HYPERGEOMETRIC SYSTEMS AND PROJECTIVE MODULES IN

HYPERTORIC CATEGORY $\mathcal{O}$

by

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DISSERTATION ABSTRACT

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We prove that indecomposable projective and tilting modules in hypertoric category $\mathcal{O}$ are obtained by applying the geometric Jacquet functor of Emerton, Nadler, and Vilonen to certain Gel'fand-Kapranov-Zelevinsky hypergeometric systems. This proves the abelian case of a conjecture of Bullimore, Gaiotto, Dimofte, and Hilburn on the behavior of generic Dirichlet boundary conditions in 3d $\mathcal{N} = 4$ SUSY gauge theories.

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For my parents who have given me so much love and support.

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CHAPTER I

INTRODUCTION

Symplectic duality, as introduced by Braden, Licata, Proudfoot, and Webster in [1, 2], is a structured equivalence between certain categories of deformation quantization modules associated to certain pairs of symplectic cones. All known symplectic dual pairs arise from physics as Higgs and Coulomb branches of the moduli space of vacua in 3d $\mathcal{N} = 4$ SUSY gauge theories. Therefore it is expected that the mathematical theory of symplectic duality and the physical theory of 3d mirror symmetry, as introduced by Intriligator and Seiberg in [3], are one and the same.

In the last year there has been substantial progress in overcoming the two major obstacles towards making this idea precise: First, on the mathematical side Braverman, Finkelberg, and Nakajima [4, 5, 6] have succeeded in giving a mathematical definition of the Coulomb branch for 3d $\mathcal{N} = 4$ theories. A similar proposal was given by the physicists Bullimore, Dimofte, and Gaiotto in [7]. Second, on the physical side Bullimore, Dimofte, Gaiotto, and Hilburn [8] have shown that a half-BPS boundary condition in a 3d $\mathcal{N} = 4$ theory gives rise to a pair of deformation quantization modules, one on the Higgs and one on the Coulomb branch, which are exchanged by symplectic duality.

This dissertation is devoted to proving, in the abelian case, a conjecture from [8, Section 3.2.4] about the Higgs branch module $\mathcal{D}^H$ associated to a generic Dirichlet boundary condition $\mathcal{D}$. Before explaining the conjecture we will review a few facts about abelian gauge theories: First by [9] the Higgs and Coulomb branches are hypertoric varieties. Second, the relevant categories of deformation

modules are the corresponding hypertoric categories $\mathcal{O}$ as defined in [10, 11]. Third, the duality between $\mathcal{O}_H$ and $\mathcal{O}_C$ exchanges indecomposable tilting and simple modules.

By the third property one expects there is a boundary condition $\mathcal{B}$ whose Higgs branch module $\mathcal{B}^H$ is an indecomposable tilting module in $\mathcal{O}_H$ and whose Coulomb branch image $\mathcal{B}^C$ is a simple module in $\mathcal{O}_C$. Indeed, in [8, Section 3.4] it was argued that the Coulomb branch image $\mathcal{D}^C$ of a generic Dirichlet boundary condition $\mathcal{D}$ is a simple module. Unfortunately, as explained in [8, Section 3.2], the Higgs branch image $\mathcal{D}^H$ is not an indecomposable tilting object. In fact, it isn't even in $\mathcal{O}_H$. For example, in the case that the Higgs branch is $T^*\mathbb{C}P^1$, so that $\mathcal{O}_H$ is equivalent to the principal block of the Bernstein-Gelfand-Gelfand category $\mathcal{O}_0$ for $\mathfrak{sl}_2$, the nondegenerate Whittaker module with the principal central character is the Higgs branch image of a generic Dirichlet boundary condition.

However, in [12], Campbell has shown that for any reductive Lie algebra $\mathfrak{g}$ one can obtain the big tilting module in $\mathcal{O}_0$ from a nondegenerate Whittaker module by applying the geometric Jacquet functor of Emerton, Nadler, Vilonen [13]. In fact, Nadler had already shown in [14] how to construct all the indecomposable tilting objects in $\mathcal{O}_0$ by applying the geometric Jacquet functor to certain sheaves known as "Morse kernels". The Whittaker D-modules are irregular and the "Morse kernels" are not perverse or even $\mathbb{C}$-constructible so it isn't obvious how to compare the two results. Very recently, another construction of the big projective in $\mathcal{O}_0$ in terms of "Morse branes" in the Fukaya category has appeared in the work of Jin [15].

Based on this evidence and more we conjectured in [8, Section 3.2.4] that applying the geometric Jacquet functor to $\mathcal{D}^H$ produces a tilting module in $\mathcal{O}_H$. In

Theorem 6.3.3 we prove a stronger result for abelian gauge theories. We will state it precisely after we introduce some further background and notation by giving a short summary of the dissertation.

In Chapter II we review basic material on Hamiltonian group actions and the category $\text{mod}(D[X], G, 0)$ of G-equivariant D-modules on a smooth variety X with an action of a group G. Then we review the how the data of an exact sequence

$$1 \to G \xrightarrow{i} (\mathbb{C}^*)^n \xrightarrow{p} F \to 1$$

of algebraic tori and a character η of G can be encoded in the data of an arrangement $\mathcal{A}$ of n-hyperplanes in $\mathfrak{f}_{\mathbb{R}}^*$. Choosing a cocharacter ξ of F determines a linear functional on $\mathfrak{f}_{\mathbb{R}}^*$. The chambers of $\mathcal{A}$ are described by sign vectors $\alpha \in \{+, -\}^n$ and when $\mathcal{A}$ is smooth the 0-dimensional flats of $\mathcal{A}$ are determined by certain subsets $b \subset \{1, \ldots, n\}$ known as bases.

In Chapter III we review the construction of hypertoric category $\mathcal{O}_\eta^\xi$. In particular we follow [16] by studying a certain full subcategory $p\mathcal{O}^\xi$ of $\text{mod}(D[\mathbb{C}^n], G, 0)$ which is equipped with a quotient functor $k_\eta^* : p\mathcal{O}^\xi \to \mathcal{O}_\eta^\xi$ known as the Kirwan functor. From [10, Theorem 6.10] and [16, Proposition 4.11] one can see that every indecomposable tilting module in $\mathcal{O}_\eta^\xi$ is the image of an indecomposable projective module in $p\mathcal{O}^\xi$ under the Kirwan functor. Thus in this paper we will show that generic Dirichlet boundary conditions give rise to projectives in $p\mathcal{O}^\xi$.

In Chapter IV we review the theory of Dirichlet boundary conditions. Both exceptional and generic Dirichlet boundary conditions are partial Fourier transforms of certain $D[\mathbb{C}^n]$-modules known as Gelfand-Kapranov-Zelevinksy hypergeometric systems [17]. These hypergeometric systems quantize G-orbits in

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$\mathbb{C}^n$ and have been extensively studied in the literature. All standard modules in $\mathcal{O}_\eta^\xi$ are images of exceptional Dirichlet boundary conditions $D_{b,\alpha}$ in $p\mathcal{O}^\xi$ under the Kirwan functor. As discussed earlier the generic Dirichlet boundary conditions $\mathcal{D}_\alpha$ are not contained in $p\mathcal{O}^\xi$.

In Section 5.1 we review the definition of the geometric Jacquet functor from [13]. Then in Section 5.3 we use ξ to define a certain degeneration $q : \overline{aO} \to \mathbb{C}$ of the closure of the G-orbit corresponding to $\mathcal{D}_\alpha$. The functor J is roughly taking nearby cycles along this degeneration so the geometry of $q^{-1}(0)$ is extremely important to understanding $J(\mathcal{D}_\alpha)$. Finally in Section 5.4 we compute the b-function of a Dirichlet boundary condition. This polynomial determines the form of the nearby cycles in the definition of J.

In Chapter VI we study the module $J(\mathcal{D}_\alpha)$. First we show that the geometry of the special fiber $q^{-1}(0)$ lets us determine the possible filtrations of $J(\mathcal{D}_\alpha)$ whose subquotients are exceptional Dirichlet boundary conditions. Then we use our knowledge of the standard filtrations on a projective module in $\mathcal{O}_\eta^\xi$ to prove our main result.

Theorem. *If α is a bounded sign vector in the arrangement $\mathcal{A}$, then $J(\mathcal{D}_\alpha)$ is the indecomposable projective cover of the simple object $\mathcal{N}_\alpha$ in $p\mathcal{O}^\xi$.*

CHAPTER II

PRELIMINARIES AND CONVENTIONS

2.1. Group actions

Suppose a complex reductive group G acts algebraically on a smooth variety X. Then G acts on functions $f \in \mathbb{C}[X]$ via the formula

$$(g \cdot f)(x) = f(g^{-1} \cdot x)$$

and on differential operators $\theta \in D[X]$ via the formula

$$(g \cdot \theta)(f) = g \cdot (\theta(g^{-1} \cdot f)).$$

Recall that $D[X]$ admits a filtration by degree whose associated graded is the coordinate ring $\mathbb{C}[T^*X]$ of the cotangent bundle to X. The G-action on X induces one on T^*X and the corresponding G-action on $\mathbb{C}[T^*X]$ coincides with the one coming from $D[X]$.

Let $X^*(G) = \mathrm{Hom}(G, \mathbb{C}^*)$ and $X_*(G) = \mathrm{Hom}(\mathbb{C}^*, G)$ be the character and cocharacter lattices of G. Because $\mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) = \mathbb{Z}$ composition of maps gives a non-degenerate pairing between $X^*(G)$ and $X_*(G)$. By differentiation we have inclusions $X_*(G) \subseteq \mathfrak{g}$ and $X^*(G) \subseteq \mathfrak{g}^*$ which will be used implicitly throughout the text.

For $\chi \in X^*(G)$ define the space of functions of weight χ by

$$\mathbb{C}[X]^\chi = \{f \in \mathbb{C}[X] \,|\, g \cdot f = \chi(g)f \text{ for all } g \in G\}.$$

Differential operators of weight χ are defined similarly. In the case that G is a torus every function (or differential operator) is a sum of weight vectors. Thus we have gradings

$$\mathbb{C}[X] = \bigoplus_{\chi \in X^*(G)} \mathbb{C}[X]^\chi$$

and

$$D[X] = \bigoplus_{\chi \in X^*(G)} D[X]^\chi.$$

When X is affine the grading on $\mathbb{C}[X]$ specifies the G-action. The grading on $\mathbb{C}[T^*X]$ coming from the degree filtration on $D[X]$ gives vector fields weight 1 and functions from X weight 0. Thus it comes from the action of $\mathbb{C}^*$ on T^*X by inverse scalar multiplication.

Differentiating the action of G on $\mathbb{C}[X]$ gives a G-equivariant homomorphism

$$\mu^G : U\mathfrak{g} \to D[X]$$

known as the infinitesimal action which satisfies

$$[\mu^G(v), \theta] = v \cdot \theta$$

for any $v \in \mathfrak{g}$ and $\theta \in D[X]$. The infinitesimal action is compatible with the degree filtration on $D[X]$ and the PBW filtration on $U\mathfrak{g}$. Thus it induces a graded map $\mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[T^*X]$. The corresponding $(G \times \mathbb{C}^*)$-equivariant map $\mu_G : T^*X \to \mathfrak{g}^*$ is known as the complex moment map for the action of G on T^*X.

Equivariant D-modules

In this section we will explain a concrete model $\mathrm{Mod}(D[X], G, \chi)$ for the unbounded derived category of χ-twisted D-modules on the Artin stack $[X/G]$ where we assume X is smooth and affine, G is reductive, and the moment map μ_G is flat and dominant. For more details see [18, 19].

Recall that if M is a coherent $D[X]$-module $\mathrm{SS}(M) \subset T^*X$ is the support of the $\mathbb{C}[T^*X]$-module obtained by taking the associated graded of any good filtration of M compatible with the degree filtration on $D[X]$. The singular support of a quasi-coherent $D[X]$-module is the union of the singular supports of its coherent submodules.

Because of the infinitesimal action $\mu^G : U\mathfrak{g} \to D[X]$ every $D[X]$-module is also a $\mathfrak{g}$-representation via the formula $v \cdot m = \mu^G(v)m$. We say that a $D[X]$-module M is G-monodromic if the singular support $\mathrm{SS}(M)$ of M is contained in $\mu_G^{-1}(0)$. It is well known that if M is regular holonomic and G-monodromic then the action of $\mathfrak{g}$ is locally finite [20, Proposition 2.1.2]. Let $\mathrm{mod}(D[X])_{G-\mathrm{mon}}^{\mathrm{r.h.}}$ be the full subcategory of $\mathrm{mod}(D[X])$ consisting of all regular holonomic G-monodromic modules.

We say that a $D[X]$-module is weakly G-equivariant if it is equipped with the structure of a algebraic G-representation satisfying

$$g \cdot (\theta m) = (g \cdot \theta)(g \cdot m).$$

Differentiating this gives a second action of $\mathfrak{g}$ on M which will be written using a $\star$ that satisfies the Leibniz rule

$$v \star (\theta m) = (v \cdot \theta)m + \theta(v \star m) = [\mu^G(v), \theta]m + \theta(v \star m)$$

for $v \in \mathfrak{g}$, $\theta \in D[X]$, and $m \in M$. The difference $v \bullet m = v \star m - v \cdot m$ defines yet a third action of $\mathfrak{g}$ which is $D[X]$-linear. Let $\mathrm{mod}(D[X], G)$ be the category of weakly G-equivariant $D[X]$-modules and G-equivariant $D[X]$-module homomorphisms. This is a Grothendieck abelian category and in particular has enough injectives. Let $\mathrm{Mod}(D[X], G)$ be the unbounded derived category of $\mathrm{mod}(D[X], G)$. It is compactly generated by objects of the form $D[X] \otimes_{\mathbb{C}} V$ where V is a finite dimensional representation of G.

