Adaptive Learning with a Unit Root: An Application to the Current Account[*]

Ronald B. Davies[†]
University of Oregon

Paul Shea[‡]
University of Oregon

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Abstract

This paper develops a simple two-country, two-good model of international trade and borrowing that suppresses all previous sources of current account dynamics. Under rational expectations, international debt follows a random walk. Under adaptive learning however, international debt behaves like either a stationary or an explosive process. Whether debt converges or diverges depends on the model's exact specification and the specific learning algorithm that agents employ. When debt diverges, a financial crisis eventually occurs to ensure that the model's transversality condition holds. Such a financial crisis causes an abrupt decrease in the debtor country's consumption and utility.

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[†] 435 PLC Building, 1285 University of Oregon, Eugene, OR 97403, USA. Phone: +1 (541) 346-4671. Email: rdavies@uoregon.edu.
[‡] Corresponding author. 435 PLC Building, 1285 University of Oregon, Eugene, OR 97403, USA. Phone: +1 (541) 346-1305. Email: pshea@uoregon.edu.

1. Introduction

Attempts to explain movements in a country's current account have been a major focus of theoretical open-economy macroeconomics. As Obstfeld and Rogoff (1996) discuss, the two dominant strains of this literature are the intertemporal approach and the overlapping generations model. The intertemporal approach uses an infinite horizon model to predict that a country experiencing a transitory, positive output shock will move towards a current account surplus while a country experiencing a transitory, negative shock will move towards a current account deficit. The current account is therefore a mechanism for intertemporal consumption smoothing. Overlapping generations models provide a different explanation. Here, the current account equals net public saving plus net private saving. If a country has a relatively young population, then a large fraction of its population will save for retirement and the current account will move towards a surplus. A country with an older population will draw on its savings and the current account will move towards a deficit.[1]

This paper uses a model similar to the intertemporal approach but suppresses all previous sources of current account dynamics. Our assumption that agents form expectations through adaptive learning instead of rational expectations alone drives the model's dynamics. The basis of our model is a Ricardian framework in which exogenous technological differences leads to complete specialization. Output depends on a serially correlated, observable technology shock that governs the translation between labor (the only input) and output. Each country chooses its level of consumption of both goods and

[1] This explanation has used by Fed chairman Benjamin Bernanke (2005) when explaining the U.S.'s large current account deficit.

its level of debt. In the model's rational expectations equilibrium (REE), international debt follows a random walk without drift. When we replace rational expectations with adaptive learning, however, the dynamics of international debt fundamentally change. Depending on the model's exact specification and agents' specific learning algorithms, debt will behave like either a stationary or an explosive process. If debt behaves like an explosive process, then a financial crisis will eventually occur to ensure that debt does not violate the model's transversality condition.

Under rational expectations, the model does not produce a unique steady state; rather, a continuum of steady states exists where any level of international debt corresponds to a different steady state. We linearize the model around its debt-free steady state, and find that in equilibrium both countries will attempt to keep their level of debt constant. As a result, the current account will depend only on a white noise error term. Under adaptive learning, debt is either stationary or follows an explosive process. Using both a baseline and a simplified version of recursive least squares learning, we identify cases where debt follows an explosive process.

Under adaptive learning, the AR(1) coefficient on debt is a function of the model's learning parameters. When the learning parameters equal their rational expectations values, the AR(1) coefficient equals one and debt follows a random walk. Adaptive learning, however, keeps the economy away from its REE and the AR(1) coefficient need not equal one. When the AR(1) coefficient on debt is a concave function, debt follows a stationary process. On the other hand, when it is a convex, debt is explosive. Different approaches to modeling learning yield different functions, some concave and some convex. Thus, under many reasonable types of learning, a free or pre-

determined variable that follows a random walk under rational expectations will not follow a random walk under adaptive learning. Therefore, along with providing a new explanation for current account movements and currency crises, our results demonstrate that introducing learning into a model that has a unit root under rational expectations may fundamentally change the model's dynamics.

Relatively few papers have analyzed the effects of learning on the dynamics of an open economy. Arifovic (1996) examines a two-country model with a continuum of steady state exchange rates. When a genetic learning algorithm replaces the assumption of rational expectations, the exchange rate appears to follow a random walk. This result differs from our model where adaptive learning eliminates the model's unit root. Kasa (2004) introduces learning into the Obstfeld (1997) "escape clause" model. Learning causes the exchange rate to follow a Markov process that helps explain recurring currency crises.

The paper proceeds as follows. Section 2 lays out the basic model. Section 3 solves the model under rational expectations. Section 4 replaces the assumption of rational expectations with the assumption that agents learn adaptively using recursive least squares. There, we find that unlike under rational expectations, debt follows an explosive process. Section 5 generalizes this result further by showing that in any model with a unit root under rational expectations, learning can cause that process to be either stationary or explosive. Section 6 discusses how explosive debt leads to currency crises and examines how the rate of learning affects the time until a crisis. Section 7 discusses alternative approaches to modeling learning that cause debt to be to stationary. Section 8 concludes.

2. A Simple General Equilibrium Model of International Trade

Our general equilibrium model builds off of the well-known Ricardian model of trade where relative technological differences across countries drive comparative advantages.[2] We consider two countries: Home and Foreign.[3] Each country can convert its exogenous stock of labor into two consumption goods, X and Y. We normalize the stock of labor in each country to one.

As is standard, production always exhibits constant returns to scale, but we assume that all unit labor requirements follow exogenous, stationary processes over time. Home's unit labor requirements for goods X and Y in period *t* are s_t^{-1} and a_t^{-1} respectively. Both of these unit labor requirements evolve according to AR(1) processes:

$$s_t = s_{t-1}^{\rho}\varepsilon_t \text{ and} \tag{2.1}$$

$$a_t = a_c a_{t-1}^{\rho}\varepsilon_t$$

where $\ln(\varepsilon_t)$ is mean-zero white noise and $\rho \in (0,1)$.

Similarly, Foreign's unit labor requirements for goods Y and X in period *t* are s_t^{*-1} and a_t^{*-1} respectively. These also evolve according to AR(1) processes:

$$s_t^* = s_{t-1}^{*\rho}\varepsilon_t^* \text{ and}$$

$$a_t^* = a_c^* a_{t-1}^{*\rho}\varepsilon_t^*$$

where $\ln(\varepsilon_t^*)$ is also mean-zero white noise.

[2] See Bhagwati, Panagariya, and Srinivasan (1998) for a presentation of the classic Ricardian model.
[3] An asterisk (*) denotes Foreign variables.

The two error terms, ε_t and ε_t^*, represent observable, country-specific, industry-neutral technology shocks. We assume that a_c and a_c^* lie between zero and one, which ensures that the following condition holds:

$$\frac{a_t}{s_t} < \frac{s_t^*}{a_t^*}.$$

This assumption implies that Home has a comparative advantage in the production of X. As is standard in the Ricardian model, with trade Home will completely specialize in the production of X, while Foreign will completely specialize in the production of Y. Therefore, in the trade equilibrium s_t denotes Home (and world) production of X, and s_t^* denotes Foreign (and world) production of Y. We assume that the autoregressive parameter, $\rho \in (0,1)$, is identical for both countries.

Consumers in both countries derive utility from the consumption of both goods. Per-period utility in each country is given by Cobb-Douglas utility functions, where $\alpha \in (0,1)$:

$$u_t = \ln(X_t^{\alpha} Y_t^{1-\alpha} \xi_t) \text{ and}$$

$$u_t^* = \ln(X_t^{*\alpha} Y_t^{*1-\alpha} \xi_t^*).$$

The variables X_t and Y_t denote Home's consumption of goods X and Y.[4] The variables $\ln(\xi_t)$ and $\ln(\xi_t^*)$ are exogenous, white noise preference shocks that affect each country's marginal utility. Incorporating these preference shocks into the model has two small but useful effects. First, under rational expectations, it causes international debt to

[4] In equilibrium, Home's consumption of good X, X_t, will be less than its production of good X, s_t.

follow a random walk rather than being constant. Second, under adaptive learning, it ensures that the learning process is persistent.

