Set Class Conceptualizations: A Pedagogical and Theoretical Approach

by

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DISSERTATION ABSTRACT

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Set Class Conceptualizations has two main goals: one, facilitate a student's learning of set classes; and two, demonstrate multiple ways in which they could benefit from doing so. The intended audience of this dissertation is music professionals and teachers. The main gist of this approach is: one, as the student studies individual set classes, they refine a "bigger picture" of how the set classes relate; two, they relate "new" set classes to ones that they are already familiar with; and three, they let their own musical interests guide them. Part I and the Appendices provides general background information and resources that can act as an aid and inspiration towards the student's development of their own "bigger picture." Part II, the heart of this pedagogy, provides tools and resources that are specifically tailored towards the learning of the set classes. Part III provides examples of how an increased knowledge of set classes may enhance various musical endeavors.

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PART 0: INTRODUCTION

Chapter 0

The following dissertation is an homage to human remarkableness. Even in as mundane of a task as purchasing a cup of coffee, we are required to navigate social situations, financial transactions, coordinating hundreds of motor movements, operating a moving vehicle through fast moving traffic, maintaining our composure, etc. In any given moment, through the mere act of being, we do/process (consciously and unconsciously) more than could be contained in a book and more than we yet know how to describe. Yet, in this maelstrom of life processes, we still find stability. In our consciousness, we delimit near-infinity into manageable chunks. Through this categorization, grouping, many of us can even "see" and find "structure and organization" where there may be none. In doing this we can then manage getting out of bed, flirting with the barista — (mis)interpreting the signals, taking comfort in our knowledge that we know where the perfect cup of coffee is, and can reasonably well time getting both there and to work (on the other side of town) before 9 am.

Similarly, the performing musician is continuously required to classify sonic and visual events; making sense of often ambiguous signals. How does the improvising musician know when their on-stage collaborator's solo will soon end? Or, what reactions from the audience to on-board mid-solo—ones that point to their continuing to build their solo, end soon, or gradually dissipate the built-up energy through a vamp? Or, alternatively, how does the accompanist ascertain when the soloist is conceptualizing the harmony differently to the lead-sheet placed in front of them both?

Without even trying (and through trying!), we are all "geniuses" at categorizing. Accordingly, questions pertaining to classification permeate every aspect of our epistemological pursuits. Learning something new (especially a theory), tends to result in our re-evaluating and potentially re-classifying various phenomena. For instance, in studying linear analysis, we are shown that a single instantiation of a chord (such as a V) can have multiple functions; it can simultaneously signal the end of a phrase, section, and an entire work. Accordingly, we may then re-evaluate and re-classify how we interpret its accompanying "embellishment." Alternatively, many students express surprise when shown how a clave is embedded in many of Scott Joplin's rag-time works (a great signifier of Americana). Those students may then begin to question (re-evaluate and re-classify) the origins of much U.S. music—noting the pervasive Afro-Caribbean influence.

My proposed pedagogy then rests on the following assumptions:

- While there are aspects of categorization that will remain below conscious control (i.e., facial recognition and our ability to identify an object, presented from two different angles, as being the same), there are benefits to focusing on those aspects that are not.
- As categorization/classification is fundamental to our engagement with music (and the world), it should be ascribed a fundamental status in music theory—a discipline devoted to understanding and explaining music.
- Categorization is a continuous act. For sufficiently complicated material, the boundaries between its various classifications should be seen as porous. A label shouldn't be seen as a given but rather as a site for continual (re)negotiation.
- We can get better at categorizing. At best, we can get better at classifying writ large; at minimum, we can get better at predicting the pros and cons of how specific theories mask and/or overemphasize specific interpretations (through categorical assignment) of what something is. By treating categorization as a creative act (not just a reductive process) and by giving constant attention to how the material is engaged, students can: refine their sense of what it means to categorize something; increase pattern recognition; disentangle "classification" from "stereotyping;" and develop a cognitive skill set/framing that can simplify the learning (and performing of) a host of musics.

It should be noted that these assumptions are part and parcel of good reflective analyses. Analytical theorists defend their interpretations of musical events structurally (at the local and global level), they situate their discussion in regards to historical and ahistorical theories on knowledge construction and signification, and they acknowledge the advantages and limitations (and sometimes failings) of their employed methods. However, the paths to these analysts' achievements are often long, circuitous, trodden by just a few, and, are setback by each genre (and, in many cases, every piece of music) having their own host of literary challenges, methodologies, and one-off technical challenges to contend with. Furthermore, like those methods utilized by instrumentalists, it may take years (if not decades) to get traction with them.

For instance, take Jack Boss's triptych on Arnold Schoenberg; it has been twenty-plus years in the making, is built off of 30+ years of experience applying both Schenkerian and post-tonal analysis, and, in multiple spoken/written languages, engages a substantial body of musical, literary, philosophical, and historical literature. Similarly, Mazzola's near 35-years in the making, multi-volume, near 2000-page treatise is translated in multiple languages, engages each of the main areas of $20^{th}/21^{st}$ century math (even quantum physics!), re-conceptualizes most domains of music, and tackles long-standing musical questions that extend back to Rameau.

Accordingly, while no general method could ever address the specific challenges that befall all musics, there is an advantage to crafting a pedagogical "Esperanza"—an approach that can minimize the time needed to learn a great variety of musical styles. Also, if this method is not positioned to be a universal language (it's just scaffolding), it can hopefully avoid pitfalls typically associated with such general approaches; that burgeoning and **false** judgement that learned abstract rationales and terminology can best explain (rather than just give an alternative explanation for) a musical practice.

Rather this proposed categorization-based method asserts that it is helpful to describe music theory that on "day one" describes theory as a way of fitting things into boxes; on "day two" introduces a terminology/unit that is abstract and adaptable enough to label the dimensions of many box types; on "day three" introduces (in a nascent form) various box types currently circulating; on "day four" makes new boxes, learns more about specific box types, plays the game "reveal the box type." Over time, the combination of your handling of "day two's" terminology and increasingly nuanced understanding of the ramifications of various adopted lenses, will:

1. Make you more critical and aware of your own biases when formally attempting to learn a, yet unfamiliar to you, musical genre.

2. Enable you to more quickly chunk foreign material. Many (by no means all though!!!) of the mysteries surrounding a different genre can be solved simply through recognition and identification of recurring patterns.
3. Prime you to notice middle-ground patterns; organizational forces that lie below the musical surface.

Nonetheless, the following dissertation does not explicitly spell this methodology out. Rather, its aim is to provide the needed information so that a teacher can then best mold the method to the specific needs/wants of a given pupil (the details of this would better fit a subsequent paper). Furthermore, as this method is musical, philosophical, and cognitive, there is already a substantial obligation to provide sufficient background on its many components.

Over the course of this dissertation, I will situate the relevant terms historically and examine historical theory treatises that have attempted to reduce the infinity of musical expression to a discrete number of learnable steps. However, none of these treatises is meant to be seen as solutions, but rather as sources of inspiration for a particular student. Suggestions on how they can be played with, explored, interrogated will be given. Secondly, I will raise questions over the larger concerns that undergird such an ambitious project. Are there bounds to what we can learn? How transferable is knowledge learned through study to knowledge enacted in performance? Finally, guidance will be given on how to learn as many of the "relevant terms" as wanted and then demonstrate ways to incorporate them into various facets of music theory pedagogy—from aural skills, to analysis, composition, and improvisation.

Finally, one last question should be addressed: why are set classes important to this pedagogy? In short, **set classes as labels** (especially Forte's set class labels) assist the learning of the various pitch collections. Their utilily as labels supersedes a slavish adherence to their definition. If one only references set classes as they are strictly defined, one could not distinguish between any representations of a set class — a C minor would be deemed equivalent to F# minor.
While there are music and/or music pedagogical instances where this is a helpful lens, for the majority of the musical situations that you may encounter, it will most likely **not** be the most helpful lens. As such, while I do introduce set classes formally, I often treat them as labels that

can take on one of a number of different meanings. For example, when there is little to no confusion, (e.g., with a symettrical set class), I just treat the label as pertaining to a scale (if ordering in that context is helpful) or a chord (if ordering in that context is not). Of course, technically, that "symmetrical" set class is neither. The set class (an unordered collection of an unordered collection of an unordered collection of pitch classes) that can be manifested by the whole tone scale is different to the T_n class (an unordered collection of an unordered collection of pitch classes) that can be manifested by a whole-tone scale (an ordered collection of pitch classes), that again is different than a specific whole-tone scale (an ordered collection of pitches).

In certain contexts, it is critical to weed through the subtle differences in terminological usage — especially, when there is some (typically, analytical or mathematical) rationale for doing so. However, there are many instances, when there is not. For instance, when one is trying to play what one hears in a performance. An analyst may later opt to describe what you performed using set classes labels as intended, with all the attending implications (e.g., it's an aleatoric piece that continually recombines various intervals that are not distinguished by direction). Or, alternatively, they may not, instead describing your piece as a "improvisation over the whole tone scale." Does it really make sense though, for you the composer/improviser to refrain from using those labels incorrectly when in the creative act? Does the creative act proscribe one to only reference terms in accordance with their proper definition?

In large part, I don't want to regulate how you engage these terms. My assumption is that once you get more familiar with these labels and associated sound objects, you can then reference them with whatever faithfulness to the original definition that you find appropriate. Ultimately, when viewing Forte's set classes predominantly as helpful labels, rather than as a suggestion (or, even worse, prescription) on how to interpret or think of a musical passage you are imagining, a lot more traction can be gotten out of their usage. Throughout out the text, I will typically distinguish between the different senses in which I'm engaging the set class label. However, if the distinction is not made explicit, default to an interpretation of that label that makes the most sense.

PART I: LITERATURE REVIEW

Part I, Section 1, Introduction to Set Classes

Part I, Section 1, Chapter 1: Defining Set Classes

Set classes are the units of this pedagogy. Each is intended to signify a relationship between musical, or even non-musical, elements. This chapter focuses on their standard definition and usage. However, as this pedagogy progresses, set classes will also be used in a looser sense—not just as means of classifying collections of pitch-classes but as indexes for other types of musical material, ranging from rhythms to musical progression. However, if one looks at this terminology through a broader lens, as a consequence of music-theoretical concepts that have been percolating In Western musical consciousness for at least 500 years, then this proposed shift in usage may not only seem justifiable, but actually warranted.

Below is a glossary of terms related to set classes.

Glossary (terms related to Set Class)

Pitch — A note such as C_4.

Pitch Class (pc) — A set of all pitches that are octave equivalent.

E.g., the pitch class C consists of $\{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, \text{etc.}\}$

Brackets —

- **{ }** refers to an unordered collection of elements.
    - E.g., $\{x_1, x_2, x_3\} = \{x_2, x_1, x_3\} = \{x_3, x_2, x_1\}$
- **< >** refers to an ordered collection of elements.
    - E.g., $<x_1, x_2, x_3> \neq <x_2, x_1, x_3> \neq <x_3, x_2, x_1>$

Chord —

1: (used in this dissertation) A means of classifying collections of pitch classes that is invariant under transposition and ordering. In post-tonal theory, the pitch names are often expressed as integers, wherein 0 = C, 1 = C#/Db, 2 = D … t = Bb, e = B. For example, {0,4,7} = {C, E, G}, {9,2,6} = {A, D, F#}, and {2,6,9} = {D, F#, A} are all representations of the same chord.

2: (not used in this dissertation) any collection of pitch or pitch classes.

Pitch Class Set (pc-set) — A collection of pitch classes.

Normal Order — The representation of a pitch class set that, when put in ascending order, has the shortest span. If there are multiple contenders for this, the contender that starts with smallest interval is selected. If again, there are multiple contenders that have the shortest span and begin with the same smallest interval, the contender with the smallest interval following the "starting interval" is selected, etc. This process is known as finding the representation that is "**most densely packed to the left**." If this process still does not yield a sole contender, choose the "most densely packed to the left" rotation that begins on the smallest integer. Notice that normal form uses brackets indicating that the collection is unordered; yet, its expressed content is ordered. E.g., to get the normal order of {F#, C, Ab, D}:

1. Convert {F#, C, Ab, D} to integer notation {6, 0, 8, 2}.
2. Put in ascending order <6, 8, 0, 2>.
3. Find the rotation of <6, 8, 0 ,2> with the shortest span.
    a. These are <0, 2, 6, 8> and <6, 8, 0 ,2>.
    b. These are equally densely packed to the left.
4. Choose the rotation that starts on the smallest integer; in this case, <0, 2, 6, 8>
    a. Finally, {0, 2, 6, 8} is the normal form of {F#, C, Ab, D}.

Set Class (sc)— A means of classifying collections of pitch classes that is invariant under transposition, intervallic inversion, and ordering. This dissertation utilizes Forte's labeling of set classes.

Prime Form — The representation of a set class that is "most densely packed to the left" (look at the definition of **Normal Order**) and starts on 0 (C).

Interval — the directed distance between two pitches or pitch classes.

- E.g., the distance from the pitches C_4 to E_5 is an ascending Maj 10^{th}. Alternatively, the distance from the pitch classes C to E is a Major 3^{rd})

Interval Class (ic) — the absolute distance between two pitch classes. Notice that, where intervals are often expressed in terms of their quality — Perf, Maj, Min, Dim, or Aug — interval classes are typically described in terms of the number of semitones that they span. E.g., the interval class of size 4 (ic4) spans 4 semitones and can be expressed as C to E, E to C, Db to F, F to Db, etc.

Interval vector — typically an ordered six tuple, <#ic1, #ic2, #ic3, #ic4, #ic5, #ic6> that tallies the number of each type of ic in a set class.

- E.g., The interval vector representing the set class that can be represented by {0, 4, 7} is <0,0,1,1,1,0>. To check this, notice that {4,7} = ic3, {0,4} = ic4, and {0,7} = ic5.

Z relation — Two set classes are Z-related if they share the same interval vector.

Aggregate — the pitch class set that contains all pitch classes.

Literal Complement — The literal complement of a collection of pitch classes is the collection of other set classes that are needed to form the aggregate. E.g, the literal complement of {0,1,2} is {3,4,5,6,7,8,9,t,e}.

Abstract Complement — The abstract complement of a pitch class set is the set-class that contains said pitch class set's literal complement.

T_n Classes — The complete collection of representations of a set class that are invariant under transposition.

- In a symmetrical set-class, there is a single T_n class. All of the set class's representations are related by transposition.
- In a non-symettrical set class, there are two T_n classes. One T_n class contains all of the representations of the set class that are transpositions of the prime form. The other T_n class contains all of the transpositions of the intervallic inversion (I) of any of the first Tn-class's representations.

Soloman's Notation:

- A set class's "a" form is the T_n class of a set class (symmetrical or non-symettrical), which contains the prime form.
- A set class's "b" form is the T_n class of a non-symmetrical set class that does not contain the prime form.
- An asterisk (*) is appended to set class labels that are symmetrical.

Odds and ends

There is also convention in atonal pedagogy of associating pitch-classes with positions on a clock. Here, the pitch-class C is assigned the value of 0 (or 12); the pitch-class C#/Db is assigned the value of 1, D is assigned the value of 2, etc. Using this framing, one can calculate the distance between any two pitch-classes. Depending on whether you consider 'D' as above or below C the difference between the two pitch-classes C (0,12) and D (2) can be either construed as 2 or -10 steps. As discussed above, set class classification considers these two distances 2 and -10 to be equivalent;[1] pc-set/chord classification does not.

Finally, how practical or meaningful is it to equate a minor and major triad? Clearly, there is long-standing historical precedent for distinguishing the two. Yet, there has also been a long-standing historical precedent for lumping them together as the most consonant chord category.[2] This question, which the introduction of set classes can prompt, is worth investigating personally—not just taking an answer for granted, one way or the other.

At a minimum, set classes have more associations; they bundle together more musical material. The ramifications of this may be both technical and philosophical. Moreover, numbers, the typical descriptor of set classes, have been (and hopefully, will continue to be) a contentious way to navigate, construct, and interpret reality — for instance, in Plato's Timaeus (Plato and Zeyl 2000). By retaining (rather than excising) many of these accumulated connotations, set classes—most simply conceived as numbers and bracket notation, can prompt near limitless creative inquiries and fruitful comparisons.

[1] Naming convention dictates choosing 2, the smallest positive integer as the representative of the absolute distance.
[2] As pointed out by Jack Boss (Boss 1985).

Part I, Section 1, Chapter 2: Theoretical Musical Roots (Music as Discrete)

As mentioned before, set classes are more than a set of descriptive terms; set classes are an inheritor to, a reaction to, a distillation of, and/or crystallization of many ideas that came prior.[3] By putting them in dialogue with preceding theories, their metaphorical and technical richness is better highlighted; and common criticisms — such as being too dry, or just numerical jargon — are more easily rebutted.

As the following exposition is not intended to be a meta-narrative or origin story of set classes, one might ask then why such an extensive historical account should be given. This is done for two main reasons: one, it will provide an essential background for the activities discussed in parts II and III—in particular, those sections that seek to ground the newly discussed techniques (and technical approaches) in historical precedence (**not** foreshadowing!); and two, which is a reason underlying the first, that it could encourage readers to try out different perspectives and reflect on how those various perspectives may or may not positively guide their interpretation, creation, or exploration, of something meaningful to them. Accordingly, judge your own music theory prowess in terms of your adaptability to new methodology and then your proficiency with, and even authority in, a select few of the described methods.

In general, I see the power of art as residing in care and imagination. Even the most abrasive music can function as an act of care. For example, take “harsh music” such as NWA’s “Fuck the Police” and Rage Against the Machine’s “Killing in the Name (a response to the L.A. riots); both have been used as a vehicle for protest; subverting aesthetic norms can help dismantle hostile political norms that are cloaked in appeals to reasonableness, order, acceptability, and even beauty. I posit that, independent of aesthetic concerns, acts designed to elicit positive structural

[3]Please note that I am neither making any indefensible strings of causal claims that will seek to explain set classes as the inevitable consequence of earlier theories, nor am I seeking any other dubious assertions that they are the culmination of anything, with all of the accompanying Hegelian-inspired trappings.

(or even interpersonal) change have the potential to be acts of care[4] (if the intent and effect are considered). By ruminating on as many other ways of framing our cherished material that we can get our hands on, at maximum we may have some new 'revolutionary' attitude; at minimum, we'll be more likely to reach across the aisle—hopefully, finding potential colleagues, rather than "threats to your authority/way of doing things."

Similarly, by practicing putting on different lenses, I suggest that our imaginative powers greatly increase. It takes imagination, listening, and care to chip away at those bigger intractable non-musical problems. Even if a new music theory pedagogical tool cannot directly lower desalination costs and help mitigate the impact of global warming in drought-ridden regions, it would be great if pre-collegiate music theory could promote the flexible thinking necessary for that future scientist/politician/advocate to balance ideologically-divided and multi-disciplinary mental landscapes—typically navigating different interests and perspectives requires navigating different terminology and discipline-specific assumptions. If music-theory training, situated as it is in technical and cultural concerns, can be more explicitly centered around this value (at least at certain stages), getting better at navigating multi-dimensional problems, it would be easier to pitch it as more than just an add on to pre-collegiate instrumental lessons.

Let's discuss an assortment of multi-dimensional music-related concerns that lurk in the background of set classes' pre-history!

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1.

Converging interests intersected in Plato's philosophy, Timaeus (Plato and Zeyl 2000); salient amongst these interests was a want to both explain the cosmos and then relate that understanding to phenomena. According to Popper (Popper 1989), the irreducibility of the square root of 2 to a rational number compelled Plato to confront a Pythagorean belief head on—number, and subsequently proportion, constituted reality. If no measurable transformation of a unit

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[4]My discussion of care is meant to signal my own biases; neither to make universal claims about what music does or does not demonstrate care.

(multiplying it by a rational number) can beget a measure as common as the square root of 2 (the hypotenuse of a unit square), how can our knowledge of the cosmos, constructed out of uniform atomic units, be understood? This monist creed convincingly explained phenomena ranging from the constitution of the heavens, to fluctuations of our passions, the forms of epistemological pursuits, the world-soul. Along these lines, Plato also specifies the order in which a single string length can (through duplication and multiplication of its length) engender the gamut (range of pitches, the Greater Perfect system) and the scale (really an idea) out of which music is born. To save the otherwise cogent explanation of reality from contradiction, Plato through his Platonic solids (dodecahedron, octahedron etc.) found 3-dimensional models for those atomic units. Built into each face were either the irreducible square root of 2 or 3. Hence, the impending crisis was (temporarily) averted with scotch tape.

Moreover, Plato used mathematical expressions to relate how we know to what we know and, in a sense, categorize our experience. Analogy, bound up in geometric methods, provided one significant way in which this was achieved—analogies are of the form a:b::c:d (Lawlor 2003, pp. 44-45; 80-81). However, in this most general form, a and b may not be relatable to c and d. Or at least, there is no indication of the transformation that a or b would have to undergo to yield either c or d. Nonetheless, other models of this analogy, particularly those containing only three terms, are more relatable:

1. The *continuous proportion* is a:b::b:c. The mediating b, is a lens through which topic a can be related to c. "I am to my father, as my father is to his."
2. The *golden proportion* is a:b::b:(a+b). 2:4::4:6, 4:6::6:10. "Me is to ye, as ye is to we."
3. The *mediating proportions* consist of those proportions with "a group of three unequal numbers such as that two of their differences are to each other in the same relationship as one of these numbers is to itself [case 1] or to one of the other two numbers [case 2 &3]."
    a. Case 1: The *arithmetic* is a-b:b-c::a:a, b:b, c:c
        i. This is linear
    b. Case 2: The *geometric* is a-b:b-c::a:b
        i. This is exponential
    c. Case 3: The *harmonic* (the proportion specific to music) is a-b:b-c::a:c.

i. As demonstrated by Plato in *Timaeus*, it is responsible (3-4:4-6::3:6) for the derivation of the P4 (diatessaron; 3:4) and P5 (diapente; 4:6) out of the octave (diapason; 3:6).
ii. As Walter Odington broaches and Riemann later hammers in, a similar logic can be used to construct major or minor triads out of the harmonic proportion (10:12:15). Minor, if we think in string lengths (10:12:15); Major, if we think in string divisions—1/10, 1/12, and 1/15 which respectively equals 30/300, 25/300, and 20/300.

