging society using a three-period OLG model. The study concludes that selective immigration policies can enhance welfare for both native and immigrant populations and mitigate the negative effects of aging, whereas non-selective policies may reduce native welfare.

Finally, Chojnicki and Ragot (2016) assess the impacts of immigration on the French welfare state through an applied general equilibrium model. Their findings indicate that high-skilled immigration positively affects GDP, investment, and welfare, while low-skilled immigration may have adverse economic effects. The study emphasizes the importance of considering the skill level of immigrants when evaluating their impact on the economy and the welfare state. These studies highlight the potential of immigration as an alternative or complementary strategy to address demographic challenges. Future research could build on these findings to explore how immigration policies might interact with fertility-enhancing measures to sustain economic growth and fiscal stability.

Mandatory Paternity Leave

Expanding our model to incorporate non-unitary households and gender-specific parental leave policies offers another avenue for addressing low fertility rates. In many developed countries, parental leave is predominantly utilized by women, while utilization by men remains significantly lower. This disparity can contribute to gender imbalances in career advancement and labor market participation, potentially influencing couples' decisions regarding childbearing. Promoting paternity leave may alleviate the perceived career disadvantages faced by women and encourage a more equitable sharing of child-rearing responsibilities, which could increase fertility rates or at least mitigate declining trends.

Several countries have implemented "use-it-or-lose-it" paternity leave policies, where a non-transferable portion of parental leave is specifically designated for fathers. While not mandatory in the sense of legal obligation, these policies provide entitlements that are forfeited if not utilized by the father, incentivizing paternal involvement in early child care. Below are examples of countries with non-transferable paternity leave policies:

Sweden (1995): Sweden introduced a "daddy quota" in 1995, setting aside a specific portion of parental leave for fathers. Currently, parents are entitled to 480 days of parental leave per child, which can be shared. However, 90 days are earmarked exclusively for each parent and are non-transferable. The leave is compensated at approximately 80% of the salary up to a certain limit for the first 390 days and at a flat rate for the remaining days. As of 2021, 46% of parents who took parental leave in Sweden were men.

Norway (1993): Norway implemented paternal quotas in its parental leave policy in 1993. Parents can choose between 49 weeks at 100% pay or 59 weeks at 80% pay, which can be shared. Within this framework, a non-transferable "daddy quota" of 15 weeks is reserved for fathers. If the father does not take his allocated quota, it is forfeited and cannot be used by the mother.

Iceland (2000): Iceland's parental leave policy includes a non-transferable three months of paternity leave at 80% of salary, intended to be used within the first year of the child's birth or adoption. This policy was implemented to promote gender equality in parental responsibilities.

Spain (2021): Spain extended its paternity leave in 2021 to match maternity leave, providing 16 weeks of non-transferable leave for each parent at 100% of the base salary. This change followed several incremental increases from the initial two weeks of fully paid paternity leave introduced in 2007.

Portugal (2009): Fathers in Portugal are entitled to a total of 25 days of paternity leave at 100% of salary. This includes 15 mandatory working days that should be taken within the first month after childbirth and an additional 10 days that may be taken during the first year.

The implementation of non-transferable paternity leave policies aims to encourage fathers to participate more actively in child-rearing, promoting gender equality in both the household and the workplace. By reducing the career opportunity costs associated with motherhood, such policies may positively influence fertility decisions. Incorporating gender-specific parental leave into our model could provide insights into how these policies affect household labor supply, fertility choices and overall economic outcomes.

7 Conclusion

This study extends the analysis presented in Chapter 2 by incorporating pension benefits into a three-period overlapping generations (OLG) model, calibrated to South Korean data. Our objective is to assess how various government policies — child allowances, parental leave payments, public education spending, pension benefits, and income tax rates — shape household fertility decisions and key macroeconomic outcomes.

Our results reconfirm that increasing child allowances is the most effective policy for raising the fertility rate in South Korea, even after accounting for pension benefits. Yet, this still comes at a cost: higher fertility reduces human capital growth and lowers output per worker. The results highlight the quantity–quality trade-off in child-rearing, where resources allocated to raising more children limit the investment in each child's education.

While policies that boost fertility may initially appear beneficial for supporting pensions — by increasing the future workforce — their long-term effect on wages and human capital can erode the tax base and thereby reduce per-recipient pension benefits. Conversely, scenarios with lower fertility can lead to higher per-recipient pension benefits over time due to wage gains and fewer beneficiaries, albeit at the risk of undermining broader economic sustainability. These complexities highlight that the interplay between fertility and pension systems is more subtle than simple linear logic might suggest.

Several extensions could further enhance our understanding of these dynamics. Incorporating immigration as a policy could illuminate whether augmenting the labor force with foreign workers helps mitigate the pressures of an aging population. Examining gender-specific policies, such as non-transferable paternity leave, could address labor market inequalities and alter fertility behavior. Finally, adopting a more realistic PAYG pension framework, where current retirees' benefits are guaranteed, would clarify how generational burdens shift under different demographic and policy scenarios.

A Chapter 1 Appendix

A.1 Model Derivations

A.1.1 Open Economy with Tradable Service

Household

Suppose $\theta \notin \{0, 1\}$ and the household chooses c^T_{st} and c^N_{st} within service sector. Then, by substituting $\lambda_t = \frac{1}{C_t} C_t^{\frac{1}{\varepsilon}} \omega_m^{\frac{1}{\varepsilon}} (c_{mt})^{-\frac{1}{\varepsilon}}$ into the equations (1), (3) and (4), one can rewrite them as follows:

$$
\begin{aligned}
p_{at} &= \left[\frac{\omega_a}{\omega_m}\left(\frac{c_{mt}}{c_{at}-\bar{c}_a}\right)\right]^{\frac{1}{\varepsilon}} \\
p^T_{st} &= \theta\left[\frac{\omega_s}{\omega_m}\left(\frac{c_{mt}}{c_{st}+\bar{c}_s}\right)\right]^{\frac{1}{\varepsilon}}\left(\frac{c_{st}}{c^T_{st}}\right)^{\frac{1}{\sigma}} \\
p^N_{st} &= (1-\theta)\left[\frac{\omega_s}{\omega_m}\left(\frac{c_{mt}}{c_{st}+\bar{c}_s}\right)\right]^{\frac{1}{\varepsilon}}\left(\frac{c_{st}}{c^N_{st}}\right)^{\frac{1}{\sigma}} \\
&\text{or} \\
c_{at}-\bar{c}_a &= \frac{\omega_a}{\omega_m} p_{at}^{-\varepsilon} c_{mt} \\
c^T_{st} &= \left(\frac{\theta}{p^T_{st}}\right)^{\sigma}\left[\frac{\omega_s}{\omega_m}\left(\frac{c_{mt}}{c_{st}+\bar{c}_s}\right)\right]^{\frac{\sigma}{\varepsilon}} c_{st} \\
c^N_{st} &= \left(\frac{1-\theta}{p^N_{st}}\right)^{\sigma}\left[\frac{\omega_s}{\omega_m}\left(\frac{c_{mt}}{c_{st}+\bar{c}_s}\right)\right]^{\frac{\sigma}{\varepsilon}} c_{st} \\
\text{where} \quad c_{st} &\equiv \left[\theta\left(c^T_{st}\right)^{\frac{\sigma-1}{\sigma}} + (1-\theta)\left(c^N_{st}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}
\end{aligned}
$$

***Euler Equation:** Using the equations (1)-(4), we can derive the Euler equation. First, by multiplying c^T_{st} to (3), c^N_{st} to (4) and add them together:

$$
\begin{aligned}
\text{LHS:} \quad & \frac{1}{C_t} C_t^{\frac{1}{\varepsilon}} \omega_s^{\frac{1}{\varepsilon}} (c_{st}+\bar{c}_s)^{-\frac{1}{\varepsilon}} (c_{st})^{\frac{1}{\sigma}} \left[\theta(c^T_{st})^{\frac{\sigma-1}{\sigma}} + (1-\theta)(c^N_{st})^{\frac{\sigma-1}{\sigma}}\right] \\
&= \frac{1}{C_t} C_t^{\frac{1}{\varepsilon}} \omega_s^{\frac{1}{\varepsilon}} (c_{st}+\bar{c}_s)^{-\frac{1}{\varepsilon}} (c_{st})^{\frac{1}{\sigma}} c_{st}^{\frac{\sigma-1}{\sigma}} \\
&= \frac{1}{C_t} C_t^{\frac{1}{\varepsilon}} \omega_s^{\frac{1}{\varepsilon}} (c_{st}+\bar{c}_s)^{-\frac{1}{\varepsilon}} c_{st} \\
\text{RHS:} \quad & \lambda_t\left(p^T_{st}c^T_{st} + p^N_{st}c^N_{st}\right) = \lambda_t p_{st} c_{st} \quad \text{where } p_{st} \equiv \frac{p^T_{st}c^T_{st} + p^N_{st}c^N_{st}}{c_{st}}
\end{aligned}
$$

Then, by setting LHS = RHS, we can derive following equation:

$$
\frac{1}{C_t} C_t^{\frac{1}{\varepsilon}} \omega_s^{\frac{1}{\varepsilon}} (c_{st}+\bar{c}_s)^{-\frac{1}{\varepsilon}} = \lambda_t p_{st} \tag{30}
$$

If one divides the equation (3) by (4), then

$$
\begin{aligned}
\frac{\theta}{1-\theta}\left(\frac{c_{st}^T}{c_{st}^N}\right)^{\frac{-1}{\sigma}} &= \frac{p_{st}^T}{p_{st}^N} \\
\Rightarrow \left(\frac{c_{st}^T}{c_{st}^N}\right)^{\frac{1}{\sigma}} &= \frac{\theta}{1-\theta}\left(\frac{p_{st}^N}{p_{st}^T}\right) \\
\Rightarrow \frac{c_{st}^T}{c_{st}^N} &= \left(\frac{\theta}{1-\theta}\right)^{\sigma}\left(\frac{p_{st}^N}{p_{st}^T}\right)^{\sigma} \\
\Rightarrow c_{st}^T &= \left(\frac{\theta}{1-\theta}\right)^{\sigma}\left(\frac{p_{st}^N}{p_{st}^T}\right)^{\sigma} c_{st}^N
\end{aligned}
$$

By substituting c_{st}^T into $p_{st} \equiv \dfrac{p_{st}^T c_{st}^T + p_{st}^N c_{st}^N}{c_{st}}$,

$$
\begin{aligned}
\text{Numerator:}\quad p_{st}^T c_{st}^T + p_{st}^N c_{st}^N &= \left[p_{st}^T\left(\frac{\theta}{1-\theta}\right)^{\sigma}\left(\frac{p_{st}^N}{p_{st}^T}\right)^{\sigma} + p_{st}^N\right]c_{st}^N \\
&= \left[\frac{p_{st}^T}{p_{st}^N}\left(\frac{\theta}{1-\theta}\right)^{\sigma}\left(\frac{p_{st}^N}{p_{st}^T}\right)^{\sigma} + 1\right]p_{st}^N c_{st}^N \\
&= \left[1 + \left(\frac{\theta}{1-\theta}\right)^{\sigma}\left(\frac{p_{st}^N}{p_{st}^T}\right)^{\sigma-1}\right]p_{st}^N c_{st}^N \\
&= \left[1 + \theta^{\sigma}(1-\theta)^{-\sigma}(p_{st}^N)^{\sigma-1}(p_{st}^T)^{1-\sigma}\right]p_{st}^N c_{st}^N \\
&= \left[(p_{st}^T)^{\sigma-1} + \theta^{\sigma}(1-\theta)^{-\sigma}(p_{st}^N)^{\sigma-1}\right]p_{st}^N c_{st}^N \times (p_{st}^T)^{1-\sigma} \\
&= \left[(1-\theta)^{\sigma}(p_{st}^T)^{\sigma-1} + \theta^{\sigma}(p_{st}^N)^{\sigma-1}\right]p_{st}^N c_{st}^N \times (p_{st}^T)^{1-\sigma} \times (1-\theta)^{-\sigma}
\end{aligned}
$$

$$
\begin{aligned}
\text{Denominator:}\quad c_{st} &= \left[\theta\left(c_{st}^T\right)^{\frac{\sigma-1}{\sigma}} + (1-\theta)\left(c_{st}^N\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \\
&= \left[\theta\left[\left(\frac{\theta}{1-\theta}\right)^{\sigma}\left(\frac{p_{st}^N}{p_{st}^T}\right)^{\sigma} c_{st}^N\right]^{\frac{\sigma-1}{\sigma}} + (1-\theta)\left(c_{st}^N\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \\
&= \left[\left(\theta\left(\frac{\theta}{1-\theta}\right)^{\sigma-1}\left(\frac{p_{st}^N}{p_{st}^T}\right)^{\sigma-1} + (1-\theta)\right)(c_{st}^N)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \\
&= \left[\theta\left(\frac{\theta}{1-\theta}\right)^{\sigma-1}\left(\frac{p_{st}^N}{p_{st}^T}\right)^{\sigma-1} + (1-\theta)\right]^{\frac{\sigma}{\sigma-1}} c_{st}^N \\
&= \left[\theta^{\sigma}(1-\theta)^{1-\sigma}(p_{st}^N)^{\sigma-1}(p_{st}^T)^{1-\sigma} + (1-\theta)\right]^{\frac{\sigma}{\sigma-1}} c_{st}^N \\
&= \left[\theta^{\sigma}(1-\theta)^{1-\sigma}(p_{st}^N)^{\sigma-1} + (1-\theta)(p_{st}^T)^{\sigma-1}\right]^{\frac{\sigma}{\sigma-1}} c_{st}^N \times (p_{st}^T)^{-\sigma}
\end{aligned}
$$

$$= \left[\theta^\sigma (p_{st}^N)^{\sigma-1} + (1-\theta)^\sigma (p_{st}^T)^{\sigma-1}\right]^{\frac{\sigma}{\sigma-1}} c_{st}^N \times (p_{st}^T)^{-\sigma} \times (1-\theta)^{-\sigma}$$

$$= \left[(1-\theta)^\sigma (p_{st}^T)^{\sigma-1} + \theta^\sigma (p_{st}^N)^{\sigma-1}\right]^{\frac{\sigma}{\sigma-1}} c_{st}^N \times (p_{st}^T)^{-\sigma} \times (1-\theta)^{-\sigma}$$

Now let's solve for p_{st}:

$$\begin{aligned}
p_{st} &= \frac{p_{st}^T c_{st}^T + p_{st}^N c_{st}^N}{c_{st}} \\
&= \frac{\left[(1-\theta)^\sigma (p_{st}^T)^{\sigma-1} + \theta^\sigma (p_{st}^N)^{\sigma-1}\right] p_{st}^N c_{st}^N \times (p_{st}^T)^{1-\sigma} \times (1-\theta)^{-\sigma}}{\left[(1-\theta)^\sigma (p_{st}^T)^{\sigma-1} + \theta^\sigma (p_{st}^N)^{\sigma-1}\right]^{\frac{\sigma}{\sigma-1}} c_{st}^N \times (p_{st}^T)^{-\sigma} \times (1-\theta)^{-\sigma}} \\
&= \frac{p_{st}^N p_{st}^T}{\left[(1-\theta)^\sigma (p_{st}^T)^{\sigma-1} + \theta^\sigma (p_{st}^N)^{\sigma-1}\right]^{\frac{1}{\sigma-1}}}
\end{aligned}$$

where the equation (30) states $\frac{1}{C_t} C_t^{\frac{1}{\varepsilon}} \omega_s^{\frac{1}{\varepsilon}} (c_{st} + \bar{c}_s)^{-\frac{1}{\varepsilon}} = \lambda_t p_{st}$.

Finally, if one multiply (1), (2) and (30) to the power $1-\varepsilon$ and add them together, we can derive the Euler equation of the model as follows:

$$\begin{aligned}
&\left(\frac{1}{C_t}\right)^{1-\varepsilon} C_t^{\frac{1-\varepsilon}{\varepsilon}} \left[\omega_a^{\frac{1}{\varepsilon}} (c_{at} - \bar{c}_a)^{-\frac{1}{\varepsilon}} + \omega_m^{\frac{1}{\varepsilon}} (c_{mt})^{-\frac{1}{\varepsilon}} + \omega_s^{\frac{1}{\varepsilon}} (c_{st} + \bar{c}_s)^{-\frac{1}{\varepsilon}}\right] = \lambda_t^{1-\varepsilon} \left(\omega_a p_{at}^{1-\varepsilon} + \omega_m + \omega_s p_{st}^{1-\varepsilon}\right) \\
&\qquad \Rightarrow \left(\frac{1}{C_t}\right)^{1-\varepsilon} C_t^{\frac{1-\varepsilon}{\varepsilon}} \left[C_t^{\frac{\varepsilon-1}{\varepsilon}}\right] = \lambda_t^{1-\varepsilon} \left(\omega_a p_{at}^{1-\varepsilon} + \omega_m + \omega_s p_{st}^{1-\varepsilon}\right) \\
&\qquad \Rightarrow \frac{1}{C_t} = \lambda_t \left(\omega_a p_{at}^{1-\varepsilon} + \omega_m + \omega_s p_{st}^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}} \\
&\qquad \Rightarrow \frac{1}{C_t} \equiv \lambda_t P_t
\end{aligned}$$

where λ_t denotes the Lagrange multiplier on the budget constraint in period t. Since λ_t is the marginal value of additional unit of consumption at time t, it implies that other terms on the right side of the equation can be interpreted as the price index of a unit of composite consumption, P_t. Using the Lagrangian multiplier, the first order conditions for $\tilde{c}_t$ and k_{t+1} provide the Euler equation:

$$(1+\rho) \frac{P_{t+1} C_{t+1}}{P_t C_t} = 1 - \delta - \nu + R_{t+1}$$

Firm

Here, suppose the tradable service sector is more labor intensive than non-traded sector, but still uses capital.

$$
\begin{aligned}
y_{at} &= (k_{at})^{\eta}(A_{at}n_{at})^{\beta}\left(\frac{L}{N_t}\right)^{1-\eta-\beta} \\
y_{mt} &= (k_{mt})^{\alpha}(A_{mt}n_{mt})^{1-\alpha} \\
y_{st}^{N} &= (k_{st}^{T})^{\alpha}(A_{st}^{T}n_{st}^{T})^{1-\alpha} \\
y_{st}^{T} &= (k_{st}^{T})^{\chi}(A_{st}^{T}n_{st}^{T})^{1-\chi} \quad \text{where } 0<\chi<\alpha<1
\end{aligned}
$$

Using FOC in each sector with respect to labor and capital, one can derive:

$$
\frac{R_t}{w_t} = \frac{\eta}{\beta}\left(\frac{n_{at}}{k_{at}}\right) = \frac{\alpha}{1-\alpha}\left(\frac{n_{mt}}{k_{mt}}\right) = \left(\frac{\chi}{1-\chi}\right)\left(\frac{n_{st}^{T}}{k_{st}^{T}}\right) = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{n_{st}^{N}}{k_{st}^{N}}\right)
$$

Then, by combining these equations together, one can obtain:

$$
\begin{aligned}
\frac{n_{at}}{k_{at}} &= \frac{\beta}{\eta}\left(\frac{\alpha}{1-\alpha}\right)\frac{n_{mt}}{k_{mt}} \\
\frac{n_{mt}}{k_{mt}} &= \frac{n_{st}^{N}}{k_{st}^{N}} \\
\frac{n_{mt}}{k_{mt}} &= \frac{1-\alpha}{\alpha}\left(\frac{\chi}{1-\chi}\right)\frac{n_{st}^{T}}{k_{st}^{T}}
\end{aligned}
$$

At the optimum, the marginal products of capital and labor must be equal to the rental rate of capital and the wage respectively, where

$$
\begin{aligned}
R_t &= \frac{\partial y_{mt}}{\partial k_{mt}} = p_{at}\frac{\partial y_{at}}{\partial k_{at}} = p_{st}^{T}\frac{\partial y_{st}^{T}}{\partial k_{st}^{T}} = p_{st}^{N}\frac{\partial y_{st}^{N}}{\partial k_{st}^{N}} \\
w_t &= \frac{\partial y_{mt}}{\partial n_{mt}} = p_{at}\frac{\partial y_{at}}{\partial n_{at}} = p_{st}^{T}\frac{\partial y_{st}^{T}}{\partial n_{st}^{T}} = p_{st}^{N}\frac{\partial y_{st}^{N}}{\partial n_{st}^{N}}
\end{aligned}
$$

Using the optimality and the FOCs together, one can derive:

$$
\begin{aligned}
p_{at} &= \left(\frac{\alpha}{\eta}\right)^{\alpha}\left(\frac{1-\alpha}{\beta}\right)^{1-\alpha}(k_{at})^{\alpha-\eta}\, n_{at}^{1-\alpha-\beta} A_{mt}^{1-\alpha} A_{at}^{-\beta}\left(\frac{N_t}{L}\right)^{1-\eta-\beta} \\
p_{st}^{T} &= \left(\frac{\alpha}{\chi}\right)^{\alpha}\left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha}\left(k_{st}^{T}\right)^{\alpha-\chi}\left(n_{st}^{T}\right)^{\chi-\alpha} A_{mt}^{1-\alpha}\left(A_{st}^{T}\right)^{\chi-1} \\
p_{st}^{N} &= \left(\frac{A_{mt}}{A_{st}^{N}}\right)^{1-\alpha}
\end{aligned}
$$

