Diagrammatics of Soergel Bimodules of type $A_1 \times A_1$

by

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DISSERTATION ABSTRACT

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There is an action of $\mathbb{Z}/2$ diagrammatic Hecke category $\mathcal{H}$ of type $A_1 \times A_1$ induced by the nontrivial automorphism of its Dynkin diagram. We consider the equivariantization category $\mathcal{H}^\tau$ with respect to this action. We construct an isotopy presentation by local generators and relations of $\mathcal{H}^\tau$ and develop a diagrammatic calculus for this category.

We also study the localization $\mathcal{Q}^\tau$ obtained from $\mathcal{H}^\tau$ by inverting the polynomial $\alpha_s\alpha_t$ and taking its Karoubi envelope. We give a diagrammatic presentation for $\mathcal{Q}^\tau$ that extends the presentation of $\mathcal{H}^\tau$. We then build a double leaves basis for the morphism spaces of $\mathcal{H}^\tau$.

This is the first step in a program to describe and study the equivariant Hecke categories that appear in [Eli17].

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DEDICATION

Para Tany, mis padres, mi tia Pancha,
Y para mi abuelo Ernesto,
- en su cumpleaños póstumo.

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INTRODUCTION

1.1 Categorification of Hecke algebras with unequal parameters

1.1.1 *The Hecke Algebra with equal parameters*

Let W be a Coxeter group and let $\Bbbk$ be a field. Let us consider the category of W-graded vector spaces $\mathbf{Vec}_W(\Bbbk)$. The objects of this category are formal direct sums $\bigoplus_{w\in W}(V)_w$ of finite dimensional vector spaces over $\Bbbk$, its morphisms are linear transformations preserving the grading, and its tensor product is given component-wise by

$$(V_1)_u \otimes (V_2)_w = (V_1 \otimes_{\Bbbk} V_2)_{uw}. \tag{1.1.1.1}$$

Since all of our vector spaces are finite dimensional this category is semisimple and its irreducible objects are of the form $(\Bbbk)_w$ for $w \in W$. Moreover, by the relation

$$(\Bbbk)_u \otimes (\Bbbk)_w = (\Bbbk)_{uw} \tag{1.1.1.2}$$

we can conclude that the Grothendieck group of $\mathbf{Vec}_W(\Bbbk)$ is isomorphic to the group algebra $\mathbb{Z}[W]$. Because of this we say that $\mathbf{Vec}_W(\Bbbk)$ *categorifies* $\mathbb{Z}[W]$.

Let (W,S) be a Coxeter system. The (equal parameter) *Hecke algebra* $\mathsf{H}(W)$ associated to W, is the $\mathbb{Z}[v^{\pm 1}]$-algebra generated by the symbols $\{\delta_s | s \in S\}$ modulo the following relations:

(a) For all $s \in S$, we have the quadratic relation

$$\delta_s^2 = (v^{-1} - v)\delta_s + 1. \tag{1.1.1.3}$$

(b) For $s,t \in S$, such that $m_{st} < \infty$ we have the braid relation

$$\underbrace{\delta_s\delta_t\cdots}_{m_{st}} = \underbrace{\delta_t\delta_s\cdots}_{m_{st}}, \tag{1.1.1.4}$$

analogous to the braid relation in W.

Let $w \in W$, let $w = s_1 s_2 \cdots s_r$ be a reduced expression for w, and define

$$\delta_w := \delta_{s_1}\cdots\delta_{s_r}. \tag{1.1.1.5}$$

By Matsumoto's theorem δ_w is independent of the choice of reduced expression. In fact, the set $\{\delta_w | w \in W\}$ is a basis of $\mathsf{H}(W)$. Notice that if we set the parameter $v = 1$, we recover the group algebra $\mathbb{Z}[W]$ of our Coxeter group. Because of this, we say that the Hecke algebra is a *deformation* of $\mathbb{Z}[W]$.

There are multiple ways of categorifying the Hecke algebra. All of them have some sort of grading, and the grading shift categorifies multiplication by the parameter v, though we will not discuss that grading here. The geometric categorification is the category of equivariant perverse sheaves over the flag variety. The category of *Soergel bimodules*, a category of bimodules over the polynomial ring $R := \Bbbk[\alpha_s | s \in S]$ (which can be seen as the symmetric tensor algebra of the geometric representation of W), also categorifies the Hecke algebra. A categorification in representation theory comes from projective functors acting on (graded) category $\mathcal{O}_0(\mathfrak{g})$, where g is a semisimple complex Lie algebra (in this setting the grading is not obvious, and came much later). Finally, the diagrammatic Hecke category $\mathcal{H}(W)$, a category whose morphisms are defined combinatorially in terms of planar diagrams, also categorifies $\mathsf{H}(W)$ [EW14]. Because the Hecke algebra is a fundamental object in representation theory, geometry, and knot theory, the study of Soergel Bimodules became relevant for these fields as well.

The Hecke algebra also has a special 'self-dual' basis $\{b_w | w \in W\}$ called the Kazhdan-Lusztig basis. For example, $b_s = \delta_s + v$ for $s \in S$ and satisfies the relation

$$b_s^2 = (v + v^{-1})b_s. \tag{1.1.1.6}$$

When $\Bbbk$ is a field of characteristic zero, the indecomposable Soergel bimodules correspond to the Kazhdan-Lusztig basis under the map to $\mathsf{H}(W)$ [Soe07]. The Kazhdan-Lusztig polynomials $h_{u,w}$ defined by

$$b_w = \sum_{u \in W} h_{u,w} \delta_u \tag{1.1.1.7}$$

are of special importance in Lie theory, since can be used to calculate the Jordan-Hölder multiplicities of irreducible modules of Verma modules.

The category of *standard R-bimodules* is the (graded) R-linear category generated under direct sum and grading shift by $\{R_w | w \in W\}$, where R_w is R with a right multiplication twisted by the action of w. Soergel bimodules have a filtration by standard bimodules, and the KL polynomials count multiplicities in this filtration. After base change to the fraction field Q of R, this filtration splits. The category standard bimodules may also be regarded as the (graded) R-linearization of $\mathbf{Vec}_W(\Bbbk)$. Thus localization is a technique for describing Soergel bimodules and morphisms between them in terms of the easy category $\mathbf{Vec}_W(Q)$. See Section 1.2 for more details.

The diagrammatic Hecke category of $\mathcal{H}(W)$ was first introduced by Elias and Khovanov for type A [EK+10], by Elias for dihedral groups [Eli11], and then for a general Coxeter group W by Elias and Williamson [EW16]. The diagrammatic Hecke category is an R-linear graded category that models many properties of the category of Soergel bimodules. Unlike the category of Soergel bimodules, $\mathcal{H}(W)$ is defined integrally and the computations in $\mathcal{H}(W)$ consist of an algorithmic reduction of planar diagrams instead of (possibly more complicated) polynomial computations. We also have that $\mathcal{H}(W)$ categorifies the Hecke algebra, even in instances where Soergel Bimodules do not. For a similar diagrammatic approach to localization, see [EW16] and [EW16].

1.1.2 Folding of finite Coxeter groups

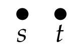
Let W be a finite Coxeter group with Coxeter system (W, S) and let G be a finite group of automorphisms of W that preserves S. Let W^G be the subgroup of G-invariant elements of W, and let S/G be the set of G-orbits of S, and for each $s \in S$ let O_s be its G-orbit. Also let w_{O_s} the longest element of the parabolic subgroup generated by O_s. Under the identification $\widehat{\iota} : S/G \longrightarrow W$ sending $O_s \mapsto w_{O_s}$, we may regard $(W^G, S/G)$ as a Coxeter system, and the inclusion $\iota : W^G \longrightarrow W$ becomes a Coxeter embedding that extends $\widehat{\iota}$. Using ι, we may restrict the length function ℓ_W of W to a weight function $L : W^G \longrightarrow \mathbb{N}$ defined by $L(w) = \ell_W(w)$ [GI14]. Note that if we take the Dynkin diagram of (W, S), identify the vertices on each orbit, and relabel the edges appropriately, we obtain the Dynkin diagram of $(W^G, S/G)$. Because of this, the Coxeter system $(W^G, S/G)$ is called a *folding* of (W, S) and the group action is called a *folding G-action*. It is important to note that the embedding $\iota : W \longrightarrow W^G$ exists whenever the parabolic subgroups generated by the orbits O_s are finite, even if W is not. In this context, the embedding $\iota : W^G \longrightarrow W$ is called a *quasi-split embedding* of Coxeter groups.

Remark 1.1.2.1. Note that a folding action of G on W also induces natural actions on both the Hecke algebra $\mathsf{H}(W)$ and on the diagrammatic Hecke category $\mathcal{H}(W)$. We will denote by $\mathsf{H}(W)^G$ to the subring of G-invariant elements of $\mathsf{H}(W)$.

Now, we will show the two primary examples of folding that we will be interested in. Refer to [Eli17, Sec. 2] for broader array of examples.

The following is the smallest example of folding. Let

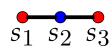
$$A_1 \times A_1 = \tag{1.1.2.1}$$



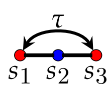
be the Dynkin diagram above with simple reflections s and t. Let τ be the automorphism of $A_1 \times A_1$ swapping the simple reflections s and t. The invariant subgroup $W(A_1 \times A_1)^\tau$ is generated by st and is isomorphic to $W(A_1)$.

Let us now consider the Dynkin diagram

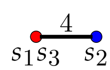
$$A_3 = \quad , \tag{1.1.2.2}$$



and let s_1, s_2, and s_3 be its simple reflections. Let τ be the automorphism of A_3 given by swapping the simple reflections s_1 and s_3. In this case, τ induces an automorphism of the group $W(A_3)$, whose invariant subgroup $W(A_3)^\tau$ is isomorphic to $W(B_2)$. Moreover, if we set t_1, t_2 to be the simple reflections of B_2, the morphism $\iota : W(B_2) \longrightarrow W(A_3)$ sending $t_1 \mapsto s_1 s_3$ and $t_2 \mapsto s_2$ (the longest elements of the parabolic subgroups of the orbits) is a Coxeter embedding. In this case, we may obtain the Dynkin diagram of B_2 by folding the Dynkin diagram of A_1. That is, we fold the diagram



into



. (1.1.2.3)

The induced weight function $L : W(B_2) \longrightarrow \mathbb{N}$ is determined by $L(t_1) = 2$ and $L(t_2) = 1$.

1.1.3 The Hecke Algebra with unequal parameters

Let (W, S) be a finite Coxeter group and let ℓ_W be its length function. Let L be a weight function on W satisfying $L(u) + L(w) = L(uw)$ whenever $\ell(u) + \ell(w) = \ell(uw)$. The *Hecke algebra with unequal parameters* $\mathsf{H}(W, L)$ is the $\mathbb{Z}[v^{\pm 1}]$-algebra generated by the symbols $\{\eta_s | s \in S\}$ modulo the following relations:

(a) For all $s \in S$, we have the quadratic relation

$$\eta_s^2 = (v^{-L(s)} - v^{L(s)})\delta_s + 1. \tag{1.1.3.1}$$

(b) For $s, t \in S$, such that $m_{st} < \infty$ we have the braid relation

$$\underbrace{\eta_s \eta_t \cdots}_{m_{st}} = \underbrace{\eta_t \eta_s \cdots}_{m_{st}}. \tag{1.1.3.2}$$

The algebra $\mathsf{H}(W, L)$ also has a Kazhdan-Lusztig basis, but its corresponding Kazhdan-Lusztig may have negative coefficients (unlike the equal parameters case) [Lus02, Sec. 12]. In this case, $b_s = \eta_s + v^{L(s)}$ and

$$b_s^2 = (v^{L(s)} + v^{-L(s)})b_s. \tag{1.1.3.3}$$

Now let (W, S) be a Coxeter system with a folding action of G, let $(W^G, S/G)$ be the Coxeter system of invariant elements under the action, and let L be the induced weight function on W^G.

Consider the folding in (1.1.2.1), where $W = W(A_1 \times A_1)$. In this case $L(st) = 2$ and the Hecke algebra with unequal parameters is

$$H(W(A_1 \times A_1)^\tau, L) = \mathbb{Z}[v^{\pm 1}][b_{st}] \Big/ \left\langle b_{st}^2 = v^2 b_{st} + v^{-2} b_{st} \right\rangle. \tag{1.1.3.4}$$

Similarly, for the folding in (1.1.2.3), we have $W = W(A_3)$ and the Hecke algebra with unequal parameters is

$$H(W(B_2), L) = \mathbb{Z}[v^{\pm 1}] \left\langle b_{t_1}, b_{t_2} \right\rangle \Big/ \left\langle \begin{matrix} b_{t_1}^2 = v^2 b_{t_1} + v^{-2} b_{t_1} \\ b_{t_2}^2 = v b_{t_1} + v^{-1} b_{t_2} \\ b_{t_1} b_{t_2} b_{t_1} b_{t_2} + 2 b_{t_2} b_{t_1} = b_{t_2} b_{t_1} b_{t_2} b_{t_1} + 2 b_{t_1} b_{t_2} \end{matrix} \right\rangle. \tag{1.1.3.5}$$

Note that the embedding of $W(B_2)$ into $W(A_3)$ does not induce a morphism of their Hecke algebras. There is no obvious relationship between $\mathsf{H}(W(B_2), L)$ and $\mathsf{H}(W(A_3))$. Instead, the relation between these two algebras only becomes clear through the use of categorification.

1.1.4 *Categorification of* $\mathsf{H}(W^G, L)$

Let $\mathcal{C}$ be an additive monoidal category with a strict action of a group G. The *G-equivariantization* $\mathcal{C}^G$ of $\mathcal{C}$ is the category whose objects are pairs $(A, \{f_g\}_{g \in G})$, where $A \in \mathcal{C}$ and $\{f_g : g(A) \longrightarrow A\}$ is a collection of morphism such that $f_{gh} = f_g \circ g(f_h)$ for all $g, h \in G$. A morphism $T : (A, \{f_g\}_{g \in G}) \longrightarrow (A', \{f'_g\}_{g \in G})$ is a morphism $T : A \longrightarrow A'$ in $\mathcal{C}$ satisfying $T \circ f_g = f'_g \circ T$. Note that we always have a *restriction functor* $\mathsf{Res} : \mathcal{C}^G \longrightarrow \mathcal{C}$ sending $(A, \{f_g\}_{g \in G}) \mapsto A$. When $G = \mathbb{Z}/2$ with generator τ, an object of the equivariantization may be seen as a pair $(A, f_\tau : \tau(A) \to A)$ satisfying $f_\tau \circ \tau(f_\tau) = \mathrm{id}_A$.

The main candidates for the categorification of $\mathsf{H}(W^G, L)$ is the G-equivariantization of the Hecke category $\mathcal{H}(W)^G$. However, the Grothendieck group of $\mathcal{H}(W)^G$ is not isomorphic to $\mathsf{H}(W^G, L)$. In type $\mathrm{A}_1 \times \mathrm{A}_1$, for example, Theorem 3.1.2.7 states that the Grothendieck group of $\mathcal{H}^\tau$ is the algebra

$$\mathcal{A} = \mathbb{Z}[v^{\pm 1}][X, Y, Z]\Big/_{K}, \text{ where } K = \left\langle \begin{array}{c} X^2 = 1,\ XY = Y \\ Y^2 = (v + v^{-1})Y + Z + XZ \\ YZ = (v + v^{-1})(Z + XZ) \\ Z^2 = (v^2 + 1 + v^{-2})Z + XZ \end{array} \right\rangle. \tag{1.1.4.1}$$

Instead, it has specializations to both $\mathsf{H}(W^G, L)$ and to the ring of invariants $\mathsf{H}(W)^G$. In the example above, setting $X = -1$ gives us $\mathsf{H}(W(\mathrm{A}_1 \times \mathrm{A}_1)^\tau)$, while setting $X = 1$ produces $\mathsf{H}(W(\mathrm{A}_1 \times \mathrm{A}_1))^\tau$. This was first shown geometrically [Lus02, Sec. 16] and then diagrammatically [Eli17, Sec. 5].

Let W be a Coxeter group with a folding G-action. The *folded Hecke category* $\mathcal{H}(W)^G$ is the equivariantization of $\mathcal{H}(W)$ under the induced G-action. Recall that $R = \Bbbk[\alpha_s | s \in S]$. Since $\mathcal{H}(W)$ is an R-linear category, $\mathcal{H}(W)^G$ is an R^G-linear category, where R^G is the subring of G-invariant polinomial of R. Elias' approach in [Eli17] was to describe $\mathcal{H}(W)^G$ in terms of the diagrammatics of the Hecke category $\mathcal{H}(W)$. Nevertheless, there has not been a systematic study of the equivariant category itself. In particular, we want to study if the equivariant category has an intrinsic diagrammatic presentation, independent of its description as an equivariant category. Given one such description, we may ask: How does the diagrammatic presentation of the category of Soergel Bimodules compare to the presentation of the equivariant category? Which desirable properties of the category of Soergel Bimodules are preserved by the equivariantization?

The main goal of this dissertation is to study the *diagrammatic Folded Hecke category* $\mathcal{H}^\tau$ of type $A_1 \times A_1$. In this case, we let $W = W(A_1 \times A_1)$ with the folding given in (1.1.2.1) We will construct a diagrammatic presentation of it by generators and relations, give a double leaves basis for its morphism spaces, and provide all the tools necessary for the reader to perform diagrammatic calculations in this context. Despite this being the smallest possible example of *equivariant Hecke category*, it is far more complex than the original category. We encourage the reader to take a glance at our presentation in Definition 3.2.2.1. The complexity of this presentation should be immediately clear.

We want to use this category as model to present other equivariant categories. For instance, $\mathcal{H}(\mathrm{A}_1 \times \mathrm{A}_1)^\tau$ is a full subcategory of the equivariant category $\mathcal{H}(A_3)^\tau$ coming from the folding in (1.1.2.3).

1.1.5 Diagrammatic Presentation of $\mathcal{H}^\tau$

Our first main result, Theorem 3.2.7.2, is to construct the diagrammatic category $\mathcal{D}_{A_1 \times A_1}$, to build a functor $\mathbf{F} : \mathcal{D}_{A_1 \times A_1} \longrightarrow \mathcal{H}(\mathrm{A}_1 \times \mathrm{A}_1)^\tau$, and then show that $\mathbf{F}$ is, in fact, an equivalence of categories. Here are a few words about the proof:

- By finding enough idempotent decompositions in $\mathcal{D}_{A_1 \times A_1}$, we check that

$$\mathcal{I} := \{\widehat{\mathbb{1}}, \widehat{\mathrm{X}}, \widehat{\mathrm{Y}}, \widehat{\mathrm{Z}}, \widehat{\mathrm{XZ}}\} \tag{1.1.5.1}$$

  is a full set of indecomposable objects of $\mathcal{D}_{A_1 \times A_1}$ (up to grading shift). This also implies that $\mathcal{D}_{A_1 \times A_1}$ is a *Krull-Schmidt category*, so every object is a finite direct sum of (grading shifts of) objects in $\mathcal{I}$.

- We show that $\mathbf{F}$ is an essentially surjective well-defined functor by verifying that our relations still hold in $\mathcal{H}(\mathrm{A}_1 \times \mathrm{A}_1)$ after applying $\mathbf{F}$.

- Using adjunctions, we show that to check $\mathbf{F}$ is full and faithful it is enough to check it on $\mathsf{Hom}(\widehat{A}, \widehat{\mathbb{1}})$ for $\widehat{A} \in \mathcal{I}$.

- Let $R := \Bbbk[\alpha_s, \alpha_t]$ and $R^\tau := \Bbbk[\alpha_s + \alpha_t, \alpha_s\alpha_t]$. Theorem 3.2.6.1 shows that every $\mathsf{Hom}(\widehat{A}, \widehat{\mathbb{1}})$ is a free R^τ-module, using a diagrammatic induction argument.

- Finally, we observe that $\mathbf{F}$ is an R^τ-linear functor mapping the basis of $\mathsf{Hom}(\widehat{A}, \widehat{\mathbb{1}})$ to the basis of its image.

It is important to note that to use this presentation it really helps to have a comprehensive list of simplifying relations including the idempotent decompositions in Proposition 3.2.5.1 and the relations in Appendix A.

1.2 Localization of the Folded Hecke Category

1.2.1 Localization of the Hecke Category

Let W be a Coxeter group, let $R = \Bbbk[\alpha_s | s \in S]$, and let W act on R by its geometric representation. Define the *standard R-bimodule* R_w for $w \in W$ to be R (as a set) with a left action given by usual multiplication and 'twisted' right action given by

$$x \cdot_w f = x(w(f)) \tag{1.2.1.1}$$

for all $x \in R_w$ and $f \in R$. Note that for R_1 is isomorphic to the bimodule R without a 'twisted' action. Let $\mathsf{StdBim}_R(W)$ be the (graded) category generated by the standard R-bimodules under direct sum and R-tensor product. Because the action of W on R is faithful, it follows the R-bimodule morphisms between R_u and R_w are trivial if $u \neq w$ and isomorphic to R if $u = w$. We also have that $R_u \otimes_R R_w \cong R_{uw}$. Hence, $\mathsf{StdBim}_R(W)$ is semi-simple and its Grothendieck group is isomorphic to the $\mathbb{Z}[v^{\pm 1}]$-group algebra $\mathbb{Z}[v^{\pm 1}][W]$.

Let $Q = \Bbbk[\alpha_s^{\pm 1} | s \in S]$ and for $w \in W$ define the *standard Q-bimodule* $Q_w := R_w \otimes_R Q$. Define the category of *standard Q-bimodules* StdBim_Q to be the (graded) category generated by the standard Q-bimodules under direct sum and Q-tensor product. It is equivalent to the localization of the category $\mathsf{StdBim}_Q(W)$ obtained by inverting α_s for all $s \in S$. Since R is an integral domain, the functor $- \otimes_R Q : \mathsf{StdBim}_R \longrightarrow \mathsf{StdBim}_Q$ is exact and $\mathbf{SBim}_Q$ must be semisimple. Moreover, left multiplication by any α_s gives us an isomorphism Q_w and its grading shift $Q_w[2]$. Thus, the Grothendieck group of $\mathsf{StdBim}_Q(W)$ is

$$\mathbb{Z}[v^{\pm 1}]\big/_{(v^2)}[W] \cong \mathbb{Z}[W \times \mathbb{Z}/2]. \tag{1.2.1.2}$$

Let $w \in W$. Soergel showed that the indecomposable Soergel bimodule B_w, has a filtration by grading shifts of standard bimodules R_u with $\ell(u) \leq \ell(w)$ and where R_w only appears once [Soe07]. Even more, when we apply a base change by $- \otimes_R Q$ the filtration splits, so the localization $B_w \otimes_R Q$ is isomorphic to a finite direct sum of standard Q-bimodules. This allows us to write morphisms between Soergel bimodules as matrices with coefficients in Q.

The objects of the diagrammatic Hecke category are not themselves R-bimodules, so we can not apply the base change to the objects directly. Instead, the *localized diagrammatic Hecke category* $\mathcal{Q}(W)$ is obtained by formally inverting the morphisms $\alpha_s \in R = \mathsf{End}(\mathbb{1})$ and then taking the Karoubi envelope. In [EW16, Sec. 5.4], Elias and Williamson show that $\mathcal{Q}(W)$ is semisimple and construct a diagrammatic presentation of $\mathcal{Q}(W)$ extending the presentation of $\mathcal{H}(W)$. By extending $\mathcal{H}(W)$, we get a full and faithful embedding of $\mathcal{H}(W)$ into $\mathcal{Q}(W)$.

Let $A, B \in \mathcal{H}(W)$ and $T : A \longrightarrow B$. Using the diagrammatics of $\mathcal{Q}(W)$ one may write inclusion and projection morphisms for the standard summands of A and B. Then, using diagrammatic calculus, one can compute the Q-valued matrix associated to T, with respect to this set of inclusions and projections. Using the machinery of double leaves, it is possible to recursively define a basis of inclusions and projections for all the objects of $\mathcal{H}(W)$. This, in turn, gives a basis to write matrices for all the morphisms of $\mathcal{H}(W)$.

Remark 1.2.1.1. In this dissertation we only consider the localization of the Hecke category of type $\mathrm{A}_1 \times \mathrm{A}_1$, but a lot of the arguments we use apply for the general case as well. We will point out whenever this is not the case. For a complete discussion on the diagrammatics of the localized diagrammatic Hecke category see [EW23].

1.2.2 *Localization of $\mathcal{H}^\tau$*

Let W be a Coxeter group with a folding G-action and let $\mathcal{H}(W)^G$ be the folded Hecke category. Let $R = \Bbbk[\alpha_s | s \in S]$ and let R^G be the subring of R of G-invariant polynomial. We have that the endomorphisms of the unit object of $\mathcal{H}(W)^G$ are in correspondence with R^G and $\mathcal{H}(W)^G$ is an R^G-linear category Let

$$\kappa := \prod_{s \in S} \alpha_s. \tag{1.2.2.1}$$

Since $\kappa \in R^G$, we may consider the *localized category* $\mathcal{Q}(W)^G$, obtained from $\mathcal{H}(W)^G$ by inverting κ and taking its Karoubi envelope. This is equivalent to the G-equivariantization of $\mathcal{Q}(W)$.

Even though every object in $\mathcal{Q}(W)^G$ is a finite direct sum of indecomposable objects, non-isomorphic indecomposable objects may have non-zero morphisms between them. Because of this, $\mathcal{Q}(W)^G$ is not semisimple and we refuse to call these objects irreducible. The orange dot $: \mathrm{X} \longrightarrow \mathbb{1}$ is an example a non-zero morphism between non-isomorphic indecomposable objects in the $A_1 \times A_1$ case. In fact, is both epic and monic, but is not an isomorphism (hence $\mathcal{Q}^\tau$ is not a *balanced category*).

Since the morphisms of $\mathcal{H}(W)^G$ are just morphisms in $\mathcal{H}(W)$ that intertwine with the action of G, they can already be written as matrices with coefficients in R. If our basis of inclusions and projections also intertwine with the G-action, they give us a way to calculate these matrix coefficients by composing morphisms in $\mathcal{Q}(W)^G$.

Remark 1.2.2.1. As we have seen, the categories $\mathbf{Vec}_W(\Bbbk)$ and $\mathcal{Q}(W)$ are not as complex as $\mathcal{H}(W)$. Similarly, the equivariantizations $\mathbf{Vec}_W(\Bbbk)^G$ and $\mathcal{Q}^\tau(W)^G$ are easier to work with than $\mathcal{H}^\tau(W)^G$. In particular, a lot of the theory of equivariantization of pointed fusion categories from [BN13] and [Bur15] may apply for $\mathcal{Q}^\tau(W)^G$.

Writing diagrammatic presentations of $\mathbf{Vec}_W(\Bbbk)^G$ and $\mathcal{Q}^\tau(W)^G$ could be used as a starting point to write diagrammatics for $\mathcal{H}^\tau(W)^G$ for a general folding action.

Let us consider the category $\mathcal{H}^\tau$, the folded Hecke category of type $\mathrm{A}_1 \times \mathrm{A}_1$, and its localization $\mathcal{Q}^\tau$. In Theorem 4.3.1.5, we show that $\mathcal{J} := \{\mathrm{Q}_{\mathbb{1}}, \mathrm{Q}_{\mathrm{X}}, \mathrm{Q}_{\mathrm{Y}}, \mathrm{Q}_{\mathrm{Z}}, \mathrm{Q}_{\mathrm{XZ}}\}$ is a full set of irreducible objects of $\mathcal{Q}^\tau$ up to grading shift, and that all the objects of $\mathcal{Q}^\tau$ are finite direct sums of irreducible objects in $\mathcal{J}$ (and their grading shifts).

Our second main result is the construction of a diagrammatic presentation of $\mathcal{Q}^\tau$, extending the presentation of $\mathcal{H}^\tau$ (Definition 4.3.1.1). In particular we may write the indecomposable objects of $\mathcal{H}^\tau$ in $\mathcal{I} := \{\mathbb{1}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}, \mathrm{XZ}\}$ and their tensor products as finite direct sums of grading shifts of objects in $\mathcal{J}$. We also provide most of the relations one may need to perform diagrammatic calculations in $\mathcal{Q}^\tau$.

Remark 1.2.2.2. We have two presentations for $\mathcal{Q}^\tau$. The first one, Definition 4.3.1.1, is an extension of the presentation of $\mathcal{H}^\tau$. The second presentation is implicitly described by using relations (4.3.1.31)-(4.3.1.34) to write the trivalent vertices of $\mathcal{H}^\tau$ as direct sums of morphisms between the irreducible objects in $\mathcal{J}$. This presentation looks like the equivariantization of the R-linearization of $\mathbf{Vec}_{\mathrm{A}_1\times\mathrm{A}_1}(\Bbbk)$ with respect to the action of τ. This is a special case of Remark 1.2.2.1.

1.3 Double Leaves basis of the Folded Hecke Category

1.3.1 Cellular Algebras and Cellular Categories

The ordinary Hecke category has a great deal of structure, and one of the most useful structures is that it is an object-adapted cellular category. Unfortunately, the folded Hecke category does not have this structure, but it has something close. Let us start by recalling the definitions around cellular categories and motivating the ideas.

A *cellular algebra* over a commutative ring R is an R-algebra $\mathfrak{A}$ together with a distinguished R-basis $\{D^\lambda_{f,e}\}$ indexed by:

- an element λ in a poset Λ, called a *cell*.
- elements $e \in \mathsf{E}(\lambda)$ and $f \in \mathsf{M}(\lambda)$, where $\mathsf{E}(\lambda)$ and $\mathsf{M}(\lambda)$ are finite sets depending on λ that are in a fixed bijection denoted by $\bar{\cdot}$ (in both directions).

Define $\mathfrak{I}_{<\lambda} := \mathsf{span}_R(\{D^\mu_{f,e} | e \in \mathsf{E}(\mu), f \in \mathsf{M}(\mu), \mu < \lambda\})$. Define $\mathfrak{I}_{\leq\lambda}$ analogously. The most important conditions on a cellular algebra are for $D_{<\lambda}$ and $D_{\leq\lambda}$ to be two-sided ideals, and that the action of $\mathfrak{I}_{\leq\lambda}/\mathfrak{I}_{<\lambda}$ on itself is roughly like the action of a matrix algebra on itself (with some re-scaling factors). More precisely, it is governed by the following *cellular formula*: for all $a \in R$ we have

$$aD^\lambda_{f,e} = \sum_{t \in \mathsf{M}(\lambda)} l_a(f,t) D^\lambda_{t,e} \mod \mathfrak{A}_{<\lambda}, \tag{1.3.1.1}$$

for some $l_a(f,t) \in R$ depending only in a, f, and t. We also require $\mathfrak{A}$ to have an algebra anti-automorphism, also denoted by $\bar{\cdot}$, sending $D^\lambda_{f,e}$ to $D^\lambda_{\bar{e},\bar{f}}$. Applying $\bar{\cdot}$ to (1.3.1.1) gives us a right version of the cellular formula. For example, the matrix algebra $\mathsf{M}_{n\times n}(\Bbbk)$ is a cellular algebra with a single cell, whose cellular basis consists of the matrices $\delta_{i,j}$ be the matrix whose entry (i,j) is 1 and the rest are zero, and $\bar{\cdot}$ corresponds to matrix transposition. For a matrix $M \in \mathsf{M}_{n\times n}(\Bbbk)$, we have that

$$M\delta_{i,j} = \sum_k M_{k,i}\delta_{k,j}, \tag{1.3.1.2}$$

which is a special case of the cellular formula. The Hecke algebra $\mathsf{H}(W)$ for W of finite type [Gec07] is also a cellular algebra (in type A the Kazhdan-Lusztig basis is a cellular basis, but not in other types). Other important examples of (diagrammatic) cellular algebras are the Temperley-Lieb algebra and the Brauer algebra [GL04].

In [Wes09], Westbury defined *cellular categories*. This generalized cellular algebras, since they may be regarded as cellular categories with one object. Cellular categories have a cellular formula of their own and a filtration by two-sided ideals. In [EL16], Elias and Lauda point out that for most cellular categories, the cellular formula is a consequence of factorization. For example, the category of finite dimensional vector spaces **Vec**$(\Bbbk)$ is a cellular category with $\Lambda = \{*\}$ and whose cellular basis are the matrices in $\mathsf{Hom}(\Bbbk^m, \Bbbk^n)$ of the form $\delta_{i,j}$. We may factor $\delta_{i,j} \in \mathsf{Hom}(\Bbbk^m, \Bbbk^n)$ through $\Bbbk$ by writing $\delta_{i,j} = e_j e_i{}^{\mathsf{T}}$, where $e_i{}^{\mathsf{T}} \in \mathsf{Hom}(\Bbbk^m, \Bbbk)$ is the standard row matrix and $e_j \in \mathsf{Hom}(\Bbbk, \Bbbk^n)$ is the standard column matrix. To account for this factorization, Elias and Lauda give the following definition.

Definition 1.3.1.1. A *strictly object adapted cellular category*, or *SOACC*, over a commutative base ring R is an R-linear category $\mathcal{C}$ together with:

- a poset of cells Λ together with a designated object $J_\lambda \in \mathcal{C}$ for each $\lambda \in \Lambda$.

- for each object $X \in \mathcal{C}$ and each J_λ we have finite sets of morphisms

$$\mathbf{LL}_X^{J_\lambda} := \{LL_p | p \in \mathsf{E}(X, J_\lambda)\} \subseteq \mathsf{Hom}(X, J_\lambda) \text{ and } \mathbf{UL}_{J_\lambda}^X := \{UL_p | p \in \mathsf{M}(J_\lambda, X)\} \subseteq \mathsf{Hom}(J_\lambda, X), \tag{1.3.1.3}$$

indexed by sets $\mathsf{E}(X, J_\lambda)$ and $\mathsf{M}(J_\lambda, X)$. We also need a fixed bijection between $\mathsf{E}(X, J_\lambda)$ and $\mathsf{M}(J_\lambda, X)$ denoted by $\bar{\cdot}$ (in both directions). For $X, Y \in \mathcal{C}$ and $\lambda \in \Lambda$, we subsequently define the sets define the sets

$$\mathbf{DL}_X^Y(\lambda) := \{U \circ L | L \in \mathbf{LL}_X^{J_\lambda}, U \in \mathbf{UL}_{J_\lambda}^Y\} \text{ and } \mathbf{DL}_X^Y := \bigcup_{\lambda \in \Lambda} \mathbf{DL}_X^Y(\lambda). \tag{1.3.1.4}$$

- there is a contravariant involution of $\mathcal{C}$, also denoted by $\bar{\cdot}$, sending LL_p to $UL_{\overline{p}}$ for all $\lambda \in \Lambda$ and $p \in \mathsf{E}(X.J_\lambda)$.

For $X, Y \in \mathcal{C}$, define

$$\mathsf{Hom}_{<\lambda}(X, Y) := \mathsf{span}_R\left(\bigcup_{\mu < \lambda} \mathbf{DL}_X^Y(\mu)\right) \text{ and } \mathsf{Hom}_{\leq\lambda}(X, Y) := \mathsf{span}_R\left(\bigcup_{\mu \leq \lambda} \mathbf{DL}_X^Y(\mu)\right). \tag{1.3.1.5}$$

Also define $\mathfrak{I}_{<\lambda}$ and $\mathfrak{I}_{\leq\lambda}$ to be the non-unital subcategories of $\mathcal{C}$ whose morphism spaces are generated by $\mathsf{Hom}_{<\lambda}(X, Y)$ and $\mathsf{Hom}_{\leq\lambda}(X, Y)$ respectively, under the operations of composition and direct sum. In order to be a SOACC, $\mathcal{C}$ must satisfy the following two conditions:

- for all $\lambda \in \Lambda$ we have that

$$\mathbf{LL}_{J_\lambda}^{J_\lambda} = \mathbf{UL}_{J_\lambda}^{J_\lambda} = \{\mathrm{id}_{J_\lambda}\}. \tag{1.3.1.6}$$

- $\mathbf{DL}_X^Y$ is an R-basis of $\mathsf{Hom}(X, Y)$. We call the collection $\{\mathbf{DL}_X^Y\}_{X, Y \in \mathcal{C}}$ a *cellular basis* of $\mathcal{C}$.

- $\mathfrak{I}_{\leq\lambda}$ is a (two-sided) monoidal ideal of $\mathcal{C}$.

Note that the usual cellular formula follows from this definition [EL16, Lemma 2.8]. Also, in a SOACC, we have an isomorphism

$$\mathsf{Hom}_{\leq\lambda}(J_\lambda, J_\lambda) \big/ \mathsf{Hom}_{<\lambda}(J_\lambda, J_\lambda) \cong R. \tag{1.3.1.7}$$

Let $\Bbbk$ be a field and $\mathcal{C}$ a semisimple category over $\Bbbk$ which is split (meaning that the endomorphism ring of each irreducible object is $\Bbbk$). Then $\mathcal{C}$ is an SOACC over $\Bbbk$, whose cells are in correspondence with irreducible objects J_λ for $\lambda \in \Lambda$, and

$\lambda \leq \mu$ if and only if $\lambda = \mu$. In this case, the cellular basis in any morphism space factors as a projection map and an inclusion map to an irreducible object J_λ. This is completely analogous to the example of $\mathbf{Vec}(\Bbbk)$ above. More generally, R-linear categories which become split semisimple after base change to a localization Q of R (like $\mathsf{H}(W)$) are prime candidates to be SOACCs. There might not be an object of $\mathcal{C}$ corresponding to a given irreducible object J_λ of the localization, but one might have an object I_λ such that

$$I_\lambda \otimes_R Q \cong J_\lambda \oplus \bigoplus_{\mu < \lambda} J_\mu^{\oplus N_{\lambda,\mu}} \tag{1.3.1.8}$$

for some $N_{\lambda,\mu} \in \mathbb{N}$. One can then rephrase the ideals $\mathfrak{I}_{<\lambda}$ of morphisms which factor through $J_\mu, \mu < \lambda$, as being the same as morphism that factor through $I_\mu, \mu < \lambda$. One might be able to lift projection maps to J_λ (defined only over Q) to maps to I_λ (defined over R) which agree $\mathfrak{I}_{<\lambda}$. This is how many SOACCs function. As expected, the Grothendieck group of any (graded) SOACC is the free $\mathbb{Z}[v^{\pm 1}]$-module generated by Λ [EW16, Prop 2.24] (this is proved more generally for fibered SOACCs).

The Hecke category $\mathsf{H}(W)$ is a SOACC whose poset Λ is the group W with the Bruhat order, and whose indecomposable objects B_w are indexed by $w \in W$[EW16]. In order to build a cellular basis, Elias and Williamson developed the technology of *diagrammatic light leaves*, a combinatorial way of encoding the inclusion and projection morphisms between a tensor product of objects and walks in the branching graph of the localization $\mathcal{Q}(W)$. They based their work on the double leaves basis proposed by Libedinsky for the category of Soergel Bimodules [Lib08].

Elias and Lauda also define a *fibered strictly object adapted cellular category*, or *fibered SOACC*. A fibered SOACC is similar to a SOACC, except the morphisms of its cellular basis have the form $U \circ N \circ L$ for $L \in \mathbf{LL}_X^{J_\lambda}, U \in \mathbf{UL}_{J_\lambda}^Y$, and $N \in \mathbf{N}(J_\lambda)$, where $\mathbf{N}(J_\lambda) \subseteq \mathsf{Hom}(J_\lambda, J_\lambda)$ is a finite set depending on λ. We also require for $\mathrm{id}_{J_\lambda} \in \mathbf{N}(J_\lambda)$ and for $\mathsf{A}_\lambda := \mathsf{span}_R \mathbf{N}(J_\lambda)$ to be closed under composition (hence A_λ is a finite R-algebra). This allows for $\mathsf{Hom}_{\leq\lambda}(J_\lambda, J_\lambda) \big/ \mathsf{Hom}_{<\lambda}(J_\lambda, J_\lambda)$ (which is isomorphic to A_λ) to be an R-algebra with basis $\mathbf{N}(J_\lambda)$, instead having the isomorphism in (1.3.1.7).

However, the folded Hecke category $\mathcal{H}(W)^G$ still does not satisfy this, as the irreducible objects in it have non-trivial morphisms between them that do not factor in lower cells (for instance, the orange dot in the $A_1 \times A_1$ case). In [EW16, Rema. 2.12] it is pointed out that sometimes it is useful to consider cellular categories $\mathcal{C}$ where the objects of Λ correspond to a class of objects of $\mathcal{C}$ with fixed *neutral morphisms* between them. In this context, we require the R-linear span of the neutral morphisms to be closed under composition (hence they form an R-linear subcategory). Elias and Williamson conjectured that the Hecke algebroid, inside of which the Hecke algebra appears as an endomorphism ring, exhibits this behavior. We conjecture that the folded Hecke category $\mathcal{H}(W)^G$ is a fibered NSOACC, whose classes correspond to all objects whose images under the restriction functor to $\mathcal{H}(W)$ are isomorphic. The isomorphism classes of objects under the restriction functor correspond to the G-orbits of indecomposable objects of $\mathcal{H}(W)$, which were partially ordered by the Bruhat order. The resulting cell poset of $\mathcal{H}(W)^G$ is the quotient poset of G-orbits induced from the Bruhat order on W. This is the motivation for the definition of a *fibered non-strict object adapted cellular category*, or *fibered NSOACC*.

In Theorem 4.3.9.2, we prove that the folded Hecke category $\mathcal{H}^\tau$ of type $\mathrm{A}_1 \times \mathrm{A}_1$ is a fibered NSOACC. The poset of classes of objects of $\mathcal{H}^\tau$ is

$$\begin{tikzcd} \{\mathrm{Z}, \mathrm{XZ}\} \arrow[d, dash] \\ \{\mathrm{Y}\} \arrow[d, dash] \\ \{\mathbb{1}, \mathrm{X}\} \end{tikzcd} . \tag{1.3.1.9}$$

The neutral morphisms of $\mathcal{H}^\tau$ are exactly the epic and monic morphisms that are not isomorphisms. They all involve the orange dot . We should point out that the theory of *non-strict object adapted cellular categories* remains underdeveloped, just like when [EW16] was published. Elias and Williamson conjecture that the Hecke algebroid is a fibered NSOACC is still not proven.

1.3.2 Branching Graphs on a Semisimple Category

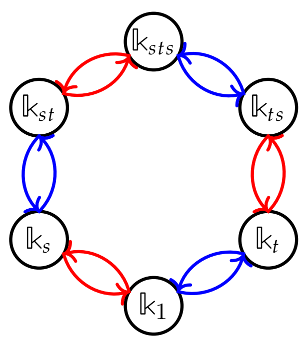
Let $\mathcal{C}$ be a semisimple monoidal category over $\Bbbk$. That is, a $\Bbbk$-linear semisimple monoidal category with finitely many irreducible objects, where the endomorphism ring of its unit object $\mathrm{Hom}(\mathbb{1}, \mathbb{1})$ is isomorphic to $\Bbbk$. Let $\mathcal{I}_{\mathcal{C}} := \{\mathbb{1} = I_1, \ldots, I_r\}$ be a full set of representatives of the isomorphism classes of irreducible objects of $\mathcal{C}$.

Let $\mathcal{A} := \{A_1, \ldots, A_m\}$ be a finite subset of objects of $\mathcal{C}$. We have expressions

$$I_i \otimes A_j \cong \bigoplus_{1 \le k \le k} I_k^{\oplus N_{i,j}^k}. \tag{1.3.2.1}$$

Here, $N_{i,j}^k$ corresponds to the multiplicity of I_k in $I_i \otimes A_j$. Define the *branching graph* $P_{\mathcal{C}}(\mathcal{A})$ of $\mathcal{C}$ with respect to $\mathcal{A}$ to be the directed graph whose nodes are indexed by $\mathcal{I}_{\mathcal{C}}$ and has $N_{i,j}^k$ edges from I_i to I_k colored by A_j. Whenever there is no room for ambiguity, we will denote the branching graph just by $P_{\mathcal{C}}$.

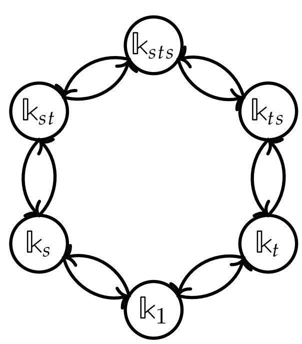
Example 1.3.2.1. For example, let $S_3 \cong W(A_2)$ be the symmetric group on 3 elements and let s_1, s_2 be its simple reflections. Consider the category $\mathbf{Vec}_{S_3}(\Bbbk)$ and the let $S = \{\Bbbk_{s_1}, \Bbbk_{s_2}\}$. The branching graph $P_{\mathbf{Vec}_{S_3}(\Bbbk)}(S)$ with respect to S is the Cayley graph of S_3 with respect to the generators $\{s_1, s_2\}$:



. (1.3.2.2)

Here, the red edges are $\Bbbk_{s_1}$ colored and the blue edges are $\Bbbk_{s_2}$ colored.

Let now $S' = \{\Bbbk_{s_1} \oplus \Bbbk_{s_2}\}$. The branching graph $P_{\mathbf{Vec}_{S_3}(\Bbbk)}(S')$ of $\mathbf{Vec}_{S_3}$ with respect to S' is



. (1.3.2.3)

It looks just like the previous graphs, except all edges are colored by $\Bbbk_{s_1} \oplus \Bbbk_{s_2}$.

Let $\mathcal{A} := \{A_1, \ldots, A_m\}$ as before. Choosing an irreducible summand of A_j is the same as choosing an A_j-colored edge of $P_{\mathcal{C}}(\mathcal{A})$ with source $\mathbb{1}$. Similarly, choosing an irreducible summand of $I_i \otimes A_j$ is the same as choosing an A_j-colored edge of

$P_C(\mathcal{A})$ with source I_j. In general, let A be tensor product of the form

$$A = A_{i_1} \otimes A_{i_2} \otimes \cdots A_{i_l} \tag{1.3.2.4}$$

for $i_1, ..., i_l \in \{1, ..., m\}$. We may choose an irreducible summand of A by first choosing an irreducible summand I_{i_1} of A_{i_1}, then choose an irreducible summand I_{i_2} of $I_{i_1} \otimes A_{i_2}$, then choose an irreducible summand I_{i_3} of $I_{i_2} \otimes A_{i_3}$ and so on. Hence, the irreducible summands of A are in correspondence with walks $p = (e_1, e_2, ..., e_l)$ on $P_C(\mathcal{A})$ satisfying the following conditions:

- $e_1, ..., e_l$ are edges of P_C where the source of e_{i+1} equals the target of e_i.
- the source of e_1 is $\mathbb{1}$. A walk starting at e_1 will be called a *based walk*.
- e_j is an A_{i_j} colored edge for all $1 \leq j \leq l$.

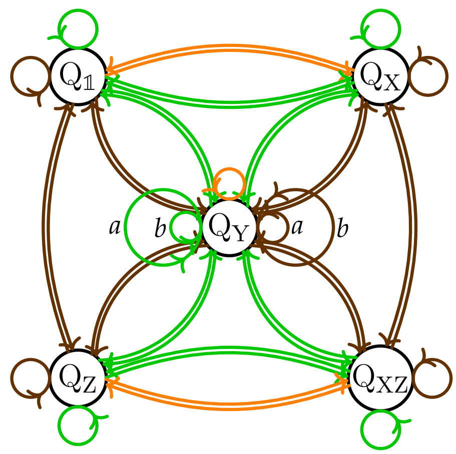
Example 1.3.2.2. Let $S' = \{\mathbb{k}_{s_1} \oplus \mathbb{k}_{s_2}\}$ and $P_{\mathbf{Vec}_{S_3}(\mathbb{k})}(S')$ as in Example 1.3.2.1. We have that

$$(\mathbb{k}_{s_1} \oplus \mathbb{k}_{s_2}) \otimes (\mathbb{k}_{s_1} \oplus \mathbb{k}_{s_2}) = \mathbb{k}_{s_1 s_1} \oplus \mathbb{k}_{s_1 s_2} \oplus \mathbb{k}_{s_2 s_1} \oplus \mathbb{k}_{s_2 s_2} \tag{1.3.2.5}$$

$$= (\mathbb{k}_1)^{\oplus 2} \oplus \mathbb{k}_{s_1 s_2} \oplus \mathbb{k}_{s_2 s_1}\,. \tag{1.3.2.6}$$

The irreducible summands of $(\mathbb{k}_{s_1} \oplus \mathbb{k}_{s_2})^{\otimes 2}$ are in correspondence with based walks of length 2. These are, in turn, in correspondence with words of length 2 on $\{s_1, s_2\}$. More generally, the irreducible summands of $(\mathbb{k}_{s_1} \oplus \mathbb{k}_{s_2})^{\otimes k}$ are in correspondence with based walks of length k on $P_{\mathbf{Vec}_{S_3}(\mathbb{k})}(S')$ and with words of length k on $\{s_1, s_2\}$.

Example 1.3.2.3. Consider $\mathcal{Q}^\tau$, the localized folded Hecke category of type $\mathrm{A}_1 \times \mathrm{A}_1$, with irreducible objects $\mathcal{J} := \{\mathrm{Q}_\mathbb{1}, \mathrm{Q}_\mathrm{X}, \mathrm{Q}_\mathrm{Y}, \mathrm{Q}_\mathrm{Z}, \mathrm{Q}_{\mathrm{XZ}}\}$. The branching graph of $\mathcal{Q}^\tau$ with respect to $\mathcal{A} := \{\mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$ is



. (1.3.2.7)

Here, the orange edges are colored by X, the green edges are colored by Y, and the brown edges are colored by Z. Note that we have double green and brown loops at the node Q_Y.

Let $\mathcal{C}, \mathcal{I}_\mathcal{C}$, and $\mathcal{A}$ as above. Let $P_\mathcal{C}(\mathcal{A})$ be the branching graph of $\mathcal{C}$ with respect to $\mathcal{A}$. The tensor product $I \otimes A$ for $I \in \mathcal{I}_\mathcal{C}, A \in \mathcal{A}$ may not have canonical inclusion and projection morphisms into its irreducible summands. Instead, want to choose inclusions and projections for the irreducible summands of $I \otimes A$ for all $I \in \mathcal{I}_\mathcal{C}, A \in \mathcal{A}$, and label the edges of $P_\mathcal{C}(\mathcal{A})$ with them. The labels of the edges $P_\mathcal{C}(\mathcal{A})$ will be called *basic inclusions* or *projections*. Consider a tensor product $A = A_1 \otimes A_2 \otimes \cdots \otimes A_r$ and a based walk $p = (e_1, \ldots, e_r)$ corresponding to an irreducible summand J of A. We can construct a projection $A \longrightarrow J$ by starting with $\mathrm{id}_\mathbb{1}$ and, at each step e_i, apply $L_i \circ (-) \otimes \mathrm{id}_{A_i}$., where L_i is the projection labeling e_i. This recursive process lets us write inclusions and projections of summands of an arbitrary tensor product. We will explore these ideas further in Section 1.3.3.

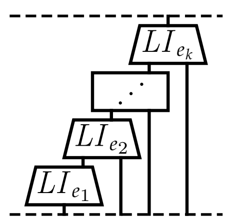
The Hecke category $\mathcal{H}(W)$ is not semisimple, only becomes semisimple after localization. The irreducible objects in the localization are not actually objects in the Hecke category, nor are the projection and inclusion maps morphisms in that category. Instead, the way in which one obtains a cellular basis is to label each edge of the branching graph of $\mathcal{Q}(W)$ with morphisms that lifts the basic projections and inclusions; morphisms which agree with those projections and inclusions after localization, modulo lower terms. This is at the heart of the construction of light leaves and double leaves found in [EW16], which we recall in Section

4.2. This is the technology we imitate for the equivariant category, but with certain necessary modifications, e.g. the addition of neutral maps between different objects in the same cell.

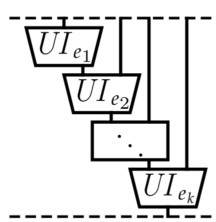
1.3.3 *Double Leaves basis of $\mathcal{H}^\tau$*

A labeling of the edges of $P_C(\mathcal{A})$ by morphisms from $I \otimes A$ to its direct summands will be called an **LI**-*labeling of* $P_C(\mathcal{A})$. The morphisms in an **LI**-labeling will be called *basic lower projections*. A labeling of the edges of $P_C(\mathcal{A})$ by morphisms from the direct summands of $I \otimes A$ to $I \otimes A$ will be called a **UI**-*labeling of* $P_C(\mathcal{A})$. The morphisms in a **UI**-labeling will be called *basic upper inclusions*.

Let $A = A_1 \otimes A_2 \otimes \cdots \otimes A_k$ for $A_1, ..., A_k \in \mathcal{A}$, let $I \in \mathcal{I}_C$, and let $p = (e_1, e_2, ..., e_k)$ be a based walk on $P_C(\mathcal{A})$ ending at I, so that each edge e_i is colored by A_i. In this case, p is said to be an $(A_1, ..., A_k)$-*colored walk*. Fix an **LI**-labeling of $P_C(\mathcal{A})$ and let LI_{e_i} be the label of the edge e_i. Define the morphism $\underline{LI}_p : A \longrightarrow J$ by

$\underline{LI}_p =$  . (1.3.3.1)

Morphisms of the form $\underline{LI}_p$ for some based walk p on $P_C(\mathcal{A})$ will be called *lower projections*. Let $\mathbf{LI}(C)$ be the set of all lower projections. Now fix a **UI**-labeling of $P_C(\mathcal{A})$ and let UI_{e_i} be the label of the edge e_i. We may define $\underline{UI}_p : J \longrightarrow A$ analogously by

$\underline{UI}_p =$  . (1.3.3.2)

Morphisms of the form $\underline{UI}_p$ for some based walk p on $P_C(\mathcal{A})$ will be called *upper inclusions*. Define $\mathbf{UI}(C)$ be the set of all upper inclusions. Finally, define $\mathbf{DI}(C)$ to be the set of morphisms of the form $U \circ L$ for $L \in \mathbf{LI}(C)$ and $U \in \mathbf{UI}(C)$.

Let $A = A_1 \otimes ... \otimes A_k$ for $A_1, ..., A_k \in \mathcal{A}_C$ and let $A' = A'_1 \otimes ... \otimes A'_l$ for $A'_1, ..., A'_l \in \mathcal{A}_C$. Let $T : A \longrightarrow A'$. Define the matrix $[T]$ with respect to $\mathbf{LI}(C)$ and $\mathbf{UI}(C)$ in the following way:

- the columns of $[T]$ are indexed by $(A_1, ..., A_k)$-colored based walks.

- the rows of $[T]$ are indexed by $(A'_1, ..., A'_l)$-colored based walks.

- the entry $[T]_{p,q}$ is the scalar corresponding to the composition

$$\underline{LI}_q \circ T \circ \underline{UI}_p, \tag{1.3.3.3}$$

where p is an $(A_1, ..., A_k)$-colored based walk and q is an $(A'_1, ..., A'_l)$-colored based walk.

Furthermore, if the matrices of the identity morphisms of the objects in $\mathcal{A}$ are invertible, we can use their inverses to find matrix representations of our morphisms that preserve composition.

Also suppose that $\mathcal{A}$ generates all the objects of C under the operations of tensor product and direct sum. Since C is a fusion category, its Hom spaces are finite dimensional. Because of this, if we show that the set $\mathbf{DI}(C)$ is linearly independent over $\mathbb{k}$, then it is a basis for C.

Remark 1.3.3.1. Let R be a commutative ring. Even though we set up the construction of lower projections $\mathbf{LI}(C)$ and upper inclusions $\mathbf{UI}(C)$ for fusion categories, they apply to any semisimple monoidal category whose Hom spaces are all free finitely generated R-modules. In this case, though, it is not enough to check the linear independence of $\mathbf{DI}(C)$ for it to be an R-basis of C.

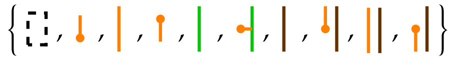
Let us consider the example of $\mathcal{Q}^\tau$. Let P be the branching graph of $\mathcal{Q}^\tau$ with respect to the set $\mathcal{A} := \{\mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$. In Definition 4.3.3.1 we define $\mathbf{LI}(\mathcal{Q}^\tau)$ and $\mathbf{UI}(\mathcal{Q}^\tau)$. These labelings give us a way to write the morphisms of $\mathcal{Q}^\tau$ as matrices with coefficients in Q. In fact, our choice of $\mathbf{LI}$ and $\mathbf{UI}$ labelings lets us write the morphisms of $\mathcal{H}^\tau$ as matrices with coefficients in R.

In $\mathcal{H}^\tau$, the indecomposable objects $\mathbb{1}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}$, and XZ each have a unique top summand in $\mathcal{Q}^\tau$; namely, $\mathrm{Q}_{\mathbb{1}}, \mathrm{Q}_{\mathrm{X}}, \mathrm{Q}_{\mathrm{Y}}, \mathrm{Q}_{\mathrm{Z}}$, and Q_{XZ} respectively. For $J \in \mathcal{J}$ define $\mathcal{H}^\tau(J) = I$, where I is the object of $\mathcal{I}$ whose top summand is J. That is,

$$\mathcal{H}^\tau(\mathrm{Q}_{\mathbb{1}}) = \mathbb{1}, \mathcal{H}^\tau(\mathrm{Q}_{\mathrm{X}}) = \mathrm{X}, \mathcal{H}^\tau(\mathrm{Q}_{\mathrm{Y}}) = \mathrm{Y}, \mathcal{H}^\tau(\mathrm{Q}_{\mathrm{Z}}) = \mathrm{Z}, \text{ and } \mathcal{H}^\tau(\mathrm{Q}_{\mathrm{XZ}}) = \mathrm{XZ}. \tag{1.3.3.4}$$

An $\mathbf{LL}$-*labeling of* P is a labeling of the edges of $P_C(\mathcal{A})$ by morphisms of the form $\mathcal{H}^\tau(J_1) \otimes A \longrightarrow \mathcal{H}^\tau(J_2)$ (instead of morphisms of the form $J_1 \otimes A \longrightarrow J_2$) for $J_1 \in \mathcal{J}, A \in \mathcal{A}$, and where J_2 is a summand of $J_1 \otimes A$ in $\mathcal{Q}^\tau$. The morphisms in an $\mathbf{LL}$-labeling will be called *basic lower leaves*. Similarly, a **UL**-*labeling of* P is a labeling of the edges of $P_C(\mathcal{A})$ by morphisms of the form $\mathcal{H}^\tau(J_2) \longrightarrow \mathcal{H}^\tau(J_1) \otimes A$ for $J_1 \in \mathcal{J}, A \in \mathcal{A}$, and where J_2 is a summand of $J_1 \otimes A$ in $\mathcal{Q}^\tau$. The morphisms in an **UL**-labeling will be called *basic upper leaves*.

Let p be a based walk on P. We can define the *lower leaf* $\underline{LL}_p$ analogously to $\underline{LI}_p$. Let $\mathbf{LL}(\mathcal{H}^\tau)$ be the set of all lower leaves. We may also define the *upper leaf* $\underline{UL}_p$ analogously to $\underline{UI}_p$. Let $\mathbf{UL}(\mathcal{H}^\tau)$ be the set of all upper leaves. The set of compositions of a lower leaf with an upper leaf doesn't give us an R^τ-basis for $\mathcal{H}^\tau$ yet: we need to account for the non-zero morphisms between irreducible objects of $\mathcal{Q}^\tau$ and for the fact that $\mathsf{Hom}(\mathrm{Q_Y}, \mathrm{Q_Y}) \cong R$ is 2-dimensional as a free R^τ-module. Define the set of *neutral morphisms* $\mathbf{N}(\mathcal{H}^\tau)$ by

$$\mathbf{N}(\mathcal{H}^\tau) :=$$

(1.3.3.5)

Define the set of *double leaves* $\mathbf{DL}(\mathcal{H}^\tau)$ to be the set of morphisms of the form $U \circ N \circ L$ for $U \in \mathbf{UL}(\mathcal{H}^\tau), N \in \mathbf{N}(\mathcal{H}^\tau)$, and $L \in \mathbf{LL}(\mathcal{H}^\tau)$.