Lawlor showed that over time these proportions came to symbolize how we as humans can construct knowledge—when there are only 3 terms in the analogy and we are "a", there is an implication that all the knowledge acquired comes through relation with oneself. It is quite enticing to relate the form of which we can know things (essentially through analogy) to the essence (the form) of what something (as manifest in number/proportion) is.

Even if Plato and other ancient philosophers' form of categorization (partitions of what is knowable and what is) is not palatable to many 21st century tastes, relatively recent re-imaginings of it are. Many musicians (and audience members alike) believe that when they are in the creative act there is a blurring between how they experience the music and how the music manifests—these artists feel (and are deemed so by certain fans) to be "tapping into something" that is then embodied in their inspired music.[5] Without asking you the reader to ascribe to this monist-leaning view, I want to suggest that you explore it more deeply—directly. What do those proportions mean to you specifically? What are the many shortcomings that you find in regards to the privileging of these proportions when seeking to align this with your own musical practice? As the book progresses, there will be bucketloads of similar questions about consciously relating, not just passively accepting and or dismissing, other models for or ideas about music. These types of questions (mediated later through set classes) are critical questions that this pedagogy wants you or the student to focus on; not to find a right answer, but to become more nuanced and personal in your response.

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[5] This type of conceptualization, discovering the world (or seeking truth) in our means of comprehension, will be discussed further in the section dealing with Kant, Hegel, Adorno, and others of that ilk.

2.

Aristoxenus' gamut, the Greater Perfect System, was also built around Plato's scale. However, his partition and interpretation of it differed. Aristoxenus focused more on how performers engaged his gamut than on how it was a testament to the cosmos' inner workings. One outcome of this orientation was the attention given to tetrachords. Akin to Aristotle's categorizing tendencies, from the 4 types of causes to ethical behavior and sentence construction, Aristoxenus classified the various performed tetrachords (4 adjacent pitches) into 3 genera, *enharmonic*, *chromatic*, and *diatonic*; *enharmonic* contains two quarter tones, and a major third (roughly), *chromatic* contains two semitones and a minor third, and *diatonic* was directly seen in Plato's inherited gamut — two 9:8 whole steps and a 256:243 half step. Unlike the named notes and ascribed proportions of the gamut (greatly extended by the Harmonicists) Aristoxenus's genera were treated in an approximate matter — it was the prototype so to speak for how performers framed various registers of the derived gamut. Where Plato introduced one construction of musical material, Aristoxenus (and his followers) introduced at least two (a hybrid) or a superimposition of performers' conception on a philosopher's.

This prompts another insight into a significant tenet of this pedagogy; be suspicious of speculative, rationalist, theories that, at least in the arts, dismiss their empirical instantiation. However, do not in turn dismiss those speculative theory's capacity to inspire creative endeavors. In efforts to reconstruct and imagine Greek music performance, Aristoxenus's efforts have provided more fruit. Aristoxenus not only attested to music's affect, associations between meaning and pitch selection, but also to shifts in Tonoi (transpositions of the gamut).

For instance, singers have a variety of ranges that cannot necessarily accommodate fixed, pre-established, pitches. If there are various singers in a performative event, it may be important to point out how the new transposition of the gamut relates to the recently tuned instrument's fixed pitches. Furthermore, when melodic ranges that span a 4^{th} (tetrachords) take precedence, it raises questions about how significantly the starting pitch in Platonic tuning practices (what we would call the tonic today) actually figured into the performer's imagination. Once the pitches were fixed on a particular instrument, the strings tuned, they like the instrument itself could act as a tool for other performative and even ideological intentions.

Naming the various tonoi is a building on and enriching of the speculative terms that Plato offered earlier. The naming the tonoi (classifying the genera) is one of a plethora of examples of how more discriminating terminology can assist practitioners (and incidentally future musicologists). While there may be diminishing returns on seeking to name every possibility in some musical system, there is a benefit to continuing to question the boundaries separating what is worth being given its own name from what isn't. What is the price paid when seeking to lessen current terminology's overburdening (in terms of having too many associations)? What is the price paid in terms of analysis vs. that of composition and/or improvisation? They are not necessarily the same. Terminology shifts may simplify a composer's thought process and simultaneously obscure an analyst's attempt to translate the composer's intentions; or worse yet, continue to propagate the myth that the analyst and composer should share the same terminology and/or that the composer's terminological schema should be deemed an *a priori* given of that same composer's musical conception. As Aristoxenus problematized Plato's scale (the proportions used were derived in the context of performed tetrachords rather than fixed absolutely), and thereby enriched it, we should problematize and then enrich our conceptions of set classes.

While our usage of set classes will eventually be more general, they still were born in a context wherein pitches were discrete entities, equal temperament was assumed, and wherein, in non-serial music, simultaneities of pitches were deemed as chords, harmonic units, rather than contingencies (possibilities) that could arise out of a more horizontally-conceived musical landscape.[6] However, in seeking to fix the pitches of keyboard instruments, dating from the late 16th century, neither equal temperament, our present layout of 7 white and 5 black notes, nor even the rationale for naming chords independently of their intervallic content was obvious.

3.

On the other hand, In a 16th century polyphonic texture, it is common, especially near cadences, to notice a dominant triad sounding with "its 7th" also occurring through passing motion in

[6] Nevertheless, serial music does typically presuppose that harmony unity can be found through the coordination of multiple iterations of rows, melodic statements. In this context, harmony could be conceived as "contingencies (possibilities) that could arise out of a more horizontally-conceived musical landscape."

another voice. However, the structural dominant 7th, as a 4-note chord, whose signification and identity is retained through its rotations, was not often (if ever) found. Furthermore, before equal temperament, the various tuning differences, incurred through changing the bass of a chord, and the differing manner in how various rotations of a 7th chord were used in practice, would have lessened the benefit of a essentializing the various rotations and tunings of 7th chord to a structural dominant 7th.

Those craftsmen that initially designed the keyboard layout then had to reconcile current burgeoning musical conceptions, the want to realize a dominant 7th chord, with the limits imposed by current tuning practices. Even though they recognized discrete pitches (as we do now), those pitches were embedded in a music practice that did not accommodate modulation as freely as it is accommodated now. As the tuning of the original key was optimized towards lower-ratio intervals (for instance, 4/3 as opposed to 685/437), going to distant keys exacted a high toll in aesthetic quality. In short, as the modulated key became more distant, more of the notes of that key disproportionally bear the “temperament burden”—the lack of alignment between intervals generated by 2, 3, and 5. No matter the size of n, 3^n [or 5^n] (where $n, m \in N$) will never equal 2^m. In short, no pitches generated the same fundamental and stacking of either M3s $(5/4)^n$ or P5s $(3/2)^n$ will ever equal $(4/2)^m$, that fundamental at higher octaves. This is a simple consequence of the powers of odd numbers (3 and 5) being odd and even numbers (2) being even. Those increasingly modulated scales were then deemed less pleasing, and in actuality were less uniform in their construction.

Vocal music at the time further reflected this—the tonoi could be shifted, but it was usually restricted to a P5th up or down. Within certain keyboard instruments (or organs) they could have different manuals (keyboards), each tuned to a different fundamental. Accordingly, if accidentals were used, they were typically either an additional one or at most two sharps or flats. There was not a practical need (and keyboards were an expensive enterprise) for seriously considering more distant accidentals.

Overall, this reflection on the trials surrounding the discovery and then the establishment of the current near-uniform single keyboard layout is intended to illustrate that there is no neutral representation of musical space. Any representation adopted results out of a negotiation that weighs pragmatic and ideological considerations. Moreover, the establishment process of certain representation often reflects an interaction between the "intellectual," facilitation of certain external (i.e., rendering music of a particular genre) end goals, the convenience afforded by building off of previous technology/methodologies, and changes in expectation (the getting used to) that accompanies familiarization with a new technology. Keyboardists primarily, but even non-keyboardists, associate "all white note" pieces with typically diatonic works that modulate little and genres such as new age and contemporary artists such as Tori Amos—even though virtually no pieces are "black-note" free. In other words, this keyboard layout or representation of musical pitch space has now become a vehicle for categorizing vast swaths of music.

Furthermore, the 16th-century criteria for partitioning pitch space could not have anticipated its present role in discriminating between various musics today. In the 16th century, extensive modulation and fixed pitches were both at odds from a technical perspective and from the aims of music practice—especially as it was accounted for in 16th/17th theory treatises and through examination of the corresponding music literature. Moreover, what in the 16th century was spun positively—enriching the sonic landscape with more resonant and singer-friendly, low ratio, intervals could be spun negatively by those in 20/21st century who associate musical richness with formal complexity—as witnessed in the variety of pitch structures employed and certain types of performance challenges. Again, categories and their signification are continually re-negotiated. The focus of this pedagogy is then to raise awareness of the student regarding the consequences of: one, how they segment ontological space; two, dispose them to considering and either keeping or rejecting many categorizations; and three, as much as possible, help them get better at making those categorization/ontological decisions that best reflect their values and interests.

4.

Another important forebearer to conceptions of set classes were pitch-classes. We now take for granted that various "C"s represent the same "C" phenomenon. One justification that I've often heard for this collapsing of pitch into pitch-class space pertains to the prototypical octave difference between the ranges of female and male voices in adult choirs. However, experience in teaching aural skills has shown me time and again that matching pitches when the teacher is of the opposite sex, independent of age, is found to be a **non**-negligible challenge for many; for instance, soprano ranged singers often have difficulty picking out the bass.

This back and forth here is meant to show that why we categorize different (in terms of octave) representations of a note as being the same is neither obvious nor an expected consequence of biology. Depending on context, different arguments can be made for differentiating between registers. Only when the reasons (in terms of signification) for differentiating registers significantly outweigh those against, may a single way of categorization be reasonably be promoted as "intuitive." Nonetheless, past a certain threshold, such as facial recognition, I posit that it is better to treat the categorization of something as a choice that is enhanced through its continually being contested.

Amongst others, the shift from pitches to pitch-classes could have been aided by: keyboards that visually treated different representations of "C" uniformly; a straight-forward simplification of the solfege system—just truncate that part of its label which locates its place within the Guidonian gamut; and an increasing awareness of chords as distinct entities whose identity remained in spite of their inversion (not intervallic inversion—rotation is a better word).[7] In the early 17th century, Lippius even likened the triad to the Trinity—the root is God; the 5th the Son; the 3rd the Holy Ghost. However, to do this he had to reconcile the change in the various pitches' register as the various rotations were cycled through.

Undoubtedly, Lippius's burgeoning notion of the triad and pitch-class has received so much attention recently due to our fascination with identifying "inventors." However, I also find it significant because it makes explicit that the significance of our triad, which is now considered neutral, at least in terms of theoretical justification, relied on factors and assumptions that would not be considered neutral today. Nonetheless, its theological derivation is a testament to the power that it must have held in Lippius's (and his colleagues') minds. By grounding what we

[7] Boss (Boss 1985) and Gregory Barnett's chapter in Cambridge History of Western Music Theory, "Tonal organization in seventeenth-century music theory,",(T. S. Christensen 2007) wherein Barnett accounts for many of the factors that could have contributed to the adoption/formation of 18th century classical tonal practice.

would now consider impartial material, in religious doctrine, Lippius may have hastened its acceptance as a musical construct, and signaled that in discussing music through the lens of a triad (new terminology) additional religious insight could be yielded. Viewed skeptically though, one could insinuate that as a career professional, Lippius was overly cautious about introducing material that could be misconstrued as threatening to doctrine.[8] However, I would rather cast Lippius's efforts, in the spirit of this pedagogy, as an example of the great dividends that can be yielded from imbuing our own "neutral" musical terms within systems of explanation that can enrich our own understanding, spurring our imagination regarding the potential of our musical resources.

It is true that certain classifications will discourage certain usages of the described material. However, those same classifications can point to unforeseen usages that, emboldened by our increased care and/or attachment towards the material, may result in something quite meaningful to the composer/performer/analyst; and, in doing so, lessen the burden of our finishing some longer musical tasks. Often, great care for a topic can increase motivation towards exploring it. In short, openly exploring a topic's significance by re-classifying, finding alternate explanations for its terminology structure, can also help circumvent "writer's block."

5.

A *Tonnetz* describes pitch (not pitch-class) space in regards to 3 intervals. The most common *Tonnetz* assigns each of a triad's absolute intervals, M3(m6), m3(M6), and P5(P4) to linearly dependent axes. Each area then bounded by the intersection of these axes (in a 2D space) yields a major triad. While any tri-chord can generate a "Tonnetz" mapping, most do not exhibit either of the two most significant properties of the regular *Tonnetz*: transformations yielded by "flipping" over the respective edges of the bounded areas that are: one, parsimonious; and, two, render intervallic inversions. Some *Tonnetze* share the latter property, no others the former. The visual simplicity and straight forwardness of this visual aid (for the triad) has inspired centuries of theorists. For example: in the 19th century, it was adapted to show, first, how close closely

[8] Or, according to Caleb Mulch (2020), it may have been crafted in the context of justifying it via the public and/or written tradition of debate.

related keys are—in just intonation and equal temperament, and second, the prototypical trajectories of harmonic progressions, conceived dually. In the 20th century, it has been redrawn as a torus, and used to chart out how progressions of triads unfold when parsimony is privileged. As the *Tonnetz* will be explored more extensively later (the first chapter of Part III), only these few (of many) examples are given.

It turns out then that the *Tonnetz* has been a powerful vehicle for explaining the “inner logic” of many tonally ambiguous harmonic passages. In such a way, it has helped fuel that widespread desire, since at least antiquity, to align fundamental, “objective”, properties of music with the culturally determined “fundamentals” of music practice. Even if these “objective” and culturally determined fundamentals cannot be perfectly aligned, it still can be inspiring to find those parallels. As such, I encourage readers to ask and then answer questions related to mappings via their practice. For instance, when you transform a chord uniformly (such as transposing it up a 3rd), what pitch levels are attained and do you view all transpositions equally?

Again, it **can** feel revelatory to notice similarities between your experience of music and some other discipline’s explanatory system. Of course, problems have arisen around certain persons, such as Schenker, asserting that both their experience (and interpretation of it) are uniquely correct and reflect both greater rationality (writ large) and truth-value; problematically invoking another discipline’s tenets (loosely and often incorrectly) as a source of authority. However, that extreme usage of this desire, to find a deep connection between the meaning and form of our statements, is just that, an extreme usage. Milder, less offensive, usages can also spur the imagination.

Along these lines, Covach (T. S. Christensen 2007, 604–7) in his Cambridge History of Western Music Theory chapter, details Josef Matthias Hauer’s philosophy regarding music. He saw it primarily as a mental and or spiritual activity; thus, setting an opposition between music as a performed act (impure) and its pure or ideal (mental) state. Intervals, conceived as colors, are gestures and units in the spiritual plane that they inhabit. A significant concern then regards transmission; how does the listener reconstruct its spiritual essence and import after its performance has defiled it. Namely, suppress its earthly attributes; don’t focus on its visceral components—such as timbre. Furthermore, choose unnatural temperaments (such as equal temperament) as it encourages one to use one's mind to improve upon the impoverished wake of an inherently derelict performance. Hauer also expresses this dichotomy in terms of rhythm and melody. Rhythm, associated with material, is considered more base; melody, associated with the spiritual, is considered more elevated.

So, in one fell swoop, Hauer's framing puts rhythmic African practices and the natural temperament of the melodically-driven Indian classical music on a less than solid footing. Nonetheless, set class theory's focus on interval vectors did arise out of this background (one could make a case that Webern could have taken it to heart; even if his music is also interesting along 'sensuous' lines pertaining to timbre and/or textural patterns)—a reverence for pitch-based structures, associations of purity and spirit with the composer's imaginings, an emphasis on the intervals (as a basic conceptual unit and a key to unlocking its spirit), and a surprisingly Medieval (legacy of St. Augustine) inclination that music—if music is experienced "too sensuously," its more spiritual reception will be threatened. Hence, according to Hauer, atonal (especially twelve-tone) music could be thought of as of a higher order than tonal music.

Viewed today, in the early 21st century, it's hard to defend any aspect of Hauer's philosophy. Rather, a contemporary critic of post-tonal theory—from its explicit theories, to whiffs of inaccessibility, may be tempted to resurrect this unflattering specter as a justification for their misgivings. Yet, I know of no current theorist who would seriously entertain this. I see these overtones as toxic and counterproductive to this pedagogy. While this pedagogy does focus on the mental aspects of music production, it is opposed to attempts either to equate the developed mental techniques with the essence of a given work, or to make claims that this modality of relating to music is in any way privileged. The want of this pedagogy is to prime its adepts to seriously explore more ways of thinking about music—not to insinuate that all but the least sensuous ways of relating to purposely abstract music (at least regarding pitch-structures) can sully you.

In short, I have touted the potential of other backgrounds as a way to positively enrich our understanding of set classes and suggested that not all of the problematic overtones of a particular theory's approach should be deemed sacrosanct and above reproach!

Part I, Section 1, Chapter 3: Towards a Systematic Treatment of the *Grundgestalt*

The following foray into another assortment of set class's historical precedents is both more direct and aimed towards explaining many of its technical features now exploited. Again, a good portion of this overview, at least in its organization, is indebted to the work of a few: in particular, Covach, Nolan, and Bernard.

1.

There is a legacy of letting chance dictate the parameters of one's composition. Imbued with the spirit of the Enlightenment—a felt belief that power of reason and scientific enquiry can explain all phenomenon (positivism), certain compositions became the site for fun, inquiry, and, in a sense, were re-imagined as mechanical contraptions. The basic premise was this: compositions were an ordering of sections and each section could be filled with one of a number of given options. Thus, a piece was transformed into a machine that produces pieces. Just "roll the die" and randomly (or not) choose which of each section's respective candidates you want to use in this particular performance.

Exploiting the well-established formal expectations on structure, such as in genre (let's say a dance work) and phrase structure, these pieces could remain coherent to the listeners in spite of each performance's randomly swapping out sections. There was a regularity to the form of the works that, perhaps donning an 18th century scientific cap, enabled the composer to predict that every possible arrangement that they allowed would be successful. Moreover, it's unlikely that the composer tested each of the potentially one-million combinations (9 sections with 5 options each) permitted. Built into this combinatoric legacy then is an implicit understanding that most important aspects of a piece need not rely on the particulars of each section's embellishment. Furthermore, the function of a piece (its ability to accommodate a potentially very long social engagement with a paucity of learned material), not just its "pleasantness," was a major selling point.

Returning to set classes, one of their most significant selling points is their abstractness—they promise to be an umbrella term for the core components (sections of a work). Furthering the long-standing notion that formal coherency lay in pitch structures, Forte even crafted an analytical methodology that showed how many recalcitrant (to an analyst's gaze) atonal pieces could be reduced to the interaction of two set classes and their complements (the "Kh-relation"). However, the particulars of his theory will be discussed later.

2.

In 1924, Eimert, a young university student, submitted a manifesto for his "systematic atonalism." While he was inspired by Hauer, adopting some of his theoretical dispositions, his motive was not spiritual. Eimert advocates for progressing by melodic aggregates (requiring all twelve pitch-classes to sound before one is repeated) (T. S. Christensen 2002a, 608–9). This restriction is enforced at the piece level (all the sounding pitches), not at the melodic level; as later theorists/composers, such as Schoenberg, often adhered to. Furthermore, as long as the melodic aggregate condition is met, it does not matter how those aggregates are partitioned (broken into chords). Displaying a combinatoric sensibility, Eimert first proceeds to derive the total number of possible aggregates and then describe how many ways those 12 pitch-classes can be distributed through a fixed number of voices—for two voices there are 11; three, 55; four; 165. In his *Topos of Music*, Mazzola explores a similar line of inquiry (Mazzola 2002, 234–38).

As will be brought up in more detail later, there is great benefit to considering constraints on arrangement (a classification scheme) alongside the more abstract constraints on the pitch material alone. Eimert's very large number, 12! (479,001,600)—all potential rows, is framed in terms of the more manageable 165 ways that 12 elements can be partitioned into 4 parts. To the excited early 20th century composer, eager to plumb the depths of possibility and forge the future of music, that relatively simple reduction, offering just a glimpse of manageability, could have anchored their future creative efforts. In short, by attending to arrangement constraints, extremely abstract terms can be better grasped. In general, when possible, it's important to put things in their most manageable forms. As the complexity of the thing being described increases, this is even more critical.

3.

Schoenberg is associated with the term *Grundgestalt*, an organizing principle, basic idea, that shapes an entire work. This has also been described as a blueprint for the work; it prototypically is the initial musical statement in a work. In his books, Boss has made detailed and compelling (and dare I say irrefutable—an admittedly impossible claim for analyses) cases for how single ideas and "problems" embedded within them became the seed out of which an entire piece of Schoenberg's unfolded. Alternatively, when the idea consisted of musical material (like a melody, or row), the piece is typically weaved out of the motivic developments of its components; its overall form is a projection of structures (sometimes extra-musical) embedded (or implied) in that initial material. However, as Boss as also shown, the overarching framework need not be "musical:" in his book on Schoenberg's atonal music, he argues that a visual image (interpreted musically) could act as the blueprint by which a musical *Grundgestalt* can be developed through a piece. Similarly, Covach attests to Krenek's theorizing about the "recession" of the row's "role as a motivic entity on the surface to one that germinates material from the background." Music is "no longer motival, but ... extra- motival (T. S. Christensen 2002a, 616)."