Home and Foreign trade where P_t represents the relative price of Y in terms of X. The Cobb-Douglas form of the utility functions requires that both countries consume positive amounts of both goods each period, otherwise utility will approach negative infinity. Therefore in equilibrium, both countries will always choose to trade with each other. In addition, one country may borrow from the other at the interest rate, r_{t+1}. The variable N_t represents Foreign's debt to Home, expressed in terms of good X. Because the model does not include capital, the only way that one country can save is to make loans to the other country. Debt evolves according to the following equation:

$$N_t = (1+r_t)N_{t-1} + s_{t-1} - X_{t-1} - P_{t-1}Y_{t-1}. \tag{2.2}$$

World consumption of good X must equal Home's production of good X, and world consumption of good Y must equal Foreign's production of good Y:

$$X_t + X_t^* = s_t \text{ and} \tag{2.3}$$

$$Y_t + Y_t^* = s_t^*. \tag{2.4}$$

Both countries discount utility at the rate β. Home's intertemporal utility maximization problem entails choosing X_t and Y_t to maximize:

$$\max_{X_t,Y_t} E_t\{\sum_{i=0}^{\infty} \beta^{t+i}[\ln(X_{t+i}^{\alpha} Y_{t+i}^{1-\alpha} \xi_{t+i})]\}.$$

Home's maximization problem is subject to Equations (2.1) and (2.2), and a No-Ponzi Games condition:

$$E_t[\lim_{i\to\infty} \beta^{t+i} N_{t+i}] \le 0.$$

The Cobb-Douglas form of each utility function ensures that both countries will spend constant shares of their total expenditures on each good. It is therefore possible to eliminate both X_t and Y_t from Home's maximization problem, and instead rely on the value of Home's consumption: $M_t = X_t + P_t Y_t$. Home's maximization problem yields an Euler Equation and a transversality condition:

$$M_t^{-1}\xi_t^{-1} = \beta(1+r_{t+1})E_t[(M_{t+1}^{-1}\xi_{t+1}^{-1}] \text{ and} \tag{2.5}$$

$$E_t[\lim_{i\to\infty} \beta^{t+i} N_{t+i}] = 0\,. \tag{2.6}$$

We assume that the rate of return on debt between periods *t* and *t+1* is specified at the time of debt's purchase. We therefore treat this rate of return, r_{t+1}, as known. Foreign's intertemporal utility maximization problem mimics that of Home and yields an additional Euler Equation. Defining the value of Foreign's consumption as $M_t^* = X_t^* + P_t Y_t^*$:

$$M_t^{*-1}\xi_t^{*-1} = \beta(1+r_{t+1})E_t^*[M_{t+1}^{*-1}\xi_{t+1}^{*-1}]\,. \tag{2.7}$$

Equations (2.2), (2.3), and (2.4) may also be re-stated in terms of M_t and M_t^*:

$$N_t = (1+r_t)N_{t-1} + s_{t-1} - M_{t-1} \text{ and} \tag{2.8}$$

$$\alpha(M_t + M_t^*) = s_t\,. \tag{2.9}$$

Equations (2.1), (2.5), (2.7), (2.8), and (2.9) fully characterize the system. By relying on the value of Home's and Foreign's consumption, we eliminate s_t^* and P_t from the system. We can now consider the model's "temporary" equilibrium for any pair of expectations, $E_t[M_{t+1}]$ and $E_t^*[M_{t+1}^*]$. Agents use their Euler Equations, (2.5) and (2.7), to determine their current level of consumption. The interest rate, r_{t+1}, endogenously adjusts to ensure that the global resource constraint, Equation (2.9), is satisfied. The debt

accumulation equation (2.8) then determines the next period's level of debt. Section 3 discusses the model where agents form expectations using rational expectations. Sections 4 through 7 discuss the model where agents form expectations using adaptive learning.

3. Solving the Model Under Rational Expectations

We define the system's steady state as $\bar{z} = [\bar{M}, \bar{M}^*, \bar{N}, \bar{s}, \bar{r}]$. Using Equation (2.1) and the assumption that $\ln(\varepsilon_t)$ is mean-zero white noise, Home's steady state production of good X, $\bar{s}$, equals one. Both Euler Equations, (2.5) and (2.7), simplify to the same expression when evaluated at their steady state:

$$\bar{r} = \beta^{-1} - 1. \tag{3.1}$$

Two equations, (2.8) and (2.9), remain to identify three steady state values: $\bar{M}$, $\bar{M}^*$, and $\bar{N}$. The model therefore does not produce a unique steady state. Instead, a continuum of steady states exists where any value of $\bar{N}$ corresponds to the following steady state values of $\bar{M}$, and $\bar{M}^*$:

$$\bar{M} = (\beta^{-1} - 1)\bar{N} + 1 \text{ and}$$

$$\bar{M}^* = -(\beta^{-1} - 1)\bar{N} + (1 - \alpha)/\alpha.$$

At any steady state, $\bar{\xi} = \bar{\xi}^* = 1$, and both countries perfectly smooth their consumption. The model's two Euler Equations show that, without preference shocks, perfect consumption smoothing occurs if and only if the interest rate equals its steady state value. Because the steady value of the interest rate does not depend on the steady state values of either debt or consumption, however, any level of debt is consistent with perfect consumption smoothing and a continuum of steady states exists. At any steady

state level of debt, both countries are content to perpetually make (or receive) interest payments on their debt (or outstanding loans). For the remainder of the paper, we will rely on the steady state where debt equals zero: $\bar{z}_o = [1, (1-\alpha)/\alpha, 0, 1, \beta^{-1} - 1]$.[5]

To analyze the model's dynamics under rational expectations, we approximate the system using a first order Taylor Series expansion around the debt-free steady state, $\bar{z}_0$. Defining $\tilde{z}_t = z_t - \bar{z}_0$, the linearized system becomes:[6]

$$\tilde{s}_t = \rho \tilde{s}_{t-1} + \tilde{\varepsilon}_t, \tag{3.2}$$

$$\tilde{M}_t = E_t[\tilde{M}_{t+1}] - \beta \tilde{r}_{t+1} - \tilde{\xi}_t, \tag{3.3}$$

$$\tilde{M}_t^* = E_t^*[\tilde{M}_{t+1}^*] - (1-\alpha)\beta \tilde{r}_{t+1}/\alpha - \tilde{\xi}_t^*, \tag{3.4}$$

$$N_t = -\tilde{M}_{t-1} + \beta^{-1} N_{t-1} + \tilde{s}_{t-1} \text{ and} \tag{3.5}$$

$$\alpha(\tilde{M}_t + \tilde{M}_t^*) = \tilde{s}_t. \tag{3.6}$$

The use of linearizations to approximate non-linear models is common in dynamic macroeconomics. In this case, it introduces two sources of error into the analysis. The first source of error is the approximation error associated with linearizing a non-linear model around any steady state. This type of error is present in any macroeconomic analysis that uses a linear approximation and increases as the model moves further away from the steady state.[7] The presence of a continuum of steady states in this model, however, introduces a second source of approximation error. The decision-

[5] Appendix 2 re-linearizes the model each period around the steady state corresponding to the current level of debt. The major conclusions of this paper do not change.

[6] It is not possible to log-linearize the system because debt's steady state value is zero. The steady states of productivity and Home's consumption equal one, therefore their linearized and log-linearized values are identical.

[7] Dotsey and Mao (1992) attempt to quantify this first type of approximation error in models with a unique steady state. They conclude that approximation errors are generally small for sufficiently small deviations from the model's steady state.

making rules of Equations (3.2) through (3.6) apply only to the debt-free steady state and are valid approximations only if the economy is sufficiently close to this steady state. International Real Business Cycle models (IRBC) also frequently produce a continuum of steady states. Letendre (2002) attempts to quantify the second source of approximation error caused by using a linear approximation of an IRBC model. He concludes that the approximation errors are small as long as the model is sufficiently close to the steady state that it is linearized around.[8]

Equation (3.6) shows that the value of Foreign's consumption is a linear combination of $\tilde{s}_t$ and $\tilde{M}_t$. It is therefore easy to eliminate $\tilde{M}_t^*$ from the system. By combining Equations (3.3) and (3.4), we also eliminate the interest rate from the system. Defining the white noise error term: $\omega_t = \alpha\tilde{\xi}_t^* - (1-\alpha)\tilde{\xi}_t$, the system now consists of Equations (3.2), (3.5), and:

$$\tilde{M}_t = (1-\alpha)E_t[\tilde{M}_{t+1}] + \alpha E_t^*[\tilde{M}_{t+1}] + (1-\rho)\tilde{s}_t + \omega_t . \tag{3.7}$$

Equations (3.2) and (3.5) define the evolution of the pre-determined variables $\tilde{s}_t$ and N_t. Equation (3.7) combines Home and Foreign's Euler Equations, relating current consumption to expected future consumption. Equation (3.7) allows Home and Foreign to have different expectations of future consumption. Under rational expectations, however, both countries necessarily form identical expectations and it is possible to re-state Equation (3.7) as:

$$\tilde{M}_t = E_t[\tilde{M}_{t+1}] + (1-\rho)\tilde{s}_t + \omega_t . \tag{3.8}$$

[8] There are two potential approaches to eliminating the second source of approximation error. The first is to directly simulate the non-linear model. This approach would eliminate both sources of approximation error. The second approach is to re-calculate agents' decision making rules each period around the steady state corresponding to that period's level of debt. We pursue the latter approach in Appendix 2 and demonstrate that this paper's major conclusions do not change.

