On the Map Induced on Hochschild Homology of Matrix Factorization Categories by the Inclusion of a Divisor

by

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DISSERTATION ABSTRACT

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Given a smooth variety X over $\mathbb{C}$, a smooth divisor $i : Y \hookrightarrow X$ and a global function W on X which vanishes on Y and on its critical locus we compute the map induced on Hochschild homology by the pushforward functor $i_* : D^b(Y) \to D^{abs}(MF(X, W))$ in terms of the Hochschild-Kostant-Rosenberg isomorphisms. We also begin working on generalizations of our formula in two directions: removing the assumption that Y is codimension one, and removing the assumption that W vanishes on Y. We give a conjecture for the most general case but only verify it in special cases.

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

For a smooth and projective complex variety X, the classical Hirzebruch-Riemann-Roch computes the Euler characteristic of a vector bundle on X in terms of its Chern character with values in singular cohomology. The Grothendieck-Riemann-Roch theorem, a generalization of the classical Hirzenbruch-Riemann-Roch, explains the interplay between Chern characters and derived push forward. The statement says that Chern characters *almost* intertwine derived pushforward and the induced map in singular cohomology; one has to multiply by Todd-classes to get the equality.

If one wishes to stay in the algebraic world there are cohomology theories one can use to replace singular cohomology. One such theory is that of Hochschild homology. It is originally a homology theory for associative algebras which was generalized to quasi projective varieties by Swan in [Swa96]. Later, Căldăraru used derived categories and Serre duality to develop the theory of Hochschild homology for smooth projective complex varieties further (see [Căl] and [Căl05]). He defined Chern characters with values in Hochschild homology, a perfect pairing called the Mukai pairing and explained how a map of smooth projective varieties $f : X \to X'$ induces a pair of maps

$$f_* : HH_\bullet(X) \to HH_\bullet(X'),\ \ f^* : HH_\bullet(X') \to HH_\bullet(X)$$

which are adjoint with respect to the Mukai pairing. To compare this structure with singular cohomology we have the Hochschild-Kostant-Rosenberg isomorphism (HKR) which states that for a smooth (not necessarily projective) complex variety X we have

$$HH_n(X) \cong \bigoplus_{p-q=n} H^p(X, \Omega^q_X).$$

As a consequence of the above, Hochschild homology is the same as Hodge cohomology up to grading convention. This on the other hand is closely related to singular cohomology (only the de Rham differential is missing). The similarities between the structure set up by Căldăraru and the corresponding structure in singular cohomology are remarkable. For example the Chern characters with values in Hochschild homology agree with the Chern characters in singular cohomology. The induced maps and the Mukai pairing *almost* match up with the corresponding structure in singular cohomology but the discrepancy has the satisfying effect that in Hochschild homology Chern character commute with push forwards on the nose and there is no need for Todd-classes in the GRR theorem for Hochschild homology.

Another approach to Hochschild homology is via dg categories. By replacing a smooth projective variety by its dg category of perfect complexes this approach recovers Căldăraru's definition of Hochschild homology. The advantages of this approach is that the theory can be extended to certain non-commutative spaces such as dg categories of matrix factorizations. It is natural to ask how much of the extra structure introduced by Căldăraru carry over to the more general setting of dg categories. Chern characters can readily be defined for arbitrary dg categories; for any object in a dg category $c \in \mathsf{Ob}(\mathscr{C})$, the endomorphism dg algebra $\mathscr{C}(c, c)$ sits naturally as a subcomplex of the standard complex $C_\bullet(\mathscr{C})$ which we use to compute Hochschild homology. The Chern character $ch(c) := [\mathrm{id}_c]$ is by definition the class of the identity morphism at c. The functorial properties in the setting of smooth projective varieties are captured by the fact that Hochschild homology for dg categories are functorial with respect to dg functors (and more generally pseudo functors). The formal nature of these definitions has the upshot that the corresponding HRR theorem for dg categories becomes an immediate consequence of the definitions. The Mukai pairing can only be defined for dg categories which are smooth and proper, but in this case the definition again is formal in nature and much of its interplay with Chern characters and dg functors is just formal consequences of the definitions. Finally, while there is no general HKR type theorem for dg categories there is one for the Hochschild homology $HH_\bullet(X, W)$ of the categories of matrix factorizations $MF(X, W)$. It states that when X is a non-singular variety and W vanishes on its critical locus we have

$$HH_\bullet(X, W) \cong R\Gamma(X, (\Omega^\bullet_X, dW\wedge)) \tag{1.1}$$

where $(\Omega^\bullet_X, -dW \wedge -)$ is the complex of sheaves whose differential is taking wedge product with $-dW$. This important result, and its preceding affine versions, spawned a series of papers whose goals were to give explicit formulas for the formal structures in Hochschild homology of dg categories in terms of this isomorphism. Chern character formulas appear in [PV12], [KP22], [CKK], [Pla] and formulas for the Mukai pairing appeared in [Kim] and [CKS]. However, formulas for the maps $f_* : HH_\bullet(Y, f^*W) \to HH_\bullet(X, W)$ induced by morphisms of pairs $f : (Y, f^*W) \to$

(X, W) are still very much mysterious. In fact, even in the setting of smooth projective varieties, formulas for $f_* : HH_\bullet(Y) \to HH_\bullet(X)$ induced by $f : Y \to X$ in terms of the classical HKR isomorphism have not been written down explicitly. The goal of this thesis is to begin the study of such formulas. The simplest case is the following. Suppose Y is a smooth divisor of a smooth complex variety X and denote by $i : Y \to X$ the inclusion. Suppose $W \in H^0(X, \mathcal{O}_X)$ is a regular function whose critical locus is contained in its vanishing locus and $W|_Y = 0$. Then we have a push forward functor $i_* : \mathrm{perf}(Y) \to MF(X, W)$ which induces a map $i_* : HH_\bullet(Y) \to HH_\bullet(X, W)$. In section 3.6 we give an explicit description of this map in terms of the HKR isomorphisms. It is related to the connecting homomorphism δ from the following short exact sequence of complexes of sheaves on X

$$0 \to (\Omega_X^\bullet, -dW \wedge (-)) \to (\Omega_X^\bullet(\log Y), -dW \wedge (-)) \to (\Omega_Y^{\bullet-1}, 0) \to 0. \tag{1.2}$$

Our final formula is summarized in the following theorem.

Theorem 1.1. *Let $i : Y \hookrightarrow X$ be a smooth divisor of a smooth complex variety X. Let $W \in \mathcal{O}_X(X)$ be a global function such that $W|_Y = 0$ and W vanishes on its critical locus. Let δ be the connecting homomorphism from the short exact sequence (1.2). Then there is a commutative diagram in $D_{Z/2}(\mathbb{C})$*

$$\begin{tikzcd}
HH_\bullet(Y) \arrow[rr, "HH_\bullet(i_*)"] \arrow[d, "\sim"] & & HH_\bullet(X, W) \arrow[d, "\sim"] \\
R\Gamma(X, \Omega_Y^\bullet) \arrow[r, "Td_{Y/X}^{-1}\wedge"] & R\Gamma(X, \Omega_Y^\bullet) \arrow[r, "R\Gamma(\delta)"] & R\Gamma(X, (\Omega_X^\bullet, -dW \wedge -)).
\end{tikzcd}$$

Again it is interesting to compare this to the classical setting of singular cohomology. There, there is an analogous short exact sequence involving holomorphic forms on X, on Y and holomorphic forms on X with logarithmic poles along Y and the map $i_* : H^\bullet_{Sing}(Y; \mathbb{C}) \to H^\bullet_{Sing}(X; \mathbb{C})$ is given precisely by the connecting map δ, without the Todd-class factor.

It is natural to ask what happens if we remove the assumption that $W|_Y = 0$. In this situation we get a map $HH_\bullet(Y, W|_Y) \to HH_\bullet(X, W)$. Our guess is that in terms of the HKR isomorphisms this map is again given by the same formula as above. However for technical reasons we can only prove this an additional technical assumption on the inclusion of pairs $(Y, W|_Y) \to (X, W)$. We will refer to this extra assumption as $(Y, W|_Y)$ having the local resolution property inside (X, W)

(see section 4.4). This assumption is met for example when $Sing(W|_Y)$ consists of finitely many points. We obtain a similar formula as in the case, $W|_Y = 0$; this is formulated more precisely in theorem 4.6.

The other direction in which one might wish to generalize our main result is to remove the assumption that $codim(Y, X) = 1$. Here the same formula doesn't make sense anymore because there is no natural analogue of $\Omega_X(\log Y)$ when Y is of higher codimension > 1. We describe a different map $HH_\bullet(Y) \to HH_\bullet(X, W)$ which agrees with the original one when $codim(Y, X) = 1$ and we conjecture that this is the correct formula in higher codimension. As evidence for the conjecture we verify it in the affine case.

1.1 Summary

In chapter 2 we discuss preliminray results. In section 2.1-2.3 we go over the basic definitions and constructions involving dg and cdg categories and their Hochschild homology. Most of this material is standard and we include it mainly to set up all the notation and slightly adjust some constructions to fit our purposes better. For example in section 2.1 we describe how Drinfeld's construction of dg quotients can be altered to get a quasi equivalent quotient where the "formal" null-homotopies commute with morphisms between acyclic objects. In section 2.4 we introduce a notion of "lax" morphisms between presheaves of cdg categories and explain how they induce maps between certain Cech-Hochschild complexes associated to them. In a series of technical lemmas we then describe some useful properties of these induced maps (unfortunately functoriality is not one of them). In sections 2.5-2.5 we recall the notion of matrix factorizations on varieties and explain in detail how to obtain the isomorphism (1.1). Again, this is not new but we include a lot of details since the computations later on heavily relies on a detailed understanding of this isomorphism. In sections 2.5 and 2.6 we adress some technincal differences that occur in the $\mathbb{Z}/2$-graded setting.

In chapter 3 we get to the main problem. In section 3.1 we describe how the data of a smooth divisor Y of a smooth complex variety X and a regular function $W \in \mathcal{O}_X(X)$ which vanishes on Y induces a map $HH_\bullet(Y) \to HH_\bullet(X, W)$. In sections 3.2 and 3.3 we explain how the dg category of perfect complexes on the divisor Y can be resolved with a Morita equivalent category whose objects consist of vector bundles on X with extra structure and we define an "HKR type" isomorphism for this new category which is compatible with the classical HKR isomorphism for the category of perfect complexes on Y. In section 3.4 we recall the definition of the todd class of a line bundle. In section 3.5 we construct a trace map that allows us to pass from the Hochschild homology of a certain cdg category (or more generally a presheaf of such) to the Hochschild homology of a cdg algebra (or a sheaf of such) which generates it in a precise sense. Finally in section 3.6 we prove our main theorem.

In chapter 4 we redo a lot of what was done in chapter 3 with some important alterations to make it work when $W|_Y \neq 0$. One technical difference is that we use Drinfeld's dg quotients instead of Chech models for our dg categories. Another difference occurs in section 4.4 where we want to resolve the category of matrix factorizations on Y by a Morita equivalent category whose objects are vector bundles on X with extra structure. Because our understanding for the category of matrix factorizations on Y is not as good as our understanding of the dg category of perfect complexes on Y we need to assume that this "locally free" category has enough objects. We give some examples when this is the case.

In chapter 5 we formulate a conjecture (see conjecture 5.5) for what happens in the most general case, where $codim(Y, X)$ is not restricted to 1 and $W|_Y$ is not necessarily zero. We verify that our conjecture is correct in the case when X is affine and Y is complete intersection and $W|_Y = 0$.

1.2 Conventions

We work over the field of complex numbers $\mathbb{C}$ although much of the preliminary set up work in greater generality.

We work mainly with cochain complexes so the differential is of degree +1. By a complex (of modules or sheaves etc.) we will always mean a cochain complex.

We make use of the Koszul sign convention.

In addition, when X is a scheme with an open cover $\mathscr{U}$ and $(\underline{K}, d_K)$ is a presheaf of complexes on X we make $\check{C}^\bullet(\mathscr{U}, \underline{K})$ into a complex by taking total direct sum complex and altering the sign of d_K. More precisely, we set

$$\check{C}(\mathscr{U}, \underline{K}) := (Tot^{\oplus}(\check{C}^\bullet(\mathscr{U}, \underline{K}^\bullet)), d_{\check{C}} + \bar{d}_K)$$

where $\bar{d}_K(\alpha) = (-1)^{deg_{\check{C}}(\alpha)} d_K(\alpha)$. If $\underline{K}$ was a sheaf of dg algebras to begin with we make $\check{C}(\mathscr{U}, \underline{K})$ into a dg algebra using the Alexander-Whitney product which is defined as follows. For $\alpha \in \check{C}^p(\mathscr{U}, \underline{K}^q)$ and $\beta \in \check{C}^r(\mathscr{U}, \underline{K}^s)$ we set

$$\alpha \cdot \beta_{i_0 \cdots i_{p+s}} := (-1)^{qr} \alpha_{i_0 \cdots i_p}|_{U_{i_0 \cdots i_{p+q}}} \cdot \beta_{i_p \cdots i_{p+q}}|_{U_{i_0 \cdots i_{p+q}}}$$

Our convention on matrices is that they act on the left. That is, when the source and target of a morphism in some category have some given finite direct sum decompositions then we may describe that morphism as a matrix whose columns are indexed by the summands of the source and the rows are indexed by the summands of the target.

CHAPTER 2

PRELIMINARIES

In this section we recall the notion of (curved) dg categories, their modules and their Hochschild homology. We refer to chapter IV of [GM03] for the definition of a triangulated category, triangulated subcategory, thick triangulated subcategory and Verdier quotients.

2.1 Dg categories

Dg categories are one of many ways to enhance triangulated categories. The idea of using dg categories as enhencements go back to work by Bondal and Kapranov (see [BK91]). Here we recall some basic definitions and well known facts regarding dg categories. We follow the notation and conventions from [PP12].

Dg categories and dg functors

Let Γ be one of the abelian groups $\mathbb{Z}$ or $\mathbb{Z}/2$. A Γ-graded *dg category* $\mathscr{C}$ is a category enriched in Γ-graded chain complexes. Here are some basic examples which we will encounter a lot. More interesting examples will be treated later.

Example 2.1. We denote by $\mathrm{com}(k)$ the category whose objects are Γ-graded chain complexes over k and whose homomorphism complexes are defined as

$$\mathrm{Hom}_{\mathrm{com}(k)}(V,W)^i = \prod_{j\in\Gamma} \mathrm{Hom}_k(V^j, W^{j+i})$$

and the differentials are given by

$$\partial(f) = d_W \circ f - (-1)^{|f|} f \circ d_V.$$

Example 2.2. A Γ-graded dg algebra can be thought of as a Γ-graded dg category with one object.

We will suppress the grading group Γ now and assume implicitly that our dg categories are Γ-graded.

Given a dg category $\mathscr{C}$ we can associate to it three new categories $Z^0(\mathscr{C})$, $H^0(\mathscr{C})$ and $H^\bullet(\mathscr{C})$ where the last one is a graded category. The objects of each of these are the same as the objects of $\mathscr{C}$. The homomorphism spaces are obtained from those in $\mathscr{C}$ by taking degree zero cycles, taking zero'th homology and taking the total homology respectively.

Let $\mathscr{C}$ and $\mathscr{C}'$ be dg categories. A *dg functor* $F : \mathscr{C} \to \mathscr{C}'$ is a functor between categories enriched in chain complexes. A dg functor $F : \mathscr{C} \to \mathscr{C}'$ induces functors $Z^0(F) : Z^0(\mathscr{C}) \to Z^0(\mathscr{C}')$, $H^0(F) : H^0(\mathscr{C}) \to H^0(\mathscr{C}'))$ and $H^\bullet(F) : H^\bullet(\mathscr{C}) \to H^\bullet(\mathscr{C}'))$. We say that F is a *quasi-equivalence* if $H^\bullet(F)$ is an equivalence.

The collection of dg functors from $\mathscr{C}$ to $\mathscr{C}'$ form the objects of a new dg category which we will denote $\mathsf{Func}_{dg}(\mathscr{C}, \mathscr{C}')$. Given two dg functors $F, G : \mathscr{C} \to \mathscr{C}'$, a *pre-natural transformation* $\alpha : F \to G$ of degree i consists of a degree i morphism $\alpha_x \in \mathscr{C}'(F(x), G(x))^i$ for every $x \in \mathsf{Ob}(\mathscr{C})$ satisfying the usual naturality squares. The homomorphism complex $\mathrm{Hom}_{\mathsf{Func}_{dg}}(F, G)$ is the graded vector space whose degree i part consists of the pre-natural transformations of degree i. The differential is given pointwise: $\partial(\alpha)_x = \partial(\alpha_x)$. The *natural transformations* are the degree zero cycles in $\mathrm{Hom}_{\mathsf{Func}_{dg}}(F, G)$.

Finally we observe that if $\mathscr{C}$ is a dg category, then there is a dg category $\mathscr{C}^{op}$ with $\mathsf{Ob}(\mathscr{C}^{op}) = \mathsf{Ob}(\mathscr{C})$ and $\mathscr{C}^{op}(c, c') = \mathscr{C}(c', c)$. Composition is defined by $f \circ^{op} g := (-1)^{|f||g|} g \circ f$.

Dg modules

We let $mod^{dg} - \mathscr{C}$ be the dg category $\mathsf{Func}_{dg}(\mathscr{C}^{op}, \mathrm{com}(k))$ and we refer to its objects as *right $\mathscr{C}$-modules*. Left modules are defined as the objects of $\mathsf{Func}_{dg}(\mathscr{C}, \mathrm{com}(k))$. There is a dg functor $Y : \mathscr{C} \to mod^{dg} - \mathscr{C}$ which we refer to as the dg-Yoneda embedding. The dg version of the usual Yoneda lemma says that there is an isomorphism of chain complexes

$$\mathrm{Hom}_{\mathsf{Func}_{dg}}(Y(x), F) \cong F(x).$$

For any dg category $\mathscr{C}$ the category $H^0(mod^{dg} - \mathscr{C})$ can be given the structure of a triangulated category. Shifts are defined pointwise: given a right module M we define $M[1](x) := M(x)[1]$. Cones of morphisms are also constructed pointwise: if $\alpha : M \to M'$ is a natural transformation we set $Cone(\alpha)(x) := Cone(\alpha_x)$. The module $Cone(\alpha)$ fits naturally into a diagram in $Z^0(\mathscr{C})$

$$M' \to Cone(\alpha) \to M[1] \to M'[1].$$

The distinguished triangles in $H^0(mod^{dg} - \mathscr{C})$ are those which are isomorphic to a sequence of the above form.

A right $\mathscr{C}$-module M is called acyclic if it is acyclic point wise. The full sub-dg category of acyclic modules is denoted by $acyc - \mathscr{C}$. We then have that $H^0(acyc - \mathscr{C})$ is a full thick triangulated sub category of $H^0(mod^{dg} - \mathscr{C})$ The *derived category* of $\mathscr{C}$, which we denote by $D(\mathscr{C})$ is the Verdier quotient $H^0(mod^{dg} - \mathscr{C})/H^0(acyc - \mathscr{C})$.

We have a canonical functor $H^0(\mathscr{C}) \to D(\mathscr{C})$. Let perf($\mathscr{C}$) be the smallest thick triangulated sub category of $D(\mathscr{C})$ which contains the essential image of $H^0(\mathscr{C}) \to D(\mathscr{C})$. Modules in perf($\mathscr{C}$) are called *perfect modules*.

The category $Z^0(mod^{dg} - \mathscr{C})$ can be equipped with an exact structure where the admissible short exact sequences are sequences

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N$$

such that for each $c \in \mathsf{Ob}(\mathscr{C})$ we have that both

$$L(c) \xrightarrow{\alpha_c} M(c) \xrightarrow{\beta_c} N(c)$$

and

$$H^\bullet(L(c)) \xrightarrow{H^\bullet(\alpha_c)} H^\bullet(M(c)) \xrightarrow{H^\bullet(\beta_c)} H^\bullet(N(c))$$

are short exact sequences in each degree.

Proposition 2.3. *If V is any chain complex of vector spaces, and $c \in \mathsf{Ob}(\mathscr{C})$ then the functor $V \otimes h_c$ is projective in the exact structure defined above.*

Proof. Consider a diagram

$$\begin{tikzcd} & X \arrow[d, two heads, "\pi"] \\ V \otimes h_c \arrow[r, "\varphi"] & M. \end{tikzcd}$$

The map φ corresponds to a degree 0 cycle

$$\hat{\varphi} \in \mathrm{Hom}_{Z^0(mod - \mathscr{C})}(h_c, \mathrm{Hom}^\bullet_k(V, M(-)) = \mathrm{Hom}_{Ch(k)}(V, M(c))$$

where $\mathrm{Hom}_{Ch(k)}(A, B)$ denotes chain maps between complexes of vector spaces. Because $X(c) \to M(c)$ and $H(X(c)) \to H(M(c))$ are both surjective the induced map $\mathrm{Hom}_{Ch(k)}(V, X(c)) \to \mathrm{Hom}_{Ch(k)}(V, M(c))$ is also surjective so we find a lift of $\hat{\varphi}$, call it $\hat{\psi}$. It corresponds to a map $\psi : V \otimes h_c \to X$ which makes the triangle above commute. $\square$

As a consequence, if $\mathscr{C}$ is small, then the exact category $Z^0(mod^{dg} - \mathscr{C})$ has enough projective objects. Indeed, the action map

$$\bigoplus_{c \in \mathsf{Ob}(\mathscr{C})} M(c) \otimes h_c \to M$$

is an admissible epimorphism because for every $c' \in \mathsf{Ob}(\mathscr{C})$ it is a split surjection with the splitting being the composite

$$M(c') \to M(c') \otimes h_{c'}(c') \hookrightarrow \bigoplus_{c \in \mathsf{Ob}(\mathscr{C})} M(c) \otimes h_c(c')$$

where the first map is defined by $m \mapsto m \otimes \mathrm{id}_{c'}$. Note however that this splitting is not natural in c'.

Bimodules and tensor products

If $\mathscr{C}$ and $\mathscr{C}'$ are two dg categories we define their tensor product $\mathscr{C} \otimes \mathscr{C}'$ to be the dg category whose objects are pairs (c, c') with $c \in \mathsf{Ob}(\mathscr{C})$ and $c' \in \mathsf{Ob}(\mathscr{C}')$. The homomorphism complexes are defined as

$$\mathrm{Hom}_{\mathscr{C} \otimes \mathscr{C}'}((c_1, c_1'), (c_2, c_2')) := \mathrm{Hom}_{\mathscr{C}}(c_1, c_2) \otimes \mathrm{Hom}_{\mathscr{C}'}(c_1', c_2').$$

The dg category of $\mathscr{C} - \mathscr{C}'$ *bimodules* is by definition the dg category $\mathscr{C} \otimes \mathscr{C}'^{op} - mod^{dg}$ and its objects will be referred to as $\mathscr{C} - \mathscr{C}'$-*bimodules*.

Now suppose that $\mathscr{C}, \mathscr{C}'$ and $\mathscr{C}''$ are small dg categories. If M is a $\mathscr{C} - \mathscr{C}'$-bimodule and N is a $\mathscr{C}' - \mathscr{C}''$-bimodule we define their *tensor product* $M \otimes_{\mathscr{C}'} N$ to be the $\mathscr{C} - \mathscr{C}''$ bimodule which assigns to a pair (c, c'') the chain complex which is the cokernel of the following map:

$$\begin{aligned} \bigoplus_{c' \in \mathsf{Ob}(\mathscr{C}')} M(c, c') \otimes \mathscr{C}'(c', c') \otimes N(c', c'') \to & \bigoplus_{c' \in \mathsf{Ob}(\mathscr{C}')} M(c, c') \otimes N(c', c'') \\ m \otimes f \otimes n \mapsto & mf \otimes n - m \otimes fn \end{aligned}$$

Note that in the special case that $\mathscr{C} = \mathscr{C}'' = k$ the tensor product $M \otimes_{\mathscr{C}'} N$ is a $k - k$-bimodule, i.e. just a chain complex.

As a consequence of proposition 2.3 and the discussion right after, the tensor product functors can be left derived. More precisely, if M is a $\mathscr{C} - \mathscr{C}'$-bimodule and N is a $\mathscr{C}' - \mathscr{C}''$-bimodule we define their derived tensor product $M \overset{L}{\otimes}_{\mathscr{C}'} N$ to be

$Tot^{\oplus}(P^{\bullet} \otimes_{\mathscr{C}'} N)$ where $P^{\bullet}$ is a projective resolution of M in $Z^0(\mathscr{C} - mod^{dg} - \mathscr{C}')$. For fixed M this can be made into a functor $D(\mathscr{C}' - mod^{dg} - \mathscr{C}'') \to D(\mathscr{C} - mod^{dg} - \mathscr{C}'')$ and similarly if we fix N. We could just as well replace N by a complex of projectives to compute $M \overset{L}{\otimes}_{\mathscr{C}'} N$.

Pretriangulated dg categories, twisted complexes and dg enhancements

Intuitively, pretriangulated dg categories are dg categories whose homotopy category is triangulated in a canonical way. More precisely, we say that a dg category $\mathscr{C}$ is *pretriangulated* if the Yoneda functor $Y : \mathscr{C} \to mod^{dg} - \mathscr{C}$ identifies $H^0(\mathscr{C})$ with a triangulated sub category of $H^0(mod^{dg} - \mathscr{C})$.

We will need to consider a slightly larger class of morphisms between dg categories than just dg functors. We say that a dg functor $F : \mathscr{C} \to mod^{dg} - \mathscr{D}$ is a *pseudo functor* if it lands in in the full dg subcategory consisting of the perfect modules. A pseudo functor will be denoted by a dashed line $F : \mathscr{C} \dashrightarrow \mathscr{D}$. We denote by $\mathsf{Func}^{ps}_{dg}(\mathscr{C}, \mathscr{D})$ the full sub dg category of $\mathsf{Func}_{dg}(\mathscr{C}, mod^{dg} - \mathscr{D})$ whose objects are pseudo functors.

We introduce now the *pretriangulated hull* of a dg category $\mathscr{C}$. It is constructed in two steps. First let $\mathscr{C}^{\oplus}$ be the dg category whose objects are formal finite direct sums of formal integer shifts of objects in $\mathscr{C}$: $\bigoplus_{i=1}^m c_i[n_i]$. A degree k morphism from $\bigoplus_{i=1}^m c_i[n_i]$ to $\bigoplus_{i=1}^{m'} c_i'[n_i']$ is a matrix $\varphi = (\varphi_{ij})_{1 \le i \le m', 1 \le j \le m}$ where $\varphi_{ij} \in (\mathscr{C}(c_j, c_i)[n_i' - n_j])^k$. Composition is defined using the usual formula for matrices and differentials are applied component wise. Now, a *twisted complex* over $\mathscr{C}$ is a pair (c, δ) where $c \in \mathsf{Ob}(\mathscr{C}^{\oplus})$ and $\delta \in \mathscr{C}^{\oplus}(c, c)^1$ is strictly lower triangular (see section 1.2 for our convention on matrices) and satisfies the Maurer-Cartan equation

$$d(\delta) + \delta \circ \delta = 0.$$

Now we define the pretrinagulated hull $\widehat{\mathscr{C}}$ of $\mathscr{C}$ to be the dg category whose objects are twisted complexes over $\mathscr{C}$. The morphism complexes $\widehat{\mathscr{C}}((c, \delta), (c', \delta'))$ are the same as those in $\mathscr{C}^{\oplus}$ as graded objects, but the differential of a matrix $\varphi \in \mathscr{C}^{\oplus}(c, c')^k$ is

$$D(\varphi) = d(\varphi) + \delta' \circ \varphi - (-1)^k \varphi \circ \delta.$$

If $\mathscr{C} = \mathrm{com}(k)$ there is a functor $Tot : \widehat{\mathscr{C}} \to \mathscr{C}$ which which is given by replacing the formal direct sums and shifts in $\widehat{\mathscr{C}}$ by the usual direct sum and shifts in

$\mathscr{C}$. More precisely, given a twisted complex $(c, \delta) = (\bigoplus_i c_i[n_i], \delta)$ over com(k) we define $Tot(c, \delta)$ to be the complex whose underlying graded object is $\bigoplus_i c_i[n_i]$ and whose differential is given $\oplus_i d_{c_i[n_i]} + \delta$. The Maurer-Cartan equation implies that this squares to zero.

A similar construction gives us a functor from the pretriangulated hull of an arbitrary dg category $\mathscr{C}$ to its module category $Tot_{\mathscr{C}} : \widehat{\mathscr{C}} \to mod^{dg} - \mathscr{C}$.

Proposition 2.4. *i) If $\mathscr{C}$ is a dg category then $\widehat{\mathscr{C}}$ is a pretriangulated dg category.*

ii) If $\mathscr{C}$ is a pretriangulated dg category, then the canonical map $\mathscr{C} \to \widehat{\mathscr{C}}$ is a quasi equivalence.

iii) The functor $Tot_{\mathscr{C}} : \widehat{\mathscr{C}} \to mod^{dg} - \mathscr{C}$ induces an exact functor $H^0(\widehat{\mathscr{C}}) \to H^0(mod^{dg} - \mathscr{C})$.

Proof. Part *i)* and *ii)* is lemma 3.28 and 3.33 of [Sei08]. Part *(iii)* is a special case of lemma 3.13 of [Sei08] □

Let $\mathcal{T}$ be a triangulated category. A *dg enhancement* of $\mathcal{T}$ is a pair $(\mathscr{C}, F)$ where $\mathscr{C}$ is a pretriangulated dg category and $F : H^0(\mathscr{C}) \to \mathcal{T}$ is an exact equivalence. Sometimes we will omit F from the notation and just say that $\mathscr{C}$ is a dg enhancement of $\mathcal{T}$.

