Perturbed Special Lagrangian Submanifolds

by

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DISSERTATION ABSTRACT

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This thesis investigates perturbed special Lagrangian submanifolds with the aim of developing a Floer theory analogous to existing theories in symplectic geometry. Special Lagrangians arise naturally in Calabi–Yau manifolds as submanifolds calibrated by the real part of the holomorphic volume form.

Following a proposal of Donaldson and Segal, we view special Lagrangians as solutions to an infinite-dimensional Lagrange multipliers problem. Perturbations of pairs of stable forms which give rise to SU(3)-structures yield perturbed special Lagrangian equations, whose solutions generalize classical special Lagrangian submanifolds.

Chapter 2 introduces a finite-dimensional Morse-theoretic model, linking the Morse homology of a constrained function to the Lagrange function of the associated Lagrange-multipliers problem. Guided loosely by analogy with the finite-dimensional case, Chapter 3 explores the Donaldosn–Segal Lagrange multipliers problem whose solutions are perturbed special Lagrangians. We prove ellipticity results, basic compactness under tameness conditions, and a natural volume bound. The main theorem is a transversality result which states that the moduli space of solutions is generically a set of isolated points.

This dissertation includes previously published material.

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CHAPTER I

INTRODUCTION

We begin this thesis with a broad overview of the history of gauge theory and calibrated geometry aimed at a rather general audience. The hasty reader can safely skip to section 1.6 for a more brief overview of the main results.

1.1 Historical origins

The simple fact that the laws of physics are the same everywhere in spacetime hints at a deep role for *symmetry* in the study of physics. This role was already apparent in the 1860's when Maxwell introduced the world to his theory of electromagnetism. However, it wasn't until 1918 when Herman Weyl attempted to expand the geometry inherent in Einstein's theory of relativity to include electromagnetism that the true importance of symmetry was recognized.

What is meant by symmetry in this context is perhaps worthy of a brief aside. In physics, particles are described by *fields* (sections of bundles). Equations representing physical laws appear to obey symmetries in the sense that they are invariant under the application of certain mathematical transformations. These symmetries give rise to conserved quantities and contribute an elegant mathematical framework for the forces of nature.

In the decades after Maxwell's equations were published, the study of these topics came to be known as *gauge theory* and gave rise to the standard model of particle physics. It is from this conceptual foundation and the attempt to explain physics beyond the standard model that my field of research emerged.

Eugene Wigner was among the first to introduce group theory, the natural language of symmetry, to theoretical physics. In 1960, Wigner published an article titled *The unreasonable effectiveness of mathematics in the natural sciences* [44].

The fundamental observation of this article is that diverse mathematical concepts have applicability far beyond the context in which they were originally conceived. The surprising and beautiful connections between pure math and the real world has been a constant source of motivation for me personally. In this section, I hope to give the reader a sense of the fascinating history which preceded the conception of this thesis in the hopes of sparking the interest of curious outsiders.

In the 70's, 80's, and 90's, gauge-theory inspired several striking developments in the geometry of 3- and 4-manifolds causing many to contemplate "the unreasonable effectiveness of *physics* in pure mathematics." The 1977 paper *Self-duality in four-dimensional Riemannian geometry* [3] by Atiyah–Hitchin–Singer contained some of the first such results. Therein, the authors prove a correspondence between self-dual connections and holomorphic structures on vector bundles. They also lay the foundations for the study of moduli spaces of these. In 1982, Donaldson, a student of Atiyah, proved his celebrated *Donaldson's theorem* [14] which states that any definite intersection form of a smooth 4-manifold must be diagonalizable. His proof involves analyzing the singular points in the interior of this moduli space. In the same year, Freedman [21] applied Donaldson's theorem to construct the *first ever example of an exotic smooth structure on* $\mathbb{R}^4$.

While the mathematical value of these results is apparent, it is fascinating to contemplate the fact that Atiyah–Hitchin–Singer were motivated to study self-dual connections due to their appearance as solutions to the Yang–Mills equations coming from the fusion of nuclear physics with group theory.

A decade after Donaldson published his diagonalization theorem, Nathan Seiberg and Edward Witten used gauge-theoretic techniques from Yang-Mills theory to define the *Seiberg–Witten invariants* [45] which turned out to provide a much more accessible and compact proof.

In 1966, physicist Hironari Miyazawa proposed a kind of symmetry called *supersymmetry* relating integer spin (bosonic) particles to half-integer spin (fermionic) particles. At the time, this idea was not taken seriously because it seemed to contradict experimental results. However, it gained popularity again in the 80's with the advent of *string theory*. In 1982, Witten applied a mathematical version of supersymmetry in order to derive classical Morse inequalities. This paper provided the crucial insights for the development of *Floer theory*. Floer, who himself had a background in gauge theory, proved the long-standing *Arnold conjecture* in 1988. His method, which is now known as Hamiltonian Floer theory, is an infinite-dimensional generalization of Morse theory. See [17, 19, 18].

Meanwhile, the world of physics was undergoing the "superstring revolution" during which physicists realized that adding an extra 6 or 7 dimensions to spacetime seemed to explain many apparent inconsistencies in quantum field theory. As a consequence, people became extremely interested in Ricci-flat, simply connected, closed 6- and 7-manifolds.

1.2 Manifolds with special holonomy

On a Riemannian manifold (M, g), the Levi-Civita connection defines linear isomorphisms between any two tangent spaces. This in turn provides a way to consistently transport geometric objects such as vectors along curves via *parallel transport maps*. Parallel transport maps, being linear isomorphisms between vector spaces, define a subgroup of $\mathrm{GL}(n, \mathbb{R})$ called the *holonomy group* of the metric g which encodes the *curvature*. The precise way in which the holonomy is related to the curvature is the content of the Ambrose–Singer theorem [2], proved in 1953.

In 1955, Marcel Berger gave a complete classification of possible holonomy groups for simply connected, irreducible, Riemannian manifolds together with information about their curvature [4]. Berger's list (in its modern form) is as follows.

Hol(g)	dim(M)	**Type of manifold**	**Comments**
SO(n)	n	Orientable manifold	—
U(n)	$2n$	Kähler manifold	Kähler
SU(n)	$2n$	Calabi–Yau manifold	Ricci-flat, Kähler
Sp(n)·Sp(1)	$4n$	Quaternionic–Kähler manifold	Einstein
Sp(n)	$4n$	Hyperkähler manifold	Ricci-flat, Kähler
G_2	7	G_2 manifold	Ricci-flat
Spin(7)	8	Spin(7) manifold	Ricci-flat

Physicists at the center of the superstring revolution immediately recognized that two manifolds on this list satisfied the exact requirements they needed to construct consistent string theories. The first being 6-manifolds with holonomy group SU(3) and the second being 7-manifolds with holonomy group G_2. Both are Ricci flat.

1.2.1 Calabi–Yau manifolds. Even before Berger's classification, manifolds with holonomy SU(n) were studied in the context of Kähler geometry. Their definition can be reformulated as follows.

Definition 1.2.1. A Calabi-Yau manifold is a compact, even-dimensional manifold M equipped with the following structures:

(i) a Riemannian metric g

(ii) a non-degenerate, closed 2-form ω (i.e. a symplectic form).

(iii) an integrable complex structure J

(iv) a holomorphic volume form Ω

These structures must satisfy the following compatibility conditions

(i) The Riemannian metric g must be a Kähler metric and ω must be its Kähler form, with respect to the complex structure J. That is,

$$g(u, v) = \omega(u, Jv)$$

(ii) Let n be the dimension of M. Then the symplectic form ω is related to the holomorphic volume form Ω as follows:

$$\frac{\omega^n}{n!} = (-1)^{n(n-1)/2} \left(\frac{i}{2}\right)^n \Omega \wedge \bar{\Omega}$$

Equivalently, a compact, n-dimensional Kähler manifold M is a Calabi-Yau manifold if

- the canonical bundle of M is trivial
- M has a Kähler metric with holonomy contained in $\mathrm{SU}(n)$.

In 1957, Eugenio Calabi conjectured that every simply-connected Kähler manifold with vanishing first Chern class has a unique, Ricci-flat metric. This conjecture was proved by Yau in 1978 [46]. More precisely, Yau's theorem says

Theorem 1.2.2 (Yau's Theorem)**.** *Let M be a compact, complex manifold with Kähler metric g and corresponding Kähler form ω. Suppose that the Ricci form ρ is a closed, real, $(1,1)$-form on M with $[\rho] = c_1(M)$. Then there exists a unique Kähler metric g' on M with Kähler form ω' such that $[\omega'] = [\omega] \in H^2(M, \mathbb{R})$ and the Ricci form of ω' is also ρ.*

Much to the pleasure of physicists, this theorem gives us a way to construct a wealth of examples of compact, simply-connected, Ricci-flat manifolds using the tools of algebraic geometry. Previously, it was not known how to do this.

1.2.2 G_2-manifolds.

Manifolds with holonomy G_2 are significantly more difficult to come by. A 7-dimensional analogue of Yau's theorem is as of now, nonexistent. Nonetheless several constructions exist. The first local (non-compact) examples weren't constructed until 1984 by Robert Bryant [7]. The first compact examples were constructed, using a generalized Kummer construction, by Dominic Joyce in 1994 [29]. A more in-depth discussion of G_2-manifolds will be left for later sections. For now, the important thing to know is that just as Calabi–Yau geometry is defined in part by a non-degenerate 2-form, it turns out that G_2-geometry can be defined in terms of a 3-form which is also "non-degenerate" in a generalized sense.

1.3 Calibrated geometry

As every physics student knows, the idea of minimizing functionals is central to understanding the world around us. This is because the *principle of least action* tells us that the universe is "lazy". A drop of water flowing down a hillside will take the path of least resistance. Bubbles are round because a sphere is the configuration with the least surface area. In 1760, perhaps motivated by this fact, Joseph-Louis Lagrange posed the question of the existence of area minimizing surfaces with a given boundary. This problem was not solved until 1930 by Jesse Douglas [16] and independently Tibor Radó [37], earning Douglas the Fields medal in 1936. Also in 1936, it was discovered that Wirtinger's inequality implies that complex submanifolds of Kähler manifolds are *volume-minimizing*. This result was significant, because the problem of finding minimal surfaces requires one to solve a 2nd order PDE, but finding a complex surface only requires solving a rather simple 1st order PDE.

This idea of obtaining solutions to a variational problem by solving a special first order PDE was also present in the self-dual instantons, which minimize the Yang-Mills action functional, at the heart of Donaldson's theorem. Building on this theme, Mikhail Gromov observed in 1985 [22] that maps from a Riemann surface into a symplectic manifold satisfying a Cauchy–Riemann-type condition encode an astounding amount of topological information. These *J-holomorphic curves* admit compactness and gluing theorems, giving rise to what is now known as Gromov-Witten theory.

In 1991, in another stunning application of theoretical physics to pure math, Candelas, de la Ossa, Green, and Parkes predicted the solutions to explicit curve-counting problems by employing *mirror symmetry* from the world of string theory [11]. Their solution was later verified by a rigorous argument using Gromov–Witten theory.

In 1982, Harvey and Lawson published their paper *Calibrated geometries* [23] which provided for the first time, a unified theory of distinguished, volume-minimizing submanifolds which behave in some respects like complex submanifolds do in the Kähler setting. The immensely useful J-holomorphic curves are only one example of calibrated submanifolds. There are many others. Below, some of the most interesting are listed.

submanifold	**(Hol(g), form)**	**equations**	**notes**
J-holomorphic curve	(M^{2n}, ω, J) $\mathrm{Hol}(g) \subseteq \mathrm{U}(n)$ calibration: ω	$\bar{\partial}_J u = 0$ equivalently $u^*\omega = \mathrm{vol}_\Sigma$	• Gromov–Witten invariants • Symplectic Lagrangian Floer homology
Special Lagrangian	(X^{2n}, ω, Ω) $\mathrm{Hol}(g) \subseteq \mathrm{SU}(n)$ calibration: $\mathrm{Re}\Omega$	$\omega\|_L = 0,\ \mathrm{Im}\Omega\|_L = 0$ equivalently $\mathrm{Re}\Omega\big\|_L = \mathrm{vol}_L$	• Moduli dimension $= b^1(L)$ • open Gromov–Witten invariants • Fukaya-category objects
Associative 3-fold	(M^7, φ) $\mathrm{Hol}(g) \subseteq G_2$ calibration: φ	$\varphi\|_L = \mathrm{vol}_L$	• Moduli dimension $= b^1(L)$ • conjectural G_2 Donaldson–Thomas type counts
Cayley 4-fold	(M^8, Φ) $\mathrm{Hol}(g) \subseteq \mathrm{Spin}(7)$ calibration: Φ	$\Phi\|_L = \mathrm{vol}_L$	• Moduli dimension $= b^2_+(L)$ • conjectural Spin(7) invariants

Our understanding of calibrated submanifolds in general is severely lacking compared to what is known about J-holomorphic curves.

1.4 Special Lagrangian submanifolds

Special Lagrangian submanifolds are the main subject of this thesis. The overall hope is that we might be able to reproduce some of the successes that were achieved in symplectic geometry by developing a Floer theory for special Lagrangian submanifolds.

Definition 1.4.1. A *special Lagrangian* submanifold L in a Calabi-Yau manifold (M, ω, Ω) with real dimension $2n$ (as in Definition Definition 1.2.1) is an n-dimensional submanifold L such that it is *Lagrangian*, that is

$$\omega\big|_L = 0$$

and such that

$$\mathrm{Im}\Omega\big|_L = 0.$$

This definition is *equivalent* to the condition that L is *calibrated* by $\mathrm{Re}\Omega$. That is,

$$\mathrm{Re}\Omega\big|_L = \mathrm{vol}_L.$$

In this thesis, we work exclusively with the $n = 3$ case. Special Lagrangians, like all calibrated submanifolds, are volume-minimizing in their homology class. Despite this, naïve attempts to "count" them run quickly into trouble. For one thing, the classification of Calabi-Yau 3-folds is a mostly open problem. Furthermore, even in the simplest non-trivial cases, constructing examples of special Lagrangians is also a highly non-trivial task. Moreover, families of special Lagrangians may degenerate, collide, or disappear under deformations of the Calabi-Yau structure [27]. Therefore any raw numerical count depends in ways that are not well-understood on various choices and tends to not be invariant.

It would therefore be highly desirable to develop a Floer-type theory for special Lagrangians in the hopes that such a theory could address some of these issues. Such a theory could potentially provide an invariant that is immune to the degenerations and bifurcations of individual families of submanifolds.

Special Lagrangians are also interesting from several other viewpoints. In *Fukaya categories*, they are expected to correspond to stable objects, making them the natural building blocks of homological mirror symmetry. The *Thomas–Yau conjecture* [42] posits that, within a given Hamiltonian isotopy class, special Lagrangians are the unique canonical representatives selected by mean-curvature flow, while the *SYZ conjecture* [41] envisions mirror symmetry as arising from dualizing special Lagrangian torus fibrations.

However desirable this program, formidable analytical and geometric obstacles remain. The *moduli spaces* of special Lagrangians, though locally modeled on harmonic 1-forms as a result of McLean's theorem [36], often develop singularities. Even when the moduli spaces is smooth, compactness issues arise. Finally, one must address the subtle choices and definitions of orientations and gradings on these moduli spaces.

This thesis takes the first steps towards addressing some of these issues, following a suggestion by Donaldson and Segal which views special Lagrangian submanifolds from the viewpoint of an infinite-dimensional Lagrange multipliers problem. From this point of view a perturbation of the defining equations leads to an elliptic deformation. We also address a basic version of compactness.

1.5 Organization

The remainder of this introduction is structured to guide the reader through the two-part layout of the dissertation. In section 1.6, we summarize a finite-dimensional Morse-theoretic model for Lagrange multipliers problems. In section 1.7 we outline the infinite-dimensional Donaldson–Segal Lagrange multipliers problem and the main results.

- Section 2.1 describes the classical Lagrange-multipliers problem using the language of vector bundles and connections.

- Section 2.2 defines the Morse complexes for the Lagrange functional with respect to a one-parameter family of metrics.

- Section 2.3 uses an adiabatic limit and an implicit function theorem argument to prove the main theorem of Chapter 2.

- Section 3.1 contains the definition of stable forms and elaborates on how they relate to metrics with special holonomy.

- Section 3.2 describes translation-invariant G_2-structures and contains important definitions. These will be used to prove ellipticity results.

- Section 3.3 reviews definitions of tamed SU(3)- and G_2-structures and describes how they relate to the notion of a tamed almost complex structure in symplectic geometry.

- Section 3.4 gives a rigorous formulation of the Donaldson–Segal Lagrange multipliers problem which allows us to view special Lagrangians as critical points of a functional.

- Section 3.5 contains a proof that the Euler-Lagrange equations of the above mentioned functional are elliptic, and shows how the 6-dimensional picture relates to 7-dimensional G_2-geometry along the way.

- Section 3.6 contains a proof that under tameness assumptions, perturbed special Lagrangians have bounded volume.

- Section 3.7 contains results about elliptic regularity and a basic compactness theorem.

- Section 3.8 applies all of the above to prove the main result of the project.

1.6 Overview of Chapter 2

Suppose that we fix the following

- a closed, Riemannian n-manifold (M, g)

- a rank-k vector bundle $\pi : E \to M$ equipped with a connection ∇^E and bundle metric h.

– a smooth section $s \in \Gamma(E)$ whose zero set $Z = s^{-1}(0)$ is a closed, codimension-k submanifold of M.

– a function $f : M \to \mathbb{R}$.

The *Lagrange multipliers problem* is to find the critical points of $f\big|_Z$. The *Lagrange multipliers theorem* tells us that these critical points are in 1-1 correspondence with the *Lagrange-function*, which are often easier to compute. In this case, the Lagrange function is given by

$$\Lambda(\lambda) = f(\pi(\lambda)) + \langle \lambda, s(\pi(\lambda)) \rangle, \qquad \lambda \in E^*.$$

In [40], the authors compared the Morse theory of Λ on the non-compact manifold E^* to the Morse theory of $f\big|_Z$ in the case where E is a trivial, rank-1 vector bundle. That is, in the case where the section s is just a function. We extend their result to this more general setting. Specifically, we prove (Corollary 2.3.14)

Theorem 1.6.1. *Under suitable Morse–Smale regularity assumptions, the Morse homology of the Lagrange function* Λ *is well-defined. Moreover,*

$$H_\bullet\left(E^*, \Lambda\right) \cong H_{\bullet - k}\left(Z, f\big|_Z\right).$$

This should be compared to Theorem 5 in [40]. The proof contained here follows the same strategy, with adaptations to the bundle setting.

1.7 Overview of Chapter 3

This chapter includes previously published work.

It was discovered by Hitchin [25, 26] that metrics with special holonomy in dimensions 6 and 7 are deeply connected to the existence and properties of differential forms whose pointwise $\mathrm{GL}(n, \mathbb{R})$-orbit is open. Such forms are called *stable*. One very interesting feature of dimension 6 is the fact that an SU(3)-structure on a 6-manifold is equivalent to a choice of a stable 3-form ρ and a stable 4-form τ satisfying certain algebraic conditions (see Definition 3.1.5). Furthermore, a Calabi-Yau structure on a 6-manifold is an SU(3)-structure (ρ, τ) where the 3-form ρ and the 4-form τ along with their Hitchin duals (see section 3.1.1) are *closed*. In the Calabi-Yau setting, the 3-form ρ is the real part of the holomorphic volume form and its Hitchin dual $\hat{\rho}$ is the imaginary part.

In a 6-dimensional Calabi-Yau manifold, both ρ and $\hat{\rho}$ are calibrations, whose calibrated submanifolds are called *special Lagrangian*. In [15], Donaldson and Segal pointed out that special Lagrangian submanifolds can be characterized as solutions to a certain Lagrange multipliers problem defined purely in terms of these stable forms. We briefly summarize this Lagrange multipliers problem in the following paragraphs. See section 3.4 for a more rigorous exposition.

Suppose that (M, ρ, τ) is a manifold equipped with a pair of closed, stable forms $(\rho, \tau) \in \Omega^3(M) \times \Omega^4(M)$. Any Calabi-Yau manifold satisfies this condition, but as we will see, we will also consider more general settings. Next, fix a 3-submanifold $L_0 \subset M$. Let L be any nearby 3-submanifold representing the same homology class as L_0 and let Q be a cobordism connecting L_0 and L. Then, roughly speaking, one can define a functional f_τ on the space of submanifolds by integrating the 4-form τ over Q

$$f_\tau(L) = \int_Q \tau.$$

Of course, this functional is not well-defined since it depends on the choice of cobordism Q. However, it is well-defined on the covering space of the space of embeddings of a 3-manifold into M as explained in section 3.4.

Next, let C_ρ be the set of 3-submanifolds in M, all diffeomorphic to L_0, representing the same homology class and satisfying $\rho\big|_L = 0$. Note that in the Calabi-Yau case, a special Lagrangian submanifold which is calibrated by the 3-form $\hat{\rho}$ always satisfies this condition. The Donaldson–Segal Lagrange-multipliers problem is to find the critical points of f_τ restricted to the set C_ρ.

In a Calabi-Yau manifold, special Lagrangian submanifolds are critical points of the Lagrange functional arising from this Lagrange multipliers problem. This fact raises the question of whether it is possible to construct a Floer theory for special Lagrangian submanifolds of a 6-dimensional Calabi-Yau manifold.

However, a result of McLean [36] says that for any special Lagrangian submanifold L, the moduli space of nearby special Lagrangians is a smooth manifold with dimension equal to the first Betti number of L. Thus, with the goal of defining a Floer theory in mind, we want to perturb the SU(3)-structure underlying the Calabi-Yau structure on our 6-manifold so that the critical points of the relevant functional are isolated.

The fact that 7 is 6+1 gives us a natural space of perturbations. This is because any 6-manifold M with an SU(3)-structure can be embedded as a hypersurface in a 7-dimensional cylinder $\mathbb{R} \times M$ equipped with a G_2-structure arising in a canonical way from the SU(3)-structure. Specifically, if $(\rho, \tau) \in \Omega^3(M) \times \Omega^4(M)$ defines an SU(3)-structure on M, then

$$\psi = dt \wedge \rho + \tau$$

is a stable 4-form on $\mathbb{R}\times M$ corresponding to a G_2-structure on $\mathbb{R}\times M$. If the SU(3)-structure on M is a Calabi-Yau structure, the metric associated to the 4-form ψ on $\mathbb{R} \times M$ will have holonomy SU(3) contained in G_2. On the other hand, if (M, ρ, τ) is a 6-manifold equipped with a pair of stable forms $(\rho, \tau) \in \Omega^3(M) \times \Omega^4(M)$, the condition that this pair gives rise to a G_2-structure on $\mathbb{R} \times M$ is a *weaker* condition than the requirement that it defines an SU(3)-structure on M. Let

$$\mathcal{R}_{G_2} = \big\{(\rho, \tau) \in \Omega^3(M) \times \Omega^4(M) \mid d\rho = 0, d\tau = 0, \text{and } \psi = dt \wedge \rho + \tau \text{ is stable on } \mathbb{R} \times M\big\}$$

be the set of *closed G_2-pairs* on a 6-manifold M.

If we start with a Calabi-Yau 6-manifold M, we can perturb the underlying SU(3)-structure (ρ, τ) in such a way that it is no longer an SU(3)-structure but nonetheless gives rise to a G_2-structure on $\mathbb{R} \times M$. The reason we require ρ and τ to be closed is to ensure that the deformation theory of the ψ-associative submanifolds in $\mathbb{R} \times M$ is governed by a *self-adjoint* operator which is not necessarily the case unless ψ is closed. Since the condition that a form be stable is an *open condition*, any small perturbation of the of the pair (ρ, τ) by closed forms remains in the space of closed G_2-pairs since such a perturbation corresponds to a small perturbation of the 4-form ψ. As long as ρ and τ, and hence the 4-form ψ, are closed, the Lagrange-multipliers problem described above can still be defined. The solutions to this more general Lagrange-multipliers problem generalize the notion of a special Lagrangian. They are no longer calibrated submanifolds since we no longer require that the Hitchin duals of ρ and τ are closed but nonetheless retain several useful features typically enjoyed by calibrated submanifolds.

We fix the following notation. Let $\iota : P \to M$ be a submanifold. Suppose that $\alpha \in \Omega^k(M)$ is a k-form such that $\iota^*\alpha \equiv 0$. Let $N\iota$ denote the normal bundle. Then α defines an $N^*\iota$-valued $(k-1)$-form α_N on ιP by letting

$$\alpha_N(v_1, \ldots, v_{k-1}) = \alpha(v_1, \ldots, v_{k-1}, \cdot).$$

The Euler-Lagrange equations corresponding to the Donaldson–Segal Lagrange-multipliers problem take the following form.

Proposition 1.7.1. *Let (M, ρ, τ) be a 6-manifold equipped with a G_2-pair (ρ, τ) where both ρ and τ are closed. A 3-submanifold $L \subset M$ is a solution to the Lagrange multipliers problem if and only if there exists $\lambda \in C^\infty(L)$ such that*

$$\begin{aligned} \tau_N + d\lambda \wedge \rho_N &= 0 \\ \rho\big|_L &= 0. \end{aligned}$$

These equations first appeared in [15] without much explanation. The details of their derivation can be found in section 3.4.2. Here, they will be referred to as the *perturbed special Lagrangian equations* (perturbed SL equations). The function λ is the *Lagrange-multiplier* and plays a similar role to the Lagrange-multipliers found in calculus. When M is a Calabi-Yau manifold with holomorphic volume form Ω and symplectic structure ω, the 3-form $\rho = \mathrm{Re}(\Omega)$ and the 4-form $\tau = \frac{1}{2}\omega^2$. In this case, the solutions to the perturbed SL equations are precisely the special Lagrangian submanifolds together with constant functions λ. When M is not necessarily Calabi-Yau, a submanifold L solving the perturbed SL equations for some function λ is called a *perturbed special Lagrangian submanifold.*

The perturbed SL equations can also be considered separately from the Lagrange-multipliers set-up. In this case, it is not necessary to require that ρ and τ be closed. One could still hope to construct a numerical invariant for 6-manifolds equipped with a G_2-pair in this way, although this is not the focus of this paper.

The key connection between solutions to the perturbed SL equations and the 7-dimensional setting is as follows. Suppose that (M, ρ, τ) is a 6-manifold with a G_2-pair and that (λ, L) is a solution to the perturbed SL equations. Then the *graph* of the function λ over $L \subset M$ is an *associative submanifold* (see Definition 3.3.6) of $\mathbb{R} \times M$. For this reason, we call the pair (λ, L) a *graphical associative.* See Lemma 3.5.10 for proof of this fact. As a consequence, the perturbed special

Lagrangian equations are *elliptic* since the deformation theory of associative submanifolds in a G_2-manifold is governed by an elliptic operator [36].

The *moduli space* $\mathcal{M}(A, P; (\rho, \tau))$ is defined to be the set of all submanifolds L in M, diffeomorphic to a particular 3-manifold P and representing a fixed homology class $A \in H_3(M; \mathbb{Z})$, which satisfy the perturbed SL equations for some λ.

The definition of a G_2-pair is flexible enough to prove transversality for $\mathcal{M}(A, P; (\rho, \tau))$. More precisely, let P be a closed, oriented 3-manifold and M a closed, oriented 6-manifold equipped with a closed G_2-pair $(\rho, \tau) \in \Omega^3(M) \times \Omega^4(M)$. Recall that the space of embeddings $\iota : P \to M$ is a Fréchet manifold carrying a natural C^∞ topology whose open subsets are obtained by considering graphs of smooth vector fields. This is often called the *Whitney*-topology. Thus, the moduli space $\mathcal{M}(A, P; (\rho, \tau))$, being a subspace of a quotient of this Fréchet manifold, inherits a quotient Whitney-topology.

Remark 1.7.2. When dealing with transversality and perturbation arguments in infinite-dimensional settings (such as spaces of sections, forms, or embeddings equipped with the C^∞-topology), classical notions of genericity defined via measure theory (e.g., "measure-zero" subsets) become inadequate or ill-defined because these spaces typically lack a meaningful measure-theoretic structure. Instead, we use the concept of "residual subsets," which captures genericity in terms of Baire category theory. Residual subsets can thus be seen as large subsets reflecting properties that hold for "typical" or "generic" choices of data.

Definition 1.7.3. A subset R of a topological space X is called *residual* if it contains a countable intersection of open dense subsets of

We prove

Theorem 1.7.4 (transversality)**.** *Fix a homology class $A \in H_3(M;\mathbb{Z})$ and a closed 3-manifold P. There is a residual subset $\mathscr{R}_{\mathrm{reg}}$ of $\mathscr{R}_{G_2}$ such that the moduli space $\mathcal{M}(A, P; (\rho, \tau))$ is a collection of isolated (with respect to the quotient topology) points whenever $(\rho, \tau) \in \mathscr{R}_{\mathrm{reg}}$.*

When the G_2-pair (ρ, τ) is of class C^ℓ, the proof of this theorem is a consequence of the implicit function theorem for Banach spaces. In order to extend the result to smooth G_2-pairs, we must apply the so-called Taubes trick which is a standard method from symplectic geometry (see for example chapter 3 of [35]). In order to utilize the Taubes trick, we must prove an elliptic regularity theorem (Theorem 3.7.5) and a compactness theorem (Theorem 3.7.6) for associative submanifolds with bounded second fundamental form and bounded volume. This is non-trivial because the solutions to the perturbed SL equations are not necessarily minimal submanifolds.

If the elements of the moduli space $\mathcal{M}(A, P; (\rho, \tau))$ are to be counted, then we must determine if or when the moduli space is compact. Towards this end, we adapt the definition of a *tamed* G_2-structure (first introduced in [15]) to define a *tamed* G_2-pair on M. This is analogous to the notion of a tamed almost complex structure first introduced by Gromov in order to control the energy of J-holomorphic curves. In our case, a G_2-pair (ρ, τ) is *tamed* by a second pair $(\rho', \omega') \in \Omega^3(M) \times \Omega^2(M)$ where both ρ' and ω' are closed and stable. See section 3.3 for details. We have

Proposition 1.7.5 (volume bounds)**.** *Let (M, ρ', ω') be a closed, 6-manifold equipped with taming forms $(\rho', \omega') \in \Omega^3(M) \times \Omega^2(M)$. Suppose that (ρ, τ) is a (ρ', ω')-tame G_2-pair. Then every element in $\mathcal{M}(A, P; (\rho, \tau))$ has a topological volume bound depending only on the cohomology class of ρ'.*

More details about tamed G_2-structures can be found in [28]. Although in general we cannot expect that the second fundamental form of a perturbed SL submanifold to be bounded, we nonetheless expect the volume bound in the previous proposition to give us a compactification of the moduli space using rectifiable currents. However, such a compactification must necessarily contain singular objects which are not entirely understood. Such singular objects were studied by Joyce in [27]. We hope to explore these topics along with other topics related to developing a Floer theory for special Lagrangian submanifolds in future papers.