Equilibrium

Given the initial capital stock k_0 and exogenous variables $\{A_{at}, A_{mt}, A_{st}^T, A_{st}^N, N_t, p_{at}, p_{st}^T\}_{t=0}^{\infty}$, the equilibrium conditions can be summarized by followings: a set of prices $\{p_{st}^N, w_t, R_t, q_t\}_{t=0}^{\infty}$, sectoral shares of employment and capital $\{n_{at}, n_{mt}, n_{st}^T, n_{st}^N, \kappa_{at}, \kappa_{mt}, \kappa_{st}^T, \kappa_{st}^N\}_{t=0}^{\infty}$ and quantities $\{k_t, y_{at}, y_{mt}, y_{st}^T, y_{st}^N, c_{at}, c_{mt}, c_{st}^T, c_{st}^N, x_{at}, x_{mt}, x_{st}\}_{t=0}^{\infty}$ which satisfy: (i) Household's optimization, (ii) Firm's optimization and (iii) Market clearing conditions. The equilibrium conditions can be summarized by the following system of 14 equations with the 14 unknowns $\{p_{st}^N, k_t, c_{at}, c_{mt}, c_{st}^T, c_{st}^N,$
$n_{at}, n_{st}^T, n_{st}^N, \kappa_{at}, \kappa_{st}^T, \kappa_{st}^N, x_{mt}, x_{st}\}_{t=0}^{\infty}$:

$$k_{t+1} = \left(\left(1-\kappa_{at}-\kappa_{st}^T-\kappa_{st}^N\right)k_t\right)^{\alpha}\left(A_{mt}\left(1-n_{at}-n_{st}^T-n_{st}^N\right)\right)^{1-\alpha} + \left(1-\delta-\nu\right)k_t - c_{mt} - x_{mt} \tag{31}$$

$$(1+\rho)\frac{P_{t+1}C_{t+1}}{P_tC_t} = 1-\delta-\nu+R_{t+1} \tag{32}$$

$$c_{at}-\bar{c}_a = \frac{\omega_a}{\omega_m}p_{at}^{-\varepsilon}c_{mt} \tag{33}$$

$$c_{st}^T = \left(\frac{\theta}{p_{st}^T}\right)^{\sigma}\left[\frac{\omega_s}{\omega_m}\left(\frac{c_{mt}}{c_{st}+\bar{c}_s}\right)\right]^{\frac{\sigma}{\varepsilon}}c_{st} \tag{34}$$

$$c_{st}^N = \left(\frac{1-\theta}{p_{st}^N}\right)^{\sigma}\left[\frac{\omega_s}{\omega_m}\left(\frac{c_{mt}}{c_{st}+\bar{c}_s}\right)\right]^{\frac{\sigma}{\varepsilon}}c_{st} \tag{35}$$

$$\frac{1-n_{at}-n_{st}^T-n_{st}^N}{1-\kappa_{at}-\kappa_{st}^T-\kappa_{st}^N} = \frac{\eta}{\beta}\left(\frac{1-\alpha}{\alpha}\right)\frac{n_{at}}{\kappa_{at}} \tag{36}$$

$$\frac{1-n_{at}-n_{st}^T-n_{st}^N}{1-\kappa_{at}-\kappa_{st}^T-\kappa_{st}^N} = \frac{n_{st}^N}{\kappa_{st}^N} \tag{37}$$

$$\frac{1-n_{at}-n_{st}^T-n_{st}^N}{1-\kappa_{at}-\kappa_{st}^T-\kappa_{st}^N} = \frac{1-\alpha}{\alpha}\left(\frac{\chi}{1-\chi}\right)\frac{n_{st}^T}{\kappa_{st}^T} \tag{38}$$

$$p_{at} = \left(\frac{\alpha}{\eta}\right)^{\alpha}\left(\frac{1-\alpha}{\beta}\right)^{1-\alpha}\left(\kappa_{at}k_t\right)^{\alpha-\eta}n_{at}^{1-\alpha-\beta}A_{mt}^{1-\alpha}A_{at}^{-\beta}\left(\frac{N_t}{L}\right)^{1-\eta-\beta} \tag{39}$$

$$p_{st}^T = \left(\frac{\alpha}{\chi}\right)^{\alpha}\left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha}\left(\kappa_{st}^Tk_t\right)^{\alpha-\chi}\left(n_{st}^T\right)^{\chi-\alpha}A_{mt}^{1-\alpha}\left(A_{st}^T\right)^{\chi-1} \tag{40}$$

$$p_{st}^N = \left(\frac{A_{mt}}{A_{st}^N}\right)^{1-\alpha} \tag{41}$$

$$c_{at} - \frac{x_{mt}}{p_{at}} - \frac{p_{st}^Tx_{st}}{p_{at}} = \left(\kappa_{at}k_t\right)^{\eta}\left(A_{at}n_{at}\right)^{\beta}\left(\frac{L}{N_t}\right)^{1-\eta-\beta} \tag{42}$$

$$c_{st}^T + x_{st} = \left(\kappa_{st}^Tk_t\right)^{\chi}\left(A_{st}^Tn_{st}^T\right)^{1-\chi} \tag{43}$$

$$c_{st}^N = \left(\kappa_{st}^Nk_t\right)^{\alpha}\left(A_{st}^Nn_{st}^N\right)^{1-\alpha} \tag{44}$$

The equations (31) and (32) denote the law of motion for the capital and the Euler equation respectively. The equations (33)-(35) are derived from the household's FOCs where equations (36)-(38) are from a firm's FOCs. The equations (39)-(41) are from the competitive price of capital and labor. Finally, the equations (42)-(44) are from the market clearing conditions.

Steady States - Derivations

This subsection provides a detailed, step-by-step derivation of the steady state outcomes presented in Section 4, under the assumption that the key conditions outlined therein are satisfied.

Step 1) Agricultural shares: κ_a^{ss} and n_a^{ss}

To analyze how input share changes in the steady states, it is simpler to rewrite variables in detrended term:

$$\hat{c}_{mt} \equiv \frac{c_{mt}}{A_{mt}}, \qquad \hat{k}_t \equiv \frac{k_t}{A_{mt}}, \qquad \psi_t \equiv p_{at} A_{at}^{\beta} A_{mt}^{\eta-1} \left(\frac{L}{N_t}\right)^{1-\eta-\beta}$$

First, using equations (39) and (42), we can rewrite the ψ_t as follows:

$$\begin{aligned}
\psi_t &\equiv p_{at} A_{at}^{\beta} A_{mt}^{\eta-1} \left(\frac{L}{N_t}\right)^{1-\eta-\beta} \\
&= \left(\frac{\alpha}{\eta}\right)^{\alpha} \left(\frac{1-\alpha}{\beta}\right)^{1-\alpha} (\kappa_{at} k_t)^{\alpha-\eta} n_{at}^{1-\alpha-\beta} A_{mt}^{1-\alpha} A_{at}^{-\beta} \left(\frac{N_t}{L}\right)^{1-\eta-\beta} \times A_{at}^{\beta} A_{mt}^{\eta-1} \left(\frac{L}{N_t}\right)^{1-\eta-\beta} \\
&= \left(\frac{\alpha}{\eta}\right)^{\alpha} \left(\frac{1-\alpha}{\beta}\right)^{1-\alpha} \kappa_{at}^{\alpha-\eta} \hat{k}_t^{\alpha-\eta} n_{at}^{1-\alpha-\beta} \\
&= \left(\frac{\alpha}{\eta}\right)^{\alpha} \left(\frac{1-\alpha}{\beta}\right)^{1-\alpha} \hat{k}_t^{\alpha-\eta} \left(\frac{\kappa_{at}}{n_{at}}\right)^{\alpha-\eta} n_{at}^{1-\eta-\beta} \\
&= \left(\frac{\alpha}{\eta}\right)^{\alpha} \left(\frac{1-\alpha}{\beta}\right)^{1-\alpha} \hat{k}_t^{\alpha-\eta} \left[\frac{\eta}{\beta}\left(\frac{1-\alpha}{\alpha}\right) \frac{1-\kappa_{at}-\kappa_{st}^T-\kappa_{st}^N}{1-n_{at}-n_{st}^T-n_{st}^N}\right]^{\alpha-\eta} n_{at}^{1-\eta-\beta} \\
&= \left(\frac{\alpha}{\eta}\right)^{\eta} \left(\frac{1-\alpha}{\beta}\right)^{1-\eta} \hat{k}_t^{\alpha-\eta} \left(\frac{1-\kappa_{at}-\kappa_{st}^T-\kappa_{st}^N}{1-n_{at}-n_{st}^T-n_{st}^N}\right)^{\alpha-\eta} n_{at}^{1-\eta-\beta}
\end{aligned}$$

Once the condition (5) holds, $\lim_{t\to\infty} \psi_t = 0$. Thus,

$$\begin{aligned}
&\lim_{t\to\infty} \psi_t = \left(\frac{\alpha}{\eta}\right)^{\eta} \left(\frac{1-\alpha}{\beta}\right)^{1-\eta} \hat{k}_t^{\alpha-\eta} \left(\frac{1-\kappa_{at}-\kappa_{st}^T-\kappa_{st}^N}{1-n_{at}-n_{st}^T-n_{st}^N}\right)^{\alpha-\eta} n_{at}^{1-\eta-\beta} = 0 \\
&\Rightarrow \; n_a^{ss} = \kappa_a^{ss} = 0 \qquad (45)
\end{aligned}$$

In the steady state, both labor and capital shares of agriculture converge to zero. Furthermore, using equations (33) and (39), we know that

$$
\begin{aligned}
c_{at} - \bar{c}_a &= \frac{\omega_a}{\omega_m} p_{at}^{-\varepsilon} c_{mt} \\
\Rightarrow \; \frac{\omega_a}{\omega_m} p_{at}^{1-\varepsilon} \hat{c}_{mt} &= p_{at} A_{mt}^{-1} \left(c_{at} - \bar{c}_a\right) \\
&= p_{at} A_{mt}^{-1} c_{at} - p_{at} A_{mt}^{-1} \bar{c}_a \\
&= p_{at} A_{mt}^{-1} \left(\left(\kappa_{at} k_t\right)^{\eta} \left(A_{at} n_{at}\right)^{\beta} \left(\frac{L}{N_t}\right)^{1-\eta-\beta} + \frac{x_{mt}}{p_{at}} + \frac{p_{st}^T x_{st}}{p_{at}} \right) - p_{at} A_{mt}^{-1} \bar{c}_a \\
&= p_{at} A_{mt}^{-1+\eta} A_{at}^{\beta} \left(\frac{L}{N_t}\right)^{1-\eta-\beta} \left(\kappa_{at} \hat{k}_t\right)^{\eta} n_{at}^{\beta} + \hat{x}_{mt} + p_{st}^T \hat{x}_{st} - p_{at} A_{mt}^{-1} \bar{c}_a \\
&= \psi_t \left(\kappa_{at} \hat{k}_t\right)^{\eta} n_{at}^{\beta} + \hat{x}_{mt} + p_{st}^T \hat{x}_{st} - p_{at} A_{mt}^{-1} \bar{c}_a
\end{aligned}
$$

Given $\kappa_a^{ss} = n_a^{ss} = 0$ and the condition (5), $\lim_{t\to\infty} \psi_t \left(\kappa_{at} \hat{k}_t\right)^{\eta} n_{at}^{\beta} = 0$. Moreover, since $\gamma_p^a < 0 < \gamma_m$, we obtain $\lim_{t\to\infty} p_{at} A_{mt}^{-1} \bar{c}_a = 0$. Consequently,

$$
\lim_{t\to\infty} \left(\frac{\omega_a}{\omega_m} p_{at}^{1-\varepsilon} \hat{c}_{mt} \right) = \lim_{t\to\infty} \left(\hat{x}_{mt} + p_{st}^T \hat{x}_{st} \right), \tag{46}
$$

and we will refer back to this result in subsequent derivations.

Step 2) Service shares: k_{ss}^T, n_{ss}^T & k_{ss}^N, n_{ss}^N

Using the equation (45), (40) and (41), one obtains:

$$
\begin{aligned}
\frac{1 - n_{ss}^T - n_{ss}^N}{1 - \kappa_{ss}^T - \kappa_{ss}^N} &= \frac{n_{ss}^N}{\kappa_{ss}^N} \qquad (47) \\
\frac{1 - n_{ss}^T - n_{ss}^N}{1 - \kappa_{ss}^T - \kappa_{ss}^N} &= \frac{1-\alpha}{\alpha} \left(\frac{\chi}{1-\chi}\right) \frac{n_s^T}{\kappa_{ss}^T} \\
\Rightarrow \; \frac{n_{ss}^N}{\kappa_{ss}^N} &= \frac{1-\alpha}{\alpha} \left(\frac{\chi}{1-\chi}\right) \frac{n_s^T}{\kappa_{ss}^T}. \qquad (48)
\end{aligned}
$$

We thus have two equations but four unknowns: $\{n_{ss}^T, n_{ss}^N, k_{ss}^T, k_{ss}^N\}$. To proceed, we treat $\{n_{ss}^T, n_{ss}^N\}$ as endogenous variables and $\{\kappa_{ss}^T, \kappa_{ss}^N\}$ as given parameters. First, rearrange (47):

$$
\begin{aligned}
\frac{1 - n_{ss}^T - n_{ss}^N}{1 - \kappa_{ss}^T - \kappa_{ss}^N} &= \frac{n_{ss}^N}{\kappa_{ss}^N} \\
\Rightarrow \; \frac{1 - n_{ss}^T - n_{ss}^N}{n_{ss}^N} &= \frac{1 - \kappa_{ss}^T - \kappa_{ss}^N}{\kappa_{ss}^N} \\
\Rightarrow \; \frac{1 - n_{ss}^T}{n_{ss}^N} - 1 &= \frac{1 - \kappa_{ss}^T}{\kappa_{ss}^N} - 1
\end{aligned}
$$

$$\Rightarrow \quad \frac{1-n_{ss}^T}{n_{ss}^N} = \frac{1-\kappa_{ss}^T}{\kappa_{ss}^N} \tag{49}$$

Next, rewrite (48) as:

$$\begin{aligned}
\frac{n_{ss}^N}{\kappa_{ss}^N} &= \frac{1-\alpha}{\alpha}\left(\frac{\chi}{1-\chi}\right)\frac{n_s^T}{\kappa_{ss}^T} \\
\Rightarrow \quad \frac{n_{ss}^N}{n_s^T} &= \frac{1-\alpha}{\alpha}\left(\frac{\chi}{1-\chi}\right)\frac{\kappa_{ss}^N}{\kappa_{ss}^T} && (50) \\
\Leftrightarrow \quad \frac{n_s^T}{n_{ss}^N} &= \frac{\alpha}{1-\alpha}\left(\frac{1-\chi}{\chi}\right)\frac{\kappa_{ss}^T}{\kappa_{ss}^N} && (51)
\end{aligned}$$

Dividing (49) by (51) yields:

$$\begin{aligned}
\frac{1-n_{ss}^T}{n_{ss}^T} &= \frac{1-\alpha}{\alpha}\left(\frac{\chi}{1-\chi}\right)\frac{1-\kappa_{ss}^T}{\kappa_{ss}^T} \\
\Rightarrow \quad \frac{1}{n_{ss}^T} - 1 &= \frac{\chi(1-\alpha)(1-\kappa_{ss}^T)}{\alpha(1-\chi)\kappa_{ss}^T} \\
\Rightarrow \quad \frac{1}{n_{ss}^T} &= \frac{\chi(1-\alpha)(1-\kappa_{ss}^T) + \alpha(1-\chi)\kappa_{ss}^T}{\alpha(1-\chi)\kappa_{ss}^T} \\
\Rightarrow \quad n_{ss}^T &= \frac{\alpha(1-\chi)\kappa_{ss}^T}{\chi(1-\alpha)(1-\kappa_{ss}^T) + \alpha(1-\chi)\kappa_{ss}^T}
\end{aligned}$$

Collecting terms in the numerator yields:

$$\begin{aligned}
\chi(1-\alpha)(1-\kappa_{ss}^T) + \alpha(1-\chi)\kappa_{ss}^T &= \left[\alpha(1-\chi) - \chi(1-\alpha)\right]\kappa_{ss}^T + \chi(1-\alpha) \\
&= (\alpha-\chi)\,\kappa_{ss}^T + \chi(1-\alpha) \\
\therefore \quad n_{ss}^T &= \frac{\alpha(1-\chi)\kappa_{ss}^T}{(\alpha-\chi)\,\kappa_{ss}^T + \chi(1-\alpha)} && (52)
\end{aligned}$$

Note that $\alpha < \chi$ is assumed in the firm's problem, but even if $\alpha > \chi$, this denominator remains positive under suitable parameter restrictions.

From (51) and (52), solve for n_{ss}^N:

$$\begin{aligned}
\frac{n_{ss}^T}{n_{ss}^N} &= \frac{\alpha}{1-\alpha}\left(\frac{1-\chi}{\chi}\right)\frac{\kappa_{ss}^T}{\kappa_{ss}^N} \\
\Rightarrow \quad n_{ss}^N &= \frac{1-\alpha}{\alpha}\left(\frac{\chi}{1-\chi}\right)\frac{\kappa_{ss}^N}{\kappa_{ss}^T} \times n_{ss}^T \\
&= \frac{1-\alpha}{\alpha}\left(\frac{\chi}{1-\chi}\right)\frac{\kappa_{ss}^N}{\kappa_{ss}^T} \times \frac{\alpha(1-\chi)\kappa_{ss}^T}{(\alpha-\chi)\,\kappa_{ss}^T + \chi(1-\alpha)} \\
\therefore \quad n_{ss}^N &= \frac{\chi(1-\alpha)\kappa_{ss}^N}{(\alpha-\chi)\,\kappa_{ss}^T + \chi(1-\alpha)} && (53)
\end{aligned}$$

Summing (52) and (53) yields:

$$
\begin{aligned}
n_{ss}^T + n_{ss}^N &= \frac{\alpha(1-\chi)\kappa_{ss}^T}{(\alpha-\chi)\,\kappa_{ss}^T + \chi(1-\alpha)} + \frac{\chi(1-\alpha)\kappa_{ss}^N}{(\alpha-\chi)\,\kappa_{ss}^T + \chi(1-\alpha)} \\
&= \frac{\alpha(1-\chi)(\kappa_{ss}^T - \kappa_{ss}^N)}{(\alpha-\chi)\,\kappa_{ss}^T + \chi(1-\alpha)} \qquad (54)
\end{aligned}
$$

Equations (52) and (53) also imply that if the tradable and non-tradable capital shares in services are equal ($\chi = \alpha$), then $n_{ss}^T = \kappa_{ss}^T$ and $n_{ss}^N = \kappa_{ss}^N$, matching Teignier (2018) for the case in which all service output is non-tradable.