Actually, a want for unity and for a piece's ability to be reduced to a single principle, was a hallmark of 19th century thought. Popper, mentioned earlier, hypothesized that this desire was rooted in a cognitive dissonance (my paraphrase of Popper's argument) partially prompted by Kant. When the credibility of scientific inquiry ceased to extend beyond what was possible to test empirically, God could no longer act as an organizing principle for the discipline. Man could not abdicate responsibility from imprinting knowledge with his own fallible footprint—neither objective data nor theories were free from this tainting. This created a crisis (the aforementioned cognitive dissonance) that later philosophers such as Hegel and Schopenhauer sought to alleviate. Through fancy footwork, redefining logic as the dialectic process, and assertions that artists were privy to intuiting and then manifesting that highest among high organizational principles, various Hegelian inspired philosophers then put tape on a shifting enlightenment world-view—one that did not require grand design nor an ultimate authority to ground pursuits into meaning. In short, theological overtones were immanent to German Idealist 19th century philosophical thought; if not explicitly in references to God, implicitly in its preservation of his role in concept formation. Albeit in a less pronounced form, perhaps these overtones were still felt in the 20th century predilection for formal unity.

An alternative (and/or additional) explanation could be that there are cognitive reasons for seeking unifying principles; it's easier to relate things back to a single idea (grasp the forest) rather than get lost in the many trees. A recent book, The Quick Fix by Jesse Singal (Singal 2022), argues that greater uncertainty in life (such as living one paycheck away from financial ruin) and dissolution over the U.S. Senate's inability to solve larger and more complicated problems, may be contributing to constituents' gravitating towards simple solutions for problems that are not simple. Along those lines, cognitive neuroscience has shown the energy costs associated with overtaxing the working memory. When compromised by long-term stress, it can be challenging to face the impossible challenge of making an impossibly complex world coherent—even with (or in spite of) an overabundance of pundits pointing the way forward.

Either way, for much avant-garde music in the early to mid-20th century, typical rebuttals to criticisms of unintelligibility or difficulty, lay in appeals to formal coherency. Moreover, set classes were introduced as a means to do just that, show harmonic coherency in atonal works. However, this pedagogy is non-partisan in that regard; while it exploits their ability to illuminate larger top-down organizational principles or even act as a *Grundgestalt*, it doesn't reduce their worth to just that; set classes can be used to show dis-unity as well as unity. In other words, acknowledge their historical association with calls for demonstrating formal unity, but only engage those associations that you find most valuable.

4.

This appeal to unity and the best practices of the western art tradition rippled through the 20/21st century. Again, Covach's stellar chapter in Cambridge History of Western Music Theory, tells us how Richard S. Hill (similar to Eimert) responds, explicitly, to concerns born out of music terminology being too abstract and, implicitly, to a critique of Schoenberg's musical incomprehensibility. His solution is dodecaphonic modes. Like the major scale, these modes are enriched by an understanding of how each tone functions in regards to the others.

Hill's offered short list of intended substitutes for all twelve-tone rows could act as a bulwark against the actual disorder that undergirds the asserted order of Schoenberg's 12-tone *dalliances*—Schoenberg (Boss 2014) and later Perle, would invoke tonal strictures as a guiding principle and justification for a subset of 12-tone works. Of course, their reasoning differed:

- from rescuing 12-tone music from oblivion (Hill),
- to exploring a way to project 12-tone coherency and perhaps characterize his music as an evolution of the musical tradition (Schoenberg),
- to asserting the best 12-tone scale (Perle)(Baker, Beach, and Bernard 2009, 8:41).

While composers contested who should be the inheritors of the tonal musical past, this spirit of this pedagogy requests to stay out of the ring. It does not want to advocate for any *universal* proscriptive measures on how a row/set class should be. Instead, this pedagogy asks the learner to center, justify, and develop theoretically *personal* proscriptive measures regarding how **they individually** think musical material should be used. Hopefully, the age of promoting single composers/academics as inheritors and guiding lights of "all of a" cultural tradition's evolution has passed.

The next generation of theorists and composers, the high modernists, experimented with and occasionally (as in the case of Babbitt) espoused theories that sought to ground musical unity in the work's structure. The question changed from what can we use to project unity, to what can't we use. In the case of Babbitt, the answer to the latter question was well-specified—at least in regard to structural (not extra-motival) concerns; one couldn't use anything in the domain of rhythm and pitch that could not be derived from the initial row. Babbitt's *Grundgestalt* was explicitly structural.

In the hands of Babbitt, as 18th and 19th century theorists respectively did, 12-tone theory was draped in scientific terminology and apparel. There was an attempt to get the rigor associated with methodologies in the Natural Sciences—such as Chemistry and Physics, to rub off on this discipline's pursuits. The result was a heavy emphasis on quantitative and deductive methods and explanations. The formality of 12-tone serialism, the abstractness of its terms, the advent of the computer, the serial works' resistance to tonal analytical methods, the structure of prestige in academia and the concert world, etc. contributed to the mid-20th century palatability of this approach. Many of its fruits, especially the mathematical work and the explorations into set classes, including their assignation to more than pitches, has grounded much of the research underpinning this pedagogy. Let me reiterate, though, that my exploration into set classes does not privilege set classes' abstractness, but rather their ability to accommodate the wealth of the student's often very concrete interests.

One example of the mathematical change in orientation is Babbitt's framing of the 12-tone row as a permutational rather than combinational system. When rows are described as permutations, any row can be described as being related to another by one or a fixed number of permutations. As permutations are reversible, "the set of rows" are closed under permutation (there is no way to permute a row that does not yield another row), and "no permutation" can be described as the identity operation, the set of rows (12 tones) and the operation permutation together form a group. A chord can then be viewed as a partition of a row, rather than another formulation that may describe the row as an aggregate of specific chords. There is a musical precedence for this in the sense that when notes of a row are presented simultaneously, the "order of those notes in terms of the row" is obscured. Furthermore, composers such as Hauer and Ralph Shapey did organize their music in terms of constituent partitions; meaning that often no order was specified for the notes within each partition. So, Babbitt's labeling a row as a permutation was both reductive (boiling all transformations down to a single principle), and it facilitated a whole-sale adoption of mathematical group theory as a way to describe 12-tone serial pitch material.[9]

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[9] In his chapter "Chord, Collection, and Set in Twentieth-Century Theory," Jonathan Bernard also gives an accounting of those ideas that were eventually consolidated into Forte's presentation of set classes (Baker, Beach, and Bernard 2009). These ideas were around: using a single collection name to describe harmonic and melodic resources, adopting a classification system that combines both inversional and transpositional equivalence, exhaustiveness, an appeal to group theory, and ordering in regard to the interval vector. The supporting context for this development was the largely the compositional and analytical interests of mid-20th century composers and theorists.
His narrative links:

- Theorists at the turn of the 20th century who, although invested in tonal analysis and literature, still sought to augment their harmonic vocabulary. These included Rene Lenormand, Arthur Hull, Bernard Ziehn, Schoenberg (circa Theory of Harmony). In their writings, voice leading considerations were still weighed.
- Soon following theorists, such as Bacon and Hauer, who were more systematic in their efforts. Bacon actually accounted for all T_n-classes.
- Less systematic theorists, such as Hába, Slonimsky (discussed later), and Cowell, who, although profusely laying out harmonic material, were not motivated to given an exhaustive accounting; instead, their curation appeared guided by musical interests.
- The more systematic, though not exhaustive: Schillinger, Hindemith, and Hanson who are described later.

Part I, Section 1, Chapter 4: Classifying Pitch Collections

The following chapter provides a non-exhaustive overview of 20th-century pitch classification methods. The goal of this chapter is then two-fold: one, to better contextualize Forte's classification system, historically and in terms of competing systems; and two, highlight the benefits of his system—ultimately, making a case for why Forte's labels best serve the aims of this dissertation.

Hindemith's classification of chords

However, in the 20th century, this predisposition towards classification, systematization, and unifying principles did not begin with Babbitt. Other examples include Joseph Schillinger (Part 3 chapter 3) and Paul Hindemith; the first, I will treat later, the last I will discuss now. Unlike Babbitt, Hindemith endeavored to account for the overtone series (he used it to derive the chromatic scale) and perceptions of consonance (and dissonance). Like Babbitt, he saw the chromatic scale as primary.

Hindemith classified chords into 6 groups. The first two groups were devoted to canonic tonal chords (triads and 7th chords); the latter two to chords that contained the tritone and those that contained maximally even sets (namely the dim 7 chord and the augmented triad). Within each group, the specific ordering of the intervals (from bass to treble) and the size of the interval were taken into account when calculating how consonant or dissonant the chord is. In a sense then, his system was also reductive; ultimately, he related everything back to the overtone series and notions of consonance and dissonance that immediately follow.

There was a precedence for explaining cognitive constructs (like consonance) in both the energeticist literature and the practices of the psychological behaviorists. In the energeticist literature, unobserved forces (akin to those alluded to by Newton; for instance, gravitational) shaped our perception of movement in the "magnetic field" generated by a fundamental and its

- The most systematic: Forte, Perle, and even Babbitt—also described later.

overtones. Similarly, cognitive/dissonance theory sought to ameliorate the downside of strictly behaviorist approaches, like that of Skinner, who endeavored to explain all cognitive constructs in terms of learned/conditioned observable behavior.[10] Alternatively, cognitive dissonance is a principle offered to explain unobserved cognitive behavior in terms similar to the biological sciences—disruption vs. homeostasis. In other words, it appeals to the rigorous findings of the biological sciences; its own establishment does not necessitate strict reference to data culled through learned conditioning experiments; and, cognitive behavior is still reduced to biological processes.

The Interval Vector

At minimum, interval vectors were a byproduct of this systematization effort and the musical and philosophical rationales provided above. Howard Hanson, a mid-20th-century music theorist whose 1960 book is a "kindred spirit" to this pedagogy, wrote that the "advantage of equal temperament is the simplicity possible in the symbolism of the pitches involved" (p.2). By extension, interval vectors, and later set classes, can also be described as simplifying (or clarifying) various complex harmonic (and other) concepts. The first slot of Hanson's "interval vector" is labeled d, the second, s; the third n, the fourth, m; the fifth, p; and the sixth, t. Lewin, through his EMB function, now substituting 1 for d, 2 for s, 3 for n etc., extends this notation to name the trichords, tetrachords, etc. contained in a set class. For example, a tetrachord contains 4 distinct triads. The "trichord" vector of that tetrachord, would list the number of occurrences (in this case adding up to 4) of each of the 12 potential trichords. My pedagogy also follows this notion; the power of musical terminology is more than its literal representational power; it lies in its ability to inspire certain connections and notice relevant patterns.

David Lewin presented interval vectors, in the format known today, in his 1959 JMT article. There he fleshed out two critical properties of interval-vectors: complementarity and Z-relations. Regarding complementarity, there is a relationship between the interval vector of a set class and its complement: the difference between the two vectors is equal to the difference in cardinality

[10] A well-known behaviorist experiment involved rats, who were conditioned through electric shock treatment to respond positively or negatively to certain stimuli (Skinner 1938).

(except for the tritone, whose value is halved). An immediate insight offered by this is that (with only one exception, 5-Z12* & 7-Z12*) a set class is contained in its larger complement. In other words, complements typically share many structural features and often the smaller set class is a good way to gain insight into the larger. On the other hand, Z-relations refer to distinct set classes that share the same interval vector. Outside of hexachords, this is a rare occurrence. This property immediately indicates shared structural similarities between two set classes that may be manifested in music very differently.

In his article, "Modalism — a "third" world, Vieru focuses on the various ways in which the very special set class 5-22*, (01378), can be recombined with itself at various transposition levels to yield the literal complement of another one of its transpositions. Vieru, analogous to other theorists, such as Perle, compensates for a perceived gap felt in the wake of reductions of harmonic space to interval vectors (and later set classes); transposition, often an important agent of harmonic organization, gets overlooked. An interval vector doesn't distinguish between an ascending or descending 2^{nd}, most music does. These concerns and others, as well as comparisons of Forte and Hanson, will be addressed in more detail as an "introduction to the second part of the dissertation." It will be fruitful to use comparisons between various approaches as a means to contextualize this pedagogy's extra efforts.

<u>Various pc-set/set class orderings and the rationales</u>

While this book will focus on Forte's ordering of set classes, his are not the only options out there. Furthermore, without knowing exactly Forte's motive for his ordering, it seems that he used interval vectors and an "alphabetical ordering" as his guide—the first set class (of a given cardinality) has the most occurrences of the smallest interval (the minor second). If two sets share the same number of smallest intervals, then the greatest number of next smallest intervals, settles which comes first. For instance, sc 3-1's interval vector is <210000>, sc 3-2's interval vector is <111000>, sc 3-3's interval vector is <101100> … etc. An issue arises with Z-related set classes; not transpositionally- or inversionally-related set classes that share the same interval vector. Forte lists these after one example of each type of interval has been listed. For example, there is only one Z-related pair amongst the set classes of cardinality 4. One element of this pair, sc 4-Z29, is listed after the last "alphabetically speaking" set class, 4-28* (the asterisk means that the transpositions of the set class represented are equivalent to the intervallic inversions). So why does one of a Z-related pair precede its counterpart? It seems that three factors weigh in, in this order of priority: 1, cardinality; 2, whether the set is symmetrical; and 3, span of the prime forms.

1. If the set's cardinality, x, is greater than 6, refer to the ordering of its complement cardinality, 12-x.
2. If one of the Z-related pair, such as is the case with sc 5-Z12*, is symmetrical, and its counterpart, sc 5-Z36, is not, the symmetrical set class, no matter the size of its span, precedes its non-symmetrical counterpart.
3. If the span of the prime form of one of the Z-related pair, for instance sc 4-Z29 (0137), 7-0, is bigger than the span of the prime form sc 4-Z15 (0146), 6-0, than the shorter span goes first.

When there are more than one pair of Z-related pairs, the "left-overs" adopt the same alphabetical ordering as their earlier counterparts. Again, I'm not asserting that my rationale for Forte's ordering is exactly Forte's rationale. However, if you choose to reconstruct it on your own, this should suffice.

What is a relatively simple and intuitive way to make sense of Forte's ordering, though? Essentially, set classes with more instances of the smaller intervals go first. So, the first set class of any cardinality is comprised of many half-steps. Put alternatively, the set classes that have very few or none of the smallest intervals (a half-step) have higher numbers. So, expect the most popular set classes, like that which contains the major scale, to have one of the highest, of its cardinality, set class numbers. Up until the number of distinct interval vectors have been exhausted, this (alternatively put) trend is followed. When attending to the "left-overs," the same trend applies. Finally, with the exception of the hexachords, the number of "left-overs" is quite small.

Larry Solomon and John Rahn have also commented on Forte's ordering. I'm fully adopting, as has already been demonstrated, Larry Solomon's amendments; adding an asterisk when a set class contains a single symmetrical component scale, and 'a' and 'b' as way to distinguish the two-symmetrically-related component scales of a set class. The prime form receives the 'a'.

Guerino Mazzola also provides a classification of chords. Again, my explanation of Mazzola's ordering may or may not coincide with his reasoning. Nonetheless, it should still enable you to reconstruct the ordering on your own.

Like Forte, Mazzola numbers set class (really M_5 classes) in accordance with their cardinality; e.g., the 3-note set classes come before the 4-note set classes. Unlike Forte, he neither gives an explicit label (like sc 7-32) to set classes greater than 6, nor does he restart the ordering at each shift in cardinality; therefore, when eyeballing one of his chord labels, let's say 28, only previous knowledge, or a reconstruction of the entire set, will indicate its potential cardinality to you.

Like Forte, one of his numeric labels includes a scale and its inverse. Unlike Forte, he associates the M_5 transform of a particular scale with the same numeric label.[11] He does this by adding a ".1." For instance, Forte's sc 4-1* is (0,1,2,3). In Mazzola's classification system, this is **17**.[12] On the other hand, in Forte's system, the M_5 transform of (0,1,2,3) is (0x5, 1x5, 2x5, 3x5) = (0, 5, 10, 15) = (0, 3, 5, 10); or, in prime form, (0, 2, 5, 7)—Forte's sc 4-23*. In Mazzola's numbering, this same M_5 transformation is named **17.1**.

Like Forte, Mazzola organizes his numbering systematically. Unlike Forte, he does not base it on the interval vector. Instead, he "alphabetizes" the transposition-level-0 of pitch-class sets themselves. He does this at multiple levels;

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[11] Mazzola's classification system bundles together pc-sets that are equivalent under canonic musical transformations. The impetus for this is to make transparent all musically-relevant endomorphisms—maps that send an element back to itself—that are associated with a given pc-set. For instance, {C,D,E}, **13**, maps to itself {C,D,E} under the following endomorphisms {1,7, -1e^8, 5e^8}. These are M_1 {1}, M_7 {7}, 3 x TI_8 {-1e^8}, 3 x M_5 {5e^8}.

[12] How 4-1* = **17**; (Mazzola numbers are given in bold). 17 = 16 (<u>1</u> + <u>1</u> + <u>5</u> + <u>9</u>) + 1

- <u>1</u> set class of cardinality 12 [**1**] +
- <u>1</u> set class of cardinality 1 [**2**] +
- <u>5</u> set classes of cardinality 2 [**3**, **3.1**, **4**, **5**, **6**, **7**] +
  - **3.1** = M_5(2-1*) = 2-5.
- <u>9</u> set classes of cardinality 3 [**8**, **8.1**, **9**, **9.1**, **10**, **10.1**, **11**, **12**, **13**, **14**, **15**, **16**]
  - **8.1** = M_5(3-1*) = 3-9*; **9.1** = M_5(3-2) = 3-7; **10.1** = M_5(3-3) = 3-11.
- So, 4-1* = **17**

- The first level is the cardinality's greatest chromatic cluster;
- The second level contains the minimal supersets of the cardinality-minus-one's greatest chromatic cluster;
- The third level contains, not yet accounted for supersets of the cardinality-minus-two's greatest chromatic cluster;
- etc.

In his demonstration of 4-note set classes:

- the first level is the "greatest chromatic cluster," (0, 1, 2, 3). As just mentioned, sc 4-1* = **17**, and its M_5 counterpart, **17.1**.
- the second level contains the supersets of cardinality 3's "greatest chromatic cluster," (0, 1, 2) or **8**. These are **18** (sc 4-2), **19** (sc 4-4), **20** (sc 4-5), and **21** (sc 4-6*) and their respective M_5 counterparts **18.1**, **19.1**, **20.1**, and **21.1**.
    - In numeric form these are, sans repetition: (0,1,2,4), (0, 1, 2, 5), (0, 1, 2, 6), and (0, 1, 2, 7). Remember, as (0, 1, 2, 8) is the inverse of (0, 1, 2, 7), it does not receive an additional number.
- the third level contains the supersets of cardinality 2's "greatest chromatic cluster" (0, 1) or **3**. These are **22-33** and their respective M_5 counterparts **22.1**, **25.1**, and **28.1**.
    - In numeric form these are, sans repetition those "chords" that contain:
        - (0, 1, 3) — such as (0, 1, 3, 4) and then (0, 1, 3, 5) — and then, when exhausted, those, yet unaccounted for, "chords" that contain
        - (0, 1, 4) — (0, 1, 4, 5) and then (0, 1, 4, 6).
- This multi-leveled process continues until all of the "chords" of any cardinality have been addressed.

To sum up: the most notable differences between the classification systems are that:

- Mazzola's numbering system does not make cardinality explicit.
- Mazzola does not make Z-relations evident. In other words, the phenotype of the interval vector, does not influence his classification.
    - It turns out that in the vast majority of cases though, the Z-relation is actually a reflection of an underlying M_5 transformation existing between it and its Z-related pair. Therefore, it is typically subsumed in the affixing of ".1" to another number.

In the case where a Z-related pair are not M_5 related (Forte's sc 5-Z12* and sc 5-Z36) each gets its own number (correspondingly 55 and 44).

- Each chord label contains a lot more stuff; not only inversion, but also M_5 relations and the complement.

The next ordering system I will examine is Elliot Carter's. Elliot Carter's ordering developed over many years. Spurred by compositional interests, he would, as a pre-compositional exercise, explore a set class of a particular cardinality exhaustively. This included showing the ways that those various set classes could be decomposed into smaller units of interest. Actually, he seemed to have brought more systematic efforts towards the illumination of set class decomposition than towards his basic ordering of the set classes. By his own admission, in conversation with John Link, he acknowledged that his ordering rationale was not consistent between cardinalities and that in certain cases, compositional interests guided his inquiry.

"You must realize I never thought about this book as becoming something pedagogical, because it isn't in what I would have considered a really logical order. The basic thing that I was unhappy about, for instance, was that the chords that contain semitones are numbered in different ways: the chord of five semitones is not numbered the same way as the chord of three semitones (Carter, Hopkins, and Link 2002, 29)."

However, systematic his efforts were overall though, greater organization could be witnessed in his (de)composition rather than ordering efforts.