By re-dating Equation (3.8), it is possible to write the model in first-order form:

$$\begin{pmatrix} \tilde{M}_t \\ N_t \\ \tilde{s}_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & (\rho - 1) \\ -1 & \beta^{-1} & 1 \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} \tilde{M}_{t-1} \\ N_{t-1} \\ \tilde{s}_{t-1} \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\mu}_t \\ \omega_t \\ \tilde{\varepsilon}_t \end{pmatrix}$$

or using vector notation:

$$\tilde{z}_t = G\tilde{z}_{t-1} + \zeta_t . \tag{3.9}$$

The term $\tilde{\mu}_t = E_{t-1}[\tilde{M}_t] - \tilde{M}_t$ represents an extraneous expectational error that may affect the system. The three eigenvalues of the matrix G are 1, β^{-1}, and ρ. Because $\beta^{-1} > 1$, Equation (3.9) represents an explosive system. The model's transversality condition, Equation (2.6), may therefore be violated and it is necessary to suppress the explosive root, β^{-1}, in order to derive a non-explosive solution. This requires factoring the G matrix so that:

$$G = S\Lambda S^{-1}.$$

The matrix S consists of J's eigenvectors, and Λ consists only of the corresponding eigenvalues along the diagonal. By defining $\tilde{k}_t = S^{-1}\tilde{z}_t$, it is possible to re-write Equation (3.9) as:

$$\tilde{k}_t = \Lambda\tilde{k}_{t-1} + S^{-1}\zeta_t .$$

To suppress the explosive root, we set the row of $\tilde{k}_t$ that corresponds to β^{-1} equal to zero. This entails setting a linear combination of the variables in the system equal to zero. This side constraint details how agents choose the free variable, $\tilde{M}_t$, and eliminates the extraneous expectational error, $\tilde{\mu}_t$, from the system. The model therefore possesses a unique solution. The relevant side constraint is:

$$\tilde{M}_t = (\beta^{-1} - 1)N_t + \tilde{s}_t + \beta\omega_t \tag{3.10}$$

Substituting Equation (3.10) into Equation (3.5) reveals that the necessary condition for the REE to be non-explosive is $N_t = N_{t-1} - \beta\omega_{t-1}$. Defining the current account as $N_t - N_{t-1}$, the current account simply equals white noise and there are no current account dynamics in the REE.[9] Imposing Equation (3.10) yields the REE's VAR(1) reduced form:

$$\begin{pmatrix} \tilde{M}_t \\ N_t \\ \tilde{s}_t \end{pmatrix} = \begin{pmatrix} 1-\beta^{-1} & \beta^{-1}(\beta^{-1}-1) & \rho-1+\beta^{-1} \\ -1 & \beta^{-1} & 1 \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} \tilde{M}_{t-1} \\ N_{t-1} \\ \tilde{s}_{t-1} \end{pmatrix} + \begin{pmatrix} 1 & \beta \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\varepsilon}_t \\ \omega_t \end{pmatrix} \tag{3.11}$$

Equation (3.11) possesses a unit root and is therefore not a stationary process. Suppose that Home experiences a positive productivity shock where $\tilde{\varepsilon}_t > 0$. For simplicity, assume that $\omega_t = 0$. The intertemporal approach to the current account predicts that Home will attempt to smooth the effects of this shock over time by lending to Foreign.[10] The current account, $N_t - N_{t-1}$, will therefore rise above zero. In our model with rational expectations, however, the current account will equal zero because of our choice of utility functions. We have normalized the price of good X to one. Home's income is therefore s_t. Foreign's income is the price of good Y multiplied by s_t^*. The price of good Y depends on the ratio of both goods' unit labor requirements and the relative weighting of each good in the utility functions:

$$P_t = (1-\alpha)s_t / (\alpha s_t^*).$$

[9] The current account equals white noise because we linearize the model around the debt free steady state. In Appendix 2, we re-linearize the model each period around the steady state corresponding to the current level of debt. While debt continues to follow a random walk, in this case it depends on both white noise preference shocks and autocorrelated productivity shocks

[10] See Obstfeld and Rogoff (1996).

A positive productivity shock has two effects. First, for any P_t, it raises Home's wealth and therefore its utility. Second, by making good Y scarcer relative to good X, it increases P_t which benefits Foreign but harms Home. With our specification, this latter terms of trade effect is large enough so that Foreign's income experiences the same proportional increase as Home's.[11] One country's saving, however, necessarily equals the other's borrowing. The equilibrium interest rate must therefore adjust to the productivity shock to ensure that global saving equals zero. Because both countries have identical incentives to save, however, this can only occur when N_t equals zero. If the model includes preference shocks, then the current account will equal white noise. In the next section, we replace the assumption of rational expectations with adaptive learning. The lack of current account dynamics in the REE allows us to isolate the effects of adaptive learning on the current account.

4. E-Stability

So far, we have assumed that both countries form rational expectations. Rational expectations assume that agents know the coefficients in the model's side constraint that sets consumption equal to a linear combination of debt, productivity, and preference shocks:

$$\tilde{M}_t = (\beta^{-1} - 1) N_t + \tilde{s}_t + \beta \omega_t . \tag{4.1}$$

An infinite number of models could generate this model's reduced form. Rational expectations is a realistic assumption if both countries agree that this model best explains

[11] This is weaker version of the immiserizing growth effect where the impact of Home's productivity shock on the terms of trade is so large that Home's utility decreases. In our model, however, both countries benefit from Home's productivity shock. For details on immiserizing growth, see Johnson (1954) and Bhagwati (1969).

the economy, know the calibrated parameter values, and are able to solve for Equation (4.1). However, if both countries do not know which model generates this reduced form, then rational expectations is not a realistic assumption.

We therefore now examine the model when agents use adaptive learning instead of rational expectations.[12] A primary goal of this section is to provide unfamiliar readers with an introduction to adaptive learning using the context of our model. There are many sensible methods for modeling adaptive learning. However, because of the unit root in debt, the method chosen leads to very different predictions regarding the behavior of the current account. This section presents our baseline approach where debt behaves like an explosive process. Section 6 discusses the model under *coordinated learning*, another, simpler approach under which debt behaves like an explosive process. Section 7 discusses alternate methods of modeling learning where debt behaves like a stationary process.[13]

In presenting our baseline case, we focus on the expectational or E-Stability of the model. Evans and Honkapohja (2001) demonstrate that under general conditions, a model is stable under adaptive learning if and only if it is E-Stable. This approach to modeling adaptive learning assumes that agents know that consumption is a linear combination of the other variables in the system, but do not know the values of the coefficients in Equation (4.1). This yields agents' *perceived law of motion* (PLM) for Home:[14]

$$\tilde{M}_t = aN_t + b\tilde{s}_t . \qquad (4.2)$$

[12] For a thorough discussion of adaptive learning algorithms, see Evans and Honkapohja (2001).
[13] Several additional types of learning are examined in Appendix 1.
[14] We assume that because agents' data is measured as deviations from the zero debt steady state, agents are able to deduce that the side constraint's intercept equals zero. They therefore employ a properly specified PLM. Including intercept terms in the model's PLM does not affect whether the model is E-Stable or whether debt behaves explosively for any of the learning approaches discussed in the body of this paper.

Because agents do not know which structural model generates Equation (4.2), they are unaware that one of their regressors follows a random walk under rational expectations.

