GRADED REPRESENTATION THEORY OF HECKE ALGEBRAS

by

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We study the graded representation theory of the Iwahori-Hecke algebra, denoted by H_d, of the symmetric group over a field of characteristic zero at a root of unity. More specifically, we use graded Specht modules to calculate the graded decomposition numbers for H_d. The algorithm arrived at is the Lascoux-Leclerc-Thibon algorithm in disguise. Thus we interpret the algorithm in terms of graded representation theory.

We then use the algorithm to compute several examples and to obtain a closed form for the graded decomposition numbers in the case of two-column partitions. In this case, we also precisely describe the 'reduction modulo p' process, which relates the graded irreducible representations of H_d over $\mathbb{C}$ at a p^{th}-root of unity to those of the group algebra of the symmetric group over a field of characteristic p.

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CHAPTER I

INTRODUCTION

The *symmetric group* Σ_d is a classical object that has been very well studied over the years. It is deeply connected to group theory in general since every finite group can be embedded into a sufficiently large symmetric group. It also plays a fundamental role in combinatorics and in Lie theory. One way to study symmetric groups is by exploring how they interact with vector spaces; that is, to study their *representation theory.* Much of the foundation in the representation theory of symmetric groups was laid down by Frobenius at the end of the nineteenth century. Since his time, work has been done by countless mathematicians in an effort to fully understand this basic and classical group.

The main goal in representation theory is to understand the irreducible modules. In more modern language, the goal is to find bases for the Grothendieck groups of the categories of finite dimensional representations and finitely generated projective modules respectively. Much work has been done toward this end in describing both the irreducible representations of Σ_d as well as the indecomposable projective Σ_d-modules. In fact, over fields of characteristic zero, both of these sets of modules are well-known and well-understood.

However, over fields of positive characteristic, the representation theory of Σ_d still contains many open questions. For example, there are no explicit bases for the irreducible representations as vector spaces. In fact, the dimensions of the irreducible representations are not even known. In trying to approach this problem of *modular representation theory* using induction, Kleshchev studied the restrictions of irreducible modules from Σ_d to Σ_{d-1}. His work in [K1]-[K6], resulted in the discovery of *Kleshchev's modular branching rules.*

These branching rules only provide partial information about the dimensions of irreducible representations. However, Lascoux, Leclerc, and Thibon [LLT] were inspired by the subtle combinatorics involved to discover deep connections between the representation theory of Σ_d and representations of *quantum Kac-Moody algebras.* They showed that the modular branching rules correspond to the *crystal graph* - in the sense of Kashiwara - of the basic module of a certain Kac-Moody algebra $\mathfrak{g}$. This observation provided the framework for many of the exciting developments currently appearing in the representation theory of Σ_d.

Other tools which are quite useful in the study of symmetric groups are the corresponding *Iwahori-Hecke algebras.* The Iwahori-Hecke algebra, $H_d := H_d(\mathbb{F}, \xi)$, of Σ_d over a field $\mathbb{F}$ with parameter $\xi \in \mathbb{F}^{\times}$ is actually a generalization of the group algebra of the symmetric group. Furthermore, since the work of Dipper and James [DJ] it has been known that the representation theory of Σ_d over a field of characteristic $p > 0$ is closely related to the representation theory of H_d over $\mathbb{C}$ when the parameter ξ is a primitive p^{th}-root of unity. Thus, one may study the representation theory of H_d in general and gain information about the modular representation theory of Σ_d by specializing to $\mathbb{F} = \mathbb{C}$ and $\xi = e^{2\pi i/p}$.

The connection between the representation theory of $H_d(\mathbb{C}, e^{2\pi i/p})$ and the representation theory of Σ_d in characteristic p can be made precise using a 'reduction modulo p' procedure. In Section VII.4, we give an example of this procedure in the case of two-column partitions. Unfortunately, reductions modulo p of irreducible modules over the Hecke algebra are not always irreducible, however, *James' Conjecture* [J4] predicts that they are in the *James region.* If the reduction modulo p of an irreducible H_d-module is not irreducible, it can still be thought of as a 'good approximation' of the corresponding irreducible module over the group algebra of the symmetric group $\mathbb{F}\Sigma_d$. It turns out that the work done by Lascoux, Leclerc, and Thibon in [LLT] leads to a good understanding of the irreducible modules over $H_d(\mathbb{C}, e^{2\pi i/p})$, at least in terms of their *characters.*

More precisely, Lascoux, Leclerc, and Thibon conjectured an explicit connection between the *canonical bases* of modules over affine Kac-Moody algebras $\mathfrak{g}$, in the sense of Lusztig [Lus] and Kashiwara [Kas1] - [Kas4], and projective indecomposable modules over the Iwahori-Hecke algebras $H_d(\mathbb{C}, e^{2\pi i/p})$. Lascoux, Leclerc, and Thibon also conjectured an explicit combinatorial algorithm for computing the *decomposition numbers* for H_d. The decomposition numbers are the multiplicities of the irreducible H_d-modules in the corresponding *Specht modules.*

One can readily show that knowing the decomposition numbers is sufficient for computing the dimensions and even characters of the irreducible H_d-modules since the characters of Specht modules are explicitly known. The Lascoux-Leclerc-Thibon algorithm yields certain Laurent polynomials with non-negative coefficients that, when evaluated at $q = 1$, conjecturally compute the decomposition numbers.

Using the powerful geometric results of Kazhdan-Lusztig [KL] and Ginzburg [CG, Chapter 8], Ariki [A] later proved the conjecture of Lascoux, Leclerc, and Thibon, thereby providing a way to compute the decomposition numbers for complex Iwahori-Hecke algebras at roots of unity. Following in Ariki's footsteps, Varagnolo and Vasserot [VV] later proved a similar theorem for Schur algebras.

One nice way to connect the Iwahori-Hecke algebras to Lie theory is through the idea of *categorification* in the sense of I. Frenkel. Let Λ_0 be the fundamental dominant weight of the Kac-Moody algebra $\mathfrak{g}$ referenced above. It turns out that the finite dimensional modules over H_d for all $d \geq 0$ categorify the irreducible highest weight module $V(\Lambda_0)$ over $\mathfrak{g}$. Notice that the categorification by Ariki and Grojnowski only categorifies $V(\Lambda_0)$ as a $\mathfrak{g}$-module, rather than as a module over the *quantized enveloping algebra* $U_q(\mathfrak{g})$.

The fact that the canonical bases appeared when evaluated at $q = 1$ implies that $U_q(\mathfrak{g})$ should play a relevant role however. Thus, the picture appears incomplete unless one categorifies a q-analog of $V(\Lambda_0)$. A standard way to approach this problem is to find a grading on the Iwahori-Hecke algebras and to study *graded representation theory*, where the action of the parameter q on the Grothendieck group corresponds to a 'grading shift.'

The existence of important, but well-hidden, gradings on the Iwahori-Hecke algebra and the group algebras of the symmetric groups was predicted by Rouquier [R1] and Turner [T]. More recently, Brundan and Kleshchev [BK2] were able to construct such gradings by constructing an explicit isomorphism between Hecke algebras and certain quiver Hecke algebras (or Khovanov-Lauda-Rouquier algebras) which were defined independently by Khovanov-Lauda [KhL, KhL2] and Rouquier [R2]. The quiver Hecke algebras are naturally $\mathbb{Z}$-graded. Thus, using the isomorphism, Brundan and Kleshchev obtained an explicit grading on the Iwahori-Hecke algebra.

In [BKW], Brundan, Kleshchev, and Wang then grade the Specht modules, which allows them to define the *graded decomposition numbers* for H_d. In [BK1, BK3], Brundan and Kleshchev prove a graded analogue of Ariki's theorem. More specifically, they prove that the graded decomposition numbers are precisely the Laurent polynomials appearing in the conjecture of Lascoux, Leclerc, and Thibon.

The grading also provides new information about H_d-modules, which is collected into a *graded character* of a module. The graded characters of Specht modules are computed explicitly in [BKW]. As in the ungraded setting, knowledge of the graded characters of Specht modules combined with knowledge of the graded decomposition numbers is equivalent to knowing the graded characters of the irreducible H_d-modules.

The goal of this paper is to understand the graded characters of the irreducible H_d-modules. In particular, the graded characters of the irreducible $H_d(\mathbb{C}, e^{2\pi i/p})$-modules also give partial information about the corresponding graded characters of the irreducible $\mathbb{F}\Sigma_d$-modules when $\operatorname{char}\mathbb{F} = p$. Thus, in view of the previous discussion, the aim of the paper is to compute the graded decomposition numbers for H_d. In fact, the main result here is a combinatorial algorithm for computing the graded decomposition numbers.

Towards this goal, the paper is organized as follows: In Chapter II, we define the Iwahori-Hecke algebra as well as all of our basic objects which we will use throughout. We also give a homogeneous presentation for the Iwahori-Hecke algebra H_d and describe an important $\mathbb{Z}$-grading on it. Then, in Chapter III we setup all of the combinatorics which we will need to describe the representation theory of H_d and our algorithm. Specifically, we define partitions, Young diagrams, tableaux, moves, and ladders and describe some nice combinatorics associated to each of them.

In Chapter IV, we give a general explanation of how studying graded representation theory can yield additional insight into the usual representation theory. We then describe what is well-known about the representation theory (both graded and ungraded) of H_d. We finish this chapter by defining graded characters and graded decomposition numbers and setting up our main problem.

In Chapter V we define the ladder weight $\boldsymbol{j}^\lambda$ (for any e-restricted partition λ) which plays a central role in our algorithm. We also prove several nice results about $\boldsymbol{j}^\lambda$ and describe an explicit multiplicity for $\boldsymbol{j}^\lambda$ in the irreducible H_d-module corresponding to λ. Then, in Chapter VI we describe the algorithm which computes the graded decomposition numbers of H_d over any field of characteristic zero.

In Chapter VII, we use the algorithm to compute a closed form for the graded decomposition numbers in the case of *two-column* partitions. In this special case, we are also able to completely describe the splitting of irreducible H_d-modules when making the 'reduction modulo p' to the symmetric group algebra in characteristic p.

Despite taking a completely different approach, using the graded representation theory of H_d, the algorithm described in Chapter V is equivalent to the one suggested in [LLT], although this equivalence is not immediately obvious. In Chapter VIII, we briefly describe the LLT algorithm and explain the equivalence of the two algorithms. Note that our approach gives a new interpretation of some of the formal objects and coefficients appearing in the LLT algorithm.

The paper concludes with the computation of several examples using our algorithm in Chapter IX. We carefully describe how to compute a complete set of graded decomposition numbers for H_d and also how to compute a particular desired graded decomposition number from scratch. In the final section we also provide several matrices of graded decomposition numbers in the cases when $e = 2, 3$, or 4.

CHAPTER II

PRELIMINARIES

In this chapter, we set up some basic notation which will appear throughout and also describe our primary object of study - the Iwahori-Hecke algebra, H_d. In Section II.1 we define the quantum characteristic and the bar-involution on the space of Laurent polynomials $\mathbb{Z}[q, q^{-1}]$, while also introducing notation for the quantum integers. Then, in Section II.2 we give the standard definition of the Iwahori-Hecke algebra by generators and relations. We also describe the Gelfand-Zeitlin subalgebra and explain how it can be thought of as a 'Cartan subalgebra' for H_d.

In Section II.3, we describe a different set of generators for H_d which are homogeneous with respect to a $\mathbb{Z}$-grading. This $\mathbb{Z}$-grading is non-obvious from the perspective of the standard set of generators and relations. In fact, this new set of generators along with the relations and compatible grading described in Section II.4, was derived from an isomorphism between H_d and various quiver Hecke algebras (or Khovanov-Lauda-Rouquier algebras) which was recently discovered by Brundan, Kleshchev, and Wang [BKW].

II.1 Basic Objects

Let $\mathbb{F}$ be an algebraically closed field and $\xi \in \mathbb{F}^{\times}$. The *quantum characteristic* is the smallest positive integer, e, such that $1 + \xi + \xi^2 + \cdots + \xi^{e-1} = 0$, where we set $e := 0$ if no such integer exists. Set $I := \mathbb{Z}/e\mathbb{Z}$. For any $i \in I$ we have a well-defined element

$$\nu(i) := \begin{cases} i & \text{if } \xi = 1 \ , \\ \xi^i & \text{if } \xi \neq 1 \ , \end{cases}$$

of $\mathbb{F}$. Throughout the paper, q is an indeterminate and we define the *bar-involution* on $\mathbb{Z}[q, q^{-1}]$ by $\overline{p(q)} = p(q^{-1})$ for all $p(q) \in \mathbb{Z}[q, q^{-1}]$. We then refer to a Laurent polynomial $p(q) \in \mathbb{Z}[q, q^{-1}]$ as *bar-invariant* if $p(q) = \overline{p(q)}$. We also denote quantum integers and quantum factorials by

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{and} \quad [n]_q^! := [n]_q[n-1]_q \cdots [1]_q.$$

II.2 The Symmetric Group and the Iwahori-Hecke Algebra

Let Σ_d denote the symmetric group on d letters and denote the simple transpositions by

$$s_r := (r, r+1) \qquad (1 \leq r < d).$$

Then $\mathbb{F}\Sigma_d$ denotes the group algebra of Σ_d over the ground field $\mathbb{F}$. Recall that $\mathbb{F}\Sigma_d$ can be generated as an algebra by these simple transpositions $s_1, \ldots, s_{d-1}$, subject only to the relations

$$s_r^2 = 1 \qquad (1 \leq r < d),$$

$$s_r s_{r+1} s_r = s_{r+1} s_r s_{r+1} \quad (1 \leq r < d-1),$$

$$s_r s_t = s_t s_r \qquad (1 \leq r, t < d, \ |r - t| > 1).$$

The *Iwahori-Hecke algebra* of Σ_d with parameter ξ, is the $\mathbb{F}$-algebra $H_d = H_d(\mathbb{F}, \xi)$ with generators $T_1, T_2, \ldots, T_{d-1}$ and relations

$$T_r^2 = (\xi - 1)T_r + \xi \qquad (1 \leq r < d),$$

$$T_r T_{r+1} T_r = T_{r+1} T_r T_{r+1} \quad (1 \leq r < d-1),$$

$$T_r T_s = T_s T_r \qquad (1 \leq r, s < d, \ |r - s| > 1).$$

We will typically use the shorter notation H_d rather than $H_d(\mathbb{F}, \xi)$, with the understanding that the field $\mathbb{F}$ and the parameter ξ are fixed. Only when $\mathbb{F}$ and ξ are not clear from context, or there is more than one pair $(\mathbb{F}, \xi)$ to consider will we specify the field and parameter explicitly.

Observe that in the case when $\xi = 1$, we may identify H_d with the group algebra $\mathbb{F}\Sigma_d$ by identifying s_r and T_r for $1 \leq r < d$. In this way we may think of H_d as a generalization of $\mathbb{F}\Sigma_d$ and may consider the generators $T_1, \dots, T_{d-1}$ as the 'simple transpositions' for H_d. Observe that in the symmetric group case when $\xi = 1$, the quantum characteristic is just the characteristic of the field, i.e. $e = \operatorname{char} \mathbb{F}$.

The *Jucys-Murphy elements* of H_d are:

$$\mathcal{L}_r = \begin{cases} (1,r) + (2,r) + \cdots + (r-1,r) & \text{if } \xi = 1 \\ \xi^{1-r} T_{r-1} \cdots T_2 T_1 T_1 T_2 \cdots T_{r-1} & \text{if } \xi \neq 1 \end{cases} \qquad (1 \leq r \leq d).$$

It is well-known and easy to check that the Jucys-Murphy elements commute, see e.g. [Ju, M]. The *Gelfand-Zeitlin* subalgebra is the commutative subalgebra $\langle \mathcal{L}_1, \dots, \mathcal{L}_d \rangle \subset H_d$ generated by the Jucys-Murphy elements. Okounkov and Vershik [OV] suggested studying the representation theory of H_d by exploiting the Gelfand-Zetlin subalgebra as an analog of a 'Cartan subalgebra' for a Lie algebra. More precisely, they suggested studying the corresponding 'generalized weight spaces' of H_d-modules. The following Lemma describes the 'weights' of the Gelfand-Zeitlin subalgebra on any finite dimensional H_d-module.

Lemma II.2.1. [G, Lemma 4.7],[K7, Lemma 7.1.2] *Let V be a finite dimensional H_d-module. Then all of the eigenvalues of $\mathcal{L}_1, \dots, \mathcal{L}_d$ on V are of the form $\nu(i)$ for $i \in I$.*

Given $\boldsymbol{i} = (i_1, \dots, i_d) \in I^d$ and a finite dimensional H_d-module V, we define the $\boldsymbol{i}$*-weight space* of V to be:

$$V_{\boldsymbol{i}} := \{ v \in V \mid (\mathcal{L}_r - \nu(i_r))^N v = 0 \text{ for } N \gg 0 \text{ and } r = 1, \dots, d \}.$$

Then, by Lemma II.2.1 we have the *weight space decomposition*

$$V = \bigoplus_{\boldsymbol{i} \in I^d} V_{\boldsymbol{i}}.$$

II.3 Homogenous Generators

Using the weight space decomposition of the left regular H_d-module, one gets a system of orthogonal idempotents

$$\{e(\boldsymbol{i}) \mid \boldsymbol{i} \in I^d\}$$

in H_d, almost all of which are zero, such that $\sum_{\boldsymbol{i}\in I^d} e(\boldsymbol{i}) = 1$, and

$$e(\boldsymbol{i})V = V_{\boldsymbol{i}} \qquad (\boldsymbol{i} \in I^d)$$

for any finite dimensional H_d-module V, cf. [BK2].

We can now define a family of nilpotent elements $y_1, \dots y_d \in H_d$ via:

$$y_r := \begin{cases} \displaystyle\sum_{\boldsymbol{i}\in I^d} \left(1 - \xi^{-i_r}\mathcal{L}_r\right) e(\boldsymbol{i}) & \text{if } \xi \neq 1, \\ \displaystyle\sum_{\boldsymbol{i}\in I^d} \left(\mathcal{L}_r - i_r\right) e(\boldsymbol{i}) & \text{if } \xi = 1. \end{cases} \qquad (1 \leq r \leq d).$$

In [BK2], Brundan and Kleshchev define explicit power series $P_r(\boldsymbol{i}), Q_r(\boldsymbol{i}) \in \mathbb{F}[[y_r, y_{r+1}]]$ such that $Q_r(\boldsymbol{i})$ has non-zero constant term. Since each y_r is nilpotent in H_d, the power series $P_r(\boldsymbol{i})$ and $Q_r(\boldsymbol{i})$ can be interpreted as elements in H_d, with $Q_r(\boldsymbol{i})$ being invertible. We refer the reader to [BK2, Section 3.3, 4.3] for precise definitions of these power series. Now set:

$$\psi_r := \sum_{\boldsymbol{i}\in I^d} (T_r + P_r(\boldsymbol{i}))Q_r(\boldsymbol{i})^{-1}e(\boldsymbol{i}) \qquad (1 \leq r < d).$$

The main result of [BK2] asserts that H_d is generated by the elements

$$\{e(\boldsymbol{i}) \mid \boldsymbol{i} \in I^d\} \cup \{y_1, \dots, y_d\} \cup \{\psi_1, \dots \psi_{d-1}\}$$

and describes a set of defining relations as well.

This homogenous presentation yields a non-obvious $\mathbb{Z}$-grading on H_d which allows one to gain new insight by studying graded representation theory. To describe the grading explicitly, it is convenient to introduce the notation of quivers.

Let Γ be the quiver with vertex set I and a directed edge from i to j whenever $j = i + 1$. So Γ is the quiver of type A_∞ when $e = 0$ or of type $A^{(1)}_{e-1}$ when $e > 0$, with a specific orientation, see Figure 1:

$$A_\infty : \qquad \cdots \longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots$$

$$A^{(1)}_{e-1} : \qquad 0 \rightleftarrows 1 \qquad \begin{tikzcd}[column sep=small] & 0 \arrow[dr] & \\ 2 \arrow[ur] & & 1 \arrow[ll] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & 1 \arrow[d] \\ 3 \arrow[u] & 2 \arrow[l] \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & 0 \arrow[dr] & \\ 4 \arrow[ur] & & 1 \arrow[dl] \\ 3 \arrow[u] & 2 \arrow[l] & \end{tikzcd} \qquad \cdots$$

Figure 1: The quivers A_∞ and $A^{(1)}_{e-1}$ for $e = 2, 3, 4,$ and 5.

The corresponding *Cartan matrix*, denoted $(a_{i,j})_{i,j\in I}$ is defined by

$$a_{i,j} := \begin{cases} 2 & \text{if } i = j, \\ 0 & \text{if } i \not\!\!- j, \\ -1 & \text{if } i \to j \text{ or } i \leftarrow j, \\ -2 & \text{if } i \leftrightarrows j. \end{cases} \tag{II.1}$$

Here the symbols $i \to j$ and $j \leftarrow i$ are both interpreted to mean that $j = i+1 \neq i-1$. The symbol $i \leftrightarrows j$ means that $j = i + 1 = i - 1$, and the symbol $i \not\!\!- j$ means that $j \neq i$ or $i \pm 1$.

II.4 Homogenous Presentation

We now have the notation necessary to state the main result of [BK2].