Like other classification systems, Carter includes inversely-related scales under the same number. Unlike other systems though, his numbers are encased within icons; 2-D equilateral polygons designate the size of the cardinality. For instance, all of his 3-note set classes' names lie within a triangle. When the polygon, due to increasing cardinalities, gets unwieldy, he uses a circle icon instead and "raises it to the power of the cardinality" by affixing the cardinality's

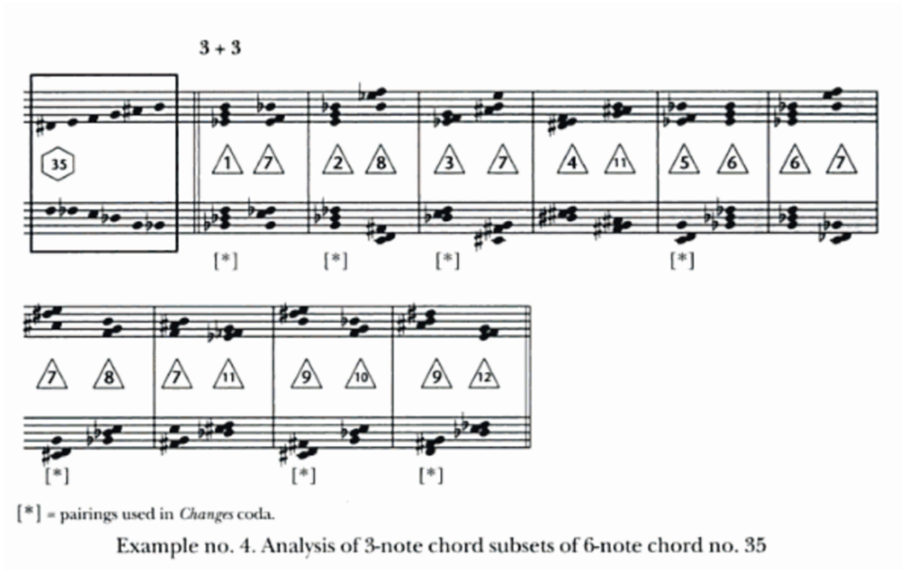


Figure 1.4

number to the upper outer right-hand side of the circle. Above (Figure 1.4.1) is an example of his notation; treble is the clef for both staves (Carter, Hopkins, and Link 2002, 14):

His ordering of the 2/10 set classes is the same as Mazzola's and Forte's; the rest diverge. Like Mazzola, who uses a single graphic of 12 horizontally adjacent small circles to relate a set class (the filled in circles) to its complement (the empty circles), Carter demonstrates each set class by using a barline to isolate the set class in question from its literal complement. For the 3/9-note set classes, he begins with the symmetrical set classes and ends with those that have two inversely-related component scales. For the rest, I, like Carter it seems, am unable to find a systematic reasoning behind the ordering.

Finally, I will look at Schillinger's listing of scales in Kaleidoscope. Unlike any of the other systems shown here, Schillinger does not engage set classes. In other words, symmetry is not a grouping variable—inversely-related scales are not linked together. Furthermore, he is neither exhaustive, attentive to complements, nor makes any reference to either 6-, 8-, nor 10-note scales.

What he does do is systematically show all possible scales (of a particular cardinality) that contain various pc-sets: the fixed pc-sets of the 3-note scales are the dyads; the fixed pc-sets of the 5-note scales are the trichords; the fixed pc-sets of the 7-note scales are the tetrads; and, the fixed chords of the 9-note scales are the pentads. Unlike the derived scales though, his selection of the fixed pc-sets is neither exhaustive nor chosen (at least as far as I can discern) systematically. Instead, it seems that musical interest guides the inclusion of the various fixed pc-sets. Plus, there is the further stipulation that the cardinality 3 fixed pc-sets do not contain chromatic tones. Also, curiously, he doesn't count the difference between the penultimate scale degree (when it's a major 7^{th}) and the octave as a chromatic tone; this is due to his scales not containing the octave. In such a way, the various derived scales are the fixed pc-sets with "the half steps filled in."

There is an indication that the ordering of the fixed pc-sets in a lower cardinality inspires the ordering of the fixed pc-sets of a higher cardinality. Moreover, the manner in which this occurs seems important; for instance, his first 4-note fixed pc-sets are enlargements of his first 3-note fixed pc-sets. Since the first 3-note fixed pc-set is CDE, the first seven 4-note fixed chords are CDE-F, CDE-F#, CDE-G, CDE-G#, etc. Since the second 3-note fixed chord is CDF, the next 4-note fixed chords are CDF-Gb, CDF-G, CDF-Gb, etc. This is not done consistently throughout though. At a certain point, it seems that that the size of bottom interval in the fixed 3-note chord takes precedence as an organizing force.

By his methodology, it appears that he doesn't see rotation as an important organizing force. While he may get through all of the scales from a particular cardinality, it's clear that their understanding as embellishments of a fixed chord take precedence over the prominent classifications of the other systems. Furthermore, from the perspective of the number of fixed chords, there is a substantial amount of redundancy in the presentation.

Forte bundled all possible 7-note scales (in 12-tet) into 38 categories. If the means of classifying the chords does not hold intervallic inversion invariant, there are 66 categories in total. Schillinger, on the other hand, bundles all of the 462 possible 7-note scales/modes into 56 fixed 4-note chord categories. However, there are only 43 distinct 4-note pc-sets.
Nonetheless, Schillinger does exhaust the 7-note scales/modes and that is quite the feat. Very likely, he could have worked out the 7-note scales first and then chose 4-note fixed pc-set labels

that best accommodated them. Nonetheless, if, for a given cardinality, the number of categories could be diminished and/or more systematically organized this would be an absolutely stellar classificatory system. However, let me not devalue his judgement to prioritize musical concerns over combinatoric concerns when choosing his classificatory labels. Within certain pedagogical contexts, his choice is not only appropriate but desired.

Schillinger's classificatory system may act as a beneficial practice aid when you are engaging the aural component of this pedagogy. I agree that one of the most effective ways to make sense of the larger set classes is through the smaller fixed-chords that frame them.

In summary, while the various contenders above have advantages, Forte's system was chosen for its lack of extra-musical overhead (e.g., unlike Hindemith's system, it does not make assertions about consonance), its being systematic (unlike Schillinger and Carter's systems); its making the cardinality of the set class explicit (unlike Mazzola), and its bundling scales and their inverses together under the same label (unlike Schillinger).

Part I, Section 2: Composition/Pedagogy/Math Texts Overview

Part I, Section 2, Chapter 5: Roots in Atonal Theory and Pedagogy

This chapter comprises the bulk of the “literature review” and it focuses on post tonal theory and pedagogy. However, this area of music theory is but one of many that influenced this dissertation. Many facets of current mathematical music theory and cognitive science — and a host of other related topics — were also critical towards the formulation and execution of this dissertation. Nonetheless, for succinctness, many of those topics and theorists have been moved to a series of appendices A - E. At the end of this section, I will provide an additional glossary, that, by area of music theory, discusses terms and concepts later referenced. Treat the appendices as a background for many of the ideas presented in this dissertation and as a resource/reservoir of ideas to employ/explore in conjunction with the pursuit of this dissertation’s goals.

<u>Josef Matthias Hauer</u>

Hauer, when partitioning the aggregate into unordered hexads, enumerated many of the T_n classes of the six-note set-classes.

<u>Oliver Messiaen</u>

In his composition manual detailing his composition techniques (Messiaen and Satterfield 1956), Messiaen introduces seven modes of limited transposition. As this dissertation treats set classes (as scales as well as unordered pitch collections), these modes are treated as scales as well as unordered pitch collections. All of these modes are symmetrical and have the trait that the size of the set containing all of these modes transpositions is less than 12.

- Mode 1 is the whole tone scale <0,2,4,6,8,t>, a representation of sc 6-35*.
- Mode 2 is the octatonic scale, <0,1,3,4,6,7,9,t>; a representation of sc 8-28*.

- Mode 3 consists of three distinct augmented triads <0,2,3,4,6,7,8,t,e>; a representation of sc 9-12*.
- Mode 4 consists of two tetrad clusters separated by a min 3 <0,1,2,5,6,7,e>; a representation of sc 8-9*.
- Mode 5 consists of two trichord clusters separated by a maj 3 <0,1,5,6,7,e>; a representation of sc 6-20*.
- Mode 6 consists of two diatonic tetrachords (W,W,H) separated by a tritone <0,2,4,5,6,8,t,e>; a representation of sc 8-25*.
- Mode 7 consists of a tetrad and pentad cluster separated by a whole step <0,1,2,3,5,6,7,8,9>; a representation of sc 10-6*.

Interval Vector

Howard Hanson, a mid-20th-century music theorist whose 1960 book is a "kindred spirit" to this pedagogy, wrote that the "advantage of equal temperament is the simplicity possible in the symbolism of the pitches involved" (p.2). By extension, interval vectors, and later set classes, can also be described as simplifying (or clarifying) various complex harmonic (and other) concepts. The first slot of Hanson's "interval vector" is labeled d, the second, s; the third n, the fourth, m; the fifth, p; and the sixth, t. Lewin, through his EMB function, now substituting 1 for d, 2 for s, 3 for n etc., extends this notation to name the trichords, tetrachords, etc. contained in a set class. For example, a tetrachord contains 4 distinct triads. The "trichord" vector of that tetrachord, would list the number of occurrences (in this case adding up to 4) of each of the 12 potential trichords. My pedagogy also follows this notion; the power of musical terminology is more than its literal representational power; it lies in its ability to inspire certain connections and notice relevant patterns.

Earlier Forte

In his 1963 article (Forte 1963), Forte examines how the dyad {7,e} can be used to generate material that is helpful in the analysis of Schoenberg's op 19. Here, Forte looks at the set of pitches that are equidistant from either of elements of {7,e} — a pitch class set, not a set class.

He identifies these sets as “A sets.” He then defines “B sets” that arise from applying different set operations (inclusion, intersection, and union) to the “A sets.” Here we see a usage of set class (in the mathematical sense) relations that bears an affinity to the way in which set classes are utilized in the pedagogy; just as often as not when discussing the subsets of the various set class, pitch class sets rather than set classes (in the musical sense) are being referenced.

John Rahn’s Basic Atonal Theory

Rahn’s 1980 introduction to atonal theory builds extensively off of the topics covered in Forte’s The Structure of Atonal Music. Since Forte’s book will be covered later in close detail, this introduction to Rahn will focus on its tone and suppositions. Rahn roots the techniques covered in either the practices of the composer or the seeking-to-be-informed listener. For instance, in his first chapter, he demonstrates how the notion of symmetry can act as a key to understanding Webern’s Symphonie Op. 21. Over the rest of the book, he defines set classes, related terminology, and provides tools helping a composer apply (and then understand the results!) of applying the canonic transformations (T_n, I, M_5) to pc-sets (set classes). In line with common compositional aims of the time (harmonic coherency), there was a professed want to assess how similar the transformed material was to the starting material.

Unlike this dissertation, Rahn does not ask the student to internalize any of the set classes. Instead, the focus is on the taught techniques, introduced theorems, the interval vector, and how they together can enable the student to manipulate any arbitrary set class towards common desired compositional ends. In other words, the craft of the composer is centered on on how they can manipulate any material, rather than the initial choice of harmonic material.

While this dissertation also aims to help the student manipulate set classes, it also sets out to help the student more quickly get inroads towards “understanding” — in a manner that could rival their appreciation of the major triad — an arbitrary set class.

Hanson

In many regards (not all!) this dissertation is of a similar spirit and indebted to Hanson's "The Harmonic Materials of Modern Music." He too saw pitch sets as more than a detached object of study; he, too, created a pedagogy to hasten their internalization. Along these lines, near the end of his book, Hanson makes the following comment:

"The student who has worked his way slowly and perhaps painfully through the preceding chapters cannot fail to be impressed, not only with the vast number of possibilities within the chromatic scale, but also with the subtleties involved in the change or the addition of one tone. He may feel overwhelmed both by the amount and the complexity of the material available to him in the apparently simple chromatic scale, and wonder how any one person can possibly arrive at a complete assimilation of this material in one lifetime.

The answer, of course, is that he cannot. For if a composer were to have a complete aural comprehension of all of the tonal relationships here presented, he would know more than all of the composers of occidental music from Bach to Bartok combined. This would be a formidable assignment for any young composer and should not be attempted in a one-year course!

The young composer should use this study rather as a means of broadening his tonal understanding and gradually and slowly increasing his tonal vocabulary. He may find one series of relationships which appeals to his esthetic tastes and set about absorbing this material until it becomes a part of himself. He will then speak in this "new" language as confidently, as naturally, and as communicatively, as Palestrina wrote in his idiom, providing, of course, that he has Palestrina's talent (Hanson 1960b, 347)."

With only a few subtle amendments, I would say that his sentiment mirrors mine. In terms of change:

- I would emphasize the word 'tonal' less;[13]
- I would emphasize more the student's capacity to acquire a finely-honed sense of how different harmonic resources interrelate;
- I would suggest that the acquired knowledge can apply to more than harmonic resources; and,

[13] Even though Hanson seems to be using the word tonal in a less restricted sense, it still carries too many overtones of 18th and 19th century harmonic practices. That being said, I too recommend that one begin this study by focusing on tonal-associated material.

- When claiming that one can never gain "a complete aural comprehension," I would cite limits in processing ability and inherent ambiguity,[14] rather than inability to exceed "all of the composers of occidental music from Bach to Bartok combined."

Over the course of his book Hanson generates all of the set classes. He does this primarily through:

- "Projection of the [various intervals]:" the Hanson algorithm that Quinn distilled (Quinn 2006, 128–29)};
- The superimposition of chords; e.g., F7 + D Maj.
    - Sometimes the superimpositions involve the same chord at different transposition levels.
        - first, at transpositions levels found in the chord—e.g., C Maj + T_5(C Maj)—a P5 is in a Maj triad; lastly, at "foreign" transposition levels;
    - At other times, different chords.
- "Projection by involution," (e.g., Perle's cyclic array) as chords built from the tonic and pairs of consecutive elements of an interpolated ascending and then descending interval series—e.g., 5-35* (C, G-F, D-B_b), 5-34* (C, G-F, A-E_b)
- Finally, he explores the relation between chords and their complements.

Below (Figure 1.5.1) is the ordering of set classes (using Forte's names for them) that he presents at the end of the book; this ordering reflects the order in which these set classes of various cardinalities first appeared in his book.

[14] Context impacts our perception of harmony and it's impossible to know all potential contexts.

1. 12-1*	1. 11-1*/1-1*	1. 10-5*/2-5*	1. 9-9*/3-9*	1. 8-23*/4-23*	1. 7-35*/5-35*	1. 6-32*
		2. 10-1*/2-1*	2. 9-7/3-7	2. 8-22/4-22	2. 7-23/5-23	2. 6-33
		3. 10-2*/2-2*	3. 9-11/3-11	3. 8-26*/4-26*	3. 7-27/5-27	3. 6-Z25/6-Z47
		4. 10-3*/2-3*	4. 9-4/3-4	4. 8-14/4-14	4. 7-29/5-29	4. 6-Z26*/6-Z48*
		5. 10-4*/2-4*	5. 9-1/3-1	5. 8-20*/4-20*	5. 7-14/5-14	5. 6-1*
		6. 10-6*/2-6*	6. 9-2/3-2	6. 8-1*/4-1*	6. 7-20/5-20	6. 6-2
			7. 9-3/3-3	7. 8-2/4-2	7. 7-1*/5-1*	7. 6-Z3/6-Z36
			8. 9-6*/3-6*	8. 8-3*/4-3*	8. 7-2/5-2	8. 6-Z4*/6-Z37*
			9. 9-10*/3-10*	9. 8-4/4-4	9. 7-3/5-3	9. 6-35*
			10. 9-12*/3-12*	10. 8-7*/4-7*	10. 7-4/5-4	10. 6-34
			11. 9-5/3-5	11. 8-21*/4-21*	11. 7-5/5-5	11. 6-22
			12. 9-8/3-8	12. 8-11/4-11	12. 7-6/5-6	12. 6-21
				13. 8-10*/4-10*	13. 7-33*/5-33	13. 6-9
				14. 8-28*/4-28*	14. 7-34*/5-34*	14. 6-8
				15. 8-27/4-27	15. 7-24/5-24	15. 6-27
				16. 8-12/4-12	16. 7-9/5-9	16. 6-Z50*/6-Z29*
				17. 8-13/4-13	17. 7-8*/5-8*	17. 6-Z13*/6-Z42*
				18. 8-18/4-18	18. 7-31/5-31	18. 6-Z23*/6-Z45*
				19. 8-17*/4-17*	19. 7-32/5-32	19. 6-Z49*/6-Z28*
				20. 8-19/4-19	20. 7-25/5-25	20. 6-20*
				21. 8-24*/4-24*	21. 7-16/5-16	21. 6-31
				22. 8-9*/4-9*	22. 7-10/5-10	22. 6-16
				23. 8-25*/4-25*	23. 7-21/5-21	23. 6-15
				24. 4-Z29/8-Z29	24. 7-30/5-30	24. 6-14
				25. 8-Z15/4-Z15	25. 7-13/5-13	25. 6-Z19/6-Z44
				26. 8-16/4-16	26. 7-26/5-26	26. 6-7*
				27. 8-8*/4-8*	27. 7-Z17*/5-Z17*	27. 6-30
				28. 8-6*/4-6*	28. 7-Z37*/5-Z37*	28. 6-18
				29. 8-5/4-5	29. 7-22*/5-22*	29. 6-5
					30. 7-7*/5-7*	30. 6-Z38*/6-Z6*
					31. 7-15*/5-15*	31. 6-Z41/6-Z12
					32. 7-19/5-19	32. 6-Z43/6-Z17
					33. 7-28/5-28	33. 6-Z46/6-Z24
					34. 7-11/5-11	34. 6-Z10/6-Z39
					35. 7-Z18/5-Z18	35. 6-Z11/6-Z40
					36. 7-13/5-13	
					37. 7-Z36/5-Z36	
					38. 7-Z12*/5-Z12*	

Figure 1.5 1

Later Forte

Had I not been introduced to Forte's Structure of Atonal Music, this proposed pedagogy would have been much smaller in scope. His numbering of set classes, clear exposition, rigor, range of ideas, and near absence of technical errors (perhaps I found one) gave me the solid foundation and inspiration that I needed to anchor my deeper dives into alternative descriptions of set classes and then develop methods by which to more quickly learn them. While there may have been other books that could have added jet fuel to my nascent inquiries, this was the one that did it. I am indebted to it. Furthermore, beyond the set class machinery that he introduced, I subscribed to the book's tone, open-ended disposition, and applauded the sheer number of learning/theoretical resources that he introduced. In a nutshell, he provided a good pedagogy that at minimum aided me, the student, in both a better understanding of a given topic by: providing effective tools that simplified it, quickening associated tasks, **and** enabling exploration into related topics yet to be fleshed out. I aim for this pedagogy to do the same.

Forte's book is divided into two parts; the first part introduces set class machinery (he calls them pitch class sets); the second part provides an analytical framework (based on subsets and supersets) for making sense of pitch relations in atonal music.

- In sections 1-1.5, he introduces set classes.
- In sections 1.6-1.8, he introduces interval vectors and hones in on those whose set classes are prominent in the literature: those that display the maximum occurrence (for a particular cardinality) of a given interval (for instance, 7-1*), those with the deep interval property — scales that contain each interval class a unique number of times (like 7-35*), and those with the all-interval property (4-Z15, 4-Z29).
- In 1.9, he introduces Z-relations (two distinct set classes that share the same interval vector).
- In 1.10-1.12, he defines subset and super sets; discusses the tabulation of subsets (or supersets) of a given set class; and then, discusses subsets (of a given set class) that are invariant under transposition—in short, the set class's interval vector shows, how many pitches are held constant by a given transposition. Lastly, he does this for subsets invariant under inversion.
- In 1.13 he defines his, discussed earlier, similarity relations; R_p, R_0, R_1, R_2. (Forte 1973, 48–50), which he then relates to binary relations.[15] A **transitive n-tuple** is a collection of

[15] Definition of binary relations:
- reflexive, R(a,a); symmetric, R(a,b) implies R(b,a); transitive, R(a,b) and R(b,c) implies R(a,c).

Similarity measures in terms of binary relations:
- R_0 is irreflexive "as a vector will never be minimally similar to itself (Forte 1973, 53)."
- "The relation R_1 is irreflexive (since, although a vector will always be identical to itself, the interchange feature cannot be realized)," symmetric, and non-transitive.
- R_2 is irreflexive, symmetric, and non-transitive.
- R_p is reflexive, symmetric, and non-transitive.

set classes where three or more set classes are in a transative relation with one another under R_0, R_1, or R_2.

- In 1.14, Forte looks at order relations (permutations, 1-1 mappings). These order relations (*ordered transpositions*) are the T_n transformations of an ordered pc-set.
    - *Ordered inversions* first invert the pc-set and then complete the transposition.
        - In both *ordered transposition* and *inversion*, the *interval succession* remains the same.
    - *Basic interval patterns* (BIPs) arrange the *interval successions* in ascending order.
        - Two re-ordered TI_n related pc-sets can have vastly different or similar bips.

In 1.15, he discusses the complements of a pc-set: such as,

- **Abstract complement** (TI_n transformation of the literal) vs **Literal complement**;
- All but one of the larger cardinality complements are a superset of their lower cardinality complements (5-Z12*/7-Z12* is an exception).
- When comparing ic-vectors between complements, all of the values of the components (except for ic6 — which must then be divided by 2) differ from the values of the others components by the difference in cardinality of the compared set classes.
- The complement of a Z hexad is its Z companion; otherwise, a hexad's complement is itself.
- If two sets are in R_0, R_1, or R_2 relation: so are their complements.
- "Sets that hold their complements invariant under inversion followed by transposition all are all-combinatorial except for 8-6* and 7-8*(Forte 1973, 82)."

In chapter 1.16, Forte discusses and then offers terms regarding segmentation—"the procedure of determining which musical units of a composition are to be regarded as analytical objects(Forte 1973, 83)." A **primary segment** is "a configuration that is isolated as a unit by conventional means, such as a rhythmically distinct melodic figure(Forte 1973, 83)."

<u>In Part 2, Forte looks at set-complexes.</u>

Here is a description of set-complexes that sticks closely to his own:

K complexes: $K(T,\bar{T})$ is understood as "the complex about T." It states that iff $S \supset\subset T \mid S \supset\subset \bar{T}$,[16] then $S/\bar{S}$[17] $\in K(T,\bar{T})$. $\#(X)$ is understood as "the cardinal number of (the set) X" and the preliminary condition of #(S) and #(T) is: one, $2 < \#(S) < 10$ & $2 < \#(T) < 10$; and two, $\#(S) \neq \#(T)$ & $\#(S) \neq \#(\bar{T})$. Notes on K complexes: "the relation determined by the definition of set-complex membership is symmetric (Forte 1973, 95)" and the inclusion shown here is abstract, not literal.

The subcomplex Kh: $S/\bar{S} \in Kh(T,\bar{T})$ iff $S \supset\subset T$ & $S \supset\subset \bar{T}$. This definition produces 4 relations for any S and T that satisfy it. $S \supset\subset T$ & $S \supset\subset \bar{T}$ iff $\bar{T} \supset\subset \bar{S}$ & $T \supset\subset \bar{S}$. "This is what is meant by the *reciprocal complement relation*."