Define the classes of $\mathcal{J}$ by

$$\mathsf{C}_{\mathbb{1}} := \{\mathrm{Q}_{\mathbb{1}}, \mathrm{Q_X}\}, \mathsf{C_Y} := \{\mathrm{Q_Y}\}, \mathsf{C_Z} := \{\mathrm{Q_Z}, \mathrm{Q_{XZ}}\}. \tag{1.3.3.6}$$

and a (pre)-order $\leq_D$ on $\mathcal{J}$ by

$$\begin{array}{c} \mathsf{C_Z} = \{\mathrm{Q_Z}, \mathrm{Q_{XZ}}\} \\ | \\ \mathsf{C_Y} = \{\mathrm{Q_Y}\} \\ | \\ \mathsf{C}_{\mathbb{1}} = \{\mathrm{Q}_{\mathbb{1}}, \mathrm{Q_X}\} \end{array} . \tag{1.3.3.7}$$

This induces a (pre)-order $\leq_D$ on the set based walks on P called the *dominance (pre)-order* (Definition 4.3.2.3). An important fact about lower/upper leaves is that their matrices are block-triangular with respect to the dominance (pre)-order. This happens because for all $J \in \mathcal{J}$ we have

$$\mathcal{H}^\tau(J) \cong J \oplus \bigoplus_{I <_D J} I^{\oplus N_I}. \tag{1.3.3.8}$$

Therefore, if $I >_D J$ then any morphism $I \longrightarrow \mathcal{H}^\tau(J)$ must be zero.

Our third main result, Theorem 4.3.9.1, shows that the double leaves $\mathbf{DL}(\mathcal{H}^\tau)$ forms an R^τ-basis for all the Hom spaces of $\mathcal{H}^\tau$. We proved the linear independence of our double leaves in the following way:

- We begin by noting that the matrices of our upper/lower leaves are block-triangular with respect to the dominance total order in the paths of P.

- Using our upper projection and lower inclusion morphisms to write the basic upper/lower leaves as R-valued matrices (the list of these matrices is in Appendix B.1).

- We note that the matrices for our basic double leaves with the same source and target are linearly independent *leading vectors*, corresponding to the paths whose class is maximal under $\leq_D$.

- Lastly, we show that the leading vector of a general leaf the tensor product of the leading vectors of its constituent basic leaves, and this tensor product preserves linear independence.

Showing that the double leaves span all of our morphism spaces requires a more involved diagrammatic argument:

- First, we show that a composition of lower/upper leaves is again an R^τ-combination of lower/upper leaves.

- Then, we write idempotent decompositions for the tensor products the objects in $\mathcal{I}$ in terms of our double leaves, and use them to reduce the problem to showing that the morphisms between indecomposable objects are in the R^τ-span of our double leaves.

- We conclude the proof showing, by exhaustion, that the double leaves span $\mathsf{Hom}(A, B)$ for $A, B \in \mathcal{I}$.

Remark 1.3.3.2. For a general W, the composition of lower/upper leaves is not necessarily in the span of lower/upper leaves. This only happens up to double leaves factoring through lower cells. This is done in [EW16, Sec 6.] for the classical case.

Theorem 4.3.9.1 states that $\mathbf{DL}(\mathcal{H}^\tau)$ is an R^τ-basis of $\mathcal{H}^\tau$. Moreover, $\mathbf{DL}(\mathcal{H}^\tau)$ is a cellular basis for $\mathcal{H}^\tau$ that factors into $\mathbf{UL}(\mathcal{H}^\tau), \mathbf{N}(\mathcal{H}^\tau)$, and $\mathbf{LL}(\mathcal{H}^\tau)$. Hence, $\mathcal{H}^\tau$ is a fibered NSOACC.

CHAPTER 2

PRELIMINARIES

2.1 Graded and Pivotal Categories

2.1.1 Graded Monoidal Categories

Definition 2.1.1.1. We say $\mathcal{C}$ is a *($\mathbb{Z}$-)graded monoidal category* if the Hom spaces in $\mathcal{C}$ have a $\mathbb{Z}$-graduation compatible with both the composition of morphisms and the tensor product.

For $X, Y \in \mathcal{C}$ we use the notation $\mathrm{Hom}^i(X,Y)$ for the i-th graded component of $\mathrm{Hom}(X,Y)$, so that

$$\mathrm{Hom}(X,Y) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}^i(X,Y). \tag{2.1.1.1}$$

We assume all our graded monoidal categories $\mathcal{C}$ are *closed under grading shift*. That is to say that $\mathcal{C}$ has a degree 1 graded autoequivalence $[1] : \mathcal{C} \longrightarrow \mathcal{C}$ with a graded pseudo-inverse $[-1] : \mathcal{C} \longrightarrow \mathcal{C}$ of degree -1. We will denote by $[d] : \mathcal{C} \longrightarrow \mathcal{C}$ to the shift functor of degree d. Define

$$\mathrm{Hom}^i(X,Y)[d] := \mathrm{Hom}^{i+d}(X,Y). \tag{2.1.1.2}$$

Note that

$$\mathrm{Hom}(X[k],Y[m]) = \mathrm{Hom}(X,Y)[m-k]. \tag{2.1.1.3}$$

Definition 2.1.1.2. We say that a graded ring $R = \bigoplus_{i\in\mathbb{Z}} R^i$ is a *local graded ring* if any sum of two homogeneous non-units is again a non-unit.

Remark 2.1.1.3. Let $\mathcal{C}$ be a graded monoidal category and let $X \in \mathcal{C}$. Then $\mathrm{End}(X)$ is a graded algebra and its idempotents only live in degree 0. In particular, if $\mathrm{End}(X)$ is non-negatively graded and $\mathrm{End}^0(X)$ is local, we have that $\mathrm{End}(X)$ is a local graded ring and X is indecomposable.

2.1.2 Pivotal Categories and Tensor Adjunction

Definition 2.1.2.1. Let $\mathcal{C}$ be a strict monoidal category and $X \in \mathcal{C}$. A *right tensor adjoint* (or just *right adjoint*) $(X^*, \bigcap_X, \bigcup_X)$ of X is a triple where $X^* \in \mathcal{C}$, $\bigcap_X : X^* \otimes X \longrightarrow \mathbb{1}$, and $\bigcup_X : \mathbb{1} \longrightarrow X \otimes X^*$ such that the compositions

$$X \xrightarrow{\bigcup_X \otimes \mathrm{id}_X} X \otimes X^* \otimes X \xrightarrow{\mathrm{id}_X \otimes \bigcap_X} X \tag{2.1.2.1}$$

and

$$X^* \xrightarrow{\mathrm{id}_{X^*} \otimes \bigcup_X} X^* \otimes X \otimes X^* \xrightarrow{\bigcap_X \otimes \mathrm{id}_{X^*}} X^* \tag{2.1.2.2}$$

are id_X and id_{X^*} respectively. We call $\bigcap_X$ the *evaluation* or *cap* morphism. We call $\bigcup_X$ the *coevaluation* or *cup* morphism.

A *left adjoint* $({}^*X, {}_X\bigcap, {}_X\bigcup)$ of X is defined analogously except ${}_X\bigcap : X \otimes {}^*X \longrightarrow \mathbb{1}$ and ${}_X\bigcup : \mathbb{1} \longrightarrow {}^*X \otimes X$.

A *biadjoint* $(X^*, \bigcap_X, \bigcup_X, {}_X\bigcap, {}_X\bigcup)$ of X is an object such that $(X^*, \bigcap_X, \bigcup_X)$ is right adjoint and $(X^*, {}_X\bigcap, {}_X\bigcup)$ is a left adjoint of X.

We say that X is self-adjoint if is biadjoint to itself for some choice of $\bigcap_X, \bigcup_X, {}_X\bigcap$, and ${}_X\bigcup$. When X is self-adjoint, $\bigcap_X = {}_X\bigcap$ and $\bigcup_X = {}_X\bigcup$, we say that $(X, \bigcap_X, \bigcup_X)$ is biadjoint to X.

The following standard result shows the name 'right/left adjoint' is well deserved.

Proposition 2.1.2.2 ([BK01, Lemma 2.1.6])**.** *Let $X \in \mathcal{C}$. The object $(X^*, \bigcap_X, \bigcup_X)$ is a right adjoint of X if and only if the functor $(-) \otimes X^*$ is a right adjoint of $(-) \otimes X$ with $\bigcup_X$ and $\bigcap_X$ being the unit and counit of adjunction respectively. We have an analogous result for left adjoint objects.*

Remark 2.1.2.3. In [TV+17, Sec. 1-2] and most of the rigid and fusion category literature, right adjoint objects are called *left duals* and left adjoint are called *right duals*.

Definition 2.1.2.4. A *two-sided adjoint structure* on $\mathcal{C}$ is a family

$$\{(X^*, \bigcap_X, \bigcup_X, {}_X\bigcap, {}_X\bigcup)\}_{X \in \mathcal{C}} \tag{2.1.2.3}$$

consisting of a choice of a biadjoint for each $X \in \mathcal{C}$. We call it a *pivotal structure* if, in addition, the natural adjunction functors $(-)^*$ and ${}^*(-)$ are equal (see [TV+17, Sec 1.7]).

By Proposition 2.1.2.2 we have the following result.

Proposition 2.1.2.5. *Let $\mathcal{C}$ be a pivotal category and let $X \in \mathcal{C}$. The functors $(-) \otimes X$ and $(-) \otimes X^*$ are biadjoint. The functors $X \otimes (-)$ and $X^* \otimes (-)$ are also biadjoint.*

The following corollary is a direct consequence of Proposition 2.1.2.5, and is frequently used to calculate the Hom spaces of additive pivotal categories.

Corollary 2.1.2.6. *Let $\mathcal{C}$ be an additive monoidal category with a right (or left) adjunction structure, in which every object is a finite direct sum of indecomposable objects. Then the* Hom *spaces of $\mathcal{C}$ are completely determined by the spaces* $\mathsf{Hom}(X, \mathbb{1})$ *(alternatively* $\mathsf{Hom}(\mathbb{1}, X)$*) for all indecomposable objects $X \in \mathcal{C}$. That is, for any $X, Y \in \mathcal{C}$ the space*

$$\mathsf{Hom}(X, Y) \cong \bigoplus_{i=1}^{k} \mathsf{Hom}(X_i, \mathbb{1}), \tag{2.1.2.4}$$

where $X_1, ..., X_k \in \mathcal{C}$ are the indecomposable summands of $X \otimes Y^*$.

Let $\mathcal{I}$ be a set of objects in $\mathcal{C}$ such that every object of $\mathcal{C}$ is a finite direct sum of the objects in $\mathcal{I}$. Corollary 2.1.2.6 suggests that to check a given strict monoidal additive functor is full or faithful, we only need to check it on $\mathsf{Hom}(X, \mathbb{1})$ for all $X \in \mathcal{I}$. The following proposition shows this.

Proposition 2.1.2.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be strict additive monoidal categories. Let $\mathcal{I}$ be a set of objects in $\mathcal{C}$ such that every object of $\mathcal{C}$ is a finite direct sum of the objects in $\mathcal{I}$. Suppose $\mathcal{C}$ has a right (or left) adjunction structure. Let $\mathbf{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be a strict monoidal functor.*

(a) If the induced map $\mathbf{F} : \mathsf{Hom}(X, \mathbb{1}) \longrightarrow \mathsf{Hom}(\mathbf{F}(X), \mathbb{1})$ is injective for all $X \in \mathcal{I}$ then $\mathbf{F}$ is faithful.

(b) If the induced map $\mathbf{F} : \mathsf{Hom}(X, \mathbb{1}) \longrightarrow \mathsf{Hom}(\mathbf{F}(X), \mathbb{1})$ is surjective for all $X \in \mathcal{I}$ then $\mathbf{F}$ is full.

(c) If the induced map $\mathbf{F} : \mathsf{Hom}(X, \mathbb{1}) \longrightarrow \mathsf{Hom}(\mathbf{F}(X), \mathbb{1})$ is bijective for all $X \in \mathcal{I}$ and $\mathbf{F}$ is essentially surjective, then $\mathbf{F}$ is an equivalence.

Proof. Clearly (a) and (b) imply (c). We will only show (a), as the proof of (b) is analogous. Note that if $(X^*, \bigcap_X, \bigcup_X)$ is a right adjoint of X then $(\mathbf{F}(X^*), \mathbf{F}(\bigcap_X), \mathbf{F}(\bigcup_X))$ is a right adjoint of $\mathbf{F}(X)$. This induces a right adjoint structure on $\mathbf{F}(\mathcal{C})$, where $\mathbf{F}(X^*) = \mathbf{F}(X)^*$. Under this identification, for all $X, Y \in \mathcal{C}$ the diagram

$$\begin{tikzcd} \mathsf{Hom}(X, Y) \arrow[r, "\mathbf{F}"] \arrow[d, "\cong"] & \mathsf{Hom}(\mathbf{F}(X), \mathbf{F}(Y)) \arrow[d, "\cong"] \\ \mathsf{Hom}(X \otimes Y^*, \mathbb{1}) \arrow[r, "\mathbf{F}"] & \mathsf{Hom}(\mathbf{F}(X) \otimes \mathbf{F}(Y)^*, \mathbb{1}) \end{tikzcd} \tag{2.1.2.5}$$

commutes. Here, the vertical isomorphisms are given by the right adjoint structures. This shows that the top morphism is injective if and only if the bottom morphism is injective. Writing

$$X \otimes Y^* \cong X_1 \oplus \ldots \oplus X_k \tag{2.1.2.6}$$

for some $X_i \in \mathcal{I}$ we reduce the problem to the assumption that $\mathbf{F}$ is injective on each $\mathsf{Hom}(X_i, \mathbb{1})$. □

2.2 Equivariantization

2.2.1 Action of a Group on a Monoidal Category

Definition 2.2.1.1. Let C be a category and G a group. An *action of* G *on* C is a triple $(G, \mathbf{F}, \gamma, \epsilon)$, where:

- $\mathbf{F}$ is a collection of functors $\mathbf{F}_g : C \longrightarrow C$ for $g \in G$.
- $\gamma_{g,h} : \mathbf{F}_{gh} \longrightarrow \mathbf{F}_g \circ \mathbf{F}_h$ is collection of natural isomorphisms of functors for all $g, h \in G$ called the associator of the action.
- $\eta : \mathbf{F}_1 \longrightarrow \mathrm{Id}_C$ is an isomorphism of functors called the unit of the action.

This collection has to satisfy the axioms in [BN13, Sec. 2.1]. In this case we say that C is a *G-category*.

If both the associator and unit are the identity morphism, we say that the G-action is *strict*.

If C is a strict monoidal category, we say that the action of G is *strict monoidal* if it is strict and all the functors $\mathbf{F}_g$ are strict monoidal functors.

If C is additive, we say that the action of G is *additive* if all the functors $\mathbf{F}_g$ are additive.

If C is a $\mathbb{k}$-linear category for some field $\mathbb{k}$, we say that the action of G is *$\mathbb{k}$-linear* if all the functors $\mathbf{F}_g$ are $\mathbb{k}$-linear.

If C is a strict graded monoidal, we say that the action of G is *graded monoidal* if it is strict and all the functors $\mathbf{F}_g$ are strict graded monoidal functors.

We will assume that when C is additive, $\mathbb{k}$-linear, or graded, so is the group action.

In this article we will suppose all our categories are strict monoidal categories and all actions are strict group actions by strict functors, meaning that $\mathbf{F}_g$

is a strict monoidal functor for all $g \in G$. These conditions simplify our previous definition into the following.

Proposition 2.2.1.2. *Let $\mathcal{C}$ be a strict monoidal category and G a group. A strict monoidal action of G on $\mathcal{C}$ is equivalent to a pair $(G, \mathbf{F})$, where $F_g : \mathcal{C} \longrightarrow \mathcal{C}$ is a strict monoidal functor for all $g \in G$ satisfying $\mathbf{F}_1 = \mathrm{Id}_{\mathcal{C}}$ and $\mathbf{F}_g \circ \mathbf{F}_h = \mathbf{F}_{gh}$.*

Note that in Proposition 2.2.1.2, we have an equality of functors (and not just an isomorphism). This happens because all of our functors are strict.

Notation 2.2.1.3. Let $g \in G$, $X, Y \in \mathcal{C}$ and a morphism $T : X \longrightarrow Y$. We will write $g(X)$ and $g(T)$ as a shorthand for $\mathbf{F}_g(X)$ and $\mathbf{F}_g(T)$ respectively.

Remark 2.2.1.4. An action can be defined in a more general way as a monoidal functor

$$\rho : \underline{G} \longrightarrow \mathsf{Aut}_{\otimes}(\mathcal{C}). \tag{2.2.1.1}$$

Here $\underline{G}$ is the monoidal category whose objects are the set G, with only identity morphisms, and $\otimes$ is given by the product in G. In this context, the action is strict if and only if ρ is a strict monoidal functor (see [Eti+16, Section 2.7]).

2.2.2 *Definition of Equivariantization*

From now on let G be a group and fix a field $\Bbbk$ such that $\mathsf{char}(\Bbbk) \nmid |G|$. Let $\mathcal{C}$ be a strict monoidal $\Bbbk$-linear category with a strict G-action by strict $\Bbbk$-linear functors.

Definition 2.2.2.1. A *G-equivariant object* of $\mathcal{C}$ is a pair (X, f) where $X \in \mathcal{C}$ and $f = (f_g)_{g \in G}$ is a collection of isomorphisms $f_g : g(X) \longrightarrow X$ where $f_1 = \mathrm{Id}_X$ and such that the diagram

$$\begin{tikzcd} gh(X) \arrow[r, "g(f_h)"] \arrow[d, "f_{gh}"'] & g(X) \arrow[dl, "f_g"] \\ X & \end{tikzcd} \tag{2.2.2.1}$$

commutes for all $g, h \in G$.

A *G-equivariant morphism* $T : (X, f) \longrightarrow (Y, f')$ is a morphism $T : X \longrightarrow Y$ that intertwines f and f', i.e. a morphism such that the diagram

$$\begin{tikzcd} g(X) \arrow[r, "g(T)"] \arrow[d, "f_g"] & g(Y) \arrow[d, "f'_g"] \\ X \arrow[r, "T"] & Y \end{tikzcd} \tag{2.2.2.2}$$

commutes for all $g \in G$.

The *equivariantization* $\mathcal{C}^G$ is the category whose objects are G-equivariant objects and whose morphisms are G-equivariant morphisms.

Proposition 2.2.2.2. *Let (X, f) and (Y, f') be equivariant objects in $\mathcal{C}$. The tensor product*

$$(X, f) \otimes (Y, f') := (X \otimes Y, f \otimes f') \tag{2.2.2.3}$$

makes $\mathcal{C}^G$ into a strict monoidal category.

Proof. This is a direct consequence of the functoriality of $\otimes$. $\square$

Remark 2.2.2.3. For the definition of equivariantization in the general (non-strict) setting see [Eti+16, Sec. 2.7].

As the next example shows, an object $X \in \mathcal{C}$ may be equipped with multiple non-isomorphic equivariant structures.

Example 2.2.2.4. Let $\Bbbk$ be a field. Let us consider the case where our category is $\mathbf{Vec}(\Bbbk)$, the category of finite dimensional vector spaces over $\Bbbk$, and G acts trivially on $\mathbf{Vec}(\Bbbk)$. The equivariantization $\mathbf{Vec}(\Bbbk)^G$ is isomorphic to $\mathbf{Rep}(G)$, the category of finite dimensional representations of G. In fact, we can see the object $(V, f) \in \mathbf{Vec}(\Bbbk)^G$ as the representation where $g(v) = f_g(v)$ (for $v \in V$ and $g \in G$). As such, we can see G-equivariant categories as a generalization of the category $\mathbf{Rep}(G)$.

Definition 2.2.2.5. Let $\mathcal{C}$ be an additive category $\mathcal{C}$. We say $\mathcal{C}$ is *Karoubian* if for every object $X \in \mathcal{C}$ and every idempotent $f \in \mathsf{Hom}(X, X)$ there is an object Y and maps $\iota_Y : Y \longrightarrow X$ and $p_Y : X \longrightarrow Y$ such that $f = \iota_Y \circ p_Y$ and $p_Y \circ \iota_Y = \mathrm{id}_Y$. That is, $\mathcal{C}$ is closed under direct summands.

In [Dri+10, Sec. 4.1.3], it is noted without a proof that the equivariantization of a Karoubian category will again be Karoubian. Here we provide a proof.

Proposition 2.2.2.6. *Let $\Bbbk$ be a field. Let $\mathcal{C}$ be a Karoubian $\Bbbk$-linear category with an additive G-action. The category $\mathcal{C}^G$ is Karoubian.*

Proof. Suppose that $\mathcal{C}$ is Karoubian. Let $(X, f) \in \mathcal{C}^G$ and let $T : X \longrightarrow X$ be an idempotent. Since $\mathcal{C}$ is Karoubian, there exists a $Y \in \mathcal{C}$, $\iota_Y : Y \longrightarrow X$ and $p_Y : X \longrightarrow Y$

such that $T = \iota_Y \circ p_Y$ and $\mathrm{Id}_Y = p_Y \circ \iota_Y$. For each $g \in G$ define $f'_g : g(Y) \longrightarrow Y$ by the composition

$$g(Y) \xrightarrow{g(\iota_Y)} g(X) \xrightarrow{f_g} X \xrightarrow{p_Y} Y. \tag{2.2.2.4}$$

We want to show that (Y, f') is a G-equivariant object and that $\iota_Y : (Y, f') \longrightarrow (X, f)$ and $p_Y : (X, f) \longrightarrow (Y, f')$ are G-equivariant morphisms. For the first statement, check that the diagram

$$\begin{tikzcd}
gh(Y) \arrow[r, "gh(\iota_Y)"] & gh(X) \arrow[r, "g(f_h)"] \arrow[ddrr, "f_{gh}"'] & g(X) \arrow[r, "g(p_Y)"] \arrow[dr, "\mathrm{Id}_{g(X)}"] & g(Y) \arrow[d, "g(\iota_Y)"] \\
 & & & g(X) \arrow[d, "f_g"] \\
 & & & X \arrow[d, "p_Y"] \\
 & & & Y
\end{tikzcd} \tag{2.2.2.5}$$

commutes.

To check that ι_Y is a G-equivariant morphism, observe that

$$\begin{tikzcd}
g(Y) \arrow[r, "g(\iota_Y)"] \arrow[dr, "g(\iota_Y)"'] & g(X) \arrow[r, "f_g"] \arrow[d, "\mathrm{Id}_{g(X)}"] & X \arrow[r, "p_Y"] \arrow[d, "\mathrm{Id}_X"] & Y \arrow[dl, "\iota_Y"] \\
 & g(X) \arrow[r, "f_g"] & X &
\end{tikzcd} \tag{2.2.2.6}$$

commutes. The diagram for p_Y is similar. □

2.2.3 *Induction and Restriction Functors*

Since G-equivariant categories generalize the category of finite dimensional representations of a group, we may expect induction and restriction functors like in the usual case. We will base our approach on [Dri+10, Sec. 4]. We will discuss these functors because they provide a way to understand the indecomposable objects of the equivariant category from the indecomposable objects of the original category.

Definition 2.2.3.1. Define the *induction functor* $\mathsf{Ind} : \mathcal{C} \longrightarrow \mathcal{C}^G$ as follows:

- We will map an object $X \in \mathcal{C}$ to $X \longmapsto (\bigoplus_{h \in G} h(X), \sigma_g)$, where σ_g just permutes the components of $\bigoplus_{h \in G} h(X)$ by multiplication by g.

- We will map a morphism $T : X \longmapsto Y$ to $\bigoplus_{h\in G} h(T)$.

Define the *restriction functor* $\mathsf{Res} : \mathcal{C}^G \longrightarrow \mathcal{C}$ to be the functor sending $(X, f) \mapsto X$ and sending $T : (X, f) \longrightarrow (Y, f')$ to $T : X \longrightarrow Y$ (in $\mathcal{C}$).

Just like in $\mathbf{Rep}(G)$, we will show that the induction and restriction functors are biadjoint. Let $X \in \mathcal{C}$ and $(Y, f') \in \mathcal{C}^G$. Define the map

$$\Phi_{X,(Y,f')} : \mathsf{Hom}(X, \mathsf{Res}(Y, f')) = \mathsf{Hom}(X, Y) \longrightarrow \mathsf{Hom}(\mathsf{Ind}(X), (Y, f')) \tag{2.2.3.1}$$

by

$$\Phi_{X,(Y,f')} : \varphi \mapsto \bigoplus_{g\in G} f'_g \circ g(\varphi). \tag{2.2.3.2}$$

Also define the map

$$\Psi_{X,(Y,f')} : \mathsf{Hom}(\mathsf{Ind}(X), (Y, f')) \longrightarrow \mathsf{Hom}(X, \mathsf{Res}(Y, f')) = \mathsf{Hom}(X, Y) \tag{2.2.3.3}$$

by the restriction

$$\Psi_{X,(Y,f')} : \psi = \left(\bigoplus_{g\in G} \psi_g\right) \mapsto \psi_1. \tag{2.2.3.4}$$

Proposition 2.2.3.2. *Let $\phi : X \longrightarrow Y$. The map $\Phi_{X,(Y,f')}(\phi)$ is G-equivariant. Therefore, $\Phi_{X,(Y,f')}$ is a well defined map.*

Proof. To check that the map $\Phi(\varphi)$ is G-equivariant, we can follow the component $gh(X)$ of $\mathsf{Ind}(X)$ for $g, h \in G$ on the diagram (2.2.2.2). That is the same as checking that the diagram

$$\begin{tikzcd} gh(X) \arrow[r, "gh(\varphi)"] \arrow[d, "\mathrm{Id}"] & gh(Y) \arrow[r, "g(f'_h)"] \arrow[dr, "f'_{gh}"] & g(Y) \arrow[d, "f'_g"] \\ gh(X) \arrow[rr, "f'_{gh}\circ gh(\varphi)"] & & Y \end{tikzcd} \tag{2.2.3.5}$$

commutes. The square clearly commutes and the triangle commutes by equation (2.2.2.1). ◻

Proposition 2.2.3.3. *The maps $\Phi_{X,(Y,f')}$ and $\Psi_{X,(Y,f')}$ are functorial and inverse to each other. Therefore, the functor $\mathsf{Ind} : \mathcal{C} \longrightarrow \mathcal{C}^G$ is left adjoint to $\mathsf{Res} : \mathcal{C}^G \longrightarrow \mathcal{C}$.*

Proof. Let $X \in \mathcal{C}$, and $(Y, f), (Y', f') \in \mathcal{C}^G$. Let $T : (Y, f) \longrightarrow (Y', f')$, $\varphi : X \longrightarrow Y$, and $\psi : \mathsf{Ind}(X) \longrightarrow (Y, f)$. The functoriality of most components is straightforward. We will show the functoriality of $\Phi_{X,(Y,f)}$ on the (Y, f) component and leave the others to the reader.

Using (2.2.2.2),

$$\Phi_{X,(Y',f')}(T \circ \varphi) = \bigoplus_{g \in G} f'_g \circ g(T \circ \varphi) \tag{2.2.3.6}$$

$$= \bigoplus_{g \in G} f'_g \circ g(T) \circ g(\varphi) \tag{2.2.3.7}$$

$$= \bigoplus_{g \in G} T \circ f_g \circ g(\varphi). \tag{2.2.3.8}$$

The following computations show that Φ and Ψ are mutually inverse.

$$\Psi_{X,(X,f)} \circ \Phi_{X,(Y,f)}(\varphi) = \Psi_{X,(X,f)}\left(\bigoplus_{g \in G} f_g \circ g(\varphi)\right) \tag{2.2.3.9}$$

$$= f_1 \circ 1(\varphi) \tag{2.2.3.10}$$

$$= \varphi. \tag{2.2.3.11}$$

$$\Phi_{X,(Y,f)} \circ \Psi_{X,(Y,f)}(\psi) = \bigoplus_{g \in G} f_g \circ g(\psi)_1. \tag{2.2.3.12}$$

Using (2.2.2.2), we obtain

$$\Phi_{X,(Y,f)} \circ \Psi_{X,(Y,f)}(\psi) = \bigoplus_{g \in G} \psi_g = \psi. \tag{2.2.3.13}$$

□

Now suppose that G is finite. The induction and restriction functors on **Rep**(G) are bi-adjoint. We will show the same for Ind and Res. Define

$$\Phi'_{(X,f),Y} : \mathsf{Hom}(\mathsf{Res}(X, f), Y) = \mathsf{Hom}(X, Y) \longrightarrow \mathsf{Hom}((X, f), \mathsf{Ind}(Y)) \tag{2.2.3.14}$$

by

$$\Phi'_{(X,f),Y} : \varphi \mapsto \begin{pmatrix} 1(\varphi \circ f_1) \\ \vdots \\ g(\varphi \circ f_{g^{-1}}) \end{pmatrix}_{g \in G}. \tag{2.2.3.15}$$

Above and elsewhere, we will describe the maps to and from $\mathsf{Ind}(X)$ by matrices indexed by G.

Also define

$$\Psi'_{(X,f),Y} : \mathsf{Hom}((X,f), \mathsf{Ind}(Y)) \longrightarrow \mathsf{Hom}(\mathsf{Res}(X,f), Y) = \mathsf{Hom}(X,Y) \tag{2.2.3.16}$$

by

$$\Psi'_{(X,f),Y} : \psi \mapsto \pi_1 \circ \psi, \tag{2.2.3.17}$$

where $\pi_1 : \bigoplus_{g\in G} g(Y) \longrightarrow Y$ is the projection to the identity component. Note that the definition of Φ' requires G to be finite. Otherwise, it might have infinitely many non-zero components.

Proposition 2.2.3.4. *The maps $\Phi'_{(X,f),Y}$ and $\Psi'_{(X,f),Y}$ are functorial and mutually inverse. Therefore, when G is finite, the functor $\mathsf{Ind} : \mathcal{C} \longrightarrow \mathcal{C}^G$ is right adjoint to $\mathsf{Res} : \mathcal{C}^G \longrightarrow \mathcal{C}$.*

The proof of this proposition is similar to the one of Proposition 2.2.3.3.

Now, for every $(X,f) \in \mathcal{C}^G$ define the morphisms $p_{(X,f)} : \mathsf{Ind}(X) \longrightarrow (X,f)$ given by

$$p_{(X,f)} := \begin{pmatrix} f_1 & \dots & f_g \end{pmatrix}_{g\in G} \tag{2.2.3.18}$$

and $\iota_{(X,f)} : (X,f) \longrightarrow \mathsf{Ind}(X)$ given by

$$\iota_{(X,f)} := \begin{pmatrix} 1(f_1) \\ \vdots \\ g(f_{g^{-1}}) \end{pmatrix}_{g\in G} . \tag{2.2.3.19}$$

Proposition 2.2.3.5. *The morphisms $p_{(X,f)} : \mathsf{Ind}(X) \longrightarrow (X,f)$ and $\iota_{(X,f)} : \mathsf{Ind}(X) \longrightarrow (X,f)$ are G-equivariant and satisfy $p_{(X,f)} \circ \iota_{(X,f)} = |G|\,\mathrm{id}_X$.*

Proof. The map $p_{(X,f)} = \Phi_{X,(X,f)}(\mathrm{id}_X)$ is the counit of the adjunction from Proposition 2.2.3.3. Similarly, $\iota_{(X,f)} = \Phi'_{(X,f),X}(\mathrm{id}_{(X,f)})$ is the unit of the adjunction from Proposition 2.2.3.4. Alternatively, we can show $p_{(X,f)}$ is equivariant by following the component $gh(X)$ of $\mathsf{Ind}(X)$ in equation (2.2.2.2). That is, checking that the diagram

$$\begin{tikzcd} gh(X) \arrow[r, "g(f_h)"] \arrow[d, "\mathrm{id}"] & g(X) \arrow[d, "f_g"] \\ gh(X) \arrow[r, "f_{gh}"] & X \end{tikzcd} \tag{2.2.3.20}$$

commutes. But that is just equation (2.2.2.1). The check for $\iota_{(X,f)}$ is similar.

Also note that

$$g(f_{g^{-1}}) = f_g^{-1} \tag{2.2.3.21}$$

by equation (2.2.2.2). Hence

$$p_{(X,f)} \circ \iota_{(X,f)} = \sum_{g \in G} f_g \circ g(f_{g^{-1}}) \tag{2.2.3.22}$$

$$= \sum_{g \in G} f_g \circ f_g^{-1} \tag{2.2.3.23}$$

$$= |G| \operatorname{id}_X . \tag{2.2.3.24}$$

□

The following corollary relies on the fact that $\mathsf{char}(\Bbbk) \nmid |G|$.

Corollary 2.2.3.6. *The map $\iota_{(X,f)} \circ p_{(X,f)}$ is a pseudo-idempotent. Moreover, in this case (X, f) is a direct summand of $\mathsf{Ind}(X)$ with inclusion $\iota_{(X,f)}$ and projection $\frac{1}{|G|} p_{(X,f)}$.*

Proof. By the previous proposition,

$$(\iota_{(X,f)} \circ p_{(X,f)})^2 = |G| \iota_{(X,f)} \circ p_{(X,f)}. \tag{2.2.3.25}$$

The scalar $|G|$ is invertible if and only if the characteristic of $\Bbbk$ does not divide the order of G. □

2.2.4 Unique Decomposition in $\mathcal{C}^G$

Definition 2.2.4.1. Let $\mathcal{C}$ be an additive category and let $X \in \mathcal{C}$. The object X is said to have *unique decomposition* if it satisfies the following properties:

- X has a finite direct sum decomposition

$$X \cong X_1 \oplus X_2 \oplus \ldots \oplus X_k \tag{2.2.4.1}$$

for some indecomposable objects $X_1, \ldots, X_k \in \mathcal{C}$.

- If

$$X \cong Y_1 \oplus Y_2 \oplus \ldots \oplus Y_m \tag{2.2.4.2}$$

is another decomposition of X into indecomposable objects, then $m = k$ and $Y_i \cong X_{\sigma(i)}$ for some permutation $\sigma \in S_k$.

The category C is said to have the *unique decomposition property* if all of its objects have unique decomposition. Alternatively we may say that C *has unique decomposition*.

One of our goals is to show that all the categories we care about, $\mathcal{H}(\mathrm{A}_1)$, $\mathcal{H}(\mathrm{A}_1 \times \mathrm{A}_1)$, and $\mathcal{H}(\mathrm{A}_1 \times \mathrm{A}_1)^\tau$, have unique decomposition. The unique decomposition property for these categories will come from the general theory for Krull-Schmidt categories. This will require us to have a description of all their indecomposable objects and show they have (graded) local endomorphism rings. The next result will allow us to get a full list of indecomposable objects for the equivariantization in all the cases we care about.

Proposition 2.2.4.2. *Let* $(Y, f) \in \mathcal{H}$*. Suppose that both* Y *and* $\mathsf{Ind}(Y)$ *have unique decomposition. Let*

$$Y \cong Y_1 \oplus Y_2 \oplus \ldots \oplus Y_k \tag{2.2.4.3}$$

be the unique decomposition of Y *into indecomposables. Then* (Y, f) *is a summand of* $\mathsf{Ind}(Y_i)$ *for some* i*. Hence, listing all the summands of* $\mathsf{Ind}(X)$ *for all indecomposable* $X \in C$ *will give us a full list of the indecomposable objects of* C^G*.*

Proof. Let (Y, f) be an indecomposable object of C^G. By Corollary 2.2.3.6, (Y, f) is a summand of $\mathsf{Ind}(Y)$. If Y is indecomposable, $k = 1$ and we are done. Otherwise, $\mathsf{Ind}(Y) \cong \mathsf{Ind}(Y_1) \oplus \ldots \oplus \mathsf{Ind}(Y_k)$. Since $\mathsf{Ind}(Y)$ has unique decomposition, (Y, f') has to be a summand of $\mathsf{Ind}(Y_i)$ for some i. □

Definition 2.2.4.3. Let C be an additive category and let $X \in C$. We say X is a has a *local decomposition* if it there exists a direct sum decomposition

$$X \cong X_1 \oplus X_2 \oplus \ldots \oplus X_k \tag{2.2.4.4}$$

for some indecomposable objects $X_1, \ldots, X_k \in C$ with local endomorphism rings. If case C is a graded category, X has a *graded local decomposition* if the endomorphism rings of the X_i's are graded local.

We say C is a *Krull-Schmidt category* (or has the *Krull-Schmidt* property) if all of its objects have local decompositions. A graded category C is a *graded Krull-Schmidt category* if all of its objects have graded local decompositions.

It is important to note that all the results here apply both in the graded and non-graded cases. The following is a standard result about Krull-Schmidt categories.

Lemma 2.2.4.4 ([Eli+20, Thm. 11.50]). *Let $\mathcal{C}$ be a Karoubian category and suppose that $X \in \mathcal{C}$ admits a local decomposition. Then X has unique decomposition. Hence, any Krull-Schmidt category has unique decomposition.*

To show that an additive (graded) category $\mathcal{C}$ is Krull-Schmidt, one usually finds a set P of objects of $\mathcal{C}$ such that:

- All the objects in $\mathcal{P}$ admit a local decomposition.
- All objects in $\mathcal{C}$ are a summand of an object of the form $P_1 \oplus P_2 \oplus ... \oplus P_k$ for some $P_1, ..., P_k \in P$ (or are a grading shift of an object in P). In other words,

$$\mathsf{Kar}(\mathsf{add}(A)) = \mathcal{C}\,. \tag{2.2.4.5}$$

This is enough because a direct summand of an object with a local decomposition also has a local decomposition. In the case of $\mathcal{C}^G$ this yields the following result.

Proposition 2.2.4.5. *Suppose $\mathcal{C}$ is a (graded) category with unique decomposition and for every indecomposable object $X \in \mathcal{C}$ the object $\mathsf{Ind}(X)$ has a (graded) local decomposition. Then $\mathcal{C}^G$ is a Krull-Schmidt category.*

Proof. Let $(Y, f) \in \mathcal{C}^G$. By Corollary 2.2.3.6, (Y, f) is a summand of $\mathsf{Ind}(Y)$. By unique decomposition, there exist indecomposable objects $X_1, ..., X_k$ such that $Y \cong X_1 \oplus ... \oplus X_k$. Hence $\mathsf{Ind}(Y) \cong \mathsf{Ind}(X_1) \oplus ... \oplus \mathsf{Ind}(X_k)$ has a (graded) local decomposition. By Lemma 2.2.4.4, $\mathsf{Ind}(Y)$ has unique decomposition. Since (Y, f) is a summand of $\mathsf{Ind}(Y)$, it also has a local decomposition. □

2.2.5 Actions of $\mathbb{Z}/2$ on a Monoidal Category

One very important special case is when the group $G = \mathbb{Z}/2$. Let τ be the generator of G. From here on, we will suppose that $\mathsf{char}(\Bbbk) \neq 2$.

A strict action of $\mathbb{Z}/2$ on a monoidal strict category $\mathcal{C}$ is the same as a strict monoidal functor $\mathbf{F}_\tau$ such that $\mathbf{F}_\tau^2 = \mathrm{Id}_{\mathcal{C}}$. Using Notation 2.2.1.3 τ will represent both the functor and the group element. In this case, an equivariant object (X, f) only depends on f_τ, as $f_1 = \mathrm{id}_X$. We will denote our equivariant objects by (X, f_τ), listing the morphism f_τ in the second coordinate. For an equivariant object (X, f_τ), equation (2.2.3.21) shows that we need

$$\tau(f_\tau) = f_\tau^{-1}. \tag{2.2.5.1}$$

Let (X, f_τ) and (X', f'_τ) be two equivariant objects. In this notation,

$$(X, f_\tau) \otimes (X', f'_\tau) = (X \otimes X', f_\tau \otimes f'_\tau). \tag{2.2.5.2}$$

Example 2.2.5.1. Consider the unit object $\mathbb{1} \in \mathcal{C}$. Since τ fixes $\mathbb{1}$, we have that both $(\mathbb{1}, \mathrm{id}_\mathbb{1})$ and $(\mathbb{1}, -\mathrm{id}_\mathbb{1})$ are equivariant objects. We will note these objects by $(\mathbb{1}, 1)$ and $(\mathbb{1}, -1)$ respectively. Let $R := \mathsf{End}(\mathbb{1})$. By our strictness assumption, $\tau(\mathbb{1}) = \mathbb{1}$. Thus, functor τ acts on R in a natural way. Define

$$R^\tau := \{r \in R | \tau(r) = r\} \tag{2.2.5.3}$$

and

$$R^{-\tau} := \{r \in R | \tau(r) = -r\}. \tag{2.2.5.4}$$

Using equation (2.2.2.2) it is easy to see that

$$\mathsf{Hom}((\mathbb{1}, 1), (\mathbb{1}, 1)) = R^\tau, \tag{2.2.5.5}$$

the set of τ-invariant elements of R. Similarly,

$$\mathsf{Hom}((\mathbb{1}, -1), (\mathbb{1}, -1)) = R^\tau, \tag{2.2.5.6}$$

and

$$\mathsf{Hom}((\mathbb{1}, 1), (\mathbb{1}, -1)) = R^{-\tau}. \tag{2.2.5.7}$$

Since $\mathsf{Hom}((\mathbb{1}, 1), (\mathbb{1}, 1)) \ncong \mathsf{Hom}((\mathbb{1}, 1), (\mathbb{1}, -1))$ we can conclude that $(\mathbb{1}, 1)$ and $(\mathbb{1}, -1)$ are not isomorphic. Nevertheless, by the monoidal unit, $(\mathbb{1}, -1) \otimes (\mathbb{1}, -1) = (\mathbb{1} \otimes \mathbb{1}, \mathrm{id}_\mathbb{1} \otimes \mathrm{id}_\mathbb{1})$ is isomorphic to $(\mathbb{1}, 1)$.

More generally, for any equivariant object (X, f_τ), the object $(X, -f_\tau)$ is also equivariant and is isomorphic to $(\mathbb{1}, -1) \otimes (X, f_\tau) = (\mathbb{1} \otimes X, -\mathrm{id}_\mathbb{1} \otimes f_\tau)$.

Example 2.2.5.2. Now let $X \in \mathcal{C}$ and $\mathsf{Ind}(X) = \left(X \oplus \tau(X), \begin{pmatrix} 0 & \mathrm{id}_X \\ \mathrm{id}_{\tau(X)} & 0 \end{pmatrix}\right)$. In contrast with Example 2.2.5.1, we claim that

$$\mathsf{Ind}(X) \cong (\mathbb{1}, -1) \otimes \mathsf{Ind}(X) = \left(X \oplus \tau(X), -\begin{pmatrix} 0 & \mathrm{id}_X \\ \mathrm{id}_{\tau(X)} & 0 \end{pmatrix}\right). \tag{2.2.5.8}$$

Let

$$T: \ (\mathbb{1}, -1) \otimes \mathsf{Ind}(X) \xrightarrow{\begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & -\mathrm{id}_{\tau(X)} \end{pmatrix}} \mathsf{Ind}(X)\,. \tag{2.2.5.9}$$

It is easy to see that $T^2 = \begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_{\tau(X)} \end{pmatrix}$. To check that T is a τ-equivariant morphism, observe that the diagram

$$\begin{tikzcd}
\tau(X \oplus \tau(X)) \arrow[r, phantom, "="] & \tau(X) \oplus X \arrow[r, "\begin{pmatrix} \mathrm{id}_{\tau(X)} & 0 \\ 0 & -\mathrm{id}_X \end{pmatrix}"] \arrow[d, "-\begin{pmatrix} 0 & \mathrm{id}_X \\ \mathrm{id}_{\tau(X)} & 0 \end{pmatrix}"'] & \tau(X) \oplus X \arrow[r, phantom, "="] \arrow[d, "\begin{pmatrix} 0 & \mathrm{id}_X \\ \mathrm{id}_{\tau(X)} & 0 \end{pmatrix}"] & \tau(X \oplus \tau(X)) \\
& X \oplus \tau(X) \arrow[r, "\begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & -\mathrm{id}_{\tau(X)} \end{pmatrix}"'] & X \oplus \tau(X) &
\end{tikzcd} \tag{2.2.5.10}$$

commutes.

Let C be a graded monoidal $\Bbbk$-linear category with unique decomposition for an algebraically closed field $\Bbbk$. Let $X \in C$ be an indecomposable object such that $\mathsf{End}^0(X) = \Bbbk\,\mathrm{id}_X$. Let G_X to the stabilizer of X, that is,

$$G_X := \{g \in G | g(X) \cong X\}. \tag{2.2.5.11}$$

Since $G = \mathbb{Z}/2$, G_X is either trivial or G. We have two cases for decomposing $\mathsf{Ind}(X)$ into indecomposables.

- If $G_X = 1$ then $\mathsf{Ind}(X) = \left(X \oplus \tau(X), \begin{pmatrix} 0 & \mathrm{id}_X \\ \mathrm{id}_{\tau(X)} & 0 \end{pmatrix}\right)$ is indecomposable. That is because, by unique decomposition, X and $\tau(X)$ are the only summands of $\mathsf{Res}(\mathsf{Ind}(X))$ and they can not have an equivariant structure.

- If $G_X = G$, let $T : \tau(X) \longrightarrow X$ be an isomorphism. Then $T \circ \tau(T) = a\,\mathrm{id}_X$ for some $a \in \Bbbk^*$. The field $\Bbbk$ is algebraically closed, so there is $b \in \Bbbk^*$ such that $b^2 = a$. Let $f_\tau = b^{-1}T$. Since $(b^{-1}T) \circ (\tau(b^{-1}T)) = \mathrm{id}_X$, the object (X, f_τ) equivariant. Furthermore, we have a decomposition $\mathsf{Ind}(X) \cong (X, f_\tau) \oplus (X, -f_\tau)$ with splittings

$$\begin{tikzcd}
(X, f_\tau) \arrow[r, bend left, "\begin{pmatrix} 1 \\ \tau(f_\tau) \end{pmatrix}"] & \mathsf{Ind}(X) \arrow[l, bend left, "\frac{1}{2}\begin{pmatrix} 1 & f_\tau \end{pmatrix}"] \arrow[r, bend left, "\begin{pmatrix} 1 \\ -\tau(f_\tau) \end{pmatrix}"] & (X, -f_\tau) \arrow[l, bend left, "\frac{1}{2}\begin{pmatrix} 1 & -f_\tau \end{pmatrix}"]
\end{tikzcd} \tag{2.2.5.12}$$

(as in Corollary 2.2.3.6).

Remark 2.2.5.3. Let C be a *fusion category*, a $\Bbbk$-linear semisimple monoidal category with finitely many irreducible objects. Let G be a group with a (not necessarily strict) group action on $\mathcal{C}$. For each $X \in \mathcal{C}$, let $\mathbf{Stab}_G(X) := \{g \in G | g(X) \cong X\}$. In [BN13, Prop. 2.6] it is shown that the irreducible objects of $\mathcal{C}^G$ are in correspondence with pairs (X, V), where $X \in \mathcal{C}$ is irreducible and V is a projective representation of $\mathbf{Stab}_G(X)$ (twisted by a cocycle induced by the associator $\gamma_{g,h}$). Elias extends this result to a *mixed graded category* $\mathcal{C}$ [Eli17, Prop 3.10], by showing that the indecomposable objects of $\mathcal{C}^G$ are in correspondence with pairs (X, V), where $X \in \mathcal{C}$ an indecomposable object and V is a projective representation of $\mathbf{Stab}_G(X)$.

2.3 Diagrammatic Hecke Category of type $A_1 \times A_1$.

2.3.1 Diagrammatic Hecke Category of type A_1

Definition 2.3.1.1. Let $R := \Bbbk[\alpha_s]$. A one color *(m, n)-diagram* consists of the following.

(a) A graph Γ embedded on the planar strip $[0, 1] \times [0, 1]$ with m strands on its lower boundary and n strands on its upper boundary, made out of trivalent vertices () and univalent vertices (also called dots) ().

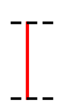
(b) Floating polynomial boxes in the regions of $[0, 1] \times [0, 1]$ partitioned by Γ, labeled by polynomials $f \in R$.

We say two (n, m)-diagrams are *isotopic* if the graph embeddings are isotopic relative to the boundary (both upper and lower) and we have the we have the same polynomial boxes in each region of $[0, 1] \times [0, 1]$.

Definition 2.3.1.2. We define the *diagrammatic Hecke Category of type A_1*, $\mathcal{H}(\mathrm{A}_1)$, to be the graded R-linear monoidal category presented in the following way. The objects of $\mathcal{H}(\mathrm{A}_1)$ are generated by B_s and its grading shifts, under the operations of direct sum and tensor product.

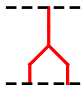
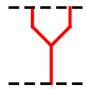
The morphisms $B_s^{\otimes m} \longrightarrow B_s^{\otimes n}$ are R-linear combinations of (m, n)-diagrams. The morphisms of $\mathcal{H}(\mathrm{A}_1)$ are generated monoidally by

(a) The identity of B_s

 $$: B_s \longrightarrow B_s, \tag{2.3.1.1}$$

of degree 0.

(b) The trivalent vertices

 $: B_s \otimes B_s \longrightarrow B_s[-1]$ and  $: B_s[1] \longrightarrow B_s \otimes B_s,$ (2.3.1.2)

of degree -1.

(c) The dots

 $: B_s \longrightarrow \mathbb{1}[1]$ and  $: \mathbb{1} \longrightarrow B_s[1],$ (2.3.1.3)

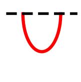
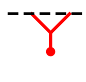
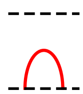
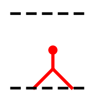
of degree 1.

(d) The polynomial boxes

$$\boxed{f} : \mathbb{1} \longrightarrow \mathbb{1}, \tag{2.3.1.4}$$

of degree $\deg(f)$ (where $\deg(\alpha_s) = 2$).

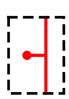
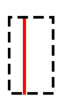
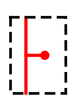
We define the cup and cap morphisms by

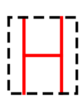
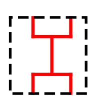
 $=$  and  $=$  respectively. (2.3.1.5)

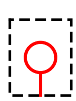
Define an action of s on R by $s(\alpha) = -\alpha$. Also define the *Demazure operator* $\partial_s : R \longrightarrow R$ by

$$\partial_s(f) = \frac{f - s(f)}{\alpha_s}. \tag{2.3.1.6}$$

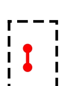
These morphisms are subject to the isotopy relations and subject to the following local relations.

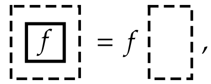
 $=$  $=$ , (2.3.1.7)

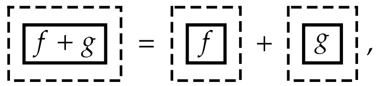
 $=$ , (2.3.1.8)

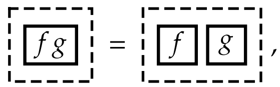
 $= 0.$ (2.3.1.9)

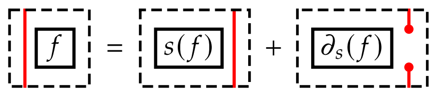
A diagram with the shape of the one in equation 2.3.1.9 will be called a *needle*.

 $= \boxed{\alpha_s}$, (2.3.1.10)

 (2.3.1.11)

 (2.3.1.12)

 (2.3.1.13)

 (2.3.1.14)

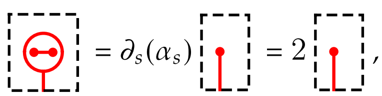
Here, the dashed boxes on the boundary indicate that this relation takes place within a neighborhood homeomorphic to D^2.

The tensor product of morphisms is given by horizontal concatenation of diagrams (resizing them to fit into $[0,1]\times[0,1]$), while the composition is given by vertical concatenation (of morphisms with matching boundaries).

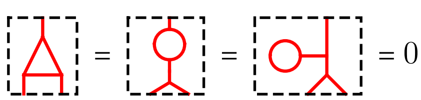
Here are some important consequences of the relations in $\mathcal{H}(\mathrm{A}_1)$:

(a) The object $(B_s, \cap, \cup)$ is biadjoint to B_s.

(b) An empty cycle is a diagram consisting exclusively of a cycle formed by trivalent vertices, with no diagrams inside its central region. While

 (2.3.1.15)

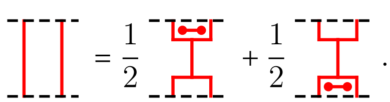
any empty cycle is zero. For example, the 3-cycle

 (2.3.1.16)

In general, we may use (2.3.1.8) repeatedly until we get an empty needle.

(c) The relation 2.3.1.14 shows us how to move polynomials to the left. We may rotate this relation in order to move polynomials to the right instead.

(d) One idempotent decomposition for $B_s^{\otimes 2}$ is

 (2.3.1.17)

Since every idempotent in this decomposition factors through B_s, then

$$B_s \otimes B_s \cong B_s[1] \oplus B_s[-1]. \tag{2.3.1.18}$$

Note that this requires $\mathsf{char}(\Bbbk) \neq 2$.

(e) Using the previous idempotent decomposition every object $X \in \mathcal{H}(\mathrm{A}_1)$ can be written as

$$X \cong \bigoplus_{i \in \mathbb{N}} \left(B_s^{\oplus a_i} \oplus \mathbb{1}^{\oplus b_i} \right)[i], \tag{2.3.1.19}$$

where $a_i, b_i \in \mathbb{N}$ and only finitely many of them are non-zero.

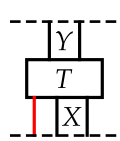
This makes B_s and $\mathbb{1}$ (and their grading shifts) the only indecomposable objects. In particular, this shows that $\mathcal{H}(\mathrm{A}_1)$ is Karoubian.

Remark 2.3.1.3. For a general Coxeter group, one typically defines the diagrammatic category of Bott-Samelson bimodules as in Definition 2.3.1.2 and takes its Karoubi envelope to get the category of Soergel Bimodules (our true object of study). However, by enumerating their indecomposable objects we obtain that $\mathcal{H}(\mathrm{A}_1)$ already Karoubian (as will be $\mathcal{H}(\mathrm{A}_1 \times \mathrm{A}_1)$). For the general definition of the Diagrammatic Hecke Category see [Eli+20, Ch. 10].

The next proposition is a consequence of Proposition 2.1.2.5, but we will prove it diagrammatically so that the reader becomes more familiar with the graphical calculus of $\mathcal{H}(\mathrm{A}_1)$.

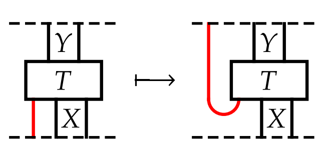
Proposition 2.3.1.4. *The functor $B_s \otimes (-)$ is self adjoint, i.e. for all $X, Y \in \mathcal{H}(\mathrm{A}_1)$ we have a natural isomorphism $\mathsf{Hom}(B_s \otimes X, Y) \cong \mathsf{Hom}(X, B_s \otimes Y)$. Moreover, the functor $B_s[k] \otimes (-)$ is bi-adjoint to $B_s[-k] \otimes (-)$. Hence, $\mathcal{H}(\mathrm{A}_1)$ is a rigid category.*

Proof. To describe this isomorphism, let



(2.3.1.20)

represent a map from $T : B_s \otimes X \longrightarrow Y$. We can pre-compose T with the cup morphism  $\otimes \operatorname{Id}_X$ and get the mapping



. (2.3.1.21)

Its inverse consists of applying the corresponding cap morphism on top.

Since all the objects of $\mathcal{H}(\mathrm{A}_1)$ are finite direct sums of grading shifts of $\mathbb{1}$ and B_s, we get that $\mathcal{H}(\mathrm{A}_1)$ is rigid. $\square$

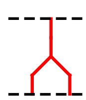
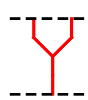
Definition 2.3.1.5. Define the rotation functor $(-)^\vee : \mathcal{H}(\mathrm{A}_1) \longrightarrow \mathcal{H}(\mathrm{A}_1)^{\mathrm{op},\otimes\mathrm{op}}$ to be the strict additive contravariant and tensor contravariant functor such that:

- $(-)^\vee$ sends the objects $B_s[k] \mapsto B_s[-k]$, $\mathbb{1}[k] \mapsto \mathbb{1}[-k]$ and extends linearly to all objects of $\mathcal{H}(\mathrm{A}_1)$.
- Sends a degree d morphism $T : X \longrightarrow Y$ to a degree $-d$ morphism $T^\vee : Y^\vee \mapsto X^\vee$ given by rotating T by $180°$.

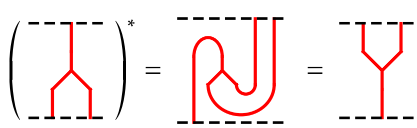
Let $\bigcap_{B_s^{\otimes k}} : B_s^{\otimes k} \otimes B_s^{\otimes k} \longrightarrow \mathbb{1}$ and $\bigcup_{B_s^{\otimes k}} : \mathbb{1} \longrightarrow B_s^{\otimes k} \otimes B_s^{\otimes k}$ be the morphism corresponding to nesting k caps or cups respectively. Given that $(B_s, \cap, \cup)$ is biadjoint to B_s, we can easily show that $\left(B_s^{\otimes k}, \bigcap_{B_s^{\otimes k}}, \bigcup_{B_s^{\otimes k}}\right)$ is a biadjoint to $B_s^{\otimes k}$. The next proposition shows that this, in fact, determines a pivotal structure on $\mathcal{C}$.

Proposition 2.3.1.6. *With the two-sided adjunction structures above we have that the functors $(-)^* = {}^*(-) = (-)^\vee$. Hence, $\mathcal{H}(\mathrm{A}_1)$ is pivotal.*

Proof. It suffices to verify this equality on the generating morphisms

 ,  ,  , and  . (2.3.1.22)

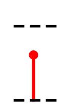
For the first generator,

 (2.3.1.23)

The rest of the calculations are just as straightforward. $\square$

In [EK+10], Elias and Khovanov used diagrammatic arguments to calculate the Hom spaces of $\mathcal{H}(W)$ for any Coxeter group W of type A. In particular, we have the following proposition.

Proposition 2.3.1.7 ([EK+10, Coro. 4.26])**.** *We have that*

$$\mathrm{Hom}(\mathbb{1}, \mathbb{1}) = R \text{ and } \mathrm{Hom}(B_s, \mathbb{1}) = R \cdot$$  . (2.3.1.24)

Both of them are free graded left R-modules with graded dimension

$$\mathrm{grdim}_R(\mathrm{Hom}(\mathbb{1},\mathbb{1})) = 1 \text{ and } \mathrm{grdim}_R(\mathrm{Hom}(B_s,\mathbb{1})) = v. \tag{2.3.1.25}$$

We will now use this to calculate the graded dimension of the endomorphism ring of B_s. This is an example on how to calculate Hom spaces in $\mathcal{H}(\mathrm{A}_1)$.

Corollary 2.3.1.8. *We have that*

$$\mathrm{grdim}_R(\mathrm{End}(B_s)) = 1 + v^2. \tag{2.3.1.26}$$

Proof. Using the adjunction in Proposition 2.3.1.4, we get

$$\begin{aligned} \mathrm{Hom}(B_s, B_s) &\cong \mathrm{Hom}(B_s \otimes B_s, \mathbb{1}) && (2.3.1.27)\\ &\cong \mathrm{Hom}(B_s[-1],\mathbb{1}) \oplus \mathrm{Hom}(B_s[1],\mathbb{1}) && (2.3.1.28)\\ &\cong \mathrm{Hom}(B_s,\mathbb{1})[1] \oplus \mathrm{Hom}(B_s,\mathbb{1})[-1]. && (2.3.1.29) \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{grdim}_R(\mathrm{End}(B_s)) &= \mathrm{grdim}_R(B_s[1]) + \mathrm{grdim}_R(B_s[-1]) && (2.3.1.30)\\ &= \mathrm{grdim}_R(B_s)v + \mathrm{grdim}_R(B_s)v^{-1} && (2.3.1.31)\\ &= 1 + v^2. && (2.3.1.32) \end{aligned}$$

□

Corollary 2.3.1.9. *The category $\mathcal{H}(\mathrm{A}_1)$ is a graded Krull-Schmidt category.*

Proof. The ring $\mathrm{End}(\mathbb{1}) = R$ is non-negatively graded with $\mathrm{End}^0(\mathbb{1}) = \Bbbk$, thus local. By Corollary 2.3.1.8, $\mathrm{End}(B_s) \cong R \oplus R[2]$ as a graded left R-module. Since $\mathrm{End}(B_s)$ is non-negatively graded and $\mathrm{End}^0(B_s) = \Bbbk$, it also is graded local. □

Corollary 2.3.1.10. *All the* Hom *spaces of $\mathcal{H}(\mathrm{A}_1)$ are free graded left R-modules.*

Proof. By Corollary 2.1.2.6, all Hom spaces are finite direct sums of grading shifts of $\mathrm{Hom}(\mathbb{1},\mathbb{1})$ and $\mathrm{Hom}(B_s,\mathbb{1})$. By Proposition 2.3.1.7, these are all free left R-modules. □

2.3.2 *Diagrammatic Hecke Category of type $A_1 \times A_1$*

The Diagrammatic Hecke Category of type $A_1 \times A_1$ is defined analogously to $\mathcal{H}(\mathrm{A}_1)$, but adding a generating object we will call B_t. Let us define the diagrams in this category.