Symmetrically, if one regards $\{n_{ss}^T, n_{ss}^N\}$ as parameters and $\{\kappa_{ss}^T, \kappa_{ss}^N\}$ as unknowns, the solution has an analogous form:

$$
\begin{aligned}
\kappa_{ss}^T &= \frac{\chi(1-\alpha)n_{ss}^T}{(\chi-\alpha)\,n_{ss}^T + \alpha(1-\chi)} \\
\kappa_{ss}^N &= \frac{\chi(1-\alpha)n_{ss}^N}{(\chi-\alpha)\,n_{ss}^T + \alpha(1-\chi)}
\end{aligned}
$$

Step 3) Non-Tradable Service Sector

From (34) and (35):

$$
\begin{aligned}
c_{st}^T &= \left(\frac{\theta}{1-\theta}\right)^{\sigma} \left(\frac{p_{st}^N}{p_{st}^T}\right)^{\sigma} c_{st}^N \\
c_{st}^N &= \left(\frac{1-\theta}{\theta}\right)^{\sigma} \left(\frac{p_{st}^T}{p_{st}^N}\right)^{\sigma} c_{st}^T
\end{aligned}
$$

Using $c_{st} = \left[\theta(c_{st}^T)^{\frac{\sigma-1}{\sigma}} + (1-\theta)(c_{st}^N)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$, one obtains:

$$
\begin{aligned}
c_{st} &= \left[\theta\left(\left(\frac{\theta}{1-\theta}\right)^{\sigma} \left(\frac{p_{st}^N}{p_{st}^T}\right)^{\sigma} c_{st}^N\right)^{\frac{\sigma-1}{\sigma}} + (1-\theta)(c_{st}^N)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \\
&= \left[\theta\left(\frac{\theta}{1-\theta}\right)^{\sigma-1} \left(\frac{p_{st}^N}{p_{st}^T}\right)^{\sigma-1} (c_{st}^N)^{\frac{\sigma-1}{\sigma}} + (1-\theta)(c_{st}^N)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \\
&= \left[(c_{st}^N)^{\frac{\sigma-1}{\sigma}}\left(1-\theta+\theta\left(\frac{\theta}{1-\theta}\right)^{\sigma-1} \left(\frac{p_{st}^N}{p_{st}^T}\right)^{\sigma-1}\right)\right]^{\frac{\sigma}{\sigma-1}} \\
&= \left[1-\theta+\theta\left(\frac{\theta}{1-\theta}\right)^{\sigma-1} \left(\frac{p_{st}^N}{p_{st}^T}\right)^{\sigma-1}\right]^{\frac{\sigma}{\sigma-1}} c_{st}^N
\end{aligned}
$$

$$
\rightarrow\ c_{st} = (F_t^N)^{\frac{\sigma}{\sigma-1}} c_{st}^N \quad \text{where } F_t^N \equiv 1-\theta+\theta\left(\frac{\theta}{1-\theta}\right)^{\sigma-1} \left(\frac{p_{st}^N}{p_{st}^T}\right)^{\sigma-1}
$$

Then, from (35): $c^N_{st} = \left(\frac{1-\theta}{p^N_{st}}\right)^\sigma \left[\frac{\omega_s}{\omega_m}\left(\frac{c_{mt}}{c_{st}+\bar{c}_s}\right)\right]^{\frac{\sigma}{\varepsilon}} c_{st}$, we derive:

$$\begin{aligned}
\left[\frac{\omega_s}{\omega_m}\left(\frac{c_{mt}}{c_{st}+\bar{c}_s}\right)\right]^{\frac{\sigma}{\varepsilon}} &= \left(\frac{p^N_{st}}{1-\theta}\right)^\sigma \frac{c^N_{st}}{c_{st}} \\
\Rightarrow \frac{\omega_s}{\omega_m}\left(\frac{c_{mt}}{c_{st}+\bar{c}_s}\right) &= \left(\frac{p^N_{st}}{1-\theta}\right)^\varepsilon \left[(F^N_t)^{\frac{-\sigma}{\sigma-1}}\right]^{\frac{\varepsilon}{\sigma}} \\
\Rightarrow \frac{\omega_s}{\omega_m} c_{mt} &= \left(\frac{p^N_{st}}{1-\theta}\right)^\varepsilon (F^N_t)^{\frac{\varepsilon}{1-\sigma}}\left(c_{st}+\bar{c}_s\right) \\
&= \left(\frac{p^N_{st}}{1-\theta}\right)^\varepsilon (F^N_t)^{\frac{\varepsilon}{1-\sigma}}\left((F^N_t)^{\frac{\sigma}{\sigma-1}} c^N_{st}+\bar{c}_s\right) \\
&= \left(\frac{p^N_{st}}{1-\theta}\right)^\varepsilon (F^N_t)^{\frac{\varepsilon-\sigma}{1-\sigma}} c^N_{st} + \left(\frac{p^N_{st}}{1-\theta}\right)^\varepsilon (F^N_t)^{\frac{\varepsilon}{1-\sigma}}\bar{c}_s
\end{aligned}$$

Using equation (44): $c^N_{st} = (\kappa^N_{st} k_t)^\alpha (A^N_{st} n^N_{st})^{1-\alpha}$,

$$\frac{\omega_s}{\omega_m} c_{mt} = \left(\frac{p^N_{st}}{1-\theta}\right)^\varepsilon (F^N_t)^{\frac{\varepsilon-\sigma}{1-\sigma}} (\kappa^N_{st} k_t)^\alpha (A^N_{st} n^N_{st})^{1-\alpha} + \left(\frac{p^N_{st}}{1-\theta}\right)^\varepsilon (F^N_t)^{\frac{\varepsilon}{1-\sigma}}\bar{c}_s$$

After switching to de-trended variables $\hat{c}_{mt} \equiv \frac{c_{mt}}{A_{mt}}$ and $\hat{k}_t \equiv \frac{k_t}{A_{mt}}$, one obtains an expression whose limit is:

$$\begin{aligned}
\left(\frac{1-\theta}{p^N_{st}}\right)^\varepsilon \frac{\omega_s}{\omega_m}\hat{c}_{mt} &= (F^N_t)^{\frac{\varepsilon-\sigma}{1-\sigma}} (\kappa^N_{st}\hat{k}_t)^\alpha (n^N_{st})^{1-\alpha} (A^N_{st})^{1-\alpha} A^{\alpha-1}_{mt} + (F^N_t)^{\frac{\varepsilon}{1-\sigma}}\bar{c}_s A^{-1}_{mt} \\
\Rightarrow (1-\theta)^\varepsilon (p^N_{st})^{1-\varepsilon}\frac{\omega_s}{\omega_m}\hat{c}_{mt} &= p^N_{st}(F^N_t)^{\frac{\varepsilon-\sigma}{1-\sigma}} (\kappa^N_{st}\hat{k}_t)^\alpha (n^N_{st})^{1-\alpha} (A^N_{st})^{1-\alpha} A^{\alpha-1}_{mt} + p^N_{st}(F^N_t)^{\frac{\varepsilon}{1-\sigma}}\bar{c}_s A^{-1}_{mt} \\
\lim_{t\to\infty}(1-\theta)^\varepsilon (p^N_{st})^{1-\varepsilon}\frac{\omega_s}{\omega_m}\hat{c}_{mt} &= \lim_{t\to\infty} p^N_{st}(F^N_t)^{\frac{\varepsilon-\sigma}{1-\sigma}} (\kappa^N_{st}\hat{k}_t)^\alpha (n^N_{st})^{1-\alpha} (A^N_{st})^{1-\alpha} A^{\alpha-1}_{mt} + \lim_{t\to\infty} p^N_{st}(F^N_t)^{\frac{\varepsilon}{1-\sigma}}\bar{c}_s A^{-1}_{mt}
\end{aligned} \tag{55}$$

We will return to this expression in later steps.

Step 4) Tradable Service Sector

Similarly, for tradable service goods, where $c_{st} = \left[\theta(c^T_{st})^{\frac{\sigma-1}{\sigma}} + (1-\theta)(c^N_{st})^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$, one can show:

$$\begin{aligned}
c_{st} &= \left[\theta(c^T_{st})^{\frac{\sigma-1}{\sigma}} + (1-\theta)\left[\left(\frac{1-\theta}{\theta}\right)^\sigma \left(\frac{p^T_{st}}{p^N_{st}}\right)^\sigma c^T_{st}\right]^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \\
&= c^T_{st}\left[\theta + (1-\theta)\left(\frac{1-\theta}{\theta}\right)^{\sigma-1}\left(\frac{p^T_{st}}{p^N_{st}}\right)^{\sigma-1}\right]^{\frac{\sigma}{\sigma-1}} \\
\to \ c_{st} &= (F^T_t)^{\frac{\sigma}{\sigma-1}} c^T_{st} \quad \text{where } F^T_t \equiv \theta + (1-\theta)\left(\frac{1-\theta}{\theta}\right)^{\sigma-1}\left(\frac{p^T_{st}}{p^N_{st}}\right)^{\sigma-1}
\end{aligned}$$

Using (34): $c_{st}^T = \left(\frac{\theta}{p_{st}^T}\right)^\sigma \left[\frac{\omega_s}{\omega_m}\left(\frac{c_{mt}}{c_{st}+\bar{c}_s}\right)\right]^{\frac{\sigma}{\varepsilon}} c_{st}$, similar manipulations lead to:

$$\frac{\omega_s}{\omega_m}c_{mt} = \left(\frac{p_{st}^T}{\theta}\right)^\varepsilon (F_t^T)^{\frac{\varepsilon-\sigma}{1-\sigma}} c_{st}^T + \left(\frac{p_{st}^T}{\theta}\right)^\varepsilon (F_t^T)^{\frac{\varepsilon}{1-\sigma}} \bar{c}_s$$

Using (43): $c_{st}^T + x_{st} = (\kappa_{st}^T k_t)^\chi (A_{st}^T n_{st}^T)^{1-\chi}$ and de-trended variables $\hat{c}_{mt}$, $\hat{k}_t$ and $\hat{x}_{st} \equiv \frac{x_{st}}{A_{mt}}$,

$$\begin{aligned}
&\frac{\omega_s}{\omega_m}\hat{c}_{mt} = \left(\frac{p_{st}^T}{\theta}\right)^\varepsilon (F_t^T)^{\frac{\varepsilon-\sigma}{1-\sigma}} (\kappa_{st}^T \hat{k}_t)^\chi (n_{st}^T)^{1-\chi} (A_{st}^T)^{1-\chi} A_{mt}^{-1+\chi} - \left(\frac{p_{st}^T}{\theta}\right)^\varepsilon (F_t^T)^{\frac{\varepsilon-\sigma}{1-\sigma}} \hat{x}_{st} + \left(\frac{p_{st}^T}{\theta}\right)^\varepsilon (F_t^T)^{\frac{\varepsilon}{1-\sigma}} \bar{c}_s A_{mt}^{-1} \\
&\Rightarrow\ \sigma^\varepsilon (p_{st}^T)^{1-\varepsilon} \frac{\omega_s}{\omega_m}\hat{c}_{mt} = p_{st}^T (F_t^T)^{\frac{\varepsilon-\sigma}{1-\sigma}} (\kappa_{st}^T \hat{k}_t)^\chi (n_{st}^T)^{1-\chi} (A_{st}^T)^{1-\chi} A_{mt}^{-1+\chi} - p_{st}^T (F_t^T)^{\frac{\varepsilon-\sigma}{1-\sigma}} \hat{x}_{st} + p_{st}^T (F_t^T)^{\frac{\varepsilon}{1-\sigma}} \bar{c}_s A_{mt}^{-1} \\
&\Rightarrow\ \lim_{t\to\infty} \sigma^\varepsilon (p_{st}^T)^{1-\varepsilon} \frac{\omega_s}{\omega_m}\hat{c}_{mt} = \lim_{t\to\infty} \Big[p_{st}^T (F_t^T)^{\frac{\varepsilon-\sigma}{1-\sigma}} (\kappa_{st}^T \hat{k}_t)^\chi (n_{st}^T)^{1-\chi} (A_{st}^T)^{1-\chi} A_{mt}^{-1+\chi} - p_{st}^T (F_t^T)^{\frac{\varepsilon-\sigma}{1-\sigma}} \hat{x}_{st} \\
&\qquad\qquad + p_{st}^T (F_t^T)^{\frac{\varepsilon}{1-\sigma}} \bar{c}_s A_{mt}^{-1} \Big]
\end{aligned} \tag{56}$$

Step 5) Net Exports of Manufacturing Sector, $\hat{x}_m^{ss}$

We have from the agricultural and non-tradable service sectors:

$$\begin{aligned}
\lim_{t\to\infty}\left(\frac{\omega_a}{\omega_m} p_{at}^{1-\varepsilon} \hat{c}_{mt}\right) &= \lim_{t\to\infty} \hat{x}_{mt} + \lim_{t\to\infty} p_{st}^T \hat{x}_{st} \\
\lim_{t\to\infty} (1-\theta)^\varepsilon (p_{st}^N)^{1-\varepsilon} \frac{\omega_s}{\omega_m}\hat{c}_{mt} &= \lim_{t\to\infty} p_{st}^N (F_t^N)^{\frac{\varepsilon-\sigma}{1-\sigma}} (\kappa_{st}^N \hat{k}_t)^\alpha (n_{st}^N)^{1-\alpha} (A_{st}^N)^{1-\alpha} A_{mt}^{\alpha-1} + \lim_{t\to\infty} p_{st}^N (F_t^N)^{\frac{\varepsilon}{1-\sigma}} \bar{c}_s A_{mt}^{-1}
\end{aligned}$$

By dividing two equations, we have:

$$\lim_{t\to\infty} \frac{\omega_a}{(1-\theta)^\varepsilon \omega_s} \left(\frac{p_{at}}{p_{st}^N}\right)^{1-\varepsilon} = \frac{\lim_{t\to\infty} \hat{x}_{mt} + \lim_{t\to\infty} p_{st}^T \hat{x}_{st}}{\lim_{t\to\infty} p_{st}^N (F_t^N)^{\frac{\varepsilon-\sigma}{1-\sigma}} (\kappa_{st}^N \hat{k}_t)^\alpha (n_{st}^N)^{1-\alpha} (A_{st}^N)^{1-\alpha} A_{mt}^{\alpha-1} + \lim_{t\to\infty} p_{st}^N (F_t^N)^{\frac{\varepsilon}{1-\sigma}} \bar{c}_s A_{mt}^{-1}}$$

Dividing the former by the latter and applying the condition $\gamma_p^a < 0 < \gamma_s^N < \gamma_m$ (which implies $\lim_{t\to\infty} \frac{p_{at}}{p_{st}^N} = 0$ under growth assumptions), we conclude the numerator on the right side must be zero:

$$\lim_{t\to\infty} \hat{x}_{mt} = -\lim_{t\to\infty} p_{st}^T \hat{x}_{st}. \tag{57}$$

Furthermore, the denominator must be non-zero in the limit. Given F_t^N and the condition $\gamma_p^s < (1-\alpha)(\gamma_m - \gamma_s^N)$, then $\lim_{t\to\infty} p_{st}^N (F_t^N)^{\frac{\varepsilon}{1-\sigma}} \bar{c}_s A_{mt}^{-1} = 0$. Thus, following equation must hold:

$$\lim_{t\to\infty} p_{st}^N (F_t^N)^{\frac{\varepsilon-\sigma}{1-\sigma}} (\kappa_{st}^N \hat{k}_t)^\alpha (n_{st}^N)^{1-\alpha} (A_{st}^N)^{1-\alpha} A_{mt}^{\alpha-1} \neq 0,$$

which implies that the input share of capital and labor must not be zero in the steady state, i.e., $\kappa_{ss}^N \neq 0$ and $n_{ss}^N \neq 0$.

Step 6) Net Exports of Service Sector, $\lim_{t\to\infty} p_{st}^T \hat{x}_{st}$

From (56) and (55), dividing one by the other yields:

$$\lim_{t\to\infty}\left(\frac{\theta}{1-\theta}\right)^{\varepsilon}\left(\frac{p_{st}^T}{p_{st}^N}\right)^{1-\varepsilon} = \frac{\lim_{t\to\infty} p_{st}^T(\kappa_{st}^T\hat{k}_t)^{\chi}(n_{st}^T)^{1-\chi}(\frac{A_{st}^T}{A_{mt}})^{1-\chi} - \lim_{t\to\infty} p_{st}^T\hat{x}_{st} + \lim_{t\to\infty} p_{st}^T(F_t^T)^{\frac{\sigma}{1-\sigma}}\bar{c}_s A_{mt}^{-1}}{\lim_{t\to\infty} p_{st}^N(\kappa_{st}^N\hat{k}_t)^{\alpha}(n_{st}^N)^{1-\alpha}(A_{st}^N)^{1-\alpha}A_{mt}^{\alpha-1} + \lim_{t\to\infty} p_{st}^N(F_t^N)^{\frac{\sigma}{1-\sigma}}\bar{c}_s A_{mt}^{-1}}$$

Under the condition $\gamma_p^s < (1-\alpha)(\gamma_m - \gamma_s^N)$, it implies $\lim_{t\to\infty}\left(\frac{\theta}{1-\theta}\right)^{\varepsilon}\left(\frac{p_{st}^T}{p_{st}^N}\right)^{\varepsilon} = 0$ and the numerator of the right-side equation must be zero in the limit:

$$\lim_{t\to\infty} p_{st}^T\hat{x}_{st} = \lim_{t\to\infty} p_{st}^T(\kappa_{st}^T\hat{k}_t)^{\chi}(n_{st}^T)^{1-\chi}(A_{st}^T)^{1-\chi}A_{mt}^{-1+\chi} + \lim_{t\to\infty} p_{st}^T(F_t^T)^{\frac{\sigma}{1-\sigma}}\bar{c}_s A_{mt}^{-1}$$

The term $\lim_{t\to\infty} p_{st}^T(F_t^T)^{\frac{\sigma}{1-\sigma}}\bar{c}_s A_{mt}^{-1}$ on the right-hand side will converge to zero under those same growth assumptions. Thus, we finalize

$$\lim_{t\to\infty} p_{st}^T\hat{x}_{st} = \lim_{t\to\infty} p_{st}^T(\kappa_{st}^T\hat{k}_t)^{\chi}(n_{st}^T)^{1-\chi}(A_{st}^T)^{1-\chi}A_{mt}^{-1+\chi} \tag{58}$$

Step 7) Back to $\hat{x}_m^{ss}$

Recall the equation (57). Substituting the equation (40) into the equation (58) will obtain:

$$\begin{aligned}
\lim_{t\to\infty}\hat{x}_{mt} &= -\lim_{t\to\infty} p_{st}^T\hat{x}_{st}\\
&= -\lim_{t\to\infty} p_{st}^T(\kappa_{st}^T\hat{k}_t)^{\chi}(n_{st}^T)^{1-\chi}(A_{st}^T)^{1-\chi}A_{mt}^{-1+\chi}\\
&= -\lim_{t\to\infty}\left[\left(\frac{\alpha}{\chi}\right)^{\alpha}\left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha}(\kappa_{st}^T k_t)^{\alpha-\chi}(n_{st}^T)^{\chi-\alpha}A_{mt}^{1-\alpha}(A_{st}^T)^{\chi-1}\right](\kappa_{st}^T\hat{k}_t)^{\chi}(n_{st}^T)^{1-\chi}(A_{st}^T)^{1-\chi}A_{mt}^{-1+\chi}\\
&= -\lim_{t\to\infty}\left[\left(\frac{\alpha}{\chi}\right)^{\alpha}\left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha}(\kappa_{st}^T\hat{k}_t)^{\alpha-\chi}(n_{st}^T)^{\chi-\alpha}A_{mt}^{1-\chi}(A_{st}^T)^{\chi-1}\right](\kappa_{st}^T\hat{k}_t)^{\chi}(n_{st}^T)^{1-\chi}(A_{st}^T)^{1-\chi}A_{mt}^{-1+\chi}\\
&= -\lim_{t\to\infty}\left(\frac{\alpha}{\chi}\right)^{\alpha}\left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha}(\kappa_{st}^T\hat{k}_t)^{\alpha}(n_{st}^T)^{1-\alpha}
\end{aligned}$$

Therefore,

$$\therefore \lim_{t\to\infty}\hat{x}_{mt} = -\lim_{t\to\infty} p_{st}^T\hat{x}_{st} = -\lim_{t\to\infty}\left(\frac{\alpha}{\chi}\right)^{\alpha}\left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha}(\kappa_{st}^T\hat{k}_t)^{\alpha}(n_{st}^T)^{1-\alpha}$$

Step 8) $\lim_{t\to\infty}\hat{x}_{st}$ and $\lim_{t\to\infty}\hat{c}_{st}^T$

Since $\lim_{t\to\infty} p_{st}^T\hat{x}_{st} = \lim_{t\to\infty} p_{st}^T(\kappa_{st}^T\hat{k}_t)^{\chi}(n_{st}^T)^{1-\chi}(A_{st}^T)^{1-\chi}A_{mt}^{-1+\chi}$ and $p_{st}^T > 0$, we have

$$\lim_{t\to\infty}\hat{x}_{st} = \lim_{t\to\infty}(\kappa_{st}^T\hat{k}_t)^{\chi}(n_{st}^T)^{1-\chi}(A_{st}^T)^{1-\chi}A_{mt}^{-1+\chi}$$

Then, from equation (43): $c_{st}^T + x_{st} = (\kappa_{st}^T k_t)^{\chi}(A_{st}^T n_{st}^T)^{1-\chi}$,

$$\begin{aligned}
\lim_{t\to\infty}\hat{c}_{st}^T &= \lim_{t\to\infty}(\kappa_{st}^T\hat{k}_t)^{\chi}(A_{st}^T n_{st}^T)^{1-\chi}A_{mt}^{-1+\chi} - \lim_{t\to\infty}\hat{x}_{st}\\
&= 0
\end{aligned}$$

Step 9) $\hat{c}_m^{ss}$

Finally, consider $\hat{c}_m^{ss} \equiv \lim_{t\to\infty} \hat{c}_{mt}$. From the tradable service sector we had:

$$\frac{\omega_s}{\omega_m}\hat{c}_{mt} = \left(\frac{p_{st}^T}{\theta}\right)^{\varepsilon} (F_t^T)^{\frac{\varepsilon-\sigma}{1-\sigma}} (\kappa_{st}^T \hat{k}_t)^{\chi} (n_{st}^T)^{1-\chi} (A_{st}^T)^{1-\chi} A_{mt}^{-1+\chi} - \left(\frac{p_{st}^T}{\theta}\right)^{\varepsilon} (F_t^T)^{\frac{\varepsilon-\sigma}{1-\sigma}} \hat{x}_{st} + \left(\frac{p_{st}^T}{\theta}\right)^{\varepsilon} (F_t^T)^{\frac{\varepsilon}{1-\sigma}} \bar{c}_s A_{mt}^{-1}$$

In the limit, the first terms vanish (under growth-rate assumptions discussed earlier), leaving

$$\lim_{t\to\infty} \frac{\omega_s}{\omega_m}\hat{c}_{mt} = \lim_{t\to\infty} \left(\frac{p_{st}^T}{\theta}\right)^{\varepsilon} (F_t^T)^{\frac{\varepsilon}{1-\sigma}} \bar{c}_s A_{mt}^{-1}.$$