Under rational expectations, we are able to eliminate M_t^* from the model. Our baseline learning method, however, focuses on *uncoordinated learning* where Foreign uses data on its own consumption to forecast its own future consumption. Equation (3.7) includes Foreign's expectation of Home's consumption. By substituting the global resource constraint, Equation (3.6), into this equation we re-write the model's forward-looking structural equation to include Foreign's expectation of its own consumption:

$$\tilde{M}_t = (1-\alpha)E_t[\tilde{M}_{t+1}] - \alpha E_t^*[\tilde{M}_{t+1}^*] + \tilde{s}_t + \omega_t . \tag{4.3}$$

Using the global resource constraint, Equation (3.6), it is also possible to re-state the model's side constraint in terms of Foreign's consumption:

$$\tilde{M}_t^* = (1-\beta^{-1})N_t + (\alpha^{-1}-1)\tilde{s}_t - \beta\omega_t . \tag{4.4}$$

We assume that Foreign's agents base their PLM on Equation (4.4):

$$\tilde{M}_t^* = c\tilde{N}_t + d\tilde{s}_t . \tag{4.5}$$

We assume that agents know the coefficients in Equations (3.2) and (3.5), which detail the evolution of the pre-determined variables. Home agents use their PLM, Equation (4.2), to form their expectations of future consumption:[15]

$$E_t[\tilde{M}_{t+1}] = aE_t[N_{t+1}] + bE_t[\tilde{s}_{t+1}],$$

$$E_t[\tilde{s}_{t+1}] = \rho\tilde{s}_t \text{ and}$$

$$E_t[\tilde{N}_{t+1}] = -E_t[\tilde{M}_t] + \beta^{-1}\tilde{N}_t + \tilde{s}_t .$$

[15] The expectations operator on contemporaneous consumption reflects the possibility that agents may not know M_t when choosing their level of consumption.

It is reasonable to assume either that Home agents know the current value of $\tilde{M}_t$ when forming expectations at time *t*, or that they only know the value of $\tilde{M}_{t-1}$. Using the former assumption, Home agents' expectation of future consumption equals:[16]

$$E_t[M_{t+1}] = a(-\tilde{M}_t + \beta^{-1}N_t + \tilde{s}_t) + b\rho\tilde{s}_t . \tag{4.6}$$

Foreign agent's expectation of their future consumption equals:

$$E_t^*[M_{t+1}^*] = c(-\tilde{M}_t + \beta^{-1}N_t + \tilde{s}_t) + d\rho\tilde{s}_t . \tag{4.7}$$

Equations (4.6) and (4.7) are not rational expectations. Instead they represent agents' best estimate of future consumption given their informational deficiencies. Inserting these expectations into Equation (4.3), a forward-looking structural equation, yields the economy's *actual law of motion* (ALM) for $\tilde{M}_t$:

$$\begin{aligned}[1+(1-\alpha)a-\alpha c]\tilde{M}_t = {} & [(1-\alpha)a\beta^{-1} - \alpha c\beta^{-1}]N_t \\ & +[(1-\alpha)(a+b\rho) - \alpha(c+d\rho)+1]\tilde{s}_t + \omega_t .\end{aligned} \tag{4.8}$$

Substituting the economy's global resource constraint, Equation (3.6), into Equation (4.8) yields the ALM for Foreign's consumption:

$$\begin{aligned}(1+(1-\alpha)a-\alpha c)\tilde{M}_t^* = {} & [\alpha c\beta^{-1} - (1-\alpha)a\beta^{-1}]N_t \\ & +[(1+(1-\alpha)a-\alpha c)\alpha^{-1} - 1 - (1-\alpha)(a+b\rho) + \alpha(c+d\rho)]\tilde{s}_t - \omega_t .\end{aligned} \tag{4.9}$$

To evaluate E-Stability, we analyze the E-Stability differential equation based on the mapping from the models' PLM to its ALM:

$$d/d\tau(a,b,c,d) = T(a,b,c,d) - (a,b,c,d) . \tag{4.10}$$

The notation $T(a,b,c,d)$ refers to the vector of coefficients from the ALM, Equations (4.8) and (4.9), corresponding to (a,b,c,d) from the PLM, Equations (4.2) and (4.5). The

[16] Appendix 1 discusses the model using the latter assumption that agents do not know the current value of $\tilde{M}_t$. Under both assumptions, debt behaves like an explosive process.

eigenvalues of the Jacobian matrix from Equation (4.10) equal –1, -1, $\beta - 1$, and $\beta\rho - 1$ when evaluated at the model's REE. Because all of these eigenvalues have real parts less than one, the PLM converges to the ALM and the model is locally E-Stable.

E-Stability implies that, for this type of PLM, the model is stable under learning for most sensible learning algorithms.[17] We now consider one such learning algorithm: recursive least squares. Under recursive least squares, Home agents run an OLS regression of $\tilde{M}$ on N and $\tilde{s}$ to obtain their initial learning parameter estimates, a_0 and b_0. Foreign agents regress $\tilde{M}^*$ on N and $\tilde{s}$ to obtain their initial learning parameter estimates, c_0 and d_0. They then update their estimates each period as new data becomes available using the following algorithm:

$$\begin{pmatrix} a_t \\ b_t \end{pmatrix} = \begin{pmatrix} a_{t-1} \\ b_{t-1} \end{pmatrix} + \gamma_t R_t^{-1} \begin{pmatrix} N_{t-1} \\ \tilde{s}_{t-1} \end{pmatrix} (\tilde{M}_{t-1} - a_{t-1} N_{t-1} - b_{t-1} \tilde{s}_{t-1}),$$

$$R_t = R_{t-1} + \gamma_t [\begin{pmatrix} N_{t-1}^2 & N_{t-1}\tilde{s}_{t-1} \\ N_{t-1}\tilde{s}_{t-1} & \tilde{s}_{t-1}^2 \end{pmatrix} - R_{t-1}],$$

$$\begin{pmatrix} c_t \\ d_t \end{pmatrix} = \begin{pmatrix} c_{t-1} \\ d_{t-1} \end{pmatrix} + \gamma_t^* R_t^* \begin{pmatrix} N_{t-1} \\ \tilde{s}_{t-1} \end{pmatrix} (\tilde{M}_{t+1}^* - c_{t-1} N_{t-1} - d_{t-1} \tilde{s}_{t-1}) \text{ and}$$

$$R_t^* = R_{t-1}^* + \gamma_t^* [\begin{pmatrix} N_{t-1}^2 & N_{t-1}\tilde{s}_{t-1} \\ N_{t-1}\tilde{s}_{t-1} & \tilde{s}_{t-1}^2 \end{pmatrix} - R_{t-1}^*].$$

The gains, γ_t and γ_t^*, represents the weights placed on the most recent observation. We consider two variations of the learning process. Under decreasing-gain learning, agents typically weigh all observations equally in their estimations, and the gain therefore equals the inverse of the sample size. As *t* approaches infinity, the gain

[17] For exact conditions for when E-Stability implies stability under adaptive learning, see Evans and Honkapohja (2001).

approaches zero and the learning process converges if the model is stable under adaptive learning. Asymptotically the model will behave identically under rational expectations and adaptive learning, implying that asymptotically the current account is a random walk.

For the remainder of the paper, however, we will assume that agents use a constant-gain learning algorithm where γ is a constant. Constant-gain learning places greater emphasis on more recent observations than earlier observations. If the model includes preference shocks, as we have so far assumed, then the learning process will be persistent under constant-gain learning. This baseline approach is locally stable under learning. When simulated, the model's learning parameters, (a_t, b_t, c_t, d_t), remain close to their rational expectations values.[18] Switching from rational expectations to adaptive learning, however, fundamentally changes the dynamics of international debt. Numerical simulations demonstrate that debt now behaves like an explosive process rather than a random walk.[19]

5. Adaptive Learning with a Unit Root in a General Model

In our model, a unit root causes debt to follow a random walk under rational expectations. This section uses a more general model to illustrate how adaptive learning profoundly changes the dynamics of a system so that a unit root under rational expectations can behave like either a stationary or explosive process under adaptive learning.

[18] This approach is not, however, globally stable under learning. Sufficiently large shocks can move the model into a region where it is not stable under adaptive learning.
[19] Simulations of this case are available upon request.

Consider a model with a persistent learning process where a free or pre-determined variable, y_t, possesses a unit root under rational expectations:[20]

$$y_t = y_{t-1} + e_t$$

where $e_t \sim N(0, \sigma^2)$.

Under adaptive learning, the autoregressive process now depends on a vector of learning coefficients, denoted χ_t. Defining χ_{RE} as the vector of rational expectations counterparts to these learning parameters, the autoregressive process may be re-written as:

$$y_t = g(\chi_t) y_{t-1} + e_t$$

where $g(\chi_{RE}) = 1$.

Adaptive learning keeps the economy out of its rational expectations equilibrium. The function $g(\chi_t)$ will therefore typically not equal one. Two potential cases are of interest. First, the cumulative product of $g(\chi_t)$ may asymptotically approach zero:

$$\prod_{i=0}^{\infty} g(\chi_{t+i}) \to 0$$

In this case, adaptive learning causes the free or pre-determined variable to behave like a stationary process around zero.