Theorem II.4.1. *The algebra H_d is generated by the elements*

$$\{e(\boldsymbol{i}) \mid \boldsymbol{i} \in I^d\} \cup \{y_1, \ldots, y_d\} \cup \{\psi_1, \ldots \psi_{d-1}\}$$

subject only to the following relations for $\boldsymbol{i}, \boldsymbol{j} \in I^d$ and all admissible r, s:

$$
\begin{aligned}
y_1^{\delta_{i_1,0}} e(\boldsymbol{i}) &= 0; \\
e(\boldsymbol{i})e(\boldsymbol{j}) &= \delta_{\boldsymbol{i},\boldsymbol{j}} e(\boldsymbol{i}); && \textstyle\sum_{\boldsymbol{i}\in I^d} e(\boldsymbol{i}) = 1; \\
y_r e(\boldsymbol{i}) &= e(\boldsymbol{i}) y_r; && \psi_r e(\boldsymbol{i}) = e(s_r \boldsymbol{i})\psi_r; \\
y_r y_s &= y_s y_r; \\
\psi_r y_s &= y_s \psi_r && \text{\textit{if} } s \neq r, r+1; \\
\psi_r \psi_s &= \psi_s \psi_r && \text{\textit{if} } |r-s| > 1;
\end{aligned}
$$

$$
\psi_r y_{r+1} e(\boldsymbol{i}) = \begin{cases} (y_r \psi_r + 1) e(\boldsymbol{i}) & \text{\textit{if} } i_r = i_{r+1}, \\ y_r \psi_r e(\boldsymbol{i}) & \text{\textit{if} } i_r \neq i_{r+1}; \end{cases}
$$

$$
y_{r+1} \psi_r e(\boldsymbol{i}) = \begin{cases} (\psi_r y_r + 1) e(\boldsymbol{i}) & \text{\textit{if} } i_r = i_{r+1}, \\ \psi_r y_r e(\boldsymbol{i}) & \text{\textit{if} } i_r \neq i_{r+1}; \end{cases}
$$

$$
\psi_r^2 e(\boldsymbol{i}) = \begin{cases} 0 & \text{\textit{if} } i_r = i_{r+1}, \\ e(\boldsymbol{i}) & \text{\textit{if} } i_r \not- i_{r+1}, \\ (y_{r+1} - y_r) e(\boldsymbol{i}) & \text{\textit{if} } i_r \rightarrow i_{r+1}, \\ (y_r - y_{r+1}) e(\boldsymbol{i}) & \text{\textit{if} } i_r \leftarrow i_{r+1}, \\ (y_{r+1} - y_r)(y_r - y_{r+1}) e(\boldsymbol{i}) & \text{\textit{if} } i_r \rightleftarrows i_{r+1}; \end{cases}
$$

$$
\psi_r \psi_{r+1} \psi_r e(\boldsymbol{i}) = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1} + 1) e(\boldsymbol{i}) & \text{\textit{if} } i_{r+2} = i_r \rightarrow i_{r+1}, \\ (\psi_{r+1}\psi_r\psi_{r+1} - 1) e(\boldsymbol{i}) & \text{\textit{if} } i_{r+2} = i_r \leftarrow i_{r+1}, \\ \big(\psi_{r+1}\psi_r\psi_{r+1} - 2y_{r+1} & \\ \qquad + y_r + y_{r+2}\big) e(\boldsymbol{i}) & \text{\textit{if} } i_{r+2} = i_r \rightleftarrows i_{r+1}, \\ \psi_{r+1}\psi_r\psi_{r+1} e(\boldsymbol{i}) & \text{\textit{otherwise.}} \end{cases}
$$

Note that these relations depend only on e rather than on the specific parameter ξ chosen. Recall that if $\xi = 1$ and $\operatorname{char}\mathbb{F} = p > 0$, then $e = p$. Observe that $e = p$ again in the case when $\xi = e^{2\pi i/p}$ and $\mathbb{F} = \mathbb{C}$. Hence $H_d(\mathbb{C}, e^{2\pi i/p})$ and $H_d(\mathbb{F}, 1) \cong \mathbb{F}\Sigma_d$ have the same quantum characteristic, which implies that they satisfy the same relations (over different fields). This is suggestive of the fact that we can define a 'reduction modulo p' procedure to pass from graded representations of $H_d(\mathbb{C}, e^{2\pi i/p})$ over $\mathbb{C}$, to those of $\mathbb{F}\Sigma_d$ over a field of characteristic p.

As alluded to earlier, this presentation is particularly useful because it is obviously homogeneous with respect to the following grading:

Corollary II.4.2. *There is a unique $\mathbb{Z}$-grading on H_d such that*

$$\deg(e(\boldsymbol{i})) = 0, \qquad \deg(y_r) = 2, \qquad \deg(\psi_r e(\boldsymbol{i})) = -a_{i_r, i_{r+1}}$$

for all admissible r and $\boldsymbol{i}$.

CHAPTER III

COMBINATORICS

In this chapter we review several combinatorial notions which will become useful in what follows. In Section III.1 we discuss the basics of partitions, including the dominance order '$\trianglelefteq$' and the definition of an *e-restricted* partition. We also describe their associated Young diagrams, and the notion of addable and removable nodes. Then, in Section III.2, we describe residue diagrams and introduce the notion of the 'moves' for a partition.

Section III.3 reviews James' notion of ladders [J2] and introduces notation for keeping track of how the ladders interact with the Young diagram of a given partition λ. Finally, in Section III.4 we recall the set of (standard) λ-tableaux for a partition λ. We also define, for each standard λ-tableau $\mathtt{T}$, an associated residue sequence $\boldsymbol{i}^{\mathtt{T}}$ and the degree of $\mathtt{T}$, denoted $\deg(\mathtt{T})$.

III.1 Partitions and Young Diagrams

A *partition* is a non-increasing sequence of non-negative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$. We set $|\lambda| := \lambda_1 + \lambda_2 + \dots$ and refer to $|\lambda|$ as the *size* of λ. If $|\lambda| = d$, then λ is said to be a *partition of d* and we denote the set of all partitions of d by $\mathcal{P}_d$. Given a partition $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{P}_d$, we define the *transpose* partition $\lambda^T = (\lambda_1^T, \lambda_2^T, \dots) \in \mathcal{P}_d$ by setting $\lambda_i^T = \max\{r \mid \lambda_r \geq i\}$.

Example III.1.1. Let $\lambda = (4,3,1,1,1) \in \mathcal{P}_{10}$. Then the transpose of λ is $\lambda^T = (5,2,2,1)$. $\diamondsuit$

Let $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{P}_d$. We define the function $\sigma_k(\lambda) = \sum_{i=1}^{k} \lambda_i$. Given $\mu \in \mathcal{P}_d$, we say μ *dominates* λ, and we write $\mu \trianglerighteq \lambda$, if $\sigma_k(\mu) \geq \sigma_k(\lambda)$ for all $k \geq 1$. This defines a partial order on $\mathcal{P}_d$ which we refer to as the *dominance order* following James [J1]. The partition λ is called *e-restricted* if $\lambda_r - \lambda_{r+1} < e$ for all $r = 1, 2, \dots$. We then let $\mathcal{RP}_d \subset \mathcal{P}_d$ be the subset of all e-restricted partitions of d.

Example III.1.2. Consider the case when $d = 4$ and $e = 2$. Then the dominance order actually forms a *total* order on $\mathcal{P}_4$. The sets $\mathcal{P}_4$ and $\mathcal{RP}_4$, written in dominance order, are as follows:

$$\mathcal{P}_4 = \{(4) \trianglerighteq (3,1) \trianglerighteq (2,2) \trianglerighteq (2,1,1) \trianglerighteq (1,1,1,1)\} \text{ and } \mathcal{RP}_4 = \{(2,1,1) \trianglerighteq (1,1,1,1)\}.$$

Note that, it is common practice to use exponents to express repeated parts of a partition and write (2^2) instead of $(2,2)$ and (1^4) instead of $(1,1,1,1)$ for the sake of brevity. $\diamond$

Given a partition $\lambda \in \mathcal{P}_d$, the *Young diagram* of λ is the set

$$\{(a,b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid 1 \le b \le \lambda_a\}.$$

The elements of this set are called the *nodes* of λ. More generally, a *node* is any element of the set $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$. Given nodes $A = (a_1, a_2)$ and $B = (b_1, b_2)$ we say A is *above* (resp. *below*) B if $a_1 < b_1$ (resp. $a_1 > b_1$).

We usually identify a partition λ with its Young diagram as they are easier to visualize. One can then reinterpret the dominance order and the function σ_k using Young diagrams. Observe that $\sigma_k(\mu)$ counts the number of nodes in rows 1 through k of μ. Thus, the condition $\sigma_k(\mu) \ge \sigma_k(\lambda)$ for all k (or $\mu \trianglerighteq \lambda$) can be interpreted as being able to move nodes of μ from earlier rows 'down' to later rows and arrive at the shape of λ. We can also more easily visualize the transpose of a partition λ. The transpose λ^T is exactly the partition which has the rows of λ as its columns.

Example III.1.3. Consider the partitions $\mu = (3, 2^2)$ and $\lambda = (3, 2, 1^2)$. Following English conventions, we draw nodes in the 4^{th} quadrant so that the node (a, b) has its bottom right corner at the point $(-a, b)$ in the xy-plane. Then, μ, λ, and λ^T have the Young diagrams in Figure 1:

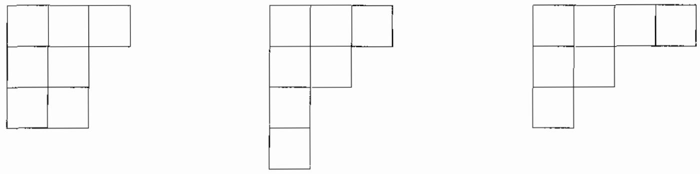

Figure 1: Young diagrams. Left: $\mu = (3, 2^2)$. Center: $\lambda = (3, 2, 1^2)$. Right: $\lambda^T = (4, 2, 1)$

Observe that we can move the node $(3, 2)$ down in μ to position $(4, 1)$ to obtain the partition λ (see Figure 2). Hence $\mu \trianglerighteq \lambda$. $\diamond$

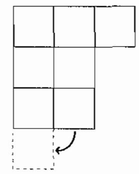

Figure 2: The graphical interpretation of moving the node $(3,2)$ down to $(4,1)$.

Let $\lambda \in \mathcal{P}_d$. A node $A \in \lambda$ is called *removable (for* λ*)* if $\lambda \backslash \{A\}$ has the shape of a partition. A node $B \notin \lambda$ is called *addable (for* λ*)* if $\lambda \cup \{B\}$ has the shape of a partition. Observe that any given row of a partition λ can have *at most* one addable node and one removable node (although it might not have either). In view of this observation, we may define the *bottom removable node* of λ to be the unique removable node which is below all other removable nodes.

Given any set of removable nodes $\boldsymbol{A} = \{A_1, A_2, \ldots, A_m\}$ for λ, we denote the partition obtained by removing these nodes from λ by $\lambda_{\boldsymbol{A}} := \lambda \setminus \{A_1, \ldots, A_m\}$. Similarly, for any set of addable nodes $\boldsymbol{B} = \{B_1, B_2, \ldots, B_m\}$ for λ, we denote the partition obtained by adding these nodes to λ by $\lambda^{\boldsymbol{B}} := \lambda \cup \{B_1, \ldots, B_m\}$.

Example III.1.4. Again consider the partition $\mu = (3, 2^2)$. Figure 3 contains the Young diagram for λ with two sets of nodes labeled.

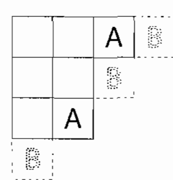

Figure 3: The Young diagram for $(3, 2^2)$ with sets of nodes labeled by $\boldsymbol{A}$ and $\boldsymbol{B}$.

The set $\boldsymbol{A} = \{(1,3), (3,2)\}$ is exactly all of the removable nodes of μ, thus the node $(3,2)$ is the bottom removable node of λ. Similarly, the set $\boldsymbol{B} = \{(1,4), (2,3), (4,1)\}$ is the set of all addable nodes for μ. From these sets we create $\mu_{\boldsymbol{A}} = (2^2, 1)$ and $\mu^{\boldsymbol{B}} = (4,3,2,1)$, see Figure 4. $\diamond$

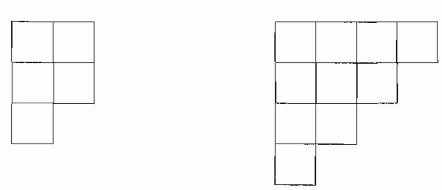

Figure 4: Young diagrams. Left: $\mu_{\boldsymbol{A}} = (2^2, 1)$. Right: $\mu^{\boldsymbol{B}} = (4,3,2,1)$.

III.2 Moves

Given the node $A = (a, b)$, the $(e\text{-})$*residue* of A is defined to be res $A := b - a \pmod{e} \in I$. We then define the *residue diagram* of a partition λ to be the Young diagram for λ where each node is labeled by its residue. For $i \in I$, a node A is called an *i-node* if res $A = i$. For $\lambda, \mu \in \mathcal{P}_d$, we say that λ and μ have the same *content*, and we write $\lambda \sim \mu$, if and only if for each $i \in I$ the number of i-nodes in λ is equal to that in μ.

Example III.2.1. Figure 5 gives the residue diagram for the partition $\lambda = (4, 3, 2, 1)$ in the case when the quantum characteristic is $e = 3$.

0	1	2	0
2	0	1	
1	2		
0			

Figure 5: Residue diagram for $\lambda = (4, 3, 2, 1)$ with $e = 3$.

λ's content is written $(4, 3, 3)$ meaning there are 4 zeros, 3 ones, and 3 twos. A quick calculation shows that the set of all partitions with content $(4, 3, 3)$ is

$\{(10), (8, 2), (7, 3), (7, 2, 1), (7, 1^3), (6, 2, 1^2), (5^2), (5, 2^2, 1), (5, 2, 1^3), (4^2, 1^2), (4, 3^2),$
$(4, 3, 2, 1), (4, 3, 1^3), (4, 2^3), (4, 2, 1^4), (4, 1^6), (3^3, 1), (3, 2, 1^5), (2^5), (2^3, 1^4), (2^2, 1^6), (1^{10})\}$ $\diamond$

Given $\mu \in \mathcal{P}_d$, we call $\lambda \in \mathcal{P}_d$ a *move for* μ if $\lambda \trianglelefteq \mu$ and $\lambda \sim \mu$. Informally, λ is a move for μ if we can move nodes down in μ to get λ while maintaining the residue content. We denote the set of moves for μ by $M(\mu)$. Put $M(\mu, \lambda) := \{\nu \in M(\mu) | \lambda \in M(\nu)\}$ for the set of nodes 'between' μ and λ. Note that if λ is not a move of μ then $M(\mu, \lambda)$ is empty.

The standard dominance order on $\mathcal{P}_d$ gives a partial order on $M(\mu)$, so it makes sense to refer to *minimal* non-trivial moves for μ. A non-trivial move λ for μ is a minimal move if and only if $M(\mu, \lambda) = \{\mu, \lambda\}$. Note that it is possible for μ to have more than one minimal non-trivial move.

For $\lambda \in M(\mu)$, we define the *distance* between λ and μ to be

$$l(\mu, \lambda) := \sum_{k \geq 1} \sigma_k(\mu) - \sigma_k(\lambda).$$

Since $\lambda \trianglelefteq \mu$, we know that $\sigma_k(\lambda) \leq \sigma_k(\mu)$ for all $k \geq 1$. So $l(\mu, \lambda) \geq 0$, with equality if and only if $\lambda = \mu$. Observe also that if $\nu \in M(\mu, \lambda)$, then $l(\mu, \nu) \leq l(\mu, \lambda)$ with equality if and only if $\nu = \lambda$.

Example III.2.2. Consider the case when $d = 5$ and $e = 2$, with the partitions $\mu = (3,2)$ and $\lambda = (2^2, 1)$. Then $M(\mu) = \{(3,2), (3,1^2), (2^2,1), (1^5)\}$, $M(\mu, \lambda) = \{(3,2), (3,1^2), (2^2,1)\}$ and $l(\mu, \lambda) = (3-2) + (5-4) + (5-5) = 2$. ◇

III.3 Ladders

Following [J2], for $m \in \mathbb{Z}_{>0}$ we define the m^{th} *ladder* L_m as the set of nodes of the form $(1+k, m-k(e-1))$ for all non-negative integers k with $k < \frac{m}{e-1}$. Informally, the ladders are straight lines with slope $\frac{1}{e-1}$. Note that our ladders are transposed to those of James, since we are using the newer Dipper-James-Mathas [DJM] notation for Specht modules. Observe that $\operatorname{res}(1+k, m-k(e-1)) = [m-k(e-1)] - [1+k] = m-1-ek \equiv m-1 \pmod{e}$, regardless of k. Thus, all of the nodes in the ladder L_m have the same residue, $m-1 \pmod{e}$, which we refer to as the *residue* of the ladder, and denote $\operatorname{res} L_m$. For a partition λ and a positive integer m, we keep track of the number of nodes in λ which lie on L_m by setting $r_m(\lambda) := |\lambda \cap L_m|$. We then denote by t_λ the maximal index such that $r_{t_\lambda}(\lambda) \neq 0$, and refer to the ladders $L_1, L_2, \dots, L_{t_\lambda}$ as the *ladders of* λ (some of them could have trivial intersection with λ).

Example III.3.1. Let $\lambda = (4,3,1,1)$ and $e = 3$. The ladders that intersect λ non-trivially can be seen in Figure 6.

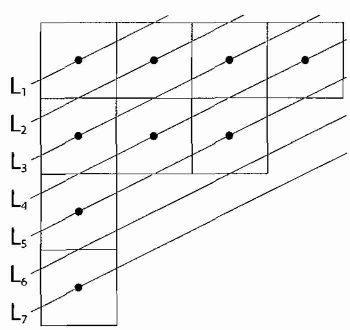

Figure 6: Ladders intersecting $\lambda = (4,3,1,1)$ non-trivially.

Observe that $t_\lambda = 7$, since the lowest ladder that intersects λ non-trivially is L_7. The intersections for each of the ladders $L_1, \dots, L_7$ are $\lambda \cap L_1 = \{(1,1)\}$, $\lambda \cap L_2 = \{(1,2)\}$, $\lambda \cap L_3 = \{(2,1),(1,3)\}$, $\lambda \cap L_4 = \{(2,2),(1,4)\}$, $\lambda \cap L_5 = \{(3,1),(2,3)\}$, $\lambda \cap L_6 = \varnothing$, and $\lambda \cap L_7 = \{(4,1)\}$. Thus $r_1(\lambda) = 1$, $r_2(\lambda) = 1$, $r_3(\lambda) = 2$, $r_4(\lambda) = 2$, $r_5(\lambda) = 2$, $r_6(\lambda) = 0$, and $r_7(\lambda) = 1$. ◇

In Example III.3.1, we observe that the bottom ladder L_7 actually intersects the bottom removable node of λ. It is true in general that the bottom ladder is exactly the ladder that intersects the bottom removable node.

A ladder L_m is *bottom complete for* λ if whenever a node $A = (a, b) \in L_m$ belongs to λ, all other nodes $(a', b') \in L_m$ with $a' > a$ also belong to λ. The following result of James describes a nice feature of the shape of any e-restricted partition.

Lemma III.3.2. [J2, 1.2] *Let $\lambda \in \mathcal{P}_d$. Then λ is e-restricted if and only if all ladders are bottom complete for λ.*

Let $\lambda \in \mathcal{P}_d$. For a removable i-node A in λ we define the *degree of A* to be:

$$\begin{aligned} d_A(\lambda) := \#\{&\text{addable } i\text{-nodes below } A\} \\ -\#\{&\text{removable } i\text{-nodes below } A\}. \end{aligned}$$

There is a notion of degree for an addable i-node $B \notin \lambda$ as well. We define the degree of B to be:

$$\begin{aligned} d^B(\lambda) := \#\{&\text{addable } i\text{-nodes above } B\} \\ -\#\{&\text{removable } i\text{-nodes above } B\}. \end{aligned}$$

We also define, for each $i \in I$,

$$d_i(\lambda) := \#\{\text{addable } i\text{-nodes of } \lambda\} - \#\{\text{removable } i\text{-nodes of } \lambda\}.$$

Example III.3.3. Consider the partition $\lambda = (9, 7, 6, 5^3, 3, 1)$ with $e = 3$.

0	1	2	0	1	2	0	1	2
2	0	1	2	0	1	2		
1	2	0	1	2	0			
0	1	2	0	1				
2	0	1	2	0				
1	2	0	1	2				
0	1	2						
2								

Figure 7: The residue diagram for the partition $\lambda = (9, 7, 6, 5^3, 3, 1)$.

Observe that if we consider the removable 2-node $A_1 = (8,1)$, then there are no addable or removable 2-nodes below A_1, hence $d_{A_1}(\lambda) = 0$. If we take the removable 2-node $A_2 = (2,7)$ instead, then there is one addable 2-node below A_2 at $(4,6)$ and three removable 2-nodes below A_2 at $(6,5)$, $(7,3)$, and $(8,1)$ respectively. Thus $d_{A_2}(\lambda) = 1 - 3 = -2$. Similarly, if we take the addable 0-node $B = (7,4)$, then there are two addable 0-nodes above B at $(1,10)$ and $(2,8)$ and one removable 0-node above B at $(3,6)$. Hence, $d^B(\lambda) = 2 - 1 = 1$. Finally, there is one addable 2-node in λ and five removable 2-nodes, which implies that $d_2(\lambda) = 1 - 5 = -4$. $\diamond$

III.4 Tableaux

Let $\lambda \in \mathcal{P}_d$. A λ-*tableau*, T, is obtained from the Young diagram of λ by labeling the nodes by the integers $1, \ldots, d$ without repeats. Each λ-tableau T then has an associated *residue sequence*

$$\boldsymbol{i}^{\mathrm{T}} = (i_1, \ldots, i_d) \in I^d,$$

where i_r denotes the residue of the node labeled by r in T $(1 \le r \le d)$.

Example III.4.1. Let $\lambda = (4,3,2,1)$ and $e = 3$. Figure 8 gives the residue diagram for λ.

0	1	2	0
2	0	1	
1	2		
0			

Figure 8: Residue diagram for $\lambda = (4,3,2,1)$ with $e = 3$ again.

3	5	2	6
4	7	9	
1	8		
10			

Figure 9: A (non-standard) λ-tableau for $\lambda = (4,3,2,1)$.

The λ-tableau T in Figure 9 has the associated residue sequence $\boldsymbol{i}^{\mathrm{T}} = (1,2,0,2,1,0,0,2,1,0)$. $\diamond$

A λ-tableau, T, is *row-strict* if its labels increase from left to right within each row. Similarly, T is *column-strict* if its labels increase from top to bottom within each column. T is *standard* if it is both row- and column-strict. The tableau in Example III.4.1 is **not** standard. We denote the set of all standard λ-tableaux by $\mathscr{T}(\lambda)$. Let $\mathtt{T}^\lambda$ denote the λ-tableau in which the nodes have been labeled sequentially by $1, \ldots, d$ along each row and from top to bottom. We then set

$$\boldsymbol{i}^\lambda := \boldsymbol{i}^{\mathtt{T}^\lambda}.$$

Note that the symmetric group Σ_d acts on the complete set of λ-tableaux on the left by permuting the labels; i.e. if $\pi \in \Sigma_d$ and $\mathtt{T}$ has label $r_{a,b}$ on each node $(a, b) \in \lambda$, then $\pi \cdot \mathtt{T}$ will be the λ-tableau with label $\pi(r_{a,b})$ on the node (a, b) for each $(a, b) \in \lambda$.

Example III.4.2. Let $\lambda = (4, 3, 2, 1)$, then Figure 10 shows the standard λ-tableau $\mathtt{T}^\lambda$.

1	2	3	4
5	6	7	
8	9		
10			

Figure 10: The standard λ-tableau, $\mathtt{T}^\lambda$, for the partition $\lambda = (4, 3, 2, 1)$.

Thus, refering again to the residue diagram in Figure 8, we have $\boldsymbol{i}^\lambda := \boldsymbol{i}^{\mathtt{T}^\lambda} = (0, 1, 2, 0, 2, 0, 1, 1, 2, 0)$. Figure 11 then shows the λ-tableau $\pi \cdot \mathtt{T}^\lambda$ in the case when $\pi = (35)(48)(67\,10) \in \Sigma_{10}$.