Since $\varepsilon(1-\alpha)(\gamma_m - \gamma_s^N) - \gamma_m < 0$ for $0 < \varepsilon < 1$, the left term also goes to zero, implying

$$\hat{c}_m^{ss} \equiv \lim_{t\to\infty} \hat{c}_{mt} = 0$$

Step 10) $\hat{k}^{ss}$

Consider the law of motion k_{t+1} from (31) where $A_{mt+1} = (1+\gamma_m)A_{mt}$:

$$\begin{aligned} k_{t+1} &= \Big(\left(1 - \kappa_{at} - \kappa_{st}^T - \kappa_{st}^N\right) k_t \Big)^{\alpha} \Big(A_{mt}\left(1 - n_{at} - n_{st}^T - n_{st}^N\right) \Big)^{1-\alpha} + (1-\delta-\nu)\, k_t - c_{mt} - x_{mt} \\ (1+\gamma_m)\hat{k}_{t+1} &= \Big(\left(1 - \kappa_{at} - \kappa_{st}^T - \kappa_{st}^N\right) \hat{k}_t \Big)^{\alpha} \Big(1 - n_{at} - n_{st}^T - n_{st}^N \Big)^{1-\alpha} + (1-\delta-\nu)\, \hat{k}_t - \hat{c}_{mt} - \hat{x}_{mt} \end{aligned}$$

Since $\kappa_a^{ss} = n_a^{ss} = 0$,

$$\begin{aligned} (1+\gamma_m)\hat{k}^{ss} &= \left(1 - \kappa_{ss}^T - \kappa_{ss}^N\right)^{\alpha} \left(1 - n_{ss}^T - n_{ss}^N\right)^{1-\alpha} (\hat{k}^{ss})^{\alpha} + (1-\delta-\nu)\, \hat{k}^{ss} - \hat{c}_m^{ss} - \hat{x}_m^{ss} \\ \Rightarrow\ (\gamma_m + \delta + \nu)\, \hat{k}_{ss} &= \left(1 - \kappa_{ss}^T - \kappa_{ss}^N\right)^{\alpha} \left(1 - n_{ss}^T - n_{ss}^N\right)^{1-\alpha} (\hat{k}^{ss})^{\alpha} - \hat{c}_m^{ss} - \hat{x}_m^{ss} \end{aligned}$$

Because $\hat{c}_m^{ss} = 0$ and $\hat{x}_m^{ss}$ is given by the negative of $\lim_{t\to\infty} p_{st}^T \hat{x}_{st}$ (which we derived), one obtains:

$$\begin{aligned} (\gamma_m + \delta + \nu)\, \hat{k}_{ss} &= \left[\left(1 - \kappa_{ss}^T - \kappa_{ss}^N\right)^{\alpha} \left(1 - n_{ss}^T - n_{ss}^N\right)^{1-\alpha} + \left(\frac{\alpha}{\chi}\right)^{\alpha} \left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha} (\kappa_{ss}^T)^{\alpha} (n_{ss}^T)^{1-\alpha} \right] (\hat{k}^{ss})^{\alpha} \\ \Rightarrow\ (\hat{k}^{ss})^{1-\alpha} &= \frac{1}{\gamma_m + \delta + \nu} \left[\left(1 - \kappa_{ss}^T - \kappa_{ss}^N\right)^{\alpha} \left(1 - n_{ss}^T - n_{ss}^N\right)^{1-\alpha} + \left(\frac{\alpha}{\chi}\right)^{\alpha} \left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha} (\kappa_{ss}^T)^{\alpha} (n_{ss}^T)^{1-\alpha} \right] \end{aligned}$$

Hence,

$$\hat{k}^{ss} = \left(\frac{1}{\gamma_m + \delta + \nu}\right)^{\frac{1}{1-\alpha}} \left[\left(1 - \kappa_{ss}^T - \kappa_{ss}^N\right)^{\alpha} \left(1 - n_{ss}^T - n_{ss}^N\right)^{1-\alpha} + \left(\frac{\alpha}{\chi}\right)^{\alpha} \left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha} (\kappa_{ss}^T)^{\alpha} (n_{ss}^T)^{1-\alpha} \right]^{\frac{1}{1-\alpha}}$$

$$\begin{aligned} \text{where } n_{ss}^T &= \frac{\alpha(1-\chi)\kappa_{ss}^T}{(\alpha-\chi)\,\kappa_{ss}^T + \chi(1-\alpha)}, \quad n_{ss}^N = \frac{\chi(1-\alpha)\kappa_{ss}^N}{(\alpha-\chi)\,\kappa_{ss}^T + \chi(1-\alpha)} \\ \kappa_{ss}^T &= \frac{\chi(1-\alpha)n_{ss}^T}{(\chi-\alpha)\,n_{ss}^T + \alpha(1-\chi)}, \quad \kappa_{ss}^N = \frac{\chi(1-\alpha)n_{ss}^N}{(\chi-\alpha)\,n_{ss}^T + \alpha(1-\chi)} \end{aligned}$$

Finally, from the equation (32) where $\kappa_m^{ss} = 1 - \kappa_{ss}^T - \kappa_{ss}^N$ and $n_m^{ss} = 1 - n_{ss}^T - n_{ss}^N$:

$$\begin{aligned}
1+\rho &= 1-\delta-\nu+R^{ss} \\
\Rightarrow\ \rho+\delta+\nu &= \alpha(1-\kappa_a^{ss}-\kappa_s^{ss})^{\alpha-1}(\hat{k}^{ss})^{\alpha-1}(1-n_a^{ss}-n_s^{ss})^{1-\alpha} \\
&= \alpha(\kappa_m^{ss})^{\alpha-1}(n_m^{ss})^{\alpha-1}(\hat{k}^{ss})^{\alpha-1} \\
\therefore\ \hat{k}^{ss} &= \left(\frac{\alpha}{\rho+\delta+\nu}\right)^{\frac{1}{1-\alpha}}\frac{n_m^{ss}}{\kappa_m^{ss}},
\end{aligned}$$

leading to

$$\begin{aligned}
\frac{\alpha}{\rho+\delta+\nu} &= \frac{1}{\gamma_m+\delta+\nu}\left[\left(1-\kappa_{ss}^T-\kappa_{ss}^N\right)^{\alpha}\left(1-n_{ss}^T-n_{ss}^N\right)^{1-\alpha}+\left(\frac{\alpha}{\chi}\right)^{\alpha}\left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha}(\kappa_{ss}^T)^{\alpha}(n_{ss}^T)^{1-\alpha}\right] \\
\Rightarrow\ \frac{\alpha(\gamma_m+\delta+\nu)}{\rho+\delta+\nu} &= \left(1-\kappa_{ss}^T-\kappa_{ss}^N\right)^{\alpha}\left(1-n_{ss}^T-n_{ss}^N\right)^{1-\alpha}+\left(\frac{\alpha}{\chi}\right)^{\alpha}\left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha}(\kappa_{ss}^T)^{\alpha}(n_{ss}^T)^{1-\alpha} \\
&= \left(\kappa_m^{ss}\right)^{\alpha}\left(n_m^{ss}\right)^{1-\alpha}+\left(\frac{\alpha}{\chi}\right)^{\alpha}\left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha}(\kappa_{ss}^T)^{\alpha}(n_{ss}^T)^{1-\alpha} \\
\Rightarrow\ \left(\kappa_m^{ss}\right)^{\alpha}\left(n_m^{ss}\right)^{1-\alpha} &= \frac{\alpha(\gamma_m+\delta+\nu)}{\rho+\delta+\nu}-\left(\frac{\alpha}{\chi}\right)^{\alpha}\left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha}(\kappa_{ss}^T)^{\alpha}(n_{ss}^T)^{1-\alpha}
\end{aligned}$$

Consequently, if $\kappa_m^{ss} = n_m^{ss}$,

$$\kappa_m^{ss} = n_m^{ss} = \frac{\alpha(\gamma_m+\delta+\nu)}{\rho+\delta+\nu}-\left(\frac{\alpha}{\chi}\right)^{\alpha}\left(\frac{1-\alpha}{1-\chi}\right)^{1-\alpha}(\kappa_{ss}^T)^{\alpha}(n_{ss}^T)^{1-\alpha}$$

A.1.2 Open Economy with Non-tradable Service

In this scenario, there is only one type of service good, i.e., $\theta = 0$. Consequently, the optimization problems for households and firms, as well as the market clearing conditions, can be simplified.

Household

The FOCs can be summarized as follows:

$$\begin{aligned}
\frac{1}{C_t} C_t^{\frac{1}{\varepsilon}} \omega_a^{\frac{1}{\varepsilon}} \left(c_{at} - \bar{c}_a\right)^{-\frac{1}{\varepsilon}} &= \lambda_t p_{at} \\
\frac{1}{C_t} C_t^{\frac{1}{\varepsilon}} \omega_m^{\frac{1}{\varepsilon}} \left(c_{mt}\right)^{-\frac{1}{\varepsilon}} &= \lambda_t \\
\frac{1}{C_t} C_t^{\frac{1}{\varepsilon}} \omega_s^{\frac{1}{\varepsilon}} \left(c_{st} + \bar{c}_s\right)^{-\frac{1}{\varepsilon}} &= \lambda_t p_{st}
\end{aligned}$$

Firm

Since $\theta = 0$, there is only one production process for the service sector, meaning that χ is not relevant in this scenario. Consequently, by applying the FOCs for each sector with respect to labor and capital, we can derive the following relationships:

$$\frac{R_t}{w_t} = \frac{\eta}{\beta}\left(\frac{n_{at}}{k_{at}}\right) = \frac{\alpha}{1-\alpha}\left(\frac{n_{mt}}{k_{mt}}\right) = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{n_{st}^N}{k_{st}^N}\right)$$

Using the optimality conditions and the FOCs together, the following price equations can be derived:

$$\begin{aligned}
p_{at} &= \left(\frac{\alpha}{\eta}\right)^{\alpha}\left(\frac{1-\alpha}{\beta}\right)^{1-\alpha} \left(k_{at}\right)^{\alpha-\eta} n_{at}^{1-\alpha-\beta} A_{mt}^{1-\alpha} A_{at}^{-\beta} \left(\frac{N_t}{L}\right)^{1-\eta-\beta} \\
p_{st} &= \left(\frac{A_{mt}}{A_{st}^N}\right)^{1-\alpha}
\end{aligned}$$

Market Clearing Conditions

Since only agricultural and manufacturing goods are tradable, the market clearing conditions can be simplified as follows:

$$\begin{aligned}
n_{at} + n_{mt} + n_{st} &= 1 \\
\kappa_{at} + \kappa_{mt} + \kappa_{st} &= 1 \quad \text{where } \kappa_{jt} \equiv \frac{k_{jt}}{k_t} \\
c_{at} + x_{at} &= y_{at}
\end{aligned}$$

$$c_{mt} + x_{mt} + k_{t+1} - (1-\delta-\nu)\, k_t = y_{mt}$$
$$c_{st} = y_{st}$$
$$p_{at}x_{at} + x_{mt} = 0$$

Equilibrium

Given the initial capital stock k_0 and exogenous variables $\{A_{at}, A_{mt}, A_{st}^T, A_{st}^N, N_t, p_{at}\}_{t=0}^{\infty}$, the equilibrium conditions can be summarized by followings: a set of prices $\{p_{st}, w_t, R_t, q_t\}_{t=0}^{\infty}$, sectoral shares of employment and capital $\{n_{at}, n_{mt}, n_{st}, \kappa_{at}, \kappa_{mt}, \kappa_{st}\}_{t=0}^{\infty}$ and quantities $\{k_t, y_{at}, y_{mt}, y_{st}, c_{at}, c_{mt}, c_{st}, x_{at}, x_{mt}\}_{t=0}^{\infty}$ which satisfy: (i) Household's optimization, (ii) Firm's optimization and (iii) Market clearing conditions. The equilibrium conditions can be summarized by the following system of 10 equations with the 10 unknowns $\{p_{st}, k_t, c_{at}, c_{mt}, c_{st}, n_{at}, n_{st}, \kappa_{at}, \kappa_{st}, x_{mt}\}_{t=0}^{\infty}$:

$$\begin{aligned}
k_{t+1} &= \Big((1-\kappa_{at}-\kappa_{st})\, k_t\Big)^{\alpha}\Big(A_{mt}\,(1-n_{at}-n_{st})\Big)^{1-\alpha} \\
&\quad + (1-\delta-\nu)\, k_t - c_{mt} - x_{mt} && (59)\\
(1+\rho)\,\frac{P_{t+1}C_{t+1}}{P_tC_t} &= 1-\delta-\nu+R_{t+1} && (60)\\
c_{at} - \bar{c}_a &= \frac{\omega_a}{\omega_m} p_{at}^{-\varepsilon} c_{mt} && (61)\\
c_{st} + \bar{c}_s &= \frac{\omega_s}{\omega_m} p_{st}^{-\varepsilon} c_{mt} && (62)\\
\frac{1-n_{at}-n_{st}}{1-\kappa_{at}-\kappa_{st}} &= \frac{\eta}{\beta}\left(\frac{1-\alpha}{\alpha}\right)\frac{n_{at}}{\kappa_{at}} && (63)\\
\frac{1-n_{at}-n_{st}}{1-\kappa_{at}-\kappa_{st}} &= \frac{n_{st}}{\kappa_{st}} && (64)\\
p_{at} &= \left(\frac{\alpha}{\eta}\right)^{\alpha}\left(\frac{1-\alpha}{\beta}\right)^{1-\alpha}(\kappa_{at}k_t)^{\alpha-\eta}\, n_{at}^{1-\alpha-\beta} A_{mt}^{1-\alpha} A_{at}^{-\beta}\left(\frac{N_t}{L}\right)^{1-\eta-\beta} && (65)\\
p_{st} &= \left(\frac{A_{mt}}{A_{st}}\right)^{1-\alpha} && (66)\\
c_{at} - \frac{x_{mt}}{p_{at}} &= (\kappa_{at}k_t)^{\eta}\,(A_{at}n_{at})^{\beta}\left(\frac{L}{N_t}\right)^{1-\eta-\beta} && (67)\\
c_{st} &= (\kappa_{st}k_t)^{\alpha}\,(A_{st}n_{st})^{1-\alpha} && (68)
\end{aligned}$$

The equations (59) and (60) denote the law of motion for the capital and the Euler equation respectively. The equations (61) and (62) are derived from a household's FOCs where equations (63) and (64) are from a firm's FOCs. The equations (65) and (66) are from the competitive price of capital and labor. Finally, the equations (67) and (68) are from the market clearing condition.

Steady States - Derivation

Similar to the previous appendix, to analyze how input shares change in the steady state, it is helpful to rewrite the variables in detrended terms:

$$\hat{c}_{mt} \equiv \frac{c_{mt}}{A_{mt}}, \quad \hat{k}_t \equiv \frac{k_t}{A_{mt}}, \quad \hat{x}_{mt} \equiv \frac{x_{mt}}{A_{mt}}, \quad \psi_t \equiv p_{at} A_{at}^{\beta} A_{mt}^{\eta-1} \frac{L}{N_t}^{1-\eta-\beta}$$

STEP 1) Agricultural shares: κ_a^{ss} and n_a^{ss}

Using equation (64) and (67), we can simplify the equation as follows:

$$\begin{aligned}
\frac{\omega_s}{\omega_m} p_{at}^{1-\varepsilon} \hat{c}_{mt} &= p_{at} \left(\kappa_{at} \hat{k}_t\right)^{\eta} n_{at}^{\beta} A_{at}^{\beta} A_{mt}^{\eta-1} \frac{L}{N_t}^{1-\eta-\beta} - \hat{x}_{mt} - p_{at} A_{mt}^{-1} \bar{c}_a \\
&= \psi_t \left(\kappa_{at} \hat{k}_t\right)^{\eta} n_{at}^{\beta} - \hat{x}_{mt} - p_{at} A_{mt}^{-1} \bar{c}_a
\end{aligned}$$

Given the condition (5) holds, then $\lim_{t\to\infty} \psi_t = 0$. If one expands the equation of ψ_t:

$$\begin{aligned}
\psi_t &\equiv p_{at} A_{at}^{\beta} A_{mt}^{\eta-1} \frac{L}{N_t}^{1-\eta-\beta} \\
&= \left(\frac{\alpha}{\eta}\right)^{\alpha} \left(\frac{1-\alpha}{\beta}\right)^{1-\alpha} (\kappa_{at} k_t)^{\alpha-\eta} n_{at}^{1-\alpha-\beta} A_{mt}^{1-\alpha} A_{at}^{-\beta} \left(\frac{N_t}{L}\right)^{1-\eta-\beta} \times A_{at}^{\beta} A_{mt}^{\eta-1} \frac{L}{N_t}^{1-\eta-\beta} \\
&= \left(\frac{\alpha}{\eta}\right)^{\eta} \left(\frac{1-\alpha}{\beta}\right)^{1-\eta} \hat{k}_t^{\alpha-\eta} \left(\frac{1-\kappa_{at}-\kappa_{st}}{1-n_{at}-n_{st}}\right)^{\alpha-\eta} n_{at}^{1-\eta-\beta}
\end{aligned}$$

Since $\lim_{t\to\infty} \psi_t = 0$ with under the condition (5), we can conclude that $\lim_{t\to\infty} n_{at} = 0$, which implies $\lim_{t\to\infty} \kappa_{at} = 0$ as well. Thus,

$$\kappa_a^{ss} = n_a^{ss} = 0$$

STEP 2) Service shares: κ_s^{ss} and n_s^{ss}

Given $\kappa_a^{ss} = n_a^{ss} = 0$, using the equation (64), we can solve for κ_s^{ss} and n_s^{ss} as below:

$$\begin{aligned}
\frac{1-n_{at}-n_{st}}{1-\kappa_{at}-\kappa_{st}} &= \frac{n_{st}}{\kappa_{st}} \\
\Rightarrow \frac{1-n_{st}}{1-\kappa_{st}} &= \frac{n_{st}}{\kappa_{st}} \\
\therefore\ \kappa_s^{ss} &= n_s^{ss}
\end{aligned}$$

STEP 3) $\hat{c}_m^{ss}$

Given $\gamma_m > \gamma_s$, using the equations (62), (66) and (68) together, one can write:

$$c_{st} + \bar{c}_s = \frac{\omega_s}{\omega_m} p_{st}^{-\varepsilon} c_{mt}$$

$$
\begin{aligned}
\Rightarrow\ \frac{\omega_s}{\omega_m} p_{st}^{1-\varepsilon} \hat{c}_{mt} &= p_{st} A_{mt}^{-1} \left(c_{st} + \bar{c}_s \right) \\
&= \left(\kappa_{st} \hat{k}_t \right)^{\alpha} n_{st}^{1-\alpha} + p_{st} A_{mt}^{-1} \bar{c}_s \\
\Rightarrow\ \lim_{t\to\infty} \frac{\omega_s}{\omega_m} p_{st}^{1-\varepsilon} \hat{c}_{mt} &= \lim_{t\to\infty} \left(\kappa_{st} \hat{k}_t \right)^{\alpha} n_{st}^{1-\alpha} + 0 \\
\Rightarrow\ \lim_{t\to\infty} \frac{\omega_s}{\omega_m} \hat{c}_{mt} &= \lim_{t\to\infty} \left(\kappa_{st} \hat{k}_t \right)^{\alpha} n_{st}^{1-\alpha} \left(\frac{1}{p_{st}} \right)^{1-\varepsilon} \\
&= 0 \\
\therefore\ \hat{c}_m^{ss} &= 0
\end{aligned}
$$

STEP 4) $\hat{x}_m^{ss}$

In the open economy model, we need to also find the steady state value of $\hat{x}_{mt}$. Using the equations (61), (65) and (67), one can write:

$$
\begin{aligned}
c_{at} - \bar{c}_a &= \frac{\omega_a}{\omega_m} p_{at}^{-\varepsilon} c_{mt} \\
\Rightarrow\ \frac{\omega_a}{\omega_m} p_{at}^{1-\varepsilon} \hat{c}_{mt} &= p_{at} A_{mt}^{-1} \left(c_{at} - \bar{c}_a \right) \\
&= p_{at} A_{mt}^{-1+\eta} A_{at}^{\beta} \left(\frac{L}{N_t} \right)^{1-\eta-\beta} \left(\kappa_{at} \hat{k}_t \right)^{\eta} n_{at}^{\beta} + \hat{x}_{mt} - p_{at} A_{mt}^{-1} \bar{c}_a \\
&= \psi_t \left(\kappa_{at} \hat{k}_t \right)^{\eta} n_{at}^{\beta} + \hat{x}_{mt} - p_{at} A_{mt}^{-1} \bar{c}_a \\
\Rightarrow\ \hat{x}_{mt} &= \frac{\omega_a}{\omega_m} p_{at}^{1-\varepsilon} \hat{c}_{mt} - \psi_t \left(\kappa_{at} \hat{k}_t \right)^{\eta} n_{at}^{\beta} + p_{at} A_{mt}^{-1} \bar{c}_a
\end{aligned}
$$