The second case occurs if the cumulative product of $g(\chi_t)$ asymptotically approaches infinity:

$$\prod_{i=0}^{\infty} g(\chi_{t+i}) \to \infty$$

[20] The results of this section do not apply to a unit root on a purely exogenous variable.

In this case, adaptive learning causes the free or pre-determined variable to behave like an explosive process. As a result, it is possible that the model's transversality condition will be violated. Whether the free variable acts like a stationary or an explosive process depends not only on the specific problem being studied, but also on the exact type of learning algorithm that agents use and agents' information sets.[21]

Very little research has focused on adaptive learning in the presence of a unit root on either a free or pre-determined variable. Other research does show, however, that adaptive learning can fundamentally change the dynamics of a system in a different context. Evans, Honkapohja, and Marimon (2001) set up an overlapping generations model where the government finances its deficit by issuing money. If the deficit is constrained as a share of GDP, then the model often possesses four steady states. Under perfect foresight, it is possible to converge to two of these steady states, including an autarky solution where money is worthless and a hyperinflation occurs. Under adaptive learning, however, the economy always converges to an interior solution and a hyperinflation can never occur.

Sections 6 and 7 apply the results of this section to our model of international debt by examining two different types of learning where debt behaves like an explosive and stationary process, respectively.

[21] This section assumes that the learning process is persistent. If a model includes sufficient uncertainty, then constant-gain learning will typically be a persistent process. Decreasing-gain learning, however, is typically not a persistent process. If the learning process is not persistent, then model will approach a random walk as the learning process converges.

6. Explosive Debt

In our baseline learning approach of Section 4, the model is stable under adaptive learning but debt behaves like an explosive process. For sufficiently small preference shocks, the model's learning parameters remain in the neighborhood of their rational expectations values, but the absolute value of debt increases over time. Because analytical results are unclear in our baseline approach, we make one additional modification in this section by modeling *coordinated learning* instead of uncoordinated learning.[22] Under coordinated learning, both countries use Home's consumption data to form their expectations. Foreign's PLM therefore becomes:

$$\tilde{M}_t = cN_t + d\tilde{s}_t \,. \tag{6.1}$$

Foreign agents use Equation (6.1) to form their expectation of Home's future consumption. They then use the global resource constraint, Equation (3.6), to convert this expectation into an expectation of Foreign's future consumption:

$$E_t^*[\tilde{M}_{t+1}] = cE_t^*[N_{t+1}] + dE_t^*[\tilde{s}_{t+1}] \text{ and}$$

$$E_t^*[\tilde{M}_{t+1}^*] = -E_t^*[\tilde{M}_{t+1}] + E_t^*[\tilde{s}_{t+1}]/\alpha \,. \tag{6.2}$$

By inserting the expectation from Equation (6.2) into Equation (4.3), we are able to obtain the economy's ALM:

$$\begin{aligned}(1+(1-\alpha)a+\alpha c)\tilde{M}_t = {} & [((1-\alpha)a+\alpha c)\beta^{-1}]N_t \\ & +[(1-\alpha)(a+b\rho)+\alpha(c+d\rho)+1-\rho]\tilde{s}_t + \omega_t \,.\end{aligned} \tag{6.3}$$

The Jacobian of $d/d\tau(a,b,c,d) = T(a,b,c,d) - (a,b,c,d)$ has eigenvalues equal to

[22] Coordinated learning has two additional technical advantages over uncoordinated learning. First, while both methods are locally stable under learning, random shocks are less likely to move the coordinated learning algorithm into a region where it is unstable under learning. Second, it is computationally easier to repeatedly simulate coordinated learning than uncoordinated learning.

-1, -1, $\beta-1$, and $\beta\rho-1$ when evaluated at the model's REE. Because all of these eigenvalues have real parts less than zero, this type of coordinated learning is locally E-Stable.

By substituting Equation (6.3) into Equation (3.5), we re-write the debt accumulation equation:[23]

$$\begin{aligned} N_{t+1} = {} & \beta^{-1}[1-(1+(1-\alpha)a_t+\alpha c_t)^{-1}((1-\alpha)a_t+\alpha c_t)]N_t \\ & +[1-(1+(1-\alpha)a_t+\alpha c_t)^{-1}((1-\alpha)(a_t+b_t\rho)+\alpha(c_t+d_t\rho)+1-\rho)]\tilde{s}_t-\tilde{\omega}_t \end{aligned} . \quad (6.4)$$

If the learning parameters equal their rational expectations values, $[a,b]=[c,d]=[\beta^{-1}-1,1]$, then Equation (6.4) reduces to Equation (3.10) and debt follows a random walk. To understand the intuition for why debt behaves explosively, suppose that the learning parameters are distributed symmetrically around their rational expectations values.[24] In the case of homogeneous learning where $[a_t,b_t]=[c_t,d_t]$, Equation (6.4) may be re-stated:

$$N_{t+1}=\beta^{-1}[1-(1+a_t)^{-1}a_t]N_t+[1-(1+a_t)^{-1}(a_t+b_t\rho+1-\rho)]\tilde{s}_t-\tilde{\omega}_t .$$

Because $d^2(\beta^{-1}(1-a_t(1+a_t)^{-1}))/da_t^2=2\beta^{-1}(1+a_t)^{-2}(1-a_t(1+a_t)^{-1})>0$ for all a_t, $(1-a_t(1+a_t)^{-1})$ is a convex function. By Jensen's Inequality:

$$E[\beta^{-1}(1-a_t(1+a_t)^{-1})]>[\beta^{-1}(1-E[a_t](1+E[a_t])^{-1})]=1 .$$

It is therefore the case that:

$$E_t[\prod_{i=1}^{\infty}(1-a_{t+i}(1+a_{t+i})^{-1})]\to\infty$$

---

[23] Equation (6.4) defines $\tilde{\omega}_t=(1+(1-\alpha)a_t+\alpha c_t)^{-1}\omega_t$.

[24] Evans and Honkapohja (2001) formally prove this result in the case of standard, constant-gain learning. Their proof does not extend, however, to include cases where the model includes a unit root. Our numerical simulations confirm, however, that the distributions of the learning parameters are sufficiently symmetric around their rational expectations values.

and debt will behave like an explosive process. Numerical simulations confirm the explosiveness of debt in this case and that it extends to cases with heterogeneous and/or uncoordinated learning (as in Section 4).[25]

We now simulate the homogeneous case. We set $N_0 = .00$, $\gamma = \gamma^* = 0.01$, $[a_0, b_0] = [c_0, d_0] = [\beta^{-1} - 1, 1]$, and $R_0 = R_0^* = I_2$.[26] We assume that $\tilde{\varepsilon}_t$ is uniformly distributed between -.005 and .005 and that $\tilde{\omega}_t$ is uniformly distributed between -.0005 and .0005.[27] Debt now behaves like an explosive process, threatening to violate the model's transversality conditions. We assume that once the absolute value of debt reaches a pre-determined level, a financial crisis occurs.[28] For this simulation, we assume that a crisis occurs when $|N_t| > .50$.[29] When a crisis occurs, the debtor country is required to pay the creditor country interest plus 5% of its debt every period. Once the debt is below half of the crisis level, the debtor country is again free to borrow and the model once again operates normally. Figure 1 charts the simulated paths of Foreign's debt to Home and Foreign's consumption over a period where Foreign's debt happens to trigger

[25] We also directly simulate the non-linear model of Section 2 where agents continue to use linear PLMs. Once again, depending on the exact type of learning, both explosive and stationary cases exist. Simulation results for both the linear and non-linear models are available on request.

[26] I_2 is the 2x2 identity matrix.

[27] For these simulations, we treat the rational expectations value of $(1+(1-\alpha)a_t + \alpha c_t)^{-1}$ as known; therefore, $\tilde{\omega}_t = \beta\omega_t$.

[28] Evans and Honkapohja (2005) employ a similar strategy in an unrelated model. Their paper examines a New Keynesian model where the central bank is passive in responding to inflationary pressures and where the government does not actively attempt to balance its intertemporal budget constraint. In that model, government debt behaves explosively for all equilibrium paths. The authors assume that if government debt exceeds a pre-determined level, then the government pursues an alternate fiscal policy that ensures that its debt behaves like a stationary process. Marcet and Nicolini (2003) assume that the government switches its policy from using the money supply to finance seignorage to an exchange rate rule if and only if inflation exceeds a certain threshold.