Let $\chi : \mathfrak{g} \to \mathbb{C}$ be a Lie algebra character. We say that a weakly G-equivariant $D[X]$-module is (G, χ)-equivariant if $v \star m = \chi(v)m$. Let $\mathrm{mod}(D[X], G, \chi)$ denote the full subcategory of $\mathrm{mod}(D[X], G)$ consisting of the (G, χ)-equivariant modules. The categories for different values of χ are often equivalent. In particular for every $\gamma \in X^*(G)$ the functor

$$- \otimes_{\mathbb{C}} \gamma : \mathrm{mod}(D[X], G, \chi) \to \mathrm{mod}(D[X], G, \chi - \gamma)$$

is an equivalence of categories. The functor

$$Q_\chi : \mathrm{mod}(D[X], G) \to \mathrm{mod}(D[X], G, \chi)$$

which sends a weakly G-equivariant $D[X]$-module to its largest (G, χ)-equivariant quotient is left adjoint to the inclusion. In [18] this functor is used to show that

$\mathrm{mod}(D[X], G, \chi)$ inherits the good homological properties of $\mathrm{mod}(D[X], G)$. In particular $\mathrm{mod}(D[X], G, \chi)$ is Grothendieck abelian and the unbounded derived category $\mathrm{Mod}(D[X], G, \chi)$ is compactly generated.

2.2. Hyperplane arrangements

Let $\mathbb{K}$ be $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{R} \oplus \mathbb{C}$ and for any abelian group A define $A_{\mathbb{K}} = A \otimes_{\mathbb{Z}} \mathbb{K}$. The integral structure on $\mathbb{R} \oplus \mathbb{C}$ is $\mathbb{Z} \otimes \{0\}$. Let

$$0 \to V \xrightarrow{a} \mathbb{Z}^n \xrightarrow{b} W \to 0$$

be an exact sequence of free abelian groups. There is a natural basis $\{h_1, \ldots, h_n\}$ for $(\mathbb{Z}^n)^*$ given by the projections $h_i : \mathbb{Z}^n \to \mathbb{Z}$. Assume that none of the h_i vanish on when restricted to V. Choose $\tilde{\eta} \in \mathbb{K}^n$, let $\eta = b(\tilde{\eta})$, and let $V_{\mathbb{K},\eta}$ be the coset in $\mathbb{K}^n / V_{\mathbb{K}}$ corresponding to η. Note that $V_{\mathbb{K},\eta}$ is a $V_{\mathbb{K}}$-torsor so the point $\tilde{\eta} \in V_{\mathbb{K},\eta}$ specifies an isomorphism $V_{\mathbb{K},\eta} \cong V_{\mathbb{K}}$. Moreover, through this isomorphism the integral structure on $V_{\mathbb{K}}$ induces one on $V_{\mathbb{K},\eta}$.

By assumption the intersection of the i-th coordinate hyperplane in $\mathbb{K}^n$ with $V_{\mathbb{K},\eta}$ gives a hyperplane H_i in $V_{\mathbb{K},\eta}$. The hyperplane is cut out by an affine functional whose linear part is h_i viewed as an element of V^* using $a^* : (\mathbb{Z}^n)^* \to V^*$. The vector h_i is called an integral coorientation of H_i.

$$H_i = \{v \in V_{\mathbb{K}} \cong V_{\mathbb{K},\eta} \,|\, h_i(v) + \tilde{\eta}_i = 0\}.$$

Thus we can recover a^* and $\tilde{\eta}$ from the (multi)-arrangement consisting of the integrally cooriented hyperplanes $\{H_1, \ldots, H_n\}$ in $V_{\mathbb{K},\eta}$. In fact this is enough to recover all of our starting data up an automorphism of W.

In the case that $\mathbb{K} = \mathbb{R} \oplus \mathbb{C}$ we can split $\eta = (\eta_\mathbb{R}, \eta_\mathbb{C})$ and hence get a pair of a real arrangement $\mathcal{A}_\mathbb{R}$ in $V_{\mathbb{R}, \eta_\mathbb{R}}$ and a complex arrangement $\mathcal{A}_\mathbb{C}$ in $V_{K, \eta_\mathbb{C}}$. There are natural projections $\pi_\mathbb{R}$ and $\pi_\mathbb{C}$ from $V_\mathbb{K}$ to $V_\mathbb{R}$ and $V_\mathbb{C}$ respectively. The arrangements are compatible in the sense that in $V_\mathbb{K}$ we have $H_i = \pi_\mathbb{R}^{-1}(H_i) \cap \pi_\mathbb{C}^{-1}(H_i)$.

For each subset $S \subseteq \{1, \ldots, n\}$ we say that $H_S = \cap_{i \in S} H_i$ is the *flat* of $\mathcal{A}$ spanned by S. A *basis* of $\mathcal{A}$ is subset $b \subseteq \{1, \ldots, n\}$ such that the coorientation vectors $\{h_i, i \in b\}$ give a basis for $(V_\mathbb{Q})^*$.

Now we will define a slew of properties that an arrangement $\mathcal{A}$ might have. An arrangement is essential if there is a 0-dimensional flat. This is equivalent to a^* being surjective which is always case for the arrangements constructed from exact sequences. We say that an arrangement is central if $\tilde{\eta} = 0$ and write $\mathcal{A}_0$ for the centralization of $\mathcal{A}$. An arrangement is integral (resp. real) if η is contained in W (resp. $W_\mathbb{R}$). An arrangement is simple if for any $S \subseteq \{1, \ldots, n\}$ where H_S is nonempty we have that the codimension of H_S is $|S|$. For simple arrangements every basis uniquely determines a 0-dimensional flat and vice versa. We say an arrangement $\mathcal{A}$ is unimodular if a^* is a totally unimodular integer matrix. This is equivalent to $\{h_i \,|\, i \in b\}$ being a basis for V^* for every basis b. Finally an arrangement is smooth if it is both simple and unimodular.

Suppose that either $\mathbb{K} = \mathbb{R}$ or that $\mathcal{A}$ is real. In particular, the following makes sense for the centralized arrangement $\mathcal{A}_0$. For for each sign vector $\alpha \in \{+, -\}^n$ we can define a chamber

$$\Delta_\alpha = \{v \in V_\mathbb{R} \subseteq V_{\mathbb{K}, \eta} \mid \alpha_i(h_i(v) + \tilde{\eta}_i) \geq 0 \text{ for all } 1 \leq i \leq n \}.$$

We say that α is feasible if Δ_α is not empty. Similarly given a sign vector α and a basis b one can define an orthant

$$O_{b,\alpha} = \{v \in V_\mathbb{R} \subseteq V_{\mathbb{K},\eta} \mid \alpha_i(h_i(v) + \tilde{\eta}_i) \geq 0 \text{ for all } i \in b\}.$$

Now consider an arrangement $\mathcal{A}$ equipped with an additional vector $\xi \in V^*$ known as a polarization. This gives an affine functional on $V_{\mathbb{K},\eta}$ that is well defined up to translation. We say that $(\mathcal{A}, \xi)$ is regular if ξ is not constant on any one dimensional flat of $\mathcal{A}_0$. We say that $\alpha \in \{+, -\}^n$ is bounded if the value of ξ on Δ_α in $\mathcal{A}_0$ is bounded from above. Similarly we say that (b, α) is bounded if the value of ξ on $O_{b,\alpha}$ in $\mathcal{A}_0$ is bounded above. Assume that $(\mathcal{A}, \xi)$ is real, simple, and regular, then for each basis $b \subseteq \{1, \ldots, n\}$ let $\alpha(b) \in \{+, -\}^n$ be the unique sign vector such that the maximum point of $\Delta_{\alpha(b)}$ is H_b.

Given an integral polarized arrangement $(\mathcal{A}, \xi)$ we define the *Gale dual* polarized arrangement $(\mathcal{A}^!, \xi^!)$ using the dual sequence

$$0 \to W^* \xrightarrow{b^*} (Z^n)^* \xrightarrow{a^*} V^* \to 0,$$

the element $\eta^! = -\xi \in V^*$, and the element $\xi^! = -\eta \in W$. By our previous discussion $\mathcal{A}^!$ is an essential arrangement of n cooriented hyperplanes $\{H_1^!, \ldots, H_n^!\}$ in $W_\mathbb{K}^*$. The many properties of Gale dual arrangements are discussed in [10, Section 2] but we will list a few here. First, Gale duality is an idempotent operation. Second $\mathcal{A}^!$ is simple if and only if $(\mathcal{A}_0, \xi)$ is regular. Third, there is a bijection between bases for $\mathcal{A}$ and bases for $\mathcal{A}^!$ given by sending a basis b to its complement $b^! = b^c$. Fourth, $\alpha \in \{+, -\}^n$ is feasible in $\mathcal{A}$ if and only if it is

bounded in $(\mathcal{A}_0^!, \xi^!)$. Now assume that $(\mathcal{A}, \xi)$, and hence also $(\mathcal{A}^!, \xi^!)$, is simple and regular. Then we have $\alpha(b) = \alpha^!(b^!)$ and that $H_b \in \Delta_\alpha$ if and only if $\Delta_\alpha^! \subseteq O_{b^!,\alpha^!(b^!)}^!$.

2.3. Tori and Arrangements

Consider the torus $T = (\mathbb{C}^*)^n$ and let

$$1 \to G \xrightarrow{i} T \xrightarrow{p} F \to 1$$

be an exact sequence of algebraic tori. This induces exact sequences of the character

$$0 \leftarrow X^*(G) \xleftarrow{i^*} X^*(T) \xleftarrow{p^*} X^*(F) \leftarrow 0$$

and cocharacter lattices

$$0 \to X_*(G) \xrightarrow{i_*} X_*(T) \xrightarrow{p_*} X_*(F) \to 0.$$

The n projection maps $(\mathbb{C}^*)^n \to \mathbb{C}^*$ give a canonical basis $\{h_1^* \ldots h_n^*\}$ for $X^*(T)$. Let $\{h_1, \ldots, h_n\}$ be the dual basis of $X_*(T)$. We will often use p_* (respectively i^*) without comment to view the h_i (respectively h_i^*) as elements of $X_*(F)$ (respectively $X^*(G)$).

In this paper the standard action of T on $\mathbb{C}^n = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]$ is the one where the coordinate x_i has weight h_i^*. The corresponding coordinate vector field ∂_i has weight $-h_i^*$. The infinitesimal action $\mu_T : U\mathfrak{t} \to D[\mathbb{C}^n]$ sends h_i to $x_i \partial_i$. The torus G acts on $\mathbb{C}^n$ through T with the same weights now though of as elements of $X^*(G)$ using i^*. The infinitesimal G-action is $\mu^G = \mu^T \circ i_*$.

As in Section 2.2 all of this data can encoded in pair of Gale dual hyperplane arrangements. From the exact sequence of character lattices we get a central essential arrangement $\mathcal{A}_{\mathbb{K},0}$ in $\mathfrak{f}^*_{\mathbb{K},0} = X^*(F)_{\mathbb{K},0}$ and from the exact sequence of cocharacter lattices we get $\mathcal{A}^!_{\mathbb{K},0}$ in $\mathfrak{g}_{\mathbb{K},0} = X_*(G)_{\mathbb{K}}$. Given $\eta \in \mathfrak{g}_{\mathbb{K}}$ and $\eta^! \in \mathfrak{f}^*_{\mathbb{K}}$ we get decentralized arrangements $\mathcal{A}_{\mathbb{K}}$ and $\mathcal{A}^!_{\mathbb{K}}$ in $\mathfrak{f}^*_{\mathbb{K},\eta}$ and $\mathfrak{g}_{\mathbb{K},\eta}$ respectively. If in fact η and $\eta^!$ were integral and we define $\xi = -\eta^!$ and $\xi^! = -\eta$ we get Gale dual polarized arrangements $(\mathcal{A}_{\mathbb{K}}, \xi)$ and $(\mathcal{A}^!_{\mathbb{K}}, \xi^!)$.

CHAPTER III

HYPERTORIC CATEGORY $\mathcal{O}$

3.1. Hypertoric varieties

Let G, T, and F be as in Section 2.3 and choose $\eta \in X^*(G)$. Using the character exact sequence we get a hyperplane arrangement $\mathcal{A}_{\mathbb{R} \oplus \mathbb{C}}$ in $\mathfrak{f}^*_{\mathbb{R} \oplus \mathbb{C}, \eta}$. As in Section 2.2 we will split this arrangement into a pair of a real arrangement $\mathcal{A}_{\mathbb{R}}$ in $\mathfrak{f}_{\mathbb{R}, \eta}$ and a central complex arrangement $\mathcal{A}_{\mathbb{C}}$ in $\mathfrak{f}_{\mathbb{C}, 0}$ In the differential geometric setting Bielawski and Dancer [21] showed how to construct a hyperkahler variety $\mathcal{M}$ known as a hypertoric variety whose geometry is governed by the combinatorics of a $\mathcal{A}_{\mathbb{R} \oplus \mathbb{C}}$. In the algebraic setting these varieties were studied by Hausel and Sturmfels [22].

Recall that we have Hamiltonian actions of T and hence G on $T^*\mathbb{C}^n$ with moment maps μ_T and μ_G respectively. There is also an additional action of $\mathbb{C}^*$ on $T^*\mathbb{C}^n$ that gives both the x_i and ∂_i weight 1. We will call this $\mathbb{C}^*$-action the conical action. It is not the same as the fiber dilation action discussed in Section 2.1. The map μ_T (respectively μ_G) can be made equivariant for the conical $\mathbb{C}^*$ action if we give any linear function $f \in \mathfrak{t} \subseteq \mathbb{C}[\mathfrak{t}^*]$ (respectively $f \in \mathfrak{g} \subseteq \mathbb{C}[\mathfrak{g}^*]$) weight 2.

We can build a hypertoric variety $\mathcal{M}$ by taking the Hamiltonian reduction of $T^*\mathbb{C}^n$ by G. This will be done in two stages. First we define the Lawrence toric variety $\mathcal{L}$ to be the GIT quotient

$$T^*\mathbb{C}^n \mathbin{/\!\!/}_\eta G = \operatorname{Proj} \bigoplus_{k \in \mathbb{N}} \mathbb{C}[T^*\mathbb{C}^n]^{k\eta}$$

with respect to the stability parameter η. Another way of defining the GIT quotient is to consider the open subset $(T^*\mathbb{C}^n)^{\eta-\mathrm{ss}} \subset T^*\mathbb{C}^n$ of η-semistable points and take the categorical quotient $(T^*\mathbb{C}^n)^{\eta-\mathrm{ss}} /\!\!/ G$. Because μ_T and $\mu_G = i^* \circ \mu_T$ are both G-invariant they descend to $\mathcal{L}$. The Lawrence toric variety $\mathcal{L}$ also inherits the conical $\mathbb{C}^*$-action and an action of F. Now we can define the hypertoric variety $\mathcal{M}$ to be $\mu_G^{-1}(0) \subset \mathcal{L}$.