Definition 2.3.2.1. Let $w_1 = (i_1, i_2, \ldots, i_m)$ and $w_2 = (j_1, j_2, \ldots, j_n)$ be words in letters s and t. Let $R := \Bbbk[\alpha_s, \alpha_t]$. A (w_1, w_2)-*Hecke diagram* (of type $A_1 \times A_1$) consists of the following.

(a) A graph Γ embedded on the planar strip $[0,1] \times (0,1)$ whose edges are labeled by s (red) or t (blue), with the following properties:

  (i) Γ has m strands on the lower boundary, labeled with $w_1 = (i_1, \ldots, i_m)$ (from left to right).

  (ii) Γ has n strands on the upper boundary, labeled with $w_2 = (j_1, \ldots, j_n)$ (from left to right).

  (iii) Γ is made of one-colored trivalent vertices ( , ) , one-colored dots ( , ) , and two color crossings ( , ).

(b) Floating polynomial boxes in the regions of $[0,1] \times (0,1)$ partitioned by Γ, labeled by polynomials $f \in R$.

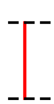
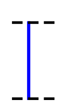
We say two (w_1, w_2)-diagrams are *isotopic* if the graph embeddings are isotopic (relative to their boundary) and we have the we have the same polynomial boxes in each region of $[0,1] \times (0,1)$.

Definition 2.3.2.2. We define the *diagrammatic Hecke category* of type $A_1 \times A_1$, $\mathcal{H}(\mathrm{A}_1 \times \mathrm{A}_1)$ (abbreviated into $\mathcal{H}$), to be the graded R-linear monoidal category presented in the following way.

The objects of $\mathcal{H}$ are generated by B_s, B_t, and their grading shifts, under the operations of direct sum and tensor product.

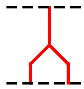
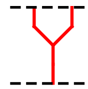
The generating morphisms are:

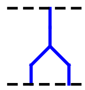
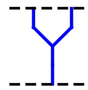
(a) The identity of B_s and B_t

 $: B_s \longrightarrow B_s,$ and  $: B_t \longrightarrow B_t$ (2.3.2.1)

and have degree 0.

(b) The trivalent vertices in red and blue

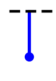
 $: B_s \otimes B_s \longrightarrow B_s[-1],$  $: B_s[1] \longrightarrow B_s \otimes B_s,$ (2.3.2.2)

 $: B_t \otimes B_t \longrightarrow B_t,$ and  $: B_t \longrightarrow B_t \otimes B_t,$ (2.3.2.3)

of degree -1.

(c) The dots

 $: B_s \longrightarrow \mathbb{1}[1],$  $: \mathbb{1} \longrightarrow B_s[1],$ (2.3.2.4)

 $: B_t \longrightarrow \mathbb{1}[1],$ and  $: \mathbb{1} \longrightarrow B_t[1],$ (2.3.2.5)

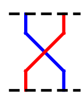
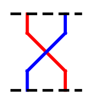
of degree 1.

(d) The polynomial boxes

$$\boxed{f} : \mathbb{1} \longrightarrow \mathbb{1}, \tag{2.3.2.6}$$

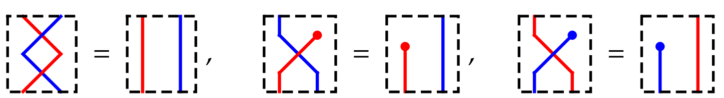
of degree $\deg(f)$ (where $\deg(\alpha_s) = 2$ and $\deg(\alpha_t) = 2$).

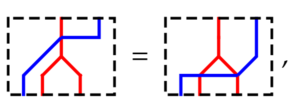
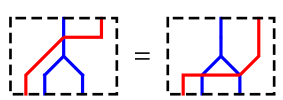
(e) The 4-valent vertices

 , and  (2.3.2.7)

of degree 0.

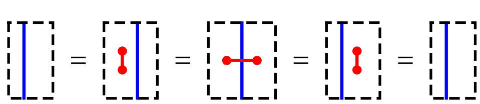
These morphisms are subject to isotopy relations, the local relations (2.3.1.7)-(2.3.1.14) in red and blue colors, and the 2-color relations

 (2.3.2.8)

 , and  . (2.3.2.9)

Define the set $\mathcal{I} = \{\mathbb{1}, B_s, B_t, B_sB_t\}$.

Note that by (2.3.2.8) and (2.3.1.10),

α_s  α_s. (2.3.2.10)

Therefore, any polynomial on α_s slides over the blue strands and any polynomial on α_t slides over the red strands.

As before, we have $B_s^2 \cong B_s[1] \oplus B_s[-1]$ and $B_t^2 \cong B_t[1] \oplus B_t[-1]$. Recall that $\mathcal{I} = \{\mathbb{1}, B_s, B_t, B_sB_t\}$. Since $B_sB_t \cong B_tB_s$, we have that

$$\begin{aligned} B_sB_t \otimes B_s &\cong B_tB_s \otimes B_s \cong B_sB_t[-1] \oplus B_sB_t[1], && (2.3.2.11)\\ B_sB_t \otimes B_t &\cong B_sB_t[-1] \oplus B_sB_t[1], && (2.3.2.12)\\ B_sB_t \otimes B_sB_t &\cong B_s \otimes B_tB_s \otimes B_t && (2.3.2.13)\\ &\cong B_s \otimes B_sB_t \otimes B_t && (2.3.2.14)\\ &\cong B_sB_t[-2] \oplus B_sB_t^{\oplus 2} \oplus B_sB_t[2]. && (2.3.2.15) \end{aligned}$$

Hence, the tensor products of any two objects in $\mathcal{I}$ decomposes again as direct sums of objects in $\mathcal{I}$. Therefore we have the following proposition.

Proposition 2.3.2.3. *Any object $X \in \mathcal{H}$ is a finite direct sum of the objects in $\mathcal{I}$ and their grading shifts.*

Like in the $\mathcal{H}(\mathrm{A}_1)$ case, $(B_s, \cap, \cup)$ is biadjoint to B_s and $(B_t, \cap, \cup)$ is biadjoint to B_t. By Proposition 2.1.2.5, we conclude that

Proposition 2.3.2.4. *The functors $B_s[k] \otimes (-)$ and $B_t[k] \otimes (-)$ are bi-adjoint to the functors $B_s[-k] \otimes (-)$ and $B_t[-k] \otimes (-)$.*

Definition 2.3.2.5. Define the rotation functor $(-)^\vee : \mathcal{H} \longrightarrow \mathcal{H}^{\mathrm{op},\otimes\mathrm{op}}$ to be the strict additive contravariant and tensor contravariant functor such that

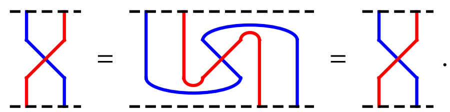
- Sends $\mathbb{1}[k] \mapsto \mathbb{1}[-k]$, $B_s[k] \mapsto B_s[-k]$, and $B_t[k] \mapsto B_t[-k]$.

- Sends a degree d morphism $T : X \longrightarrow Y$ to a degree $-d$ morphism $T^\vee : Y^\vee \mapsto X^\vee$ given by rotating T by $180°$.

By nesting cups/caps, we can extend the biadjoints of B_s and B_t to a two-sided adjoint structure on $\mathcal{H}$.

Proposition 2.3.2.6. *Let us fix the two-sided adjoint structure defined above. The functors $(-)^* = {}^*(-) = (-)^\vee$. Hence, $\mathcal{H}$ is pivotal.*

Proof. In the proof of Proposition 2.3.1.6, we showed that these functors agree on the one-color generators. For the 2-color generators, notice that



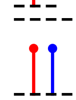
(2.3.2.16)

□

As before, we need the diagrammatic argument in [EK+10] to get a description of the Hom spaces in $\mathcal{H}$.

Proposition 2.3.2.7 ([EK+10, Coro. 4.26])**.** *We have that*

$$\mathrm{Hom}(\mathbb{1}, \mathbb{1}) = R, \qquad \mathrm{Hom}(B_s, \mathbb{1}) = R \cdot$$  , (2.3.2.17)

$$\mathrm{Hom}(B_t, \mathbb{1}) = R \cdot$$  , $$\mathrm{Hom}(B_s B_t, \mathbb{1}) = R \cdot$$  . (2.3.2.18)

All of them are free graded left R-modules with graded dimension

$$\mathrm{grdim}_R(\mathrm{Hom}(\mathbb{1}, \mathbb{1})) = 1, \tag{2.3.2.19}$$

$$\mathrm{grdim}_R(\mathrm{Hom}(B_s, \mathbb{1})) = v, \tag{2.3.2.20}$$

$$\mathrm{grdim}_R(\mathrm{Hom}(B_t, \mathbb{1})) = v, \tag{2.3.2.21}$$

$$\mathrm{grdim}_R(\mathrm{Hom}(B_s B_t, \mathbb{1})) = v^2. \tag{2.3.2.22}$$

We can check like in Corollary 2.3.1.8, that the endomorphism rings of $\mathbb{1}, B_s, B_t$, and $B_s B_t$ are graded local. Thus we have the following corollary.

Corollary 2.3.2.8. *The category $\mathcal{H}(\mathrm{A}_1 \times \mathrm{A}_1)$ is a Krull-Schmidt category. The set $\mathcal{I} = \{\mathbb{1}, B_s, B_t, B_s B_t\}$ is a full set of isomorphism classes of indecomposable of $\mathcal{H}$ (up to grading shift).*

Like Corollary 2.3.1.10, the following is a consequence of Corollary 2.1.2.6 and Proposition 2.3.2.7.

Corollary 2.3.2.9. *All the* Hom *spaces of $\mathcal{H}$ are free graded left R-modules.*

CHAPTER 3

THE FOLDED FOLDED HECKE CATEGORY OF TYPE $A_1 \times A_1$

3.1 The Folded Hecke Category $\mathcal{H}^\tau$

3.1.1 Equivariantization of $\mathcal{H}(A_1 \times A_1)$ with respect to τ.

Definition 3.1.1.1. Let $R = \Bbbk[\alpha_s, \alpha_t]$. Define $\tau_R : R \longrightarrow R$ to be the algebra morphism permuting α_s and α_t. It is clear that τ_R has order 2.

Definition 3.1.1.2. Define the functor $\tau : \mathcal{H} \longrightarrow \mathcal{H}$ to be the strict monoidal functor that:

(a) sends $B_s \mapsto B_t$ and $B_t \mapsto B_s$.

(b) τ acts on diagrams by exchanging the two colors and acts on polynomials by τ_R.

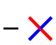
Proposition 3.1.1.3. *The functor τ is its own inverse and it defines a strict action of $\mathbb{Z}/2$ on $\mathcal{H}$.*

Proof. This is clear. $\square$

Given the functor τ, consider the equivariantization $\mathcal{H}^\tau$. As seen in Section 2.2.5, an object in $\mathcal{H}^\tau$ is an object $A \in \mathcal{H}$ together with a morphism $f : \tau(A) \longrightarrow A$ where $f \circ \tau(f) = \mathrm{Id}_A$. We will start by decomposing $\mathsf{Ind}(\mathbb{1}), \mathsf{Ind}(B_s) = \mathsf{Ind}(B_t)$, and $\mathsf{Ind}(B_sB_t)$ into indecomposables. Consider the equivariant structures $(\mathbb{1}, 1)$ and $(B_sB_t,$  $)$. From the computation in (2.2.5.12), we get

$$\mathsf{Ind}(\mathbb{1}) \cong (\mathbb{1}, 1) \oplus (\mathbb{1}, -1) \tag{3.1.1.1}$$

and

$\mathsf{Ind}(B_sB_t) \cong (B_sB_t,$  $) \oplus (B_sB_t, -$  $)\,.$ (3.1.1.2)

Since B_s and B_t, are not τ-invariant, the object

$$\mathsf{Ind}(B_s) = \mathsf{Ind}(B_t) = \left(B_s \oplus B_t, \begin{pmatrix} 0 & | \\ | & 0 \end{pmatrix} : B_t \oplus B_s \longrightarrow B_s \oplus B_t\right) \tag{3.1.1.3}$$

is indecomposable.

From now on we will use the following notation:

$$\mathbb{1} := (\mathbb{1}, 1), \tag{3.1.1.4}$$

$$\mathrm{X} := (\mathbb{1}, -1), \tag{3.1.1.5}$$

$$\mathrm{Y} := \left(B_s \oplus B_t, \begin{pmatrix} 0 & \textcolor{red}{|} \\ \textcolor{blue}{|} & 0 \end{pmatrix}\right), \tag{3.1.1.6}$$

$$\mathrm{Z} := (B_s B_t, \times), \tag{3.1.1.7}$$

$$\mathrm{XZ} := (B_s B_t, -\times). \tag{3.1.1.8}$$

This makes sense since $\mathrm{X} \otimes \mathrm{Z} \cong \mathrm{XZ}$. Also define

$$\mathcal{I} := \{\mathbb{1}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}, \mathrm{XZ}\}. \tag{3.1.1.9}$$

Lemma 3.1.1.4. *The objects $\mathsf{Ind}(\mathbb{1}), \mathsf{Ind}(B_s)$, and $\mathsf{Ind}(B_s B_t)$ have a graded local decomposition, and therefore have unique decomposition. Moreover, $\mathsf{Ind}(A)$ has a graded local decomposition for all $A \in \mathcal{H}$.*

Proof. In Proposition 3.1.3.3 we will show that the endomorphism rings of objects in $\mathcal{I}$ are graded local. The object $\mathsf{Ind}(B_s) = Y$ is already indecomposable and has a graded local endomorphism ring. In (3.1.1.1) and (3.1.1.2) we showed graded local decompositions for $\mathsf{Ind}(\mathbb{1})$ and $\mathsf{Ind}(B_s B_t)$. Since $\mathcal{H}$ is a graded Krull-Schmidt category, $\mathsf{Ind}(A)$ is a direct sum of copies of $\mathsf{Ind}(\mathbb{1}), \mathsf{Ind}(B_s)$, and $\mathsf{Ind}(B_s B_t)$ (and their grading shifts). Hence $\mathsf{Ind}(A)$ also has a graded local decomposition. □

Proposition 3.1.1.5. *The set $\mathcal{I} = \{\mathbb{1}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}, \mathrm{XZ}\}$ and its grading shifts make a full set of indecomposable objects of $\mathcal{H}^\tau$.*

Proof. By Proposition 2.2.4.2, every indecomposable object of $\mathcal{H}^\tau$ is a summand of $\mathsf{Ind}(\mathbb{1}), \mathsf{Ind}(B_s)$, or $\mathsf{Ind}(B_s B_t)$. By Lemma 3.1.1.4, these objects have unique decomposition, so they can not have any summands not in $\mathcal{I}$. □

Corollary 3.1.1.6. *Every object in $\mathcal{H}^\tau$ is a direct sum of objects in $\mathcal{I}$ (and their grading shifts). Hence, the category $\mathcal{H}^\tau$ is a graded Krull-Schmidt category.*

Proof. Let $(A, f) \in \mathcal{H}^\tau$. By Lemma 3.1.1.4, $\mathsf{Ind}(A)$ has a graded local decomposition. By Corollary 2.2.3.6, (A, f) is a direct summand of $\mathsf{Ind}(A)$ so it also has a local decomposition. □

3.1.2 The Grothendieck Ring of $\mathcal{H}^\tau$

Our next goal is to describe the Grothendieck ring of $\mathcal{H}^\tau$. We will do this by writing each tensor products of two indecomposable objects in $\mathcal{H}^\tau$ as a direct sum of indecomposables.

We will write all the morphisms involved in the following notation.

Notation 3.1.2.1. The tensor products of $\mathbb{1}, B_s \oplus B_t$, and $B_s B_t$ can be decomposed naively into a direct sum of objects of the form $B_{i_1} \otimes \ldots \otimes B_{i_m}$ for $i_j \in \{s, t\}$ and $m \in \mathbb{N}$. For example,

$$(B_s \oplus B_t) \otimes (B_s \oplus B_t) = B_s \otimes B_s \oplus B_s \otimes B_t \oplus B_t \otimes B_s \oplus B_t \otimes B_t. \tag{3.1.2.1}$$

By associating the object $B_{i_1} \otimes \ldots \otimes B_{i_m}$ with the tuple $(i_1, \ldots, i_m)$ we can order summands with the same number of tensor factors lexicographically (with $s > t$). This order allows us to represent morphisms between direct sums of objects of this form by matrices of morphisms between the summands. For instance, a morphism $\phi : Y \otimes Y \longrightarrow Y \otimes Y$ can be represented by a matrix of the form

$$\begin{pmatrix} B_s \otimes B_s \to B_s \otimes B_s & B_s \otimes B_t \to B_s \otimes B_s & B_t \otimes B_s \to B_s \otimes B_s & B_t \otimes B_t \to B_s \otimes B_s \\ B_s \otimes B_s \to B_t \otimes B_s & B_s \otimes B_t \to B_t \otimes B_s & B_t \otimes B_s \to B_t \otimes B_s & B_t \otimes B_t \to B_t \otimes B_s \\ B_s \otimes B_s \to B_s \otimes B_t & B_s \otimes B_t \to B_s \otimes B_t & B_t \otimes B_s \to B_s \otimes B_t & B_t \otimes B_t \to B_s \otimes B_t \\ B_s \otimes B_s \to B_t \otimes B_t & B_s \otimes B_t \to B_t \otimes B_t & B_t \otimes B_s \to B_t \otimes B_t & B_t \otimes B_t \to B_t \otimes B_t \end{pmatrix}. \tag{3.1.2.2}$$

To simplify our notation further, we will only write the non-zero entries of our matrices.

Example 3.1.2.2. We will start by calculating the decomposition of $(B_s \oplus B_t) \otimes (B_s \oplus B_t)$ inside $\mathcal{H}$.

$$(B_s \oplus B_t) \otimes (B_s \oplus B_t) = (B_s^{\otimes 2} \oplus B_t^{\otimes 2}) \oplus (B_s B_t \oplus B_t B_s) \tag{3.1.2.3}$$

$$= (B_s[-1] \oplus B_s[1] \oplus B_t[-1] \oplus B_t[1]) \oplus (B_s B_t \oplus B_t B_s) \tag{3.1.2.4}$$

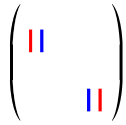
$$= (B_s \oplus B_t)[-1] \oplus (B_s \oplus B_t)[1] \oplus (B_s B_t \oplus B_t B_s). \tag{3.1.2.5}$$

This suggests that $Y \otimes Y \cong Y[-1] \oplus Y[+1] \oplus \mathsf{Ind}(B_s B_t)$.

Let us consider the morphism

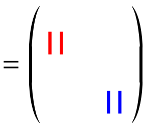
$$\mathrm{id}_Y = \begin{pmatrix} || & & & \\ & || & & \\ & & || & \\ & & & || \end{pmatrix}. \tag{3.1.2.6}$$

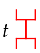
Observe that the restriction

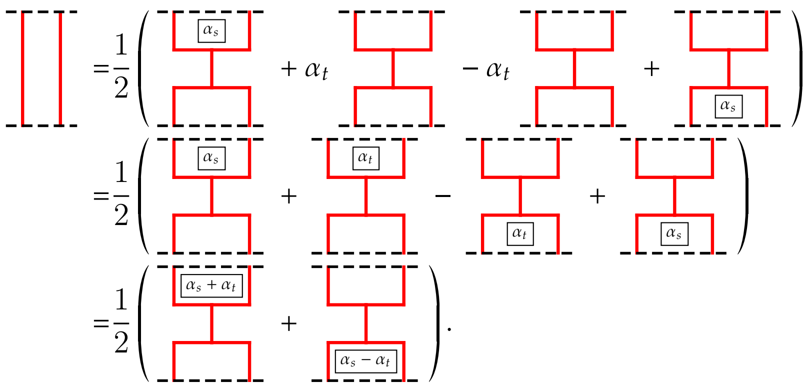
$$\mathrm{id}_{\mathrm{Y}}|_{B_sB_t \oplus B_tB_s} =$$  $$= \mathrm{id}_{\mathsf{Ind}(B_sB_t)}\,. \tag{3.1.2.7}$$

This is an idempotent factoring through $\mathsf{Ind}(B_sB_t)$, so $\mathsf{Ind}(B_sB_t)$ is a summand of $\mathrm{Y} \otimes \mathrm{Y}$.

Now consider

$$\mathrm{id}_{\mathrm{Y}}|_{B_sB_s \oplus B_tB_t} =$$  $$. \tag{3.1.2.8}$$

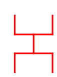
Notice that (2.3.1.17) gives us a decomposition of $\mathrm{id}_{B_s \otimes B_s}$, but its idempotents are not τ-invariant. To fix this, we add an extra α_t  as follows:



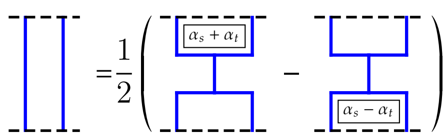
(3.1.2.9)

(3.1.2.10)

(3.1.2.11)

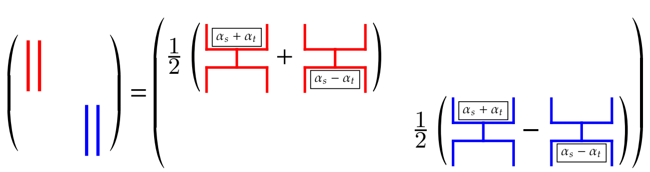
Note that we used (2.3.2.10) to slide α_t inside of  .

Similarly,



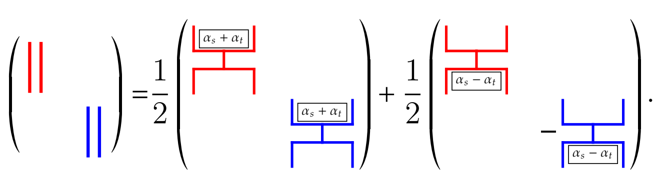
(3.1.2.12)

Using (3.1.2.11) and (3.1.2.12) we get



(3.1.2.13)

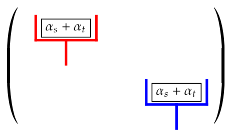
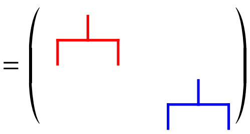
Hence we get



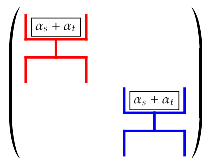
(3.1.2.14)

Define

$$\iota_{Y,-1} : Y[-1] \longrightarrow B_s \otimes B_s \oplus B_t \otimes B_t \quad \text{and} \quad p_{Y,-1} : B_s \otimes B_s \oplus B_t \otimes B_t \longrightarrow Y[-1] \tag{3.1.2.15}$$

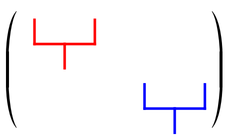
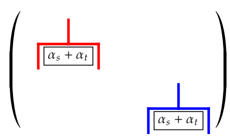
$\iota_{Y,-1} := \frac{1}{2}$  $\qquad p_{Y,-1} :=$  . (3.1.2.16)

It is clear that

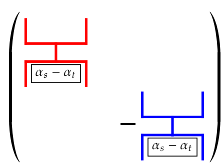
$\iota_{Y,-1} \circ p_{Y,-1} = \frac{1}{2}$  . (3.1.2.17)

By (2.3.1.14), we also have $p_{Y,-1} \circ \iota_{Y,-1} = \mathrm{id}_{Y[-1]}$. Similarly, if we define

$$\iota_{Y,1} : Y[1] \longrightarrow B_s \otimes B_s \oplus B_t \otimes B_t \quad \text{and} \quad p_{Y,1} : B_s \otimes B_s \oplus B_t \otimes B_t \longrightarrow Y[1] \tag{3.1.2.18}$$

$\iota_{Y,1} :=$  $\qquad p_{Y,1} :=$  , (3.1.2.19)

we have

$\iota_{Y,1} \circ p_{Y,1} = \frac{1}{2}$  (3.1.2.20)

and $p_{Y,1} \circ \iota_{Y,1} = \mathrm{id}_{Y[1]}$.

Together, these explicit inclusions and projections show that $\mathrm{id}_{B_s \otimes B_s \oplus B_t \otimes B_t}$ decomposes into idempotents factoring through $Y[-1]$ and $Y[-1]$ respectively. Hence

$$Y \otimes Y \cong Y[-1] \oplus Y[1] \oplus \mathsf{Ind}(B_s B_t) \cong Y[-1] \oplus Y[1] \oplus Z \oplus XZ\,. \tag{3.1.2.21}$$

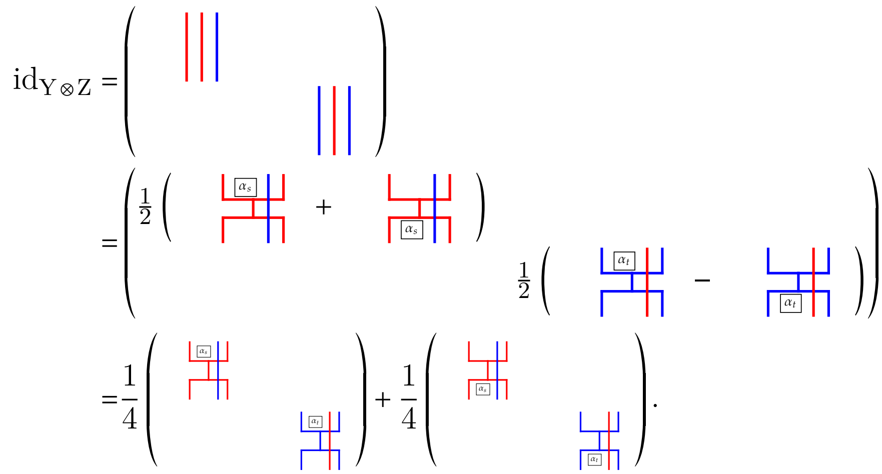
Example 3.1.2.3. We will now calculate the decomposition of $Y \otimes Z$. Inside $\mathcal{H}$ we have

$$\begin{aligned}
(B_s \oplus B_t) \otimes (B_s B_t) &\cong B_s \otimes B_s \otimes B_t \oplus B_t \otimes B_s \otimes B_t && (3.1.2.22)\\
&\cong B_s \otimes B_s \otimes B_t \oplus B_t \otimes B_t \otimes B_s && (3.1.2.23)\\
&\cong B_s B_t[-1] \oplus B_s B_t[1] \oplus B_t B_s[-1] \oplus B_t B_s[1] && (3.1.2.24)\\
&\cong (B_s B_t \oplus B_t B_s)[-1] \otimes (B_s B_t \oplus B_t B_s)[1]. && (3.1.2.25)
\end{aligned}$$

This suggests that

$$Y \otimes Z \cong \mathsf{Ind}(B_s B_t)[-1] \oplus \mathsf{Ind}(B_s B_t)[1] \cong (Z \oplus XZ)[-1] \otimes (Z \oplus XZ)[1]. \tag{3.1.2.26}$$

We may write



(3.1.2.27)

(3.1.2.28)

(3.1.2.29)

Like in Example 3.1.2.2, we may write inclusion and projection morphisms that factor through $\mathsf{Ind}(B_sB_t)[-1]$ and $\mathsf{Ind}(B_sB_t)[1]$ respectively. Therefore,

$$\mathrm{Y}\otimes\mathrm{Z} \cong \mathsf{Ind}(B_sB_t)[-1]\oplus\mathsf{Ind}(B_sB_t)[1] \cong (\mathrm{Z}\oplus\mathrm{XZ})[-1]\otimes(\mathrm{Z}\oplus\mathrm{XZ})[1]. \tag{3.1.2.30}$$

Example 3.1.2.4. Similar to Examples 3.1.2.2 and 3.1.2.3, we can show that

$$\mathrm{Z}\otimes\mathrm{Z} \cong \mathrm{Z}[-2]\oplus\mathrm{Z}\oplus\mathrm{Z}[2]\oplus\mathrm{XZ}\,. \tag{3.1.2.31}$$

We leave writing the explicit inclusion and projection morphisms as an exercise to the reader. In (3.2.5.2) we write this decomposition our new diagrammatic presentation for $\mathcal{H}^\tau$.

Remark 3.1.2.5. Equations (3.2.5.1), (3.2.5.2), and (3.2.5.3) give explicit idempotent decompositions of $\mathrm{Y}\otimes\mathrm{Y}$, $\mathrm{Z}\otimes\mathrm{Z}$, and $\mathrm{Y}\otimes\mathrm{Z}$ in the diagrammatic presentation we propose for $\mathcal{H}^\tau$. We may use the functor $\mathbf{F}:\mathcal{D}_{A_1\times A_1}\longrightarrow\mathcal{H}^\tau$ to write explicit idempotent decompositions for these tensor products, that do not involve $\mathsf{Ind}(B_sB_t)$.

The following lemma summarizes all the decompositions we have.

Lemma 3.1.2.6. *We have the following isomorphisms:*

$$\mathrm{X}\otimes\mathrm{X}\cong\mathbb{1}, \tag{3.1.2.32}$$

$$\mathrm{X}\otimes\mathrm{Y}\cong\mathrm{Y}\cong\mathrm{Y}\otimes\mathrm{X}, \tag{3.1.2.33}$$

$$\mathrm{X}\otimes\mathrm{Z}\cong\mathrm{XZ}\cong\mathrm{Z}\otimes\mathrm{X}, \tag{3.1.2.34}$$

$$\mathrm{Y}\otimes\mathrm{Y}\cong\mathrm{Y}[-1]\oplus\mathrm{Y}[1]\oplus\mathrm{Z}\oplus\mathrm{XZ}, \tag{3.1.2.35}$$

$$\mathrm{Y} \otimes \mathrm{Z} \cong \mathrm{Z}[-1] \oplus \mathrm{Z}[1] \oplus \mathrm{XZ}[-1] \oplus \mathrm{XZ}[1] \cong \mathrm{Z} \otimes \mathrm{Y}, \tag{3.1.2.36}$$

$$\mathrm{Z} \otimes \mathrm{Z} \cong \mathrm{Z}[-2] \oplus \mathrm{Z} \oplus \mathrm{Z}[2] \oplus \mathrm{XZ}\,. \tag{3.1.2.37}$$

In particular, the tensor product in $\mathcal{H}^\tau$ is commutative.

Proof. The isomorphisms (3.1.2.32) and (3.1.2.34) are clear. The equation (3.1.2.33) follows from (2.2.5.8). The remaining isomorphisms correspond to Examples 3.1.2.2, 3.1.2.3, and 3.1.2.4. □

Hence, we have the following theorem.

Theorem 3.1.2.7. *The Grothendieck ring of $\mathcal{H}^\tau$ is*

$$\mathbf{A} := \mathbb{Z}[v^{\pm 1}][X, Y, Z] / K, \tag{3.1.2.38}$$

where K is the ideal generated by the relations

$$X^2 = 1 \tag{3.1.2.39}$$

$$XY = Y \tag{3.1.2.40}$$

$$Y^2 = (v + v^{-1})Y + Z + XZ \tag{3.1.2.41}$$

$$YZ = (v + v^{-1})(Z + XZ) \tag{3.1.2.42}$$

$$Z^2 = (v^2 + 1 + v^{-2})Z + XZ. \tag{3.1.2.43}$$

Remark 3.1.2.8. We want to note two important specializations of **A**:

- Setting $X = 1$, where **A** specializes to the τ-invariant ring of Hecke algebra of type $A_1 \times A_1$.

- Setting $X = -1$, where **A** specializes to Hecke algebra with unequal parameters of $A_1 \times A_1$ folded by τ (see [Lus02]).

This is a special case of [Eli17, Prop. 5.1 & Thm. 5.4].

3.1.3 Hom *spaces in* $\mathcal{H}^\tau$

In this section we will calculate the Hom spaces of $\mathcal{H}^\tau$ from its irreducible objects to $\mathbb{1}$. By adjunction, this will completely determine the morphisms between any two objects of $\mathcal{H}^\tau$. We will then calculate the endomorphism rings of $\mathbb{1}, X, Y, Z,$ and XZ, which were needed for the proof of Lemma 3.1.1.4.

Recall $R = \Bbbk[\alpha_s, \alpha_t]$. Using the notation in Example 2.2.5.1, we get

$$R^\tau = \Bbbk[\alpha_s + \alpha_t, \alpha_s\alpha_t] \tag{3.1.3.1}$$

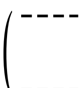
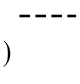
is the ring of symmetric polynomials in α_s and α_t, and that

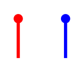
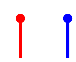
$$R^{-\tau} = (\alpha_s - \alpha_t)R^\tau \tag{3.1.3.2}$$

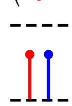
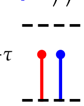
is the τ anti-invariant polynomials in α_s and α_t. We also have that

$$R = R^\tau \oplus (\alpha_s - \alpha_t)R^\tau = R^\tau \oplus R^{-\tau}. \tag{3.1.3.3}$$

Proposition 3.1.3.1. *The following* Hom *spaces are free* R^τ*-modules with the generators given below:*

$\mathsf{Hom}(\mathbb{1}, \mathbb{1}) = R^\tau \cdot \Bigg($  $\Bigg)$, $\mathsf{Hom}(X, \mathbb{1}) = R^\tau \cdot \Bigg((\alpha_s - \alpha_t)$  $\Bigg)$, (3.1.3.4)

$\mathsf{Hom}(Y, \mathbb{1}) = R^\tau \cdot \Big($  $\Big) \oplus R^\tau \cdot \Big((\alpha_s - \alpha_t)\Big($  $\Big)\Big)$, (3.1.3.5)

$\mathsf{Hom}(Z, \mathbb{1}) = R^\tau \cdot$  , *and* $\mathsf{Hom}(XZ, \mathbb{1}) = R^{-\tau}$  . (3.1.3.6)

The graded dimensions of these spaces over R^τ *are*

$$\mathsf{grdim}^\tau_R(\mathsf{Hom}(\mathbb{1}, \mathbb{1})) = 1, \tag{3.1.3.7}$$

$$\mathsf{grdim}^\tau_R(\mathsf{Hom}(X, \mathbb{1})) = v^2, \tag{3.1.3.8}$$

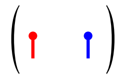
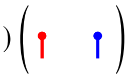
$$\mathsf{grdim}^\tau_R(\mathsf{Hom}(Y, \mathbb{1})) = 1 + v^2, \tag{3.1.3.9}$$

$$\mathsf{grdim}^\tau_R(\mathsf{Hom}(Z, \mathbb{1})) = v^2, \tag{3.1.3.10}$$

$$\mathsf{grdim}^\tau_R(\mathsf{Hom}(XZ, \mathbb{1})) = v^4. \tag{3.1.3.11}$$

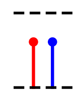
Proof. We already showed this for $\mathsf{Hom}(\mathbb{1}, \mathbb{1})$ and $\mathsf{Hom}(X, \mathbb{1})$ in Example 2.2.5.1. By Proposition 2.2.3.3 and Proposition 2.3.1.7 we have

$$\mathsf{Hom}(Y, \mathbb{1}) = \mathsf{Hom}(\mathsf{Ind}(B_s), \mathbb{1}) \cong \mathsf{Hom}(B_s, \mathbb{1}) \cong R. \tag{3.1.3.12}$$

Since $\mathrm{grdim}_{R^\tau}(R) = 1 + v^2$, the morphisms  and $(\alpha_s - \alpha_t)$  are free generators of $\mathrm{Hom}(Y, \mathbb{1})$ as an R^τ-module.

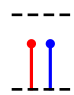
From Proposition 2.3.2.7, we have that $\mathrm{Hom}(B_s B_t, \mathbb{1}) = R \cdot$ . By (2.2.2.2), we get that a morphism in $\mathrm{Hom}(Z, \mathbb{1})$ must be a τ-invariant morphism in $\mathrm{Hom}(B_s B_t, \mathbb{1})$, hence

$$\mathrm{Hom}(Z, \mathbb{1}) = R^\tau \cdot \quad (3.1.3.13)$$



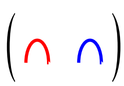
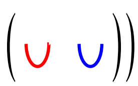
Similarly, a morphism in $\mathrm{Hom}(XZ, \mathbb{1})$ must be a τ-anti invariant and

$$\mathrm{Hom}(XZ, \mathbb{1}) = R^{-\tau} \cdot \quad (3.1.3.14)$$



□

Here we give a biadjoint to each object in $\mathcal{I}$ not equal to $\mathbb{1}$:

- $(X, \mathrm{id}_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}})$ is biadjoint to X.
- $(Y,$ , $)$ is biadjoint to Y.
- $(Z,$ , $)$ is biadjoint to Z.
- $(XZ,$ , $)$ is biadjoint to XZ.

By Proposition 2.1.2.5, we have the following result.

Proposition 3.1.3.2. *The functors $X[k] \otimes (-)$, $Y[k] \otimes (-)$, and $Z[k] \otimes (-)$ are bi-adjoint to $X[-k] \otimes (-)$, $Y[-k] \otimes (-)$, and $Z[-k] \otimes (-)$ respectively.*

Proposition 3.1.3.3. *We have that*

$$\mathrm{End}(\mathbb{1}) \cong \mathrm{End}(X) \cong R^\tau, \quad (3.1.3.15)$$

$$\mathrm{End}(Y) \cong R \oplus R^{\oplus 2}[2], \quad (3.1.3.16)$$

$$\mathrm{End}(Z) \cong \mathrm{End}(XZ) \cong R^\tau \oplus R^\tau[2] \oplus R^\tau[4]^{\oplus 2}. \quad (3.1.3.17)$$

As free graded R^τ-modules. They are all graded local rings.

Proof. In Example 2.2.5.1, we calculated that

$$\mathrm{End}(\mathbb{1}) \cong \mathrm{End}(X) \cong R^\tau. \quad (3.1.3.18)$$

Using Proposition 2.2.3.3, we get that

$$\mathsf{Hom}(Y, Y) = \mathsf{Hom}(\mathsf{Ind}(B_s), \mathsf{Ind}(B_s)) \tag{3.1.3.19}$$
$$\cong \mathsf{Hom}(B_s, B_s \oplus B_t) \tag{3.1.3.20}$$
$$\cong \mathsf{Hom}(B_s, B_s) \oplus \mathsf{Hom}(B_s, B_t) \tag{3.1.3.21}$$
$$\cong R \oplus R[2] \oplus R[2] \tag{3.1.3.22}$$
$$= R \oplus R^{\oplus 2}[2]. \tag{3.1.3.23}$$

Since, all these endomorphism rings are non-negatively graded and their degree 0 component is $\Bbbk$, they are graded local rings. □

The graded dimension of these Hom spaces as free R^τ-modules is as follows.

$$\mathsf{grdim}(\mathsf{Hom}(\mathbb{1}, \mathbb{1})) = 1$$

$$\mathsf{grdim}(\mathsf{Hom}(\mathbb{1}, X)) = v^2$$

$$\mathsf{grdim}(\mathsf{Hom}(\mathbb{1}, Y)) = v + v^3$$

$$\mathsf{grdim}(\mathsf{Hom}(\mathbb{1}, Z)) = v^2$$

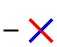
$$\mathsf{grdim}(\mathsf{Hom}(\mathbb{1}, XZ)) = v^4.$$

Definition 3.1.3.4. Define the standard trace $\epsilon : \mathbf{A} \longrightarrow \mathbb{Z}[v^{\pm 1}]$ to be the linear map sending $\mathbb{1} \mapsto 1$, $X \mapsto v^2$, $Y \mapsto v + v^3$, $Z \mapsto v^2$, and $XZ \mapsto v^4$.

Proposition 3.1.3.5. *Let $A, B \in \mathcal{H}^\tau$ The graded dimension*

$$\mathsf{grdim}(\mathsf{Hom}(A, B)) = \epsilon(AB).$$

3.2 The Diagrammatic Folded Hecke Category of type $A_1 \times A_1$

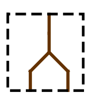
In Section 3.1.1 we showed that the category $\mathcal{H}^\tau$ has indecomposable objects $\mathbb{1} := (\mathbb{1}, 1)$, X = $(\mathbb{1}, -1)$, Y = $(B_s \oplus B_t, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$, Z = $(B_sB_t,$  $)$, and XZ = $(B_sB_t, -$  $)$ (up to grading shift). We also showed $\mathcal{H}^\tau$ is a Krull-Schmidt category.

In Section 3.2.1, we will define a 'free' model for our category $\mathcal{D}_{\mathsf{pre}}$ and a functor $\mathbf{F} : \mathcal{D}_{\mathsf{pre}} \longrightarrow \mathcal{H}^\tau$. In Section 3.2.2 we define our diagrammatic category $\mathcal{D}_{A_1 \times A_1}$, as a quotient of $\mathcal{D}_{\mathsf{pre}}$ by a given set of relations. We then show that the relations defining $\mathcal{D}_{A_1 \times A_1}$ are satisfied through the functor $\mathbf{F}$, so this functor descends to a functor $\widetilde{\mathbf{F}} : \mathcal{D}_{A_1 \times A_1} \longrightarrow \mathcal{H}^\tau$. In Sections 3.2.4 and 3.2.5 we show the general polynomial forcing formulas, as well as the idempotent decompositions in $\mathcal{D}_{A_1\times A_1}$. In Section 3.2.3, we show some manipulations to be able to slide orange strands (corresponding to the identity of the invertible element $\widehat{\mathrm{X}}$), which simplify our diagrams. Finally, in Section 3.2.6 we prove Theorem 3.2.6.1. Some proofs may require the relations derived in Appendix A, derived from the ones given in Definition 3.2.2.1.

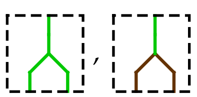
3.2.1 *The Free Diagrammatic Category* $\mathcal{D}_{\mathsf{pre}}$

Definition 3.2.1.1. Define the *free diagrammatic category*, $\mathcal{D}_{\mathsf{pre}}$ to be the graded strict pivotal R^τ-linear category, generated by the self-adjoint objects $\widehat{\mathrm{X}}$, $\widehat{\mathrm{Y}}$, and $\widehat{\mathrm{Z}}$, with respect to direct sums, tensor products and degree shifts. The unit object of $\mathcal{D}_{\mathsf{pre}}$ will be denoted by $\widehat{\mathbb{1}}$. The identity morphisms of $\widehat{\mathrm{X}}$, $\widehat{\mathrm{Y}}$, and $\widehat{\mathrm{Z}}$ are represented by the diagrams  , , and respectively. The evaluation and coevaluation morphisms are represented cup and cap diagrams in the appropriate color. The following is a list of the local generators for the morphism diagrams of $\mathcal{D}_{\mathsf{pre}}$ listed by their degree:

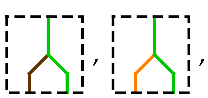
Degree -2:

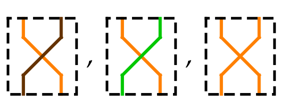
 (3.2.1.1)

Degree -1:

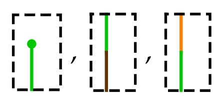
 (3.2.1.2)

Degree 0:

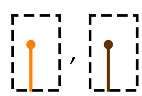
 (3.2.1.3)

 (3.2.1.4)

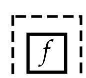
Degree 1:

 (3.2.1.5)

Degree 2:

 (3.2.1.6)

Degree $\deg(f)$ for $f \in R^\tau$:



(3.2.1.7)

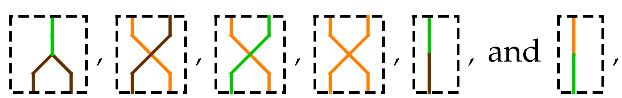
We also require polynomial boxes to satisfy relations (2.3.1.12) and (2.3.1.13). We set $\deg(\alpha_s) = 2$ and $\deg(\alpha_t) = 2$.

The diagrams in (3.2.1.5) joining two different colored strands will be called *bivalent vertices*. The green and orange trivalent vertex in (3.2.1.3) will be called an *orange landing*. The diagrams in (3.2.1.4) will be called *orange crossings*. The trivalent (resp. bivalent) vertices in green or brown (with no orange strands) will be called *GB trivalent (resp. bivalent) vertices*.

Here composition is given by vertical stacking and tensor product by horizontal pasting of diagrams.

A diagram in $\mathcal{D}_{\mathsf{pre}}$ will be called an *equivariant Hecke diagram of type $A_1 \times A_1$*.

Remark 3.2.1.2. In Section 3.2.2 we will see that the generators

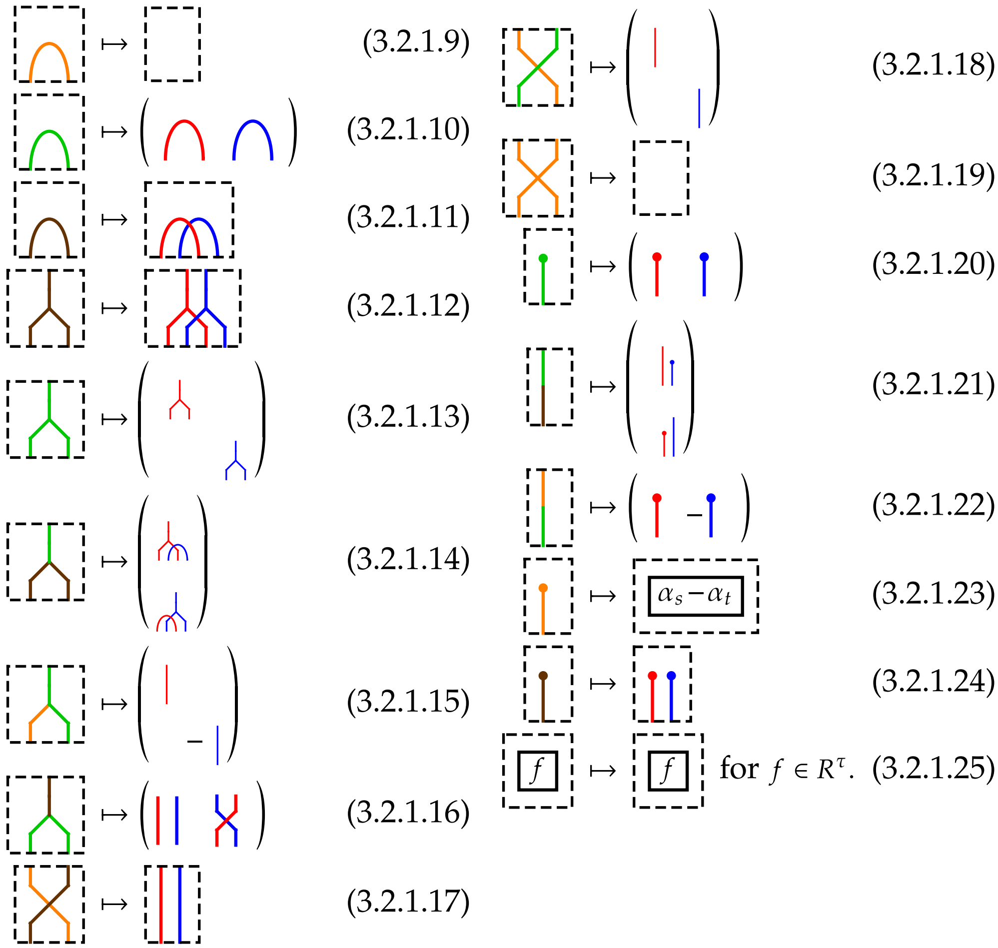


(3.2.1.8)

can be expressed in terms of the other generators in the quotient category $\mathcal{D}_{A_1 \times A_1}$. In fact, their definition is given in relations 3.2.2.2 and 3.2.2.4 . We choose to include these generators because they make the notation easier and facilitate the diagrammatic argument used in Section 3.2.6.

In this chapter, we will use the hat notation to differentiate the objects $\widehat{\mathbb{1}}, \widehat{\mathrm{X}}, \widehat{\mathrm{Y}}, \widehat{\mathrm{Z}} \in \mathcal{D}_{\mathsf{pre}}$ from the objects $\mathbb{1}, \mathrm{X}, \mathrm{Y}, \mathrm{Z} \in \mathcal{H}^{\tau}$. We will also define $\widehat{\mathrm{XZ}} := \widehat{\mathrm{X}} \otimes \widehat{\mathrm{Z}}$ and $\widehat{\mathcal{I}} := \{\widehat{\mathbb{1}}, \widehat{\mathrm{X}}, \widehat{\mathrm{Y}}, \widehat{\mathrm{Z}}, \widehat{\mathrm{XZ}}\}$.

Definition 3.2.1.3. We can now define a strict monoidal functor $\widetilde{\mathbf{F}} : \mathcal{D}_{\mathsf{pre}} \longrightarrow \mathcal{H}^{\tau}$ that sends $\widehat{\mathrm{X}} \mapsto \mathrm{X}$, $\widehat{\mathrm{Y}} \mapsto \mathrm{Y}$, and $\widehat{\mathrm{Z}} \mapsto \mathrm{Z}$, and sends



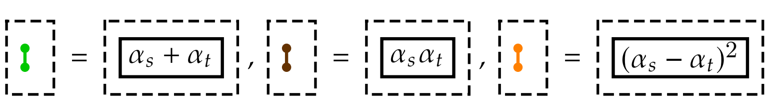
(3.2.1.9) (3.2.1.10) (3.2.1.11) (3.2.1.12) (3.2.1.13) (3.2.1.14) (3.2.1.15) (3.2.1.16) (3.2.1.17) (3.2.1.18) (3.2.1.19) (3.2.1.20) (3.2.1.21) (3.2.1.22) (3.2.1.23) (3.2.1.24) (3.2.1.25)

This functor is well defined since, by construction, the isotopy relations of $\mathcal{D}_{\mathsf{pre}}$ also hold in $\mathcal{H}^\tau$.

3.2.2 *Presentation*

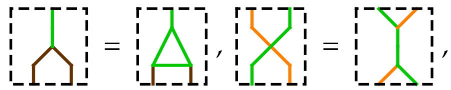
Definition 3.2.2.1. Define the *diagrammatic equivariant Hecke category of type* $A_1 \times A_1$, which we denote by $\mathcal{D}_{A_1 \times A_1}$ (and abbreviate to $\mathcal{D}$), to be the quotient of $\mathcal{D}_{\mathsf{pre}}$ by the following relations.

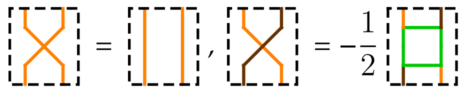
Barbell relations:

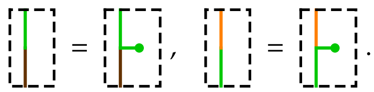


(3.2.2.1)

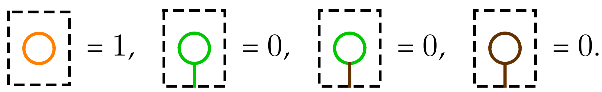
Vertex definitions:

 (3.2.2.2)

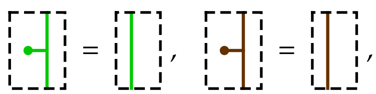
 (3.2.2.3)

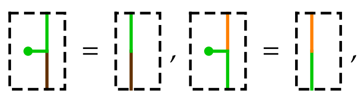
 (3.2.2.4)

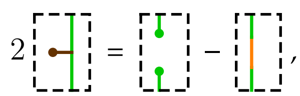
Circle and Needle relations:

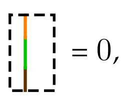
 (3.2.2.5)

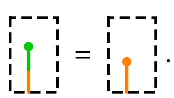
Unit relations:

 (3.2.2.6)

 (3.2.2.7)

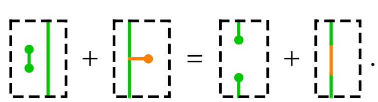
 (3.2.2.8)

 (3.2.2.9)

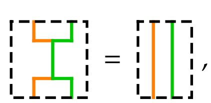
 (3.2.2.10)

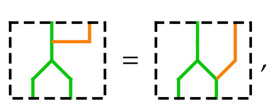
Note that (3.2.2.4) and (3.2.2.7) are mirror images of each other (and we need them both).

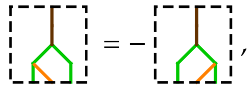
Polynomial forcing relation:

 (3.2.2.11)

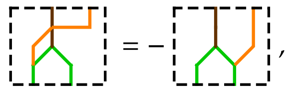
Sliding of the orange landing:

 (3.2.2.12)

 (3.2.2.13)

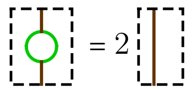


(3.2.2.14)

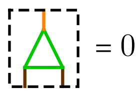


(3.2.2.15)

Bigon and triangle relations:

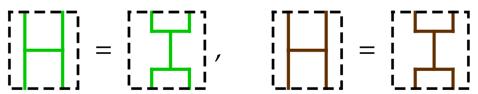


(3.2.2.16)

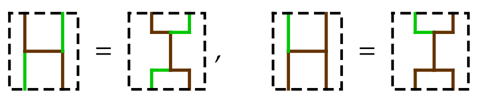


(3.2.2.17)

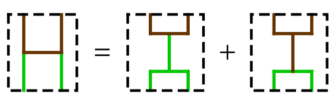
H=I relations:



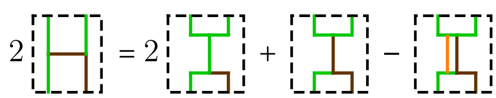
(3.2.2.18)



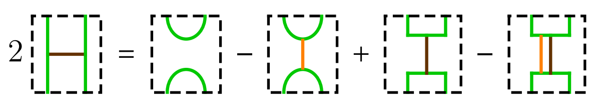
(3.2.2.19)



(3.2.2.20)



(3.2.2.21)



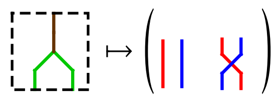
(3.2.2.22)

Remark 3.2.2.2. The relations (3.2.2.1)-(3.2.2.22) constitute a generating set of relations for $\mathcal{D}$. They are not an extensive list of the relations used to reduce diagrams in $\mathcal{D}$. In the following sections and in Appendix A we deduce a number of additional relations from the ones stated above.

Lemma 3.2.2.3. *The relations in Definition 3.2.2.1 hold after we apply the functor* $\widetilde{\mathbf{F}}$: $\mathcal{D}_{\mathsf{pre}} \longrightarrow \mathcal{H}^{\tau}$ *(defined in Section 3.2.1). Thus,* $\widetilde{\mathbf{F}}$ *descends to an essentially surjective functor* $\mathbf{F} : \mathcal{D} \longrightarrow \mathcal{H}^{\tau}$.

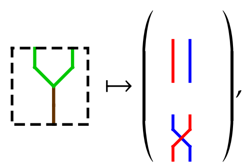
Proof. The fact that the relations hold comes down to diagrammatic calculations in $\mathcal{H}^\tau$. Let us show (3.2.2.11),(3.2.2.16),(3.2.2.21), and (3.2.2.22). We will leave the rest for the reader to verify.

(3.2.2.16) Since



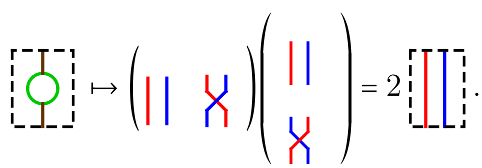
(3.2.2.23)

and



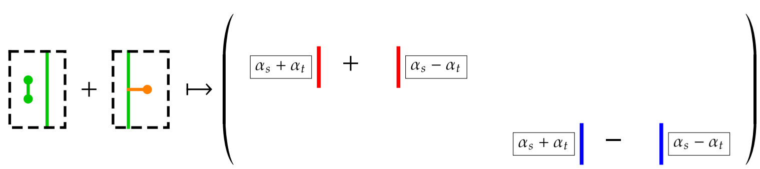
(3.2.2.24)

then

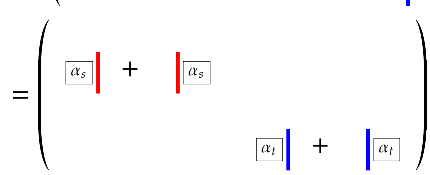


(3.2.2.25)

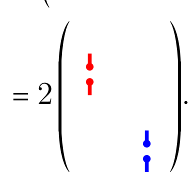
(3.2.2.11) The left hand side of the equation is sent to



(3.2.2.26)

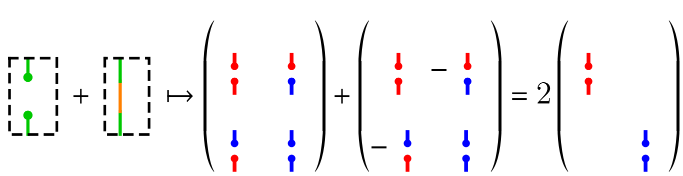


(3.2.2.27)



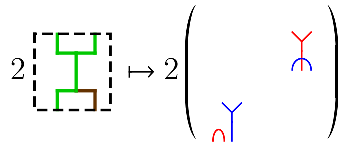
(3.2.2.28)

The right hand side of the equation is sent to

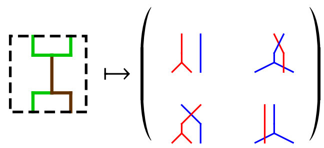


(3.2.2.29)

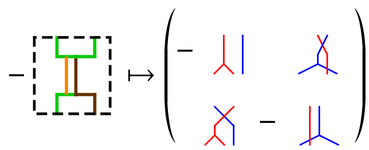
(3.2.2.21) The summands of the right hand side are sent to



(3.2.2.30)

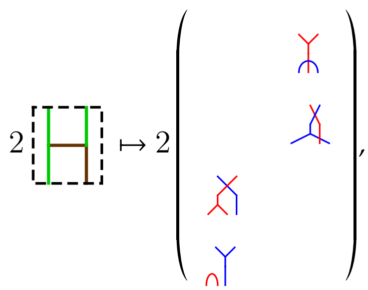


(3.2.2.31)



(3.2.2.32)

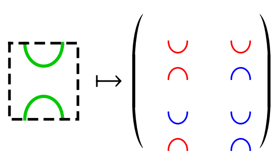
Rotating equation (3.2.2.31) we get



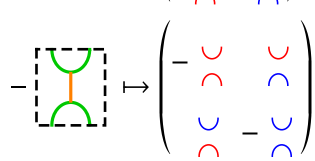
(3.2.2.33)

which is exactly the sum of equations (3.2.2.30)-(3.2.2.32).

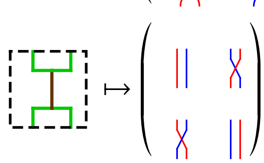
(3.2.2.22) The summands of the right hand side are sent to



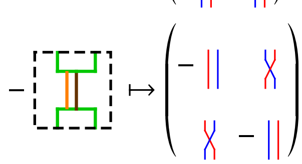
(3.2.2.34)



(3.2.2.35)

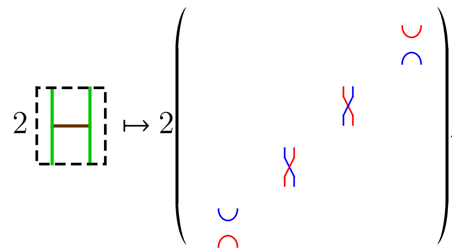


(3.2.2.36)



(3.2.2.37)

Rotating equation (3.2.2.36), we get



(3.2.2.38)

which is exactly the sum of equations (3.2.2.34)-(3.2.2.37).