CHAPTER II

MORSE THEORY FOR LAGRANGE MULTIPLIERS

In this chapter, we consider Morse theory in the context of the classical Lagrange multipliers problem. Specifically, we compare the Morse homology of a function constrained to a submanifold defined by the zero set of a section of a vector bundle to the Morse homology of the associated Lagrange function. This chapter will serve as a finite-dimensional analogue for the material in Chapter 2. Beyond serving as inspiration, this chapter is independent of the rest of the material in this dissertation.

The main result of this chapter is Corollary 2.3.14 which states that the Morse homology of the Lagrange function is isomorphic to the Morse homology of the compact manifold Z with grading shifted by k, the rank of the vector bundle E. The proof of this theorem closely follows the proof of Theorem 5 in [40]. The main difference is that the constraint considered here is given as the zero set of a section of an arbitrary, rank k vector-bundle while the constraint in [40] is given as the zero set of a function. In the present chapter, we leave out some of the more tedious details that do not differ much from [40] and focus instead on the parts of the proof which must be reworked in the bundle case.

2.1 The Lagrange multipliers problem

In this section, we prove the Lagrange multipliers theorem where the constraint is given as the zero set of a section of a vector bundle. Along the way, we define the Lagrange function and prove that it is Morse whenever the constrained function is. The material in this section is mostly standard (see Supplement 3.5A of [1]).

We start by describing the setup and fixing some notation. Suppose that $\pi : E \to M$ is a rank k vector bundle over a closed, n-dimensional manifold M. Let $s : M \to E$ be a section of E and $f : M \to \mathbb{R}$ a Morse function. Define $Z = s^{-1}(0)$. Suppose that ∇^E is a connection on E and let $TE \cong HE \oplus VE$ be the corresponding splitting. Let $ds : TM \to TE$ denote the full derivative. Then $\nabla^E s = \pi_V \circ ds$ where π_V is the projection of TE onto $TV = \ker d\pi$. By the implicit function theorem, if $\nabla^E s_p$ is constant rank r along Z so that $\nabla^E s_p$ has the same rank for each $p \in Z$, then Z is a properly embedded manifold of codimension r in M.

The *Lagrange multipliers problem* refers to the problem, in the above setup, of finding the critical points of $f\big|_Z$.

We start with a standard linear algebra fact.

Lemma 2.1.1. *Suppose that V and W are vector spaces and $A : V \to W$ is a linear map. Denote the dual map by $A^* : W^* \to V^*$. Then $\operatorname{Im} A^* = (\ker A)^0$, the annihilator of $\ker A$.*

Proof. The first isomorphism theorem tells us there is an isomorphism $\overline{A}$: $V/\ker A \to \operatorname{Im} A$ satisfying $A = \overline{A} \circ \pi$ where π is the quotient map. Furthermore, $\pi = \overline{A}^{-1} \circ A$. Let $g \in V^*$. Then

$$\begin{aligned} T \in (\ker A)^0 \quad &\iff \quad T(v) = 0 \ \forall\, v \in \ker A \\ &\iff \quad \ker A \subset \ker T \\ &\iff \quad T \text{ factors through } \pi. \end{aligned}$$

In other words, $T \in (\ker A)^0$ if and only if there exists $\overline{T} : V/\ker A \to \mathbb{R}$ satisfying $T = \overline{T} \circ \pi$. Thus, we have

$$T = \overline{T} \circ \pi = \overline{T} \circ \left(\overline{A}^{-1} \circ A\right) = \left(\overline{T} \circ \overline{A}^{-1}\right) \circ A.$$

Then $\overline{L} = \overline{T} \circ \overline{A}^{-1}$ is a linear functional on $\operatorname{Im} A$ which may be extended (non-uniquely) to a linear functional L on W satisfying $T = L \circ A$. Thus $T \in (\ker A)^0$ if and only if $T \in \operatorname{Im} A^*$. $\square$

Note that the lemma is equivalent to the statement: $\ker A \subseteq \ker g \iff g = f \circ A$ for some $f \in W^*$.

Remark 2.1.2. Two functionals L_1 and L_2 extending $\overline{L}$ must agree on $\operatorname{Im} A$. If A is surjective, then $\operatorname{Im} A = W$ so there is a unique extension. If A has rank $r < \dim W$ then the space of possible extensions has the same dimension as $\ker A$ which by the rank-nullity theorem is equal to $\dim W - r$.

Theorem 2.1.3 (Lagrange multipliers theorem)**.** *In the context described above, a point $p \in Z \subset M$ is a critical point of $f\big|_Z$ if and only if there exists $\lambda \in E^*$ such that*

$$df_p = -\lambda \circ \nabla^E s_p \tag{2.1}$$

The set of λ's satisfying equation 2.1 is an affine space of dimension $(k - r)$ where k is the rank of E and r is the rank of $\nabla^E s_p$ along Z.

Proof. Note that p is a critical point of $f\big|_Z$ implies that $\ker \nabla^E s_p \subseteq \ker df_p$. Therefore, by the lemma, p is a critical point of $f\big|_Z$ if and only if $df_p = \lambda \circ \nabla^E s_p$ for some $\lambda \in E^*$. The last statement is a direct consequence of the remark. $\square$

Definition 2.1.4. The *Lagrange function* is a function on E^* defined by:

$$\Lambda(\lambda) = f(\pi(\lambda)) + (\lambda \circ s \circ \pi)(\lambda)$$

For convenience, let $S(\lambda) = (\lambda \circ s \circ \pi)(\lambda)$. Note that S is a section of E^{**}.

Lemma 2.1.5. *Let Λ be the Lagrange function. $\lambda \in E^*$ is a critical point of Λ if and only if $p = \pi(\lambda)$ is a critical point of $f\big|_Z$ and λ satisfies $df_p = -\lambda \circ \nabla^E s_p$.*

Proof. First, we calculate $d\Lambda : TE^* \to \mathbb{R}$. For any $\lambda \in E^*$ and any $v \in T_\lambda E^*$, we have

$$d\Lambda_\lambda(v) = (df \circ d\pi)(v) + \pi_V(v) \circ s(\pi(v)) + \left(\lambda \circ \nabla^E s \circ d\pi\right)(v). \tag{2.2}$$

Now, assume that $p = \pi(\lambda) \in \operatorname{Crit} f\big|_Z$ and that $df_p = -\lambda \circ \nabla^E s_p$. Then clearly $s(\pi(v)) = 0$ since $p \in Z = s^{-1}(0)$. Therefore

$$d\Lambda_\lambda = \left(df_p + \lambda \circ \nabla^E s_p\right) \circ d\pi_p = 0$$

so λ is also a critical point of Λ.

Conversely, suppose that $\lambda \in \operatorname{Crit} \Lambda$. Then $d\Lambda_\lambda(v) = 0$ for all $v \in T_\lambda E^*$. In particular $d\Lambda_\lambda(v) = 0$ for $v \in \ker d\pi$. In this case, $\pi_V(v) = v$ which means $s(\pi(\lambda)) = 0$. Therefore $df_p + \lambda \circ Ds_p = 0$ and so p is a critical point of $f\big|_Z$ by Theorem 2.1.3. □

Next, we will investigate the Hessian of Λ.

Definition 2.1.6. The *Hessian* of a differentiable function $f : M \to \mathbb{R}$ on a Riemannian manifold (M, g) is

$$\nabla df$$

where ∇ is the Levi-Civita connection associated to the metric g. We denote the Hessian of f at a point p by $H(f, p)$.

Fact 2.1.7. Suppose that $X, Y \in T_pM$. Then you can show that

$$\nabla df(X, Y) = X(Y(f)) - df(\nabla_X Y).$$

Therefore, if p is a critical point of f,

$$H(f, p)(X, Y) = X(Y(f)) = Y(X(f))$$

is a bilinear form on T_pM that does *not* depend on the connection. In fact, some authors define the Hessian by $H(f,p)(X,Y) = X(Y(f))$ only at the critical points of f for this reason.

Definition 2.1.8. A function $f \in C^2(M,\mathbb{R})$ is called a *Morse function* if for every $p \in \mathrm{Crit}\, f$ the Hessian $H(f,p)$ is nondegenerate.

Lemma 2.1.9. *Let ∇^E be a connection on E and λ a critical point of Λ. Let $TE^* = HE^* \oplus VE^*$ be the corresponding splitting. With respect to this splitting, the Hessian of Λ at λ takes the form*

$$H(\Lambda,\lambda) = \begin{pmatrix} H_M(\Lambda,\lambda) & \nabla^E s_p^* \\ \nabla^E s_p & 0 \end{pmatrix} \tag{2.3}$$

where $H_M(\Lambda,\lambda)$ denotes the matrix given by the restriction of $H(\Lambda,\lambda)$ to $H_\lambda E^* \otimes H_\lambda E^*$ and $p = \pi(\lambda)$.

Proof. Suppose that (u,v) are a pair of vertical vectors in $T_\lambda E^*$. Then, recalling equation 2.2, $d\Lambda$ depends only on the base point. Therefore, in this case, $H(\Lambda,\lambda)(u,v) = 0$.

If u is a vertical vector but v is a horizontal vector, then $H(\Lambda,\lambda)(u,v) = u \circ \nabla^E s_p(v)$. If instead, u is horizontal, and v is vertical, we just switch the roles of u and v. Thus in the first case, the map is $\nabla^E s_p$ while in the second case, the map is $\nabla^E s_p^*$.

The remaining block represents the case where $H(\Lambda,\lambda)$ acts on a pair of horizontal vectors. Using the connection, we identify $H_\lambda E$ with $T_{\pi(\lambda)}M$ and denote this block by $H_M(\Lambda,\lambda)$ where the subscript M should remind us that it's the Hessian of Λ "along M". $\square$

Lemma 2.1.9 allows us to view $H(\Lambda, \lambda)$ in a familiar matrix format as we are used to seeing it. This format should be compared the so-called bordered Hessian from calculus. Next, we prove a lemma that allows us to extract useful information about the indices of critical points from the Hessian.

Lemma 2.1.10. *Suppose that $\nabla^E s_z$ is surjective for $z \in Z$, λ is a critical point of Λ, and let $\pi(\lambda) = p$. Also suppose that $f\big|_Z$ is a Morse function. Then Λ is also a Morse function. Furthermore,*

$$\operatorname{index}(\lambda, \Lambda) = \operatorname{index}(p, f\big|_Z) + k$$

Proof. Let $T_\lambda E^* = N_p Z \oplus T_p Z \oplus E_p^*$ be a splitting of $T_\lambda E^*$ with respect to any metric on M. Here $N_p Z$ is the normal bundle of Z at p and we have used the connection on E. Then the Hessian of Λ at λ written as a matrix with respect to this splitting has the form

$$H(\Lambda, \lambda) = \begin{pmatrix} A & B & D \\ B^\intercal & C & 0 \\ D^\intercal & 0 & 0 \end{pmatrix}$$

Here, we have used Lemma 2.1.9 along with the fact that $T_p Z \subset \ker \nabla^E s_p$. Note that $C = H(f\big|_Z, p)$ Next, suppose that $H(\Lambda, \lambda)$ has 0 as an eigenvalue with eigenvector $\langle n, z, v \rangle$. Then

$$\begin{pmatrix} A & B & D \\ B^\intercal & C & 0 \\ D^\intercal & 0 & 0 \end{pmatrix} \begin{pmatrix} n \\ z \\ v \end{pmatrix} = \begin{pmatrix} An + Bz + Dv \\ B^\intercal n + Cz \\ D^\intercal n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

but since $\nabla^E s_p$ is surjective, this implies that $n = 0$. In turn, we see that $Cz = H(f\big|_Z, p)(z) = 0$ which (for $z \neq 0$) cannot be true since $f\big|_Z$ is Morse.

These observations are independent of the metric on M. Although the specific forms of the blocks A, B, C, and D may depend on the metric, the nondegeneracy of the matrix in equation 2.3 does not. Next recall that the signature of a matrix is equal to the number of negative eigenvalues minus the number of positive eigenvalues. Thus, if r is the rank of a matrix, A, then its signature, σ_A is related to its index

$$\operatorname{index} A = \frac{1}{2}(r - \sigma_A).$$

Let σ_Λ denote the signature of $H(\Lambda, \lambda)$. The claim that $\operatorname{index}(\lambda, \Lambda) = \operatorname{index}(p, f\big|_Z) + k$ is therefore equivalent to

$$\begin{aligned} \frac{1}{2}((n+k) - \sigma_\Lambda) &= \frac{1}{2}\left((n-k) - \sigma_C\right) + k \\ n + k - \sigma_\Lambda &= n - k - \sigma_C + 2k \\ \sigma_\Lambda &= \sigma_C. \end{aligned}$$

Thus, it suffices to show that $\sigma_\Lambda = \sigma_C$. Although it is not necessary, we may always choose coordinates so that $D = D^\intercal = I_{k\times k}$, the $k \times k$ identity matrix. Performing simultaneous row and column operations preserve the signature. We have

$$\begin{pmatrix} A & B & I \\ B^\intercal & C & 0 \\ I & 0 & 0 \end{pmatrix} \xleftrightarrow{C_2 - C_3 B} \begin{pmatrix} A & 0 & I \\ B^\intercal & C & 0 \\ I & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 - B^\intercal R_3} \begin{pmatrix} A & 0 & I \\ 0 & C & 0 \\ I & 0 & 0 \end{pmatrix}$$

$$\xleftrightarrow{C_1 - \frac{1}{2} C_3 A} \begin{pmatrix} \frac{1}{2}A & 0 & I \\ 0 & C & 0 \\ I & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 - \frac{1}{2} A R_3} \begin{pmatrix} 0 & 0 & I \\ 0 & C & 0 \\ I & 0 & 0 \end{pmatrix}.$$

It is now easy to see that $C = H(p, f\big|_Z)$ and $H(\Lambda, \lambda)$ have the same signature. □

2.2 The Morse Complex

As before, suppose that $\pi : E \to M$ is a rank-k vector bundle over a closed manifold M. Let $g : TM \times_M TM \to \mathbb{R}$ be a Riemannian metric on M. Let $h : E \times_M E \to \mathbb{R}$ be a bundle metric on E. Let ∇^E be a connection on E. Then ∇^E induces a splitting $TE = VE \oplus HE$ of TE into vertical and horizontal sub bundles. Let π_V be the projection onto VE along HE. We have isomorphisms

$$TE \cong VE \oplus HE \cong \pi^* E \oplus \pi^* TM \cong \pi^* (E \oplus TM).$$

Thus the triple (g, h, ∇^E) determines a metric, denoted by g_1, on the total space E. Let a be any positive real number. Let g_a denote the metric on the total space of E determined by $\left(g, a^{-2}h, \nabla^E\right)$. Let ∇^a be the Levi-Civita connection of g_a. When there's no confusion, we will use the same notations for the analogous structures on E^*.

Throughout, we will also make the following assumptions.

Assumption 2.2.1. The function $f : M \to \mathbb{R}$ is Morse.

Assumption 2.2.2. There is a section $s : M \to E$ where $\nabla^E s_p$ is rank k for all $p \in Z = s^{-1}(0)$. Thus Z is a codimension k submanifold of M and 0 is a regular value of s.

Assumption 2.2.3. The restriction $f\big|_Z$ is Morse.

There are two aims of this section.

1. The Morse homology for the Lagrange function Λ (Definition 2.1.4) with respect to each metric g_a is well-defined.

2. The Morse homology of (E^*, Λ, g_a) does not depend on the choice of a.

We address the first aim first. It's easy to see that if a manifold is not compact, the Morse complex might fail to be well-defined. This is because it can happen that a sequence of flow lines between critical points might not converge to a broken flow line. Nonetheless, if a Morse function on a noncompact manifold satisfies the following condition, the Morse complex can still be defined.

Definition 2.2.4. Suppose that (M, g) is a (possibly noncompact) Riemannian manifold. A function $f : M \to \mathbb{R}$ is said to satisfy the *Palais-Smale condition* if any sequence of points $x_i \in M$ for which $f(x_i)$ is bounded and $|\operatorname{grad} f(x_i)| \to 0$ has a convergent subsequence.

Lemma 2.2.5. *For each $a \in \mathbb{R}^+$, the Lagrange function Λ satisfies the Palais-Smale condition with respect to the gradient of g_a.*

Proof. Let $\lambda_i = (x_i, l_i)$ be a sequence of points in E^*. Here, $x_i = \pi(\lambda_i)$ and $l_i = \pi_V(\lambda_i)$. Suppose also that $\Lambda(\lambda_i)$ is bounded and that $|\operatorname{grad}^a \Lambda(\lambda_i)| \to 0$. Assume that λ_i does *not* have a convergent subsequence. Then, since M is compact, it must be the case that $|l_i| \to \infty$. But since

$$\Lambda(\lambda_i) = f(x_i) + \lambda_i\left(s(x_i)\right),$$

and since we assume that $\Lambda(\lambda_i)$ is bounded, this means that $s(x_i) \to 0$. However, note that 0 is a regular value of s so $|\operatorname{grad}^a \Lambda(\lambda_i)|$ cannot have arbitrarily small norm (see for instance equation 2.2) contradicting our assumptions. □

Note that by Theorem 2.1.3, the function Λ has only finitely many critical points (since $\operatorname{Crit}\Lambda$ is in one-to-one correspondence with $\operatorname{Crit} f\big|_Z$). The above lemma tells us that the moduli space of gradient flow lines can be compactified more or less as usual. Next, we define the stable and unstable manifolds of critical points in E^*. For each $a \in \mathbb{R}^+$, the metric g_a the negative gradient $-\operatorname{grad}^a \Lambda$ defines a one-parameter group of diffeomorphisms $\Psi_t^a : E^* \to E^*$.

Definition 2.2.6. Let $\lambda \in \operatorname{Crit} \Lambda$ and $a \in \mathbb{R}^+$. Define the *stable submanifold* of E^* for λ and a.

$$W^s(\lambda, a) = \left\{ \mu \in E^* \mid \lim_{t \to +\infty} \Psi^a(t, \mu) = \lambda \right\}.$$

Define the *unstable submanifold* of E^* for λ and a

$$W^u(\lambda, a) = \left\{ \mu \in E^* \mid \lim_{t \to -\infty} \Psi^a(t, \mu) = \lambda \right\}.$$

For any interval $I \subseteq \mathbb{R}$, we similarly define the *stable submanifold* of $E^* \times \mathbb{R}$ for λ and I

$$W^s(\lambda, I) = \{(\mu, a) \in E^* \times I \mid \mu \in W^s(\lambda, a)\}.$$

Define the *unstable submanifold* of $E^* \times \mathbb{R}$ for λ and I

$$W^u(\lambda, I) = \{(\mu, a) \in E^* \times I \mid \mu \in W^u(\lambda, a)\}.$$

Next we define the moduli spaces. Recall that if λ_+ and λ_- are two critical points, a g_a-*flow line* from λ_- to λ_+ is a path $\gamma : \mathbb{R} \to E^*$ which satisfies

$$\operatorname{grad}^a \Lambda\,(\gamma(t)) + \gamma'(t) = 0$$

and

$$\lim_{t \to -\infty} \gamma(t) = \lambda_- \qquad \lim_{t \to +\infty} \gamma(t) = \lambda_+.$$

The group $\mathbb{R}$ acts on the set of g_a-flow lines from λ_- to λ_+ by precomposition with translations of $\mathbb{R}$.

Definition 2.2.7. Let $\widetilde{\mathcal{M}}^a(\lambda_-, \lambda_+)$ denote the moduli space of g_a-flow lines from λ_- to λ_+. Equivalently,

$$\widetilde{\mathcal{M}}^a(\lambda_-, \lambda_+) = W^u(\lambda_-, a) \cap W^s(\lambda_+, a).$$

Let

$$\mathcal{M}^a(\lambda_-, \lambda_+) = \widetilde{\mathcal{M}}^a(\lambda_-, \lambda_+)/\mathbb{R}$$

denote the moduli space of g_a-flow lines modulo translation.

We also have family versions of these moduli spaces.

Definition 2.2.8. Define

$$\widetilde{\mathcal{M}}^I(\lambda_-, \lambda_+) = W^u(\lambda_-, I) \cap W^s(\lambda_+, I)$$

and

$$\mathcal{M}^I(\lambda_-, \lambda_+) = \widetilde{\mathcal{M}}^I(\lambda_-, \lambda_+)/\mathbb{R}.$$

In the standard Morse theory setting, we usually require that the Morse function and the metric satisfy the *Morse-Smale* condition. That is, for every pair of critical points (p, q), the unstable manifold for p is transverse to the stable manifold for q. In this case, since we are considering a family of metrics, we cannot ask that this condition is satisfied for all values of a. Instead, we require the following.

Assumption 2.2.9. For all $\lambda_-, \lambda_+ \in \operatorname{Crit}\Lambda$, assume that $W^s(\lambda_+, \mathbb{R}^+)$ and $W^u(\lambda_-, \mathbb{R}^+)$ intersect transversally.

The above assumption implies that for all but a discrete set of $a \in \mathbb{R}^+$, $W^s(\lambda_+, a)$ and $W^u(\lambda_-, a)$ intersect transversally.

Definition 2.2.10. Let A^{reg} denote the set of $a \in \mathbb{R}^+$ such that $W^s(\lambda_+, a)$ and $W^u(\lambda_-, a)$ intersect transversally for all $\lambda_-, \lambda_+ \in \operatorname{Crit}\Lambda$.

The above discussion shows that the Morse complex of (E^*, g_a, Λ) can be defined whenever $a \in A^{\mathrm{reg}}$. Let C^a denote this complex. That is, C^a_i is the free $\mathbb{Z}$-module generated by the set of g_a-critical points of Λ with Morse index i. The boundary operators ∂^a are defined as usual, by counting g_a-gradient flow lines. By the Lagrange multipliers theorem, the generators of C^a are the same for all a. However, the boundary operator might change as a passes through a non-regular value. Nonetheless, we show that the Morse homology H^a *is the same for all* $a \in A^{\mathrm{reg}}$.

We start by computing how the gradient flow lines depend on the parameter a.

Lemma 2.2.11. *Any g_a-flow line $\lambda(t) = (x(t), l(t))$ satisfies*

$$\begin{aligned} x'(t) &= -\left(\operatorname{grad} f\right)_{x(t)} - \left(l(t) \circ \left(\nabla^E s\right)_{x(t)}\right)^\sharp \\ l'(t) &= -a^2 s(x(t)) \end{aligned}$$

where $\sharp$ in the first equation is the metric isomorphism with respect to the metric g on M.

Proof. We compute the horizontal and vertical components separately. Fix $\lambda \in E^*$ and write $\lambda = (x, l)$ where $x \in M$ and $l \in E^*_x$. Given any vector $V \in T_\lambda E^*$, write

$$V = V^h + V^v$$

where $V^h \in H_\lambda E^*$ and $V^v \in V_\lambda E^*$. Next, using the identification $T_\lambda E^* \cong E^*_x \oplus T_x M$ given by the connection ∇^{E^*}, write

$$\begin{aligned} V^h &= X \in T_x M \\ V^v &= W \in E^*_x. \end{aligned}$$

Let $\gamma_h : \mathbb{R} \to E^*$ be a horizontal curve such that $\gamma_h(0) = \lambda$ and $\gamma_h'(0) = V^h = X$. Set $\pi(\gamma_h(t)) = x(t)$ and $\pi_V(\gamma_h(t)) = l(t)$. Compute

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \Lambda(\gamma_h(t)) &= \left.\frac{d}{dt}\right|_{t=0} (f(x(t)) + l(t) \circ s(x(t))) \\ &= df(X) + \left.\frac{d}{dt}\right|_{t=0} \langle l(t), s(x(t))\rangle \\ &= df(X) + \langle \nabla^{E^*}_X \gamma_h, s(x)\rangle + \langle \lambda, \nabla^E_X s\rangle \end{aligned}$$

where we have used the Leibniz rule in the last step. The expression $\nabla^{E^*}_X \gamma_h$ requires some explanation. By assuming that γ_h is a "horizontal path", what we mean is that γ_h is a *horizontal lift* of the curve $x(t) = \pi(\gamma_h(t)) : \mathbb{R} \to M$. Therefore, we may identify $\gamma_h(t)$ with a section of the pullback bundle $x^* E^*$. The connection ∇^{E^*} can also be pulled back. When we write $\nabla^{E^*}_X \gamma_h$, we mean the covariant derivative with respect to this pulled back connection. Furthermore, by the definition of horizontal lift, we see that

$$\nabla^{E^*}_X \gamma_h = 0.$$

Thus, we have

$$\left.\frac{d}{dt}\right|_{t=0} \Lambda(\gamma_h(t)) = df(X) + \lambda \circ \nabla^E_X s.$$

Writing this equation in terms of the gradient now amounts to taking its metric dual. Note that both $df(X)$ and $\lambda \circ \nabla^E_X s$ depend only on the vector $X \in M$. Thus the metric dual can be taken with respect to the metric g on M and is not affected by the parameter a. We have

$$\begin{aligned} x'(t) &= -\pi_H(\operatorname{grad}^a \Lambda(\gamma(t))) \\ &= -\left(\operatorname{grad} f\right)_{x(t)} - \left(l(t) \circ \left(\nabla^E s\right)_{x(t)}\right)^\sharp \end{aligned}$$

as desired.

Next, we consider a vertical path $\gamma_v : \mathbb{R} \to E^*$ such that $\gamma_v(0) = \lambda = (x, l)$ and $\gamma'(0) = V^v = W \in E_x^*$. In this case, we have

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \Lambda(\gamma_v(t)) &= \left.\frac{d}{dt}\right|_{t=0} \left(f(x(t)) + l(t) \circ s(x(t))\right) \\ &= W(s(x)). \end{aligned}$$

Note that functional on E^* which sends the vector W to $W(s(x))$ exactly corresponds to $s(x)$ under the standard identification $(E_x^*)^* \cong E_x$. All that is left is to raise the indices. In this case, the parameter "a" *does* come into play since we require that the gradient vector v must satisfy, for all $W \in E_x^*$,

$$a^{-2}h^*\left(v, W\right) = W(s(x)) \Rightarrow h^*\left(v, W\right) = a^2 W(s(x)).$$

Thus the vertical component of a negative gradient flow must satisfy

$$\begin{aligned} l'(t) &= -\pi_V\left(\operatorname{grad}^a \Lambda(\gamma(t))\right) \\ &= -a^2 s(x(t)) \end{aligned}$$

as desired. $\square$

Definition 2.2.12. The *energy* of a g_a-flow line γ is defined to be

$$\mathcal{E}(\gamma) = \int_{\mathbb{R}} \|\gamma'(t)\|_{g_a}^2 dt = \int_{\mathbb{R}} \|x'(t)\|_g^2 dt + \int_{\mathbb{R}} \|l'(g)\|_{g_a}^2 dt$$

Note that if $\gamma(+\infty) = \lambda_+$ and $\gamma(-\infty) = -\lambda$ for $\lambda_\pm \in \operatorname{Crit}\Lambda$, then

$$\mathcal{E}(\gamma) = \Lambda(\lambda_-) - \Lambda(\lambda_+).$$

Therefore the energy of any g_a-flow line between two critical points is finite.

Lemma 2.2.13. *For any $C > 0$, there exists a constant $K_C > 0$ such that for $\lambda_\pm \in \operatorname{Crit}\Lambda$, $W^u(\lambda_-, (0, C]) \cap W^s(\lambda_+, (0, C])$ is a subset of $M \times [-K_C, K_C] \times [0, C]$.*

Proof. Suppose the statement is false. Then there exists $\lambda_\pm \in \operatorname{Crit}\Lambda$, a sequence a_i and a sequence $\mu_i = (x_i, l_i)$ such that

– $\lim_{i\to\infty} a_i = a_\infty \in [0, C]$

– $\mu_i = (x_i, l_i) \in W^u(\lambda_-, a_i) \cap W^s(\lambda_+, a_i)$ for each i

– $\lim_{i\to\infty} |l_i| = +\infty$.

Since each point in E^* lies on a flow line, unique for whatever metric we choose, we may find a sequence $\gamma_i = (x_i(t), l_i(t)) : \mathbb{R} \to E^*$ where $\gamma_i(t)$ is a g_{a_i}-flow line and satisfies $\gamma_i(0) = \mu_i = (x_i, l_i)$ for each i. Next, note that since a_i is bounded, $|l_i(t)| \to +\infty$ uniformly on $-T \le t \le T$ for any $T > 0$. Furthermore, by assumption,

$$\begin{aligned} \Lambda(\lambda_-) &\ge \Lambda(\gamma_i(t)) \ge \Lambda(\lambda_+) \\ \Rightarrow \Lambda(\lambda_-) &\ge f(x_i(t)) + l_i(t) \circ s(x_i(t)) \ge \Lambda(\lambda_-). \end{aligned}$$

Therefore, $s(x_i(t)) \to 0$ uniformly on $-T \le t \le T$. Then, since 0 is a regular value of s, $\left\| \left(\nabla^E s\right)_{x_i(t)} \right\|_{g_{a_i}}$ is bounded away from zero on $-T \le t \le T$ for large i. Thus we have have

$$\begin{aligned} \Lambda(\lambda_-) - \Lambda(\lambda_+) &= \int_T^T \|\gamma_i'(t)\|_{g_{a_i}}^2 dt \\ &\ge \int_{-T}^T \|x_i'(t)\|_g^2 dt \\ &= \int_{-T}^T \left\| (\operatorname{grad} f)_{x_i(t)} + \left(l_i(t) \circ \left(\nabla^E s\right)_{x_i(t)} \right)^\sharp \right\|_g^2 dt \to \infty \end{aligned}$$

which is a contradiction. $\square$

The key point in the above lemma is that the set $M \times [-K_C, K_C] \times [0, C]$ is compact. This allows us to show the following.