If we restrict μ_T to $\mathcal{M}$ we find that it factors through $\mathfrak{f}^* \subset \mathfrak{t}^*$. The resulting map, which we call μ_F, is the moment map for the F-action and is also equivariant for the conical $\mathbb{C}^*$-action. Recall that the hyperplane H_i in $\mathcal{A}_{\mathbb{C},0}$ is precisely the vanishing set of the function $h_i \in \mathfrak{f}$. Thus

$$\mu_F^{-1}(H_i) = V(\mu^F(h_i)) = V(x_i \partial_i) \subseteq \mathcal{M}.$$

There is a similar story for the real arrangement $\mathcal{A}_{\mathbb{R}}$. After introducing the real moment map $\mu_{F,\mathbb{R}} : \mathcal{M} \to \mathfrak{f}_{\mathbb{R},\eta}^*$ one finds that

$$\mu_{F,\mathbb{R}}^{-1}(H_i) = V(\mu^{F,\mathbb{R}}(h_i)) = V(|x_i|^2 - |\partial_i|^2) \subseteq \mathcal{M}.$$

Combining these, one gets $\mu_{F,\mathbb{R}\oplus\mathbb{C}} : \mathcal{M} \to \mathfrak{f}_{\mathbb{R}\oplus\mathbb{C},\eta}^*$ satisfying

$$\mu_{F,\mathbb{R}\oplus\mathbb{C}}^{-1}(H_i) = \mu_{F,\mathbb{R}}^{-1}(H_i) \cap \mu_F^{-1}(H_i) = V(x_i, \partial_i) \subseteq \mathcal{M}.$$

There is an important Lagrangian subvariety

$$\mathcal{E} = \mu_F^{-1}(0) = V(x_1 \partial_1, \ldots, x_n \partial_n)$$

of $\mathcal{M}$ known as the *extended core*. For each sign vector $\alpha \in {+,-}^n$ define

$$\mathcal{N}_\alpha^{\mathrm{cl}} = V(\{\partial_i \,|\, \alpha_i = +\} \cup \{x_i \,|\, \alpha_i = -\}) \subseteq T^*\mathbb{C}^n$$

The irreducible components of $\mathcal{E}$ are precisely the GIT quotients

$$\mathcal{N}_\alpha^{H,\mathrm{cl}} = \mathcal{N}_\alpha^{\mathrm{cl}} /\!\!/_\eta G \subseteq \mathcal{M}.$$

From the last paragraph it is clear that $\mathcal{N}_\alpha^{H,\mathrm{cl}} = \mu_{F,\mathbb{R}}^{-1}(\Delta_\alpha) \cap \mathcal{E}$. In particular $\mathcal{N}_\alpha^{H,\mathrm{cl}}$ is non-empty if and only if α is feasible. In fact $\mathcal{N}_\alpha^{H,\mathrm{cl}}$ is exactly the toric variety associated to the polyhedron Δ_α.

A choice of $\xi \in X_*(F)$ gives a Hamilonian $\mathbb{C}^*$-action on $\mathcal{M}$ and a polarization of the arrangement $\mathcal{A}$. Assume that the polarization is regular or equivalent that the $\mathbb{C}^*$-action has finitely many fixed points. Then the extended core

$$\mathcal{M}^+ = \{m \in M \,|\, \lim_{t \to 0} \xi(t) \cdot m \text{ exists}\}$$

of $\mathcal{M}$ is Lagrangian and contained in $\mathcal{E}$. It is a union of precisely the $\mathcal{N}_\alpha^{H,\mathrm{cl}}$ such that α is bounded.

Let $\mathcal{L}_0$ and $\mathcal{M}_0$ be the varieties associated to the central arrangement $\mathcal{A}_0$. The inclusion of

$$(T^*\mathbb{C}^n)^{\eta-\mathrm{ss}} \hookrightarrow (T^*\mathbb{C}^n)^{0-\mathrm{ss}} = T^*\mathbb{C}^n$$

induces projective $(F \times \mathbb{C}^*)$-equivariant maps $\pi_\mathcal{L} : \mathcal{L} \to \mathcal{L}_0$ and $\pi_\mathcal{M} : \mathcal{M} \to \mathcal{M}_0$. We will assume $\mathcal{A}$ is smooth, so both $\mathcal{L}$ and $\mathcal{M}$ are smooth and the map $\pi_\mathcal{M}$ is a conical symplectic resolution.

3.2. The hypertoric enveloping algebra

The ring $U = D[\mathbb{C}^n]^G$ is known as the *hypertoric enveloping algebra* in [11, Section 3]. For each Lie algebra character χ of $\mathfrak{g}$ there is a central quotient U_χ that quantizes the Poisson algebra $\mathbb{C}[\mathcal{M}] = \mathbb{C}[\mathcal{M}_0]$.

First we will give a presentation for $\mathbb{C}[\mathcal{M}]$ that is mirror to the presentation of the coordinate ring of the Coulomb branch of an abelian 3d $\mathcal{N} = 4$ theory given in [7, 5, 4]. Recall that the F-action induces a grading

$$\mathbb{C}[\mathcal{M}] = \oplus_{\lambda \in X^*(F)} \mathbb{C}[\mathcal{M}]^\lambda.$$

Using the moment map μ_F we can identify $\mathbb{C}[\mathfrak{f}^*]$ with $\mathbb{C}[\mathcal{M}]^0$. The affine functionals $h_i = x_i \partial_i \in \mathbb{C}[\mathfrak{f}^*]$ cutting out the hyperplanes in the arrangement $\mathcal{A}_{\mathbb{C},0}$ will be extremely important in what follows.

For each $\lambda \in X^*(F)$ let $\lambda_i = \langle \lambda, h_i \rangle$ be the i-th coordinate of $p^*(\lambda)$. Define a monomial $m^\lambda \in \mathbb{C}[\mathcal{M}]^\lambda$ by the formula

$$m^\lambda = x^{\lambda_+} \partial^{\lambda_-}$$

where $(\lambda_+)_i = \max(\lambda_i, 0)$, $(\lambda_-)_i = \max(-\lambda_i, 0)$, and we are using multi-index notation. It is easy to see that the m^λ form a basis for $\mathbb{C}[\mathcal{M}]$ as a free $\mathbb{C}[\mathfrak{f}^*]$-module and in this basis the multiplication is given by

$$m^\lambda m^\mu = m^{\lambda+\mu} \prod_{\mu_i \lambda_i < 0} h_i^{\min(|\lambda_i|, |\mu_i|)}.$$

Notice that $\lambda_i \mu_i < 0$ if and only if λ and μ are on opposite sides of the hyperplane H_i in $\mathcal{A}_{\mathbb{C},0}$.

Now we will describe the analogous presentation for U_χ. Recall that U_χ is defined by quantum Hamiltonian reduction of $D[\mathbb{C}^n]$ by G. First we define the module

$$M_\chi = D[\mathbb{C}^n]/(\mu^G(v) - \chi(v)|v \in \mathfrak{g})$$

which inherits the degree filtration from $D[\mathbb{C}^n]$ and has associated graded $\mathbb{C}[\mu_G^{-1}(0)]$. Then we define the hypertoric enveloping algebra to be

$$U_\chi = (M_\chi)^G$$

which also inherits the degree filtration and has associated graded $\mathbb{C}[\mathcal{M}]$.

The torus F acts on U_χ as before but we only have that $\mathbb{C}[\mathfrak{f}^*]$ is the associated graded of $(U_\chi)^0$ with respect to the degree filtration. If $v \in U_\chi$ lifts an element $\bar{v} \in \mathfrak{f}$ we have

$$[v, u] = \bar{v} \cdot u$$

for all $u \in U_\chi$.

The monomials $h_i = x_i \partial_i$ cannot be lifted unambiguously to U_χ because x_i and ∂_i do not commute. Define

$$[h_i]^m = \begin{cases} \prod_{j=0}^{m-1}(x_i\partial_i - j) & m \geq 0 \\ \prod_{j=0}^{|m|-1}(\partial_i x_i + j) & m < 0 \end{cases}.$$

Note that the infinitesimal action is given by $\mu^T(h_i) = [h_i]^1$. The monomials m^λ can be lifted unambiguously to U_χ and multiplication in U_χ is determined by the

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relation

$$m^\lambda \, m^\mu = \prod_{\substack{\lambda_i \mu_i < 0 \\ |\lambda_i| \le |\mu_i|}} [h_i]^{\lambda_i} \; m^{\lambda+\mu} \prod_{\substack{\lambda_i \mu_i < 0 \\ |\lambda_i| > |\mu_i|}} [h_i]^{-\mu_i}.$$

The representation theory of U_χ is controlled by a *quantized hyperplane arrangement* in the sense of [11, Section 2.3]. Let $q\mathfrak{f}^* = \operatorname{Spec} U_\chi^0$. Then we have an arrangement $q\mathcal{A}$ of $2n$ hyperplanes H_i^+ and H_i^- in $q\mathfrak{f}^*$ given by the vanishing of $[h_i]^1$ and $[h_i]^{-1}$ respectively. Note that in[11] they also require the choice of an integral lattice $\Lambda \subset q\mathfrak{f}^*$. One can generalize all the concepts from Section 2.2 to quantized arrangements as is done in [11, Section 2.3].

3.3. Localization

Now we would like to describe an analogue of the classical Beilinson-Bernstein theorem [23]. In particular we would like to identify the category of U_χ-modules with the category $\mathrm{DQ}_\chi(\mathcal{M})$ of deformation quantization modules on $\mathcal{M}$. For our purposes it is best to use the category $\mathrm{mod}(D[\mathbb{C}^n], G, \chi)$ of (G, χ)-modules as an intermediary.

First we will relate $\mathrm{mod}(U_\chi)$ and $\mathrm{mod}(D[\mathbb{C}^n], G, \chi)$. The $D[\mathbb{C}^n]$-module M_χ is manifestly (G, χ)-equivariant and has a right U_χ-action coming from right multiplication. Therefore we have the adjunction

$$\mathrm{mod}(U_\chi) \underset{\mathrm{Hom}_G(M_\chi, -)}{\overset{M_\chi \otimes_{U_\chi} (-)}{\rightleftarrows}} \mathrm{mod}(D[\mathbb{C}^n], G, \chi)$$

Because G is reductive and connected the functor $\mathrm{Hom}_G(M_\chi, -) = (-)^G$ is exact.

Next we will define $\text{DQ}_\chi(\mathcal{M})$ to be a quotient of $\text{mod}(D[\mathbb{C}^n], G, \chi)$. Given $\eta \in X^*(G)$ we have a complementary open and closed embeddings

$$(T^*\mathbb{C}^n)^{\eta-\text{ss}} \xrightarrow{k_\eta} T^*\mathbb{C}^n \xleftarrow{j_\eta} (T^*\mathbb{C}^n)^{\eta-\text{uns}}$$

of the η-semistable and η-unstable sets respectively. By [19, Corollary 1.3] this induces a recollement

$$\text{Mod}(D[\mathbb{C}^n], G, \chi)_{\eta-\text{uns}} \underset{\substack{\xleftarrow{j_\eta^*} \\ \xrightarrow{j_{\eta,*}} \\ \xleftarrow{j_\eta^!}}}{} \text{Mod}(D[\mathbb{C}^n], G, \chi) \underset{\substack{\xleftarrow{k_{\eta,!}} \\ \xrightarrow{k_\eta^*} \\ \xleftarrow{k_{\eta,*}}}}{} \text{DQ}_\chi(\mathcal{M}). \qquad (3.3.1)$$

The functor k_η^* is known as the Kirwan functor and realizes $\text{DQ}_\chi(\mathcal{M})$ as the quotient of $\text{Mod}(D[\mathbb{C}^n], G, \chi)$ by the subcategory $\text{Mod}(D[\mathbb{C}^n], G, \chi)_{\eta-\text{uns}}$ consisting of complexes whose cohomology has singular support contained in the η-unstable locus. The left adjoint $k_{\eta,!}$ was constructed in [1, Section 5.4] where it was used to show that this definition of $\text{DQ}_\chi(\mathcal{M})$ coincides with the usual one from [24]. All of these functors preserve coherence and compactness. Note that the equivalences $- \otimes_\mathbb{C} \gamma$ from Section descend to $\text{DQ}_\chi(\mathcal{M})$. In particular the category $\text{DQ}_\chi(\mathcal{M})$ only depends on the class of χ in $\mathfrak{g}^*/X^*(G)$.

By combining these two constructions we get an adjunction

$$\text{Mod}(U_\chi) \underset{\Gamma_\chi(\mathcal{M},-)}{\overset{\text{Loc}_\chi}{\rightleftarrows}} \text{DQ}_\chi(\mathcal{M})$$

where $\text{Loc}_\chi = k_\eta^*(M_\chi \otimes_{U_\chi} (-))$ and $\Gamma_\chi(\mathcal{M}, -) = \text{Hom}_G(M_\chi, k_{\eta,*}(-))$. We say that *localization holds at χ* if these functors are t-exact equivalences. When $\mathcal{A}$ is smooth [1, Corollary B.1] tells us that for most positive integers k localization holds at $\chi + k\eta$. In particular one can always find a representative of the class of χ in $\mathfrak{g}^*/X^*(G)$

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where localization holds. This coarse result will be enough for our purposes, but [11, Theorem 6.1] gives an explicit criterion, in terms of the quantized arrangement $q\mathcal{A}$, for checking when localization holds.