Since the objects $\mathbb{1}, X, Y, Z$, and XZ are in the image of $\mathbf{F}$, by Corollary 3.1.1.6, $\mathbf{F}$ is essentially surjective. $\square$

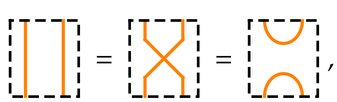
3.2.3 *Sliding Orange Strands*

One of the tools we will need to prove Theorem 3.2.6.1 in Section 3.2.6, is to get orange strands, as much as possible, out of the way of the non-orange elements of our diagrams. In this section we will prove all the tools needed for this.

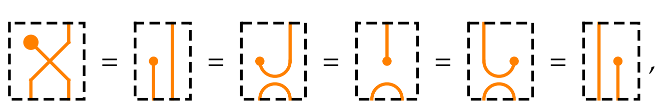
Our first goal is to show that orange strands slide over any diagram. To do this, we must show that the orange strand slides over all other strands, as well as over dots, trivalent vertices and polynomials.

The following lemma shows the interactions of an orange strand with other orange strands and dots. Its follows easily from the definition of  in (3.2.2.3) and the orange barbell in 3.2.2.1.

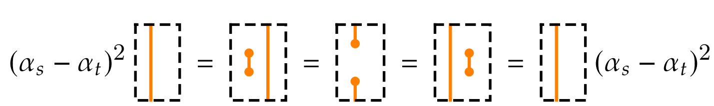
Lemma 3.2.3.1. *The orange strand slides over an orange strand, dot, or barbell. That is, the following relations hold:*



(3.2.3.1)



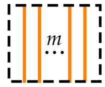
(3.2.3.2)

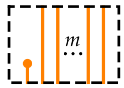


(3.2.3.3)

Corollary 3.2.3.2. *Let D be a diagram containing only orange strands and dots, with k boundary strands. Define* Or_k *by*

$\mathsf{Or}_k :=$

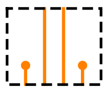
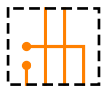
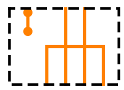
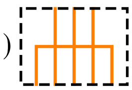
 *if* $k = 2m$

 *if* $k = 2m+1$

. (3.2.3.4)

Then D is an R^τ-multiple of Or_k *up to adjunction.*

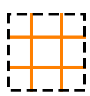
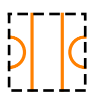
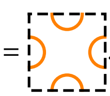
Proof. By (3.2.3.2) and (3.2.3.3),

 $=$  $=$  $= (\alpha_s - \alpha_t)$  . (3.2.3.5)

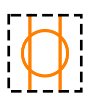
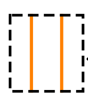
If D has more than one orange dot, we can use this process to merge them into a factor of $(\alpha_s - \alpha_t)^2$. Hence, we may suppose that D has at most one orange dot. Moreover, (3.2.3.2) allows us to move that orange dot to the bottom left orange boundary strand.

By (3.2.3.1), we may replace any  by . Thus, we may suppose that D has no orange crossings.

We can also use Lemma 3.2.3.1 along with the orange circle relation in (3.2.2.5), to break any orange cycle in D, e.g.

 $=$  $=$  . (3.2.3.6)

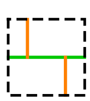
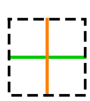
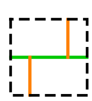
We may also slide any orange circle out of D, e.g.

 $=$  $=$  . (3.2.3.7)

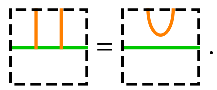
Because of this, we can suppose further that D does not have any orange cycles. Therefore, the diagram D is a non-crossing matching of its boundary points, plus an orange boundary dot if k is odd.

Finally, (3.2.3.1) allows us to transform any perfect matching into the one in (3.2.3.4). □

Let us consider the relations

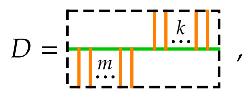
 $=$  $=$  , (3.2.3.8)

which are rotations of (3.2.2.2). This shows that we can split orange crossings and slide orange landings on opposite sides of a green strand past each other. When we have two orange landings on the same side of a green strand we get,

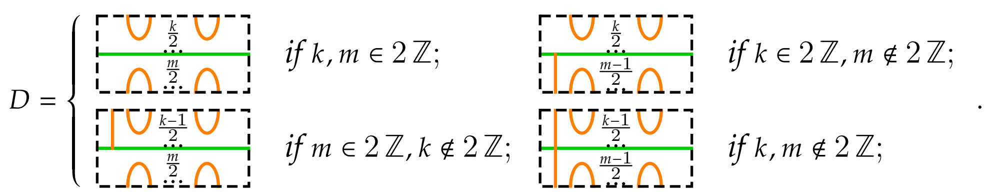

(3.2.3.9)

instead. This is a rotation of (3.2.2.12). Applying (3.2.3.8) and (3.2.3.9) repeatedly, we have the following lemma.

Lemma 3.2.3.3. *Let D be a diagram of the form*


(3.2.3.10)

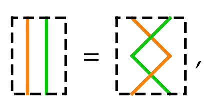
with m orange landings from one side and k from the other. Then

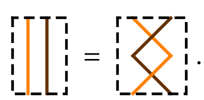


(3.2.3.11)

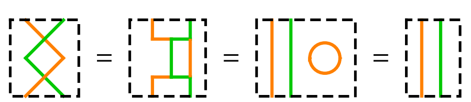
Hence, we can transform any green edge with an arbitrary number of orange crossings and landings into an edge that has at most one crossing or landing.

Proposition 3.2.3.4. *The orange strand slides over the green and brown strands. That is, the following relations hold:*

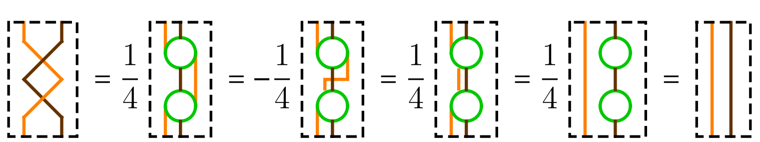

(3.2.3.12)


(3.2.3.13)

Proof. Using the orange circle in (3.2.2.5), (3.2.3.1), and (3.2.2.12) we can conclude that

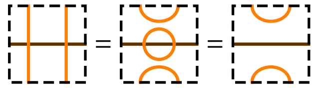

(3.2.3.14)

Similarly, (3.2.2.15) and (3.2.2.17) allow us to write


(3.2.3.15)

□

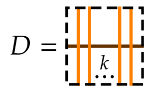
As a consequence of (3.2.3.13), we have



(3.2.3.16)

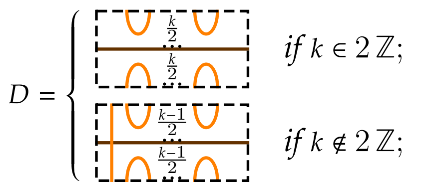
Applying this relation repeatedly, we have the following lemma.

Lemma 3.2.3.5. *Let* D *be a diagram of the form*



(3.2.3.17)

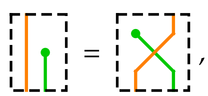
with k *orange crossings. Then*



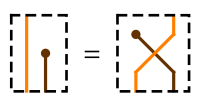
(3.2.3.18)

Hence, we can transform any brown edge with an arbitrary number of orange crossings into an edge that has at most one crossing.

Proposition 3.2.3.6. *The orange strand slides over green and brown dots. That is, the following relations hold:*

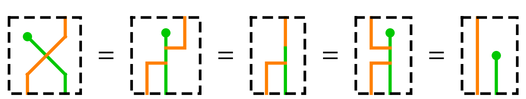


(3.2.3.19)



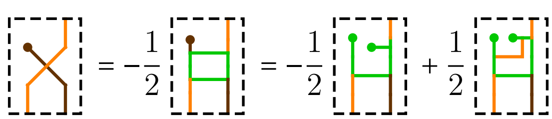
(3.2.3.20)

Proof. For (3.2.3.19), let us use (3.2.2.4), (3.2.2.7), and (3.2.2.12) to write

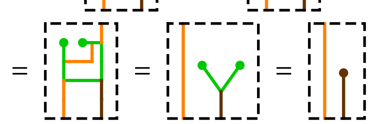


(3.2.3.21)

For (3.2.3.20), let us use (3.2.2.2), (3.2.2.8), and (A.1.0.9) to write



(3.2.3.22)

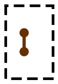
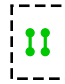
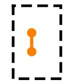


(3.2.3.23)

□

An easy consequence of Proposition 3.2.3.6 is that the orange strand slides over barbells.

Since $R^\tau = \Bbbk[\alpha_s + \alpha_t, \alpha_s\alpha_t]$, $= \alpha_s + \alpha_t$, and $= \alpha_s\alpha_t$, we may write any polynomial box in terms of green and brown barbells. We must point out the following algebraic dependence between green, brown, and orange barbells:

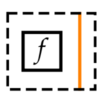
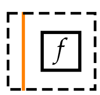
4  $= \boxed{\boxed{4\alpha_s\alpha_t}} = \boxed{\boxed{(\alpha_s + \alpha_t)^2}} - \boxed{\boxed{(\alpha_s - \alpha_t)^2}} =$  $-$ . (3.2.3.24)

Altogether, we have the following proposition.

Proposition 3.2.3.7. *Any polynomial box* $\boxed{f}$ *, for* $f \in R^\tau$*, may be written as a polynomial on the green and brown barbells. Alternatively,* $\boxed{f}$ *may also be written as a polynomial on the green and orange barbells.*

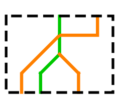
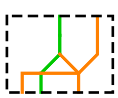
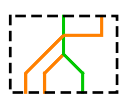
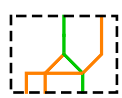
From Proposition 3.2.3.6 and Proposition 3.2.3.7 we have the following corollary.

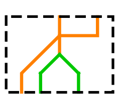
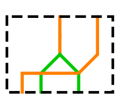
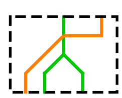
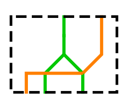
Corollary 3.2.3.8 (Orange polynomial forcing). *The orange strand slides over polynomial boxes. That is,*

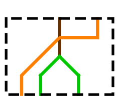
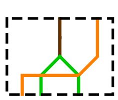
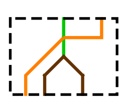
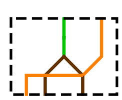
 =  *for all* $f \in R^\tau$. (3.2.3.25)

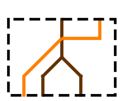
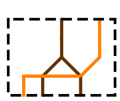
The following proposition shows that the orange strand slides over all trivalent vertices.

Proposition 3.2.3.9. *The orange strand slides over trivalent vertices. That is, the following relations hold:*

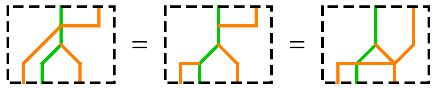
 = , (3.2.3.26)  = , (3.2.3.27)

 = , (3.2.3.28)  = , (3.2.3.29)

 = , (3.2.3.30)  = , (3.2.3.31)

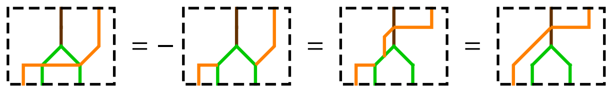
 = . (3.2.3.32)

Proof. The proof of (3.2.3.26)-(3.2.3.28) are all similar. We will only show (3.2.3.26) and leave the rest as an exercise for the reader. By (3.2.2.14) and (3.2.3.8),



(3.2.3.33)

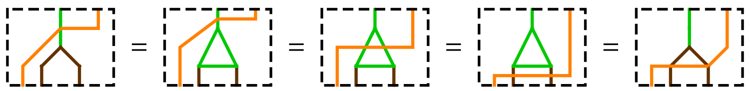
For (3.2.3.30), let us use (3.2.2.14) and (3.2.2.15) to write



(3.2.3.34)

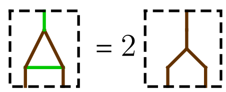
The proof of (3.2.3.29) is similar.

For (3.2.3.31), use (3.2.2.2), (3.2.3.29), and (3.2.3.30) to write



(3.2.3.35)

To show (3.2.3.32) we must show the relation



(3.2.3.36)

and then use relations (3.2.3.29) and (3.2.3.31). We leave the verification of (3.2.3.36) as an exercise to the reader (use (A.2.0.20)). □

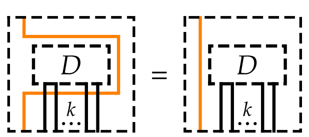
As bivalent vertices and crossings are defined in terms of trivalent vertices and dots, orange strands also slide over them. The following corollary follows directly from Proposition 3.2.3.6 and Proposition 3.2.3.9.

Corollary 3.2.3.10. *The orange strand slides over bivalent vertices and crossings.*

Notation 3.2.3.11. From now on, we will use purple strands as a shorthand for places where we can place a brown or green strand indistinctively, and black strands for places where we can place an orange, brown, or green strand indistinctively.

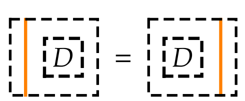
Adding all the results in this section we have the following theorem.

Theorem 3.2.3.12. *The orange strand slides over any diagram. That is, for a diagram D with k boundary strands*



(3.2.3.37)

In particular, when $k = 0$,



(3.2.3.38)

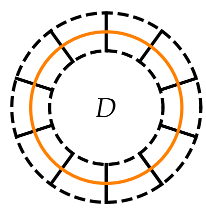
Definition 3.2.3.13. Let D be a diagram. Define the *orange encircling of D*, $\mathrm{circ}_O(D)$, to be the diagram obtained from D in the following way:

(a) Start with an annulus A consisting of an orange circle around its center.

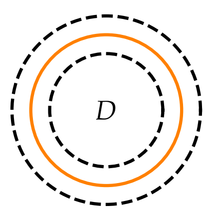
(b) For each boundary strand of D, add (sequentially) to A a strand of the same color and an orange crossing connecting the outer and inner boundary of A.

(c) Paste D in the center of A (noting that its inner boundary is the same as the boundary of D).

That is, add an orange circle encircling D. Diagrammatically,

$\mathrm{circ}_O(D) =$  , (3.2.3.39)

where the number and color of the boundary strands corresponds exactly with the boundary of D. In particular, if D has empty boundary then

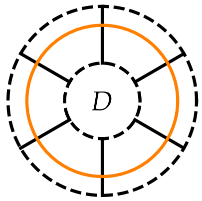
$\mathrm{circ}_O(D) =$  . (3.2.3.40)

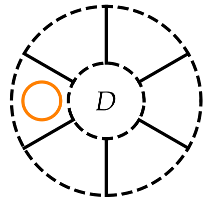
We say that a diagram D is an *orange encircled diagram* if there exists a diagram D' contained in D such that $\mathrm{circ}_O(D')$ is exactly D.

Using all the slide relations shown in Proposition 3.2.3.9 along with the slide relations (3.2.3.2)-(3.2.3.13), we can conclude that the orange strands slide over any diagram. Hence, we have the following corollary.

Corollary 3.2.3.14 (Circle Sliding)**.** *Let D be a diagram with no orange boundary strands. Then $\mathrm{circ}_O(D) = D$. Hence, erasing closed orange circles from a diagram D does not change the underlying morphism in $\mathcal{D}$.*

Proof. Let D be a diagram without any orange boundary strands. By Theorem 3.2.3.12, we can slide the orange circle on the outside of $\mathrm{circ}_O(D)$ to a region where it does not intersect D, and then use the orange circle relation to erase it. Diagrammatically,

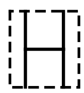
$\mathrm{circ}_{\mathsf{O}}(D) =$ 

$=$ 

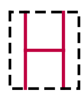
$= D.$ (3.2.3.41)

□

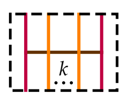
Definition 3.2.3.15. A *crossbar diagram* is a diagram made exclusively of two trivalent vertices joined by a single strand and where the other 4 strands are in the boundary. That is, a diagram of the form


. (3.2.3.42)

A crossbar made only with GB trivalent vertices will be called a *GB crossbar*. That is, a diagram of the form

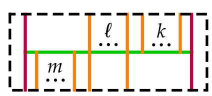

. (3.2.3.43)

A *GB comb* is a diagram made by starting with a GB crossbar and any adding number of non-crossing orange strands, each intersecting the crossbar once on the middle edge (at a crossing or landing). Diagrammatically, a *GB comb* either of the form



, (3.2.3.44)

for some $k \in \mathbb{N}$, or obtained from

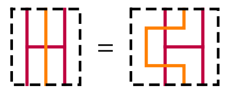


, (3.2.3.45)

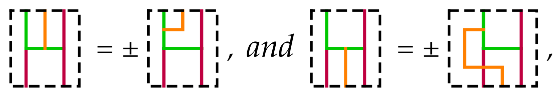
for some $k, \ell, m \in \mathbb{N}$, by permuting its orange crossings and landings. A *single GB comb* is a GB comb which has only one orange crossing or landing.

An important part of the proof of Lemma 3.2.6.7 requires us to apply and $H = I$ relation on GB cycles (until they eventually become needles). To be able to apply an $H = I$ relation, we need a diagram containing a GB crossbar. To make a GB comb into a diagram containing a GB crossbar, we need to remove all the orange crossings and landings out of its central edge. The following lemma shows that we can do for most GB combs.

Lemma 3.2.3.16. *The following relations hold:*

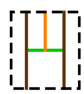


(3.2.3.46)



(3.2.3.47)

where the sign is positive if the bottom left strand is green and negative it its brown. Hence, any single GB comb that is not of the form



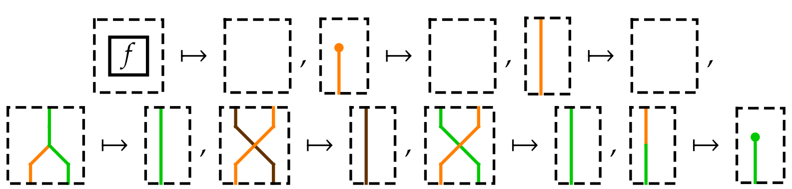
(3.2.3.48)

(or its rotation), is equal to a diagram (with exactly two GB trivalent vertices) containing a GB crossbar.

Remark 3.2.3.17. Let D be a GB comb. By Lemma 3.2.3.3 and Lemma 3.2.3.5, every GB comb can be transformed into the union of a single GB comb or GB crossbar E and some number of orange cups and caps. By Lemma 3.2.3.16, if E is not of the form , equals a diagram containing a GB crossbar.

The only tool we have left to define is the restriction to one or more colors. This operation erases all polynomials and edges that are not of that color and is defined as follows.

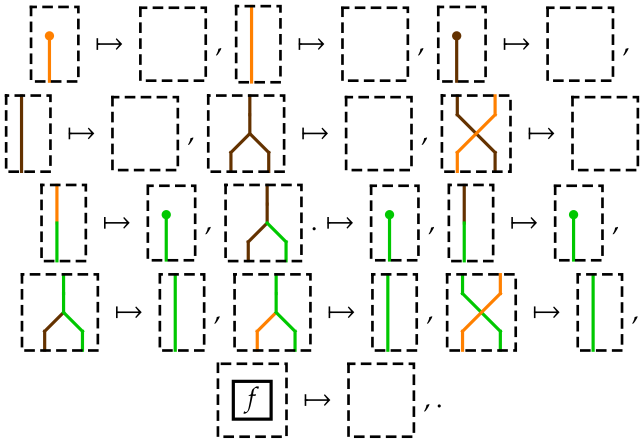
Definition 3.2.3.18. Let Res_{GB} be the operation diagrams of $\mathcal{D}$ sending all the generators on green and brown colors to themselves and sending



(3.2.3.49)

This operation takes a diagram and erases all orange strands, orange dots and polynomials. We will call this operation the *green-brown restriction*. We will refer to the connected components of $\mathsf{Res}_{\mathrm{GB}}(D)$ as its *GB components*.

Let Res_G be the diagrams of $\mathcal{D}$ sending all the green generators to themselves and sending



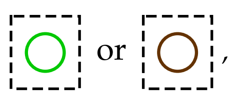
(3.2.3.50)

This operation takes a diagram and erases all orange or brown strands, all orange or brown dots, and all polynomials. We will call this operation the *green restriction*. We will refer to the connected components of $\mathsf{Res}_G(D)$ as its *green components*.

For a given diagram D, let ν_D^{GB} (resp. ν_D^G) be the embedding of the graph of $\mathsf{Res}_{\mathrm{GB}}(D)$ (resp. $\mathsf{Res}_G(D)$) into D.

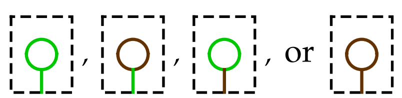
Let S be a subset of the graph of $\mathsf{Res}_{\mathrm{GB}}(D)$ (resp. $\mathsf{Res}_G(D)$) and t a point in D corresponding to an orange crossing or landing. We say *t is over S* if $t \in \nu_D^{GB}(S)$ (resp. $t \in \nu_D^G(S)$).

We say that D is a *GB circle* if $\mathsf{Res}_{\mathrm{GB}}(D)$ is



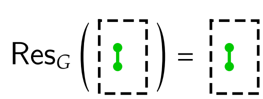
(3.2.3.51)

and D is a *GB needle* if $\mathsf{Res}_{\mathrm{GB}}(D)$ is



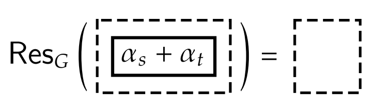
(3.2.3.52)

Remark 3.2.3.19. The operations Res_G and $\mathsf{Res}_{\mathrm{GB}}$ do not define a functor. For instance,



(3.2.3.53)

but



(3.2.3.54)

These operations are well defined for a specific diagram, but do not preserve the relations of $\mathcal{D}$.

Having defined the green components of a diagram, we can state the following definition.

Definition 3.2.3.20. A diagram E of $\mathcal{D}$ is said to be *orange simplified* if it has no orange circles, each green component of E has at most one orange landing over it, and each GB region has at most one orange dot in it. We say that E is *orange reduced* if E is *orange simplified* and has at most one orange boundary strand.

Proposition 3.2.3.21. *For any diagram D there is an orange simplified diagram E such that $D = E$.*

Proof. We may use Corollary 3.2.3.14 to slide out any orange circles and merge tow orange dots in the same GB region with (3.2.3.3). Note that the orange landing slide relations (3.2.3.29) and (3.2.2.13)-(3.2.2.15) only change the number of orange crossings. Because of this, we can slide all the orange landings in the same connected components to a single green edge without creating any new orange landings. We can finally use (3.2.3.8) and (3.2.3.9) to change any pair of orange landings into a crossing or a cup/cap. □

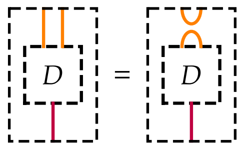
We will use the following lemma to deal with diagrams that have two or more orange boundary strands.

Lemma 3.2.3.22. *Let D be a diagram with two or more orange boundary strands and at most one green or brown boundary strand. Then there is a diagram E such that $D = E$ and diagrams O, N contained in E satisfying:*

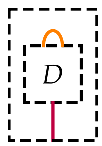
- $E = O \circ N$.
- $\mathsf{Res}_{\mathrm{GB}}(D) = \mathsf{Res}_{\mathrm{GB}}(E) = \mathsf{Res}_{\mathrm{GB}}(N)$.
- *N is an orange reduced diagram.*
- *O is a rotation of Or_{2k} for some $k \in \mathbb{N}$.*

That is, we can make D into a diagram E that factors into a diagram of the form Or_{2k} and a diagram with at most one orange boundary strand.

Proof. Let us only consider the case where $\mathsf{Res}_{\mathrm{GB}}(D)$ has exactly one boundary strand, as the case where $\mathsf{Res}_{\mathrm{GB}}(D)$ has empty boundary is similar. We will use relation (3.2.3.1) in D to build a diagram E, containing a diagram N that has at most one orange boundary strand. Note that we can slide any pair of nearby orange boundary strands next to each other and merge them into . If D has exactly two orange boundary strands, we write

$D =$  . (3.2.3.55)

If we set E to be the diagram on the right in (3.2.3.55), then we can set

$N :=$  . (3.2.3.56)

Now it becomes clear that E and N satisfy all the conditions required.

If D has more than two orange strands, we can apply (3.2.3.55) until we are left with at most one orange strands coming out of D. Let E be the diagram obtained at the end of this process, and define N to be the diagram contained in E that only excludes the edges connecting orange boundary points of E and the (at most two) boundary edges of D that were not changed. Then N has at most one orange boundary strand and $\mathsf{Res}_{\mathrm{GB}}(D) = \mathsf{Res}_{\mathrm{GB}}(E) = \mathsf{Res}_{\mathrm{GB}}(N)$. Hence, E and N satisfy all the conditions needed. $\square$

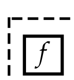
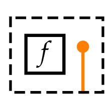
The following corollary is a special case of Lemma 3.2.3.22.

Corollary 3.2.3.23. *Let D be a GB circle/needle. Then $D = O \circ N$, where N is some orange reduced GB circle/needle and O is a rotation of Or_{2k} for some $k \in \mathbb{N}$.*

3.2.4 Polynomial forcing relations

Recall that $R^\tau = \Bbbk[\alpha_s + \alpha_t, \alpha_s\alpha_t]$. By (3.2.2.1), the generator $\alpha_s + \alpha_t$ corresponds to . Also recall that and $R = R^\tau \oplus (\alpha_s - \alpha_t)R^\tau$, and the dots and go to the polynomial $\alpha_s - \alpha_t$ under the functor $\mathbf{F}$. Hence, the orange dot plays the role that the polynomial $\alpha_s - \alpha_t$ played in $\mathcal{H}^\tau$. Because of this we must consider the orange dots, in addition to the green and brown barbells, as the generators of our polynomials in $\mathcal{D}$. This motivates the following definition.

Definition 3.2.4.1. We will call the diagrams , , , and the *polynomial generators* of $\mathcal{D}$. Define an *extended polynomial diagram of $\mathcal{D}$* to be a diagram of the form

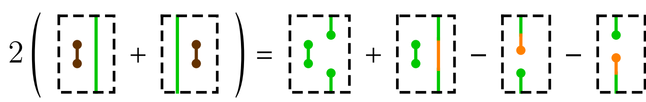
 , or  (3.2.4.1)

for some $f \in R^{\tau}$.

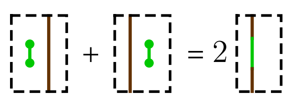
A *basic polynomial forcing relation* in the context of $\mathcal{D}$ writes a polynomial generator on the right region of an orange, green, or brown strand as a sum of diagrams with either no polynomial boxes or polynomial generators, or with polynomial generators only in its leftmost region. Analogously, a general *polynomial forcing relation* writes an extended polynomial diagram on the right region of an orange, green, or brown strand as a sum of diagrams with either no polynomial boxes or polynomial generators, or with an extended polynomial diagram only in its leftmost region.

We already have the polynomial forcing relations (3.2.2.11), (3.2.3.2), and (3.2.3.25). We will now show the rest of the basic polynomial forcing relations for the green and brown strands.

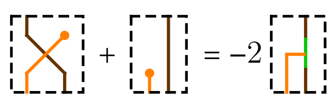
Proposition 3.2.4.2 (Basic polynomial forcing relations)**.** *The following relations follow from the polynomial forcing relation* (3.2.2.11) *and the bigon relations* (3.2.2.16) *and* (3.2.2.17).



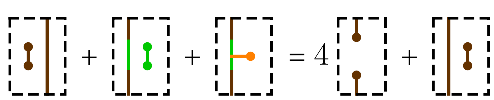
(3.2.4.2)



(3.2.4.3)



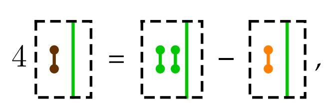
(3.2.4.4)



(3.2.4.5)

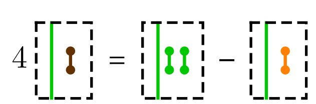
Proof. We will show (3.2.4.2) and (3.2.4.3). The others are similar and will be left to the reader as an exercise.

(3.2.4.2) By (3.2.3.24) we have



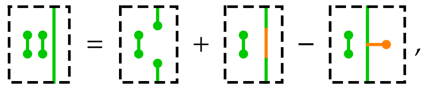
(3.2.4.6)

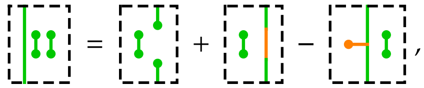
and

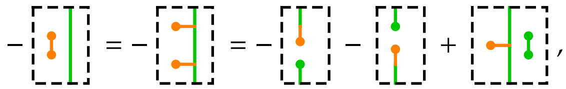


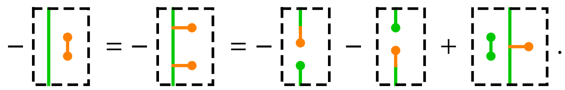
(3.2.4.7)

Using (3.2.2.11) (and orange landing slide relations) we get

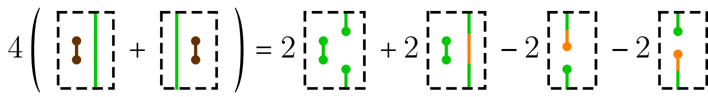
 (3.2.4.8)

 (3.2.4.9)

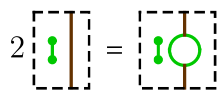
 (3.2.4.10)

 (3.2.4.11)

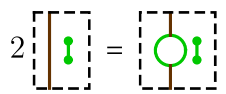
Combining the previous equations we get

 (3.2.4.12)

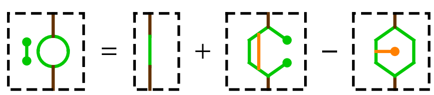
(3.2.4.3) By (3.2.2.16), we have

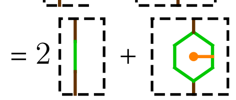
 (3.2.4.13)

and

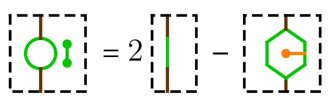
 (3.2.4.14)

By (3.2.2.11) (and orange landing slide relations) we have

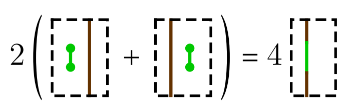
 (3.2.4.15)

 (3.2.4.16)

Similarly,

 (3.2.4.17)

Combining the last two equations, we get

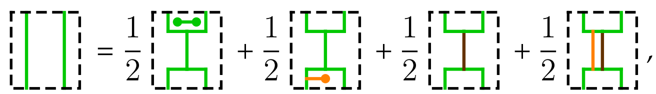
 (3.2.4.18)

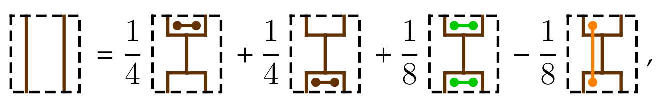
□

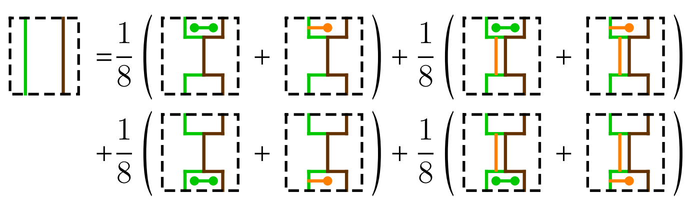
In Appendix A.3, we show the general polynomial forcing and needle relations for general polynomial boxes. These general relations follow from the basic relations by writing polynomial boxes in terms of barbells and applying the basic relations recursively. We must emphasize that the general polynomial forcing relations look very complicated compared to the basic ones.

The following proposition gives explicit idempotent decompositions for the objects $\widehat{\mathrm{Y}} \otimes \widehat{\mathrm{Y}}$, $\widehat{\mathrm{Y}} \otimes \widehat{\mathrm{Z}}$ and $\widehat{\mathrm{Z}} \otimes \widehat{\mathrm{Z}}$.

Proposition 3.2.5.1 (Idempotent decompositions)**.** *The following are decompositions of the identity morphisms of $\widehat{\mathrm{Y}} \otimes \widehat{\mathrm{Y}}$, $\widehat{\mathrm{Z}} \otimes \widehat{\mathrm{Z}}$, and $\widehat{\mathrm{Y}} \otimes \widehat{\mathrm{Z}}$, into orthogonal idempotent summands.*


(3.2.5.1)

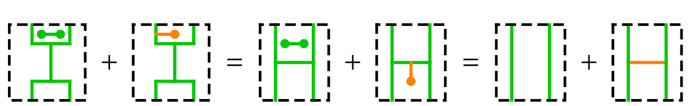

(3.2.5.2)


(3.2.5.3)

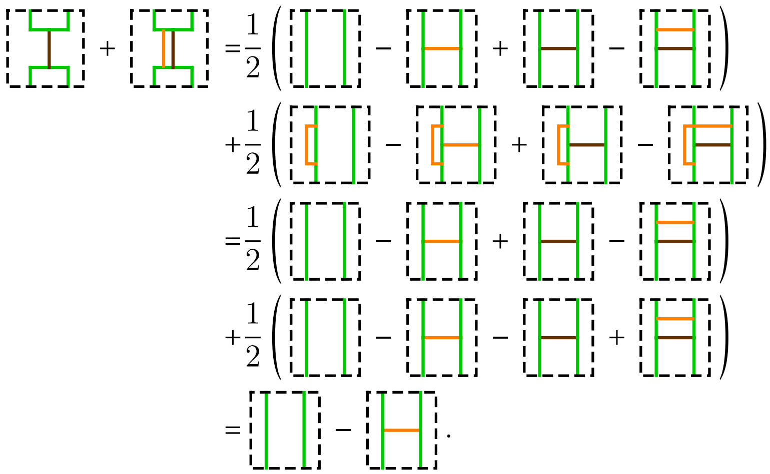
Each of these idempotents factors through the objects in $\widehat{\mathcal{I}} = \{\widehat{\mathbb{1}}, \widehat{\mathrm{X}}, \widehat{\mathrm{Y}}, \widehat{\mathrm{Z}}, \widehat{\mathrm{XZ}}\}$ (up to grading shift).

Proof. We will show (3.2.5.1) and (3.2.5.3). Showing (3.2.5.2) is similar and relies on using an H=I relation and one of the polynomial forcing relations in Proposition 3.2.4.2. We will also leave to the reader to check that the summands are, in fact, orthogonal idempotents. All of them are bigons that can be reduced with an $H = I$ relation (see Appendix A for the full list of relations).

(3.2.5.1) Start by using (3.2.2.11) and (3.2.2.18) on

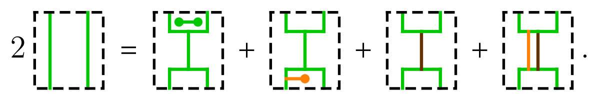

(3.2.5.4)

Now use (3.2.2.22)



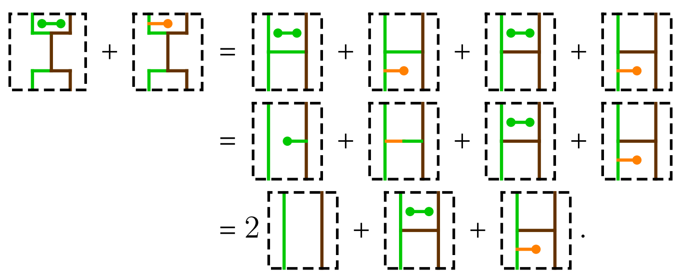
(3.2.5.5)

Combining (3.2.5.4) and (3.2.5.5) we get



(3.2.5.6)

(3.2.5.3) Using (3.2.2.20) and (3.2.2.11) we get

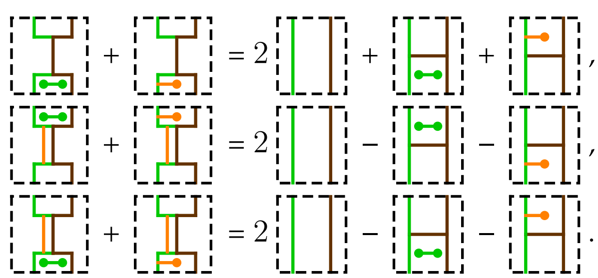


(3.2.5.7)

(3.2.5.8)

(3.2.5.9)

Similarly,

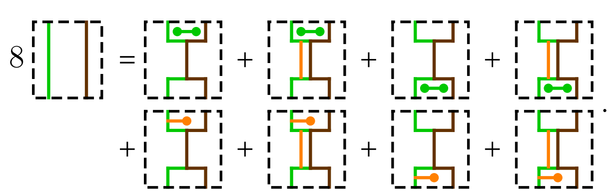


(3.2.5.10)

(3.2.5.11)

(3.2.5.12)

Adding (3.2.5.7)-(3.2.5.12), we get



(3.2.5.13)

□

The following is an analog of Lemma 3.1.2.6 for $\mathcal{D}$.

Theorem 3.2.5.2. *We have the following isomorphisms:*

$$\widehat{\mathrm{X}} \otimes \widehat{\mathrm{X}} \cong \mathbb{1}, \tag{3.2.5.14}$$

$$\widehat{\mathrm{X}} \otimes \widehat{\mathrm{Y}} \cong \widehat{\mathrm{Y}} \cong \widehat{\mathrm{Y}} \otimes \widehat{\mathrm{X}}, \tag{3.2.5.15}$$

$$\widehat{\mathrm{X}} \otimes \widehat{\mathrm{Z}} \cong \widehat{\mathrm{XZ}} \cong \widehat{\mathrm{Z}} \otimes \widehat{\mathrm{X}}, \tag{3.2.5.16}$$

$$\widehat{\mathrm{Y}} \otimes \widehat{\mathrm{Y}} \cong \widehat{\mathrm{Y}}[-1] \oplus \widehat{\mathrm{Y}}[1] \oplus \widehat{\mathrm{Z}} \oplus \widehat{\mathrm{XZ}}, \tag{3.2.5.17}$$

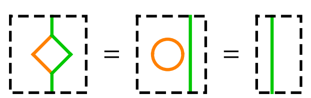
$$\widehat{\mathrm{Y}} \otimes \widehat{\mathrm{Z}} \cong \widehat{\mathrm{Z}}[-1] \oplus \widehat{\mathrm{Z}}[1] \oplus \widehat{\mathrm{XZ}}[-1] \oplus \widehat{\mathrm{XZ}}[1] \cong \widehat{\mathrm{Z}} \otimes \widehat{\mathrm{Y}}, \tag{3.2.5.18}$$

$$\widehat{\mathrm{Z}} \otimes \widehat{\mathrm{Z}} \cong \widehat{\mathrm{Z}}[-2] \oplus \widehat{\mathrm{Z}} \oplus \widehat{\mathrm{Z}}[2] \oplus \widehat{\mathrm{XZ}}. \tag{3.2.5.19}$$

Hence, every object in $\mathcal{D}$ is a direct sum of grading shifts of objects in $\widehat{\mathcal{I}}$.

Proof. (3.2.5.14) The diagrams $\cap$ and $\cup$ correspond to mutually inverse morphisms between $\widehat{\mathrm{X}} \otimes \widehat{\mathrm{X}}$ and $\mathbb{1}$. This is shown in equations (3.2.2.5) and (3.2.3.1).

(3.2.5.15) The diagrams $\curlywedge$ and Y correspond to mutually inverse morphisms between $\widehat{\mathrm{X}} \otimes \widehat{\mathrm{Y}}$ and $\widehat{\mathrm{Y}}$. To check this, note that (3.2.2.12) implies $\mathrm{Y} \circ \curlywedge =$ $||$ and



(3.2.5.20)

(3.2.5.16) The diagrams $\times$ and $\times$ correspond to mutually inverse morphisms between $\widehat{\mathrm{X}} \otimes \widehat{\mathrm{Z}}$ and $\widehat{\mathrm{Z}} \otimes \widehat{\mathrm{X}}$. This is shown in (3.2.3.13).

The isomorphisms (3.2.5.17), (3.2.5.17), and (3.2.5.19) follow from Proposition 3.2.5.1.

□

3.2.6 *Reduction of Diagrams*

The goal of this section is to carefully prove the diagram reduction theorem (Theorem 3.2.6.1) with the use of diagrammatic techniques. This theorem is needed to show that the functor $\mathbf{F} : \mathcal{F} \longrightarrow \mathcal{H}^{\tau}$, defined in Section 3.2.2 is an equivalence of categories.

Theorem 3.2.6.1 (Diagram Reduction). *The spaces* $\mathsf{Hom}\left(\widehat{\mathbb{1}}, \widehat{\mathbb{1}}\right)$, $\mathsf{Hom}\left(\widehat{\mathrm{X}}, \widehat{\mathbb{1}}\right)$, $\mathsf{Hom}\left(\widehat{\mathrm{Y}}, \widehat{\mathbb{1}}\right)$, $\mathsf{Hom}\left(\widehat{\mathrm{Z}}, \widehat{\mathbb{1}}\right)$, *and* $\mathsf{Hom}\left(\widehat{\mathrm{XZ}}, \widehat{\mathbb{1}}\right)$ *have the following R^{τ}-spanning sets:*

(a) $\mathsf{Hom}\left(\widehat{\mathbb{1}}, \widehat{\mathbb{1}}\right)$ *is spanned by the empty diagram* .

(b) $\mathsf{Hom}\left(\widehat{\mathrm{X}}, \widehat{\mathbb{1}}\right)$ *is spanned by* *.*

(c) $\mathsf{Hom}\left(\widehat{\mathrm{Y}}, \widehat{\mathbb{1}}\right)$ *is spanned by* *and* *.*

(d) $\mathsf{Hom}\left(\widehat{\mathrm{Z}}, \widehat{\mathbb{1}}\right)$ *is spanned by* *.*

(e) $\mathsf{Hom}\left(\widehat{\mathrm{XZ}}, \widehat{\mathbb{1}}\right)$ *is spanned by*  *.*

By Proposition 3.2.3.21, any diagram is equal to an orange simplified one, i.e. they have no orange circles, at most one orange landing in each green component, and at most one orange dot in each GB region. Unless otherwise stated, wee will assume that all the diagrams in this section are orange simplified.

Definition 3.2.6.2. We say a diagram D is a *tree* if the complement of the underlying graph is connected. That is, and doesn't contain any cycles (it may still have polynomial boxes).

We say that D is a *tree without boundary* if D is a tree and doesn't have any boundary strands (e.g. a barbell). We say D is a *based tree* if D is a tree and contains a single boundary strand.

We say that D is a *forest* if all of its connected components are trees. We say that D is a *based forest* if all of its connected components are either based trees or trees without boundary.

We say that D is a *GB tree (resp. based tree, forest, based forest)* if $\mathsf{Res}_{GB}(D)$ is a tree (resp. based tree, forest, based forest).

Note that a based forest does not partition the strip $[0,1]\times[0,1]$ into more than one region. Hence, every based forest is an R^{τ}-multiple of a based forest with no polynomials.

Lemma 3.2.6.3. *Let D be a GB tree.*

(a) *If D has no boundary strands, D is in the R^{τ}-span of* $\{$$\}$.

(b) *If the boundary of D is one orange strand, D is in the R^{τ}-span of* $\{$$\}$.

(c) *If the boundary of D is one green strand, D is in the R^{τ}-span of* $\{$, $\}$.

(d) *If the boundary of* D *is one green strand and one orange strand,* D *is in the* R^τ*-span of* $\{$, $\}$.

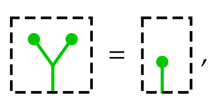
(e) *If the boundary of* D *is one brown strand,* D *is in the* R^τ*-span of* $\{$$\}$.

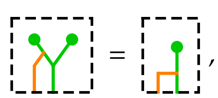
(f) *If the boundary of* D *is one brown strand and one orange strand,* D *is in the* R^τ*-span of* $\{$$\}$.

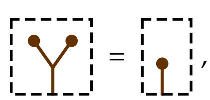
Proof. Let D be a green-brown tree whose boundary is either empty, an orange strand, a green strand, a brown strand, an orange and a green strand, or an orange and a brown strand. Note that $\mathrm{Res}_{\mathrm{GB}}(D)$ has only one region, so we may suppose that D has no polynomial boxes (as they can be pulled out). Also, D has at most one orange dot since it is orange simplified.

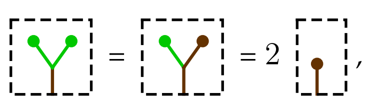
This proof will go by induction on the number of trivalent vertices of $\mathrm{Res}_{\mathrm{GB}}(D)$, after replacing any bivalent vertices by a corresponding trivalent vertex with a dot. When $\mathrm{Res}_{\mathrm{GB}}(D)$ has no trivalent vertices, then $\mathrm{Res}_{\mathrm{GB}}(D)$ is either empty, , or . If $\mathrm{Res}_{\mathrm{GB}}(D)$ is empty then D can't have any orange landings. Since we assumed that D is orenge simplified, it must be a polynomial or an R^τ-multiple of . Similarly, if $\mathrm{Res}_{\mathrm{GB}}(D)$ is a green or a brown dot then D must be an R^τ-multiple of , , , , , or . This constitutes our base case.

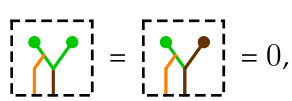
Now suppose that $\mathrm{Res}_{\mathrm{GB}}(D)$ has at least one trivalent vertex. Then, the tree $\mathrm{Res}_{\mathrm{GB}}(D)$ must have a trivalent vertex with two dots (leaves). Let T be a closed neighborhood (in D) that only includes this trivalent vertex. By Proposition 3.2.3.9, we may slide out any orange crossings on T. Since we assume that T is orange simplified, then T is one of the following:

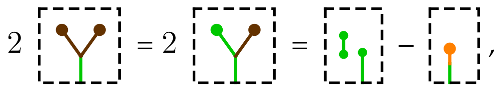
 (3.2.6.1)

 (3.2.6.2)

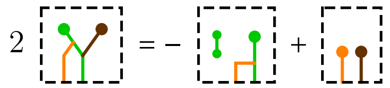
 (3.2.6.3)

 (3.2.6.4)

 (3.2.6.5)



(3.2.6.6)



(3.2.6.7)

In each case, we write that trivalent vertex T as a sum of diagrams with no GB trivalent vertices. After replacing T by the corresponding expression with no GB trivalent vertices, we are left with a diagram D' with one less GB trivalent vertex. By induction the result follows. $\square$

Corollary 3.2.6.4. *Let D be a GB based forest. Then $D = f_1 D_1 + \ldots + f_k D_k$ for some $f_i \in R^\tau$ and diagrams D_i and such that $\mathsf{Res}_{\mathrm{GB}}(D_i)$ is a union of boundary dots. In particular, if $\mathsf{Res}_{\mathrm{GB}}(D)$ has empty boundary, then $D = f\,\mathsf{Or}_k$ for some $f \in R^\tau$ and k is the number of orange boundary strands of D.*

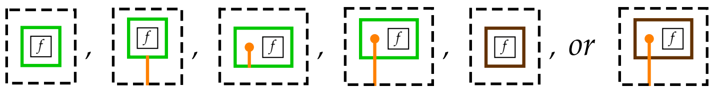
Proof. It suffices to prove this when D is a GB based tree, since any based forest can be split into finitely many based trees.

By Lemma 3.2.3.22, we may suppose that D is the composition of a diagram N with at most one orange boundary strand, and a diagram O that is a rotation of Or_{2k} for some $k \in \mathbb{N}$. By Lemma 3.2.6.3, N is in the R^τ-span of the sets { }, { }, { , } , { , } , { } , or { }. Let $N = f_1 D_1 + f_2 D_2$ where D_1' and D_2' are both in one of the generating sets before and $f_1, f_2 \in R^\tau$, so that $D = f_1 D_1 \circ O + f_2 D_2 \circ O$. Since $\mathsf{Res}_{\mathrm{GB}}(D_1 \circ O) = \mathsf{Res}_{\mathrm{GB}}(D_1)$ and $\mathsf{Res}_{\mathrm{GB}}(D_2 \circ O) = \mathsf{Res}_{\mathrm{GB}}(D_2)$ are either empty or a boundary dot, the result follows.

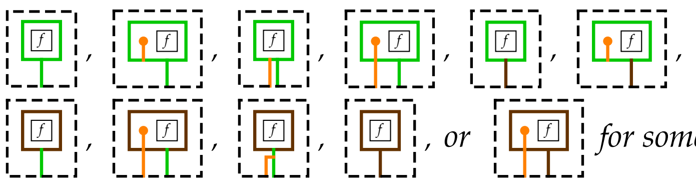
Suppose now that D is a based GB forest with k orange boundary strands, such that $\mathsf{Res}_{\mathrm{GB}}(D)$ has empty boundary. Applying our result to D, we can write $D = f_1 D_1 + \ldots + f_k D_k$ for some $f_i \in R^\tau$ and diagrams D_i such that $\mathsf{Res}_{\mathrm{GB}}(D_i)$ is the empty diagram for all i. Since each D_i is entirely made of orange strands and dots, each D_i must be an R^τ-multiple of Or_k. Hence, $D = f\,\mathsf{Or}_k$ for some $f \in R^\tau$. $\square$

The argument to prove Theorem 3.2.6.1 will go by induction, first on the total number of GB bivalent and trivalent vertices, and then on the number of GB components of our diagram (given it has the same total number of GB bivalent and trivalent vertices). For our base case, we need to prove the theorem for diagrams with at most one GB component and at most one GB bivalent or trivalent vertex. Since we already showed the theorem for GB forests, the base cases left to check are orange reduced GB circles and needles.

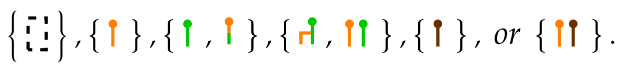
Lemma 3.2.6.5. *Let D be a GB connected orange reduced diagram with at most 1 GB trivalent vertex. If D has no trivalent vertices, then D is an R^τ-multiple of a diagram of the form*

 *, for some* $f \in R^\tau$. (3.2.6.8)

If D has exactly one trivalent vertex, then D is (up to rotation) an R^τ-multiple of a diagram of the form

 *for some* $f \in R^\tau$. (3.2.6.9)

All of these diagrams are in the R^τ-span of

 (3.2.6.10)

Proof. Since D is not a GB forest, $\mathrm{Res}_{\mathrm{GB}}(D)$ must be a circle or a needle. The diagrams listed, correspond to the all possible circles and needles, possibly adding an orange boundary strand or an orange dot. Proposition A.3.0.4 shows that the diagrams in (3.2.6.8) are in the span of the generators in (3.2.6.10). Proposition A.3.0.3 shows that the diagrams in (3.2.6.9) are in the span of the generators in (3.2.6.10). □

The following is a generalization of GB circles and needles.

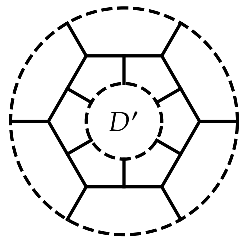
Definition 3.2.6.6. An *n-cycle with spokes* is a pair of diagrams (D, D') where D' is contained in the interior of D and the annulus obtained by excising D' from D forms an n-cycle entirely made by:

(a) Bivalent vertices.

(b) Trivalent vertices with a strand to the boundary of D, which we will call the *outside spokes*.

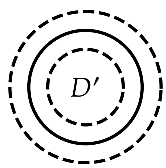
(c) Trivalent vertices with a strand to the boundary of D', which we will call the *inside spokes*.

This implies that the boundary of D' corresponds to the inside spokes. We will refer to D' as the *inside* of (D, D'). Diagramatically,

$D =$ 

. (3.2.6.11)

We note that when $k = 0$, D is of the form

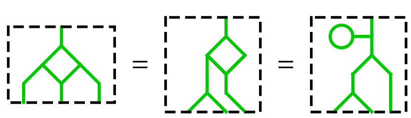
$D =$ 

. (3.2.6.12)

We say that our cycle with spokes is *empty* if D' is empty. In particular, (D, D') has no inside spokes.

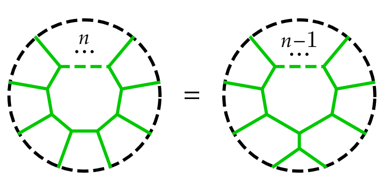
We say that our cycle with spokes is *innermost* if D' is a based forest. Abusing notation, we say that D is a cycle with spokes (without specifying the interior) if there exists a diagram D' contained in D such that (D, D') is a cycle with spokes. We say that D is a *GB cycle with spokes (resp. empty/innermost cycle with spokes)* if $(\mathrm{Res}_{GB}(D)$ is a cycle with spokes (resp. empty/innermost cycle with spokes).

We want to show that any empty GB cycle with spokes D is equal to an R^τ-combination of GB forests. Before we do the general case, let our initial diagram be the green 4-cycle and apply the green $H = I$ relation repeatedly:



. (3.2.6.13)

Observe how we end up getting a tree with a needle attached to it. In general, when we apply the $H = I$ relation to a green n-cycle with spokes we get

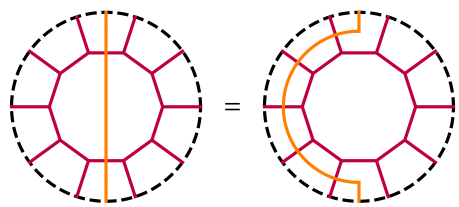


. (3.2.6.14)

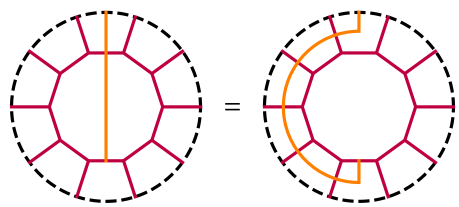
The right hand side becomes a cycle with spokes of size $n - 1$ attached to a trivalent vertex. If we apply the $H = I$ relation repeatedly, we will eventually get a tree and a needle (as in the case for $n = 4$). That is the main idea for the proof of the following lemma.

Lemma 3.2.6.7. *Let D be an empty GB cycle with spokes. Then $D = D_1 + \ldots + D_k$ for some diagrams D_i such that $\mathsf{Res}_{\mathrm{GB}}(D_i)$ is a forest with fewer trivalent vertices than $\mathsf{Res}_{\mathrm{GB}}(D)$, or the same number of trivalent vertices and fewer GB components than $\mathsf{Res}_{\mathrm{GB}}(D)$.*

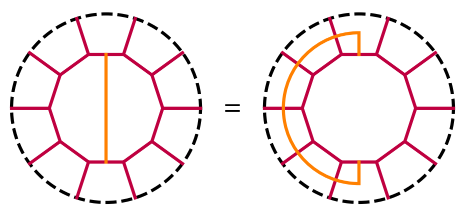
Proof. Let D be an empty GB cycle with spokes. We can assume that D has no orange strands splitting the interior of its cycle into 2 different regions. That is because we can slide any orange strand outside the cycle. Here are the three possible cases.



(3.2.6.15)



(3.2.6.16)

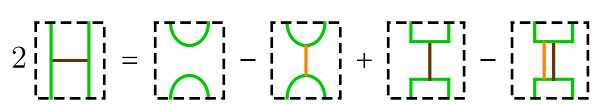


(3.2.6.17)

The proof will go by induction on the size of the cycle of $\mathsf{Res}_{\mathrm{GB}}(D)$. We showed our base case in Lemma 3.2.6.5.

Now let $\mathsf{Res}_{\mathrm{GB}}(D)$ be a cycle of size 2 or greater, writing any GB bivalent vertices in D by their definition as a trivalent vertex with a dot on the outside of the cycle. We will now make use of all the $H = I$ relations in Appendix A.2.

In most cases, when we apply an $H = I$ relation to the outside cycle of D we get that $D = D_1 + \ldots + D_r$ where each D_i satisfies that $\mathsf{Res}_{\mathrm{GB}}(D_i)$ is an empty $n-1$-cycle with spokes attached to a trivalent vertex (unlike equation (3.2.6.14) we get a sum of diagrams of this form). The one exception is (3.2.2.22),



(3.2.6.18)

whose first two summands break the cycle, making them trees with 2 fewer GB trivalent vertices right away. Thus, if we show that our diagram D equals a GB cycle with spokes E containing a GB crossbar (to be able to apply an $H = I$ relation), we are done.

If the central cycle has a brown edge, we can use Lemma 3.2.3.16 to get E. Otherwise, the central cycle in D is made entirely of green edges, all in the same green component. Since we assume that D is orange simplified, there is at most one orange landing over its green component, and we can apply Lemma 3.2.3.16 to an edge that doesn't have an orange landing. This concludes the proof. $\square$

Corollary 3.2.6.8. *Let D be an innermost RG cycle with spokes. Then $D = E_1 + \dots + E_k$ for diagrams E_i such that each $\mathsf{Res}_{\mathrm{GB}}(E_i)$ is a forest with fewer trivalent vertices than $\mathsf{Res}_{\mathrm{GB}}(D)$.*

Proof. Let D be a diagram such that $\mathsf{Res}_{\mathrm{GB}}(D)$ is an innermost cycle with spokes and let D' be the inside of D. Since $\mathsf{Res}_{\mathrm{GB}}(D')$ is a based forest, by Corollary 3.2.6.4, we can write $D' = D'_1 + \dots + D'_m$, where each $\mathsf{Res}_{\mathrm{GB}}(D'_j)$ is a union of boundary dots. Thus, $D = D_1 + \dots + D_m$, where each $\mathsf{Res}_{\mathrm{GB}}(D_j)$ is a cycle with spokes whose inside is a union of boundary dots.

We now want to apply unit relations to the dots in the inside of each D_j. There are three types of unit relations:

- (3.2.2.4),(3.2.2.6),(3.2.2.7), (A.1.0.6), and (A.1.0.8), that merge a dot into a green or brown strand, or into a green-brown bivalent vertex. After using one of these relations on D_j, we end up with a GB cycle with spokes whose interior has one fewer boundary dot. If D_j only has boundary dots of these types, we can use our relations to make D_j into an empty GB cycle with spokes.

- (3.2.2.8), that splits the trivalent vertex into an R^τ-combination of  and  . This relation transforms D_j into a GB tree, as it breaks the cycle with spokes (with at least one trivalent vertex less than D). Note that using this more than once will result in a forest with more than one GB tree.

- (A.1.0.7), which makes the diagram equal to zero.

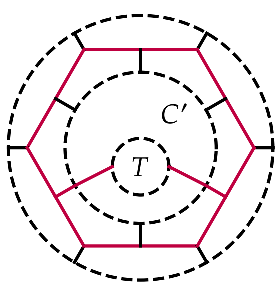
In the last two cases, D_j becomes a forests right away. In the first case, instead, we can use Lemma 3.2.6.7 to write $D_j = E_1 + \dots + E_k$ for some diagrams

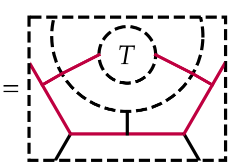
$E_1, \ldots, E_k$, where each $\mathrm{Res}_{\mathrm{GB}}(E_i)$ is a forest and has fewer trivalent vertices or fewer GB components than $\mathrm{Res}_{\mathrm{GB}}(D_j)$. This completes the proof. □

Lemma 3.2.6.9. *Let D be a diagram such that D is not a GB forest. Then there is a diagram N contained in D such that N is an innermost GB cycle with spokes.*

Proof. Let $\mathbf{C}$ be the set of all GB cycles with spokes contained in D. The set $\mathbf{C}$ is partially ordered by containment. Since D is not a GB forest, the set $\mathbf{C}$ is non-empty. Let (C, C') be a minimal element of $\mathbf{C}$. By minimality, C' must be a GB forest. We claim that C' is a GB based forest and (C, C') is an innermost GB cycle with spokes.

Let us now suppose that $\mathrm{Res}_{\mathrm{GB}}(C')$ is a GB forest, but not a based forest. This means that $\mathrm{Res}_{\mathrm{GB}}(C')$ contains a green-brown tree with at least 2 boundary strands connecting to C. Let T be such tree. Let P be a diagram inside C containing T and also containing all the edges in the outer cycle connecting the two boundary strands of T, but excluding at least one trivalent vertex in the outer cycle of C. Diagrammatically

$C =$ 

, and $P =$ 

. (3.2.6.19)

But P contains a GB cycle with spokes and this contradicts the minimality of C. □

Now we will prove the theorem.

Proof of Theorem 3.2.6.1. Let D be a diagram such that $\mathrm{Res}_{\mathrm{GB}}(D)$ has at most one boundary strand and D has at most one orange boundary strand. Hence, we will assume that D is an orange reduced diagram. The proof will go by induction on the total number of bivalent and trivalent vertices of $\mathrm{Res}_{\mathrm{GB}}(D)$ first, and then on the number of GB components of D.