1	2	5	8
3	7	10	
4	9		
6			

Figure 11: The λ-tableau $\pi \cdot \mathtt{T}^\lambda$ for $\lambda = (4, 3, 2, 1)$ and $\pi = (35)(48)(6710) \in \Sigma_{10}$.

This new λ-tableau happens to be standard, and $\boldsymbol{i}^{\pi \cdot \mathtt{T}^\lambda} = (0, 1, 2, 1, 2, 0, 0, 0, 2, 1)$. Note, however, that if $\mathtt{T}$ is standard and $\pi \in \Sigma_d$, then $\pi \cdot \mathtt{T}$ is not standard in general. $\diamond$

Let $\mathtt{T} \in \mathscr{T}(\lambda)$. For $s \in \mathbb{Z}_{\geq 0}$ we denote by $\mathtt{T}_{\leq s}$ and $\mathrm{sh}(T_{\leq s})$ the tableau obtained by retaining the nodes of $\mathtt{T}$ labeled by the numbers $1, \ldots, s$ and its shape respectively. Observe that since $\mathtt{T}$ is standard, it follows that $\mathrm{sh}(\mathtt{T}_{\leq s})$ is a partition in $\mathcal{P}_s$ for all $0 \leq s \leq d$. Thus, $\mathtt{T}_{\leq s} \in \mathscr{T}(\mathrm{sh}(\mathtt{T}_{\leq s}))$ for $0 \leq s \leq d$.

Example III.4.3. Let $\lambda = (4,3,2,1)$ and let $\mathtt{T}$ be the λ-tableau in Figure 12.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 8 \\ \hline 3 & 7 & 10 \\ \cline{1-3} 4 & 9 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$$

Figure 12: A standard λ-tableau for $\lambda = (4,3,2,1)$.

Then $\mathtt{T}_{\leq 7}$ is the tableau in Figure 13.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 7 \\ \cline{1-2} 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}$$

Figure 13: The tableau $\mathtt{T}_{\leq 7}$ as created from the tableau $\mathtt{T}$ in Figure 12.

It has shape $\nu = \mathrm{sh}(\mathtt{T}_{\leq 7}) = (3,2,1^2)$. Observe that residue sequence $\boldsymbol{i}^{\mathtt{T}_{\leq 7}} = (0,1,2,1,2,0,0)$ is exactly $\boldsymbol{i}^{\mathtt{T}} = (0,1,2,1,2,0,0,0,2,1)$ with the final three residues removed. $\diamond$

Let $\mathtt{T} \in \mathscr{T}(\lambda)$ and let A be the node of λ labeled by d in $\mathtt{T}$. Following [BKW], we define the *degree* of $\mathtt{T}$ inductively by:

$$\deg(\mathtt{T}) = \begin{cases} d_A(\lambda) + \deg(\mathtt{T}_{\leq d-1}) & \text{if } d > 0, \\ 0 & \text{if } d = 0. \end{cases}$$

Example III.4.4. Consider the partition $\lambda = (3,2,1^2)$ and the λ-tableau $\mathtt{T}$ given in Figure 14.

$$\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 2 & 0 \\ \cline{1-2} 1 \\ \cline{1-1} 0 \\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 7 \\ \cline{1-2} 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}$$

Figure 14: Left: The residue diagram for $\lambda = (3,2,1^2)$. Right: A standard λ-tableau $\mathtt{T}$.

To calculate the degree of tableau $\mathtt{T}$ we must inductively calculate the degree of each node in $\mathtt{T}$ starting with the node labeled by 7 and working backward.

- Starting with the 0-node $A_7 = (2,2)$ which is labeled by 7, we can see that there is one removable 0-node below $(2,2)$ and no addable 0-nodes below it. Hence $d_{A_7} = -1$.
- We then remove this node from the diagram and consider the 0-node $A_6 = (4,1)$ labeled by 6. There are no addable or removable 0-nodes below this, hence $d_{A_6} = 0$.
- Next, we remove A_6 and consider the 2-node $A_5 = (1,3)$ labeled by 5. There are no addable or removable 2-nodes below A_5, hence $d_{A_5} = 0$ also.
- After removing A_5 we must consider the 1-node $A_4 = (3,1)$ labeled by 4. There are no addable or removable 1-nodes below A_4 that are left in the diagram (since we have removed A_5, A_6, and A_7), thus $d_{A_4} = 0$.
- Now we remove A_4 and consider the 2-node $A_3 = (2,1)$ labeled by 3. Once again, there are no addable or removable 2-nodes below A_3, so $d_{A_3} = 0$.
- Next we have the 1-node $A_2 = (1,2)$ labeled by 2. There are no addable or removable 1-nodes below A_2, thus $d_{A_2} = 0$.
- Finally, we are left with the single 0-node $A_1 = (1,1)$ labeled by 1. There are no addable or removable 0-nodes below A_1, hence $d_{A_1} = 0$.

Having gone through this process, the degree of the tableau T is exactly

$$\deg(\mathtt{T}) = \sum_{i=1}^{7} d_{A_i} = -1.$$

◇

CHAPTER IV

GRADED REPRESENTATION THEORY

In this chapter we lay the foundation for stating the main problem which this paper strives to solve. In Section IV.1, we discuss the graded representation theory of a $\mathbb{Z}$-graded algebra in general. In particular, we explain the connection between graded representation theory and the usual one. In doing so, we discuss the categories of finite dimensional graded representations and finitely generated projective graded representations.

Then, in Section IV.2 we study the (ungraded) representation theory of H_d by defining a set of explicitly defined Specht modules and explaining that the complete set of finite dimensional irreducible H_d-modules can be realized as quotients of these Specht modules. We add the $\mathbb{Z}$-grading to the story in Section IV.3 and explain how to 'grade' the Specht modules and the finite dimensional irreducible modules so that we may study their structures as *graded* H_d-modules.

In Section IV.4, we define the graded character of a graded H_d-module. We also describe the graded character of each explicit Specht module. Then, in Section IV.5 we define the graded decomposition numbers for H_d and state some known results about them. We then carefully state the main problem that this paper solves.

IV.1 A General Picture

The main drive of this paper is to study the *graded representation theory* of the group algebra of the symmetric group and the Iwahori-Hecke algebra. It is known that all of the irreducible modules over finite dimensional $\mathbb{Z}$-graded algebras are 'gradable.' Thus, in studying the graded irreducible modules we 'do not lose any information,' but actually gain additional insight into the structure of the ungraded irreducible modules. To be more precise about this, we will briefly discuss graded representation theory in more generality.

We will always use the term 'grading' to refer to a $\mathbb{Z}$-grading. Let H be a graded $\mathbb{F}$-algebra and let H-Mod denote the abelian category whose objects form the set of all graded left H-modules, and whose morphisms are *degree-preserving* module homomorphisms, denoted by Hom. Let $\mathrm{Rep}(H)$ denote the abelian subcategory of all finite dimensional graded H-modules and $\mathrm{Proj}(H)$ denote the additive subcategory of all finitely generated projective graded H-modules. We denote the corresponding Grothendieck groups by $[\mathrm{Rep}(H)]$ and $[\mathrm{Proj}(H)]$. We can view $[\mathrm{Rep}(H)]$ and $[\mathrm{Proj}(H)]$ as $\mathbb{Z}[q, q^{-1}]$-modules by setting

$$q^m[V] := [V\langle m\rangle],$$

where $V\langle m\rangle$ denotes the graded H-module obtained by shifting the grading up by m so that

$$V\langle m\rangle_n = V_{n-m},$$

for each $m, n \in \mathbb{Z}$.

For any graded H-modules, V and W, and any integer $n \in \mathbb{Z}$ we let

$$\mathrm{Hom}_H(V,W)_n := \mathrm{Hom}_H(V\langle n\rangle, W) = \mathrm{Hom}_H(V, W\langle -n\rangle)$$

denote the space of all homomorphisms that are homogeneous of degree n. Thus, if $\theta \in \mathrm{Hom}_H(V,W)_n$, then $\theta(V_m) \subseteq W_{n+m}$ for all $m \in \mathbb{Z}$. We then set

$$\mathrm{HOM}_H(V,W) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_H(V,W)_n.$$

Given any finite dimesional graded vector space $V = \bigoplus_{m\in\mathbb{Z}} V_m$, we define its *graded dimension* by an explicit Laurent polynomial

$$\mathrm{qdim}\, V := \sum_{m\in\mathbb{Z}} (\dim V_m) q^m \in \mathbb{Z}[q, q^{-1}].$$

Then, there is a canonical *Cartan pairing*

$$\langle\, .\, ,\, .\rangle := [\mathrm{Proj}(H)] \times [\mathrm{Rep}(H)] \longrightarrow \mathbb{Z}[q, q^{-1}], \qquad \langle [P], [V]\rangle = \mathrm{qdim}\, \mathrm{HOM}_H(P, V).$$

Note that the Cartan pairing is *sesquilinear* (meaning that it is anti-linear in the first position and linear in the second).

We denote the category of finite dimensional ungraded H-modules (resp. finitely generated projective ungraded H-modules) by $\underline{\mathrm{Rep}}(H)$ (resp. $\underline{\mathrm{Proj}}(H)$) and we denote the morphisms in these categories by $\underline{\mathrm{Hom}}$. Given a graded H-module V, we write $\underline{V}$ for the ungraded module obtained by forgetting the grading. Then for $V, W \in \mathrm{Rep}(H)$ we have

$$\underline{\mathrm{Hom}}_H(\underline{V}, \underline{W}) = \underline{\mathrm{HOM}_H(V, W)}.$$

In addition to this nice fact, the following standard lemmas show, informally, that in studying graded representation we do not lose any information from ungraded representation theory, but actually gain information from the additional structure.

Lemma IV.1.1. *[NV, Theorem 4.4.6 and Remark 4.4.8] If V is any finitely generated graded H-module, then the radical of $\underline{V}$ is a graded submodule of V.*

Lemma IV.1.2. *[NV, Theorem 4.4.4(v)] If $L \in \mathrm{Rep}(H)$ is irreducible, then $\underline{L} \in \underline{\mathrm{Rep}}(H)$ is irreducible as an ungraded module as well.*

Lemma IV.1.3. *[NV, Theorem 9.6.8], [BGS, Lemma 2.5.3] Assume that H is finite dimensional. If $K \in \underline{\mathrm{Rep}}(H)$ is irreducible, then there exists an irreducible $L \in \mathrm{Rep}(H)$ such that $\underline{L} \cong K$. Moreover, L is unique of to isomorphism and grading shift.*

IV.2 Representation Theory of H_d

In [GL], Graham and Lehrer introduced the concept of cellular algebras. They also showed that the Kazhdan-Lusztig basis for H_d is a cellular basis thereby proving that H_d is a cellular algebra. Dipper, James, and Mathas [DJM], then exploited the cellular structure to reconstruct a special family of H_d-modules $\{S(\mu) \mid \mu \in \mathcal{P}_d\}$, labeled by the partitions of d, called *Specht modules.* The original construction goes back to Dipper-James [DJ], although the Specht modules defined in [DJ] are different from those in [DJM]. Here we follow the conventions of [DJM].

For a more complete treatment of the following, please refer to [DJM]. There is a cellular basis for H_d as an $\mathbb{F}$-module, different from that defined by Kazdhan and Lusztig,

$$\mathcal{M} := \{m_{\mathtt{T},\mathtt{U}} \mid \mathtt{T}, \mathtt{U} \in \mathscr{T}(\mu); \mu \in \mathcal{P}_d\},$$

which is in bijection with pairs of standard μ-tableaux for each $\mu \in \mathcal{P}_d$. In the special case when $\mathtt{T}$ and $\mathtt{U}$ are both taken to be $\mathtt{T}^\mu$, we denote the basis element $m_{\mathtt{T}^\mu,\mathtt{T}^\mu}$ by m_μ.

For each partition $\mu \in \mathcal{P}_d$, we define an $\mathbb{F}$-submodule of H_d denoted by $\overline{N}^\mu$ that is spanned by $\{m_{\mathtt{T},\mathtt{U}} \mid \mathtt{T}, \mathtt{U} \in \mathscr{T}(\lambda); \lambda \in \mathcal{P}_d; \lambda \trianglerighteq \mu\}$. Set $z_\mu = (m_\mu + \overline{N}^\mu)/\overline{N}^\mu$, to be the coset representative of m_μ in $H_d/\overline{N}^\mu$. The Specht module, $S(\mu)$, is then the submodule of $H_d/\overline{N}^\mu$ given by $S(\mu) = z_\mu H_d$.

The Specht modules are examples of *cell modules* which implies the existence of a nice cellular basis

$$\{C_\mathtt{T} \mid \mathtt{T} \in \mathscr{T}(\mu)\} \qquad (\mu \in \mathcal{P}_d),$$

parametrized, in this case, by the set of standard μ-tableaux. In Section IV.3, we will see a different cellular basis which is well-suited to the task of 'grading' the Specht modules.

Since H_d is a cellular algebra, the cellular structure also provides an explicit bilinear form $\langle\, , \rangle$ which acts on cell modules, hence on Specht modules. Let $\operatorname{rad} S(\mu)$ denote the radical of $S(\mu)$ under this bilinear form. We define $D(\mu) := S(\mu)/\operatorname{rad} S(\mu)$. In the case when $D(\mu) \neq 0$, the quotient $D(\mu)$ forms the irreducible head of $S(\mu)$.

If $e = 0$, then $D(\mu) = S(\mu)$ for all $\mu \in \mathcal{P}_d$ and the set of Specht modules, $\{S(\mu) \mid \mu \in \mathcal{P}_d\}$, is a complete irredundant set of the irreducible H_d-modules. In the more interesting case $e > 0$, the head $D(\mu)$ of $S(\mu)$ is non-zero (and therefore irreducible), provided $\mu \in \mathcal{RP}_d$, and the set $\{D(\mu) \mid \mu \in \mathcal{RP}_d\}$ is a complete irredundant set of the irreducible H_d-modules.

IV.3 Graded Representation Theory of H_d

As seen in Section II.4, the algebra H_d has an explicit grading (see also [BK1]). Therefore we may speak of graded H_d-modules. Theorem II.4.1 also implies that H_d has a *graded* anti-automorphism

$$\circledast : H_d \longrightarrow H_d, \qquad e(\boldsymbol{i}) \mapsto e(\boldsymbol{i}), \quad y_r \mapsto y_r, \quad \psi_s \mapsto \psi_s$$

for all admissible r, s, and $\boldsymbol{i}$. In [BK3, §2.7], this anti-automorphism leads to the introduction of a *graded duality*, denoted by $\circledast$, on the set of finite dimensional graded H_d-modules, sending each module V to

$$V^{\circledast} := \mathrm{HOM}_{\mathbb{F}}(V, \mathbb{F}),$$

with the action defined by

$$(xf)(v) = f(vx^{\circledast}) \qquad (v \in V, \quad f \in V^{\circledast}, \quad x \in H_d).$$

Since H_d is a finite dimensional graded algebra, it follows from the discussion in Section IV.1 that each irreducible module $D(\mu)$ can be graded uniquely up to a grading shift. Khovanov and Lauda [KhL, §3.2] point out that there is a preferred choice of grading which makes each module self-dual with respect to the graded duality $\circledast$. This leads to the following result:

Theorem IV.3.1. *[BK3, Theorem 4.11] For each $\mu \in \mathcal{RP}_d$, there exists a unique grading on $D(\mu)$ which makes it into a graded H_d-module such that*

$$D(\mu)^{\circledast} \cong D(\mu).$$

Moreover, the set of modules

$$\{D(\mu)\langle m \rangle \mid \mu \in \mathcal{RP}_d, \quad m \in \mathbb{Z}\}$$

forms a complete irredundant set of the finite dimensional irreducible graded H_d-modules.

In [BKW], Brundan, Kleshchev, and Wang exhibit a new explicit (cellular) basis for each Specht module

$$\{v_{\mathrm{T}} \mid \mathrm{T} \in \mathscr{T}(T)\} \qquad (\mu \in \mathcal{P}_d).$$

There are two major advantages which the vectors v_{T} enjoy, which the natural cellular basis vectors C_{T} do not. First, the vectors v_{T} are actually weight vectors:

Lemma IV.3.2. *[BKW] Let $\mu \in \mathcal{P}_d$ and $\mathrm{T} \in \mathscr{T}(\mu)$. Then v_{T} is an element of the weight space $S(\mu)_{\boldsymbol{i}^{\mathrm{T}}} = e(\boldsymbol{i}^{\mathrm{T}})S(\mu)$.*

Secondly, they are homogeneous with respect to a grading of $S(\mu)$ as an H_d-module. Recall the notion deg(T) which defined the degree of a standard μ-tableau (see Section III.4). We can then define the *degree* of v_{T} to be

$$\deg(v_{\mathrm{T}}) := \deg(\mathtt{T}).$$

With the grading defined on $S(\mu)$ in this way, the natural projection $S(\mu) \twoheadrightarrow D(\mu)$ is actually a degree zero map for all $\mu \in \mathcal{RP}_d$. The following result also shows that this grading of $S(\mu)$ as a vector space is compatible with the grading on H_d.

Lemma IV.3.3. [BKW] *Let $\mu \in \mathcal{P}_d$ and $\mathtt{T} \in \mathscr{T}(\mu)$. For each r, the vectors $y_r v_{\mathrm{T}}$ and $\psi_r v_{\mathrm{T}}$ are homogeneous, and we have that*

$$\begin{aligned} e(\boldsymbol{i})v_{\mathrm{T}} &= \delta_{\boldsymbol{i},\boldsymbol{i}^{\mathrm{T}}} v_{\mathrm{T}} && (\boldsymbol{i} \in I^d), \\ \deg(y_r v_{\mathrm{T}}) &= \deg(y_r) + \deg(v_{\mathrm{T}}) && (1 \le r \le d), \\ \deg(\psi_r v_{\mathrm{T}}) &= \deg(\psi_r e(\boldsymbol{i}^{\mathrm{T}})) + \deg(v_{\mathrm{T}}) && (1 \le r < d). \end{aligned}$$

In particular, our grading makes $S(\mu)$ into a graded H_d-module.

While we may realize the finite dimensional irreducible H_d modules as quotients of the explicit Specht modules, there is, as yet, no nice way of describing irreducible modules independently. More specifically, there is no known basis for $D(\mu)$, nor is even the dimension of $D(\mu)$ known in general. Certainly we would like to find answers to these questions. One nice way of approaching these problems is through a study of graded characters.

IV.4 Graded Characters

Recall that for $V = \bigoplus_{m\in\mathbb{Z}} V_m \in \mathrm{Rep}(H_d)$, we defined the graded dimension of V to be $\operatorname{qdim} V := \sum_{m\in\mathbb{Z}}(\dim V_m)q^m \in \mathbb{Z}[q,q^{-1}]$. Let $\mathscr{C}$ be the free $\mathbb{Z}[q,q^{-1}]$-module on I^d. Given a finite dimensional graded H_d-module V, we define its *graded character* to be

$$\operatorname{ch}_q V := \sum_{\boldsymbol{i}\in I^d} (\operatorname{qdim} V_{\boldsymbol{i}}) \cdot \boldsymbol{i} \in \mathscr{C}.$$

The graded character of an H_d-module formally keeps track of the graded dimension of each weight space of the module. Observe that grading shifts can also be kept track of via the graded character as $\mathrm{ch}_q\, V\langle m\rangle = q^m \mathrm{ch}_q\, V$ for all $V \in \mathrm{Rep}(H)$. The graded character of $D(\mu)$ keeps track of more information than simply the dimension of $D(\mu)$. Thus, we will answer the first of our main questions by finding the graded character of $D(\mu)$ for each $\mu \in \mathcal{RP}_d$.

The homogeneous basis and associated grading of Specht modules described in Section IV.3 implies that the graded characters of Specht modules are as follows:

Theorem IV.4.1. [BKW, §4.3] *Let $\mu \in \mathcal{P}_d$. Then*

$$\mathrm{ch}_q\, S(\mu) = \sum_{\mathtt{T} \in \mathscr{T}(\mu)} q^{\deg(\mathtt{T})} \boldsymbol{i}^{\mathtt{T}}.$$

Remark IV.4.2. In particular, observe that $\mathrm{ch}_q\, S(\mu)$ depends only on μ and e as these are enough to completely determine the set $\mathscr{T}(\mu)$ and the degrees $\deg(\mathtt{T})$ for each $\mathtt{T} \in \mathscr{T}(\mu)$. Thus, given two Hecke algebras with the same quantum characteristic e, the graded character $\mathrm{ch}_q\, S(\mu)$ will be exactly the same for each.

Example IV.4.3. Let $\mu = (2, 1^4)$. Then there are exactly 5 standard μ-tableaux as parametrized by the label which appears on the node $(1, 2)$, see Figure 1 below.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 6 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \quad \Bigg| \quad \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1} 0 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array}$$

Figure 1: Left: The set $\mathscr{T}(\mu)$ for $\mu = (2, 1^4)$. Right: The residue diagram for $\mu = (2, 1^4)$.

On the right of Figure 1 we also have the residue diagram for μ. Labeling these tableaux $\mathtt{T}_1$, $\mathtt{T}_2$, $\mathtt{T}_3$, $\mathtt{T}_4$, and $\mathtt{T}_5$ from left to right then the residue sequences for these tableaux are $\boldsymbol{i}^{\mathtt{T}_1} = (0, 1, 2, 1, 0, 2)$, $\boldsymbol{i}^{\mathtt{T}_2} = (0, 2, 1, 1, 0, 2)$, $\boldsymbol{i}^{\mathtt{T}_3} = (0, 2, 1, 1, 0, 2)$, $\boldsymbol{i}^{\mathtt{T}_4} = (0, 2, 1, 0, 1, 2)$ and $\boldsymbol{i}^{\mathtt{T}_5} = (0, 2, 1, 0, 2, 1)$. We may also calculate the degree of each tableau, following Example III.4.4, to find that $\deg(\mathtt{T}_1) = 0$, $\deg(\mathtt{T}_2) = 1$, $\deg(\mathtt{T}_3) = -1$, $\deg(\mathtt{T}_4) = 0$, and $\deg(\mathtt{T}_5) = 1$. Thus, the graded character for $S(\mu)$ is

$$\mathrm{ch}_q\, S(\mu) = (0,1,2,1,0,2) + \left(q + q^{-1}\right)(0,2,1,1,0,2) + (0,2,1,0,1,2) + q(0,2,1,0,2,1). \qquad \diamond$$

By extending the bar-involution from $\mathbb{Z}[q,q^{-1}]$ to $\mathscr{C}$ so that $\bar{\boldsymbol{i}} = \boldsymbol{i}$ for all $\boldsymbol{i} \in I^d$, we may interpret graded duality $\circledast$ as the bar-involution on graded characters, i.e $\mathrm{ch}_q\, V^{\circledast} = \overline{\mathrm{ch}_q\, V}$. The fact that each irreducible module $D(\mu)$ was self-dual implies that:

Theorem IV.4.4. [BK3, Theorem 4.18(3)] *Let* $\lambda \in \mathcal{RP}_d$. *Then* $\mathrm{ch}_q\, D(\lambda)$ *is bar-invariant.*

Since each Specht module is an explicit finite dimensional graded H_d-module, it is also natural to ask how each Specht module decomposes into a sum of irreducible $D(\mu)$'s. Answering this question is equivalent to finding the graded decomposition numbers.