- 3-1* ⊂ 4-1* & 3-1* ⊂ 8-1* hence
- 8-1* ⊂ 9-1* & 4-1* ⊂ 9-1*

In his analyses, Forte makes a chart (the Kh table) that compares all of the set classes in a composition to each other, showing whether each is in either in the less restrictive complex (K) about the other or the more restrictive subcomplex (Kh) about the other. One determines if certain sets are in either the K or Kh relation to all other **and** whether or not those sets are a primary segment.

The size of a set complex about T is the number of all of set classes that are either a subset of T or its complement.

- The ***set-complex*** **of T** refers to all S in "the complex *about* T," such that $S/\bar{S} \in K(T,\bar{T})$

[16] The compound symbol ⊃⊂ is read "can contain **or** can be contained in."
[17] $S/\bar{S}$ is read "S and its complement."

  - $\#K(T,\bar{T})$ — For a given set class T, the number of S that are either contained in T or $\bar{T}$.
- The use of ***nexus set*** signifies a Kh set-complex. "We will now refer to T (or, alternatively $\bar{T}$) as the nexus set and say that S and $\bar{S}$ belong to the set complex about T (Forte 1973, 101)."
  - $\#Kh(T,\bar{T})$ — For a given set class T, the number of S, if #(S) < #(T), that are both contained in T and contain $\bar{T}$. Or put alternatively, if #(T) < #(S), the number of $\bar{S}$ that are both contained in T and contain $\bar{T}$.

In the last portions of his book, Forte looks at:

- ***The closure property*** — when "every member [of some subset consisting of set classes—one representative set class of each cardinality] of a set complex about a (nexus) set is in the set-complex relation with every other member [of that subset] (Forte 1973, 95)(101)" and invariance within the set-complex.
- Invariance within the set-complex
- Similarity relations within the set complex.
- The challenges faced by using a nexus set designation as a descriptor of smaller and larger harmonic regions.

Robert Morris

Morris's great familiarity with set classes, most likely inspired and facilitated by Hanson's pedagogy, infuses so much of Morris's theory. For the rest of this section, I will examine Robert Morris's Composition with Pitch Classes (CPC), its reduction (for a graduate course at Eastman) in his Class Notes for Atonal Theory, and various articles. Not only has Morris introduced many new concepts to this field, but his pedagogical interests and disposition, like Hanson's, resonate with this dissertation's: he situates his theoretical discussions in the creative demands of compositional interest.

"A compositional theory for something larger than a small segment of today's music needs to be explicit and general at once, since, although it may be designed to help generate a certain species of music based on models of preexisting contemporary music, it must not specify, in any but a rather tentative manner, the stylistic component of the music …. This kind of theory may initially seem to be speculative because it does not lead immediately to the solution of specific compositional problems (R. Morris 1987, 3)."

To fully engage much of Morris's works then, requires building the associated skill sets and applying myriad concepts to one's own creative pursuits. In this dissertation's pedagogy, as with Hanson, the primary "technique" developed is set class fluency. However, the primary aim, like with Morris, is to use "techniques" (in this case those associated with set class fluency) to promote insight into one's own compositional, analytical, theoretical, and performative interests.

In CPC's chap 1, Morris introduces *compositional design*, "pre-compositional plans" that take the form of an ***array*** and facilitate the creation of non-canonic musical forms. As an example of working with an ***array***, Morris:

- First takes a sequence of numbers, let's call them A, to represent an idea.
- Makes this array out of various copies of A, each starting at different time points.
- Explores forms more complex than a canon by:
  - Utilizing the canonic transformation on A, e.g., T_3IA.
  - Utilizing some aspect of the sequence A (perhaps order, perhaps the numbers themselves) to signify arrangement.

Consistent with other literature on serial composition, Morris also gives suggestions on exploiting shared pitches between overlapping, not identical, transformations of A.

In the second chapter, Morris lays the groundwork for and establishes properties for *contour space*—an inestimably big contribution to atonal theory. "Def 1.1. A c-space of order n, is a pitch-space of n elements, called c-pitches (cps). C-pitches are numbered in order from low to high, beginning with 0 up to n-1. The intervallic distance between the cps is ignored and left undefined." This is set in contrast to u-spaces (non-equidistant pitch spaces), m-spaces (collapsed u-spaces, like the mean-tone major scale), p-spaces (the chromatic equal-tempered scale), and pc-spaces (collapsed p-spaces).

- *Cpsets* are unordered sets of cps.
- *Contours* are ordered cps.

- A contour's *comparison matrices* are square matrices comparing a contour with itself. Each intersection describes the relation (+, -, 0) between any two cps. *Contours* are deemed equivalent if their comparison matrices are the same.

He goes on to: defining inversion and retrograde on contours; looking at psegs—ordered sets of pitches that can be classified by the distances between consecutive pitches; and finally, he groups psegs into a non-exhaustive list of loosely defined spacing types. For instance, one spacing type describes psegs whose intervals between their pitches declines. Another looks at psegs whose smallest intervals are in the middle of the pseg.

In chapter 3, he introduces posets (partially ordered sets). These are partitioned psegs such that each partition is treated as a representation of a set class (meaning unordered and invariant under intervallic symmetry). Note that by systematically listing all the potential posets of a given pseg, one can derive both EMB (X,Y), the number of subsets of X that are contained in Y, and COV, the number of distinct subsets of X that are contained in Y. Morris uses posets as a way to construct chains of psegs such that: one, each link of the chain is a representation of one of the poset's partitions; and, two, adjacent links have at least one pc in common. Not only is this a practical compositional aid for developing melodies with an eye towards motivic and harmonic consistency, but it is later used to give a melodic-based description of Forte's K and KH complexes:

- (TH. 3.9.1.1) any partition of the aggregate into non overlapping subsets, A|X|B, where X|B is the complement of A (A') yields a literal K relation between A|A' and B|B'.
- (TH. 3.9.1.2) any partition of the aggregate into non overlapping subsets, A|X|Y|FA [F is a TTO(a canonic twelve-tone operation], where A|X equals B and Y|FA equals B', yields a literal Kh relation between A|A' and B|B'.

Regarding literal inclusion, a K (or Kh) complex about a nexus set is the graph of all of the set classes that are either in K (or Kh) relation with a set class. Regarding an abstract inclusion Morris gives an alternative description[18] (R. Morris 1987, 99–103):

- A K relation between S|S' and T|T' means that EMB(S,T) > 0.
- A Kh relation between S|S' and T|T' means that EMB(S,T) and EMB(FS,T') > 0.

In the following pages of the chapter, Morris looks at: pc-segs of rows (how they may or may map onto each other), rows as permutations of each other (in ordering), how similar one row is to another, and both explains and shows how much information can be gleaned from a matrix comparing either a row and itself or a row and its inverse [what the diagonals, rows, columns, and some patterns indicate].

In chapter 4, Morris digs into TTO's (combinations of I, M, and T), detailing an algebra for "solving" the myriad ways in which they can be concatenated:

- Reframing them in terms of the cycles generated by how they act upon each pc—for instance T_4 is (0, 4, 8), (1, 5, 9), (2,6,A), and (3,7,B),
- Investigating the interaction between a TTO, the powers that it may be raised to, and the length of its cycles,
- Offering methods to construct lists of cycles that, when applied to a particular pc-seg, are invariant (it contains complete cycles) or returns its complement.

Afterwards, Morris describes compositional designs wherein form is articulated by TTO's (or, if hierarchically construed TTO's of [TTO's of [TTO's etc.).

Morris then extends his discussion to include M (and not all of the TTO's); breaking the T_nM transformations into 3 classes, and the T_nMI transformations into 4. For the next portion of chapter, he looks at:

[18] Morris's descriptions of K and Kh relations have taken a very abstract idea and operationalized them in such a way that students can build greater familiarity through relatively straight-forward practice routines.

1. The groups (and subgroups) formed by interrelating (regarding the aforementioned partitions) all of the TTOs.
2. Cyclic vs. non-cyclic groups
3. Conditions under which various transformations commute with one another.
4. Cosets of subgroups and the transformations that map one subgroup of transformations into another.
5. Ways to overlap transformations of the same pc-seg when seeking to complete the aggregate.
6. Compositional designs that utilize these groups (and related structures)

The final portion of the chapter is dedicated to working with permutation operators—operators that also are expressed as cycles but are applied slightly differently and to the ordering rather than the elements of pc-segs.[19]

On the other hand, Chapter 7 turns its attention to time (rather than pitch). It extends (or supplements) tools and concepts presented earlier.

- The time-based analogue to c-space is given;
    - Sequential time (s-time—measured time is m-time) is a series of time points wherein the distance between each time point isn't specified.[20]
- Consideration is given to the audibility of certain compositional realizations of contours and arrays.

The final portion of the chapter is devoted to "Time and Pitch-space Isomorphisms." The units in m-time are time-points. Transposition adds a constant to each time point. Inversion multiples the time point by -1. The operator M multiplies each time point by a constant. Modular time, mod n, is a series of time-points mod n. The ordered duration from one time point to another is the second minus the first; the unordered duration is its absolute value. Finally, he gives a method,

[19] It should be noted that the machinery built here seems to be aimed at the serial composer; it offers a lens onto common serial techniques — a perspective that both makes them easier to generate and then navigate, contextualizing them within more general principles.
[20] Similarly, dynamics, timbre-color, envelope, etc. are also well expressed with an ordinal scale—although, not all dimensions of the qualities being put on a scale are independent.

using matrices and the lowest common denominator to relate p-segs of different groupings (i.e. 3:2).

In *Compositional Spaces and Other Territories*, Morris amends his model of composition. Where his earlier suggested models of the potentially 3-fold compositional process relied on compositional designs first, and then improvisation (on the design) and/or the drafting of a score, he now introduces a new category, *compositional spac*e, that subsumes compositional design and free improvisation.

Compositional spaces, abstract (e.g., grammars and partition graphs) and literal (realizations of a grammar, two partitions, etc.), acknowledge an intermediate stage in one's compositional process; a stage where the larger conception is not yet a fixed template out of which the rest of the compositional process proceeds. These intermediate stages/spaces can act as a playground for working out the eventual form. Morris's basic requirement of a *compositional space* is that a set of objects are related in at least one way.

How vague the conception of this space is falls along a spectrum. Accordingly, looking at another's graphic instantiation of the compositional process is not enough to ascertain whether it is a compositional space or design; intent plays a role. A sufficiently complicated *compositional space* may not yet be a compositional design; the composer may not yet have committed to a single design.

In the first portion of the article, Morris demonstrates various ways that a compositional design (an array) can be adapted (additional partitions etc.) and realized (in terms of arrangement, rhythmic placement, dynamics, articulations etc.). In the second portion, he shows various compositional spaces, typically a graphic description of some musical element (a row, a motive, etc.) and various iterations of it.

Over the rest of the article, Morris details *compositional spaces* of great interest and complexity that have been either used or discovered by composers and/or theorists. These playgrounds, and associated rules about how to navigate them (grammars), can exhibit mathematical properties.

For instance, its elements may combine to form a group and/or it may be constructed so that no matter how its navigated (again, following a grammar/algorithm) some musical element, such as an aggregate or a specific pc-set, is formed. One may even find these spaces, ingeniously wrought, beautiful, and inspiring—as a template from which to work. An example of a compositional space (not mentioned by Morris) that has inspired composers and theorists for centuries is the *Tonnetz*.

In "Some Musical Application of Minimal Graph cycles" Morris provides a characterization of graphs that both enables comparison between different types (K-nets, *Tonnetz*, etc.) and can yield insight into more complex graphs. Relevant observations include: "Graphs of three or more rows are probably too complicated to present visually in any format.[21]" A list of distinct minimal cycles may be more fruitful. This problem is also encountered when seeking to compare many set classes. As well as, "We can model sophisticated musical practice with a network approach, but when we do, the representation of its structure is often too complex to be handled by the conscious mind. Yet learning small portions of the network is manageable and a knowledge base of many portions may suggest the network's overall nature (R. Morris 2010, 115)." Morris's approach resonates with this pedagogy's in the sense that there is recommendation towards learning many set classes by first learn smaller, more manageable, chunks of them.

In his article, "Recommendations for Atonal Music Pedagogy in General; recognizing and Hearing Set classes in Particular," Morris gives a background in Atonal Music Pedagogy, surveys competing foci in atonal pedagogy, and then offers advice on how to learn hexachords. In noting a shift from a compositional to analytic approach, he states that:

> "the pedagogy of compositional theory must be distinguished from that of analytic theory. Compositional theory must always respect the indeterminacy of compositional choice—choices that remain open until the composition is completed. Analytic theory addresses the way in which extant pieces are related in a repertoire, often to an ideal piece or an archetype (R. D. Morris 1994, 79)."

[21] This is a running concern of this pedagogy; no adequate visual representation exists that can encapsulate the more complex ways that we can engage musical material.

This shift has been promulgated by theory's emerging function in undergraduate music departments as a critical tool and aid for performance rather than a resource for composers (R. D. Morris 1994, 80). Exacerbated by its near-exclusive focus on pitch structures, atonal (as a composition-oriented pedagogy) theory pedagogy has now been cast as cerebral, not based on "practical and audible relations (R. D. Morris 1994, 80)."

The last part of the paper focuses on hearing and identifying pc-sets and set classes. He suggests accomplishing this in 5 steps:

1. Get fluent with identifying intervals. First on their own; second, as the outer voices of larger chords and/or melodic passages.
2. Learn to identify aurally, and in writing, the trichords. Four suggested aids:
    a. Devising practice routines wherein the trichord set class's component scales, such as 3-11a vs. 3-11b, are differentiated;
    b. Introducing particular trichords in their various "figured-bass" (the two upper pitches are respectively x and y semitones above the base) configuration.
    c. Teach the most contrasting pc-sets first; the more similar second.
    d. Supplement that aural work with written work that harmonizes either pitches or melodies with trichords.
3. Use these tri-chord set classes as a site for identifying canonical transformations (that act upon them).
    a. Put alternatively, identify the canonical transformation that describes the relation between two tri-chords.
    b. For instance,
        i. Learn to identify how much (exactly) a tri-chord is transposed;
        ii. Learn to hear if the sum of two adjacent intervals are the same; is the relation transposition, inversion, or both, etc.
    c. Exercises offered include two-part singing exercises wherein one part is the literal inversion of the other and gradated exercises that ask the student to assess whether or not consecutive dyads share the same axis of symmetry; increasingly obscuring the sequenced intervals using inversion and/or octave displacement.

4. Identify some of the larger set classes. These set classes are chosen for either their prevalence in the literature, their special properties (all-interval, CUP, etc.), and/or their familiarity as subsets of the diatonic scale.
5. "Identifying the Hexachordal Set classes (R. D. Morris 1994, 115)." Essentially, he breaks this into three parts: building off of the ten that the reader student will have learned by this point, learning them in terms of the smaller set classes, and finally, learning groups of them that share certain properties. The groups include:
    a. Cohn's *transpositional combination*—the hexachordal sets that are comprised of two transpositionally related subsets; e.g., 6-19 can be decomposed into two disjunct instances of 3-11.
    b. Morris's *complement union property* (CUP), a set class, p, has CUP if can be decomposed into two disjunct (sub)set classes, q and r, such that the union of any T_nI instantiation of q and/or r yields p.
    c. Set classes that are subsets of notable larger set classes such as 7-35*, 8-9*, 8-28*, and 9-12*.
    d. All-combinatorial and Z-related and/or M-related.

Finally, I am going to discuss Morris's "Class Notes for Advanced Tonal Music Theory." As mentioned before, it mostly condenses most of the topics discussed in *Composition with Pitch Classes*. Major differences include:

- An absence of discussion on compositional design.
- It acts as a survey of important concepts in the field (circa 2001). Not an introduction to the many lens that can aid the implementation of and enrich one's understanding of various compositional designs.
- It reflects the most recent advances in the field (circa 2001).
- There is significantly less discussion overall.
- In a bid to increase clarity and reduce redundancy, there may be more (or less) examples (and sometimes they are better) on a given topic. It's clear that Morris has refined his pedagogy in the ~ 15 years between the two books.

- There is a significantly expanded discussion on K-nets and various 12-tone topics—especially Mosaics, different types of rows and partitions of the aggregate, and exploring invariance between a row and its transform (on either its pcs or its order).

Part I, Section 2, Chapter 6: Julian Hook and Why There Are 29 Tetrachords

In Hook's article "Why are There Twenty-Nine Tetrachords? A Tutorial on Combinatorics and Enumeration in Music Theory," he makes a difficult topic, unfamiliar to many musicians, palatable. My aim in this summary is to show his steps; I will use them to frame the second portion of this dissertation. In the first section, Hook defines combinatorics as the study of "methods of counting (Hook 2007, 8)" and then relates its helpfulness in addressing various music queries. Here he explains and then shows algebraic shortcuts for answering such questions as: how many permutations are there of a set of n elements, [p(n)]? n!. Or, how many ways can you choose subsets of size k from a set of size n, [C(n,k)]? $\frac{n!}{k!(n-k)!}$. This last formula also shows that C(n,k) = C(n,n-k). Pascal's triangle can also be used to reveal C(n,k); just look at the meeting point of the n^{th} row and k^{th} diagonal. Furthermore, the sum of the n^{th} row in Pascal's triangle is 2^n and the numbers in Pascal's triangle are associated with the coefficients of the binomial $(x+y)^n$. For instance, for $(x+y)^4$, one gets $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$. Notice how the sum of the powers of the **placeholder variables** always sum to n—4+0, 3+1, 2+2, 1+3, 0+4; this means that the polynomial is *homogenous of size n*.

In part 2, Hook builds on part 1 to find the number of set classes associated with a given cardinality. Take 12-tet: while there are sixty-six intervals, there are only six 2-note set classes; this reduction occurs due to the great number of equivalencies; if two intervals are related by transposition or intervallic inversion—there is only one tritone, not six; and there are only five other intervals (m3s are not distinguished from M6s), not 60 (12 x 5). In other words, one must find a way to not overcount. To do this, just count the number of orbits[22] (six in the example given) rather than the number of instances of a k-note configuration (sixty-six in the example given).

[22] E.g., a "semitone" represents an orbit—all semitones found in 12-tet; whereas as the singular interval <E,F> does not. Moreover, i1 also represents an orbit—all semitones and Major 7ths in pc space.

After associating transformations with the "types of permutations" that act on a set, Hook then introduces Burnside's lemma to shows how to find a single overall count of orbits. Here is the formula:

$$\frac{1}{|G|}\sum_{g\in G}\psi(g)$$

G = the cardinality of the group of actions (in the above case, there are twenty-four—twelve transpositions, $T_{1\text{-}11}$, and 12 inversions, $TI_{1\text{-}11}$). $\psi(g)$ = the number of configurations that are fixed by each $g \in G$. The summation sign counts them up. In short "Burnside's Lemma says, therefore, that the number of orbits is equal to the average number of configurations fixed by an element of the transformation group (Hook 2007, 44)."

Then, introducing cycle notation and permutations, T_1 = (0 1 2 3) means 0→1→2→3→0, Hook provides an example $\mathbf{D_4}$—the dihedral group of size 4, or the symmetries associated with the square, which includes the squares four transpositions [rotation by 90°] 0-4 and inversion [flipping by 180°] (0) (1 3) (2). "The ***cycle type*** is a shorthand accounting of the number of cycles of each length (Hook 2007, 51)." On the other hand, "the **cycle index** $\mathbf{p_g}$ **of a permutation g** of N objects is an expression potentially involving N variables …[and] the **cycle index** ***P_G* *of a permutation group G*** is a more complex polynomial in *N* variables, defined as the average of the cycle indices of all the permutations in *G*:

- $P_G(t_1, t_2, \ldots, t_N) = \frac{1}{|G|}\sum_{g\in G} p_g(t_1, t_2, \ldots, t_N)$ where $p_g(t_1, t_2, \ldots, t_N) = t_1^{\lambda_1(g)} t_2^{\lambda_2(g)} \ldots\ t_N^{\lambda_N(g)}$

… $\boldsymbol{\lambda_j(g)}$ is the number of *j*-cycles in the cycle representation of *g* (Hook 2007, 51)."

Hook then broaches the idea of weight—differentiation between elements. In this way, he describes a pc-set as having two colors (weights) called *yes* and *no*; depending on whether a certain pc-set contains that pc.[23] Each pc-set consists of a weight x_1 compiled of noes and weight x_2 compiled of yeses. "In the general setting in which there are *k* colors from which to choose, we shall write $w_1, w_2, \ldots, w_k$ for the weights assigned to the colors;" The *weight of a*

[23] Why are there 2^{12} colorings of the pitch classes? In short, there are 12 possible pcs and two possibilities for whether each set class contains a particular pitch class—e.g., it either contains a pc G or it doesn't.

configuration is then the product of the weights of the colors of its objects [and] thus describes the multiplicities of its colors. (Hook 2007, 54)" For instance, the weight of any tetrachord is x^8y^4.

Having established the necessary background information, Hook then provides a definition of Polya's theorem.

"Suppose a transformation group *G* acts on configurations of *N* objects colored with *k* colors. Let the weights assigned to the colors be $w_1, w_2, \ldots, w_k$, and define

$$\alpha_1 = W_1 + W_2 + \cdots + W_k,$$
$$\alpha_2 = W_1^2 + W_2^2 + \cdots + W_k^2,$$
$$\vdots$$
$$\alpha_N = W_1^N + W_2^N + \cdots + W_k^N$$

Let P_G be the cycle index of the group *G*. Then the sum of the weights of all the orbits, counting each orbit once, is $P_G(\alpha_1, \alpha_2, \ldots, \alpha_N)$ (Hook 2007, 56)."