Suppose that D has no barbells, since we can use the barbell relations (3.2.2.1) to obtain a diagram with one fewer GB component. Similarly, if $\mathrm{Res}_{\mathrm{GB}}(D)$ is a GB forest, we can apply Lemma 3.2.6.4, so suppose that D is not a forest.

First suppose that D has at most one GB trivalent vertex and at most one GB component. Since D is not a forest, Lemma 3.2.6.5 shows our base case.

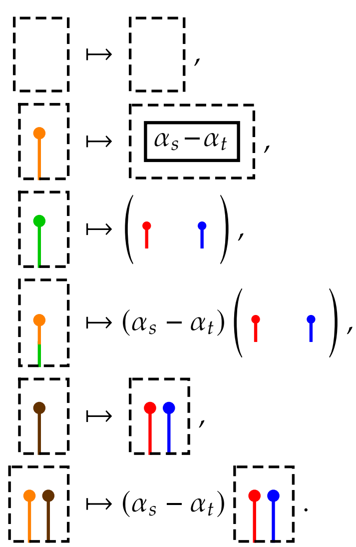
Now suppose that $\mathrm{Res}_{\mathrm{GB}}(D)$ has more than one trivalent vertex. Since D is not a GB forest, by Lemma 3.2.6.9, D has an innermost cycle with spokes. Let C be an innermost cycle with spokes contained in D. Use Corollary 3.2.6.8 to write $C = E_1 + \dots + E_k$ for diagrams E_i, such that $\mathrm{Res}_{\mathrm{GB}}(E_i)$ is a forest with either fewer trivalent vertices than $\mathrm{Res}_{\mathrm{GB}}(C)$, or with the same number of trivalent vertices but fewer GB components. Hence $D = D_1 + \dots + D_k$, where D_i is the diagram obtained by replacing C for E_i. Each D_i either has fewer trivalent vertices than D, or has the same number of trivalent vertices as D but has fewer GB components. Hence we can apply our induction hypothesis to all the D_i's. This concludes our proof. $\square$

3.2.7 *Equivalence between $\mathcal{D}$ and $\mathcal{H}^\tau$*

We will now prove that the functor $\mathbf{F}$ is an equivalence. This requires the use of Theorem 3.2.6.1, as well as Theorem 3.2.5.1.

Proposition 3.2.7.1. *The functor* $\mathbf{F}$ *induces an isomorphism on* $\mathrm{Hom}\left(\widehat{\mathbb{1}},\widehat{\mathbb{1}}\right)$, $\mathrm{Hom}\left(\widehat{\mathrm{X}},\widehat{\mathbb{1}}\right)$, $\mathrm{Hom}\left(\widehat{\mathrm{Y}},\widehat{\mathbb{1}}\right)$, $\mathrm{Hom}\left(\widehat{\mathrm{Z}},\widehat{\mathbb{1}}\right)$, *and* $\mathrm{Hom}\left(\widehat{\mathrm{XZ}},\widehat{\mathbb{1}}\right)$. *In particular, these are all free R^τ-modules.*

Proof. Recall that $\mathcal{I} = \{\mathbb{1}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}, \mathrm{XZ}\}$ and $\widehat{\mathcal{I}} = \{\widehat{\mathbb{1}}, \widehat{\mathrm{X}}, \widehat{\mathrm{Y}}, \widehat{\mathrm{Z}}, \widehat{\mathrm{XZ}}\}$. Let $A \in \mathcal{I}$ and $\widehat{A} \in \widehat{\mathcal{I}}$. Proposition 3.1.3.1 shows that $\mathrm{Hom}(A, \mathbb{1})$ is a free R^τ-module and gives an R^τ-basis of it. In Theorem 3.2.6.1 we gave a generating set of $\mathrm{Hom}(\widehat{A}, \widehat{\mathbb{1}})$. We have that $\mathbf{F}$ sends



(3.2.7.1)

(3.2.7.2)

(3.2.7.3)

(3.2.7.4)

(3.2.7.5)

(3.2.7.6)

That is, $\mathbf{F}$ sends the generators in Theorem 3.2.6.1 to the basis elements in Proposition 3.1.3.1. Hence, $\mathbf{F}$ induces an isomorphism on $\mathsf{Hom}(\widehat{A}, \widehat{\mathbb{1}})$. □

Theorem 3.2.7.2. *The functor* $\mathbf{F}$ *is an equivalence of categories.*

Proof. We have to show that $\mathbf{F}$ is full and faithful (we already have that $\mathbf{F}$ is essentially surjective). By Theorem 3.2.5.1, every object of $\mathcal{D}$ is isomorphic to a direct sum of grading shifts of objects in $\widehat{\mathcal{I}}$. By Proposition 3.2.7.1, $\mathbf{F}$ is bijective on $\mathsf{Hom}(\widehat{A}, \widehat{\mathbb{1}})$ for all $A \in \widehat{\mathcal{I}}$. By Proposition 2.1.2.7, $\mathbf{F}$ is an equivalence. □

CHAPTER 4

LOCALIZATION OF THE FOLDED HECKE CATEGORY

4.1 Categorical Preliminary: Object-Adapted Cellular Categories

In Section 1.3.1, we introduced cellular algebras and cellular categories. In this section we will give the precise definition of *strictly object adapted cellular categories*, or *SOACCs*, *fibered strictly object adapted cellular categories*, or *fibered SOACCs*, and *fibered non-strictly object adapted cellular categories*, or *fibered NSOACCs*. Definitions 4.1.0.1 and 4.1.0.3 are given by Elias and Lauda in [EL16], while Definition 4.1.0.5 is a variation of the previous ones to accommodate for $\mathcal{H}^{\tau}$. We must remark that Definition 4.1.0.5 in the non-fibered context is suggested in [EW16, Rema. 2.12]. See [EL16] and [Eli+20, Sec 11.3] for an in-depth treatment of the properties of these categories.

Definition 4.1.0.1 (Strictly object adapted cellular category)**.** Let R be a graded commutative ring and let $\mathcal{C}$ be an R-linear graded category. A *(graded) strictly object adapted cellular category (SOACC)* is a category $\mathcal{C}$ together with the following data:

- there is a set of objects $\{J_\lambda\}_{\lambda\in\Lambda}$ of $\mathcal{C}$ indexed by a poset Λ that satisfies the descending chain condition.

- for each object $X \in \mathcal{C}$ and J_λ, we have two finite sets of homogeneous morphisms

$$\mathbf{LL}_X^{J_\lambda} := \{LL_p | p \in \mathsf{E}(X, J_\lambda)\} \subseteq \mathsf{Hom}(X, J_\lambda) \text{ and } \mathbf{UL}_{J_\lambda}^{X} := \{UL_p \,|p \in \mathsf{M}(J_\lambda, X)\} \subseteq \mathsf{Hom}(J_\lambda, X) \tag{4.1.0.1}$$

  indexed by sets $\mathsf{E}(X, J_\lambda)$ and $\mathsf{M}(J_\lambda, X)$ respectively. For $X, Y \in \mathcal{C}$ and $\lambda \in \Lambda$, we subsequently define the sets define the sets

$$\mathbf{DL}_X^Y(\lambda) := \{U \circ L | L \in \mathbf{LL}_X^{J_\lambda}, U \in \mathbf{UL}_{J_\lambda}^Y\} \text{ and } \mathbf{DL}_X^Y := \bigcup_{\lambda\in\Lambda} \mathbf{DL}_X^Y(\lambda). \tag{4.1.0.2}$$

- there are a pair of mutually inverse maps (of sets) between $\mathsf{E}(X, J_\lambda)$ and $\mathsf{M}(J_\lambda, X)$ that we denote by $\bar{\cdot}$ (in both directions).

- there is a contravariant involution of $\mathcal{C}$ also denoted by $\bar{\cdot}$.

For $X, Y \in \mathcal{C}$, define

$$\mathsf{Hom}_{<\lambda}(X,Y) := \mathsf{span}_R\left(\bigcup_{\mu<\lambda} \mathbf{DL}_X^Y(\mu)\right) \text{ and } \mathsf{Hom}_{\leq\lambda}(X,Y) := \mathsf{span}_R\left(\bigcup_{\mu\leq\lambda} \mathbf{DL}_X^Y(\mu)\right). \quad (4.1.0.3)$$

Also define $\mathfrak{J}_{<\lambda}$ and $\mathfrak{J}_{\leq\lambda}$ to be the non-unital subcategories of $\mathcal{C}$ whose morphism spaces are generated by $\mathsf{Hom}_{<\lambda}(X,Y)$ and $\mathsf{Hom}_{\leq\lambda}(X,Y)$ respectively, under the operations of composition and direct sum. Note that we do not require the identity morphisms of all the objects of $\mathcal{C}$ to be in $\mathfrak{J}_{\leq\lambda}$.

This data is subject to the following conditions:

(a) the set $\mathbf{DL}_X^Y$ is an R-basis of $\mathsf{Hom}(X,Y)$.

(b) the contravariant involution $\bar{\cdot}$ extends the maps on $\mathsf{E}(X, J_\lambda)$ and $\mathsf{M}(J_\lambda, X)$. That is, for all $p \in \mathsf{E}(X, J_\lambda)$ we have

$$\overline{LL_p} = UL_{\overline{p}}\,. \quad (4.1.0.4)$$

(c) for all $J_\lambda \in \Lambda$, the sets $\mathsf{E}(J_\lambda, J_\lambda)$ and $\mathsf{M}(J_\lambda, J_\lambda)$ are singletons and

$$\mathbf{LL}_{J_\lambda}^{J_\lambda} = \mathbf{UL}_{J_\lambda}^{J_\lambda} = \{\mathrm{id}_{J_\lambda}\}. \quad (4.1.0.5)$$

(d) $\mathfrak{J}_{\leq J_\lambda}$ is a double-sided monoidal ideal of $\mathcal{C}$. That is, $\mathfrak{J}_{\leq J_\lambda}$ is closed under left and right composition by morphisms in $\mathcal{C}$.

The following lemma from [EL16] shows that any SOACC satisfies the following version of the *cellular formula*.

Lemma 4.1.0.2 ([EL16, Lemma 2.8]). *Let $\mathcal{C}$ be a potential SOACC satisfying conditions (a)-(c). Then $\mathfrak{J}_{\leq\lambda}$ is a double-sided monoidal ideal of $\mathcal{C}$ if and only if for all $UL_p \in \mathbf{UL}_{J_\lambda}^X$ and all $T : X \longrightarrow Y$ there are $\ell_T(f,p,p') \in R$ such that*

$$T \circ UL_p = \sum_{p' \in \mathsf{M}(\lambda, Y)} \ell_T(f,p,p')\, UL_{p'} \mod \mathfrak{J}_{<\lambda}\,. \quad (4.1.0.6)$$

A fibered SOACC is similar to a SOACC, while also allowing for the quotients $\mathsf{Hom}_{\leq\lambda}(J_\lambda, J_\lambda) / \mathsf{Hom}_{<\lambda}(J_\lambda, J_\lambda)$ to be finite dimensional algebras over R. It has the following definition.

Definition 4.1.0.3 (Fibered strictly object adapted cellular category)**.** Let R be a graded commutative ring and let $\mathcal{C}$ be an R-linear graded category. A *(graded) strictly object adapted cellular category (fibered SOACC)* is a category $\mathcal{C}$ together with the following data:

- there is a set of objects $\{J_\lambda\}_{\lambda\in\Lambda}$ of $\mathcal{C}$ indexed by a poset Λ that satisfies the descending chain condition.
- for each J_λ we have a finite set of homogeneous morphisms

$$\mathbf{N}(J_\lambda) := \{N_p \,|p \in \mathsf{A}(J_\lambda)\} \subseteq \mathsf{Hom}(J_\lambda, J_\lambda) \tag{4.1.0.7}$$

  indexed by a finite set $\mathsf{A}(J_\lambda)$. Define

$$A_\lambda := \mathsf{span}_R\, \mathbf{N}(J_\lambda) \tag{4.1.0.8}$$

- for each object $X \in \mathcal{C}$ and J_λ, we have two finite sets of homogeneous morphisms

$$\mathbf{LL}_X^{J_\lambda} := \{LL_p|p \in \mathsf{E}(X, J_\lambda)\} \subseteq \mathsf{Hom}(X, J_\lambda) \text{ and } \mathbf{UL}_{J_\lambda}^X := \{UL_p \,|p \in \mathsf{M}(J_\lambda, X)\} \subseteq \mathsf{Hom}(J_\lambda, X) \tag{4.1.0.9}$$

  indexed by sets $\mathsf{E}(X, J_\lambda)$ and $\mathsf{M}(J_\lambda, X)$ respectively.

  For $X, Y \in \mathcal{C}$ and $\lambda \in \Lambda$, we subsequently define the sets define the sets

$$\mathbf{DL}_X^Y(\lambda) := \{U \circ N \circ L|L \in \mathbf{LL}_X^{J_\lambda}, N \in \mathbf{N}(J_\lambda), U \in \mathbf{UL}_{J_\lambda}^Y\} \text{ and } \mathbf{DL}_X^Y := \bigcup_{\lambda\in\Lambda} \mathbf{DL}_X^Y(\lambda). \tag{4.1.0.10}$$

- there are a pair of mutually inverse maps (of sets) between $\mathsf{E}(X, J_\lambda)$ and $\mathsf{M}(J_\lambda, X)$ that we denote by $\bar{\cdot}$ (in both directions).
- there is an involution of $\mathsf{A}(J_\lambda)$ also denoted by $\bar{\cdot}$.

Define

$$\mathsf{Hom}_{\leq\lambda}(X, Y) := \mathsf{span}_R\left(\bigcup_{\mu\leq\lambda} \mathbf{DL}_X^Y(J_\mu)\right). \tag{4.1.0.11}$$

Define $\mathfrak{I}_{\leq\lambda}$ to be the non-unital subcategory of $\mathcal{C}$ whose morphism spaces are generated by $\mathsf{Hom}_{\leq\lambda}(X, Y)$ under the operations of composition and direct sum. Note that we do not require the identity morphisms of all the objects of $\mathcal{C}$ to be in $\mathfrak{I}_{\leq\lambda}$.

This data is subject to the following conditions:

(a) the identity $\mathrm{id}_{J_\lambda} \in \mathbf{N}(J_\lambda)$ and the set A_λ is closed under composition. Hence, A_λ is a finite R-algebra.

(b) the set $\mathbf{DL}_X^Y$ is an R-basis of $\mathsf{Hom}(X, Y)$.

(c) there is a contravariant involution $\bar{\cdot}$ on $\mathcal{C}$ that extends the maps on $\mathsf{E}(X, J_\lambda), \mathsf{M}(J_\lambda, X)$, and $\mathsf{A}(J_\lambda)$. That is, for all $p \in \mathsf{E}(X, J_\lambda)$ we have

$$\overline{LL_p} = UL_{\overline{p}}, \tag{4.1.0.12}$$

and for all $q \in \mathsf{A}(J_\lambda)$ we have

$$\overline{N_q} = N_{\overline{q}}. \tag{4.1.0.13}$$

(d) for all $\lambda \in \Lambda$, the sets $\mathsf{E}(J_\lambda, J_\lambda)$ and $\mathsf{M}(J_\lambda, J_\lambda)$ are singletons and

$$\mathbf{LL}_{J_\lambda}^{J_\lambda} = \mathbf{UL}_{J_\lambda}^{J_\lambda} = \{\mathrm{id}_{J_\lambda}\}. \tag{4.1.0.14}$$

(e) $\mathfrak{I}_{\le\lambda}$ is a two-sided monoidal ideal of $\mathcal{C}$.

Remark 4.1.0.4. In [EL16], the definition of a fibered SOACC is given in terms of the finite algebras A_λ, without the need to specify a basis. This definition is equivalent to the one in [EL16] because given an involution $\bar{\cdot}$ of A_λ, one can always find a basis $B \subseteq A_\lambda$ where $\bar{\cdot}$ acts by permuting its elements.

Definition 4.1.0.5 (Fibered non-strictly object adapted cellular category)**.** Let R be a graded commutative ring and let $\mathcal{C}$ be an R-linear graded category. A *(graded) fibered non-strictly object adapted cellular category (fibered NSOACC)* is a category $\mathcal{C}$ together with the following data:

- there is a poset Λ with partial order $\le$ satisfying the descending chain condition.
- for each $\lambda \in \Lambda$, there is a finite set $\mathsf{C}(\lambda)$ of objects of $\mathcal{C}$.
- for all $I, J \in \mathsf{C}(\lambda)$ we have a finite set of homogeneous morphisms

$$\mathbf{N}_I^J :=:= \{N_p \,|p \in \mathsf{A}(I, J)\} \subseteq \mathsf{Hom}(I, J) \tag{4.1.0.15}$$

indexed by the set $\mathsf{A}(I, J)$. Also define

$$A(\lambda) := \bigoplus_{I,J \in \mathsf{C}(\lambda)} \mathsf{span}_R\, \mathbf{N}_I^J. \tag{4.1.0.16}$$

- for each object $X \in \mathcal{C}$ and $J \in \mathsf{C}(\lambda)$, we have two finite sets of homogeneous morphisms

$$\mathbf{LL}_X^J := \{LL_p | p \in \mathsf{E}(X,J)\} \subseteq \mathsf{Hom}(X,J) \text{ and } \mathbf{UL}_J^X := \{UL_p \,|p \in \mathsf{M}(J,X)\} \subseteq \mathsf{Hom}(J,X) \tag{4.1.0.17}$$

  indexed by sets $\mathsf{E}(X,J)$ and $\mathsf{M}(J,X)$ respectively. Also, for $\lambda \in \Lambda$ define

$$\mathbf{LL}_X^\lambda := \bigcup_{J \in \mathsf{C}(\lambda)} \mathbf{LL}_X^J \text{ and } \mathbf{UL}_\lambda^X := \bigcup_{J \in \mathsf{C}(\lambda)} \mathbf{UL}_J^X. \tag{4.1.0.18}$$

- there is a pair of mutually inverse maps (of sets) between $\mathsf{A}(I,J)$ and $\mathsf{A}(J,I)$ denoted by $\bar{\cdot}$ (in both directions).

- there is a pair of mutually inverse maps (of sets) between $\mathsf{E}(X,L)$ and $\mathsf{M}(L,X)$ also denoted by $\bar{\cdot}$.

For $X, Y \in \mathcal{C}$ and $\lambda \in \Lambda$, define the set $\mathbf{DL}_X^Y(\lambda)$ to be the set of all morphisms of the form $U \circ N \circ L$ such that $L \in \mathbf{LL}_X^\lambda$, $N \in \mathbf{N}(\lambda)$, and $U \in \mathbf{UL}_\lambda^Y$. Also define

$$\mathbf{DL}_X^Y := \bigcup_{\lambda \in \Lambda} \mathbf{DL}_X^Y(\lambda) \text{ and } \mathsf{Hom}_{\leq \lambda}(X,Y) := \mathsf{span}_R\left(\bigcup_{\mu \leq \lambda} \mathbf{DL}_X^Y(\mu)\right). \tag{4.1.0.19}$$

Define $\mathfrak{J}_{\leq \lambda}$ to be the non-unital subcategory of $\mathcal{C}$ whose morphism spaces are generated by $\mathsf{Hom}_{\leq I}(X,Y)$ under the operations of composition and direct sum. As before, we do not require the identity morphisms of all the objects of $\mathcal{C}$ to be in $\mathfrak{J}_{\leq \lambda}$.

This data is subject to the following conditions:

(a) the morphism $\mathrm{id}_J \in \mathbf{N}(J,J)$ for all $J \in \mathsf{C}(\lambda)$ and the collection $A(\lambda)$ is closed under composition. Hence, $A(\lambda)$ forms a subcategory.

(b) the set $\mathbf{DL}_X^Y$ is an R-basis of $\mathsf{Hom}(X,Y)$.

(c) we have $\mathrm{id}_I \in \mathbf{N}_I^I$ for all $\lambda \in \Lambda, I \in \mathsf{C}(\lambda)$.

(d) there is an R-linear contravariant involution $\bar{\cdot}$ of $\mathcal{C}$ extends the maps on $\mathsf{E}(X,J), \mathsf{M}(J,X)$, and $\mathsf{A}(I,J)$. That is, for all $p \in \mathsf{E}(X,J)$ we have

$$\overline{LL_p} = UL_{\overline{p}} \tag{4.1.0.20}$$

and for all $q \in \mathsf{A}(I,J)$ we have

$$\overline{N_p} = N_{\overline{p}}. \tag{4.1.0.21}$$

(e) for all $J \in \Lambda$, the sets $\mathsf{E}(J,J)$ and $\mathsf{M}(J,J)$ are singletons and

$$\mathbf{LL}^J_J = \mathbf{UL}^J_J = \{\mathrm{id}_J\}. \tag{4.1.0.22}$$

(f) $\mathfrak{J}_{\leq J}$ is a two-sided ideal of $\mathcal{C}$.

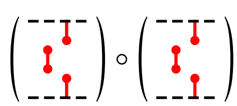
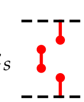
Compared to a fibered SOACC, a fibered NSOACC allows us to associate to each element of the poset Λ with a finite set of objects of $\mathcal{C}$ that have isomorphic endomorphism rings. By the end of this chapter, we will show that $\mathcal{H}^\tau$ is a fibered NSOACC.

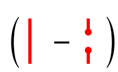
4.2 The Localized Hecke Category $\mathcal{Q}$

This section reviews the construction of double leaves basis of Hecke category $\mathcal{H}$ of type $\mathrm{A}_1 \times \mathrm{A}_1$, as context for the construction of double leaves of $\mathcal{H}^\tau$. We will be following Section 5.4 and Section 6 of [EW16], taking ideas from [EW23] as well. We should also note that the proofs shown for type $\mathrm{A}_1 \times \mathrm{A}_1$ are much simpler for a general Coxeter group. Nevertheless, the techniques outlined here are enough to show that the double leaves are a basis of $\mathcal{H}^\tau$.

4.2.1 Localization of the Hecke Category of type $A_1 \times A_1$

In Section 2.3.2 we saw that $\mathcal{H}$ is Karoubian. Nevertheless, the relation

 $= \alpha_s$  (4.2.1.1)

makes both and  pseudo-idempotents. We have an analogous relation in the blue color. Hence, if we localize the Hom spaces of $\mathcal{H}$ by inverting $\alpha_s\alpha_t$, we get new idempotent in our category. The goal of this section is to show that the diagrammatic category $\mathcal{Q}$ we will define below corresponds to the Karoubi envelope of the localization of $\mathcal{H}$ at $\alpha_s\alpha_t$.

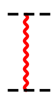
Definition 4.2.1.1. Define the ring $Q := R[\alpha_s^{-1}, \alpha_t^{-1}]$ to be the localization of R obtained by inverting $\alpha_s\alpha_t$.

We define *diagrammatic localized Hecke category of type $A_1 \times A_1$*, $\mathcal{Q}(A_1 \times A_1)$ (abbreviated into $\mathcal{Q}$) to be the graded Q-linear monoidal category presented in the following way:

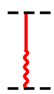
The objects of $\mathcal{Q}$ are generated by the generators B_s, B_t, Q_s, Q_t and their grading shifts, under the operations of direct sum and tensor product. The unit object of $\mathcal{Q}$ will be denoted by $Q_{\mathbb{1}}$ and the tensor products $Q_s \otimes Q_t$ and $Q_t \otimes Q_s$ will be denoted by Q_{st} and Q_{ts} respectively. Also define the set $\mathcal{I}_Q := \{Q_{\mathbb{1}}, Q_s, Q_t, Q_{st}\}$.

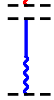
The generating morphisms of $\mathcal{Q}$ extend the generators (2.3.2.1)-(2.3.2.7) of $\mathcal{H}$ by allowing polynomial boxes (2.3.2.6) to have polynomials in Q, in addition to the following morphisms:

(a) The identity morphisms

 $: Q_s \longrightarrow Q_s,$  $: Q_t \longrightarrow Q_t\,.$ (4.2.1.2)

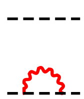
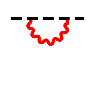
(b) The localized bivalent vertices

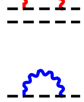
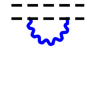
 $: Q_s \longrightarrow B_s,$  $: B_s \longrightarrow Q_s,$ (4.2.1.3)

 $: Q_t \longrightarrow B_t,$  $: B_t \longrightarrow Q_t,$ (4.2.1.4)

of degree 1.

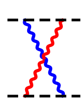
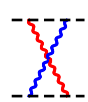
(c) The localized cup and cap morphisms

 $: Q_s \otimes Q_s \longrightarrow Q_{\mathbb{1}},$  $: Q_{\mathbb{1}} \longrightarrow Q_s \otimes Q_s,$ (4.2.1.5)

 $: Q_t \otimes Q_t \longrightarrow Q_{\mathbb{1}},$  $: Q_{\mathbb{1}} \longrightarrow Q_t \otimes Q_t,$ (4.2.1.6)

of degree 0.

(d) The localized 4-valent vertices

 $: Q_s \otimes Q_t \longrightarrow Q_t \otimes Q_s,$  $: Q_t \otimes Q_s \longrightarrow Q_s \otimes Q_t,$ (4.2.1.7)

of degree 0.

These generators are subject to the relations defining $\mathcal{H}$ ((2.3.1.7)-(2.3.1.14),(2.3.2.8)) plus the following relations:

(a) Definition of the localized cups and caps:

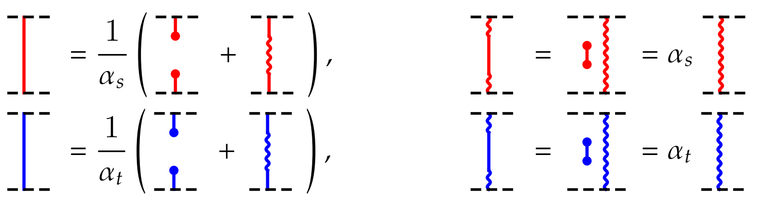
(4.2.1.8)

(4.2.1.9)

(b) Definition of the localized 4-valent vertices:

(4.2.1.10)

(c) The idempotent decomposition relations.



(4.2.1.11)

(4.2.1.12)

Notation 4.2.1.2. For clarity, we will write $\mathrm{Hom}_{\mathcal{H}}$ for the Hom spaces of $\mathcal{H}$ and $\mathrm{Hom}_{\mathcal{Q}}$ for the ones in $\mathcal{Q}$.

Remark 4.2.1.3. The relations (4.2.1.8)-(4.2.1.10) imply that the localized cups, caps and 4-valent vertices are superfluous. We included the other relations to make the superfluous generators to make our graphical calculus easier.

The following are some consequences of the relations in $\mathcal{Q}$.

(a) Because α_s and α_t are invertible in Q, the unit object $Q_{\mathbb{1}}$ is isomorphic to its grading shift $Q_{\mathbb{1}}[2]$. In general, the isomorphism type of a grading shift of a homogeneous object in $\mathcal{Q}$ will only depend on the parity of its grading.

(b) Cup and cap relations (both in red and blue).

(4.2.1.13)

(4.2.1.14)

(c) The dot annihilates the bivalent vertex.

(4.2.1.15)

(d) The localized needle relations hold:

 (4.2.1.16)

(e) The isomorphism relations for $\mathrm{Q_s} \otimes \mathrm{Q_s} \cong \mathrm{Q}_{\mathbb{1}}$ and $\mathrm{Q_t} \otimes \mathrm{Q_t} \cong \mathrm{Q}_{\mathbb{1}}$ hold:

 (4.2.1.17)

 (4.2.1.18)

(f) The relations between the two 4-valent vertices.

 (4.2.1.19)

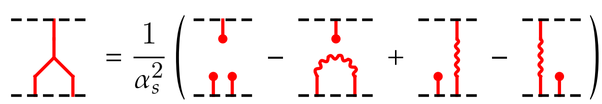
(g) Let $f \in Q$. The extension to Q of the polynomial forcing relation (2.3.1.14) holds, and we also have the relations:

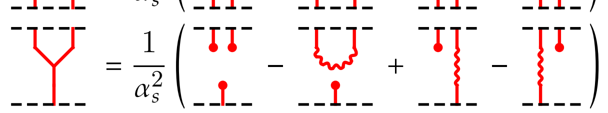
 (4.2.1.20)

(h) The relations between the localized and classical 4-valent vertices.

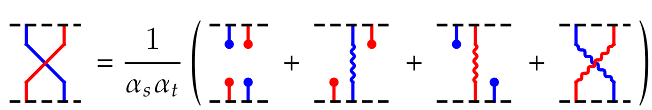
 (4.2.1.21)

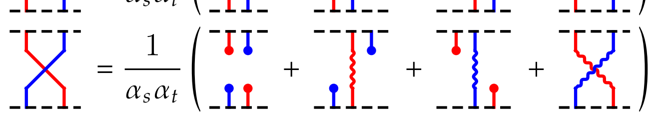
(i) The trivalent vertices may be written as:

 (4.2.1.22)

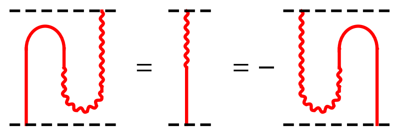
 (4.2.1.23)

(j) The regular 4-valent vertices may be written by:

 (4.2.1.24)

 (4.2.1.25)

Remark 4.2.1.4. A very important remark is that $\mathcal{Q}$ is NOT pivotal with the usual cups and caps. In fact, by (4.2.1.13), the generator satisfies



(4.2.1.26)

If we instead set $(\mathrm{Q}_{\mathrm{s}}, -\ , -\ )$ to be the left adjoint of Q_{s} (while maintaining $(\mathrm{Q}_{\mathrm{s}},\ ,\ )$ as its right adjoint) the adjunction functors coincide. Analogously, we may set $(\mathrm{Q}_{\mathrm{t}}, -\ , -\ )$ to be the left adjoint of Q_{t}. This defines a pivotal structure on $\mathcal{Q}$. While we can still rotate diagrams in $\mathcal{Q}$, we need to keep track of the signs appearing every time we use the left adjoints of Q_{s} and Q_{t}.

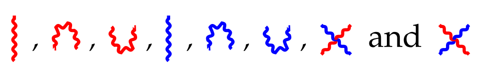
Remark 4.2.1.4 has the following corollary.

Corollary 4.2.1.5. *The tuple* $(\mathrm{Q}_{\mathrm{s}},\ ,\ , -\ , -\ )$ *is biadjoint to* Q_{s} *and* $(\mathrm{Q}_{\mathrm{t}},\ ,\ , -\ , -\ )$ *is biadjoint to* Q_{t}. *We also have that* $(\mathrm{Q}_{\mathrm{st}},\ ,\ )$ *is biadjoint to* Q_{st}. *These, together with the pivotal structure of* $\mathcal{H}$ *make* $\mathcal{Q}$ *into a pivotal category.*

Remark 4.2.1.6. Observe that we can recover the trivalent and 4-valent vertices in the original category from dots and bivalent vertices. While defining $\mathcal{Q}$ in terms of $\mathcal{H}$ has many advantages, we may also define $\mathcal{Q}$ in the following way:

(a) The objects of $\mathcal{Q}$ are generated by $\mathrm{Q}_{\mathrm{s}}, \mathrm{Q}_{\mathrm{s}}[1], \mathrm{Q}_{\mathrm{t}}$, and $\mathrm{Q}_{\mathrm{t}}[1]$ under the operations of direct sum and tensor product.

(b) The morphisms of $\mathcal{Q}$ are monoidally generated by



(4.2.1.27)

(c) The endomorphism space $\mathrm{End}(\mathrm{Q}_{\mathbb{1}}) \cong Q$, generated by the empty diagram as a free left Q-module.

(d) The generating morphisms are subject to relations (4.2.1.17)-(4.2.1.20).

Observe that Q_{s} and Q_{t} the braid relation $\mathrm{Q}_{\mathrm{s}} \otimes \mathrm{Q}_{\mathrm{t}} \cong \mathrm{Q}_{\mathrm{t}} \otimes \mathrm{Q}_{\mathrm{s}}$, and the relations $\mathrm{Q}_{\mathrm{s}} \otimes \mathrm{Q}_{\mathrm{s}} \cong \mathrm{Q}_{\mathrm{t}} \otimes \mathrm{Q}_{\mathrm{t}} \cong \mathrm{Q}_{\mathbb{1}}$, resemble the relations in the group algebra $\mathbb{Z}[W]$. Since we also have the shifted objects $\mathrm{Q}_{\mathrm{s}}[1]$ and $\mathrm{Q}_{\mathrm{t}}[1]$, we get that $\mathcal{Q}$ categorifies the group algebra $\mathbb{Z}[W \times \mathbb{Z}/2]$.

Definition 4.2.1.7. Define an *$\mathcal{H}$-word* $w = (A_1, ..., A_k)$ to be a word on the set $\{B_s, B_t\}$. Define the *Bott-Samelson object* B_w associated to w to be

$$\mathsf{B}_w := A_1 \otimes A_2 \otimes \cdots \otimes A_k. \tag{4.2.1.28}$$

The next lemma is a direct consequence of relations (4.2.1.8)-(4.2.1.10) and will help us prove Theorem 4.2.1.10.

Lemma 4.2.1.8. *Let u, w be two $\mathcal{H}$-words and let $D \in \mathrm{Hom}_{\mathcal{Q}}(\mathsf{B}_u, \mathsf{B}_w)$ be a diagram in $\mathcal{Q}$. Then D may be written as a Q-linear combination of diagrams in $\mathrm{Hom}_{\mathcal{H}}(\mathsf{B}_u, \mathsf{B}_w)$.*

Proof. Let $D \in \mathrm{Hom}_{\mathcal{Q}}(\mathsf{B}_u, \mathsf{B}_w)$. We may use relations (4.2.1.8)-(4.2.1.10) to replace localized cups, caps, and 4-valent vertices by their definition in terms of bivalent vertices and the generators of $\mathcal{H}$. After doing this, we must observe that any bivalent vertex must be composed with another opposite bivalent vertex (since the boundaries are all in $\mathcal{H}$). We may use the relations


(4.2.1.29)

to write any pair of composed bivalent vertices in terms of the generators of $\mathcal{H}$. Finally, we may use the polynomial forcing relation (2.3.1.14) to move any polynomials as coefficients to the left. ◻

Corollary 4.2.1.9.

(a) *Let $D \in \mathrm{Hom}_{\mathcal{Q}}(\mathrm{Q}_{\mathbb{1}}, \mathrm{Q}_{\mathbb{1}})$ be a diagram in $\mathcal{Q}$. Then D is a Q-multiple of* .

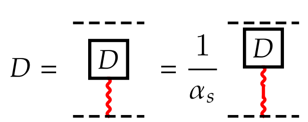
(b) *The* Hom *spaces* $\mathrm{Hom}_{\mathcal{Q}}(\mathrm{Q}_{\mathrm{s}}, \mathrm{Q}_{\mathbb{1}}), \mathrm{Hom}_{\mathcal{Q}}(\mathrm{Q}_{\mathrm{t}}, \mathrm{Q}_{\mathbb{1}})$, *and* $\mathrm{Hom}_{\mathcal{Q}}(\mathrm{Q}_{\mathrm{st}}, \mathrm{Q}_{\mathbb{1}})$ *are zero.*

Proof. Let $D \in \mathrm{Hom}_{\mathcal{Q}}(\mathrm{Q}_{\mathbb{1}}, \mathrm{Q}_{\mathbb{1}})$. By Lemma 4.2.1.8, D may be written as

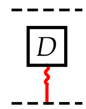
$$D = f_1 T_1 + f_2 T_2 + \cdots + f_r T_r \tag{4.2.1.30}$$

with $f_i \in Q$ and $T_i \in \mathrm{Hom}_{\mathcal{H}}(\mathbb{1}, \mathbb{1})$. By Proposition 2.3.2.7, each T_i is an R-multiple of . Hence, D is a Q-multiple of .

Now let $D \in \mathrm{Hom}_{\mathcal{Q}}(\mathrm{Q}_{\mathrm{s}}, \mathrm{Q}_{\mathbb{1}})$. Using (4.2.1.11), we may write


(4.2.1.31)

Define

$$T := $$  $$, \qquad (4.2.1.32)$$

so that $D = \frac{1}{\alpha_s} T \circ$.

By Lemma 4.2.1.8, T is a Q-linear combination of diagrams in $\mathrm{Hom}_{\mathcal{H}}(B_s, \mathbb{1})$. By Proposition 2.3.2.7, T is a Q-multiple of , so the composition $\frac{1}{\alpha_s} T \circ$ is zero.

Showing that $\mathrm{Hom}_{\mathcal{Q}}(\mathrm{Q_t}, \mathrm{Q}_{\mathbb{1}})$ and $\mathrm{Hom}_{\mathcal{Q}}(\mathrm{Q_{st}}, \mathrm{Q}_{\mathbb{1}})$ are zero is analogous. □

Observe that the idempotent decomposition relations (4.2.1.11) imply that

$$B_s \cong \mathrm{Q_s}[1] \oplus \mathrm{Q}_{\mathbb{1}}[1] \text{ and } B_t \cong \mathrm{Q_t}[1] \oplus \mathrm{Q}_{\mathbb{1}}[1]. \qquad (4.2.1.33)$$

Also, relation (4.2.1.17) implies that

$$\mathrm{Q_s} \otimes \mathrm{Q_s} \cong \mathbb{1} \cong \mathrm{Q_t} \otimes \mathrm{Q_t}, \qquad (4.2.1.34)$$

and relation (4.2.1.19) implies

$$\mathrm{Q_s} \otimes \mathrm{Q_t} \cong \mathrm{Q_t} \otimes \mathrm{Q_s}. \qquad (4.2.1.35)$$

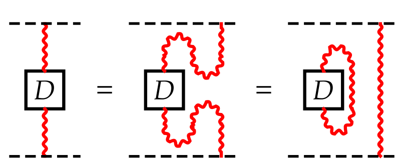
We will now show that $\mathcal{Q}$ is a semisimple category and the set $\mathcal{I}_{\mathcal{Q}} = \{\mathrm{Q}_{\mathbb{1}}, \mathrm{Q_s}, \mathrm{Q_t}, \mathrm{Q_{st}}\}$ is a full set of irreducible objects of $\mathcal{Q}$ (up to the parity of their grading shift).

Theorem 4.2.1.10. *The objects $\mathrm{Q}_{\mathbb{1}}, \mathrm{Q_s}, \mathrm{Q_t}, \mathrm{Q_{st}}$ (and their grading shifts by 1) are irreducible and any object of $\mathcal{Q}$ admits a (unique) decomposition into a finite direct sum of these objects. Let I_1, I_2 be two of these irreducible objects. The morphism space $\mathrm{Hom}_{\mathcal{Q}}(I_1, I_2)$ is either trivial if $I_1 \not\cong I_2$ or isomorphic to Q as a left Q-module. Hence, $\mathcal{Q}$ is semisimple.*

Proof. The fact that any object can be written as a finite direct sum of these irreducible objects follows from equations (4.2.1.33)-(4.2.1.35).

For the second part, we will only show that any morphism in $\mathrm{Hom}_{\mathcal{Q}}(\mathrm{Q_s}, \mathrm{Q_s})$ is a Q-multiple of and that $\mathrm{Hom}_{\mathcal{Q}}(\mathrm{Q_{st}}, \mathrm{Q_s})$ is trivial. The argument for other Hom spaces is analogous and is left as an exercise to the reader.

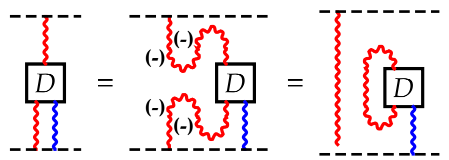
Let D be a diagram in $\mathrm{Hom}_{\mathcal{Q}}(\mathrm{Q_s}, \mathrm{Q_s})$. Using adjunction and relation (4.2.1.34), we write



$$. \qquad (4.2.1.36)$$

By Corollary 4.2.1.9, the diagram on the left is a polynomial in Q. Hence, D can be written as a Q-multiple of .

Let D instead be a diagram in $\mathrm{Hom}_{\mathcal{Q}}(\mathrm{Q}_s, \mathrm{Q}_{st})$. As before, we may use left adjunction (which comes with a sign) to write



(4.2.1.37)

By Corollary 4.2.1.9, the diagram on the right is zero. □

Corollary 4.2.1.11. *The category $\mathcal{Q}$ is the Karoubian envelope of the localization of $\mathcal{H}$ given by inverting $\alpha_s \alpha_t$.*

Proof. The localization of $\mathcal{H}$ inverting $\alpha_s \alpha_t$ corresponds to applying the functor $Q \otimes_R (-)$ to the Hom spaces of $\mathcal{H}$. This is achieved by allowing polynomials in Q in our diagrams. By Theorem 4.2.1.10, $\mathcal{Q}$ is Karoubian. By Remark 4.2.1.3, we only need the idempotent relations (4.2.1.11) and (4.2.1.12) (together with the relations in $\mathcal{H}$) to derive all our other relations. Hence, $\mathcal{Q}$ is the Karoubian envelope of this localization. □

Definition 4.2.1.12. Define the localization functor $\Lambda : \mathcal{H} \longrightarrow \mathcal{Q}$ to be the functor sending $B_s \mapsto B_s$, $B_t \mapsto B_t$, and sending a diagram in $\mathcal{H}$ to its corresponding diagram in $\mathcal{Q}$.

Corollary 4.2.1.13. *The functor Λ is faithful, so Λ is an embedding of $\mathcal{H}$ into $\mathcal{Q}$.*

Proof. By Proposition 2.1.2.7, we only have to check that Λ is injective on

$$\mathrm{Hom}_{\mathcal{H}}(\mathbb{1}, \mathbb{1}), \mathrm{Hom}_{\mathcal{H}}(B_s, \mathbb{1}), \mathrm{Hom}_{\mathcal{H}}(B_t, \mathbb{1}), \text{ and } \mathrm{Hom}_{\mathcal{H}}(B_{st}, \mathbb{1}). \tag{4.2.1.38}$$

By Lemma 4.2.1.8,

$$\mathrm{Hom}(\Lambda(\mathbb{1}), \Lambda(\mathbb{1})) = \mathrm{Hom}(\mathrm{Q}_{\mathbb{1}}, \mathrm{Q}_{\mathbb{1}}) \cong Q. \tag{4.2.1.39}$$

The function $\Lambda : \mathrm{Hom}_{\mathcal{H}}(\mathbb{1}, \mathbb{1}) \longrightarrow \mathrm{Hom}_{\mathcal{Q}}(\Lambda(\mathbb{1}), \Lambda(\mathbb{1}))$ corresponds to the inclusion of $R \longrightarrow Q$. Therefore, it is injective.

Similarly,

$\mathrm{Hom}(\Lambda(B_s), \Lambda(\mathbb{1})) = \mathrm{Hom}(\mathrm{Q}_{\mathbb{1}}[1] \oplus \mathrm{Q}_s[1], \mathrm{Q}_{\mathbb{1}}) \cong \mathrm{Hom}(\mathrm{Q}_{\mathbb{1}}[1], \mathrm{Q}_{\mathbb{1}}) = Q$  . (4.2.1.40)

The function $\Lambda : \mathrm{Hom}_{\mathcal{H}}(B_s, \mathbb{1}) \longrightarrow \mathrm{Hom}_{\mathcal{Q}}(\Lambda(B_s), \Lambda(\mathbb{1}))$ sends $\mapsto$. Since we already saw that it sends the polynomial coefficients $R \longrightarrow Q$ injectively, it is injective.

Checking that Λ is injective on $\mathrm{Hom}_{\mathcal{H}}(B_t, \mathbb{1})$ and $\mathrm{Hom}_{\mathcal{H}}(B_{st}, \mathbb{1})$ is analogous. □

Definition 4.2.1.14. Define a partial order $\leq_D$ on $\mathcal{I}_{\mathcal{Q}}$ by the lattice

$$\begin{tikzcd} & \mathrm{Q_{st}} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathrm{Q_s} \arrow[dr, dash] & & \mathrm{Q_t} \arrow[dl, dash] \\ & \mathrm{Q}_{\mathbb{1}} & \end{tikzcd} \quad . \tag{4.2.1.41}$$

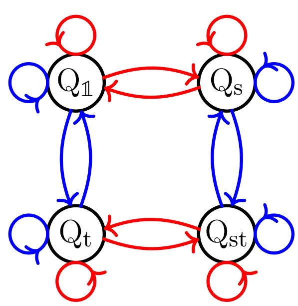
This corresponds to the Bruhat order on the elements of $A_1 \times A_1$ labeling each irreducible. For $J \in \mathcal{I}_{\mathcal{Q}}$, define $\mathcal{H}(J)$ to be the indecomposable object $I \in \mathcal{I}$ such that J is its top summand with respect to $\leq_D$. That is,

$$\mathcal{H}(\mathrm{Q}_{\mathbb{1}}) = \mathbb{1}, \mathcal{H}(\mathrm{Q_s}) = B_s, \mathcal{H}(\mathrm{Q_t}) = B_t, \text{ and } \mathcal{H}(\mathrm{Q_{st}}) = B_s B_t. \tag{4.2.1.42}$$

Note that for $I, J \in \mathcal{Q}$, we have $J \leq_D I$ if and only if J is a summand of $\mathcal{H}(I)$ (in $\mathcal{Q}$).

4.2.2 Walks on the branching graph of $\mathcal{Q}$

Definition 4.2.2.1. Define the *branching graph* $\mathrm{P}_{\mathcal{Q}}$ *of* $\mathcal{Q}$ *with respect to the objects* B_s, B_t to be the following directed multigraph (admitting loops):



$$. \tag{4.2.2.1}$$

The nodes of $\mathrm{P}_{\mathcal{Q}}$ correspond to the irreducible objects of $\mathcal{Q}$, which in turn are in correspondence with the indecomposable objects of $\mathcal{H}$. The edges of $\mathrm{P}_{\mathcal{Q}}$ are colored by the generators B_s, B_t of $\mathcal{H}$. In (4.2.2.1), the red edges are B_s-colored and the blue edges are B_t-colored. Given two irreducible objects I_1, I_2 of $\mathcal{Q}$ and $A \in \{B_s, B_t\}$, the graph $\mathrm{P}_{\mathcal{Q}}$ has an edge from I_1 to I_2 of color A for each time I_2 appears as a summand of $I_1 \otimes A$ regardless of their degree shift. For instance, $\mathrm{Q}_\mathrm{t} \otimes B_s \cong \mathrm{Q}_\mathrm{t}[1] \oplus \mathrm{Q}_\mathrm{st}[1]$, so Q_t has a B_s-colored loop and a B_s-colored edge to Q_st.

Given an edge e of $\mathrm{P}_\mathcal{Q}$, we denote its source by $\mathbf{s}(e)$, its target by $\mathbf{t}(e)$, and its color by $\mathbf{c}(e)$.

Notice that for each indecomposable $\mathbb{1}, B_s, B_t, B_{st}$ of $\mathcal{H}$ we have a corresponding irreducible object $\mathrm{Q}_\mathbb{1}, \mathrm{Q}_\mathrm{s}, \mathrm{Q}_\mathrm{t}, \mathrm{Q}_\mathrm{st}$. Define $\mathbf{s}_\mathcal{H}(e) = \mathcal{H}(\mathbf{s}(e))$ and $\mathbf{t}_\mathcal{H}(e) = \mathcal{H}(\mathbf{s}(e))$ (the corresponding indecomposable in $\mathcal{H}$ that they are a top summand of).

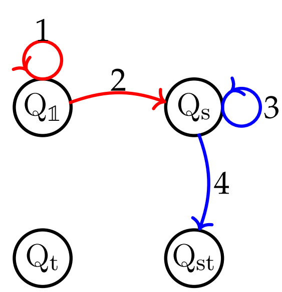
Definition 4.2.2.2. Define an *$\mathcal{H}$-walk* p to be a walk on the graph $\mathrm{P}_\mathcal{Q}$. That is, $p = (e_1, e_2, \ldots, e_r)$ is a tuple of edges of $\mathrm{P}_\mathcal{Q}$ where $\mathbf{s}(e_{i+1}) = \mathbf{t}(e_i)$. We define the *start of* p by $\mathbf{s}(p) := \mathbf{s}(e_1)$, and the *end of* p is $\mathbf{t}(p) := \mathbf{t}(e_r)$. Also define $\mathbf{s}_\mathcal{H}(p) = \mathcal{H}(\mathbf{s}(p))$ and $\mathbf{t}_\mathcal{H}(p) = \mathcal{H}(\mathbf{t}(p))$. We say that p is a *based walk* if it starts at $\mathrm{Q}_\mathbb{1}$. We will additionally consider the empty walk ϕ a based walk.

Define the *$\mathcal{H}$-word* $\mathbf{w}(p)$ associated to p to be the word $(\mathbf{c}(e_1), \ldots, \mathbf{c}(e_r))$ consisting of the colors of the edges of p.

Given an $\mathcal{H}$ word w, we say that p is *w-colored* if $\mathbf{w}(p) = w$. Define $\mathsf{W}(w)$ to be the set of all w-colored based walks on $\mathrm{P}_\mathcal{Q}$. For an irreducible object $J \in \mathcal{Q}$, we define $\mathsf{W}^J(w)$ to be the set of all w-colored based walks on $\mathrm{P}_\mathcal{Q}$ ending at J.

Define the *Bott-Samelson object* B_p corresponding to p to be $\mathsf{B}_{\mathbf{w}(p)}$.

Notation 4.2.2.3. Instead of giving a name to each of the edges, we will describe the walks on $\mathrm{P}_\mathcal{Q}$ by the target of each edge, colored by the color of each edge For example, the tuple $(\mathrm{Q}_\mathbb{1}, \mathrm{Q}_\mathrm{s}, \mathrm{Q}_\mathrm{s}, \mathrm{Q}_\mathrm{st})$ represents the walk



. (4.2.2.2)

We will use a further shorthand by only writing the element of $A_1 \times A_1$ that each object corresponds to. In the example above, we will write $(\mathbb{1}, s, s, st)$ instead.

Additionally, to each $I \in \mathcal{I}$ we will assign $\mathsf{o}(I) = \mathsf{o}(\mathrm{id}_I)$ be the root path of I. Similarly, let $J \in \mathcal{I}_\mathcal{Q}$ and let $\mathcal{H}(J) \in \mathcal{I}$ be its associated indecomposable object in $\mathcal{H}$. Then we set $\mathsf{o}(I) = \mathsf{o}(\mathrm{id}_{\mathcal{H}(J)})$. More explicitly,

$$\mathsf{o}(\mathrm{Q}_\mathbb{1}) = \phi, \mathsf{o}(\mathrm{Q}_\mathrm{s}) = (s), \mathsf{o}(\mathrm{Q}_\mathrm{t}) = (t), \text{ and } \mathsf{o}(\mathrm{Q}_\mathrm{st}) = (s, st). \tag{4.2.2.3}$$

Definition 4.2.2.4. Let w be an $\mathcal{H}$-word. Define an order $\leq_D$ on $\mathsf{W}(w)$ in the following way: Let $p = (e_1, ..., e_r)$ and $q = (e'_1, ..., e'_r)$ in $\mathsf{W}(w)$. We say $p \leq_D q$ if $\mathbf{t}(e_i) \leq_D \mathbf{t}(e'_i)$, where i is the first index where $\mathbf{t}(e_i) \neq \mathbf{t}(e'_i)$. If we fix a starting node and an edge coloring, the targets of all edges with that starting node and color are all comparable. Thus $\leq_D$ is a total order. We call $\leq_D$ the *dominance order* on $\mathsf{W}(w)$.

Proposition 4.2.2.5. *Let w be an $\mathcal{H}$-word and B_w its Bott-Samelson object. We have a bijective correspondence between the set of w-colored based walks $\mathsf{W}(w)$ and the irreducible summands of B_w in $\mathcal{Q}$. The multiplicity of an irreducible $J \in \mathcal{Q}$ in B_w corresponds to the size of $\mathsf{W}^J(w)$ (the number of based walks ending at J).*

Proof. This follows from the distributive property of the tensor product $\otimes$ over the direct sum $\oplus$. To show how this works, suppose that $w = (B_s, B_s)$ and $\mathsf{B}_w = B_s \otimes B_s$. Since $B_s \cong \mathrm{Q}_{\mathbb{1}}[1] \oplus \mathrm{Q}_{\mathrm{s}}[1]$, we have that

$$\mathsf{B}_w = B_s \otimes B_s \cong \mathrm{Q}_{\mathbb{1}}[1] \otimes B_s \oplus \mathrm{Q}_{\mathrm{s}}[1] \otimes B_s. \tag{4.2.2.4}$$

Thus, the summands of B_w (ignoring the grading) are either summands of $B_s[1] \cong \mathrm{Q}_{\mathbb{1}}[2] \otimes \mathrm{Q}_{\mathrm{s}}[2]$ or summands of $\mathrm{Q}_{\mathrm{s}}[1] \otimes B_s \cong \mathrm{Q}_{\mathrm{s}}[2] \otimes \mathrm{Q}_{\mathbb{1}}[2]$. In general, let $w = (B_{i_1}, ..., B_{i_k})$ (for $i_j \in \{s, t\}$). In our first step, we are picking a summand of B_{i_1}, then tensoring it with B_{i_2} and picking a new summand, and continue the process until the k-th step. These choices of summands are in correspondence with based w-colored walks on $\mathrm{P}_{\mathcal{Q}}$. ◻

4.2.3 Labeling on the branching graph of $\mathcal{Q}$

Definition 4.2.3.1. Let $\mathrm{P}_{\mathcal{Q}}$ be the branching graph of $\mathcal{Q}$. Let e be and edge of $\mathrm{P}_{\mathcal{Q}}$ colored by $\mathbf{c}(e)$, with source and target $\mathbf{s}(e)$ and $\mathbf{t}(e)$. Let $\mathbf{s}_{\mathcal{H}}(e) = \mathcal{H}(\mathbf{s}(e))$ and $\mathbf{t}_{\mathcal{H}}(e) = \mathcal{H}(\mathbf{t}(e))$.

- Define a *lower $\mathcal{H}$-label* of e to be a morphism $LL_e : \mathbf{s}_{\mathcal{H}}(e) \otimes \mathbf{c}(e) \longrightarrow \mathbf{t}_{\mathcal{H}}(e)$ in $\mathcal{H}$.
- Define a *lower $\mathcal{Q}$-label* of e to be a morphism $LI_e : \mathbf{s}(e) \otimes \mathbf{c}(e) \longrightarrow \mathbf{t}(e)$ in $\mathcal{Q}$.
- Define an *upper $\mathcal{H}$-label* of e to be a morphism $UL_e : \mathbf{t}_{\mathcal{H}}(e) \longrightarrow \mathbf{s}_{\mathcal{H}}(e) \otimes \mathbf{c}(e)$ in $\mathcal{H}$.
- Define an *upper $\mathcal{Q}$-label* of e to be a morphism $UI_e : \mathbf{t}(e) \longrightarrow \mathbf{s}(e) \otimes \mathbf{c}(e)$ in $\mathcal{Q}$.
- Define an **LL**-*labeling* of $\mathrm{P}_{\mathcal{Q}}$ to be an assignment $e \mapsto LL_e$ of a lower $\mathcal{H}$-label to every edge of $\mathrm{P}_{\mathcal{Q}}$. We will call the labels our *basic lower leaves*.

- Define an **LI**-*labeling* to be an assignment $e \mapsto LI_e$ of a lower $\mathcal{Q}$-label to every edge of $\mathrm{P}_{\mathcal{Q}}$. We will call the labels our *basic lower projections*.

- Define an **UL**-*labeling* of $\mathrm{P}_{\mathcal{Q}}$ to be an assignment $e \mapsto UL_e$ of an upper $\mathcal{H}$-label to every edge of $\mathrm{P}_{\mathcal{Q}}$. We will call the labels our *basic upper leaves*.

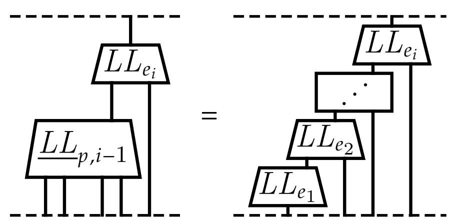
- Define an **UI**-*labeling* of $\mathrm{P}_{\mathcal{Q}}$ to be an assignment $e \mapsto UI_e$ of an upper $\mathcal{Q}$-label to every edge of $\mathrm{P}_{\mathcal{Q}}$. We will call the labels our *basic upper inclusions*.

Definition 4.2.3.2. Let $p = (e_1, e_2, \ldots, e_k)$ be a based walk on $\mathrm{P}_{\mathcal{Q}}$. For $1 \leq i \leq k$, let $p_i = (e_1, e_2, \ldots, e_i)$, B_{p_i} be its corresponding Bott-Samelson object, and $J_i = \mathbf{t}_{\mathcal{H}}(p_i)$.

Fix an **LL**-labeling on $\mathrm{P}_{\mathcal{Q}}$. Define the morphisms $\underline{LL}_{p,i}$ for $0 \leq i \leq k$ inductively in the following way:

- Set $\underline{LL}_{p,0} = \mathrm{id}_{\mathbb{1}} : \mathbb{1} \longrightarrow \mathbb{1}$.

- Then set $\underline{LL}_{p,i} : \mathsf{B}_{p_i} \longrightarrow J_i$ to be the composition $\underline{LL}_{p,i} = LL_{e_i} \circ (\underline{LL}_{p,i-1} \otimes \mathrm{id}_{\mathbf{c}(e_i)})$. Diagramatically,

$\underline{LL}_{p,i} =$ 

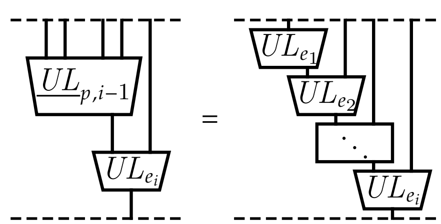
. (4.2.3.1)

Also define $\underline{LL}_p = \underline{LL}_{p,k}$. We define a *lower leaf morphisms* (with respect to the standard **LL**-labeling) to be a morphism of the form $\underline{LL}_p$.

Fix a **UL**-labeling on $\mathrm{P}_{\mathcal{Q}}$. Define the morphisms $\underline{UL}_{p,i}$ for $0 \leq i \leq k$ inductively in the following way:

- Set $\underline{UL}_{p,0} = \mathrm{id}_{\mathbb{1}} : \mathbb{1} \longrightarrow \mathbb{1}$.

- Then set $\underline{UL}_{p,i} : J_i \longrightarrow \mathsf{B}_{p_i}$ to be the composition $\underline{UL}_{p,i} = (\underline{UL}_{p,i-1} \otimes \mathrm{id}_{\mathbf{c}(e_i)}) \circ UL_{e_i}$. Diagramatically,

$\underline{UL}_{p,i} =$ 

(4.2.3.2)

(like $\underline{LL}_{p,i}$ flipped upside-down).

We define an *upper leaf morphisms* (with respect to the standard **UL**-labeling) to be a morphism of the form $\underline{UL}_p$.

We define a *double leaf morphism* to be a morphism of the form $\underline{UL}_q \circ \underline{LL}(p)$, where p, q are based walks on $\mathrm{P}_{\mathcal{Q}}$ with $\mathfrak{t}(p) = \mathfrak{t}(q)$.

We define the morphisms $\underline{LI}_{p,i}$ and $\underline{UI}_{p,i}$ (in $\mathcal{Q}$) analogously by fixing a **LI**-*labeling* or **UI**-*labeling* instead. The morphisms of the form $\underline{LI}_p$ will be called *lower projection morphisms*, and the ones of the form $\underline{UI}_{p,i}$ will be called *upper inclusion morphisms*.

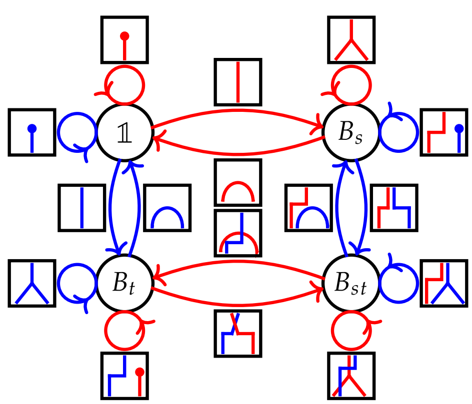
Definition 4.2.3.3. Let L be a lower leaf morphism. Let p be a based walk on $\mathrm{P}_{\mathcal{Q}}$ such that $L = \underline{LL}_p$. We define the *root path* $\mathsf{o}^{LL}(L) = p$.