First, we derive $\lim_{t\to\infty} \hat{c}_{mt} = \hat{c}_m^{ss} = 0$. Moreover, if the condition (5) holds, $\lim_{t\to\infty} p_{at} A_{mt}^{-1} \bar{c}_a = 0$. Thus, we can conclude $\lim_{t\to\infty} \hat{x}_{mt} = 0$, i.e.,

$$
\hat{x}_m^{ss} = 0
$$

STEP 5) $\hat{k}^{ss}$

Next, using the equation (60), we can derive the steady state level of detrended capital from the equation (60) as follows:

$$
\begin{aligned}
(1+\rho) &= 1 - \delta - \nu + R^{ss} \\
\Rightarrow\ \rho + \delta + \nu &= \alpha (1 - \kappa_a^{ss} - \kappa_s^{ss})^{\alpha-1} (\hat{k}^{ss})^{\alpha-1} (1 - n_a^{ss} - n_s^{ss})^{1-\alpha} \\
&= \alpha (\hat{k}^{ss})^{\alpha-1} \\
\therefore\ \hat{k}^{ss} &= \left(\frac{\alpha}{\rho + \delta + \nu} \right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

STEP 6) κ_m^{ss} **&** n_m^{ss}

Finally, given $\kappa_a^{ss} = n_a^{ss} = 0$, $\kappa_s^{ss} = n_s^{ss}$ and $\hat{c}_m^{ss} = \hat{x}_m^{ss} = 0$, using the equation (59) one can derive the steady state level of capital and labor share in the service sector as follows:

$$
\begin{aligned}
k_{t+1} &= \left((1-\kappa_{at}-\kappa_{st})\,k_t\right)^\alpha \left(A_{mt}\,(1-n_{at}-n_{st})\right)^{1-\alpha} + (1-\delta-\nu)\,k_t - c_{mt} - x_{mt} \\
\Rightarrow\ \frac{k_{t+1}}{A_{mt}} &= \left((1-\kappa_{at}-\kappa_{st})\,\hat{k}_t\right)^\alpha (1-n_{at}-n_{st})^{1-\alpha} + (1-\delta-\nu)\,\hat{k}_t - \hat{c}_{mt} - \hat{x}_{mt} \\
\Rightarrow\ (1+\gamma_m)\hat{k}_{t+1} &= (1-\kappa_{at}-\kappa_{st})^\alpha \left(\hat{k}_t\right)^\alpha (1-n_{at}-n_{st})^{1-\alpha} + (1-\delta-\nu)\,\hat{k}_t - \hat{c}_{mt} - \hat{x}_{mt} \\
\Rightarrow\ (1+\gamma_m)\hat{k}^{ss} &= (1-\kappa_a^{ss}-\kappa_s^{ss})^\alpha \left(\hat{k}^{ss}\right)^\alpha (1-n_a^{ss}-n_s^{ss})^{1-\alpha} + (1-\delta-\nu)\,\hat{k}^{ss} - \hat{c}_m^{ss} - \hat{x}_m^{ss} \\
\Rightarrow\ (\gamma_m+\delta+\nu)\hat{k}^{ss} &= (1-\kappa_s^{ss}) \left(\hat{k}^{ss}\right)^\alpha \\
\Rightarrow\ 1-\kappa_s^{ss} &= (\gamma_m+\delta+\nu) \left(\hat{k}^{ss}\right)^{1-\alpha} \\
\therefore\ \kappa_s^{ss} &= n_s^{ss} = 1 - \frac{\alpha(\gamma_m+\delta+\nu)}{\rho+\delta+\nu}
\end{aligned}
$$

Given $\kappa_a^{ss} + \kappa_m^{ss} + \kappa_s^{ss} = 1$ and $n_a^{ss} + n_m^{ss} + n_s^{ss} = 1$,

$$
\begin{aligned}
\kappa_m^{ss} &= 1 - \kappa_a^{ss} - \kappa_s^{ss} = 1 - \kappa_s^{ss} \\
&= 1 - \left(1 - \frac{\alpha(\gamma_m+\delta+\nu)}{\rho+\delta+\nu}\right) \\
\therefore\ \kappa_m^{ss} &= n_m^{ss} = \frac{\alpha(\gamma_m+\delta+\nu)}{\rho+\delta+\nu}
\end{aligned}
$$

A.2 Data

Groningen Growth and Development Centre (GGDC)

- Data Sources (GGDC 10-sector Database)

  1. Employment (EMP)

  - The number of persons employed engaged in thousands.

  2. Hours Worked (H-EMP)

  - The total hours worked by persons engaged (in millions). This measurement is only available to 8 OECD countries.

  3. Nominal Value Added (VA)

  - Gross value added at current prices (in each country's currency)

  4. Real Value Added (VA-Q95 or VA-K)

  - Gross value added at constant prices (in each country's currency and own base year).

- Sector assignment

  1. Agriculture - The sum of International Standard Industrial Classification Revision 3 (ISIC Rev.3) sections 01-05 (Agriculture, Forestry and Fishing).
  2. Manufacturing - The sum of ISIC Rev.3 sections 10-37 and 45 (Mining and Quarrying; Manufacturing; and Construction).
  3. Services - The sum of ISIC Rev.3 sections 40-41 and 50-55, 60-99 (Public Utilities; Wholesale and Retail Trade, Hotels and Restaurants; Transport, Storage, and Communication; Finance, Insurance, and Real Estate; Community, Social and Personal Services; Government Services)

World Bank

To analyze the dynamics of three tradable sectors, it is necessary to obtain trade statistics that distinguish among agriculture, manufacturing, and services, as well as differentiate between exports and imports. Although the World Bank provides data on "Goods and services" and "Services" separately, there is no further breakdown of the "Goods" category into agriculture and manufacturing. Likewise, while indicators for "Trade," "Merchandise trade," and "Trade in services" exist, they do not divide agriculture from manufacturing or separate exports from imports.

Given these data constraints, we implemented the following steps for data collection and sectoral definitions.

- Data Sources (World Bank)
    1. GDP (current US $)
    2. Merchandise exports (current US$)
        (a) Agricultural raw materials exports (% of merchandise exports)
        (b) Food exports (% of merchandise exports)
        (c) Fuel exports (% of merchandise exports)
        (d) Ores and metals exports (% of merchandise exports)
        (e) Manufactures exports (% of merchandise exports)
    3. Merchandise imports (current US$)
        (a) Agricultural raw materials imports (% of merchandise imports)
        (b) Food exports (% of merchandise imports)
        (c) Fuel exports (% of merchandise imports)
        (d) Ores and metals imports (% of merchandise imports)
        (e) Manufactures imports (% of merchandise imports)
    4. Commercial service exports (current US$)
        (a) Computer, communications and other services (% of commercial service exports)
        (b) Transport services (% of commercial service exports)
        (c) Travel services (% of commercial service exports)
        (d) Insurance and financial services (% of commercial service exports)

5. Commercial service imports (current US$)
    (a) Computer, communications and other services (% of commercial service exports & imports)
    (b) Transport services (% of commercial service exports & imports)
    (c) Travel services (% of commercial service exports & imports)
    (d) Insurance and financial services (% of commercial service exports & imports)

- Sector assignment

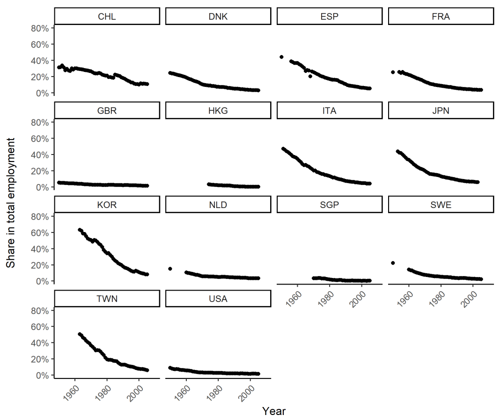
  * Below, "Trade $\equiv$ Exports + Imports". In order to unify the different units of indicators in terms of % of GDP, we multiplied each sector's values by either $\frac{\text{Merchandise}}{\text{GDP}}$ or $\frac{\text{Commercial service}}{\text{GDP}}$ where components of fractions are all measured in current US dollar.

  1. Agriculture $\equiv$ Agricultural raw materials Trade + Food Trade
  2. Manufacturing $\equiv$ Manufactures Trade + Ores & metals Trade + Fuel Trade
  3. Services $\equiv$ The sum of Trades in Computer, communication & other services, Transport services, Travel services and Insurance & financial services.

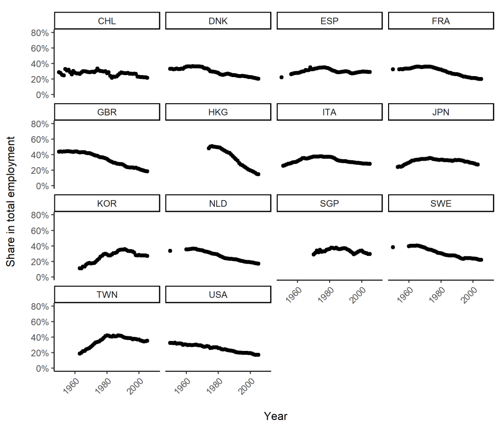
A.3 Figures by Year

¡High Income Countries¿

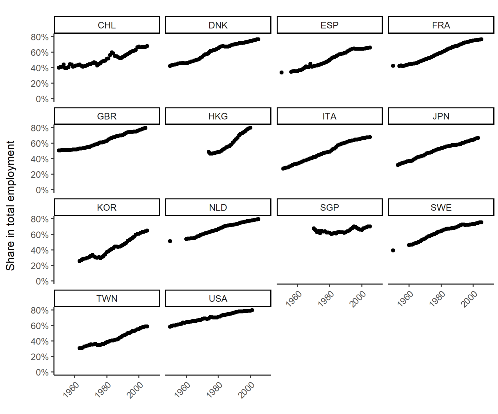
Agriculture



Manufacturing

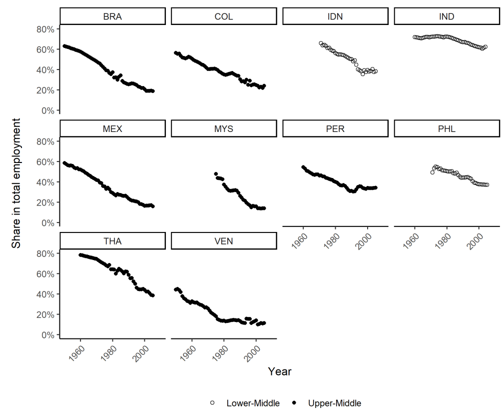


Service

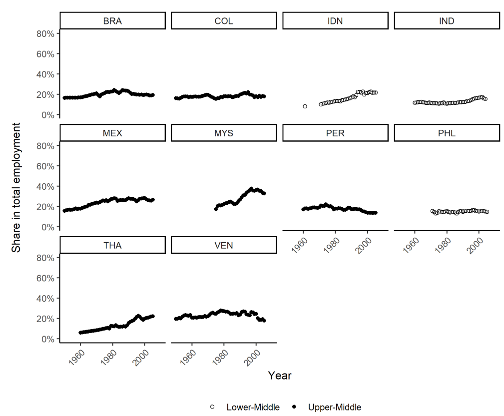


Middle Income Countries

Agriculture



Manufacturing



Service

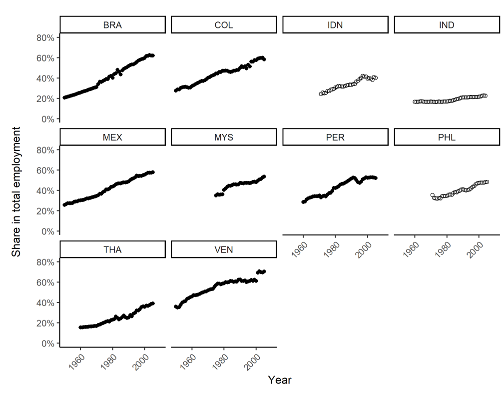
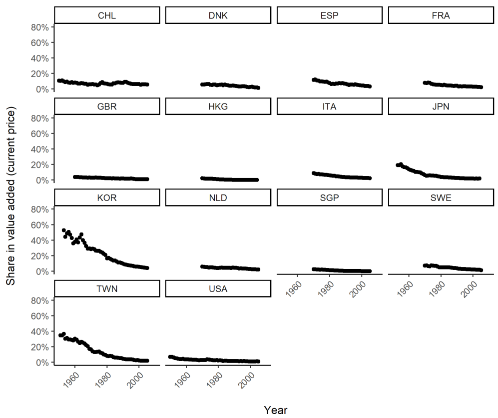


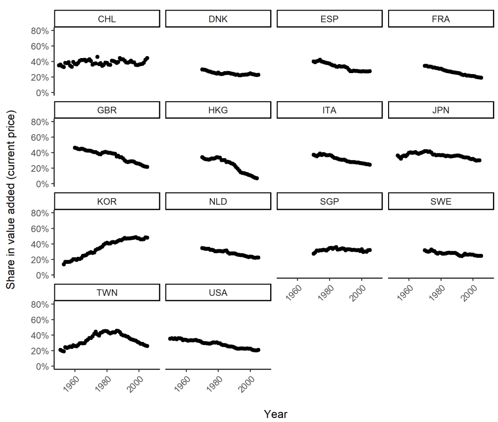
Figure 17: Sectoral Shares of Total Employment with Year - Countries by Income Level

High Income Countries

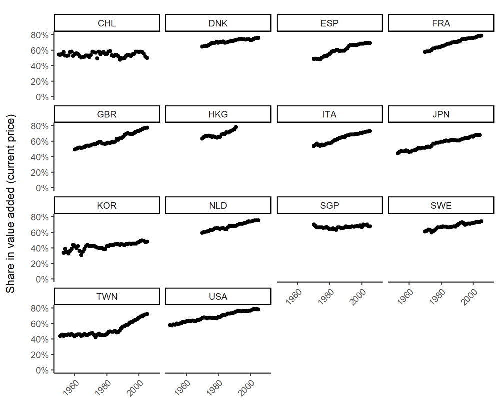
Agriculture



Manufacturing

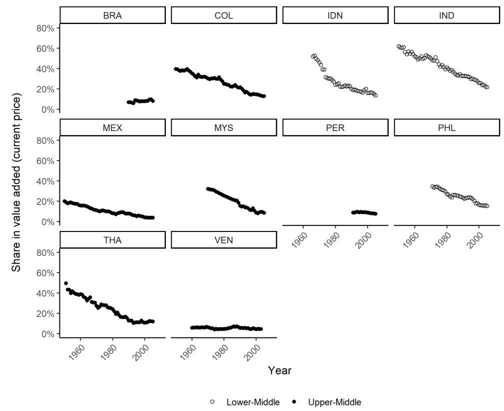


Service

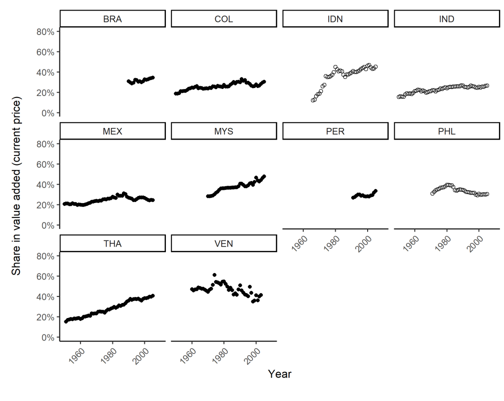


Middle Income Countries

Agriculture



Manufacturing



Service

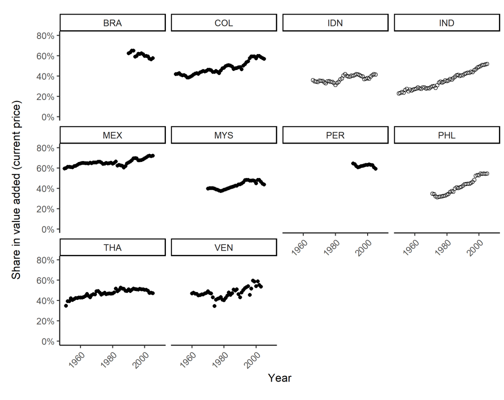


Figure 18: Sectoral Shares of Nominal VA with Year - Countries by Income Level

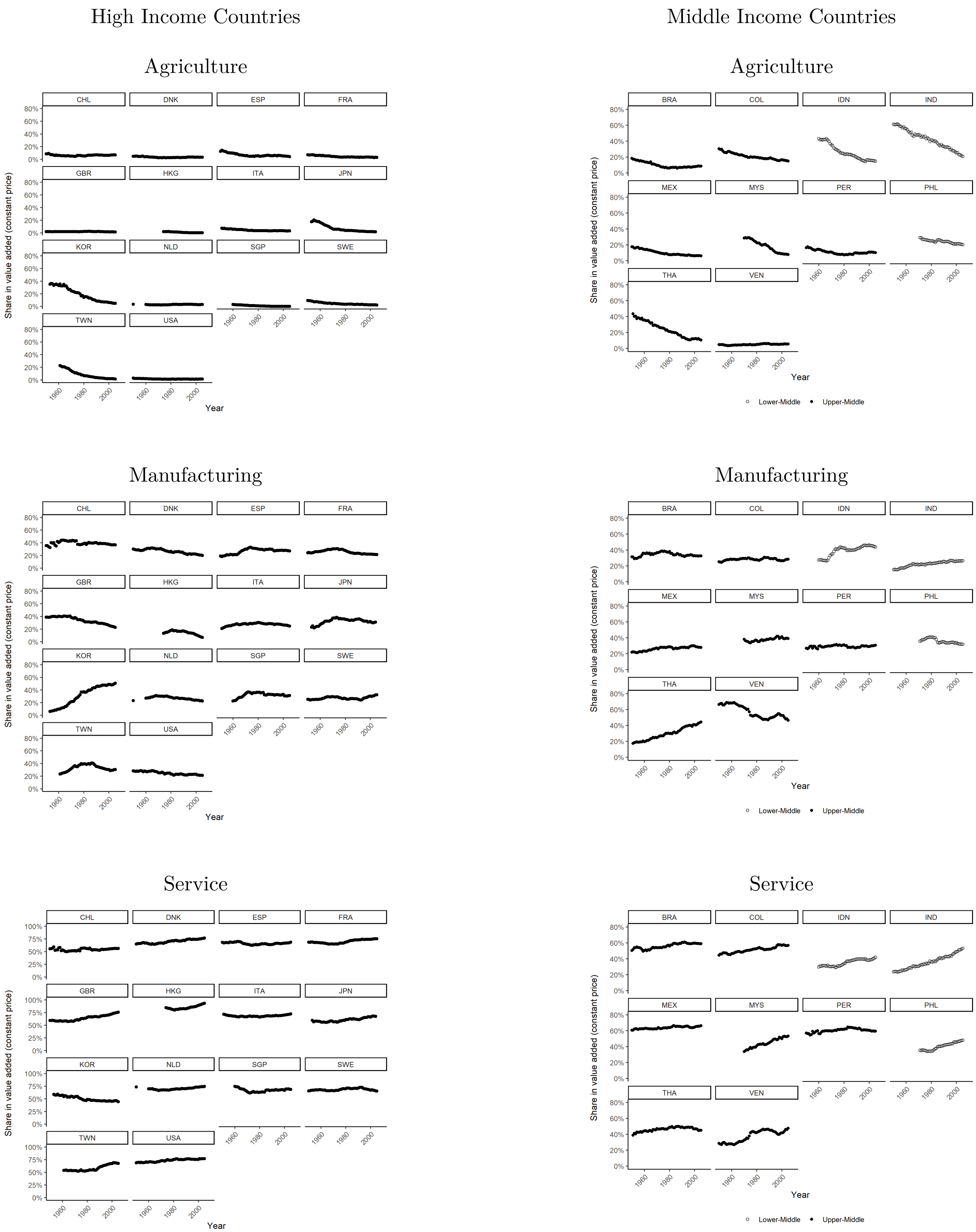


Figure 19: Sectoral Shares of Real VA with Year - Countries by Income Level

B Chapter 2 Appendix

B.1 Data

Shares of Education Expenditure (δ)

In our model, the human capital function is $h_{t+1} = a(e_0 + e_t)^{\delta} g_t^{1-\delta}$, where δ represents the share of education expenditure by parents and $1 - \delta$ by the government. We derive the parameter δ using data from KOSIS (2023) and OECD's "Education at a Glance" reports (OECD, 2023a).[44] We calculate the average value of δ using data from 2007 to 2020.

It is important to note that the OECD's data on private sector contributions to public education exclude parents' expenditures on private education, such as tutoring or extracurricular activities. Instead, they include payments made by parents for public education, like tuition and school fees. Since our parameter δ encompasses both public education fees and private education spending, we also incorporate data from KOSIS (2023)[45] on total private education expenditures by parents. We calculate the shares of education expenditure as follows:

$$\text{by Parents} = \frac{\text{Pub Edu}|_{\text{private}} + \text{Priv Edu Exp}}{\text{Pub Edu} + \text{Priv Edu Exp}}$$
$$\text{by Government} = \frac{\text{Pub Edu}|_{\text{public}}}{\text{Pub Edu} + \text{Priv Edu Exp}}$$

Fixed Time Cost to raise a child (ν)

Households allocate their available time between child-rearing and working within one unit of time. The fixed time cost per child, ν, is sourced from survey data by KIHSA (2021).[46] Table 4 summarizes total childcare time by respondent type, differentiated by weekday and weekend, and employment status. Assuming 8 hours of sleep, individuals have 16 hours (960 minutes) per day for work or childcare.