Proposition 2.2.14. *For any $a_1, a_2 \in A^{\text{reg}}$, there is a canonical isomorphism $\Phi_{a_1,a_2} : H^{a_1} \to H^{a_2}$ such that for $a_1, a_2, a_3 \in A^{\text{reg}}$, $\Phi_{a_2,a_3} \circ \Phi_{a_1,a_2} = \Phi_{a_1,a_3}$.*

Proof. The previous lemma allows us to apply standard methods of *continuation maps* in order to construct the desired isomorphism. See, for example, chapter 4 of [32]. □

2.3 The limit as $a \to \infty$

In this section, we show that the Morse homology of Λ on E^*, with grading shifted by k (where k is the rank of E), is isomorphic to the Morse homology of $f\big|_Z$. We do this by constructing a bijection between $\widetilde{\mathcal{M}}^a(\lambda_\pm)$ for sufficiently large a and $\widetilde{\mathcal{M}}_Z(x_\pm)$, the space of flow lines between $x_+ = \pi(\lambda_+)$ and $x_- = \pi(\lambda_-)$ in $\operatorname{Crit} f\big|_Z$. Our proof closely follows the first proof of Theorem 5 in [40] and comes in two parts. In the first part, we show that trajectories in $\widetilde{\mathcal{M}}^a$ with a sufficiently large can be approximated by trajectories in $\widetilde{\mathcal{M}}_Z$, where M is identified with the zero section of E^*.

2.3.1 Approximation of $\lambda(t)$ by $x(t)$.

We start with several lemmas.

Lemma 2.3.1. *Let B_ϵ be the ball of radius ϵ in $\mathbb{R}^k$ and let $U_\epsilon = s^{-1}(B_\epsilon)$. For any $\epsilon > 0$, there exists a number A_ϵ such that for all $a > A_\epsilon$,*

$$\lambda(t) \in \widetilde{\mathcal{M}}^a(\lambda_\pm) \quad \Rightarrow \quad x(t) = (\pi \circ \lambda)(t) \in U_\epsilon.$$

Proof. Let $\lambda(t) \in \widetilde{\mathcal{M}}^a(\lambda_\pm)$. We have

$$\begin{aligned}
\mathcal{E} = \mathcal{E}(\lambda) &= \int_{\mathbb{R}} \|x'(t)\|_g^2 dt + \int_{\mathbb{R}} \|l'(t)\|_{g_a}^2 dt \\
&= \|x'\|_{L^2}^2 + \int_{\mathbb{R}} \left\|-a^2 s(x(t))\right\|_{g_a}^2 dt \\
&= \|x'\|_{L^2}^2 + \int_{\mathbb{R}} (a^2 a^{-1})^2 \|s(x(t))\|_{g_1}^2 dt \\
&= \|x'\|_{L^2}^2 + a^2 \|s(x)\|_{L^2}^2
\end{aligned}$$

where the L^2 norms are taken with respect to g on M and g_1 on E. Also, the notation $\|s(x(t))\|_{g_a}$ is slightly abusive. What we mean here is actually the fiberwise length of the vector $s(x(t))$ measured along the unique vertical geodesic in the fiber with respect to the bundle metric $a^{-2}h$. In any case, we have

$$\|s(x)\|_{L^2} \leq \frac{\mathcal{E}^{1/2}}{a}.$$

On the other hand,

$$\left\|\frac{d}{dt} s(x(t))\right\|_{L^2} \leq \|ds\|_{L^\infty} \|x'\|_{L^2}$$

so $s(x(t)) \in W^{1,2}$. Suppose that a_i is a sequence with $a_i \to \infty$. Let $\lambda_i(t) \in \widetilde{\mathcal{M}}^{a_i}(\lambda_\pm)$ and let $x_i(t) = (\pi \circ \lambda_i)(t)$. Then the compact Sobolev embedding $W^{1,2} \to C^0$ implies that there is a subsequence x_j such that $s(x_j)$ converges in C^0. Since $\|s(x_j)\|_{L^2} \to 0$ as $a \to \infty$, we know that $s(x_j)$ converges to zero uniformly. In other words, the statement of the lemma follows. $\square$

Lemma 2.3.2. *Suppose that $a_i \to \infty$ and that $\lambda_i = (x_i, l_i) \in \widetilde{\mathcal{M}}^{a_i}(\lambda_\pm)$. Then $l_i(t)$ is uniformly bounded.*

Proof. Choose $\epsilon > 0$ and let A_ϵ, U_ϵ be as in the previous lemma. We can define a section ζ of $E^*\big|_{U_\epsilon}$ by

$$\nabla^E s \left(\operatorname{grad} f(x) + \left(\zeta(x) \circ \left(\nabla^E s\right)_x\right)^\sharp\right) = 0. \tag{2.4}$$

Note that when $x_0 \in \operatorname{Crit} f\big|_Z$, then Theorem 2.1.3 implies that $df_{x_0} = \lambda \circ \left(\nabla^E s\right)_{x_0}$ for some $\lambda \in E^*$. Therefore, at $x_0 \in \operatorname{Crit} f\big|_Z$, we have

$$\begin{aligned}
\nabla^E s\left(\operatorname{grad} f(x) + \left(\zeta(x) \circ \left(\nabla^E s\right)_x\right)^\sharp\right) &= \nabla^E s\left(\left(\lambda \circ \left(\nabla^E s\right)_{x_0}\right)^\sharp + \left(\zeta(x_0) \circ \left(\nabla^E s\right)_{x_0}\right)^\sharp\right) \\
&= \nabla^E s\left[\left((\lambda(x_0) + \zeta(x_0)) \circ \left(\nabla^E s\right)_{x_0}\right)^\sharp\right] \\
&= 0.
\end{aligned}$$

Then, since $x_0 \in Z$, recall that $\left(\nabla^E s\right)_{x_0}$ is surjective by assumption. Therefore we conclude that

$$\lambda(x_0) = -\zeta(x_0)$$

for any $x_0 \in \operatorname{Crit} f\big|_Z$.

Next, let $\lambda_i = (x_i, l_i) \in \widetilde{\mathcal{M}}^{a_i}(\lambda_\pm)$. Note that $\zeta \circ x_i : \mathbb{R} \to E^*$ is uniformly bounded since M is compact.

Since λ_i is a g_{a_i}-flow line, it must satisfy

$$x_i'(t) + \operatorname{grad} f(x_i(t)) + \left(l_i(t) \circ \left(\nabla^E s\right)_{x_i(t)}\right)^\sharp = 0.$$

Acting on the above by $\nabla^E s$ gives

$$\nabla^E s\left(x_i'(t)\right) + \nabla^E s\left[\left(l_i(t) - \zeta(x_i(t))\right) \circ \left(\nabla^E s\right)_{x_i(t)}\right]^\sharp = 0 \tag{2.5}$$

by the definition of ζ. Also, since Lemma 2.2.11 implies that $l_i' + a_i^2 s(x_i) = 0$, we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{a_i}\left(l_i - \zeta(x_i)\right)\right) + a_i(s(x_i)) &= \frac{1}{a_i} l_i' + a_i s(x_i) - \frac{1}{a_i}\frac{d}{dt}\zeta(x_i) &(2.6)\\
&= -\frac{1}{a_i}\frac{d}{dt}\zeta(x_i). &(2.7)
\end{aligned}$$

Next, define a fiberwise endomorphism

$$\begin{aligned}
\Phi(t) : E^*_{x(t)} &\to E_{x(t)} \\
\mu &\mapsto \left(\nabla^E s_t\right)\left(\left(\mu \circ \nabla^E_t s\right)^\sharp\right)
\end{aligned}$$

where we have introduced the notation ∇_t^E to indicate the covariant derivative along $x(t)$. Note that since $\nabla_t^E s$ is surjective, $\Phi(t)$ is invertible for each t. Then, for any $a_i \geq A_\epsilon$, define a linear map $D : W^{1,2} \to L^2$ by

$$D_i = \begin{pmatrix} \nabla_t^E & 0 \\ 0 & \frac{d}{dt} \end{pmatrix} + a_i \begin{pmatrix} 0 & \Phi(t) \\ 1 & 0 \end{pmatrix}.$$

The key property of this operator is that it is *bounded below* by Ca_i where C is a uniform constant. Let D_i act on

$$\begin{pmatrix} s(x_i(t)) \\ a_i^{-1}\left(l_i(t) - \zeta(x_i(t))\right) \end{pmatrix}$$

and then repeat the argument given in the proof of Proposition 7 in [40] to conclude that there exists a uniform constant $C > 0$ such that

$$\|s(x_i)\|_{W^{1,2}} + a_i^{-1}\|l_i - \zeta(x_i)\|_{W^{1,2}} \leq Ca^{-1}.$$

The Sobolev embedding $W^{1,2} \to C^0$ implies that l_i is uniformly bounded since $\zeta(x_i)$ is. $\square$

Lemma 2.3.3. *Let $\lambda_i(t)$ be as before with $x_i = (\pi \circ \lambda_i)(t)$. Then there exists a subsequence $x_j \to x$ in C^0_{loc} where x satisfies*

$$x'(t) = -\operatorname{grad}^Z f\big|_Z(x(t)).$$

That is, $x(t) \in \widetilde{m}_Z(x_\pm)$.

Proof. Let $p : NZ \to Z$ denote the normal bundle of Z. Identify the set U_ϵ with a subset of NZ. Then we can identify $U_\epsilon \cong s^{-1}(0) \times B_\epsilon$ for some $\epsilon > 0$ such that for any $x_i(t) \in U_\epsilon$, we have

$$x_i(t) = \left(\overline{x_i}(t), s(x_i(t))\right).$$

Let $p(x_i) = \overline{x_i}$. Since $\lambda_i(t)$ is a g_{a_i}-flow line, we have

$$\begin{aligned}
\overline{x_i}'(t) + f\big|_Z(\overline{x_i}(t)) &= p\left(-\operatorname{grad} f(x_i(t)) - \left(l_i(t)\circ\left(\nabla^E s\right)_{x_i(t)}\right)^\sharp\right) + \operatorname{grad}^Z f\big|_Z(\overline{x_i}(t)) \\
&= p\left(-\operatorname{grad} f\left(x_i(t)\right)\right) + \operatorname{grad}^Z f\big|_Z\left(\overline{x_i}(t)\right)
\end{aligned}$$

where we have used the fact that locally, span $\left(\nabla^E s\right)^\sharp$ is normal to Z. Thus we have

$$\begin{aligned}
\left|\overline{x_i}'(t) + \operatorname{grad}^Z f\big|_Z\left(\overline{x_i}(t)\right)\right| &\leq C|p\left(\operatorname{grad} f(\overline{x_i}(t)) - \operatorname{grad} f(x_i(t))\right)| \\
&\leq C|\text{distance}(x_i(t), \overline{x_i}(t))| \\
&\leq C|s(x_i(t))|.
\end{aligned}$$

Thus the result follows from the Sobolev embedding $W^{1,2} \to C^0$ together with Lemma 2.3.2. □

The reparameterization trick, as applied in the standard compactification of the moduli space of flow lines by broken trajectories, together with Lemmas **??** imply

Proposition 2.3.4. *Suppose that $a_i \to \infty$ and that $\lambda_i = (x_i, l_i) \in \widetilde{\mathcal{M}}^{a_i}(\lambda_\pm)$. Then there is exists a subsequence of λ_i such that*

1. *there exists $t_i^n \in \mathbb{R}$ with $i = 1, \ldots, I$*
2. *there exists $x^n \in \widetilde{\mathcal{M}}_Z(p_n, p_{n+1})$ with $x_+ = p_1$, $x_- = p_{I+1}$*
3. *$f\big|_Z(p_1) > \cdots > f\big|_Z(p_{I+1})$*

and

$$x_i^n = x_i(\cdot + t_i^n) \to x^n \in \widetilde{\mathcal{M}}_Z(p_n, p_{n+1}) \text{ in } C^1_{\text{loc}}.$$

2.3.2 Applying the implicit function theorem.

Suppose that the Morse indices of $\lambda_+, \lambda_- \in \operatorname{Crit}\Lambda$ differ by 1. For large enough a, Proposition 2.3.4 provides us with a map $\mathscr{F}^a : \widetilde{\mathcal{M}}^a(\lambda_\pm) \to \widetilde{\mathcal{M}}_Z(x_\pm)$. We would like to prove that this map is a bijection, for large enough a. As before, the material in this section closely follows [40]. Therefore, we do not provide every detail of the proofs, but instead make clear how to adapt the proofs in [40].

2.3.2.1 Preliminaries. In this section, we describe a framework that will be useful for thinking of flow lines in the context of Banach spaces.

Definition 2.3.5. Let $\overline{\mathbb{R}}$ denote the bounded manifold $\mathbb{R} \cup \{\pm\infty\}$. Let x, y be two points in a manifold M. Define $C^\infty_{x,y}(\overline{\mathbb{R}}, M)$ be the set of smooth, compact curves connecting x and y. That is,

$$C^\infty_{x,y}\left(\overline{\mathbb{R}}, M\right) = \left\{\gamma \in C^\infty(\overline{\mathbb{R}}, M) \mid \gamma(-\infty) = x, \gamma(+\infty) = y\right\}.$$

Definition 2.3.6. Let ξ be a smooth, rank-k vector bundle over $\overline{\mathbb{R}}$. Let $\varphi : \xi \to \overline{\mathbb{R}} \times \mathbb{R}^k$ be a smooth trivialization. Then using the induced one-to-one mapping φ_* between the associated vector spaces of sections, define a functor

$$W^{p,q}_{\mathbb{R}}(\xi) = \varphi_*^{-1}\left(W^{p,q}\left(\mathbb{R}, \mathbb{R}^k\right)\right) = \left\{\varphi_*^{-1}(s) \mid s \in W^{p,q}\left(\mathbb{R}, \mathbb{R}^k\right)\right\}.$$

When $p = 0$ and $q = 2$, we call the functor $L^2_{\mathbb{R}}$.

The vector space $W^{1,2}_{\mathbb{R}}(\xi)$ is a Banach space whose topology does not depend on the trivialization φ. Next, let $p \in M$. Then let $D \subset TM$ be an open subset such that $\exp_p$ is defined for every $p \in \pi(D)$. Recall that $\exp_p : v \mapsto \gamma(1)$ where γ is the unique geodesic in M such that $\gamma(0) = p$ and $\gamma'(0) = v$.

Definition 2.3.7. Let γ be a smooth, compact curve in M. Define

$$\exp_\gamma : W^{1,2}_{\mathbb{R}}\left(\gamma^* D\right) \to C^0\left(\overline{\mathbb{R}}, M\right)$$

by

$$\exp_\gamma(X)(t) = \exp_{\gamma(t)}\left(X(t)\right).$$

Also define the set of $W^{1,2}$ curves with endpoints $x, y \in M$.

$$\mathcal{B}^{1,2}_{x,y} = \left\{\exp_\gamma(X) \in C^0\left(\overline{\mathbb{R}}, M\right) \mid \gamma \in C^\infty_{x,y}(\overline{\mathbb{R}}, M), X \in W^{1,2}_{\mathbb{R}}(\gamma^* D)\right\}$$

and, using the functor $L^2_{\mathbb{R}}$,

$$\mathcal{L}^2_{x,y} = L^2_{\mathbb{R}}\left(\left(\mathcal{B}^{1,2}_{x,y}\right)^* TM\right) = \bigcup_{s \in \mathcal{B}^{1,2}_{x,y}} L^2_{\mathbb{R}}\left(s^* TM\right).$$

The tangent space $T\mathcal{B}^{1,2}_{x,y}$ is

$$T\mathcal{B}^{1,2}_{x,y} = W^{1,2}_{\mathbb{R}}\left(\left(\mathcal{B}^{1,2}_{x,y}\right)^* TM\right) = \bigcup_{s \in \mathcal{B}^{1,2}_{x,y}} W^{1,2}_{\mathbb{R}}\left(s^* TM\right).$$

We have the following standard result.

Proposition 2.3.8. *The set of curves $\mathcal{B}^{1,2}_{x,y}$ with endpoints $x, y \in M$ is a Banach manifold with atlas*

$$\left(W^{1,2}(\gamma^* D), \exp_\gamma\right)$$

where $\gamma \in C^\infty_{x,y}(\overline{\mathbb{R}}, M)$. Furthermore

$$C^\infty_{x,y}(\overline{\mathbb{R}}, M) \overset{\text{dense}}{\subset} \mathcal{B}^{1,2}_{x,y}\left(\mathbb{R}, M\right) \overset{\text{dense}}{\subset} C^0_{x,y}(\overline{\mathbb{R}}, M).$$

Also, $\mathcal{L}^2_{x,y}$ is a Banach bundle over $\mathcal{B}^{1,2}_{x,y}$.

2.3.2.2 Banach space setup. Next, we apply the material of the previous section to the Lagrange multiplier problem. Start with $\lambda_+, \lambda_- \in \operatorname{Crit}\Lambda$, such that the Morse indices of λ_+ and λ_- differ by 1. As usual, let $x_\pm = \pi(\lambda_\pm)$. Define the following

- Let $\mathcal{B}^Z = \mathcal{B}^{1,2}_{x_\pm}$ be the Banach manifold of $W^{1,2}$ curves in Z with endpoints $x_\pm$.
- Let $\mathcal{L}^Z = L^2_{\mathbb{R}}\left(\left(\mathcal{B}^Z\right)^* TZ\right)$ be the Banach bundle over $\mathcal{B}^Z$.
- Let $\mathcal{B} = \mathcal{B}^{1,2}_{\lambda_\pm}$ be the Banach manifold of $W^{1,2}$ curves in E^* with endpoints $\lambda_\pm$.
- Let $\mathcal{L} = L^2_{\mathbb{R}}\left(\mathcal{B}^* TE^*\right)$ be the Banach bundle over $\mathcal{B}$.

The gradient of $f\big|_Z$ induces a section of $\mathcal{L}^Z$ via

$$\begin{aligned} \bar{\mathcal{S}} : \mathcal{B}^Z &\to \mathcal{L}^Z \\ y &\mapsto y' + \operatorname{grad} f\big|_Z(y) \end{aligned}$$

so that $\widetilde{\mathcal{M}}_Z(x_\pm) = \bar{\mathcal{S}}^{-1}(0)$. The linearization of $\bar{\mathcal{S}}$ at x is

$$\begin{aligned} \bar{D}_x : T_x\mathcal{B}^Z &\to \mathcal{L}^Z_x \\ V &\mapsto \nabla^Z_{x'} V + \nabla^Z \operatorname{grad}^Z f\big|_Z. \end{aligned}$$

Recall the section $\zeta : M \to E^*$ defined in equation 2.4 by

$$\nabla^E s \left(\operatorname{grad} f + \left(\zeta \circ \nabla^E s\right)^\sharp\right) = 0.$$

Note that by definition, $\operatorname{grad} f + \left(\zeta \circ \nabla^E s\right)^\sharp$ is a vector field on M whose restriction to Z is $\operatorname{grad}^Z f\big|_Z$. Thus, we can write

$$\begin{aligned} \overline{D}_x &: T_x\mathcal{B}^Z \to \mathcal{L}^Z_x \\ V &\mapsto \nabla^Z_{x'} V + \nabla^Z_V \left(\operatorname{grad} f + (\zeta \circ \nabla^E s)^\sharp\right). \end{aligned}$$

We assume that $\left(f\big|_Z, g\big|_Z\right)$ is Morse-Smale. Therefore, $\bar{D}$ has a bounded right inverse which we call $\bar{Q}$.

Similarly, for each $a \in \mathbb{R}^+$, the gradient of Λ with respect to the metric g_a induces a section of $\mathcal{L}$ via

$$\begin{aligned} \mathcal{S}^a : \mathcal{B} &\rightarrow \mathcal{L} \\ \lambda &\mapsto \lambda' + \operatorname{grad}^a \Lambda(\lambda) \end{aligned}$$

Writing λ in horizontal and vertical components yields

$$\mathcal{S}^a : (x, l) \mapsto \Big(x' + \operatorname{grad} f(x) + \left(\lambda \circ \nabla^E s\right)^\sharp (x), l' + a^2 s(x) \Big) .$$

For each $a \in A^{\mathrm{reg}}$, $\widetilde{\mathcal{M}}^a(\lambda_\pm) = \left(\mathcal{S}^a\right)^{-1}(0)$.

2.3.2.3 The implicit function theorem. Let $x(t) \in \widetilde{\mathcal{M}}_Z(x_\pm)$ be any g-flow line. We define *an approximate solution* to the equation $\mathcal{S}^a(\lambda) = 0$ by

$$\gamma(t) = (x(t), \zeta(x(t))) . \tag{2.8}$$

If we can show that the hypotheses of the implicit function theorem are satisfied, then we will know that a genuine solution exists, thus allowing us to construct a map $(\mathcal{F}^a)^{-1} : \widetilde{\mathcal{M}}_Z(x_\pm) \rightarrow \widetilde{\mathcal{M}}^a(\lambda_\pm)$.

The linearization of $\mathcal{S}^a$ at $\gamma = (x, l)$ is

$$\begin{aligned} D^a_\gamma : T_\gamma \mathcal{B} &\rightarrow \mathcal{L}_\gamma \\ \begin{pmatrix} V \\ U \end{pmatrix} &\mapsto \begin{pmatrix} \nabla_{x'} V + \nabla_V \left(\operatorname{grad} f + \left(l \circ \nabla^E s\right)^\sharp \right) + \left(U \circ \nabla^E s\right)^\sharp \\ \nabla^E_{x'} U + a^2 \nabla^E s(V) \end{pmatrix} \end{aligned}$$

Next, we split $T_\gamma \mathcal{B}$ and $\mathcal{L}_\gamma$ in a convenient way to help us compare the operators $\bar{D}_x$ and D^a_γ. Let $K = \ker(\nabla^E s)$. Let $K^\perp$ be the orthogonal complement of K inside TM, and let $L = \ker(d\pi)$. Then define

$$W_K = W^{1,2}_{\mathbb{R}}(x^* K) \qquad W_{K^\perp} = W^{1,2}_{\mathbb{R}}(x^* K^\perp) \qquad W_L = W^{1,2}_{\mathbb{R}}(\gamma^* L) \tag{2.9}$$

$$\mathcal{L}_K = L^2_{\mathbb{R}}(x^* K) \qquad \mathcal{L}_{K^\perp} = L^2_{\mathbb{R}}(x^* K^\perp) \qquad \mathcal{L}_L = L^2_{\mathbb{R}}(\gamma^* L). \tag{2.10}$$

Now that we have this splitting, we rescale the L^2 and $W^{1,2}$ norms as follows. For $V' \in W_{K^\perp}$, define

$$\|V'\|_{W_a^\perp} = a\|V'\|_{L^2} + \|\nabla_t V'\|_{L^2}$$

For $U \in W_L$, define

$$\|U\|_{W_a^L} = \|U\|_{L^2} + a^{-1}\left\|\nabla_t^E U\right\|_{L^2}$$

and

$$\|U\|_{L_a^L} = a^{-1}\|U\|_{L^2}$$

Then, define on $W_{K^\perp} \oplus W_L$

$$\|(V', U)\|_{W^a} = a\|V'\|_{L^2} + \|\nabla_t V'\|_{L^2} + \|U\|_{L^2} + a^{-1}\left\|\nabla_t^E U\right\|_{L^2}$$

$$\|(V', U)\|_{L_a} = \|V'\|_{L^2} + a^{-1}\|U\|_{L^2}$$

With respect to this splitting, D_γ^a will take the form of a three by three matrix. Let $V = (\overline{V}, V', U)$ be any vector in $T_\gamma\mathcal{B}$ written in components with respect to the above splitting. We have

$$D_\gamma^a(V) = \begin{pmatrix} D_1 & A & B \\ E & D_2 & C \\ F & G & D_3 \end{pmatrix} \begin{pmatrix} \overline{V} \\ V' \\ U \end{pmatrix} = \begin{pmatrix} \overline{W} \\ W' \\ Y \end{pmatrix}$$

Let's see how D_γ^a acts on each component of V.

Case 1: Suppose that $V = (\overline{V}, 0, 0)$. We have:

$$\begin{aligned} D^a_\gamma(V) &= \left(\nabla_{x'}\overline{V} + \nabla_{\overline{V}}\left(\operatorname{grad} f + (\zeta \circ \nabla^E s)^\sharp\right), a^2\nabla^E s(\overline{V})\right) \\ &= \left(\nabla_{x'}\overline{V} + \nabla_{\overline{V}}(\operatorname{grad} f + (\zeta \circ \nabla^E s)^\sharp), 0\right) \end{aligned}$$

with respect to the horizontal/vertical splitting. $\overline{W}$ will be the component of $D^a_\gamma(V)$ that is tangent to Z. In other words, it's:

$$\bar{D}_x(\overline{V}) = \overline{\nabla}_{x'}\overline{V} + \overline{\nabla}_{\overline{V}}\left(\operatorname{grad} f + (\zeta \circ \nabla^E s)^\sharp\right) = \overline{\nabla}_{x'}\overline{V} + \overline{\nabla}_{\overline{V}}(\operatorname{grad}^Z f\big|_Z) \in \mathcal{L}_K$$

$$\Rightarrow D_1 = \bar{D}$$

W' will be the component of $D^a_\gamma(V)$ that is normal to Z. We can express this using the second fundamental form which is defined by

$$\begin{aligned} \mathrm{II}(v, w) &= (\nabla_v w)^\perp = \text{projection of } \nabla_v w \text{ onto the normal bundle} \\ \Rightarrow &\quad E(\overline{V}) = \mathrm{II}(x', \overline{V}) + \mathrm{II}(\overline{V}, \operatorname{grad} f + (\zeta \circ \nabla^E s)^\sharp) = 0 \end{aligned}$$

since II is bilinear and symmetric and since $x(t)$ is assumed to be a gradient flow line inside Z. Lastly, Y will be the component of D^a_γ orthogonal to TM. Since $\overline{V} \in \ker \nabla^E s$ by assumption, $F = 0$.

Case 2: Suppose that $V = (0, V', 0)$. Then

$$D^a_\gamma(V) = \left(\nabla_{x'}V' + \nabla_{V'}(\operatorname{grad} f + (\zeta \circ \nabla^E s)^\sharp), a^2\nabla^E s(V')\right)$$

As before, $\overline{W}$ will be the component of $\nabla_{x'}V' + \nabla_{V'}(\operatorname{grad} f + (\zeta \circ \nabla^E s)^\sharp)$ that is tangent to Z. Thus, if $p : NZ \to Z$, we have:

$$A(V) = dp(\nabla_{x'}V' + \nabla_{V'}(\operatorname{grad} f + (\zeta \circ \nabla^E s)^\sharp))$$

Similarly, W' will be the component that is normal to Z:

$$D_2(V) = \pi_N\left(\nabla_{x'}V' + \nabla_{V'}(\operatorname{grad} f + (\zeta \circ \nabla^E s)^\sharp)\right)$$

and lastly, Y is the component of $D^a_\gamma(V)$ perpendicular to TM. Thus

$$G(V) = a^2\nabla^E s(V')$$

Case 3: Suppose that $V = (0, 0, U)$. Then

$$D^a_\gamma(V) = \left((U \circ \nabla^E s)^\sharp, \nabla^E_{x'}U\right)$$

Since $(U \circ \nabla^E s)^\sharp \in NZ$, $\overline{W} = 0$, so $B = 0$ and $W' = (U \circ \nabla^E s)^\sharp \Rightarrow C = (\cdot \circ \nabla^E s)^\sharp$ and $Y = \nabla^E_{x'}U$ so $D_3 = \nabla_{x'}(\cdot)$.

All together, the matrix looks like

$$D^a_\gamma = \begin{pmatrix} \bar{D}_x & A & 0 \\ 0 & D_2 & (\cdot \circ \nabla^E s)^\sharp \\ 0 & a^2\nabla^E s(\cdot) & \nabla^E_{x'}(\cdot) \end{pmatrix} \tag{2.11}$$

In order to apply the Implicit Function Theorem, we need to show that D^a_γ has a bounded right inverse. Since we know that $\bar{D}_x$ has a bounded right inverse, we just need to have some control over A, and show that the lower right block has a bounded right inverse. To that end, let

$$D'_\gamma = \begin{pmatrix} D_2 & (\cdot \circ \nabla^E s)^\sharp \\ a^2\nabla^E s(\cdot) & \nabla^E_{x'}(\cdot) \end{pmatrix}$$

After trivializing the sub-bundle of TE^* generated by K and L locally, we can rewrite D'_γ as:

$$\frac{d}{dt} + \begin{pmatrix} c(t) & b(t) \\ a^2b(t) & 0 \end{pmatrix} = \frac{d}{dt} + B(t)$$

Then for large a, $B(t)$ has one positive and one negative eigenvalue bounded away from zero by Ca where C is a constant independent of a. Thus, D'_γ has a unique bounded right inverse

$$Q'_\gamma : \mathcal{L}_{K^\perp \oplus L} \to W_{K^\perp \oplus L}, \qquad \qquad \left\| Q'_\gamma \right\| \leq c_1$$

for some constant $c_1 > 0$.

Remark 2.3.9. To see that D' has a bounded right inverse, consider the one-dimensional problem of inverting an operator of the form $\frac{d}{dt} + ag(t)$ for some function g and some constant a.

Define an approximate right inverse for the operator D^a_γ:

$$Q = \begin{pmatrix} \overline{Q}_\gamma & 0 \\ 0 & Q'_\gamma \end{pmatrix}$$

Lemma 2.3.10. *There exists a number $A_0 > 0$ such that for all $a > A_0$,*

$$\left\| Id - D^a_\gamma Q \right\|_{L_a} < \frac{1}{2}$$

Proof. First, note that $D^a_\gamma Q - \mathrm{Id} = (A_x, 0) \circ Q'_\gamma$. Next, note that

$$\|A_x(V')\|_{L_a} = \|V'\|_{L^2} \leq \frac{1}{a} \|V'\|_{W_a}.$$

So we have:

$$\left\| (A_x, 0) \circ Q'_\gamma(V', U) \right\|_{L_a} \leq \frac{1}{a} \left\| Q'_\gamma(V', U) \right\|_{W_a} \leq \frac{c_1}{a} \|(V', U)\|_{L_a}$$

□

Therefore, D^a_γ is invertible with a right inverse that is uniformly bounded by some constant independent of a. Let the right inverse for D^a_γ be denoted by Q^a_γ.

The following statement of the implicit function theorem was taken from Appendix A of [35].