Drinfeld quotients

For triangulated categories we have the notion of Verdier quotients. Given a triangulated category $\mathcal{T}$ and a thick triangulated subcategory $\mathcal{B} \subset \mathcal{T}$, the *Verdier quotient*, $\mathcal{T}/\mathcal{B}$ is by definition the localization of $\mathcal{T}$ at the class of morphisms in $\mathcal{T}$ whose cone lie in $\mathcal{B}$. The assumption that $\mathcal{B}$ is thick implies that this is a localizing class and so we have some understanding of what the quotient looks like.

This construction can be lifted to the dg category level. Such quotients were first constructed by Keller in [Kel04] and then Drinfeld gave an easier to work with construction, in [Dri04], which we now describe. Suppose $\mathscr{C}$ is a pretriangulated dg category and $\mathscr{D} \subset \mathscr{C}$ is a full subcategory such that $H^0(\mathscr{D}) \subset H^0(\mathscr{C})$ is a thick triangulated subcategory. Then the *Drinfeld quotient* is by definition $\widehat{\mathscr{C}/\mathscr{D}}$ where $\mathscr{C}/\mathscr{D}$ is the dg category whose objects are the same as in $\mathscr{C}$. The homomorphism complexes are obtained by formally adding null homotopies $t_z \in \mathscr{C}(z, z)$ for every

$z \in \mathsf{Ob}(\mathscr{D})$. More precisely, we define

$$\mathrm{Hom}_{\mathscr{C}/\mathscr{D}}(x,y) = \bigoplus_{n\geq 0} Hom^{(n)}_{\mathscr{C}/\mathscr{D}}(x,y)$$

where

$$Hom^{(n)}_{\mathscr{C}/\mathscr{D}}(x,y) := \bigoplus_{z_1,\dots,z_n \in \mathsf{Ob}(\mathscr{D})} \mathscr{C}(z_n,y)\otimes k\cdot t_{z_n}\otimes \mathscr{C}(z_{n-1},z_n)\otimes k\cdot t_{z_{n-1}}\otimes \dots \otimes \mathscr{C}(x,z_1).$$

The differential is given by the differential in $\mathscr{C}$ and $t_z \mapsto \mathrm{id}_z$ for $z \in \mathsf{Ob}(\mathscr{D})$. Drinfeld proves the following proposition.

Proposition 2.5.

$$H^0(\widehat{\mathscr{C}/\mathscr{D}}) \simeq H^0(\mathscr{C})/H^0(\mathscr{D}).$$

Proof. This is [Dri04]. □

We will need a slight alteration of this. Inside $\mathscr{C}/\mathscr{D}$ consider the two-sided ideal I generated by morphisms of the form

$$\varphi \otimes t_{z_1} \otimes \mathrm{id}_{z_1} - (-1)^{|\varphi|}\mathrm{id}_{z_2} \otimes t_{z_2} \otimes \varphi$$

where $\varphi \in \mathscr{C}(z_1,z_2)$ and $z_1, z_2 \in \mathsf{Ob}(\mathscr{D})$. Let $(\mathscr{C}/\mathscr{D})' := (\mathscr{C}/\mathscr{D})/I$.

Proposition 2.6. *The quotient map $\mathscr{C}/\mathscr{D} \to (\mathscr{C}/\mathscr{D})'$ is a quasi equivalence.*

Proof. We will construct a quasi inverse to the quotient map $\pi : (\mathscr{C}/\mathscr{D})(x,y) \to (\mathscr{C}\mathscr{D})_c(x,y)$. We first define it as a map $(\mathscr{C}/\mathscr{D})(x,y) \to (\mathscr{C}\mathscr{D})(x,y)$ by

$$\mathrm{Hom}^{(n)}_{\mathscr{C}/\mathscr{D}}(x,y) \ni \varphi_n \otimes t_{z_n} \otimes \varphi_{n-1} \otimes t_{z_{n-1}} \otimes \dots \otimes t_{z_1} \otimes \varphi_0 \mapsto$$

$$(-1)^{\epsilon}\varphi_n\varphi_{n-1}\cdots\varphi_1 \otimes t_{z_1} \otimes \mathrm{id}_{z_1} \otimes t_{z_1} \otimes \mathrm{id}_{z_1} \otimes \dots \otimes t_{z_1} \otimes \varphi_0$$

where

$$\epsilon = \sum_{i=1}^{n-1} i|\varphi_{n-i}|.$$

This is a chain map which vanishes on $I(x,y)$ and therefore descends to a map $\nu : (\mathscr{C}/\mathscr{D})'(x,y) \to (\mathscr{C}/\mathscr{D})(x,y)$. We have $\pi\nu = 1$. We will show that $\nu\pi \simeq 1$. Such a homotopy can be constructed using that

$$d(\mathrm{id}_{z_2}\otimes t_{z_2}\otimes\varphi\otimes t_{z_1}\otimes\mathrm{id}_{z_1}) = \varphi\otimes t_{z_1}\otimes\mathrm{id}_{z_1} - \mathrm{id}_{z_2}\otimes t_{z_2}\otimes d(\varphi)\otimes t_{z_1}\otimes\mathrm{id}_{z_1} - (-1)^{|\varphi|}\mathrm{id}_{z_2}\otimes t_{z_2}\otimes\varphi.$$

□

2.2 Cdg categories

Cdg categories should be thought of as deformations of dg categories. Cdg algebras were originally introduced by Positselski to generalize Koszul duality to the non-homogeneous setting. Since then they have been used also to study matrix factorizations which can be thought of as modules over certain cdg algebras.

Definitions and examples

As before, Γ denotes one of the abelian groups $\mathbb{Z}$ or $\mathbb{Z}/2$. Recall that a *curved differential graded category* $\mathscr{C}$ (*cdg category* for short) is a triple $(\mathscr{C}, d, h)$ where $\mathscr{C}$ is a category enriched over Γ-graded vector spaces, d is an assignment of a degree 1 operator (called *differential*) $d_{x,y} : \mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathrm{Hom}_{\mathscr{C}}(x, y)[1]$ for every pair of objects x and y in $\mathsf{Ob}(\mathscr{C})$ and h is the assignment of a degree 2 endomorphism $h_x \in \mathrm{End}_{\mathscr{C}}(x)$ (called *curvature*) for every object $x \in \mathsf{Ob}(\mathscr{C})$. This structure d and h must satisfy the following, d is a derivation for the composition, $d_{x,x}(h_x) = 0$ and $d^2_{x,y}(f) = h_y \circ f - f \circ h_x$ for all $f \in \mathrm{Hom}_{\mathscr{C}}(x, y)$. If $\mathscr{C}$ is a cdg category we will denote by $\mathscr{C}^{\#}$ the underlying Γ-graded category.

Here are some examples that will be important to us.

Example 2.7. A dg category can be thought of as a cdg category with $h_x = 0$ for all objects x.

Example 2.8. Let A be a graded algebra and let h be an element of the center of A. Then $(A, 0, h)$ is a cdg algebra with only one object.

Example 2.9. Let $Com_{cdg}(k)$ denote the cdg algebra whose objects are pairs $(C^\bullet, \delta_C)$ where $C^\bullet$ is a graded vector space and $\delta_C : C^\bullet \to C^\bullet[1]$ is a degree 1 operator. We define $d_{C,C'}(f) := \delta_{C'} \circ f - (-1)^{|f|} f \circ \delta_C$ and $h_C := \delta_C^2$. If $(C^\bullet, \delta_C)$ is an object of $Com_{cdg}(k)$ we define the shift $(C^\bullet[1], \delta_C[1] = -\delta_C)$ as usual.

Let $\mathscr{C}$ and $\mathscr{C}'$ be two cdg categories. A functor $F : \mathscr{C} \to \mathscr{C}'$ of categories enriched over graded vector spaces is called a *quasi cdg functor* if $F_{x,y} \circ d_{x,y} = d_{F(x),F(y)} \circ F_{x,y}$ for all objects x and y. If in addition we have $F(h_x) = h_{F(x)}$ for all objects x then we call F a *cdg functor*.

Remark 2.10. What we just defined is sometimes called a strict functor but we omit the prefix strict since we will not need the notion of ”non-strict” cdg functors.

If $F, G : \mathscr{C} \to \mathscr{C}'$ are two cdg functors, then a *pre natural transformation* $\alpha : F \implies G$ is a natural transformation of functors between Γ-graded categories. A pre natural transformation is called a natural transformation if $d_c(\alpha_c) = 0$ for all $c \in \mathsf{Ob}(\mathscr{C})$. A natural isomorphism is a degree 0 natural transformation which is an isomorphism.

Finally, if $\mathscr{C}$ is a cdg category there is a cdg category $\mathscr{C}^{op}$ objects and homomorphism complexes are defined analogously to the case of dg categories in section 2.1. The curvature elements are defined to be $h_c^{op} := -h_c$.

Cdg and qdg modules

Cdg (quasi-)modules over a cdg category $\mathscr{C}$ are by definition (quasi) cdg functors $F : \mathscr{C} \to Com_{cdg}(k)$. If F is a quasi functor we denote by $F[n]$ the quasi functor obtained by composing $[n] \circ F$. The collection of quasi modules themselves form a cdg category that we will denote by $mod^{qdg} - \mathscr{C}$. The objects are quasi modules. A degree n morphisms $\alpha \in \mathrm{Hom}^n_{mod^{qdg}-\mathscr{C}}(F, F')$ is a pre natural transformation $\alpha : F \implies F'[n]$ between functors of categories enriched over graded vector spaces (defined just like in the case of dg categories). The differentials $d_{F,F'}$ are given by $d_{F,F'}(\alpha)_x := \delta_{F'(x)} \circ \alpha_x - (-1)^n \alpha_x \circ \delta_{F(x)}$ and the curvatures are by definition $h_F(x) := h_{F(x)} - F(h_x)$. The full subcategory of modules inside the cdg category of quasi modules forms a dg-category which we denote by $mod^{cdg} - \mathscr{C}$. If $\mathscr{C}$ is a dg category we have $mod^{dg} - \mathscr{C} = mod^{cdg} - \mathscr{C}$.

The category $Z^0(mod^{cdg} - \mathscr{C})$ is abelian and it has enough projective objects. However, we will be more interested in projective objects which are also projective as Γ-graded modules. We say that a cdg module (or qdg module) P is *graded projective* if it is projective as a Γ-graded $\mathscr{C}^{\#}$-module. Recall that this is equivalent to $P^{\#}$ being a direct summand of a free $\mathscr{C}^{\#}$-module. We say that P is *finitely generated graded projective* if $P^{\#}$ is a summand of a finitely generated free module. Then we denote by $mod^{cdg}_{fgp} - \mathscr{C}$ the full sub dg category $mod^{cdg} - \mathscr{C}$ whose objects are finitely generated graded projective. Similarly we define $mod^{qdg}_{fgp} - \mathscr{C}$ as the full sub cdg category of $mod^{qdg} - \mathscr{C}$ whose objects are finitely generated graded projective. The following proposition is part of Theorem 3.6 in [Pos11]. We include a proof here for convenience and for us to refer back to later.

Proposition 2.11. *The category $Z^0(mod^{cdg} - \mathscr{C})$ has enough projective modules which are graded projective.*

Proof. Let M be a cdg module over $\mathscr{C}$. Let $\pi : P \to M^\#$ be a surjection in the abelian category of Γ-graded $\mathscr{C}^\#$-modules from a projective object there. We define a cdg modules $\tilde{P}$ whose underlying Γ-graded module is $P \oplus P[-1]$. The differential is given in matrix form by

$$\delta_{\tilde{P}} := \begin{pmatrix} 0 & P(h) \\ \mathrm{id} & 0 \end{pmatrix}$$

and the action of $\mathscr{C}$ on $\tilde{P}$ is defined by

$$\mathscr{C}(x,y) \ni f \mapsto \tilde{P}(f) := \begin{pmatrix} P(f) & -(-1)^{|f|}P(d(f)) \\ 0 & (-1)^{|f|}P(f) \end{pmatrix} \in \mathrm{Hom}(\tilde{P}(y), \tilde{P}(x)).$$

Note that this defines a cdg functor as $[\delta_{\tilde{P}}, \tilde{P}(f)] = \tilde{P}(df)$. Now define $\tilde{\pi} : \tilde{P} \to M$ in matrix notation by

$$\tilde{\pi} := \begin{pmatrix} \pi & \delta_M \circ \pi \end{pmatrix}.$$

Since π was already surjective in the category of Γ-graded modules $\tilde{\pi}$ is surjective in the category $Z^0(mod^{cdg} - \mathscr{C})$. The module $\tilde{P}$ is clearly graded projective. It remains to prove that $\tilde{P}$ is projective. Consider a diagram in $Z^0(mod^{cdg} - \mathscr{C})$

$$\begin{tikzcd} & \tilde{P} \arrow[dl, dashed, "\tilde{\varphi}"] \arrow[d] & \\ L \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

where the bottom row is exact and we have to construct the map $\tilde{\varphi}$. Since P was projective as a Γ-graded module we obtain a map of $\varphi : P \to L^\#$ so that the diagram

$$\begin{tikzcd} & P \arrow[dl, "\varphi"] \arrow[d] & \\ L^\# \arrow[r] & N^\# \arrow[r] & 0 \end{tikzcd}$$

Setting $\tilde{\varphi} = \begin{pmatrix} \varphi & \delta_L \varphi \end{pmatrix}$ completes the proof. □

Just like for dg categories $H^0(mod^{cdg} - \mathscr{C})$ is a triangulated category and the collection of distinguished triangles is defined analogously to the dg setting in section 2.1. However, as there is no homology in this more general setting we want to find some other notion of acyclic modules. There are different notions which play different roles. Here we describe the one we will be using. We say that module M is *absolutely acyclic* if it is in the smallest thick triangulated subcategory of $H^0(mod^{cdg} - \mathscr{C})$ which contains all totalizations of short exact sequences in $Z^0(mod^{cdg} - \mathscr{C})$. This subcategory is denoted $Acyc^{abs}(mod^{cdg} - \mathscr{C})$. We define the *absolutely derived category* to be the Verdier quotient

$$D^{abs}(mod^{cdg} - \mathscr{C}) := H^0(mod^{cdg} - \mathscr{C})/Acyc^{abs}(mod^{cdg} - \mathscr{C}).$$

Analogous constructions work for left modules.

Pseudo equivalences

Let $\mathscr{D}$ be a cdg category. We say that an object $x \in \mathsf{Ob}(\mathscr{D})$ is the *direct sum* of two objects $y \oplus z$ if there are degree zero morphisms $\iota_y : y \to x$ and $\iota_z : z \to x$ which induce isomorphisms of graded groups $\mathrm{Hom}(y, w) \times \mathrm{Hom}(z, w) \cong \mathrm{Hom}(x, w)$ and we require that $d(\iota_y) = 0$ and $d(\iota_z) = 0$.

If $y \in \mathsf{Ob}(\mathscr{D})$ and $\tau \in \mathrm{Hom}^1(y, y)$ we say that an object $x \in \mathsf{Ob}(\mathscr{D})$ is a *twist* of y by τ (and we write $x = y(\tau)$) if there are degree 0 morphisms $i : y \to x$ and $j : x \to y$ such that $ij = \mathrm{id}_x$, $ji = \mathrm{id}_y$ and $jd(i) = \tau$.

Finally we call an object $x \in \mathsf{Ob}(\mathscr{D})$ a *degree n shift* of another object $y \in \mathsf{Ob}(\mathscr{D})$ if there are morphisms $i : y \to x$ and $j : x \to y$ of degree n and $-n$ respectively such that $ij = \mathrm{id}_x$, $ji = \mathrm{id}_y$, $d(i) = 0$ and $d(j) = 0$.

We call a cdg functor $F : \mathscr{C} \to \mathscr{D}$ *pseudo equivalence* if it is fully faithful and any object in $\mathscr{D}$ can be obtained from objects in the image of F in finitely many steps by taking direct sums, twsting, shifting or passing to direct summands.

Here are two important examples of pseudo equivalences from [PP12]. If $\mathscr{C}$ is a cdg category, we have a Yoneda embedding $I : \mathscr{C}^{op} \to mod^{qdg}_{fgp} - \mathscr{C}$. We also have an inclusion functor $R : mod^{cdg}_{fgp} - \mathscr{C} \to mod^{qdg}_{fgp} - \mathscr{C}$.

Proposition 2.12. *The functors I and R are pseudo equivalences.*

Proof. This is [PP12] lemma 1.5 A. □

Bimodules and tensor products

Let $\mathscr{C}$ and $\mathscr{C}'$ be two cdg categories. We define their tensor product $\mathscr{C} \otimes \mathscr{C}'$ to be the cdg category whose objects and homomorphism complexes are defined analogously to the case of dg categories in section 2.1. The curvature elements are defined to be $h_{(c,c')} := h_c \otimes 1 + 1 \otimes h_{c'}$.

The cdg category of *quasi* $\mathscr{C}-\mathscr{C}'$*-bimodules* is by definition the cdg category $\mathscr{C}\otimes\mathscr{C}'^{op}-mod^{qdg}$. The dg category of $\mathscr{C}\otimes\mathscr{C}'$*-bimodules* is the by definition $\mathscr{C}\otimes\mathscr{C}'^{op}-mod^{cdg}$.

If M is a quasi $\mathscr{C}-\mathscr{C}'$-bimodule and N is a quasi $\mathscr{C}'-\mathscr{C}''$-bimodule we define their *tensor product* $M\otimes_{\mathscr{C}'}N$ just as in the case of dg modules in section 2.1. It is a quasi $\mathscr{C}-\mathscr{C}''$-bimodules. Moreover, if M and N are modules then their tensor product again is a module.

We explain how to derive the tensor product structure in the case when $\mathscr{C}=\mathscr{C}''=k$. To not confuse it with the derived tensor product of dg modules we call it the derived tensor product of the second kind. Let M is a right $\mathscr{C}'$-module and N is a left $\mathscr{C}'-$module. We define their *derived tensor product of the second kind* $M\overset{L,II}{\otimes}_{\mathscr{C}'}N:=Tot^{\Pi}(P^\bullet\otimes_{\mathscr{C}'}N)$ where $P^\bullet$ is resolution of M in $Z^0(\mathscr{C}'^{op}-mod^{cdg})$ and by projective modules P^i whose underlying Γ-graded modules $(P^i)^\#$ are projective. Note that such a resolution exists by proposition 2.11.

Proposition 2.13. *If we fix M, the derived tensor product of the second kind $M\overset{L,II}{\otimes}_{\mathscr{C}'}(-)$ defines an exact functor $D^{abs}(\mathscr{C}'-mod)\to D(k)$. Up to isomorphism of functors this is independent of the choice of projective resolution of M.*

Proof. The functor $Tot^{\Pi}(P^\bullet\otimes_{\mathscr{C}'}(-)):Z^0(\mathscr{C}'-mod)\to Z^0(k-mod)$ preserves the homotopy relation on morphisms and therefore descends to a functor $H^0(\mathscr{C}'-mod)\to H^0(k-mod)$ which commutes with the cone constructions and therefore is an exact functor. Now if $N=Tot^{\oplus}(L_1\to L_2\to L_3)$ is the totalization of a short exact sequence. We have canonically

$$\begin{aligned}&Tot^{\Pi}(P^\bullet\otimes_{\mathscr{C}'}N)\cong\\ Tot^{\oplus}(Tot^{\Pi}(P^\bullet\otimes_{\mathscr{C}'}L_1)\to &Tot^{\Pi}(P^\bullet\otimes_{\mathscr{C}'}L_2)\to Tot^{\Pi}(P^\bullet\otimes_{\mathscr{C}'}L_3)).\end{aligned}$$

Because the P^i's are projective as Γ-graded modules the sequence above is exact so $Tot^{\Pi}(P^\bullet\otimes_{\mathscr{C}'}N)$ i acyclic. Now if we pick another projective resolution of M, say $Q^\bullet$ then there is a quasi isomorphism $\alpha:P^\bullet\to Q^\bullet$ and $C^\bullet:=Cone(\alpha)$ is a bounded above exact sequence of objects in $Z^0(\mathscr{C}'^{op}-mod^{cdg})$ whose underlying graded modules are projective. We can identify the cone of the induced map as follows

$$Cone(Tot^{\Pi}(P^\bullet\otimes_{\mathscr{C}'}N)\to Tot^{\Pi}(Q^\bullet\otimes_{\mathscr{C}'}N))\cong Tot^{\Pi}(C^\bullet\otimes_{\mathscr{C}'}N)$$

which is acyclic because $C^\bullet \otimes_{\mathscr{C}'} N$ is acyclic in the $\mathbb{Z}$ direction and bounded above. Here we are using that we are taking the *product* totalization rather than the direct sum totalization. $\square$

To compute the derived tensor product we could also resolve N by projectives.

2.3 Two kinds of Hochschild homology

We recall here two types of Hochschild homology. We begin with the first kind which is only defined for dg categories.

Hochschild homology of the first kind

Let $\mathscr{C}$ be a dg category. We define the following two special bimodules

$$\Delta^l \in \mathscr{C} \otimes \mathscr{C}^{op} - mod^{dg},\ \Delta^l(c, c') := \mathscr{C}(c', c)$$

and

$$\Delta^r \in mod^{dg} - \mathscr{C} \otimes \mathscr{C}^{op},\ \Delta^r(c, c') := \mathscr{C}(c, c').$$

The *Hochschild homology* of $\mathscr{C}$ is by definition the object $HH_\bullet(\mathscr{C}) := \Delta^r \overset{L}{\otimes}_{\mathscr{C}\otimes\mathscr{C}^{op}} \Delta^l \in D(k)$.

There is a *standard complex* which computes the Hochschild homology of a dg category $\mathscr{C}$. Set

$$C_n(\mathscr{C}) := \bigoplus_{c_0,\dots,c_n \in \mathscr{C}} \mathscr{C}(c_1, c_0) \otimes \mathscr{C}(c_2, c_1) \otimes \dots \otimes \mathscr{C}(c_0, c_n).$$

Consider the direct sum totalization of this graded object; its k'th, for $k \in \Gamma$ is

$$\bigoplus_{n-m=k} C_m(\mathscr{C})^n.$$

On this graded object we introduce two differentials:

$$d_1\big(a_0[a_1|\cdots|a_n]\big) = \sum_{i=0}^{n} (-1)^{i+|a_0|+\dots+|a_{i-1}|} a_0[\cdots|d_{\mathscr{C}}(a_i)|\cdots]$$

and

$$d_2\big(a_0[a_1|\cdots|a_n]\big) = (-1)^{|a_0|} a_0 a_1[a_2|\cdots|a_n]$$

$$+\sum_{i=1}^{n-1}(-1)^{i+|a_0|+...+|a_i|}a_0[\cdots|a_ia_{i+1}|\cdots]$$
$$+(-1)^{1+(|a_n|+1)(|a_0|+...+|a_{n-1}|+n-1)}a_na_0[a_1|\cdots|a_{n-1}].$$

One can check that d_1+d_2 square to zero so we get a (cohomologically graded) chain complex which we will denote by $C_\bullet(\mathscr{C})$.

To see that this indeed represents the object $\Delta^r \overset{L}{\otimes}_{\mathscr{C}\otimes\mathscr{C}^{op}} \Delta^l \in D(k)$ we first recall the following standard way to resolve $\Delta^l_{\mathscr{C}}$ by projective left $\mathscr{C}\otimes\mathscr{C}^{op}$-modules, called the bar resolution which we will denote by $\mathrm{Bar}^\bullet(\mathscr{C})$. It is defined by

$$\mathrm{Bar}^n(\mathscr{C}) := \Big((x,y)\mapsto \bigoplus_{c_0,\dots,c_n\in\mathscr{C}}\mathscr{C}(c_0,x)\otimes\mathscr{C}(c_1,c_0)\otimes ...\otimes\mathscr{C}(y,c_n)\Big)$$

for $n\geq 1$. For each $n\geq 1$ we have the map of left $\mathscr{C}\otimes\mathscr{C}^{op}$ modules $d^n:\mathrm{Bar}^n(\mathscr{C})\to\mathrm{Bar}^{n-1}(\mathscr{C})$ defined by

$$f_0|f_1|\cdots|f_{n+1}\mapsto\sum_{i=0}^{n+1}(-1)^i f_0|\cdots|f_if_{i+1}|\cdots|f_{n+1}$$

and we have the map $d^0:\mathrm{Bar}^0(\mathscr{C})\to\Delta^l_{\mathscr{C}}$ defined by $f_0|f_1\mapsto f_0f_1$. Note that each $\mathrm{Bar}^n(\mathscr{C})$ is projective as a left $\mathscr{C}\otimes\mathscr{C}^{op}$ module by lemma 2.3. The fact that this is a resolution in the exact structure follows from the fact that for any pair (x,y) we have a contracting homotopy (which is not natural in x and y however)

$$\mathrm{Bar}^n(\mathscr{C})\to\mathrm{Bar}^{n+1}(\mathscr{C}),\ \ f_0|\cdots|f_{n+1}\mapsto\mathrm{id}_x|f_0|\cdots|f_{n+1}.$$

Then we have the following proposition.

Proposition 2.14. *For each n we have an isomorphism of chain complexes*

$$C_n(\mathscr{C})\cong\Delta^r_{\mathscr{C}}\otimes_{\mathscr{C}\otimes\mathscr{C}^{op}} Bar^n(\mathscr{C}).$$

Moreover, this isomorphism is compatible with the Hochschild and Bar differentials and so it gives an isomorphism

$$C_\bullet(\mathscr{C})\cong Tot^{\oplus}(\Delta^r_{\mathscr{C}}\otimes_{\mathscr{C}\otimes\mathscr{C}^{op}} Bar^\bullet(\mathscr{C})).$$

Proof. The complex on the right is the cokernel of

$$\bigoplus_{(x,y),(x',y')}\mathscr{C}(x,y)\otimes(\mathscr{C}(y',y)\otimes\mathscr{C}(x,x'))\otimes\mathrm{Bar}^n(\mathscr{C})(y',x')\to\bigoplus_{(x,y)}\mathscr{C}(x,y)\otimes\mathrm{Bar}^n(y,x)$$

$$f \otimes (\alpha \otimes \beta) \otimes B \mapsto (-1)^{|\alpha||f|} \alpha f \beta \otimes B - (-1)^{(|B|+|\beta|)|\alpha|} f \otimes \beta B \alpha.$$

The isomorphism is given induced by

$$\bigoplus_{(x,y)} \mathscr{C}(x,y) \otimes \mathrm{Bar}^n(y,x) \to C_n(\mathscr{C})$$

$$f \otimes b_0|b_1|\cdots|b_{n+1} \mapsto (-1)^{|b_{n+1}|(|b_0|+\ldots+|b_n|+|f|)} b_{n+1} f b_0[b_1|\cdots b_n]$$

and its inverse sends $a_0[a_1|\cdots|a_n]$ to the image of $a_0 \otimes 1|a_1|...|a_n|1$ in the cokernel.

□

Finally we discuss invariance of Hochschild homology. Both of the following propositions follow from lemma 3.4 in [Kel99].

Proposition 2.15. *If $\mathscr{C} \to \mathscr{D}$ is a quasi equivalence then the induced map $HH_\bullet(\mathscr{C}) \to HH_\bullet(\mathscr{D})$.*

Proposition 2.16. *If $\mathscr{C}$ is a dg category, then the Yoneda functor $\mathscr{C} \to perf(\mathscr{C})$ induces an isomorphism in Hochschild homology $HH_\bullet(\mathscr{C}) \to HH_\bullet(perf(\mathscr{C}))$.*

As a consequence of the above propositions we see that Hochschild homology is functorial with respect to pseudo functors. Indeed, a pseudo functor $F : \mathscr{C} \dashrightarrow \mathscr{D}$ induces the following map in Hochschild homology

$$HH_\bullet(\mathscr{C}) \xrightarrow{F_*} HH_\bullet(\mathrm{perf}(\mathscr{D})) \xrightarrow{Y_*^{-1}} HH_\bullet(\mathscr{D}).$$

Hochschild homology of the second kind

Now we consider a cdg category $\mathscr{C}$. We define the following two special bimodules:

$$\Delta^l \in \mathscr{C} \otimes \mathscr{C}^{op} - mod^{cdg},\ \Delta^l(c,c') := \mathscr{C}(c',c)$$
$$\Delta^r \in mod^{cdg} - \mathscr{C} \otimes \mathscr{C}^{op},\ \Delta^r(c,c') := \mathscr{C}(c,c').$$

The Hochschild homology of the second kind is by definition the object $HH_\bullet^{II} := \Delta^r \overset{L,II}{\otimes}_{\mathscr{C}'} \Delta^l \in D(k)$. Again there is a standard complex which we use to compute Hochschild homology of the second kind. Recall the Γ-graded complexes $C_n(\mathscr{C})$

defined in section 2.3. We define a Γ-graded vector space $C^{II}_\bullet(\mathscr{C})$ by taking direct product totalization; its n'th term, for $n \in \Gamma$ is given by

$$C^{II}_n(\mathscr{C}) := \prod_{j-i=n} C_i(\mathscr{C})^j.$$

Equip this Γ-graded vector space with the differential $d_0 + d_1 + d_2$ where d_1 and d_2 are defined by the same formulas as in section 2.3 and d_0 is given by the formula

$$d_0(a_0[a_1|\cdots|a_n]) := \sum_{i=0}^{n} (-1)^{i+|a_0|+\ldots+|a_i|} a_0[a_1|\cdots|a_i|h|a_{i+1}|\cdots]$$

The axioms of a cdg category imply that this differential square to zero and the resulting chain complex is the standard complex of the second kind, which we will denote just by $C^{II}_\bullet(\mathscr{C})$.