[29] This cutoff is similar to the sudden stop/ "Wile E. Coyote" literature in open-economy macroeconomics. See Krugman (2006) for a discussion.

two crises.[30] For anecdotal comparison, compare this to Figure 2, which plots the actual values of debt and consumption for Argentina during its recent financial crisis. As can be seen, the two look remarkably similar.

The vast majority of research into adaptive learning assumes that all agents use the same learning algorithm. Under constant-gain learning, however, it is not obvious which gain agents should use.[31] It is reasonable to assume that different agents may use different gains. In this section, we will therefore examine cases where both Home and Foreign use the same gain and where they use different gains. Only a few papers have studied adaptive learning where agents use different gains in their constant-gain learning algorithms. Negroni (2003) divides the population into two groups, each of which uses a different gain. Heterogeneous gains make his model less likely to be stable under adaptive learning. Honkapohja and Mitra (2005) obtain similar results when agents use different learning algorithms.

We further evaluate the model's dynamics by repeating the previous simulation for several different pairs of gains. We simulate the learning process 5,000 times for nine gain combinations where γ and γ^* equal .01, .02, or .03. If a country's gain equals .01, then that country is a slow learner. If a country's gain equals .03, then that country is a fast learner. In all 45,000 simulations, debt behaves like an explosive process, threatening to violate the model's transversality conditions.[32] Table 1 summarizes the average time

[30] The time between crises is highly variable and does not always follow the pattern of Figure 1. For this simulation, the level of debt where the debtor country is able to borrow again, .25, is high enough to ensure that the debtor country rarely becomes the lender country. Lower values for this threshold increase the frequency of these switches.

[31] Evans and Ramey (2005) endogenize the gain in a simpler model with constant-gain learning.

[32] The model is only locally stable under adaptive learning. In some simulations, unobservable shocks drive the economy into a region where it is unstable under adaptive learning. These simulations are discarded.

until a crisis for all nine combinations of gains and yields the following three sets of hypotheses.

Null Hypothesis 1: $d(\gamma+\tau, \gamma^*) - d(\gamma, \gamma^*) = 0$ if $\gamma \geq \gamma^*$, for $\tau > 0$.

Null Hypothesis 1a: $d(\gamma, \gamma^*+\tau) - d(\gamma, \gamma^*) = 0$ if $\gamma^* \geq \gamma$, for $\tau > 0$.

These hypotheses state that if the faster learner uses a larger gain (learns even faster), then the time until a crisis is unaffected. We fail to reject these null hypotheses for all applicable gain combinations.

Null Hypothesis 2: $d(\gamma+\tau, \gamma^*) - d(\gamma, \gamma^*) = 0$ if $\gamma^* > \gamma$, for $\tau > 0$.

Null Hypothesis 2a: $d(\gamma, \gamma^*+\tau) - d(\gamma, \gamma^*) = 0$ if $\gamma > \gamma^*$, for $\tau > 0$.

These hypotheses state that if the slower learner uses a larger gain (learns faster), then the time until a crisis is unaffected. We reject these null hypotheses for all applicable gain combinations in favor of the alternative that increasing the gain of the slower learner decreases the time until a crisis.

Null Hypothesis 3: $d(\gamma, \gamma^*) - d(\gamma+\tau, \gamma^*+\tau) - d(\gamma, \gamma^*) = 0$ for, $\tau > 0$.

This hypothesis states that if both countries become faster learners, then the time until a crisis is unaffected. Graphically, this implies moving down and to the right in Table 1. We reject this null hypothesis for all applicable gain combinations in favor of the alternative that the time until a crisis decreases.

Null Hypothesis 4: $d(\gamma, \gamma^*) - d(\gamma^*, \gamma) = 0$.

This hypothesis states that Table 1 must be a symmetric matrix. When α equals one-half, both $E_t[\tilde{M}_{t+1}]$ and $E_t^*[\tilde{M}_{t+1}]$ enter Equation (3.7) in exactly the same manner. It therefore cannot matter whether Home or Foreign is the faster learner. We fail to reject this null hypothesis for all nine combinations of gains.

The first two sets of hypotheses yield two interesting conclusions. First, the gain of the slower learner has a larger effect on the time until a crisis occurs than the gain of the faster learner. Second, if both countries become faster learners, then the speed of divergence increases. If both gains equal zero, then adaptive learning and rational

expectations are identical in these simulations. Debt will display no tendency to diverge and the time until a crisis will be very long. In these simulations, the rational expectations values of the learning parameters are constant. Larger gains move the model further from its rational expectations equilibrium, strengthening the tendency to diverge, and shortening the time until a crisis.

7. Stationary Debt

Sections 4 and 6 demonstrate that a particular set of assumptions may cause debt to behave like an explosive process instead of a random walk. This section discusses alternate assumptions that cause debt to behave like a stationary process. So far, we have assumed that agents attempt to learn the model's side constraint, Equation (4.1), that sets consumption equal to a linear combination of debt, productivity, and preference shocks. It is also reasonable to assume that agents attempt to learn the model's VAR(1) reduced form, Equation (3.11), that sets consumption equal to a linear combination of the model's lagged variables and current shocks. If we also replace recursive least squares with stochastic gradient learning, then debt behaves like a stationary process regardless of whether or not agents know the current values of $\tilde{M}_t$ and whether agents use coordinated or uncoordinated learning.[33] Appendix 1 provides additional details on this type of learning.

8. Conclusion

This paper develops a simple general equilibrium model that suppresses all previous explanations of current account dynamics. The effects of adaptive learning on

[33] For a discussion of stochastic gradient learning, see Evans, Honkapohja, and Williams (2005).

the current account are therefore isolated. Under rational expectations, debt follows a random walk. Adaptive learning, however, fundamentally alters the models dynamics so that debt behaves like either an explosive or stationary process. Whether debt tends to diverge or converge depends on agents' specific learning algorithm.

This paper's conclusions regarding learning may be applicable to other models. International Real Business Cycle models also tend to exhibit an endogenous unit root under rational expectations. Given the similarities between our model and an IRBC model, we would expect these results to carry over to that class of models. Furthermore, this paper's general result may also extend to models that do not include a unit root. Consider a model that contains an explosive process under rational expectations due to an eigenvalue with an absolute value greater than one. If the absolute value of this eigenvalue is sufficiently close to one, then adaptive learning could cause the model to behave like a stationary process. Likewise, if a model is stationary under rational expectations, then adaptive learning could cause the model to behave like an explosive process if it contained an eigenvalue with an absolute value sufficiently close to one.

This paper's purpose is not to argue that previous explanations of the current account are invalid. Rather, its goal is to examine the effect of a new factor, adaptive learning, in a simplified environment. It is possible that the effect of learning on the current account will differ if previous explanations are also included in the model. Learning's role in a more complex model remains a rich area for further research.

Appendix 1: Different Types of Learning in the Model of International Debt

In our model, debt follows a random walk under rational expectations but behaves as either an explosive or stationary process under adaptive learning. This section discusses three assumptions that can affect debt's dynamics and reports debt's behavior for each permutation.

Assumption 1: Not Knowing $\tilde{M}_t$ vs. Knowing $\tilde{M}_t$

The learning algorithm of Section 4 assumes that at time *t*, agents know $\tilde{M}_t$ and $\tilde{M}_t^*$. If agents know $\tilde{M}_{t-1}$ and $\tilde{M}_{t-1}^*$, but not $\tilde{M}_t$ and $\tilde{M}_t^*$ at time *t*, the results potentially may change. Under this assumption, Home and Foreign's expectations of their future consumption equal:

$$E_t[M_{t+1}] = a(\beta^{-1} - a)N_t + [a(1-b) + b\rho]\tilde{s}_t \text{ and}$$

$$E_t^*[\tilde{M}_{t+1}^*] = c(\beta^{-1} + c)N_t + [c(1-\alpha^{-1} + d) + d\rho]\tilde{s}_t .$$

Inserting these expectations into Equation (4.3) yields the economy's ALM:

$$\begin{aligned}\tilde{M}_t &= [(1-\alpha)a(\beta^{-1} - a) - \alpha c(\beta^{-1} + c)]N_t \\ &\quad + [(1-\alpha)(a - ab + b\rho) - \alpha(c - c\alpha^{-1} + cd + d\rho) + 1]\tilde{s}_t + \omega_t\end{aligned} \text{ and}$$

$$\begin{aligned}\tilde{M}_t^* &= [-(1-\alpha)a(\beta^{-1} - a) + \alpha c(\beta^{-1} + c)]N_t \\ &\quad + [-(1-\alpha)(a - ab + b\rho) + \alpha(c - c\alpha^{-1} + cd + d\rho) - 1 + \alpha^{-1}]\tilde{s}_t + \omega_t\end{aligned} .$$

The results for this model, however, do not depend on whether or not agents know the current values of $\tilde{M}_t$ and $\tilde{M}_t^*$ for any of the types of learning that we consider.