Theorem 2.3.11. *Let X and Y be Banach spaces, $U \subset X$ an open set, and $f : U \to Y$ a continuously differentiable map. Let $x_0 \in U$ be such that $D := df(x_0) : X \to Y$ is surjective and has bounded right inverse $Q : Y \to X$. Choose positive constants δ and c such that $\|Q\| \leq c$, $B_\delta(x_0, X) \subset U$, and*

$$\|x - x_0\| < \delta \Rightarrow \|df(x) - D\| \leq \frac{1}{2c}.$$

Suppose that $x_1 \in X$ satisfies

$$\|f(x_1)\| < \frac{\delta}{4c} \qquad\qquad \|x_1 - x_0\| < \frac{\delta}{8}.$$

Then there exists a unique $x \in X$ such that

$$f(x) = 0, \quad x - x_1 \in \mathrm{Im}(Q), \quad \|x - x_0\| < \delta.$$

Moreover,

$$\|x - x_1\| \leq 2c\|f(x_1)\|.$$

In order to apply the IFT, we trivialize $\mathcal{L} \to \mathcal{B}$ locally near a point γ. The exponential map $\exp_\gamma$ (Definition 2.3.7) identifies a neighborhood $\mathcal{U}_\gamma$ of γ with a W_a-small ball centered at the origin of $T_\gamma\mathcal{B}$. Let $\rho : (B_\epsilon \subset T_\gamma\mathcal{B}) \to (\mathcal{U}_\gamma \subset \mathcal{B})$ denote this identification. Then, for any small vector $V \in T_\gamma\mathcal{B}$, we also have an associated parallel transport map

$$\Phi_V : \mathcal{L}_{\rho(V)} \to \mathcal{L}_\gamma$$

which is an isomorphism. Composing $\mathcal{S}^a$ with Φ_V and ρ yields a map between Banach spaces given by

$$\Phi_\cdot \circ \mathcal{S}^a \circ \rho : T_\gamma\mathcal{B} \to \mathcal{L}_\gamma.$$

Denote by D_V the linearization of this map at V. We have

Lemma 2.3.12. *There exists $\epsilon, c > 0$ independent of a, such that for all $V \in T_\gamma\mathscr{B}$ satisfying $\|V\|_{W_a} < \epsilon$,*

$$\left\|D_V - D_\gamma^a\right\| \leq c\|V\|_{W_a}.$$

Proof. This follows from three facts which can be checked by considering the matrix form of D_γ^a given in equation 2.11.

1. Break $\mathscr{S}^a : \mathscr{B} \to \mathscr{L}$ into two components. Let $\mathcal{S}_1 = x' \operatorname{grad} f + (l \circ \nabla^E s)^\sharp$ be the horizontal component, which does not depend on a. Next, suppose that $W = \left(\overline{W}, W', 0\right)$ is a vector in $T_\gamma\mathscr{B}$ with components according to the splitting equation 2.10. We have

$$\|D\mathcal{S}_1(W) - D\mathcal{S}_1(\gamma)\| \leq c\|W\|_{W_a}.$$

2. Similarly, let $\mathcal{S}_2^a$ denote the vertical component of $\mathscr{S}^a$, which does depend on a. As before, let $W = \left(\overline{W}, W', 0\right)$. Let $V = \left(\overline{V}, V', U\right) \in T_\gamma\mathscr{B}$ be arbitrary. Then

$$D\mathcal{S}_2^a(W)(V) = a^2 U.$$

3. This time, let $W = (0, 0, Y)$ and suppose that $W_0 \in T_\gamma\mathscr{B}$ is W_a-small. Then

$$\|D_{W+W_0} - D_{W_0}\|_{L_a} \leq c\|W\|_{W_a}\|V\|_{W_a}.$$

$\square$

Furthermore, since $\gamma = (x(t), \zeta(x(t)))$ for a g-flow line in Z, it's clear that

$$\|\mathscr{S}^a(\gamma)\|_{L_a} = a^{-1}\left\|\frac{d}{dt}\zeta(x(t))\right\|_{L^2}.$$

Thus, taking f to be $\Phi \circ \mathscr{S}^a \circ \rho$ and $x_0 = x_1 = \gamma$ in Theorem 2.3.11, we conclude that there exists $\epsilon > 0$ such that for large enough a, there exists a unique $V_a \in T_\gamma\mathscr{B}$

satisfying

$$\|W_a(x)\|_{W_a} \leq \varepsilon, \qquad V_a(x) \in \operatorname{Im} Q^a_\gamma, \qquad \mathcal{S}^a\left(\exp_\gamma V_a(x)\right) = 0$$

Moreover, there exists $c > 0$ such that

$$\|V_a(x)\|_{W_a} \leq a^{-1} c$$

We conclude

Theorem 2.3.13. *There exists a number $A \in \mathbb{R}^+$ such that for all $a > 0$, there is a 1-1, surjective map*

$$\begin{aligned} \mathcal{F}^a : \widetilde{\mathcal{M}}_Z(x_\pm) &\leftrightarrow \widetilde{\mathcal{M}}^a(\lambda_\pm) \\ x(t) &\mapsto \exp_\gamma\left(V_a(x(t))\right) \end{aligned}$$

where $\pi(\lambda_\pm) = x_\pm$, $\pi : E^* \to M$.

Corollary 2.3.14. *For each $a \in A^{\text{reg}}$, there is an isomorphism*

$$H^a_\bullet \cong H_{\bullet - k}$$

where H^a is the Morse homology of Λ with respect to the metric g_a and H is the Morse homology of Z.

CHAPTER III

PERTURBED SPECIAL LAGRANGIAN SUBMANIFOLDS

This chapter is comprised mostly of previously publsihed work. Results that attributed to someone other than the author are cited.

In this chapter we shift from the finite-dimensional Morse–theoretic model of Chapter 2 to the geometric arena in which special Lagrangian submanifolds live. Rather than working directly with metrics or potentials, we encode the Calabi–Yau and G_2-structures in terms of *stable* differential forms and use algebraic data to reformulate the special Lagrangian condition as an infinite-dimensional Lagrange multiplier problem. By viewing a 6–manifold equipped with a pair of closed stable forms as the "hypersurface" in a seven–dimensional G_2–cylinder, we obtain a natural perturbation which gives rise to an elliptic deformation theory. We begin in Section 3.1 with the definition and local classification of stable forms in dimensions 6 and 7, and show how they underlie SU(3)– and G_2–structures.

3.1 Stable forms

Definition 3.1.1. Let V be an n-dimensional vector space. A form $\phi \in \Lambda^p V^*$ is called *stable* if its $\mathrm{GL}(V)$-orbit in $\Lambda^p V^*$ is open.

In [25, 26], Hitchin studied stable 3-forms on manifolds of dimension 6, 7 and 8. These considerations were motivated by the desire to understand the moduli space of complex structures on Calabi-Yau 3-folds but led to an entirely new approach for studying metrics with special holonomy and the discovery of a new type of geometry in dimension 8. The existence of stable forms on manifolds of dimension 6 and their connection to Calabi-Yau structures is in some sense the starting point for this entire project. Therefore in this section, we give a detailed overview of this concept. Most of the material in this section is not original and can be found in [25, 26]. Further sources include [8], [20], and [33].

The definition of a stable form generalizes the notion of a *non-degenerate* 2-form because in dimension $2n$, a non-degenerate 2-form is always stable with stabilizer $\mathrm{Sp}(2n,\mathbb{R})$. On the other hand, a 2-form in dimension $2n+1$ might be stable, but is always non-degenerate. Note that stable forms in any given degree don't usually exist since $\dim \mathrm{GL}(V) = n^2$ is usually much smaller than $\dim \Lambda^p V^* = n!/p!(n-p)!$. In this paper, we will only be concerned with stable forms in dimensions 6 and 7.

3.1.1 Dimension 6. Let V be an oriented, 6-dimensional vector space. Stable forms are most abundant in dimension 6. Of course, stable 2-forms are possible in dimension 6 but stable 3-forms and 4-forms are also possible. Note that

$$\begin{aligned} \dim \mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R}) = \dim \mathrm{SL}(3,\mathbb{C}) &= 16 \\ \dim \Lambda^3 V^* &= 20 \\ \dim \mathrm{GL}(V) &= 36 \end{aligned}$$

and $16 + 20 = 36$. Furthermore one can check that, given a basis $e^1, \ldots, e^6$ for V^*, the $\mathrm{GL}(V)$-stabilizer of

$$\rho_- = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6$$

is $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$. On the other hand, the stabilizer of

$$\rho_+ = e^1 \wedge e^3 \wedge e^5 - e^1 \wedge e^4 \wedge e^6 - e^2 \wedge e^3 \wedge e^6 - e^2 \wedge e^4 \wedge e^5 \tag{3.1}$$

is $\mathrm{SL}(3,\mathbb{C})$. In fact if $\rho \in \Lambda^3 V^*$ has stabilizer $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$, then there exists a basis for V^* in which ρ has the form of ρ_-. Similarly if the stabilizer of ρ is $\mathrm{SL}(3,\mathbb{C})$, then there exists a basis for V^* in which ρ has the form of ρ_+.

Definition 3.1.2. Let $\rho \in \Lambda^3 V^*$ be stable. If the $\mathrm{GL}(V)$-stabilizer of ρ is $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$ then ρ is called *negative*. If the stabilizer of ρ is $\mathrm{SL}(3,\mathbb{C})$ then ρ is called *positive*.

In this paper, we will only be interested in *positive* 3-forms. Thus from now on, if a 3-form is referred to as stable, the reader may assume that it is in fact positive.

3.1.2 Dimension 7.

Let V be an oriented, 7-dimensional vector space and fix an identification $V \cong \mathrm{Im}\mathbb{O}$ with the imaginary octonions. Then the multiplication on $\mathbb{O}$ endows V with

1. an inner product g

$$g(u,v) := -\mathrm{Re}(uv)$$

2. a cross-product

$$u \times v := \mathrm{Im}(uv)$$

3. a 3-form

$$\varphi(u,v,w) := g(u \times v, w)$$

4. an *associator* $[\cdot,\cdot,\cdot] : \Lambda^3\mathbb{R}^7 \to \mathbb{R}^7$

$$[u,v,w] := (u \times v) \times w + \langle v,w \rangle u - \langle u,w \rangle v \tag{3.2}$$

5. and a 4-form

$$\psi(u,v,w,z) := g\left([u,v,w],z\right).$$

One can always choose coordinates $e^1,\ldots,e^7$ on V so that the 3-form φ above can be written as

$$\varphi_0 = e^1 \wedge e^3 \wedge e^5 - e^1 \wedge e^4 \wedge e^6 - e^2 \wedge e^3 \wedge e^6 - e^2 \wedge e^4 \wedge e^5 \\ - e^1 \wedge e^2 \wedge e^7 - e^5 \wedge e^6 \wedge e^7 - e^3 \wedge e^4 \wedge e^7. \quad (3.3)$$

To shorten notation, we often write e^{ijk} to mean $e^i \wedge e^j \wedge e^k$.

Definition 3.1.3. The subgroup of $\mathrm{GL}(7, \mathbb{R})$ which fixes φ_0 is the compact, connected, simple Lie group G_2

$$G_2 = \{A \in \mathrm{GL}(7, \mathbb{R}) \mid A^*(\varphi_0) = \varphi_0\}.$$

The group G_2 also preserves the standard inner product and orientation when acting on V. In particular, G_2 also fixes the 4-form $\psi_0 = *\varphi_0$, where the Hodge star is taken with respect to the metric determined by φ_0. These forms are stable with respect to Definition 3.1.1 since the Lie group G_2 is 14-dimensional and $\dim\mathrm{GL}(7, \mathbb{R}) - \dim\Lambda^3(\mathbb{R}^7) = 49 - 35 = 14$, implying that their $\mathrm{GL}(7, \mathbb{R})$-orbit is open. There are also 3-forms in 7-dimensions whose stabilizer is the Lie group *split* G_2. These are analogous to the negative stable 3-forms in dimension 6, and will not be considered in this paper. Throughout, a stable 3-form on a 7-dimensional manifold will always be one whose pointwise stabilizer is the compact real Lie group G_2.

It should be noted that the stabilizer of the 4-form $\psi = *\varphi$ is actually $\pm G_2 = G_2 \cup -\mathrm{Id}G_2$. Therefore a stable 4-form on a 7-dimensional vector space determines an inner-product but *not* an orientation. This is not an issue if we assume that V is oriented from the beginning.

3.1.3 Volume forms and Hitchin duals.

Each stable form preserves a volume form which is chosen in a standard way. We will now review the standard volume forms for each of the relevant cases. If α is any stable form, the associated volume form will be denoted by $\mathrm{vol}(\alpha)$. Also any vector spaces mentioned will be assumed to be oriented and the volume forms will be constructed so that they are positive with respect to this orientation.

– Let V be 6-dimensional and $\omega \in \Lambda^2 V^*$ be stable (non-degenerate). Then

$$\mathrm{vol}(\omega) = \frac{1}{6}\omega^3.$$

This volume form is sometimes called the *Liouville volume form* in symplectic geometry literature. Since any element in $\mathrm{Sp}(V)$ preserves ω, $\mathrm{vol}(\omega)$ is also preserved.

– Let V be again 6-dimensional and let $\rho \in \Lambda^3 V^*$ be a *positive* 3-form. That is, the suppose the stabilizer of ρ is $\mathrm{SL}(3, \mathbb{C})$. Define an operation $K_\rho : V \to \Lambda^5 V^*$ by

$$K_\rho(v) = (v \lrcorner \rho) \wedge \rho.$$

We also have the canonical isomorphism

$$A : \Lambda^5 V^* \cong V \otimes \Lambda^6 V^*.$$

Thus, given any element $w \in \Lambda^6 V$, we get a corresponding isomorphism

$$\begin{array}{ccc} \Lambda^5 V^* & \to & V \\ \gamma & \mapsto & \gamma \lrcorner w \end{array}$$

which, when composed with K_ρ, determines an endomorphism

$$\mathrm{End}_w : V \xrightarrow{K_\rho} \Lambda^5 V^* \xrightarrow{\cdot \lrcorner w} V.$$

This construction thus determines a linear map $\Lambda^6 V \to \mathbb{R}$ by sending w to $\mathrm{tr}\,(\mathrm{End}_w)$. We also obtain a bilinear form given by sending $w_1, w_2 \in \Lambda^6 V$ to $\mathrm{tr}\,(\mathrm{End}_{w_1} \circ \mathrm{End}_{w_2})$. This bilinear form is denoted by $\mathrm{tr}(K^2)$ and it is an element of $\Lambda^6 V^* \otimes \Lambda^6 V^*$. One can check ([25], Proposition 2) that, despite the name, positive 3-forms are characterized by the condition $\mathrm{tr}(K^2) < 0$ with respect to the orientation on V. We define the volume form associated to the positive 3-form ρ by

$$\mathrm{vol}(\rho) = \sqrt{-\mathrm{tr}(K_\rho^2)}.$$

As before, this volume form is invariant under the action of $\mathrm{SL}(3, \mathbb{C})$.

On an oriented, 6-dimensional vector space V, the map $A \circ K_\rho : V \to V \otimes \Lambda^6 V^*$ defines a complex structure $I_\rho : V \to V$ which can be written explicitly as

$$I_\rho = \frac{1}{\sqrt{-\frac{1}{6}\mathrm{tr}\ K_\rho^2}} K_\rho. \tag{3.4}$$

– This time, let V be a 7-dimensional vector space and let $\varphi \in \Lambda^3 V^*$ be a 3-form with stabilizer G_2. Given $v, w \in V$ we define a symmetric, bilinear form on V with values in $\Lambda^7 V^*$ by sending $u, v \in V$ to

$$(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi \in \Lambda^7 V^*.$$

This in turn gives rise to a linear map $K_\varphi : V \to V^* \otimes \Lambda^7 V^*$. Recall that, given two n-dimensional vector spaces V, W, the determinant of a linear map from V to W is an element of $\Lambda^n V^* \otimes \Lambda^n W$. Thus

$$\det\ K_\varphi \in \Lambda^7 V^* \otimes \Lambda^7 \left(V^* \otimes \Lambda^7 V^* \right).$$

Also recall that if W is 1-dimensional, then $\Lambda^k (V \otimes W) \cong \Lambda^k V \otimes W^k$. So

$$\det\ K_\varphi \in \left(\Lambda^7 V^* \right)^9.$$

Define

$$\mathrm{vol}(\varphi) = |\det\ K_\varphi|^{1/9}.$$

As stated before, the group G_2 preserves the standard inner-product while acting on V. This inner-product can be written explicitly in terms of φ as

$$g_\varphi(u,v) = \frac{1}{6\mathrm{vol}(\varphi)}\,(u\lrcorner\varphi)\wedge(v\lrcorner\varphi)\wedge\varphi$$

and agrees with the inner-product coming from octonionic multiplication from above. Furthermore, using this metric, we may define a two-fold cross product on V by taking the metric dual of $\varphi(u,v,\cdot)$. That is

$$u\times_\varphi v = (\varphi(u,v,\cdot))^\flat.$$

– Let V be again 7-dimensional and let $\psi \in \Lambda^4V^*$ have stabilizer G_2. Then, using the isomorphism $\Lambda^4V^* \cong \Lambda^3V\otimes\Lambda^7V^*$, identify ψ with $w\otimes\sigma \in \Lambda^3V\otimes\Lambda^7V^*$. We form a bilinear map similar to the previous case. Let $\alpha,\beta\in V^*$ and obtain an element in $(\Lambda^7V^*)^2$ as follows

$$((\alpha\lrcorner w)\wedge(\beta\lrcorner w)\wedge w)\otimes(\beta)^3.$$

Thus we have a linear map

$$K_\psi : V^* \to V\otimes\left(\Lambda^7V^*\right)^2$$

and

$$\det\, K_\psi \in \left(\Lambda^7V^*\right)^{12}.$$

We define

$$\mathrm{vol}(\psi) = |\det K_\psi|^{1/12}$$

This 4-form ψ agrees with he 4-form obtained from the octonionic associator above.

If an oriented, n-dimensional vector space V admits a stable p-form, then, once a convention for the volume form is fixed, it also admits a stable $(n-p)$-form with the same stabilizer. This $(n-k)$-form is called a *Hitchin dual*. In the next paragraphs we describe in detail how to obtain these. Let V be an oriented, n-dimensional vector space and let $\sigma \in \Lambda^p V^*$ be any stable p-form. Let U be its open $\mathrm{GL}(V)$-orbit. Then the associated volume form determines a $\mathrm{GL}(V)$-invariant map

$$\mathrm{vol} : U \to \Lambda^n V^*.$$

If A is a diagonal $n \times n$ matrix whose entries are all the same constant a, then $A^*\mathrm{vol}(\sigma) = a^n \mathrm{vol}(\sigma)$. On the other hand, $\mathrm{vol}(A^*\sigma) = \mathrm{vol}(a^p\sigma)$. Thus, the map vol is *homogeneous* of degree $\frac{n}{p}$.

The derivative of vol at σ is a map

$$D_\sigma \mathrm{vol} : \Lambda^p V^* \to \Lambda^n V^*$$

or in other words, $D_\sigma \mathrm{vol} \in (\Lambda^p V^*)^* \otimes \Lambda^n V^*$. Next, recalling the isomorphism $(\Lambda^p V^*)^* \cong \Lambda^{n-p} V^* \otimes \Lambda^n V$, regard $D_\sigma\mathrm{vol}$ as an element of $\Lambda^{n-p}V^*$. Then we see that there is a *unique* $\hat{\sigma} \in \Lambda^{n-p}V^*$ for which

$$D_\sigma \mathrm{vol}(\dot{\sigma}) = \hat{\sigma} \wedge \dot{\sigma}.$$

By Euler's formula for homogeneous functions, and taking $\dot{\sigma} = \sigma$, we have

$$\frac{n}{p}\mathrm{vol}(\sigma) = \hat{\sigma} \wedge \sigma.$$

Definition 3.1.4. In the construction described above, take $\dot{\sigma} = \sigma$. Define the *Hitchin dual* of σ to be $\hat{\sigma}$.

We now write the Hitchin duals of all the stable forms relevant to this paper.

– For a stable (non-degenerate) 2-form ω on a 6-dimensional vector space, $\hat{\omega} = \frac{1}{2}\omega^2$.

– For a positive 3-form ρ on a 6-dimensional vector space, $\hat{\rho}$ is the unique 3-form such that $\rho + i\hat{\rho}$ is a nowhere vanishing complex volume form, with respect to the complex structure I_ρ.

– For a stable 4-form τ on a 6-dimensional vector space, $\hat{\tau}$ is the unique stable 2-form satisfying $\frac{1}{2}\hat{\tau}^2 = \tau$.

– For a positive 3-form φ on a 7-dimensional vector space $\hat{\varphi} = *\varphi$ with respect to the metric associated to φ.

– Similarly, for a stable 4-form ψ, $\hat{\psi} = *\psi$.

3.1.4 SU(3)-structures. Suppose that M is a 6-manifold equipped with a positive 3-form ρ and a stable 2-form ω. Let F be the principal bundle over M with fiber $\mathrm{GL}(6,\mathbb{R})$. Since the pointwise stabilizer of ρ is $\mathrm{SL}(3,\mathbb{C})$, the form ρ determines an $\mathrm{SL}(3,\mathbb{C})$-subbundle of F whose fiber at each point is the subset of $\mathrm{GL}(6,\mathbb{R})$ preserving ρ. This subbundle is called an $\mathrm{SL}(3,\mathbb{C})$-*structure.* The stable form ω likewise determines an $\mathrm{Sp}(6,\mathbb{R})$-*structure* on M. Next, recall that

$$\mathrm{SL}(3,\mathbb{C}) \cap \mathrm{Sp}(6,\mathbb{R}) \cong \mathrm{SU}(3).$$

Thus, we might be tempted to conclude that M is automatically equipped with an SU(3)-structure as well. However, the above isomorphism depends on the way in which $\mathrm{SL}(3,\mathbb{C})$ and $\mathrm{Sp}(6,\mathbb{R})$ are included in $\mathrm{GL}(6,\mathbb{R})$. Therefore the pointwise stabilizer of the pair (ρ,ω) is SU(3) only under certain algebraic conditions. In the case where these conditions are satisfied, in a slight abuse of notation, we refer to the pair (ρ,ω) as an SU(3)-*structure.*

Definition 3.1.5. Let M be an oriented, 6-dimensional manifold. A pair (ρ, ω) consisting of a positive 3-form ρ and a stable 2-form ω is called an SU(3)-*structure* on M if

1. $\omega \wedge \rho = 0$
2. and $\frac{1}{6}\omega^3 = \frac{1}{4}\rho \wedge \hat{\rho}$.

Using the concept of a Hitchin dual, we can obtain the same reduction of the structure group from a pair (ρ, τ) where τ is a stable 4-form and letting $\omega = \hat{\tau}$ in the conditions above.

Since $\mathrm{SU}(3) \subset \mathrm{SO}(6)$, an SU(3)-structure determines a metric on the manifold M. Two choices of SU(3)-structures (ρ, ω) and (ρ', ω') can determine the same metric. In fact, the set of pairs (ρ, ω) which determine the same metric on M is isomorphic to $\mathrm{SO}(6)/\mathrm{SU}(3) \cong \mathbb{RP}^7$.

The first condition has two interpretations. The pointwise complex structure defined by equation 3.4 determines an almost complex structure on M. From the perspective of this almost complex structure, condition (1) implies that ω is a $(1,1)$-form. On the other hand, if ω is closed so that it is symplectic, then condition (1) implies that ρ is *primitive*. Condition (2) ensures that $\mathrm{vol}(\omega)$ and $\mathrm{vol}(\rho)$ agree with the metric volume form.

In this paper, the symbol ω will *always* denote a stable 2-form, ρ will *always* denote a stable 3-form, and τ will *always* denote a stable 4-form unless otherwise stated. Various conditions may be placed on the stable forms, most of which have been studied in some detail. Suppose that (M, ρ, τ) is a manifold with an SU(3)-structure.

1. If both ρ and $\hat{\rho}$ are closed then M is a *complex threefold with a trivial canonical bundle*. Manifolds such as these are not necessarily Calabi-Yau manifolds because they are not necessarily Kähler. These manifolds were studied in [43] where they are referred to as non-Kähler Calabi-Yau manifolds. There are many simple examples such as $S^1 \times S^3$.

2. If both ρ and τ are closed, then M is said to have a *half-flat* SU(3)-structure. Half-flat structures are important to the study of hypersurfaces in $\mathbb{R}^7$ and were first studied by Calabi. They have since showed up again in physics and geometry. A slightly more restricted sub-class of six manifolds with half-flat SU(3)-structures are the *nearly Kähler* manifolds whose Riemannian cones have holonomy equal to G_2. See [20] for more details about nearly Kähler manifolds and [34] for an explanation of half-flat SU(3)-structures.

3. If $\hat{\tau}$ is closed, then M is a *symplectic manifold with a compatible almost-complex structure*. Note that $\hat{\tau}$ being closed also implies that τ is closed since $\tau = \frac{1}{2}\hat{\tau}^2$.

4. If $\tau, \hat{\tau}, \rho$, and $\hat{\rho}$ are all closed, then M is a *Calabi-Yau* manifold.

Note that we could also drop the requirement that (ρ, τ) forms an SU(3) structure and simply study manifolds that carry a pair of stable forms. In this paper, we will usually consider pairs of stable forms that determine a co-closed G_2 structure in $6 + 1$ dimensions. More details can be found in the following sections.

Remark 3.1.6. Recall that the existence of an almost complex structure implies the existence of a stable 2-form, since one can be constructed out of an almost complex structure and any Riemannian metric. This is true in any dimension. Therefore in dimension 6, the existence of a stable 3-form implies the existence of a stable 2-form since a stable 3-form defines an almost complex structure.

3.1.5 G_2-structures.

Suppose that X is a 7-dimensional manifold equipped with a positive 3-form. Similarly to the 6-dimensional case, φ determines a G_2-structure on X. The inner-product g_φ, twofold cross-product $\times_\varphi$, associator $[\cdot,\cdot,\cdot]_\varphi$, and 4-form $\psi = *_\varphi\varphi$ from the linear algebra setting are all carried over to the manifold setting. As in the SU(3) setting, the set of positive 3-forms corresponding to a given metric g_φ is isomorphic to $SO(7)/G_2 \cong \mathbb{RP}^7$. We use the following terminology in this context.

Definition 3.1.7. Suppose that X is a 7-manifold equipped with a positive 3-form. Then X is called a *G_2-manifold* or a manifold with a *G_2-structure*. If φ is closed, then φ is called a *closed G_2-structure*. If $\psi = *_\varphi\varphi$ is closed, then φ is called a *co-closed G_2-structure*. If φ *and* ψ are both closed, then φ is called a *torsion-free* G_2-structure. The same definitions may be applied to a 7-manifold X with a stable 4-form ψ. In this case however, X must also be equipped with a fixed orientation.

Lemma 3.1.8 (See Lemma 11.5 of [39])**.** *Let X be a 7-manifold equipped with a G_2-structure φ. The metric g_φ has holonomy contained in G_2 if and only if $d\varphi = 0$ and $d *_\varphi \varphi = 0$.*

If g_φ is a metric with holonomy contained in G_2 corresponding to the positive 3-form φ, then $\nabla\varphi = 0$ where ∇ is the Levi-Civita connection. The form $\nabla\varphi$ is called the *torsion* of the G_2-structure. For more details about G_2-structures and manifolds with G_2 holonomy see for example [30] or [6].

3.2 Translation-Invariant G_2-structures

In this section, we study *translation-invariant* G_2-structures on 7-manifolds of the form $\mathbb{R} \times M$, where M is a 6-manifold equipped with a pair of stable forms (ρ, ω) or (ρ, τ).

Definition 3.2.1. Let M be a 6-manifold. Let $\mathbb{R}$ have coordinate t. A 3-form γ on $\mathbb{R} \times M$ is called *translation-invariant* if it has the form

$$\varphi = dt \wedge \alpha + \beta$$

where $\alpha \in \Omega^2(M)$ and $\beta \in \Omega^3(M)$.

The Lie group $\mathrm{SU}(3)$ is a subgroup of the Lie group G_2. Indeed, if one chooses a nonzero vector v_0 in a 7-dimensional vector space V, then the subgroup of G_2 that preserves v_0 is isomorphic to $\mathrm{SU}(3)$. Therefore there are interesting relationships between manifolds with G_2-structures and manifolds with $\mathrm{SU}(3)$-structures. The following lemma provides the first such relationship.

Lemma 3.2.2 (Proposition 11.1.2 in [30])**.** *Suppose that (M, ρ, ω) is an oriented 6-manifold equipped with an $\mathrm{SU}(3)$-structure. Let $\mathbb{R}$ have coordinate t. Define a 3-form on $\mathbb{R} \times Y$ by*

$$\varphi = dt \wedge \omega + \rho.$$

Then φ is stable on $\mathbb{R} \times M$. The corresponding metric g_φ is the product metric $dt^2 + g_{(\rho,\omega)}$ on $\mathbb{R} \times M$. Furthermore, in terms of g_φ, the Hitchin dual of φ is

$$\psi = *_\varphi \varphi = \hat{\omega} - dt \wedge \hat{\rho}.$$

If (ρ, ω) is a Calabi-Yau structure (so that $d\rho = d\omega = d\hat{\rho} = d\hat{\omega} = 0$), then φ is a torsion-free G_2-structure with $\mathrm{Hol}(g_\varphi) \subseteq \mathrm{SU}(3)$.

Note that a sort of converse to this lemma is also clearly true. That is,

Lemma 3.2.3. *Let $X = \mathbb{R} \times M$ and suppose that φ is a stable 3-form on X whose corresponding metric g_φ is a product metric $dt^2 + g_M$. Let ∂_t be the g_φ-dual of dt. Then the pair*

$$\omega = \partial_t \lrcorner \varphi \qquad\qquad \rho = \varphi\big|_M$$

is an SU(3)*-structure on M. Furthermore, if φ is closed and co-closed, then (ρ, ω) determines a Calabi-Yau structure on M.*

The stable 3-forms in the above lemmas represent the most well-behaved translation-invariant 3-forms built from pairs (ρ, ω) on M. The following simple examples illustrate the existence of less well-behaved cases.