Proposition 2.17. *The complex $C^{II}_\bullet(\mathscr{C})$ represents the object $HH^{II}_\bullet(\mathscr{C})$ in $D(k)$.*

Proof. See [PP12] proposition 2.4 B. □

In [PP12] they also showed that Hochschild homology of the second kind is invariant under pseudo equivalences.

Proposition 2.18. *If $\mathscr{C} \to \mathscr{D}$ is a pseudo equivalence then the induced map $HH^{II}_\bullet(\mathscr{C}) \to HH^{II}_\bullet(\mathscr{D})$ is a quasi isomorphism.*

2.4 Presheaves of cdg categories

We will also need presheaf versions of the Hochschild complexes. But first, a *presheaf of cdg categories* $\underline{\mathscr{E}}$ on a scheme X is a contravariant functor from the category of open subsets of X and inclusions of such to the category of small cdg categories with cdg functors as morphisms. If for all U $\underline{\mathscr{E}}(U)$ has only one object we call it a presheaf of cdg algebras and if $U \mapsto \mathrm{End}(\underline{\mathscr{E}}(U))$ is a sheaf we call it a sheaf of cdg algebras. If for all U $\underline{\mathscr{E}}(U)$ is a dg category then we call it presheaf of dg categories. Here is an example that will be important to us later.

Example 2.19. If W is a global function on X then $U \mapsto (\mathcal{O}_X(U), 0, W|_U)$ is a sheaf of cdg algebras which we will denote by $\mathcal{O}_W$.

A *morphism of presheaves of cdg categories* $\varphi : \underline{\mathscr{E}} \to \underline{\mathscr{E}}'$ is the data a cdg functor $\varphi_U : \underline{\mathscr{E}}(U) \to \underline{\mathscr{E}}'(U)$ for each open set $U \subset X$ such that whenever $V \subset U \subset X$ the following diagram commutes on the nose

$$\begin{tikzcd}
\underline{\mathscr{E}}(U) \arrow[r, "\varphi_U"] \arrow[d, "res"] & \underline{\mathscr{E}}'(U) \arrow[d, "res"] \\
\underline{\mathscr{E}}(V) \arrow[r, "\varphi_V"] & \underline{\mathscr{E}}'(V).
\end{tikzcd}$$

If $\underline{\mathscr{E}}$ is a presheaf of cdg categories we define $\underline{C}^{II}_{\bullet}(\underline{\mathscr{E}})$ to be the presheaf of Γ-graded chain complexes

$$\underline{C}^{II}_{\bullet}(\underline{\mathscr{E}})(U) := C^{II}_{\bullet}(\underline{\mathscr{E}}(U)).$$

Similarly one defines $\underline{C}_{\bullet}(\underline{\mathscr{E}})$. Any morphism $\varphi : \underline{\mathscr{C}} \to \underline{\mathscr{D}}$ of presheaves of cdg categories gives rise to a map of presheaves of complexes $\varphi_* : \underline{C}^{?}_{\bullet}(\underline{\mathscr{C}}) \to \underline{C}^{?}_{\bullet}(\underline{\mathscr{D}})$ (where $? = I, II$).

We will need a weaker notion of morphism between presheaves of cdg categories where the cdg functors are only asked to commute with the restriction functors up to compatible natural isomorphisms. Here is the exact definition. Let $\underline{\mathscr{C}}$ and $\underline{\mathscr{D}}$ be two presheaves of cdg categories on a scheme X.

Definition 2.20. A lax morphism $\varphi : \underline{\mathscr{C}} \to \underline{\mathscr{D}}$ is a collection of cdg functors $\{\varphi_U : \underline{\mathscr{C}}(U) \to \underline{\mathscr{D}}(U) |\ U \subset X \text{ open}\}$ together with isomorphisms of functors $\{\alpha_{UV} : \varphi_V \circ \mathrm{Res}_{UV} \implies \mathrm{Res}_{UV} \circ \varphi_U |\ U \supset V \supset X \text{ open }\}$ which satisfies the following cocycle condition, for all $c \in \mathsf{Ob}(\underline{\mathscr{C}})$ and all $U \supset V \supset W$ we have

$$\alpha_{UW,c} = \mathrm{Res}_{VW}(\alpha_{UV,c}) \circ \alpha_{VW,\mathrm{Res}_{UV}(c)}$$

Suppose $X = \cup_{i=1,\dots,n} U_i$ is a finite union of affine schemes. We will write $\check{C}(\mathscr{U}, \underline{C}^{?}_{\bullet}(\underline{\mathscr{C}}))$ for the total complex obtained by taking the total complex of the bicomplex $\check{C}^p(\mathscr{U}, \underline{C}^{?}_{-q}(\underline{\mathscr{C}}))$ where $? = I, II$ (see section 1.2 for our conventions on the differentials in this situation). We will see that lax morphisms induce chain maps on Cech-Hochschild complexes associated to a presheaf of cdg categories as above. To understand this we first introduce some notation. Let (φ, α) be a lax morphism. If $I = \{i_0 < \dots < i_p\}$, $J = \{j_1, \dots, j_q\}$ and $J \cap I = \emptyset$ then we set $J_s := \{j_1, j_2, \dots, j_s, i_0, \dots, i_p\}$ for $0 \leq s \leq q$. With this notation we define the following maps $h^q : \check{C}(\mathscr{U}, \underline{C}^{?}_{\bullet}(\underline{\mathscr{C}})) \to \check{C}(\mathscr{U}, \underline{C}^{?}_{\bullet}(\underline{\mathscr{D}}))$ by

$$C_k(\underline{\mathscr{C}}(U_I)) \ni a_0[a_1|\cdots|a_k] \mapsto$$

$$\sum_J (-1)^{\epsilon+\sigma(I,J)+q} \alpha \circ \varphi(a_0|_{J_q}) \Big[\cdots \Big| \alpha^{-1} \Big| \varphi(a_{l_1+1}|_{J_{q-1}})|_{J_q} \Big| \cdots \Big| \alpha^{-1} \Big| \varphi(a_{l_q+1})|_{J_q} \Big| \cdots \Big] \\ \in \bigoplus_J C_{k+q}(\underline{\mathscr{C}}(U_{I\cup J})). \tag{2.1}$$

where the sum is over all ordered $J = (j_1, ..., j_q)$ disjoint from I and all $0 \leq l_1 \leq l_2 \leq ... \leq l_q \leq k$. The sign is given by

$$\epsilon = (|a_0| + ... + |a_{l_1}| + l_1) + ... + (|a_0| + ... + |a_{l_q}| + l_q)$$

where $\sigma(I, J)$ is the sign of the permutation which arranges $(j_1, ..., j_q, i_0, ..., i_p)$ in order. For example if

$$X_0 \xleftarrow[a_0]{} X_1 \xleftarrow[a_1]{} X_2 \xleftarrow[a_2]{} X_0$$

are three composable arrows in $\underline{\mathscr{C}}(U_I)$ and $j \notin I$ then the terms in $h^1(a_0[a_1|a_2])$ over $U_{I\cup j}$ are in bijection with paths of length 4 in the diagram

$$\begin{tikzcd}
& \varphi_{U_I}(X_1)|_{U_{I\cup j}} \arrow[d, "\alpha^{-1}"] & \varphi_{U_I}(X_2)|_{U_{I\cup j}} \arrow[l, "\varphi_{U_I}(a_1)|_{U_{I\cup j}}"'] \arrow[d, "\alpha^{-1}"] & \varphi_{U_I}(X_3)|_{U_{I\cup j}} \arrow[l, "\varphi_{U_I}(a_2)|_{U_{I\cup j}}"'] \arrow[d, "\alpha^{-1}"] \\
\varphi_{U_I}(X_0)|_{U_{I\cup j}} & \varphi_{U_{I\cup j}}(X_1|_{U_{I\cup j}}) \arrow[l, "\alpha\circ\varphi_{U_{I\cup j}}(a_0|_{U_{I\cup j}})"'] & \varphi_{U_{I\cup j}}(X_2|_{U_{I\cup j}}) \arrow[l, "\varphi_{U_{I\cup j}}(a_1|_{U_{I\cup j}})"'] & \varphi_{U_{I\cup j}}(X_0|_{U_{I\cup j}}) \arrow[l, "\varphi_{U_{I\cup j}}(a_2|_{U_{I\cup j}})"']
\end{tikzcd}$$

We define $\check{C}(\varphi, \alpha) : \check{C}(\mathscr{U}, \underline{C}^?_\bullet(\mathscr{C})) \to \check{C}(\mathscr{U}, \underline{C}^?_\bullet(\mathscr{C}))$ by

$$C_k(\underline{\mathscr{C}}(U_I)) \ni \bar{a} := a_0[a_1| \cdots |a_k] \mapsto \sum_{q\geq 0} h^q(\bar{a})$$

and extending over infinite sums in the case $? = II$.

Proposition 2.21. *The map $\check{C}(\varphi, \alpha)$ is a chain map and fits into a commutative diagram in $D_{Z/2}(\mathbb{C})$*

$$\begin{tikzcd}
C^?_\bullet(\underline{\mathscr{C}}(X)) \arrow[r, "(\varphi_X)_*"] \arrow[d] & C^?_\bullet(\underline{\mathscr{D}}(X)) \arrow[d] \\
\check{C}(\mathscr{U}, \underline{C}^?_\bullet(\mathscr{C})) \arrow[r, "\check{C}(\varphi,\alpha)"] & \check{C}(\mathscr{U}, \underline{C}^?_\bullet(\mathscr{D}))
\end{tikzcd}$$

where the vertical arrows are induced by the restriction functors.

Proof. We first check that $\check{C}(\varphi, \alpha)$ is a chain map. This follows from the following lemma

Lemma 2.22. *(i)* $d_{\check{C}} \circ h^{q-1} + \bar{d}_2 \circ h^q = h^q \circ \bar{d}_2 + h^{q-1} \circ d_{\check{C}}$ *for* $q \geq 1$,
(ii) $\bar{d}_1 \circ h^q = h^q \circ \bar{d}_1$ *for all* $q \geq 0$,
(iii) $\bar{d}_0 \circ h^q = h^q \circ \bar{d}_0$ *for all* $q \geq 0$.

Proof of lemma. For (i) we note that $\bar{d}_2 \circ h^q(\bar{a})$ will consist of terms of the following form

$$(I)\ \alpha\varphi(a_0|_{U_{I\cup J}})\alpha^{-1}\Big[\cdots\Big]$$

$$(II)\ \alpha\varphi(a_0|_{U_{I\cup J}})\Big[\cdots\Big|\varphi((a_r a_{r+1})|_{U_{J_s}})|_{U_{I\cup J}}\Big|\cdots\Big]$$

$$(III)\ \alpha\varphi(a_0|_{U_{I\cup J}})\Big[\cdots\Big|\varphi(a_{l_s}|_{U_{J_{q-s+1}}})|_{U_{I\cup J}}\alpha^{-1}|_{U_{I\cup J}}\Big|\cdots\Big]$$

$$(IV)\ \alpha\varphi(a_0|_{U_{I\cup J}})\Big[\cdots\Big|\alpha^{-1}|_{U_{I\cup J}}\varphi(a_{l_s+1}|_{U_{J_{q-s}}})|_{U_{I\cup J}}\Big|\cdots\Big]$$

$$(V)\ \alpha\varphi(a_0|_{U_{I\cup J}})\Big[\cdots\Big|\alpha^{-1}|_{U_{I\cup J}}\alpha^{-1}|_{U_{I\cup J}}\Big|\cdots\Big]$$

$$(VI)\ \alpha^{-1}|_{U_{I\cup J}}\alpha\varphi(a_0|_{U_{I\cup J}})\Big[\cdots\Big]$$

$$(VII)\ \varphi(a_k)|_{U_{I\cup J}}\alpha\varphi(a_0|_{U_{I\cup J}})\Big[\cdots\Big]$$

The terms in (I) cancel with the terms in $d_{\check{C}} \circ h^{q-1}(\bar{a})$, the terms in (II) cancel with terms in $h^q \circ \bar{d}_2(\bar{a})$, the terms in (III) cancel with the terms in (IV), the terms in (V) cancel with themselves, the terms in (VI) cancel with the terms from $h^{q-1} \circ d_{\check{C}}$ and finally the terms (VII) cancel with the remaining terms in $h^q \circ \bar{d}_2(\bar{a})$.

Part (ii) and (iii) is also just a matter of checking. $\square$

Now for the second part of the proposition we note that the two composites are related by a homotopy H which we now describe. On an element $\bar{a} = a_0[a_1|\cdots|a_k]$

it is defined to be

$$\sum_{q\geq 1} \tilde{h}^q(\bar{a})$$

where $\tilde{h}^q : C_k(\underline{\mathscr{C}}(X)) \to \check{C}^{q-1}(\mathscr{U}, \underline{C}_{k+q})$ are defined exactly like h^q in (2.1) with the convention that $p = -1$. $\square$

We say that two lax-morphisms $(\varphi, \alpha), (\psi, \beta) : \underline{\mathscr{C}} \to \underline{\mathscr{D}}$ are isomorphic if there are natural isomorphisms $\tau_U : \varphi_U \overset{\sim}{\Longrightarrow} \psi_U$ such that for any $V \subset U$ we have a commutative diagram of natural isomorphisms

$$\begin{array}{ccc} \mathrm{Res}^U_V \circ \varphi_U & \xrightarrow{\mathrm{Res}^U_V(\tau_U)} & \mathrm{Res}^U_V \circ \psi_U \\ {\scriptstyle \alpha^U_V}\uparrow & & {\scriptstyle \beta^U_V}\uparrow \\ \varphi_V \circ \mathrm{Res}^U_V & \xrightarrow{\tau_V} & \psi_V \circ \mathrm{Res}^U_V. \end{array}$$

Proposition 2.23. *If (φ, α) and (ψ, β) are isomorphic lax morphisms of presheaves of dg categories $\underline{\mathscr{C}} \to \underline{\mathscr{D}}$ then the induced maps are homotopic $\check{C}(\varphi, \alpha) \simeq \check{C}(\psi, \beta)$.*

Proof. Let $\tau : (\varphi, \alpha) \to (\psi, \beta)$ be an isomorphism. The following is a chain homotopy between $\check{C}(\varphi, \alpha)$ and $\check{C}(\psi, \beta)$:

$$\sum_{q\geq 0} H^q, \text{ where } H^q(a_0[a_1|\cdots|a_k]) :=$$

$$\sum\sum(-1)^{\epsilon'}\tau|_{U_{I\cup J}}\alpha\varphi(a_0|_{U_{I\cup J}})\Big[\cdots\Big|\alpha^{-1}\Big|\varphi(a_{l_i+1}|_{U_{J_{q-i}}})|_{U_{I\cup J}}\Big|\cdots$$
$$\Big|\tau|^{-1}_{U_{I\cup J}}\Big|\cdots\Big|\beta^{-1}|_{U_{I\cup J}}\Big|\psi(a_{l_{i+1}+1})|_{U_{I\cup J}}\Big|\cdots\Big]$$

where the outer sum is indexed as in the definition of h^q (see equation (2.1) above) and the inner sum runs from 0 to $k+q$ picking out the position for τ^{-1}. The sign is

$$\epsilon' = \epsilon + |a_0| + ... + |a_r| + r + s$$

where ϵ is as in the definition of h^q, a_r is the last morphism among $a_0, ..., a_k$ that appear before τ^{-1} and s is the number of α^{-1}'s that appear before τ^{-1}. $\square$

If $\underline{\mathscr{C}}$ is a presheaf of dg algebras (recall that this means $\underline{\mathscr{C}}(U)$ only has one object for all $U \subset X$), then any two lax morphisms of the form $(\varphi, \alpha), (\varphi, \beta)$: $\underline{\mathscr{C}} \to \underline{\mathscr{D}}$ are isomorphic. Indeed, lets call the object in $\underline{\mathscr{C}}(X)$ c. Then we can define $\tau_{U,c|_U} := (\beta^X_U)^{-1} \circ \alpha^X_U$. Now if φ is an actual morphism of presheaves of dg categories then (φ, id) is a lax morphism and if $\underline{\mathscr{C}}$ is a presheaf of dg algebras then any other lax morphism (φ, α) is isomorphic to (φ, id). Note that a morphism of presheaves of dg categories φ induces a morphism

$$\check{C}(\varphi) : \check{C}(\mathscr{U}, \underline{C_\bullet}(\mathscr{C})) \to \check{C}(\mathscr{U}, \underline{C_\bullet}(\mathscr{D}))$$

by taking $(\varphi_{U_I})_*$ on the summand $C_\bullet(\underline{\mathscr{C}}(U_I))$.

Lemma 2.24. *If $\varphi : \underline{\mathscr{C}} \to \underline{\mathscr{D}}$ is a morphism of presheaves of dg categories, then the maps $\check{C}(\varphi)$ and $\check{C}(\varphi, \mathrm{id})$ are chain homotopic.*

Proof. Note that with α being the identity natural transformation all the h^q's in the definition of $\check{C}(\varphi, \mathrm{id})$, with $q > 1$, vanish. Therefore $\check{C}(\varphi, \mathrm{id}) - \check{C}(\varphi) = h^1$. Let $\bar{a} = a_0[a_1|\cdots|a_k] \in C_k(\underline{\mathscr{C}}(U_I))$, where $I = \{i_0 < ... < i_p\}$ and define

$$H(\bar{a}) = \sum_{j,l_1,l_2} (-1)^\tau \varphi(a_0)[\cdots|\varphi(a_{l_1})|\mathrm{id}|\cdots|\varphi(a_{l_2})|\mathrm{id}|\cdots] \in \bigoplus C_{k+2}(\underline{\mathscr{D}}(U_{I\cup j}))$$

where the sum is over all $j \notin I$, and $0 \le l_1 \le l_2 \le k$ and the sign is given by

$$\tau = (|a_0| + ... + |a_{l_1}| + l_1) + (|a_0| + ... + |a_{l_1}| + l_2) + \sigma(I, j).$$

Then $(\bar{d}_1 + \bar{d}_2 + d_{\check{C}})H + H(\bar{d}_1 + \bar{d}_2 + d_{\check{C}}) = -h^1 = \check{C}(\varphi) - \check{C}(\varphi, \mathrm{id})$. $\square$

In section 4.6 we will need the following lemma.

Lemma 2.25. *Let $(\varphi, \alpha), (\psi, \beta) : \mathscr{C} \to \mathscr{D}$ be weak morphisms of presheaves of dg categories. Assume that for each $U \subset X$ and $c \in \mathscr{C}(U)$ we have*

i) a natural homotopy equivalence $\eta_{U,c} : \psi_U(c) \to \varphi_U(c)$
ii) a natural homotopy inverse $\eta_{U,c}^{-1} : \varphi_U(c) \to \psi_U(c)$
iii) a natural homotopy $b_{U,c} : \eta_{U,c}^{-1}\eta_{U,c} \simeq \mathrm{id}$
iv) a natural homotopy $a_{U,c} : \eta_{U,c}\eta_{U,c}^{-1} \simeq \mathrm{id}$

and that this structure satisfies the following conditions, for each $V \subset U$,

$$I)\ \eta_U|_V \circ \beta_{UV} = \alpha_{UV} \circ \eta_V$$
$$II)\ \eta_U^{-1}|_V \circ \alpha_{UV} = \beta_{UV} \circ \eta_V^{-1}$$
$$III)\ b_U|_V \circ \beta_{UV} = \beta_{UV} \circ b_V$$
$$IV)\ a_U|_V \circ \alpha_{UV} = \alpha_{UV} \circ a_V.$$

Then $\check{C}(\psi, \beta) \simeq \check{C}(\varphi, \alpha)$.

Proof. If $I = \{i_0 < ... < i_p\}$, $J = (j_1, ..., j_q)$ and $J \cap I = \emptyset$ then as before we set $J_s := \{j_1, j_2, ..., j_s, i_0, ..., i_p\}$ for $0 \le s \le q$. With this notation we define the following homotopies H^q

$$
\begin{aligned}
&C_k(\underline{\mathscr{C}}(U_I)) \ni a_0[a_1|\cdots|a_k] \mapsto \\
&\sum(-1)^{\epsilon_1}\eta^{-1}|_{J_q}\alpha\circ\varphi(a_0|_{J_q})\Big[\cdots\Big|\alpha^{-1}\Big|\varphi(a_{l_1+1}|_{J_{q-1}})|_{J_q}\Big|\cdots\Big|\alpha^{-1}\Big|\varphi(a_{l_s+1}|_{J_{q-s-1}})|_{J_q}\Big|\cdots \\
&\cdots\Big|\varphi(a_{l_{s+1}}|_{J_{q-s-1}})|_{J_q}\Big|\eta|_{J_q}\Big|\psi(a_{l_{s+1}+1}|_{J_{q-s-1}})|_{J_q}\Big|\cdots\Big|\beta^{-1}\Big|\psi(a_{l_{s+2}+1}|_{J_{q-s-2}})|_{J_q}\Big|\cdots\Big]+ \\
&\sum(-1)^{\epsilon_2}a|_{J_q}\alpha\circ\varphi(a_0|_{J_q})\Big[\cdots\Big|\alpha^{-1}\Big|\varphi(a_{l_1+1}|_{J_{q-1}})|_{J_q}\Big|\cdots\Big|\alpha^{-1}\Big|\varphi(a_{l_q+1})|_{J_q}\Big|\cdots\Big]+ \\
&\sum(-1)^{\epsilon_3}b|_{J_q}\beta\circ\psi(a_0|_{J_q})\Big[\cdots\Big|\beta^{-1}\Big|\psi(a_{l_1+1}|_{J_{q-1}})|_{J_q}\Big|\cdots\Big|\beta^{-1}\Big|\psi(a_{l_q+1})|_{J_q}\Big|\cdots\Big]
\end{aligned}
$$

where the first sum is over all ordered $J = (j_1, ..., j_q)$ and all $0 \le l_1 \le ... \le l_{q+1} \le k$, the second and third sum are over all ordered J and all $0 \le l_1 \le ... \le l_q \le k$. The signs are

$$
\begin{aligned}
\epsilon_1 &= (|a_0| + ... + |a_{l_1}| + l_1) + ... + (|a_0| + ... + |a_{l_{q+1}}| + l_{q+1}), \\
\epsilon_2 = \epsilon_3 &= (|a_0| + ... + |a_{l_1}| + l_1) + ... + (|a_0| + ... + |a_{l_q}| + l_q)
\end{aligned}
$$

Then we have

$$(\bar{d}_2 + \bar{d}_1 + d_{\check{C}})H^q + H^q(\bar{d}_1 + \bar{d}_2 + d_{\check{C}}) = h^q(\varphi, \alpha) - h^q(\psi, \beta).$$

□

2.5 Matrix factorizations

In this section we introduce matrix factorizations and recall the Hochschild homology for some categories of matrix factorizations. Matrix factorizations were originally introduced by Eisenbud in [Eis80] as a tool to do homological algebra over a hypersurface singularity. Later, due to work by Orlov (see for example [Orl12]), they became an important tool to understand the singularity category of a hypersurface singularity.

Affine matrix factorizations

Let A be a commutative ring and $h \in A$ and consider the cdg algebra $A_h = (A, 0, h)$ from example 2.8. Then a matrix factorization of h is nothing but an object in $A_h - mod^{cdg}_{fgp}$. More precisely, such an object consists of the following data

$$P^0 \underset{\delta^1}{\overset{\delta^0}{\rightleftarrows}} P^1$$

where P^0, P^1 are projective A-modules, δ^0, δ^1 are A-module homomorphisms such that the composites $\delta^1\delta^0$ and $\delta^0\delta^1$ are both multiplication by h.

If A is a commutative ring and $a, b \in A$, then we denote by $\{a, b\}$ the following matrix factorization of ab

$$\left[\overset{deg1}{A} \underset{b}{\overset{a}{\rightleftarrows}} \overset{deg0}{A}\right].$$

The usual definition of tensor product of complexes of modules over commutative rings generalize to an external tensor product of matrix factorizations

$$\otimes : A_h - mod^{cdg}_{fgp} \times A_{h'} - mod^{cdg}_{fgp} \to A_{h+h'} - mod^{cdg}_{fgp}.$$

Global matrix factorizations

Now let X be a smooth quasi compact scheme and $W \in \mathcal{O}_X(X)$ a global function. We will define the derived category of coherent matrix factorizations and a dg enhancement of this. First, consider the dg category $\mathrm{fact}(X, W)$ whose objects consist of the following data

$$\mathcal{F}^\bullet = \mathcal{F}^0 \underset{\delta^1}{\overset{\delta^0}{\rightleftarrows}} \mathcal{F}^1$$

where $\mathcal{F}^0, \mathcal{F}^1$ are coherent $\mathcal{O}_X$-modules and δ^0, δ^1 are $\mathcal{O}_X$-module homomorphisms such that $\delta^0\delta^1$ and $\delta^1\delta^0$ both equal multiplication by W. The hom spaces are defined as

$$\mathrm{Hom}_{\mathrm{fact}(X,W)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) := \mathrm{Hom}^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$$

with differential $\partial(\alpha) := \delta_{\mathcal{G}^\bullet}\alpha - (-1)^{|\alpha|}\alpha\delta_{\mathcal{F}^\bullet}$. The category $Z^0(\mathrm{fact}(X, W))$ is abelian and $H^0(\mathrm{fact}(X, W))$ is triangulated. We define $D(\mathrm{fact}(X, W))$ as the Verdier quotient of $H^0(\mathrm{fact}(X, W))$ by the full thick triangulated subcategory generated by all totalizations of short exact sequences in $Z^0(\mathrm{fact}(X, W))$.

Next we define $\mathrm{vect}(X, W)$ to be the full dg subcategory of $\mathrm{fact}(X, W)$ on the objects whose underlying sheaves $\mathcal{F}^0, \mathcal{F}^1$ are vector bundles. If $X = \mathrm{Spec}(A)$ is affine then this is the category $mod^{cdg}_{fgp} - A_W$ defined earlier and if X is smooth and affine then $H^0(\mathrm{vect}(X,W)) \simeq D(\mathrm{fact}(X,W))$. Under some extra assumptions on W this gives an important example where the two kinds of Hochschild homology agree.

Proposition 2.26. *If X is a smooth affine variety and $W|_{X\setminus V(W)}$ is smooth then the natural map $C_\bullet(vect(X,W)) \to C_\bullet^{II}(vect(X,W))$ is a quasi isomorphism.*

Proof. This is [PP12], corollary A section 4.7. □

If X not affine the category $\mathrm{vect}(X,W)$ will not work as a dg enhancement for $D(\mathrm{fact}(X,W))$. To resolve this we introduce the dg category $\mathrm{vect}_{\check{C}}(X,W)$. Let $\mathscr{U}$ denote a finite affine cover of X. Then we define $\mathrm{vect}_{\check{C}}(X,W)$ to have objects same as $\mathrm{vect}(X,W)$ but for the homomorphism complexes we take

$$\mathrm{Hom}_{\mathrm{vect}_{\check{C}}(X,W)}(\mathcal{P}^\bullet, \mathcal{Q}^\bullet) := Tot(\check{C}^\bullet(\mathscr{U}, \underline{\mathrm{Hom}}^\bullet(\mathcal{P}^\bullet, \mathcal{Q}^\bullet))). \tag{2.2}$$

Proposition 2.27. *The dg category $vect_{\check{C}}(X,W)$ is a dg enhancement of $D^{abs}(fact(X,W))$.*

Proof. This is [LP] Proposition 2.11. □

Using Drinfeld quotients we obtain an alternative dg enhancement of $D^{abs}(\mathrm{fact}(X,W))$, which we will call $\mathrm{fact}_Q(X,W)$.

We will also need the cdg category $\mathrm{qfact}(X,W)$. Its objects are defined exactly as the objects of $\mathrm{fact}(X,W)$ except we don't require anything about the composites $\delta^0\delta^1$ or $\delta^1\delta^0$. The hom-spaces and the differentials are defined in the same way as for $\mathrm{fact}(X,W)$ and the curvature of an object $\mathcal{F}^\bullet$ is the endomorphism $\delta^2 - \rho_W$. We define $\mathrm{qvect}(X,W)$ to be the full sub cdg category of $\mathrm{qfact}(X,W)$ whose underlying sheaves are vector bundles. We define the cdg category $\mathrm{qvect}_{\check{C}}(X,W)$ by altering the homomorphism complexes as in (2.2).

Finally we will also need presheaf versions of some of these. We define the following presheaf of dg categories

$$\underline{\mathrm{vect}}_{\check{C}}(X,W)(U) := \mathrm{vect}_{\check{C}}(U, W|_U).$$

The presheaves of cdg categories $\underline{\mathrm{qvect}}_{\check{C}}(X,W)$, $\underline{\mathrm{vect}}(X,W)$ and $\underline{\mathrm{qvect}}(X,W)$ are defined similarly.

Remark 2.28. Note that a lot of the dg and cdg categories introduced in this section are not small. In order to still make sense of the standard complexes computing their Hochschild homology we therefore have to restrict to some small subcategories containing at least one object from each isomorphism class. In what follows we will do this implicitly and will assume all categories to be small.