Assumption 2: Learning the Side Constraint vs. Learning the VAR(1) with Stochastic Gradient Learning

The discussion of adaptive learning in Section 4 assumes that agents attempt to learn the model's side constraint by estimating current consumption as a function of

current debt and productivity. Alternatively, agents could attempt to learn the model's VAR(1) reduced form, represented by Equation (3.11), by regressing current consumption on lagged consumption, lagged debt, and lagged productivity. Consider the simpler case of coordinated learning where the PLM for Home and Foreign equals

$$\tilde{M}_t = a\tilde{M}_{t-1} + bN_{t-1} + c\tilde{s}_{t-1} + \tilde{\varepsilon}_t \text{ and}$$

$$\tilde{M}_t = d\tilde{M}_{t-1} + eN_{t-1} + f\tilde{s}_{t-1} + (\alpha^{-1} - 1)\tilde{\varepsilon}_t .$$

Assuming that agents know the current values of $\tilde{M}_t$ and $\tilde{M}_t^*$, both countries use this PLM to form their expectations of future consumption:

$$E_t[\tilde{M}_{t+1}] = a\tilde{M}_t + bN_t + c\tilde{s}_t \text{ and}$$

$$E_t^*[\tilde{M}_{t+1}] = d\tilde{M}_t + eN_t + f\tilde{s}_t .$$

Inserting these expectations into Equation (3.7) yields the economy's ALM:

$$\begin{aligned}[1-(1-\alpha)a-\alpha d]\tilde{M}_t &= [(1-\alpha)b+\alpha e]N_t \\ &\quad +[(1-\alpha)c+\alpha f+1-\rho]\tilde{s}_t + \omega_t \end{aligned} \text{ and}$$

$$\tilde{M}_t^* = -\tilde{M}_t + \tilde{s}_t / \alpha .$$

To ensure that the model is stable under learning, we assume that agents use stochastic gradient learning instead of recursive least squares.[34] Under stochastic gradient learning, agents do not update their variance-covariance matrices:

$$R_t = R_t^* = \chi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where χ is an arbitrary constant.

[34] For a discussion of stochastic gradient learning, see Evans, Honkapohja, and Williams (2005).

Assuming that the learning parameters are symmetrically distributed around their rational expectations values, the debt accumulation equation may be re-stated in the case of heterogeneous learning where $[a_t, b_t, c_t] = [d_t, e_t, f_t]$:

$$N_{t+1} = [\beta^{-1} - (1-a_t)^{-1} b_t] N_t + [1-(1-a_t)^{-1}(c_t + 1 - \rho)]\tilde{s}_t - (1-a_t)^{-1}\omega_t .$$

Because $d^2(\beta^{-1} - b_t (1-a_t)^{-1})$ is negative semidefinite when evaluated close to the learning parameters' rational expectations value, $(\beta^{-1} - b_t (1-a_t)^{-1})$ is a concave function. Because ω_t and $\tilde{s}_t$ are exogenous processes and $d^2(\beta^{-1} - b_t (1-a_t)^{-1})$ has one eigenvalue less than zero when evaluated close to the learning parameters' rational expectations values:

$$E[(\beta^{-1} - b_t (1-a_t)^{-1})] < [(\beta^{-1} - E[b_t](1-E[a_t])^{-1})] = 1 .$$

It is therefore the case that:

$$E_t[\prod_{i=1}^{\infty}(\beta^{-1} - b_t (1-a_t)^{-1})] \to 0$$

and debt will behave like a stationary process if the learning parameters are symmetrically distributed around their rational expectations values.

Numerical simulations confirm that this type of VAR(1) learning as while as heterogeneous and/or uncoordinated VAR(1) learning cause debt to behave like a stationary process.[35]

[35] These simulations also show, however, that under stochastic gradient learning the learning parameters typically are not symmetrically distributed around their rational expectations values. Instead, the learning parameters usually become "stuck" in an area that corresponds to an AR(1) coefficient on debt less than one. Asymmetric distributions, as well as concavity, may therefore induce stationary debt.

Assumption 3: Coordinated Learning vs. Uncoordinated Learning

Section 4 discusses *uncoordinated learning* where both countries' PLMs depend on their own levels of consumption. Section 6 relies on *coordinated learning* where both countries' PLMs depend on Home's consumption data. This distinction does not affect whether debt behaves like an explosive or stationary process.

Table 2 summarizes the results for the model under all combinations of these four assumptions:

Table 2
Behavior of Debt Under Different Types of Adaptive Learning

SC/VAR	CONT/PRI	COR/UNCOR	RESULT
SC	CONT	COR	Explosive
SC	CONT	UNCOR	Explosive
SC	PRI	COR	Explosive
SC	PRI	UNCOR	Explosive
VAR	CONT	COR	Stationary
VAR	CONT	UNCOR	Stationary
VAR	PRI	COR	Stationary
VAR	PRI	UNCOR	Stationary

Notes: SC and VAR refer to learning the side constraint and learning the VAR(1) respectively. CONT and PRI refer to knowing and not knowing the current value of consumption. COR and UNCOR refer to coordinated and uncoordinated learning respectively.

Appendix 2: Non-Linearities

So far, we have approximated our model by using a first order Taylor Series expansion around the debt-free steady state. Because debt follows a random walk and the model possesses a continuum of steady states, it is not obvious that our linear approximation is valid. Under rational expectations, the model displays no tendency to converge towards the debt-free steady state. It is therefore equally valid to linearize our model around any other steady state. In this appendix, we iteratively re-linearize the model around the steady state corresponding to the level of debt in each period. The central conclusions of this paper are unaffected. Under rational expectations, debt continues to follow a random walk. Under adaptive learning, however, debt behaves like either an explosive or a stationary process.

For any steady state level of debt, the model's steady state equals:

$$[\bar{M}, \bar{M}^*, \bar{N}, \bar{s}, \bar{r}] = [(\beta^{-1}-1)\bar{N}+1, -(\beta^{-1}-1)\bar{N}+(1-\alpha)/\alpha, \bar{N}, 1, \beta^{-1}-1].$$

We now take a first order Taylor Series expansion around the steady state for any level of debt. We continue to define $\tilde{X}_t$ as the deviation of X from its debt-free steady state rather than the steady state corresponding to $\bar{N}$. This choice ensures that the units of measurement for M_t and M_t^* do not change along with the steady state value of debt.[36] To simplify the model, we limit preference shocks to Home's Euler Equation. The model's linearized Euler Equations become:

$$\tilde{M}_t = E_t[\tilde{M}_{t+1}] - \beta\bar{M}\tilde{r}_{t+1} - \tilde{\xi}_t \text{ and} \qquad (8.1)$$

[36] For productivity and the interest rate, this distinction is trivial because their steady state values do not depend on $\bar{N}$.

$$\tilde{M}_t^* = E_t^*[\tilde{M}_{t+1}^*] - \beta \bar{M}^* \tilde{r}_{t+1}. \tag{8.2}$$

The model's debt accumulation equation now includes a term allowing for the deviation of the interest rate from its steady state:

$$N_{t+1} - \bar{N} = -\tilde{M}_t + (\beta^{-1} - 1)\bar{N} + \beta^{-1}(N_t - \bar{N}) + \tilde{s}_t + \bar{N}\tilde{r}_{t+1}. \tag{8.3}$$