"The cycle index of the group *G*, it will be recalled, is a polynomial in *N* variables, $P_G(t_1, t_2, \ldots, t_N)$, calculated by averaging together the cycle indices of the various permutations in the group. Pólya's theorem instructs us to substitute $\alpha_1, \alpha_2, \ldots, \alpha_N$ for the variables $t_1, t_2, \ldots, t_N$ in this expression. Because $\alpha_1, \alpha_2, \ldots, \alpha_N$ are defined as sums of powers of the weights $w_1, w_2, \ldots, w_k$, the resulting expression $P_G(\alpha_1, \alpha_2, \ldots, \alpha_N)$ is ultimately a polynomial in $w_1, w_2, \ldots, w_k$; it is known as the *Pólya polynomial*. The theorem says that Pólya's polynomial is the enumeration function: by calculating this polynomial in full, we will see the number of orbits of each weight as the coefficient of the appropriate term (Hook 2007, 57)."

Hook finishes by using Pólya's theorem to count the number of set classes. When detailing why it is so cumbersome to calculate the contribution of the identity transformation, he offers this helpful comment: "the point of the calculation is to enumerate the configurations fixed by each transformation, and the identity fixes everything—more than any other transformation (Hook 2007, 59)." The other calculations are simpler.

In the end of the article, he gives the general formula for finding the various cycles lengths for a given formula (Euler's function); it is the coprime factors.

Part I, Section 2, Chapter 7: Glossary for Terms, relevant to Part II, that are discussed in Appendices A-C

Appendix A — Terms related to Lewin and Neo Riemannian theory

GIS (Lewin 2007, 26)**, the Generalized Interval System**, "as an ordered triple (S, IVLS, int), where

- **S**, the *space* of the GIS, is a family of elements {a set},
- **IVLS,** the *group* of intervals for the GIS, is a mathematical group, and
- ***int*** is a function mapping S x S into IVLS, all subject to the two conditions (A) and (B) following.

(A): For all r,s, and t in S, int(r,s)int(s,t) = int(r,t).

(B): For every s in S and every i in IVLS, there is a unique t in S which lies the interval i from s, that is a unique t which satisfies the equation int(s,t) = i.

Under a **relation R**, the elements a and b are deemed equivalent if:

1. a ~ a (the reflexive property)
2. a ~ b implies that b ~ a (symmetric property)
3. a ~ b and b ~ c implies that a ~ c. (transitive property)

Commutative operation: An operation such that a * b = b * a; where a and b are elements of a set and * is a binary operation on that set.

Binary operation: a binary operation associates two elements in a set with a one element. E.g., a*b = c, where * represents the binary operator.

IFUNC tallies the number and type of intervals between two elements in the underlying set of a GIS; these could be pitches, rhythms, etc.

Embedding (EMB) function—a function that tallies the number of appearances of one chord in another.

INJ counts how much f(X) is in Y, although there is no requirement that this "function" is either 1-1 or onto.

STRANS is a recasting of IVLS as an "indexing" function that one-to-one maps STRANS to an ordered list.

Tonnetz, a lattice of pitch classes (or pitches), can be referenced to indicate how a triad (represented as a triangle) parsimoniously relates to other instantiations of it. Adjacent instantiations share one or two pitches. To "transform" a triad of three vertices connected by two lines (sides) into one of its neighbors is described by flipping one instantiation over one of its sides (or single pitches). These bi-directional flips over a side are called **L**, **P**, **R**.

- E.g., L (*Leittonwechsel*) maps a C major triad to an E minor triad;
- E.g., R (*relative*) maps a C major triad to an A minor triad; and
- E.g., P (*parallel*) maps a C major triad to a C minor triad.

Appendix B — Terms related to Similarity Measures

Similarity measures — assessing the distance between two set classes. Schuijer, in Analyzing Atonal Music, bundles them into three general approaches: one, interval-vector; two, subset inclusion; and three, "Absolute" distance.

Interval vector:

- Teitelbaum's **APIC** calculates a similarity index by taking the square root of the sums of the squared differences between the two interval vectors' components.
- Morris's **SIM**, which calculates the difference vector between compared interval vectors by summing the differences between the corresponding interval vectors components.
- Forte's **R_p, R_0, R_1, and R_2 relations**:

- **R_0** (minimal similarity): When there are no interval components in common
- **R_1** (first-order maximal similarity): First-order maximally similar requires that four components be identical and the remaining two components, in opposite order, be the same.
- **R_2** (second-order maximal similarity): Second-order maximally similar requires that four components be identical and does not require that the remaining two components, in opposite order, be the same.
- **R_p**: When two set classes share a common subset of cardinality N-1.

<u>Shared Subsets approaches</u>:

- Regener's **Common-note vector** (of length 12): a vector whose components indicate the number of shared pitches between the two pc-sets as the transposition-level of one of those pc-sets is shifted 12 times
- Regener's **Partition-vector**: a vector which tallies up how many times 3 common tones, 2 common-tones etc., are found in a common-note vector.

<u>Absolute Approaches</u>:

- Morris's **ASIM** "standardizes" his measurement **SIM**. Here, Morris controls for set class size, operationalized as a set class's total number of intervals. It is a measure that ranges from 0, when the compared vectors are the same, to 1, when they are maximally different—when there are no intervals in common.
- Rahn introduces **TMEMB** that counts the number of appearances of shared subsets of cardinality 3 or greater.
- Rahn's **ATMEMB** normalizes **TMEMB** by placing the results between 0 and 1.
- Isaacson's **IcVSIM** ("Interval-class Vector Similarity"), adopts a statistical tool, standard deviation. By examining interval vectors' differences in regard to a normal distribution, the cardinality-related restriction in ranges could cease to obscure the significance (the likeliness or unlikeliness) of the various scores.

Genera — "higher-order taxonomic categories … [that are] organized around privileged, highly symmetric chord species (prototypes), [that can be] quite distant from one another in quality space. The taxonomic categories are structured by virtue of chords being qualitatively close ("similar") to the prototypes in quality space (Quinn 2006, 121)."

Hanson's algorithm for generating prototypes (of a given cardinality):

1. Take a pitch
2. Add a pitch that is an interval *i* above it.
3. Repeat, until you either get a previous pitch or stop.
4. When you get a previous pitch, raise it by a half-step,
5. Repeat, until all 12 pitches are exhausted or you're satisfied.

FOURPROPS should be considered as 1) the characteristic ME (Maximally Even) species of a qualitative genus, and 2) not having the property by which it is named. It's better to think of a chord as **being balanced** in regard to the property by which it is named; not as equaling that property. A [02468A] does not balance FOURPROP(6) (the whole tone scale), [0145] does. Accordingly, transposition and inversion do not affect the balance.

The Discrete Fourier Transform (DFT) locates a chord's coordinates, first presented in 12-dimensional harmonic space, in "quality space," which describes how said chord is located in regard to each of the two-dimensional Fourier (FOURPROP) Balances.

Appendix C — Terms related to Diatonic Set theory

- **Clens:** the distance between two scale degrees in units of the underlying temperament.
  - E.g., The clen between scale degrees 2 and 4 of a major scale is 3.
- **Dlens** (generic interval)**:** the distance between two scale degrees in scale degree units
  - E.g., The dlen (generic interval) between scale degrees 2 and 4 (or 3 and 5, 4 and 6 etc.) of any scale is 2.
- **Spectrum:** all of the clens associated with a scale's generic interval of a particular size.

  - E.g., In a major scale, the generic interval "a 2nd" is associated with a half-step and a whole step.
- **Spec($M_{c,d}$)** a list of all of the spectra (remember, each spectrum is itself a list) that are associated with a particular $M_{c,d}$.
- **CV,** cardinality equals variety: the condition that no two modes of the same scale are identical.
  - This condition is met in regards to the diatonic scale, but not the whole tone scale.
- **Myhill's property**: none of a scale's spectra have more than two clens.
  - The harmonic minor scale does not have this property; its 2nd includes 1, 2, and 3 half-steps.
- **Rounded**: the spectrum of a scale's *step*, is a set of two consecutive integers.
  - As shown above, the diatonic scale has this property
- **CP** (consecutive property)**:** a scale's spectrum consists solely of consecutive integers.
  - All *diatonic* and *pentatonic* sets have the CP.
- A ***g-generated*** maximally even set:
  - One can arrange a g-generated $M_{c,d}$ set so that at least all-but-1 of the clens between successive pitches are identical.
  - E.g., C major can be expressed as B-E-A-D-G-C-F-(B). Except for F-B, all pairs of adjacent pitches are 7 semitones apart. Hence it is a **g-generated** maximally even set.

Appendix E — Terms related to Cognitive Psychology and Learning

- **Executive control processes** allow one to actively manipulate information (such as adding in your head) and execute retrieval strategies that retrieve memories from long term memory (LTM), maintain them, and then manipulate them
- **Short term memory** (STM) is posited as transient storage space for information that is above some level of activation, is readily available, and your attention may not be focused on it
- **Long term memory** (LTM) is considered a long-term storage space. Typically, it is divided into declarative and non-declarative memory.

  - **Declarative memories** are those memories that you are consciously aware of and can talk about.
    - **Semantic memory** is the store for general knowledge about the world.
    - **Episodic memory** is the store for specific episodes and events.
  - **non-Declarative memories** are those which appear more in terms of your performance and/or behavior; you do not always have conscious access to them.
    - **Conditioning** is a learned association between different stimuli or between behaviors and specific rewards and/or punishments.
    - **Skill-learning** reflects the learning of and demonstration of a skill.
    - **Priming** is when exposure to a stimulus influences your response.
- **Implicit learning** is an umbrella term for any learning that takes place without focus.
- **Explicit learning** is intentional; it occurs through directed attention.
- **Heuristics** are cognitive biases that influence our decision-making processes. Viewed most critically, these biases impair our abilities to make the best decisions; viewed more positively, they are decision-making strategies that take into account the cost of deliberation.
- A **mental model** "has been defined as a mental structure that reflects the user's understanding of a system … A mental model may be created spontaneously by the user or carefully formed and structured through training (Wickens et al. 2013, 236)."
- **Chunking**: A chunk "can be defined as a set of adjacent stimulus units that are tied together by associations in the subject's long-term memory."

PART II: LEARNING THE SET CLASSES

Part II, Chapter 8: 2.0

Part II presents the heart of this paper; the pedagogy proposed. However, the open-ended nature of this endeavor, learning all set classes and their interaction, precludes this pedagogy from being overly prescriptive. Seeking "mastery of everything" not only borders on the impossible, but it can be an obstacle to better aims of this pedagogy—increasing fluency with the musical material that you care about, exponentially increasing your music-related processing speed, swimming (drowning even) in an ocean of inspiration, and removing barriers between you and your appreciation of certain types of musical complexity; either because you now recognize a particular chord or have a way in to making sense of it. There is more to the "Rite of Spring" than the fact that it is populated by octatonic scale segments, but if one can quickly recognize those scale segments—by sight or ear, the prospect of digging into other pitch related analytical inquires may seem less imposing. Therefore, when working through this pedagogy, aim high but try to anchor it in music and musical questions that you may have. Make it relevant! Those who anchor this course in their own musical interests may be the best positioned to achieve that "elusive" task of mastering everything. Every pitch structure can be related to another somehow and in general, when we relate new things to musical material that we are already familiar with or appreciate, it sticks better. So, again, as much as possible, try to anchor your study of this in what you care about and your experience. There is not a best position from which to engage this material.

Part I was meant both as literature review and as a resource for you. After a first read through, feel free to find connections between Part I and II; seeing if your accumulating familiarity with certain set classes can aid your understanding of those theories. Similarly, feel free to continually refresh Part II, as you go through those case studies in Part III that best speak to you.

The supposition guiding this pedagogy is that the best way to learn the set classes and how they interrelate is to simultaneously refine your understanding of the macro, how they all interrelate, while building increasing familiarity with the micro, the smaller set classes. Familiarity entails more than just visual recognition; hopefully, it also entails auditory recognition. Am I implying that by the end of this process that you will be able to aurally identify every set class, in any possible manifestation? I wish; but, unfortunately, no. Instead, what I am requesting is that you take the aural component seriously, seeking to identify the smaller and then bigger set classes that most intrigue you by ear and by sight. View that bigger complement as a window into how the smaller complement can be traced through 12-tone space.

To bring this objective into further clarity, the following sections are going to accompanied by a mathematical metaphor — the polynomial expression $(x + y)^n$, where x and y are complements of each other, and ultimately, placeholders in the manner suggested by Hook's article on counting set classes. The unfolding of this polynomial at different degrees counts all possible combinations of those recombined elements, differentiating them by type. For example, take FOIL, a mnemonic for how to square binomials (firsts, outsides, insides, lasts). In the case of a pitch-set of size 2 in 2-tet, let's say C and F# and firsts yield 'not hearing anything,' outsides yield 'hearing F#,' insides yield 'hearing C,' and lasts yield hearing C and F# together. In this case, there is one instance of hearing no pitches, two of hearing a single pitch, and one of hearing both pitches together. One also observes this in the coefficients of $(x + y)^2 = x^2 + 2xy + y^2$; respectively, the coefficients are 1, 2, and 1. In this space, 2-tet, one element, is the complement of F# (x^2y^0), the other element, and "hearing nothing, (x^2y^0)," is the complement of "hearing everything," C and F# (x^0y^2). There are two instances of hearing one element but not the other.

The advantage of this lens is two-fold:

1. It suggests that there may be a significance to the order in which we receive these partial impressions of a pitch set; and
2. It frames how our understanding of larger set classes (the content of their constituent elements) is related hierarchically to our understanding of smaller set classes (also constructed out of combinations of their constituent elements).

This is pertinent to this pedagogy's goal of helping the student develop intuitions about how various set classes relate to each other.

Finally, understanding and then learning the set classes in this pedagogy is a bi-directional process; the premise being that you use the big picture to interpret the details and you in turn use the details to inform your understanding of the big picture. Towards this end, each section will begin with another symbol (A_x, R_y). Here A refers to audio (listening), and R refers to reading; respectively, x and y specify the cardinality of the pitch collections being discussed. We begin by Reading the set classes with the highest cardinality and or listening to (Audio) the set classes with the lowest cardinality. This ordering presumes that it's easier to focus on the big picture with one's eyes, and to start learning the smaller set classes by ear. However, just as it's difficult to extricate (if even possible) what we know about music from how we hear it, this given distinction is not meant to be perfect and absolute. Rather, the signage is just meant to indicate, my suggestions on how to engage the material:

1. Learn the smaller sets of pitch material as thoroughly as possible (up to and including, if possible, recognition through listening).
2. Learn more general principles about the larger set classes and how they relate to their subsets and other set classes of the same cardinality.
3. Finally, after completing Part II for the first time, begin to put the two approaches in dialogue.

In general, the aim is for you to develop a sense of how those musical elements that you care about connect and have a more refined sense, not needing pen and paper in front of you, of how various pitch-based transformations will impact your starting material. Again, the goal is to "get a feel" for how these many things connect while getting quicker at mentally, or through your playing/improvisation, navigating those connections during a performance. A master in this method is not the one who has memorized everything by the end — memorization alone does not eliminate the great mental processing needed to "choose the best deformation possible" out of 1000 options (rather, that requires instinct or a gut reaction that has been shaped by an interaction between this approach and your interests over a decade or more). Instead, a master of this pedagogy is one who is comfortable applying it as a lens when exploring compatible (to this approach) areas of their musical interests.

Well, if "mastery" takes 10-plus years to achieve, how is this really a help? A dedicated student can achieve almost anything in 10 years. Well, the research for this book took upward of a decade, and I don't believe that the level of fluency that this book promotes can easily be stumbled upon; at least not by other than once-in-a-century musicians. As such, your availing yourself of these resources, which can guide, inspire, and simplify your pursuits, could still be a boon; even if you only embrace a fraction of this pedagogy.

When pursuing my master's in Composition, I remember being advised on how I could go about developing my harmonic language: hunker down with certain pitch material, or constellations of pitch material (networks of similar chords), for long enough that my usage of that material would someday distinguish me from other composers. At the time,[24] I felt that this request was laced with arbitrariness as I knew absolutely nothing about any chords other than the standard fare, 7th chords and triads. I had to determine that there was a "there there" with some random pitch collection and then dedicate my future musical journey to it. Hopefully, I wouldn't waste a few years on the wrong material! While my professor's lesson was undoubtably more nuanced than my current reduction of his words would imply, the raised concern may still resonate with you. Does choosing unknown pitch material feel like a "Hail Mary"; maybe you'll hit gold,[25] maybe you won't? Can the pitch structures that you already know help you understand other pitch structures? In terms meaningful to you, is there a way to better assess the consequences of focusing on certain pitch material before committing to that pitch material long term? Is it possible to experience a similar fluency to what one has with the "standard fare" with other sets of pitch collections? What would this look like? Can you get to a place wherein you can assess this near instantaneously?[26] Plus, as a performer, quickly recognizing more pitch material by

[24] The following personal anecdote is a recollection of how I felt about composing at that time. It's not meant to be taken as a universal statement on what either the act of composition should be or how a composer should relate to said act. Actually, my thoughts on composing now differ to the ones being conveyed in this anecdote.
[25] Here, "hit gold" means choosing material that perfectly fits whatever your lofty musical intentions require.
[26] To the last question, the answer is sometimes. So much scholarship has been done on triads that we can now say that only 3-3 has similar voice leading properties to 3-11, and it's not the same. So, if you want to use one set class exactly as you use another, you may immediately know that you can't. However, rarely is a structural determinant, unless the composer is dogmatic about following their methodology/algorithm to the letter 't,' a barrier to projecting a more important idea. So, what was initially a precise question then degenerates into a fuzzy one. What is "behaves similarly enough" so as to be useful? Again, the best way to get at this, without a computer, is through a developed sensibility around how the set of pitch material that you care about, and various degrees of closely related neighbors, interacts with itself along multiple dimensions.

sight (as a listening improviser, by sound) felt promising as a way forward in the interpretation of (and even memorization) of scores that I then found intractable; for instance, certain atonal and serial works.

This portion of this dissertation provides the nuts and bolts of the pedagogy.

Section (A_0, R_{12}); $(x+y)^0$.

R:

This section focuses on this pedagogy's most top-down perspective, the frame so to speak, the single 12-note pitch collection in 12-tone equal temperament space, Forte sc 12-1* (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B) It has no rotations; is symmetrical; provides all of the available pitch content in 12-tone equal temperament space; is a symbol for what is possible harmonically; and, is a frame within which all 12-tone equal temperament harmonic motion occurs. However, no harmonic motion exists between it and transformations of itself.[27]

As mentioned before, for Hauer, it could be seen as "the spiritual plane" that pure intervals inhabit. However, this "plane" need not be considered as a two-dimensional plane. For instance, both Tymozcko and Quinn discuss 12^{12} space, wherein every pitch-class collection is located by a 12 x 1 vector. How that vector engages 12^{12} is partially determined by how each of its components, numbers, evenly divide 12. Below (Figure 2.0.1), Quinn shows the fold by which the six FOURPROP vectors in 12^{12} space are projected onto a single line in Quality space; that space whose elements are measured in regard to FOURPROP vectors (Quinn 2007, 60).

[27] For an extensive exploration of musical spaces, I highly recommend reading Julian Hook's *Exploring Music Spaces: A Synthesis of Mathematical Approaches*.

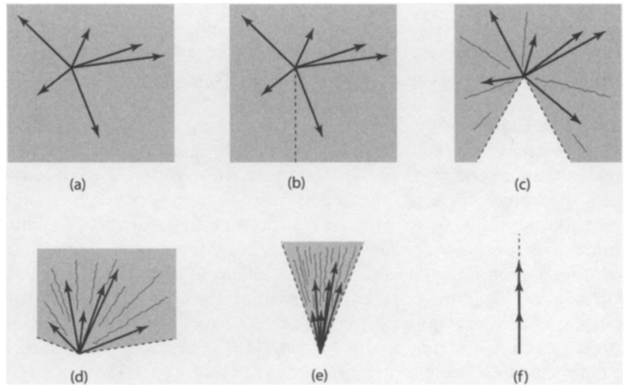


Figure 2.0 1

On the other hand, Mazzola offers, 3 (min 3rds) x 4 (major 3rds), a torus, as an optimal space by which to locate a harmony. Transposition retains the shape, and rotation flips it. For instance, take the table below Figure 2.0.2; it's a projection of the torus onto a 2D plane—respectively, the top row and left column are connected to the bottom row and right column and both the chromatic scale and sequence of P5's unfold without a change in the torus's side. All major triads are of the form over one and then up one; and all minor triads are of the inverse form, up one and then over one. Chromatic movement is attained by moving diagonally to the southeast; P5 movement by moving diagonally to the northeast (Mazzola 2002, 318).

A	C#	F
Gb	Bb	D
Eb	G	B
C	E	G#

Figure 2.0 2

Alternatively, imagine a piano keyboard that is 6 x 2, meaning that there are 6 white notes and 6 black notes that are juxtaposed within each octave. Were this the case, all chords and their transpositions would be reduced to two forms; those starting on a black note and those starting on a white one. Inversions of a chord would "flip the shape," reversing the proportions of white notes to black.

However, whether it's designed in a manner that directly engages the symmetry in 12-tone equal temperament space or not, each instrument provides a space, a frame, by which one can view different combinations of tones in 12-tone spaces—including the voice! Not all instruments may enable all twelve pitch-classes or allow the performer to play chords of all shapes, but each provides a lens through which those so dispositioned can reflect on how chords are related. I argue that to some extent, our instruments of choice, influence both the types of connections that we make between different set classes and the degree of motivation that we may have for asking the types of questions that this pedagogy seeks to answer.

There are also non-twelve-tone equal temperament spaces through which to view the chords that we are familiar with. This general area of inquiry, or compositional interest, is called Xenharmonica. In Hook's article on enumerating the set classes, he derived the number of set classes of each cardinality in 19-tone-temperament. Clough and Douthett (Clough and Douthett 1991) and other diatonic theory scholars (Clough and Myerson 1985 & Clough, Engebretsen, and Kochavi 1999) raised questions such as under what conditions, meaning in what equal-tempered spaces, do diatonic properties arise. Most notably, when do triads and 7th chords arise and when do they interrelate in "diatonic" ways? For example, in 24-tet, quarter-tone space, there is still a circle of fifths and if parsimony is defined as two or four quarter tones, as opposed to one or two semitones, the non-functional voice leading properties of the triad, called L, P and R by neo-Riemannian theory, are still maintained.