Example 3.2.4. Let $M = \mathbb{R}^6$ with coordinates $x_1, \ldots, x_6$. Let

$$\begin{aligned} \rho &= dx_{135} + dx_{632} + dx_{254} + dx_{416} \\ \omega &= dx_{63} + dx_{25} + dx_{41}. \end{aligned}$$

Here, we use the shorthand $dx_{i_1} \wedge \cdots \wedge dx_{i_k} = dx_{i_1 \ldots i_k}$. One easily checks that both ρ and ω are stable by verifying that ρ is ρ_+ (from equation 3.1) and that ω is non-degenerate. However one can also check that

$$\omega \wedge \rho = dx_{63254} + dx_{25416} + dx_{41632} \neq 0$$

so (ρ, ω) is *not* an SU(3)-structure since it fails condition (1) from Definition 3.1.5. Next, let t be an extra $\mathbb{R}$ coordinate. Let g_0 denote the standard metric on $\mathbb{R} \times \mathbb{R}^6$ and let ∂_t be the g_0-dual of dt. Define

$$\varphi = \rho + dt \wedge \omega.$$

Then

$$\partial_t \lrcorner \varphi = \omega$$

so

$$(\partial_t \lrcorner \varphi) \wedge (\partial_t \lrcorner \varphi) \wedge \varphi = \omega \wedge \omega \wedge \varphi = 0.$$

This implies that g_φ is *not* nondegenerate. Thus φ is not stable.

This result was expected. The next example shows that there are pairs of stable which, while not SU(3)-structures, still behave well with respect to the G_2 geometry on $\mathbb{R} \times M$.

Example 3.2.5. As before, let $M = \mathbb{R}^6$ with coordinates $x_1, \ldots, x_6$. Let

$$\rho = dx_{135} + dx_{632} + dx_{254} + dx_{416} + Kdx_{123}$$

where K is a small constant. Note that this is just ρ_+ plus Kdx_{123}. Since the orbit of stable forms is open, ρ is also stable for small enough K. Let ω be the standard symplectic form on $\mathbb{R}^6$. That is

$$\omega = dx_{12} + dx_{34} + dx_{56}.$$

As before, let x_0 be an extra $\mathbb{R}$-coordinate. On $\mathbb{R} \times \mathbb{R}^6$, we have

$$\begin{aligned} \varphi &= \rho + dx_0 \wedge \omega \\ &= (\rho_+ + Kdx_{123}) \wedge \omega \\ &= Kdx_{12356} \neq 0 \end{aligned}$$

so (ρ, ω) is *not* an SU(3)-structure. On the other hand, φ is still stable, since

$$\varphi = \varphi_0 + Kdx_{123}$$

where φ_0 is as in equation 3.3. In other words, φ is a small perturbation of a stable form, and is therefore still stable and defines a G_2-structure on $\mathbb{R}^7$. Note that this does not contradict Lemma 3.2.3 because the metric g_φ is *not* a product metric.

Indeed,

$$(\partial_{x_3} \lrcorner \varphi) \wedge (\partial_{x_0} \lrcorner \varphi) \wedge \varphi \neq 0$$

but, for any metric of the form $g = dx_0^2 + g_M$ where g_M is a metric on M, $g(\partial_{x_3}, \partial_{x_0}) = dx_0^2(0, \partial_{x_0}) + g_M(\partial_{x_3}, 0) = 0$.

With these considerations in mind, we give the following definitions.

Definition 3.2.6. A pair of stable forms on a 6-manifold M that define a G_2-structure on $\mathbb{R} \times M$ is called a *G_2-pair.* These pairs can consist of a stable 3-form and a stable 2-form *or* a stable 3-form and a stable 4-form. Set

$$\mathscr{R}_{G_2} = \left\{ (\rho, \omega) \in \Omega^3_{\text{stable}}(M) \times \Omega^2_{\text{stable}}(M) \mid \varphi = \rho + dt \wedge \omega \in \Omega^3_{\text{stable}}(X) \right\}$$

or

$$\mathscr{R}_{G_2} = \left\{ (\rho, \tau) \in \Omega^3_{\text{stable}}(M) \times \Omega^4_{\text{stable}}(M) \mid \psi = dt \wedge \rho + \tau \in \Omega^4_{\text{stable}}(X) \right\}.$$

The pair in Example 3.2.5 falls into this category.

Definition 3.2.7. Similarly, we will let $\mathscr{R}_{\mathrm{SU}(3)}$ denote the set of SU(3)-structures on M. That is

$$\mathscr{R}_{\mathrm{SU}(3)} = \left\{ (\rho, \omega) \in \Omega^3_{\text{stable}}(M) \times \Omega^2_{\text{stable}}(M) \mid (\rho, \omega) \text{ is an SU(3)-structure} \right\}$$

or

$$\mathscr{R}_{\mathrm{SU}(3)} = \left\{ (\rho, \tau) \in \Omega^3_{\text{stable}}(M) \times \Omega^4_{\text{stable}}(M) \mid (\rho, \tau) \text{ is an SU(3)-structure} \right\}.$$

Note that $\mathscr{R}_{\mathrm{SU}(3)} \subset \mathscr{R}_{G_2}$. However, as shown by Example 3.2.5, they are not the same.

Remark 3.2.8. If $(\rho, \omega) \in \mathscr{R}_{G_2}$ or $\mathscr{R}_{\mathrm{SU}(3)}$ then so is $(\hat{\rho}, \hat{\omega})$. Furthermore, as long as $\mathbb{R} \times M$ is oriented, the set of matrices in $\mathrm{GL}(7, \mathbb{R})$ preserving (ρ, ω) also preserves $(\hat{\rho}, \hat{\omega})$ by the definition of a Hitchin dual. Thus, the set of SU(3)-structures and the set of G_2-structures is the same whether or not we choose a stable 2-form ω or a stable 4-form τ.

Lemma 3.2.9. *Suppose that $(\rho, \omega) \in \mathscr{R}_{G_2}$, but not necessarily contained in $\mathbb{R}_{\mathrm{SU}(3)}$. Then there exists a stable 2-form ω' such that the pair $(\rho, \omega') \in \mathscr{R}_{\mathrm{SU}(3)}$.*

Proof. This follows from the fact that $M = \{0\} \times M$ is a hypersurface in $\mathbb{R} \times M$. Let $\varphi = \rho + dt \wedge \omega$ as before and let $\times_\varphi$ denote the cross-product associated to φ. Then, given any unit normal vector field n along M, we can define an almost complex structure

$$J_n v = n \times_\varphi x \qquad \text{for all } v \in TM.$$

Let ω' be defined by

$$\omega'(u, v) = g_\varphi(u, -J_n v) \qquad \text{for all } u, v \in TM.$$

The pair (ρ, ω') defines an SU(3)-structure. The SU(3)-structures of this type were first studied in [10] and more recently by in [9]. We emphasize that, in Example 3.2.5, the vector field ∂_{x_0} is *not* normal to M with respect to the metric g_φ. In general, ω' is not the same as ω. □

Lemma 3.2.10. *The space $\mathscr{R}_{\mathrm{SU}(3)}$ is a deformation-retract of $\mathscr{R}_{G_2}$.*

Proof. Clearly $\mathscr{R}_{\mathrm{SU}(3)} \subseteq \mathscr{R}_{G_2}$. Let $(\rho, \omega) \in \mathscr{R}_{G_2}$. Let t denote the $\mathbb{R}$-coordinate on $\mathbb{R} \times M$ and set $\varphi = \rho + dt \wedge \omega$ as usual. Choose a unit vector field n along $\{0\} \times M$ that is normal to $\{0\} \times M$ with respect to the metric g_φ. Note that even if g_φ is not a product metric on $\mathbb{R} \times M$, the vector field ∂_t is nonvanishing, and therefore homotopic to n. Let $\{n_s\}_{s=0}^{1}$ be a smooth family of nowhere-vanishing vector fields satisfying $n_0 = \partial_t$ and $n_1 = n$. If φ *does* define the product metric, simply set $n = \partial_t = n_s$ for all $s \in [0,1]$. Then for each s define

$$\rho = \iota^*\varphi \qquad J_s(\cdot) = n_s \times_\varphi \cdot \qquad \omega_s(\cdot,\cdot) = g_\varphi\left(\cdot, -J_s(\cdot)\right).$$

Since there exists a normal vector field for each pair $(\rho,\omega) \in \mathcal{R}_{G_2}$, this construction defines a map

$$F : \mathcal{R}_{G_2} \times I \to \mathcal{R}_{G_2}$$

by

$$F\left((\rho,\omega),s\right) = (\rho,\omega_s).$$

Clearly $F\left((\rho,\omega),0\right) = (\rho,\omega)$. On the other hand, $F\left((\rho,\omega),1\right) \in \mathcal{R}_{\mathrm{SU}(3)}$ (see also proposition 4.1 of [9]). Clearly if $(\rho,\omega) \in \mathcal{R}_{\mathrm{SU}(3)}$, then $F\left((\rho,\tau),1\right)$ is the identity. Thus, F is a deformation retract as desired. $\square$

3.3 Tamed structures

In this section, we define what it means for one G_2-pair to be *tamed* by another. This is inspired by the definition of an ω-*tame* almost complex structure on a symplectic manifold (M,ω) which was first introduced by Gromov [22] in order to control the energy of J-holomorphic curves. The structure of this section is as follows. First, we review the definition of a *calibrated* submanifold. Calibrated geometry was introduced in [23] and provides a framework for unifying several concepts in the study of manifolds with special holonomy. Next, we review the symplectic case. Then, we define tamed structures in the SU(3) and G_2 cases in 6- and 7-dimensions respectively. Finally, we discuss how tamed G_2-pairs related to tamed almost complex structures.

3.3.1 Calibrated geometry.

Definition 3.3.1. Let (M, g) be a Riemannian manifold of dimension n. A k-form α is called a *calibration* if

1. $d\alpha = 0$
2. and for any $x \in M$, any oriented k-plane $V \subset T_x M$ satisfies $\alpha\big|_V = \lambda \mathrm{vol}_V$ with $\lambda \leq 1$. Here vol_V is the volume form on V with respect to the metric g.

A k-submanifold N of M is said to be *calibrated by* α if each of its tangent planes $T_y N$ is calibrated.

The primary advantage of this definition is that one can easily see that calibrated submanifolds are volume-minimizing in their homology class. Indeed, suppose that N is calibrated by a calibration α. Let N' be any submanifold in the same homology class. Then

$$\int_N \mathrm{vol}_N = \int_N \alpha = \int_{N'} \alpha \leq \int_{N'} \mathrm{vol}_{N'}.$$

The first equality holds by the definition of calibrated. The second equality is Stokes' theorem. The last inequality holds by the definition of a calibration. As we will see, stable forms which are closed are calibrations whose calibrated submanifolds are extremely interesting.

3.3.2 Tamed almost complex structures.

Definition 3.3.2. Let (M, ω) be an even-dimensional manifold equipped with a stable 2-form ω. An almost complex structure $J : TM \to TM$ is called ω*-tame* if

$$\omega(v, Jv) > 0$$

for all nonzero $v \in TM$. It is called ω*-compatible* if it is ω-tame and

$$\omega(Ju, Jv) = \omega(u, v)$$

for all $u, v \in TM$.

If the 2-form ω is closed, then M is called *symplectic*. A symplectic manifold with a *compatible* almost complex structure is sometimes called *almost-Kähler*. A symplectic manifold with an integrable, compatible complex structure is Kähler and has holonomy contained in $\mathrm{U}(n)$. The definition of a tame almost complex structure is essential for understanding *J-holomorphic curves*. Many results that hold for Kähler manifolds also hold for manifolds with tame almost complex structures.

Every ω-tame almost complex structure determines a Riemannian metric via the formula

$$g_J(u,v) = \frac{1}{2}\left(\omega\left(u, Jv\right) + \omega\left(v, Ju\right)\right).$$

The central objects of interest in symplectic geometry are *J-holomorphic curves*.

Definition 3.3.3. Let (M, J) be an almost complex manifold and (Σ, j) be a Riemann surface. A smooth map $u : \Sigma \to M$ is called a *J-holomorphic curve* if the differential du satisfies

$$J \circ du = du \circ j.$$

In many ways, the submanifolds studied in this paper are analogues of J-holomorphic curves. In other ways, they differ substantially. The primary difficulty when one attempts to define an invariant using J-holomorphic curves has to do with proving that the relevant moduli space is compact. In order to attack compactness problems, one must have a notion of *energy*.

Definition 3.3.4. Let (M, ω) be an even-dimensional manifold with a stable 2-form. Let J be an ω-tame almost complex structure on M. Let $(\Sigma, j, \mathrm{vol}_\Sigma)$ be a compact Riemann surface. The *energy* of a smooth map $u : \Sigma \to M$ is defined by

$$E(u) = \frac{1}{2} \int_\Sigma |du|_J^2 \mathrm{vol}_\Sigma$$

where the norm $|du|_J$ is taken with respect to the metric g_J. More precisely,

$$|du|_J = |v|^{-1} \sqrt{|du(v)|_J^2 + |du(j(v))|_J^2}$$

for all nonzero $v \in T_s\Sigma$.

For more details about J-holomorphic curves and these definitions, see chapters 2-4 of [35].

Lemma 3.3.5 (Lemma 2.2.1 of [35])**.** *Let ω be a stable 2-form on a smooth even-dimensional manifold M. If J is an ω-tame almost complex structure, then every J-holomorphic curve $u : \Sigma \to M$ satisfies the energy identity*

$$E(u) = \int_\Sigma u^*\omega.$$

If J is ω-compatible, then every smooth map $u : \Sigma \to M$ satisfies

$$E(u) = \int_\Sigma \left|\overline{\partial}_J(u)\right|_J^2 \mathrm{vol}_\Sigma + \int_\Sigma u^*\omega$$

Therefore if ω is closed, J is ω-compatible, and Σ is closed, then J-holomorphic curves are energy-minimizing in their homology class. This is not necessarily true if J is merely ω-tame. In either case, however, the energy of a J-holomorphic curve is a topological invariant. Note also that in this case, J-holomorphic curves are *calibrated* by ω in the sense of Definition 3.3.1.

3.3.3 Tamed G_2-structures and tamed G_2-pairs.

A similar notion of *taming* and *tamed* structures was introduced in [15] for manifolds with stable forms in dimensions 6, 7, and 8. This definition is most easily stated for manifolds with G_2-structures. We will adapt this definition to 6-manifolds equipped with a

G_2-pair. In the 7-dimensional setting, the role of J-holomorphic curves is played by *associative submanifolds* which we now define.

Definition 3.3.6. Suppose that (X, ψ) is an oriented 7-manifold equipped with a stable 4-form ψ. Let $[\cdot,\cdot,\cdot]_\psi : \Lambda^3 TX \to TX$ denote the associator (equation 3.2) corresponding to ψ. Let $\varphi = *\psi$. Let P be a 3-dimensional manifold and $\iota : P \to X$ an embedding. We call ι *associative* if $\iota^*[\cdot,\cdot,\cdot]_\psi \equiv 0$ and $\iota^*\varphi > 0$. Then an *associative submanifold* of X is an equivalence class $[\iota] \in \mathrm{Emb}(P, X)/\mathrm{Diff}^+(P)$ where ι is an associative embedding of some 3-manifold P. Here, $\mathrm{Emb}(P, X)$ is the space of embeddings of P into X and $\mathrm{Diff}^+(P)$ is the group of orientation-preserving diffeomorphisms of P.

If $V \cong \mathrm{Im}\mathbb{O}$ is a 7-dimensional vector space identified with the imaginary octonions, then the standard associator $[\cdot,\cdot,\cdot]$ vanishes precisely on the 3-dimensional subspaces of V on which the octonionic multiplication is associative, hence the name. In the case where X is equipped with a *torsion-free* G_2-structure (i.e. $d\psi = 0$ and $d\varphi = 0$), an associative submanifold is *calibrated* by φ and is therefore volume-minimizing in its homology class. That is, the condition $\iota^*[\cdot,\cdot,\cdot]_\psi \equiv 0$ is equivalent to the condition that $\iota^*\varphi = \mathrm{vol}_\iota$ where vol_ι is the volume form on P induced by the pullback metric $\iota^* g_\varphi$.

Now, as with J-holomorphic curves, we want to work with associative submanifolds in manifolds that have G_2-structures that are not necessarily torsion-free, but retain volume-boundedness.

Definition 3.3.7 (Donaldson–Segal)**.** Suppose that (X, ψ) is an oriented 7-manifold equipped with a stable 4-form ψ. Then we say that a 3-form φ' *tames* ψ if for all $x \in X$, and for all ψ-associative, oriented 3-planes $V \subset T_x X$, we have $\varphi'\big|_V > 0$.

Note that $\varphi = *\psi$ always tames ψ. Note that the definition does not require φ' to be stable. However, the taming condition does imply this, as shown by the next lemma.

Lemma 3.3.8. *Let (X, ψ) be a 7-manifold equipped with a stable 4-form ψ. Suppose that the 3-form φ' tames ψ. Then φ' is stable.*

Proof. It suffices to prove this for $X = \mathbb{R}^7$. Let ψ be a stable 4-form on $\mathbb{R}^7$. Suppose that a 3-form φ' tames ψ, but is *not* a stable form. Recall that a stable 3-form φ determines a symmetric, bilinear form on $\mathbb{R}^7$ via the formula

$$6g_\varphi(u,v)\mathrm{vol}_\varphi = (u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi.$$

This form is positive-definite precisely when φ is positive, and negative-definite precisely when φ is negative, so if φ' is not stable, then $g_{\varphi'}$ is neither positive definite nor negative definite. That means there exists a nonzero $u \in \mathbb{R}^7$ such that

$$(u \lrcorner \varphi') \wedge (u \lrcorner \varphi') \wedge \varphi' = 0.$$

We show that this contradicts the assumption that φ' tames ψ. Note that if $u \lrcorner \varphi' = 0$ then φ' clearly does not tame ψ since u is contained in some ψ-associative 3-plane. There are two remaining cases.

Case 1. Suppose that $(u \lrcorner \varphi') \wedge (u \lrcorner \varphi') = 0$. Let $\{e_i\}_{i=1}^7$ be a $g_{\varphi'}$-orthogonal basis for $\mathbb{R}^7$ with $e_1 = u$ and let $\{e^i\}_{i=1}^7$ be the $g_{\varphi'}$-dual basis. Then $(u \lrcorner \varphi') \wedge (u \lrcorner \varphi') = 0$ implies that $u \lrcorner \varphi'$ must be indecomposable. That is, $u \lrcorner \varphi' = Ae^{ij}$ for some constant A with $i, j \neq 1$. Next, let $v \in \mathbb{R}^7$ be a nonzero vector orthogonal to the 3-plane spanned by u, e_i, and e_j with respect to the metric $g_{\varphi'}$. Then $V = \operatorname{span}\{u, v, u \times_\psi v\}$, where $\times_\psi$ is the cross-product defined by the 4-form ψ, is ψ-associative. But $v \lrcorner u \lrcorner \varphi' = v \lrcorner A e^{ij} = 0$ since v is orthogonal to e_i and e_j. This means that $\varphi'\big|_V = 0$ so φ' does not tame ψ.

Case 2. Suppose that $(u \lrcorner \varphi') \wedge (u \lrcorner \varphi') \wedge \varphi' = 0$ but $(u \lrcorner \varphi') \wedge (u \lrcorner \varphi') \neq 0$. Let $\beta = (u \lrcorner \varphi') \wedge (u \lrcorner \varphi')$. This is impossible because the map sending β to $\beta \wedge \varphi'$ is an isomorphism (see Proposition 2.2.1 in [31]).

$\square$

Now, if $\iota : P \to X$ is a ψ-associative embedding, and if ψ is tamed by the closed form φ', then ι has a topological volume bound. That is, it follows from the definitions that

$$\mathrm{Vol}_{g_\psi}(\iota P) \leq \int_P \iota^* \varphi' = K \langle [\varphi'], [\iota] \rangle$$

for some positive constant K. Here $[\iota]$ is the homology class represented by ι, $[\varphi']$ is the cohomology class represented by φ, and $\langle \cdot, \cdot \rangle$ is the natural pairing between them. When $\varphi' = *\psi$ and ψ is an integrable G_2 structure, equality holds. This can be compared to the energy identity in the symplectic setting.

We now extend this definition to 6-manifolds with G_2-pairs.

Definition 3.3.9. Let $(\rho, \tau) \in \Omega^3_{\text{stable}}(M) \times \Omega^4_{\text{stable}}(M)$ be a G_2 pair as in Definition 3.2.6. We say that the pair $(\rho', \omega') \in \Omega^3_{\text{stable}}(M) \times \Omega^2_{\text{stable}}(M)$ *tames* (ρ, τ) if

$$\begin{aligned} \psi &= \tau + dt \wedge \rho \\ \varphi' &= \rho' + dt \wedge \omega' \end{aligned}$$

comprises a *tamed* G_2-structure as in Definition 3.3.7.

It is therefore natural to wonder if there is a natural calibrated submanifold of a 6-manifold M equipped with a G_2-pair similar to associatives in manifolds with G_2-structures or J-holomorphic curves in symplectic manifolds. In the case where (ρ, τ) forms a Calabi-Yau structure, there is an obvious candidate. Namely, *special Lagrangian* submanifolds which are calibrated by $\hat{\rho}$. The purpose of section 5 is to explain how to generalize the definition of a special Lagrangian submanifold in such a way that it makes sense for manifolds with G_2-pairs.

Before delving into the next topic, we take some time to explain how tamed G_2-pairs interacted with tamed almost complex structures.

3.3.4 Tamed G_2-pairs and almost complex structures.

Suppose that (M, ρ, ω) is a 6-manifold equipped with a positive 3-form ρ and a stable 2-form ω. Since the 3-form ρ determines an almost complex structure $J_\rho : TM \to TM$ via equation 3.4 it is natural to ask, under what conditions is J_ρ ω-compatible? What about ω-tame? In this section, we give an answer to this question.

Proposition 3.3.10. *Suppose that (M, ρ, ω) is a 6-manifold equipped with a stable 2-form ω and a positive 3-form ρ. The almost complex structure J_ρ defined by equation 3.4 is ω-compatible if and only if $(\rho, \omega) \in \mathscr{R}_{\mathrm{SU}(3)}$.*

Proof. Note that a pair (ρ, ω) where J_ρ is ω-compatible always determines a U(3)-structure. On the other hand, at every point $x \in M$, ρ determines a nowhere-vanishing complex volume form. So this U(3)-structure reduces to an SU(3)-structure. These implications go both directions. □

Proposition 3.3.11. *Suppose that (M, ρ, ω) is a 6-manifold equipped with a stable 2-form ω and a positive 3-form ρ. The almost complex structure J_ρ is ω-tame if and only if $(\rho, \omega) \in \mathscr{R}_{\mathrm{G}_2}$.*

Proof. First we assume that $(\rho, \omega) \in \mathscr{R}_{G_2}$ and show that J_ρ is ω-tame. It suffices to prove this for $M = \mathbb{R}^6$. Let $\mathbb{R}^7 = \mathbb{R} \times \mathbb{R}^6$ and let t be the first $\mathbb{R}$-coordinate. Write ∂_t for the vector field dual to dt *with respect to the standard inner-product* g_0. Let the G_2-structure determined by (ρ, ω) be

$$\varphi = \rho + dt \wedge \omega.$$

Let $\times_\varphi$ be the cross-product determined by φ. Keep in mind that the inner-product determined by φ is *not* necessarily g_0. Recall that for any two linearly independent vectors $u, v \in \mathbb{R}^7$, the triple

$$\{u, v, u \times_\varphi v\}$$

spans a φ-associative plane in $\mathbb{R}^7$. In other words,

$$\varphi(u \times_\varphi v, u, v) > 0.$$

Therefore in particular, we have

$$\varphi(u \times_\varphi J_\rho u, u, J_\rho u) > 0$$

Write $w = u \times_\varphi J_\rho u$. Since φ is not the standard G_2-structure, ∂_t is not necessarily orthogonal to $\mathbb{R}^6 = \{0\} \times \mathbb{R}^6 \subset \mathbb{R}^7$. Therefore, w has a component that is parallel to ∂_t and a component that is contained in $\mathbb{R}^6$. Write

$$w = w_H + w_t$$

where $w_H \in \mathbb{R}^6$ and $w_t = A\partial_t$ for some constant A. This splitting is orthogonal with respect to g_0. Next, compute

$$\begin{aligned} \varphi(w, u, J_\rho u) &= \rho(w, u, J_\rho u) + dt \wedge \omega(w, u, J_\rho u) \\ &= \rho(w_H, u, J_\rho u) + \rho(w_t, u, J_\rho u) + dt \wedge \omega(w_H, u, J_\rho u) + dt \wedge \omega(w_t, u, J_\rho u) \\ &= \rho(w_H, u, J_\rho u) + A\omega(u, J_\rho u) > 0 \end{aligned}$$

where the second and third terms in the second line vanish for dimensional reasons. Furthermore, one can see by inspecting equation 3.4 that $\rho(v, u, J_\rho u) = 0$ for all vectors v. Therefore,

$$\omega(u, J_\rho u) > 0$$

as desired. Note that we know $\omega(u, J_\rho u) > 0$ instead of < 0 because φ must induce the same orientation on $\mathbb{R}^7$ as the standard one. If not, then we simply change dt to $-dt$.

Conversely, suppose that J_ρ is ω-tame. We want to show that $\varphi = \rho + dt \wedge \omega$ is a positive 3-form. Note that by Lemma 3.3.8, it is enough to show that φ tames $\psi_0 = *_{g_0}\varphi_0$ (in the sense of Definition 3.3.7) where $\varphi_0 = \rho + dt \wedge \omega_0$ and ω_0 is defined by $g_0\left(J_\rho u, v\right)$. Let $\{u, v, u \times_0 v\}$ be a basis for a ψ_0-associative 3-plane. Then we want to show that $\varphi(u, v, u \times_0 v) > 0$.

In fact, we can simplify the problem even further. Consider again φ_0 in its standard form, using coordinates $e_7, e_1, \ldots, e_6$ on $\mathbb{R} \times \mathbb{R}^6$.

$$\varphi_0 = e^{135} - e^{145} - e^{236} - e^{245} - e^{127} - e^{567} - e^{457}.$$

Observe that each term is a g_0-volume form for some 3-plane V. Thus we only need to check two cases.

Case 1: Assume that V is a ψ_0-associative 3-plane lying entirely in $\mathbb{R}^6$. In this case, $\varphi_0(u, v, w) = \rho(u, v, w) = \varphi(u, v, w)$. So clearly $\varphi(u, v, w) > 0$ since V is φ_0-associative.

Case 2: Assume that V is a ψ_0-associative 3-plane where $u = Ae_7$. In this case, $w = Ae_7 \times v$ for $v \in \mathbb{R}^6$. Therefore, $w = J_\rho v$. We have

$$\varphi(A\partial_t, v, J_\rho v) = A\omega(v, J_\rho v) > 0$$

since J_ρ is ω-tame.

These considerations show that φ tames φ_0 as a G_2-structure. Therefore, φ is a positive 3-form. $\square$

3.4 Special Lagrangians as critical points

The purpose of this section is to describe the Lagrange multipliers problem, first mentioned by Donaldson and Segal [15], that is at the center of this dissertation. As we will see, the solutions to this Lagrange multipliers problem correspond to special Lagrangian submanifolds in the special case where we start with a Calabi-Yau manifold.

3.4.1 Setup. Much of the notation in what follows is taken from [38] and adapted to this 6-dimensional setting. Suppose that M is a closed 6-manifold equipped with a G_2-pair $(\rho, \tau) \in \Omega^3_{\text{stable}}(M) \times \Omega^4_{\text{stable}}(M)$. Suppose also that both ρ and τ are closed, but do not require that their Hitchin duals are closed. Fix a closed, oriented, 3-manifold P and homology class $A \in H_3(M; \mathbb{Z})$. Let

$$\mathscr{F} = \{\iota : P \to M \mid \iota \text{ is smooth embedding }, \iota_*[P] = A,\ \iota^*\hat{\rho} > 0\}\,.$$

The tangent space to $\mathscr{F}$ at ι is

$$T_\iota \mathscr{F} = \Gamma\left(\iota^* TM\right).$$

Let $\mathscr{G}$ denote the group of orientation-preserving diffeomorphisms of P and define

$$\mathscr{S} = \mathscr{F}/\mathscr{G}$$

which can be identified with the space of oriented, 3-dimensional submanifolds of M diffeomorphic to P along which $\hat{\rho}$ is positive. Let $[\iota]$ denote the equivalence class in $\mathscr{S}$ of an element $\iota \in \mathscr{F}$. The tangent space $T_{[\iota]}\mathscr{S}$ is the quotient

$$T_{[\iota]}\mathscr{S} = \frac{\Gamma(\iota^* TM)}{\{d\iota \circ X \mid X \in \Gamma(TP)\}}.$$

Since (ρ, τ) determines a G_2 structure on $\mathbb{R} \times M$, it also determines a metric on $\mathbb{R} \times M$ and therefore an induced metric on M. This tangent space therefore may be identified with space of normal vector fields on ιP in M. We will let $N\iota$ denote the normal bundle of ιP whenever we want to take this perspective. Next, fix a particular embedding ι_0 of P into M. Let $\tilde{\mathcal{F}}$ denote the *universal cover* of $\mathcal{F}$ based at ι_0. That is,

$$\tilde{\mathcal{F}} = \{\tilde{\iota} : [0,1] \times P \to M \mid \tilde{\iota}(0,\cdot) = \iota_0,\ \tilde{\iota}(t,\cdot) = \iota_t \in \mathcal{F}\ \forall\, t \in [0,1]\} \,/ \sim$$

where $\tilde{\iota} \sim \tilde{\iota}'$ if $\tilde{\iota}$ and $\tilde{\iota}'$ have the same endpoints and are smoothly homotopic. The group of orientation-preserving diffeomorphisms $\mathcal{G}$ also has a covering space $\tilde{\mathcal{G}}$, which is the group of smooth isotopies from $[0,1]$ to $\mathcal{G}$ starting at the identity. Let

$$\tilde{\mathcal{S}} = \tilde{\mathcal{F}}/\tilde{\mathcal{G}}.$$

Define a functional, $f_\tau : \tilde{\mathcal{F}} \to \mathbb{R}$ by

$$f_\tau(\tilde{\iota}) = \int_{[0,1]\times P} \tilde{\iota}^* \tau.$$

This functional is well-defined since τ is closed. Its derivative df_τ is a one-form on $\mathcal{F}$ given by

$$\left(df_\tau\right)_\iota (n) = \int_P \iota^* \left(n \lrcorner \tau\right)$$

for all $n \in T_\iota \mathcal{F}$. Note that both f_τ and its differential are gauge invariant in the sense $f_\tau\left(\tilde{g}^* \tilde{\iota}\right) = f_\tau(\tilde{\iota})$ for any $\tilde{g} \in \tilde{\mathcal{G}}$. Similarly, $\left(df_\tau\right)_{g^*\iota} (g^* n) = \left(df_\tau\right)_\iota (n)$ for any $g \in \mathcal{G}$. Also note that if n and n' are in the same equivalence class in $T_{[\iota]}\mathcal{S}$, then $\left(df_\tau\right)_\iota (n) = \left(df_\tau\right)_\iota (n')$ since $\left(df_\tau\right)_\iota (v) = 0$ for all $v \in \Gamma(T\iota P)$. Thus f_τ descends to a functional on $\tilde{\mathcal{S}}$ and df_τ descends to a one-form on $\mathcal{S}$.