Similarly, given an upper leaf morphism $U = \underline{UL}_p$, we define its *root path* $\mathsf{o}^{UL}(L) = p$. The definitions of the *root path functions* o^{LI} and o^{UI} for lower projections and upper inclusions are analogous.

Whenever there is no ambiguity, we will just denote the root path of a morphism by $\mathsf{o}(L)$.

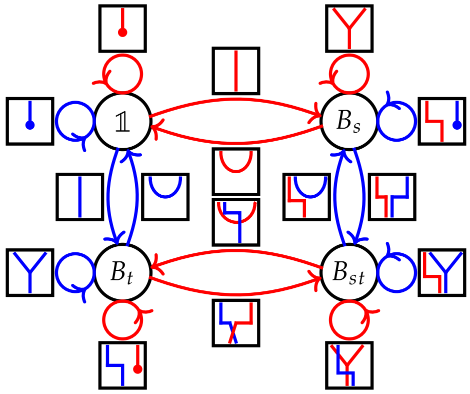
The reader should note that, so far, we haven't imposed any conditions on the morphisms in an **LL**, **UL**, **LI**, or **UI**-labeling (other than having the appropriate source and target). This is a general construction, and will be applied to our folded category in Section 4.3.3. Here are the labeling we will use.

Definition 4.2.3.4. Let us fix the following **LL**-labeling on $\mathrm{P}_{\mathcal{Q}}$:



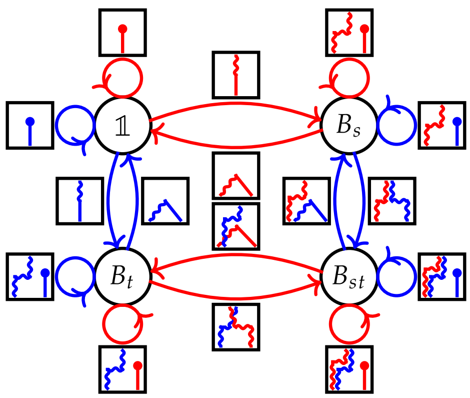
(4.2.3.3)

Let us fix the following **UL**-labeling on P_Q:



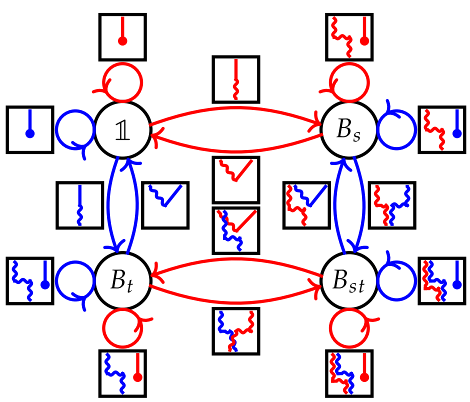
(4.2.3.4)

Let us fix the following **LI**-labeling on P_Q:



(4.2.3.5)

Let us fix the following **UI**-labeling on P_Q:



(4.2.3.6)

Definition 4.2.3.5. Let u, w be $\mathcal{H}$-words and $J \in \mathcal{I}_Q$.

Define the set $LL_u^J := \{\underline{LL}_p \,|p \in \mathsf{W}^J(u)\}$ to be the set of lower leaves with source u and target J. We denote by $\mathbf{LL}_u^J$ to the R-span of LL_u^J.

Define the set $UL_J^w := \{\underline{UL}_q \,|q \in \mathsf{W}^J(w)\}$ to be the set of upper leaves with source J and target w. We denote by $\mathbf{UL}_J^w$ to the R-span of UL_J^w.

Define the set $DL_u^w := \{\underline{UL}_q \circ \underline{LL}_p \,|p \in \mathsf{W}(u), q \in \mathsf{W}(w) \text{ s.t. } \mathbf{t}(p) = \mathbf{t}(q)\}$ to be the set of double leaves with source u and target w. We denote by $\mathbf{DL}_u^w$ to the R-span of DL_u^B.

Define the set $LI_u^J := \{\underline{LI}_p \,|p \in \mathsf{W}^J(u)\}$ to be the set of lower projections with source u and target J. We denote by $\mathbf{LI}_u^J$ to the Q-span of LI_u^J.

Define the set $UI_J^w := \{\underline{UI}_q \,|q \in \mathsf{W}^J(w)\}$ to be the set of upper inclusions with source J and target w. We denote by $\mathbf{UI}_J^w$ to the Q-span of UI_J^w.

Remark 4.2.3.6. Note that since the identity morphism , , and are part of our **LL**-labeling, all of the basic lower leaves are, in fact, lower leaves. Analogously, all basic upper leaves are upper leaves.

4.2.4 Writing the morphisms in $\mathcal{H}$ with matrices

By Corollary 4.2.1.13, we may identify any morphism $T : A \longrightarrow B$ in $\mathcal{H}$ with its image $\Lambda(T)$ in $\mathcal{Q}$. By the semisimplicity of $\mathcal{Q}$, we may $\Lambda(T)$ as a matrix with coefficients in Q, where each entry is a morphism between the summands irreducible summands of A and B. We should think of our **LI** and **UI**-labelings as a way to fix a Q-basis for $\mathsf{Hom}_{\mathcal{Q}}(A, B)$ in $\mathcal{Q}$ to write our matrix in. In turn, our **LL** and **UL**-labelings will help us for R-bases for $\mathsf{Hom}_{\mathcal{Q}}(A, B)$ where A and B are Bott-Samelson objects in $\mathcal{H}$.

Definition 4.2.4.1. Let $I \in \mathcal{I}_{\mathcal{Q}}$ and let $T \in \mathsf{Hom}_{\mathcal{Q}}(I, I)$. By Theorem 4.2.1.10, we have that T there is an $f_T \in R$ such that $T = f_T \,\mathrm{id}_I$. Define the function $\mathbf{r} : \mathsf{Hom}_{\mathcal{Q}}(I, I) \longrightarrow R$ given by $\mathbf{r}(T) := f_T$. Extend the definition of $\mathbf{r}$ to all $T \in \mathsf{Hom}_{\mathcal{Q}}(I, J)$ for $I, J \in \mathcal{I}_{\mathcal{Q}}$ by setting $\mathbf{r}(T) := 0$ when $I \neq J$.

Let u, w be two $\mathcal{H}$-words, let $p \in \mathsf{W}(u)$ and let $q \in \mathsf{W}(w)$. Let $T \in \mathsf{Hom}_{\mathcal{H}}(\mathsf{B}_u, \mathsf{B}_w)$. Define the functional $\mathbf{r}_p^q : \mathsf{Hom}_{\mathcal{H}}(\mathsf{B}_u, \mathsf{B}_w) \longrightarrow R$ by $\mathbf{r}_p^q(T) := \mathbf{r}(\underline{LI}_q \circ T \circ \underline{UI}_p)$.

Define the *non-reduced matrix* $[T]$ *of* T (with respect to the standard **LI** and **UI** labelings) in the following way:

(a) The columns of $[T]$ are indexed by $\mathsf{W}(u)$.

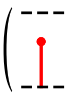
(b) The rows of $[T]$ are indexed by $\mathsf{W}(w)$.

(c) For $p \in \mathsf{W}(u)$ and $q \in \mathsf{W}(w)$, the entry $[T]_{p,q} = \mathbf{r}_p^q(T)$.

Define $[T]_J$ for $J \in \mathcal{J}$ analogously, by restricting ourselves to walks in $\mathrm{W}^J(u)$ and $\mathrm{W}^J(w)$. It is clear from the definition that

$$[T] = \bigoplus_{J \in \mathcal{I}_Q} [T]_J. \tag{4.2.4.1}$$

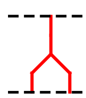
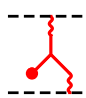
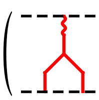
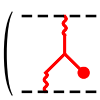
Example 4.2.4.2. Let us calculate some non-reduced matrices. Let us recall Notation 4.2.2.3, where we write our walks in P_Q by listing the colored target of their edges. Let write the matrix for the diagram , which has source word $w_1 = (B_s)$ (and empty target). We have the two w-colored walks $(\mathrm{Q}_{\mathbb{1}})$ and (Q_s). To calculate the coefficients, we write

$\mathbf{r}_{(\mathrm{Q}_{\mathbb{1}})}\Big($  $\Big) =$  $= \alpha_s$ and $\mathbf{r}_{(\mathrm{Q}_s)}\Big($  $\Big) =$  $= 0.$ (4.2.4.2)

Hence,

$[$  $] = \begin{array}{c|cc} & (\mathbb{1}) & (s) \\ \hline \phi & \alpha_s & 0 \end{array}.$ (4.2.4.3)

Similarly,

$\mathbf{r}^{(s)}_{(\mathbb{1},s)}\Big($  $\Big) = \mathbf{r}\Big($  $\Big) = \alpha_s$ and $\mathbf{r}^{(s)}_{(s,s)}\Big($  $\Big) = \mathbf{r}\Big($  $\Big) = \alpha_s.$ (4.2.4.4)

Hence,

$\Big[$  $\Big] = \begin{array}{c|cccc} & (\mathbb{1},\mathbb{1}) & (s,\mathbb{1}) & (\mathbb{1},s) & (s,s) \\ \hline (\mathbb{1}) & \alpha_s & \alpha_s & 0 & 0 \\ (s) & 0 & 0 & \alpha_s & \alpha_s \end{array}.$ (4.2.4.5)

Remark 4.2.4.3. The non-reduced matrices of the identity morphisms of the indecomposable objects in $\mathcal{H}$ are

$$[\mathrm{id}_{\mathbb{1}}] = [1], [\mathrm{id}_{B_s}] = \begin{array}{c|cc} & (\mathbb{1}) & (s) \\ \hline (\mathbb{1}) & \alpha_s & 0 \\ (s) & 0 & \alpha_s \end{array}, [\mathrm{id}_{B_t}] = \begin{array}{c|cc} & (\mathbb{1}) & (t) \\ \hline (\mathbb{1}) & \alpha_t & 0 \\ (t) & 0 & \alpha_t \end{array}, \tag{4.2.4.6}$$

$$\text{and } [\mathrm{id}_{B_{st}}] = \begin{array}{c|cccc} & (\mathbb{1},\mathbb{1}) & (s,s) & (\mathbb{1},t) & (s,st) \\ \hline (\mathbb{1},\mathbb{1}) & \alpha_s\alpha_t & 0 & 0 & 0 \\ (s,s) & 0 & \alpha_s\alpha_t & 0 & 0 \\ (\mathbb{1},t) & 0 & 0 & \alpha_s\alpha_t & 0 \\ (s,st) & 0 & 0 & 0 & \alpha_s\alpha_t \end{array}. \tag{4.2.4.7}$$

Hence, the non-reduced matrices do not preserve composition. Note, though, that all of these matrices are invertible over Q. Since $[\mathrm{id}_{B_s}]$ and $[\mathrm{id}_{B_t}]$ are invertible, all their tensor products are invertible as well. Hence, $[\mathrm{id}_{\mathsf{B}_w}]$ is invertible for any $\mathcal{H}$-word w. In order for them to preserve composition, we need to multiply by the inverse of the non-reduced identity matrix of the middle object.

Definition 4.2.4.4. In Remark 4.2.4.3, we saw that the matrices of our identity morphism are invertible over Q. Let u, w be two $\mathcal{H}$-words. Let $T \in \mathsf{Hom}_{\mathcal{H}}(\mathsf{B}_u, \mathsf{B}_w)$. Define the *lower-reduced matrix of* T by $\underline{[T]} = [T] \circ [\mathrm{id}_{\mathsf{B}\,u}]^{-1}$, and the *upper-reduced matrix of* T by $\overline{[T]} = [\mathrm{id}_{\mathsf{B}\,w}]^{-1} \circ [T]$. Define $\underline{[T]}_J$ and $\overline{[T]}_J$ for $J \in \mathcal{I}_Q$ analogously.

Let w be an $\mathcal{H}$-word and B_w its corresponding Bott-Samelson object. Let $J \in \mathcal{I}_Q$. Proposition 4.2.2.5 tells us that the multiplicity of J as a summand of B_w is in correspondence with the size of $\mathsf{W}(w)$. Define the morphism

$$\iota_J^w = \bigoplus_{p \in \mathsf{W}^J(w)} \underline{UI}_p : \bigoplus_{p \in \mathsf{W}^J(w)} J \longrightarrow \mathsf{B}_w \tag{4.2.4.8}$$

to be the coproduct of all the upper inclusion morphisms $\underline{UI}_p$, and

$$\rho_w^J = \bigoplus_{p \in \mathsf{W}^J(w)} \underline{LI}_p : \mathsf{B}_w \longrightarrow \bigoplus_{p \in \mathsf{W}^J(w)} J \tag{4.2.4.9}$$

to be the product of the lower projection morphisms $\underline{LI}_p$. By definition, we have the equation

$$\rho_w^J \circ \iota_J^w = [\mathrm{id}_{\mathsf{B}_w}]_J \otimes \mathrm{id}_J\,. \tag{4.2.4.10}$$

The matrix $[\mathrm{id}_{\mathsf{B}_w}]_J$ is invertible in Q because $[\mathrm{id}_{\mathsf{B}_w}]$ is invertible. Define $\eta_J : \mathsf{B}_w \longrightarrow \mathsf{B}_w$ by the composition

$$\eta_w^J = \iota_J^w \circ ([\mathrm{id}_{\mathsf{B}_w}]_J^{-1} \otimes \mathrm{id}_J) \circ \rho_w^J. \tag{4.2.4.11}$$

Note that η_w^J is idempotent. Moreover, since $[\mathrm{id}_{\mathsf{B}\,w}]$ is invertible we get the idempotent decomposition

$$\mathrm{id}_{\mathsf{B}\,w} = \sum_{J \in \mathcal{I}_Q} \eta_w^J. \tag{4.2.4.12}$$

This result helps us prove the following proposition.

Proposition 4.2.4.5. *Let u, v, w be $\mathcal{H}$-words, let $T \in \mathsf{Hom}_{\mathcal{H}}(\mathsf{B}_u, \mathsf{B}_v)$, and let $T' \in \mathsf{Hom}_{\mathcal{H}}(\mathsf{B}_v, \mathsf{B}_w)$. Then*

$$[T' \circ T] = [T'] \circ \overline{[T]} = \underline{[T']} \circ [T] = \underline{[T']} \circ [\mathrm{id}_{B_v}] \circ \overline{[T]}. \tag{4.2.4.13}$$

Proof. Let u, v, w, T, T' as above. Let $J \in \mathcal{I}_Q$. We will show the result for $[T]_J$ and the general case will follow. By (4.2.4.10),

$$\rho_u^J \circ T \circ T' \iota_J^v = [T' \circ T]_J \otimes \mathrm{id}_J\,. \tag{4.2.4.14}$$

Using (4.2.4.12) and the fact that $\mathsf{Hom}_Q(I, J) = 0$ whenever $I \neq J$ (for $I \in \mathcal{I}_Q$), we get

$$\begin{aligned}
\rho_w^J \circ T \circ T' \iota_J^u &= \rho_w^J \circ T \circ \mathrm{id}_{\mathsf{B}\,v} \circ T' \iota_J^u && (4.2.4.15)\\
&= \sum_{I \in \mathcal{I}_Q} \rho_w^J \circ T \circ \eta_v^I \circ T' \circ \iota_J^u && (4.2.4.16)\\
&= \rho_w^J \circ T \circ \eta_v^J \circ T' \circ \iota_J^u && (4.2.4.17)\\
&= \rho_w^J \circ T \circ \iota_J^v \circ ([\mathrm{id}_{\mathsf{B}_w}]_J^{-1} \otimes \mathrm{id}_J) \circ \rho_v^J \circ T' \circ \iota_J^u && (4.2.4.18)\\
&= ([T']_J \circ \overline{[T]}_J) \otimes \mathrm{id}_J = (\underline{[T']}_J \circ [T]_J) \otimes \mathrm{id}_J\,. && (4.2.4.19)
\end{aligned}$$

□

4.2.5 Linear idependence of double leaves

The goal of this section is to show the linear independence for the double leaves DL_u^w for some $\mathcal{H}$-words u, w. In order to do this, we will show how to calculate the non-reduced matrices from Section 4.2.4 and observe that their matrices are linearly independent over Q. We now define the leading coefficient of a lower/upper leaf morphism.

Definition 4.2.5.1. Let L be a lower leaf morphism and let $p = \mathsf{o}(L)$ be its root path. Let $J = \mathbf{t}(p)$ and let $q = \mathsf{o}(J)$. Let $\overline{[L]}$ be the upper reduced matrix of L. Define the *leading coefficient* $\mathsf{lead}^{LL}(L)$ *of* L to be the entry $\mathsf{lead}^{LL}(L) = \overline{[L]}_{p,q}$.

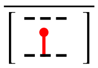
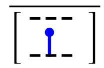
Similarly, let U be an upper leaf morphism and let $p' = \mathsf{o}(U)$ be its root path. Let $J = \mathbf{t}(p')$ and let $q = \mathsf{o}(J)$. Let $\underline{[U]}$ be the lower reduced matrix of U. Define the *leading coefficient* $\mathsf{lead}^{UL}(U)$ *of* U to be the entry $\mathsf{lead}^{UL}(U) = \overline{[U]}_{q,p'}$.

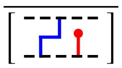
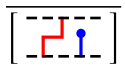
Now let D be a double leaf morphism. Let L be a lower leaf morphism and U an upper leaf morphism such that $D = U \circ L$. Let $p = \mathsf{o}(L)$ and $p' = \mathsf{o}(U)$. Define the *leading coefficient* $\mathsf{lead}^{DL}(D)$ *of* D to be the entry $\mathsf{lead}^{DL}(D) = [D]_{p,p'}$.

Whenever there is no ambiguity, we will just write lead for the leading coefficient.

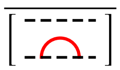
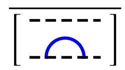
In light of Proposition 4.2.4.5, we must compute the upper-reduced matrices of all our basic lower leaves. Note that for a based walk p in P_Q, the matrix $\underline{\left[UL_p\right]} = \overline{\left[LL_p\right]}^{\mathsf{T}}$, so it suffices to compute these for our lower leaves.

Proposition 4.2.5.2. *The matrices for the (non-identity) basic lower leaves are the following:*

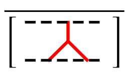
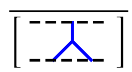
 $= \begin{array}{c|cc} & (\mathbb{1}) & (s) \\ \hline \phi & \alpha_s & 0 \end{array}$  $= \begin{array}{c|cc} & (\mathbb{1}) & (t) \\ \hline \phi & \alpha_t & 0 \end{array}$ (4.2.5.1)

 $= \begin{array}{c|cccc} & (\mathbb{1},\mathbb{1}) & (\mathbb{1},s) & (t,t) & (t,st) \\ \hline (\mathbb{1}) & \alpha_s & 0 & 0 & 0 \\ (t) & 0 & 0 & \alpha_s & 0 \end{array}$  $= \begin{array}{c|cccc} & (\mathbb{1},\mathbb{1}) & (\mathbb{1},t) & (s,s) & (s,st) \\ \hline (\mathbb{1}) & \alpha_t & 0 & 0 & 0 \\ (t) & 0 & 0 & \alpha_t & 0 \end{array}$

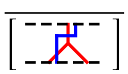
(4.2.5.2)

 $= \begin{array}{c|cccc} & (\mathbb{1},\mathbb{1}) & (s,\mathbb{1}) & (\mathbb{1},s) & (s,s) \\ \hline \phi & 1 & 1 & 0 & 0 \end{array}$  $= \begin{array}{c|cccc} & (\mathbb{1},\mathbb{1}) & (t,\mathbb{1}) & (\mathbb{1},t) & (t,t) \\ \hline \phi & 1 & 1 & 0 & 0 \end{array}$

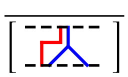
(4.2.5.3)

 $= \begin{array}{c|cccc} & (\mathbb{1},\mathbb{1}) & (s,\mathbb{1}) & (\mathbb{1},s) & (s,s) \\ \hline (\mathbb{1}) & 1 & 1 & 0 & 0 \\ (s) & 0 & 0 & 1 & 1 \end{array}$  $= \begin{array}{c|cccc} & (\mathbb{1},\mathbb{1}) & (t,\mathbb{1}) & (\mathbb{1},t) & (t,t) \\ \hline (\mathbb{1}) & 1 & 1 & 0 & 0 \\ (t) & 0 & 0 & 1 & 1 \end{array}$

(4.2.5.4)

 $= \begin{array}{c|cccccccc} & (\mathbb{1},\mathbb{1},\mathbb{1}) & (s,s,\mathbb{1}) & (\mathbb{1},\mathbb{1},s) & (s,s,s) & (\mathbb{1},t,t) & (s,st,t) & (\mathbb{1},t,st) & (s,st,st) \\ \hline (\mathbb{1},\mathbb{1}) & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ (s,s) & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ (\mathbb{1},t) & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ (s,st) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}$

(4.2.5.5)

 $= \begin{array}{c|cccccccc} & (\mathbb{1},\mathbb{1},\mathbb{1}) & (\mathbb{1},t,\mathbb{1}) & (s,s,s) & (s,st,s) & (\mathbb{1},\mathbb{1},t) & (\mathbb{1},t,t) & (s,s,st) & (s,st,st) \\ \hline (\mathbb{1},\mathbb{1}) & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ (s,s) & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ (\mathbb{1},t) & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ (s,st) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}$

(4.2.5.6)

In particular, note that their leading coefficients are all non-zero.

Proof. The proof goes by direct computation and is left to the reader as an exercise. □

Proposition 4.2.5.3. *Let w be an $\mathcal{H}$-word, let $p \in \mathsf{W}(w)$, and let $J = \mathbf{t}(p)$. Consider the lower leaf morphism $\underline{LL}_p$. Then $\underline{LL}_p$ has non-zero leading coefficient and for any $q \in \mathsf{W}(w)$ such that $q \not\leq_D p$ we have that $\underline{LL}_p \circ \underline{LI}_q = 0$. In particular, the column $[\underline{LL}_p]_{q,-} = 0$.*

Since $[\underline{UL}_p] = [\underline{LL}_p]^{\mathsf{T}}$, the leading coefficient of $\underline{LL}_p$ is also non-zero and the row $[\underline{UL}_p]_{-,q} = 0$.

Proof. In Proposition 4.2.5.2, we showed that the basic leaves have non-zero leading coefficients. For the general case, let $w, p,$ and J as above. First suppose that $p = (e_1, e_2)$. By Proposition 4.2.4.5 we have that the matrix

$$\overline{\left[\underline{LL}_p\right]} = \overline{\left[LL_{e_2}\right]} \circ \overline{\left[LL_{e_1} \otimes \mathrm{id}_{\mathbf{c}(e_1)}\right]} = \overline{\left[LL_{e_2}\right]} \circ \overline{\left[LL_{e_1}\right]} \otimes_Q \overline{\left[\mathrm{id}_{\mathbf{c}(e_1)}\right]}. \tag{4.2.5.7}$$

Since $\overline{\left[\mathrm{id}_{\mathbf{c}(e_1)}\right]}$ is just an identity matrix, the leading coefficient $\mathsf{lead}(\underline{LL}_p) = \mathsf{lead}(LL_{e_1})\,\mathsf{lead}(LL_{e_2})$. Thus, it is non-zero. In general, if we write write $p = (e_1, \ldots, e_r)$, the leading coefficient $\mathsf{lead}(\underline{LL}_p) = \mathsf{lead}(LL_{e_1})\,\mathsf{lead}(LL_{e_2}) \cdots \mathsf{lead}(LL_{e_r})$.

Now let $q \in \mathsf{W}(w)$ such that $q \not\leq_D p$. Write $q = (e'_1, \ldots, e'_r)$. Let i be the first index where $\mathbf{t}(e'_i) \not\leq_D \mathbf{t}(e_i)$. Then, $\mathbf{t}(e'_i)$ is not a summand of $\mathbf{t}_{\mathcal{H}}(e_i)$ in $\mathcal{Q}$ and the composition $\underline{LL}_{p,i} \circ \underline{LI}_{q,i} = 0$. Hence $\underline{LL}_p \circ \underline{LI}_q = 0$. □

Corollary 4.2.5.4. *Let D be a double leaf morphism. Write $D = U \circ L$ for some lower leaf L and upper leaf U. Let $p = \mathsf{o}(L)$ and $p' = \mathsf{o}(U)$. Let $J = \mathbf{t}(p) = \mathbf{t}(p')$, let $q = \mathsf{o}(J)$, and let $I = \mathcal{H}(J)$. Then*

$$\mathsf{lead}^{DL}(D) = \mathsf{lead}^{UL}(U)[\mathrm{id}_I]_{q,q}\,\mathsf{lead}^{LL}(L). \tag{4.2.5.8}$$

Hence, it is non-zero.

Also, the entry $[D]_{r,r'} = 0$ if either $r \not\leq_D p$ or $r' \not\leq_D p'$.

Proof. This follows immediately from Proposition 4.2.4.5 and Proposition 4.2.5.3.

□

Theorem 4.2.5.5. *Let u, w be two $\mathcal{H}$-words. The set $DL_u^w \subseteq \mathsf{Hom}_{\mathcal{H}}(\mathsf{B}_u, \mathsf{B}_w)$ of double leaves is linearly independent over Q.*

Proof. Let u, w be two $\mathcal{H}$-words. Let

$$f_1 D_1 + f_2 D_2 + \cdots + f_k D_k = 0 \tag{4.2.5.9}$$

be an expression of minimal length k with $f_i \in Q$ and $D_i \in DL_u^w$. Let us write each $D_i = \underline{UL}_{q_i} \circ \underline{LL}_{p_i}$ for some $p_i \in \mathsf{W}(u)$ and $q_i \in \mathsf{W}(w)$ with $\mathbf{t}(p_i) = \mathbf{t}(q_i)$. Without loss of generality, suppose that p_1 is maximal among $p_1, \ldots, p_k$ with respect to $\leq_D$. Also suppose without loss of generality, that q_1 is maximal among the set $\{q_i | p_i = p_1\}$ with respect to $\leq_D$. By Corollary 4.2.5.4, we have that $[D_1]_{p_1,q_1} \neq 0$ and $[D_i]_{p_1,q_1} = 0$ for all $i \geq 2$. Thus $f_1 = 0$, which contradicts the minimality of k. □

4.3 The Localized Folded Hecke Category of $\mathcal{Q}^\tau$ of type $A_1 \times A_1$

4.3.1 Presentation of $\mathcal{Q}^\tau$

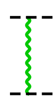
Definition 4.3.1.1. Define the ring $Q^\tau := R^\tau[(\alpha_s\alpha_t)^{-1}]$ to be the localization of R^τ obtained by inverting $\alpha_s\alpha_t$.

We define *diagrammatic localized folded Hecke category of type $A_1 \times A_1$*, $\mathcal{Q}(A_1 \times A_1)^\tau$ (abbreviated into $\mathcal{Q}^\tau$) to be the graded Q-linear monoidal category presented in the following way:

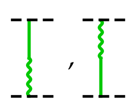
The objects of $\mathcal{Q}$ are generated by the generators $\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \mathrm{Q_Y}$, and $\mathrm{Q_Z}$ and their grading shifts, under the operations of direct sum and tensor product. The unit object of $\mathcal{Q}^\tau$ will be denoted by $\mathrm{Q}_{\mathbb{1}}$ and we will use the alternative notation $\mathrm{Q_X}$ for X when referring to the object in $\mathcal{Q}^\tau$. Also, the tensor product $\mathrm{Q_X} \otimes \mathrm{Q_Z}$ will be denoted by $\mathrm{Q_{XZ}}$.

The generating morphisms of $\mathcal{Q}^\tau$ extend the generators (3.2.1.2)-(3.2.1.7) of $\mathcal{H}^\tau$ by allowing polynomial boxes (2.3.2.6) to have polynomials in Q^τ, in addition to the following morphisms:

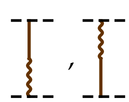
(a) The identity morphisms

$\mathrm{id}_{\mathrm{Q_Y}} :=$  $\mathrm{id}_{\mathrm{Q_Z}} :=$  . (4.3.1.1)

(b) The localized bivalent vertices

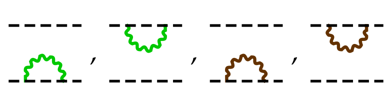
 (4.3.1.2)

of degree 1; and

 (4.3.1.3)

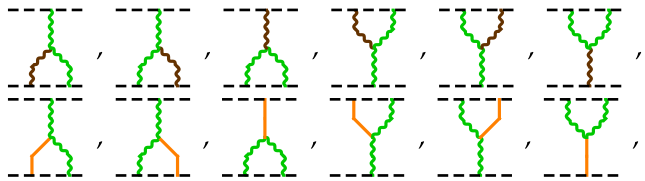
of degree 2.

(c) The cup and cap morphisms

 (4.3.1.4)

of degree 0.

(d) The localized trivalent vertices



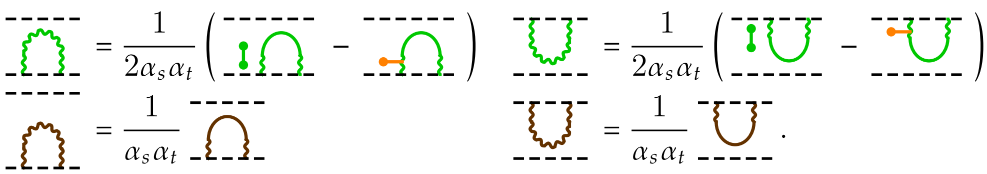
(4.3.1.5)

(4.3.1.6)

of degree 0.

These generators are subject to the relations defining $\mathcal{H}$ ((2.3.1.7)-(2.3.1.14) plus the following relations:

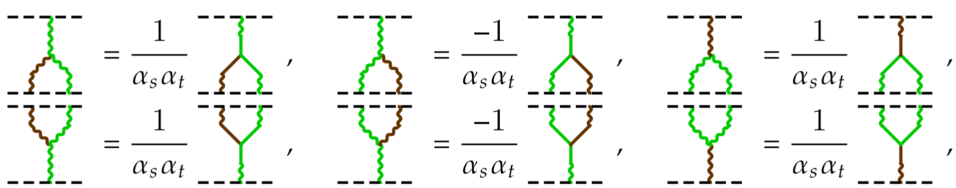
(a) Definition of the localized cups and caps:



(4.3.1.7)

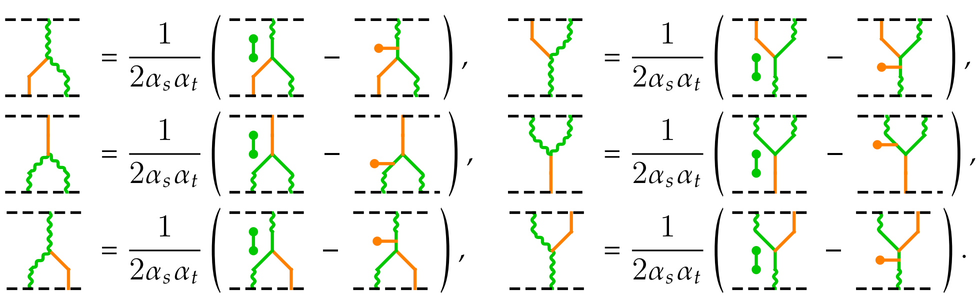
(4.3.1.8)

(b) Definition of the localized trivalent vertices:



(4.3.1.9)

(4.3.1.10)

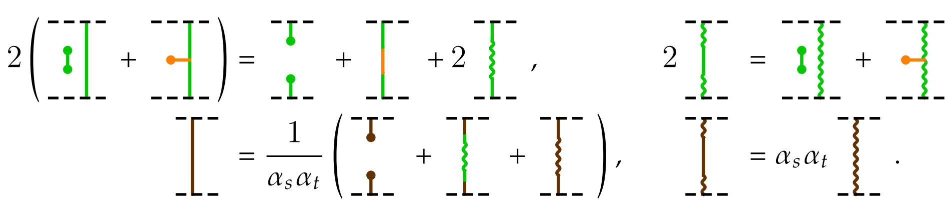


(4.3.1.11)

(4.3.1.12)

(4.3.1.13)

(c) The idempotent decomposition relations.



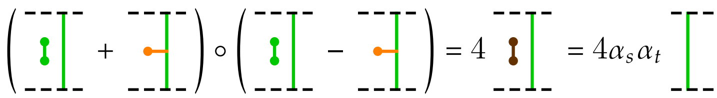
(4.3.1.14)

(4.3.1.15)

The following are some consequences of the relations in Q^{τ}.

(a) Because $\alpha_s\alpha_t$ is invertible in Q^τ, the unit object $Q_{\mathbb{1}}$ is isomorphic to its grading shift $Q_{\mathbb{1}}[2]$. In general, the isomorphism type of a grading shift of a homogeneous object in $\mathcal{Q}^\tau$ will only depend on the parity of its grading.

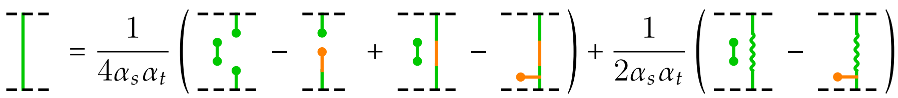
(b) The relation



(4.3.1.16)

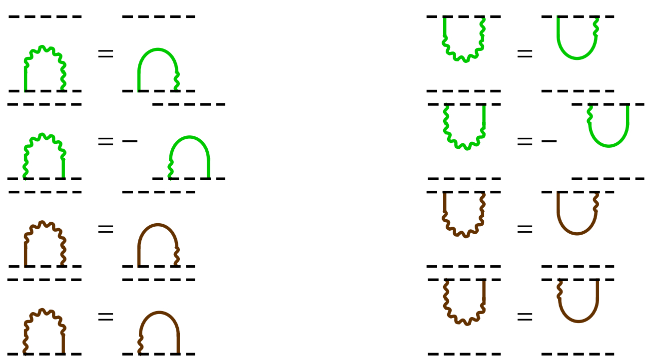
holds. Hence both  and  are invertible.

(c) The actual idempotent decomposition equation



(4.3.1.17)

(d) Cup and cap relations.



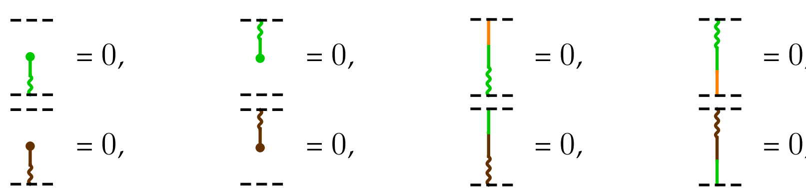
(4.3.1.18)

(4.3.1.19)

(4.3.1.20)

(4.3.1.21)

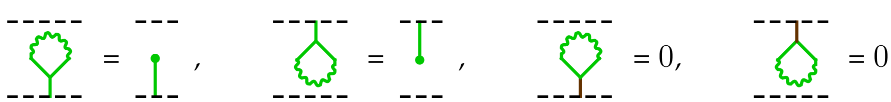
(e) Generators annihilated by the localized bivalent vertices.



(4.3.1.22)

(4.3.1.23)

(f) The localized needle relations hold:



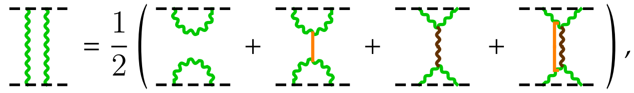
(4.3.1.24)

(g) The interaction between the regular and localized trivalent vertices

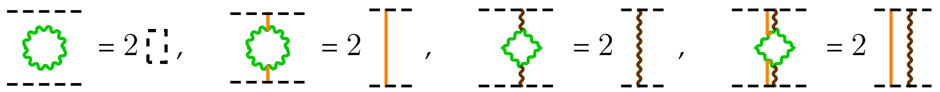
(4.3.1.25)

and their rotations hold.

(h) The idempotent decomposition relations for $Q_Y \otimes Q_Y$ hold:



(4.3.1.26)

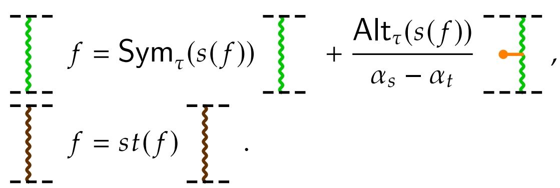


(4.3.1.27)

(i) The isomorphism relations for $Q_Y \otimes Q_Z \cong Q_Y$ hold:

(4.3.1.28)

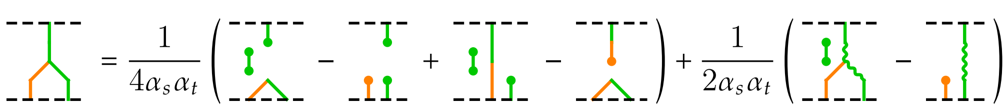
(j) Let $f \in Q^\tau$. The extension to Q^τ of the general polynomial forcing relations (A.3.0.4)-(A.3.0.7) hold. We also have the relations:



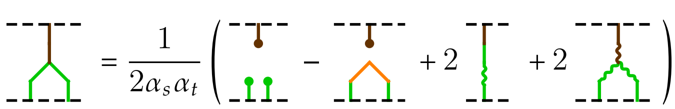
(4.3.1.29)

(4.3.1.30)

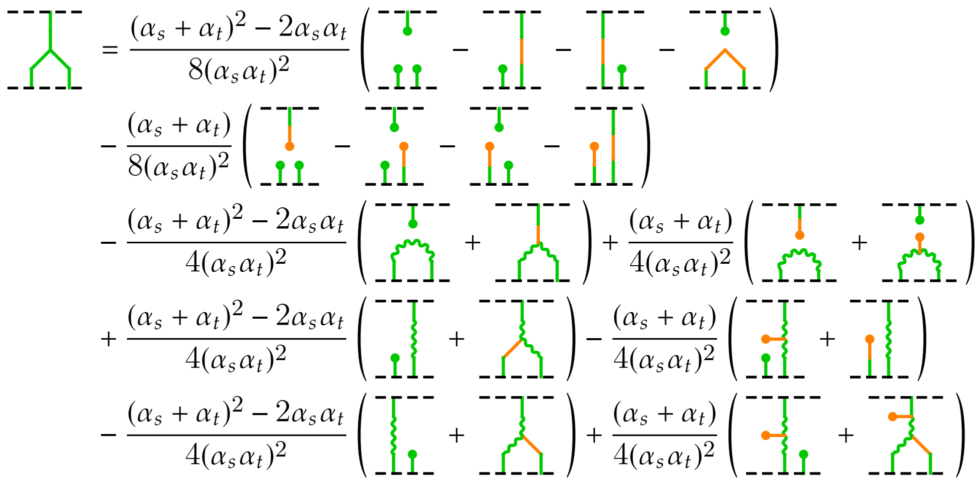
(k) The trivalent vertices may be written by



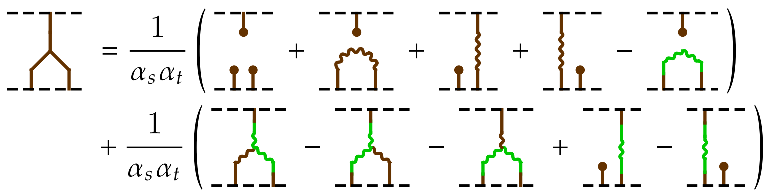
(4.3.1.31)



(4.3.1.32)



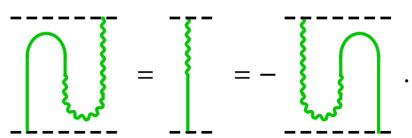
(4.3.1.33)



(4.3.1.34)

and their rotations.

Remark 4.3.1.2. Like in $\mathcal{Q}$, the bivalent vertex satisfies



(4.3.1.35)

Because of this, we may set $(\mathrm{Q}_\mathrm{Y}, -\ , -\ )$ to be the left adjoint of Q_Y (while maintaining $(\mathrm{Q}_\mathrm{Y}, \ , \ )$ as its right adjoint) to make $\mathcal{Q}^\tau$ pivotal. As with $\mathcal{Q}$, we can rotate diagrams in $\mathcal{Q}^\tau$ by multiplying by -1 every time we use the localized green cup or cap to the left.

Corollary 4.3.1.3. *The tuple* $(\mathrm{Q}_\mathrm{Z}, \ , \ )$ *is biadjoint to* Q_Z*. The tuple* $(\mathrm{Q}_\mathrm{Y}, \ , \ , -\ , -\ )$ *is biadjoint to* Q_Y*. These, together with the pivotal structure of* $\mathcal{H}^\tau$ *make* $\mathcal{Q}^\tau$ *into a pivotal category.*

Lemma 4.3.1.4.

(a) Let $D \in \mathrm{Hom}_{\mathcal{Q}^\tau}(\mathrm{Q}_\mathbb{1}, \mathrm{Q}_\mathbb{1})$ *be a diagram in* $\mathcal{Q}^\tau$*. Then* D *may be written as a* Q^τ*-linear combination of diagrams in* $\mathcal{H}^\tau$*. Moreover, by Theorem 3.2.6.1 we must have that* D *is a* Q^τ*-multiple of* .

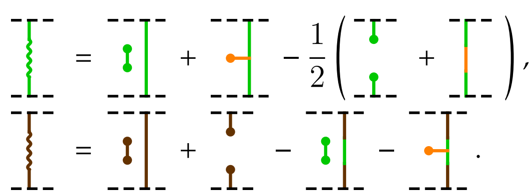
(b) *Let $D \in \mathrm{Hom}_{\mathcal{Q}^\tau}(\mathrm{Q}_\mathrm{X}, \mathrm{Q}_\mathbb{1})$ be a diagram in $\mathcal{Q}^\tau$. Then D may be written as a Q^τ-linear combination of diagrams in $\mathcal{H}^\tau$. Moreover, by Theorem 3.2.6.1 we must have that T is a Q^τ-multiple of .*

(c) *Let $D \in \mathrm{Hom}_{\mathcal{Q}^\tau}(\mathrm{Q}_\mathrm{Y}, \mathrm{Q}_\mathbb{1})$ be a diagram in $\mathcal{Q}^\tau$. Then D may be written as a composition $T \circ$, where T is a Q^τ-linear combination of diagrams in $\mathcal{H}^\tau$. Moreover, by Theorem 3.2.6.1 we must have that T is a Q^τ-multiple of and , so D is zero.*

(d) *Let $D \in \mathrm{Hom}_{\mathcal{Q}^\tau}(\mathrm{Q}_\mathrm{Z}, \mathrm{Q}_\mathbb{1})$ be a diagram in $\mathcal{Q}^\tau$. Then D may be written as a composition $T \circ$, where T is a Q^τ-linear combination of diagrams in $\mathcal{H}^\tau$. Moreover, by Theorem 3.2.6.1 we must have that T is a Q^τ-multiple of , so D is zero.*

(e) *Let $D \in \mathrm{Hom}_{\mathcal{Q}^\tau}(\mathrm{Q}_\mathrm{XZ}, \mathrm{Q}_\mathbb{1})$ be a diagram in $\mathcal{Q}$. Then D may be written as a composition $T \circ$, where T is a Q^τ-linear combination of diagrams in $\mathcal{H}^\tau$. Moreover, by Theorem 3.2.6.1 we must have that T is a Q^τ-multiple of , so D is zero.*

Proof. Let D be a diagram as above. As noted in Remark 4.2.1.3, We may use relations (4.3.1.7)-(4.3.1.13) to replace localized cups, caps, and trivalent vertices by their definition in terms of localized bivalent vertices and diagrams in $\mathcal{H}^\tau$. After doing this, we must observe that any bivalent vertex is either at the boundary of D or is composed with another localized bivalent vertex. We may use the relations



(4.3.1.36)

(4.3.1.37)

to write any bivalent vertices that are not in the boundary in terms of diagrams in $\mathcal{H}^\tau$. Finally, we may use the polynomial forcing relations (Proposition A.3.0.2) to move any polynomials as coefficients to the left. □

Observe that the idempotent decomposition relations (4.3.1.14) and (4.3.1.15) imply that

$$\mathrm{Y} \cong \mathrm{Q}_\mathrm{Y}[1] \oplus \mathrm{Q}_\mathrm{X}[1] \oplus \mathrm{Q}_\mathbb{1}[1], \tag{4.3.1.38}$$

$$\mathrm{Z} \cong \mathrm{Q}_\mathrm{Z}[1] \oplus \mathrm{Q}_\mathrm{Y}[1] \oplus \mathrm{Q}_\mathbb{1}[1], \tag{4.3.1.39}$$

$$\mathrm{XZ} \cong \mathrm{Q}_\mathrm{XZ}[1] \oplus \mathrm{Q}_\mathrm{Y}[1] \oplus \mathrm{Q}_\mathrm{X}[1]. \tag{4.3.1.40}$$

Also, relation (4.3.1.26) implies that

$$\mathrm{Q}_{\mathrm{Y}} \otimes \mathrm{Q}_{\mathrm{Y}} \cong \mathrm{Q}_{\mathbb{1}} \oplus \mathrm{Q}_{\mathrm{X}} \oplus \mathrm{Q}_{\mathrm{Z}} \oplus \mathrm{Q}_{\mathrm{XZ}}, \tag{4.3.1.41}$$

and relations (4.3.1.28) and (4.3.1.15) give us

$$\mathrm{Q}_{\mathrm{Y}} \otimes \mathrm{Q}_{\mathrm{Z}} \cong \mathrm{Q}_{\mathrm{Y}}, \qquad \mathrm{Q}_{\mathrm{Z}} \otimes \mathrm{Q}_{\mathrm{Z}} \cong \mathrm{Q}_{\mathbb{1}}\,. \tag{4.3.1.42}$$

We will now show that $\mathcal{Q}^{\tau}$ is a graded category whose indecomposable objects (up to the parity of their grading shift) are $\mathrm{Q}_{\mathbb{1}}, \mathrm{Q}_{\mathrm{X}}, \mathrm{Q}_{\mathrm{Y}}, \mathrm{Q}_{\mathrm{Z}}$, and Q_{XZ}. We define $\mathcal{J} := \{\mathrm{Q}_{\mathbb{1}}, \mathrm{Q}_{\mathrm{X}}, \mathrm{Q}_{\mathrm{Y}}, \mathrm{Q}_{\mathrm{Z}}, \mathrm{Q}_{\mathrm{XZ}}\}$. An important remark is that we are not calling the objects in $\mathcal{J}$ irreducible since they have non-zero morphisms between them. We may regard them as irreducible if we strengthen the notion of a sub-object in our category, but this is not very relevant for the results proved further.

Theorem 4.3.1.5. *The set $\mathcal{J}$ is a full set of indecomposable objects of $\mathcal{Q}^{\tau}$ (up to grading shift). Every objects of $\mathcal{Q}^{\tau}$ admits a decomposition into a finite direct sum objects of $\mathcal{J}$. We have the following morphism spaces:*

$$\mathsf{Hom}_{\mathcal{Q}^{\tau}}(\mathrm{Q}_{\mathbb{1}}, \mathrm{Q}_{\mathbb{1}}) = Q^{\tau}\,, \qquad \mathsf{Hom}_{\mathcal{Q}^{\tau}}(\mathrm{Q}_{\mathbb{1}}, \mathrm{Q}_{\mathrm{X}}) = Q^{\tau}\,, \tag{4.3.1.43}$$

$$\mathsf{Hom}_{\mathcal{Q}^{\tau}}(\mathrm{Q}_{\mathrm{X}}, \mathrm{Q}_{\mathbb{1}}) = Q^{\tau}\,, \qquad \mathsf{Hom}_{\mathcal{Q}^{\tau}}(\mathrm{Q}_{\mathrm{X}}, \mathrm{Q}_{\mathrm{X}}) = Q^{\tau}\,, \tag{4.3.1.44}$$

$$\mathsf{Hom}_{\mathcal{Q}^{\tau}}(\mathrm{Q}_{\mathrm{Z}}, \mathrm{Q}_{\mathrm{Z}}) = Q^{\tau}\,, \qquad \mathsf{Hom}_{\mathcal{Q}^{\tau}}(\mathrm{Q}_{\mathrm{Z}}, \mathrm{Q}_{\mathrm{XZ}}) = Q^{\tau}\,, \tag{4.3.1.45}$$

$$\mathsf{Hom}_{\mathcal{Q}^{\tau}}(\mathrm{Q}_{\mathrm{XZ}}, \mathrm{Q}_{\mathrm{Z}}) = Q^{\tau}\,, \qquad \mathsf{Hom}_{\mathcal{Q}^{\tau}}(\mathrm{Q}_{\mathrm{XZ}}, \mathrm{Q}_{\mathrm{XZ}}) = Q^{\tau}\,, \tag{4.3.1.46}$$

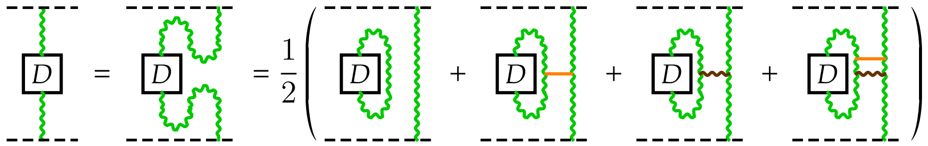
$$\mathsf{Hom}_{\mathcal{Q}^{\tau}}(\mathrm{Q}_{\mathrm{Y}}, \mathrm{Q}_{\mathrm{Y}}) = Q^{\tau} \oplus Q^{\tau} \cong Q. \tag{4.3.1.47}$$

All the other Hom*-spaces between objects in $\mathcal{J}$ are trivial. Hence $\mathcal{Q}^{\tau}$ is a graded Krull-Schmidt category.*

Proof. The fact that any object can be written as a finite direct sum of objects in $\mathcal{J}$ follows from (4.3.1.38)-(4.2.1.35). If we show that the morphism spaces are as described, they would be local graded rings (like in 3.1.3.3). Thus, by Lemma 2.2.4.4, the category $\mathcal{Q}^{\tau}$ has unique decomposition.

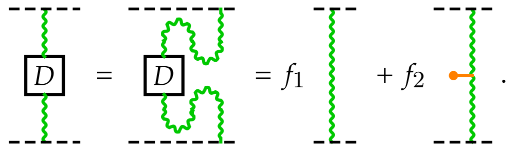
We will show that $\mathsf{Hom}_{\mathcal{Q}^{\tau}}(\mathrm{Q}_{\mathrm{Y}}, \mathrm{Q}_{\mathrm{Y}})$ is a free $\mathcal{Q}^{\tau}$ module spanned by $\{\ ,\ \}$, and that $\mathsf{Hom}_{\mathcal{Q}^{\tau}}(\mathrm{Q}_{\mathrm{Y}}, \mathrm{Q}_{\mathrm{Z}})$ is trivial. The argument for other Hom spaces is analogous and is left as an exercise to the reader.

Let D be a diagram in $\mathrm{Hom}_{Q^\tau}(\mathrm{Q_Y}, \mathrm{Q_Y})$. Using adjunction and relation (4.3.1.26), we write

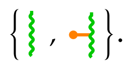


(4.3.1.48)

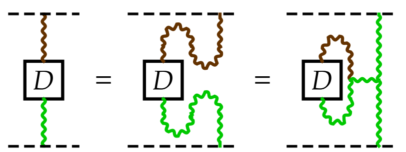
By Lemma 4.3.1.4, the two last summands are zero and there are $f_1, f_2 \in Q^\tau$ such that



(4.3.1.49)

Hence, D is in the span of .

Let D instead be a diagram in $\mathrm{Hom}_{Q^\tau}(\mathrm{Q_Y}, \mathrm{Q_Z})$. As before, we may use adjunction to write



(4.3.1.50)

By Lemma 4.3.1.4, the diagram is zero. □

Corollary 4.3.1.6. *The category Q^τ is the Karoubian envelope of the localization of $\mathcal{H}^\tau$ given by inverting $\alpha_s \alpha_t$.*

Proof. By Theorem 4.3.1.5, Q is Karoubian. Since we only used the idempotent relations (4.3.1.14)-(4.3.1.15) (together with the relations in $\mathcal{H}^\tau$) to derive all our other relations the result follows. □

Given the result in Theorem 4.3.1.5, we will group all the objects in $\mathcal{J}$ that have non-zero morphisms between them.

Definition 4.3.1.7. Define the following partition of $\mathcal{J}$:

$$\mathsf{C}_{\mathbb{1}} := \{\mathrm{Q}_{\mathbb{1}}, \mathrm{Q_X}\}, \mathsf{C}_{\mathrm{Y}} := \{\mathrm{Q_Y}\}, \mathsf{C}_{\mathrm{Z}} := \{\mathrm{Q_Z}, \mathrm{Q_{XZ}}\}. \tag{4.3.1.51}$$

This defines an equivalence relation on $\mathcal{J}$ where $I_1 \sim I_2$ if and only if $\mathrm{Hom}_{Q^\tau}(I_1, I_2) \neq 0$. The denote the class of an object $I \in \mathcal{J}$ by $\mathsf{C}(I)$.

Also define the dominance partial order $\leq_D$ on the classes of $\mathcal{J}$ by the lattice:

$$\begin{tikzcd} C_{Z} = \{Q_Z, Q_{XZ}\} \arrow[d, dash] \\ C_{Y} = \{Q_Y\} \arrow[d, dash] \\ C_{\mathbb{1}} = \{Q_{\mathbb{1}}, Q_X\}, \end{tikzcd} \tag{4.3.1.52}$$

This, of course, defines a preorder on $\mathcal{J}$ similarly noted by $\leq_D$. This order corresponds to the quotient of the Bruhat order of $W(A_1 \times A_1)$ by the action of τ.

We also have that the all the objects in $\mathcal{I} = \{\mathbb{1}, X, Y, Z, XZ\}$ have a unique maximal summand in $\mathcal{Q}^\tau$ with respect to $\leq_D$. For $J \in \mathcal{J}$, define $\mathcal{H}^\tau(J)$ to be the object in $\mathcal{I}$ for which J is the top summand. That is,

$$\mathcal{H}^\tau(Q_{\mathbb{1}}) = \mathbb{1}, \mathcal{H}^\tau(Q_X) = X, \mathcal{H}^\tau(Q_Y) = Y, \mathcal{H}^\tau(Q_Z) = Z, \text{ and } \mathcal{H}^\tau(Q_{XZ}) = XZ. \tag{4.3.1.53}$$

Definition 4.3.1.8. Define the localization functor $\Lambda^\tau : \mathcal{H}^\tau \longrightarrow \mathcal{Q}^\tau$ to be the functor sending $X \mapsto X$, $Y \mapsto Y$, $Z \mapsto Z$, and sending a diagram in $\mathcal{H}^\tau$ to its corresponding diagram in $\mathcal{Q}^\tau$.

Corollary 4.3.1.9. *The functor Λ^τ is faithful, so Λ^τ is an embedding of $\mathcal{H}^\tau$ into $\mathcal{Q}^\tau$.*

Proof. By Proposition 2.1.2.7, we only have to check that Λ^τ is injective on

$$\mathrm{Hom}_{\mathcal{H}^\tau}(\mathbb{1}, \mathbb{1}), \mathrm{Hom}_{\mathcal{H}^\tau}(X, \mathbb{1}), \mathrm{Hom}_{\mathcal{H}^\tau}(Y, \mathbb{1}), \mathrm{Hom}_{\mathcal{H}^\tau}(Z, \mathbb{1}), \text{ and } \mathrm{Hom}_{\mathcal{H}^\tau}(XZ, \mathbb{1}). \tag{4.3.1.54}$$

By Lemma 4.3.1.4,

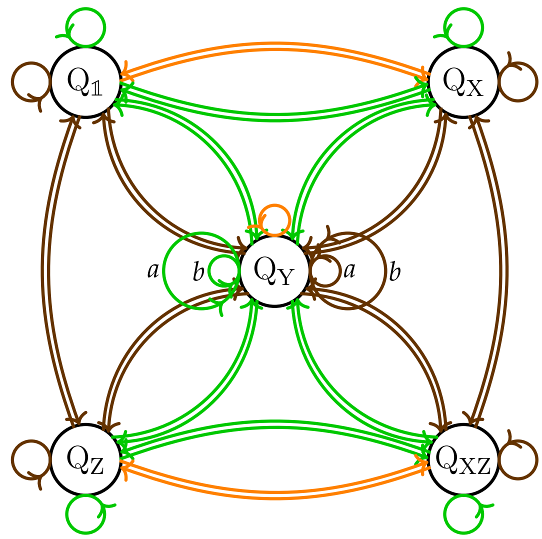
$$\mathrm{Hom}_{\mathcal{Q}^\tau}(\Lambda^\tau(\mathbb{1}), \Lambda^\tau(\mathbb{1})) = \mathrm{Hom}_{\mathcal{Q}^\tau}(Q_{\mathbb{1}}, Q_{\mathbb{1}}) = \mathcal{Q}^\tau \tag{4.3.1.55}$$

and

$$\mathrm{Hom}_{\mathcal{Q}^\tau}(\Lambda^\tau(X), \Lambda^\tau(\mathbb{1})) = \mathrm{Hom}_{\mathcal{Q}^\tau}(Q_X, Q_{\mathbb{1}}) = \mathcal{Q}^\tau \,. \tag{4.3.1.56}$$

Observe that the functor Λ^τ sends the generators , , , , , and to themselves. Hence, it is injective on $\mathrm{Hom}_{\mathcal{H}^\tau}(\mathbb{1}, \mathbb{1})$, $\mathrm{Hom}_{\mathcal{H}^\tau}(X, \mathbb{1})$, $\mathrm{Hom}_{\mathcal{H}^\tau}(Y, \mathbb{1})$, $\mathrm{Hom}_{\mathcal{H}^\tau}(Z, \mathbb{1})$, and $\mathrm{Hom}_{\mathcal{H}^\tau}(XZ, \mathbb{1})$. $\square$

Definition 4.3.2.1. Define the *branching graph P of $\mathcal{Q}^{\tau}$ with respect to* $\{\mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$ to be the following directed multigraph (admitting loops):



. (4.3.2.1)

The nodes of P correspond to the set of indecomposable objects $\mathcal{J}$ of $\mathcal{Q}^{\tau}$, which in turn are in correspondence with the indecomposable objects of $\mathcal{H}^{\tau}$. The edges of P are colored by the set of objects $\{\mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$. In (4.3.2.1), the orange edges are X-colored, the green edges are Y-colored, and the brown edges are Z-colored. Given two indecomposable objects $I_1, I_2 \in \mathcal{J}$ and let $A \in \{\mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$. The graph P has an edge from I_1 to I_2 of color A for each time I_2 appears as a summand of $I_1 \otimes A$ regardless of their degree shift. For instance,

$$\mathrm{Q}_{\mathrm{Y}} \otimes \mathrm{Y} \cong \mathrm{Q}_{\mathbb{1}}[1] \oplus \mathrm{Q}_{\mathrm{X}}[1] \oplus {\mathrm{Q}_{\mathrm{Y}}}^{\oplus 2}[1] \oplus \mathrm{Q}_{\mathrm{Z}}[1] \oplus \mathrm{Q}_{\mathrm{XZ}}[1], \tag{4.3.2.2}$$

so Q_{Y} has two Y-colored loops and Y-colored edges to $\mathrm{Q}_{\mathbb{1}}, \mathrm{Q}_{\mathrm{X}}, \mathrm{Q}_{\mathrm{Z}}$, and Q_{XZ}. We distinguish the two green and the two brown loops by the letter labels on them.

As before, given an edge e of P, we denote its source by $\mathbf{s}(e)$, its target by $\mathbf{t}(e)$, and its color by $\mathbf{c}(e)$.

Notice that for each indecomposable in $\mathcal{J}$ we have a corresponding indecomposable object in $\mathcal{I}$. Define $\mathbf{s}_{\mathcal{H}^{\tau}}(e) = \mathcal{H}^{\tau}(\mathbf{s}(e))$ and $\mathbf{t}_{\mathcal{H}^{\tau}}(e) = \mathcal{H}^{\tau}(\mathbf{t}(e))$.