Hochschild homology of category of matrix factorizations

The Hochschild homology of $\mathrm{vect}_{\check{C}}(X,W)$ will be denoted $HH_\bullet(X,W)$. For a smooth variety X where $W|_{X\setminus V(W)}$ is smooth this was computed in [Efi18], it is derived global sections of the complex of sheaves $(\Omega_X^\bullet, -dW\wedge(-))$. Here $\Omega_X^\bullet$ denotes the object of the $\mathbb{Z}/2$-graded derived category $\bigoplus_{i\geq 0}\wedge^i(\Omega_X)[i]$. Below is a slight adaption of the computation in [Efi18].

There is a sequence of maps which we will show are all quasi isomorphisms.

$$\begin{tikzcd}
C_\bullet(\mathrm{vect}_{\check{C}}(X,W)) \arrow[r, "(1)"] & \check{C}(\mathscr{U}, \underline{C}_\bullet(\underline{\mathrm{vect}}_{\check{C}}(X,W))) & \check{C}(\mathscr{U}, \underline{C}_\bullet(\underline{\mathrm{vect}}(X,W))) \arrow[l, "(2)"'] \arrow[dll, "(3)"] \\
\check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\underline{\mathrm{vect}}(X,W))) \arrow[r, "(4)"] & \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\underline{\mathrm{qvect}}(X,W))) & \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\mathcal{O}_{-W})) \arrow[l, "(5)"'] \arrow[dll, "(6)"] \\
\check{C}(\mathscr{U}, (\Omega^\bullet_X, -dW\wedge -)). & &
\end{tikzcd} \tag{2.3}$$

We will first discuss (6) (See [CT13], theorem 4.2 b).

Proposition 2.29. *The following defines a quasi isomorphism*

$$HKR_{(X,W)} : \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\mathcal{O}_W)) \to \check{C}(\mathscr{U}, (\Omega^\bullet_X, dW\wedge -)),$$

$$C^{II}_\bullet(\mathcal{O}_W(U_{i_0\cdots i_p})) \ni a_0[a_1|\cdots|a_k] \mapsto \frac{1}{k!}a_0 da_1\wedge \ldots \wedge da_k \in \Omega^k_X(V\cap U_{i_0\cdots i_p})$$

To prove this we will need two lemmas

Lemma 2.30. *Suppose that we have an unbounded exact chain complex of inverse systems of abelian groups*

$$S_\bullet := (\ldots \to S_n \to S_{n-1} \to S_{n-2} \to \ldots).$$

and that each term is Mittag-Leffler. Then the complex

$$... \to \lim S_n \to \lim S_{n-1} \to \lim S_{n-2} \to ...$$

is exact.

Proof of lemma. Split the long exact sequence into short exact sequences

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & \\
 & & & B_n = Z_{n-1} \arrow[ur] \arrow[dr] & & & \\
... \arrow[rr] \arrow[dr] & & S_n \arrow[ur] \arrow[rr] & & S_{n-1} \arrow[rr] \arrow[dr] & & S_{n-2} \\
 & B_{n+1} = Z_n \arrow[ur] \arrow[dr] & & & & B_{n-1} = Z_{n-2} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

Now the inverse systems $B_k = Z_{k-1}$ are quotients of Mittag-Leffler systems and are therefore themselves Mittag-Leffler. It follows that after applying $\varprojlim$ termwise the diagonal short exact sequences remain exact and then it follows that the middle long exact sequence remains exact. $\square$

Lemma 2.31. *Let $A^{\bullet,\bullet}$ and $B^{\bullet,\bullet}$ be two bicomplexes whose vertical and horizontal differentials both decrease the degrees. For each n let $A^{\leq n,\bullet} \subset A^{\bullet,\bullet}$ denote the subcomplex truncated horizontally at n. Let $A^{\geq n,\bullet}$ be the quotient of $A^{\bullet,\bullet}$ by $A^{\leq n-1,\bullet}$. Suppose further that for each n, the diagonals of $A^{\geq n,\bullet}$ and $B^{\geq n,\bullet}$ only have finitely many non zero terms. If $\alpha : A^{\bullet,\bullet} \to B^{\bullet,\bullet}$ is a morphism of bicomplexes which is a quasi isomorphism with respect to the vertical differentials, then $Tot^{\Pi}(\alpha) : Tot^{\Pi}(A^{\bullet,\bullet}) \to Tot^{\Pi}(B^{\bullet,\bullet})$ is a quasi isomorphism.*

Proof. Consider the inverse systems of chain complexes

$$S(A) := \{... \to \mathrm{Tot}^{\oplus} A^{\geq -2,\bullet} \to \mathrm{Tot}^{\oplus} A^{\geq -1,\bullet} \to \mathrm{Tot}^{\oplus} A^{\geq 0,\bullet}\}$$
$$S(B) := \{... \to \mathrm{Tot}^{\oplus} B^{\geq -2,\bullet} \to \mathrm{Tot}^{\oplus} B^{\geq -1,\bullet} \to \mathrm{Tot}^{\oplus} B^{\geq 0,\bullet}\}$$

where the maps are induced by canonical quotient maps of bicomplexes. The map α induces a morphism $S(\alpha) : S(A) \to S(B)$ of complexes of inverse systems. The assumption on α and the Mapping Theorem for filtered modules ([Lan95],theorem XI.3.4) imply that $S(\alpha)$ is a termwise quasi isomorphism, in other words a quasi isomorphism of complexes of inverse systems. But then $Cone(S(\alpha))$ is an acyclic complex of inverse systems of abelian groups, all of whose terms are Mittag-Leffler and therefore, by the previous lemma,

$$Cone(\lim_{\leftarrow} S(\alpha)) = \lim_{\leftarrow} Cone(S(\alpha)) = 0.$$

We see that $\lim\limits_{\leftarrow} S(\alpha) : \mathrm{Tot}^{\prod} A^{\bullet,\bullet} \to \mathrm{Tot}^{\prod} B^{\bullet,\bullet}$ is a quasi isomorphism. $\square$

Proof of proposition. The affine case follows from the classical HKR theorem and by applying the previous lemma to the bicomplexes

$$A^{\bullet,\bullet} := \begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\ \cdots & \mathcal{O}^{\otimes 3} \arrow[l] \arrow[d, "d_2"] & \mathcal{O}^{\otimes 2} \arrow[l, "d_0"'] \arrow[d, "d_2"] & \mathcal{O} \arrow[l, "d_0"'] \\ \cdots & \mathcal{O}^{\otimes 2} \arrow[l] \arrow[d, "d_2"] & \mathcal{O} \arrow[l, "d_0"'] & \\ \cdots & \mathcal{O} \arrow[l] & & \end{tikzcd}$$

and

$$B^{\bullet,\bullet} := \begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\ \cdots & \Omega^2 \arrow[l] \arrow[d, "0"] & \Omega \arrow[l, "dW\wedge"'] \arrow[d, "0"] & \mathcal{O} \arrow[l, "dW\wedge"'] \\ \cdots & \Omega \arrow[l] \arrow[d, "0"] & \mathcal{O} \arrow[l, "dW\wedge"'] & \\ \cdots & \mathcal{O} \arrow[l] & & \end{tikzcd}$$

For the non-affine case, think of $\mathrm{Tot}^{\prod} A^{\bullet,\bullet}$ and $\mathrm{Tot}^{\prod} B^{\bullet,\bullet}$ as presheaves of complexes. We want to show that

$$\check{C}^p(\mathscr{U}, (\mathrm{Tot}^{\prod} A^{\bullet,\bullet})_q) \to \check{C}^p(\mathscr{U}, (\mathrm{Tot}^{\prod} B^{\bullet,\bullet})_q)$$

is a quasi isomorphism. By considering total complexes and filtering this in the p-direction we obtain a map of filtered complexes. It then follows from the Mapping

Theorem of filtered complexes and the affine case already considered that this map of bicomplexes gives rise to a quasi isomorphism of total complexes. This proves the proposition. □

Now (5) is a quasi isomorphism because for any $U_I = U_{i_0 \cdots i_p}$ the map $C^{II}_\bullet(\mathcal{O}_{U_I, -W|_{U_I}}) \to C^{II}_\bullet(\mathrm{qvect}(U_I, W|_{U_I}))$ is a quasi isomorphism by propositions 2.18 and 2.12. The map (4) is a quasi isomorphism for exactly the same reasons. The map (3) is a quasi isomorphism because each $C_\bullet(\mathrm{vect}(U_I, W|_{U_I})) \to C^{II}_\bullet(\mathrm{vect}(U_I, W|_{U_I}))$ is a quasi isomorphism by proposition 2.26. The map (2) is a quasi isomorphism because for each U_I we have a quasi equivalence $\mathrm{vect}(U_I, W|_{U_I}) \to \mathrm{vect}_{\check{C}}(U_I, W|_{U_I})$ which induce quasi isomorphisms on standard complexes. The proof that the map (1) is a quasi isomorphism is similar to the proof of proposition 5.1 in [Efi18]. We include the proof here with some adjustments to make it fit our situation.

Proposition 2.32. *Let $\mathscr{C}$ be one of the two presheaves of dg categories $\underline{perf}_{\check{C}}(X)$ or $\underline{vect}_{\check{C}}(X, W)$. The natural map $C_\bullet(\mathscr{C}(X)) \to \check{C}(\mathscr{U}, \underline{C}_\bullet(\mathscr{C}))$ is a quasi isomorphism.*

Proof. We prove it for $\mathscr{C} = \underline{\mathrm{vect}}_{\check{C}}(X, W)$. Recall that $\mathscr{U}$ is referring to a fixed affine open covering $X = U_1 \cup ... \cup U_n$. Let $U = U_n$ and $V = U_1 \cup ... \cup U_{n-1}$. Let $Z = X \setminus U$. For any open $V' \subset X$ let $\mathscr{C}(V')_{Z \cap V'} \subset \mathscr{C}(V')$ denote the full sub dg category on the objects which become acyclic in $\mathscr{C}(V' \cap U)$. Let $\mathscr{V}$ denote the open covering $V = U_1 \cup U_3 \cup ... \cup U_{n-1}$. Let $\mathscr{C}'$ denote the presheaf of dg categories on V

$$V \supset V' \mapsto \mathrm{vect}_{\check{C}(\mathscr{V})}(V', W|_{V'}).$$

We have a natural functor $\mathscr{R}es : \mathscr{C}(X) \to \mathscr{C}'(V)$ which restricts objects on X to V and which is defined on homomorphism complexes by

$$\check{C}(\mathscr{U}, \underline{\mathrm{Hom}}^\bullet_X(\mathcal{F}^\bullet, \mathcal{G}^\bullet)) \to \check{C}(\mathscr{U} \cap V, \underline{\mathrm{Hom}}^\bullet_V(\mathcal{F}|^\bullet_V, \mathcal{G}|^\bullet_V)) \twoheadrightarrow \check{C}(\mathscr{V}, \underline{\mathrm{Hom}}^\bullet_V(\mathcal{F}|^\bullet_V, \mathcal{G}|^\bullet_V)).$$

After passing to homotopy categories this functor fits into a commutative diagram

$$\begin{tikzcd} H^0(\mathscr{C}(X)) \arrow[r, "H^0(\mathscr{R}es)"] \arrow[d, "\sim"] & H^0(\mathscr{C}'(V)) \arrow[d, "\sim"] \\ D(\mathrm{fact}(X, W)) \arrow[r, "Res"] & D(\mathrm{fact}(V, W|_V)) \end{tikzcd}.$$

We have the following diagram in $D_{Z/2}(\mathbb{C})$

$$\begin{tikzcd}
C_\bullet(\mathscr{C}(X)_Z) \arrow[r] \arrow[d] & C_\bullet(\mathscr{C}(X)) \arrow[r] \arrow[d, "(\mathscr{R}es)_*"] & C_\bullet(\mathscr{C}(U)) \arrow[r] \arrow[d] & C_\bullet(\mathscr{C}(X)_Z)[1] \arrow[d] \\
C_\bullet(\mathscr{C}'(V)_Z) \arrow[r] & C_\bullet(\mathscr{C}'(V)) \arrow[r] & C_\bullet(\mathscr{C}'(V\cap U)) \arrow[r] & C_\bullet(\mathscr{C}'(V)_Z[1])
\end{tikzcd}$$

where the rows are distinguished. (See [Efi18], proposition 5.1 and its proof.) Since $Z \subset V$ the left most and the right most vertical arrows are quasi isomorphisms. Therefore we get a Mayer-Viertoris exact triangle

$$\begin{tikzcd}
C_\bullet(\mathscr{C}(X)) \arrow[r] & C_\bullet(\mathscr{C}(U)) \oplus C_\bullet(\mathscr{C}'(V)) \arrow[r] & C_\bullet(\mathscr{C}'(V\cap U)) \arrow[dll] \\
C_\bullet(\mathscr{C}(X))[1] \arrow[r] & ...
\end{tikzcd}$$

We now claim that there is an exact triangle of the form

$$\begin{tikzcd}
\check{C}(\mathscr{U},\underline{C}_\bullet(\mathscr{C})) \arrow[r, "\alpha"] & \check{C}(\mathscr{U}\cap U,\underline{C}_\bullet(\mathscr{C})) \oplus \check{C}(\mathscr{V},\underline{C}_\bullet(\mathscr{C}')) \arrow[r, "\beta"] & \check{C}(\mathscr{V}\cap U,\underline{C}_\bullet(\mathscr{C}')) \arrow[dll] \\
\check{C}(\mathscr{U},\underline{C}_\bullet(\mathscr{C}))[1] \arrow[r] & ...
\end{tikzcd}$$

to which the first exact triangle naturally maps. Assuming this claim for now the rest of the proof goes as follows. Because the open covering $\mathscr{U} \cap U$ of U has U itself as one of its open sets the map $C_\bullet(\mathscr{C}(U)) \to \check{C}(\mathscr{U}\cap U,\underline{C}_\bullet(\mathscr{C}))$ is a quasi isomorphism. By induction on the number of open sets the maps $C_\bullet(\mathscr{C}'(V)) \to \check{C}(\mathscr{V},\underline{C}_\bullet(\mathscr{C}'))$ and $C_\bullet(\mathscr{C}'(V\cap U)) \to \check{C}(\mathscr{V}\cap U,\underline{C}_\bullet(\mathscr{C}'))$ are quasi isomorphisms. The proposition then follows by the five lemma.

To construct the second distinguished triangle, we first define what the two maps α and β are. First, let $A \subset \check{C}(\mathscr{U},\underline{C}_\bullet(\mathscr{C}))$ be the subcomplex consisting of all summands $C_\bullet(\mathscr{C}(U_I))$ with $n \in I$ and $B = C_\bullet(\mathscr{C}(U_I))/A$ so that as a vector space we have $C_\bullet(\mathscr{C}(U_I)) = A \oplus B$. Similarly let $E \subset \check{C}(\mathscr{U}\cap U,\underline{C}_\bullet(\mathscr{C}))$ be the sub complex consisting of the summands $C_\bullet(\mathscr{C}(U_I \cap U_n))$ with $n \in I$ and $F = \check{C}(\mathscr{U}\cap U,\underline{C}_\bullet(\mathscr{C}))/E$ and $G = \check{C}(\mathscr{V},\underline{C}_\bullet(\mathscr{C}'))$ so that as vector spaces we have $\check{C}(\mathscr{U}\cap U,\underline{C}_\bullet(\mathscr{C})) \oplus \check{C}(\mathscr{V},\underline{C}_\bullet(\mathscr{C}')) = E \oplus F \oplus G$. Finally let $H = \check{C}(\mathscr{V}\cap U,\underline{C}_\bullet(\mathscr{C}'))$. In this notation, the maps α and β can be described by matrices

$$\alpha = \begin{bmatrix} \mathrm{id} & 0 & 0 \\ 0 & \alpha_0 & \alpha_1 \end{bmatrix} : A \oplus B \to E \oplus F \oplus G$$

$$\beta = \begin{bmatrix} 0 \\ \beta_0 \\ \beta_1 \end{bmatrix} : E \oplus F \oplus G \to H$$

where α_0 is made up by the restrictions $C_\bullet(\mathscr{C}(U_I)) \to C_\bullet(\mathscr{C}(U_I \cap U))$ $(n \notin I)$, α_1 is a quasi isomorphism which is induced by the quasi equivalences $\mathscr{C}(U_I) \to \mathscr{C}'(U_I)$ $(n \notin I)$. The map β_0 is also a quasi isomorphism, it's induced by the quasi equivalences $\mathscr{C}(U_I \cap U) \to \mathscr{C}'(U_I \cap U)$ $(n \notin I)$ and β_1 is -1 times the map induced by the restrictions $\mathscr{C}'(U_I) \to \mathscr{C}'(U_I \cap U)$ $(n \notin I)$. In this situation one can show that the canonical maps $Cone(\alpha) \to H$ is a quasi isomorphism and therefore we obtain the required distinguished triangle. □

Matrix factorizations of $W = 0$

When $W = 0$ we have two intresting categories. On the one hand the $\mathbb{Z}$-graded dg-category $\mathrm{perf}_{\check{C}}(X)$ whose objects are finite $\mathbb{Z}$-graded complexes of vector bundles and whose morphism complexes are defined as

$$\mathrm{Hom}_{\mathrm{perf}_{\check{C}}(X)}(\mathcal{P}^\bullet, \mathcal{Q}^\bullet) := Tot(\check{C}^\bullet(\mathscr{U}, \underline{\mathrm{Hom}}^\bullet_X(\mathcal{P}^\bullet, \mathcal{Q}^\bullet))).$$

We can then think of $\mathrm{perf}_{\check{C}}(X)$ as a $\mathbb{Z}/2$-graded category by forgetting the $\mathbb{Z}$-grading on the homomorphism complexes and only remembering the parity of each degree.

On the other hand we can consider the $\mathbb{Z}/2$-graded category $\mathrm{vect}_{\check{C}}(X, 0)$ defined in section 2.5. Note that we have a fully faithful dg functor functor $\mathrm{perf}_{\check{C}}(X) \to \mathrm{vect}_{\check{C}}(X, 0)$ which forgets the $\mathbb{Z}$-grading on the objects and only remembers the parity of each term of a complex. In fact, we have the following proposition.

Proposition 2.33. *If X is a nonsingular variety, then any object in $vect_{\check{C}}(X, 0)$ is homotopy equivalent to an object in $perf_{\check{C}}(X)$ and hence they are quasi equivalent.*

Proof. Let $\mathcal{F}^\bullet := \left[\overset{deg1}{\mathcal{F}^0} \underset{\delta^1}{\overset{\delta^0}{\rightleftarrows}} \overset{deg0}{\mathcal{F}^1} \right]$ be an object of $\mathrm{vect}_{\check{C}}(X, 0)$. Let $\mathcal{K}^i = \ker(\delta^i)$ and $\mathcal{Q}^i = \mathrm{im}(\delta^i)$. Then the following is an exact triangle in $D(\mathrm{fact}(X, 0))$

$$\begin{tikzcd}
\mathcal{K}^0 \arrow[r] \arrow[d, shift left, "0"] & \mathcal{F}^0 \arrow[r] \arrow[d, shift left, "\delta^0"] & \mathcal{Q}^0 \arrow[r] \arrow[d, shift left, "0"] & \mathcal{K}^1 \arrow[d, shift left, "0"] \\
\mathcal{K}^1 \arrow[r] \arrow[u, shift left, "0"] & \mathcal{F}^1 \arrow[r] \arrow[u, shift left, "\delta^1"] & \mathcal{Q}^1 \arrow[r] \arrow[u, shift left, "0"] & \mathcal{K}^0 \arrow[u, shift left, "0"]
\end{tikzcd}$$

Both $(\mathcal{K}^\bullet, 0)$ and $(\mathcal{Q}^\bullet, 0)$ are isomorphic to objects from $H^0(\mathrm{perf}_{\check{C}}(X))$. The reason is because X is non-singular the components of $\mathcal{K}^\bullet$ and $\mathcal{Q}^\bullet$ have finite resolutions by locally free sheaves. It follows that $\mathcal{F}^\bullet$ too is isomorphic to an object in $H^0(\mathrm{perf}_{\check{C}}(X))$. $\square$

Now let's consider the sequence

$$\begin{tikzcd}
C_\bullet(\mathrm{perf}_{\check{C}}(X)) \arrow[r] & \check{C}(\mathscr{U}, \underline{C}_\bullet(\underline{\mathrm{perf}}_{\check{C}}(X))) & \check{C}(\mathscr{U}, \underline{C}_\bullet(\underline{\mathrm{perf}}(X))) \arrow[l] \arrow[dll, "="'] \\
\check{C}(\mathscr{U}, \underline{C}_\bullet(\underline{\mathrm{perf}}(X))) \arrow[r, "="] & \check{C}(\mathscr{U}, \underline{C}_\bullet(\underline{\mathrm{perf}}(X))) & \check{C}(\mathscr{U}, \underline{C}_\bullet(\mathcal{O}_X)) \arrow[l] \arrow[dll] \\
\check{C}(\mathscr{U}, \Omega^\bullet_X) & &
\end{tikzcd}$$

From each term in the above sequence there is a natural map to the corresponding term of (2.3) forming commutative squares, the map $\check{C}(\mathscr{U}, \Omega^\bullet_X) \to \check{C}(\mathscr{U}, \Omega^\bullet_X)$ being the identity. All of the maps above are quasi isomorphisms and this forces all of the maps from this sequences to (2.3) to be quasi isomorphisms.

2.6 Derived pushforward in the $\mathbb{Z}/2$-graded setting

In theorem 1.1, a little care must be taken as how to define $R\Gamma$ because we are in the $\mathbb{Z}/2$-graded setting. Let X be a non-singular variety of dimension d. Let $I \subset Qcoh(X)$ be the subcategory of injective quasi coherent sheaves on X. Let $D(I) \subset D(Qcoh(X))$ be the full subcategory of the derived category consisting of objects whose terms lie in I. Let $K(I) \subset K(Qcoh(X))$ be the subcategory of the homotopy category on the same collection of objects.

Lemma 2.34. *The inclusion $\iota : D(\mathcal{I}) \to D(Qcoh(X))$ is an equivalence.*

Proof. Let $\mathcal{F}^\bullet = (\mathcal{F}^0, \mathcal{F}^1, \delta^0_{\mathcal{F}}, \delta^1_{\mathcal{F}})$ be an object in the derived category. We will show that there is an object $(\mathcal{I}^0, \mathcal{I}^1, \delta^0_{\mathcal{I}}, \delta^1_{\mathcal{I}}) \in D(I)$ and a chain map which is a term wise injection $\varphi^\bullet \mathcal{F}^\bullet \to \mathcal{I}^\bullet$. Include $\mathcal{F}^i$ in an injective sheaf $\alpha^i : \mathcal{F}^i \hookrightarrow \mathcal{J}^i$ for $i = 0, 1$. Then take $\mathcal{I}^i = \mathcal{J}^0 \oplus \mathcal{J}^1$, $\delta^0_{\mathcal{I}} = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{pmatrix}$, $\delta^1_{\mathcal{I}} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{id} \end{pmatrix}$. The desired chain map $\varphi^\bullet$ is then given by $\varphi^0 = \begin{pmatrix} \alpha^0 \\ \alpha^2 \delta^0_{\mathcal{F}} \end{pmatrix}$ and $\varphi^1 = \begin{pmatrix} \alpha^0 \delta^1 \\ \alpha^1 \end{pmatrix}$.

Now inductively we can construct an exact sequence of complexes

$$\mathcal{F}^\bullet \to \mathcal{I}_0^\bullet \to \mathcal{I}_1^\bullet \to ...$$

If we truncate this after $2d+1$'th steps, we get an exact sequence

$$\mathcal{F}^\bullet \to \mathcal{I}_0^\bullet \to \mathcal{I}_1^\bullet \to ... \to \mathcal{I}_{2d}^\bullet \to C \to 0.$$

I claim that the terms of C are still injective. To see why this is, note first that the spectral sequence $R^q\Gamma(X, \mathcal{E}xt^p(\mathcal{F}^0, \mathcal{H})) \implies \mathrm{Ext}^{p+q}(\mathcal{F}^0, \mathcal{H})$ implies that $\mathrm{Ext}^{2d+1}(\mathcal{F}^0, \mathcal{H}) = 0$ for all quasi coherent $\mathcal{H}$ and similarly for $\mathcal{F}^1$. Then we only have to observe that $\mathrm{Ext}^1(C^0, \mathcal{H}) = \mathrm{Ext}^{2d+1}(\mathcal{F}^0) = 0$ and similarly for C^1. This gives a quasi isomorphism

$$\mathcal{F}^\bullet \xrightarrow{\sim} Tot^\oplus(\mathcal{I}_0^\bullet \to \mathcal{I}_1^\bullet \to ... \to \mathcal{I}_{2d-1}^\bullet \to C).$$

$\square$

Lemma 2.35. *The naive functor* $\Gamma : K(I) \to D(k)$ *descends to a functor* $\Gamma' : D(I) \to D(k)$.

Proof. Let $A^\bullet \in K^\bullet(I)$ be an acyclic complex. We have to show that $\Gamma(A^\bullet)$ is acyclic. Consider the sequence

$$\Gamma(\ker(d_A^0)) = \ker(\Gamma(d_A^0)) \longrightarrow \Gamma(A_A^0) \xrightarrow{\Gamma(d_A^0)} \Gamma(A^1) \xrightarrow{\Gamma(d_A^1)} \Gamma(A^0) \xrightarrow{\Gamma(d_A^0)} \Gamma(A^1) \xrightarrow{\Gamma(d_A^1)} ...$$

This complex is either exact or has homology in infinitely many degrees by periodicity. The latter option is not possible because $R^i\Gamma(\ker(d^0)) = 0$ for $i > \dim(X)$. $\square$

Now we define $R\Gamma := \Gamma' \circ \iota^{-1}$. We will now describe a natural map $\check{C}^\bullet(\mathscr{U}, \mathcal{F}^\bullet) \to R\Gamma(X, \mathcal{F})$. First, let us point out that the resolution

$$\mathcal{F}^\bullet \to \mathcal{I}_0^\bullet \to \mathcal{I}_1^\bullet \to ...$$

in the proof of lemma 2.34 is an injective resolution in the abelian category of $\mathbb{Z}/2$ graded complexes with chain maps as morphisms. We obtain a commutative diagram in this category

$$\begin{array}{ccccccccccccc}
\mathcal{F}^\bullet & \longrightarrow & \underline{\check{C}}^0(\mathscr{U}, \mathcal{F}^\bullet) & \longrightarrow & \underline{\check{C}}^1(\mathscr{U}, \mathcal{F}^\bullet) & \longrightarrow & ... & \longrightarrow & \underline{\check{C}}^n(\mathscr{U}, \mathcal{F}^\bullet) & \longrightarrow & 0 & \longrightarrow & ... \\
\downarrow{=} & & \downarrow{c^0} & & \downarrow{c^1} & & & & \downarrow{c^n} & & \downarrow{0} & & \\
\mathcal{F}^\bullet & \longrightarrow & \mathcal{I}_0^\bullet & \longrightarrow & \mathcal{I}_1^\bullet & \longrightarrow & ... & \longrightarrow & \mathcal{I}_n^\bullet & \longrightarrow & \mathcal{I}_{n+1}^\bullet & \longrightarrow & ...
\end{array}$$

Lemma 2.36. *The map* $Tot^{\oplus}(\Gamma(\tau^{\leq n}c^\bullet)) : \check{C}^\bullet(\mathscr{U}, \mathcal{F}^\bullet) \to R\Gamma(X, \mathcal{F}^\bullet)$ *is a quasi isomorphism.*

Proof. As in the proof of proposition 2.33 we can fit $\mathcal{F}^\bullet$ into a short exact sequence of complexes $\mathcal{K}^\bullet \to \mathcal{F}^\bullet \to \mathcal{Q}^\bullet$ where $\mathcal{K}^\bullet$ and $\mathcal{Q}^\bullet$ have vanishing differentials. We can think of the map in the lemma as a natural transformation between exact functors $D(Qcoh(X)) \to D(k)$. It follows from chapter III.4 of [Har77] that our map is a quasi isomorphism at $\mathcal{K}^\bullet$ and $\mathcal{Q}^\bullet$ and therefor must be so also at $\mathcal{F}^\bullet$. $\square$

CHAPTER 3

MAIN PROBLEM

In this section we discuss the main problem.

3.1 Setup

Let X be a smooth complex variety and let us fix a finite affine covering $\mathscr{U}$ of X. Let $W \in H^0(X, \mathcal{O}_X)$ and suppose $Y \subset X$ is a smooth divisor such that $W|_Y = 0$. We assume also that $W|_{X\setminus W^{-1}(0)}$ is smooth. The categories we are mostly interested in are $D^b(\mathrm{coh}Y)$ (usual derived category) and $D(\mathrm{fact}(X,W))$.