Next, we define the constraint. Let $c : \mathcal{F} \to \Omega^3(P)$ denote the function given by

$$c(\iota) = \iota^* \rho.$$

Similarly, let $\tilde{c} : \tilde{\mathcal{F}} \to \Omega^3(P)$ denote the function given by

$$\tilde{c}(\tilde{\iota}) = \iota_1^*(\rho).$$

Then let

$$C_\rho = c^{-1}(0) \quad \text{and} \quad \tilde{C}_\rho = \tilde{c}^{-1}(0).$$

Note that C_ρ is also gauge invariant in the sense that if $\iota^*\rho = 0$, then all elements in the equivalence class $[\iota]$ also satisfy this condition. Let

$$\mathcal{C}_\rho = C_\rho/\mathcal{G} \quad \text{and} \quad \tilde{\mathcal{C}}_\rho = \tilde{C}_\rho/\tilde{\mathcal{G}}.$$

Next, we want to show that $\mathcal{C}_\rho$ is a submanifold of $\mathcal{S}$. First, a lemma.

Lemma 3.4.1. *Suppose that $[\iota] \in \mathcal{C}_\rho$ so that $\iota^*\rho = 0$ and $\iota^*\hat{\rho} > 0$. Then for every $p \in P$, the map $N_{\iota(p)}\iota \to \Lambda^2\left(T_p^*P\right)$ given by*

$$n \mapsto \iota^*(n \lrcorner \rho)$$

is an isomorphism.

Proof. Let (ρ, ω') be any SU(3)-structure on M and let g be the corresponding metric with respect to this SU(3)-structure. There is a g-orthogonal splitting of the k-forms on M. We fix the following notation for this splitting throughout the proof. Let N denote the normal bundle of ιP and T denote the tangent bundle. Let $p \in P$ and $x = \iota(p)$. Define

$$\begin{aligned}
\Lambda^2\left(T_x^*M\right) &= \Lambda^2(N_x^*) \oplus (N_x^* \otimes T_x^*) \oplus \Lambda^2\left(T_x^*\right) \\
&= \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2} \\
\Lambda^3\left(T_x^*M\right) &= \Lambda^3(N_x^*) \oplus \left(\Lambda^2\left(N_x^*\right) \otimes T_x^*\right) \oplus \left(N_x^* \otimes \Lambda^2\left(T_x^*\right)\right) \oplus \Lambda^3\left(T_x^*\right) \\
&= \Lambda^{3,0} \oplus \Lambda^{2,1} \oplus \Lambda^{1,2} \oplus \Lambda^{0,3} \\
\Lambda^4\left(T_x^*M\right) &= \left(\Lambda^3\left(N_x^*\right) \otimes T_x^*\right) \oplus \left(\Lambda^2\left(N_x^*\right) \otimes \Lambda^2\left(T_x^*\right)\right) \oplus \left(N_x^* \otimes \Lambda^3\left(T_x^*\right)\right) \\
&= \Lambda^{3,1} \oplus \Lambda^{2,2} \oplus \Lambda^{1,3}.
\end{aligned}$$

In this notation, the symbol $\Lambda^{p,q}$ does *not* refer to a decomposition with respect to a complex structure as it usually does. Instead, it refers to $\Lambda^p(N_x^*) \oplus \Lambda^q(T_x^*)$ as written. We may write ρ_x in components as follows

$$\rho_x = \rho^{3,0} + \rho^{2,1} + \rho^{1,2} + \rho^{0,3}.$$

The following observations are immediately apparent.

1. $\rho^{0,3} = 0$ since $\iota^*\rho = 0$

2. $\rho^{3,0} \neq 0$ since $\iota^*\hat{\rho} > 0$.

Since N_x and $\Lambda^2\left(T_x^*\right)$ have the same dimension, it suffices to show that the kernel of the map $n \mapsto \iota(n \lrcorner \rho)$ is trivial. Suppose that there exists a non-zero $n \in N_x$ such that $\iota^*(n \lrcorner \rho) = 0$. Then it must be the case that

$$n \lrcorner \rho \in \Lambda^{1,1} \oplus \Lambda^{2,0}.$$

Let J denote the almost complex structure on T_xM determined by ρ. Then a formula from [20] tells us that

$$\begin{aligned} *(n \lrcorner \rho) &= -Jn \wedge \rho \in \Lambda^{2,2} \oplus \Lambda^{1,3} \\ &\Rightarrow Jn \wedge \rho^{3,0} = 0 \\ &\Rightarrow Jn \in N_x. \end{aligned}$$

Here, when we write $-Jn \wedge \rho$ we mean the wedge product of the metric dual of $-Jn$ with ρ. This contradicts the fact that $\rho + i\hat{\rho}$ must be a complex volume form since ρ is a stable 3-form. $\square$

Note that this lemma is not true if $\iota^*\hat{\rho}$ vanishes at p. If ιP is a special Lagrangian submanifold in a Calabi-Yau manifold, then this condition is automatically satisfied.

Lemma 3.4.2. *For all $[\iota] \in \mathcal{C}_\rho$, the cokernel of $dc_{[\iota]} : T_{[\iota]}\mathcal{S} \to \Omega^3(P)$ is isomorphic to $\mathbb{R}$.*

Proof. Note that

$$dc_{[\iota](n)} = \iota^* (\mathcal{L}_n \rho) = \iota^* (d(n \lrcorner \rho))$$

since ρ is closed and where $\mathcal{L}$ denotes the Lie derivative. So any 3-form in the image of $dc_{[\iota]}$ is clearly exact. Furthermore Lemma 3.4.1 says that the image of $dc_{[\iota]}$ is *all* exact 3-forms on P. Therefore since any 3-form on P is closed, P being 3-dimensional, the result follows. □

Finally, we can state the Lagrange multipliers problem: Find the critical points of $f_\tau\big|_{\tilde{C}\rho}$.

3.4.2 The perturbed SL equations.

In this section, we derive the Euler-Lagrange equations for the above Lagrange multipliers problem. These equations also appeared in [15] but without much explanation. We define the Lagrange functional Λ by analogy with the finite-dimensional setting. Note that $C^\infty(P)$ is dual to $\Omega^3(P)$ in the sense that there is a pairing

$$\begin{aligned} C^\infty(P) \times \Omega^3(P) &\to \mathbb{R} \\ (f, \alpha) &\mapsto \int_P f\alpha. \end{aligned}$$

Thus, the Lagrange functional $\Lambda : C^\infty(P) \times \tilde{\mathcal{F}} \to \mathbb{R}$ is given by

$$\Lambda(\lambda, \tilde{\iota}) = \int_{[0,1]\times P} \tilde{\iota}^* \tau + \int_P \lambda \iota^* \rho.$$

Here, $\iota = \tilde{\iota}(1, \cdot)$. The tangent space of $C^\infty(P) \times \mathcal{F}$ at (λ, ι) is

$$T_{(\lambda,\iota)} \left(C^\infty(P) \times \mathcal{F}\right) = C^\infty(P) \times \Gamma(\iota^* TM).$$

As with f_τ, the functional Λ is $\tilde{\mathcal{G}}$-invariant and therefore descends to a functional on $\tilde{\mathcal{S}}$. Furthermore, its derivative $d\Lambda$ is a $\mathcal{G}$-invariant one-form on $C^\infty(P) \times \mathcal{F}$ given by

$$d\Lambda_{(\lambda,\iota)}(l, n) = \int_P \iota^*(n \lrcorner \tau) + \int_P \lambda \iota^*(n \lrcorner d\rho) + \int_P l \iota^* \rho.$$

Note that since ρ is closed, Stokes' theorem gives

$$d\Lambda_{(\lambda,\iota)}(l,n) = \int_P \iota^*(n \lrcorner \tau) + \int_P d\lambda \wedge \iota^*(n \lrcorner \rho) + \int_P l\iota^* \rho.$$

If $v \in \Gamma(T\iota P)$, then $d\Lambda_{(\lambda,\iota)}(l,v) = d\Lambda_{(\lambda,\iota)}(l,0)$. Therefore $d\Lambda$ descends to a one-form on $C^\infty(P) \times \mathcal{S}$. The following notation was used in [15] and will also be used in this paper.

Definition 3.4.3. Let α be a k-form on a manifold M. Let $\iota : P \to M$ be an embedding of a manifold P into M and suppose that $\iota^*\alpha = 0$. Then α defines an ι^*T^*M-valued $(k-1)$-form on P called α_N given by

$$\alpha_N(v_1, \ldots, v_{k-1}) = \alpha(\iota_* v_1, \ldots, \iota_* v_{k-1}, \cdot)$$

for any $(v_1, \ldots, v_{k-1}) \in TP$. In the presence of a metric, α_N takes values in $N^*\iota$.

The above construction shows the following.

Proposition 3.4.4. *We have $d\Lambda_{(\lambda,\iota)}(l,n) = 0$ for all $(l,n) \in T_{(\lambda,\iota)}(C^\infty(P) \times \mathcal{F})$ if and only if the following two equations are satisfied*

$$\tau_N + d\lambda \wedge \rho_N = 0 \tag{3.5}$$

$$\iota^*\rho = 0. \tag{3.6}$$

Definition 3.4.5. Let (M, ρ, τ) be a 6-dimensional manifold equipped with a G_2-pair $(\rho, \tau) \in \Omega^3(M) \times \Omega^4(M)$. Let P be an oriented, closed, 3-manifold. A pair $(\lambda, \iota) \in C^\infty(P) \times \mathcal{F}$ which solves equation **??** is called a *graphical associative* for reasons that will become clear in the next section.

These are the *Euler-Lagrange* equations for the Lagrange multipliers problem and will henceforth be referred to as the *perturbed special Lagrangian (SL)* *equations.* The pair (λ, ι) is a *critical point* of the functional Λ if and only if (λ, ι) satisfies these equations. By the Lagrange multipliers theorem, a pair (λ, ι) is a critical point of Λ if and only if there exists $\tilde{\iota} \in \tilde{\mathscr{F}}$ such that $\iota = \tilde{\iota}(1, \cdot)$ and $\tilde{\iota}$ is a critical point of $f_\tau\big|_{\tilde{C}_\rho}$. Note that the perturbed SL equations are $\mathscr{G}$-invariant. Thus the solutions may be thought of up to diffeomorphism. The following lemma appeared in [23] and shows how the above setup relates to Calabi-Yau manifolds.

Lemma 3.4.6 (Harvey–Lawson). *If (M, J, g, Ω) is a 6-(real) dimensional Calabi-Yau manifold, then* $\mathrm{Im}(\Omega)$ *is a calibration whose calibrated submanifolds are called special Lagrangians. If ω is the Kähler form for the Calabi-Yau metric g, then a 3-submanifold L admits an orientation making it special Lagrangian if and only if L satisfies*

$$\begin{aligned} \omega\big|_L &= 0 \\ \mathrm{Re}(\Omega)\big|_L &= 0. \end{aligned}$$

Therefore it is easy to see that if, in the Calabi-Yau case, we let $\tau = \frac{1}{2}\omega^2$ and $\rho = \mathrm{Re}(\Omega)$, a submanifold ιP is special Lagrangian if and only if the embedding ι together with $\lambda =$ constant is a critical point of Λ (see grass). Due to this fact we hope to define a Floer theory for the critical points of Λ.

However, we will need to consider solutions to equations **??** other than those for which λ is a constant because of the following theorem from [36] which shows that the critical points of Λ and therefore $f_\tau\big|_{\tilde{C}_\rho}$ cannot usually be isolated.

Theorem 3.4.7 (McLean). *Suppose that M is a Calabi-Yau manifold and $L \subset M$ is a special Lagrangian submanifold. Then the moduli space of nearby special*

Lagrangians is a smooth manifold of dimension equal to the first Betti number of L.

Of course, if we think of $f_\tau\big|_{\tilde{C}_\rho}$ as a functional on embeddings rather than on submanifolds (i.e. embeddings up to diffeomorphism) the critical points are not isolated since if ι is a critical point, so is $g^*\iota$ for any $g \in \mathcal{G}$. In the following section, we will prove that if one perturbs the special Lagrangian equations the critical points of Λ will become isolated (up to diffeomorphism).

3.5 Ellipticity and the G_2-cylinder

In this section,we express the perturbed SL equations in terms of a section of an infinite-dimensional vector bundle. We define the moduli space of the solutions as the zero set of this section. Finally, we prove that the linearization of the section is an elliptic operator by exploiting the relationship between the 6- and 7-dimensional geometries. Throughout, M will be a closed 6-manifold equipped with

- a smooth G_2-pair $(\rho', \omega') \in \Omega^3(M) \times \Omega^2(M)$ such that both ρ' and ω' are *closed*.
- a smooth G_2-pair $(\rho, \tau) \in \Omega^3(M) \times \Omega^4(M)$ such that both ρ and τ are *closed* and *tamed* by (ρ', ω').

Since $\varphi' = \rho' + dt \wedge \omega'$ determines a G_2-structure on $X = \mathbb{R} \times M$, it also determines a metric on $\mathbb{R} \times M$, and therefore an induced metric on $M = \{0\} \times M \subset \mathbb{R} \times M$. We call this metric g.

3.5.1 Moduli spaces. As before, let P be a closed 3-manifold. Let $A \in H_3(M, \mathbb{Z})$ be a fixed homology class. Define

$$\mathcal{F} = \{\iota : P \to M \mid \iota \text{ is a smooth embedding}, \quad \iota_*[P] = A, \quad \iota^*\rho, \iota^*\rho' > 0\}.$$

Next, consider the vector bundle

$$\mathscr{E} \to C^\infty(P) \times \mathscr{F}$$

whose fiber at (λ, ι) is

$$\mathscr{E}_{(\lambda,\iota)} = \Omega^3(P) \times \Omega^3(P, \iota^* T^* M).$$

Solutions to the perturbed SL equations then correspond to the zero set of the following section of this bundle.

$$L : C^\infty(P) \times \mathscr{F} \quad \to \quad \mathscr{E} \tag{3.7}$$

$$(\lambda, \iota) \quad \mapsto \quad (\iota^* \rho, \tau_N + d\lambda \wedge \rho_N)\,. \tag{3.8}$$

Definition 3.5.1. Let $\tilde{\mathcal{N}}\,(A, P; (\rho, \tau))$ denote the *moduli space* of solutions to the perturbed SL equations. That is,

$$\tilde{\mathcal{N}} = \tilde{\mathcal{N}}\,(A, P; (\rho, \tau)) = \{(\lambda, \iota) \mid \iota \in \mathscr{F}, \iota^* \rho = 0, \tau_N + d\lambda \wedge \rho_N = 0\}\,.$$

Then $\tilde{\mathcal{N}}$ can be identified with the zero set of L

$$\tilde{\mathcal{N}} = \tilde{\mathcal{N}}\,(A, P; (\rho, \tau)) = L^{-1}(0).$$

As in the finite-dimensional case, we often want to neglect the Lagrange multiplier λ. Let $\pi_2 : C^\infty(P) \times \mathscr{F} \to \mathscr{F}$ be the projection and let

$$\tilde{\mathcal{M}} = \pi_2(\tilde{\mathcal{N}}).$$

This is the *moduli space of perturbed special Lagrangians.* A few remarks about this definition of the moduli space are in order.

1. The tilde above $\mathcal{M}$ is there to remind us that we have not yet taken the quotient with $\mathscr{G}$.

2. At the moment, we do not need to keep track of the parameters A, P and (ρ, τ) so we drop them from the notation. Later on, we will want to add them back in.

3. Also at the moment, we have defined the moduli space in such a way that all its elements are smooth. In later sections we will want to allow for non-smooth elements. The results of section 3.7 will imply that if (ρ, τ) are of class C^ℓ then there exists a diffeomorphism ϕ such that if (λ, ι) is a C^3 solution to the perturbed SL equations, $(\lambda \circ \phi, \iota \circ \phi)$ is also of class C^ℓ.

The group of orientation-preserving diffeomorphisms of P acts freely on $C^\infty(P) \times \mathcal{F}$ by composition (see [12] for details) and $\mathcal{S} = \mathcal{F}/\mathcal{G}$ is a Fréchet manifold. However, we will later need to work with Banach spaces and will want to identify elements in $(C^\infty(P) \times \mathcal{F})/\mathcal{G}$ near a particular element with normal vector fields of varying regularity. Restricting to a slice of the action of $\mathcal{G}$ removes this complication.

The metric g corresponding to the G_2-pair (ρ', ω') induces an inner-product on the tangent spaces of $C^\infty(P) \times \mathcal{F}$ as follows. Suppose that $n_1, n_2 \in T_\iota\mathcal{F} = \Gamma(\iota^*TM)$ and $l_1, l_2 \in T_\lambda C^\infty(P)$. Define

$$\langle n_1, n_2 \rangle = \int_P g(n_1, n_2)\iota^*\rho'$$

and

$$\langle l_1, l_2 \rangle = \int_P l_1 l_2 \iota^*\rho'.$$

These equations define a $\mathcal{G}$-invariant metric on $C^\infty(P) \times \mathcal{F}$.

Definition 3.5.2. Using the exponential map corresponding to the metric g on M, a neighborhood $U_{(\lambda,\iota)}$ of (λ, ι) can be identified with an open set in

$T_{(\lambda,\iota)}\left(C^\infty(P)\times\mathcal{F}\right)$. Under this identification, let $S_{(\lambda,\iota)}\subset U_{(\lambda,\iota)}$ denote the subset corresponding to the orthogonal complement of the $\mathcal{G}$-orbit of (λ,ι). We refer to $S_{(\lambda,\iota)}$ as a *local slice* for (λ,ι) and it consists of pairs (l,n) where n is a normal (with respect to g) vector field along ιP.

This definition makes sense because $S_{(\lambda,\iota)}$ is transverse to the $\mathcal{G}$-orbit of (λ,ι) in $C^\infty(P)\times\mathcal{F}$.

Definition 3.5.3. Let $\mathcal{N}^S$ denote the *local moduli space of graphical associatives* given by

$$\mathcal{N}^S=\mathcal{N}^S(A,P;(\rho,\tau))=\left\{(\lambda,\iota)\in L^{-1}(0)\mid(\lambda,\iota)\in S\right\}$$

where S is a local slice as described above. As before, also define

$$\mathcal{M}^S=\pi_2(\mathcal{N}^S).$$

3.5.2 The symbol of $DL\big|_S$.

Let S be a local slice as in Definition 3.5.2. Let $L\big|_S$ denote the restriction of the section L (equation 3.8) to S. We want to prove that the linearization $DL\big|_S$ is a self-adjoint, elliptic operator. First, we review the definition of an elliptic operator.

Definition 3.5.4. Let E,F be vector bundles over a manifold M and let $T:\Gamma(E)\to\Gamma(F)$ be a differential operator of order k. Let $(x,v)\in T^*M$ and $e\in E_x$ be given. Find a smooth function $g(x)$ and a section $f\in\Gamma(E)$ such that $dg_x=v$ and $f(x)=e$. Then *the principal symbol of* T is defined by

$$\sigma(T)(x,v)e=L\left((g-g(x))^kf\right)(x)\in F_x$$

Example 3.5.5. Suppose T is the exterior derivative and let $e \in \Lambda^p(T_xX)$. Then:

$$\begin{aligned}\sigma(T)(x,v)e &= d\left((g-g(x))f\right)(x)\\ &= \left(dg\wedge f\right)(x)\\ &= v\wedge e\end{aligned}$$

Definition 3.5.6. A symbol σ is called *elliptic* if for all $(x,v)\in T^*M$ with $v\neq 0$, the linear map $\sigma(x,v) : E_x \to F_x$ is an isomorphism. A differential operator T of order k is called *elliptic* if its principal symbol is elliptic.

Next, we calculate the linearization of $L\big|_S$ and compute its symbol. Let (λ,ι) be a graphical associative contained in a local slice S. Let $N\iota$ be the normal bundle of ιP with respect to the metric determined by (ρ',ω'). Then the linearization of $L\big|_S$ is given by

$$D : C^\infty(\iota P)\times\Gamma(N\iota)\to\Omega^3(P)\times\Omega^3(P,N^*\iota)$$

$$D_{(\lambda,\iota)}(l,n) = \left(\iota^* d(n\lrcorner\rho), d\left(n\lrcorner(\tau+d\lambda\wedge\rho)\right)_N + dl\wedge\rho_N\right).$$

In matrix form

$$D_{(\lambda,\iota)}(l,n) = \begin{pmatrix}(d(\,\cdot\,)\wedge\rho)_N & d(\,\cdot\,\lrcorner(\tau+d\lambda\wedge\rho))_N\\ 0 & \iota^*d(\,\cdot\,\lrcorner\rho)\end{pmatrix}\begin{pmatrix}l\\ n\end{pmatrix}.$$

Note that D is a first order operator. Let $(x,\alpha_x)\in T^*\iota P$ and choose $(k,n_x)\in\mathbb{R}\times N_x\iota$. Let $\sigma_{\tau,\rho}$ denote the symbol of D. Furthermore, let $\eta=\tau+d\lambda\wedge\rho$. Then Definition 3.5.4 gives

$$\sigma_{\tau,\rho}(x,\alpha_x)(k,n_x) = \left(\alpha_x\wedge(n_x\lrcorner\rho_x), (\alpha_x\wedge(n_x\lrcorner\eta_x))_N + k\alpha_x\wedge\rho_{N_x}\right)$$

or, in matrix form

$$\sigma_{\tau,\rho}(x,\alpha_x) = \begin{pmatrix} \cdot\ (\alpha_x \wedge \rho_N) & \alpha_x \wedge (\cdot\ \lrcorner(\tau + d\lambda \wedge \rho))_N \\ 0 & (\ \cdot\ \lrcorner\rho) \wedge \alpha \end{pmatrix} \begin{pmatrix} k \\ n \end{pmatrix}.$$

In order to check that $\sigma_{\rho,\tau}$ is an isomorphism for all choices of $(x,\alpha_x) \in T^*\iota P$, we view our problem from the perspective of the G_2-structure on $\mathbb{R} \times M$. This requires us to consider the analogous construction in 7-dimensions.

3.5.3 The associative action functional. Just as special Lagrangian submanifolds are the critical points of a functional, associative submanifolds are also critical points of a functional. In fact, the situation is much simpler in this case since we do not have any constraints. Therefore the idea is to write the symbol of $\sigma_{\tau,\rho}$ in terms of the symbol of the action functional for associative submanifolds which is easily seen to be elliptic.

A detailed description of the following material was given in [38]. Here it is slightly generalized. Suppose that (X,ψ,φ') is a 7-dimensional manifold equipped with a closed, stable 3-form φ' and a closed φ'-tame 4-form ψ. Let $A \in H_3(X;\mathbb{Z})$. Let P be a closed, oriented, 3-dimensional manifold. Then define

$$\mathscr{F}(X) = \{\iota : P \to X \mid \iota \text{ is a smooth embedding}, \quad \iota_*[P] = A \quad \iota^*\varphi' > 0\}.$$

We also have the covering space versions of these objects exactly as in section 3.4. The 4-form ψ defines a $\mathcal{G}$-invariant action functional on $\tilde{\mathscr{F}}(X)$ by integration:

$$f_\psi(\tilde{\iota}) = \int_{[0,1]\times P} \tilde{\iota}^*\psi.$$

This descends to a $\mathcal{G}$-invariant, horizontal one-form on $\mathscr{F}(X)$

$$(df_\psi)_\iota(n) = \int_P \iota^*(n \lrcorner \psi).$$

Similar to the 6-dimensional case, ι is a critical point of f_ψ if and only if $\psi_N = 0$. In other words, if and only if ιP is an associative submanifold of X.

Therefore, it's easy to see that the critical points of f_ψ are exactly the associative submanifolds of X. As before, we can write down the symbol for the operator associated to the equation $\psi_N = 0$ and restrict ourselves to a local slice of the action of the diffeomorphism group. In this case, the linearization $(D_\psi)_\iota : \Gamma(N\iota) \to \Omega^3(P, N^*\iota)$ where $\iota : P \to X$ is associative is

$$(D_\psi)_\iota(n) = d\,(n \lrcorner \psi)_N \,.$$

To calculate its symbol, let $\alpha_x \in T^*_x \iota P$ and $n_x \in N_x\iota$. Then

$$\sigma_\psi(\alpha_x)(n_x) = (\alpha_x \wedge (n_x \lrcorner \psi))_N \,.$$

See Lemma 2.13 of [28] for an alternative proof of the following.

Lemma 3.5.7. *Let (X, ψ) be an oriented 7-manifold equipped with a closed, stable 4-form ψ. Suppose that $\iota : P \to X$ is a ψ-associative submanifold. Then the operator $(D_\psi)_\iota$ is elliptic and self-adjoint.*

Proof. To simplify notation, we will write D for $(D_\psi)_\iota$ for the duration of this proof. Note that $D : \Gamma \to \Omega^3(P, N^*\iota)$. However, by integrating and using the metric isomorphism, we can view D as an operator from $\Gamma(N\iota)$ to itself. Therefore it makes sense to consider whether D is self-adjoint. First, we show that D is elliptic. Let $p \in P$. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for $T_p\iota P$. Let $\alpha = a_1 e^1 + a_2 e^2 + a_3 e^3$ be an arbitrary nonzero 1-form on ιP at p. Let $n \in N_p\iota$. Then

$$\begin{aligned}
(e_3 \lrcorner e_2 \lrcorner e_3 \lrcorner (\alpha \wedge (n \lrcorner \psi)))^\flat &= a_1(e_1 \lrcorner e_2 \lrcorner n \lrcorner \psi)^\flat + a_2(e_3 \lrcorner e_1 \lrcorner n \lrcorner \psi)^\flat + a_3(e_2 \lrcorner e_1 \lrcorner n \lrcorner \psi)^\flat \\
&= a_1(e_3 \times n) + a_2(e_2 \times n) + a_3(e_1 \times n) \\
&= \alpha \times n.
\end{aligned}$$

Thus $\sigma_\psi(\alpha) : N_p\iota \to N_p\iota$ is clearly an isomorphism for each nonzero α. Next, we show that D is self-adjoint. Let $n, m \in \Gamma(N\iota)$. We have

$$\langle n, Dm \rangle = \int_P \iota^* \left(n \lrcorner d(m \lrcorner \psi) \right)$$

and similarly for $\langle Dn, m \rangle$. We calculate

$$\begin{aligned}
\int_P \iota^* \left([n, m] \lrcorner \psi \right) &= \int_P \iota^* \left(\mathcal{L}_n \left(m \lrcorner \psi \right) - m \lrcorner \left(\mathcal{L}_n \psi \right) \right) \\
&= \int_P \iota^* \left(n \lrcorner d \left(m \lrcorner \psi \right) + d \left(n \lrcorner m \lrcorner \psi \right) - m \lrcorner d \left(n \lrcorner \psi \right) \right) \\
&= \langle n, Dm \rangle - \langle Dn, m \rangle
\end{aligned}$$

where $\mathcal{L}$ is the Lie derivative. We have also used the general fact that for vector fields X, Y we have $X \lrcorner Y \lrcorner \cdot = [\mathcal{L}_X, Y \lrcorner \cdot]$, Cartan's magic formula, and Stokes' theorem. Next, note that since ιP is an associative submanifold, $\int_P \iota^* \left([n, m] \lrcorner \psi \right) = 0$. Therefore we have shown

$$\langle n, Dm \rangle = \langle Dn, m \rangle.$$

Thus D is self-adjoint. $\square$

Remark 3.5.8. In the above proof, expressions of the form $\mathcal{L}_n \psi$ require some explanation. Since n is a section of a vector field along ιP, we first need to extend it to an open neighborhood U of ιP in order for $\mathcal{L}_n$ to make sense. Such an extension depends on a choice of metric. However, it turns out that the above expressions do not depend on this choice.

3.5.4 The connection between 6- and 7-dimensions. Next, we show that $\sigma_{\rho,\tau}$ is elliptic. In order to accomplish this, we relate the solutions to the perturbed SL equations to associative submanifolds of $\mathbb{R} \times M$. First, a definition.

Definition 3.5.9. Suppose that $(\lambda, \iota) \in C^\infty(P) \times \mathscr{I}$. Define

$$\begin{aligned} \iota_\lambda : P &\to \mathbb{R} \times M \\ \iota_\lambda(p) &= (\lambda(p), \iota(p))\,. \end{aligned}$$

Then $\iota_\lambda P$ is the graph of the function λ over ιP. The following lemma finally allows us to justify our term *graphical associative*.

Lemma 3.5.10. *For any graphical associative (λ, ι), $\iota_\lambda P$ as defined above is an associative submanifold of $X = \mathbb{R} \times M$ with respect to the G_2 4-form $\psi = \tau + dt \wedge \rho$.*

Proof. Note that any vector field tangent to $\iota_\lambda P$ can be written in the form $u = \overline{u} + d\lambda(\overline{u})\partial_t$ where $\overline{u}$ is a vector field on ιP. Let (u, v, w) be three such vector fields tangent to $\iota_\lambda P$ and calculate $\iota_\lambda^*(u \lrcorner v \lrcorner w \lrcorner \psi)$. It will be easy to see that since $\iota^*(\rho) = 0$ and $(\tau + d\lambda \wedge \rho)_N = 0$, $\iota_\lambda^*(u \lrcorner v \lrcorner w \lrcorner \psi)$ also vanishes. $\square$

The above lemma also allows us to prove that when M is an actual Calabi-Yau manifold, then the *only* solutions to the perturbed SL equations are special Lagrangians.