The model's global resource constraint and the AR(1) productivity shock are unchanged from the original linearization:

$$\alpha(\tilde{M}_t + \tilde{M}_t^*) = \tilde{s}_t \text{ and} \tag{8.4}$$

$$\tilde{s}_t = \rho \tilde{s}_{t-1} + \tilde{\varepsilon}_t.$$

By substituting Equation (8.4) into Equation (8.2) and combining the resulting equation with Equation (8.1), we are able to derive a forward-looking structural equation that determines the current value of $\tilde{M}_t$:

$$\tilde{M}_t = \alpha \bar{M} E_t[\tilde{M}_{t+1}] - \alpha \bar{M}^* E_t^*[\tilde{M}_{t+1}^*] + \bar{M}\tilde{s}_t + \omega_t \text{ and} \tag{8.5}$$

$$\omega_t = -\alpha \bar{M}^* \tilde{\xi}_t. \tag{8.6}$$

To approximate the model, we iteratively re-linearize the model each period by substituting $\bar{N} = N_t$ into Equations (8.3) and (8.5). By using Equation (8.1), we are able to eliminate the interest rate from the system and re-write the debt accumulation equation:

$$N_{t+1} = -\tilde{M}_t + \beta^{-1} N_t + \tilde{s}_t - \beta^{-1}\bar{N}(E_t[\tilde{M}_{t+1}] - \tilde{M}_t + \omega_t / \alpha \bar{M}^*)\bar{M}^{-1}.$$

This approach allows us to evaluate two significant effects of the model's non-linearities. First, the debt accumulation equation now considers fluctuations in the interest rate. Second, the weights given to each expectation in Equation (8.5) differ from those in Equation (4.3). We use the techniques of Section 3 to evaluate the model under rational

expectations. We obtain a side constraint that ensures that the model's transversality condition is not violated and a resulting equation for the evolution of debt:

$$\tilde{M}_t = (\beta^{-1} - 1)N_t + s_t + \beta^{-1}(1-\rho)(\beta^2 - 1)(\beta\rho - 1)^{-1} N_t s_t + \beta\omega_t + f(\alpha, \beta)\omega_t N_t \text{ and } (8.7)$$

$$N_{t+1} = N_t + (1-\rho)\beta^{-1}[1-(1-\rho)(\beta^2 - 1)(\beta\rho - 1)^{-1}]N_t \tilde{s}_t - \beta\omega_t + h(\alpha, \beta)\omega_t N_t .$$

Although this model is non-linear, its non-linearities are limited to pre-determined or exogenous variables and it is therefore easy to simulate. Under rational expectations, debt continues to follow a random walk without drift. The current account ($N_{t+1} - N_t$), however, is no longer white noise but instead depends both on the products of debt and productivity and of debt and preference shocks. The current account is therefore both serially correlated and heteroskedastic.

We now analyze the iteratively re-linearized model under adaptive learning. We assume that Home agents use the following perceived law of motion (PLM):

$$\tilde{M}_t = aN_t + bN_t \tilde{s}_t .$$

The coefficient on lagged debt in Equation (8.7) is unchanged from Section 3. It is therefore unsurprising that simulations confirm that our iterative re-linearization does not affect the model's behavior under adaptive learning. Although debt follows a random walk under rational expectations, the types of learning from Sections 4 and 6 continue to cause debt to behave explosively. Under the type of learning from Section 7 with a stochastic discount factor, debt continues to act like a stationary process.

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Figures and Tables

Figure 1
Debt and Foreign's Consumption During Simulated Crises

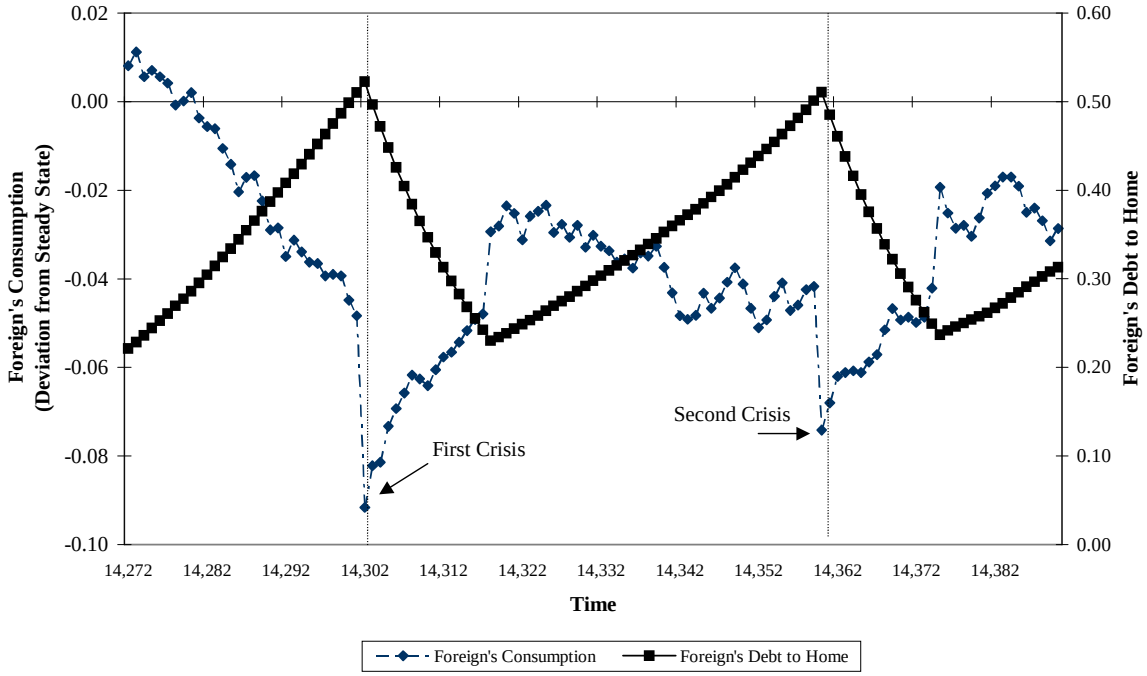
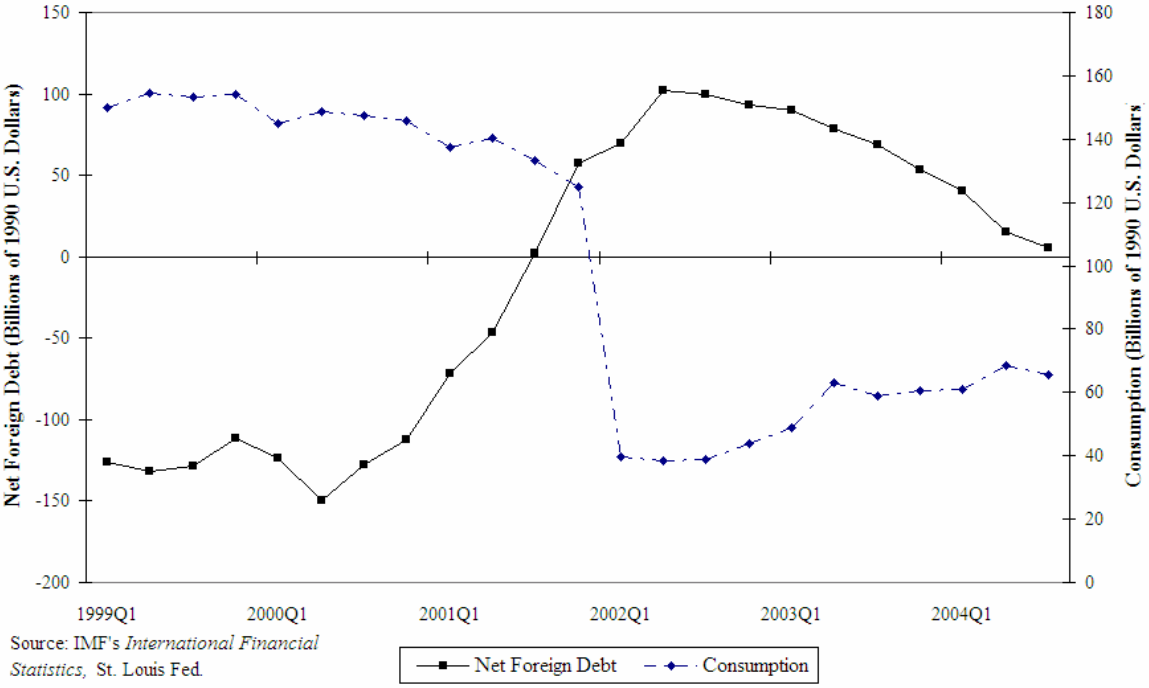


Figure 2
Net Foreign Debt and Consumption During Argentina's Financial Crisis
1999 - 2005



Source: IMF's *International Financial Statistics*, St. Louis Fed.

Table 1
Average Time Until Crisis, d(γ ,γ*), for Different Gains
(St. Error)

		γ* = 0.01	γ* = 0.02	γ* = 0.03
γ	0.01	6467 (112.8)	6257 (106.0)	6585 (114.4)
	0.02	6358 (110.3)	4789 (94.7)	4705 (91.9)
	0.03	6538 (113.3)	4653 (90.6)	3961 (84.4)