We have dg enhancements for both $D^b(\mathrm{coh}(Y))$ and $D(\mathrm{fact}(X,W))$. For D^b(Y) we take the category $\mathrm{perf}_{\check{C}}(Y)$ and for $D(\mathrm{fact}(X,W))$ we take $\mathrm{vect}_{\check{C}}(X,W)$. We have a functor $i_* : D^b(\mathrm{coh}Y) \to D(\mathrm{fact}(X,W))$. It lifts to a pseudo functor $\underline{i}_* : \mathrm{perf}_{\check{C}}(Y) \dashrightarrow \mathrm{vect}_{\check{C}}(X,W)$ by which we mean that the following diagram commutes

$$\begin{tikzcd}
 & H^0(\mathrm{perf}(\mathrm{vect}_{\check{C}}(X,W))) & \\
H^0(\mathrm{perf}_{\check{C}}(Y)) \arrow[ur, "H^0(\underline{i}_*)"] \arrow[d, "\sim"] & & H^0(\mathrm{vect}_{\check{C}}(X,W)) \arrow[ul, "Yoneda"'] \arrow[dl, "\sim"] \\
D^b(\mathrm{coh}Y) \arrow[r, "i_*"] & D(\mathrm{fact}(X,W)) \arrow[uu, "Yoneda"] &
\end{tikzcd}.$$

The quasi functor $\underline{i}_*$ induces a map in Hochschild homology. We will compute this map in terms of the identifications

$$HH_\bullet(Y) := HH_\bullet(\mathrm{perf}_{\check{C}}(Y)) \simeq \check{C}(\mathscr{U} \cap Y, \Omega_Y^\bullet)$$

and

$$HH_\bullet(X,W) := HH_\bullet(\mathrm{vect}_{\check{C}}(X,W)) \simeq \check{C}(\mathscr{U}, (\Omega_X^\bullet, -dW \wedge (-)))$$

from sections 2.5 and 2.5 respectively.

3.2 Replacing i_* by a dg functor

Consider the following matrix factorization

$$P := \left[\overset{deg1}{\mathcal{O}_X(-Y)} \underset{\cdot g}{\overset{\cdot x}{\rightleftarrows}} \overset{deg0}{\mathcal{O}_X} \right]$$

where $x \in H^0(X, \mathcal{O}_X(Y))$ is an equation for Y and $g \in H^0(X, \mathcal{O}_X(-Y))$ is such that $gx = W \in H^0(X, \mathcal{O}_X)$. In $D(\mathrm{fact}(X, W))$ we have $i_*\mathcal{O}_Y \cong P$. Indeed, the kernel of the canonical map $P \to i_*\mathcal{O}_Y$ is the matrix factorization

$$\Big[\overset{deg1}{\mathcal{O}_X(-Y)} \underset{\cdot f}{\overset{\cdot 1}{\rightleftarrows}} \overset{deg0}{\mathcal{O}_X(-Y)} \Big]$$

which is nullhotomopic.

Consider also the sheaf of dg algebras $\mathcal{A} := [\mathcal{O}_X(-Y) \xrightarrow{x} \mathcal{O}_X]$. It is quasi isomorphic to $\mathcal{O}_Y$ as sheaves of dg algebras on X. The quasi isomorphism $\mathcal{A} \to \mathcal{O}_Y$ also induces a quasi isomorphism on Hochschild complexes for any affine open $V \subset X$, $C_\bullet(\mathcal{A}|_V) \xrightarrow{\sim} C_\bullet(\mathcal{O}_Y|_V)$.

We define the dg category $\mathrm{perf}_{\check{C}}(\mathcal{A})$ to have objects, same as $\mathrm{perf}(X)$. The homomorphism complexes are given by

$$\mathrm{Hom}_{\mathrm{perf}_{\check{C}}(\mathcal{A})}(\mathcal{V}, \mathcal{W}) := \check{C}(\mathscr{U}, \underline{\mathrm{Hom}}^\bullet_{\mathcal{A}}(\mathcal{V} \otimes \mathcal{A}, \mathcal{W} \otimes \mathcal{A})).$$

We now introduce a dg functor $\mathrm{Ind} : \mathrm{perf}_{\check{C}}(\mathcal{A}) \to \mathrm{perf}_{\check{C}}(Y)$. On objects it is defined by

$$\mathrm{Ind}(\mathcal{V}) = \mathcal{V} \otimes i_*\mathcal{O}_Y.$$

The chain maps on homomorphism complexes are induced by the following morphism of complexes of sheaves

$$\begin{aligned} \underline{\mathrm{Hom}}^\bullet_{\mathcal{A}}(\mathcal{V} \otimes \mathcal{A}, \mathcal{W} \otimes \mathcal{A}) &\xrightarrow{-\otimes_{\mathcal{A}} i_*\mathcal{O}_Y} \underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_Y}(\mathcal{V} \otimes \mathcal{A} \otimes_{\mathcal{A}} i_*\mathcal{O}_Y, \mathcal{W} \otimes \mathcal{A} \otimes_{\mathcal{A}} i_*\mathcal{O}_Y) \\ &\xrightarrow{\cong} \underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_Y}(\mathcal{V} \otimes i_*\mathcal{O}_Y, \mathcal{W} \otimes i_*\mathcal{O}_Y). \end{aligned}$$

Lemma 3.1. *The functor Ind defined above induces an isomorphism in $D_{Z/2}(\mathbb{C})$*

$$Ind_* : C_\bullet(perf_{\check{C}}(\mathcal{A})) \to C_\bullet(perf_{\check{C}}(Y)).$$

Proof. For a dg category $\mathscr{C}$, let $\widehat{\mathscr{C}}$ denote its pretriangulated hull (see section 2.1). We get a strictly commutative diagram of dg functors

$$\begin{array}{ccc} \widehat{\mathrm{perf}_{\check{C}}(\mathcal{A})} & \xrightarrow{\widehat{\mathrm{Ind}}} & \widehat{\mathrm{perf}_{\check{C}}(Y)} \\ \uparrow & & \uparrow \\ \mathrm{perf}_{\check{C}}(\mathcal{A}) & \xrightarrow{Ind} & \mathrm{perf}_{\check{C}}(Y) \end{array}$$

where the vertical functors induce isomorphism in Hochschild homology ([Kel99]) and the right vertical arrow is a quasi equivalence since $\mathrm{perf}_{\check{C}}(Y)$ is pretriangulated. Therefore it is enough to check that $\widehat{\mathrm{Ind}}$ induces an isomorphism in Hochschild homology. We show that $\widehat{\mathrm{Ind}}$ is in fact a quasi equivalence. To see that $\widehat{\mathrm{Ind}}$ induces a quasi isomorphism on homomorphism complexes we note that Ind applied to hom-complexes can be factored as a composite of quasi isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{perf}_{\check{C}}(\mathcal{A})}(\mathcal{V}^\bullet \otimes \mathcal{A}, \mathcal{W}^\bullet \otimes \mathcal{A}) &= \check{C}(\mathscr{U}, \underline{\mathrm{Hom}}^\bullet_{\mathcal{A}}(\mathcal{V}^\bullet \otimes \mathcal{A}, \mathcal{W}^\bullet \otimes \mathcal{A})) \\
&\simeq \check{C}(\mathscr{U}, \underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_X}(\mathcal{V}^\bullet, \mathcal{W}^\bullet \otimes \mathcal{A})) \\
&\simeq \check{C}(\mathscr{U}, (\mathcal{V}^\vee)^\bullet \otimes \mathcal{W}^\bullet \otimes \mathcal{A}) \\
&\xrightarrow{\sim} \check{C}(\mathscr{U}, (\mathcal{V}^\vee)^\bullet \otimes \mathcal{W}^\bullet \otimes \mathcal{O}_Y) \\
&\simeq \check{C}(\mathscr{U}, \underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_X}(\mathcal{V}^\bullet, \mathcal{W}^\bullet \otimes \mathcal{O}_Y)) \\
&\simeq \check{C}(\mathscr{U}, \underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_Y}(\mathcal{V}^\bullet \otimes \mathcal{O}_Y, \mathcal{W}^\bullet \otimes \mathcal{O}_Y)) \\
&\simeq \mathrm{Hom}_{\mathrm{perf}_{\check{C}}(Y)}(\mathrm{Ind}(\mathcal{V}^\bullet \otimes \mathcal{A}), \mathrm{Ind}(\mathcal{W}^\bullet \otimes \mathcal{A}))
\end{aligned}$$

This means that $H^0(\mathrm{Ind})$ is fully faithful and therefore so is $H^0(\widehat{\mathrm{Ind}})$. Therefore, to show that $H^0(\widehat{\mathrm{Ind}})$ is is essentially surjective it is enough to check that the smallest triangulated subcategory of $H^0(\widehat{\mathrm{perf}_{\check{C}}(Y)})$ containing the image of $H^0(\widehat{\mathrm{Ind}})$ is all of $H^0(\widehat{\mathrm{perf}_{\check{C}}(Y)})$. Let $\mathcal{O}_X(1)$ denote a very ample line bundle on X and let $\mathcal{O}_Y(1) = \mathcal{O}_X(1)|_Y$. We know that $\{\mathcal{O}_Y(n)\}_{n\in\mathbb{Z}}$ classically generates $H^0(\mathrm{perf}_{\check{C}}(Y)) \simeq H^0(\widehat{\mathrm{perf}_{\check{C}}(Y)})$. Essential surjectivity then follows from the observation that

$$\mathcal{O}_Y(n) = \mathrm{Ind}(\mathcal{O}_X(n)).$$

□

Let $\mathcal{A}^\#$ denote the underlying sheaf of graded algebras of $\mathcal{A}$ and $P^\#$ denote the underlying sheaf of $\mathbb{Z}/2$-graded $\mathcal{O}_X$-modules of P. As graded $\mathcal{O}_X$-modules we have $\mathcal{A}^\# = P^\#$ and this gives us a multiplication $\mu : \mathcal{A}^\# \otimes P^\# \to P^\#$. If $\mathcal{M}$ is an $\mathcal{A}$-module we can thus make sense of the tensor product $\mathcal{M} \otimes_{\mathcal{A}} P$ as a graded $\mathcal{O}_X$-module. In fact, the morphism

$$\mathcal{M} \otimes \mathcal{A} \otimes P \to \mathcal{M} \otimes P,\ m \otimes a \otimes p \mapsto ma \otimes p - m \otimes ap$$

whose cokernel by definition is the tensor product $\mathcal{M} \otimes_{\mathcal{A}} P$ is a morphism of matrix factorizations of W. Therefore the product $\mathcal{M} \otimes_{\mathcal{A}} P$ is naturally a matrix factorization of W. We get a dg functor

$$- \otimes_{\mathcal{A}} P : \mathrm{perf}_{\check{C}}(\mathcal{A}) \to \mathrm{vect}_{\check{C}}(X, W) \tag{3.1}$$

which on objects is defined by $\mathcal{V} \mapsto \mathcal{V} \otimes_X P$ and the chain maps on hom complexes is induced by the morphism of complexes of sheaves

$$\begin{tikzcd} \underline{\mathrm{Hom}}^\bullet_{\mathcal{A}}(\mathcal{V}\otimes\mathcal{A},\mathcal{W}\otimes\mathcal{A}) \arrow[r, "-\otimes_{\mathcal{A}}P"] & \underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_Y}(\mathcal{V}\otimes\mathcal{A}\otimes_{\mathcal{A}}P,\mathcal{W}\otimes\mathcal{A}\otimes_{\mathcal{A}}P) \arrow[dl, "\cong"] \\ \underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_Y}(\mathcal{V}\otimes P,\mathcal{W}\otimes P). & \end{tikzcd}$$

Lemma 3.2. *The following diagram commutes up to natural quasi isomorphism*

$$\begin{tikzcd} perf_{\check{C}}(\mathcal{A}) \arrow[r, "\otimes_{\mathcal{A}}P"] \arrow[d, "Ind"] & vect_{\check{C}}(X,W) \arrow[d, "Yoneda"] \\ perf_{\check{C}}(Y) \arrow[r, "i_*"] & perf(vect_{\check{C}}(X,W)) \end{tikzcd}$$

Proof. For any $Q \in \mathrm{vect}_{\check{C}}(X,W)$, the natural map

$$\mathrm{Hom}_{\mathrm{vect}_{\check{C}}(X,W)}(Q,\mathcal{V}\otimes P) \to \mathrm{Hom}_{\mathrm{fact}_{\check{C}}(X,W)}(Q,\mathcal{V}\otimes\mathcal{O}_Y)$$

is a quasi isomorphism where $\mathrm{fact}_{\check{C}}(X,W)$ is the DG category with objects same as $\mathrm{fact}(X,W)$ and whose homomorphism complexes are defined in the same way as in $\mathrm{vect}_{\check{C}}(X,W)$. $\square$

In fact, we can define similar functors

$$\mathrm{Ind}_U : \mathrm{perf}_{\check{C}}(\mathcal{A}|_U) \to \mathrm{perf}_{\check{C}}(Y\cap U),$$
$$-\otimes_{\mathcal{A}|_U} P|_U : \mathrm{perf}_{\check{C}}(\mathcal{A}|_U) \to \mathrm{vect}_{\check{C}}(U,W|_U)$$

for any open set U and assemble them into lax morphisms of presheaves of dg categories.

Proposition 3.3. *There are natural isomorphisms*

$$\alpha_{U,V,\mathcal{F}} : Ind_V(\mathcal{F}|_V) \implies Ind_U(\mathcal{F})|_V$$

and

$$\beta_{U,V,\mathcal{F}} : \mathcal{F}|_V \otimes_{\mathcal{A}|_V} P|_V \implies (\mathcal{F}\otimes_{\mathcal{A}|_U} P|_U)|_V$$

making

$$(Ind,\alpha) : \underline{perf}_{\check{C}}(\mathcal{A}) \to \underline{perf}_{\check{C}}(Y)$$

and

$$(-\otimes_{\mathcal{A}} P,\beta) : \underline{perf}_{\check{C}}(\mathcal{A}) \to \underline{vect}_{\check{C}}(X,W)$$

into lax morphisms of presheaves of dg categories.

Proof. For *Ind* this is just the statement that for $W \subset U \subset V$ we have a commutative diagram of natural isomorphisms

$$\begin{tikzcd}
\mathcal{V}|_W \otimes_W \mathcal{O}_{Y\cap W} & (\mathcal{V}|_V \otimes_V \mathcal{O}_{Y\cap V})|_W \arrow[l, "\cong"'] \\
& (\mathcal{V} \otimes_X \mathcal{O}_Y)|_W. \arrow[u, "\cong"] \arrow[ul, "\cong"]
\end{tikzcd}$$

It may be easier to see that the following diagram commutes

$$\begin{tikzcd}
(j_W)_*(\mathcal{V}|_W \otimes_W \mathcal{O}_{Y\cap W}) & (j_V)_*(\mathcal{V}|_V \otimes_V \mathcal{O}_{Y\cap V}) \arrow[l] \\
& \mathcal{V} \otimes_X \mathcal{O}_Y \arrow[u] \arrow[ul]
\end{tikzcd}$$

where $j_W : W \to U$ and $j_V : V \to U$ are the inclusions.

The case of $\otimes_{\mathcal{A}} P$ is similar. $\square$

3.3 HKR type map for $\mathcal{A}$

Let $\Omega^k_X(\log Y)$ denote the algebraic k-forms on X with logarithmic poles along Y. It is a subsheaf of $\Omega^k_X(Y)$ whose sections locally look like $\omega + \frac{dx}{x} \wedge \omega'$ where $\omega \in \Omega^k_X$, $\omega' \in \Omega^{k-1}_X$ and x is a local equation for Y. There are short exact sequences on X

$$0 \longrightarrow \Omega^k_X \longrightarrow \Omega^k_X(\log Y) \longrightarrow \Omega^{k-1}_Y \longrightarrow 0$$

where the first map is the inclusion and the second map is locally defined by $\beta + \frac{dx}{x}\alpha \mapsto \alpha|_Y$. (See [Staa] and [Stab] for details). To simplify notation later on we will define $\Omega^\bullet_X := \wedge^\bullet(\Omega_X[1])$ and similarly for $\Omega^\bullet_X(\log Y)$ and $\Omega^\bullet_Y$. On $\Omega^\bullet_X$ and $\Omega^\bullet_X(\log Y)$ we introduce a differential on $\Omega^\bullet_X$

$$-dW \wedge (-) : \Omega^k_X \to \Omega^{k+1}_X$$

and similarly on $\Omega^\bullet_X(\log Y)$. Note that according to our grading convention on $\Omega^\bullet_X$ and $\Omega^\bullet_X(\log Y)$ the differential defined above decreases degree by 1. This will not cause a problem because in the end we will think of them as $\mathbb{Z}/2$-graded complexes. These $\mathbb{Z}/2$-graded complexes will be denoted $\Omega^\bullet_{X,-W}$ and $\Omega^\bullet_{X,-W}(\log Y)$ respectively. The short exact sequences above, can be assembled into a short exact sequence of complexes:

$$0 \longrightarrow \Omega^\bullet_{X,-W}[-1] \xrightarrow{L} \Omega^\bullet_{X,-W}(\log Y)[-1] \longrightarrow (\Omega^\bullet_Y, 0) \longrightarrow 0.$$

(3.2)

It gives rise to a quasi isomorphism

$$\Omega^\bullet_{\mathcal{A}} := \mathrm{Cone}(L) \xrightarrow{\sim} \Omega^\bullet_Y. \tag{3.3}$$

We will construct a commutative diagram

$$\begin{tikzcd} \check{C}(\mathscr{U}, \underline{C}_\bullet(\mathcal{O}_Y)) \arrow[d, "HKR_Y"] & \check{C}(\mathscr{U}, \underline{C}_\bullet(\mathcal{A})) \arrow[l] \arrow[d, "HKR_{\mathcal{A}}"] \\ \check{C}(\mathscr{U}, \Omega^\bullet_Y) & \check{C}(\mathscr{U}, \Omega^\bullet_{\mathcal{A}}). \arrow[l] \end{tikzcd}$$

(3.4)

The horizontal arrows are induced by the quasi isomorphisms $\mathcal{A} \to \mathcal{O}_Y$ and (3.3) respectively and the left vertical arrow comes from the classical HKR theorem. It remains to define the map $HKR_{\mathcal{A}}$. Let $x_i \in \mathcal{O}_X(U_i)$ be local equations for $Y \subset X$. Let $u_{ij} = x_j x_i^{-1}$ on the intersections U_{ij}. Then we identify $\mathcal{A}|_{U_j} \cong [\mathcal{O}_{U_i}\epsilon_i \to \mathcal{O}_{U_i}]$ where ϵ_i is in degree -1 and the differential maps ϵ_i to x_i. Note that on the intersections we have $u_{ij}\epsilon_i = \epsilon_j$. Let $U_I = U_{i_0\cdots i_p}$ and let $\bar{a} := a_0[a_1|\cdots|a_k] \in \mathcal{A}^0(U_I)^{\otimes k+1} \subset C_k(\mathcal{A}(U_I))$. Then define

$$\begin{aligned} HKR_{\mathcal{A}}(\bar{a}) :=& \frac{1}{k!} a_0 \frac{dx_{i_0}}{x_{i_0}} \wedge da_1 \wedge ... \wedge da_k \\ &+ \frac{1}{k!} a_0 dx_{i_0} \wedge dg_{i_0} \wedge da_1 \wedge ... \wedge da_k \\ &- \sum_{j<i_0} \frac{(-1)^p}{k!} a_0 \frac{du_{i_0 j}}{u_{i_0 j}} \wedge da_1 \wedge ... \wedge da_k \\ &\in \Gamma\Big(U_I, \Omega^{k+1}_X(\log Y)) \oplus \Gamma(U_I, \Omega^{k+2}_X) \oplus (\bigoplus_{j<i_0} \Gamma(U_{j\cup I}, \Omega^{k+1}_X)\Big) \end{aligned}$$

and

$$HKR_{\mathcal{A}}(a_0[a_1|\cdots|a_l\epsilon_{i_0}|\cdots|a_k]) := \frac{(-1)^l}{k!} a_0 dx_{i_0} \wedge da_1 \wedge ... \wedge da_k.$$

and $HKR_{\mathcal{A}}$ vanishes on elements which have two or more factors from $\mathcal{A}^{-1}$.

Proposition 3.4. *The following is true about the map $HKR_{\mathcal{A}}$*
1) $HKR_{\mathcal{A}}$ is a chain map.
2) The diagram (4.4) commutes.
3) $HKR_{\mathcal{A}}$ is a quasi isomorphism.

Proof. We check first that $HKR_{\mathcal{A}}$ is a chain map. We do this first on elements of internal degree zero. As before let $U_I = U_{i_0 \cdots i_p}$ and let $\bar{a} := a_0[a_1|\cdots|a_k] \in \mathcal{A}^0(U_I)^{\otimes k+1} \subset C_k(\mathcal{A}(U_I))$.

$$
\begin{aligned}
HKR_{\mathcal{A}}(\bar{d}_2\bar{a}) &= 0\\
HKR_{\mathcal{A}}(\bar{d}_1\bar{a}) &= 0\\
HKR_{\mathcal{A}}(d_{Cech}\bar{a}) &= HKR_{\mathcal{A}}\Big(\sum_{j\notin I}(-1)^{Sgn(j,I)}\bar{a}|_{U_{I\cup j}}\Big)\\
&= HKR_{\mathcal{A}}\Big(\sum_{j<i_0}\bar{a}|_{U_{I\cup j}} + \sum_{j\notin I,\ j>i_0}(-1)^{Sgn(j,I)}\bar{a}|_{U_{I\cup j}}\Big)\\
&= \frac{1}{k!}\sum_{j<i_0} a_0\frac{dx_j}{xj}\wedge da_1\wedge ...\wedge da_k|_{j\cup I}\\
&+ \frac{1}{k!}\sum_{j<i_0} a_0 dx_j\wedge dg_j\wedge da_1\wedge ...\wedge da_k|_{j\cup I}\\
&- \frac{1}{k!}\sum_{j'<j<i_0}(-1)^{p+1}a_0\frac{du_{jj'}}{u_{jj'}}\wedge da_1\wedge ...\wedge da_k|_{j'\cup j\cup I}\\
&+ HKR_{\mathcal{A}}\Big(\sum_{j\notin I,\ j>i_0}(-1)^{Sgn(j,I)}\bar{a}|_{U_{I\cup j}}\Big).
\end{aligned}
\tag{3.5}
$$

On the other hand we have

$$
\begin{aligned}
d_{Cech}HKR_{\mathcal{A}}(\bar{a}) &= \frac{1}{k!}\sum_{j<i_0} a_0\frac{dx_{i_0}}{x_{i_0}}\wedge da_1\wedge ...\wedge da_k|_{j\cup I}\\
&+ \frac{1}{k!}\sum_{j<i_0} a_0 dx_{i_0}\wedge dg_{i_0}\wedge da_1\wedge ...\wedge da_k|_{j\cup I}\\
&- \frac{1}{k!}\sum_{j<j'<i_0}(-1)^p a_0\Big(\frac{du_{i_0j'}}{u_{i_0j'}} - \frac{du_{i_0j}}{u_{i_0j}}\Big)\wedge da_1\wedge ...\wedge da_k|_{j\cup j'\cup I}\\
&+ \sum_{j\notin I, j>i_0}(-1)^{Sgn(j,I)}HKR_{\mathcal{A}}(\bar{a})|_{j\cup I}
\end{aligned}
\tag{3.6}
$$

and

$$\begin{aligned}\bar{d}_{Cone}HKR_{\mathcal{A}}(\bar{a}) &= \frac{(-1)^p}{k!}a_0 dW \wedge \frac{dx_{i_0}}{x_{i_0}} \wedge da_1 \wedge ... \wedge da_k \\ &+ \frac{(-1)^p}{k!}L(a_0 dx_{i_0} \wedge dg_{i_0} \wedge da_1 \wedge ... \wedge da_k) \\ &- \sum_{j<i_0} \frac{1}{k!}a_0 dW \wedge \frac{du_{i_0 j}}{u_{i_0 j}} \wedge da_1 \wedge ... \wedge da_k|_{j\cup I} \\ &+ \sum_{j<i_0} \frac{1}{k!}L(a_0 \frac{du_{i_0 j}}{u_{i_0 j}} \wedge da_1 \wedge ... \wedge da_k)|_{j\cup I}\end{aligned} \tag{3.7}$$

From here one can check that $(\bar{d}_{Cone}+d_{Cech})HKR_{\mathcal{A}}(\bar{a}) = HKR_{\mathcal{A}}(d_{Cech}+\bar{d}_1+\bar{d}_2)(\bar{a})$. Indeed the first sum in (3.5) cancels with with the first sum in (3.6) and the last sum in (3.7). The second sum in (3.5) cancels with the second sum in (3.6) and the sum on the third row of (3.7). The third sum in (3.5) cancels with the third sum in (3.6). The last sum in (3.5) cancels with the last sum in (3.6). Finally the first two terms in (3.7) cancel with each other.

Next, let's check that $(\bar{d}_{Cone} + d_{Cech})HKR_{\mathcal{A}}(\bar{b}) = HKR_{\mathcal{A}}(d_{Cech} + \bar{d}_1 + \bar{d}_2)(\bar{b})$ where $\bar{b} = b_0[b_1|\cdots|b_l\epsilon_{i_0}|\cdots|b_k]$ has internal degree -1. We have

$$\begin{aligned}HKR_{\mathcal{A}}(\bar{d}_2\bar{b}) &= 0 \\ HKR_{\mathcal{A}}(\bar{d}_1\bar{b}) &= (-1)^{p+l}\frac{1}{k!}b_0\frac{dx_{i_0}}{x_{i_0}} \wedge db_1 \wedge ... \wedge d(b_l x_{i_0}) \wedge ... \wedge db_k \\ &+ (-1)^{p+l}\frac{1}{k!}b_0 dx_{i_0} \wedge dg_{i_0} \wedge db_1 \wedge ... \wedge d(b_l x_{i_0}) \wedge ... \wedge db_k \\ &- \sum_{j<i_0}\frac{(-1)^l}{k!}b_0\frac{du_{i_0 j}}{u_{i_0 j}} \wedge db_1 \wedge ... \wedge d(b_l x_{i_0}) \wedge ... \wedge db_k \\ HKR_{\mathcal{A}}(d_{Cech}\bar{b}) &= \sum_{j\notin I,\ j>i_0} (-1)^{l+Sgn(j,I)}\frac{1}{k!}b_0 dx_{i_0} \wedge db_1 \wedge ... \wedge db_l \wedge ... \wedge db_k \\ &+ \sum_{j<i_0}(-1)^l\frac{1}{k!}b_0 dx_j \wedge db_1 \wedge ... \wedge d(u_{ji_0}b_l) \wedge ... \wedge db_k\end{aligned}$$

On the other hand we have

$$\begin{aligned}d_{Cech}HKR_{\mathcal{A}}(\bar{a}) &= \sum_{j\notin I}(-1)^{Sgn(j,I)+l}\frac{1}{k!}b_0 dx_{i_0} \wedge db_1 \wedge ... \wedge db_l \wedge ... \wedge db_k \\ \bar{d}_{Cone}HKR_{\mathcal{A}}(\bar{b}) &= (-1)^{p+1+l}\frac{1}{k!}b_0 dW \wedge dx_{i_0} \wedge db_1 \wedge ... \wedge db_l \wedge ... \wedge db_k \\ &+ (-1)^{p+l}\frac{1}{k!}L(b_0 dx_{i_0} \wedge db_1 \wedge ... \wedge db_l \wedge ... \wedge db_k)\end{aligned}$$

Again the relevant equality can be checked from these formulas. This proves the first part of the proposition. The last thing one has to check for the first part is that $HKR_{\mathcal{A}} \circ \bar{d}_1$ vanishes on elements with two factors from $\mathcal{A}^{-1}$.