Even though this pedagogy is not focused on viewing set classes of interest through many of the lenses just mentioned, as some of them may either be familiar to or intrigue you, I encourage you to explore those that are enticing in tandem. Again, the goal of this pedagogy is open-ended, meaning that should you continue to pursue some of its aims, there will always be further to go. As such, things will fare best when you align it with your more open-ended musical interests.

This is also a good space to discuss the top-down approach more generally.

The hardest part of learning larger set classes is just recognizing them. Not accounting for octave displacement, where there are 3! = 6 ways to arrange a tri-chord, there are 9! = 362,880 ways to arrange a 9-note chord. Plus, each tri-chord and 9-note chord can typically be presented in 12 keys. In the former case (trichords), this results in 72 different presentations (6 * 12) — a very trying number for many music theory fundamental students. In the latter case (9-note chords), it's 4,354,560 different presentations, which is trying for most likely anyone. Furthermore, where it is relatively easy to manipulate three things in one's head, nine tests the short-term and/or working memory of most. Even ruling out one 9-note chord from the other can take time and puts a barrier between achieving the type of automaticity that one may have come to enjoy with triads.

Furthermore, no amount of theory can make up for lack of exposure. Take the octatonic scale, whose symmetries result in its only being in 3 distinct pc-sets; it still has 120,960 representations. Nonetheless, many musicians still learn to recognize the octatonic scale without fail—especially as there may only be 3 instances that most music readers are typically exposed to. So, at least in this case, continued exposure to a larger harmonic set can help with identifying it. On the other hand, without utilizing arbitrary segmentation, there are just not as many opportunities to find 9-4 out in the wild. Plus, even if 9-4 is found; it's most likely not found enough, even in one piece, to be labeled a recognizable unit.

While a good teacher, ensconced in Post-Tonal literature, could eventually find some salient examples of sc 9-4, I will refrain from doing so here (plus, others like Hanson have already done this!). This is for one main reason: just because an example of sc 9-4 can be found, that doesn't mean that that chord was a meaningful unit to the composer — e.g., it may just be a temporary result of voice-leading processes.

In general, I want to avoid giving the impression that composers "thought in terms of larger set classes" — although some did! (e.g., Messiaen when using mode 3 of his limited modes of transposition)— or that being able to label them may offer an advantage to a performer realizing them in a specific performance context. While, I do believe that being able to label set classes is extremely helpful; I believe that it is only helpful when sufficient support is given to understanding how that label enriches one's understanding of the piece or spurs the imagination of a composer/improvisor (harmonically, melodically, rhythmically — discussed later, etc.). This pedagogy is largely then about providing that additional support. An aim that is simply beyond all introductory Post-Tonal texts that I have encountered; those texts typically are constrained to introducing post-tonal literature and the associated methodologies and terminology, associated with it. This pedagogy, on the other hand will not spend any words showing you how to fill out a matrix.

So, most students will not get familiar with sc 9-4, see it in the literature that they perform, and even if they do, not have the taught background, from studying various textbooks, to illuminate just how helpful that label may be. This of course is not an indictment against the amazing post-tonal textbooks out there, but just an acknowledgement that this goal is often overlooked. The biggest counterexamples to this "overlooking" come from Hanson and Morris's texts.

So, summing up, what are the three big challenges with learning larger set classes are: one, the vast number of ways in which they can be presented; two, the lack of exposure to them; and three, the little attention to or value placed on learning/internalizing them in many educational contexts. So, how do we work around this?

The standard way is to take advantage of the complement theorem. If you can identify the notes of the aggregate that are not contained in the chord/set class, you can quickly identify the set class/chord in question. Regarding chords, one hiccup to this is that, while in most cases, if the complement is the "b" component,[28] then the chord is question is the "a" component of its respective set class. However, this is not always the case. Nonetheless, the exceptions to the aforementioned tendency can easily be memorized; there are not too many of them.

What is the downside to this? Focusing on a chord by determining what it's not is less satisfying than focusing on a chord by determining what it is. Presumably, when one selects a chord when improvising or composing, one typically selects it for what it is, rather than, at the level of the complement, what it is not. Plus, by the time that you are able to quickly recognize, in any of its manifestations, any single 9 note-set class, quickly recognizing many of the smaller set classes that it contains will be a trivial concern.

Further helpful strategies will be discussed in the second portion of this chapter.

[28] The "a" and "b" components are the two T_n classes associated with a given set class.

A:

This section will interpret auditory units of the zero dimension as those relations that we construe from what we hear, but do not actually hear. For instance, parsimony. As mentioned earlier, two chords are deemed parsimoniously related if they are of the same cardinality and if you change either of them by a half, or whole, step motion in one voice you will obtain the other. Moving forward, I will confine the definition of parsimony to half-step motion.[29]

It can be argued that the easiest relationship to identify between two chords is a parsimonious one. Not only is there a single change, but it is generally easy to determine how it occurred; both the changed note and, in memory, the not-yet-changed note are readily identified. We may even impose a connection between the two: hearing the motion as a stepping stone in a larger melody that adjoins them. Along these lines, I will occasionally offer a parsimonious map that connects all of the set classes of a given cardinality.

In general, this pedagogy acts under the assumption that the best way to learn the various set classes is in their relationships to other set classes. Practicing them from the just-mentioned maps, foregrounds this; the idea that the best way to learn a new set class is through comparing it to a set class one already knows. The smaller the difference is between the two set classes compared, the more advantageous this approach is. Over the course of this pedagogy then, you will grow your reservoir of chord relationships. As your networked understanding of set classes becomes increasingly enriched, so does the speed with which you can then learn new set classes—typically, by plugging into networks built around parsimony.

However, parsimony is just one "auditory unit of dimension 0." Other such units could be a LIP (linear intervallic pattern) or schemas writ large (e.g., the Romanesca), that also can frame our interpretation of series of chords/set classes. To be clear, right now I am distinguishing between the labeling/notation of musical events ("auditory units of dimension zero") and sounded musical

[29] The other definition acknowledges that in a 2nd order maximally even space (12,7), dlens—or put alternatively, differences between scale degrees—can either be a half or whole step. As such, when examining how diatonic chords map to each other in the context of a diatonic scale, it can be helpful to expand the definition of parsimony to refer to half and whole steps. However, in this wide-ranging pedagogy, this diatonic assumption is rarely met; as such, it's easier for mathematical and practical reasons to define it in regard to half-steps exclusively.

events.[30] Furthermore, by viewing your set classes of interest through the lens of certain musical (and even non-musical!) schema, you have an additional means by which you can hasten your learning of unfamiliar set classes.

For instance, a 10-5 linear intervallic pattern (LIP) shown below (Figure 2.0.3) — a term found in Schenkerian analysis, only specifies the outer voices in a sequence; depending on the inner voices, a host of non-diatonic set classes could arise.

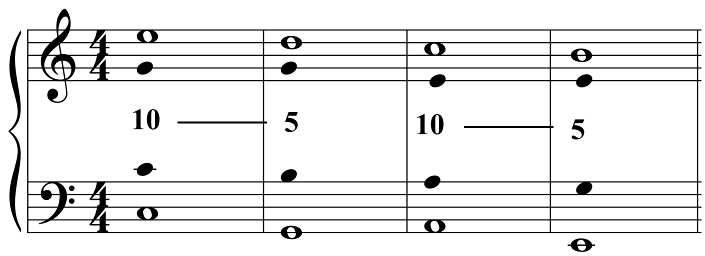


Figure 2.0 3

Find an "off the beaten track" LIP that you like; what set classes does it consist of? How "far" is it from a diatonic LIP? You may be able to get familiarity with most of the 4-note set classes by simply learning a handful of such LIPs.

As set classes can also be associated with Dualism, let's briefly look at a couple of its major proponents, historically speaking, and at least breach the question about what does it mean to hear a chord and its inverse as being the same. In short, Dualism rose in response to an assumption made by Rameau, that symmetry related core components of music; the three cadences and the major and minor triad. Cadences were defined in regard to bass motion (sometimes explicit, sometime inferred through subposition):

- the authentic cadence, down a P5;
- the irregular, up by a P5; and
- the deceptive, both up and down by a 5th.

[30] While schema will not be explored directly in Part II, they will come up in Part III.

He also posited that the minor triad, expressed in regard to a harmonic undertone series, 0:2/3:4/5 (C, F, Ab) was the parallel of the major triad, 0:3/2:5/4 (C, G, E), defined in regard to the overtone series.

Nonetheless, Rameau's theoretical observation on triads, as straight-forward as it appears, was not heard in 18th century acoustic experiments.[31] Rameau's related theories were then perceived to have shortcomings and various theorists, from Hauptmann to Riemann, attempted to resolve this. For instance,

- Hauptmann and Oettingen both abandoned the overtone/undertone rationale. Instead,
    - Hauptmann invoked Hegel and how we perceive these triads—the major triad in relation to its fundamental, the minor in regard to its $P5^{th}$. On the other hand,
    - Oettingen coined the terms *tonicity* and *phonicity*.
        - *Tonicity* refers to a major triad's component pitches being evaluated in regard to their shared fundamental; *phonicity*, in regard to a minor triad's component pitches being evaluated in regard to their lowest shared partial.
- Finally, Riemann eventually noticed that the 3-fold ratio assigned to major triads, when string lengths were used, was identical to the 3-fold ratio assigned to minor triads, when string divisions were used.
    - Riemann also built on Rameau's appeal to symmetry in harmonic syntax (Rameau's attention to cadences). Riemann explored the major and minor triad's inverse relationship in certain harmonic progressions/networks.

Synthesizing the above paragraph, Dualism posits symmetry as an organizing force behind the construction, perception, and utilization of our musical units. Each of the above theorists made complex arguments justifying this. Most likely, this complexity of argumentation was a byproduct of both how unobservable this symmetry typically is in music practice and 18th century experiments. Nonetheless, when looking at "all possible" classifications of pitch collections in 12 tone space, the complete set of canonic transformations, or even the "complete"

[31] Nonetheless, ways have since been found to generate undertones.

set of klangs that Riemann defines, this lens of symmetry is helpful. All chords are either symmetrical or can be paired with their intervallic inverse.

Actually, group theory, the mathematical language later invoked to deal with these complete sets of transformations and/or or musical objects, initially arose out of investigations into symmetrical mathematical structures. In a similar way, Neo-Riemannian theory is a formalization of Riemann's theories, Rameau's assignation of importance to symmetry, and an abandoning of earlier theorists' mandates/assumptions that contemporaneous musical practice must evolve out of "overtone" considerations.

Nonetheless, should you choose to learn/imagine vast swaths of the set classes introduced here, it is helpful to bundle them into symmetrical pairs or single units (set classes) and seek out other, perhaps higher-level structures that also exhibit overall symmetry. For instance,

- sc 4-21* (0, 2, 4, 6), sc 4-24* (0, 2, 4, 8), and sc 4-25* (0, 2, 6, 8) are the 4-note subsets of the sc 6-35*, which is represented by the whole tone scale.
    - 4-21* and 4-25* reflect complementary ways in which one can select two dyads from a particular whole tone scale.
    - 4-24* is the only way one can pair a trichord and a not-adjoined pitch from a particular whole tone scale.
- Again, even though in practice, two instances of sc 4-25* and sc 4-21* may not make the just mentioned complementary/symmetrical relation apparent; audiating this auditory unit of size 0, a label/pointer, may still be helpful.

This takes us to an even larger point. Set classes can be thought of as a both an audible chord and a label or trace/record of the structure of 12-tone space. sc 4-28*, a diminished 7th chord, can be heard and is an indicator of an underlying 4-fold symmetry that underlies 12-tone space. Quinn, following in Lewin's footsteps exploits this in his "solution" to similarity relations (Quinn 2006 & Quinn 2007). Classifying all chords/set classes of a given cardinality in regard to the proportion to which they exhibit 12-tone space's underlying symmetries, provides a reliable metric by which to compare chords/set classes. Similarly, this proposed pedagogy will prompt you to associate set classes with both types of designations.

Part II, Chapter 9: <u>2.1</u>

Section (A_1, R_{11}); $(x+y)^1$.

R:

This is the last section where only a single chord/set class is considered. Unlike the last section, which viewed its single chord/set class as a frame for what is possible, this section (and the following ones) focuses on what that set class, 11-1* (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A), is. It turns out that there are 12 different transpositions and 11 different rotations of 11-1.

Due to 11-1*'s having transpositions, this is a good place to discuss the overtone series (or its approximation) and ways in which this pedagogy can incorporate it. As a fundamental changes, so do the pitches associated with its overtone series. Accordingly, an overtone series' proportions represent relations between a fundamental's (a string length) partials (fractions of that string length), not fixed pitches (such as A or F). So, in regard to a single fundamental, how many partials are needed to represent all of the pitches within an octave? In the 5-limit just intonation rendering of the overtone series the pitches/proportions are:

	C	C#	Db	D	D#	Eb	E	E#	F	F#	Gb	G	G#	Ab	A	A#	Bb	B	B#	C
Ratio	1	25/24	16/15	9/8	75/64	6/5	5/4	32/25	4/3	45/32	36/25	3/2	25/16	8/5	5/3	125/72	9/5	15/8	48/25	2

Figure 2.1 1

To reduce this to a 12-tone scale, l have retained only the enharmonic spellings whose associated numerator is lowest. So, between C# (25/24) and Db (16/15), only Db is retained. The reduced scale's highest partial is 25 (culled from the 5^{th} octave above the fundamental). However, all of the ratios given below reside in the span of one octave:

	C	Db	D	Eb	E	F	Gb	G	Ab	A	Bb	B	C
Ratio	1	16/15	9/8	6/5	5/4	4/3	36/25	3/2	8/5	5/3	9/5	15/8	2

Figure 2.1 2

One could then rank the transpositions of sc 11-1*. The higher the numerator of the missing pitch is, the lower the rank of the transposition. For instance, if the transposition of 11-1 is G (7,8,9,A,B,0,1,2,3,4,5), then the missing pitch is 35/25 — the ratio whose numerator is highest. Therefore, the G transposition is assigned the lowest rank. Here is the complete ranking of the transpositions, from lowest to highest: G, D, C, B, Eb, Ab, E, F, Bb, Ab, Db, Gb. Why did I construct this ordering? As far as I know, there is no precedent for this in the literature. It's introduction here is simply to encourage you to think about set classes more flexibly and foreshadow a topic further explored in the A: section of this chapter — the overtone series.

11-1* as a label can represent a chord; however, it can also represent a melody, a harmonic field, a chord whose tunings may vary, and even a chord, shown above, whose different transpositions have different levels of signification associated with them. Viewed in this light, a set class label can offer a helpful inroad towards making sense of pitch collections. Not by divorcing one's sense of set classes from all but their technical definition but instead by embracing one's broader sense of them. Internalization requires deeper exploration; and, the vehicle for this deeper exploration can come in as many shapes as you can imagine. For example, this point of view asserts that your sense of a major triad is enriched by, not corrupted by, how familiar you may be with it.

Relatedly, musical spaces can also be construed as consisting of one or more copies (e.g., sc 11-1* and correspondingly, the Tonnetz) of single chord, an algorithm (e.g., Perle's cyclic arrays), or even a pitch (e.g., in Klumpenhouwer networks — each node of the network is a single pitch, 1-1*.

About the notation used prominently in this pedagogy:

This pedagogy will widely adopt jazz terminology. Jazz nomenclature is a helpful tool for describing set classes. It frames set classes in terms of familiar 7th chords. Then, each rotation of a set class's component scales (e.g., sc 7-32's component scales are harmonic minor and harmonic major) has its own label. For instance, mode 1 of 7-32a is MinMaj^{2,4,5,b6}.

Relatedly, how helpful is figured bass notation? Jazz nomenclature is actually rooted in figured bass notation. One significant conceptual difference is that figured bass notation doesn't imply a harmonic field; it just specifies which, potentially altered, scale degrees (of some diatonic scale) should be sounded in regard to a bass. If additional specification is needed, information is also provided on how those scale degrees should resolve.

Taking a cue from figured bass notation, set classes do not need to be thought of as 12 (or 24) potential transpositions or inversions of a chord (or two chords). Instead, their labels can be interpreted as a range of "acceptable" altered or unaltered scale degrees in the context of some underlying scale. Diatonic set theory explores this;[32] here, the underlying scale is the diatonic scale. However, should you adopt this approach though, the 7th chord labels will be less specific. Nonetheless, my above argument stands—broader understandings of a set class should be seen as enriching, not corrupting, one's understanding of the set class; I still recommend (especially when inspired to do so!!) learning the set class headings (4-1*, 5-3, etc.) and letting them be enriched by an alternate figured bass interpretation.

While it may seem cleaner to adopt an adaptation of jazz nomenclature that doesn't specify a 7th chord and relates everything to Major—as such, describing sc 11-1* as {b2,2,b3,3,4,5,b6,6,b7,7},[33] a significant downside (I contend that it is too significant) is the longer string of numbers to remember. We would have to specify at least 10 numbers for sc 11-

[32] For example, there are 4 different ways to select three notes from an equal-tempered scale of cardinality 7 {0,1,2,3,4,5,6}: **{0,1,2}** = {1,2,3} (a transposition) = {2,3,4} (a transposition) etc. **{0,1,3}** = {1,2,4} (a transposition) = **{0,2,3}** (an intervallic inverse) = {1,3,4} (a transposed intervallic inverse) etc., {0,1,4} = {1,3,4}; **{0,2,4}**. Now project these projected these set class labels (in an equal-tempered scale of cardinality 7):

- **{0,1,2}** can be associated with 3-2ab and 3-6*.
- **{0,1,3}** can be associated with 3-4ab, 3-7ab, and 3-8ab.
- **{0,1,4}** can be associated with 3-5ab and 3-9*.
- **{0,2,4}** can be associated with 3-10* and 3-11ab.

[33] I'm assuming that the first scale degree never varies.

1* rather than 8. In terms of getting these set classes to be accessible and memorizable/learnable, two fewer numbers to memorize (instead, they are bundled into the chord heading) makes a huge difference. Also, note that this jazz nomenclature indicates to the reader that there are no inflections of the 1st, 3rd, and 7th degree; only of the 2nd, 4th, 5th, and 6th degrees.

Ultimately, once you gain familiarity with a chord, it can start to gain an identity of its own in your creative consciousness. Forte's article on "Liszt's experimental idiom and Music of the Early Twentieth Century," shows how Liszt's usage of certain symmetrical chords changed over his life. In earlier periods, circa 1840's, they were sonorities—appearing to be the consequence of voice leading motion and contextual harmonic/phrase considerations. By the 1870's, Forte argued that these sonorities began to act as agents in their own right; even being used thematically (Forte 1987). In other words, some set classes may be bundled up with various tonal associations now, but if you continue to create music with these set classes (or even similar ones), you may notice that the tonal connections cease to be the only or even the most significant associations that you may build with them.

In short, get these set classes into your body; and then let your creativity work its magic. I see no benefit in attempting to regulate how you think about this musical material. Furthermore, if your interest in chords or set classes does not extend beyond their applicability to tonal music, that too is your choice. Therefore, mandating how you should think about these set classes, in order to "properly" understand them — for example, asking you to completely disassociate these set classes from their tonal associations — is both wrong-headed and not helpful; plus, it makes learning these set classes even more difficult. If you play Jazz big band charts, it's okay if every time you see some 5 or 6-note chord, you are able to test yourself on the name of the associated 5 or 6-note set class, quickly noticing many associations perhaps hidden before between these complex voicings' structures. Noticing these previously hidden associations may even inspire a new composition of yours or give additional insight into the harmonic logic of Wayne Shorter's works.

Figures describing sc 11-1* and 1-1* (Figures 2.1.3a, 2.1.3b)

Other name	10 note subsets	Maj7	MinMaj7	Dom 7
2-2*(1st)(1➔11)	1*,2*,3*,4*,5*,6*	(**2**:b2,2,#4,5,b6,6)	(**8**:b2,2,4,#4,5,b6,6,#6)	(**1**:b2,2,#2,4,#4,5,b6,6)
		(**3**:b2,2,#2,4,#4,5,b6,#6)		
		(**4**:b2,2,#2,4,#4,5,6,#6)		
		(**5**:b2,2,#2,4,#4,#5,6,#6)		
		(**6**:b2,2,#2,4,5,b6,6,#6)		
		(**7**:b2,2,#2,#4,5,b6,6,#6)		
		(**9**:b2,2,4,#4,5,b6,6,#6)		
		(**10**:b2,#2,4,#4,5,b6,6,#6)		
		(**11**:2,#2,4,#4,5,b6,6,#6)		

Other name	
A single pitch	(**1**:7)
	(**2**:#6)
	(**3**:6)
	(**4**:b6)
	(**5**:5)
	(**6**:#4)
	(**7**:4)
	(**8**:3)
	(**9**:#2)
	(**10**:2)
	(**11**:b2)

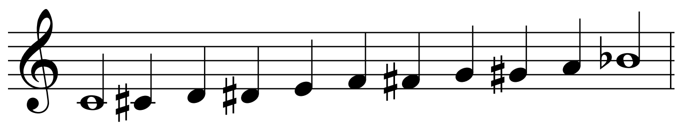
Figure 2.1 3(a-b)

How do we read this chart?[34] In Figure 2.1.3.a, we have a description of sc 11-1* in all of its modes/rotations. In Figure 2.1.3.b, we have a description of sc 1-1*. Let's focus on the left side, sc 11-1*, first.