During weekdays, married women (MW) and their partners (P) spend a combined average of 387 minutes per day on childcare. For employed individuals, the combined average is 329 minutes. Married women's average childcare time is 291.4 minutes per day, decreasing to

[44] The OECD's reports provide data from three years prior to publication. Refer to the section "Total expenditure on educational institutions as a percentage of GDP by source of funds—for primary to tertiary levels—final funds (after transfers between public and private sectors)".

[45] KOSIS (2023) includes only students from elementary to high school and it does not account for private education expenditures for children before elementary school nor students retaking the College Scholastic Ability Test (CSAT).

[46] Data restructured by KOSIS can be accessed at URL (Korean).

230.0 minutes for those employed. Since our model represents a household without gender distinction, we use the average weekday time spent on childcare by employed individuals.

	Married Woman (19-49)				Partner			
(minutes)	**Weekday**	Weekend	Respondents	Avg	**Weekday**	Weekend	Respondents	Avg
Total	**291.4**	373.1	2,701	314.7	**96.2**	257.7	2,701	142.3
Employed	**230.0**	348.0	1,312	263.7	**99.3**	256.3	1,312	144.2
Unemployed	349.3	396.8	1,389	362.9	93.3	259.1	1,389	140.6

Table 4: Childcare Time for Married woman & Partners

Note: The survey targets individuals with children in elementary school or younger.

$$\text{Fixed Time costs}|_{emp} = \frac{\text{Time}_{MW,emp} + \text{Time}_{P,emp}}{960 \times 2} = \frac{329.3}{1920} \approx 0.17$$

Tax rate

The tax rate τ is set to 0.216, based on the average tax revenue as a percentage of GDP in South Korea from 2000 to 2022, as reported by NABO (2023). The data provide two different types of tax rates: (1) total tax revenue as a percentage of GDP, which includes both the tax burden rate (tax revenue/GDP) and the social security burden rate (social security contributions/GDP), and (2) total tax revenue (excluding social security contributions) as a percentage of GDP, calculated as tax revenue (national and local taxes)/nominal GDP. We chose the first measure, which includes social security contributions.

Family Support Policy

In the model, there are two parameters for family policies supported by the government: parental leave payments (σ_p) and child allowances (σ_c). Parental leave payments and child allowances are sourced from NABO (2021), which analyzes the family policies based on the categories of recipients. These categories include eight groups: child-rearing households, mothers and newborns, infants, children and youth outside of early childhood, young adults and newly-weds, workplace culture and system/environment development, societal awareness and education, and others. For parental leave payments, we select "Maternity protection benefits" within Child-rearing households category. For child allowances, we include "Home visit health support" within Mothers and newborns, "Childcare & Education expenses" and "cash allowances" within Infants category.

unit: 100M KRW	2017	2018	2019	2020	2021
Parental Leave Payments					
Maternity Protection benefits	10,657	12,871	14,919	15,895	15,699
Child Allowances					
Home visit health support	377	339	777	968	1165
Childcare & Education expenses	69,653	70,455	69,539	72,382	73,336
Cash allowances	12,242	17,988	30,551	30,991	29,802
Sum	82,272	88,782	100,867	104,341	104,303

Table 5: Parental Leave Payments & Child Allowances

B.2 Equilibrium Derivations

1) Household Optimization

We begin by simplifying the household's optimization problem by substituting the budget constraints and the human capital accumulation function into the utility function:

$$\max_{s_t,n_t,e_t} \log\Big((1-\tau)(1-(\nu-\alpha_t^p)n_t)w_t h_t - s_t - (e_t+\varepsilon-\alpha_t^c)n_t\Big)$$
$$+\gamma\log\Big(n_t^\theta a^{1-\theta}(e_0+e_t)^{\delta(1-\theta)}g_t^{\mu(1-\theta)}\Big)+\beta\log\Big((1+r_{t+1})s_t\Big).$$

First-Order Conditions

The first-order conditions (FOCs) with respect to savings s_t, number of children n_t, and education expenditure by parents e_t are:

$$\frac{\partial\mathcal{L}}{\partial s_t}: \quad c_{t+1}^o = \beta(1+r_{t+1})c_t^y \tag{69}$$

$$\frac{\partial\mathcal{L}}{\partial n_t}: \quad (1-\tau)(\nu-\alpha_t^p)w_t h_t n_t + (e_t+\varepsilon-\alpha_t^c)n_t = \gamma\theta c_t^y \tag{70}$$

$$\frac{\partial\mathcal{L}}{\partial e_t}: \quad n_t(e_0+e_t) = \gamma\delta(1-\theta)c_t^y, \tag{71}$$

where c_t^y and c_{t+1}^o represent consumption in the young and old age, respectively. Using the budget constraints, we can express savings as:

$$s_t = \frac{\beta}{1+\beta}\Big[(1-\tau)w_t h_t - (1-\tau)(\nu-\alpha_t^p)n_t w_t h_t - (e_t+\varepsilon-\alpha_t^c)n_t\Big]. \tag{72}$$

Education Expenditure, e_t

Dividing FOC (70) by FOC (71), we eliminate c_t^y and solve for e_t:

$$
\begin{aligned}
\frac{(1-\tau)(\nu-\alpha_t^p)w_t h_t n_t + (e_t+\varepsilon-\alpha_t^c)n_t}{n_t(e_0+e_t)} &= \frac{\gamma\theta c_t^y}{\gamma\delta(1-\theta)c_t^y} \\
\Rightarrow\ e_t &= \frac{\delta(1-\theta)(1-\tau)(\nu-\alpha_t^p)w_t h_t}{\theta+\theta\delta-\delta} + \frac{\delta(1-\theta)(\varepsilon-\alpha_t^c)-\theta e_0}{\theta+\theta\delta-\delta} \qquad (73)
\end{aligned}
$$

Fertility rate, n_t

Substituting c_t^y and s_t from (72) into the equation (70), we can solve for n_t:

$$
n_t = \frac{\gamma\theta(1-\tau)w_t h_t}{(1+\beta+\gamma\theta)\Big((1-\tau)(\nu-\alpha_t^p)w_t h_t + e_t + \varepsilon - \alpha_t^c\Big)} \qquad (74)
$$

First, if $e_t = 0$, i.e., $w_t h_t < \hat{E}_t$, then

$$
n_t = \frac{\gamma\theta(1-\tau)w_t h_t}{(1+\beta+\gamma\theta)\Big((1-\tau)(\nu-\alpha_t^p)w_t h_t + \varepsilon - \alpha_t^c\Big)}
$$

Second, if $e_t > 0$, i.e., $w_t h_t \geq \hat{E}_t$, then we can substitute e_t into the equation (74):

$$
n_t = \frac{\gamma(\theta+\theta\delta-\delta)(1-\tau)w_t h_t}{(1+\beta+\gamma\theta)\Big((1-\tau)(\nu-\alpha_t^p)w_t h_t + \varepsilon - \alpha_t^c - e_0\Big)} \qquad (75)
$$

Saving, s_t

By substituting n_t from the equation (74) into the equation (72):

$$
\begin{aligned}
s_t &= \frac{\beta}{1+\beta}\left[(1-\tau)w_t h_t - n_t\Big((1-\tau)(\nu-\alpha_t^p)w_t h_t + e_t + \varepsilon - \alpha_t^c\Big)\right] \\
&= \frac{\beta}{1+\beta}\left[(1-\tau)w_t h_t - \frac{\gamma\theta(1-\tau)w_t h_t}{(1+\beta+\gamma\theta)}\right] \\
&= \frac{\beta}{1+\beta}\left[\frac{(1+\beta)(1-\tau)w_t h_t}{(1+\beta+\gamma\theta)}\right] \\
\therefore\ s_t &= \frac{\beta(1-\tau)}{1+\gamma\theta+\beta}w_t h_t
\end{aligned}
$$

It is noted that s_t remains the same regardless of $w_t h_t \gtreqless \hat{E}_t$.

2) Government Budget Constraint

Assuming a balanced government budget, the total tax revenue equals total government expenditure:

$$\tau\left(1-(\nu-\alpha_t^p)n_t\right)w_t h_t N_t^y = \alpha_t^c n_t N_t^y + \alpha_t^p n_t w_t h_t N_t^y + g_t N_{t+1}^y, \tag{76}$$

where N_t^y is the population of young workers. We define the government's expenditure shares as σ^p (parental leave payments), σ^c (child allowances), and σ^e (public education), such that $\sigma^p+\sigma^c+\sigma^e=1$. Using these shares, we express the policy parameters:

$$\begin{aligned}
\text{(1) Parental leaves: } \alpha_t^p n_t w_t h_t N_t^y &= \sigma^p \times \tau(1-(\nu-\alpha_t^p)n_t)w_t h_t N_t^y\\
\Rightarrow\ \alpha_t^p n_t &= \sigma^p\tau(1-(\nu-\alpha_t^p)n_t)\\
\therefore\ \alpha_t^p &= \frac{\sigma^p\tau(1-\nu n_t)}{(1+\sigma^p\tau)n_t} &(77)\\
\text{(2) Child allowance: } \alpha_t^c n_t N_t^y &= \sigma^c \times \tau(1-(\nu-\alpha_t^p)n_t)w_t h_t N_t^y\\
\Rightarrow\ \alpha_t^c &= \sigma^c\tau(1-(\nu-\alpha_t^p)n_t)w_t h_t n_t^{-1}\\
\therefore\ \alpha_t^c &= \frac{\sigma^c\tau(1-\nu n_t)w_t h_t}{(1+\sigma^p\tau)n_t} &(78)\\
\text{(3) Public education: } g_t N_{t+1}^y &= \sigma^e \times \tau(1-(\nu-\alpha_t^p)n_t)w_t h_t N_t^y\\
\Rightarrow\ g_t &= \sigma^e\tau(1-(\nu-\alpha_t^p)n_t)w_t h_t n_t^{-1}\\
\therefore\ g_t &= \frac{\sigma^e\tau(1-\nu n_t)w_t h_t}{(1+\sigma^p\tau)n_t} &(79)
\end{aligned}$$

3) Simplification

Here we assume $e_t>0$, i.e., $w_t h_t \geq \hat{E}_t$.

Human Capital Evolution

The human capital of the next generation is given by:

$$h_{t+1} = a\left(e_0+e_t\right)^{\delta} g_t^{1-\delta}. \tag{80}$$

Substituting (73) and (79) into (80), we express h_{t+1} as a function of n_t and w_t. First, we simplified e_0+e_t:

$$\begin{aligned}
e_0+e_t &= e_0 + \frac{\delta(1-\theta)(1-\tau)(\nu-\alpha_t^p)w_t h_t + \delta(1-\theta)(\varepsilon-\alpha_t^c) - \theta e_0}{\theta+\theta\delta-\delta}\\
&= \frac{\delta(1-\theta)}{\theta+\theta\delta-\delta}\left((1-\tau)(\nu-\alpha_t^p)w_t h_t - \alpha_t^c\right) + \frac{\delta(1-\theta)(\varepsilon-e_0)}{\theta+\theta\delta-\delta}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\delta(1-\theta)}{\theta+\theta\delta-\delta}\left[(1-\tau)(\nu-\alpha_t^p) - \frac{\sigma^c\tau(1-\nu n_t)}{(1+\sigma^p\tau)n_t}\right] w_t h_t + \frac{\delta(1-\theta)(\varepsilon - e_0)}{\theta+\theta\delta-\delta} \\
&\simeq \frac{\delta(1-\theta)}{\theta+\theta\delta-\delta}\left[(1-\tau)(\nu-\alpha_t^p) - \frac{\sigma^c\tau(1-\nu n_t)}{(1+\sigma^p\tau)n_t}\right] w_t h_t \\
&\equiv Z(n_t) w_t h_t
\end{aligned}
$$

Given $g_t = \sigma^e\tau(1-(\nu-\alpha_t^p)n_t)w_t h_t n_t^{-1}$, we can rewrite:

$$
\begin{aligned}
h_{t+1} &= a(e_0+e_t)^\delta g_t^{1-\delta} \\
&= a\Big(Z(n_t)w_t h_t\Big)^\delta \left(\sigma^e\tau\left(\frac{1-(\nu-\alpha_t^p)n_t}{n_t}\right) w_t h_t\right)^{1-\delta} \\
\Rightarrow \quad \frac{h_{t+1}}{h_t} &= aZ(n_t)^\delta \left[\sigma^e\tau\left(\frac{1-(\nu-\alpha_t^p)n_t}{n_t}\right)\right]^{1-\delta} w_t
\end{aligned}
$$

Substituting the equation (77), where $(1-\tau)(\nu-\alpha_t^p) = \frac{(1-\tau)(1+2\sigma^p\tau)\nu n_t - (1-\tau)\sigma^p\tau}{(1+\sigma^p\tau)n_t}$, we can rewrite $Z(n_t)$:

$$
\begin{aligned}
Z(n_t) &= \frac{\delta(1-\theta)}{\theta+\theta\delta-\delta}\left[\frac{(1-\tau)(1+2\sigma^p\tau)\nu n_t - (1-\tau)\sigma^p\tau}{(1+\sigma^p\tau)n_t} - \frac{\sigma^c\tau(1-\nu n_t)}{(1+\sigma^p\tau)n_t}\right] \\
&= \frac{\delta(1-\theta)}{\theta+\theta\delta-\delta}\left[\frac{((1-\tau)(1+2\sigma^p\tau)+\sigma^c\tau)\nu n_t - \tau((1-\tau)\sigma^p+\sigma^c)}{(1+\sigma^p\tau)n_t}\right]
\end{aligned}
$$

Additionally, since $\frac{1-(\nu-\alpha_t^p)n_t}{n_t} = \frac{(1+2\sigma^p\tau)(1-\nu n_t)}{(1+\sigma^p\tau)n_t}$, we can simplify $\frac{h_{t+1}}{h_t}$:

$$
\begin{aligned}
\frac{h_{t+1}}{h_t} &= aZ(n_t)^\delta\left[\sigma^e\tau\left(\frac{1-(\nu-\alpha_t^p)n_t}{n_t}\right)\right]^{1-\delta} w_t \\
&= a(\sigma^e\tau)^{1-\delta}\left[\frac{\delta(1-\theta)}{\theta+\theta\delta-\delta}\left(\frac{((1-\tau)(1+2\sigma^p\tau)+\sigma^c\tau)\nu n_t - \tau((1-\tau)\sigma^p+\sigma^c)}{(1+\sigma^p\tau)n_t}\right)\right]^\delta \\
&\quad \times \left(\frac{(1+2\sigma^p\tau)(1-\nu n_t)}{(1+\sigma^p\tau)n_t}\right)^{1-\delta} w_t \\
&= a(\sigma^e\tau)^{1-\delta}\left[\frac{\delta(1-\theta)\left[((1-\tau)(1+2\sigma^p\tau)+\sigma^c\tau)\nu n_t - \tau((1-\tau)\sigma^p+\sigma^c)\right]}{(\theta+\theta\delta-\delta)(1+2\sigma^p\tau)}\right]^\delta \frac{(1+2\sigma^p\tau)(1-\nu n_t)^{1-\delta}w_t}{(1+\sigma^p\tau)n_t} \\
&= a(\sigma^e\tau)^{1-\delta}\left(\frac{1+2\sigma^p\tau}{1+\sigma^p\tau}\right)\left[\frac{\delta(1-\theta)}{(\theta+\theta\delta-\delta)(1+2\sigma^p\tau)}\right]^\delta \\
&\quad \times \left[\Big((1-\tau)(1+2\sigma^p\tau)+\sigma^c\tau\Big)\nu n_t - \tau\Big((1-\tau)\sigma^p+\sigma^c\Big)\right]^\delta \times \frac{(1-\nu n_t)^{1-\delta}w_t}{n_t}
\end{aligned}
$$

Thus, the growth rate of human capital, $\frac{h_{t+1}}{h_t}$ can be reduced:

$$
\frac{h_{t+1}}{h_t} = \frac{H_1\left(H_2 n_t - H_3\right)^\delta\left(1-\nu n_t\right)^{1-\delta} w_t}{n_t}, \tag{81}
$$

$$
\text{where } H_1 = \frac{a(\sigma^e\tau)^{1-\delta}(1+2\sigma^p\tau)}{1+\sigma^p\tau}\left[\frac{\delta(1-\theta)}{(\theta+\theta\delta-\delta)(1+2\sigma^p\tau)}\right]^\delta \tag{82}
$$

$$H_2 = ((1-\tau)(1+2\sigma^p\tau) + \sigma^c\tau)\nu \tag{83}$$

$$H_3 = \tau((1-\tau)\sigma^p + \sigma^c) \tag{84}$$

where H_1, H_2, and H_3 are constants defined by model parameters and policy variables.

Fertility rate, n_t

Substituting the equations (77) and (78) into (75), we can simplify n_t:

$$n_t = \frac{\gamma(\theta+\theta\delta-\delta)(1-\tau)w_t h_t}{(1+\beta+\gamma\theta)\left((1-\tau)\left(\frac{(1+2\sigma^p\tau)\nu n_t - \sigma^p\tau}{(1+\sigma^p\tau)n_t}\right)w_t h_t + \varepsilon - \frac{\sigma^c\tau(1-\nu n_t)w_t h_t}{(1+\sigma^p\tau)n_t} - e_0\right)}$$

$$\Rightarrow\ n_t = \frac{\left[\gamma(\theta+\theta\delta-\delta)(1-\tau)(1+\sigma^p\tau) + (1+\beta+\gamma\theta)\left((1-\tau)\sigma^p\tau + \sigma^c\tau\right)\right]w_t h_t}{(1+\beta+\gamma\theta)(1+\sigma^p\tau)(\varepsilon - e_0) + (1+\beta+\gamma\theta)\left((1-\tau) + 2(1-\tau)\sigma^p\tau + \sigma^c\tau\right)\nu w_t h_t}$$

$$\equiv \frac{C_1 w_t h_t}{A_1 + B_1 w_t h_t}$$

where $A_1 = (1+\beta+\gamma\theta)(1+\sigma^p\tau)(\varepsilon - e_0)$

$$B_1 = (1+\beta+\gamma\theta)\left((1-\tau) + 2(1-\tau)\sigma^p\tau + \sigma^c\tau\right)\nu$$

$$C_1 = \gamma(\theta+\theta\delta-\delta)(1-\tau)(1+\sigma^p\tau) + (1+\beta+\gamma\theta)\left((1-\tau)\sigma^p\tau + \sigma^c\tau\right)$$

In the long run, w_t converge to the steady state $\bar{w}$ and h_t will keep growing indefinitely. Thus, if we divide both numerator and denominator by $w_t h_t$, then $\frac{A_1}{h_t}$ will converge to zero. Consequently, in the long run, n_t will converge to:

$$\bar{n} = \frac{C_1}{B_1}$$

Physical capital per effective unit of labor, k_t

Using $s_t = \frac{\beta(1-\tau)w_t h_t}{1+\beta+\gamma\theta}$ and $k_t \equiv \frac{K_t}{L_t}$, the equilibrium equation for k_t can be simplified:

$$\begin{aligned} k_{t+1} &= \frac{s_t}{(1-\nu n_{t+1})h_{t+1}n_t} \\ &= \frac{1}{(1-\nu n_{t+1})h_{t+1}n_t} \times \frac{\beta(1-\tau)w_t h_t}{1+\beta+\gamma\theta} \\ &= \frac{\beta(1-\tau)w_t}{(1+\beta+\gamma\theta)(1-\nu n_{t+1})n_t} \times \frac{h_t}{h_{t+1}} \end{aligned}$$

$$\therefore \; k_{t+1} = \frac{D_1 w_t}{(1-\nu n_{t+1})n_t} \times \frac{h_t}{h_{t+1}}$$
$$\text{where } D_1 = \frac{\beta(1-\tau)}{1+\beta+\gamma\theta}$$

In the long run, $n_t = n_{t+1} = \bar{n}$ and $\frac{h_{t+1}}{h_t}$ will be a constant. As a result,

$$\bar{k} = \frac{D_1}{H_1(H_2\bar{n} - H_3)^{\delta}(1-\nu\bar{n})^{1-\delta}(1-\nu\bar{n})}$$

4) Threshold Fertility Rate for Human Capital Growth

We analyze the effect of fertility on the growth rate of human capital by examining the derivative of $\frac{h_{t+1}}{h_t}$ with respect to n_t in (81):

$$\begin{aligned}
\frac{\partial\left(\frac{h_{t+1}}{h_t}\right)}{\partial n_t} &= H_1\Bigg(\delta(H_2 n_t - H_3)^{\delta-1} H_2 (1-\nu n_t)^{1-\delta}\frac{w_t}{n_t} - (1-\delta)\nu(H_2 n_t - H_3)^{\delta}(1-\nu n_t)^{-\delta}\frac{w_t}{n_t} \\
&\quad - (H_2 n_t - H_3)^{\delta}(1-\nu n_t)^{1-\delta}\frac{w_t}{n_t^2}\Bigg) \\
&= H_1\left(\frac{H_2 n_t - H_3}{1-\nu n_t}\right)^{\delta}\frac{w_t}{n_t^2}\left[\frac{n_t(\delta H_2 - \delta\nu H_3 - H_2) + H_3}{H_2 n_t - H_3}\right]
\end{aligned}$$

Simplifying, if $n_t > \frac{H_3}{H_2}$, the sign of the derivative depends on:

$$\frac{\partial\left(\frac{h_{t+1}}{h_t}\right)}{\partial n_t} \gtrless 0 \quad \text{if} \quad n_t \lesseqgtr n^*,$$

where n^* is the threshold fertility rate given by:

$$n^* = \frac{H_3}{(1-\delta)H_2 + \delta\nu H_3}.$$

This threshold represents the point at which the effect of increasing fertility on human capital growth changes sign, illustrating the quantity-quality trade-off in childbearing. In the model, the minimum n_t, must be greater than 0.22. Additionally, if $n_t > 0.37$, an increase in n_t leads to a decrease in $\frac{h_{t+1}}{h_t}$. Conversely if $n_t < 0.37$, an increase in n_t results in an increase in $\frac{h_{t+1}}{h_t}$.