The second part can be checked from the definitions. The last part then follows from the second because all the other arrows in the diagram are quasi isomorphisms. $\square$

3.4 Multiplying by the Todd class

We have the wedge product map $\Omega_X^\bullet \otimes \Omega_X^\bullet \to \Omega_X^\bullet$ which we can use to make $\check{C}(\mathscr{U}, \Omega_X^\bullet)$ into a dg algebra (see section 1.2 for the precise formula). The natural map $\check{C}(\mathscr{U}, \Omega_X^\bullet) \to \check{C}(\mathscr{U}, \Omega_Y^\bullet)$ is a dg algebra homomorphism. We make $\check{C}(\mathscr{U}, \Omega_{\mathcal{A}}^\bullet)$ into a left dg $\check{C}(\mathscr{U}, \Omega_X^\bullet)-$module as follows. For $\alpha \in \check{C}^p(\mathscr{U}, \Omega_X^q)$, $\beta \in \check{C}^a(\mathscr{U}, \Omega_X^b)$ and $\gamma \in \check{C}^{a'}(\mathscr{U}, \Omega_X^{b'}(\log Y))$ we define

$$(\alpha \cdot \beta)_{i_0 \cdots i_{p+a}} = (-1)^{qa} \alpha_{i_0 \cdots i_p}|_{U_{i_0 \cdots i_{p+a}}} \wedge \beta_{i_p \cdots i_{p+a}}|_{U_{i_0 \cdots i_{p+a}}}$$

and

$$(\alpha \cdot \gamma)_{i_0 \cdots i_{p+a'}} = (-1)^{q(a'+1)} \alpha_{i_0 \cdots i_p}|_{U_{i_0 \cdots i_{p+a'}}} \wedge \gamma_{i_p \cdots i_{p+a'}}|_{U_{i_0 \cdots i_{p+a'}}}.$$

The Cech cocycle $(u_{ij}^{-1})_{j<i} \in \check{C}^1(\mathscr{U}, \mathcal{O}_X^\times)$ corresponds to the line bundle $\mathcal{O}_X(Y)$. The Chern class $c_1(-Y)$ is the Cech cocycle $(u_{ij}^{-1} du_{ij})_{j<i} \in \check{C}^1(\mathscr{U}, \Omega_X^1)$. The relative Todd class of $Y \subset X$, denoted $Td_{Y/X}$ is defined as the formal power series $\frac{x}{1-e^{-x}}$ evaluated at $c_1(-Y)$. It's inverse is

$$\mathrm{Td}_{Y/X}^{-1} := -\sum_{q \geq 0} \frac{c_1(-Y)^{\wedge q}}{(q+1)!}.$$

Proposition 3.5. *The map*

$$Td_{Y/X}^{-1} \wedge (-) : \check{C}(\mathscr{U}, \Omega_{\mathcal{A}}^\bullet) \to \check{C}(\mathscr{U}, \Omega_{\mathcal{A}}^\bullet)$$

commutes with the differentials and gives rise to a commutative diagram

$$\begin{tikzcd}
\check{C}(\mathscr{U}, \Omega_Y^\bullet) \arrow[d, "Td_{Y/X}^{-1} \wedge (-)"] & \check{C}(\mathscr{U}, \Omega_{\mathcal{A}}^\bullet) \arrow[l] \arrow[d, "Td_{Y/X}^{-1} \wedge (-)"] \\
\check{C}(\mathscr{U}, \Omega_Y^\bullet) & \check{C}(\mathscr{U}, \Omega_{\mathcal{A}}^\bullet) \arrow[l]
\end{tikzcd}$$

Proof. This is just a matter of keeping track of the signs. $\square$

3.5 Trace map for a small cdg category of quasi modules

Let $\{P_1, ..., P_r\}$ be a collection of objects in qvect(X, W) and let $\mathscr{C}$ be the presheaf of cdg categories defined by $U \mapsto \langle P_1|_U, ..., P_r|_U\rangle$ where the latter refers to the full sub cdg category of qvect$(U, W|_U)$ on the objects $P_1|_U, ..., P_r|_U$. Assume that we have trivialisations $P_i|_{U_j} \cong \mathcal{O}_{U_j}^{n_i} \oplus \mathcal{O}_{U_j}^{m_i}$. Our goal is to define a map

$$\phi : \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\mathscr{C})) \to \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\mathcal{O}_{-W})) \tag{3.8}$$

such that if $(P_i, \delta_{P_i}) = (\mathcal{O}_X, 0)$ for some $1 \le i \le r$ then ϕ is a quasi inverse to the quasi isomorphism

$$\check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\mathcal{O}_{-W})) \to \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\mathscr{C})) \tag{3.9}$$

induced by the Yoneda functor.
The map ϕ is made up of three components and to define it we need to first introduce some intermediate presheaves of cdg categories. For $i = 1, ..., r$ let $\overline{P}_i$ denote the object in qvect(X, W) whose underlying graded $\mathcal{O}_X$-module is the same as P_i but whose predifferential is zero and whose curvature is $-W \in \mathrm{End}_X(P_i)$. Let $\overline{\mathscr{C}}$ be the presheaf of cdg categories $U \mapsto \langle \overline{P}_1, ..., \overline{P}_r\rangle$. Now, for $\bar{\alpha} := \alpha_0[\alpha_1|\cdots|\alpha_k] \in C^{II}_\bullet(\mathscr{C}(V))$ define

$$Sh(\underbrace{[\delta|\cdots|\delta]}_{n factors}, \bar{\alpha}) := \sum_{i_0+...+i_k=n} \alpha_0[\delta^{i_0}|\alpha_1|\delta^{i_1}|\alpha_2|\cdots|\alpha_k|\delta^{i_k}]$$

where in the formula on the right δ^i means $\delta^{\otimes i}$.

Lemma 3.6. *The map*

$$SH : \bar{\alpha} \mapsto \sum_{n\ge 0}(-1)^n Sh(\underbrace{[\delta|\cdots|\delta]}_{n factors}, \bar{\alpha})$$

is defines a morphism of presheaves of chain complexes $\underline{C}^{II}_\bullet(\mathscr{C}) \to \underline{C}^{II}_\bullet(\overline{\mathscr{C}})$

Proof. We have to show that $(d_0 + d_1 + d_2)SH = SH(d_0 + d_1 + d_2)$. Let $SH^n(\alpha)$, denote the $n'th$ term in the sum above. Then $d_2SH^n(\bar{\alpha})$ will consist of terms of the form

$$\alpha_0[\delta^{i_0}|\alpha_1|\delta^{i_1}|\alpha_2|\cdots|\alpha_j\alpha_{j+1}|\cdots|\alpha_k|\delta^{i_k}],$$
$$\alpha_0[\delta^{i_0}|\alpha_1|\delta^{i_1}|\alpha_2|\cdots|\delta\alpha_j|\cdots|\alpha_k|\delta^{i_k}],$$

$$\alpha_0[\delta^{i_0}|\alpha_1|\delta^{i_1}|\alpha_2|\cdots|\alpha_j\delta|\cdots|\alpha_k|\delta^{i_k}],$$
$$\alpha_0[\delta^{i_0}|\alpha_1|\delta^{i_1}|\alpha_2|\cdots|\delta\delta|\cdots|\alpha_k|\delta^{i_k}].$$

Now the terms of the first kind above give us precisely $SH^n(d_2(\bar{\alpha}))$. The terms of the second and third kind above, will give us precisely $SH^{n-1}(d_1(\bar{\alpha}))$. The terms of the last kind above, will give us $SH^{n-2}(d_0(\bar{\alpha})) - d_0 SH^{n-2}(\bar{\alpha})$ because $\delta_P^2 = h_P + W$. $\square$

The morphism SH from the previous lemma gives rise to a map $\check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\mathscr{C})) \to \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\overline{\mathscr{C}}))$ which we will also call SH.

We will also need the preasheaf of cdg categories $\mathscr{F}$ which to an open set U assigns the subcategory of $\mathrm{qvect}(U, W|_U)$ on all objects of the form

$$\left[\mathcal{O}_U^{\oplus n} \underset{0}{\overset{0}{\rightleftarrows}} \mathcal{O}_U^{\oplus m}\right].$$

We will define a collection of maps

$$h^q : \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\overline{\mathscr{C}})) \to \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\mathscr{F})).$$

First some notation: Given a morphism $\alpha : P_i(U_l) \to P_j(U_l)$ we will denote by $(\alpha)_l$ the matrix defined by

$$\mathcal{O}_{U_l}^{n_i} \oplus \mathcal{O}_{U_l}^{m_i} \cong P_i \to P_j \cong \mathcal{O}_{U_l}^{n_j} \oplus \mathcal{O}_{U_l}^{m_j}.$$

Let g_{kl} denote the change of basis matrix such that

$$g_{kl}(\alpha)_l g_{kl}^{-1} = (\alpha)_k$$

on $U_k \cap U_l$. Now, let $\bar{\alpha} := \alpha_0[\alpha_1|\cdots|\alpha_k] \in C^{II}_\bullet(\overline{\mathscr{C}})(U_{i_0\cdots i_p})$ and define

$$\begin{aligned} h^q(\bar{\alpha}) := \\ \sum (-1)^{\epsilon+pq+\binom{q}{2}} g_{i_0 i_{-q}}(\alpha_0)_{i_{-q}}\Big[(\alpha_1)_{i_{-q}}\Big|\cdots\Big|(\alpha_{l_1})_{i_{-q}}\Big|g^{-1}_{i_{1-q}i_{-q}}\Big|(\alpha_{l_1+1})_{i_{1-q}}\Big|\cdots \\ \cdots\Big|(\alpha_{l_2})_{i_{1-q}}\Big|g^{-1}_{i_{2-q}i_{1-q}}\Big|(\alpha_{l_2+1})_{i_{2-q}}\Big|\cdots\Big|(\alpha_{l_q})_{i_{-1}}\Big|g^{-1}_{i_0 i_{-1}}\Big|(\alpha_{l_q+1})_{i_0}\Big|\cdots\Big] \end{aligned}$$

where the sum is over all q-tuples $1 \leq i_{-q} < i_{1-q} < i_{2-q} < ... < i_{-1} < i_0$ and all q-tuples $0 \leq l_1 \leq ... \leq l_q \leq k$. The sign is given by

$$\epsilon = \sum_{i=1}^{q}\left(l_i + \sum_{j=0}^{l_i}|a_j|\right)$$

Lemma 3.7. *We have*

$$\bar{d}_2 h^q + d_{Cech} h^{q-1} = h^{q-1} d_{Cech} + h^q \bar{d}_2.$$

Proof. We apply both sides to $\bar{\alpha} = \alpha_0[\alpha_1|\cdots|\alpha_k]$ which is a section over $U_{i_0 \ldots i_p}$. First, $\bar{d}_2 h^q(\bar{\alpha})$ will consist of terms of the form

$$\begin{aligned}
&(-1)^{\epsilon_1+pq+\binom{q}{2}+p+q+|a_0|} g_{i_0 i_{1-q}} (\alpha_0)_{i_{1-q}} \Big[(\alpha_1)_{i_{1-q}} \Big| \cdots \Big], \\
&(-1)^{\epsilon_2+pq+\binom{q}{2}+p+q+|a_0|+\ldots+|a_l|+l+j} g_{i_0 i_{-q}} (\alpha_0)_{i_{-q}} \Big[\cdots \Big| g^{-1}_{i_{j+1-q} i_{j-1-q}} \Big| \cdots \Big], \\
&(-1)^{\epsilon_3+pq+\binom{q}{2}+p+q+|a_0|+\ldots+|a_r|+r+j} g_{i_0 i_{-q}} (\alpha_0)_{i_{-q}} \Big[\cdots \Big| (\alpha_r \alpha_{r+1})_{j-q} \Big| \cdots \Big], \\
&(-1)^{\epsilon_4+pq+\binom{q}{2}+p+q+|a_0|+\ldots+|a_{l_j}|+l_j+j} g_{i_0 i_{-q}} (\alpha_0)_{i_{-q}} \Big[\cdots \Big| g^{-1}_{i_{j-q} i_{j-1-q}} (\alpha_{l_j+1})_{i_{j-q}} \Big| \cdots \Big] \\
&(-1)^{\epsilon_5+pq+\binom{q}{2}+p+q+|a_0|+\ldots+|a_{l_j+1}|+l_j+j} g_{i_0 i_{-q}} (\alpha_0)_{i_{-q}} \Big[\cdots \Big| (\alpha_{l_j+1})_{i_{j-1-q}} g^{-1}_{i_{j-q} i_{j-1-q}} \Big| \cdots \Big] \\
&(-1)^{\epsilon_6+pq+\binom{p}{q}+p+q+1+|a_0|+\ldots+|a_k|+k+q-1} g_{i_{-1} i_{-q}} (\alpha_0)_{i_{-q}} \Big[\cdots \Big], \\
&(-1)^{\epsilon_7+pq+\binom{q}{2}+p+q+1+(|a_k|+1)(|a_0|+\ldots+|a_{k-1}|+q+k-1)} g_{i_0 i_{-q}} (\alpha_k \alpha_0)_{i_{-q}} \Big[\cdots \Big]
\end{aligned} \tag{3.10}$$

Next, $d_{Cech} h^{q-1}$ will consist of terms of the form

$$\begin{aligned}
&(-1)^{\epsilon_8+p(q-1)+\binom{q-1}{2}} g_{i_0 i_{1-q}} (\alpha_0)_{i_{1-q}} \Big[(\alpha_1)_{i_{1-q}} \Big| \cdots \Big], \\
&(-1)^{\epsilon_9+p(q-1)+\binom{q-1}{2}+j} g_{i_0 i_{-q}} (\alpha_0)_{i_{-q}} \Big[\cdots \Big| g^{-1}_{i_{j+1-q} i_{j-1-q}} \Big| \cdots \Big], \\
&(-1)^{q+l} h^{q-1}(\bar{\alpha})|_{i_{1-q} \ldots i_0 \ldots i_l j i_{l+1} \ldots i_p}
\end{aligned} \tag{3.11}$$

Also, $h^{q-1} d_{Cech}$ will consist of terms of the form

$$\begin{aligned}
&(-1)^{l+1} h^{q-1}(\bar{\alpha}|_{i_0 \ldots i_l j i_{l+1} \ldots i_p}) \\
&(-1)^{\epsilon_{10}+(p+1)q+\binom{q}{2}} g_{i_{-1} i_{-q}} (\alpha_0)_{i_{-q}} \Big[\cdots \Big].
\end{aligned} \tag{3.12}$$

Finally, $h^q \bar{d}_2$ will contain terms of the form

$$\begin{aligned}
&(-1)^{\epsilon_{11}+pq+\binom{q}{2}+p+|\alpha_0|+\ldots+|\alpha_r|+r} g_{i_0 i_{-q}} (\alpha_0)_{i_{-q}} \Big[\cdots \Big| (\alpha_r \alpha_{r+1})_{j-q} \Big| \cdots \Big] \\
&(-1)^{\epsilon_{12}+pq+\binom{q}{2}+p+1+(|\alpha_k|+1)(|\alpha_0|+\ldots+|\alpha_{k-1}|+k-1)} g_{i_0 i_{-q}} (\alpha_k \alpha_0)_{i_{-q}} \Big[\cdots \Big]
\end{aligned} \tag{3.13}$$

Note that, the ϵ's in the signs are all different. The first term in (3.10) will cancel with the first term in (3.11) because $\epsilon_1 = \epsilon_8 + |\alpha_0|$. Similarly, by keeping track of the signs we see that the second term in (3.10) cancels with the second term in

(3.11). The third term in (3.10) cancels with the first term in (3.13). The fourth and fifth terms in (3.10) cancel. The sixth term in (3.10) cancels with the second term in (3.12). The last term in (3.10) cancels with the last term in (3.13). Finally the last term in (3.11) cancels with the first term in (3.12) $\square$

Clearly the h^q commute with $\bar{d}_0$ and since there is no d_1's in $\overline{\mathscr{C}}$ or $\mathscr{F}$ the previous lemma in fact tells us that $\sum_{q\geq 0} h^q$ defines a chain map $\check{C}(\mathscr{U}, \underline{C}^{II}_{\bullet}(\overline{\mathscr{C}})) \to \check{C}(\mathscr{U}, \underline{C}^{II}_{\bullet}(\mathscr{F}))$.

Now we introduce the last ingredient to the map (3.8). Suppose R is a commutative ring and $P_0, ..., P_m$ are a collection of $\mathbb{Z}/2$-graded free modules. Pick homogeneous bases $\{e_i^j\}$ for each P_i. Let $F^0 \in \mathrm{Hom}(P_1, P_0)$, $F^1 \in \mathrm{Hom}(P_2, P_1)$,..., $F^m \in \mathrm{Hom}(P_0, P_m)$ be endomorphisms which we can think of as matrices with the fixed choice of bases. We define

$$sTr(F^0 \otimes F^1 \otimes ... \otimes F^m) := \sum (-1)^{\sigma} F^0_{j_0 j_1} \otimes F^1_{j_1 j_2} \otimes ... \otimes F^m_{j_m j_0}$$

where the sum is over all $j_0, ..., j_m$ and the sign is given by

$$\sigma = (m+1)|e_0^{j_0}| + |e_1^{j_1}| + ... + |e_m^{j_m}|.$$

Extending this over infinite sums gives a map of presheaves $\underline{C}^{II}_{\bullet}(\mathscr{F}) \to \underline{C}^{II}_{\bullet}(\mathcal{O}_{-W})$ from which we obtain a chain map $\check{C}(\mathscr{U}, \underline{C}^{II}_{\bullet}(\mathscr{F})) \to \check{C}(\mathscr{U}, \underline{C}^{II}_{\bullet}(\mathcal{O}_W))$.

Definition 3.8. We define

$$\phi : \check{C}(\mathscr{U}, \underline{C}^{II}_{\bullet}(\mathscr{C})) \to \check{C}(\mathscr{U}, \underline{C}^{II}_{\bullet}(\mathcal{O}_{-W}))$$

by

$$\phi = \sum_{n,q\geq 0} (-1)^n sTr\Big(h^q\big(Sh(\underbrace{[\delta|\cdots|\delta]}_{n \ factors}, -)\big)\Big)$$

Theorem 3.9. *(i) ϕ is a chain map.*
(ii) If $(P_i, \delta_i) = (\mathcal{O}_X, 0)$ for some i then ϕ is a quasi inverse to the quasi isomorphism (3.9).

Proof. For the first part, we simply note that it is the composite of the three chain maps described earlier in this section.

One can compute that the composite $\underline{\check{C}}(\mathscr{U}, \underline{C}^{II}_{\bullet}(\mathcal{O}_{-W})) \to \underline{\check{C}}(\mathscr{U}, \underline{C}^{II}_{\bullet}(\mathscr{C})) \to \underline{\check{C}}(\mathscr{U}, \underline{C}^{II}_{\bullet}(\mathcal{O}_{-W})) \to \underline{\check{C}}(\mathscr{U}, (\Omega^{\bullet}_X, -dW \wedge -))$ equals the map $HKR_{(X,-W)}$ from proposition 2.29. From this the second part follows. $\square$

Recall that $Y \subset X$ is locally cut out by $x_i \in \mathcal{O}_X(U_i)$ and that $u_{ij}x_i = x_j$ over U_{ij}. Recall also the matrix factorization P from section 3.2. Over each of the open sets U_i we can identify $P = \mathcal{O}_X\epsilon_i \oplus \mathcal{O}_X$. Suppose F_i is a matrix describing the action of an element of $\mathrm{End}(P|_{U_i})$ in this basis. If F_j denotes the matrix of the same endomorphism but with respect to the basis $P = \mathcal{O}_X\epsilon_j \oplus \mathcal{O}_X$ over U_{ji} then $g_{ij}F_jg_{ij}^{-1} = F_i$ where the change of basis matrix is

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & u_{ij} \end{bmatrix}.$$

Lemma 3.10. *The following diagram commutes in $D_{Z/2}(\mathbb{C})$*

$$\begin{tikzcd}
\check{C}(\mathscr{U}, \underline{C}_\bullet(\mathcal{A})) \arrow[r, "can"] \arrow[d, "HKR_{\mathcal{A}}"] & \check{C}(\mathscr{U}, \underline{C}_\bullet(\mathcal{E}nd(P^\bullet))) \arrow[r, "\phi"] & \check{C}(\mathscr{U}, \underline{C}_\bullet^{II}(\mathcal{O}_{-W})) \arrow[d, "HKR_{\mathcal{O}_{-W}}"] \\
\check{C}(\mathscr{U}, \Omega^\bullet_{\mathcal{A}}) \arrow[r, "Td^{-1}_{Y/X}\wedge"] & \check{C}(\mathscr{U}, \Omega^\bullet_{\mathcal{A}}) \arrow[r, "\delta"] & \check{C}(\mathscr{U}, \Omega^\bullet_X, -dW\wedge)
\end{tikzcd}$$

Proof. Start with an element $a_0[a_1|\cdots|a_k]$ of internal degree zero over U_I, $I = \{i_0 < i_1 < ... < i_p\}$. Note that the only terms from ϕ which do not vanish under $HKR_{\mathcal{O}_{-W}}$ are those where $n = 0$ and $q \geq 1$ or $n = 2$ and $q \geq 0$. These are mapped to

$$\sum_{q\geq 1}\sum \frac{(-1)^{1+pq+\binom{q}{2}}}{k!q!} u_{i_0i_{-q}}a_0du^{-1}_{i_{1-q}i_{-q}} \wedge ... \wedge du^{-1}_{i_0i_{-1}} \wedge da_1 \wedge ... \wedge da_k$$

$$\sum_{q\geq 0}\sum \frac{(-1)^{pq+\binom{q}{2}}}{k!(q+1)!} u_{i_0i_{-q}}a_0du^{-1}_{i_{1-q}i_{-q}} \wedge ... \wedge du^{-1}_{i_0i_{-1}} \wedge dx_{i_0} \wedge dg_{i_0} \wedge da_1 \wedge ... \wedge da_k$$

where the second sums are over all $\{(i_{-q}, ..., i_{-1})|i_{-q} < i_{1-q} < ... < i_{-1} < i_0\}$. This is precisely what we get when going down first and then to the right twice.

Now let's start with an element of with precisely one factor from $\mathcal{A}^{-1}$, $\bar{b} = b_0[b_1|\cdots|b_l\epsilon_{i_0}|\cdots|b_k]$ and first go to the right twice and then down. This time, the only terms in the definition of ϕ which don't vanish under $HKR_{\mathcal{O}_{-W}}$ is $n = 1$ and $q \geq 0$. We get

$$\sum \frac{(-1)^{l+pq+\binom{q}{2}}}{k!(q+1)!} u_{i_0i_{-q}}b_0du^{-1}_{i_{1-q}i_{-q}} \wedge ... \wedge du^{-1}_{i_0i_{-1}} \wedge dx_{i_0} \wedge db_1 \wedge ... \wedge db_k$$

which agrees with what we get going down and then to the right.

Finally if we start with an element with two or more factors from $\mathcal{A}^{-1}$, then going either direction is zero. this is clear if we go down first because $HKR_{\mathcal{A}}$ vanishes on such elements by definition. For the other direction note that the only terms in the definition of ϕ which are non-zero on such an element are the ones where n equals the number of factors from $\mathcal{A}^{-1}$ so $n \geq 2$. Each such term will contain two or more factors x_i and will therefore vanish when we apply $HKR_{\mathcal{O}_{-W}}$ because $dx_i \wedge dx_i = 0$. $\square$

Lemma 3.11. *There is a commutative diagram in $D_{Z/2}(\mathbb{C})$*

$$\begin{tikzcd}
C_\bullet(perf_{\check{C}}(Y)) \arrow[d, "\sim"] & C_\bullet(perf_{\check{C}}(\mathcal{A})) \arrow[l, "\sim"'] \arrow[d, "\sim"] \\
\check{C}(\mathscr{U}, \Omega^\bullet_Y) & \check{C}(\mathscr{U}, \Omega^\bullet_{\mathcal{A}}) \arrow[l, "\sim"']
\end{tikzcd}$$

Proof. We will divide the diagram into smaller commutative pieces where each arrow is a chain map

$$\begin{tikzcd}
C_\bullet(\mathrm{perf}_{\check{C}}(Y)) \arrow[d] & C_\bullet(\mathrm{perf}_{\check{C}}(\mathcal{A})) \arrow[l] \arrow[d] \arrow[dl, phantom, "(A)"] \\
\check{C}(\mathscr{U}, \underline{C}_\bullet(\underline{\mathrm{perf}}_{\check{C}}(Y))) & \check{C}(\mathscr{U}, \underline{C}_\bullet(\underline{\mathrm{perf}}_{\check{C}}(\mathcal{A}))) \arrow[l] \\
\check{C}(\mathscr{U}, \underline{C}_\bullet(\mathcal{O}_Y)) \arrow[u] \arrow[d] & \check{C}(\mathscr{U}, \underline{C}_\bullet(\mathcal{A})) \arrow[l] \arrow[u] \arrow[d] \arrow[ul, phantom, "(B)"] \arrow[dl, phantom, "(C)"] \\
\check{C}(\mathscr{U}, \Omega^\bullet_Y) & \check{C}(\mathscr{U}, \Omega^\bullet_{\mathcal{A}}) \arrow[l]
\end{tikzcd}$$

Note first that all the vertical arrows on the left are quasi isomorphisms by the discussion in section 2.5. We will see below that the horizontal arrows in the squares $(A) - (C)$ are quasi isomorphisms and then it follows from commutativity (which is also dealt with below) that the vertical arrows on the right too are quasi isomorphisms.

In the square labeled (A) the top arrow is a quasi isomorphism by lemma 3.1. The bottom arrow in this square is $\check{C}(\mathrm{Ind}, \alpha)$ where (Ind, α) is the lax morphism of presheaves of dg categories from proposition 3.3. Commutativity of (A) is precisely proposition 2.21. The bottom arrow in this square can be shown to be a quasi isomorphism by filtering both complexes by Cech degree and using the mapping theorem ([Lan95],theorem XI.3.4).

To see that the square (B) commutes, note that that the top arrow is induced by a lax morphism of presheaves of dg categories and the other three by actual morphisms of presheaves of dg categories. Commutativity no follows from lemma 2.24 and the discussion right before it.

Commutativity of the square (C) is part of proposition 3.4.

□

Lemma 3.12. *There is a commutative diagram in $D_{Z/2}(\mathbb{C})$*

$$\begin{tikzcd}
C_\bullet(perf_{\check{C}}(\mathcal{A})) \arrow[rr] \arrow[d, "\sim"] & & C_\bullet(vect_{\check{C}}(X,W)) \arrow[d, "\sim"] \\
\check{C}(\mathscr{U}, \underline{C}_\bullet(\mathcal{A})) \arrow[r, "can"] & \check{C}(\mathscr{U}, \underline{C}_\bullet(\underline{\mathrm{End}}(P))) \arrow[r, "\phi"] & \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\mathcal{O}_{-W}))
\end{tikzcd}$$

Proof. We will divide the diagram into smaller commutative pieces where each arrow is a chain map:

$$\begin{tikzcd}
C_\bullet(\mathrm{perf}_{\check{C}}(\mathcal{A})) \arrow[rr] \arrow[d, "\sim"] \arrow[drr, phantom, "(A)"] & & C_\bullet(\mathrm{vect}_{\check{C}}(X,W)) \arrow[d, "\sim"] \\
\check{C}(\mathscr{U}, \underline{C}_\bullet(\underline{\mathrm{perf}}_{\check{C}}(\mathcal{A}))) \arrow[rr] & & \check{C}(\mathscr{U}, \underline{C}_\bullet(\underline{\mathrm{vect}}_{\check{C}}(X,W))) \\
\check{C}(\mathscr{U}, \underline{C}_\bullet(\mathcal{A})) \arrow[rr] \arrow[u, "\sim"'] \arrow[urr, phantom, "(B)"] \arrow[drr, phantom, "(C)"] & & \check{C}(\mathscr{U}, \underline{C}_\bullet(\underline{\mathrm{vect}}(X,W))) \arrow[u, "\sim"'] \arrow[d, "\sim"] \\
\check{C}(\mathscr{U}, \underline{C}_\bullet(\mathcal{A})) \arrow[r] \arrow[u, "="] \arrow[d, "="] \arrow[dr, phantom, "(D)"] & \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\mathscr{C})) \arrow[r] \arrow[dr, shift left, "\phi"] \arrow[dr, phantom, "(E)"'] & \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\underline{\mathrm{qvect}}(X,W))) \\
\check{C}(\mathscr{U}, \underline{C}_\bullet(\mathcal{A})) \arrow[r] & \check{C}(\mathscr{U}, \underline{C}_\bullet(\mathcal{E}nd(P))) \arrow[r, "\phi"] \arrow[u] & \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\mathcal{O}_{-W})) \arrow[u, "\sim", "(F)"'] \arrow[ul, shift left]
\end{tikzcd} .$$

We already know that the vertical arrows on the left and right are quasi isomorphisms. Indeed, on the left hand side this was dealt with in the proof of the previous lemma. Those on the right hand side were dealt with in section 2.5.

The bottom arrow of the square (A) is $\check{C}(- \otimes_{\mathcal{A}} P, \beta)$ where $(- \otimes_{\mathcal{A}} P, \beta)$ is the lax morphisms of presheaves of dg categories from proposition 3.3. Commutativity of the square (A) is exactly proposition 2.21. Commutativity of (B) is similar to commutativity of the square (B) in the previous lemma. Note that all arrows except the top horizontal one are induced by morphisms of presheaves of dg categories and the top map is induced by a lax morphism. In the square (C), $\mathscr{C}$ denotes the presheaf of cdg categories $\langle P, \mathcal{O}_X \rangle \subset \underline{\text{qvect}}(X, W)$. All the arrows in (C) are induced by morphisms of presheaves of cdg categories and the diagram commutes already at the level of presheaves of cdg categories. Same holds for the square (D). In the triangle (F) all the arrows are quasi isomorphisms and the two diagonal arrows are quasi inverses to eachother. If we choose the diagonal arrow going up then (F) clearly commutes because it comes from a commutative diagram of presheaves of cdg categories and morphisms of such. Finally the triangle (E) commutes if we choose the diagonal arrow going down by the definition of the map ϕ from section 3.5 $\square$

Now we are ready to prove that the map on Hochschild homology induced by the pushforward functor $i_* : D^b(\text{coh}Y) \to D(\text{fact}(X, W))$ is given by first multiplying by the inverse Todd class described in section 3.4 and then applying the connecting homomorphism δ coming from the short exact sequence (3.2). We formulate this in a precise way in the following theorem (which is the same as theorem 1 from the introduction).

Theorem 3.13. *Let $i : Y \hookrightarrow X$ be a smooth divisor of a smooth complex variety X. Let $W \in \mathcal{O}_X(X)$ be a global function such that $W|_Y = 0$ and W vanishes on its critical locus. Let δ be the connecting homomorphism from the short exact sequence (3.2). Then there is a commutative diagram in $D_{Z/2}(\mathbb{C})$*

$$\begin{tikzcd}
HH_\bullet(Y) \arrow[rr, "HH_\bullet(i_*)"] \arrow[d, "\sim"] & & HH_\bullet(X, W) \arrow[d, "\sim"] \\
R\Gamma(X, \Omega_Y^\bullet) \arrow[r, "Td_{Y/X}^{-1} \wedge"] & R\Gamma(X, \Omega_Y^\bullet) \arrow[r, "R\Gamma(\delta)"] & R\Gamma(X, (\Omega_X^\bullet, -dW \wedge -)).
\end{tikzcd}$$

Proof. Again we break it into smaller commutative pieces

$$
\begin{tikzcd}
 & & C^{II}_{\bullet}(\mathrm{vect}(X,W)) \arrow[dddd] \\
C_{\bullet}(\mathrm{perf}_{\check{C}}(Y)) \arrow[urr] \arrow[d, "\sim"] & C_{\bullet}(\mathrm{perf}_{\check{C}}(\mathcal{A})) \arrow[l, "\sim"'] \arrow[ur] \arrow[d, "\sim"] & \\
\check{C}(\mathscr{U}, \Omega^{\bullet}_{Y}) \arrow[d, "-Td^{-1}_{Y/X}\wedge"] & \check{C}(\mathscr{U}, \Omega^{\bullet}_{\mathcal{A}}) \arrow[l, "\sim"'] \arrow[d, "-Td^{-1}_{Y/X}\wedge"] & \\
\check{C}(\mathscr{U}, \Omega^{\bullet}_{Y}) \arrow[drr, "R\Gamma(\delta)"] & \check{C}(\mathscr{U}, \Omega^{\bullet}_{\mathcal{A}}) \arrow[l, "\sim"'] \arrow[dr] & \\
 & & \check{C}(\mathscr{U}, (\Omega^{\bullet}_{X}, -dW\wedge))
\end{tikzcd}
$$

The top triangle commutes by lemma 3.2. The top left square commutes by lemma 3.11. The bottom left square commutes by lemma 3.5. The bottom triangle commutes by definition of the connecting morphism δ. The part on the right commutes by lemmas 3.10 and 3.12. $\square$

CHAPTER 4

THE CASE $W|_Y \neq 0$

In this section we try and generalize the formula from theorem 3.13 to the case where $W|_Y \neq 0$. We only succeed in doing so under an extra technical assumption about the inclusion of pairs $(Y, W|_Y) \to (X, W)$ which is described later.