Hypertoric Category $\mathcal{O}$

Now we would like to define the geometric version of category $\mathcal{O}_\chi^\xi$ following the approach in [16, Section 3]. Consider the variety

$$p\mathcal{M}^+ = \bigcup_{\alpha \text{ is } \xi\text{-bounded}} \mathcal{N}_\alpha^{\mathrm{cl}} \subseteq \mu_T^{-1}(0)$$

known as the *extended pre-core*. Lift $\xi \in \mathfrak{f}$ to $\tilde{\xi} \in \mathfrak{t}$ and then use μ^T to get an element of $D[\mathbb{C}^n]$. We say that a $D[\mathbb{C}^n]$-module is ξ-*regular* if $\tilde{\xi}$ acts locally finitely. Define $p\mathcal{O}_\chi^\xi$ be the full subcategory of $\mod D[\mathbb{C}^n], G, \chi$ consisting of finitely generated ξ-regular modules M supported on $p\mathcal{M}^+$. Every such module is holonomic and T-monodromic. Then define $\mathcal{O}_\chi^\xi$ to be the image of $p\mathcal{O}_\chi^\xi$ under the Kirwan functor k_η^*. Both of these categories only depend on the class of $\chi \in \mathfrak{g}^*/X^*(G)$. By [16, Proposition 2.10] and the fact the functors in the recollement (3.3.1) preserve coherence we have the recollement

$$D^b(p\mathcal{O}_\chi^\xi)^{\eta-\mathrm{uns}} \overset{\overset{j_\eta^*}{\underset{j_{\eta,*}}{\longrightarrow}}}{\underset{\underset{j_\eta^!}{\longleftarrow}}{\longleftarrow}} D^b(p\mathcal{O}_\chi^\xi) \overset{\overset{k_{\eta,!}}{\underset{k_\eta^*}{\longrightarrow}}}{\underset{\underset{k_{\eta,*}}{\longleftarrow}}{\longleftarrow}} D^b(\mathcal{O}_\chi^\xi). \tag{3.3.2}$$

The following proposition is a specialization of [19, Lemma 3.3].

Lemma 3.3.1. *Let M be an object of $D^b(\mathcal{O}_\chi^\xi)$ and suppose that $\overline{M}$ is any extension to an object of $D^b(p\mathcal{O}_\chi^\xi)$. In other words, that that $k_\eta^* \overline{M} \cong M$. If $Ext(\overline{M}, N) = 0$ for all objects $N \in D^b(p\mathcal{O}_\chi^\xi)$ for which $k_\eta^* N = 0$, then $k_{\eta,!} M \cong \overline{M}$.*

Category $p\mathcal{O}_\chi^\xi$ consists of holonomic modules so it is Artinian and Noetherian. The simple objects are extensions of integrable local systems on T-orbits in $\mathbb{C}^n$. In particular by [16, Proposition 3.2] when χ is integral the simple objects correspond to bounded sign vectors. More specifically if α is bounded the module $\mathcal{N}_\alpha = i_{\alpha,*}\mathbb{C}[\mathbb{C}_\alpha]$, where i_α is the inclusion of $\mathbb{C}_\alpha = V(x_i \,|\, \alpha_i = -)$, is simple. This is compatible with the notation in Section 3.1 since $\mathrm{SS}(\mathcal{N}_\alpha) = \mathcal{N}_\alpha^{\mathrm{cl}}$. By construction $k_\eta^* \mathcal{N}_\alpha = 0$ if and only if α is infeasible in $\mathcal{A}_{\mathbb{R}}$. The quotient category $\mathcal{O}_\chi^\xi$ is highest weight by [10, Theorem 5.23] and Koszul by [10, Theorem 5.24]. In particular for every bounded feasible α we have surjections

$$\mathcal{P}_\alpha \to \mathcal{V}_\alpha \to \mathcal{N}_\alpha$$

where $\mathcal{P}_\alpha$ is the irreducible projective cover of $\mathcal{N}_\alpha$ and the module $\mathcal{V}_\alpha$ is called a *standard* module. One key relationship between standards and projectives is that, by the proof of [10, Theorem 5.23], the projective $\mathcal{P}_\alpha$ admits a filtration whose associated graded is

$$\bigoplus_{H_{b^!}^! \in \Delta_\alpha^!} \mathcal{V}_{\alpha^!(b^!)}.$$

One can also define a purely algebraic version of category $a\mathcal{O}_\chi^\xi$ as in [11, Section 4]. Recall from our discussion in Section 3.2 that we have a surjection $\mu^T : U\mathfrak{t} \to (U_\chi)^0$. We say that a U_χ-module is a weight module if it decomposes into a direct sum of generalized eigenspaces for the $\mathfrak{t}$-action. Define hypertoric category $\mathcal{O}_\chi^\xi$ to be the full subcategory of $\mathrm{mod}(U_\chi)$ consisting of finitely generated weight modules such that the generalized eigenvalues of the $\tilde{\xi}$-action are bounded from above. By [2, Corollary 3.19], whenever localization holds at χ we have that $\mathcal{O}_\chi^\xi \cong a\mathcal{O}_\chi^\xi$.

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CHAPTER IV

DIRICHLET BOUNDARY CONDITIONS

In [8] Bullimore, Dimofte, Gaiotto, and showed a boundary condition in an abelian 3d $\mathcal{N} = 4$ gauge theories give rise to a pair of DQ-modules on Gale dual hypertoric varieties. In this paper we will only be concerned with the Higgs branch image. Our focus will be on the Dirichlet boundary conditions which are constructed from partial Fourier transforms of GKZ hypergeometric systems so we will begin with a review of these topics.

4.1. Partial Fourier Transform

Let V be a vector space with coordinates $x_1, \ldots, x_n$ and let $y_1, \ldots, y_n$ be the dual coordinates on V^*. There is an isomorphism $\mathbb{F} : D[V] \cong D[V]$ called the *Fourier transform* given by $\frac{\partial}{\partial x_i} \mapsto y_i$ and $x_i \mapsto -\frac{\partial}{\partial y_i}$. The classical limit of $\mathbb{F}$ induces a symplectomorphism $T^*V \cong T^*V^*$. We will often be interested in *partial Fourier transforms*. Notice that $D[V_1 \oplus V_2] \cong D[V_1] \otimes_\mathbb{C} D[V_2]$. Thus we can apply the Fourier transform to the first tensor factor to get a morphism $\mathbb{F}_{V_1} : D[V_1 \oplus V_2] \cong D[V_1^* \oplus V_2]$.

Now suppose that we are in the setting of Section 2.3. We would like to study how the Fourier transform $\mathbb{F} : D[\mathbb{C}^n] \cong D[T_0 \mathbb{C}^n]$ interacts with the torus actions. Partial Fourier transforms along other T-invariant decompositions of $\mathbb{C}^n$ will behave in a similar manner.

Remark 4.1.1. *At first glance it might seem confusing that the dual coordinates to the x_i are the ∂_i. Fortunately $\mathbb{F}(\partial_i) = \partial_i$ so there is no notational confusion.*

We will also use $\mathbb{F}$ to write the coordinate vector fields on $T_0^*\mathbb{C}^n$ as $-x_i$ instead of something awful like $\frac{\partial}{\partial(\partial_i)}$.

First note that $\mathbb{F}$ is T-invariant. However the infinitesimal T-actions are not intertwined. In particular, the infinitesimal action of h_i on $T_0^*\mathbb{C}^n$ is $\partial_i x_i$ instead of $x_i \partial_i$. The discrepancy disappears after taking associated graded so $\mathbb{F}$ intertwines the moment maps for the T-actions on $T^*\mathbb{C}^n$ and $T^*(T_0^*\mathbb{C}^n)$.

Now note that the action of T on $T_0^*\mathbb{C}^n$ gives ∂_i weight $-h_i^*$. This is not the case for the actions that we have encoded using hyperplane arrangements in Section 2.3. Let $\iota : T \to T$ be the inverse map. Inverting the action of T on $T_0^*\mathbb{C}^n$ has the effect of changing i to $\iota \circ i$ and p to $p \circ \iota$. Thus the arrangement encoding the actions of G and F on $T_0^*\mathbb{C}^n$ should be related to the one encoding the actions on $\mathbb{C}^n$ by reversing all of the coorientations. More generally doing a partial Fourier transform of the variables x_i and ∂_i reverses the coorientation of H_i. In particular one can use partial Fourier transforms to change the sign vector associated to a chamber.

4.2. GKZ hypergeometric systems

Let G, T, F, etc. be as in Section 2.3 and choose a $\chi \in \mathfrak{g}^*$. Recall that we have an exact sequence

$$0 \leftarrow X^*(G) \xleftarrow{i^*} X^*(T) \xleftarrow{p^*} X^*(F) \leftarrow 0.$$

To this data Ge′lfand, Graev, Kapranov, and Zelevinsky [25, 17] associated a certain holonomic D-module M_{i*}^χ known as a *GKZ-hypergeometric system.*

Define M_{i*}^{χ} to be the cyclic $D[\mathbb{C}^n]$-module with relations

$$\partial^{\lambda_+} = \partial^{\lambda_-} \tag{4.2.1}$$

for $\lambda \in X^*(F)$ and

$$\mu^G(v) = \chi(v) \tag{4.2.2}$$

for $v \in \mathfrak{g}$. Recall that by Fourier transform we can identify $D[\mathbb{C}^n] \cong D[T_0^*(\mathbb{C})^n]$. Under this identification first class of relations cut out the closure of the orbit $O = G \cdot (1, \ldots, 1) \subset T_0^* \mathbb{C}^n$. The second class of relations ensure M_{i*}^{χ} is holonomic and (G, χ)-equivariant.

Let $o : G \cong O \hookrightarrow T_0^*(\mathbb{C})^n$ be inclusion of the orbit O. Choose a basis $\{t_1 \ldots t_d\}$ for $X^*(G) \subset \mathbb{C}[G]$. Define the χ-twisted structure sheaf to be the cyclic $D[G]$-module $\mathbb{C}[G]t^{-\chi-1}$ with relations

$$\frac{\partial}{\partial t_i} t_i + \chi_i = 0 \tag{4.2.3}$$

for $i \in \{1, \ldots, d\}$. Notice that as $\mathbb{C}[G]$-module $\mathbb{C}[G]t^{-\chi-1} \cong \mathbb{C}[G]$ and that as a $D[G]$-module $\mathbb{C}[G]t^{-\chi-1}$ only depends on the class of χ in $\mathfrak{g}^*/X^*(G)$. The module $o_*(\mathbb{C}[G] \cdot t^{-\chi-1})$ is known as an χ-*twisted Gauss-Manin system.*

In [26, Proposition 2.1] Schulze and Walther show that $K_{i*}^{\chi}[(\partial_i \ldots \partial_n)^{-1}]$, the localized Euler-Koszul complex of M_{χ}^{i*}, is isomorphic to $\mathbb{F}(o_*(\mathbb{C}[G] \cdot t^{-\chi-1}))$. The main result of [26, Theorem 3.6] gives a description of the χ for which localization is unnecessary. Since a description of the χ for which the Euler-Koszul complex is a resolution of M_{χ}^{i*} was worked out in [27, Theorem 6.6], this is enough to give a criterion for when $M_{\chi}^{i*} \cong \mathbb{F}(o_*(\mathbb{C}[G] \cdot t^{-\chi-1}))$.

Before stating this criterion we need to review some graded algebra. Recall that the action of G on $\mathbb{C}[\overline{O}]$ gives a $X^*(G)$-grading. Let M be a finitely generated graded module over $\mathbb{C}[\overline{O}]$. The set $\deg(M)$ of *true degrees* of M is the set of all $\gamma \in X^*(G)$ such that $M^\gamma \neq 0$. Let $\operatorname{qdeg}(M)$ be the Zariski closure of $\deg(M)$ in $\mathfrak{g}^*$. Let

$$\operatorname{sRes}_l(i^*) = \{\gamma \in \mathfrak{g}^* \mid -\gamma \in \operatorname{qdeg}(\mathbb{C}[\overline{O}]/\partial_l) + (\mathbb{N}+1)h_l^*\}.$$

Note that $\operatorname{sRes}_l(i^*)$ is empty whenever ∂_l is invertible. Define the set of *strongly resonant parameters* be

$$\operatorname{sRes}(i^*) = \bigcup_{j=1}^{n} \operatorname{sRes}_j(i^*).$$

Now [26, Corollary 3.8] states that under the hypothesis that the semigroup $\mathbb{N} \operatorname{im} i^*$ is pointed

$$\mathbb{F}(o_*(\mathbb{C}[G] \cdot t^{-\chi-1})) \cong M_\chi^{i^*}$$

if and only if χ is not strongly resonant. It is clear from their proof that the if direction still holds if we drop the hypothesis that $\mathbb{N} \operatorname{im} i^*$ is pointed. A priori it is unclear whether there exist any such parameters but according [26, Corollary 3.9] one can find a representative of any equivalence class in $\mathfrak{g}^*/X^*(G)$ that is not strongly resonant. Moreover, according to [28, Lemma 1.11], if $\mathbb{N} \operatorname{im} i^*$ is saturated 0 is not strongly resonant.

4.3. Exceptional Dirichlet boundary conditions

In [8, Section 4] Bullimore, Dimofte, Gaiotto, and I studied *exceptional Dirichlet* boundary conditions $\mathcal{D}_{b,\alpha}$ in abelian 3d $\mathcal{N} = 4$ gauge theories. We showed that these boundary conditions are classified by pairs consisting of a T-invariant

Lagrangian in $T^*\mathbb{C}^n$ and a basis b of $\mathcal{A}$. The former are exactly the $\mathcal{N}_\alpha^{\mathrm{cl}}$ associated to a sign vector $\alpha \in \{+, -\}^n$ in Section 3.1.

As discussed in Section 4.1 one can use partial Fourier transform to change which T-invariant Lagrangian is the base of $T^*\mathbb{C}^n$. For studying a given exceptional Dirichlet boundary condition one convenient choice is to arrange that

$$\alpha_i = \begin{cases} - & \text{if } i \in b, \\ + & \text{if } i \notin b \end{cases}$$

Furthermore by reordering hyperplanes one may always arrange that $b = \{1, \ldots, k\} \subseteq \{1, \ldots, n\}$. In this case define

$$D_{b,\alpha} = \mathbb{F}_{\{0\} \times \mathbb{C}^{n-k}}(i_{b,*}\mathbb{C}[G]t^{-1-\chi})$$

where $i_b : G \cong \{0\} \times (\mathbb{C}^*)^{n-k} \hookrightarrow \mathbb{C}^k \times \mathbb{C}^{n-k} \cong \mathbb{C}^n$.