IV.5 Graded Decomposition Numbers

For $\mu \in \mathcal{P}_d$ and $\lambda \in \mathcal{RP}_d$, we define the corresponding *graded decomposition number* to be the Laurent polynomial

$$d_{\mu,\lambda} = d_{\mu,\lambda}(q) := \sum_{m \in \mathbb{Z}} a_m q^m \in \mathbb{Z}_{\geq 0}[q, q^{-1}],$$

where a_m is the multiplicity of $D(\lambda)\langle m \rangle$ in a graded composition series of $S(\mu)$. We refer to $\{d_{\mu,\lambda} \mid \mu \in \mathcal{P}_d, \lambda \in \mathcal{RP}_d\}$ as the graded decomposition numbers for H_d. We are now prepared to state the main problem which this paper solves.

Main Problem IV.5.1. *Calculate the graded decomposition numbers for* H_d*:*

$$\{d_{\mu,\lambda} \mid \mu \in \mathcal{P}_d,\ \lambda \in \mathcal{RP}_d\}.$$

In Section VI.2 we describe an algorithm for solving this problem in general. We can relate the graded characters of $S(\mu)$ and $D(\lambda)$ for $\lambda \in \mathcal{RP}_d$ via the system of equations:

$$\mathrm{ch}_q\, S(\mu) = \sum_{\lambda \in \mathcal{RP}_d} d_{\mu,\lambda} \mathrm{ch}_q\, D(\lambda) \qquad (\mu \in \mathcal{P}_d). \tag{IV.1}$$

Note that the graded decomposition numbers are q-analogs of their ungraded counterparts, meaning that $d_{\mu,\lambda}(1)$ is the usual (or ungraded) decomposition number. The following result then easily follows from well-known facts in the ungraded setting and the fact that the natural map $S(\mu) \twoheadrightarrow D(\mu)$ is of degree zero. Recall that $M(\mu)$ is the set of moves for μ (see Section III.2), then:

Theorem IV.5.2. *Let* $\lambda \in \mathcal{RP}_d$ *and* $\mu \in \mathcal{P}_d$. *Then*

(i) $d_{\mu,\lambda} = 0$ *unless* $\lambda \in M(\mu)$.

(ii) $d_{\lambda,\lambda} = 1$.

By Theorem IV.5.2, the graded decomposition matrix $(d_{\mu,\lambda})$ is unitriangular. Thus, the knowledge of the graded decomposition numbers actually implies the knowledge of the graded characters of the irreducible H_d-modules. The converse is also true since the graded characters of the irreducible H_d-modules are linearly independent, see e.g. [KhL, Theorem 3.17]). Thus, in decomposing the Specht modules, we will also be finding the graded characters and dimensions of the irreducible H_d-modules.

The following key fact is special for the case $\operatorname{char} \mathbb{F} = 0$. This is the important additional fact which allows us to run our algorithm in characteristic zero, but not in characteristic p.

Theorem IV.5.3. [BK3, Theorem 3.9 and Corollary 5.15] *Let* $\operatorname{char} \mathbb{F} = 0$, $\lambda \in \mathcal{RP}_d$ *and* $\mu \in \mathcal{P}_d$. *If* $\mu \neq \lambda$, *then* $d_{\mu,\lambda} \in q\mathbb{Z}_{\geq 0}[q]$.

CHAPTER V

THE LADDER WEIGHT

In this chapter we define the ladder weight for an e-restricted partition λ and describe its nice properties. To do so, we start in Section V.1, by defining the bottom removable sequence for λ. Then, in Section V.2 we prove the main dominance lemma for e-restricted partitions and their bottom removable sequences. It is this structural lemma which provides the basis for the useful properties of the ladder weights.

In Section V.3, we give the explicit definition of the ladder weight, $\boldsymbol{j}^\lambda$, for any e-restricted partition λ and then prove that it has several nice properties. In particular, we also define the explicit Laurent polynomial r_λ which gives the exact multiplicity of $\boldsymbol{j}^\lambda$ within the graded characters of both the Specht module $S(\lambda)$ and the irreducible module $D(\lambda)$. Finally, in Section V.4, we set up the notation for the multiplicity of $\boldsymbol{j}^\lambda$ in any graded H_d-module V and collect the nice properties of $\boldsymbol{j}^\lambda$ into statements about its multiplicity in various graded characters.

V.1 The Bottom Removable Sequence of $\boldsymbol{\lambda}$

Equation IV.1, which we hope to solve for the graded decomposition numbers, is actually a system of equations for $\mu \in \mathcal{P}_d$ and the various weight spaces $\boldsymbol{i} \in I^d$. The complete system is somewhat difficult to get a handle on, so we need a way to reduce to special weight spaces which make the system of equations simpler, but still keep track of the decomposition information. More precisely, we need to reduce Equation IV.1, which relates graded characters, to an equation which only relates Laurent polynomials. The special weights which allow us to do this are called the "ladder weights."

Recall the notion of ladders laid out in Section III.3. For $\lambda \in \mathcal{RP}_d$, let $t = t_\lambda$ be the index of its bottom ladder, and $r_t(\lambda) = |\lambda \cap L_t|$. Denote the set of nodes of λ that lie on ladder L_t by

$\lambda \cap L_t = \{A_1, \ldots, A_{r_t(\lambda)}\}$. Order the nodes of this set so that A_u is below A_s whenever $u < s$. Observe that all of the nodes in this set are actually removable. In fact, the bottom node in this set, A_1, is the bottom removable node of λ (see Section III.1).

For the purpose of induction, it is natural to remove this bottom removable node. However, if λ is e-restricted, then the new partition $\lambda_{A_1} = \lambda \setminus \{A_1\}$ may not be. Looking at a few examples, we can quickly see that λ_{A_1} will still be e-restricted if and only if A_1 is the only node in the set $\lambda \cap L_t$. It follows that, if we wish to maintain the "e-restrictedness" while removing the bottom removable node, then we must actually remove the entire set of nodes $\lambda \cap L_t$. We therefore refer to the sequence $\boldsymbol{A} = (A_1, \ldots, A_{r_t(\lambda)})$ as the *bottom removable sequence of* λ. Recall that since all of these nodes lie on a single ladder, L_t, they all have the same residue which we refer to as the *residue of the bottom removable sequence.*

Example V.1.1. Let $\lambda = (8, 7^3, 6, 5, 3, 1)$ with $e = 3$. Figure 1 shows the residue diagram for λ.

0	1	2	0	1	2	0	1
2	0	1	2	0	1	2	
1	2	0	1	2	0	1	
0	1	2	0	1	2	0	
2	0	1	2	0	1		
1	2	0	1	2			
0	1	2					
2							

Figure 1: The residue diagram for the partition $\lambda = (8, 7^3, 6, 5, 3, 1)$.

Let $A_1 = (8, 1)$, $A_2 = (7, 3)$, and $A_3 = (6, 5)$ be the 2-nodes in the bottom ladder. Thus, the bottom removable sequence for λ is $\boldsymbol{A} = (A_1, A_2, A_3)$ and the residue of the bottom removable sequence is 2. Observe that if we wish to remove A_1, the new partition λ_{A_1} will no longer be e-restricted because $(\lambda_{A_1})_7 - (\lambda_{A_1})_8 = 3 - 0 = 3 \not< e$. This can seemingly be 'fixed' by removing the next 2-node A_2 as well. However, this simply shifts the problem from the difference between rows 7 and 8 to that between rows 6 and 7. Thus, we must also remove the final 2-node, A_3, as well if we wish to maintain the property of being e-restricted. $\diamond$

V.2 A Dominance Lemma

The bottom removable sequence for an e-restricted partition $\lambda \in \mathcal{RP}_d$ is compatible with the standard dominance order '$\trianglelefteq$' in a fundamental way. We can make this statement more precise with the following technical result, which generalizes Lemma 1.1 in [KS]. As alluded to at the beginning of this chapter, it is this fundamental result which provides the foundation for all of the useful properties of the ladder weights.

Lemma V.2.1. *Let $\lambda \in \mathcal{RP}_d$ and $\mu \in \mathcal{P}_d$, with $\lambda \not\trianglelefteq \mu$. Let $\boldsymbol{A} = (A_1, \ldots, A_r)$ be the bottom removable sequence for λ, and i be its residue. If $\boldsymbol{B} = \{B_1, \ldots, B_r\}$ is any set of r removable i-nodes for μ then $\lambda_{\boldsymbol{A}} \not\trianglelefteq \mu_{\boldsymbol{B}}$*

Proof. Let $\lambda_{\boldsymbol{A}} \trianglelefteq \mu_{\boldsymbol{B}}$. We need to show that $\lambda \trianglelefteq \mu$. Let A_m be in row j_m of λ, and let B_m be in row l_m of μ for $m = 1, \ldots, r$. By our convention $j_m = j_1 - (m-1)$ for $1 \leq m \leq r$. We may also assume that $l_1 > \cdots > l_r$.

Let $\lambda^m = \lambda \backslash \{A_{m+1}, \ldots, A_r\}$ and $\mu^m = \mu \backslash \{B_{m+1}, \ldots, B_r\}$ for $0 \leq m \leq r$. Then it suffices to show by induction on $m = 0, 1, \ldots, r$ that $\lambda^m \trianglelefteq \mu^m$, with the induction base case, $m = 0$, being our assumption.

Let $m > 0$ and assume by induction that $\lambda^{m-1} \trianglelefteq \mu^{m-1}$. Note that

$$\sigma_k(\lambda^m) = \begin{cases} \sigma_k(\lambda^{m-1}) & \text{if } k < j_m, \\ \sigma_k(\lambda^{m-1}) + 1 & \text{if } k \geq j_m, \end{cases}$$

$$\sigma_k(\mu^m) = \begin{cases} \sigma_k(\mu^{m-1}) & \text{if } k < l_m, \\ \sigma_k(\mu^{m-1}) + 1 & \text{if } k \geq l_m. \end{cases}$$

Since $\lambda^{m-1} \trianglelefteq \mu^{m-1}$, we deduce that

$$\sigma_k(\mu^m) \geq \sigma_k(\lambda^m) \qquad (\text{for } \ k \geq l_m \text{ or } k < j_m).$$

Observe that since $\boldsymbol{A}$ was the bottom removable sequence, it follows that j_1 is the bottom non-empty row in λ. Since λ is e-restricted, we know also that $\operatorname{res}(j_1 + 1, 1) \neq i$. Furthermore, we have that row $j_1 + 1$ is empty in $\lambda_{\boldsymbol{A}}$, which implies that it is empty in $\mu_{\boldsymbol{B}}$ as well since $\lambda_{\boldsymbol{A}} \trianglelefteq \mu_{\boldsymbol{B}}$.

Since B_1 is an addable i-node for $\mu_{\boldsymbol{B}}$, it follows that $l_1 < j_1 + 1$. Since $j_m = j_1 - (m-1)$ and $l_n < l_{n-1}$ for all $1 \le n \le r$, it follows that $l_m \le j_m$. Thus, for all k, we have that either $k < j_m$ or $k \ge l_m$, which gives us that $\sigma_k(\mu^m) \ge \sigma_k(\lambda^m)$ for all $k > 0$. Hence $\mu^m \trianglerighteq \lambda^m$ which completes the proof. □

Informally, the bottom removable sequence $\boldsymbol{A} = (A_1, \ldots, A_r)$ for λ is the 'lowest addable' sequence of (exactly r) i-nodes for $\lambda_{\boldsymbol{A}}$. That is, there is no way to add exactly r i-nodes to $\lambda_{\boldsymbol{A}}$ and arrive at a partition ν with $\nu \triangleleft \lambda$. From this point of view, the lemma is quite natural. If $\lambda_{\boldsymbol{A}} \trianglelefteq \mu_{\boldsymbol{B}}$ then certainly we should have $\lambda \trianglelefteq \mu$ since the set $\boldsymbol{A}$ is the lowest addable sequence of r i-nodes.

V.3 Definition and Properties of j^λ

Let $\lambda \in \mathcal{RP}_d$ have bottom removable sequence $\boldsymbol{A} = (A_1, A_2, \ldots, A_r)$ and define the *ladder weight* $\boldsymbol{j}^\lambda = (j_1, \ldots, j_d)$ inductively as follows: $j_d = \operatorname{res} A_r$ and $(j_1, \ldots, j_{d-1}) = \boldsymbol{j}^{\lambda_{A_r}}$. Thus, if $i = \operatorname{res} \boldsymbol{A}$ is the residue of the bottom removable sequence, then $\boldsymbol{j}^\lambda$ has i as its final r residues. Note that the idea that ladders play a fundamental role also appears in [LLT, §6.2].

Example V.3.1. In practice, we find the ladder weight $\boldsymbol{j}^\lambda$ by reading off the residues along each ladder, starting with the top ladder and working our way down. That is, if $L_1, \ldots, L_t$ is the set of ladders for λ, $r_m = r_m(\lambda) = |\lambda \cap L_m|$, and $i_m = \operatorname{res} L_m$ for each $m = 1, \ldots t$, then the ladder weight $\boldsymbol{j}^\lambda$ will be

$$\boldsymbol{j}^\lambda = (i_1, i_2, \ldots, \underbrace{i_m, \ldots, i_m}_{r_m \text{ times}}, i_{m+1}, \ldots),$$

where the residue i_m appears r_m consecutive times for each m.

Recall the partition $\lambda = (4, 3, 2, 1)$ which has the residue diagram given in Figure 2.

0	1	2	0
2	0	1	
1	2		
0			

Figure 2: The residue diagram for the partition $\lambda = (4, 3, 2, 1)$.

By reading off the residues along each ladder from top to bottom, we end up with the ladder weight $\boldsymbol{j}^\lambda = (0, 1, 2, 2, 0, 0, 1, 1, 2, 0)$. ◇

Of course, we are interested in the ladder weight $\boldsymbol{j}^\lambda$ because it has a few remarkable properties. Informally, the set of ladder weights $\{\boldsymbol{j}^\lambda \mid \lambda \in \mathcal{RP}_d\}$ is actually enough to encompass all of the graded decomposition information. To see this more explicitly, we first need the following result, which shows that $\boldsymbol{j}^\lambda$ exhibits a nice triangularity condition.

Theorem V.3.2. *Let $\lambda \in \mathcal{RP}_d$ and $\mu \in \mathcal{P}_d$, with $\mu \not\trianglerighteq \lambda$. Then $\boldsymbol{j}^\lambda$ does not appear in $\mathrm{ch}_q\, S(\mu)$. In particular, if μ is e-restricted, then $\boldsymbol{j}^\lambda$ does not appear in $\mathrm{ch}_q\, D(\mu)$.*

Proof. We apply induction on d, the base case $d = 0$ being clear. Let $d > 0$ and suppose for a contradiction that $\boldsymbol{i}^{\mathrm{T}} = \boldsymbol{j}^\lambda$ for some $\mathrm{T} \in \mathscr{T}(\mu)$. Let $\boldsymbol{A} = (A_1, \ldots, A_r)$ be the bottom removable sequence of λ and let $i = \mathrm{res}\,\boldsymbol{A}$ be its residue. Let $\boldsymbol{B} = \{B_1, \ldots, B_r\}$ be the nodes of μ labeled in T with $d, d-1, \ldots, d-r+1$. Since $\boldsymbol{i}^{\mathrm{T}} = \boldsymbol{j}^\lambda$ we have that $\mathrm{res}\, B_1 = \cdots = \mathrm{res}\, B_r = i$. Thus, $\boldsymbol{B}$ is a set of r removable i-nodes in μ. Let $\mathrm{T}' \in \mathscr{T}(\mu_{\boldsymbol{B}})$ be the tableau obtained from T by removing $B_1, \ldots, B_r$. Then $\boldsymbol{i}^{\mathrm{T}'} = \boldsymbol{j}^{\lambda_{\boldsymbol{A}}}$, whence $\boldsymbol{j}^{\lambda_{\boldsymbol{A}}}$ appears in $\mathrm{ch}_q\, S(\mu_{\boldsymbol{B}})$. By the inductive assumption, $\mu_{\boldsymbol{B}} \trianglerighteq \lambda_{\boldsymbol{A}}$. Now, by Lemma V.2.1, $\mu \trianglerighteq \lambda$, which is a contradiction. □

This triangularity condition works in conjunction with the triangularity condition on the graded decomposition numbers to imply that, in fact, $\boldsymbol{j}^\lambda$ does appear in $\mathrm{ch}_q\, D(\lambda)$. The hope is thus to use various ladder weights as 'indicators' of whether $\mathrm{ch}_q\, D(\lambda)$ appears in the decomposition of $\mathrm{ch}_q\, S(\mu)$. Unfortunately, the weight $\boldsymbol{j}^\lambda$ *can* (and does) appear in other graded characters $\mathrm{ch}_q\, D(\nu)$ for $\nu \trianglerighteq \lambda$, so we actually need a little bit more information. More specifically, we would like to know the multiplicity of $\boldsymbol{j}^\lambda$ in $\mathrm{ch}_q\, D(\lambda)$ explicitly. An explicit multiplicity is obtainable, and follows from the following technical result.

Lemma V.3.3. *Let $\lambda \in \mathcal{RP}_d$ and set $t = t_\lambda$, $r_m = r_m(\lambda)$, $R_m := r_1 + \cdots + r_m$, and $\lambda(m) := \lambda \cap (L_1 \cup \cdots \cup L_m)$ for $m > 0$. If $\mathrm{T} \in \mathscr{T}(\lambda)$ has $\boldsymbol{i}^{\mathrm{T}} = \boldsymbol{j}^\lambda$ then for each $m > 0$ we have $\mathrm{sh}(T_{\leq R_m}) = \lambda(m)$.*

Proof. We apply induction on $m > 0$ with the induction base $m = 1$ being clear as $r_1 = 1$. Let $m > 1$ and assume that $\mathrm{sh}(T_{\leq R_{m-1}}) = \lambda(m-1)$. Letting $\boldsymbol{B}$ denote the set of nodes in $\mathrm{sh}(T_{\leq R_m}) \setminus \mathrm{sh}(T_{\leq R_{m-1}})$, it suffices to prove that $\boldsymbol{B} = \lambda \cap L_m$. Since $|\boldsymbol{B}| = |\lambda \cap L_m| = r_m$, it is enough to prove that $\boldsymbol{B} \subseteq \lambda \cap L_m$. Observing that $\boldsymbol{B}$ is contained in λ it then remains to show that $\boldsymbol{B} \subseteq L_m$. Observe that the nodes of $\boldsymbol{B}$ must have residue $\mathrm{res}\, L_m$ since $\boldsymbol{i}^{\mathrm{T}} = \boldsymbol{j}^\lambda$. We know also that none of the nodes in $\boldsymbol{B}$ belong to any of the ladders $L_1, \ldots, L_{m-1}$ since $\mathrm{sh}(T_{\leq R_{m-1}}) = \lambda(m-1)$. We conclude that $\boldsymbol{B} \subseteq L_m$, completing the inductive step. □

This technical result describes how a standard λ-tableau, $\mathtt{T} \in \mathscr{T}(\lambda)$, must be labeled in order to have its residue sequence satisfy $\boldsymbol{i}^{\mathtt{T}} = \boldsymbol{j}^{\lambda}$. Any such tableau must label the nodes so that if $A \in \lambda \cap L_m$ and $B \in \lambda \cap L_n$ with $n > m$, then the label on A must be less than the label on B. In other words, while labeling a λ tableau $\mathtt{T}$, the ladder L_m must be completely labeled (in any way) by the lowest labels available, before moving on to the next ladder L_{m+1}.

Let $\lambda \in \mathcal{P}_d$ and set $t = t_\lambda$ and $r_m = r_m(\lambda)$ for $m > 0$. We define

$$r_\lambda := [r_1]_q^! \, [r_2]_q^! \cdots [r_t]_q^!. \tag{V.1}$$

Theorem V.3.4. *If $\lambda \in \mathcal{RP}_d$ then $\boldsymbol{j}^\lambda$ has multiplicity r_λ in $\operatorname{ch}_q S(\lambda)$.*

Proof. We must describe the complete set of λ-tableau $\{\mathtt{T} \in \mathscr{T}(\lambda) \mid \boldsymbol{i}^{\mathtt{T}} = \boldsymbol{j}^{\lambda}\}$. Lemma V.3.3 implies that for any tableau $\mathtt{T}$ in this set, the numbers $d, d-1, \ldots, d - r_t + 1$ must appear in the bottom removable sequence $\boldsymbol{A} = \{A_1, \ldots, A_{r_t}\}$ for λ. Moreover, within this ladder, all $r_t!$ possible permutations of those labels are valid. Since λ is e-restricted, Lemma III.3.2 implies that all ladders are bottom complete for λ, and thus the possible labelings of $\boldsymbol{A}$ give a contribution of exactly $[r_t]_q^!$ to the multiplicity of $\boldsymbol{j}^\lambda$ in $\operatorname{ch}_q S(\lambda)$. The result then follows by induction on d. $\square$

Corollary V.3.5. *If $\lambda \in \mathcal{RP}_d$ then $\boldsymbol{j}^\lambda$ has multiplicity r_λ in $\operatorname{ch}_q D(\lambda)$.*

Proof. By Theorem V.3.4, $\boldsymbol{j}^\lambda$ appears in $\operatorname{ch}_q S(\lambda)$ with multiplicity r_λ, and by Lemma V.3.2, $\boldsymbol{j}^\lambda$ does not appear in $\operatorname{ch}_q D(\mu)$ for $\mu \lhd \lambda$. By Theorem IV.5.2, the composition factors of $S(\lambda)$ are of the form $D(\mu)$ for $\mu \trianglelefteq \lambda$, thus the result follows. $\square$

In two special cases, where we have further restrictions on the shape of the partition λ, we can be more explicit.