4.1 The setup

Let $i : Y \hookrightarrow X$ be a smooth divisor in a smooth variety and $W \in H^0(X, \mathcal{O}_X)$. We have a push forward functor $i_* : D^{abs}(fact(Y, W|_Y)) \to D^{abs}(\mathrm{fact}(X, W))$ which lifts to a dg functor $\mathrm{fact}_Q(Y, W|_Y) \to \mathrm{fact}_Q(X, W)$ (recall that $\mathrm{fact}_Q(X, W)$ is the dg enhancement of $D^{abs}(\mathrm{fact}(X, W))$ obtained using Drinfeld quotients). We will explain how $HH_\bullet(\mathrm{fact}_Q(Y, W|_Y))$ and $HH_\bullet(\mathrm{fact}_Q(X, W))$ can be identified with $\check{C}(\mathscr{U} \cap Y, (\Omega^\bullet_Y, -dW|_Y \wedge))$ and $\check{C}(\mathscr{U}, (\Omega^\bullet_X, -dW \wedge))$ and give a formula for the resulting map $i_* : HH_\bullet(\mathrm{fact}_Q(Y, W|_Y)) \to HH_\bullet(\mathrm{fact}_Q(X, W))$ in terms of these identifications.

4.2 The HKR isomorphism for $\mathrm{fact}_Q(X, W)$

As in section 2.5 there is a sequence of natural quasi isomorphisms

$$\begin{tikzcd}
C_\bullet(\mathrm{fact}_Q(X,W)) \arrow[r, "(1)"] & \check{C}(\mathscr{U}, \underline{C_\bullet}(\underline{\mathrm{fact}_Q}(X,W))) & \check{C}(\mathscr{U}, \underline{C_\bullet}(\underline{\mathrm{vect}}(X,W))) \arrow[l, "(2)"'] \arrow[dll, "(3)"] \\
\check{C}(\mathscr{U}, \underline{C^{II}_\bullet}(\underline{\mathrm{vect}}(X,W))) \arrow[r, "(4)"] & \check{C}(\mathscr{U}, \underline{C^{II}_\bullet}(\underline{\mathrm{qvect}}(X,W))) & \check{C}(\mathscr{U}, \underline{C^{II}_\bullet}(\mathcal{O}_{-W})) \arrow[l, "(5)"'] \arrow[dll, "(6)"] \\
\check{C}(\mathscr{U}, (\Omega^\bullet_X, -dW \wedge -)). & &
\end{tikzcd} \tag{4.1}$$

All maps except for (1) and (2) above have been dealt with in section 2.5. The proof that (1) is a quasi isomorphism is completely analogous to that of proposition 2.32. The fact that the map (2) above is a quasi isomorphism follows from a spectral sequence argument together with the fact that $\mathrm{vect}(U, W|_U) \to \mathrm{fact}_Q(U, W|_U)$ is a quasi equivalence for any affine $U \subset X$.

4.3 The cdg algebra $\mathcal{A}_W$

We define $\mathcal{A}_W$ do be the sheaf of cdg algebras $(\mathcal{A}, d_{\mathcal{A}}, W)$ where $(\mathcal{A}, d_{\mathcal{A}})$ is the sheaf of dg algebras introduced in section 3.2 and $W \in \mathcal{A}(X)$ is the central element which is the image of $W \in \mathcal{O}_X(X)$ under the inclusion $\mathcal{O}_X \to \mathcal{A}$.

Lemma 4.1. *The map $\mathcal{A} \to i_*\mathcal{O}_Y$ induces a quasi isomorphism*

$$\check{C}(\mathscr{U}, \underline{C}^{II}_{\bullet}(\mathcal{A}_W)) \to \check{C}(\mathscr{U} \cap Y, \underline{C}^{II}_{\bullet}(\mathcal{O}_{Y,W|_Y})).$$

Proof. By a spectral sequence argument it is enough to assume that X itself is affine. In this case, consider the bicomplexes

$$A^{\bullet,\bullet} := \begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longleftarrow & (\mathcal{A}^0)^{\otimes 3} \oplus \mathcal{A}^{-1} \otimes \mathcal{A}^0 \oplus \mathcal{A}^0 \otimes \mathcal{A}^{-1} & \xleftarrow[d_0]{} & (\mathcal{A}^0)^{\otimes 2} \oplus \mathcal{A}^{-1} & \xleftarrow[d_0]{} & \mathcal{A}^0 \\ & & \downarrow{\scriptstyle d_2+d_{\mathcal{A}}} & & \downarrow{\scriptstyle d_2+d_{\mathcal{A}}} & & \\ \dots & \longleftarrow & (\mathcal{A}^0)^{\otimes 2} \oplus \mathcal{A}^{-1} & \xleftarrow[d_0]{} & \mathcal{A}^0 & & \\ & & \downarrow{\scriptstyle d_2+d_{\mathcal{A}}} & & & & \\ \dots & \longleftarrow & \mathcal{A}^0 & & & & \end{array}$$

and

$$B^{\bullet,\bullet} := \begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longleftarrow & \mathcal{O}_Y^{\otimes 3} & \xleftarrow[d_0]{} & \mathcal{O}_Y^{\otimes 2} & \xleftarrow[d_0]{} & \mathcal{O}_Y \\ & & \downarrow{\scriptstyle d_2} & & \downarrow{\scriptstyle d_2} & & \\ \dots & \longleftarrow & \mathcal{O}_Y^{\otimes 2} & \xleftarrow[d_0]{} & \mathcal{O}_Y & & \\ & & \downarrow{\scriptstyle d_2} & & & & \\ \dots & \longleftarrow & \mathcal{O}_Y & & & & \end{array}.$$

The map $\mathcal{A} \to i_*\mathcal{O}_Y$ induces a map of bicompexes $A^{\bullet,\bullet} \to B^{\bullet,\bullet}$. This lemma then follows from lemma 2.31.

□

4.4 Resolving the category $\mathrm{fact}_Q(Y, W|_Y)$

We will define a presheaf of dg categories, associated to $\mathcal{A}_W$ in a seemingly ad hoc way. Given $U \subset X$ open, let $\underline{\mathrm{perf}}_X(\mathcal{A}, W)(U)$ be the full sub dg category of $mod^{cdg} - \mathcal{A}_W|_U$ consisting of objects isomorphic to ones of the following form:

$$\{x|_U, g\} \otimes_U P, \;\; g \in H^0(U, \mathcal{O}_X(-Y)) \text{ and } P \in \mathrm{vect}(U, W|_U - x|_U \cdot g).$$

See section 2.5 for a description of the notation $\{x, g\}$.

We define a dg functor $T_X : \mathrm{perf}_X(\mathcal{A}, W) \to \mathrm{vect}_Q(Y, W)$ by

$$T_X(\mathcal{M}) := \mathcal{O}_Y \otimes_{\mathcal{A}} M$$

for $M \in \underline{\mathrm{perf}}_X(\mathcal{A}, W)(X)$. In fact, this extends to a weak morphism of presheaves of dg categories $(T, \alpha) : \underline{\mathrm{perf}}_X(\mathcal{A}, W) \to \underline{\mathrm{vect}}_Q(Y, W)$ where $\alpha_{U,V}$ is the canonical isomorphism

$$(\mathcal{O}_Y|_V \otimes_{\mathcal{A}} M|_V) \xrightarrow{\sim} (\mathcal{O}_Y \otimes_{\mathcal{A}} M)|_V.$$

Lemma 4.2. *If $M \in perf_X(\mathcal{A}, W)$ and $M \cong \{x, g\} \otimes_X P$, then we have*

$$\mathcal{O}_Y \otimes_{\mathcal{A}} M \cong P|_Y$$

Proof. There is a map $\mathcal{O}_Y \otimes_X \{x, g\} \otimes_X P \to \mathcal{O}_Y \otimes_X P$ induced by

$$\{x, g\} = \begin{tikzcd} \mathcal{O}_X \arrow[r] \arrow[d, shift left, "g"] & \mathcal{O}_Y \\ \mathcal{O}_X(-Y) \arrow[u, shift left, "x"] & \end{tikzcd}$$

and one can check that this is the cokernel of the morphism

$$\mathcal{O}_Y \otimes_X \mathcal{A} \otimes_X \{x, g\} \otimes_X P \to \mathcal{O}_Y \otimes_X \{x, g\} \otimes_X P$$

whose cokernel by definition is the tensor product $\mathcal{O}_Y \otimes_{\mathcal{A}} \{x, g\} \otimes_X P$. $\square$

Lemma 4.3. *If $U \subset X$ is affine and $\mathcal{O}_X(-Y)|_U$ is trivial, then*

$$T_U : \underline{perf}_X(\mathcal{A}, W)(U) \to vect(U \cap Y, W|_{U \cap Y})$$

is quasi fully faithful.

Proof. To simplify notation we assume that X itself is affine. Let $\mathcal{M}_1, \mathcal{M}_2 \in \mathsf{Ob}(\underline{\mathrm{perf}}_X(\mathcal{A}, W)(X))$ and write

$$\mathcal{M}_i = \{x, g_i\} \otimes P_i, \ i = 1, 2.$$

We have the following short exact sequence in $Z^0(\mathcal{A}_{xg_2} - mod^{cdg})$

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "x"] \arrow[d, shift left, "xg_2"] & \mathcal{O}_X \arrow[r] \arrow[d, shift left, "g_2"] & \mathcal{O}_Y \arrow[d, shift left] \\
\mathcal{O}_X \arrow[r, "1"] \arrow[u, shift left, "1"] & \mathcal{O}_X \arrow[r] \arrow[u, shift left, "x"] & 0 \arrow[u, shift left]
\end{tikzcd}$$

where $\mathcal{A}$ acts on the first object via

$$\epsilon \mapsto \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \in \mathrm{End}_X^{-1}(\mathcal{O}_X \oplus \mathcal{O}_X[1])$$

and on the second one via

$$\epsilon \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathrm{End}_X^{-1}(\mathcal{O}_X \oplus \mathcal{O}_X[1]).$$

The first object is null homotopic via

$$h = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Tensoring with P_2 we get a short exact sequence

$$\{1, xg_2\} \otimes P_2 \longrightarrow \{x, g_2\} \otimes P_2 \longrightarrow P_2|_Y = T_X(\mathcal{M}_2)$$

in $Z^0(\mathcal{A}_W - mod^{cdg})$ where the first term is null homotopic. Applying $\mathrm{Hom}_{\mathcal{A}_W}(\mathcal{M}_1, -)$ we see that the natural map

$$\mathrm{Hom}_{\mathcal{A}_W}(\mathcal{M}_1, \mathcal{M}_2) \to \mathrm{Hom}_{\mathcal{A}_W}(\mathcal{M}_1, T_X(\mathcal{M}_2))$$

is a quasi isomorphism. Next we have an isomorphism of complexes

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}_W}(\mathcal{M}_1, T_X(\mathcal{M}_2)) \cong & \mathrm{Hom}_{\mathcal{O}_{Y,W|_Y}}(\mathcal{O}_Y \otimes_{\mathcal{A}} \mathcal{M}_1, T_X(\mathcal{M}_2)) \\ = & \mathrm{Hom}_{\mathcal{O}_{Y,W|_Y}}(T_X(\mathcal{M}_1), T_X(\mathcal{M}_2)).\end{aligned}$$

□

Now we describe the technical assumption mentioned in the beginning of this section. We say that $(Y, W|_Y)$ has *the local resolution property inside* (X, W) if we can cover X with affine open subsets so that on each affine open U in the covering, the functor $H_0(\widehat{T}_U) : H^0(\widehat{\underline{\mathrm{perf}}_X(\mathcal{A}}, W)(U)) \to H^0(\mathrm{vect}(Y \cap U, W|_{Y \cap U}))$ is essentially surjective.

Here is one example of when this assumption is satisfied.

Proposition 4.4. *If $W|_Y$ has singular locus consisting of finitely many points, then $(Y, W|_Y)$ has the local resolution property inside (X, W).*

Proof. Cover X with affine open sets U such that each U contains at most one singular point of p of $V(W|_Y)$ and such that $k(p)$ admits a Koszul resolution $Kos(\bar{x}_2, ..., \bar{x}_n) \xrightarrow{\sim} k(p)$ for some regular sequence $\bar{x}_2, ..., \bar{x}_n \in \mathcal{O}_Y(U)$. Let $x_i \in \mathcal{O}_X(U)$ denote a lift of $\bar{x}_i$ and set $x_1 = x$ be an equation for Y. By the standing assumptions that $W|_Y$ vanishes on its singular locus we know that $W \in (x_1, ..., x_n)$ so we can write $W = \sum_{i=1}^n x_i g_i$ and then $W|_Y = \sum_{i=2}^n \bar{x}_i \bar{g}_i$. By [Dyc] the Koszul matrix factorization $\{\bar{x}_2, \bar{g}_2\} \otimes ... \otimes \{\bar{x}_n, \bar{g}_n\}$ is a generator for $\mathrm{vect}(U \cap Y, W|_{U \cap Y})$. The proposition then follows from the observation that $\{\bar{x}_2, \bar{g}_2\} \otimes ... \otimes \{\bar{x}_n, \bar{g}_n\} = T_U(\{x_1, g_1\} \otimes ... \otimes \{x_n, g_n\})$. $\square$

4.5 HKR type map for $\mathcal{A}_{-W}$

There is a short exact sequence of complexes

$$0 \longrightarrow (\Omega^\bullet_X, -dW\wedge)[1] \xrightarrow{L} (\Omega^\bullet_X(\log Y), -dW\wedge)[1] \longrightarrow (\Omega^\bullet_Y, -dW|_Y\wedge) \longrightarrow 0 \tag{4.2}$$

where the first map is the inclusion and the second map is locally defined by $\beta + \frac{dx}{x}\alpha \mapsto \alpha|_Y$. It gives rise to a quasi isomorphism

$$\Omega^\bullet_{\mathcal{A},-W} := \mathrm{Cone}(L) \xrightarrow{\sim} (\Omega^\bullet_Y, -dW|_Y \wedge (-)). \tag{4.3}$$

We will construct a commutative diagram

$$\begin{tikzcd} \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\mathcal{O}_{Y,-W|_Y})) \arrow[d, "HKR_{Y,-W|_Y}"] & \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\mathcal{A}_{-W})) \arrow[l] \arrow[d, "HKR_{\mathcal{A},-W}"] \\ \check{C}(\mathscr{U}, (\Omega^\bullet_Y, -dW|_Y\wedge)) & \check{C}(\mathscr{U}, \Omega^\bullet_{\mathcal{A},-W}). \arrow[l] \end{tikzcd}$$

(4.4)

The top vertical arrow is the quasi isomorphism from lemma 4.1. The bottom horizontal arrow is the quasi isomorphism 4.3. The left vertical map is the quasi isomorphism from proposition 2.29. It remains to define the map $HKR_{\mathcal{A}}$. Let $x_i \in \mathcal{O}_X(U_i)$ be local equations for $Y \subset X$. As in section 3.3, let $u_{ij} = x_j x_i^{-1}$ on the intersections U_{ij}. Then we identify $\mathcal{A}|_{U_j} \cong [\mathcal{O}_{U_i}\epsilon_i \to \mathcal{O}_{U_i}]$ where ϵ_i is in degree -1 and the differential maps ϵ_i to x_i. Note that on the intersections we have $u_{ij}\epsilon_i = \epsilon_j$. Let $U_I = U_{i_0\cdots i_p}$ and let $\bar{a} := a_0[a_1|\cdots|a_k] \in \mathcal{A}^0(U_I)^{\otimes k+1} \subset C^{II}_\bullet(\mathcal{A}(U_I))$. Then define

$$\begin{aligned} HKR_{\mathcal{A}}(\bar{a}) :=& \frac{1}{k!}a_0\frac{dx_{i_0}}{x_{i_0}} \wedge da_1 \wedge ... \wedge da_k \\ &- \sum_{j<i_0} \frac{(-1)^p}{k!}a_0\frac{du_{i_0j}}{u_{i_0j}} \wedge da_1 \wedge ... \wedge da_k \\ &\in \Gamma\Big(U_I, \Omega_X^{k+1}(\log Y)) \oplus \Gamma(U_I, \Omega_X^{k+2}) \oplus (\bigoplus_{j<i_0} \Gamma(U_{j\cup I}, \Omega_X^{k+1})\Big) \end{aligned}$$

and

$$HKR_{\mathcal{A}}(a_0[a_1|\cdots|a_l\epsilon_{i_0}|\cdots|a_k]) := \frac{(-1)^l}{k!}a_0 dx_{i_0} \wedge da_1 \wedge ... \wedge da_k.$$

and $HKR_{\mathcal{A}}$ vanishes on elements which have two or more factors from $\mathcal{A}^{-1}$. The proof of the following proposition is very similar to that of proposition 3.4.

Proposition 4.5. *The following is true about the map $HKR_{\mathcal{A}}$*
1) $HKR_{\mathcal{A}}$ is a chain map.
2) The diagram (4.4) commutes.
3) $HKR_{\mathcal{A}}$ is a quasi isomorphism.

4.6 Main theorem in the setting $W|_Y \neq 0$

Recall that $Y \subset X$ is locally cut out by $x_i \in \mathcal{O}_X(U_i)$ and that $u_{ij}x_i = x_j$ over U_{ij}. Recall also the sheaf of cdg algebras $\mathcal{A}_W$ from section 4.3. Over each of the open sets U_i we can identify $\mathcal{A} = \mathcal{O}_X\epsilon_i \oplus \mathcal{O}_X$. Suppose F_i is a matrix describing the action of an element of $\mathrm{End}_X(\mathcal{A}|_{U_i})$ in this basis. If F_j denotes the matrix of the same endomorphism but with respect to the basis $\mathcal{A} = \mathcal{O}_X\epsilon_j \oplus \mathcal{O}_X$ over U_{ji} then $g_{ij}F_jg_{ij}^{-1} = F_i$ where the change of basis matrix is

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & u_{ij} \end{bmatrix}.$$

Lemma 4.6. *The following diagram commutes in* $D_{Z/2}(\mathbb{C})$

$$\begin{tikzcd}
\check{C}(\mathscr{U},\underline{C}^{II}_{\bullet}(\mathcal{A}_{-W})) \arrow[r, "can"] \arrow[d, "HKR_{\mathcal{A}}"] & \check{C}(\mathscr{U},\underline{C}^{II}_{\bullet}(\mathcal{E}nd(\mathcal{A}_W))) \arrow[r, "\phi"] & \check{C}(\mathscr{U},\underline{C}^{II}_{\bullet}(\mathcal{O}_{-W})) \arrow[d, "HKR_{\mathcal{O}_{-W}}"] \\
\check{C}(\mathscr{U},\Omega^{\bullet}_{\mathcal{A}}) \arrow[r, "-Td^{-1}_{Y/X}\wedge"] & \check{C}(\mathscr{U},\Omega^{\bullet}_{\mathcal{A}}) \arrow[r, "\delta"] & \check{C}(\mathscr{U},\Omega^{\bullet}_{X},-dW\wedge)
\end{tikzcd}$$

Proof. Start with an element $a_0[a_1|\cdots|a_k]$ of internal degree zero over U_I, $I = \{i_0 < i_1 < ... < i_p\}$. Note that the only terms from ϕ which do not vanish under $HKR_{\mathcal{O}_{-W}}$ are those where $n = 0$ and $q \geq 1$ or $n = 2$ and $q \geq 0$. These are mapped to

$$\sum_{q\geq 1}\sum \frac{(-1)^{1+pq+\binom{q}{2}}}{k!q!} u_{i_0 i_{-q}} a_0 du^{-1}_{i_{1-q}i_{-q}} \wedge ... \wedge du^{-1}_{i_0 i_{-1}} \wedge da_1 \wedge ... \wedge da_k$$

where the second sum is over all $\{(i_{-q},...,i_{-1})|i_{-q} < i_{1-q} < ... < i_{-1} < i_0\}$. This is precisely what we get when going down first and then to the right twice.

Now let's start with an element of with precisely one factor from $\mathcal{A}^{-1}$, $\bar{b} = b_0[b_1|\cdots|b_l\epsilon_{i_0}|\cdots|b_k]$ and first go to the right twice and then down. This time, the only terms in the definition of ϕ which don't vanish under $HKR_{\mathcal{O}_{-W}}$ is $n = 1$ and $q \geq 0$. We get

$$\sum \frac{(-1)^{l+pq+\binom{q}{2}}}{k!(q+1)!} u_{i_0 i_{-q}} b_0 du^{-1}_{i_{1-q}i_{-q}} \wedge ... \wedge du^{-1}_{i_0 i_{-1}} \wedge dx_{i_0} \wedge db_1 \wedge ... \wedge db_k$$

which agrees with what we get going down and then to the right.

Finally if we start with an element with two or more factors from $\mathcal{A}^{-1}$, then going either direction is zero. this is clear if we go down first because $HKR_{\mathcal{A}}$ vanishes on such elements by definition. For the other direction note that the only terms in the definition of ϕ which are non-zero on such an element are the ones where n equals the number of factors from $\mathcal{A}^{-1}$ so $n \geq 2$. Each such term will contain two or more factors x_i and will therefore vanish when we apply $HKR_{\mathcal{O}_{-W}}$ because $dx_i \wedge dx_i = 0$. $\square$

Lemma 4.7. *There is a commutative diagram in $D_{\mathbb{Z}/2}(\mathbb{C})$*

$$
\begin{tikzcd}
C_\bullet(fact_{Dr}(Y,W|_Y)) \arrow[rr] \arrow[d, "\sim"] & & C_\bullet(fact_{Dr}(X,W)) \arrow[d, "\sim"] \\
\check{C}(\mathscr{U}, \underline{C_\bullet}(\underline{fact}_{Dr}(Y,W|_Y))) \arrow[rr] \arrow[u, phantom, shift left=8, "(A)"] & {} \arrow[u, phantom, "(A)"] & \check{C}(\mathscr{U}, \underline{C_\bullet}(\underline{fact}_{Dr}(X,W))) \\
\check{C}(\mathscr{U}, \underline{C_\bullet}(\underline{vect}(Y,W|_Y))) \arrow[u, "\sim"] \arrow[d, "\sim"] & \check{C}(\mathscr{U}, \underline{C_\bullet}(\underline{perf}_X(\mathcal{A}_W))) \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\sim"] \arrow[u, phantom, "(B)"] \arrow[dl, phantom, "(C)"] & \check{C}(\mathscr{U}, \underline{C_\bullet}(\underline{vect}(X,W))) \arrow[u, "\sim"'] \arrow[d, "\sim"] \arrow[dl, phantom, "(F)"] \\
\check{C}(\mathscr{U}, \underline{C_\bullet^{II}}(\underline{vect}(Y,W|_Y))) \arrow[d, "\sim"] & \check{C}(\mathscr{U}, \underline{C_\bullet^{II}}(\underline{vect}(\mathcal{A}_W))) \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\sim"] \arrow[dl, phantom, "(D)"] & \check{C}(\mathscr{U}, \underline{C_\bullet^{II}}(\underline{vect}(X,W))) \arrow[d, "\sim"] \arrow[dl, phantom, "(G)"] \\
\check{C}(\mathscr{U}, \underline{C_\bullet^{II}}(\underline{qvect}(Y,W|_Y))) & \check{C}(\mathscr{U}, \underline{C_\bullet^{II}}(\underline{qvect}(\mathcal{A}_W))) \arrow[l, "\sim"'] \arrow[r] & \check{C}(\mathscr{U}, \underline{C_\bullet^{II}}(\underline{qvect}(X,W))) \\
\check{C}(\mathscr{U}, \underline{C_\bullet^{II}}(\mathcal{O}_{Y,W|_Y})) \arrow[u, "\sim"] & \check{C}(\mathscr{U}, \underline{C_\bullet^{II}}(\mathcal{A}_W)) \arrow[l, "\sim"'] \arrow[r] \arrow[u, "\sim"] \arrow[ul, phantom, "(E)"] & \check{C}(\mathscr{U}, \underline{C_\bullet^{II}}(\mathcal{O}_{X,W})) \arrow[u, "\sim"'] \arrow[ul, phantom, "(H)"]
\end{tikzcd}
$$

Proof. The square (A) commutes because both horizontal arrows are induced by the strict morphism of presheaves of dg categories i_*.

In the pentagon shaped diagram (B), all arrows are induced by strict morphisms of presheaves of dg categories except one; the morphism $\check{C}(\mathscr{U}, \underline{C_\bullet}(\underline{\mathrm{perf}}_X(\mathcal{A}_W))) \to \check{C}(\mathscr{U}, \underline{C_\bullet}(\underline{\mathrm{vect}}(Y,W|_Y)))$ is induced by the week morphism of presheaves of dg categories (T,α) from section 4.4. Consider now the two morphisms $(i_* \circ T, i_*(\alpha)), (F,1) : \underline{\mathrm{perf}}_X(\mathcal{A}_W) \to \underline{\mathrm{fact}_Q}(X,W)$, where F is the forgetful functor, forgetting the $\mathcal{A}$-module structure. There is a natural map $\eta_{U_I,M} : F(M) \to i_*(T(M))$ whose cone is acyclic. Let $t_{\eta_{U_I,M}}$ be the distinguished endomorphism of $Cone(\eta_{U_I,M})$ such that $d(t_{U_I,M}) = \mathrm{id}_{Cone(\eta_{U_I,M})}$. Take $\eta^{-1}_{U_I,M}$ to be the composite

$$i_*T(M) \longrightarrow Cone(\eta_{U_I,M}) \xrightarrow{t_{\eta_{U_I,M}}} Cone(\eta_{U_I,M})[-1] \longrightarrow F(M).$$

Then take $a_{U_I,M}$ be the composite

$$i_*(T(M)) \longrightarrow Cone(\eta_{U_I,M}) \xrightarrow{t_{\eta_{U_I,M}}} Cone(\eta_{U_I,M})[-1] \longrightarrow i_*(T(M))[-1]$$

and $b_{U_I,M}$ to be the composite

$$F(M) \longrightarrow Cone(\eta_{U_I,M})[-1] \xrightarrow{t_{\eta_{U_I,M}}} Cone(\eta_{U_I,M})[-2] \longrightarrow F(M)[-1].$$

Then the structure (η, η^{-1}, a, b) satisfies the assumptions of lemma 2.25 so that $\check{C}(i_* \circ T, i_*(\alpha)) \simeq \check{C}(F, 1)$. Then we use lemma 2.24 to obtain commutativity of (B). The bottom left arrow in the pentagon shaped diagram (B) is a quasi isomorphism precisely because $(Y, W|_Y)$ is assumed to have the local resolution property inside (X, W).

The square (C) commutes because for any open in the cover $\mathrm{perf}_X(\mathcal{A}_W)$ is a full sub category of $\mathrm{vect}(\mathcal{A}_W)$. Note that the left vertical arrow is a quasi isomorphisms by proposition 2.18. Therefore the right vertical arrow will also be a a quasi isomorphism once we have shown that the bottom arrow in this square is a quasi isomorphism.

The square (D) commutes as $\mathrm{vect}(\mathcal{A}_W)$ and $\mathrm{vect}(Y, W|_Y)$ are full subcategories of $\mathrm{qvect}(\mathcal{A}_W)$ and $\mathrm{qvect}(Y, W|_Y)$ respectively. Both vertical arrows are quasi isomorphisms by proposition 2.12. Therefore the top horizontal arrow is a quasi isomorphism assuming that the bottom horizontal arrow is.

The square (D) commutes. The vertical arrows are again quasi isomorphisms by proposition 2.12. The bottom horizontal arrow is a quasi isomorphism by lemma 4.1. It then follows that the top horizontal map too is a quasi isomorphism.

In it not hard to see that the squares (F) and (G) commute.