By the discussion in Section 4.2 on can always find a $\gamma \in \mathfrak{g}$ such that γ is not strongly resonant and $\gamma - \chi \in X^*(G)$. Then one sees that

$$D_{b,\alpha} \cong i'_{b,*}M_{a*}^\gamma \otimes_\mathbb{C} (\gamma - \chi)$$

as (G, χ)-modules, where $i'_b : \{0\} \times \mathbb{C}^{n-k} \hookrightarrow \mathbb{C}^n$ and $a : G \cong (\mathbb{C}^*)^{n-k}$. Moreover, since $\mathcal{A}$ is unimodular, $\mathbb{N}\,\mathrm{im}\,i^*$ is saturated and we can and will choose $\gamma = 0$ when $\chi \in X^*(G)$.

Then from the presentation of a hypergeometric system in Section 4.2 and the formula for the partial Fourier transform in Section 4.1 we see that $\mathcal{D}_{b,\alpha}$ is a cyclic

$D[\mathbb{C}^n]$-module with relations

$$0 = \begin{cases} \partial_i & i \in b \text{ and } \alpha_i = + \\[1em] x_i & i \in b \text{ and } \alpha_i = - \\[1em] x_i\partial_i - \gamma(h_i) & i \notin b \text{ and } \alpha_i = + \\[1em] \partial_i x_i + \gamma(h_i) & i \notin b \text{ and } \alpha_i = - \end{cases} \tag{4.3.1}$$

It is easy to see that $\mathcal{D}_{b,\alpha} \in p\mathcal{O}$ if and only if $O_{b,\alpha}$ is bounded.

From this presentation it is easy to see that these modules had already appeared in [11, Section 4.4]. In particular Braden-Licata-Proudfoot-Webster define the standard module $\mathcal{V}_\alpha \in \mathcal{O}_\eta^\xi$ to be $k_\eta^* D_{b,\alpha}$ for an appropriately chosen b. We would like to prove the following proposition that is well known to experts but does not appear in the literature.

Proposition 4.3.1. *Assume χ is integral and let b be the unique basis of $\mathcal{A}_\mathbb{R}$ such that $O_{b,\alpha}$ is bounded and H_b is the ξ-maximal point of Δ_α. Then*

$$k_{\eta,!}(\mathcal{V}_\alpha) = \mathcal{D}_{b,\alpha}.$$

The proof will rely on the following lemma.

Lemma 4.3.2. *In the category $Mod(\mathbb{C}[\mathbb{C}^n], G, 0)$ we have that*

$$Ext_G^k(D_{b,\alpha}, \mathcal{N}_\beta) = \begin{cases} \mathbb{C} & \text{if } \Delta_\beta^! \subset O_{b',\alpha}^! \text{ and } k = |\{i \in b \,|\, \alpha_i \neq \beta_i\}|, \\[1em] 0 & \text{else.} \end{cases}$$

Proof of lemma. Relabeling hyperplanes and use the partial Fourier transform so that $b = \{1, \ldots, k\} \subseteq \{1, \ldots, n\}$ and

$$\alpha_i = \begin{cases} - & \text{if } i \in b, \\ + & \text{if } i \notin b \end{cases}$$

Let $\mathbb{C}_\beta$ be the vanishing set of $\{x_i \,|\, \beta_i = -\}$

Consider the diagram

$$
\begin{array}{ccc}
G \cong \{0\} \times (\mathbb{C}^*)^{n-r} & \xrightarrow{\;i_b\;} & \mathbb{C}^r \times \mathbb{C}^{n-r} \cong \mathbb{C}^n \\
\bar{j}_\beta \uparrow & & j_\beta \uparrow \\
\mathbb{C}_\beta \cap G & \xrightarrow{\;\;\;\;\bar{i}_b\;\;\;\;} & \mathbb{C}_\beta
\end{array}
$$

By definition $\mathcal{N}_\beta = j_{\beta,*}\mathbb{C}[\mathbb{C}_\beta]$ and by direct inspection our presentation shows that $D_{b,\alpha} = i_{b,!}\mathbb{C}[G]$. Then by base change

$$\operatorname{Ext}^*_G(D_{b,\alpha}, \mathcal{N}_\beta) = \operatorname{Ext}^*_G(\mathbb{C}[G], i^!_b j_{\alpha,*}\mathbb{C}[\mathbb{C}_\beta]) \cong \operatorname{Ext}^*_G(\mathbb{C}[G], \bar{j}_{\beta,*}\bar{i}^!_b\mathbb{C}[\mathbb{C}_\beta]).$$

The intersection $G \cap \mathbb{C}_\beta$ is nonempty if and only $\Delta^!_\beta$ is contained in the orthant $O^!_{b',\alpha}$. In this case $G \cap \mathbb{C}_\beta = G$ so

$$\bar{j}_{\beta,*}\bar{i}^!_b\mathbb{C}[\mathbb{C}_\beta] \cong \mathbb{C}[G][\dim \mathbb{C}_\beta - \dim G].$$

Hence

$$\operatorname{Ext}^*_G(D_{b,\alpha}, \mathcal{N}_\beta) = H^*_G(G)[\dim G - \dim \mathbb{C}_\beta].$$

Note that $\dim \mathbb{C}_\beta - \dim G$ is precisely $|\{i \in b \,|\, \alpha_i \neq \beta_i\}|$. $\qquad\square$

Proof. By Lemma 3.3.1 we just need to show that $\text{Ext}^*(\mathcal{D}_{\alpha,b}, N)$ for all $N \in p\mathcal{O}$ with $\text{SS}(N)$ contained in the unstable locus. Every such module is an extension of simples $\mathcal{N}_\beta$ with β infeasible in $\mathcal{A}$.

Lemma 4.3.2 implies that $\text{Ext}^*(\mathcal{D}_{\alpha,b}, \mathcal{N}_\beta)$ vanishes unless $\Delta_\beta^!$ is contained in $O_{b^!,\alpha}^!$. But our assumptions imply that $(b^!, \alpha)$ is bounded so such β must be bounded in $\mathcal{A}^!$. Gale duality exchanges boundedness and feasibility so these β are feasible in $\mathcal{A}$. $\square$

Corollary 4.3.3. *The Kirwan functor $k_{\eta,!} : D^b(\mathcal{O}_\chi^\xi) \to D^b(p\mathcal{O}_\chi^\xi)$ is t-exact on the subcategory of $\mathcal{O}_\chi^\xi$ consisting of objects admitting a standard filtration. In particular it sends modules with a standard filtration to modules with an exceptional Dirichlet filtration.*

4.4. Generic Dirichlet Boundary Conditions

In [8, Section 6.2.3] Bullimore, Dimofte, Gaiotto and I showed that *generic Dirichlet* boundary conditions are classified by the data of a T-invariant Lagrangian $\mathcal{N}_\alpha^{\text{cl}}$ in $T^*\mathbb{C}^n$.

As explained in Section 4.1 we may use a partial Fourier transform in order to make our chosen Lagrangian $\mathcal{N}_{+\dots+}$. Define

$$\mathcal{D} = \mathcal{D}_{+\dots+} = \mathbb{F}(o_*(\mathbb{C}[G] \cdot t^{-\chi-1})).$$

As discussed in the last section this is a (G, χ)-equivariant $D[\mathbb{C}^n]$-module. Except when $G = T$, the G-orbit closure $\overline{O}$ is not a the closure of a T-orbit in $T_0^*(\mathbb{C}^n)$. Thus $\mathcal{D}$ is not T-monodromic and hence is not contained in $p\mathcal{O}_\chi^\xi$.

The presentation for a hypergeometric system in Section 4.2 gives us a presentation of $\mathcal{D}$. We can deal with strongly resonant parameters as in Section 4.3: we can find $\gamma \in \mathfrak{g}^*$ with $\gamma - \chi \in X^*(G)$ such that γ is not strongly resonant and

$$\mathbb{F}(o_*(\mathbb{C}[G] \cdot t^{-\chi - 1})) \cong \mathbb{F}(o_*(\mathbb{C}[G] \cdot t^{-\gamma - 1})) \otimes_{\mathbb{C}} (\gamma - \chi) \cong M_{i*}^{\gamma} \otimes_{\mathbb{C}} (\gamma - \chi)$$

as (G, χ)-modules. Again we can and will choose $\gamma = 0$ when χ is integral. Then, using the partial Fourier transform, we can get a presentation for any $\mathcal{D}_\alpha$.

Now using the presentation for $\mathcal{D}_\alpha$ and the formula $\mathcal{D}_\alpha^H = \text{Hom}(M_\chi, \mathcal{D}_\alpha)$ is a cyclic U_χ-module with relations

$$m^\lambda = \prod_{\alpha_i \lambda_i > 0} [h_i]^{\lambda_i} \tag{4.4.1}$$

for $\lambda \in X^*(F)$ as claimed in [8]. Let $\rho_{\lambda, \alpha} \in U_\chi^0$ be the right-hand side of equation (4.4.1). Note that these relations are not homogeneous with respect to the F-action. It is clear that $\mathcal{D}_\alpha^H \cong U_\chi^0$ as a U_χ^0-module.

We would like to understand the classical limit of $\mathcal{D}$ but it turns out that neither the conical (Bernstein) or the degree filtration on $D[\mathbb{C}^n]$ is well suited for this task. Instead since $\mathcal{D}$ is a Fourier transform it is most natural to consider the filtration on $D[\mathbb{C}^n]$ given by taking the Fourier transform of the degree filtration on $D[T_0^*\mathbb{C}^n]$. Call this the $+ \ldots +$-filtration. Using the partial Fourier transform one can define the α-filtration in a similar fashion.

The α-filtrations is good because the associated graded modules $\mathcal{D}_\alpha^{\text{cl}}$ and $\mathcal{D}_\alpha^{H,\text{cl}}$ have simple presentations. For example, $\mathcal{D}^{\text{cl}}$ is is a cyclic $\mathbb{C}[T^*\mathbb{C}^n]$ module

generated with relations

$$\partial^{\lambda_+} = \partial^{\lambda_-} \tag{4.4.2}$$

for $\lambda \in X^*(F)$ and

$$\mu^G(v) = 0 \tag{4.4.3}$$

for $v \in \mathfrak{g}$. Similarly the cyclic $\mathbb{C}[M]$-module $\mathcal{D}_\alpha^{H,\mathrm{cl}}$ has relations

$$m^\lambda = \prod_{\alpha_i \lambda_i > 0} h_i^{|\lambda_i|} \tag{4.4.4}$$

for all $\lambda \in X^*(F)$ and is isomorphic to $\mathbb{C}[\mathfrak{f}^*]$ as a $\mathbb{C}[\mathfrak{f}^*]$-module. From these formulas it is clear that the support of $\mathcal{D}_\alpha^H$ is a section of the integrable system $\mu_F : \mathcal{M}_0 \to \mathfrak{f}^*$ as explained in [8, Section 2.5.1].

In [8, Section 2.5.6] we asserted that when $+ \ldots +$ is bounded $\mathcal{D}^H$ is finitely generated as a module for the subalgebra of U_χ generated by $U_\chi^- = \oplus_{l=1}^\infty U_\chi^{-l\xi}$. This is necessary for the algebraic Jacquet functor of [29], see Remark 5.1.1 for more details, to be well behaved. The proof relies on the following lemma.

Lemma 4.4.1. *If $+ \ldots +$ is bounded, then the $+ \ldots +$ degree of $m^\lambda \in \mathbb{C}[T^*\mathbb{C}^n]$ is positive whenever $\langle \lambda, \xi \rangle < 0$.*

Proof. The $+ \ldots +$-degree of m^λ is $\sum_{\lambda_i > 0} \lambda_i$. Thus it is nonnegative and is only 0 when $\lambda_i \leq 0$ for all $1 \leq i \leq n$. Since $+ \ldots +$ is bounded there exists $\tilde{\xi} \in X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ with $p_*(\tilde{\xi}) = \xi$ and $\tilde{\xi}_i \leq 0$ for all $1 \leq i \leq n$. Thus if $\langle \xi, \lambda \rangle = \sum_{i=1}^n \tilde{\xi}_i \lambda_i < 0$ then one of the λ_i must be positive. $\square$

Now we will prove a stronger classical result.

Proposition 4.4.2. *If $+ \ldots +$ is bounded, the module $\mathcal{D}^{H,cl}$ is finitely generated as a module for the subagebra of $\mathbb{C}[\mathcal{M}]$ generated by $\mathbb{C}[\mathcal{M}]^- = \mathbb{C} \oplus_{l=1}^\infty \mathbb{C}[\mathcal{M}]^{-l\xi}$.*

Proof. Let $\{\lambda^1, \ldots, \lambda^d\}$ be the collection of $\lambda \in X^*(F)$ such that λ is a primitive generator of a 1-dimensional flat in $\mathcal{A}_0$ and $\langle \lambda, \xi \rangle < 0$. By embedding generators and then acting on $1 \in \mathcal{D}^{H,\mathrm{cl}} \cong \mathbb{C}[\mathfrak{f}^*]$ we get a morphism

$$q^* : \mathbb{C}[r_1, \ldots, r_d] \to \mathbb{C}[\mathfrak{f}^*]$$

which r_i to $\rho^{\mathrm{cl}}_{\lambda^i}$. This morphism is graded if we give $\mathbb{C}[\mathfrak{f}^*]$ the $+ \ldots +$-grading and give r_i the same $+ \ldots +$-degree as m^{λ^i}. We want to show that $\mathbb{C}[\mathfrak{f}^*]$ is finite as a $\mathbb{C}[r_1, \ldots, r_d]$-module or equivalently that the map $q : \mathfrak{f}^* \to \mathbb{C}^d$ is projective with discrete fibers.