Corollary V.3.6. *Let $\lambda \in \mathcal{P}_d$ be a partition with $\lambda_1 < e$. Then $\boldsymbol{j}^\lambda$ has multiplicity 1 in $\operatorname{ch}_q S(\lambda)$ and in $\operatorname{ch}_q D(\lambda)$.*

Proof. Observe that $\lambda_1 < e$ implies that $r_m \leq 1$ for all $1 \leq m \leq t_\lambda$. The result then follows directly from Lemma V.3.4 and Corollary V.3.5. $\square$

Corollary V.3.7. *Let $\lambda = (l_1^{k_1}, l_2^{k_2}, \ldots)$ be an e-restricted partition with $l_1 = e$ and $l_i > l_{i+1}$ for $i \geq 1$. Then the multiplicity of $\boldsymbol{j}^\lambda$ in $\operatorname{ch}_q S(\lambda)$ and in $\operatorname{ch}_q D(\lambda)$ is $([2]_q)^{k_1}$.*

Proof. Use Lemma V.3.4 and Corollary V.3.5 and the fact that λ has exactly k_1 ladders of size 2, the remaining ladders of λ having size at most 1. □

For each $\lambda \in \mathcal{RP}_d$, we now have an explicit multiplicity r_λ for $\boldsymbol{j}^\lambda$ in $\mathrm{ch}_q\, S(\lambda)$ and $\mathrm{ch}_q\, D(\lambda)$. This will allow us to take the explicitly calculated multiplicity of $\boldsymbol{j}^\lambda$ in $\mathrm{ch}_q\, S(\mu)$ and find complete copies of r_λ within it. Recall the system defined by equation IV.1 which expresses the decomposition of $S(\mu)$ in terms of graded characters. Using this idea, we now reduce this system to solving a single equation for each of the distinguished ladder weight spaces.

V.4 Ladder Weight Multiplicity

Let V be a finite dimensional graded H_d-module. For any $\lambda \in \mathcal{RP}_d$ define $m_\lambda(V)$ to be the multiplicity of $\boldsymbol{j}^\lambda$ in $\mathrm{ch}_q\, V$. We then collect the important properties of the function m_λ.

Theorem V.4.1. *Let $\lambda \in \mathcal{RP}_d$, $\mu \in \mathcal{P}_d$, and V be a finite dimensional graded H_d-module. Then:*

(i) $m_\lambda(V) \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$;

(ii) *if* $m_\lambda(V) = 0$ *then* $[V : D(\lambda)] = 0$;

(iii) $m_\lambda(S(\lambda)) = m_\lambda(D(\lambda)) = r_\lambda$;

(iv) $m_\lambda(S(\mu)) = 0$ *unless* $\lambda \in M(\mu)$;

(v) *if* $\mu \in \mathcal{RP}_d$, *then* $m_\lambda(D(\mu)) = 0$ *unless* $\lambda \in M(\mu)$;

(vi) $m_\lambda(S(\mu)) = \displaystyle\sum_{\nu \in \mathcal{RP}_d \cap M(\mu,\lambda),\ \nu \neq \lambda} d_{\mu,\nu} m_\lambda(D(\nu)) + d_{\mu,\lambda} r_\lambda.$

Proof. Recall that the graded character of a graded H_d-module is a formal sum of weights with the coefficient of a weight $\boldsymbol{i}$ equal to the graded dimension of the weight space $V_{\boldsymbol{i}}$. As such, it is clear that the multiplicity of any particular weight must be in $\mathbb{Z}_{\geq 0}[q, q^{-1}]$ proving (i).

Part (iii) is a restatement of Theorem V.3.4 and Corollary V.3.5. Now, (i) and (iii) imply that if $D(\lambda)$ is a composition factor of V, then $\boldsymbol{j}^\lambda$ must appear in $\mathrm{ch}_q\, V$ proving (ii).

To see (iv), assume that $\lambda \notin M(\mu)$. Then either $\lambda \not\sim \mu$ or $\lambda \not\trianglelefteq \mu$. In the first case it follows from Theorem IV.5.2 that $m_\lambda(S(\mu)) = 0$, and in the second case the same follows from Theorem V.3.2. Now (v) follows directly from (iv) and (i).

Finally, (vi) follows from (i), (iii), (v), and Theorem IV.5.2 □

Theorem V.4.1(ii) and (iii) make precise the way in which we may think of the ladder weight $\boldsymbol{j}^\lambda$ as an 'indicator' for the irreducible module $D(\lambda)$. Note that part (vi) is exactly the reduction of equation IV.1 to the j^λ-weight space that we were after. In the next section, we introduce our main algorithm which allows us to solve this system of equations for the graded decomposition numbers by induction.

CHAPTER VI

THE ALGORITHM

In this chapter we describe the Main Algorithm which solves the problem of computing the graded decomposition numbers for H_d in general. Our algorithm gives an inductive process by which we may reduce the problem to solving a single equation in the ladder weight space $\boldsymbol{j}^\lambda$. This type of equation can be solved more generally as explained in Section VI.1 below.

VI.1 Basic Algorithm

Our Main Algorithm will ultimately reduce to solving the following basic problem, thus we will begin by describing an easy algorithm for solving this problem.

Problem VI.1.1. *Suppose $d(q) \in q\mathbb{Z}[q]$, and $m(q), r(q) \in \mathbb{Z}[q, q^{-1}]$ are such that $\overline{m(q)} = m(q)$, $\overline{r(q)} = r(q)$, and $r(q) \neq 0$. If $d(q)r(q) + m(q)$ is known and $r(q)$ is known, find $d(q)$ and $m(q)$.*

Remark VI.1.2. It is easy to see that Problem VI.1.1 has a unique solution.

We now explain an inductive algorithm to solve Problem VI.1.1. Clearly, if we have $d(q)r(q) + m(q) = 0$, then $d(q) = m(q) = 0$ by uniqueness. If $d(q)r(q) + m(q) \neq 0$ then we can write it in the form

$$d(q)r(q) + m(q) = \sum_{n=-N}^{M} a_n q^n \quad (-N \leq M\ ;\ a_{-N} \neq 0, a_M \neq 0).$$

Note that the assumptions imply that $M \geq 0$ and $M \geq N$ (but we might have $N < 0$ if $m(q) = 0$). The algorithm proceeds by induction on the pairs of non-negative integers $(M, M + N)$ ordered lexicographically. The induction base is the pair $(0, 0)$ where $d(q) = 0$ and $m(q) = a_0$ by uniqueness. Let $(M, M + N) > (0, 0)$. This implies $M > 0$. We denote the top term of $r(q)$ by bq^R, and note that $R \geq 0$ since $r(q)$ is bar-invariant. We now consider two cases.

Case 1: $M > N$. As $m(q)$ is bar-invariant, the term $a_M q^M$ must come from $d(q)r(q)$. Thus $\frac{a_M}{b} q^{M-R}$ is a term in $d(q)$. Setting

$$d'(q) := d(q) - \frac{a_M}{b} q^{M-R},$$

we are reduced to solving the basic problem for $d'(q)r(q) + m(q)$. Note that the conditions required in Problem VI.1.1 still hold since $d'(q) \in q\mathbb{Z}[q]$, but for this new equation we have $M' < M$, so we are finished by induction.

Case 2: $M = N$. As $d(q) \in q\mathbb{Z}[q]$, the term $a_{-N} q^{-N}$ must therefore come from $m(q)$. Since $m(q)$ is bar-invariant, $a_{-N} q^N$ must be a term in $m(q)$ also. Setting

$$m'(q) := m(q) - (a_{-N} q^{-N} + a_{-N} q^N),$$

we are reduced to solving the problem for $d(q)r(q) + m'(q)$. Note that the conditions of Problem VI.1.1 still hold, since $\overline{m'(q)} = m'(q)$, but we now have $M' \leq M$ and $M' + N' < M + N$, so we are again finished by induction.

Example VI.1.3. Suppose that $r(q) = q + q^{-1}$, $d(q) \in q\mathbb{Z}[q]$, $\overline{m(q)} = m(q)$ and that

$$d(q)r(q) + m(q) = 6q^{-3} + 2q^{-2} + q^{-1} + 4 + 2q + 6q^2 + 7q^3.$$

To start, $M = N = 3$, thus the term $6q^{-3}$ must come from $m(q)$. Since $m(q)$ is bar-invariant, the term $6q^3$ is also part of $m(q)$. Setting $m'(q) = m(q) - (6q^{-3} + 6q^3)$ and we are left with solving

$$d(q)r(q) + m'(q) = 2q^{-2} + 4 + q + 6q^2 + q^3.$$

Now $M = 3$ and $N = 2$, so the top term q^3 must come from $d(q)r(q)$, so $\frac{q^3}{q} = q^2$ is a term in $d(q)$. Thus, the term $q^2(q + q^{-1}) = q^3 + q$ is a term in $d(q)r(q)$. Setting $d'(q) = d(q) - q^2$ we are left with solving

$$d'(q)r(q) + m'(q) = 2q^{-2} + 4 + 6q^2.$$

Now $M = N = 2$, so the term $2q^{-2}$ must come from $m'(q)$, meaning that the bar-invariant term $2q^{-2} + 2q^2$ is in $m'(q)$. Setting $m''(q) = m'(q) - (2q^{-2} + 2q^2)$ we are left with solving

$$d'(q)r(q) + m''(q) = 4 + 4q^2.$$

Now $M = 2$ and $N = 1$, so $4q^2$ must come from $d'(q)r(q)$. Meaning that $4q$ is a term in $d'(q)$. Thus $4q(q+q^{-1}) = 4q^2 + 4$ is a term in $d'(q)r(q)$. Setting $d''(q) = d'(q) - 4q$ we are left with $d''(q)r(q) + m''(q) = 0$. Thus $d''(q) = 0 = m''(q)$ by uniqueness. We therefore have $d(q) = q^2 + 4q$ and $m(q) = 6q^{-3} + 2q^{-2} + 2q^2 + 6q^3$. $\diamond$

Remark VI.1.4. The condition that a given Laurent polynomial $a(q) \in \mathbb{Z}[q, q^{-1}]$ can be decomposed in the form $a(q) = d(q)r(q) + m(q)$ where $d(q)$, $r(q)$, and $m(q)$ satisfy the conditions of Problem VI.1.1 is non-trivial. More specifically, if $r(q)$ is taken to be a bar-invariant Laurent polynomial that is *not* a constant polynomial, then most arbitrary Laurent polynomials $a(q)$ will not satisfy this condition and therefore cannot have the algorithm applied to them.

VI.2 Main Algorithm

From now on we assume that $\operatorname{char} \mathbb{F} = 0$. If $e = 0$, then the Specht modules are irreducible, so we also assume that we are in the interesting case $e > 0$. That is, we deal with the case of the Iwahori-Hecke algebra over a field of characteristic zero with parameter a primitive e^{th} root of unity. Under these assumptions, we now describe an algorithm for computing the graded decomposition numbers of H_d.

Remark VI.2.1. The algorithm relies heavily on Theorem IV.5.3, which is why we need the critical assumption that $\operatorname{char} \mathbb{F} = 0$.

Let $\mu \in \mathcal{P}_d$ and $\lambda \in \mathcal{RP}_d$. We will compute $d_{\mu,\lambda}$ by induction. However, this induction requires us to keep track of some extra information. By Theorem IV.5.2(i), $\lambda \notin M(\mu)$ implies $d_{\mu,\lambda} = 0$, so we assume $\lambda \in M(\mu)$. We now calculate $d_{\mu,\lambda}$ and $m_\lambda(D(\mu))$ by induction on the distance $l(\mu, \lambda)$ (see Section III.2). Of course, $m_\lambda(D(\mu))$ only makes sense when μ is e-restricted, so we interpret this term as zero if $\mu \notin \mathcal{RP}_d$. Induction begins when $l(\mu, \lambda) = 0$, hence $\mu = \lambda$, and we have $d_{\mu,\mu} = 1$ by Theorem IV.5.2(ii) and $m_\mu(S(\mu)) = r_\mu$ by Theorem V.4.1.

Let $l(\mu, \lambda) > 0$, so $\mu \rhd \lambda$. By induction, we know the graded decomposition numbers $d_{\mu,\nu}$ for all $\nu \in \mathcal{RP}_d \cap M(\mu, \lambda)$ with $\nu \neq \lambda$ and the multiplicities $m_\lambda(D(\nu))$ for all $\nu \in \mathcal{RP}_d \cap M(\mu, \lambda)$ with $\nu \neq \mu$. To make the inductive step we need to compute $d_{\mu,\lambda}$ and, if μ is e-restricted, $m_\lambda(D(\mu))$.

If μ is not e-restricted, then by Theorem V.4.1(vi), we have

$$d_{\mu,\lambda} = \frac{1}{r_\lambda} \left(m_\lambda(S(\mu)) - \sum_{\nu \in \mathcal{RP}_d \cap M(\mu,\lambda),\ \nu \neq \lambda} d_{\mu,\nu} m_\lambda(D(\nu)) \right),$$

where all the terms in the right hand side are known by induction and Theorem IV.4.1.

Let μ be e-restricted. By Theorem V.4.1(vi), we have

$$m_\lambda(D(\mu)) + d_{\mu,\lambda} r_\lambda = m_\lambda(S(\mu)) - \sum_{\nu \in \mathcal{RP}_d \cap M(\mu,\lambda),\ \nu \neq \lambda,\ \nu \neq \mu} d_{\mu,\nu} m_\lambda(D(\nu)),$$

where all terms in the right hand side are known by induction and Theorem IV.4.1. Note r_λ is non-zero and bar-invariant, $d_{\mu,\lambda} \in q\mathbb{Z}_{\geq 0}[q]$ by Theorem IV.5.3, and $m_\lambda(D(\mu))$ is bar-invariant by Theorem IV.4.4. Hence, we are in the assumptions of Problem VI.1.1 with $m(q) = m_\lambda(D(\mu))$, $d(q) = d_{\mu,\lambda}$, and $r(q) = r_\lambda$. Now we apply the Basic Algorithm described in the previous section to calculate $m_\lambda(D(\mu))$ and $d_{\mu,\lambda}$ and complete the inductive step.

Remark VI.2.2. In our algorithm, to calculate $d_{\mu,\lambda}$ and $m_\lambda(D(\mu))$ one only ever needs to compute $d_{\nu,\kappa}$ and $m_\kappa(D(\nu))$ for pairs (ν, κ) such that $l(\nu, \kappa) < l(\mu, \lambda)$ and $\nu, \kappa \in M(\mu, \lambda)$. For an example of this type of calculation, see Section IX.2.

CHAPTER VII

APPLICATIONS

The problem of calculating a closed form for the graded decomposition numbers is quite difficult in general. However, we can calculate the graded decomposition numbers in a few special cases. Here we will specifically treat the case of 2-column partitions which has also been studied (in the ungraded setting) by James [J1]. In fact, in this situation we can also compute the *adjustment matrix* which describes how to make precise the 'reduction modulo p' from the graded decomposition numbers of H_d to those of the symmetric group algebra $\mathbb{F}\Sigma_d$. Note that work has been done using this algorithm in the case of 3-column and 4-column partitions as well. For examples of this work we refer the interested reader to [L].

In Section VII.1 we review the language of moves for two-column partitions and introduce some new notation for this specific situation. More specifically, we introduce the definitions of the *size* of a move as well as *good* and *bad* moves. We also introduce notation for keeping track of various important residues. Then, in Section VII.2 we prove a few important technical results regarding the multiplicity $m_\lambda(S(\mu))$ when λ is a good move or bad move for μ.

Section VII.3 contains the description and proof of the graded decomposition numbers for the 2-column partitions in the case of $H_d(\mathbb{F}, \xi)$ when $\operatorname{char} \mathbb{F} = 0$ and ξ is an e^{th}-root of unity. Finally, in Section VII.4 we use some results of James [J1] for the group algebra of the symmetric group in characteristic p to make sense of the 'reduction modulo p' alluded to in Section II.3. We are thus able to describe exactly how the irreducible modules over $H_d(\mathbb{C}, e^{2\pi i/p})$ split when passing to $\mathbb{F}\Sigma_d$ (where $\operatorname{char} \mathbb{F} = p$). This means that we can fully describe the graded decomposition numbers for $\mathbb{F}\Sigma_d$ in this situation as well.

VII.1 Two-column Partitions

A *2-column partition* $\mu \in \mathcal{P}_d$ is a partition whose Young diagram has at most two non-empty columns, thus it can be written in the form $\mu = (2^j, 1^{d-2j})$ for some $0 \leq j \leq \frac{d}{2}$. Observe that we choose to include the partition $\mu = (1^d)$ in the set of 2-column partitions. Given any 2-column partition $\mu = (2^j, 1^{d-2j})$, we let $i_1(\mu) := \text{res}\,(d-j, 1)$ and $i_2(\mu) := \text{res}\,(j, 2)$ denote the residues at the bottom of columns 1 and 2 respectively. It is also useful to keep track of the number $b(\mu)$ defined from $0 \leq b(\mu) < e$ and $b(\mu) \equiv i_2(\mu) - i_1(\mu) \pmod{e}$.

Observe that, if $\mu = (2^j, 1^{d-2j})$ is a 2-column partition, then all of the moves for μ are also 2-column partitions and moreover, each move must be of the form $\lambda = (2^k, 1^{d-2k})$ for $k \leq j$. Thus we may think of each move $\lambda \in M(\mu)$ in terms of the number of nodes moved from column 2 in μ to column 1 in λ. We define the *size* of the move λ to be $|\mu, \lambda| := j - k$. The set of 2-column partitions actually forms a totally ordered set under the standard dominance order. Thus, given $\lambda \in M(\mu)$, the set of moves between λ and μ is exactly the set of moves for μ of smaller (or equal) size, i.e. $M(\mu, \lambda) = \{\nu \in M(\mu) \mid |\mu, \nu| \leq |\mu, \lambda|\}$.

We say that $\lambda \in M(\mu)$ is a *good* move if $|\mu, \lambda| \not\equiv 0 \pmod{e}$, otherwise we say λ is a *bad* move. Observe that $|\mu, \lambda| \equiv i_1(\lambda) - i_1(\mu) \pmod{e}$. Thus a move λ for μ is a good move if and only if $i_1(\lambda) \neq i_1(\mu)$. Note that every 2-column partition μ admits a bad move, namely μ itself, but not every partition admits a good move. The fact that the set of 2-column partitions is a totally ordered set implies that if μ admits a non-trivial move (i.e. $M(\mu) \neq \{\mu\}$) then there is a unique non-trivial minimal move for μ. We refer to this unique non-trivial minimal move as the *smallest* move for μ. Moreover, if μ admits a good move, which is necessarily non-trivial, then the smallest move for μ is a good move.

VII.2 A Few Technical Lemmas

To apply our algorithm to the case of 2-column partitions we need a few technical lemmas describing the multiplicity $m_\lambda(S(\mu))$ as dependent on whether λ is a good or bad move for μ. To start, given any e-restricted 2-column partition λ, we describe a major portion of the labeling for any 2-column tableau $\mathtt{T}$ such that $\boldsymbol{i}^{\mathtt{T}} = \boldsymbol{j}^{\lambda}$ in the following corollary.

Corollary VII.2.1. *Let $\lambda := (2^k, 1^{d-2k})$ be e-restricted, with $t := t_\lambda$, $r_m := r_m(\lambda)$, $R_m := r_1 + \cdots + r_m$, and $\lambda(m) = \lambda \cap (L_1 \cup \cdots \cup L_m)$ for $m > 0$ (as in Lemma V.3.3). Let $\mu \in \mathcal{P}_d$ be a 2-column partition, with $\mu \trianglerighteq \lambda$ and $\mu(m) = \mu \cap (L_1 \cup \cdots \cup L_m)$ for $m > 0$. Set $n := 1 + k(e-1)$. If $\mathtt{T} \in \mathscr{T}(\mu)$ is such that $\boldsymbol{i}^{\mathtt{T}} = \boldsymbol{j}^\lambda$ then $\mathrm{sh}(\mathtt{T}_{\leq R_m}) = \mu(m)$ for all $1 \leq m \leq n$.*

Proof. The important observation to make is that n is the exact index such that L_n passes through the node $(k+1, 1)$. Thus, for all $1 \leq m \leq n$ we have $\mu(m) = \lambda(m)$. The result is now a direct consequence of Lemma V.3.3. □

Remark VII.2.2. Corollary VII.2.1 implies specifically that for any 2-column partition μ and any standard μ-tableau $\mathtt{T}$ with $\boldsymbol{i}^{\mathtt{T}} = \boldsymbol{j}^\lambda$, we have $\mathrm{sh}(\mathtt{T}_{\leq 2k+1}) = (2^k, 1)$.

Let μ be a 2-column partition which admits a good move and let ν be the smallest good move. By computing a few examples, one can then observe that any other good move λ for μ is necessarily a bad move for ν. Conversely, any bad move for ν is, in fact, a good move for μ as well. The smallest good move ν for μ turns out to play a dominant role in the decomposition of $S(\mu)$. The following result makes this statement more precise by describing a way to relate the multiplicity $m_\lambda(S(\mu))$, for an arbitrary good move λ, to the multiplicity $m_\lambda(S(\nu))$, where ν denotes the smallest good move for μ.

Lemma VII.2.3. *Let μ be a 2-column partition that admits a good move. Let ν be the smallest good move for μ and let $\lambda = (2^k, 1^{d-2k})$ be a bad move for ν. Then $m_\lambda(S(\mu)) = q \cdot m_\lambda(S(\nu))$.*

Proof. Set $b := b(\lambda)$ and note that since $\lambda \neq (2^{d/2})$, the final b residues in $\boldsymbol{j}^\lambda$ are exactly $i_2(\lambda) - 1, i_2(\lambda) - 2, \ldots, i_1(\lambda)$. Thus, for any standard λ-tableau which gives the residue sequence $\boldsymbol{j}^\lambda$, it must be the case that the numbers $d, d-1, \ldots, d-b+1$ appear in the final b nodes of column 1. So we define $\lambda^b := (2^k, 1^{d-2k-b})$.