Definition 4.3.2.2. Define an *$\mathcal{H}^{\tau}$-word* $w = (A_1, \ldots, A_k)$ to be a word on the set $\{\mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$. Define the *Bott-Samelson object* B_w associated to w to be

$$\mathsf{B}_w := A_1 \otimes A_2 \otimes \cdots \otimes A_k. \tag{4.3.2.3}$$

Definition 4.3.2.3. Define an *$\mathcal{H}^\tau$-walk* p to be a walk on the graph P. That is, $p = (e_1, e_2, \ldots, e_r)$ is a tuple of edges of P where $\mathbf{s}(e_{i+1}) = \mathbf{t}(e_i)$. We say that the *start* $\mathbf{s}(p)$ *of* p is $\mathbf{s}(e_1)$ and its *end* $\mathbf{s}(p)$ is $\mathbf{t}(e_r)$. Also define $\mathbf{s}_{\mathcal{H}^\tau}(p)$ and $\mathbf{t}_{\mathcal{H}^\tau}(p)$ to be $\mathbf{s}_{\mathcal{H}^\tau}(e_1)$ and $\mathbf{t}_{\mathcal{H}^\tau}(e_r)$. We say that p is a *based walk* if it starts at $\mathrm{Q}_{\mathbb{1}}$.

Define the *$\mathcal{H}^\tau$-word* $\mathbf{w}(p)$ associated to p to be the word $(\mathbf{c}(e_1), \ldots, \mathbf{c}(e_r))$ consisting of the colors of the edges of p.

Given an $\mathcal{H}^\tau$-word w, we say that p is *w-colored* if $\mathbf{w}(p) = w$. Define $\mathsf{W}(w)$ to be the set of all w-colored based walks on P. For an object $J \in \mathcal{J}$, we define $\mathsf{W}^J(w)$ to be the set of all w-colored based walks on P ending at J. We also define $\mathsf{W}^{\mathsf{C}(J)}(w)$ to be the set of all w-colored based walks on P ending at an object of class $\mathsf{C}(J)$.

Define the *Bott-Samelson object* B_p corresponding to p to be $\mathsf{B}_{\mathbf{w}(p)}$.

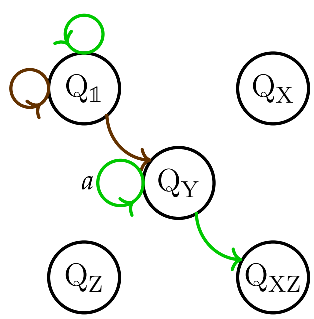
Similarly, define the *class of the walk* p, to be the tuple of classes

$$\mathsf{C}(p) := (\mathsf{C}(\mathbf{t}(e_1)), \mathsf{C}(\mathbf{t}(e_2)), \ldots, \mathsf{C}(\mathbf{t}(e_r))). \tag{4.3.2.4}$$

We say that two based walks p, p' on P have the same class if $\mathsf{C}(p) = \mathsf{C}(p')$. Define the *walk class* of p $\mathsf{C}\,\mathsf{W}(p) := \{q \in \mathsf{W}(\mathbf{w}(p)) | \, \mathsf{C}(p) = \mathsf{C}(q)\}$ to be the set of walks with the same class as p.

Let $p = (e_1, \ldots, e_r)$ and $p' = (e'_1, \ldots, e'_r)$ in $\mathsf{W}(w)$. We say that $p \leq_D p'$ if and only if $\mathsf{C}(\mathbf{t}(e_i)) \leq_D \mathsf{C}(\mathbf{t}(e'_i))$ for $1 \leq i \leq r$. This is called the *dominance (pre)order* $\leq_D$ on $\mathsf{W}(w)$.

Notation 4.3.2.4. Just like in the $\mathcal{H}$ case, we will describe the walks on P by the $\mathcal{H}^\tau$ target of each edge, colored by the color of each edge. Whenever our walk includes a green or brown loop at Q_{Y}, we will use their label subindex to indicate the loop we are walking through. For example, the tuple $(\mathbb{1}, \mathbb{1}, \mathrm{Y}, \mathrm{Y}_{\mathrm{a}}, \mathrm{XZ})$ represents the walk



. (4.3.2.5)

Proposition 4.3.2.5. *Let w be an $\mathcal{H}^\tau$-word and B_w its Bott-Samelson object. We have a bijective correspondence between the set of w-colored based walks $\mathsf{W}(w)$ and the indecomposable summands of B_w in $\mathcal{Q}^\tau$. The multiplicity of an indecomposable $J \in \mathcal{J}$ in B_w corresponds to the size of $\mathsf{W}^J(w)$ (the number of based walks ending at J).*

Proof. This is analogous to the proof of Proposition 4.2.2.5. □

4.3.3 Light leaves basis for $\mathcal{H}^\tau$

Definition 4.3.3.1. Let P be the branching graph of $\mathcal{Q}^\tau$ with respect to $\{X, Y, Z\}$. Let e be and edge of P colored by $\mathbf{c}(E)$, with source and target $\mathbf{s}(e)$ and $\mathbf{t}(e)$ respectively. Let $\mathbf{s}_{\mathcal{H}^\tau}(e)$ and $\mathbf{t}_{\mathcal{H}^\tau}(e)$ be the corresponding objects in $\mathcal{I}$ that have $\mathbf{s}(e)$ and $\mathbf{t}(e)$ as top summands.

- Define a *lower $\mathcal{H}^\tau$-labeling* of e to be a labeling of e by a morphism $LL_e : \mathbf{s}_{\mathcal{H}^\tau}(e) \otimes \mathbf{c}(e) \longrightarrow \mathbf{t}_{\mathcal{H}^\tau}(e)$ in $\mathcal{H}^\tau$.

- Define a *lower $\mathcal{Q}^\tau$-labeling* of e to be a labeling of e by a morphism $LI_e : \mathbf{s}(e) \otimes \mathbf{c}(e) \longrightarrow \mathbf{t}(e)$ in $\mathcal{Q}^\tau$.

- Define an *upper $\mathcal{H}^\tau$-labeling* and an *upper $\mathcal{Q}^\tau$-labeling* analogously (see Definition 4.2.3.1).

- Define an $\mathcal{H}^\tau$-**LL**-*labeling* of P to be a labeling of the edges of P by lower $\mathcal{H}^\tau$-labelings.

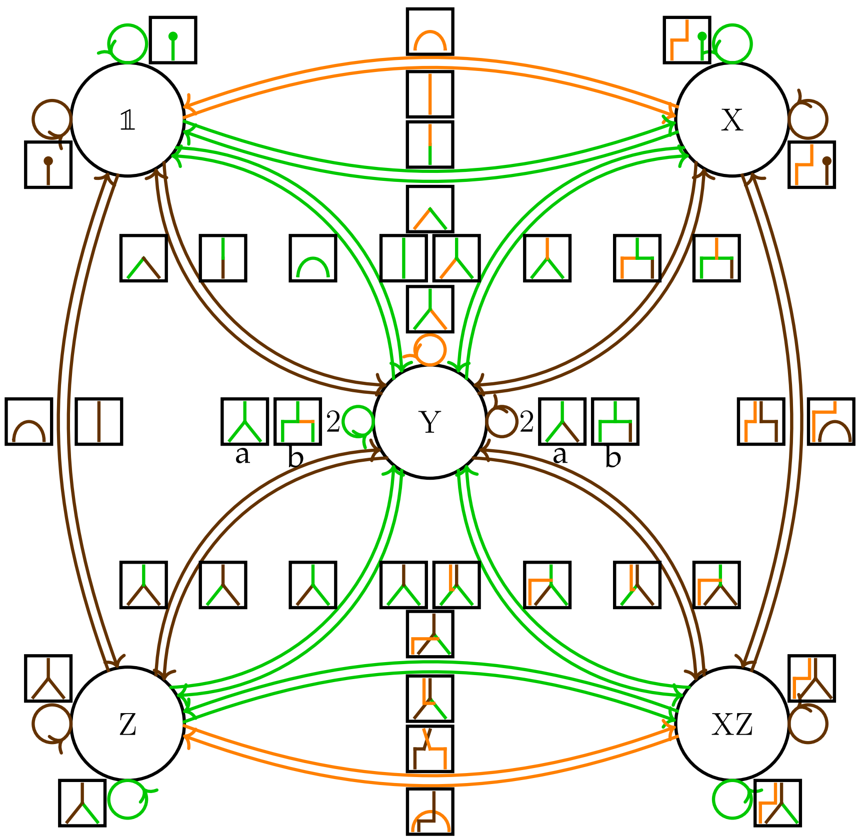
- Define an $\mathcal{H}^\tau$-**LI**-*labeling* of P to be a labeling of the edges of P by lower $\mathcal{Q}^\tau$-labelings.

- Define an $\mathcal{H}^\tau$-**UL**-*labeling* of P to be a labeling of the edges of P by upper $\mathcal{H}^\tau$-labelings.

- Define an $\mathcal{H}^\tau$-**UI**-*labeling* of P to be a labeling of the edges of P by upper $\mathcal{Q}^\tau$-labelings.

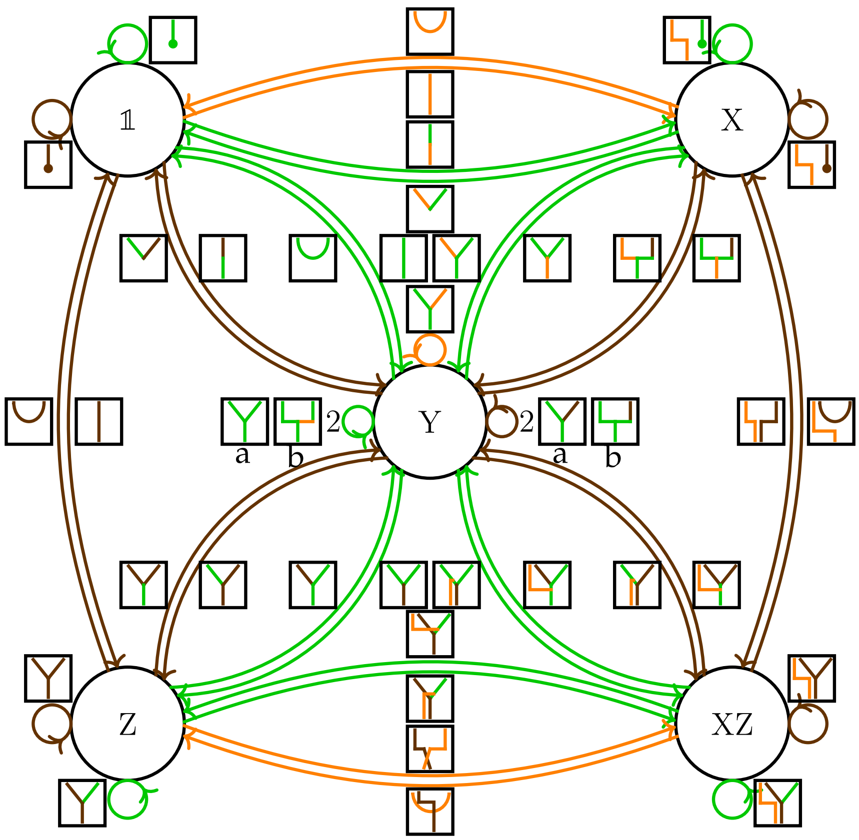
Definition 4.3.3.2. We will call the following labelings the *standard $\mathcal{H}^\tau$-labelings*.

Standard **LL**-*labeling*:



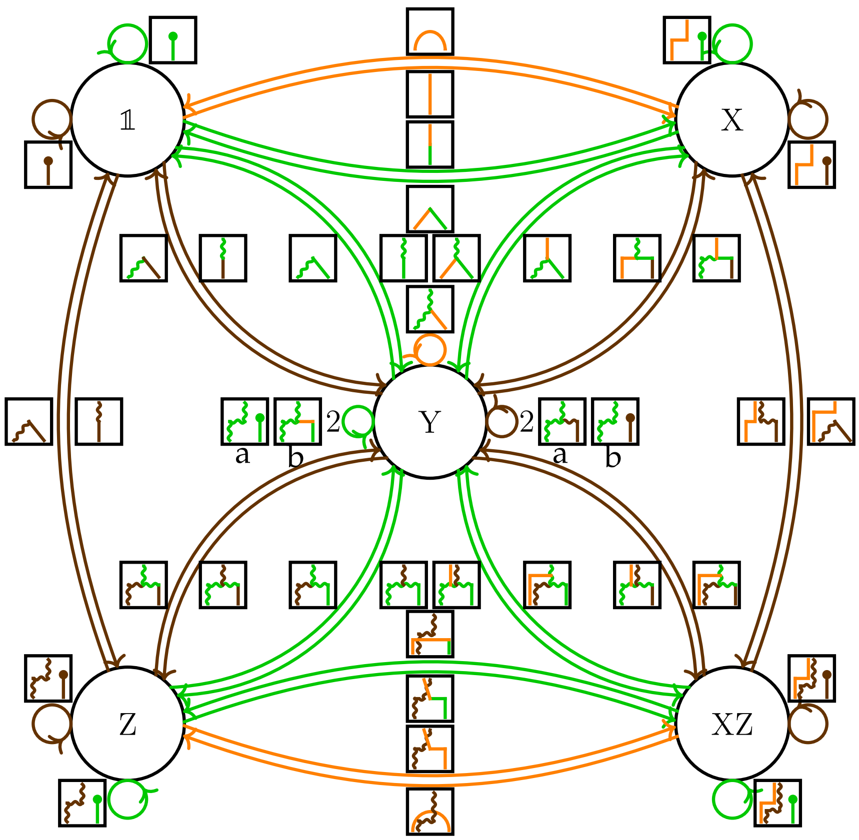
(4.3.3.1)

Standard **UL**-*labeling*:



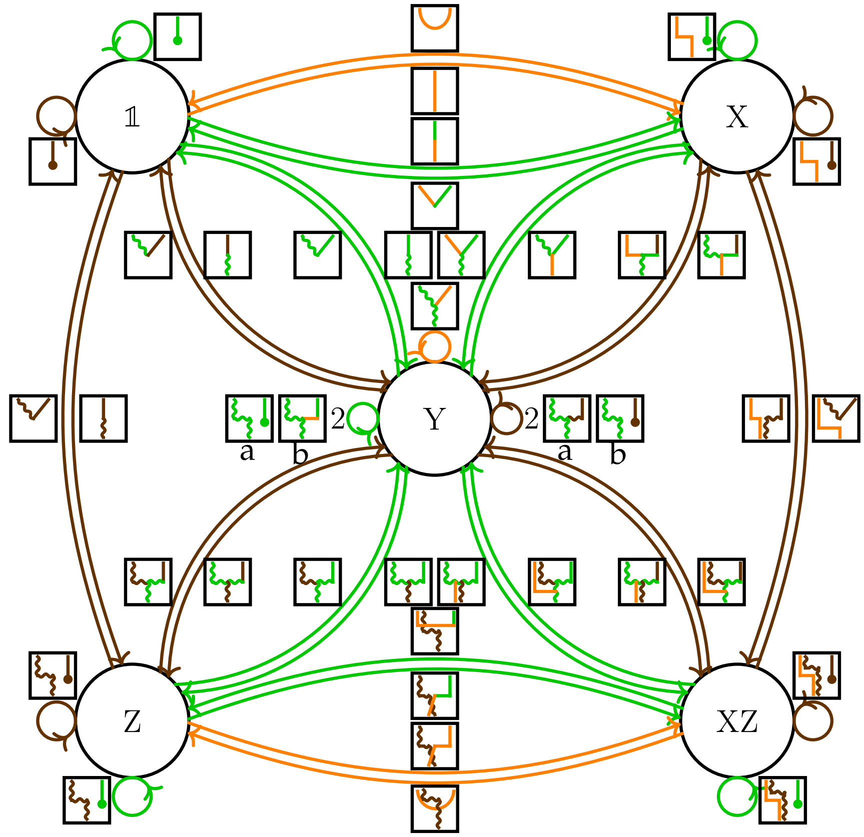
(4.3.3.2)

Standard **LI**-*labeling*:



(4.3.3.3)

Standard **UI**-*labeling*:



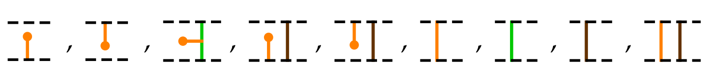
(4.3.3.4)

Definition 4.3.3.3. Let $p = (e_1, e_2, \ldots, e_r)$ be a based walk on P. Fix the standard labelings from Definition 4.3.3.2.

Define the morphisms lower leaf, upper leaf, lower projection and upper inclusion morphism $\underline{LL}_{p,i}$, $\underline{UL}_{p,i}$, $\underline{LI}_{p,i}$, and $\underline{UI}_{p,i}$ (for $0 \leq i \leq r$) according to Definition 4.2.3.2 (with respect to the standard $\mathcal{H}^\tau$-labelings). As before, define $\underline{LL}_p = \underline{LL}_{p,r}$, $\underline{UL}_p = \underline{UL}_{p,r}$, $\underline{LI}_p = \underline{LI}_{p,r}$, and $\underline{UI}_p = \underline{UI}_{p,r}$. In this case we say that p is the *root path* of $\underline{LL}_p$, $\underline{UL}_p$, $\underline{LI}_p$, and $\underline{UI}_p$.

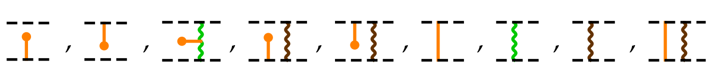
Given a lower leaf morphism L, we denote its root path by $\mathsf{o}(L)$. For $J \in \mathcal{Q}^\tau$, define its root path $\mathsf{o}(J) := \mathsf{o}(\mathrm{id}_{\mathcal{H}^\tau(J)})$.

Definition 4.3.3.4. We will call the following our $\mathcal{H}^\tau$*-neutral morphisms*:



(4.3.3.5)

Similarly, we will call the following our $\mathcal{Q}^\tau$*-neutral morphisms*:



(4.3.3.6)

An *extended neutral morphism* is a morphism of the form $N \otimes \mathrm{id}_A$ for some $\mathcal{H}^\tau$-word A.

An *extended lower leaf* is a morphism of the form $N \circ L$ for some $\mathcal{H}^\tau$-neutral morphism N and lower leaf morphism L.

An *extended upper leaf* is a morphism of the form $U \circ N$ for some $\mathcal{H}^\tau$-neutral morphism N and upper leaf morphism U.

A *double leaf* is a morphism of the form $U \circ N \circ L$ for some $\mathcal{H}^\tau$-neutral morphism N, upper leaf morphism U, and lower leaf morphism L.

Definition 4.3.3.5. Let u, w be $\mathcal{H}^\tau$-words and $J, I \in \mathcal{J}$.

We denote by $\mathbf{N}_J^I$ to the set of neutral morphisms with source $\mathcal{H}^\tau(J)$ and target $\mathcal{H}^\tau(I)$.

We denote by $\mathbf{LL}_u^J$ to the set of extended lower leaves with source u and target $\mathcal{H}^\tau(J)$.

We denote by $\mathbf{UL}_J^u$ to the set of extended upper leaves with source $\mathcal{H}^\tau(J)$ and target u.

We denote by $\mathbf{DL}_u^w$ to the set of double leaves with source u and target w.

We denote by $\mathbf{LI}_u^J$ to the set of lower inclusions with source u and target J.

We denote by $\mathbf{UI}_J^u$ to the set of upper projections with source J and target u.

4.3.4 *Writing the morphisms in $\mathcal{H}^\tau$ with matrices*

The category $\mathcal{H}^\tau$ embeds into $\mathcal{Q}^\tau$ (like $\mathcal{H}$ into $\mathcal{Q}$). We may write any morphism $T : A \longrightarrow B$ in $\mathcal{H}^\tau$ in terms of morphisms between the indecomposable summands of A and B (in $\mathcal{Q}^\tau$). This will, in turn, give us a matrix with coefficients in Q. Even though this process is very similar to the one for $\mathcal{H}$, we will need to take into account the non-zero morphisms between non-isomorphic indecomposable objects of $\mathcal{Q}^\tau$.

Definition 4.3.4.1. Let $I, J \in \mathcal{J}$ such that $\mathsf{C}(I) = \mathsf{C}(J)$. Define the function $\mathbf{r} : \mathsf{Hom}_{\mathcal{Q}^\tau}(I, J) \longrightarrow Q$ in the following way: By Theorem 4.3.1.5, any $T \in \mathsf{Hom}_{\mathcal{Q}^\tau}(I, J)$ is either

(a) of the form $T = f_T\, U$ for $f_t \in Q^\tau$ and

$U \in \Big\{$  $\Big\}$. (4.3.4.1)

In this case define $\mathbf{r}(T) = f_T$.

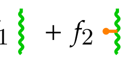
(b) of the form $T = f_T\, U$ for $f_t \in Q^\tau$ and

$$U \in \{\ \ \} . \tag{4.3.4.2}$$

In this case define $\mathbf{r}(T) = f_T(\alpha_s - \alpha_t)$.

(c) of the form

$$T = f_1 \ \ + f_2 \ \ \tag{4.3.4.3}$$



for some $f_1, f_2 \in Q^\tau$. In this case define $\mathbf{r}(T) = f_1 + f_2(\alpha_s - \alpha_t)$.

Extend the definition of $\mathbf{r}$ to all $T \in \mathsf{Hom}_{Q^\tau}(I, J)$ for $I, J \in \mathcal{J}$ by setting $\mathbf{r}(T) := 0$ when $I \neq J$.

Let u, w be two $\mathcal{H}^\tau$-words, let $p \in \mathsf{W}(u)$ and let $q \in \mathsf{W}(w)$. Let $T \in \mathsf{Hom}_{\mathcal{H}^\tau}(\mathsf{B}_u, \mathsf{B}_w)$. Define the functional $\mathbf{r}_p^q : \mathsf{Hom}_{\mathcal{H}^\tau}(\mathsf{B}_u, \mathsf{B}_w) \longrightarrow R$ by $\mathbf{r}_p^q(T) := \mathbf{r}(\underline{LI}_q \circ T \circ \underline{UI}_p)$.

Define the *non-reduced matrix* $[T]$ *of* T (with respect to the standard **LI** and **UI** labelings) in the following way:

(a) The columns of $[T]$ are indexed by $\mathsf{W}(u)$.

(b) The rows of $[T]$ are indexed by $\mathsf{W}(w)$.

(c) For $p \in \mathsf{W}(u)$ and $q \in \mathsf{W}(w)$, the entry $[T]_{p,q}$ is $\mathbf{r}_p^q(T)$.

Define $[T]_{\mathsf{C}(J)}$ for $J \in \mathcal{J}$ analogously, by restricting ourselves to walks in $\mathsf{W}^{\mathsf{C}(J)}(u)$ and $\mathsf{W}^{\mathsf{C}(J)}(w)$. It is clear from the definition that

$$[T] = [T]_{\mathsf{C}_{\mathbb{1}}} \oplus [T]_{\mathsf{C}_{\mathrm{Y}}} \oplus [T]_{\mathsf{C}_{\mathrm{Z}}}. \tag{4.3.4.4}$$

Remark 4.3.4.2. Just like with $\mathcal{H}$, we need to multiply by the inverse of the non-reduced identity matrices for them to preserve composition. Here

$$[\mathrm{id}_{\mathbb{1}}] = [\mathrm{id}_{\mathrm{X}}] = [1], [\mathrm{id}_{B_{\mathrm{Y}}}] = \begin{array}{c|ccc} & (\mathbb{1}) & (\mathrm{X}) & (\mathrm{Y}) \\ \hline (\mathbb{1}) & \alpha_s + \alpha_t & \alpha_s - \alpha_t & 0 \\ (\mathrm{X}) & \alpha_s - \alpha_t & \alpha_s + \alpha_t & 0 \\ (\mathrm{Y}) & 0 & 0 & 2\alpha_s \end{array}, \tag{4.3.4.5}$$

$$[\mathrm{id}_{\mathrm{Z}}] = \begin{array}{c|ccc} & (\mathbb{1}) & (\mathrm{Y}) & (\mathrm{Z}) \\ \hline (\mathbb{1}) & \alpha_s\alpha_t & 0 & 0 \\ (\mathrm{Y}) & 0 & \alpha_s\alpha_t & 0 \\ (\mathrm{Z}) & 0 & 0 & \alpha_s\alpha_t \end{array}, \text{ and } [\mathrm{id}_{\mathrm{XZ}}] = \begin{array}{c|ccc} & (\mathrm{X},\mathrm{X}) & (\mathrm{X},\mathrm{Y}) & (\mathrm{X},\mathrm{XZ}) \\ \hline (\mathrm{X},\mathrm{X}) & \alpha_s\alpha_t & 0 & 0 \\ (\mathrm{X},\mathrm{Y}) & 0 & \alpha_s\alpha_t & 0 \\ (\mathrm{X},\mathrm{XZ}) & 0 & 0 & \alpha_s\alpha_t \end{array}. \tag{4.3.4.6}$$

Definition 4.3.4.3. In Remark 4.3.4.2, we saw that the matrices of our identity morphism are invertible over Q. Let u, w be two $\mathcal{H}^\tau$-words. Let $T \in \mathsf{Hom}_{\mathcal{H}^\tau}(\mathsf{B}_u, \mathsf{B}_w)$. Define the *lower-reduced matrix of* T by $\underline{[T]} = [T] \circ [\mathrm{id}_{\mathsf{B}\,u}]^{-1}$, and the *upper-reduced matrix of* T by $\overline{[T]} = [\mathrm{id}_{\mathsf{B}\,w}]^{-1} \circ [T]$. Define $\underline{[T]}_{\mathsf{C}(J)}$ and $\overline{[T]}_{\mathsf{C}(J)}$ for $J \in \mathcal{J}$ analogously.

The proof of the following proposition is analogous to the proof of Proposition 4.2.4.5.

Proposition 4.3.4.4. *Let* u, v, w *be* $\mathcal{H}^\tau$*-words, let* $T \in \mathsf{Hom}_{\mathcal{H}^\tau}(\mathsf{B}_u, \mathsf{B}_v)$*, and let* $T' \in \mathsf{Hom}_{\mathcal{H}^\tau}(\mathsf{B}_v, \mathsf{B}_w)$*. Then*

$$[T' \circ T] = [T'] \circ \overline{[T]} = \underline{[T']} \circ [T] = \underline{[T']} \circ [\mathrm{id}_{B_v}] \circ \overline{[T]}. \tag{4.3.4.7}$$

Using this, we compute the matrices of all the upper-reduced lower leaves and neutral morphisms in Proposition B.1.0.1 and Proposition B.1.0.2.

4.3.5 Linear independence of double leaves

Definition 4.3.5.1. Let v, w be two $\mathcal{H}^\tau$-words, let $V \subseteq \mathsf{W}(v)$ and $W \subseteq \mathsf{W}(w)$. Let $T \in \mathsf{Hom}(B_v, B_w)$. Define $[T]_V$ to be the submatrix of $[T]$ restricted to the columns $p' \in V$ and $[T]^W$ to be the submatrix of $[T]$ restricted to the rows $q' \in W$. Also define $[T]_V^W$ the submatrix of $[T]$ restricted to columns $p' \in V$ and the rows $q' \in W$. We have analogous definitions for $\underline{[T]}_V, \underline{[T]}^W, \underline{[T]}_V^W, \overline{[T]}_V, \overline{[T]}^W$, and $\overline{[T]}_V^W$.

When $U = \{p\}$ or $W = \{q\}$ we will write $[T]_p, [T]^q$, and $[T]_p^q$ instead.

Definition 4.3.5.2. Let L be a lower leaf morphism, let $p = \mathsf{o}(L)$ be its root path, and let $V = \mathsf{C\,W}(p)$ be the walk class of p. Let $J = \mathbf{t}(p)$ be the end of p and let $r = \mathsf{o}(J)$ be its root path. Define the *leading vector* $\mathsf{lead}^{LL}(L)$ *of* L to be the row matrix $\overline{[T]}_U^r$, that restricts $\overline{[T]}$ to the row r and the columns in walk class of p.

Let U be an upper leaf morphism, let $q = \mathsf{o}(U)$ be its root path, and let $W = \mathsf{C\,W}(q)$ be the walk class of q. Let $J = \mathbf{t}(q)$ be the end of q and let $r = \mathsf{o}(J)$ be its root path. Define the *leading vector* $\mathsf{lead}^{UL}(U)$ *of* U to be the column matrix $\underline{[T]}_r^W$, that restricts $\underline{[U]}$ to the column r the rows in the same walk class of q.

Now let D be a double leaf morphism. Let $D = U \circ N \circ L$, for an upper leaf morphism U, neutral morphism N, and lower leaf morphism L. As before, let $p = \mathsf{o}(L)$, let $V = \mathsf{C\,W}(p)$, let $q = \mathsf{o}(U)$, and $W = \mathsf{C\,W}(q)$. Define the *leading matrix* $\mathsf{lead}^{DL}(D)$ *of* D to be $\mathsf{lead}^{DL}(D) = [D]_V^W$.

Whenever there is no ambiguity, we will just write lead for the leading vector/matrix without its superscript.

The following proposition follows directly from Proposition 4.3.4.4.

Proposition 4.3.5.3.

(a) Let $p = (e_1, e_2, \dots, e_r)$ *be a walk on* P *and let* $\underline{LL}_p$ *be its corresponding lower leaf. Let* $LL_{e_1}, \dots, LL_{e_r}$ *be the basic lower leaves of* $\underline{LL}_p$*. Then*

$$\mathsf{lead}^{LL}(\underline{LL}_p) = \mathsf{lead}^{LL}\, LL_{e_1} \otimes_Q \mathsf{lead}^{LL}\, LL_{e_2} \otimes_Q \dots \otimes_Q \mathsf{lead}^{LL}\, LL_{e_r}. \tag{4.3.5.1}$$

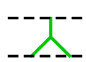
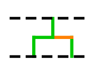
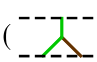
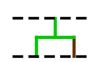
Here the tensor product corresponds to the usual Kronecker product of matrices.

The leading vector of an upper leaf is also the tensor product of the leading vectors of its basic upper leaves.

(b) Let D *be a double leaf morphism. Let* $D = U \circ N \circ L$ *for an upper leaf morphism* U*, neutral morphism* N*, and lower leaf morphism* L*. We have that*

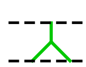
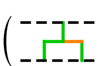
$$\mathsf{lead}^{DL}(D) = \mathsf{lead}^{UL}(U) \circ [N] \circ \mathsf{lead}^{LL}(L). \tag{4.3.5.2}$$

From now on, we will be using the matrices for lower leaves and neutral morphisms calculated in Proposition B.1.0.1 and Proposition B.1.0.2. In (4.3.3.1), observe that the pairs

( , ) and ( , ) (4.3.5.3)

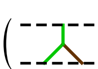
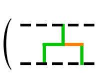
have the same source and target.

We can directly verify that the leading vectors of each of these pairs are linearly independent. For the first case,

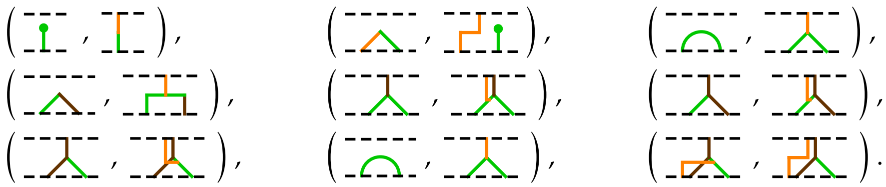
lead () $= (1, 1)$, lead () $= (-(\alpha_s + \alpha_t), -(\alpha_s - \alpha_t))$, and (4.3.5.4)

$$\det \begin{pmatrix} 1 & 1 \\ -(\alpha_s + \alpha_t) & -(\alpha_s - \alpha_t) \end{pmatrix} = 2\alpha_t, \tag{4.3.5.5}$$

which is invertible in Q. For the second one, it is clear that

lead () $= (0, \alpha_t)$ and lead () $= (\alpha_t, 0)$ (4.3.5.6) are linearly independent.

Now consider the pairs



(4.3.5.7)

(4.3.5.8)

(4.3.5.9)

They have the same source and their leading vectors are also linearly independent (even though they have different targets). The following lemma generalizes this to general lower leaves.

Lemma 4.3.5.4. *Let v be an $\mathcal{H}^{\tau}$-word and let $J \in \mathcal{J}$. Let $V = \mathsf{W}^{\mathsf{C}(J)}(v)$ bt the set of v-colored walks ending in $\mathsf{C}(J)$. Let $\mathsf{lead}(V) = \{\mathsf{lead}(\underline{LL}_{p'}) | p' \in V\}$. The set $\mathsf{lead}(V)$ is linearly independent over Q.*

We have an analogous statement for upper leaves.

Proof. We can verify that any pair of basic leaves in (4.3.5.3)-(4.3.5.9) has linearly independent leading vectors by checking the matrices in Proposition B.1.0.2. By Proposition 4.3.5.3(a), the leading vector of a general lower leaf is the tensor product of the leading vectors of its basic lower leaves. Since the tensor product of linearly independent sets is again linearly independent, the result follows. □

Corollary 4.3.5.5. *Let v, w be two $\mathcal{H}^{\tau}$-words and let $J \in \mathcal{Q}$. Define the set of matrices*

$$\mathsf{lead}(\mathsf{W}^{\mathsf{C}(J)}(v), \mathsf{W}^{\mathsf{C}(J)}(w)) = \{\mathsf{lead}^{UL}(\underline{UL}_q) \circ \mathsf{lead}^{LL}(\underline{LL}_p) | p \in \mathsf{W}^{\mathsf{C}(J)}(v), q \in \mathsf{W}^{\mathsf{C}(J)}(w)\}. \tag{4.3.5.10}$$

The set $\mathsf{lead}(\mathsf{W}^{\mathsf{C}(J)}(v), \mathsf{W}^{\mathsf{C}(J)}(w))$ is linearly independent over Q.

Moreover, the set

$$\mathsf{lead}(\mathsf{W}(v), \mathsf{W}(w)) = \{\mathsf{lead}^{UL}(\underline{UL}_q) \circ \mathsf{lead}^{LL}(\underline{LL}_p) | p \in \mathsf{W}^{\mathsf{C}(J)}(v), q \in \mathsf{W}^{\mathsf{C}(J)}(w), J \in \mathcal{J}\} \tag{4.3.5.11}$$

is linearly independent over Q.

Proof. Let

$$f_1 M_1 + f_2 M_2 + \ldots + f_r M_r = 0 \tag{4.3.5.12}$$

be an expression of minimal r for $M_1, \ldots, M_r \in \mathsf{lead}(\mathsf{W}^{\mathsf{C}(J)}(v), \mathsf{W}^{\mathsf{C}(J)}(w))$ that are different from each other, and $f_1, \ldots, f_r \in Q$. Write each $M_i = \mathsf{lead}(\underline{UL}_{q_i}) \circ \mathsf{lead}(\underline{LL}_{p_i})$

for some $p_i \in \mathsf{W}^{\mathsf{C}(J)}(v)$ and some $q_i \in \mathsf{W}^{\mathsf{C}(J)}(w)$. Note that each M_i is a matrix of rank 1 whose columns are multiples of $\mathsf{lead}^{UL}(\underline{UL}_{q_i})$ and whose rows are multiples of $\mathsf{lead}^{LL}(\underline{LL}_{p_i})$. By Lemma 4.3.5.4, the set of leading vectors the lower or upper leaves are linearly independent. If $p_1 \notin \{p_2, \ldots, p_r\}$, by linear independence, we must have that $f_1 = 0$. This contradicts the minimality of r. Hence we may suppose that $p_1 = p_2 = \cdots = p_r$. By linear independence of the columns, we must have $f_1 = f_2 = \ldots = f_r = 0$. This shows the matrices $\mathsf{lead}(\mathsf{W}^{\mathsf{C}(J)}(v), \mathsf{W}^{\mathsf{C}(J)}(w))$ are linearly independent. □

Theorem 4.3.5.6. *Let v, w be two $\mathcal{H}^\tau$-words. The set of double leaves DL_v^w is linearly independent over Q^τ.*

Proof. By Corollary 4.3.5.5, the set of matrices $\mathsf{lead}(\mathsf{W}(v), \mathsf{W}(w))$ is linearly independent over Q. Since $\{1, \alpha_s - \alpha_t\}$ is a basis of Q over Q^τ, the tensor product $\{1, \alpha_s - \alpha_t\} \otimes_{Q^\tau} \mathsf{lead}(\mathsf{W}(v), \mathsf{W}(w))$ is linearly independent over Q^τ. Finally, note that by Proposition 4.3.5.3, the leading matrices of our double leaves be in the set $\{1, \alpha_s - \alpha_t\} \otimes_{Q^\tau} \mathsf{lead}(\mathsf{W}(v), \mathsf{W}(w))$. In fact, if the neutral morphism of a double leaf D is the identity then

$$\mathsf{lead}^{DL}(D) \in \mathsf{lead}(\mathsf{W}(v), \mathsf{W}(w)), \tag{4.3.5.13}$$

and if it is not the identity then

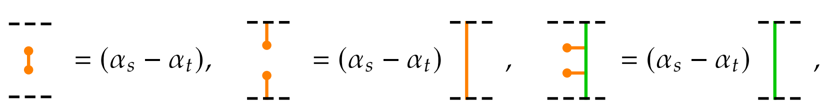
$$\mathsf{lead}^{DL}(D) \in (\alpha_s - \alpha_t)\, \mathsf{lead}(\mathsf{W}(v), \mathsf{W}(w)). \tag{4.3.5.14}$$

□

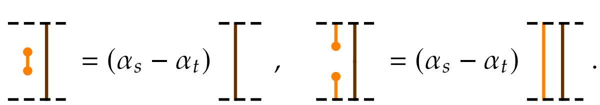
4.3.6 Composition of leaves

Lemma 4.3.6.1. *The composition of two neutral morphisms is an R^τ-multiple of an identity morphism.*

Proof. The statement follows from relations:



(4.3.6.1)

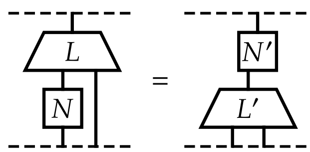


(4.3.6.2)

□

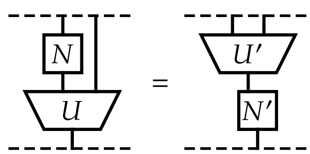
Lemma 4.3.6.2.

(a) *Let N be a neutral morphism. Let L be a basic lower leaf so that the composition $L \circ (N \otimes \mathrm{id}_A)$ is defined. Then $L \circ (N \otimes \mathrm{id}_A) = \pm N' \circ L'$ for some basic lower leaf L' and some neutral morphism N'. That is,*



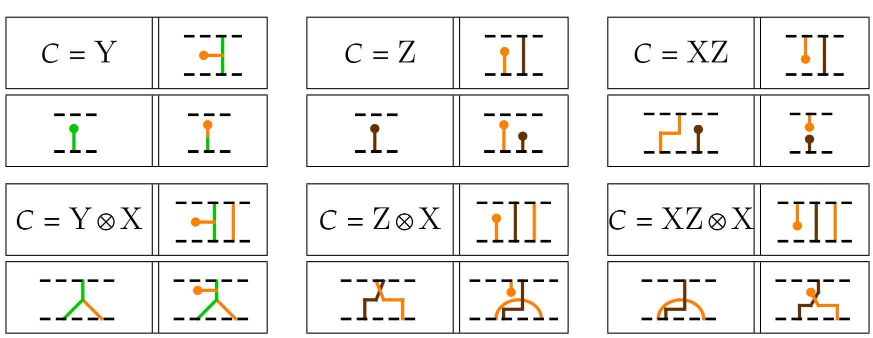
(4.3.6.3)

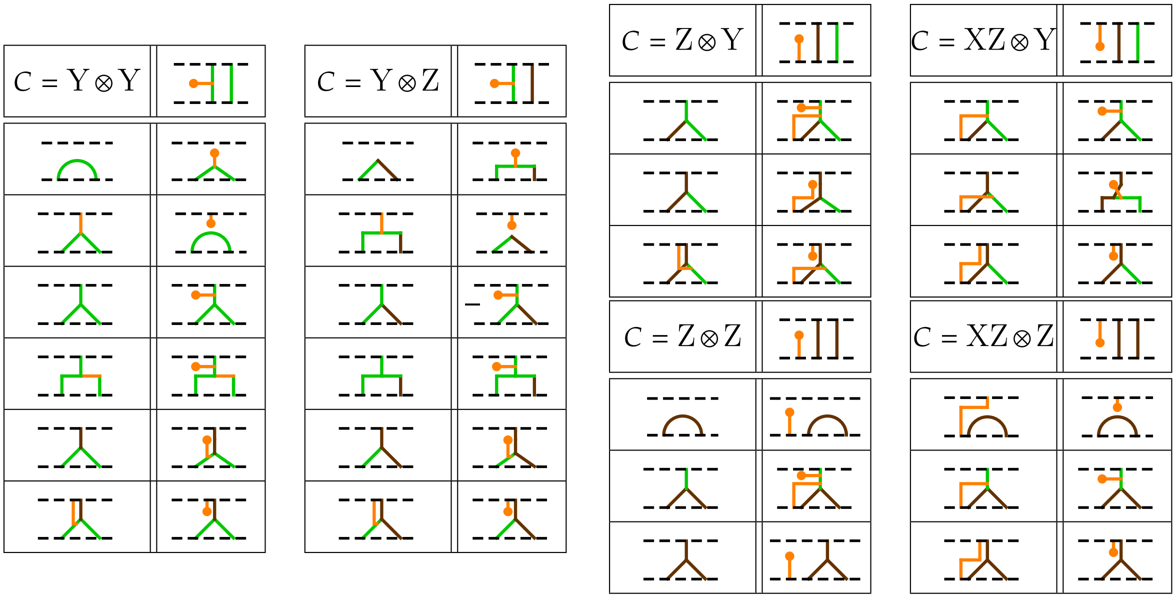
(b) *Let N be a neutral morphism with source I for $I \in \mathcal{I}$. Let U be a basic upper leaf with source target word $I \cdot A$, so that the composition $(N \otimes \mathrm{id}_A) \circ U$ is defined. Then $(N \otimes \mathrm{id}_A) \circ U = U' \circ N'$ for some basic upper leaf U' and some neutral morphism N'. That is,*



(4.3.6.4)

Proof. We will only show (a), as (b) is analogous. Let L and N as above. Let $C = I \cdot A$ be the source of L. The following composition tables show the statement for the different possible values of C:





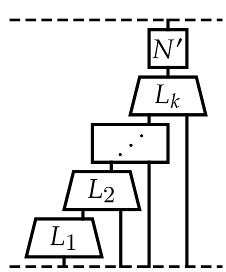
The corresponding relations hold for upper leaves. □

Proposition 4.3.6.3.

(a) Let N *be a neutral morphism with target* I *for* $I \in \mathcal{I}$*. Let* L *be an extended lower leaf so that the composition* $L \circ (N \otimes \mathrm{id}_A)$ *is defined. Then we can write* $L \circ (N \otimes \mathrm{id}_A)$ *as an* R^τ*-multiple of an extended lower leaf.*

(b) Let N *be a neutral morphism with source* I *for* $I \in \mathcal{I}$*. Let* U *be an extended upper leaf so that the composition* $(N \otimes \mathrm{id}_A) \circ U$ *is defined. Then we can write* $(N \otimes \mathrm{id}_A) \circ U$ *as an* R^τ*-multiple of an extended upper leaf.*

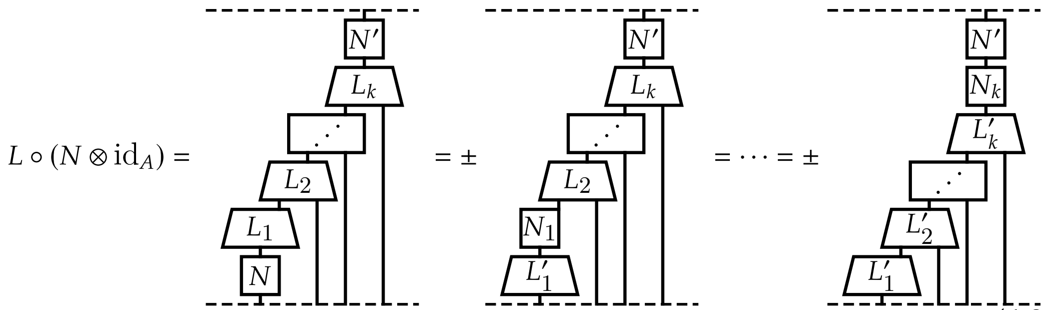
Proof. We will only prove (a) as (b) is analogous. Let N and L as above. We may write

$L =$ 

(4.3.6.5)

for some basic leaves $L_1, \ldots, L_k$ and a neutral morphism N'. Applying equation

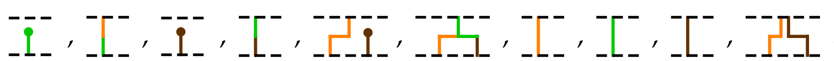
(4.3.6.3) repeatedly we may write



(4.3.6.6)

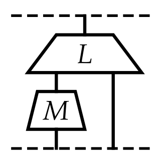
for some basic leaves $L'_1, \ldots, L'_k$ and neutral morphisms $N_1, \ldots, N_k$. Finally, by Lemma 4.3.6.1, the composition $N' \circ N_k$ is an R^τ-multiple of a neutral morphism. □

Definition 4.3.6.4. An *original lower leaf* L is a lower leaf with source I for some $I \in \mathcal{I}$. The following is a list of the possible original lower leaves:



(4.3.6.7)

A *semi-original lower leaf* is a leaf of the form



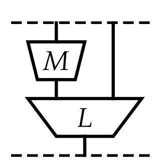
(4.3.6.8)

for some original lower leaf M and non-original basic leaf L.

An *original upper leaf* U is an upper leaf with target word $I \in \mathcal{I}$. The following is a list of the possible original upper leaves:

(4.3.6.9)

A *semi-original upper leaf* is a leaf of the form



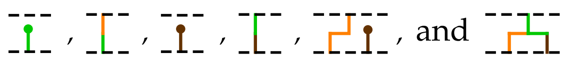
(4.3.6.10)

for some original upper leaf M and non-original basic upper leaf U.

Remark 4.3.6.5. Since every $\mathcal{H}^\tau$-word starts with X, Y, Z, or XZ, every lower lower leaf starts with an original morphism. Similarly, every upper leaf ends with an original morphism.

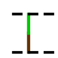
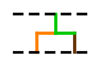
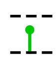
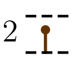
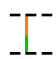
Lemma 4.3.6.6. *The composition of original lower/upper leaves is again an R^τ-multiple of an original leaf.*

Proof. We will show this for lower leaves, since the statement for upper leaves is analogous. The non-identity original leaves are



(4.3.6.11)

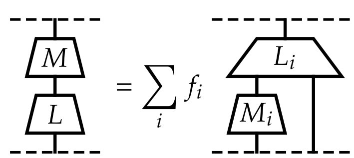
Let M_1 be a non-identity original leaf with source $I \in \mathcal{I}$. Let M_2 be a non-identity original leaf with target I, so that the composition $M_1 \circ M_2$ is defined. The only possible compositions are the ones where $I = \mathrm{Y}$, and are shown in the following composition table:

$I = \mathrm{Y}$		
	2 	0
	0	2 

∎

Lemma 4.3.6.7.

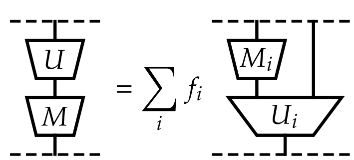
(a) *Let M be an original lower leaf with source $I \in \mathcal{I}$. Let L be a basic lower leaf with target I, so that the composition $M \circ L$ is defined. Then $M \circ L$ is an R^τ-combination of semi-original lower leaves. That is,*



(4.3.6.12)

for some $f_i \in R^\tau$, original lower leaves M_i and basic lower leaves L_i.

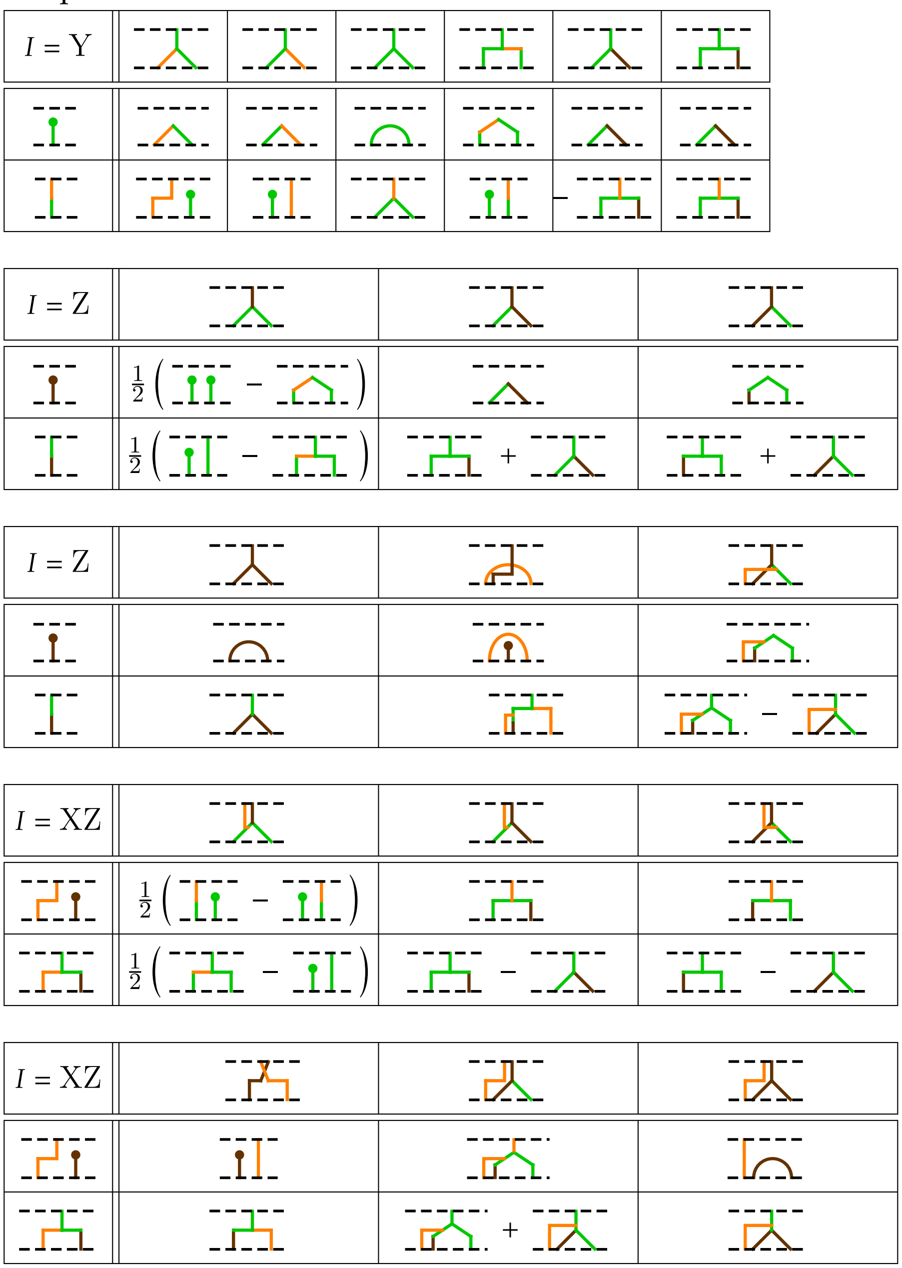
(b) *Let M be an original upper light leaf with target I,. Let U be a basic upper leaf with source I, so that the composition $U \circ M$ is defined. Then $U \circ M$ is an R^τ-combination of semi-original upper leaves. That is,*



(4.3.6.13)

for some $f_i \in R^\tau$, original lower light leaves M_i and basic upper leaves U_i.

Proof. We will only prove (a) as (b) is analogous. The proof goes by inspection. Let M and L as above. We may assume that neither L nor M is an identity morphism. Additionally, if L is an original leaf, Lemma 4.3.6.7 ensures the composition is again original. Let I be the source of M. Under these constraints, I can only be Y, Z, or XZ. The following are composition tables for all the basic light leaves for each possible value of I.



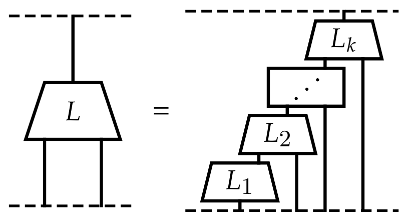
Each one of these corresponds to an R^{τ}-combination of semi-original lower leaves. ∎

Proposition 4.3.6.8.

(a) *Let M' be an original lower leaf with source $I \in \mathcal{I}$. Let L be a lower leaf with target I, so that the composition $M' \circ L$ is defined. Then $M' \circ L$ is an R^τ-combination of lower leaves.*

(b) *Let M' be an original upper leaf with target $I \in \mathcal{I}$. Let U an upper leaf with source $I \in \mathcal{I}$, so that the composition $U \circ M'$ is defined. Then $U \circ M'$ is an R^τ-combination of upper leaves.*

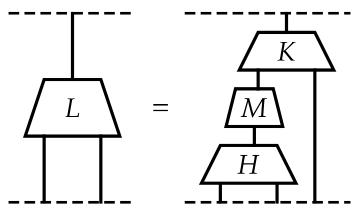
Proof. We will only show (a) as (b) is analogous. Let L and M' as above. Let



(4.3.6.14)

for some basic lower leaves $L_1, \dots, L_k$. Let M be the last L_i that is an original leaf.

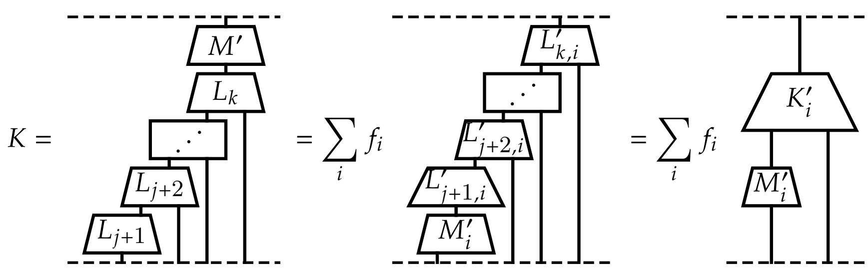
We may write L as by the composition



(4.3.6.15)

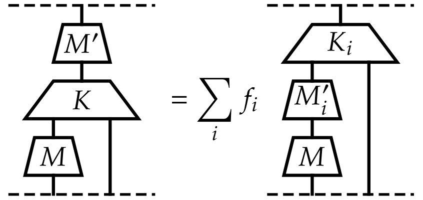
for some original leaf M and lower leaves K, H.

Applying Lemma 4.3.6.7 repeatedly, we may write



(4.3.6.16)

for some polynomials $f_i \in R^\tau$, original leaves M'_i, and lower leaves K'_i. Hence,



(4.3.6.17)

By Lemma 4.3.6.6, the composition $M_i \circ M$ is an original leaf. Thus, $M \circ L$ is an R^τ-combination of lower leaves. □

The reader should observe the similarity between Proposition 4.3.6.3 and Proposition 4.3.6.8. The first one lets us pre-compose lower leaves with neutral morphisms and still get a combination of (extended) lower leaves, while the second one lets us post-compose with original morphisms get combination of lower leaves. Together, they gives us the following corollary.

Corollary 4.3.6.9.

(a) *Let L' be an extended lower leaf with target word I. Let L be an extended lower leaf with source word $I \cdot A$ for $I \in \mathcal{I}$, so that the composition $L \circ (L' \otimes \mathrm{id}_A)$ is defined. Then $L \circ (L' \otimes \mathrm{id}_A)$ is an R^τ-combination of extended upper leaves.*

(b) *Let U' be an extended upper leaf with source word I. Let U be an extended upper leaf with target word $I \cdot A$ for $I \in \mathcal{I}$, so that the composition $(U' \otimes \mathrm{id}_A) \circ U$ is defined. Then $(U' \otimes \mathrm{id}_A) \circ U$ is an R^τ-combination of extended upper leaves.*

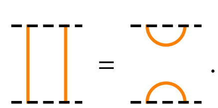
Proof. Follows directly from applying Proposition 4.3.6.3 and Proposition 4.3.6.8. □

Remark 4.3.6.10. It is important to remark again that in general, we should not expect to have the composition of lower/upper leaves to be a linear combination of lower/upper leaves. In the Hecke category $\mathcal{H}(W)$ for a general Coxeter group, the composition lower/upper leaves is a linear combination lower/upper leaves only up to double leaves factoring through lower terms.

4.3.7 Breaking Identity Morphisms into Double Leaves

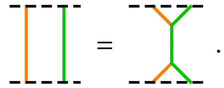
In this section we will write the idempotent decomposition relations from Proposition 3.2.5.1 in terms of our double leaves basis. We will use these relations to show that the identity morphism of the Bott-Samelson object B_w is in the R^τ-span of our double leaves basis for any $\mathcal{H}^\tau$-word w.

Proposition 4.3.7.1. *The following relations write the identity morphism fo the given tensor product as an R^τ-combination of double leaves:* $\mathrm{X} \otimes \mathrm{X}$



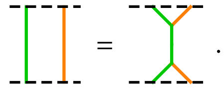
(4.3.7.1)

X ⊗ Y



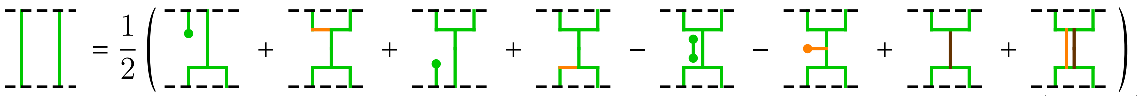
(4.3.7.2)

Y ⊗ X



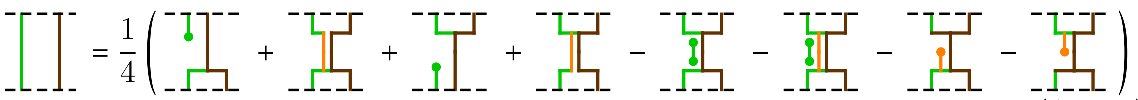
(4.3.7.3)

Y ⊗ Y



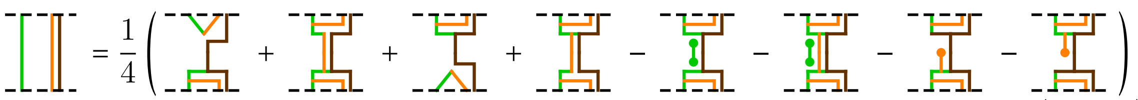
(4.3.7.4)

Y ⊗ Z



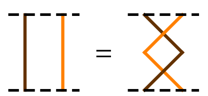
(4.3.7.5)

Y ⊗ XZ



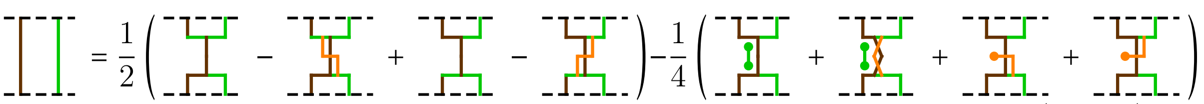
(4.3.7.6)

Z ⊗ X



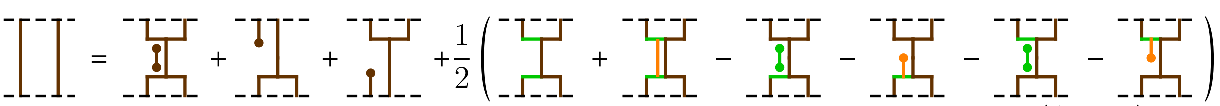
(4.3.7.7)

Z ⊗ Y



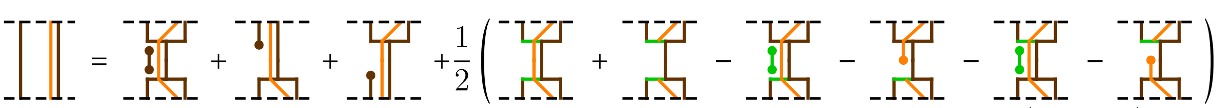
(4.3.7.8)

Z ⊗ Z



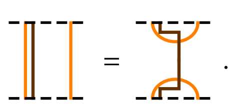
(4.3.7.9)

Z ⊗ XZ



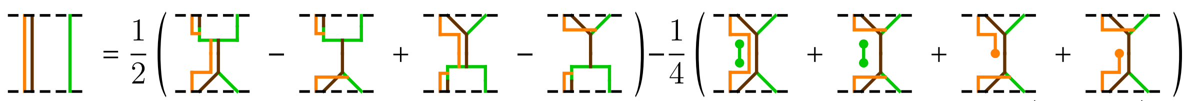
(4.3.7.10)

XZ ⊗ X



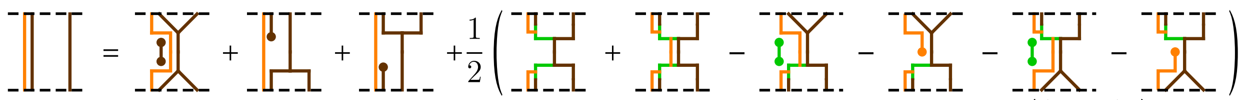
(4.3.7.11)

XZ⊗Y



(4.3.7.12)

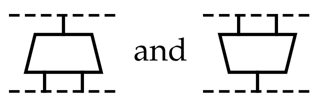
XZ⊗Z



(4.3.7.13)

These relations write all the products of expressions in $\mathcal{I}$ as R^τ-combinations of double leaves.