To show the former will show that we can find a projective morphism $\tilde{q} : X \to Y$ such that q is the pullback of $\tilde{q}$ along a map $\mathbb{C}^n \to Y$. An obvious compactification of $\mathfrak{f}^*$ is the projective space $X = \mathrm{Proj}\,\mathbb{C}[\mathfrak{f}^*][s_0]$ where the linear functions in $\mathfrak{f}$ and s_0 have degree 1. We will compactify $\mathbb{C}^d$ by embedding it as the distinguished subset where $s_0 \neq 0$ in the the weighted projective space $Y = \mathrm{Proj}\,\mathbb{C}[s_0, \ldots, s_d]$ where the degree of s_0 is 1 and the degree of s_i is the $+ \ldots +$-degree of m^{λ^i} for $1 \leq i \leq n$. Then our desired map $\tilde{q}$ is precisely the one induced by the graded morphism $\mathbb{C}[s_0, \ldots, s_d] \to \mathbb{C}[\mathfrak{f}^*][s_0]$ that sends s_0 to s_0 and sends s_i to $\rho^{\mathrm{cl}}_{\lambda^i}$ for $1 \leq i \leq n$.

Since m^* is graded the map $q : \mathfrak{f}^* \to \mathbb{C}^d$ is $\mathbb{C}^*$-equivariant. By our lemma the weights of this $\mathbb{C}^*$-action on $\mathbb{C}^d$ are all positive. Thus for any point $p \in \mathbb{C}^d$ we have $\lim_{z \to \infty} z \cdot p = 0$. Furthermore, we know that for projective morphisms fiber dimension is upper semicontinuous on the target. Hence if the dimension of $m^{-1}(0)$ is 0 so is the dimension of $m^{-1}(p)$.

By definition $m^{-1}(0)$ is just the vanishing set of all the $\rho^{\mathrm{cl}}_{\lambda^i} = \prod_{\lambda^i_j > 0} h_i^{\lambda^i_j}$. Thus it is a union of flats in the arrangement $\mathcal{A}_0$. Note that $\langle \lambda^i, h_j \rangle = \lambda^i_j$ so $\rho^{\mathrm{cl}}_{\lambda^i}$

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does not vanish identically on $\mathbb{C}\lambda_i$. In particular $m^{-1}(0)$ cannot contain any one dimensional flat since they are all of the form $\mathbb{C}\lambda^i$. Thus $m^{-1}(0) = \{0\}$ the unique 0-dimensional flat in the central essential arrangement $\mathcal{A}_0$. $\qquad\square$

CHAPTER V

THE GEOMETRIC JACQUET FUNCTOR

5.1. Definition

Lets review the definition and basic properties of the geometric Jacquet
functor of [13]. We will use a slight generalization to what are called constant limit
schemes in the much more general work [30].

Let X be a smooth affine variety equipped with a $\mathbb{C}^*$-action and let $a : \mathbb{C}^* \times X \to X$ be the action map. Since a is smooth the renormalized pullback $a^\dagger = a^![-1] = a^*[1]$ is exact. In particular $a^\dagger$ is just the pullback of underlying $\mathbb{C}[X]$-module which has a canonical $D[\mathbb{C}^* \times X]$-module structure. Let $q : \mathbb{C} \times X \to \mathbb{C}$ be the projection and consider the diagram

$$X \overset{a}{\leftarrow} \mathbb{C}^* \times X \overset{r}{\hookrightarrow} \mathbb{C} \times X \leftarrow q^{-1}(0) \cong X$$

The geometric Jacquet functor

$$J : \mathrm{mod}(D[X]) \to \mathrm{mod}(D[X])$$

is given by $J(M) = \Psi_q r_* a^\dagger M$ where Ψ_q is the functor of nearby cycles. Since $J(M)$
is a nearby cycles sheaf it is equipped an endomorphism known as the *logarithm of
monodromy.*

In this paper we will be interested in the case that $X = \mathbb{C}^n$ equipped with the
$\mathbb{C}^*$-action coming from a lift $\tilde{\xi} \in X_*(T)$ of our cocharacter $\xi \in X^*(F)$. This action
commutes with the action of our torus G and by [30, Remark 4.19] the geometric

Jacquet functor lifts to a functor

$$J : \mathrm{mod}(D[\mathbb{C}^n], G, \chi) \to \mathrm{mod}(D[\mathbb{C}^n], G, \chi).$$

Remark 5.1.1. *Although we will not make use of this fact, it is possible to make sense of the Jacquet functor at the level of DQ-modules on the hypertoric variety $\mathcal{M}$ or even at the level of U_χ-modules. In particular, even though it is harder to compute, the V-filtration, which we will discuss in the next station, still makes sense in these contexts [20]. As claimed in [8, Section 2.5.6] we expect that one can easily generalize the main theorem of [13] and identify the geometric Jacquet functor in the context of U_χ-modules with the dual algebraic Jacquet functor of [29].*

5.2. Nearby cycles and the V-filtration

Now we will review the relationship between nearby cycles and the V-filtration which is due Kashiwara [31]. A good reference for this material in the equivariant setting is [20]. We will specialize to the case when X is affine and we are trying to compute nearby cycles along the projection $q : \mathbb{C} \times X \to \mathbb{C}$ since this is all that is necessary for this paper.

We can define a decreasing $\mathbb{Z}$-filtration on $D[\mathbb{C} \times X]$ by setting

$$V_k D[\mathbb{C} \times X] = \{P \in D[\mathbb{C} \times X] \,|\, P \cdot (q)^r \subseteq (q)^{r+k} \text{ for all } r \in \mathbb{Z}\}$$

where the ideal $(q)^r = \mathbb{C}[\mathbb{C} \times X]$ for $r \leq 0$. It is easy to see that

$$\mathrm{gr}_\bullet^V D[\mathbb{C} \times X] \cong D[\mathbb{C} \times X]$$

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with $D[X]$ in degree 0, q in degree 1, and the coordinate vector field $\frac{\partial}{\partial q}$ in degree -1. The corresponding $\mathbb{C}^*$-action on $\mathbb{C} \times X$ has infinitesimal action generated by $\theta = q\frac{\partial}{\partial q}$.

Let M be a coherent $D[\mathbb{C} \times X]$-module equipped with a descending exhaustive $\mathbb{Z}$-filtration $F_\bullet M$ satisfying

$$(V_k D[\mathbb{C} \times X]) \cdot F_l M \subset F_{k+l} M.$$

We say that the filtration $F_\bullet M$ is V-*good* in addition each $F_k M$ is coherent as a $V_0 D[\mathbb{C} \times X]$-module and

$$(V_k D[\mathbb{C} \times X]) \cdot F_l M = F_{k+l} M.$$

when $k \geq 0$ and $l \gg 0$ and when $k \leq 0$ and $l \ll 0$. These are precisely the conditions necessary for $\mathrm{gr}^F_\bullet M$ to be a coherent $\mathrm{gr}^V_\bullet D[\mathbb{C} \times X]$-module. In particular the filtration $V_\bullet D[\mathbb{C} \times X]$ is V-good.

The following is a special case of [31, Theorem 1].

Proposition 5.2.1. *Let M be coherent $D[\mathbb{C} \times X]$-module and assume that there exists a coherent $V_0 D[\mathbb{C} \times X]$-submodule S of M and a nonzero polynomial $b \in \mathbb{C}[s]$ such that*

1. *$b(\theta) \cdot S \subseteq (V_1 D[\mathbb{C} \times X]) \cdot S$,*

2. *$D[\mathbb{C} \times X] \cdot S = M$.*

Then there is unique V-good filtration $V_\bullet M$ on M such that there exists a nonzero polynomial $b' \in \mathbb{C}[s]$ where the roots of b' have real part in the interval $[0, 1)$ and

$$b'(\theta - k)V_k M \subset V_{k+1}M$$

for all $k \in \mathbb{Z}$.

Note that if that M is cyclic on a generator m an obvious candidate for N is $V_0 D[\mathbb{C} \times X]m$. In this case the minimal nonzero monic polynomial $b(s) \in \mathbb{C}[s]$ satisfying the hypotheses of Proposition 5.2.1 is called the *b-function* of m.

We say that a coherent $D[\mathbb{C} \times X]$-module M is *specializable* if it satisfies the conclusion of Proposition 5.2.1. By [31, Theorem 2] we have that the *nearby cycles* of M along $\{0\} \times M$ are

$$\Psi_q M = \mathrm{gr}_0^V M$$

and the *logarithm of monodromy* is given by the action of θ. Although we will not use it in this paper, there is a deep result of Kashiwara and Kawai [32] that implies that every regular holonomic $D[\mathbb{C} \times X]$-module is specializable.

5.3. A toric degeneration

Recall from Section 4.4 that the generic Dirichlet boundary condition $\mathcal{D}$ is the Fourier transform of the twisted Gauss-Manin system

$$o_*(\mathbb{C}[G] \cdot t^{-\chi-1})$$

where $o : G \hookrightarrow T_0^* \mathbb{C}^n$ is the inclusion of the G-orbit O throught $(1, \ldots, 1)$. In order to compute $J(\mathcal{D})$ it will be helpful to have a similar description of $a\mathcal{D} = r_* a^\dagger \mathcal{D}$.

The relevant geometric object involved will be a toric degeneration $q : \overline{aO} \to \mathbb{C}$ of the orbit closure $\overline{O}$ whose special fiber is a collection of coordinate hyperplanes $T_0^* \mathbb{C}^n$. The geometry of the special fiber will be our main tool for understanding $J(\mathcal{D})$ in Section VI.

Let $\bar{a} : \mathbb{C}^* \times T_0^* \mathbb{C}^n \to T_0^* \mathbb{C}^n$ be the action map corresponding to $\tilde{\xi} \in X_*(T)$ and let $\bar{o}$ be the pullback of o along $\bar{a}$. There is a unique automorphism $\tilde{a} : \mathbb{C}^* \times T_0^* \to \mathbb{C}^* \times T_0^*$ which is the identity on the $\mathbb{C}^*$ factor and makes $\bar{a} \circ \tilde{a} : \mathbb{C}^* \times T_0^* \mathbb{C}^n \to T_0^* \mathbb{C}^n$ equal to the projection. Then one can see that $\bar{o}$ is isomorphic to the composition

$$\mathbb{C}^* \times G \xrightarrow{1_{\mathbb{C}^*} \times o} \mathbb{C}^* \times T_0^* \mathbb{C}^n \xrightarrow{\tilde{a}} \mathbb{C}^* \times T_0^* \mathbb{C}^n.$$

Now a simple application of base change tells us that

$$\bar{a}^\dagger o_* (\mathbb{C}[G] \cdot t^{-\chi - 1}) \cong \bar{o}_* (\mathbb{C}[\mathbb{C}^*] \otimes_{\mathbb{C}} \mathbb{C}[G] \cdot t^{-\chi - 1})$$

In particular, if $\bar{r} : \mathbb{C}^* \times T_0^* \mathbb{C}^n \hookrightarrow \mathbb{C} \times T_0^* \mathbb{C}^n$ is the inclusion, we have

$$a\mathcal{D} \cong F((\bar{r} \circ \bar{o})_* (\mathbb{C}[\mathbb{C}^*] \otimes_{\mathbb{C}} \mathbb{C}[G] \cdot t^{-\chi - 1}))$$

where F is the partial Fourier transform along the second factor of $\mathbb{C} \times T_0^* \mathbb{C}^n$.

In order to use results from the rest of the paper it will be helpful to extract an exact sequence of tori from $\bar{o}$. Note that $\bar{o}$ factors through the dense torus of $\mathbb{C}^* \times T_0^* \mathbb{C}^n$ which can be identified with $\mathbb{C}^* \times T$ using the action. We will write $\bar{i}$ for

$$\mathbb{C}^* \times G \xrightarrow{1_{\mathbb{C}^*} \times i} \mathbb{C}^* \times T \xrightarrow{\tilde{a}|_{\mathbb{C}^* \times T}} \mathbb{C}^* \times T.$$

the resulting inclusion of tori. Let $\bar{p}$ be the cokernel of $\bar{i}$. The codomain of $\bar{p}$ can be identified with F in such a way that the exact sequence of cocharacter lattices becomes

$$0 \to \mathbb{Z} \oplus X_*(G) \xrightarrow{\begin{bmatrix} 1 & 0 \\ -\tilde{\xi} & i_* \end{bmatrix}} \mathbb{Z} \oplus X_*(T) \xrightarrow{\begin{bmatrix} \xi & p_* \end{bmatrix}} X_*(F) \to 0.$$

Now we would like to get an explicit presentation for $a\mathcal{D}$. Since we are not interested in the G-action we can assume that χ is strongly resonant as discussed in Section 4.4. Hence we can get a presentation of $a\mathcal{D}$ by applying an inverse partial Fourier transform to the hypergeometric system associated to the exact sequence

$$1 \to \mathbb{C}^* \times G \xrightarrow{\bar{i}} \mathbb{C}^* \times T \xrightarrow{\bar{p}} F \to 1.$$

We find that $a\mathcal{D}$ is a cyclic $D[\mathbb{C} \times \mathbb{C}^n]$-module with relations

$$q^{\langle \lambda, \xi \rangle_+} \partial^{\lambda_+} = q^{\langle \lambda, \xi \rangle_-} \partial^{\lambda_-} \tag{5.3.1}$$

for $\lambda \in X^*(F)$,

$$\mu^G(v) = \chi(v) \tag{5.3.2}$$

for $v \in \mathfrak{g}$, and

$$\frac{\partial}{\partial q} q = -\mu_T(\tilde{\xi}). \tag{5.3.3}$$

Note that the relation (5.3.2) implies that $a\mathcal{D}$ didn't depend on our choice of lift of ξ.

Let $aO = \bar{a}^{-1}(O) = \tilde{o}(\mathbb{C}^* \times \mathbb{C})$. The ideal cutting out the closure $\overline{aO}$ in $\mathbb{C} \times T_0^* \mathbb{C}^n$ is generated by exactly the relations (5.3.1). The projection $q : \overline{aO} \to \mathbb{C}$ is a toric morphism. In particular the generic fibers of q are all isomorphic to the toric variety $\overline{O} \cong q^{-1}(1)$ and the special fiber $\overline{aO}_0 = q^{-1}(0)$ is a union of toric varieties along their toric strata. Since $J(\mathcal{D}) = \Psi_q a\mathcal{D}$ understanding the geometry of $\overline{aO}_0$ is very important. In what follows we will be using [33] as our reference on toric geometry.