As λ is a bad move for ν, we have $i_1(\lambda) = i_1(\nu)$ and $i_2(\lambda) = i_2(\nu)$. If there exists a tableau $\mathtt{T} \in \mathscr{T}(\nu)$ with $\boldsymbol{i}^{\mathtt{T}} = \boldsymbol{j}^\lambda$ then the numbers $d, d-1, \ldots, d-b+1$ must appear in the nodes $(\nu_1^T, 1), (\nu_1^T - 1, 1), \ldots, (\nu_1^T - b + 1, 1)$ respectively. As these are all nodes in column 1, it follows that $\deg(\mathtt{T}) = \deg(\mathtt{T}_{\leq d-b})$ for all such $\mathtt{T}$. If we denote $\nu' := \mathrm{sh}(\mathtt{T}_{\leq d-b})$, then the previous discussion shows that $m_\lambda(S(\nu)) = m_{\lambda^b}(S(\nu'))$

As ν is a good move for μ, we have $i_1(\nu) = i_2(\mu)$, $i_2(\nu) = i_1(\mu)$, and $i_1(\nu) \neq i_2(\nu)$. Note that since $i_1(\lambda) = i_1(\nu)$ and $i_2(\lambda) = i_2(\nu)$, we have $b > 0$ and moreover, b is exactly $|\mu, \nu|$. If there exists a $\mathtt{U} \in \mathscr{T}(\mu)$ with $\boldsymbol{i}^{\mathtt{U}} = \boldsymbol{j}^{\lambda}$ then the numbers $d, d-1, \ldots, d-b+1$ must appear in the nodes $(\mu_2^T, 2), (\mu_2^T - 1, 2), \ldots, (\mu_2^T - b+1, 2)$ respectively (the final b nodes in column 2). Observe that these nodes contain the residue $i_2(\lambda) - 1 = i_1(\mu) - 1$ exactly once. Hence we have $\deg(\mathtt{U}) = q \cdot \deg(\mathtt{U}_{\leq d-b})$ for all such $\mathtt{U}$. If we denote $\mu' := \mathrm{sh}(\mathtt{U}_{\leq d-b})$, then this shows that $m_\lambda(S(\mu)) = q \cdot m_{\lambda^b}(S(\mu'))$. Since $b = |\mu, \nu|$, it follows that $\nu' = \mu'$. Hence $m_\lambda(S(\mu)) = q \cdot m_\lambda(S(\nu))$. □

Lemma VII.2.4. *Let $\lambda = (2^k, 1^{d-2k})$ and $\mu = (2^j, 1^{d-2j})$ be partitions with λ a bad move for μ. Then $m_\lambda(S(\mu))$ is bar-invariant.*

Proof. Since λ is a bad move for μ, we have that $i_1(\lambda) = i_1(\mu)$ and $i_2(\lambda) = i_2(\mu)$. Let $\mathtt{T} \in \mathscr{T}(\mu)$ be such that $\boldsymbol{i}^{\mathtt{T}} = \boldsymbol{j}^{\lambda}$. From Corollary VII.2.1 we know that $\mathrm{sh}(\mathtt{T}_{\leq 2k+1}) = (2^k, 1)$. Moreover, the possible ways of labeling $(2^k, 1)$ and obtaining the proper start to $\boldsymbol{j}^{\lambda}$ produces the bar-invariant coefficient $r_{(2^k,1)}$. Thus it suffices to show that $\deg(\mathtt{T}) = \deg(\mathtt{T}_{\leq 2k+1})$.

Let $a := i_2(\lambda) + k \bmod e$. Observe that res $(k+1, 2) = i_2(\lambda) - 1$ and this residue does not appear in the set of a residues, $\{\overline{-k-1}, \ldots, \overline{-k-a} = i_2(\mu)\}$, where $\overline{k}$ denotes $k \pmod e$. This forces the numbers $2k+2, \ldots, 2k+1+a$ to appear in the nodes $(k+2, 1), \ldots, (k+1+a, 1)$ of $\mathtt{T}$ respectively. Since these are all in column 1, we have that $\deg(\mathtt{T}_{\leq 2k+1+a}) = \deg(\mathtt{T}_{\leq 2k+1})$. Observe that starting with the $(2k+1+a)+1^{th}$ residue, the sequence $\boldsymbol{j}^{\lambda}$ proceeds sequentially starting with the residue $i_2(\mu) - 1 = i_2(\lambda) - 1$ and continuing downward.

Set $b := b(\mu)$ and note that $b = b(\lambda)$ also. Thus the final b residues of $\boldsymbol{j}^{\lambda}$ are exactly $i_2(\mu) - 1, i_2(\mu) - 2, \ldots, i_1(\mu)$. Hence, the numbers $d, \ldots, d-b+1$ must appear in the nodes $(\mu_1^T, 1), \ldots, (\mu_1^T - b + 1, 1)$ of $\mathtt{T}$ respectively (the final b nodes in column 1). Since these are in the first column, $\deg(\mathtt{T}) = \deg(\mathtt{T}_{\leq d-b})$.

Since λ is a bad move for μ, by definition $|\mu, \lambda| \equiv 0 \pmod e$. It follows that the nodes $(k+1, 2), (k+2, 2), \ldots, (j, 2)$, form a series of *e-bricks* with residues $(i_2(\lambda) - 1, i_2(\lambda) - 2, \ldots, i_2(\lambda))$. Note that the nodes $(k+1+a+1, 1), (k+1+a+2, 1), \ldots, (d-j-b, 1)$, form a series of e-bricks with the residues $(i_2(\lambda) - 1, \ldots, i_2(\lambda))$ as well.

These two sets of e-bricks contain all of the nodes in $\mathtt{T}_{\leq d-b} \setminus \mathtt{T}_{\leq 2k+1+a}$. Since $\boldsymbol{j}^{\lambda}$ proceeds sequentially, $\boldsymbol{i}^{\mathtt{T}} = \boldsymbol{j}^{\lambda}$ implies that each e-brick must contain consecutive numbers in $\mathtt{T}$. Thus by Lemma 3.1.15 part (ii) from [BKW] (applied here in the transpose setting), none of these complete e-bricks affect the degree of $\mathtt{T}$. Therefore, we have shown $\deg(\mathtt{T}) = \deg(\mathtt{T}_{\leq d-b}) = \deg(\mathtt{T}_{\leq 2k+1+a}) = \deg(\mathtt{T}_{\leq 2k+1})$ as required. □

VII.3 Graded Decomposition Numbers for Two-column Partitions

Lemma VII.2.3 and Lemma VII.2.4 and the algorithm in Section VI.2 allow us to prove the following result in the case where ξ is an e^{th}-root of unity (recall that the algorithm requires that we have $\operatorname{char}\mathbb{F} = 0$):

Theorem VII.3.1. (Graded Decomposition Numbers for 2-column Partitions)

Let μ be an e-restricted 2-column partition. Then

$$\mathrm{ch}_q S(\mu) = \begin{cases} \mathrm{ch}_q D(\mu) & \textit{if } \mu \textit{ admits no good moves,} \\ \mathrm{ch}_q D(\mu) + q\,\mathrm{ch}_q D(\nu) & \textit{if } \mu \textit{ admits a good move,} \end{cases}$$

where in the second case, ν denotes the smallest good move for μ.

Proof. We prove the theorem using our algorithm from Section VI.2. Specificially, we show that for any e-restricted $\lambda \trianglelefteq \mu$ we have

$$d_{\mu,\lambda} = \begin{cases} q & \text{if } \lambda \text{ is the smallest good move for } \mu, \\ 1 & \text{if } \lambda = \mu, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$m_\lambda(D(\mu)) = \begin{cases} 0 & \text{if } \lambda \text{ is a good move for } \mu, \\ m_\lambda(S(\mu)) & \text{if } \lambda \text{ is a bad move for } \mu. \end{cases}$$

As in our algorithm, we proceed by induction on the distance $l(\mu, \lambda)$. The induction base is $\lambda = \mu$ where we have $m_\lambda(D(\lambda)) = m_\lambda(S(\lambda)) = r_\lambda$ by Theorem V.4.1, and $d_{\lambda,\lambda} = 1$ by Theorem IV.5.2.

Let $\lambda \neq \mu$. Theorem V.4.1(v) gives us that

$$m_\lambda(D(\mu)) + d_{\mu,\lambda} r_\lambda = m_\lambda(S(\mu)) - \sum_{\nu \in \mathcal{RP}_d \cap M(\mu,\lambda),\ \nu \neq \lambda,\ \nu \neq \mu} d_{\mu,\nu} m_\lambda(D(\nu)). \tag{VII.1}$$

Now we consider various cases:

Case 1: λ is a bad move for μ. Let $\nu \in \mathcal{RP}_d \cap M(\mu, \lambda)$ with $\mu \rhd \nu \rhd \lambda$. This implies that $l(\mu,\nu) < l(\mu,\lambda)$. Thus, we know $d_{\mu,\nu}$ and $m_\lambda(D(\nu))$ by induction. If ν is a bad move for μ, then $d_{\mu,\nu} = 0$ so none of these terms will appear in the sum. If ν is a good move for μ, then we know that λ is a good move for ν as well. Thus, we have that $m_\lambda(D(\nu)) = 0$, so none of these terms appear in the sum either.

The above argument reduces equation VII.1 to $m_\lambda(D(\mu)) + d_{\mu,\lambda} r_\lambda = m_\lambda(S(\mu))$. From Lemma VII.2.4, we know that $m_\lambda(S(\mu))$ is bar-invariant and thus our Basic Algorithm from Section VI.1 implies that $d_{\mu,\lambda} = 0$ and $m_\lambda(D(\mu)) = m_\lambda(S(\mu))$.

Case 2: λ is the smallest good move for μ. Since the smallest good move for μ must have size $< e$, there are no $\nu \in \mathcal{RP}_d \cap M(\mu,\lambda)$ with $\mu \rhd \nu \rhd \lambda$. Thus, there are no terms in the summation in equation VII.1 and we are left with $m_\lambda(D(\mu)) + d_{\mu,\lambda} r_\lambda = m_\lambda(S(\mu))$.

Applying Lemma VII.2.3 with $\nu = \lambda$ gives us that $m_\lambda(S(\mu)) = q \cdot m_\lambda(S(\lambda)) = q \cdot r_\lambda$. Thus $d_{\mu,\lambda} = q$ and $m_\lambda(D(\mu)) = 0$ by the uniqueness of the solution to Problem VI.1.1.

Case 3: λ is a good move for μ with $|\mu,\lambda| > e$. Let $\nu \in \mathcal{RP}_d \cap M(\mu,\lambda)$ with $\mu \rhd \nu \rhd \lambda$. If ν is a bad move for μ, then by induction $d_{\mu,\nu} = 0$ and none of these appear in the sum. If ν is a good move for μ and λ is a good move for ν, then by induction $m_\lambda(D(\nu)) = 0$, so these do not appear in the sum either.

Suppose ν is a good move for μ and λ is a bad move for ν. If ν is not the smallest good move, then by induction $d_{\mu,\nu} = 0$, so take ν to be the smallest good move for μ. By induction we have that $d_{\mu,\nu} = q$, so equation VII.1 may be reduced to

$$m_\lambda(D(\mu)) + d_{\mu,\lambda} r_\lambda = m_\lambda(S(\mu)) - q \cdot m_\lambda(D(\nu)).$$

Since ν is the smallest good move for μ, Lemma VII.2.3 implies $m_\lambda(S(\mu)) = q \cdot m_\lambda(S(\nu))$. Moreover, since λ is a bad move for ν, by induction $m_\lambda(D(\nu)) = m_\lambda(S(\nu))$. Thus equation VII.1 may be reduced to $m_\lambda(D(\mu)) + d_{\mu,\lambda} r_\lambda = 0$ which gives $m_\lambda(D(\mu)) = 0$ and $d_{\mu,\lambda} = 0$ by the uniqueness of the solution to Problem VI.1.1. □

The one situation not considered in Theorem VII.3.1 is the only case where a 2-column partition is *not* e-restricted, i.e. when $\mu = (2^{d/2})$ and $e = 2$. In this case, of course, $D(\mu)$ does not make sense. In fact, in this situation our algorithm and the discussion in the proof above imply that $\mathrm{ch}_q\, S\left(2^{d/2}\right) = q \cdot \mathrm{ch}_q\, D\left(2^{d/2-1}, 1^2\right)$.

VII.4 Two-column Partitions for $\mathbb{F}S_d$ in Characteristic p

Let $\mathbb{F}$ be a field of characteristic p. Recall that the group algebra of the symmetric group, $\mathbb{F}\Sigma_d = H_d(\mathbb{F}, 1)$, and the Iwahori-Hecke algebra over $\mathbb{C}$ at a p^{th}-root of unity, $H_d(\mathbb{C}, e^{2\pi i/p})$, both have quantum characteristic $e = p$. From the discussion above in Section IV.2, we know that in both situations there is a set of Specht modules $\{S(\mu) \mid \mu \in \mathcal{P}_d\}$ and that they have irreducible heads $D(\mu)$ which are non-zero if and only if μ is e-restricted. Thus, we have the sets $\{D(\lambda) \mid \lambda \in \mathcal{RP}_d\}$ (for $H_d(\mathbb{C}, e^{2\pi i/p})$) and $\{\overline{D(\lambda)} \mid \lambda \in \mathcal{RP}_d\}$ (for $\mathbb{F}\Sigma_d$) that are complete irredundant sets of the finite dimensional irreducible representations for each algebra. Finally, we know from Remark IV.4.2 that the graded characters for our Specht modules are the same for each algebra as well.

We can thus make sense of the irreducible representations $\overline{D(\lambda)}$ for $\mathbb{F}\Sigma_d$ as "reductions modulo p" of the irreducible representations $D(\lambda)$ for $H_d(\mathbb{C}, e^{2\pi i/p})$. In this reduction, some of the irreducible modules for $H_d(\mathbb{C}, e^{2\pi i/p})$ split. In the 2-column case, we can use the work of James in [J1] as well as our results above to write down an explicit formula for these splittings. The *adjustment matrix* is a matrix which, when multiplied by the graded decomposition matrix for $H_d(\mathbb{C}, e^{2\pi i/p})$, gives the graded decomposition matrix for $\mathbb{F}\Sigma_d$. Thus, by describing the splitting, we will have described the portion of the adjustment matrix corresponding to the 2-column partitions. In this way, we will also have found the graded decomposition numbers for 2-column partitions in the case of the symmetric group in characteristic p.

First we recall some combinatorial definitions from James in [J1]. Given two non-negative integers a and b we write their *prime power decompositions*

$$a = a_0 + a_1 p + \cdots + a_r p^r \quad (\text{ for } \ 0 \le a_i < p \ , \ a_r \neq 0),$$

$$b = b_0 + b_1 p + \cdots + b_s p^s \quad (\text{ for } \ 0 \le b_i < p \ , \ b_s \neq 0).$$

Then we say that a *contains* b *to base* p if $m < n$ and for each i we have $b_i = 0$ or $b_i = a_i$. We then for any non-negative integers n and m we define the function

$$f_p(n,m) = \begin{cases} 1 & \text{if } n+1 \text{ contains } m \text{ to base } p, \\ 0 & \text{otherwise.} \end{cases}$$

A result of James regarding the case of the symmetric group $\mathbb{F}_p\Sigma_d$ is the following theorem which appears also in [J2] and[J3]. Here we state the result in the transpose setting where it still holds since [J1, Theorem 8.15] implies that $D(\lambda) \cong D(\lambda^T) \otimes \operatorname{sgn}$, where sgn is the well-known sign representation for Σ_d.

Lemma VII.4.1. [J1, Theorem 24.15] *The multiplicity of* $D\left(2^k, 1^{d-2k}\right)$ *as a factor of* $S\left(2^j, 1^{d-2j}\right)$ *is* $f_p(d-2k, j-k)$.

In the case of the Iwahori-Hecke algebra at a p^{th}-root of unity, $H_d(\mathbb{C}, e^{2\pi i/p})$, Theorem VII.3.1 shows that the decomposition of any Specht module $S(\mu)$ has at most two factors. However, for $\mathbb{F}\Sigma_d$, Lemma VII.4.1, shows that many Specht modules $S(\mu)$ have more than two factors in their decomposition. Since the Specht module $S(\mu)$ has the same weight spaces in each case (although, over different fields), it follows that some of the irreducible modules in the decomposition of $S(\mu)$ over $H_d(\mathbb{C}, e^{2\pi i/p})$, must 'split' when viewed as $\mathbb{F}\Sigma_d$-modules. Our goal now, is to describe this splitting. To do so, we need a little bit more notation.

For simplicity, given two 2-column partitions $\lambda = (2^k, 1^{d-2k})$ and $\nu = (2^j, 1^{d-2j})$ such that $k \le j$ we define

$$s_p(\nu, \lambda) = f_p(d-2k, j-k).$$

So $s_p(\nu, \lambda) = 1$ if and only if $\overline{D(\lambda)}$ appears in the (ungraded) decomposition of $S(\nu)$. We set

$$\mathcal{B}(\nu) = \{\lambda \in \mathcal{RP}_d \cap M(\nu) \mid \lambda \text{ is a bad move for } \nu \text{ and } s_p(\nu, \lambda) = 1\}.$$

The following result makes precise the reduction modulo p for 2-column partitions.

Theorem VII.4.2. (Reduction Modulo p for 2-column Partitions)

For $\nu \in \mathcal{RP}_d$ *we have*

$$\operatorname{ch}_q D(\nu) = \sum_{\lambda \in \mathcal{B}(\nu)} \operatorname{ch}_q \overline{D(\lambda)}.$$

Proof. We denote the graded decomposition numbers for $H_d(\mathbb{C}, e^{2\pi i/p})$ by $d_{\nu,\lambda}$, while denoting their counterparts for $\mathbb{F}\Sigma_d$ by $\overline{d}_{\nu,\lambda}$. Let $\lambda \in \mathcal{B}(\nu)$, then $s_p(\nu, \lambda) = 1$, which implies that $\overline{D(\lambda)}$ appears in the decomposition of $S(\nu)$ as an $\mathbb{F}\Sigma_d$-module. Thus $\overline{d}_{\nu,\lambda}\operatorname{ch}_q \overline{D(\lambda)}$ appears in $\operatorname{ch}_q S(\nu)$ and $\overline{d}_{\nu,\lambda} \neq 0$. Since $m_\lambda(\overline{D(\lambda)}) \neq 0$ we now have that $m_\lambda(S(\nu)) \neq 0$. But λ is a bad move for ν, thus Lemma VII.2.4 implies that $m_\lambda(S(\nu))$ is bar-invariant.

We now have that each $\mathrm{ch}_q\ D(\mu)$ is bar-invariant and that $m_\lambda(S(\nu))$ and $m_\lambda(\overline{D(\lambda)})$ are bar-invariant. It follows that $\overline{d}_{\nu,\lambda}$ is bar-invariant which implies that $\overline{d}_{\nu,\lambda}\mathrm{ch}_q\ \overline{D(\lambda)}$ is bar-invariant. Thus $\overline{d}_{\nu,\lambda}\mathrm{ch}_q\ \overline{D(\lambda)}$ appears in $\mathrm{ch}_q\ D(\nu)$ and not in $\mathrm{ch}_q\ \overline{D(\nu)}$.

Evaluating $\overline{d}_{\nu,\lambda}$ when $q = 1$ gives the ungraded decomposition number which is known from [J1] to be 1. The fact that $\overline{d}_{\nu,\lambda}$ is bar-invariant now implies that $\overline{d}_{\nu,\lambda} = 1$ as well. This proves that $\mathrm{ch}_q\ \overline{D(\lambda)}$ appears in $\mathrm{ch}_q\ D(\nu)$ and not in $\mathrm{ch}_q\ \overline{D(\nu)}$.

Conversely, if λ is a good move for ν, then the proof of Theorem VII.3.1 gives us that $m_\lambda(D(\nu)) = 0$. Thus $\mathrm{ch}_q\ \overline{D(\lambda)}$ cannot appear in $\mathrm{ch}_q\ D(\nu)$. Similarly, if $s_p(\nu, \lambda) = 0$, then the ungraded character $\mathrm{ch}\ \overline{D(\lambda)}$ does not appear in $\mathrm{ch}\ S(\nu)$, which implies that $\mathrm{ch}_q\ \overline{D(\lambda)}$ does not appear in $\mathrm{ch}_q\ S(\nu)$ either. $\square$

By knowing the graded decomposition numbers for $H_d(\mathbb{C}, e^{2\pi i/p})$ and the way each irreducible module $D(\lambda)$ splits when 'reducing modulo p', we have effectively computed the graded decomposition numbers for the symmetric group $\mathbb{F}\Sigma_d$ in characteristic p. Specifically, the matrix of graded decomposition numbers for $\mathbb{F}\Sigma_d$ is the product of the matrix for $H_d(\mathbb{C}, e^{2\pi i/p})$ and the adjustment matrix which describes the splitting. Theorem VII.4.2 proves that, in the 2-column case, this adjustment matrix is an integer matrix with all entries being either 0 or 1. This observation supports the conjecture that the adjustment matrix is an integer matrix in general.

CHAPTER VIII

CONNECTION TO LLT

There is a famous algorithm due to Lascoux, Leclerc, and Thibon [LLT] for calculating global crystal bases for various Fock spaces. It was later proved by Ariki [A] to also calculate the graded decomposition numbers for the Iwahori-Hecke algebra. While our algorithm is different from the LLT algorithm, it is actually equivalent. Our approach comes directly from the graded representation theory of H_d rather than more formal calculations within Fock spaces.

In this chapter, we briefly describe the LLT algorithm and discuss how the two algorithms are related. The advantage to our approach is that it allows us to interpret several formal objects that appear in the LLT algorithm in terms of the graded representation theory of H_d. We make this statement more precise in what follows.

In Section VIII.1, we define the quantized enveloping algebra and revisit the Grothendieck groups of the categories of finite dimensional and finitely generated projective graded representations (see Section IV.1). For the Grothendieck groups, we look this time at the more specific case of the Iwahori-Hecke algebra H_d. We also reinterpret the graded decomposition numbers in terms of the isomorphism classes of modules within these Grothendieck groups.

Next, in Section VIII.2 we define the Fock space which plays a fundamental role in [LLT]. We give a basis for it as a $\mathbb{Q}(q)$-vector space and describe an action of the quantized enveloping algebra on it, making it into a module. Then, in Section VIII.3 we describe a special set of graded projective H_d-modules which are graded representation theoretic analogs of the formal 'first approximations' $A(\lambda)$ appearing in the LLT algorithm. Finally, in Section VIII.4, we explain how our algorithm is equivalent to the LLT algorithm despite our very different approach using graded representation theory. We also interpret a family of formal coefficients which appear in the LLT algorithm as multiplicities of various ladder weights.