Interpretation

If $n_t < 0.37$, an increase in fertility leads to higher human capital growth, indicating that the marginal benefit of an additional child outweighs the dilution effect on resources. Conversely, if $n_t > 0.37$, additional children reduce the growth rate of human capital due to resource constraints.

B.3 Dynamic Trajectories and Phase Diagrams

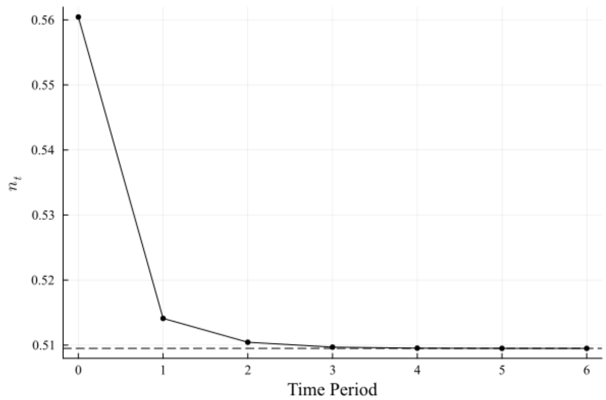
Time Paths Following Deviations from the Steady State

We examine the dynamic responses when the initial values of n_t, k_t, and g_t are set 10 percentage points (pp) above and below their steady-state values. Due to the rapid increase of h_t, we fix h_t at its initial value to focus on the dynamics of the other variables. Figure 22 presents the trajectories of each variable over time. The left-hand panels correspond to initial values 10pp above the steady state, while the right-hand panels represent 10pp below. Each row displays n_t, k_t, and g_t from top to bottom, respectively. The results indicate that for deviations of 10pp above and below the steady state, n_t, k_t, and g_t converge to within less than 1% of their steady-state values after one period ($T = 1$), two periods ($T = 2$), and three periods ($T = 3$), respectively.

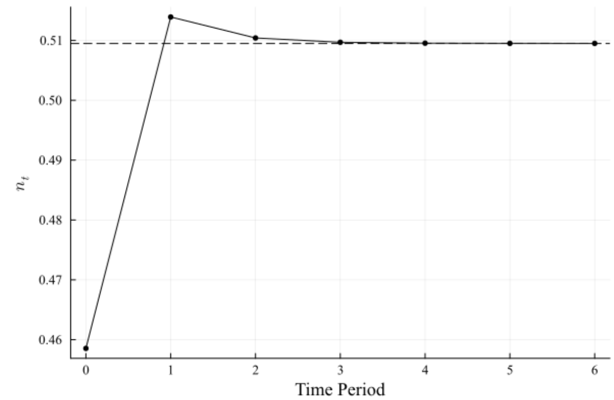
Phase Diagrams

Figure 23 illustrates the phase diagrams for n_t, k_t, and g_t under the same initial conditions. The left-hand panels depict the case with initial values 10pp above the steady state, and the right-hand panels show 10pp below. The trajectories exhibit spiral convergence toward the steady state, consistent across different initial conditions.

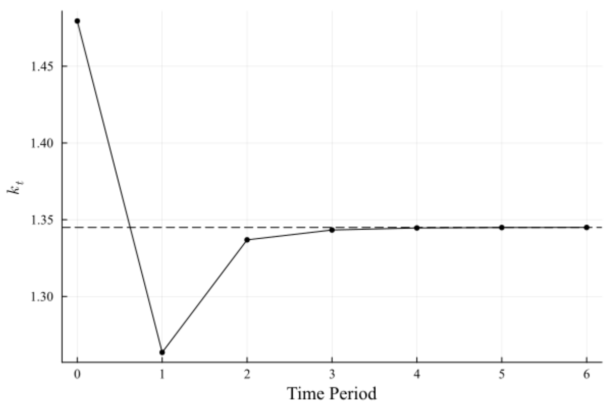
The rapid convergence of n_t reflects households' sensitivity to economic variables in their fertility decisions. The stability of the steady state implies the economy's ability to return to equilibrium after small disturbances, indicating robustness in its structure. We observe that the spiral path unfolds as follows: first, the TFR stabilizes in the first period; then, capital per effective labor, influenced by the previous fertility rate, converges in the next period; finally, the growth rate of human capital converges after the stabilization of both n_t and k_t. This sequence highlights the intertemporal dependencies among the variables. The persistence of low fertility rates, despite increasing income, suggests that additional policy measures may be necessary to address demographic challenges, which we explored in Section 5.



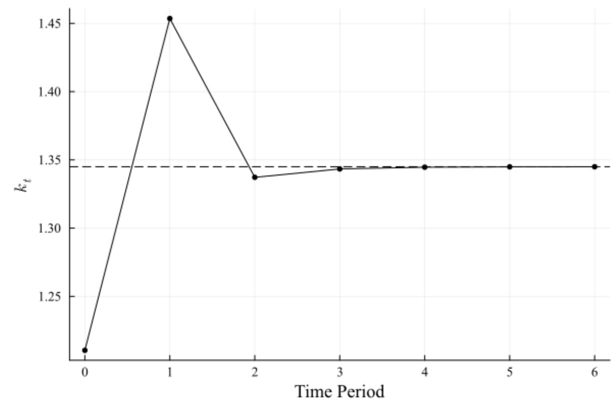
TFR



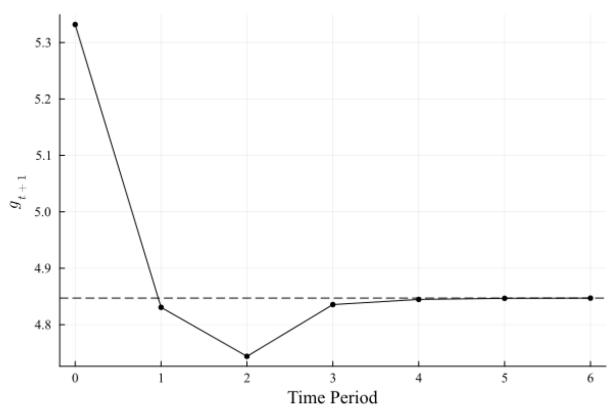
TFR



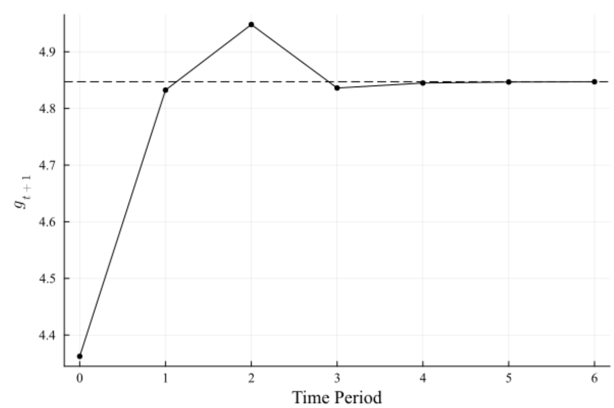
Capital per effective Labor



Capital per effective Labor



Growth rate of Human capital

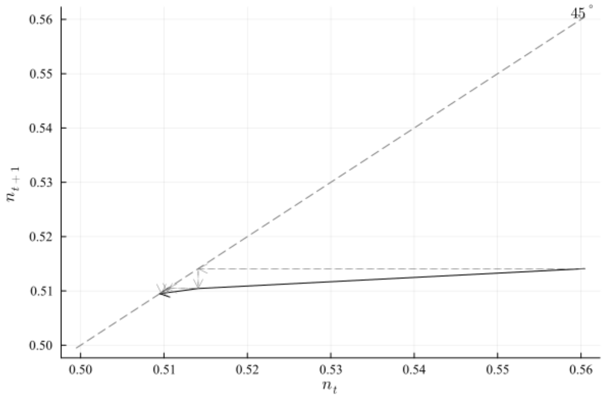


Growth rate of Human capital

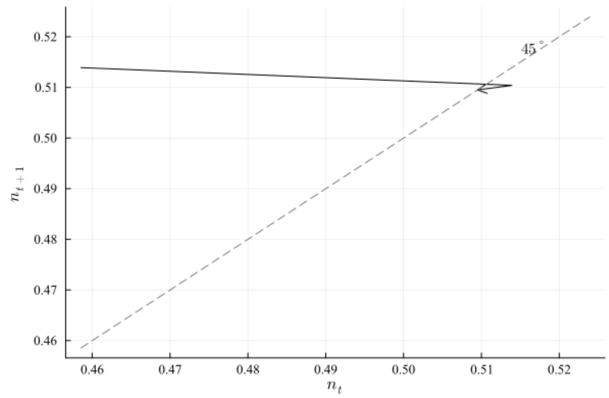
10pp Above the Steady State

10pp Below the Steady State

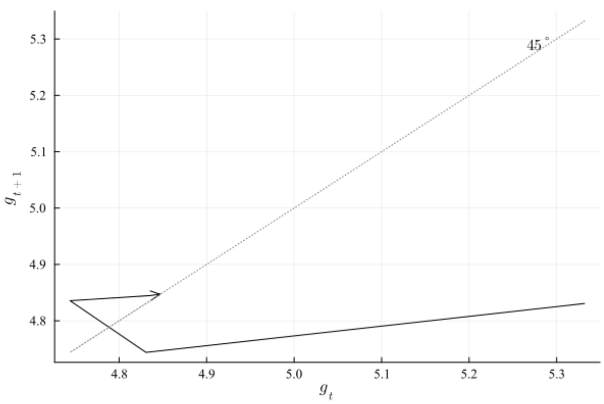
Figure 20: Time Paths for n_t, g_t, k_t: $\pm$ 10pp Deviations



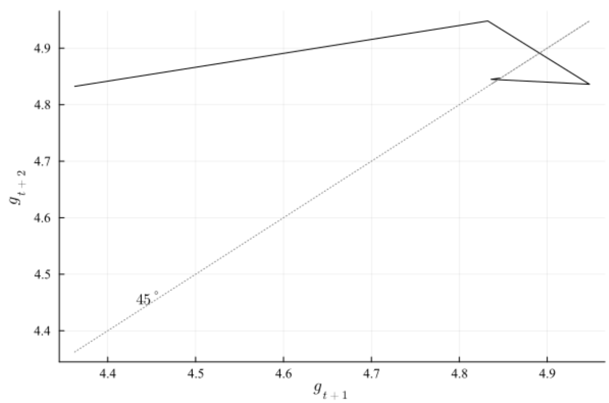
TFR



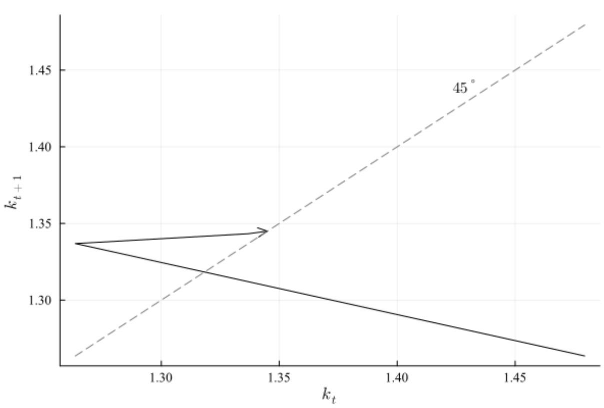
TFR



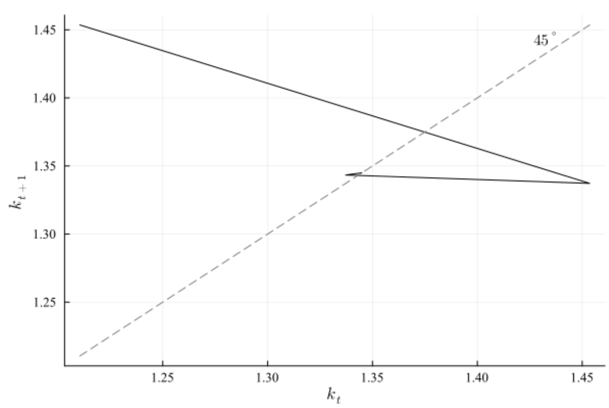
Growth rate of Human capital



Growth rate of Human capital



Capital per effective Labor



Capital per effective Labor

10pp Above the Steady State **10pp Below the Steady State**

Figure 21: Phase Diagrams for n_t, g_t, k_t: $\pm$ 10pp Deviations

C Chapter 3 Appendix

C.1 Data

In our model, the elderly population receives pension benefits σ_η from the government. The data for these benefits are sourced from NPS (2022). Similar to family policies, we assume that the government allocates a fixed proportion of its revenue to pension benefits. According to the National Pension Statistics reported by NPS (2022), national pensions are categorized into sub-pensions, including old-age pensions and disability pensions. Between 2012 and 2022, old-age pensions comprised approximately 81% to 87% of the total national pension disbursements. Consequently, our model incorporates both the overall national pension and the specific old-age pension data. To determine the pension amounts on a per-capita basis, we divided the total pension disbursements by the number of recipients. As presented in Table 6, there is no significant difference between these two measures.

	2017	2018	2019	2020
Amount (1B KRW)				
σ^p (Parental leave payments)	1065.7	1287.1	1491.9	1589.5
σ^c (Child allowances)	8227.2	8878.2	10086.7	10434.1
σ^e (Public education)	69632.4	76265	81623.8	82338.7
σ^η (National pension)	17588.0	19237.2	21294.0	24328.8
σ^η (Old-age pension)	15697.2	17191.5	18914.9	21702.1
Ratio - Without Pension				
σ^p	0.014	0.016	0.017	0.017
σ^c	0.109	0.108	0.114	0.111
σ^e	0.877	0.876	0.869	0.873
Sum	1.000	1.000	1.000	1.000
Ratio - National Pension				
σ^p	0.011	0.012	0.013	0.013
σ^c	0.085	0.084	0.088	0.088
σ^e	0.721	0.722	0.713	0.694
σ^η	0.182	0.182	0.186	0.205
Sum	1.000	1.000	1.000	1.000
Ratio - Old-age Pension				
σ^p	0.011	0.012	0.013	0.014
σ^c	0.087	0.086	0.09	0.09
σ^e	0.736	0.736	0.728	0.709
σ^η	0.166	0.166	0.169	0.187
Sum	1.000	1.000	1.000	1.000

Table 6: Government Family Policies

C.2 Dynamic Trajectories and Phase Diagrams

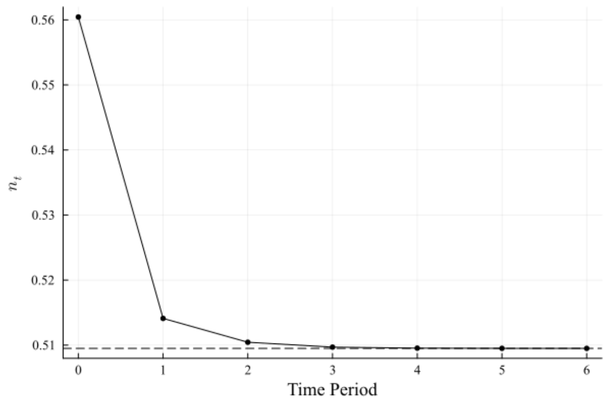
Time Paths Following Deviations from the Steady State

We examine the dynamic responses when the initial values of n_t, k_t, and g_t are set 10 percentage points (pp) above and below their steady-state values. Due to the rapid increase of h_t, we fix h_t at its initial value to focus on the dynamics of the other variables. Figure 22 presents the trajectories of each variable over time. The left-hand panels correspond to initial values 10pp above the steady state, while the right-hand panels represent 10pp below. Each row displays n_t, k_t, and g_t from top to bottom, respectively. The results indicate that for deviations of 10pp above and below the steady state, n_t, k_t, and g_t converge to within less than 1% of their steady-state values after one period ($T = 1$), two periods ($T = 2$), and three periods ($T = 3$), respectively.

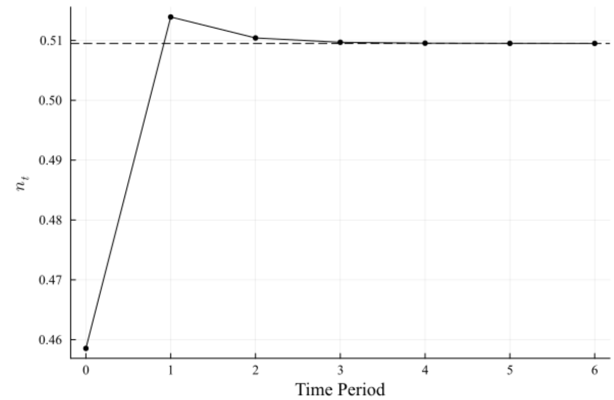
Phase Diagrams

Figure 23 illustrates the phase diagrams for n_t, k_t, and g_t under the same initial conditions. The left-hand panels depict the case with initial values 10pp above the steady state, and the right-hand panels show 10pp below. The trajectories exhibit spiral convergence toward the steady state, consistent across different initial conditions.

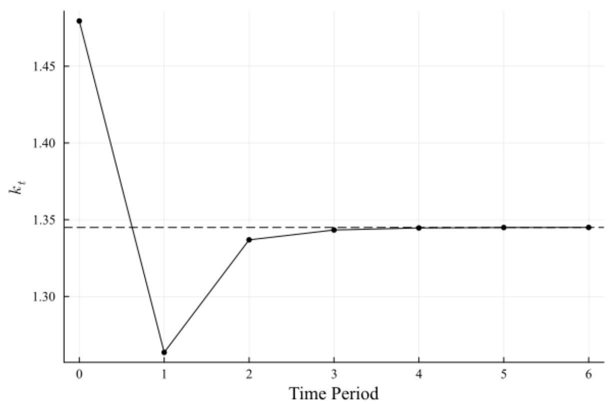
The rapid convergence of n_t reflects households' sensitivity to economic variables in their fertility decisions. The stability of the steady state implies the economy's ability to return to equilibrium after small disturbances, indicating robustness in its structure. We observe that the spiral path unfolds as follows: first, the TFR stabilizes in the first period; then, capital per effective labor, influenced by the previous fertility rate, converges in the next period; finally, the growth rate of human capital converges after the stabilization of both n_t and k_t. This sequence highlights the intertemporal dependencies among the variables. The persistence of low fertility rates, despite increasing income, suggests that additional policy measures may be necessary to address demographic challenges, which we explored in Section 5.