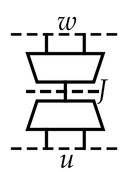
Notation 4.3.7.2. In the following sections we will denote any extended lower and upper leaf morphisms by the diagrams



(4.3.7.14)

respectively.

Also, any label in the top or bottom of our diagrams by either an $\mathcal{H}^\tau$-word or an object of $\mathcal{H}^\tau$ will indicate the source/target of that morphism. Similarly, the label of a dashed horizontal line cutting through an identity morphisms will indicate the object corresponding to this morphism. For example, let u, w be $\mathcal{H}^\tau$-words and let $J \in \mathcal{I}$. The diagram

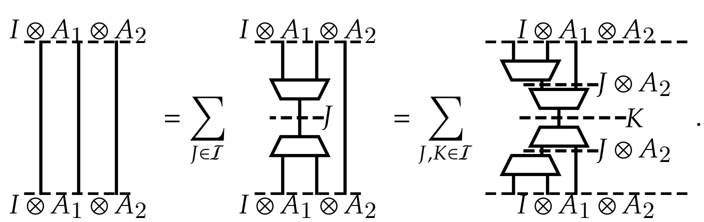


(4.3.7.15)

corresponds to a double leaf in $\mathbf{DL}_u^w$ that factors through J.

Corollary 4.3.7.3. *Let w by an $\mathcal{H}^\tau$-word. The identity morphism $\mathrm{id}_{\mathsf{B}_w}$ can be written as an R^τ-combination of double leaves.*

Proof. Proposition 4.3.7.1 shows this statement when $\mathsf{B}_w = I \otimes A$ for $I \in \mathcal{I}$ and $A \in \{\mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$. Suppose $\mathsf{B}_w = I \otimes A_1 \otimes A_2$ for $I \in \mathcal{I}$ and $A_1, A_2 \in \{\mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$. Using Proposition 4.3.7.1 twice, we may write



(4.3.7.16)

By Corollary 4.3.6.9, the composition of extended lower/upper leaves is again an R^τ-combination of lower/upper leaves. $\square$

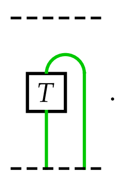
4.3.8 The Double Leaves Span $\mathcal{H}^\tau$

In this section we will show that the double leaves basis R^τ-span the morphism spaces $\mathsf{Hom}(I, J)$ for all $I, J \in \mathcal{I}$. We will then use this fact to show that the double leaves span all the Hom spaces of $\mathcal{H}^\tau$.

Proposition 4.3.8.1. *Let $I, J \in \mathcal{I}$. Any morphism in $\mathsf{Hom}(I, J)$ is an R^τ-combination of double leaves in $\mathbf{DL}_I^J$.*

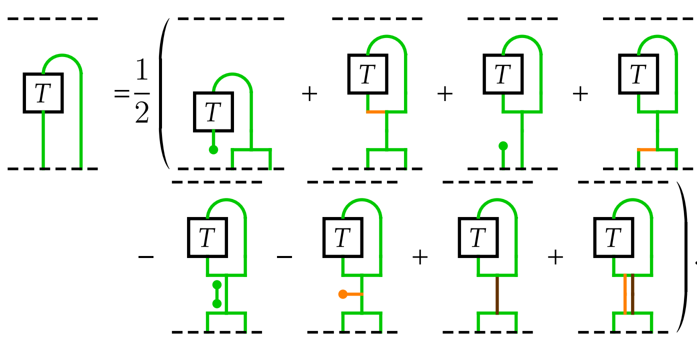
Proof. Theorem 3.2.6.1 shows R^τ-generators for $\mathsf{Hom}(I, \mathbb{1})$ and $\mathsf{Hom}(I, \mathrm{X})$ for $I \in \mathcal{I}$, each of which is in $\mathbf{DL}_I^{\mathbb{1}}$.

Let us show the statement for $\mathsf{Hom}(\mathrm{Y}, \mathrm{Y})$. Let $T : \mathrm{Y} \longrightarrow \mathrm{Y}$. Let us consider the morphism



(4.3.8.1)

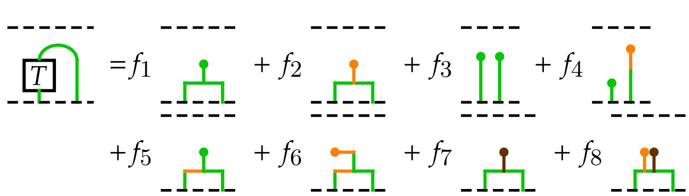
By (4.3.7.4),



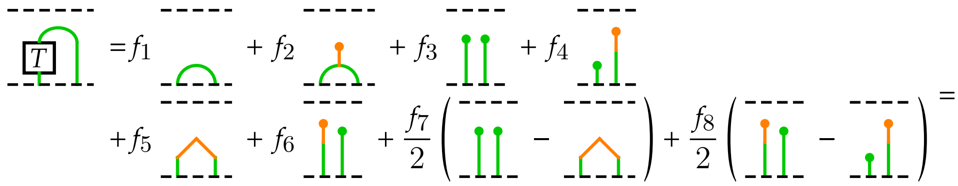
(4.3.8.2)

(4.3.8.3)

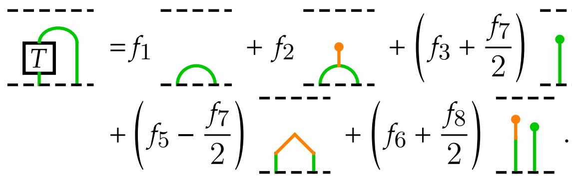
Note that all of these diagrams factor through Y, Z, or XZ. By Theorem 3.2.6.1 we can write each of the top morphisms as an R^τ-combination of its generators. Hence, there are $f_1, \ldots, f_8 \in R^\tau$ with



(4.3.8.4)

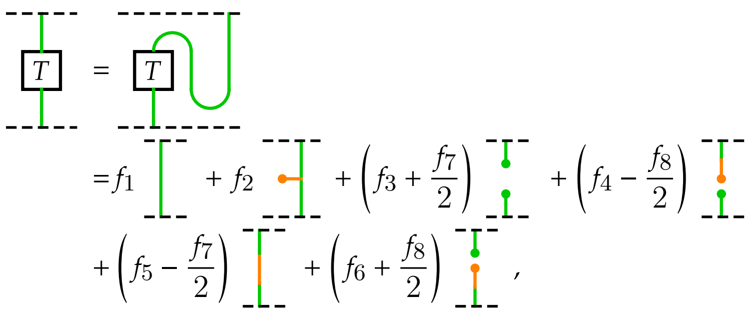


(4.3.8.5)



(4.3.8.6)

Using adjunction we may write



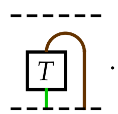
(4.3.8.7)

(4.3.8.8)

(4.3.8.9)

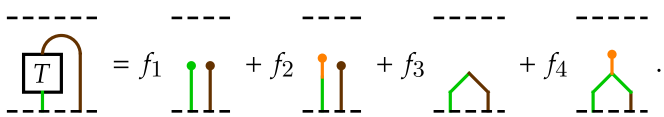
which is a combination of double leaves.

Let us show the statement for $\mathrm{Hom}(\mathrm{Y}, \mathrm{Z})$. Let $T : \mathrm{Y} \longrightarrow \mathrm{Z}$. Let us consider the morphism



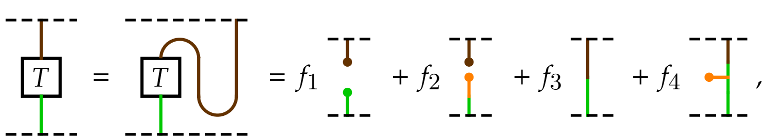
(4.3.8.10)

By (4.3.7.5), the identity of $\mathrm{Y} \otimes \mathrm{Z}$ factors through Z and XZ. Hence, there are $f_1, \ldots, f_4 \in R^{\tau}$ with



(4.3.8.11)

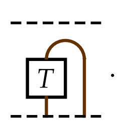
Using adjunction we may write



(4.3.8.12)

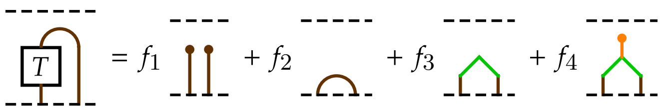
which is a combination of double leaves.

Let us show the statement for $\mathrm{Hom}(\mathrm{Z}, \mathrm{Z})$. Let $T : \mathrm{Z} \longrightarrow \mathrm{Z}$. Let us consider the morphism



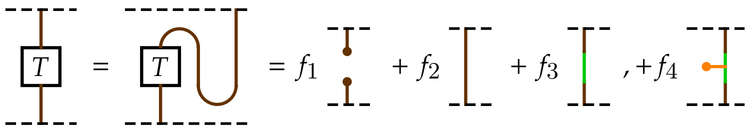
(4.3.8.13)

By (4.3.7.9), the identity of $Z \otimes Z$ factors through Z and XZ. Hence, there are $f_1, \ldots, f_4 \in R^\tau$ with



(4.3.8.14)

Using adjunction we may write



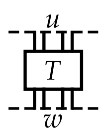
(4.3.8.15)

which is a combination of double leaves.

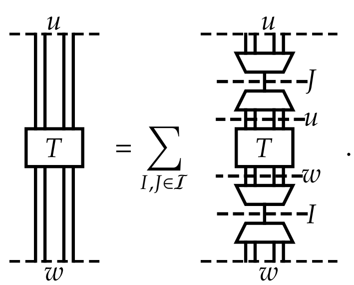
The other cases are analogous. □

Theorem 4.3.8.2. *Let u, w be $\mathcal{H}^\tau$-words. The space* $\mathrm{Hom}(\underline{w}, \underline{u})$ *is spanned over R^τ by* $\mathbf{DL}^u_w$.

Proof. Let

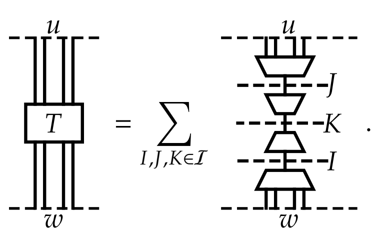
$f =$  $\in \mathrm{Hom}(\underline{w}, \underline{u}).$ (4.3.8.16)

By Corollary 4.3.7.3, we may write



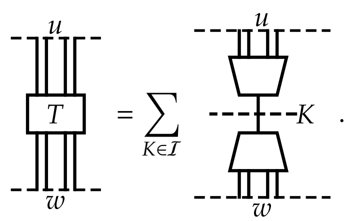
(4.3.8.17)

By Proposition 4.3.8.1, the morphism from I to J is in the span of $\mathbf{DL}^J_I$, thus



(4.3.8.18)

By Proposition 4.3.6.8, the morphisms from K to $\underline{w}$ and $\underline{u}$ are again a combination of light leaves. Thus, we get the expression



(4.3.8.19)

Hence T is an R^τ-combination of double leaves. □

4.3.9 Consequences

By Theorems 4.3.5.6 and 4.3.8.2, we get the following theorem.

Theorem 4.3.9.1. *Let u, w be $\mathcal{H}^\tau$-words. The set of double leaves $\mathbf{DL}_u^w$ is an R^τ-basis of* $\mathsf{Hom}(\underline{w}, \underline{u})$.

Moreover, we are able to show that $\mathcal{H}^\tau$ is a fibered NSOACC.

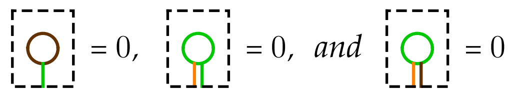
Theorem 4.3.9.2. *The category $\mathcal{H}^\tau$ is a fibered NSOACC.*

APPENDIX A

RELATIONS IN $\mathcal{H}^{\tau}$

A.1 Additional Needle and Unit relations

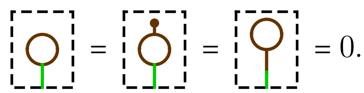
Proposition A.1.0.1. *The needle relations*



(A.1.0.1)

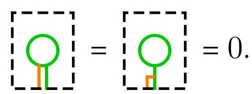
hold.

Proof. Checking these is straightforward. By (3.2.2.19) and (3.2.2.5),



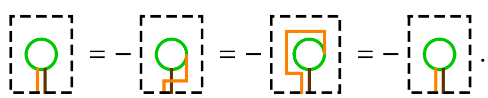
(A.1.0.2)

By (3.2.2.13) and (3.2.2.5),



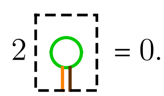
(A.1.0.3)

By (3.2.2.14) and Theorem 3.2.3.12,



(A.1.0.4)

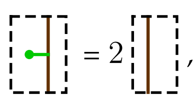
Hence,



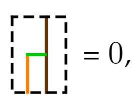
(A.1.0.5)

□

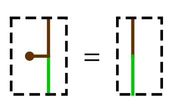
Proposition A.1.0.2. *The unit relations*



(A.1.0.6)

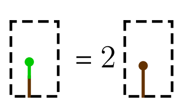


(A.1.0.7)



(A.1.0.8)

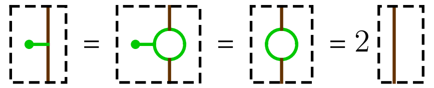
and



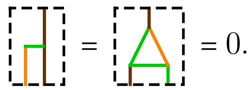
(A.1.0.9)

hold.

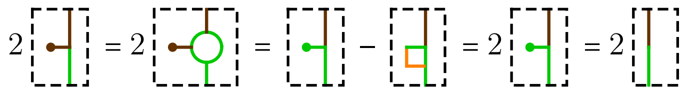
Proof. (A.1.0.6) By (3.2.2.16) and (3.2.2.6) we have

 (A.1.0.10)

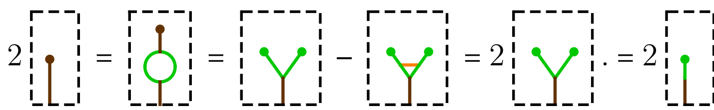
(A.1.0.7) By (3.2.2.17),

 (A.1.0.11)

(A.1.0.8) By (3.2.2.8) and (3.2.2.14)

 (A.1.0.12)

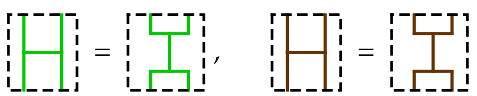
(A.1.0.9) By (3.2.2.16) and (3.2.2.14) we have

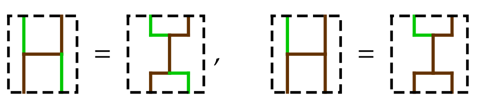
 (A.1.0.13)

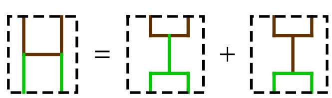
□

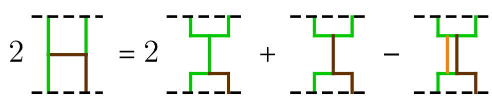
A.2 Additional $H = I$ relations

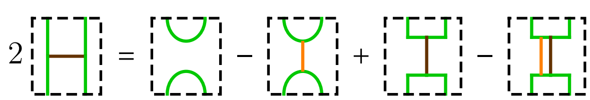
The $H = I$ relations in Definition 3.2.2.1 are:

 (A.2.0.1)

 (A.2.0.2)

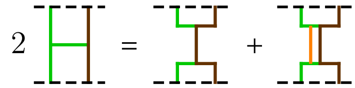
 (A.2.0.3)

 (A.2.0.4)

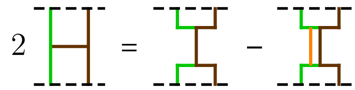
 (A.2.0.5)

We wish to show the rest of the $H = I$ relations, since we need all of them to prove Theorem 3.2.6.1.

Lemma A.2.0.1. *The following follow from* (3.2.2.20) *and all other non-$H = I$ relations.*

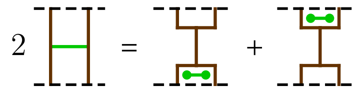


(A.2.0.6)



(A.2.0.7)

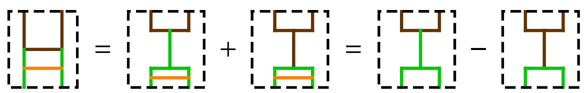
The relation



(A.2.0.8)

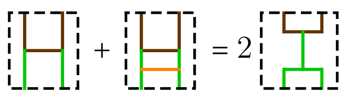
also requires (3.2.2.18).

Proof. By (3.2.2.20) we get



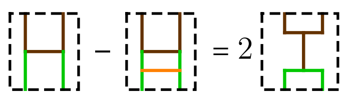
(A.2.0.9)

Adding and subtracting (3.2.2.20) and (A.2.0.9) we get



(A.2.0.10)

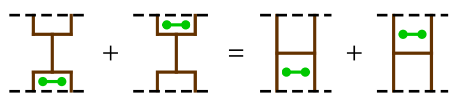
and



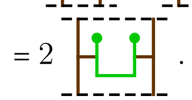
(A.2.0.11)

These are rotated versions of (A.2.0.6) and (A.2.0.7) respectively.

For (A.2.0.8),

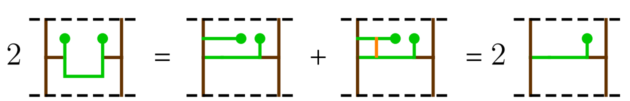


(A.2.0.12)



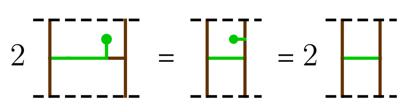
(A.2.0.13)

Applying (A.2.0.7) on the left side of the diagram and using (3.2.2.17) to get rid of the middle term we get,



(A.2.0.14)

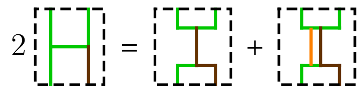
Aplying the same procedure on the right side,



(A.2.0.15)

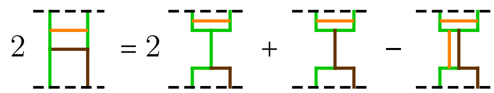
□

Lemma A.2.0.2. *The following follows from* (3.2.2.21) *and all other non-$H = I$ relations.*

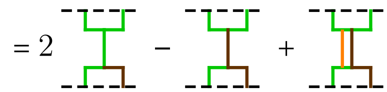


(A.2.0.16)

Proof. By (3.2.2.21) we have

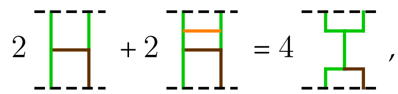


(A.2.0.17)



(A.2.0.18)

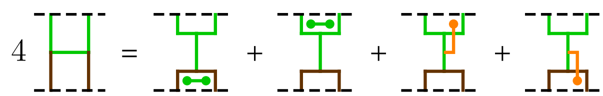
Adding (3.2.2.21) and (A.2.0.18) we get



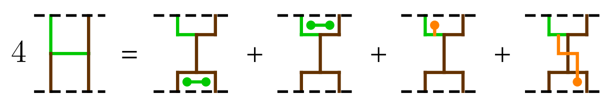
(A.2.0.19)

which is a rotation of (3.2.2.21). ◻

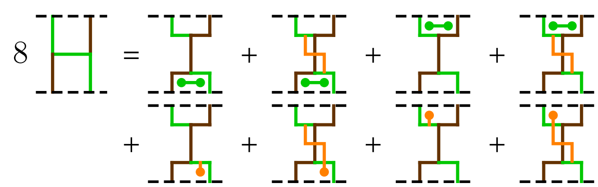
Lemma A.2.0.3. *We have the following relations:*



(A.2.0.20)

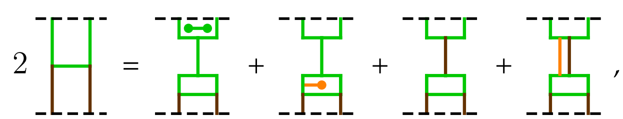


(A.2.0.21)

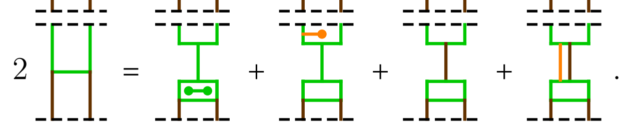


(A.2.0.22)

Proof. We will show (A.2.0.20) and leave the rest to the reader. From 3.2.5.1 and its rotation we get

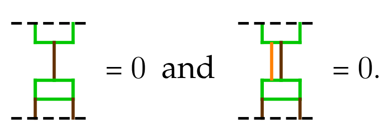


(A.2.0.23)



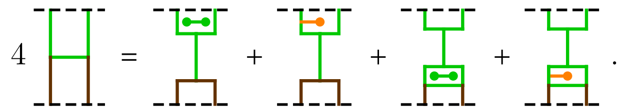
(A.2.0.24)

Using (3.2.2.22), we can show that



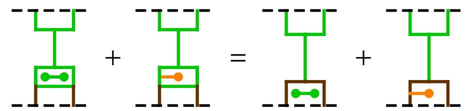
(A.2.0.25)

Adding (A.2.0.23) and (A.2.0.24) we get



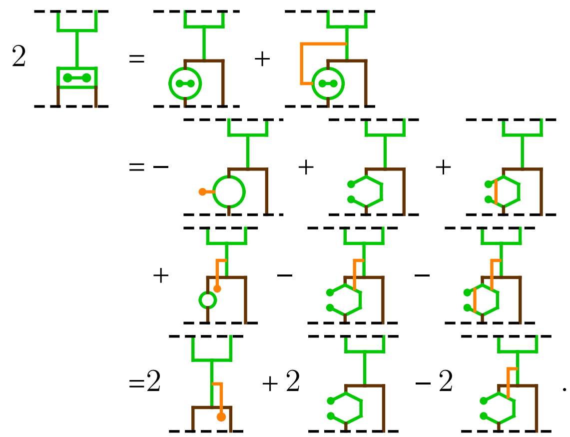
(A.2.0.26)

It remains to show that



(A.2.0.27)

Using (A.2.0.16),

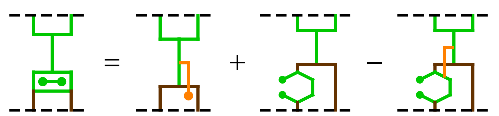


(A.2.0.28)

(A.2.0.29)

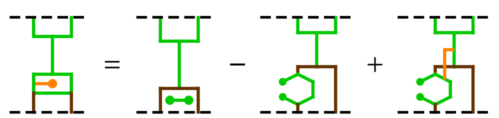
(A.2.0.30)

Hence,



(A.2.0.31)

A similar calculation yields



(A.2.0.32)

Adding (A.2.0.31) and (A.2.0.32) we get (A.2.0.27).

□

A.3 General polynomial forcing and needle relations

Definition A.3.0.1. Let $f \in R$. Define

$$\mathsf{Sym}_\tau(f) := \frac{f + \tau(f)}{2} \tag{A.3.0.1}$$

$$\mathsf{Alt}_\tau(f) := \frac{f - \tau(f)}{2} \tag{A.3.0.2}$$

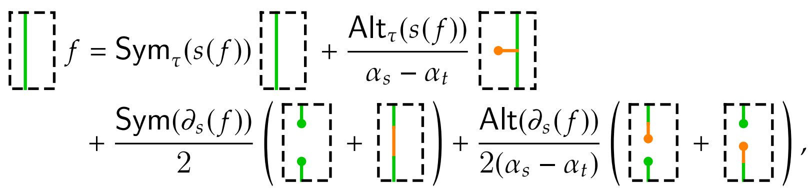
to be the τ*-invariant* and τ*-anti invariant parts of* f.

Notice that $\mathrm{Sym}_\tau(f) \in R^\tau$, $\mathrm{Alt}_\tau(f) \in R^{-\tau}$, and

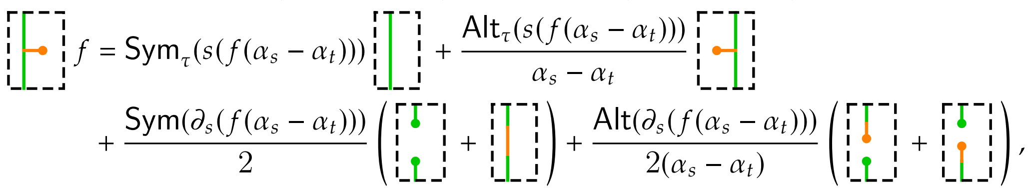
$$f = \mathrm{Sym}_\tau(f) + \mathrm{Alt}_\tau(f). \tag{A.3.0.3}$$

The following polynomial forcing formulas will be shown by induction by 'forcing' one green barbell or orange dot at a time.

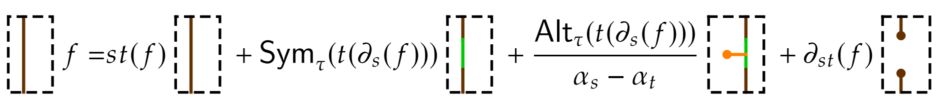
Proposition A.3.0.2 (General polynomial forcing). *Let $f \in R^\tau$. The following relations hold:*



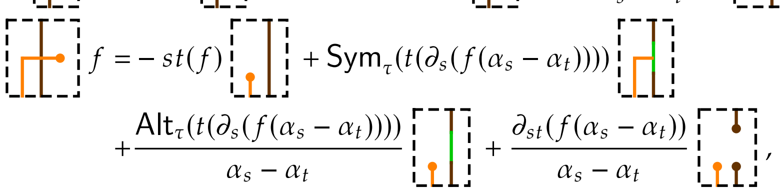
(A.3.0.4)



(A.3.0.5)



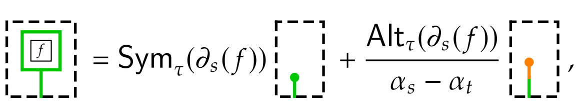
(A.3.0.6)



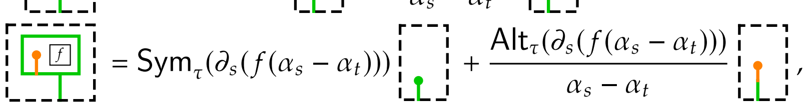
(A.3.0.7)

Applying the general polynomial forcing relations to a needle and circle diagrams and then applying the corresponding needle relation to any empty needle, we get the following propositions. The proof is a straightforward calculation and is left as an exercise to the reader.

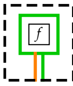
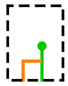
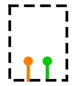
Proposition A.3.0.3 (General needle relations). *Let $f \in R^\tau$. The following relations hold:*

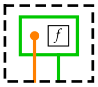
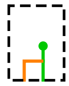
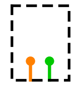


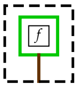
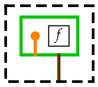
(A.3.0.8)

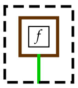
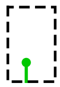
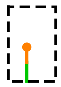


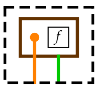
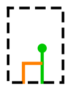
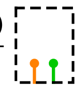
(A.3.0.9)

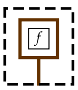
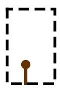
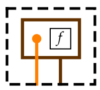
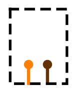
 $= \mathrm{Sym}_\tau(\partial_s(f))$  $+ \dfrac{\mathrm{Alt}_\tau(\partial_s(f))}{\alpha_s - \alpha_t}$  $,$ (A.3.0.10)

 $= \mathrm{Sym}_\tau(\partial_s(f(\alpha_s - \alpha_t)))$  $+ \dfrac{\mathrm{Alt}_\tau(\partial_s(f(\alpha_s - \alpha_t)))}{\alpha_s - \alpha_t}$  $,$ (A.3.0.11)

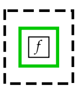
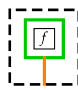
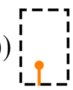
 $= 0,$  $= 0,$ (A.3.0.12)

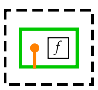
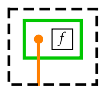
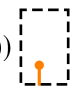
 $= \dfrac{\partial_{st}(f)}{2} \Bigg((\alpha_s + \alpha_t)$  $+$  $\Bigg),$ (A.3.0.13)

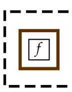
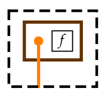
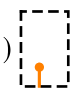
 $= \dfrac{\partial_{st}(f(\alpha_s - \alpha_t))(\alpha_s - \alpha_t)}{2}$  $+ \dfrac{(\alpha_s + \alpha_t)\partial_{st}(f(\alpha_s - \alpha_t))}{2(\alpha_s - \alpha_t)}$  $,$ (A.3.0.14)

 $= \partial_{st}(f)$  $,$  $= \dfrac{\partial_{st}(f(\alpha_s - \alpha_t))}{\alpha_s - \alpha_t}$  $.$ (A.3.0.15)

Proposition A.3.0.4 (General circle relations)**.** *Let $f \in R^\tau$. The following relations hold:*

 $= 2\,\mathrm{Sym}_\tau(\partial_s(f))(\alpha_s + \alpha_t),$  $= 2\,\mathrm{Sym}_\tau(\partial_s(f))$  $,$ (A.3.0.16)

 $= 2\,\mathrm{Sym}_\tau(\partial_s(f(\alpha_s - \alpha_t)))(\alpha_s + \alpha_t),$  $= 2\,\mathrm{Sym}_\tau(\partial_s(f(\alpha_s - \alpha_t)))$  $,$ (A.3.0.17)

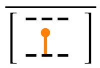
 $= \partial_{st}(f)(\alpha_s\alpha_t),$  $= \dfrac{\partial_{st}(f(\alpha_s - \alpha_t))}{\alpha_s - \alpha_t}(\alpha_s\alpha_t)$  $.$ (A.3.0.18)

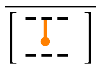
APPENDIX B

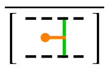
RELATIONS IN Q^{τ}

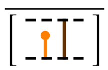
B.1 Matrices for basic lower leaves

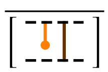
Proposition B.1.0.1. *We have the following upper reduced matrices for the neutral morphisms:*

 $= [\alpha_s - \alpha_t]$ (B.1.0.1)

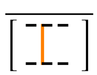
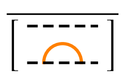
 $= [\alpha_s - \alpha_t]$ (B.1.0.2)

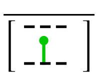
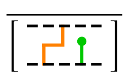
 $= \begin{array}{c|ccc} & (\mathbb{1}) & (X) & (Y) \\ \hline (\mathbb{1}) & \alpha_s - \alpha_t & 0 & 0 \\ (X) & 0 & \alpha_s - \alpha_t & 0 \\ (Y) & 0 & 0 & \alpha_s - \alpha_t \end{array}$ (B.1.0.3)

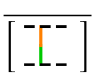
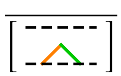
 $= \begin{array}{c|ccc} & (X,X) & (X,Y) & (X,XZ) \\ \hline (\mathbb{1}) & \alpha_s - \alpha_t & 0 & 0 \\ (Y) & 0 & \alpha_s - \alpha_t & 0 \\ (Z) & 0 & 0 & \alpha_s - \alpha_t \end{array}$ (B.1.0.4)

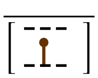
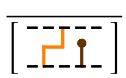
 $= \begin{array}{c|ccc} & (\mathbb{1}) & (Y) & (Z) \\ \hline (X,X) & \alpha_s - \alpha_t & 0 & 0 \\ (X,Y) & 0 & \alpha_s - \alpha_t & 0 \\ (X,Z) & 0 & 0 & \alpha_s - \alpha_t \end{array}$ (B.1.0.5)

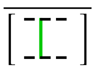
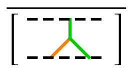
Proposition B.1.0.2. *We have the following upper reduced matrices for the basic lower leaves:*

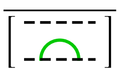
 $= [1]$  $= [1]$ (B.1.0.6)

 $= \begin{array}{c|ccc} & (\mathbb{1}) & (X) & (Y) \\ \hline \phi & \alpha_s + \alpha_t & \alpha_s - \alpha_t & 0 \end{array}$  $= \begin{array}{c|ccc} & (X,\mathbb{1}) & (X,X) & (X,Y) \\ \hline (X) & \alpha_s - \alpha_t & \alpha_s + \alpha_t & 0 \end{array}$ (B.1.0.7)

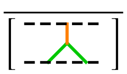
 $= \begin{array}{c|ccc} & (\mathbb{1}) & (X) & (Y) \\ \hline (X) & \alpha_s - \alpha_t & \alpha_s + \alpha_t & 0 \end{array}$  $= \begin{array}{c|ccc} & (X,\mathbb{1}) & (X,X) & (X,Y) \\ \hline \phi & \alpha_s + \alpha_t & \alpha_s - \alpha_t & 0 \end{array}$ (B.1.0.8)

 $= \begin{array}{c|ccc} & (\mathbb{1}) & (Y) & (Z) \\ \hline \phi & \alpha_s\alpha_t & 0 & 0 \end{array}$  $= \begin{array}{c|ccc} & (X,X) & (X,Y) & (X,XZ) \\ \hline (X) & \alpha_s\alpha_t & 0 & 0 \end{array}$ (B.1.0.9)

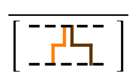
 $= \begin{array}{c|ccc} & (\mathbb{1}) & (X) & (Y) \\ \hline (\mathbb{1}) & 1 & 0 & 0 \\ (X) & 0 & 1 & 0 \\ (Y) & 0 & 0 & 1 \end{array}$  $= \begin{array}{c|ccc} & (X,\mathbb{1}) & (X,X) & (X,Y) \\ \hline (\mathbb{1}) & 1 & 0 & 0 \\ (X) & 0 & 1 & 0 \\ (Y) & 0 & 0 & 1 \end{array}$ (B.1.0.10)

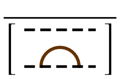
 $= \begin{array}{c|cccccc} & (\mathbb{1},\mathbb{1}) & (X,\mathbb{1}) & (Y,\mathbb{1}) & (\mathbb{1},X) & (X,X) & (Y,X) \\ \hline \phi & \alpha_s+\alpha_t & \alpha_s+\alpha_t & \alpha_s+\alpha_t & \alpha_s-\alpha_t & \alpha_s-\alpha_t & \alpha_s-\alpha_t \end{array}$ (B.1.0.11)

$$\begin{array}{c|cccccc} & (\mathbb{1},Y) & (X,Y) & (Y,Y_a) & (Y,Y_b) & (Y,Z) & (Y,XZ) \\ \hline \phi & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \quad \text{(B.1.0.12)}$$

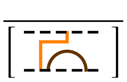
 $= \begin{array}{c|cccccc} & (\mathbb{1},\mathbb{1}) & (X,\mathbb{1}) & (Y,\mathbb{1}) & (\mathbb{1},X) & (X,X) & (Y,X) \\ \hline (X) & \alpha_s-\alpha_t & \alpha_s-\alpha_t & \alpha_s-\alpha_t & \alpha_s+\alpha_t & \alpha_s+\alpha_t & \alpha_s+\alpha_t \end{array}$ (B.1.0.13)

$$\begin{array}{c|cccccc} & (\mathbb{1},Y) & (X,Y) & (Y,Y_a) & (Y,Y_b) & (Y,Z) & (Y,XZ) \\ \hline (X) & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \quad \text{(B.1.0.14)}$$

 $= \begin{array}{c|ccc} & (\mathbb{1}) & (Y) & (Z) \\ \hline (\mathbb{1}) & 1 & 0 & 0 \\ (Y) & 0 & 1 & 0 \\ (Z) & 0 & 0 & 1 \end{array}$  $= \begin{array}{c|ccc} & (X,X) & (X,Y) & (X,Z) \\ \hline (X,X) & 1 & 0 & 0 \\ (X,Y) & 0 & 1 & 0 \\ (X,Z) & 0 & 0 & 1 \end{array}$ (B.1.0.15)

 $= \begin{array}{c|cccccc} & (\mathbb{1},\mathbb{1}) & (Y,\mathbb{1}) & (Z,\mathbb{1}) & (Y,X) & (\mathbb{1},Y) & (Y,Y_a) \\ \hline \phi & 1 & 2 & 1 & 0 & 0 & 0 \end{array}$ (B.1.0.16)

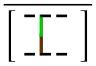
$$\begin{array}{c|cccccc} & (Y,Y_b) & (Z,Y) & (\mathbb{1},Z) & (Y,Z) & (Z,Z) & (Y,XZ) \\ \hline \phi & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \quad \text{(B.1.0.17)}$$

 $= \begin{array}{c|cccccc} & (X,Y,\mathbb{1}) & (X,X,X) & (X,Y,X) & (X,XZ,X) & (X,X,Y) & (X,Y,Y_a) \\ \hline (X) & 0 & 1 & 2 & 1 & 0 & 0 \end{array}$

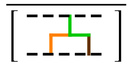
(B.1.0.18)

$$\begin{array}{c|cccccc} & (X,Y,Y_b) & (X,XZ,Y) & (X,Y,Z) & (X,X,XZ) & (X,Y,XZ) & (X,XZ,XZ) \\ \hline (X) & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

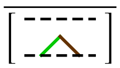
(B.1.0.19)



$$= \begin{array}{c|ccc} & (\mathbb{1}) & (\mathrm{Y}) & (\mathrm{Z}) \\ \hline (\mathbb{1}) & \frac{\alpha_s+\alpha_t}{2} & 0 & 0 \\ (\mathrm{X}) & \frac{\alpha_s-\alpha_t}{2} & 0 & 0 \\ (\mathrm{Y}) & 0 & \alpha_t & 0 \end{array}$$

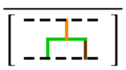


$$= \begin{array}{c|ccc} & (\mathrm{X},\mathrm{X}) & (\mathrm{X},\mathrm{Y}) & (\mathrm{X},\mathrm{XZ}) \\ \hline (\mathbb{1}) & \frac{\alpha_s-\alpha_t}{2} & 0 & 0 \\ (\mathrm{X}) & \frac{\alpha_s+\alpha_t}{2} & 0 & 0 \\ (\mathrm{Y}) & 0 & \alpha_t & 0 \end{array} \tag{B.1.0.20}$$



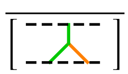
$$= \begin{array}{c|cccccc} & (\mathbb{1},\mathbb{1}) & (\mathrm{Y},\mathbb{1}) & (\mathrm{X},\mathrm{X}) & (\mathrm{Y},\mathrm{X}) & (\mathbb{1},\mathrm{Y}) & (\mathrm{X},\mathrm{Y}) \\ \hline \phi & 2\alpha_s\alpha_t & 2\alpha_s\alpha_t & 0 & 0 & 0 & 0 \end{array} \tag{B.1.0.21}$$

$$\begin{array}{c|cccccc} & (\mathrm{Y},\mathrm{Y_a}) & (\mathrm{Y},\mathrm{Y_b}) & (\mathbb{1},\mathrm{Z}) & (\mathrm{Y},\mathrm{Z}) & (\mathrm{X},\mathrm{XZ}) & (\mathrm{Y},\mathrm{XZ}) \\ \hline \phi & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \tag{B.1.0.22}$$

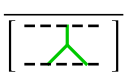


$$= \begin{array}{c|cccccc} & (\mathbb{1},\mathbb{1}) & (\mathrm{Y},\mathbb{1}) & (\mathrm{X},\mathrm{X}) & (\mathrm{Y},\mathrm{X}) & (\mathbb{1},\mathrm{Y}) & (\mathrm{X},\mathrm{Y}) \\ \hline (\mathrm{X}) & 0 & 0 & 2\alpha_s\alpha_t & 2\alpha_s\alpha_t & 0 & 0 \end{array} \tag{B.1.0.23}$$

$$\begin{array}{c|cccccc} & (\mathrm{Y},\mathrm{Y_a}) & (\mathrm{Y},\mathrm{Y_b}) & (\mathbb{1},\mathrm{Z}) & (\mathrm{Y},\mathrm{Z}) & (\mathrm{X},\mathrm{XZ}) & (\mathrm{Y},\mathrm{XZ}) \\ \hline (\mathrm{X}) & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \tag{B.1.0.24}$$

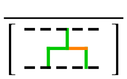


$$= \begin{array}{c|ccc} & (\mathrm{X},\mathbb{1}) & (\mathbb{1},\mathrm{X}) & (\mathrm{Y},\mathrm{Y}) \\ \hline (\mathbb{1}) & 1 & 0 & 0 \\ (\mathrm{X}) & 0 & 1 & 0 \\ (\mathrm{Y}) & 0 & 0 & 1 \end{array} \tag{B.1.0.25}$$



$$= \begin{array}{c|cccccc} & (\mathbb{1},\mathbb{1}) & (\mathrm{X},\mathbb{1}) & (\mathrm{Y},\mathbb{1}) & (\mathbb{1},\mathrm{X}) & (\mathrm{X},\mathrm{X}) & (\mathrm{Y},\mathrm{X}) \\ \hline (\mathbb{1}) & 1 & 1 & 1 & 0 & 0 & 0 \\ (\mathrm{X}) & 0 & 0 & 0 & 1 & 1 & 1 \\ (\mathrm{Y}) & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \tag{B.1.0.26}$$

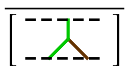
$$\begin{array}{c|cccccc} & (\mathbb{1},\mathrm{Y}) & (\mathrm{X},\mathrm{Y}) & (\mathrm{Y},\mathrm{Y_a}) & (\mathrm{Y},\mathrm{Y_b}) & (\mathrm{Y},\mathrm{Z}) & (\mathrm{Y},\mathrm{XZ}) \\ \hline (\mathbb{1}) & 0 & 0 & 0 & 0 & 0 & 0 \\ (\mathrm{X}) & 0 & 0 & 0 & 0 & 0 & 0 \\ (\mathrm{Y}) & 1 & 1 & 1 & 1 & 0 & 0 \end{array} \tag{B.1.0.27}$$



$$= \begin{array}{c|cccccc} & (\mathbb{1},\mathbb{1}) & (\mathrm{X},\mathbb{1}) & (\mathrm{Y},\mathbb{1}) & (\mathbb{1},\mathrm{X}) & (\mathrm{X},\mathrm{X}) & (\mathrm{Y},\mathrm{X}) \\ \hline (\mathbb{1}) & 0 & \alpha_s+\alpha_t & 0 & 0 & \alpha_s-\alpha_t & 0 \\ (\mathrm{X}) & \alpha_s-\alpha_t & 0 & 0 & \alpha_s+\alpha_t & 0 & 0 \\ (\mathrm{Y}) & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \tag{B.1.0.28}$$

	$(\mathbb{1},\mathrm{Y})$	(X,Y)	$(\mathrm{Y},\mathrm{Y_a})$	$(\mathrm{Y},\mathrm{Y_b})$	(Y,Z)	(Y,XZ)
$(\mathbb{1})$	0	0	0	0	0	0
(X)	0	0	0	0	0	0
(Y)	0	0	$-(\alpha_s+\alpha_t)$	$-(\alpha_s-\alpha_t)$	0	0

(B.1.0.29)

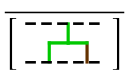
 =

	$(\mathbb{1},\mathbb{1})$	$(\mathrm{Y},\mathbb{1})$	(X,X)	(Y,X)	$(\mathbb{1},\mathrm{Y})$	(X,Y)
$(\mathbb{1})$	$\frac{\alpha_s+\alpha_t}{2}$	$\frac{\alpha_s+\alpha_t}{2}$	$\frac{\alpha_s-\alpha_t}{2}$	$\frac{\alpha_s-\alpha_t}{2}$	0	0
(X)	$\frac{-(\alpha_s-\alpha_t)}{2}$	$\frac{-(\alpha_s-\alpha_t)}{2}$	$\frac{-(\alpha_s+\alpha_t)}{2}$	$\frac{-(\alpha_s+\alpha_t)}{2}$	0	0
(Y)	0	0	0	0	α_t	$-\alpha_t$

(B.1.0.30)

	$(\mathrm{Y},\mathrm{Y_a})$	$(\mathrm{Y},\mathrm{Y_b})$	$(\mathbb{1},\mathrm{Z})$	(Y,Z)	(X,XZ)	(Y,XZ)
$(\mathbb{1})$	0	0	0	0	0	0
(X)	0	0	0	0	0	0
(Y)	α_t	0	0	0	0	0

(B.1.0.31)

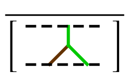
 =

	$(\mathbb{1},\mathbb{1})$	$(\mathrm{Y},\mathbb{1})$	(X,X)	(Y,X)	$(\mathbb{1},\mathrm{Y})$	(X,Y)
$(\mathbb{1})$	$\frac{\alpha_s+\alpha_t}{2}$	$\frac{\alpha_s+\alpha_t}{2}$	$\frac{\alpha_s-\alpha_t}{2}$	$\frac{\alpha_s-\alpha_t}{2}$	0	0
(X)	$\frac{-(\alpha_s-\alpha_t)}{2}$	$\frac{-(\alpha_s-\alpha_t)}{2}$	$\frac{-(\alpha_s+\alpha_t)}{2}$	$\frac{-(\alpha_s+\alpha_t)}{2}$	0	0
(Y)	0	0	0	0	α_t	α_t

(B.1.0.32)

	$(\mathrm{Y},\mathrm{Y_a})$	$(\mathrm{Y},\mathrm{Y_b})$	$(\mathbb{1},\mathrm{Z})$	(Y,Z)	(X,XZ)	(Y,XZ)
$(\mathbb{1})$	0	0	0	0	0	0
(X)	0	0	0	0	0	0
(Y)	0	α_t	0	0	0	0

(B.1.0.33)

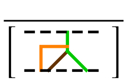
 =

	$(\mathbb{1},\mathbb{1})$	$(\mathrm{Y},\mathbb{1})$	$(\mathbb{1},\mathrm{X})$	(Y,X)	$(\mathbb{1},\mathrm{Y})$	$(\mathrm{Y},\mathrm{Y_a})$
$(\mathbb{1})$	$\frac{\alpha_s+\alpha_t}{2}$	$\frac{\alpha_s+\alpha_t}{2}$	$\frac{\alpha_s-\alpha_t}{2}$	$\frac{\alpha_s-\alpha_t}{2}$	0	0
(X)	$\frac{-(\alpha_s-\alpha_t)}{2}$	$\frac{-(\alpha_s-\alpha_t)}{2}$	$\frac{-(\alpha_s+\alpha_t)}{2}$	$\frac{-(\alpha_s+\alpha_t)}{2}$	0	0
(Y)	0	0	0	0	0	α_t

(B.1.0.34)

	$(\mathrm{Y},\mathrm{Y_b})$	(Z,Y)	(Y,Z)	(Z,Z)	(Y,XZ)	(Z,XZ)
$(\mathbb{1})$	0	0	0	0	0	0
(X)	0	0	0	0	0	0
(Y)	$-\alpha_t$	α_t	0	0	0	0

(B.1.0.35)

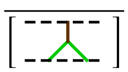
 =

	$(\mathrm{X},\mathrm{X},\mathbb{1})$	$(\mathrm{X},\mathrm{Y},\mathbb{1})$	$(\mathrm{X},\mathrm{X},\mathrm{X})$	$(\mathrm{X},\mathrm{Y},\mathrm{X})$	$(\mathrm{X},\mathrm{X},\mathrm{Y})$	$(\mathrm{X},\mathrm{Y},\mathrm{Y_a})$
$(\mathbb{1})$	$\frac{-(\alpha_s+\alpha_t)}{2}$	$\frac{-(\alpha_s+\alpha_t)}{2}$	$\frac{-(\alpha_s-\alpha_t)}{2}$	$\frac{-(\alpha_s-\alpha_t)}{2}$	0	0
(X)	$\frac{\alpha_s+\alpha_t}{2}$	$\frac{\alpha_s+\alpha_t}{2}$	$\frac{\alpha_s-\alpha_t}{2}$	$\frac{\alpha_s-\alpha_t}{2}$	0	0
(Y)	0	0	0	0	0	α_t

(B.1.0.36)

	(X, Y, Y_b)	(X, XZ, Y)	(X, Y, Z)	(X, XZ, Z)	(X, Y, XZ)	(X, XZ, XZ)
$(\mathbb{1})$	0	0	0	0	0	0
(X)	0	0	0	0	0	0
(Y)	$-\alpha_t$	α_t	0	0	0	0

(B.1.0.37)

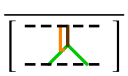
 =

	$(\mathbb{1}, \mathbb{1})$	$(X, \mathbb{1})$	$(Y, \mathbb{1})$	$(\mathbb{1}, X)$	(X, X)	(Y, X)
$(\mathbb{1})$	2	-2	0	0	0	0
(Y)	0	0	0	0	0	0
(Z)	0	0	0	0	0	0

(B.1.0.38)

	$(\mathbb{1}, Y)$	(X, Y)	(Y, Y_a)	(Y, Y_b)	(Y, Z)	(Y, XZ)
$(\mathbb{1})$	0	0	0	0	0	0
(Y)	1	-1	1	-1	0	0
(Z)	0	0	0	0	2	0

(B.1.0.39)

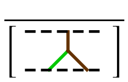
 =

	$(\mathbb{1}, \mathbb{1})$	$(X, \mathbb{1})$	$(Y, \mathbb{1})$	$(\mathbb{1}, X)$	(X, X)	(Y, X)
$(\mathbb{1})$	-2	2	0	0	0	0
(Y)	0	0	0	0	0	0
(Z)	0	0	0	0	0	0

(B.1.0.40)

	$(\mathbb{1}, Y)$	(X, Y)	(Y, Y_a)	(Y, Y_b)	(Y, Z)	(Y, XZ)
$(\mathbb{1})$	0	0	0	0	0	0
(Y)	1	-1	1	-1	0	0
(Z)	0	0	0	0	0	2

(B.1.0.41)

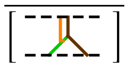
 =

	$(\mathbb{1}, \mathbb{1})$	$(Y, \mathbb{1})$	(X, X)	(Y, X)	$(\mathbb{1}, Y)$	(X, Y)
$(\mathbb{1})$	2	2	0	0	0	0
(Y)	0	0	0	0	2	0
(Z)	0	0	0	0	0	0

(B.1.0.42)

	(Y, Y_a)	(Y, Y_b)	$(\mathbb{1}, Z)$	(Y, Z)	(X, XZ)	(Y, XZ)
$(\mathbb{1})$	0	0	0	0	0	0
(Y)	1	1	0	0	0	0
(Z)	0	0	2	2	0	0

(B.1.0.43)

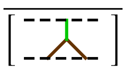
 =

	$(\mathbb{1},\mathbb{1})$	$(Y,\mathbb{1})$	(X,X)	(Y,X)	$(\mathbb{1},Y)$	(X,Y)
(X,X)	2	2	0	0	0	0
(X,Y)	0	0	0	0	2	0
(X,XZ)	0	0	0	0	0	0

(B.1.0.44)

	(Y,Y_a)	(Y,Y_b)	$(\mathbb{1},Z)$	(Y,Z)	(X,XZ)	(Y,XZ)
(X,X)	0	0	0	0	0	0
(X,Y)	1	-1	0	0	0	0
(X,XZ)	0	0	0	0	2	2

(B.1.0.45)

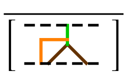
 =

	$(\mathbb{1},\mathbb{1})$	$(Y,\mathbb{1})$	$(Z,\mathbb{1})$	(Y,X)	$(\mathbb{1},Y)$	(Y,Y_a)
$(\mathbb{1})$	$\frac{\alpha_s+\alpha_t}{2}$	$\frac{\alpha_s+\alpha_t}{2}$	$\frac{\alpha_s+\alpha_t}{2}$	0	0	0
(X)	$\frac{-(\alpha_s-\alpha_t)}{2}$	$\frac{-(\alpha_s-\alpha_t)}{2}$	$\frac{-(\alpha_s-\alpha_t)}{2}$	0	0	0
(Y)	0	0	0	0	α_t	α_t

(B.1.0.46)

	(Y,Y_b)	(Z,Y)	$(\mathbb{1},Z)$	(Y,Z)	(Z,Z)	(Y,XZ)
$(\mathbb{1})$	0	0	0	0	0	0
(X)	0	0	0	0	0	0
(Y)	α_t	α_t	0	0	0	0

(B.1.0.47)

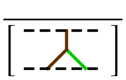
 =

	$(X,Y,\mathbb{1})$	(X,X,X)	(X,Y,X)	(X,XZ,X)	(X,X,Y)	(X,Y,Y_a)
$(\mathbb{1})$	0	$\frac{-(\alpha_s-\alpha_t)}{2}$	$\frac{-(\alpha_s-\alpha_t)}{2}$	$\frac{-(\alpha_s-\alpha_t)}{2}$	0	0
(X)	0	$\frac{\alpha_s+\alpha_t}{2}$	$\frac{\alpha_s+\alpha_t}{2}$	$\frac{\alpha_s+\alpha_t}{2}$	0	0
(Y)	0	0	0	0	α_t	α_t

(B.1.0.48)

	(X,Y,Y_b)	(X,XZ,Y)	(X,Y,Z)	(X,X,XZ)	(X,Y,XZ)	(X,XZ,XZ)
$(\mathbb{1})$	0	0	0	0	0	0
(X)	0	0	0	0	0	0
(Y)	$-\alpha_t$	α_t	0	0	0	0

(B.1.0.49)

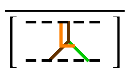
 =

	$(\mathbb{1},\mathbb{1})$	$(Y,\mathbb{1})$	$(\mathbb{1},X)$	(Y,X)	$(\mathbb{1},Y)$	(Y,Y_a)
$(\mathbb{1})$	2	2	0	0	0	0
(Y)	0	0	0	0	1	2
(Z)	0	0	0	0	0	0

(B.1.0.50)

	(Y,Y_b)	(Z,Y)	(Y,Z)	(Z,Z)	(Y,XZ)	(Z,XZ)
$(\mathbb{1})$	0	0	0	0	0	0
(Y)	0	1	0	0	0	0
(Z)	0	0	2	2	0	0

(B.1.0.51)

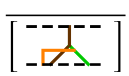
 =

	$(\mathbb{1},\mathbb{1})$	$(Y,\mathbb{1})$	$(\mathbb{1},X)$	(Y,X)	$(\mathbb{1},Y)$	(Y,Y_a)
(X,X)	0	0	2	2	0	0
(X,Y)	0	0	0	0	0	2
(X,XZ)	0	0	0	0	0	0

(B.1.0.52)

	(Y,Y_b)	(Z,Y)	(Y,Z)	(Z,Z)	(Y,XZ)	(Z,XZ)
(X,X)	0	0	0	0	0	0
(X,Y)	1	−1	0	0	0	0
(X,XZ)	0	0	0	0	2	2

(B.1.0.53)

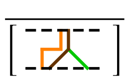
 =

	$(X,X,\mathbb{1})$	$(X,Y,\mathbb{1})$	(X,X,X)	(X,Y,X)	(X,X,Y)	(X,Y,Y_a)
$(\mathbb{1})$	2	2	0	0	0	0
(Y)	0	0	0	0	1	0
(Z)	0	0	0	0	0	0

(B.1.0.54)

	(X,Y,Y_b)	(X,XZ,Y)	(X,Y,Z)	(X,XZ,Z)	(X,Y,XZ)	(X,XZ,XZ)
$(\mathbb{1})$	0	0	0	0	0	0
(Y)	2	−1	0	0	0	0
(Z)	0	0	−2	2	0	0

(B.1.0.55)

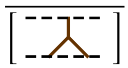
 =

	$(X,X,\mathbb{1})$	$(X,Y,\mathbb{1})$	(X,X,X)	(X,Y,X)	(X,X,Y)	(X,Y,Y_a)
(X,X)	0	0	2	2	0	0
(X,Y)	0	0	0	0	1	2
(X,XZ)	0	0	0	0	0	0

(B.1.0.56)

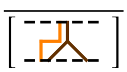
	(X,Y,Y_b)	(X,XZ,Y)	(X,Y,Z)	(X,XZ,Z)	(X,Y,XZ)	(X,XZ,XZ)
(X,X)	0	0	0	0	0	0
(X,Y)	0	1	0	0	0	0
(X,XZ)	0	0	0	0	2	2

(B.1.0.57)

 =

$$\begin{array}{c|cccccc} & (\mathbb{1},\mathbb{1}) & (\mathrm{Y},\mathbb{1}) & (\mathrm{Z},\mathbb{1}) & (\mathrm{Y},\mathrm{X}) & (\mathbb{1},\mathrm{Y}) & (\mathrm{Y},\mathrm{Y_a}) \\ \hline (\mathbb{1}) & 1 & 2 & 1 & 0 & 0 & 0 \\ (\mathrm{Y}) & 0 & 0 & 0 & 0 & 1 & 1 \\ (\mathrm{Z}) & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \tag{B.1.0.58}$$

$$\begin{array}{c|cccccc} & (\mathrm{Y},\mathrm{Y_b}) & (\mathrm{Z},\mathrm{Y}) & (\mathbb{1},\mathrm{Z}) & (\mathrm{Y},\mathrm{Z}) & (\mathrm{Z},\mathrm{Z}) & (\mathrm{Y},\mathrm{XZ}) \\ \hline (\mathbb{1}) & 0 & 0 & 0 & 0 & 0 & 0 \\ (\mathrm{Y}) & 1 & 1 & 0 & 0 & 0 & 0 \\ (\mathrm{Z}) & 0 & 0 & 1 & 2 & 1 & 0 \end{array} \tag{B.1.0.59}$$

 =

$$\begin{array}{c|cccccc} & (\mathrm{X},\mathrm{Y},\mathbb{1}) & (\mathrm{X},\mathrm{X},\mathrm{X}) & (\mathrm{X},\mathrm{Y},\mathrm{X}) & (\mathrm{X},\mathrm{XZ},\mathrm{X}) & (\mathrm{X},\mathrm{X},\mathrm{Y}) & (\mathrm{X},\mathrm{Y},\mathrm{Y_a}) \\ \hline (\mathrm{X},\mathrm{X}) & 0 & 1 & 2 & 1 & 0 & 0 \\ (\mathrm{X},\mathrm{Y}) & 0 & 0 & 0 & 0 & 1 & 1 \\ (\mathrm{X},\mathrm{XZ}) & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \tag{B.1.0.60}$$

$$\begin{array}{c|cccccc} & (\mathrm{X},\mathrm{Y},\mathrm{Y_b}) & (\mathrm{X},\mathrm{XZ},\mathrm{Y}) & (\mathrm{X},\mathrm{Y},\mathrm{Z}) & (\mathrm{X},\mathrm{X},\mathrm{XZ}) & (\mathrm{X},\mathrm{Y},\mathrm{XZ}) & (\mathrm{X},\mathrm{XZ},\mathrm{XZ}) \\ \hline (\mathrm{X},\mathrm{X}) & 0 & 0 & 0 & 0 & 0 & 0 \\ (\mathrm{X},\mathrm{Y}) & 1 & 1 & 0 & 0 & 0 & 0 \\ (\mathrm{X},\mathrm{XZ}) & 0 & 0 & 0 & 1 & 2 & 1 \end{array} \tag{B.1.0.61}$$

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