First notice that the dual cone $\sigma^\vee \subseteq \mathbb{R} \times \mathfrak{g}_{\mathbb{R}}^*$ associated to the affine toric variety $\overline{aO}$ is precisely the cone generated by $\bar{i}$. Note that by construction the columns of $\bar{i}$ correspond to the functions $q, \partial_1, \ldots, \partial_n$ on $\overline{aO}$. Now as in Section 2.3 let $h_1^* \ldots h_n^*$ in $X^*(G)$ be the columns of i^*. We can use the formula

$$
\bar{i} = \begin{bmatrix} 1 & -(\tilde{\xi})^T \\ 0 & i^* \end{bmatrix}.
$$

to express the ∂_i in therms of the h_i^* and q. Then we see that the $\sigma \subseteq \mathbb{R} \times \mathfrak{g}_{\mathbb{R}}$ associated to $\overline{aO}$ is given the set of $w \in \mathbb{R} \times \mathfrak{g}_{\mathbb{R}}$ satisfying the following inequalities

$$
\langle q, w \rangle \geq 0 \tag{5.3.4}
$$

$$
\langle \partial_i, w \rangle = -\tilde{\xi}_i \langle q, w \rangle + \langle h_i^*, w \rangle \geq 0 \tag{5.3.5}
$$

Recall that the cone for the toric variety $\mathbb{C}$ is $\mathbb{R}_\geq \subseteq \mathbb{R}$. The morphism $q : \overline{aO} \to \mathbb{C}$ corresponds to the map of cones induced by the projection $\pi : \mathbb{R} \times \mathfrak{g}_{\mathbb{R}} \to \mathbb{R}$. It is surjective, and hence a *toric degeneration* in the sense of [34], if and only if $\pi(\sigma) = \mathbb{R}_\geq$. In [34] it is shown that affine toric degenerations whose special fibers have dense torus G are classified by strongly convex rational polyhedra $\Delta \subset$

$\mathfrak{g}_{\mathbb{R}}$. Note that Δ is not the Delzant polyhedron because the Delzant polyhedron is contained in $\mathfrak{g}_{\mathbb{R}}^*$. The polyhedron corresponding to $q : \overline{aO} \to \mathbb{C}$ is precisely $\sigma \cap (\{1\} \times \mathfrak{g}_{\mathbb{R}})$. Notice that equation (5.3.5) and the discussion in Section 2.3 immediately imply the following.

Proposition 5.3.1. *The polyhedron $\sigma \cap (\{1\} \times \mathfrak{g}_{\mathbb{R}})$ associated to the toric degeneration $q : \overline{aO} \to \mathbb{C}$ is precisely the chamber $\Delta^!_{+\ldots+}$ in the arrangement $\mathcal{A}^!_{\mathbb{R}}$. This is nonempty (and hence q is surjective) if and only if $+\ldots+$ is bounded in $\mathcal{A}_{\mathbb{R},0}$.*

Now we will describe the special fiber of q in terms of the combinatorics of $\Delta^! = \Delta^!_{+\ldots+}$. By the orbit-cone correspondence prime toric divisors in $\overline{aO}$ that are contained in $q^{-1}(0)$ are in bijection with rays in σ that are not contained in $\{0\} \times \mathfrak{g}_{\mathbb{R}}$. Furthermore, vertices of the polyhedron $\Delta^!$ are in bijection with such rays via the map $v \mapsto \rho_v = \mathbb{R}_{\geq 0} \cdot (1, v)$. Let D_v be the divisor corresponding to a vertex v of $\Delta^!$. The following proposition is proved in [34, Section 3].

Proposition 5.3.2.

1. *The special fiber $q^{-1}(0)$ is reduced and $q^{-1}(0) = \cup_{v \in \Delta^!} D_v$.*

2. *The toric variety D_v has cone $\sigma_v = \mathbb{R}_{\geq 0} \cdot (\Delta^! - v) \subseteq \mathfrak{g}$.*

We would like to understand how $q^1(0)$ sits inside $T_0^* \mathbb{C}^n$. By basic toric geometry D_v has coordinate ring given by $\mathbb{C}[\rho_v^{\perp} \cap \sigma^{\vee}]$. Thus the function ∂_i vanishes identically on D_v if and only if $\langle \partial_i, (1, v) \rangle \neq 0$. The latter condition is precisely that $v \notin H^!_i$. In particular we have proven the following.

Proposition 5.3.3. *For $S \subset \{1, \ldots, n\}$ let $\partial^S = \prod_{i \in S} \partial_i$.*

1. *The special fiber $q^{-1}(0)$ is contained in the vanishing set of*

$$\{\partial^S \mid H_S^! \cap \Delta^! = \emptyset\} \tag{5.3.6}$$

2. *If $\mathcal{A}^!$ is smooth we have equality above. Moreover D_v is the vanishing set of*

$$\{\partial_i \mid v \notin H_i^!\}. \tag{5.3.7}$$

The ideal $I[\Delta^!]$ spanned by the relations (5.3.6) is known as the Stanley-Reisner ideal of the polyhedron $\Delta^!$. This ideal will be make another appearance in 6.1.

Let $(\mathcal{N}^!)^H$ be the toric variety whose Delzant polyhedron is $\Delta^!$. Notice that it's maximal torus is isomorphic to the dual torus G^* of G. It is well known [33, Theorem 12.4.14] that the Stanley-Reisner ring

$$S[\Delta^!] = \mathbb{C}[\partial_1, \dots, \partial_n]/I[\Delta^!]$$

is isomorphic to the equivariant cohomology $H_{G^*}^*((\mathcal{N}^!)^H)$.

Corollary 5.3.4. *We have that*

$$q^{-1}(0) \cong Spec\, H_{G^*}^*((\mathcal{N}^!)^H).$$

5.4. The b-function of a Dirichlet boundary condition

In practice the easiest way to find the V-filtration on a module is to prove that some naturally occurring filtration admits a nonzero polynomial b' satisfying the conclusions of Proposition 5.2.1. Then by uniqueness you know that this filtration must be the V-filtration. To this end we will compute the b-function of

$1 \in a\mathcal{D}$ and show that it admits a particular nice geometric description. Since the b-function does not depend on the G-action we may assume that χ is not strongly resonant or even, in the integral case, that $\chi = 0$.

The relation (5.3.2) tells us that the action of $D[\mathbb{C} \times \mathbb{C}^n]^0$ on $a\mathcal{D}$ factors through U^0_χ. Thus we can use the geometry of the quantized arrangement $q\mathcal{A}$ in $q\mathfrak{f}^* = \operatorname{Spec} U^0_\chi$ to find the b-function. By relation (5.3.3) the opeartor $\theta = q\frac{\partial}{\partial_q}$ defines a function on $q\mathfrak{f}$. Recall that the b-function is the minimal monic nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that

$$b(\theta) \cdot 1 \subseteq V_1 D[\mathbb{C} \times \mathbb{C}^n] \tag{5.4.1}$$

Our first observation is that the relation (5.3.1) implies that

$$q^{\langle \lambda, \xi \rangle} - m^\lambda = q^{\langle \lambda, \xi \rangle +} \prod_{\lambda_j > 0} [h_i]^{\lambda_j} = q^{\langle \lambda, \xi \rangle +} \rho_\lambda.$$

In particular we see that if $\langle \lambda, \xi \rangle < 0$ then $\rho_\lambda \in V_1 D[\mathbb{C} \times \mathbb{C}^n] \cdot 1 \subset a\mathcal{D}$. As in the proof of Proposition 4.4.2 let $\{\lambda^1, \ldots, \lambda^d\}$ be the collection of $\lambda \in X^*(F)$ such that λ is a primitive generators of a 1-dimensional flat in $q\mathcal{A}$ and $\langle \lambda, \xi \rangle < 0$. Let $V \subseteq q\mathfrak{f}^*$ be the vanishing set of the ideal I generated by the ρ_{λ^i}. When $q\mathcal{A}$ is regular V is reduced but this is not always true in cases of interest. Let V^{red} be the corresponding reduced variety.

Now we see that any polynomial $c(s) \in \mathbb{C}[s]$ with $c(\theta) \in I$ satisfies equation (5.4.1) and hence must be divisible by the b-function. The ideal J of of such polynomials has a geometric description in terms of the function $\theta : q\mathfrak{f}^* \to \mathbb{C} = \operatorname{Spec} \mathbb{C}[s]$.

Proposition 5.4.1. *The ideal*

$$J = \{c(s) \in \mathbb{C}[s] \,|\, c(\theta) \in I\}$$

cuts out the scheme theoretic image of V under θ.

Note that if only want the roots of the b-function and not their multiplicities it is enough to consider V^{red}.

The following Proposition is similar to an observation made during the last step of Proposition 4.4.2.

Proposition 5.4.2. *V^{red} is the union of 0-dimensional flats in $q\mathcal{A}$.*

Proof. Since $q\mathcal{A}$ is unimodular all the λ_j^i are in the set $\{-1, 0, 1\}$. In particular the reduced vanishing set of ρ_{λ^i} is a union of hyperplanes of the form H_j^+ in $q\mathcal{A}$. Thus V^{red} must be a flat in $q\mathcal{A}$. Furthermore it is clear that ρ_{λ^i} does not vanish identically on any affine subspace of the form $w + \mathbb{C}\lambda^i$ for $w \in q\mathfrak{f}^*$. In particular V cannot contain any flat of positive dimension. $\qquad\square$

Now we would like to determine precisely which 0-dimensional flats are contained in V. Recall that every 0-dimensional flat in $q\mathcal{A}$ is of the form $H_b = \cap_{l \in b} H_l$ for some basis $b \subset \{1, \ldots, n\}$.

Proposition 5.4.3. *A 0-dimensional flat $H_b \in V^{red}$ if and only if $(b, +\ldots+)$ is bounded.*

Proof. Choose a basis b. Since $q\mathcal{A}$ is essential and unimodular the vectors $\{h_j \,|\, j \in b\}$ form a basis for $X_*(F)$. We can find a rational lift $\tilde{\xi} \in X_*(T) \otimes \mathbb{Q}$ such that $\tilde{\xi}_j = 0$ for $j \notin b$.

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If $(b, + \ldots +)$ is a bounded we can also arrange that $\tilde{\xi}_j \leq 0$ for all $j \in b$. Thus since every λ^i satisfies $\langle \xi, \lambda^i \rangle = \sum \tilde{\xi}_j \lambda^i_j < 0$ we must have that $\lambda^i_j > 0$ for some $j \in b$. Thus every ρ_{λ^i} vanishes on H_b.

If If $(b, + \ldots +)$ is not bounded then there must exist some $j \in b$ such that $\tilde{\xi}_j > 0$. However the primitive generator λ^i of the 1-dimensional flat $\cap_{l \in b - \{j\}} H_l$ with $\langle \lambda^i, \xi \rangle < 0$ must have $\lambda^i_l = 0$ for all $l \in b - \{j\}$ and $\lambda^i_j < 0$. In particular ρ_{λ^j} does not vanish on H_b. $\qquad \square$

By the discussion in Section 2.2 we have the following.

Corollary 5.4.4. *A 0-dimensional flat $H_b \in V^{red}$ if and only $H^!_{b^!}$ is a vertex of $\Delta^!$.*

Combining Proposition 5.4.1 and Proposition 5.4.3 gives us the following corollary.

Corollary 5.4.5. *If $+ \ldots +$ is unbounded V^{red} is empty and hence $b(s) = 1$. In particular $J(\mathcal{D}) = 0$.*

If $\chi = 0$, V^{red} is either empty or consists of the single point where all the hyperplanes H_i intersect. The relation (5.3.3) implies that θ takes the value -1 on this point. Thus we have proved the following

Corollary 5.4.6. *Suppose $\chi = 0$.*

1. *The b-function of $1 \in a\mathcal{D}$ is $b(s) = (s+1)^l$ for some $l \in \mathbb{N}$.*

2. *The V-filtration on $a\mathcal{D}$ is given by*

$$V_k a\mathcal{D} = (V_{k+1}\mathbb{C}[\mathbb{C} \times \mathbb{C}^n]) \cdot 1.$$

Remark 5.4.7. *In Remark 5.1.1 we claimed that one could make sense of the V-filtration after Hamiltonian reduction. The proofs of Propositions 5.4.1 and 5.4.3 go through word for word to compute the b-function of the $D[\mathbb{C}] \otimes_{\mathbb{C}} U_\chi$-module $(a\mathcal{D})^H$. Unfortunately after reducing one does not have the freedom to set $\chi = 0$ so computing the V-filtration on $(a\mathcal{D})^H$ is difficult.*

CHAPTER VI

THE STRUCTURE OF $J(\mathcal{D})$

6.1. A presentation

Recall that the geometric Jacquet functor is given by

$$J(\mathcal{D}) = \Psi_q r_* a^\dagger \mathcal{D} \cong \mathrm{gr}_0^V a\mathcal{D}.$$

For the rest of the section assume $\chi = 0$ and that $+ \ldots +$ is bounded. Then by Corollary 5.4.6 we have $V_0 a\mathcal{D} = (V_0 \mathbb{C}[\mathbb{C} \times \mathbb{C}^n]) \cdot q$ and $V_1 a\mathcal{D} = (V_0 \mathbb{C}[\mathbb{C} \times \mathbb{C}^n]) \cdot q^2$. Since elements of $D[\mathbb{C}^n]$ commute with q computing a presentation for $J(\mathcal{D})$ as a cyclic $\mathrm{gr}_0^V D[\mathbb{C} \times \mathbb{C}^n] = D[\mathbb{C}^n][\theta]$-module will be completely straightforward.