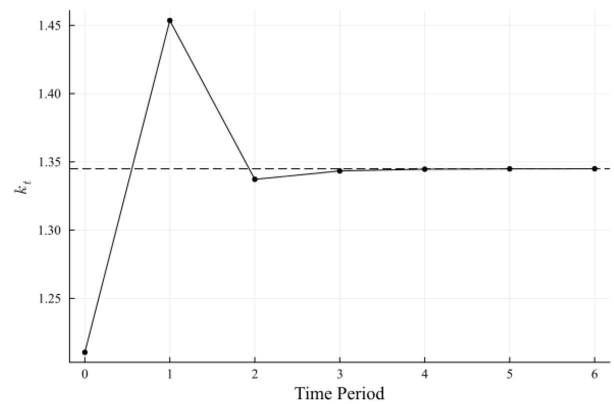
TFR



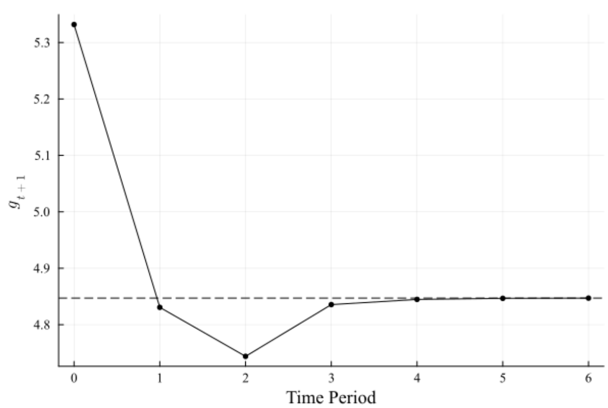
TFR



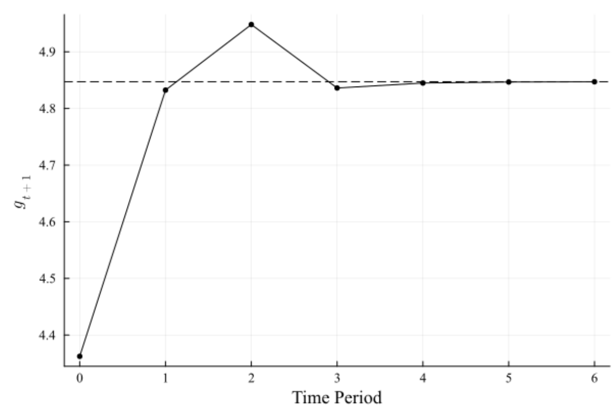
Capital per effective Labor



Capital per effective Labor



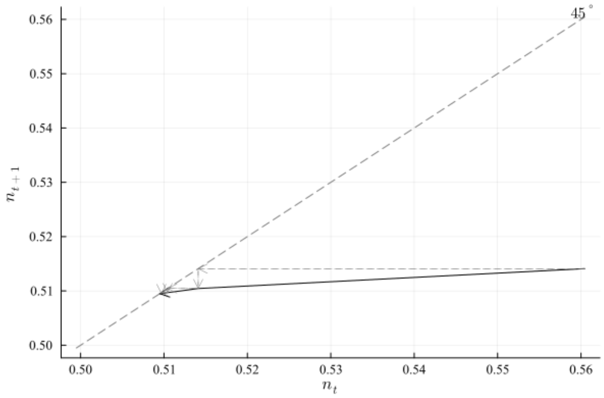
Growth rate of Human capital



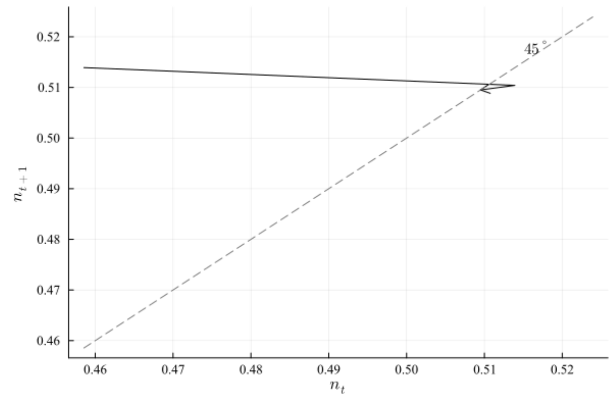
Growth rate of Human capital

10pp Above the Steady State **10pp Below the Steady State**

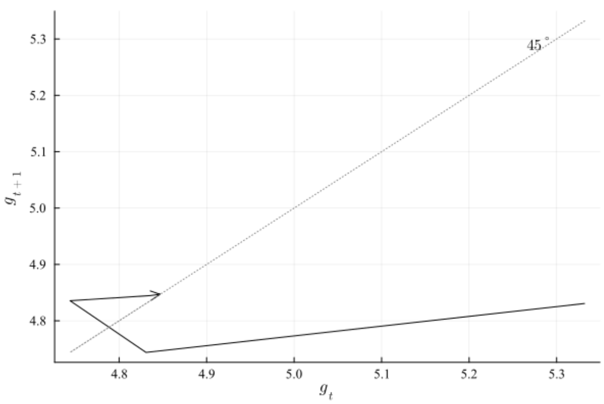
Figure 22: Time Paths for n_t, g_t, k_t: $\pm$ 10pp Deviations



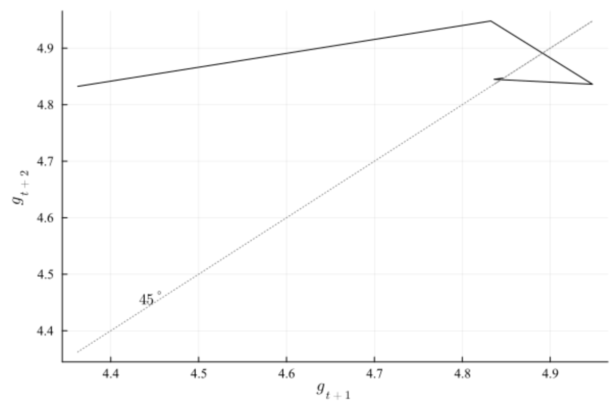
TFR



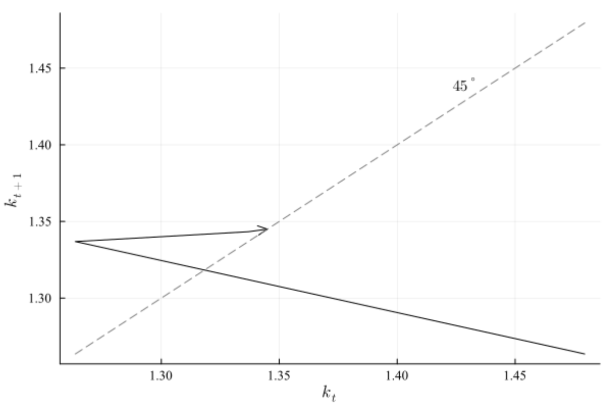
TFR



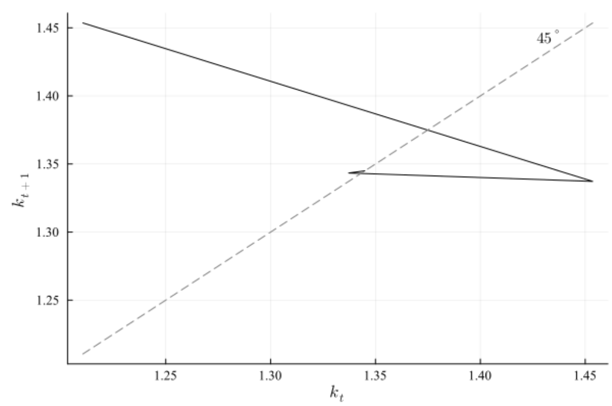
Growth rate of Human capital



Growth rate of Human capital



Capital per effective Labor



Capital per effective Labor

10pp Above the Steady State

10pp Below the Steady State

Figure 23: Phase Diagrams for n_t, g_t, k_t: ± 10pp Deviations

References

Aaronson, D., F. Lange, and B. Mazumder (2014): "Fertility transitions along the extensive and intensive margins," *American Economic Review*, 104(11), 3701–3724.

Acemoglu, D., and V. Guerrieri (2008): "Capital deepening and nonbalanced economic growth," *Journal of political Economy*, 116(3), 467–498.

BAIK (2021): "Status of response for the change of demographic structure," Audit report, The Board of Audit and Inspection of Korea, Seoul, retrieved from https://dl.nanet.go.kr/search/searchInnerDetail.do?controlNo=MONO12023000051177.

Barr, N., and P. Diamond (2006): "The Economics of Pensions," *Oxford Review of Economic Policy*, 22(1), 15–39.

Barro, R. J., and G. S. Becker (1989): "Fertility choice in a model of economic growth," *Econometrica: journal of the Econometric Society*, pp. 481–501.

Baudin, T., D. De La Croix, and P. E. Gobbi (2015): "Fertility and childlessness in the United States," *American Economic Review*, 105(6), 1852–1882.

Berger, J., T. Davoine, P. Schuster, and L. Strohner (2016): "Cross-country differences in the contribution of future migration to old-age financing," *International Tax and Public Finance*, 23, 1160–1184.

Betts, C., R. Giri, and R. Verma (2016): "Trade, reform, and structural transformation in South Korea," .

Billari, F., and H.-P. Kohler (2004): "Patterns of low and lowest-low fertility in Europe," *Population studies*, 58(2), 161–176.

Bloom, D. E., D. Canning, and J. Sevilla (2001): "Economic growth and the demographic transition," .

BLS (2024): "American Time Use Survey," U.S. Bureau of Labor Statistics, data retrieved from https://www.bls.gov/news.release/atus.htm#tus_tu_nr10.f.1.

Buera, F. J., and J. P. Kaboski (2012a): "The rise of the service economy," *The American Economic Review*, 102(6), 2540–2569.

——— (2012b): “Scale and the origins of structural change,” *Journal of Economic Theory*, 147(2), 684–712.

Cheng, Y.-h. A. (2020): “Ultra-low fertility in East Asia,” *Vienna Yearbook of Population Research*, 18, 83–120.

Choi, S.-w., A. J. Yellow Horse, and T.-C. Yang (2018): “Family policies and working women’s fertility intentions in South Korea,” *Asian Population Studies*, 14(3), 251–270, https://doi.org/10.1080/17441730.2018.1512207.

Chojnicki, X., and L. Ragot (2016): “Impacts of immigration on an ageing welfare state: An applied general equilibrium model for France,” *Fiscal Studies*, 37(2), 258–284.

Cipriani, G. P. (2014): “Population aging and PAYG pensions in the OLG model,” *Journal of population economics*, 27(1), 251–256.

Coile, C., and J. Gruber (2007): “Future social security entitlements and the retirement decision,” *The review of Economics and Statistics*, 89(2), 234–246.

Cutler, D. M., J. M. Poterba, L. M. Sheiner, L. H. Summers, and G. A. Akerlof (1990): “An aging society: opportunity or challenge?,” *Brookings papers on economic activity*, 1990(1), 1–73.

D’Addio, A., and M. Mira d’Ercole (2005): “Trends and Determinants of Fertility Rates: The Role of Policies,” OECD Social, Employment and Migration Working Papers 27, OECD Publishing, Paris, https://doi.org/10.1787/880242325663.

Delogu, M., F. Docquier, and J. Machado (2018): “Globalizing labor and the world economy: the role of human capital,” *Journal of Economic Growth*, 23, 223–258.

Diamond, P. (2004): “Social security,” *American Economic Review*, 94(1), 1–24.

Diamond, P. A. (1965): “National debt in a neoclassical growth model,” *The American Economic Review*, 55(5), 1126–1150.

Doepke, M. (2004): “Accounting for fertility decline during the transition to growth,” *Journal of Economic growth*, 9, 347–383.

Drago, R., K. Sawyer, K. M. Shreffler, D. Warren, and M. Wooden (2011): “Did Australia’s baby bonus increase fertility intentions and births?,” *Population research and policy review*, 30, 381–397.

Fanti, L., and L. Gori (2013): "Fertility-related pensions and cyclical instability," *Journal of Population Economics*, 26, 1209–1232.

Feenstra, R. C., R. Inklaar, and M. P. Timmer (2015): "The Next Generation of the Penn World Table," *American Economic Review*, 105(10), 3150–3182.

Feldstein, M., and J. B. Liebman (2002): "Social security," *Handbook of public economics*, 4, 2245–2324.

Feldstein, M., and A. Samwick (1998): "The transition path in privatizing social security," in *Privatizing social security*, pp. 215–264. University of Chicago Press.

Fougère, M., and M. Mérette (1999): "Population Ageing and Economic Growth in Seven OECD Countries," *Economic Modelling*, 16(3), 411–427.

Galor, O., and D. N. Weil (1996): "The gender gap, fertility, and growth," *American Economic Review*, 86, 374–387.

Gauthier, A. H., and J. Hatzius (1997): "Family benefits and fertility: An econometric analysis," *Population studies*, 51(3), 295–306.

Gervais, A., and J. B. Jensen (2019): "The tradability of services: Geographic concentration and trade costs," *Journal of International Economics*, 118, 331–350.

Goldin, C. (2006): "The Quiet Revolution That Transformed Women's Employment, Education, and Family," *American Economic Review*, 96(2), 1–21.

Greenwood, J., Z. Hercowitz, and P. Krusell (1997): "Long-run implications of investment-specific technological change," *The American economic review*, pp. 342–362.

Gruber, J., and D. A. Wise (2002): "Social security programs and retirement around the world: Micro estimation," .

Hayashi, F., and E. C. Prescott (2008): "The depressing effect of agricultural institutions on the prewar Japanese economy," *Journal of political Economy*, 116(4), 573–632.

Heer, B., and A. Irmen (2014): "Population, pensions, and endogenous economic growth," *Journal of Economic Dynamics and Control*, 46, 50–72.

Heer, B., V. Polito, and M. R. Wickens (2020): "Population aging, social security and fiscal limits," *Journal of Economic Dynamics and Control*, 116, 103913.

Heijdra, B. J., and W. E. Romp (2009): "Retirement, pensions, and ageing," *Journal of Public Economics*, 93(3-4), 586–604.

Herrendorf, B., C. Herrington, and A. Valentinyi (2015): "Sectoral technology and structural transformation," *American Economic Journal: Macroeconomics*, 7(4), 104–133.

Herrendorf, B., R. Rogerson, and A. Valentinyi (2014): "Growth and structural transformation," in *Handbook of economic growth*, vol. 2, pp. 855–941. Elsevier.

Hock, H., and D. N. Weil (2012): "On the dynamics of the age structure, dependency, and consumption," *Journal of population economics*, 25(3), 1019–1043.

Holzmann, R. (2005): "Old-age income support in the 21st century: An international perspective on pension systems and reform," *World Bank*.

Hong, S. C., Y. I. Kim, J. Y. Lim, and M. Y. Yeo (2016): "Pro-natalist cash grants and fertility: A panel analysis," *The Korean Economic Review*, 32(2), 331–354.

Jones, C. I. (2022): "The end of economic growth? Unintended consequences of a declining population," *American Economic Review*, 112(11), 3489–3527.

Jones, G. W. (2019): "Ultra-low fertility in East Asia: Policy responses and challenges," *Asian Population Studies*, 15(2), 131–149, https://doi.org/10.1080/17441730.2019.1594656.

Kaganovich, M., and I. Zilcha (2012): "Pay-as-you-go or funded social security? A general equilibrium comparison," *Journal of Economic Dynamics and Control*, 36(4), 455–467.

KIHSA (2021): "National Family and Fertility Survey," Korea Institute for Health and Social Affairs, data retrieved from https://kosis.kr/statHtml/statHtml.do?orgId=331&tblId=DT_331001_2021BF003&conn_path=I2.

Kim, D. (2004): "Accounting for Economic Growth by Sector and International Comparison," Discussion Paper 2004-07, Korean Development Institute, Sejong, South Korea.

Kim, J., and S. Choi (2023): "The impact of regional income and housing price on fertility rate," *Journal of Korea Real Estate Society*, 41(1), 141–160, https://www.riss.kr/link?id=A108531244.

Kim, S., M. Tertilt, and M. Yum (2024): "Status externalities in education and low birth rates in Korea," *American Economic Review*, 114(6), 1576–1611.

Kongsamut, P., S. Rebelo, and D. Xie (2001): "Beyond balanced growth," *The Review of Economic Studies*, 68(4), 869–882.

KOSIS (2022a): "Parental Leave Statistics - Use Rate," Korean Statistical Information Service, data retrieved from https://kosis.kr/statHtml/statHtml.do?orgId=101&tblId=DT_CC2020D001&conn_path=I2.

——— (2022b): "Total Fertility Rates and Age-Specific Fertility Rates for Provinces," Korean Statistical Information Service, data retrieved from https://kosis.kr/statHtml/statHtml.do?orgId=101&tblId=DT_1B81A21&conn_path=I2.

——— (2023): "Private Education Expenditures Survey of Elementary, Middle, and High School Students," Korean Statistical Information Service, data retrieved from https://kosis.kr/statHtml/statHtml.do?orgId=101&tblId=DT_1PE003&conn_path=I2.

Kuznets, S., and J. T. Murphy (1966): *Modern economic growth: Rate, structure, and spread*, vol. 2. Yale University Press New Haven.

Lacomba, J. A., and F. Lagos (2006): "Population aging and legal retirement age," *Journal of Population Economics*, 19, 507–519.

Laroque, G., and B. Salanié (2014): "Identifying the response of fertility to financial incentives," *Journal of Applied Econometrics*, 29(2), 314–332.

Lee, C.-H., and J.-C. Noh (2021): "A Study on the Improvement Approaches of Immigration Workers' Legal System to Introduce Foreign Workers according to change the Population Structure (Low Fertility and Aging)," *Industry Promotion Research*, 6(1), 79–86.

Lee, R., and A. Mason (2009): "Fertility, human capital, and economic growth over the demographic transition," *European Journal of Population*, 26(2), 159.

Lee, S.-S. (2009): "Low fertility and policy responses in Korea," *The Japanese Journal of Population*, 7(1), 57–70.

Leers, T., L. Meijdam, and H. A. Verbon (2004): "Ageing, migration and endogenous public pensions," *Journal of Public Economics*, 88(1-2), 131–159.

Luci-Greulich, A., and O. Thévenon (2014): "Does economic advancement 'cause'a re-increase in fertility? An empirical analysis for OECD countries (1960–2007)," *European Journal of Population*, 30(2), 187–221.

Ma, L. (2014): "Labor force participation, family policy change and second birth rates in South Korea," *Stockholm Research reports in Demography*, 10, 1–31.

Matsukura, R., S. Shimizutani, N. Mitsuyama, S.-H. Lee, and N. Ogawa (2018): "Untapped work capacity among old persons and their potential contributions to the "silver dividend" in Japan," *The Journal of the Economics of Ageing*, 12, 236–249.

Matsuyama, K. (1992): "Agricultural productivity, comparative advantage, and economic growth," *Journal of economic theory*, 58(2), 317–334.

——— (2009): "Structural change in an interdependent world: A global view of manufacturing decline," *Journal of the European Economic Association*, 7(2-3), 478–486.

Milligan, K. (2005): "Subsidizing the Stork: New Evidence on Tax Incentives and Fertility," *Review of Economics and Statistics*, 87(3), 539–555.

Mulligan, C. B., and X. Sala-i Martin (1999): "Social security in theory and practice (I): Facts and political theories," Working paper 7118 (National Bureau of Economic Research, Cambridge, MA).

NABO (2021): "Analysis on the Policy Responses to Low Fertility," Assessment report, National Assembly Budget Office, https://viewer.nabo.go.kr/streamdocs/view/sd;streamdocsId=72059330699327227.

——— (2023): "Total tax revenue (excluding social security) and Total tax revenue as percentage of GDP," National Assembly Budget Office, data retrieved from https://www.nabostats.go.kr/portal/stat/easyStatPage/T188383000323998.do.

Ngai, L., and C. A. PIssarides (2007): "Structural change in a multisector model of growth," *The American Economic Review*, 97(1), 429–443.

NPS (2022): "National Pension Statistics," data retrieved from https://kosis.kr/statHtml/statHtml.do?orgId=322&tblId=DT_32202_B043&conn_path=I2.

OECD (2023a): "Education at a Glance 2023," *OECD Indicators*, OECD Publishing, Paris, https://doi.org/10.1787/e13bef63-en.

——— (2023b): "Fertility Rates," *OECD Family Database*, data retrieved from http://www.oecd.org/els/family/database.htm.

Okamoto, A. (2021): "Immigration policy and demographic dynamics: Welfare analysis of an aging Japan," *Journal of the Japanese and International Economies*, 62, 101168.

Pierson, P. (1996): "The new politics of the welfare state," *World politics*, 48(2), 143–179.

Samuelson, P. A. (1958): "An exact consumption-loan model of interest with or without the social contrivance of money," *Journal of political economy*, 66(6), 467–482.

Seo, M. (2013): "Do House Price Changes Affect Household Birth Rates?," *The Journal of Women and Economics*, 10(1), 63–79, https://www.riss.kr/link?id=A99678120.

Solow, R. M. (1956): "A contribution to the theory of economic growth," *The quarterly journal of economics*, 70(1), 65–94.

Sposi, M. (2012): "Evolving Comparative Advantage, Structural Change, and the Composition of Trade," *Manuscript, University of Iowa*.

Storesletten, K. (2000): "Sustaining fiscal policy through immigration," *Journal of political Economy*, 108(2), 300–323.

Tabata, K. (2005): "Population aging, the costs of health care for the elderly and growth," *Journal of Macroeconomics*, 27(3), 472–493.

Teignier, M. (2018): "The role of trade in structural transformation," *Journal of Development Economics*, 130, 45–65.

Thévenon, O. (2011): "Family policies in OECD countries: A comparative analysis," *Population and development review*, 37(1), 57–87.

Uy, T., K.-M. Yi, and J. Zhang (2013): "Structural change in an open economy," *Journal of Monetary Economics*, 60(6), 667–682.

Van Groezen, B., T. Leers, and L. Meijdam (2003): "Social security and endogenous fertility: pensions and child allowances as Siamese twins," *Journal of public economics*, 87(2), 233–251.

van Groezen, B., and L. Meijdam (2008): "Growing old and staying young: population policy in an ageing closed economy," *Journal of Population Economics*, 21, 573–588.

Wigger, B. U. (1999): "Pay-as-you-go financed public pensions in a model of endogenous growth and fertility," *Journal of Population Economics*, 12, 625–640.

Wise, D. A. (2012): *Social security programs and retirement around the world: Historical trends in mortality and health, employment, and disability insurance participation and reforms*. University of Chicago Press.

World Bank (2023): "World Development Indicators," data retrieved from https://databank.worldbank.org/source/world-development-indicators.

Yoo, S. H. (2016): "Postponement and recuperation in cohort marriage: The experience of South Korea," *Demographic Research*, 35, 1045–1078.