To see that the square (H) commutes, consider the bigger diagram

$$\begin{tikzcd}
\check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\underline{\mathrm{qvect}}(\mathcal{A}_W))) \arrow[rr] & & \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\underline{\mathrm{qvect}}(X, W))) \\
& \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\mathscr{C})) \arrow[ur] \arrow[dr, shift left, "\phi"] & \\
\check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\mathcal{A}_W)) \arrow[uu, "\sim"] \arrow[ur] \arrow[r] & \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\mathcal{E}nd_X(\mathcal{A}_W))) \arrow[u] \arrow[r, "\phi"] & \check{C}(\mathscr{U}, \underline{C}^{II}_\bullet(\mathcal{O}_{X,W})) \arrow[uu, "\sim"'] \arrow[ul, shift left, "\psi"]
\end{tikzcd}$$

where $\mathscr{C}$ is the presheave of cdg categories on the two objects $\{F(\mathcal{A}_W), \mathcal{O}_X\}$. The triangle involving the two maps ϕ commutes by the definition of the map ϕ (see definition 3.8). The right most triangle (involving ϕ) commutes because it does so if we replace ϕ by ψ and they are inverse quasi isomorphisms by theorem 3.9.

□

Theorem 4.8. *Let $i : Y \hookrightarrow X$ be a smooth divisor of a smooth complex variety X. Let $W \in \mathcal{O}_X(X)$ be a global function such that W and $W|_Y$ vanish on their* *critical loci. We assume also that the pair $(Y, W|_Y)$ has the local resolution property inside (X, W). Let δ be the connecting homomorphism associated to the short exact sequence (4.2). Then there is a commutative diagram in $D_{Z/2}(\mathbb{C})$*

$$\begin{tikzcd}
HH_\bullet(Y, W|_Y) \arrow[rr, "HH_\bullet(i_*)"] \arrow[d, "\sim"] & & HH_\bullet(X, W) \arrow[d, "\sim"] \\
R\Gamma(X, (\Omega^\bullet_Y, -dW|_Y \wedge)) \arrow[r, "Td^{-1}_{Y/X} \wedge"] & R\Gamma(X, (\Omega^\bullet_Y, -dW|_Y \wedge)) \arrow[r, "R\Gamma(\delta)"] & R\Gamma(X, (\Omega^\bullet_X, -dW \wedge -)).
\end{tikzcd}$$

Proof. This now follows from combining the diagrams in lemma 4.6 and lemma 4.7 and proposition 4.5. $\square$

CHAPTER 5

HIGHER CODIMENSION

In this section we try and generalize our formula in an other direction. We will consider what happens if the codimension of Y inside X is greater than 1. But we assume again that $W|_Y = 0$. In this setting we still have a map

$$HH_\bullet(Y) \to HH_\bullet(X, W)$$

induced by the functor $i_* : \mathrm{perf}_{\check{C}}(Y) \to \mathrm{vect}_{\check{C}}(X, W)$ and again our goal is to describe this map in terms of the HKR isomorphisms from sections 2.5 and 2.5 respectively. In this setting however the formula from theorem 1.1 does not make sense anymore because there is no analogue of $\Omega_X(\log Y)$ when Y is not a divisor. Below, we introduce the Gysin map

$$G : R\Gamma(Y, \Omega_Y^\bullet) \to R\Gamma(X, (\Omega_X^\bullet, -dW\wedge)) \tag{5.1}$$

which is a generalization of the connecting map δ to higher codimension. Then we formulate a conjecture about the relation between $HH_\bullet(i_*)$ and the Gysin map and verify this conjecture in the affine case.

5.1 Gysin map without the potential

Let $i : Y \hookrightarrow X$ be a smooth subvariety of a smooth variety X of codimension c and assume that it is a local complete intersection. Cover X with finitely many open affines $U_i = \mathrm{Spec}(A_i)$, $i = 1, 2, ..., n$ and pick functions $f_1^{(i)}, ..., f_c^{(i)} \in A_i$ such that $Y_i := U_i \cap Y = V(f_1^{(i)}, ..., f_c^{(i)})$. Our goal in this section is to describe certain maps

$$G^p : i_*\Omega_Y^p \to \Omega_X^{p+c}[c].$$

Each one of these is the composite of two maps

$$\alpha^p : i_*\Omega_Y^p \to \mathcal{E}xt_X^c(i_*\mathcal{O}_Y, \Omega_X^{p+c})$$

and

$$\beta^p : \mathcal{E}xt_X^c(i_*\mathcal{O}_Y, \Omega_X^{p+c}) \to \Omega_X^{p+c}[c].$$

To define these we first need an explicit description of $\mathcal{E}xt_X^c(i_*\mathcal{O}_Y, \Omega_X^{p+c})$. Over each U_i we have a Koszul resolution of $i_*\mathcal{O}_{Y_i}$ that we denote by $\mathrm{Kos}(f_1^{(i)}, ..., f_c^{(i)})$. For

$1 \leq i < j \leq c$, over $U_i \cap U_j$ we can write

$$f_l^{(j)} = \sum_k a_{kl}^{(ij)} f_k^{(i)}.$$

Let $P_{j\to i} = (a_{kl}^{(ij)})_{kl}$. Recall that $\mathrm{Kos}^{-s}(f_1^{(i)}, ..., f_c^{(i)}) = \wedge^s(A_i^{\oplus c})$. Over $U_i \cap U_j$ we have a quasi isomorphism

$$\wedge(P_{j\to i}) : \mathrm{Kos}(f_1^{(j)}, ..., f_c^{(j)}) \to \mathrm{Kos}(f_1^{(i)}, ..., f_c^{(i)})$$

given termwise by $\wedge^s(P_{j\to i})$ and it fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{Kos}(f_1^{(j)}, ..., f_c^{(j)}) \arrow[r, "\wedge(P_{j\to i})"] \arrow[dr, "\sim"] & \mathrm{Kos}(f_1^{(i)}, ..., f_c^{(i)}) \arrow[d, "\sim"] \\
& i_*\mathcal{O}_{Y_ij}.
\end{tikzcd}$$

Notice that in degree $-c$ the map $\wedge(P_{j\to i})$ is given by multiplication by $\det(P_{j\to i})$. The dual complexes $\mathrm{Kos}(f_1^{(j)}, ..., f_c^{(j)})^\vee$ are related (over each double intersection) by the dual quasi isomorphisms

$$\wedge(P_{j\to i})^\vee : \mathrm{Kos}(f_1^{(i)}, ..., f_c^{(i)})^\vee \to \mathrm{Kos}(f_1^{(j)}, ..., f_c^{(j)})^\vee$$

and now these are given in degree c by multiplication by $\det(P_{j\to i})$. Finally we tensor the dual complexes by $\Omega_{U_i}^{p+c}$ and these complexes are related over the double intersections by the quasi isomorphisms $\wedge(P_{j\to i})^\vee \otimes \mathrm{id}_{\Omega^{p+c}}$. These fit into commutative diagrams

$$\begin{tikzcd}
\mathrm{Kos}(f_1^{(i)}, ..., f_c^{(i)})^\vee \otimes \Omega_{U_{ij}}^{p+c} \arrow[r, "\sim"] \arrow[d, "\sim"] & \mathrm{Kos}(f_1^{(j)}, ..., f_c^{(j)})^\vee \otimes \Omega_{U_{ij}}^{p+c} \arrow[d, "\sim"] \\
i_*\mathcal{O}_{Y_{ij}} \otimes \Omega_{U_{ij}}^{p+c}[-c] \arrow[r, "\det(P_{j\to i})\cdot"] & i_*\mathcal{O}_{Y_{ij}} \otimes \Omega_{U_{ij}}^{p+c}[-c].
\end{tikzcd}$$

We see that the maps

$$i_*\mathcal{O}_{Y_{ij}} \otimes \Omega_{U_{ij}}^{p+c} \xrightarrow{\det(P_{j\to i})\cdot} i_*\mathcal{O}_{Y_{ij}} \otimes \Omega_{U_{ij}}^{p+c}$$

give gluing data for the sheaf $\mathcal{E}xt_X^c(i_*\mathcal{O}_Y, \mathcal{O}_X) \otimes \Omega_X^{p+c} = \mathcal{E}xt_X^c(i_*\mathcal{O}_Y, \Omega_X)$. Now we define $\tilde{\alpha} : \Omega_X^p \to \mathcal{E}xt_X^c(i_*\mathcal{O}_Y, \Omega_X)$ locally over each U_i as follows,

$$\omega \mapsto [df_1^{(i)} \wedge ... \wedge df_c^{(i)} \wedge \omega]$$

where the brackets denote the class in

$$\operatorname{coker}\Big((\mathrm{Kos}^{1-c}(f_1^{(i)},...,f_c^{(i)}))^\vee \otimes \Omega_{U_i}^{p+c} \to (\mathrm{Kos}^{-c}(f_1^{(i)},...,f_c^{(i)}))^\vee \otimes \Omega_{U_i}^{p+c}\Big).$$

These glue to give a map $\tilde{\alpha}$ because

$$df_1^{(j)} \wedge ... \wedge df_c^{(j)} = \det(P_{j\to i}) \cdot df_1^{(i)} \wedge ... \wedge df_c^{(i)}.$$

Now the map $\tilde{\alpha}$ descends to give the desired map $\alpha : i_*\Omega_Y \to \mathcal{E}xt^c_X(i_*\mathcal{O}_Y, \Omega_X^{p+c})$.

The map β is the counit map

$$i_*i^!\Omega_X^{p+c} = \mathcal{E}xt^c_X(i_*\mathcal{O}_Y, \Omega_X^{p+c})[-c] \to \Omega_X^{p+c}.$$

5.2 Gysin map with a potential

Now let $W \in H^0(X, \mathcal{O}_X)$ be a function that vanishes on Y. Our goal then is to define a map in the $\mathbb{Z}/2$-graded derived category

$$G : i_*\Omega_Y^\bullet \to (\Omega_X^\bullet, -dW\wedge)$$

where we recall that $\Omega_Y^\bullet = \bigoplus_{i\geq 0} \wedge^i(\Omega_Y)[i]$ and similarly for $\Omega_X^\bullet$. Again it will be the composite of two maps

$$\alpha' : \Omega_Y^\bullet \to (\bigoplus_{j\geq -c} \mathcal{E}xt^c_X(i_*\mathcal{O}_Y, \Omega_X^{j+c})[j], -dW\wedge)$$

and

$$\beta' : (\bigoplus_{j\geq -c} \mathcal{E}xt^c_X(i_*\mathcal{O}_Y, \Omega_X^{j+c})[j], -dW\wedge) \to (\Omega_X^\bullet, -dW\wedge).$$

The first map α' obtained by noting that in the $\mathbb{Z}/2$-graded setting the maps α^i assemble to give a chain map α' as above.

To define β' is a little trickier. Consider the following object in the usual $\mathbb{Z}$-graded derived category

$$\mathcal{F} = \Big[\mathcal{O}_X \xrightarrow{-dW\wedge} \Omega_X \xrightarrow{-dW\wedge} ... \xrightarrow{-dW\wedge} \Omega_X^n \longrightarrow 0\Big]$$

where $\mathcal{O}_X$ sits in degree 0. Let π be the functor from the $\mathbb{Z}$-graded derived category to the $\mathbb{Z}/2$-graded dervied category which only remembers the parity of the complexes that form the objects in the derived category. Then

$$\pi(i_*i^!\mathcal{F}) = (\bigoplus_{j\geq -c} \mathcal{E}xt^c_X(i_*\mathcal{O}_Y, \Omega_X^{j+c})[j], -dW\wedge)$$

and we take β' to be the map $\pi(\tilde{\beta})$ where $\tilde{\beta} : i_*i^!\mathcal{F} \to \mathcal{F}$ is the counit.

5.3 Local computation

Let's assume now that $X = \mathrm{Spec}(R)$ is affine and that $Y = \mathrm{Spec}(R/I)$ where I is generated by a regular sequence $f_1, ..., f_c$. Then the wedge product defines a dg algebra structure on the complex $\mathrm{Kos}(f_1, ..., f_c)$. We will call this dg algebra A. There is a quasi isomorphism of dg algebras $A \to \mathcal{O}_Y = R/I$. Let $W = \sum_i r_i f_i$ be a regular function on X that vanishes on Y. Our goal is to prove the following theorem

Theorem 5.1. *There is a commutative diagram in $D_{\mathbb{Z}/2}(\mathbb{C})$*

$$\begin{tikzcd} HH_\bullet(Y) \arrow[r, "i_*"] \arrow[d, "HKR"] & HH_\bullet(X, W) \arrow[d, "HKR"] \\ \Omega^\bullet_Y \arrow[r, "G"] & \Omega^\bullet_{X,-W}. \end{tikzcd}$$

In the next three lemmas we will break this diagram into smaller commutative pieces.

Lemma 5.2. *There is a commutative diagram*

$$\begin{tikzcd} C_\bullet(\mathcal{O}_Y) \arrow[d, "HKR"] & C_\bullet(A) \arrow[l, "\sim"'] \arrow[r, "\varphi"] & Kos(f_1, ..., f_c)[-c] \otimes \Omega^\bullet_{X,-W} \arrow[r] \arrow[d, "\sim"] & \Omega^\bullet_{X,-W} \\ \Omega^\bullet_Y \arrow[rr, "\alpha'"] & & Ext^c_X(\mathcal{O}_Y, \Omega^\bullet_{X,-W})[-c] \arrow[ur, "\beta'"] & \end{tikzcd}$$

Proof. First we define the map φ. Suppose $\bar{a} = a_0 e_{I_0}[a_1 e_{I_1}| \cdots |a_k e_{I_k}] \in A^{\otimes k+1}$. Then let

$$\begin{aligned} \varphi(\bar{a}) := \sum_I (-1)^\epsilon e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_q} \wedge e_{I_0} \wedge ... \wedge e_{I_k} \otimes \\ a_0 dr_{i_1} \wedge ... \wedge dr_{i_q} \wedge df_1 \wedge ... \wedge df_c \wedge da_1 \wedge ... \wedge da_k \\ \in \mathrm{Kos}^\bullet(f_1, ..., f_c)[-c] \otimes \Omega^\bullet_{X,-W}. \end{aligned}$$

where the sum is over all subsets $I = \{i_1 < i_2 < ... < i_q\} \subset \{1, 2, ..., c\}$ and the sign is given by

$$\epsilon = |I_k|k + |I_{k-1}|(k-1) + ... + |I_1| + d(c+q) + \binom{q}{2}$$

where $d = \sum_{i=0}^k |I_i|$.

Let's first check that φ is a chain map. Let $dF =: df_1 \wedge ... \wedge df_c$, $d_j := |I_j|$, $dA = da_1 \wedge ... \wedge da_k$ and $dR_I = dr_{i_1} \wedge ... \wedge dr_{i_q}$. We have

$$(\varphi \circ d_1)(\bar{a}) = \sum_{j,I} \frac{(-1)^{|d_0+...+d_{j-1}+j+\epsilon}}{k!} e_I \wedge e_{I_0} \wedge ... \wedge e_{I_k} \otimes a_0 dR_I \wedge dF \wedge dA$$

On the other hand we have

$$(d_{Kos}[-c] \otimes 1)(\varphi(\bar{a})) =$$

$$\sum_I \frac{(-1)^{\epsilon+c}}{k!} d_{\mathrm{Kos}}(e_I) \wedge e_{I_0} \wedge ... \wedge e_{I_k} \otimes a_0 dR_I \wedge dF \wedge dA +$$

$$\sum_{I,j} \frac{(-1)^{\epsilon+q+d_0+...+d_{j-1}+c}}{k!} e_I \wedge e_{I_0} \wedge ... \wedge d_{\mathrm{Kos}}(e_{I_j}) \wedge ... \wedge e_{I_k} \otimes a_0 dR_I \wedge dF \wedge dA$$

and

$$(1 \otimes -dW \wedge (-))\varphi(\bar{a}) =$$

$$\sum_I \sum_{j \notin I} \frac{(-1)^{\epsilon+q+d+c+1+\sigma(j,I)}}{k!} e_I \wedge e_{I_0} \wedge ... \wedge e_{I_k} \otimes f_j a_0 \wedge dR_{I \cup j} \wedge dF \wedge dA.$$

From these formulas one can check that

$$\varphi(d_1 + d_2) = (d_{Kos}[-c] \otimes 1 - 1 \otimes dW \wedge (-))\varphi.$$

Commutativity is immediate from the definitions.

□

Now define the Koszul matrix factorization P. As a $\mathbb{Z}/2$-graded object P is the same as the A but the differential has been perturbed in the following way:

$$d_P = d_A + \Big(\sum_{i=1}^{c} r_i e_i\Big) \wedge -.$$

One can check that the wedge product map $\mu : A \otimes P \to P$ is compatible with the differentials so that the map $\hat{\mu} : A \to \mathrm{End}_X(P)$, $\alpha \mapsto \mu(\alpha, -)$ is a dg algebra morphism. Let ρ_α denote the matrix for $\mu(\alpha, -)$ in the usual basis for P.

Lemma 5.3. *There is a commutative diagram*

$$\begin{tikzcd} C_\bullet(A) \arrow[r, "\hat{\mu}"] \arrow[d, "\varphi"] & C_\bullet(\mathrm{End}_X(P)) \arrow[r, "\phi"] & C_\bullet^{II}(\mathcal{O}_{-W}) \arrow[d, "HKR"] \\ Kos^\bullet(f_1, ..., f_c)[-c] \otimes \Omega^\bullet_{X,-W} \arrow[rr] & & \Omega^\bullet_{X,-W} \end{tikzcd}$$

where ϕ is the map from definition 3.8.

Proof. The bottom vertical map is the canonical one, let's call it *can* in this proof. Note that if we start with an element

$$\bar{a} := a_0 e_{I_0}[a_1 e_{I_1}|\cdots|a_k e_{I_k}]$$

and go down and then to the right we get

$$\begin{cases} \frac{(-1)^\alpha}{k!} a_0 da_1 \wedge ... \wedge da_k \wedge df_1 \wedge ... \wedge df_c \wedge dr_{i_q} \wedge dr_{i_{q-1}} \wedge ... \wedge dr_{i_1} \\ \qquad \text{if } (\coprod_{i=0}^k I_k) \coprod \{i_1, ..., i_q\} = \{1, 2, ..., c\}, \\ 0 \text{ else.} \end{cases}$$

The sign is given by

$$\alpha = \underbrace{d_k k + d_{k-1}(k-1) + ... + d_1 + d(c+q) + \binom{q}{2}}_{from\ \varphi} + \underbrace{\sigma(I, I_0, ..., I_k)}_{from\ can}$$

where $d_j = |I_j|$ and $d = d_0 + ... + d_k$. We have to show that we get the same expression from $HKR \circ \phi \circ \hat{\mu}(\bar{a})$. We prove this in steps.

In order to see this we will need to understand the terms of $Str \circ Sh([\delta_P^n], \rho_{\bar{a}})$. First thing to observe is that P has a basis indexed by subsets $I \subset \{1, 2, ..., c\}$ and in this basis, the matrix corresponding to multiplication by a basis element is

$$(\rho_{e_I})_{J,K} = \begin{cases} (-1)^{|\sigma|} \text{ if } J = K \coprod I \\ 0 \text{ otherwise} \end{cases}$$

where σ is the permutation that writes (I, K) in order. The matrix of δ_P in this basis is

$$(\delta_P)_{J,K} = \begin{cases} (-1)^{|\gamma|} f_i \text{ if } K = J \coprod \{i\}, \\ (-1)^{|\tau|} r_i \text{ if } J = K \coprod \{i\}, \\ 0 \text{ otherwise} \end{cases}$$

γ is the permutation which writes (i, J) in order and τ is the permutation which writes (i, K) in order.

Let us fix one term in $Sh([\delta_P^n], \rho_{\bar{a}})$

$$T = \rho_{a_0 e_{I_0}}[\cdots|\delta_P|\cdots|\rho_{a_i e_{I_i}}|\cdots]$$

Now the terms of $Str(T)$ correspond to sequences subsets (see section 3.5 for the definition of Str):

$$(J_0, J_{k+n}, ..., J_2, J_1, J_0).$$

Based on the description of the matrices above, the non zero terms, correspond to sequences as above where each J_l is obtained from J_{l-1} by either adding some $i \notin J_{l-1}$, removing some $j \in J_{l-1}$ or adding a whole set I of indices which is disjoint from J_{l-1}. So the non-zero terms can be described as certain sequences

$$(Op_{n+k+1}, Op_{n+k}, ..., Op_2, Op_1, J_0)$$

where each Op_i refers to one of the operations described above and if we start with J_0 and perform all the operations (from right to left) we get back J_0. Note that Op_{n+k+1} is always the operation of adding I_0. Now given any non-zero term, corresponding to a sequence as above, there is a smallest starting set $\tilde{J}_0$ such that

$$(Op_{n+k+1}, Op_{n+k}, ..., Op_2, Op_1, \tilde{J}_0)$$

also gives a non-zero term in $Str(T)$. Indeed, we can take $\tilde{J}_0$ as the intersection of all J which give a non-zero term.

Now let $\bar{a}$ and T be as above.

Claim 1: $HKR \circ Str(T) = 0$ if $I_s \cap I_t \neq \emptyset$ for some $0 \leq s < t \leq k$.

Proof of claim 1: Let $(Op_{n+k+1}, ..., Op_1, J_0)$ correspond to a non-zero term of $Str(T)$. Let $i \in I_s \cap I_t$. Then i gets added in at least two of the operations in the sequence above. Since composing the operations should return the same set i must also get removed in two of the operations. Each time i gets removed, we get a factor df_i in $HKR \circ Str(T)$ and since $df_i \wedge df_i = 0$ the claim follows.

Claim 2: $HKR \circ Str \circ Sh([\delta^n_P], \rho_{\bar{a}})$=0 unless $n = 2c - |I_0| - |I_1| - ... - |I_k|$.

Proof of claim: Let $d = |I_0| + ... + |I_k|$. Consider again a non-zero term of $Str(T)$ and the corresponding sequences of operations:

$$(Op_{n+k+1}, ..., Op_1, J_0).$$

As in the proof of claim 1, if any index gets removed in two or more of the operations then that term goes to zero after applying HKR. We see that all indices that get removed or added in any of the operations must get added precisely once and removed precisely once to give a non zero term in $HKR \circ Str(T)$. So let us assume that this is the case. We can write the union of all indices that gets removed and added as

$$I := I_0 \cup I_1 \cup ... \cup I_k \cup \hat{I}$$

where the unions are disjoint. Note also that $n = d + 2|\hat{I}|$. We will cal an index $i \in I$ positive if it gets added before it gets removed in the sequence of operations (read from right to left) and negative otherwise. We can write $I = I^- \cup I^+$ where I^- and I^+ consist of the negative and positive indices respectively. With this notation we can explicitly describe the smallest subset $\tilde{J}_0$ which gives a non-zero term in $Str(T)$ with the given sequence of operations. Indeed $\tilde{J}_0 = I^-$. Now any subset J for which the sequence

$$(Op_{n+k+1}, ..., Op_1, J)$$

gives a non-zero term in $Str(T)$ can be written as $I^- \cup (J \backslash I^-)$ where $(J \backslash I^-) \cap I = \emptyset$. All these terms are the same up to as sign which only depends on the size of $|J|$. Therefore, if $I \neq \{1, 2, ..., c\}$ then all these terms cancel out because

$$\sum_{l=0}^{|I^C|} (-1)^l \binom{|I^C|}{l} = 0.$$

So $Str(T) = 0$ unless $I = \{1, ..., c\}$. Note that when this is the case we have

$$n = d + 2|\hat{I}| = d + 2(c - d) = 2c - d.$$

Now it remains to compute $(-1)^{2c-d} HKR \circ Str \circ Sh([\delta_P^{2c-d}], \rho_{\bar{a}})$. By carefully keeping track of the signs one can check that $HKR \circ Str$ gives the same value on any of the terms in $Sh([\delta_P^{2c-d}], \rho_{\bar{a}})$. So we consider

$$a_0 \rho_{e_{I_0}} [\delta_P | \cdots | \delta_P | a_1 \rho_{e_{I_1}} | \cdots | a_k \rho_{e_{I_k}}].$$

Also by just keeping track of the signs one can check that any sequence of operations

$$(Op_{n+k+1}, ..., Op_1, J)$$

that gives a non-zero term in Str will result in the same terms after also applying HKR. Write $\{1, 2, ..., c\} \setminus (I_0 \cup ... \cup I_k) = \{i_1, i_2, ..., i_q\}$. We may choose the sequence of operations that starts with the set $I_0 \cup \{i_1, ..., i_q\}$, then adds $I_k, I_{k-1}, ..., I_1$ one by one in this order, then removes all indices $\{1, 2, ..., c\}$ one by one in decreasing order, and then adds $i_q, ..., i_1$ in this order and finally adds I_0. The resulting term will be

$$\frac{1}{(n+k)!} (-1)^\beta a_0 da_1 \wedge ... \wedge da_k \wedge df_1 \wedge ... \wedge df_c \wedge dr_{i_q} \wedge dr_{i_{q-1}} \wedge ... \wedge dr_{i_1}.$$

It remains to check that the sign is the same as in the very beginning of this proof and that we get $\frac{(n+k)!}{k!}$ copies of this term. The number of copies is easy; there are

$\binom{n+k}{k}$ ways to choose how to shuffle in n copies of δ_P, and for each of them there are $n!$ ways of choosing in which order we should do the n distinct operations of removing indices $1, 2, ..., c$ or adding indices $i_1, i_2, ..., i_q$. In total we have $\frac{(n+k)!}{k!}$ terms. For the sign we have

$$\begin{aligned} \beta = \\ n + (k+1+c+q)(q+d_0) + (q+d_0+d_k) + \\ (q+d_0+d_k+d_{k-1}) + ... + (q+d_0+...+d_1) \\ +\sigma(I_1, ..., I_k, I_0 \cup \{i_1, ..., i_q\}) + \binom{q}{2} + q + \sigma(I_0, \{i_1, ..., i_q\}). \end{aligned}$$

Using that $n = 2c - d$ and $q = c - d$ one can check that this is indeed the same sign as the sign α from the beginning of this proof.

□

Lemma 5.4. *There is a commutative diagram in the derived category*

$$\begin{tikzcd}
HH_\bullet(\mathit{perf}(Y)) \arrow[rrr, "HH_\bullet(i_*)"] \arrow[d, "\sim"] & & & HH_\bullet(\mathit{vect}(X,W)) \arrow[d, "\sim"] \\
HH_\bullet(\mathcal{O}_Y) & HH_\bullet(A) \arrow[l, "\sim"'] \arrow[r] & HH_\bullet(\mathrm{End}_X(P)) \arrow[r, "\phi"] & HH^{II}_\bullet(\mathcal{O}_{X,-W})
\end{tikzcd}$$

Proof. We break it into smaller pieces where each map is a chain map:

$$\begin{tikzcd}
 & & HH_\bullet(\mathrm{perf}(\mathrm{vect}(X,W))) & \\
HH_\bullet(\mathrm{perf}(Y)) \arrow[urr, "HH_\bullet(i_*)"] & HH_\bullet(\mathrm{perf}(A)) \arrow[l, "\sim"'] \arrow[rr, "HH_\bullet(T_P)"] & & HH_\bullet(\mathrm{vect}(X,W)) \arrow[ul, "\sim"] \arrow[d, "\sim"] \\
 & & HH^{II}_\bullet(\{P, \mathcal{O}_{X,-W}\}) \arrow[r] \arrow[dr, shift left, "\sim"] & HH^{II}_\bullet(q\mathrm{vect}(X,W)) \\
HH_\bullet(\mathcal{O}_Y) \arrow[uu, "\sim"] & HH_\bullet(A) \arrow[l, "\sim"'] \arrow[r] \arrow[uu, "\sim"] & HH_\bullet(\mathrm{End}_X(P)) \arrow[u] \arrow[r, "\phi"] & HH^{II}_\bullet(\mathcal{O}_{X,-W}). \arrow[u, "\sim"] \arrow[ul, shift left, "\sim"]
\end{tikzcd}$$

In the diagram T_P is the functor $(-) \otimes_A P : \mathrm{perf}(A) \to \mathrm{vect}(X,W)$ and the diagonal map going downward in the bottom right corner is another instance of the map ϕ from definition 3.8. Commutativity of each square is very similar to (but simpler than) then the corresponding checks in the proofs of lemmas 3.11 and 3.12. □

proof of theorem 5.1. This now follows by combining the commutative diagrams in lemmas 5.2, 5.3 and 5.4. □

5.4 A guess for the general formula

Now if X is smooth, not necessarily affine variety and $Y \subset X$ a locally complete intersection of codimension c with $W \in \mathcal{O}_X(X)$ a global function such that W and $W|_Y$ vanish on their critical loci. Then let G denote the Gysin map defined in section 5.2. The same formula for G makes sense when $W|_Y \neq 0$ and in this case gives a map in $D_{\mathbb{Z}/2}(\mathrm{coh}X)$

$$G : i_*\Omega^{\bullet}_{Y,-W} \to \Omega^{\bullet}_{X,-W}.$$

Also we can consider the Todd class of the vector bundle $\mathcal{I}_Y/\mathcal{I}_Y^2$ on Y which we will denote $Td_{Y/X}$. Based on the computations in this thesis we make the following conjecture.

Conjecture 5.5. *There is a commutative diagram in $D_{\mathbb{Z}/2}(\mathbb{C})$*

$$\begin{tikzcd}
HH_\bullet(Y, W|_Y) \arrow[rr, "HH_\bullet(i_*)"] \arrow[d, "\sim"] & & HH_\bullet(X, W) \arrow[d, "\sim"] \\
R\Gamma(\mathscr{U}, (\Omega^\bullet_Y, -dW|_Y\wedge)) \arrow[r, "Td^{-1}_{Y/X}\wedge"] & R\Gamma(\mathscr{U}, (\Omega^\bullet_Y, -dW|_Y\wedge)) \arrow[r, "R\Gamma(G)"] & R\Gamma(\mathscr{U}, (\Omega^\bullet_X, -dW \wedge -)).
\end{tikzcd}$$

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