Lemma 3.5.11. *Suppose that M is a Calabi-Yau manifold and equip $\mathbb{R} \times M$ with the product G_2-structure. Then the only solutions (λ, ι) to the perturbed SL equations are those for which ιP is a special Lagrangian submanifold of M and $d\lambda = 0$.*

Proof. We already know that solutions with $d\lambda = 0$ correspond to special Lagrangian submanifolds in M, so it suffices to show that any solution (λ, ι) satisfies $d\lambda = 0$. However, note that if $\iota_\lambda P$ is a graphical associative submanifold in $\mathbb{R} \times M$, it must be volume-minimizing in its homology class. On the other hand, note that the volume of $\iota_0 P$ where $\iota_0(p) = (0, \iota(p))$ is always less than or equal to the volume of ι_λ where equality holds if and only if λ is constant. So the fact that $\iota_\lambda P$ is associative means that λ must be constant. $\square$

Now suppose that $X = \mathbb{R} \times M$ where M is a 6-dimensional manifold with a closed, tamed, G_2 pair. Let (λ, ι) be a critical point of Λ. Then note that $N_{(t,x)}\iota_\lambda \cong \mathbb{R} \oplus N_x\iota$.

Proposition 3.5.12. *Under the identification $N_{(t,x)}\iota_\lambda \cong \mathbb{R} \oplus N_x\iota$, we have*

$$\sigma_{\tau,\rho}(\overline{\alpha})(k, \overline{n}) = \sigma_\psi(\alpha)(\overline{n} + k\partial_t)$$

where $\overline{\alpha}$ is an arbitrary 1-form on ιP and α is the corresponding 1-form on $\iota_\lambda P$

Proof. Note that the image of $\sigma_\psi(\alpha)$ lies in

$$\Lambda^3(T^*_{(x,t)}\iota_\lambda P) \otimes N^*_{(x,t)}\iota_\lambda \cong \Lambda^3(T^*_{(x,t)}\iota_\lambda P) \oplus \left(\Lambda^3(T^*_{(x,t)}\iota_\lambda P) \otimes N^*_x\iota\right).$$

So we can write σ_ψ as a matrix with respect to this splitting:

$$\sigma_\psi(\alpha)(k, \overline{n}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} k \\ \overline{n} \end{pmatrix}.$$

Let α be an arbitrary 1-form on $\iota_\lambda P$. When $\overline{n} = 0$, we have

$$\begin{aligned} \sigma_\psi(\alpha)(k, 0) &= \alpha \wedge (k\partial_t \lrcorner \psi)_N \\ &= k\alpha \wedge \rho_N \end{aligned}$$

which is an $N^*_{(x,t)}\iota_\lambda$-valued 3-form on $\iota_\lambda P$. However, note that ρ is defined on ιP. So actually, we can view this as an $N^*_x\iota$-valued 3-form on $\iota_\lambda P$. Therefore, we can conclude that

$$\begin{aligned} A &= \cdot\, \alpha \wedge \rho_N \\ C &= 0. \end{aligned}$$

Furthermore, when $k = 0$, we have:

$$\sigma_\psi(\alpha)(0, \overline{n}) = \alpha \wedge (\overline{n} \lrcorner \tau + \overline{n} \lrcorner (dt \wedge \rho))_N \,.$$

We know that B is an $N_x^* \iota$ -valued 3-form on $\iota_\lambda P$. If $h : \iota P \to \mathbb{R} \times M$ by $h(x) = (\lambda(x), x)$, then $B(\overline{n}) = h^*(\alpha \wedge (\overline{n} \lrcorner \tau + \overline{n} \lrcorner (dt \wedge \rho))_N)$. On the other hand, D is a 3-form on $\iota_\lambda P$. To see what D is, we can contract A with the volume form on $\iota_\lambda P$. The term which has a dt remaining will be D. We have

$$\begin{aligned} B &= \alpha \wedge (\cdot \lrcorner \tau + \cdot \lrcorner (d\lambda \wedge \rho))_N \\ D &= \alpha \wedge (\cdot \lrcorner \rho). \end{aligned}$$

After comparing this to equation (3.5.2), we see that the claim is proved.

□

Theorem 3.5.13. *The linearization of $L\big|_S$ at a point $(\lambda, \iota) \in \mathcal{N}^S$ is elliptic and self-adjoint.*

Proof. This theorem follows from Proposition 3.5.12 and Lemma 3.5.7. Proposition 3.5.12 tells us that $\sigma_{\tau,\rho}$ and σ_ψ are the same. Thus, by Lemma 3.5.7, $(D_{\rho,\tau})_{(\lambda,\iota)}$ is elliptic. The proof that $(D_{\rho,\tau})_{(\lambda,\iota)}$ is self-adjoint follows from the same calculation as in Lemma 3.5.7 combined with the identification of $N_{(t,x)}\iota_\lambda$ with $\mathbb{R} \oplus N_x\iota$. □

3.6 Volume bounds

In light of Lemma 3.5.10, we now have a natural way to identify solutions to the Lagrange multipliers problem with associative graphs in a cylinder $\mathbb{R} \times M$. If (X, ψ, φ') is a closed 7-manifold with a closed, tamed G_2-structure, and $\iota : P \to X$

is a ψ-associative submanifold of X, then Definition 3.3.7 implies

$$\mathrm{Vol}_{g_\psi}(\iota P) \leq K\langle[\varphi'], \iota_*[P]\rangle$$

for some positive constant K, where $[P]$ is the fundamental class of P. We have an analogous volume bound for graphical associatives.

Lemma 3.6.1. *Suppose that (M, ρ', ω') is a closed 6-dimensional manifold with a G_2-pair $(\rho', \omega') \in \Omega^3_{\text{closed}}(M) \times \Omega^2_{\text{closed}}(M)$. Also suppose that (ρ, τ) is a (ρ', ω')-tame G_2 pair and that $(\lambda, \iota) \in \mathcal{N}(A, P; (\rho, \tau))$. Then both the volume of ιP and $\|d\lambda\|_{L^2}$ are bounded.*

Proof. Suppose that (λ, ι) is a graphical associative. Let ι_λ denote the associated embedding $\iota_\lambda : P \to \mathbb{R} \times M$, as before. We have the following metrics.

- g_M : the metric on M given by the G_2-pair (ρ, τ)
- g_X : the metric on $X = \mathbb{R} \times M$ given by the G_2-structure $\psi = \tau + dt \wedge \rho$
- $g = dt^2 + g_M$: the product metric on $\mathbb{R} \times M$.

Since (ρ, τ) is not necessarily an SU(3)-structure, g_X does not necessarily equal g. Fix a point $p \in P$ and let $\{e_i\}_{i=1}^3$ be a basis for T_pP. Also let

$$\begin{aligned} \{v_i\}_{i=1}^3 &= \{d\iota(e_i)\} \\ \{u_i\}_{i=1}^3 &= \{d\iota_\lambda(e_i)\}. \end{aligned}$$

Choose $\{e_i\}$ so that v_i and $\{u_i\}$ are orthonormal with respect to g_M and g respectively. Denote the g_M-dual of v_i by v^i. Similarly, denote the g-dual of u_i by u^i. Let μ be the function on ιP defined by $\lambda \circ \iota^{-1}$. Observe that

$$u_i = v_i + d\mu(v_i)dt.$$

Let $\mathrm{vol}_{\iota_\lambda}$ denote the g-volume form on $\iota_\lambda P$ and vol_ι the g_M-volume form on ιP. Then we have $(\mathrm{vol}_{\iota_\lambda})_{(t,x)} = u^1 \wedge u^2 \wedge u^3$. We compute

$$\begin{aligned} u^1 \wedge u^2 \wedge u^3 &= \big(v^1 + d\mu(v_1)dt\big) \wedge \big(v^2 + d\mu(v_2)dt\big) \wedge + \big(v^3 + d\mu(v_3)dt\big) \\ &= v^1 \wedge v^2 \wedge v^3 + d\mu(v_1)v^2 \wedge v^3 \wedge dt - d\mu(v_2)v^1 \wedge v^3 \wedge dt + d\mu(v_3)v^1 \wedge v^2 \wedge dt. \end{aligned}$$

Next, let $u : \iota P \to \mathbb{R} \times M$ be the map defined by $u(x) = (\mu(x), x)$. Then the above implies

$$u^* \left(\mathrm{vol}_{\iota_\lambda}\right) = \mathrm{vol}_\iota + *_\iota d\mu \wedge d\mu = \mathrm{vol}_\iota \left(1 + |d\mu|^2\right)$$

Where $*_\iota$ denotes the Hodge star on ιP with respect to the metric induced by g_M. Next, since $\iota_\lambda P$ is associative by assumption, and since (ρ', ω') tames (ρ, τ), there exists a constant $K > 0$ such that for all $(t, x) \in \mathbb{R} \times M$,

$$(\mathrm{vol}_{\iota_\lambda})_{(t,x)} \leq K\varphi'\big|_{T_{(t,x)}\iota_\lambda P}.$$

Note that this is still true even though we are using the metric g instead of g_X to define the volume form $\mathrm{vol}_{\iota_\lambda}$. Since $u^*(\varphi') = d\mu \wedge \omega' + \rho'$, we have

$$\begin{aligned} \int_{\iota P} \mathrm{vol}_\iota \left(1 + |d\mu|^2\right) &\leq K \int_{\iota P} d\mu \wedge \omega' + \rho' \\ \Rightarrow \mathrm{Vol}_{g_M}(\iota P) + \|d\mu\|^2_{L^2} &\leq K\langle [d\mu \wedge \omega'], \iota_*[P]\rangle + K\langle [\rho'], \iota_*[P]\rangle \\ &= K\langle [\rho'], \iota_*[P]\rangle. \end{aligned}$$

Clearly the L^2-norm of $d\mu$ is bounded if and only if the L^2-norm of $d\lambda$ is bounded.

□

3.7 Elliptic regularity

In this section, we prove some regularity results that will be used in the proofs of the main theorems. The results in this section mainly follow from standard elliptic bootstrapping methods. We will work exclusively in the 7-dimensional setting for this section. In section 3.8 we will apply these to solutions to the perturbed SL equations.

3.7.1 Setup on $\mathbb{R}^7$.

We begin by reviewing the so-called associator equation from [23] which is a nonlinear PDE whose solutions are functions whose graphs are associative with respect to the standard G_2-structure on $\mathbb{R}^7$. We then generalize this equation for non-flat G_2-structures. Let $\mathbb{H}$ denote the quaternions and $\mathbb{O}$ denote the octonions. Then identify

$$\mathbb{R}^7 \cong \mathrm{Im}\mathbb{H} \times \mathbb{H} \cong \mathrm{Im}\mathbb{O}.$$

Furthermore, let $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{e}, \mathbf{ie}, \mathbf{jk}, \mathbf{ke}$ be the standard basis for $\mathbb{O}$. In particular, any point $x \in \mathbb{H}$ has the form

$$x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}.$$

Definition 3.7.1. Suppose that x, y and z are octonions. Define the two- and three-fold *cross products* to be

$$x \times y = -\frac{1}{2}\left(\bar{x}y - \bar{y}x\right)$$

and

$$x \times y \times z = \frac{1}{2}\left(x(\bar{y}z) - z(\bar{y}x)\right).$$

There are two important operators which are ingredients in the associator equation.

Definition 3.7.2. Let U be a domain in $\mathrm{Im}\mathbb{H}$ and let $f : U \to \mathbb{H}$ be a C^1 map. Then the *Dirac operator* on f is defined to be

$$D(f) = -\frac{\partial f}{\partial x_1}\mathbf{i} - \frac{\partial f}{\partial x_2}\mathbf{j} - \frac{\partial f}{\partial x_3}\mathbf{k}.$$

The *first order Monge-Ampére operator* on f is defined to be

$$\sigma(f) = \frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_2} \times \frac{\partial f}{\partial x_3}.$$

Theorem 3.7.3 (Harvey–Lawson)**.** *Let $f : U \to \mathbb{H}$ be a C^1 map. Then the graph of f is an associative submanifold of* $\mathrm{Im}\mathbb{H} \times \mathbb{H}$ *if and only if f satisfies*

$$D(f) = \sigma(f).$$

This equation is known as the *associator equation* and is a first order, nonlinear PDE. It is well-known that C^1 minimal submanifolds are smooth, and that associative submanifolds are minimal. Therefore a corollary to this theorem is that the graph of an associative, C^1 map $f : U \to \mathbb{H}$ is smooth.

Let ψ_0 denote the standard G_2 4-form on $\mathbb{R}^7$. Let β be a 4-form on $\mathbb{R}^7$ which is small enough so that $\psi = \psi_0 + \beta$ is still stable. We now explain how to formulate a modified associator equation with respect to this new 4-form. To state this generalization we require the following constructions. Let

- g_0 be the G_2 metric on $\mathbb{R}^7$ determined by ψ_0 and

- g_ψ be the G_2 metric on $\mathbb{R}^7$ determined by ψ.

Also fix an orientation on $\mathbb{R}^7$. To each stable 4-form, we also associate an *associator* $[\cdot,\cdot,\cdot]_\psi : \Lambda^3(\mathbb{R}^7) \to \mathbb{R}^7$ by requiring that

$$g_\psi\left([v_1, v_2, v_3]_\psi, v_4\right) = \psi(v_1, v_2, v_3, v_4) \quad \text{for all} \quad v_1, v_2, v_3, v_4 \in \mathbb{R}^7.$$

Let $[\cdot,\cdot,\cdot]^0_\psi : \Lambda^3(\mathbb{R}^7) \to \mathbb{R}^7$ denote the map determined by replacing g_ψ with g_0 in the above formula. That is

$$g_0([v_1, v_2, v_3]^0_\psi, v_4) = \psi(v_1, v_2, v_3, v_4) \quad \text{for all} \quad v_1, v_2, v_3, v_4 \in \mathbb{R}^7.$$

Then define $\hat{\beta}(\cdot,\cdot,\cdot) : \Lambda^3(\mathbb{R}^7) \to \mathbb{R}^7$ by the equation

$$\begin{aligned} \psi(v_1, v_2, v_3, v_4) &= \psi_0(v_1, v_2, v_3, v_4) + \beta(v_1, v_2, v_3, v_4) \\ &= g_0\left([v_1, v_2, v_3]_{\psi_0}, v_4\right) + g_0\left(2\hat{\beta}(v_1, v_2, v_3), v_4\right) \end{aligned}$$

so that

$$[\cdot,\cdot,\cdot]^0_\psi = [\cdot,\cdot,\cdot]_{\psi_0} + 2\hat{\beta}(\cdot,\cdot,\cdot).$$

Finally, thinking of $\mathrm{Im}\mathbb{O}$ as $\mathrm{Im}\mathbb{H} \oplus \mathbb{H}$, define projection operators $\pi_{\mathbb{H}}$ and $\pi_{\mathrm{Im}\mathbb{H}}$.

$$\begin{aligned} \pi_{\mathbb{H}} : \mathrm{Im}\mathbb{O} &\to \mathbb{H} \\ \pi_{\mathrm{Im}\mathbb{H}} : \mathrm{Im}\mathbb{O} &\to \mathrm{Im}\mathbb{H}. \end{aligned}$$

Then set

$$\begin{aligned} \hat{\beta}_{\mathbb{H}} &= \pi_{\mathbb{H}} \circ \hat{\beta} \\ \hat{\beta}_{\mathrm{Im}\mathbb{H}} &= \pi_{\mathrm{Im}\mathbb{H}} \circ \hat{\beta}. \end{aligned}$$

Next, suppose that $f : U \to \mathbb{H}$ is a C^1 map. Let f^i denote $\frac{\partial f}{\partial x_i}$ for simplicity. Then the tangent space of the graph of f at each point is spanned by

$$\begin{aligned} u &= \mathbf{i} + f^1\mathbf{e} \\ v &= \mathbf{j} + f^2\mathbf{e} \\ w &= \mathbf{k} + f^3\mathbf{e}. \end{aligned}$$

Therefore we may think of $\hat{\beta}_{\mathbb{H}}$ as a differential operator on f by defining

$$\hat{\beta}_{\mathbb{H}}(f)\mathbf{e} = \hat{\beta}_{\mathbb{H}}(u, v, w).$$

Theorem 3.7.4. *As before, let U be a domain in* $\mathrm{Im}\mathbb{H}$*. Let $f : U \to \mathbb{H}$ be a C^1 map. Then the graph of f is associative with respect to $\psi = \psi_0 + \beta$ if and only if it satisfies the equation*

$$D(f) - \sigma(f) + \hat{\beta}_{\mathbb{H}}(f) = 0.$$

Proof. After importing the above definitions, this proof proceeds similarly to the proof of the original theorem in [23]. By definition, the graph of a function f is ψ-associative if and only if

$$[u, v, w]_\psi = 0$$

where u, v, w are vectors spanning the tangent space of the graph of f. In this case, since

$$g_\psi([x_1, x_2, x_3]_\psi, x_4) = \psi(x_1, x_2, x_3, x_4) = g_0([x_1, x_2, x_3]^0_\psi, x_4)$$

for all $v_1, v_2, v_3, v_4 \in \mathbb{R}^7$, the ψ-associative equation is equivalent to

$$[u, v, w]^0_\psi = [u, v, w]_{\psi_0} + 2\hat{\beta}(u, v, w) = 0.$$

Also recall that $\frac{1}{2}[x, y, z]_{\psi_0} = \mathrm{Im}\,(x \times y \times z)$. Therefore, if the graph of f is ψ-associative, we have

$$\begin{aligned} \frac{1}{2}[u, v, w]^0_\psi &= \mathrm{Im}(u \times v \times w) + \hat{\beta}(u, v, w) \\ &= \mathrm{Im}\left(\mathbf{i}\left(f^2 \times f^3\right) + \mathbf{j}\left(f^3 \times f^1\right) + \mathbf{k}\left(f^1 \times f^2\right)\right) + \hat{\beta}_{\mathrm{Im}\mathbb{H}}(f) \\ &+ (\sigma(f) - D(f))\,\mathbf{e} + \hat{\beta}_{\mathbb{H}}(f)\mathbf{e} = 0 \end{aligned}$$

by formulas in [23]. In particular,

$$\sigma(f) - D(f) + \hat{\beta}_{\mathbb{H}}(f) = 0$$

as desired. Conversely, suppose that $\sigma(f) - D(f) + \hat{\beta}_{\mathbb{H}}(f) = 0$. Then

$$\frac{1}{2}[u, v, w]^0_\psi = \mathrm{Im}\left(\mathbf{i}\left(f^2 \times f^3\right) + \mathbf{j}\left(f^3 \times f^1\right) + \mathbf{k}\left(f^1 \times f^2\right)\right) + \hat{\beta}_{\mathrm{Im}\mathbb{H}}(f).$$

But

$$g_0([u,v,w]^0_\psi, z) = g_\psi\left([u,v,w]_\psi, z\right) = 0$$

whenever $z = u, v$ or w since $[u,v,w]_\psi$ is orthogonal to u, v, and w with respect to the metric g_ψ. Since

$$u = \mathbf{i} + f^1\mathbf{e},$$

this means that

$$g_0\left([u,v,w]^0_\psi, u\right) = g_0\left([u,v,w]^0_\psi, \mathbf{i}\right) + g_0\left([u,v,w]^0_\psi, f^1\mathbf{e}\right) = 0.$$

Similarly for v and w. But since each term in equation 3.7.1 is contained in $\mathrm{Im}\mathbb{H}$ we conclude that

$$g_0\left([u,v,w]^0_\psi, \mathbf{i}\right) = g_0\left([u,v,w]^0_\psi, \mathbf{j}\right) = g_0\left([u,v,w]^0_\psi, \mathbf{k}\right) = 0$$

Therefore

$$[u,v,w]^0_\psi = 0$$

so the graph of f is ψ-associative as desired. □

3.7.2 Regularity and compactness. Next, we apply elliptic bootstrapping methods to the modified associator equation to prove a regularity theorem and a compactness theorem for embeddings with bounded second fundamental form. Both theorems will play a crucial part in the transversality results to follow. Since the G_2-structures we are interested in are not necessarily torsion-free, their corresponding associative submanifolds are not necessarily minimal. This is why we cannot rely on regularity results about minimal submanifolds.

Recall that a subset S of a Riemannian manifold M is a C^ℓ submanifold if it can be covered by C^ℓ charts of M. It is a standard result that a subset of a C^ℓ

manifold M, where $\ell \geq 1$, is a C^ℓ submanifold if and only if it is the image of a C^ℓ embedding. Furthermore, if S is the image of a C^1 embedding $\iota : P \to M$ and if S is also a C^ℓ submanifold, then there exists a C^1 diffeomorphism ϕ of P such that $\iota \circ \phi$ is a C^ℓ embedding. See Chapter 3 of [24] for more details.

Theorem 3.7.5 (regularity)**.** *Let $\ell \geq 2$ be an integer. Let (X, ψ) be a closed, oriented 7-manifold equipped with a stable C^ℓ 4-form ψ. Let $\iota : P \to X$ be an embedding of a closed 3-manifold P into X so that ιP is a C^3 associative submanifold with respect to ψ. Then ιP is a C^ℓ submanifold. In particular, if ψ is smooth, then so is ιP.*

Proof. First note that at every point $x \in \iota P$, there is a small neighborhood U of x in ιP which can be regarded as a graph over $T_x \iota P$. With this in mind, identify $T_x \iota P$ with $\mathrm{Im}\mathbb{H}$, and $N_x \iota$ with $\mathbb{H}$. Let r be the distance from the origin in $T_x X$. Let $\alpha \in (0, 1)$ be a number. It suffices to prove that a $C^{2,\alpha}$ function $f : U \to \mathbb{H}$ whose graph is associative with respect to a 4-form $\psi = \psi_0 + \beta$ on $\mathrm{Im}\mathbb{H} \oplus \mathbb{H}$, where β is $O(r)$ and $C^{\ell-1,\alpha}$, is in $C^{\ell,\alpha}$. Assume that f is such a function. Then f satisfies

$$D(f) - \sigma(f) + \hat{\beta}_{\mathbb{H}}(f) = 0$$

due to the fact that ιP is associative. Applying D to both sides of this equation yields

$$\Delta(f) - (D \circ \sigma)(f) + (D \circ \hat{\beta}_{\mathbb{H}})(f) = 0$$

where $\Delta(f)$ is the Laplacian. Let $L_1 = (D \circ \sigma)(f)$ which can be calculated explicitly

$$\begin{aligned} L_1(f) \;&=\; -\left(f^{11} \times f^2 \times f^3\right)\mathbf{i} - \left(f^1 \times f^{12} \times f^3\right)\mathbf{i} - \left(f^1 \times f^2 \times f^{13}\right)\mathbf{i} \\ &-\; \left(f^{21} \times f^2 \times f^3\right)\mathbf{j} - \left(f^1 \times f^{22} \times f^3\right)\mathbf{j} - \left(f^1 \times f^2 \times f^{23}\right)\mathbf{j} \\ &-\; \left(f^{31} \times f^2 \times f^3\right)\mathbf{k} - \left(f^1 \times f^{32} \times f^3\right)\mathbf{k} - \left(f^1 \times f^2 \times f^{33}\right)\mathbf{k}. \end{aligned}$$

We see that L_1 is a nonlinear, second order operator with smooth coefficients. However, we can associate a *linear* second order operator L_1^f to L_1 with coefficients in $C^{1,\alpha}$ *depending on the first derivatives of* f simply by declaring the parts of $L_1(f)$ that are first derivatives of f to be coefficients.

Next, we address the operator $(D \circ \hat{\beta}_{\mathbb{H}})$. It is useful to first get a better understanding of the operator $\hat{\beta}_{\mathbb{H}}$. For the moment, let $\{e_i\}$ be a basis for $\mathbb{R}^7$ and let $\{e^i\}$ be its dual basis. Then any $C^{\ell-1,\alpha}$ 4-form on $\mathbb{R}^7$ can be written as

$$\beta = \sum A_{ijkl} e^{ijkl}$$

where $A_{ijkl} : \mathbb{R}^7 \to \mathbb{R}$ are $C^{\ell-1,\alpha}$ functions. On the other hand, $\hat{\beta} : \Lambda^3(\mathbb{R}^7) \to \mathbb{R}^7$ is given by

$$\hat{\beta} = \sum A_{ijkl} e_l^{ijk}.$$

Next, let $\mathbb{R}^7 \cong \mathrm{Im}\mathbb{H} \oplus \mathbb{H}$ as usual. Let $x = (x_1, x_2, x_3)$ be coordinates on $\mathrm{Im}(\mathbb{H})$. At a point $(x, f(x))$, we have

$$\hat{\beta}(f)(x) = \sum A_i(x, f(x)) B_i(x) e_i$$

where A_i come from the coefficients of β and B_i are products of the first partial derivatives of f. We see that

$$\hat{\beta}(f) : \mathbb{R}^3 \to \mathbb{R}^7.$$

Projecting onto the last 4 coordinates gives us the map

$$\begin{array}{rcl} \hat{\beta}_{\mathbb{H}}(f) : \mathbb{R}^3 & \to & \mathbb{R}^4 \\ x & \mapsto & \displaystyle\sum_{i=4}^{7} A_i(x, f(x)) B_i(x) e_i. \end{array}$$

By the Leibniz rule, $(D \circ \hat{\beta}_{\mathbb{H}})$ is a nonlinear differential operator with first and second order parts. Let L_2 denote the second order part and L_3 denote the first order part. Both L_2 and L_3 are nonlinear with coefficients in $C^{\ell-2,\alpha}$. Similar to above, we can associate a *linear* second order operator L_2^f to L_2 whose coefficients depend on the first derivatives of f. In this case however, the fact that the equation for $\hat{\beta}_{\mathbb{H}}(f)$ involves the composition of f (which is $C^{2,\alpha}$) with the function A_i (which is $C^{\ell,\alpha}$) means that the linear, second order operator L_2^f has coefficients in C^{1,α^2}. The linear, first order operator L_3^f also has coefficients in C^{1,α^2}.

Altogether, let

$$L^f = \Delta - L_1^f + L_2^f + L_3^f.$$

We must check that L^f is elliptic. The principal symbol of L^f is a 3 by 3 symmetric matrix. Its diagonal components are

$$\begin{aligned} B_{11} &= 1 + \left(\cdot \times f^1 \times f^2\right)\mathbf{i} + b_{11} \\ B_{22} &= 1 + \left(f^1 \times \cdot \times f^3\right)\mathbf{j} + b_{22} \\ B_{33} &= 1 + \left(f^1 \times f^2 \times \cdot\right)\mathbf{k} + b_{33}. \end{aligned}$$

Its off-diagonal components are

$$\begin{aligned} B_{12} &= \left(f^1 \times \cdot \times f^3\right)\mathbf{i} + \left(\cdot \times f^2 \times f^3\right)\mathbf{j} + b_{12} \\ B_{13} &= \left(f^1 \times f^2 \times \cdot\right)\mathbf{i} + \left(\cdot \times f^2 \times f^3\right)\mathbf{k} + b_{13} \\ B_{23} &= \left(f^1 \times f^2 \times \cdot\right)\mathbf{j} + \left(f^1 \times \cdot \times f^3\right)\mathbf{k} + b_{23}. \end{aligned}$$

The terms b_{ij} come from L_2^f and depend on the coefficients of β (which are $C^{\ell-1,\alpha^2}$ and $O(r)$) and the first derivatives of f which can be made as small as we like since we can choose the neighborhood in the tangent plane of ιP over which we are graphing to be as small as we like. Thus we see that the principal symbol of L^f is a small perturbation of the identity matrix and L^f is therefore elliptic with coefficients in C^{1,α^2}.

Standard results from Schauder theory and elliptic regularity allow us to conclude that since f solves $L^f(f) = 0$, there is a number $\alpha \in (0,1)$ for which f is a $C^{3,\alpha}$ function on U. That means that all the first derivatives of f are in fact contained in $C^{2,\alpha}$ for some $\alpha \in (0,1)$. So the same argument implies that f is in $C^{4,\alpha}$ which means that L^f has coefficients in $C^{3,\alpha}$. This process can be repeated up until the first derivatives of f are contained in $C^{\ell-2,\alpha}$ and f is in $C^{\ell-1,\alpha}$. Then f is in fact in C^{ℓ}. Further bootstrapping is prevented by the regularity of the coefficients of β.

$\square$

Next we prove a compactness result for embeddings with bounded second fundamental form. This will be invoked when applying the "Taubes trick" in section 3.8.

Suppose that X is a 7-manifold and let $\iota : P \to X$ be an embedding. Fix any Riemannian metric on X. For any $x, y \in \iota P$ let $d_X(x,y)$ denote the distance between x and y in X. That is, the distance between x and y as points in X. On the other hand, let $d_{\iota P}(x,y)$ denote the distance between x and y in ιP. Let $\mathrm{II}(\iota)$ denote the second fundamental form of ιP and $\mathrm{vol}(\iota P)$ denote the volume with respect to the metric.

Theorem 3.7.6 (compactness)**.** *Let $\ell \geq 1$ be an integer, $\alpha \in (0,1)$ a number, and let $p \geq \frac{3}{1-\alpha}$. Let X be a closed 7-manifold equipped with sequence of stable 4-forms ψ_a converging to a stable 4-form ψ in the $C^{\ell,\alpha}$ topology. Suppose also that $\iota_a : P \to X$ is a sequence of $W^{3,p}$ embeddings such that for all $x, y \in P$,*

(i)

$$d_X(\iota_a(x), \iota_a(y)) \geq \frac{1}{K} d_{\iota_a P}(\iota_a(x), \iota_a(y))$$

(ii)

$$\|\mathrm{II}(\iota_a)\|_{L^p} \leq K$$

(iii)

$$\mathrm{vol}(\iota_a P) \leq K$$

for large enough a and some fixed constant K. Also assume that ι_a is ψ_a-associative for each a. Then there exists a subsequence ι_b of ι_a and a sequence of diffeomorphisms ϕ_b of P such that $\iota_b \circ \phi_b$ converges in the $C^{\ell,\alpha}$-topology to ι, an embedding also satisfying (i) and (ii).

Proof. The Sobolev embedding theorem guarantees that $\iota_a \in C^{2,\alpha}$. Thus, Theorem 3.7.5 implies that there is a sequence ϕ_a of diffeomorphisms such that $\iota_a \circ \phi_a \in C^{\ell+1,\alpha}$. In order to simplify notation, we therefore just assume that $\iota_a \in C^{\ell+1,\alpha}$. The main theorem in [5] implies that there exists an embedding $\iota : P \to X$ satisfying conditions (i) and (ii), a subsequence ι_b of ι_a, and a sequence of diffeomorphisms ϕ_b such that $\iota_b \circ \phi_b$ converges to ι in the C^2-topology. The condition (i) above guarantees that ι is also an embedding. Furthermore, ι is ψ-associative and therefore also in $C^{\ell+1}$. We just need to show that ι_a converges in the $C^{\ell,\alpha}$-topology.