VIII.1 The Quantized Enveloping Algebra $U_q(\mathfrak{g})$ and Grothendieck Groups

Let $\mathfrak{g}$ be the Kac-Moody algebra corresponding to the Cartan matrix (II.1) described in Section II.3, so $\mathfrak{g} = \widehat{\mathfrak{sl}}_e(\mathbb{C})$ if $e > 0$ and $\mathfrak{g} = \mathfrak{sl}_\infty(\mathbb{C})$ if $e = 0$. Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$. So $U_q(\mathfrak{g})$ is the $\mathbb{Q}(q)$-algebra generated by the Chevalley generators E_i, F_i, and $K_i^{\pm 1}$ for $i \in I$, subject only to the usual *quantum* Serre relations (for all admissible $i, j \in I$):

$$\begin{aligned}
K_iK_j &= K_jK_i, & K_iK_i^{-1} &= 1,\\
K_iE_jK_i^{-1} &= q^{a_{i,j}}E_j & K_iF_jK_i^{-1} &= q^{-a_{i,j}}F_j,\\
[E_i, F_j] &= \delta_{i,j}\frac{K_i - K_i^{-1}}{q - q^{-1}}, & &\\
(\mathrm{ad}_q\, E_i)^{1-a_{j,i}}(E_j) &= 0 \qquad (i \neq j), & &\\
(\mathrm{ad}_q\, F_i)^{1-a_{j,i}}(F_j) &= 0 \qquad (i \neq j), & &
\end{aligned}$$

where $(\mathrm{ad}_q\, x)^n(y)$ is defined by:

$$(\mathrm{ad}_q\, x)^n y := \sum_{m=0}^{n} (-1)^m \begin{bmatrix} \mathrm{n} \\ \mathrm{m} \end{bmatrix}_q x^{n-m} y x^m.$$

Now, to compare the two algorithms, we must first recast some notions from [LLT] and [BK3]. Recall that $[\mathrm{Rep}(H_d)]$ (resp. $[\mathrm{Proj}(H_d)]$) denotes the Grothendieck group of the category of finite dimensional (resp. finitely generated projective) graded H_d-modules. $[\mathrm{Rep}(H_d)]$ is a free $\mathbb{Z}[q, q^{-1}]$-module with basis $\{[D(\lambda)] \mid \lambda \in \mathcal{RP}_d\}$ (the isomorphism classes of irreducible graded H_d-modules). For each $\lambda \in \mathcal{RP}_d$, let $P(\lambda)$ be the projective cover of $D(\lambda)$; in particular there exists a degree preserving surjection $P(\lambda) \twoheadrightarrow D(\lambda)$. Then $[\mathrm{Proj}(H_d)]$ is a free $\mathbb{Z}[q, q^{-1}]$-module with basis $\{[P(\lambda)] \mid \lambda \in \mathcal{RP}_d\}$.

Recall that the Cartan pairing is a natural pairing of $[\mathrm{Proj}(H_d)]$ and $[\mathrm{Rep}(H_d)]$ and is defined by $([P], [M]) = \mathrm{qdim}\, \mathrm{HOM}_{H_d}(P, M)$ for $P \in \mathrm{Proj}(H_d)$ and $M \in \mathrm{Rep}(H_d)$. We denote $[\mathrm{Rep}(H)] = \bigoplus_{d \geq 0}[\mathrm{Rep}(H_d)]$ and $[\mathrm{Proj}(H)] = \bigoplus_{d \geq 0}[\mathrm{Proj}(H_d)]$ and then extend the pairing to $[\mathrm{Proj}(H)] \times [\mathrm{Rep}(H)]$ so that $([\mathrm{Proj}(H_c)], [\mathrm{Rep}(H_d)]) = 0$ for $c \neq d$.

By definition of the graded decomposition numbers $d_{\mu,\lambda}$, for every $\mu \in \mathcal{P}_d$, we have

$$[S(\mu)] = \sum_{\lambda \in \mathcal{RP}_d} d_{\mu,\lambda}[D(\lambda)]$$

in $[\mathrm{Rep}(H_d)]$. Moreover, from [BK3, Theorem 3.14, Theorem 5.13] it follows that we have:

$$[P(\lambda)] = \sum_{\mu \in \mathcal{P}_d} d_{\mu,\lambda}[S(\mu)] \qquad (\lambda \in \mathcal{RP}_d). \tag{VIII.1}$$

in $[\mathrm{Rep}(H_d)]$ as well. These two statements can be interpreted as (graded) Brauer reciprocity.

VIII.2 Fock Spaces

For more complete treatment of the following, see [BK3]. Given a partition $\mu \in \mathcal{P}_d$, recall the definitions of $d_A(\mu)$, $d^B(\mu)$, and $d_i(\mu)$ from Section III.3. Following the work of Hayashi [H] and Misra and Miwa [MM], we define the *Fock space*, $\mathfrak{F}$, to be the $\mathbb{Q}(q)$-vector space on basis

$$\{M_\mu \mid \mu \in \bigoplus_{d \geq 0} \mathcal{P}_d\},$$

which is referred to as the *monomial basis.* The Fock space can be made into a $U_q(\mathfrak{g})$-module by defining the action of the generators via:

$$\begin{aligned} E_i M_\mu &:= \sum_B q^{-d^B(\mu)} M_{\mu_B}, \\ F_i M_\mu &:= \sum_A q^{d_A(\mu)} M_{\mu^A}, \\ K_i M_\mu &:= q^{d_i(\mu)} M_\mu, \end{aligned}$$

where the first sum is over all removable i-nodes B for μ, and the second sum is over all addable i-nodes A for μ. We denote the divided powers of these generators by

$$E_i^{(r)} = \frac{E_i^r}{[r]_q}, \qquad F_i^{(r)} = \frac{F_i^r}{[r]_q}, \qquad \text{and} \quad K_i^{(r)} = \frac{K_i^r}{[r]_q}.$$

In general, $\mathfrak{F}$ is not irreducible, but the submodule of $\mathfrak{F}$ generated by the vector $M_\varnothing$

(corresponding to the empty partition) is the irreducible highest weight $U_q(\mathfrak{g})$-module $V(\Lambda_0)$. There is a canonical $U_q(\mathfrak{g})$-module homomorphism $\pi : \mathfrak{F} \twoheadrightarrow V(\Lambda_0)$, see [BK3, (3.29)]. We can, and always will, identify

$$[\mathrm{Rep}(H)] \bigotimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) = V(\Lambda_0) \subset \mathfrak{F}.$$

This identification is the result of a categorification of the module $V(\Lambda_0)$. Under this identification, we have $\pi(\mu) = [S(\mu)]$ for each μ.

Given any e-restricted $\lambda \in \mathcal{RP}_d$ with $t = t_\lambda$ and $r_m = r_m(\lambda)$ for $m > 0$, we define

$$F_\lambda := F_{t-1}^{(r_t)} F_{t-2}^{(r_{t-1})} \cdots F_1^{(r_2)} F_0^{(r_1)}.$$

Observe that the residue sequence formed by taking the lower indices of F_λ in reverse order is $(0^{r_1}, 1^{r_2}, \ldots, (t-1)^{r_t})$ which is exactly $\boldsymbol{j}^\lambda$. Moreover, the product of the denominators coming from the divided powers is exactly r_λ. Hence, we may write $F_\lambda = \frac{1}{r_\lambda} F_{t-1}^{r_t} F_{t-2}^{r_{t-1}} \cdots F_1^{r_2} F_0^{r_1}$. Following [LLT] we define, for each $\lambda \in \mathcal{RP}_d$, the 'first approximation' $A(\lambda)$ by

$$A(\lambda) = F_\lambda M_\varnothing.$$

Observe that if we identify each basis vector M_μ with the Young diagram for μ, then it is possible to keep track of extra information while applying F_λ. Each successive F_i has the effect of adding an i-node to each μ in the sum. If we keep track of the order in which nodes are added, then we may think of each particular partition μ as a standard μ-tableau.

Example VIII.2.1. We can better explain this idea within the framework of an example. Consider the case of $d = 5$ and $e = 2$ with the partition $\lambda = (2^2, 1)$. Then $\boldsymbol{j}^{(2^2,1)} = (0, 1, 1, 0, 0)$ which means that $F_{(2^2,1)} = \frac{1}{r_{(2^2,1)}} F_0 F_0 F_1 F_1 F_0$, where $r_{(2^2,1)} = [2]_q \cdot [2]_q$. When applying $F_{(2^2,1)}$ to $M_\varnothing$ we first apply F_0. With only one way to add a 0-node to the empty partition we immediately obtain $F_0 M_\varnothing = M_{(1)}$. Next, apply F_1 to the previous result $M_{(1)}$. There are two ways to add a 1-node. Taking degrees into account we have,

$$F_1 M_{(1)} = q M_{(2)} + M_{(1^2)}.$$

Applying F_1 again there is exactly one way to add a 1-node to each partition. Hence,

$$F_1\left(qM_{(2)} + M_{(1^2)}\right) = (q + q^{-1})M_{(2,1)} = [2]_q M_{(2,1)}.$$

Now we apply F_0. There are three ways to add 0-nodes to the partition $(2,1)$. Thus,

$$F_0\left([2]_q M_{(2,1)}\right) = q^2[2]_q M_{(3,1)} + q[2]_q M_{(2^2)} + [2]_q M_{(2,1^2)}.$$

Finally, we apply F_0 one more time. There are two ways to add 0-nodes to each partition. Thus,

$$F_0\left(q^2[2]_q M_{(3,1)} + q[2]_q M_{(2^2)} + [2]_q M_{(2,1^2)}\right) = q^2[2]_q[2]_q M_{(3,2)} + q[2]_q[2]_q M_{(3,1^2)} + [2]_q[2]_q M_{(2^2,1)}.$$

Observe that each term does, in fact, contain a factor of $r_{(2^2,1)}$. Hence, after dividing by this factor, we find that $A(2^2,1) = F_{(2^2,1)}M_{\varnothing} = q^2M_{(3,2)} + qM_{(3,1^2)} + M_{(2^2,1)}$.

Now, if we keep track of the order in which nodes are added we can view each term in the sum as a standard tableau. Figures 1 - 3 show the various tableau generated by this process, starting with the first application of F_1. At each stage, the tableau are ordered so that all those generated from the same tableau in the previous step will be grouped together.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \qquad \rightsquigarrow \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

Figure 1: Left: The tableau of $F_1F_0M_{\varnothing}$. Right: The tableau of $F_1F_1F_0M_{\varnothing}$.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

Figure 2: The tableau of $F_0F_1F_1F_0M_{\varnothing}$.

By construction, the set of tableaux in Figure 3 is exactly the complete set of standard tableaux of any shape which have residue sequence $\mathbf{j}^{(2^2,1)}$. Moreover, one can check that, for each tableau $\mathtt{T}$, the degree $\deg(\mathtt{T})$ is exactly the coefficient obtained through the application of $F_{(2^2,1)}$.

$$\begin{array}{ccc}1&2&4\\3&5&\end{array}\quad\begin{array}{ccc}1&2&5\\3&4&\end{array}\quad\begin{array}{ccc}1&2&5\\3&&\\4&&\end{array}\quad\begin{array}{ccc}1&3&4\\2&5&\end{array}\quad\begin{array}{ccc}1&3&5\\2&4&\end{array}\quad\begin{array}{ccc}1&3&5\\2&&\\4&&\end{array}$$

$$\begin{array}{ccc}1&2&4\\3&&\\5&&\end{array}\quad\begin{array}{cc}1&2\\3&4\\5&\end{array}\quad\begin{array}{cc}1&2\\3&5\\4&\end{array}\quad\begin{array}{ccc}1&3&4\\2&&\\5&&\end{array}\quad\begin{array}{cc}1&3\\2&4\\5&\end{array}\quad\begin{array}{cc}1&3\\2&5\\4&\end{array}$$

Figure 3: The tableau of $F_0F_0F_1F_1F_0M_\varnothing$.

$\diamond$

From this perspective, it follows that, in general, $F_\lambda M_\varnothing$ exactly produces the set of standard tableaux of any shape with corresponding residue sequence $\boldsymbol{j}^\lambda$. In practice, the basis vectors $\{M_\mu\}$ only encode partition data, not particular standard tableau. However, in view of Example VIII.2.1, it is easy to see that

$$A(\lambda) = \frac{1}{r_\lambda} \sum_{\mu \in \mathcal{P}_d} m_\lambda(S(\mu))M_\mu \in \mathfrak{F}. \tag{VIII.2}$$

VIII.3 Projective H_d-modules

Recall from Section II.3 that for each $\boldsymbol{i} \in I^d$ there is a unique idempotent $e(\boldsymbol{i}) \in H_d$ (possibly zero) such that $e(\boldsymbol{i})V = V_{\boldsymbol{i}}$ for any finite dimensional H_d-module. Moreover, for each weight $\boldsymbol{i}$ such that $e(\boldsymbol{i}) \neq 0$, we have a corresponding projective module $H_d e(\boldsymbol{i})$. In the special case when $\boldsymbol{i}$ is a ladder weight, we have:

Lemma VIII.3.1. *If $\lambda \in \mathcal{RP}_d$, then $H_d e(\boldsymbol{j}^\lambda) = \bigoplus_{\mu \in \mathcal{RP}_d} m_\lambda(D(\mu))P(\mu)$.*

Proof. Recall that if M and N are graded H_d-modules, then $\mathrm{HOM}_{H_d}(M, N)$ is the graded vector space which consists of *all*, and not necessarily homogeneous, H_d-homomorphisms from M to N (see Section IV.1). Now, the graded multiplicity of $P(\mu)$ in $H_d e(\boldsymbol{j}^\lambda)$ is exactly equal to result of the Cartan pairing of $P(\mu)$ with $H_d e(\boldsymbol{j}^\lambda)$. Thus, the graded multiplicity is

$$\langle P(\mu), H_d e(\boldsymbol{j}^\lambda)\rangle = \mathrm{qdim}\,\mathrm{HOM}_{H_d}(H_d e(\boldsymbol{j}^\lambda), D(\mu)) = \mathrm{qdim}\, e(\boldsymbol{j}^\lambda)D(\mu) = m_\lambda(D(\mu)),$$

as required. $\square$

It turns out that the Grothendieck group element $[H_d e(j^\lambda)]$ can be expressed in several different ways, as evidenced by the following corollary.

Corollary VIII.3.2. *For $\lambda \in \mathcal{RP}_d$, we have in* $[\mathrm{Rep}(H_d)]$*:*

(i) $[H_d e(j^\lambda)] = \sum_{\mu \in \mathcal{RP}_d} m_\lambda(D(\mu))[P(\mu)]$,

(ii) $[H_d e(j^\lambda)] = \sum_{\mu \in \mathcal{P}_d} m_\lambda(S(\mu))[S(\mu)]$,

(iii) $[H_d e(j^\lambda)] = \sum_{\mu \in \mathcal{RP}_d} m_\lambda(P(\mu))[D(\mu)]$.

Proof. (i) is a restatement of Lemma VIII.3.1 in the Grothendieck group. (ii) follows from (i) and equation (VIII.1) since:

$$\sum_{\mu \in \mathcal{RP}_d} m_\lambda(D(\mu))[P(\mu)] = \sum_{\mu \in \mathcal{RP}_d} m_\lambda(D(\mu)) \sum_{\nu \in \mathcal{P}_d} d_{\nu,\mu}[S(\nu)]$$

$$= \sum_{\nu \in \mathcal{P}_d} \left(\sum_{\mu \in \mathcal{RP}_d} m_\lambda(D(\mu)) d_{\nu,\mu} \right) [S(\nu)] = \sum_{\nu \in \mathcal{P}_d} m_\lambda(S(\nu))[S(\nu)].$$

Similarly, (iii) follows from (ii) and equation (VIII.1) since:

$$\sum_{\mu \in \mathcal{P}_d} m_\lambda(S(\mu))[S(\mu)] = \sum_{\mu \in \mathcal{P}_d} m_\lambda(S(\mu)) \sum_{\nu \in \mathcal{RP}_d} d_{\mu,\nu}[D(\nu)]$$

$$= \sum_{\nu \in \mathcal{RP}_d} \left(\sum_{\mu \in \mathcal{P}_d} m_\lambda(S(\mu)) d_{\mu,\nu} \right) [D(\nu)] = \sum_{\nu \in \mathcal{RP}_d} m_\lambda(P(\nu))[S(\nu)].$$

This completes the proof. □

Using equation VIII.2 and Corollary VIII.3.2(ii) we may connect the first approximation $A(\lambda)$ to the element $[H_d e(j^\lambda)]$ via the equation

$$\pi(r_\lambda A(\lambda)) = [H_d e(j^\lambda)].$$

Thus, we have an explicit connection between the formal sum $A(\lambda) \in \mathfrak{F}$ and the graded representation theory of H_d.

VIII.4 Comparing the Algorithms

Consider the following system of equations over $\mathbb{Z}[q, q^{-1}]$:

$$m_\lambda(S(\mu)) = \sum_{\nu \in \mathcal{RP}_d} d_{\mu,\nu} m_\lambda(D(\nu)) \quad (\lambda \in \mathcal{RP}_d,\ \mu \in \mathcal{P}_d), \tag{VIII.3}$$

with unknowns $d_{\mu,\nu}$ and $m_\lambda(D(\nu))$ for $\mu \in \mathcal{P}_d$ and $\lambda, \nu \in \mathcal{RP}_d$. Note that $m_\lambda(S(\mu))$ are known from Theorem IV.4.1.

The algorithm described in Section VI.2 allows us to solve system (VIII.3) and relies on the fact that it has a unique solution under the following conditions:

(i) all $m_\lambda(D(\nu)) \in \mathbb{Z}[q, q^{-1}]$ are bar-invariant and $m_\lambda(D(\lambda)) = r_\lambda$;

(ii) $d_{\mu,\nu} = 0$ unless $\nu \trianglelefteq \mu$, $d_{\nu,\nu} = 1$, and $d_{\mu,\nu} \in q\mathbb{Z}[q]$ for $\mu \neq \nu$.

It turns out that the LLT algorithm also relies on being able to solve system (VIII.3). In [LLT], the vector $A(\lambda)$ for $\lambda \in \mathcal{RP}_d$ is a first approximation to the canonical basis element $G(\lambda)$, which calculates a *column* of the decomposition matrix:

$$G(\lambda) = \sum_{\mu \in \mathcal{P}_d} d_{\mu,\lambda} M_\mu. \tag{VIII.4}$$

Since the set $\{G(\lambda) \mid \lambda \in \mathcal{RP}_d\}$ forms a basis for the Fock space, one may write

$$r_\lambda A(\lambda) = \sum_{\nu \in \mathcal{RP}_d} b_{\lambda,\nu} G(\nu) \tag{VIII.5}$$

for some bar-invariant coefficients $b_{\lambda,\nu}$. Combining equations VIII.2, VIII.4, and VIII.5 we have:

$$\sum_{\mu \in \mathcal{P}_d} m_\lambda(S(\mu)) M_\mu = \sum_{\nu \in \mathcal{RP}_d} b_{\lambda,\nu} \sum_{\mu \in \mathcal{P}_d} d_{\mu,\nu} M_\mu.$$

Now, fixing μ (i.e. fixing a *row* of the matrix) we are left with solving

$$m_\lambda(S(\mu)) = \sum_{\nu \in \mathcal{RP}_d} b_{\lambda,\nu} d_{\mu,\nu}$$

for each $\mu \in \mathcal{P}_d$ and $\lambda \in \mathcal{RP}_d$. Since conditions analogous to (i) and (ii) are known to hold in the

Lascoux-Leclerc-Thibon setup [LLT], we are left with solving the same system of equations under the same conditions. Moreover, the $b_{\lambda,\nu}$ appearing in the LLT algorithm can now be interpreted as the graded weight space multiplicities $m_\lambda(D(\nu))$ for each $\lambda, \nu \in \mathcal{RP}_d$.

CHAPTER IX

EXAMPLES

In this chapter we use the algorithm to compute a few examples. In Section IX.1 we describe a step by step process to compute the full matrix of decomposition numbers for H_d in the case when $d = 6$ and $e = 3$. Then, in Section IX.2, we describe how to compute a particular decomposition number. Finally, in Section IX.3 we give the matrix of decomposition numbers for H_d in several cases.

IX.1 Graded Decomposition Numbers for H_d with $d = 6$ and $e = 3$

To compute the matrix of graded decomposition numbers $(d_{\mu,\lambda})_{\mu\in\mathcal{P}_6;\,\lambda\in\mathcal{RP}_6}$ using our algorithm, recall that at each stage we calculate $d_{\mu,\lambda}$ and (if μ is 3-restricted) $m_\lambda(D(\mu))$. Recall also that $d_{\lambda,\lambda} = 1$ for all $\lambda \in \mathcal{RP}_d$ while $d_{\mu,\lambda} = 0$ unless $\lambda \in M(\mu)$ thanks to Theorem IV.5.2. We order the matrix in reverse dominance order with all 3-restricted partitions appearing before any that are **not** 3-restricted. Under this ordering, we only need to compute the entries that are strictly below the diagonal. The complete set of partitions in $\mathcal{P}_6$ is

$$\{(6), (5,1), (4,2), (4,1^2), (3^2), (3,2,1), (3,1^3), (2^3), (2^2,1^2), (2,1^4), (1^6)\},$$

while $\mathcal{RP}_6 = \{(4,2), (3,2,1), (3,1^3), (2^3), (2^2,1^2), (2,1^4), (1^6)\}$ is the set of 3-restricted partitions.

The algorithm proceeds in reverse dominance order, starting with the partition (1^6). The set of moves for (1^6) is $M(1^6) = \{(1^6)\}$ and the ladder weight is $\boldsymbol{j}^{(1^6)} = (0,2,1,0,2,1)$. In fact,

$$d_{(1^6),(1^6)} = 1 \qquad \text{and} \qquad m_{(1^6)}(D(1^6)) = m_{(1^6)}(S(1^6)) = r_{(1^6)} = 1.$$

Of course, Theorem IV.5.2 was sufficient to impart the knowledge of all entries in the first row of the matrix. Here we simply illustrated some of the information we must keep track of as the algorithm proceeds. For one, we must keep track of the ladder weights $\boldsymbol{j}^{\lambda}$ for each $\lambda \in \mathcal{RP}_d$. From here on, we will omit discussion of the entries on the 'diagonal' and will only maintain information that is important for the induction.

Next, we proceed to the partition $(2,1^4)$. The set of moves is $M(2,1^4) = \{(2,1^4),(1^6)\}$. The ladder weight is $\boldsymbol{j}^{(2,1^4)} = (0,1,2,1,0,2)$ and $m_{(2,1^4)}(D(2,1^4)) = r_{(2,1^4)} = 1$. One calculates that $m_{(1^6)}(S(2,1^4)) = q$. Hence, using the Basic Algorithm, we find that

$$d_{(2,1^4),(1^6)} = q \qquad \text{and} \qquad m_{(1^6)}(D(2,1^4)) = 0,$$

completing the second row of the matrix.

In the case of the partition $(2^2,1^2)$, we have that $M(2^2,1^2) = \{(2^2,1^2)\}$. The ladder weight is $\boldsymbol{j}^{(2^2,1^2)} = (0,1,2,0,1,0)$ and $r_{(2^2,1^2)} = 1$. The fact that there are no non-trivial moves for $(2^2,1^2)$ implies that the Specht module $S(2^2,1^2)$ is, in fact, irreducible. Hence, all of the off-diagonal entries in the third column of the matrix are 0.

The partition (2^3) has ladder weight $\boldsymbol{j}^{(2^3)} = (0,1,2,0,1,2)$ and $r_{(2^3)} = 1$. The moves for (2^3) are $M(2^3) = \{(2^3),(2,1^4),(1^6)\}$. One computes that $m_{(2,1^4)}(S(2^3)) = q$. Hence, the Basic Algorithm implies that

$$d_{(2^3),(2,1^4)} = q \qquad \text{and} \qquad m_{(2,1^4)}(D(2^3)) = 0.$$

Next, one computes that $m_{(1^6)}(S(2^3)) = 0$, which implies that

$$d_{(2^3),(1^6)} = 0 \qquad \text{and} \qquad m_{(1^6)}(D(2^3)) = 0,$$

completing the fourth row.