First, notice that in $J(\mathcal{D})$ we essentially set $q = 0$ in (5.3.1). Thus the resulting relations are are the Fourier transform of those that cut out the special fiber of the degeneration $q : \overline{aO} \to \mathbb{C}$. By Proposition 5.3.3 the ideal spanned by these relations is the Stanley-Reisner ideal $I[\Delta^!]$. Second, the relations (5.3.2) don't involve q and hence persist without change in $J(\mathcal{D})$. Finally the relation (5.3.3) becomes $\theta = -\mu_T(\tilde{\xi})$ since $V_0 a\mathcal{D}$ is generated by q. In summary we have the following proposition.

Proposition 6.1.1. *The module $D[\mathbb{C}^n][\theta]$-module $J(\mathcal{D})$ is cyclic with relations*

$$\partial^S = 0 \tag{6.1.1}$$

for $S \subseteq \{1, \ldots, n\}$ such that $H_S^! \cap \Delta^! = \emptyset$,

$$\mu^G(v) = 0 \qquad (6.1.2)$$

for $v \in \mathfrak{g}$, and

$$\theta = -\mu_T(\tilde{\xi}). \qquad (6.1.3)$$

The relations (6.1.2) are exactly those necessary that are necessary for $J(\mathcal{D})$ to be G-equivariant. Thus a fancier way of stating Proposition 6.1.1 is the following. Let

$$\mathbb{C}[\Delta^!] = \mathbb{C}[\partial_1, \ldots, \partial_n]/I[\Delta^!]$$

be the Stanley-Reisner ring with the obvious T-action. Let

$$Q_0 : \mathrm{mod}(D[X], G) \to \mathrm{mod}(D[X], G, 0)$$

be the functor described in that sends a weakly equivariant G-module to its largest G-equivariant quotient.

Corollary 6.1.2. *We have*

$$J(\mathcal{D}) \cong Q_0(D[\mathbb{C}^n] \otimes_{\mathbb{C}[\partial_1, \ldots, \partial_n]} \mathbb{C}[\Delta^!])$$

as G-equivariant $D[\mathbb{C}^n]$-modules. In fact this is even an isomorphism of weakly T-equivariant modules and the action of θ is given by the $D[\mathbb{C}^n]$-linear action of $\tilde{\xi} \in \mathfrak{t}$ discussed in Section .

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6.2. Clean filtrations of the Stanley-Reisner Ring

By Corollary 6.1.2 the module $J(\mathcal{D})$ is closely related to the Stanley-Reisner ring $\mathbb{C}[\Delta^!]$. In this section we will review some facts about certain filtrations on $\mathbb{C}[\Delta^!]$ that will be used in Section 6.3.

Let $R = \mathbb{C}[\partial_1, \ldots, \partial_n]$ with its T-action and let M be a finitely generated T-equivariant R-module. An increasing $\mathbb{N}$-filtration $F_\bullet M$ of M by T-equivariant R-submodules is called a *prime filtration* if

$$\mathrm{gr}_i^F M \cong R/P_i \otimes_{\mathbb{C}} \gamma_i \tag{6.2.1}$$

where P_i is a monomial prime ideal and $\gamma_i \in X^*(T)$ for all $i \in \mathbb{N}$. The *support* of the prime filtration F is the set of primes appearing in (6.2.1). We say that F is *clean* if the minimal associated primes of M are exactly those in the support of F.

We would like to study clean filtrations of $\mathbb{C}[\Delta^!]$. We will assume that $\mathcal{A}_{\mathbb{R}}^!$ is smooth so that we may identify vertices of $\Delta^!$ with bases and so that the minimal associated primes of $\mathbb{C}[\Delta^!]$ are precisely the ideals

$$P_{b^!} = (\partial_i \,|\, i \notin b^!) = (\partial_i \,|\, i \in b)$$

associated to vertices in Proposition 5.3.3. For each $i \in b^!$ the flat $H_{b^!-\{i\}}^!$ in $\mathcal{A}^!$ is 1-dimensional. Let $\mu^{i,b^!} \in X_*(G)$ be the primitive generator of $H_{b^!-\{i\}}^!$ pointing into $\Delta^!$. Fix a regular polarization $\xi^! \in X^*(G)$ of $\mathcal{A}_{\mathbb{R}}^!$. Define a monomial

$$\partial^{b^!} = \prod_{\substack{i \in b^! \\ \langle \xi^!, \mu^{i,b^!} \rangle > 0}} \partial_i \tag{6.2.2}$$

and let $\gamma(b^!) \in X^*(T)$ be its weight.

Remark 6.2.1. *Let $(\mathcal{N}^!)^H$, G^*, etc. be as in the discussion preceding Corollary 5.3.4. The polarization $\xi^!$ gives a character of G^* hence a $\mathbb{C}^*$-action on $(\mathcal{N}^!)^H$. Since $\xi^!$ is regular the fixed points of this action correspond to the vertices of $\Delta^!$. The G^*-equivariant fundamental class of*

$$\overline{\{n \in (\mathcal{N}^!)^H \mid \lim_{t \to \infty} \xi^!(t) \cdot n = b^!\}}$$

is precisely the monomial $\partial^{b^!} \in \mathbb{C}[\Delta^!]$. When $\Delta^!$ is compact these classes give a basis for the G^*-equivariant cohomology of $(\mathcal{N}^!)^H$.

Using $\xi^!$ we can partially order the vertices of $\Delta^!$ by the value of $\xi^!$. Let $b^!_1, \ldots, b^!_r$ be a linearization of this order. By a special case of a theorem of Dress [35] this determines a clean filtration of $\mathbb{C}[\Delta^!]$. Note that in [35] Dress considers a larger class of Stanley-Reisner rings built from simplicial complexes and states the following Proposition in terms of the combinatorial data of a shelling of this simplicial complex.

Proposition 6.2.2. *The filtration*

$$0 = F_0\mathbb{C}[\Delta^!] \subseteq \ldots \subseteq F_r\mathbb{C}[\Delta^!] = \mathbb{C}[\Delta^!]$$

defined by

$$F_i\mathbb{C}[\Delta^!] = \left(\bigcap_{j=i+1}^{r} P_{b^!_j}\right) / I[\Delta^!] \tag{6.2.3}$$

satisfies

$$gr_i^F\mathbb{C}[\Delta^!] \cong (R/P_{b^!_i}) \otimes_\mathbb{C} \gamma(b^!_i) \tag{6.2.4}$$

and hence is clean.

6.3. Exceptional Dirichlet filtrations of $J(\mathcal{D})$

A regular polarization $\xi^!$ of the smooth arrangement $\mathcal{A}_{\mathbb{R}}^!$ specifies an ordering of the vertices of $\Delta^!$ and hence a clean filtration $F_\bullet \mathbb{C}[\Delta^!]$. We have induced filtrations, which we will also call F, on $D[\mathbb{C}^n] \otimes_R \mathbb{C}[\Delta]$ and $J(\mathcal{D}) = Q_0(D[\mathbb{C}^n] \otimes_R \mathbb{C}[\Delta])$. Since $D[\mathbb{C}^n]$ is free as a right R-module Propositon 6.2.2 tells us that

$$\mathrm{gr}_i^F D[\mathbb{C}^n] \otimes_R \mathbb{C}[\Delta] = D[\mathbb{C}^n] \otimes_R \mathrm{gr}_i^F \mathbb{C}[\Delta] = (D[\mathbb{C}^n] \otimes_R R/P_{b_i^!}) \otimes_{\mathbb{C}} \gamma(b_i^!).$$

We would like to compute $\mathrm{gr}_\bullet^F J(\mathcal{D})$.

Proposition 6.3.1. *Let $b^!$ be a vertex of $\Delta^!$ and let $P_{b^!}$ be as in Section 6.2. Then*

$$\mathbb{L}Q_0((D[\mathbb{C}^n] \otimes_R R/P_{b^!}) \otimes_{\mathbb{C}} \gamma(b^!)) \cong D_{b,\alpha^!(b^!)} \otimes_{\mathbb{C}} \gamma(b^!)$$

Proof. By reordering hyperplanes we may assume that $b^! = \{1, \ldots, k\} \subseteq \{1, \ldots, n\}$. Since is $b^!$ is a basis of $\mathcal{A}_{\mathbb{R}}^!$ the vectors $\{h_i^* \mid i \in b^!\}$ are a basis for $X^*(G)$. Let $\{v_1, \ldots, v_k\}$ be the dual basis in $X_*(G)$. In this basis we our exact sequence of cocharacter lattices from Section 2.3 takes the form

$$0 \to X_*(G) \xhookrightarrow{\ i_* = \begin{bmatrix} 1 \\ A \end{bmatrix}\ } X_*(T) \xrightarrow{\ p_*\ } X_*(F)$$

so

$$\mu^G(v_i) = x_i \partial_i + \sum_{j=1}^{n-k} A_{j,i} x_{k+j} \partial_{k+j}.$$

In particular we see that $\partial_{k+1}, \ldots, \partial_n$ is a regular sequence in $\mathbb{C}[\mu_G^{-1}(0)]$.

Let

$$M = (D[\mathbb{C}^n] \otimes_R R/P_{b^!}) \otimes_{\mathbb{C}} \gamma(b^!) \cong D[\mathbb{C}^n]/(\partial_i \mid i \notin b^!) \otimes_{\mathbb{C}} \gamma(b^!)$$

Recall that $Q_0(M) = M/\sum_{i=1}^k v_i \bullet M$. But M is cyclic so we just need to examine

$$v_i \bullet 1 = \langle \gamma(b^!), v_i \rangle - \mu^G(v_i) = \gamma(b^!)_i - x_i \partial_i.$$

If we can show

$$\gamma(b^!)_i = \begin{cases} 0 & \alpha(b)_i = + \\ -1 & \alpha(b)_i = - \end{cases}$$

we are done. First, recall that $\Delta_{\alpha^!(b^!)}$ is the unique chamber with $H_{b^!}^!$ as it's $\xi^!$-maximal point. Then, looking back at equation (6.2.2) we see that $\gamma(b^!)_i = -1$ if and only if $\langle \xi^!, \mu^{i,b^!} \rangle > 0$. But this happens precisely $\Delta_{+\ldots+}^!$ and $\Delta_{\alpha^!(b^!)}^!$ are on opposite sides of $H_i^!$.

Now just need to show that $\mathbb{L}Q_0(M)$ has no higher cohomology. Since $\partial_{k+1}, \ldots, \partial_n$ commute in $D[\mathbb{C}^n]$ we can form the Koszul complex $K_\bullet(b^!)$ which is a free resolution of M as a G-equivariant $D[\mathbb{C}^n]$-module. Note that the associated graded of this complex with respect to the degree filtration on $D[\mathbb{C}^n]$ is just the ordinary Koszul complex for $\partial_{k+1}, \ldots, \partial_n$ in $\mathbb{C}[T^*\mathbb{C}^n]$. Similarly the associated graded of $Q_0(K_\bullet(b^!))$ with respect to the degree filtration is the Koszul complex for $\partial_{k+1}, \ldots, \partial_n$ in $\mathbb{C}[\mu_G^{-1}(0)]$ which has no higher cohomology. $\square$

Corollary 6.3.2. *We have that*

$$gr_i^F J(\mathcal{D}) = D_{b_i, \alpha(b_i)} \otimes_{\mathbb{C}} \gamma(b_i^!).$$

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In particular

$$gr_r^F J(\mathcal{D}) \cong \mathcal{D}_{b_r, +\ldots+}.$$

Now we are ready to prove our main theorem.

Theorem 6.3.3. *Suppose that* $+\ldots+$ *is bounded in* $\mathcal{A}_0$ *and that* $\chi = 0$*. Then* $J(\mathcal{D})$ *is the indecomposable projective cover* $\mathcal{P}$ *of* $\mathcal{N} = \mathcal{N}_{+\ldots+}$ *in* $p\mathcal{O}_\chi^\xi$.

Proof. Choose a regular polarization of $\xi^!$ such that $+\ldots+$ is bounded in $\mathcal{A}_\mathbb{R}^!$. Recall that $\eta = -\xi^!$ determines a smooth decentralization $\mathcal{A}_\mathbb{R}$ of $\mathcal{A}_{\mathbb{R},0}$ such that $+\ldots+$ is feasible in $\mathcal{A}_\mathbb{R}$. Let $b_1^!, \ldots, b_r^!$ be the ordering ordering on the vertices of $\Delta^!$. Note that since $+\ldots+$ is bounded $b_r^!$ is the $\xi^!$-maximal point of $\Delta^!$ so $\alpha(b_r) = \alpha^!(b_r^!) = +\ldots+$.

Let $F_\bullet J(\mathcal{D})$ be the exceptional Dirichlet filtration coming from $\xi^!$. Then by Corollary 6.3.2 and Proposition 4.3.1 we have that

$$\mathrm{gr}_r^F \mathcal{D} \cong \mathcal{D}_b \cong k_{\eta,!}\mathcal{V}$$

where $\mathcal{D}_b = \mathcal{D}_{b,+\ldots+}$ and $\mathcal{V} = \mathcal{V}_{+\ldots+}$.

Let P be the indecomposable projective cover of $\mathcal{V}$ in $\mathcal{O}_\chi^\xi$. Since $k_{\eta,!}$ is left adjoint to the exact functor k_η^* we know that it must send projectives in $\mathcal{O}_\chi^\xi$ to projectives in $p\mathcal{O}_\chi^\xi$. Thus $\mathcal{P} = k_{\eta,!}P$ is the indecomposable projective cover of $\mathcal{D}_b$ and hence also $\mathcal{N}$. Moreover, by Corollary 4.3.3 and the discussion in Section the projective $\mathcal{P}$ has an exceptional Dirichlet filtration whose subquotients are precisely those that appear in $\mathrm{gr}_\bullet^F J(\mathcal{D})$.

By the definition of projective cover we have a morphism $m : \mathcal{P} \to J(\mathcal{D})$ coming from the surjection $J(\mathcal{D}) \to \mathcal{D}_b$. Every element of $J(\mathcal{D})$ that maps to the

generator of $\mathcal{D}_b$ is a generator so m must be surjective. But we already knew that

$[J(\mathcal{D})] = [\mathcal{P}] \in K_0(p\mathcal{O}_\chi^\xi)$ so m is an isomorphism. $\qquad\square$

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