Let $x \in \iota P$ and let $U \subset \iota P$ be an open set in ιP containing x small enough so that it can be represented as a graph of a $C^{\ell+1,\alpha}$ function f. Let $V = \iota^{-1}(U)$ and $U_a = \iota_a(V)$. Then, since ι_a converges to ι, each U_a can be represented as a graph of a function f_a over $T_x\iota P$. Section 6 in [5] together with the assumption (ii) guarantees that there is a uniform bound on the C^2 norm of f_a. Therefore there is a uniform bound on the $C^{0,\alpha}$ norm of the first derivatives of f_a. This in turn implies that there is a uniform bound on the $C^{0,\alpha}$ norm of the coefficients of the operator L^{f_a}. Thus, the Schauder estimates imply that there is a uniform bound on the $C^{2,\alpha}$ norm of f_a. This in turn implies that there is in fact a uniform bound on the $C^{1,\alpha}$ norm of the coefficients of L^{f_a}. This process can be repeated to obtain a uniform bound on the $C^{\ell+1,\alpha}$ norm for f_a. Therefore, after passing to a subsequence f_a converges to f in the $C^{\ell,\alpha}$ topology. Thus ι_a also converges to ι in the $C^{\ell,\alpha}$-topology as desired.

□

3.8 Transversality

In this section we prove that for a generic choice of a G_2-pair, the moduli space $\mathcal{M}$ is a collection of isolated points.

3.8.1 Another slice. The group $\mathbb{R}$ acts on $C^\infty(P)/\mathcal{G}$ by addition of a constant. Let O_λ denote the $\mathbb{R}$-orbit of $\lambda \in C^\infty(P)/\mathcal{G}$. Note that the L^2-orthogonal complement to the space of constant functions is

$$I(P) = \left\{ h \in C^\infty(P)/\mathcal{G} \mid \int_P h \ \mathrm{vol}_P = 0 \right\}.$$

The quotient space $C^\infty(P)/\sim$ where $f \sim g$ if and only if $f = g + c$ for some constant c can therefore be identified with $I(P)$.

Let S be a local slice for the action of $\mathcal{G}$ on $C^\infty(P) \times \mathcal{F}$ as in Definition 3.5.2. Let $I \times S$ denote the set of pairs (λ, ι) where $\iota \in \mathcal{F}/\mathcal{G}$ and where $\lambda \in C^\infty(P)/\mathcal{G}$ is *also* a representative of an element in $I(P)$, as above. Note that if (λ, ι) is a graphical associative, so is $(\lambda + c, \iota)$ where c is any constant. Therefore restricting to the slice I amounts to neglecting solutions up to translation in the G_2 cylinder.

Let $\mathcal{E}$ denote the bundle over $I \times S$ whose fiber at (λ, ι) is $\Omega^3(P) \times \Omega^3(P, N^*\iota)$. Define

$$L\big|_{I\times S} : I \times S \to \mathcal{E} \tag{3.9}$$

$$(\lambda, \iota) \mapsto (\iota^*\rho, \tau_N + d\lambda \wedge \rho_N)\,. \tag{3.10}$$

Definition 3.8.1. The *local moduli space of perturbed special Lagrangians* is defined to be

$$\mathcal{M}^{I\times S} = \mathcal{M}^{I\times S}(A, P; (\rho, \tau)) = L\big|_{I\times S}{}^{-1}(0).$$

Let $D_{(\lambda,\iota)}$ denote the linearization of $L\big|_{I\times S}$. We make the following observations about $D_{(\lambda,\iota)}$.

Observation 3.8.2. Note that before restricting to the slice I, the operator $D_{(\lambda,\iota)}$ always had a kernel of dimension *at least* 1 since the set $\{(c, 0)\}$ for constants c was always contained in the kernel. This set is no longer in the domain of $D_{(\lambda,\iota)}$ after restricting the section L to the slice I.

Observation 3.8.3. On the other hand, the kernel of $D_{(\lambda,\iota)}$ at least sometimes contains other elements. Suppose that (M, ρ, τ) is an actual Calabi-Yau manifold. Then $(0, \iota) \in I \times S$ where $\iota : P \to M$ is a special Lagrangian submanifold are solutions to the perturbed SL equations. In this case, elements (l, n) in the kernel of $D_{(\lambda,\iota)}$ satisfy

1. $d\iota^*(n \lrcorner \rho) = 0$

2. $d(n \lrcorner \tau)_N = 0$.

Note that the first equation is true whenever $n \lrcorner \rho$ is a closed 2-form on ιP. The second equation can be re-written using the formulas in [20] as

$$\begin{aligned} n \lrcorner \tau &= *(n^\sharp \wedge \omega) \\ &= (Jn)^\sharp \wedge \omega. \end{aligned}$$

Then, $d\left((Jn)^\sharp \wedge \omega\right) = d(Jn)^\sharp \wedge \omega + (Jn)^\sharp \wedge d\omega$. This vanishes whenever $(Jn)^\sharp$ is a closed 1-form since in the Calabi-Yau case, ω is already closed.

These considerations motivate the following definitions.

Definition 3.8.4. Let

$$\mathcal{R} = \left\{(\rho, \tau) \in \mathcal{R}_{G_2} \subset \Omega^3(M) \times \Omega^4(M) \mid d\rho = 0, \quad d\tau = 0, \quad (\omega', \rho') \text{ tames } (\rho, \tau)\right\}$$

be the *space of parameters*.

Definition 3.8.5. A pair $(\rho, \tau) \in \mathcal{R}$ is called *regular* (depending on A and P) if $D_{(\lambda,\iota)}$ is surjective for all $(\lambda, \iota) \in \mathcal{M}^{I,S}(A, P; (\rho, \tau))$. Let $\mathcal{R}_{\text{reg}}$ denote the set of all regular pairs $(\rho, \tau) \in \mathcal{R}$.

The space $\mathcal{R}$ should be compared to the space $\mathcal{J}$ of ω-tame almost complex structures in the symplectic setting. Note, that we do not require that ρ and τ form an SU(3)-structure. We require that the forms ρ and τ are closed so that the operator governing the deformation theory of the associative submanifolds in $\mathcal{R} \times M$ is self-adjoint and therefore has index zero. These conditions is also needed in order for the Lagrange multipliers problem described in section 3.4 to apply but are not needed to define the moduli space.

The theorems in this section are consequences of the implicit function and Sard-Smale theorems which only apply in the Banach space setting. We address that next.

3.8.2 Banach space setup.

Let $W^{k,p}_{I\times S}$ denote the $W^{k,p}$ completion of $I \times S$. Similarly, let $\mathcal{E}^{k,p}$ denote the $W^{k,p}$ completion of the bundle over $I \times S$ whose fiber at (λ, ι) is $\Omega^3(P) \times \Omega^3(P, N^*\iota)$.

Remark 3.8.6. Suppose that $(\lambda, \iota) \in C^\infty(P) \times \mathcal{F}$ and $S_{(\lambda,\iota)}$ is a local slice for (λ, ι). Then since each element of $S_{(\lambda,\iota)}$ corresponds to a normal vector field along $\iota_\lambda P$, the $W^{k,p}$ completion of $S_{(\lambda,\iota)}$ consists of $W^{k,p}$ normal vector fields along $\iota_\lambda P$. Therefore if I is a local slice for $\lambda \in C^\infty(P)/\mathcal{G}$, $W^{k,p}_{I\times S}$ is identified with a subset of $W^{k,p}(P, N\iota \times \mathbb{R})$.

Let

$$\mathcal{M} = \mathcal{M}\left(A, P; (\rho, \tau)\right) = \tilde{\mathcal{M}}/\sim$$

be the quotient space obtained by modding out by the action of $\mathbb{R} \times \mathcal{G}$ on $C^\infty(P) \times \mathcal{F}$ which is given by

$$(c, \phi) \cdot (\lambda, \iota) = \left((\lambda \circ \phi) + c, \iota \circ \phi\right).$$

This definition of $\mathcal{M}$ corresponds to our previous definition, because after restricting to the slice I, the projection from $\mathcal{N}$ to $\mathcal{M}$ is a bijection. Endow $\mathcal{M}$ with the quotient subspace topology coming form the C^∞ topology on $C^\infty(P) \times \mathcal{F}$.

The following is a consequence of the implicit function theorem for Banach spaces.

Theorem 3.8.7. *The moduli space $\mathcal{M}\left(A, P; (\rho, \tau)\right)$ is a set of isolated points whenever $(\rho, \tau) \in \mathcal{R}_{\mathrm{reg}}$.*

Proof. Fix a local slice $I \times S$ for a particular element $(\lambda_0, \iota_0) \in \mathcal{M}$. We first prove the result for the local moduli space $\mathcal{M}^{I\times S}$. Let $L : I \times S \to \mathcal{E}$ denote the map defined by equation 3.10, dropping the restriction to $I \times S$ from the notation. Since we want to apply the implicit function theorem, extend L to a map, also called L:

$$L : W^{k,p}_{I\times S} \to \mathcal{E}^{k-1,p}$$

where $k \geq 3$ is an integer and $p > 3$. If $(\lambda, \iota) \in L^{-1}(0)$, then the Sobolev embedding theorem implies that $(\lambda, \iota) \in C^{2,\alpha}$ so Theorem 3.7.5 implies that (λ, ι) is smooth. In this way, the local moduli space $\mathcal{M}^{I\times S}$ is identified with the subset $L^{-1}(0)$ in $W^{k,p}_{I\times S}$. Next, note that any Riemannian metric on M induces a trivialization of $\mathcal{E}^{k-1,p}$ over $W^{k,p}_{I\times S}$ via parallel transport along geodesics in M. Choose any such metric. For any $(\lambda, \iota) \in W^{k,p}_{I\times S}$, let the parallel transport maps be denoted by

$$T_0(\lambda, \iota) : \mathcal{E}^{k-1,p}_{(\lambda_0,\iota_0)} \to \mathcal{E}^{k-1,p}_{(\lambda,\iota)}$$

which is an isomorphism. Each fiber of $\mathcal{E}^{k-1,p}$ is a Banach space. Therefore the map $\bar{L}$ given by

$$\begin{aligned} \bar{L} : W^{k,p}_{I\times S} &\to \mathcal{E}^{k-1,p}_{(\lambda_0,\iota_0)} \\ \bar{L}(\lambda, \iota) &= T_0(\lambda, \iota)^{-1} \left(L(\lambda, \iota)\right) \end{aligned}$$

is a smooth map between Banach spaces. Also,

$$\bar{L}(\lambda, \iota) = 0 \quad \Longleftrightarrow \quad L(\lambda, \iota) = 0 \quad \Longleftrightarrow \quad (\lambda, \iota) \in \mathcal{M}.$$

By the definition of $\mathcal{R}_{\text{reg}}$, 0 is a regular value of L and therefore also of $\bar{L}$. So the implicit function theorem says that $\mathcal{M}^{I\times S}$ is a Banach manifold with

$$T_{(\lambda,\iota)}\mathcal{M} = D\bar{L}(\lambda, \iota).$$

Since the operator $D_{(\lambda,\iota)}$ is Fredholm with index zero whenever $(\lambda, \iota) \in \mathcal{M}$, this shows that $\mathcal{M}^{I\times S}$ is a set of points isolated in the $W^{k,p}$-topology. It remains to show that (λ, ι) is isolated with respect to the C^∞ topology. Suppose that $(\lambda, \iota) \in \mathcal{M}^{I\times S}$ is not isolated in the C^∞ topology. Then every C^∞ neighborhood of (λ, ι)

contains a second point $(\lambda', \iota') \in \mathcal{M}^{I\times S}$. This is a contradiction since every $W^{k,p}$ neighborhood of (λ, ι) contains a C^∞ neighborhood.

Since the above is true for every local slice $I \times S$, the result follows for $\mathcal{M}$.

$\square$

In the next theorem, we will want to work with $C^{\ell,\alpha}$ parameters. Define

Definition 3.8.8.

$$\mathcal{R}^{\ell,\alpha} = \left\{(\rho, \tau) \in C^{\ell,\alpha}\left(M, \Lambda^3 T^*M \oplus \Lambda^4 T^*M\right) \mid d\rho = 0, \quad d\tau = 0, \quad (\omega', \rho') \quad \text{tames} \quad (\rho, \tau)\right\}$$

where $\alpha \in (0, 1)$.

Definition 3.8.9. Define the *local universal moduli space* $\mathcal{M}^{I\times S}\left(A, P; \mathcal{R}^{\ell,\alpha}\right)$ to be the set of pairs $((\lambda, \iota), (\rho, \tau))$ such that $(\rho, \tau) \in \mathcal{R}^{\ell,\alpha}$ and (λ, ι) is a solution to the perturbed SL equations contained in the $C^{\ell,\alpha}$ completion of the local slice $I \times S$.

Next, we prove that $\mathcal{M}^{I\times S}\left(A, P; \mathcal{R}^{\ell,\alpha}\right)$ is a Banach submanifold of $W^{k,p}_{I\times S} \times \mathcal{R}^{\ell,\alpha}$. First, a Lemma.

Lemma 3.8.10. *Let $(\rho, \tau) \in \mathcal{R}^{\ell,\alpha}$ and suppose that $(\lambda, \iota) \in C^{\ell,\alpha}$ solves the perturbed SL equations with respect to (ρ, τ). Also assume that $\iota^*\rho > 0$ as usual. Then for every nonzero $(l, n) \in C^{\ell,\alpha}(P, \mathbb{R} \times N\iota)$, there exists a pair (α, β) where α is a closed 3-form on M and β is a closed 4-form on M such that*

$$\int_P \iota(n \lrcorner \beta) + \int_P d\lambda \wedge \iota^*(n \lrcorner \alpha) + \int_P l\iota^*\alpha > 0.$$

Proof. **Case 1:** ($n \neq 0$ but $l = 0$) In this case, we may choose $\alpha = 0$ and show that the first term is nonzero, which follows from exactly the same arguments used to prove proposition A.2 of [13]. That is, choose $p \in P$. Let U be a neighborhood of p and let V be a tubular neighborhood of U. Since $n \neq 0$, it must be nonzero somewhere in U. Therefore, we can let f be a function supported in V such that $df(n) \geq 0$ and $df(n) > 0$ somewhere. Let η be a 3-form on M with $\iota^*\eta\big|_U = \mathrm{vol}_P\big|_U$ and $n \lrcorner d\eta\big|_V = 0$. Then, set

$$\beta = d(f\eta) = df \wedge \eta + f d\eta.$$

We have:

$$\int_P \iota^*(n \lrcorner (df \wedge \eta + f d\eta)) = \int_P df(n)\mathrm{vol}_P > 0$$

as desired.

Case 2: ($l \neq 0$ but $n = 0$). In this case, only the last term matters. But this case is easy since we can just choose U and V small enough so that l never vanishes. Then let f be a function on M such that $\iota^*(f)\big|_U = \frac{1}{l}\big|_U$. Choose the 2-form ν on M such that $df \wedge \nu$ is vol_P when pulled back to U. Then let $\alpha = d(f\nu)$.

Case 3: ($l \neq 0$ and $n \neq 0$). This is the same as in Case 1 since we can choose $\alpha = 0$ and guarantee that the first term is positive. $\square$

Proposition 3.8.11. *Let $I \times S$ be a local slice. Let $\ell \geq 3$ and $3 \leq k \leq \ell$ be integers. Let $\alpha \in (0,1)$ be a number and $p \geq \frac{3}{1-\alpha}$. Then $\mathcal{M}^{I\times S}\left(A, P; \mathcal{R}^{\ell,\alpha}\right)$ is a separable Banach submanifold of $W^{k,p}_{I\times S} \times \mathcal{R}^{\ell,\alpha}$.*

Proof. Note that when $3 \leq k \leq \ell$, $\mathcal{M}^{I\times S}\left(A, P; \mathcal{R}^{\ell,\alpha}\right)$ can be regarded as a subset of $W^{k,p}_{I\times S} \times \mathcal{R}^{\ell,\alpha}$. For $k \geq 1$, consider the (trivial) Banach space bundle

$$\mathcal{E}^{k-1,p} \to W^{k,p}_{I\times S} \times \mathcal{R}^{\ell,\alpha}$$

whose fiber over $((\lambda,\iota),(\rho,\tau))$ is

$$\mathcal{E}^{k-1,p}_{((\lambda,\iota),(\rho,\tau))} = W^{k-1,p}\left(P, \Lambda^3(T^*P) \oplus \left(\Lambda^3(T^*P) \otimes N^*\iota\right)\right)$$

Let L be a section given by

$$\begin{aligned} L : W^{k,p}_{I\times S} \times \mathcal{R}^{\ell,\alpha} &\to \mathcal{E}^{k-1,p} \\ ((\lambda,\iota),(\rho,\tau)) &\mapsto (\iota^*\rho, \tau_N + d\lambda \wedge \rho_N)\,. \end{aligned}$$

When $k \geq 3$, Theorem 3.7.5 applies, so $\mathcal{M}^{I\times S}\left(A,P;\mathcal{R}^{\ell,\alpha}\right) = L^{-1}(0)$. In order to show that the universal moduli space is a submanifold, we need to show that the differential

$$\begin{aligned} DL\left((\lambda,\iota),(\rho,\tau)\right) : W^{k,p}\left(P,\mathbb{R}\oplus N\iota\right) \times C^{\ell,\alpha}\left(M,\Lambda^3(T^*M)\oplus\Lambda^4(T^*M)\right) \\ \to W^{k-1,p}\left(P,\Lambda^3(T^*P)\oplus\left(\Lambda^3(T^*P)\otimes N^*\iota\right)\right) \end{aligned}$$

is surjective at every point $((\lambda,\iota),(\rho,\tau)) \in L^{-1}(0)$. The differential of L can be written in two components which are denoted by

$$DL\left((\lambda,\iota),(\rho,\tau)\right) = D_{(\lambda,\iota)} + D_{(\rho,\tau)}.$$

The same arguments from section 3.5 show that $D_{(\lambda,\iota)}$ is a Fredholm operator. Therefore $DL\left((\lambda,\iota),(\rho,\tau)\right)$ has a closed image when $((\lambda,\iota),(\rho,\tau)) \in L^{-1}(0)$, so it suffices to show that its image is dense.

We first consider the case $k = 1$. Although in this case, $L^{-1}(0)$ cannot be identified with the universal moduli space since the bootstrapping arguments of section 3.7 don't apply, $D_{(\lambda,\iota)}$ is still Fredholm so we can still prove that the operator DL is surjective on $L^{-1}(0)$. Then, it will follow from elliptic regularity that DL is surjective for larger values of k.

Suppose that the image of DL at $((\lambda,\iota),(\rho,\tau)) \in L^{-1}(0)$ is *not* dense. Then there exists a nonzero

$$\eta = (\eta_1,\eta_2) \in L^q\left(P,\Lambda^3(T^*P)\oplus\left(\Lambda^3(T^*P)\otimes N^*\iota\right)\right)$$

where $\frac{1}{p}+\frac{1}{q} = 1$, such that

(i)

$$\int_P \langle \eta, D_{(\lambda,\iota)}(l,n)\rangle \iota^* \rho' = 0 \qquad\qquad \forall\, (l,n) \in W^{1,p}\left(P, \mathbb{R}\oplus N\iota\right)$$

(ii) and

$$\int_P \langle \eta, D_{(\rho,\tau)}(\alpha,\beta)\rangle \iota^* \rho' = 0 \qquad\qquad \forall\, (\alpha,\beta) \in C^{\ell,\alpha}\left(M, \Lambda^3(T^*M)\oplus \Lambda^3(T^*M)\right)$$

Now, (i) implies that $\eta \in W^{1,p}$ since $D_{(\lambda,\iota)}$ is Fredholm and self-adjoint. Furthermore, $D^*\eta = 0$. In particular, the Sobolev embedding theorem implies that η is continuous.

On the other hand, for any $(\alpha,\beta) \in T_{(\rho,\tau)}\mathscr{R}^{\ell,\alpha}$,

$$D_{(\rho,\tau)}(\alpha,\beta) = \left(\iota^*\alpha, \beta_N + d\lambda \wedge \alpha_N\right).$$

Therefore, letting l the metric dual of η_1 on P and letting n be the normal vector field corresponding to η_2, we have

$$\int_P \langle (\eta_1,\eta_2), D_{(\rho,\tau)}(\alpha,\beta)\rangle \iota^*\rho' = \int_P \iota^*(n\lrcorner\beta) + \int_P d\lambda\wedge\iota^*(n\lrcorner\alpha) + \int_P l\iota^*\alpha = 0.$$

by (ii). But Lemma 3.8.10 shows that this is a contradiction.

The fact that $DL\left((\lambda,\iota),(\rho,\tau)\right)$ is surjective whenever $\left((\lambda,\iota),(\rho,\tau)\right) \in L^{-1}(0)$ for $k \geq 2$ follows from elliptic regularity. As mentioned before, as long as $3 \leq k \leq \ell$, the universal moduli space is equal to $L^{-1}(0)$. Therefore the result follows from the implicit function theorem. Since $W^{k,p}_{I\times S} \times \mathscr{R}^{\ell,\alpha}$ is separable, so is $\mathcal{M}^{I\times S}\left(A, P; \mathscr{R}^{\ell,\alpha}\right)$.

□

Lastly, we prove that the set of parameters for which the moduli space $\mathcal{M}\left(A, P; (\rho,\tau)\right)$ is a zero dimensional manifold is "generic". More precisely,

Theorem 3.8.12. *The set $\mathscr{R}_{\mathrm{reg}}$ is a residual subset of $\mathscr{R}$.*

Proof. Let $\mathscr{R}^{\ell,\alpha}_{\mathrm{reg}}$ be the subset of $\mathscr{R}^{\ell,\alpha}$ for which the associated operator $D_{(\lambda,\iota)}$ from Proposition 3.8.11 is surjective whenever (λ,ι) solves the perturbed SL equations. First, we show that $\mathscr{R}^{\ell,\alpha}_{\mathrm{reg}}$ is residual is $\mathscr{R}^{\ell,\alpha}$. For that purpose, consider the projection

$$\begin{aligned} \pi : \mathcal{M}^{I\times S}\left(A, P; \mathscr{R}^{\ell,\alpha}\right) &\rightarrow \mathscr{R}^{\ell,\alpha} \\ ((\lambda,\iota),(\rho,\tau)) &\mapsto (\rho,\tau). \end{aligned}$$

By Proposition 3.8.11, this is a map between separable Banach manifolds. Furthermore,

$$\begin{aligned} d\pi\left((\lambda,\iota),(\rho,\tau)\right) : T_{((\lambda,\iota),(\rho,\tau))}\mathcal{M}^{I\times S}\left(A, P; \mathscr{R}^{\ell,\alpha}\right) &\rightarrow T_{(\rho,\tau)}\mathscr{R}^{\ell,\alpha} \\ ((l,n),(\alpha,\beta))\,. &\mapsto (\alpha,\beta) \end{aligned}$$

Note that if $((l,n),(\alpha,\beta))$ is in the tangent space of the local universal moduli space, then

$$D_{(\lambda,\iota)}(l,n) + D_{(\rho,\tau)}(\alpha,\beta) = 0.$$

Therefore both the kernel and cokernel of $d\pi\left((\lambda,\iota),(\rho,\tau)\right)$ are the same as that of $D_{(\lambda,\iota)}$. So $d\pi\left((\lambda,\iota),(\rho,\tau)\right)$ is surjective if and only if $D_{(\lambda,\iota)}$ is surjective. Therefore, $\mathscr{R}^{\ell,\alpha}_{\mathrm{reg}}$ is precisely the set of regular values of π. The above holds for any local slice $I \times S$. Thus, for large enough ℓ, the Sard-Smale theorem implies that $\mathscr{R}^{\ell,\alpha}_{\mathrm{reg}}$ is residual.

In order to extend the argument to smooth parameters, we must use the so-called Taubes trick, similar to what is done in order to prove that the set of regular ω-tame almost complex structures is residual in the symplectic case. Let $K > 0$ be a constant. Consider the set

$$\mathscr{R}_{\mathrm{reg},K} \subset \mathscr{R}$$

of all smooth, stable, (ρ', ω')-tame, G_2 pairs (ρ,τ) such that $D_{(\lambda,\iota)}$ is surjective for every graphical associative (λ,ι) which satisfies the following two conditions.

(i) For any two points $q, q' \in \iota P$ let $d_M(q, q')$ denote the distance between them with respect to the metric induced by (ρ', ω') on M. Similarly, let $d_{\iota P}(q, q')$ denote the distance between them with respect to the induced metric on ιP. Then the first condition for (λ, ι) to be contained in $\mathscr{R}_{\mathrm{reg},K}$ is

$$d_M(\iota(x), \iota(y)) \geq \frac{1}{K} d_{\iota P}(\iota(x), \iota(y))$$

(ii) Next, let $p > 0$ be a number. Let $\mathrm{II}(\iota_\lambda)$ denote the second fundamental form of the embedding $\iota_\lambda : P \to \mathbb{R} \times M$ as in Definition 3.5.9. The second condition is that

$$\|\mathrm{II}(\iota_\lambda)\|_{L^p} \leq K.$$

Note that the conditions (i) and (ii) are both $\mathscr{G}$-invariant. Also note that since (ρ, τ) is (ρ', ω')-tame, the volume of every graphical associative is bounded by the same constant. Since we are no longer directly relying on Banach spaces, we no longer need to restrict ourselves to a slice of the action of $\mathscr{G}$. Also, every graphical associative (λ, ι) satisfies (i) and (ii) for some constant $K > 0$. Therefore

$$\mathscr{R}_{\mathrm{reg}} = \bigcap_{K>0} \mathscr{R}_{\mathrm{reg},K}.$$

Therefore we must prove that each $\mathscr{R}_{\mathrm{reg},K}$ is open and dense in the C^∞-topology. First we prove that each is open by proving that its complement is closed. Assume that a sequence (ρ_a, τ_a) converging to (ρ, τ) in the C^∞ topology is contained in the complement of $\mathscr{R}_{\mathrm{reg},K}$. Then for each a there exists a (ρ_a, τ_a) graphical associative (λ_a, ι_a) satisfying (i) and (ii) such that $D_{(\lambda_a, \iota_a)}$ is not surjective. Each graphical associative (λ, ι) has an associated associative embedding $(\iota_a)_{\lambda_a}$ and the conditions (i) and (ii) ensure that Theorem 3.7.6 applies.

Thus, there exists a subsequence (λ_b, ι_b) of (λ_a, ι_a) and a sequence of diffeomorphisms ϕ_b such that

$$(\iota_b)_{\lambda_b} \circ \phi_b \to \iota_\lambda$$

in the C^∞ topology. The condition (i) guarantees that not only is ι an embedding, but ι_λ is also *graphical* since we used the distance in M instead of the distance in $\mathbb{R} \times M$. The limit also satisfies both (i) and (ii). Furthermore $D_{(\lambda,\iota)}$ is also *not* surjective. Therefore (ρ, τ) is not contained in $\mathscr{R}_{\text{reg},K}$. Therefore $\mathscr{R}_{\text{reg},K}$ is closed for any $K > 0$.

Finally, we prove that $\mathscr{R}_{\text{reg},K}$ is dense in $\mathscr{R}$ with respect to the C^∞ topology. At this point, the argument is almost identical to the same argument in the symplectic case. See for example section 3.2 in [35]. We include it here for completion. First, let $\mathscr{R}^{\ell,\alpha}_{\text{reg},K}$ be the obvious $C^{\ell,\alpha}$-version of $\mathscr{R}_{\text{reg},K}$. Note that

$$\mathscr{R}_{\text{reg},K} = \mathscr{R}^{\ell,\alpha} \cap \mathscr{R}.$$

Then Theorem 3.7.6 still applies, so $\mathscr{R}^{\ell,\alpha}_{\text{reg},K}$ is open in $\mathscr{R}^{\ell,\alpha}$ with respect to the $C^{\ell,\alpha}$ topology. Let $(\rho_0, \tau_0) \in \mathscr{R}$. Since $\mathscr{R}^{\ell,\alpha}_{\text{reg}}$ is dense in $\mathscr{R}^{\ell,\alpha}$ by the Sard-Smale theorem, then there exists a sequence $(\rho_\ell, \tau_\ell) \in \mathscr{R}^{\ell,\alpha}_{\text{reg}}$ (here the index ℓ depends on (ℓ, α)) such that for all $\ell \geq \ell_0$ for large enough ℓ_0,

$$\|(\rho_0, \tau_0) - (\rho_\ell, \tau_\ell)\|_{C^{\ell,\alpha}} \leq 2^{-\ell}.$$

Since $(\rho_\ell, \tau_\ell) \in \mathscr{R}^{\ell,\alpha}_{\text{reg},K}$ and since $\mathscr{R}^{\ell,\alpha}_{\text{reg},K}$ is open in the $C^{\ell,\alpha}$-topology, there exists $\epsilon_\ell > 0$ depending on (ℓ, α) such that for every $(\rho, \tau) \in \mathscr{R}^{\ell,\alpha}$,

$$\|(\rho, \tau) - (\rho_\ell, \tau_\ell)\|_{C^{\ell,\alpha}} < \epsilon_\ell \qquad \Rightarrow \qquad (\rho, \tau) \in \mathscr{R}^{\ell,\alpha}_{\text{reg},K}.$$

Choose $(\rho'_\ell, \tau'_\ell) \in \mathscr{R}$ to be any smooth element such that

$$\|(\rho, \tau) - (\rho_\ell, \tau_\ell)\|_{C^{\ell,\alpha}} < \min\left\{\epsilon_\ell, 2^{-\ell}\right\}.$$

Then

$$(\rho'_\ell, \tau'_\ell) \in \mathscr{R}^{\ell,\alpha}_{\mathrm{reg},K} \cap \mathscr{R} = \mathscr{R}_{\mathrm{reg},K}.$$

Therefore the sequence (ρ'_ℓ, τ'_ℓ) converges to (ρ_0, τ_0) in the C^∞ topology. Therefore we have shown that $\mathscr{R}_{\mathrm{reg}}$ is the intersection of a countable number of open, dense sets. Therefore it is residual, as desired.

$\square$

This is a sample citation: [15].

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