The partition $(3,1^3)$ has ladder weight $\boldsymbol{j}^{(3,1^3)} = (0,1,2,2,0,1)$ and $r_{(3,1^3)} = q + q^{-1}$. The moves for $(3,1^3)$ are $M(3,1^3) = \{(3,1^3),(2,1^4),(1^6)\}$ and one computes that $m_{(2,1^4)}(S(3,1^3)) = q$ and $m_{(1^6)}(S(3,1^3)) = 0$. Hence, similar to the previous case,

$$d_{(3,1^3),(2,1^4)} = q;\ m_{(2,1^4)}(D(3,1^3)) = 0 \qquad \text{and} \qquad d_{(3,1^3),(1^6)} = 0;\ m_{(1^6)}(D(3,1^3)) = 0.$$

The partition $(3,2,1)$ has ladder weight $\boldsymbol{j}^{(3,2,1)} = (0,1,2,2,0,1)$ and $r_{(3,1^3)} = q + q^{-1}$. The moves for $(3,2,1)$ are $M(3,2,1) = \{(3,2,1),(3,1^3),(2^3),(2,1^4),(1^6)\}$. One calculates that $m_{(3,1^3)}(S(3,2,1)) = q^2+1$. This is exactly $q \cdot r_{(3,1^3)} = q(q+q^{-1})$, meaning that the Basic Algorithm implies that $d_{(3,2,1),(3,1^3)} = q$ and $m_{(3,1^3)}(D(3,2,1)) = 0$. For the other moves one can compute that $m_{2^3}(S(3,2,1)) = q$, $m_{(2,1^4)}(S(3,2,1)) = q^2$, and $m_{(1^6)}(S(3,2,1)) = q$. Hence,

$$d_{(3,2,1),(2^3)} = d_{(3,2,1),(1^6)} = q \qquad \text{and} \qquad d_{(3,2,1),(2,1^4)} = q^2,$$

while $m_\lambda(D(3,2,1)) = 0$ for all $\lambda \in M(3,2,1) \setminus \{(3,2,1)\}$. This completes the sixth row.

The next 3-restricted partition is $(4,2)$. It has ladder weight $\boldsymbol{j}^{(4,2)} = (0,1,2,2,0,0)$ and $r_{(4,2)} = q^2 + 2 + q^{-2}$. As it turns out $M(4,2) = \{(4,2)\}$, which implies that the Specht module $S(4,2)$ is irreducible. Hence, all of the off-diagonal entries in the seventh row are zero.

Now we move onto the partitions which are not 3-restricted, again in reverse dominance order. None of these partitions will have associated ladder weights or 'diagonal' entries in the matrix. The partition (3^2) has moves $M(3^2) = \{(3^2),(3,2,1),(3,1^3),(2^3),(2,1^4),(1^6)\}$. One computes that $m_{(3,2,1)}(S(3^2)) = q^2+1 = q(q+q^{-1}) = q \cdot r_{(3,2,1)}$, while $m_{(3,1^3)}(S(3^2)) = 0$, $m_{(2^3)}(S(3^2)) = 0$, $m_{(2,1^4)}(S(3^2)) = 0$, and $m_{(1^6)}(S(3^2)) = q^2$. Hence,

$$d_{(3^2),(3,2,1)} = q, \qquad d_{(3^2),(3,1^3)} = d_{(3^2),(2^3)} = d_{(3^2),(2,1^4)} = 0, \qquad \text{and } d_{(3^2),(1^6)} = q^2.$$

Of course, since (3^2) is not 3-restricted, $D(3^2)$, and therefore $m_\lambda(D(3^2))$, do not make sense.

The partition $(4,1^2)$ has moves $M(4,1^2) = \{(4,1^2),(3,2,1),(3,1^3),(2^3),(2,1^4),(1^6)\}$. One computes that $m_{(3,2,1)}(S(4,1^2)) = q^2+1 = q(q+q^{-1}) = q \cdot r_{(3,2,1)}$ and $m_{(3,1^3)}(S(4,1^2)) = q^3+q = q^2(q+q^{-1}) = q^2 \cdot r_{(3,1^3)}$, while $m_{(2^3)}(S(4,1^2)) = m_{(2,1^4)}(S(4,1^2)) = m_{(1^6)}(S(4,1^2)) = 0$. Hence,

$$d_{(4,1^2),(3,2,1)} = q, \qquad d_{(4,1^2),(3,1^3)} = q^2, \qquad \text{and } d_{(4,1^2),(2^3)} = d_{(4,1^2),(2,1^4)} = d_{(4,1^2),(1^6)} = 0.$$

The partition $(5,1)$ has $M(5,1) = \{(5,1),(4,1^2),(3^2),(3,2,1),(3,1^3),(2^3),(2,1^4),(1^6)\}$. Note that we need not check $(4,1^2)$ or (3^2) as these are not 3-restricted. One calculates that $m_{(3,2,1)}(S(5,1)) = q^3+q = q^2(q+q^{-1}) = q^2 \cdot r_{(3,2,1)}$ and $m_{(2^3)}(S(5,1)) = q$, and also that

$m_{(3,1^3)}(S(5,1)) = m_{(2,1^4)}(S(5,1)) = m_{(1^6)}(S(5,1)) = 0$. Hence,

$$d_{(5,1),(3,2,1)} = q^2, \qquad d_{(5,1),(2^3)} = q, \qquad \text{and } d_{(5,1),(3,1^3)} = d_{(5,1),(2,1^4)} = d_{(5,1),(1^6)} = 0.$$

Finally, we must check the partition (6). It has moves $M(6) = \{(6),(5,1),(4,1^2),(3^2),(3,2,1),(3,1^3),(2^3),(2,1^4),(1^6)\}$. Actually, $\mathrm{ch}_q\, S(6) = q^2(0,1,2,0,1,2)$ which is exactly $q^2 j^{(2^3)}$. Hence, $m_{(2^3)}(S(6)) = q^2$, while all other multiplicities are zero. Hence,

$$d_{(6),(2^3)} = q^2 \qquad \text{and} \qquad d_{(6),(3,2,1)} = d_{(6),(3,1^3)} = d_{(5,1),(2,1^4)} = d_{(5,1),(1^6)} = 0.$$

The matrix of graded decomposition numbers is then given in Table 1.

$\mu \diagdown \lambda$	(1^6)	$(2,1^4)$	$(2^2,1^2)$	(2^3)	$(3,1^3)$	$(3,2,1)$	$(4,2)$
(1^6)	1	0	0	0	0	0	0
$(2,1^4)$	q	1	0	0	0	0	0
$(2^2,1^2)$	0	0	1	0	0	0	0
(2^3)	0	q	0	1	0	0	0
$(3,1^3)$	0	q	0	0	1	0	0
$(3,2,1)$	q	q^2	0	q	q	1	0
$(4,2)$	0	0	0	0	0	0	1
(3^2)	q^2	0	0	0	0	q	0
$(4,1^2)$	0	0	0	0	q^2	q	0
$(5,1)$	0	0	0	q	0	q^2	0
(6)	0	0	0	q^2	0	0	0

Table 1: The matrix of graded decomposition numbers with $d = 6$ and $e = 3$.

Remark IX.1.1. The previous example was somewhat simplified because of the fact that, in this case, $m_\lambda(D(\mu)) = 0$ whenever $\lambda \neq \mu$. This occurs only in cases where d is 'small' - a notion which depends directly on e. In the following example we will see how having $m_\lambda(D(\mu)) \neq 0$ makes the computation slightly more complicated.

IX.2 Computing a Particular Graded Decomposition Number

Consider the case when $d = 8$ and $e = 3$ and suppose one wants to compute the graded decomposition number $d_{(3,2^2,1),(1^8)}$. The moves for the partition $(3, 2^2, 1)$ are $M(3, 2^2, 1) = \{(3, 2^2, 1), (3, 2, 1^3), (2^3, 1^2), (1^8)\}$. Using the algorithm here we proceed to induct on the distance between the partitions. Hence we start with the closest move to $(3, 2^2, 1)$, namely $(3, 2, 1^3)$. One computes that $m_{(3,2,1^3)}(S(3, 2^2, 1)) = q^2 + 1 = q(q + q^{-1}) = qr_{(3,2,1^3)}$. Since $j^{(3,2,1^3)}$ could not have come from any other moves, it follows that

$$d_{(3,2,1^3),(3,2^2,1)} = q \qquad \text{and} \qquad m_{(3,2,1^3)}(D(3, 2^2, 1)) = 0.$$

Next, consider the move $(2^3, 1^2)$. One computes that $m_{(2^3,1^2)}(S(3, 2^2, 1)) = q^2 + 1$, while $r_{(2^3,1^2)} = 1$. It is possible that $j^{(2^3,1^2)}$ appears in $\mathrm{ch}_q\, D(3, 2, 1^3)$, hence we must also calculate $m_{(2^3,1^2)}(D(3, 2, 1^3))$. Calculating $m_{(2^3,1^2)}(S(3, 2, 1^3)) = q$, one finds that $m_{(2^3,1^2)}(D(3, 2, 1^3)) = 0$. Hence, $j^{(2^3,1^2)}$ does not appear in $\mathrm{ch}_q\, D(3, 2, 1^3)$, and we have

$$d_{(2^3,1^2),(3,2^2,1)} = q^2 \qquad \text{and} \qquad m_{(2^3,1^2)}(D(3, 2^2, 1)) = 1.$$

Observe that the weight $j^{(2^3,1^2)}$ actually appears in the graded character of $D(3, 2^2, 1)$ with a non-zero (and necessarily bar-invariant) coefficient.

Finally, we must consider the desired move (1^8). One computes that $m_{(1^8)}(S(3, 2^2, 1) = q^2$. Again, it is possible that $j^{(1^8)}$ appears in either $D(3, 2, 1^3)$ or $D(2^3, 1^2)$. One computes that $m_{(1^8)}(S(3, 2, 1^3)) = 2q$ and that $m_{(1^8)}(D(2^3, 1^2)) = 1$, which implies that $m_{(1^8)}(D(3, 2, 1^3)) = 0$ and $m_{(1^8)}(D(2^3, 1^2)) = 1$. Hence, the factor $q^2\mathrm{ch}_q\, D(2^3, 1^2)$ appearing in the decomposition of $\mathrm{ch}_q\, S(3, 2^2, 1)$ actually contains the copy of $q^2 j^{(1^8)}$ appearing in $\mathrm{ch}_q\, S(3, 2^2, 1)$. Thus,

$$d_{(1^8),(3,2^2,1)} = 0 \qquad \text{and} \qquad m_{(1^8)}(D(3, 2^2, 1)) = 0,$$

despite the fact that $m_{(1^8)}(S(3, 2^2, 1))$ was non-zero and **not** bar-invariant.

IX.3 Matrices of Graded Decomposition Numbers

$\mu\backslash\lambda$	(1^2)
(1^2)	1
(2)	q

$\mu\backslash\lambda$	(1^3)	$(2,1)$
(1^3)	1	0
$(2,1)$	0	1
(3)	q	0

$\mu\backslash\lambda$	(1^4)	$(2,1^2)$
(1^4)	1	0
$(2,1^2)$	0	1
(2^2)	q	0
$(3,1)$	q	q^2
(4)	q^2	0

$\mu\backslash\lambda$	(1^5)	$(2,1^3)$	$(2^2,1)$
(1^5)	1	0	0
$(2,1^3)$	0	1	0
$(2^2,1)$	0	0	1
$(3,1^2)$	q	0	q
$(3,2)$	0	0	q^2
$(4,1)$	0	q	0
(5)	q^2	0	0

Table 2: The matrices of graded decomposition numbers with $e=2$ and $d=2,3,4,5$.

$\mu\backslash\lambda$	(1^6)	$(2,1^4)$	$(2^2,1^2)$	$(3,2,1)$
(1^6)	1	0	0	0
$(2,1^4)$	q	1	0	0
$(2^2,1^2)$	0	q	1	0
$(3,2,1)$	0	0	0	1
(2^3)	0	0	q	0
$(3,1^3)$	q	q^2	q	0
(3^2)	0	0	q^2	0
$(4,1^2)$	q^2	q	q^2	0
$(4,2)$	0	q^2	q^3	0
$(5,1)$	q^2	q^3	0	0
(6)	q^3	0	0	0

Table 3: The matrix of graded decomposition numbers with $e=2$ and $d=6$.

$\mu\backslash\lambda$	(1^2)	(2)
(1^2)	1	0
(2)	0	1

$\mu\backslash\lambda$	(1^3)	$(2,1)$
(1^3)	1	0
$(2,1)$	q	1
(3)	0	q

Table 4: The matrices of graded decomposition numbers with $e = 3$ and $d = 2, 3$.

$\mu\backslash\lambda$	(1^4)	$(2,1^2)$	(2^2)	$(3,1)$
(1^4)	1	0	0	0
$(2,1^2)$	0	1	0	0
(2^2)	q	0	1	0
$(3,1)$	0	0	0	1
(4)	0	0	q	0

$\mu\backslash\lambda$	(1^5)	$(2,1^3)$	$(2^2,1)$	$(3,1^2)$	$(3,2)$
(1^5)	1	0	0	0	0
$(2,1^3)$	0	1	0	0	0
$(2^2,1)$	0	q	1	0	0
$(3,1^2)$	0	0	0	1	0
$(3,2)$	q	0	0	0	1
$(4,1)$	0	0	0	0	q
(5)	0	0	q	0	0

Table 5: The matrices of graded decomposition numbers with $e = 3$ and $d = 4, 5$.

$\mu\backslash\lambda$	(1^6)	$(2,1^4)$	$(2^2,1^2)$	(2^3)	$(3,1^3)$	$(3,2,1)$	$(4,2)$
(1^6)	1	0	0	0	0	0	0
$(2,1^4)$	q	1	0	0	0	0	0
$(2^2,1^2)$	0	0	1	0	0	0	0
(2^3)	0	q	0	1	0	0	0
$(3,1^3)$	0	q	0	0	1	0	0
$(3,2,1)$	q	q^2	0	q	q	1	0
$(4,2)$	0	0	0	0	0	0	1
(3^2)	q^2	0	0	0	0	q	0
$(4,1^2)$	0	0	0	0	q^2	q	0
$(5,1)$	0	0	0	q	0	q^2	0
(6)	0	0	0	q^2	0	0	0

Table 6: The matrix of graded decomposition numbers with $e = 3$ and $d = 6$.

$\mu\setminus\lambda$	(1^2)	(2)
(1^2)	1	0
(2)	0	1

$\mu\setminus\lambda$	(1^3)	$(2,1)$	(3)
(1^3)	1	0	0
$(2,1)$	0	1	0
(3)	0	0	1

Table 7: The matrices of graded decomposition numbers with $e = 4$ and $d = 2, 3$.

$\mu\setminus\lambda$	(1^4)	$(2,1^2)$	(2^2)	$(3,1)$
(1^4)	1	0	0	0
$(2,1^2)$	q	1	0	0
(2^2)	0	0	1	0
$(3,1)$	0	q	0	1
(4)	0	0	0	q

$\mu\setminus\lambda$	(1^5)	$(2,1^3)$	$(2^2,1)$	$(3,1^2)$	$(3,2)$	$(4,1)$
(1^5)	1	0	0	0	0	0
$(2,1^3)$	0	1	0	0	0	0
$(2^2,1)$	q	0	1	0	0	0
$(3,1^2)$	0	0	0	1	0	0
$(3,2)$	0	0	q	0	1	0
$(4,1)$	0	0	0	0	0	1
(5)	0	0	0	0	q	0

Table 8: The matrices of graded decomposition numbers with $e = 4$ and $d = 4, 5$.

$\mu\setminus\lambda$	(1^6)	$(2,1^4)$	$(2^2,1^2)$	(2^3)	$(3,1^3)$	$(3,2,1)$	(3^2)	$(4,1^2)$	$(4,2)$
(1^6)	1	0	0	0	0	0	0	0	0
$(2,1^4)$	0	1	0	0	0	0	0	0	0
$(2^2,1^2)$	0	q	1	0	0	0	0	0	0
(2^3)	q	0	0	1	0	0	0	0	0
$(3,1^3)$	0	0	0	0	1	0	0	0	0
$(3,2,1)$	0	0	0	0	0	1	0	0	0
(3^2)	0	0	q	0	0	0	1	0	0
$(4,1^2)$	0	0	0	0	0	0	0	1	0
$(4,2)$	0	0	0	q	0	0	0	0	1
$(5,1)$	0	0	0	0	0	0	0	0	q
(6)	0	0	0	0	0	0	q	0	0

Table 9: The matrix of graded decomposition numbers with $e = 4$ and $d = 6$.

APPENDIX

LIST OF SYMBOLS

A, B	Nodes in the Young diagram of a partition λ	14
$A(\lambda)$	The 'first approximation' to the canonical basis element $G(\lambda)$	56
$\boldsymbol{A}$	A set of removable nodes for a partition	15
	The bottom removable sequence for λ	33
$\boldsymbol{B}$	A set of addable or removable nodes for a partition	15
$b(\mu)$	Difference in the lengths of columns 1 and 2 of μ, taken $\pmod e$	45
$B(\nu)$	Partitions labeling the splitting of $D(\nu)$ when reducing modulo e	51
$\mathrm{ch}_q\, V$	The graded character of V	28
$d_A(\lambda)$	The degree of a removable node A in λ	18
$d^B(\lambda)$	The degree of an addable node B for λ	18
$d_i(\lambda)$	A degree function on λ for each $i \in I$	18
$d_{\mu,\lambda}$	Graded decomposition number for $\mu \in \mathcal{P}_d$ and $\lambda \in \mathcal{RP}_d$.	30
$\overline{d}_{\nu,\lambda}$	Graded decomposition numbers for $\mathbb{F}\Sigma_d$ when $\mathrm{char}\,\mathbb{F} = p$	51
$\deg(\mathtt{T})$	The degree of a standard tableau $\mathtt{T}$	21
$D(\lambda)$	The irreducible H_d-module corresponding to $\lambda \in \mathcal{RP}_d$	26
$\overline{D(\lambda)}$	The irreducible $\mathbb{F}\Sigma_d$-module when $\mathrm{char}\,\mathbb{F} = p$	50
e	The quantum characteristic given the choice of $\xi \in \mathbb{F}^\times$	6
$e(\boldsymbol{i})$	One of a system of orthogonal idempotents in H_d	9
$\{E_i, F_i, K_i^{\pm 1}\}$	The Chevalley Generators for $U_q(\mathfrak{g})$	54
$\mathbb{F}$	An algebraically closed field	6

$l(\mu, \lambda)$	The distance between μ and λ	16
L_m	The m^{th} ladder	17
$\mathcal{L}_r$	One of the Jucys-Murphy elements in H_d	8
$m_\lambda(V)$	The multiplicity of $\boldsymbol{j}^\lambda$ in $\mathrm{ch}_q\, V$	38
$M(\mu)$	The set of moves for μ	16
M_μ	A basis element for the Fock space $\mathfrak{F}$	55
$M(\mu, \lambda)$	The set of moves for μ which have λ as a move	16
H-Mod	The abelian category of all graded left H-modules	24
$\lvert\mu, \lambda\rvert$	The size of the move λ for μ in the 2-column case	45
$[n]_q$	Quantum integer corresponding to n	7
$\nu(i)$	The form of an eigenvalue for a Jucys-Murphy element, L_r	6
$\mathcal{P}_d$	The set of partitions of d	13
$P(\lambda)$	The (graded) projective cover of $D(\lambda)$	54
π	The projection from $\mathfrak{F}$ onto $V(\Lambda_0)$	56
$\mathrm{Proj}(H)$	Additive category of finitely generated projective graded H-modules	24
$[\mathrm{Proj}(H)]$	The Grothendieck group of the category $\mathrm{Proj}(H)$	24
ψ_r	Part of a different generating set for H_d	9
q	An indeterminate	7
qdim	The graded dimension function on $\mathbb{Z}$-graded vector spaces	24
r_λ	The explicit multiplicity of $\boldsymbol{j}^\lambda$ in $\mathrm{ch}_q\, D(\lambda)$	37
$r_m(\lambda)$	The number of nodes in the intersection of λ and L_m	17
R_m	The number of nodes of λ appearing in the first m ladders	36
res A	The e-residue of a node A	16
$\mathrm{Rep}(H)$	The abelian category of all finite dimensional graded H-modules	24
$[\mathrm{Rep}(H)]$	The Grothendieck group of the category $\mathrm{Rep}(H)$	24

$\mathcal{RP}_d$	The set of e-restricted partitions of d	13
s_r	The simple transposition $(r, r+1) \in \Sigma_d$	7
$s_p(\nu, \lambda)$	The number of copies of $\overline{D(\lambda)}$ appearing in a decomposition of $S(\nu)$	51
$\mathrm{sh}(\mathtt{T}_{\leq s})$	The associated partition shape of the tableau $\mathtt{T}_{\leq s}$	20
σ_k	A function on partitions defined by $\sigma_k(\lambda) = \sum_{i=1}^{k} \lambda_i$	13
Σ_d	The symmetric group on d letters	7
$S(\mu)$	The Specht module corresponding to the partition μ	29
t_λ	The index of the lowest ladder which intersects λ non-trivially	17
$\mathtt{T}$	A λ-tableau for a partition λ	19
$\mathtt{T}^\lambda$	The 'row sequential' standard λ-tableau	20
$\mathtt{T}_{\leq s}$	A standard tableau obtained from a standard tableau $\mathtt{T}$	20
T_r	One of the standard generators of H_d	7
$\mathscr{T}(\lambda)$	The set of standard λ-tableaux	20
$\mathtt{U}$	A standard λ-tableau	47
$U_q(\mathfrak{g})$	The quantized enveloping algebra of $\mathfrak{g}$	54
$v_{\mathtt{T}}$	An element of an explicit basis for $S(\mu)$ when $\mathtt{T} \in \mathscr{T}(\mu)$	27
V	A finite dimensional (graded) H_d-module	8
$V^{\circledast}$	The graded dual of the graded H_d-module V	27
$V_{\boldsymbol{i}}$	The $\boldsymbol{i}$-weight space of V	8
$V\langle m \rangle$	The graded H_d-module obtained from V by shifting the grading by m	24
$V(\Lambda_0)$	The irreducible highest weight $\mathfrak{F}$-submodule generated by $M_\varnothing$	56
ξ	A parameter in $\mathbb{F}^\times$	6
y_r	A nilpotent element in H_d that is part of a different generating set	9
$\trianglerighteq$	The usual dominance order on the set of partitions $\mathcal{P}_d$	13

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