The "other name" provides a description of the prime form of sc 11-1*; showing that it is bounded by the "1st inversion" of sc 2-2* (0, t), which is a major 7th dyad (2, 0) — here, I am

[34] In his Complete Thesaurus of Musical Scales, Yamaguchi also lists the various Forte-labeled set classes in regard to their rotation/mode and scale degree—with the major scale as a reference (Yamaguchi 2006, 6–63). Unlike in this pedagogy, Yamaguchi provides each set class's: associated interval vector; all of its rotations; and only some of the maximal subsets.

interpreting the label 2-2* as an ordered scale or chord, not as an unordered set class; “(1➔11)” indicates that an 11-note cluster lies between this ordered collection’s lowest (first) and highest (last, eleventh) note (look at figure 2.1.4). Note that “1” (not “0”) is chosen to refer to the lowest pitch. This choice is in reference to the convention of labeling the primary rotation of a scale as mode 1 rather than mode 0. This short-hand expression of prime form proves helpful as the number of learned set classes increase — it also provides a short-hand way of describing less familiar chords in terms of more familiar ones. Note that, in later chapters, the “other name” signification is sometimes adapted.[35]



2-2* is read as a major second. In this case, Bb-C. When taking the “1st inversion” of the dyad Bb-C, one gets C-Bb, the outer notes of the cluster that 11-1* (that too is presented as an ordered scale— **not** an unordered set class) spans.

Figure 2.1 4

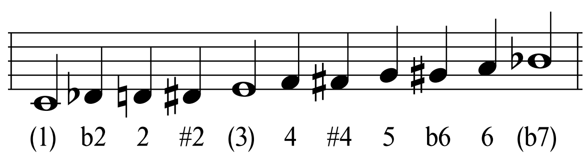
[35] Most notably, I often describe a set class in terms of a well-known superset. For instance,

- 5-33* can be described as the first 5 degrees of the whole tone (WT) scale WT(2)(1➔5);
    - In this case, the numbers in (1➔5) refer to scale degrees in the superset, rather than order position in the described set class.
    - The ‘(2)’ indicates that the super set is navigated by step.

When the “other name” navigates the superset by step, the “other name” is equivalent to prime form. When the “other name” navigates by other generic intervals, either a (3) or (4), the result must be converted into prime form.

- 5-34*, described as Maj (3)(5➔6), is to be interpreted as starting on the 5th degree and then ending on the 6th degree of the major scale
    - In this case, the set class is navigated in 3rds: 5-7-2-4-6 of any major scale.
    - The ‘(3)’ indicates that the super set is navigated in 3rds.
    - To convert the result into prime form, reorder the scale degrees into 4-**5**-**6**-7-2
        - This will always be the pattern, 4 generic steps followed by a generic third. The outer pitches in (5➔6), 5 and 6, respectively, are in order positions 2 and 3 of the associated prime form.
- 5-28A, described as Mm (4)(1➔6), is to be interpreted as starting on the 1st degree and then ending on the 6th degree of the melodic minor scale.
    - In this case, the set class is navigated in 4ths: 1-4-7-3-6 of any melodic minor scale.
    - The ‘(4)’ indicates that the super set is navigated in 4ths.
    - To convert the result into prime form, reorder the scale degrees into **6**-7-**1**-3-4
        - Except for one exception given below, this will always be the pattern; 3 steps separated from a following 2 steps by a 3rd.
        - In the case of 5-30a, take the 3rd inversion of the given pattern to get its prime form: (3rd)3-4-5-7-1 = 7-1-3-4-5 (5-30A’s prime form).
            - In the description of Mm(4)(5➔3)+, the + is given to signify that it is the exception.

The labels Maj7, MinMaj7, and Dom 7 can be used to describe the rotations/modes[36] of sc 11-1* listed in the rows below. These "framing" tri-chords—in 'C' respectively, 3-4a (C,E,B), 3-3a (C,Eb,B), and 3-8a (C,E,Bb), account for the $\hat{1}$, $\hat{3}$, and $\hat{7}$ scale degrees of the described chords. Notice that $\hat{5}$ is never assumed. For example, the "first rotation" of sc 11-1*, its prime form, is described in the Dom 7 column. You can tell this from the far-left number in bold, which is followed by a colon. This chord's label is then (**1**:b2,2,#2,4,#4,5,b6,6) signifies a Dom 7^{th} with a b9, 9, #9, 11, #11, 5, b13, and 13. It also signifies a Dom 7^{th}, with a b2, 2, #2, 4, #4, 5, b6, and 6 (see Figure 2.1.5). You choose at which octave you want to think of scale degrees/extensions. Ultimately, the goal is to help you relate new chords/set classes to chords/set classes that you already know well. I argue that this is the quickest way to learn new material. It's up to you how many associations, such as their typical usage in a particular genre, that you want to bring forth.



Note that this could be given at any transposition level

Figure 2.1 5

Furthermore, the usage of these labels is meant to help you build on chord classification methods in which you are likely already very well versed in; it does not reflect an agenda to (or not to) import any tonal associations. Like the keyboard itself, whose white keys represent a major scale, this labelling's usage of scale degrees defaults to those scale degrees associated with the major scale:

[36] "Mode" and "rotation" are both used as there is an additional ambiguity in the usage of these set class labels — are they being used as chords or as scales. When they are construed as scales, the term mode is the best fit; when construed as chords, rotations (itself, a substitute for the ambiguous term "inversion") is the better fit. While, it may be clear cut that the terms scale may better apply to 11-1*, in other cardinalities, such as with the hexads, it is less clear; e.g., it is common to find six-note chords and melodic patterns.

- the MinMaj heading specifies $\hat{1}$, b$\hat{3}$, and $\hat{7}$ so that the 1st and 7th scale degrees are the same as Major and the 3rd is flatted; similarly,
- the Dom 7 heading specifies $\hat{1}$, $\hat{3}$, and b$\hat{7}$. The other labels used,

Dim7 and Min7, are discussed in later chapters.

- Min7 specifies $\hat{1}$, b$\hat{3}$, and b$\hat{7}$,and
- Dim 7 specifies $\hat{1}$, b$\hat{3}$, and bb$\hat{7}$.

Respectively, Min7 and Dim7 are represented by the set classes 3-7a and 3-10*.

Finally, at the bottom of each of these charts are a listing of the prime forms associated with each "a" and "b" form of the various set classes. Note that the form given in the "other form" column is typically identical to the prime form. However, this is not always the case. The main exceptions regard the 5 note set classes that are described as either stacked 3rds or 4ths and those set classes, whose given prime form has two asterisks next to them. In the latter case, those normal orders associated with the "b" representation are not a reflection, about the outer pitches, of their respective, "a" representation's prime form. Note that, when the associated prime/normal order has two asterisks beside it, the ordering of the modes listed are in regard to the "other form" given in the chart.

About sc 11-1*:

A significant take-away from this introduction to sc 11-1* (Figure 2.1.3a) is that it utilizes all but one of the inflections in scale degrees associated with the given chord heading (such as MinMaj). By tracking the various modes/rotations, one will soon notice that as the number of the rotation ascends, the missing inflected scale degree descends. In other words, the right hand-side of the chart, detailing sc 1-1*, documents this explicitly.[37] I recommend memorizing sc 1-1*'s

[37] Note though, that, in the chart, 1-1* is being expressed as the complement of 11-1* rather than a transposition of a pitch. Therefore, the rotation is "the entire chromatic scale in adapted jazz nomenclature minus a specific inflected scale degree." Since a single note isn't enough to reference any 7th chord, I've arbitrarily assigned a Maj 7 column. However, there is no implied trichord! The occasional parentheses around $\hat{1}$, $\hat{3}$, and $\hat{7}$, simply indicate that those scales are contained within the associated (but not sounded!) Maj 7 trichord.

“rotations” immediately. As will be later fleshed out, it offers an additional classification structure to help memorize and understand the 10-note set classes and their many associated rotations. This will be further discussed in the next section.

Where can we find sc 11-1* or other similarly large representations of musical space that may or may not be sounded simultaneously? The Melakarta is one example. The Melakarta is a classification system for South Indian ragas. It associates ragas with 1 of 72 melas (7 note scales). The first 36 melas can utilize all pitches but scale degree #4 (sc 11-1* rotation 6); the last 36 melas can utilize all pitches but scale degree 4 (sc 11-1* rotation 7). While there are additional considerations pertaining to how to derive a 7 note mela from these two rotations of sc 11-1*, the essential point is that in totality, each set of 36 melas covers a single rotation of 11-1*.

I recommend learning each rotation/mode of sc 11-1*. Furthermore, learn to recite these rotations as fast as possible. When the speed goes beyond what you can vocally articulate, mentally recite them. If you can, rotate them via a visual image of how you would execute this on your instrument—piano is a helpful image. As every other set class is a subset of this one, doing this will make all other set classes easier to say. Plus, there is much information to be gleaned from exactly how the rotations of those subsets manifest as portions of sc 11-1*s (when construed as a component scale) rotations/modes. While there is a logic to how rotations of a subset relate to the rotations of a superset, it’s not always obvious or easy to see. Nonetheless, in some cases it is very easy to see, and the same easy-to-see logic may be shared amongst groups of subsets.

For instance, track the 10-note subsets of sc 11-1* (discussed in the next chapter) that have a rotation/mode with (b2,2), (b2,#2), and (2,#2). These are sc 10-3* (rotations/modes 6, 7, 10), sc 10-4* (rotations/modes 5, 6, 7 and 8, 9, 10), sc 10-5* (rotations/modes 4, 5, 6 and 8, 9, 10), and sc 10-6* (rotations/modes 3, 4, 5 and 8, 9, 10). In each of these cases, the sequential ordering of their appearance stays the same. Why this occurs can easily be deduced from a discussion broached in the next section. Moreover, over time, combinations of numbers, such as 4, 5, b6, #6 will cease to be seen as random numbers. Rather, they will just be seen as a common familiar label that adjoins families of set class numbers of various cardinalities. You will know this label so well and may even be able to audiate it. You will likely have developed many associations with what “it means” when a set class contains that combination of numbers.

For most people, how fast you can say something, reflects how fast you can think it. If it takes you 20 seconds as opposed to 1 to mentally recite a set class label in jazz nomenclature, it may take you several minutes rather than a few seconds to compare larger set classes. The burden of the slow processing time could dissuade someone from learning to do such and severely limit the number of set classes compared and the quality of that comparison. Furthermore, the more automatic your conjuring of these set classes is, the more concrete these set classes may feel.

For many, at least me, the more concrete something “feels,” the more accessible I consider it to be. For instance, take a car. It’s a modern engineering marvel. Increasingly, with the advance of technology, its inner workings are impenetrable; professional training is required to both diagnose its problems and partake in the fixing of them. Yet to most who use it to parallel park, navigate city streets, and brave highway traffic, the car feels straightforward. First off, how we function in regard to it—mostly, two pedals, a shift, and a steering wheel, can avoid any reflection on the engine’s internal composition. However, there is still nothing simple about the amount of processing our brain needs to do to avoid a car wreck each time we commute to work; whether it is reflecting on how combustion works or how to avoid the errant pedestrian.

Along those lines, we often calibrate difficulty by the amount of attentional resources we devote to something. Even though driving is an example of mental dexterity, as so much of it is “automatic,” experienced drivers may not feel that it is as challenging as it is. Similarly, for those who have not internalized the triads through playing them, recognizing them in their various voicing configurations is typically extremely difficult—what key is it in? what is the rotation? And, is it in major or minor? If that is not how you feel now, it’s likely partially due to how automatic triad recognition has become for you.

A:

As this section focuses on hearing a single pitch, a skill that I will assume that all readers here already have, let's instead enrich your notion of that single pitch—evaluating it as a fundamental and spectra of partials. Stockhausen's piece Stimmung, for six voices and microphones, does just this, the entire piece explores the overtones of an inaudible Bb pitch. Alternatively, how have theorists sought to intentionally embed musical practice in the overtones, the latent potential, of a single pitch? Below is a short review.

While there have been maps that assigned fixed ratios to each potential pitch—for instance, the harmonists, who were reported as having attempted to reconcile the variation of musical practice with a map assigning each potential pitch a fixed ratio—prior to Rameau, there have not been any theorists, as far as we know, who intended to frame all pitch-based musical activity as a consequence of the overtone series[38] (corps sonore). Rameau posited that all melodic and harmonic motion could be explained in regards to the intervals—a P5, M3, and m3 —that constitute a major triad. His commitment led him further though. Not prioritizing an "overtone" over an "undertone" description of practice,[39] could have initiated the dualistic practices discussed earlier. Rameau used this "dualistic" theoretical perspective to explain: cadences; how to understand non-triadic tones (such as the 9th and 13th); derive the major and minor dissonances (M7 and m7), and introduce functional theory—explaining all triads in relation to the triple proportion (1:3:9). "1" = IV is complementary to "9" = V about "3" = I. Ultimately, his musical analyses focused exclusively on how triads, consonances, and dissonances were juxtaposed (Rameau 1971; T. S. Christensen 2002b).

Vogler, considered the first scale step theorist, focused on the optimal scale's tuning. He endeavored to root all musical proportions in the corps sonore. First, using the 8th to 16th partials, he constructed the "natural" major. Out of this scale and Rameau's use of the triple proportion, he develops the "artificial" major and minor scales. For instance, the A minor scale is composed of Dm, Am, and Em triads. Every triad (even the diminished) was treated as a

[38] Previous theorists such as Zarlino and Gaffurius had described pitches in regard to string divisions.
[39] The 5^{th} is 3:2 in regard to the fundamental and the fundamental is 2:3 in regard to the 5^{th}, 3:2; $3/2 \, x \, 2/3$ = 1.

fundamental element of the scale—everything else as a deformation; and modulation, especially chromatic modulation, was treated as the exploitation of ambiguities in a particular chord's function (T. S. Christensen 2002b).

Similarly, Gottfried Weber created a 2D map indicating how closely, in terms of tonal function, these triads were related. On the x axis, two adjacent chords are either in parallel or relative relation to each other (C:c vs. C:a). On the y axis, two adjacent chords are a P5 apart. Later, given below (Figure 2.1.6), Schoenberg essentially reinterpreted this same map substituting closely related keys for closely related triads (T. S. Christensen 2002b, 804). As mentioned in section 2.1, these maps are useful sites upon which to explore your understanding of certain set classes.

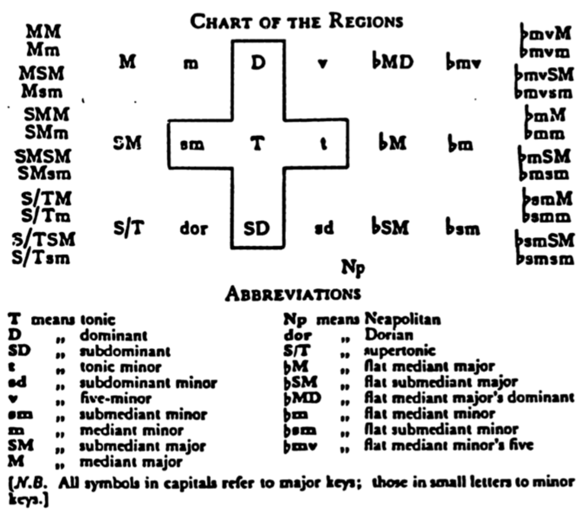


Plate 25.2 Schoenberg's "Chart of the Regions" from *Structural Functions of Harmony*, p. 20.

Figure 2.1 6

Like Rameau, Schenker sought to root "good" musical practice in the overtone series more generally, and the triad more specifically. However, unlike Rameau, Schenker's rationale for the primacy of the triad—everything arises from its unfolding, appears more metaphysical than "scientific." The *Urlinie* aligns a fundamental bass motion with descending scale fragments—whose ranges coincide with the ruling triad's intervals. Conceived in an energeticist context, the *Urlinie* is meant to be a guidepost which "conceals within itself the seeds of all forces that shape tone life (T. S. Christensen 2002b, 937)." It imparts life to motive and melody and it signifies motion and striving towards a goal. It also projects the logic behind how different tones behave in a tonal context.

Additionally, in his book Harmony, Schenker further establishes the importance of the major triad by: prioritizing non-shadow intervals from the overtone series (progenitor to partial rather than partial to partial); forgoing the consideration of overtones (7, 11, 13, and 14) that are "rejected by the ear;" anchoring harmonic motion in fundamental bass movement by a P5 (the first non-tonic pitch in the overtone series); and, after admitting one descending 5th motion, constructing the major scale out of fixed scale-degree triads that are a P5th apart. These triads are a compromise between the various P5-related major triads, and the needs of a community of seven tones. If a tone of a scale degree's major triad is not found in the tonic, subdominant, and dominant triads, then it is altered so that it is.

Finally, Hindemith, also seeking to uncover natural laws, treats the chromatic scale as primary, not the major scale, and he seeks to explain all harmonic music (tonal and non-tonal) resulting from the chromatic scale. The crux of Hindemith's theory is a method for rating consonance. It's based on the relative consonance of the given pitches in regards to a fundamental (series 1) and the relative consonance of the intervals themselves (series 2).

Series 1 consists of the fundamental's first 6 partials (3 pitches) and the first 6 partials of other fundamentals associated with the fundamental's first 6 partials. Hindemith uses series 1, and the series 1 of nearby fundamentals, to locate the best pitches for his default chromatic scale. However, shortness of derivation is only one factor; he also considers relative spacing between adjacent pitches and the equality of enharmonic spellings. To order series 2, Hindemith considers combination tones. Combination tones are tones that sound when an interval is played. If the interval is described as two partials of a shared fundamental, the combination tone can be determined by taking the difference between the two partials' ordering in the overtone series. Translate that difference in terms of the same fundamental's overtone series and one gets the combination tone. The closer an interval (condensed into a single octave) is to the beginning of series 2, the lower its two partials are. Intervals are grouped into pairs with their complements.

Part II, Chapter 10: 2.2

Section (A_2, R_{10}); $(x+y)^2$.

R:

This and the remaining chapters in part II will focus more on the learning and understanding of the set classes than on larger questions of framing. **As with sc 11-1*, I recommend memorizing the 10-note set classes as soon as possible. Aim for automaticity**—even before moving onto the next section.

General notes:

1. Notice that none of the "prime" rotations of the chords associated with each set class fall under the Maj 7th heading — this is due to the definition of prime—the "most compact" version of a chord.
    a. Since each 10-note chord/set class is two shy of the aggregate, there will always be at least one rotation that does not include the highest pitch(eleven). Hence, the prime form, at minimum, will not will **not** be represented by some type of Maj 7th chord.
2. I have also included the 9-note subsets that are contained in each 10-note set class.
    a. The number of repetitions of a subset indicates the number of distinct ways that that subset can be culled from the superset.
        i. For instance, under the heading sc 10-1*, there are two examples of 1* (meaning 9-1*) and one instance of 2 (sc 9-2), 3(sc 9-3), 4(sc 9-4), and 5(sc 9-5).[40]

[40] In the just mentioned instance, why are there only 6 represented set classes, when there are 10 ways to remove a note, hence creating a 9-note subset, from a 10-note set class? This is due to a few reasons.

1. First off, 10-1*, a chromatic cluster, is symmetrical.
    a. This means that there exists at least one representation (a rotation and spacing) of 10-1* such that pairs of its elements (a_n, a_{9-n}), where $0 \leq n \leq 9$, sum up to the same constant.
        i. In the case of 10-1*, whose elements expressed generally are (a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9), one symmetrical representation is <0, 1, 2, 3, 4, 5, 6, 7, 8, 9>. Here, 0 + 9 = 1 + 8 = 2 + 7 = 3 + 6 = 4 + 5.

   ii. In total then, 10-1*, a symmetrical set class contains ten 9-note subsets: 9-1* twice, 9-2a, 9-2b, 9-3a, 9-3b, 9-4a, 9-4b, 9-5a, and 9-5b. Nonetheless, in order to save space, I've only listed 6.

3. However, when accounting for the n-1 subsets of a non-symmetrical set class, there can be up to n distinct subsets. We will encounter this in the next chapter.
   a. Unlike the 10-note set classes, which are all symmetrical,[41] many of the 9-note set classes include pairs of inversely-related component scales.
   b. In these cases, if one of the 9-note component scales (e.g., 9-2b) of a non-symmetrical 9-note set class (e.g., 9-2) contains an 8-note component scale (e.g., 8-11b) of a non-symmetrical 8-note set class (e.g., 8-11), then the other inversely-related 9-note component scale (e.g., 9-2a) contains the other inversely-related 8-note component scale (e.g., 8-11a) of the 8-note set class (e.g., 8-11) in question. Further explanation and examples will be given in the following chapters.

My recommendation now is to examine the set classes listed, but do not worry about internalizing or even learning those set classes yet; they will be better discussed in the following sections. Actually, due to the format of part II, the subsets of the set classes greater than 6 will always be discussed in the chapter following their presentation. Integrating your understanding of various subsets and their supersets can become a greater focus in your second and third etc. review of these sections/associated charts.

Notice that the "other names" may refer to many set classes that you are not yet familiar with. Let me provide those here. The presentations given are the prime forms. The rotations/modes given are in regard to these prime forms.

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2. This, in turn, implies that, when treated as scales, the 9-note subsets (9-1*), <0, 1, 2, 3, 4, 5, 6, 7, 8> and <1, 2, 3, 4, 5, 6, 7, 8, 9> are inversely related; they retain the same axis of symmetry and their differences from 10-1* are mirror images of each other.
   a. The first subset loses the last pitch of the provided symmetrical representation of 10-1*; the second subset loses the first pitch.
3. For all of the non-symmetrical 9-note subsets of this symmetrical set class 10-1*—9-2, 9-3, 9-4, and 9-5, there will instead be two inversely-related component scales/chords.

[41] The complement theorem can help explain this. Complements share certain structural properties with each other—namely there is a correspondence between the number of each type of interval that they contain. Notably, if one is symmetrical, so is the other (Hook 2023, 273–74). The complements of the 10-note set classes are the dyads, all of which are symmetrical.

- Regarding 10-1*:
    - 2-3* should first be learned as <0,3>;
    - 2-3*(1^{st}) is <3,0>;
    - 2-3*(1^{st})(1➔10) is then <3,4,5,6,7,8,9,t,e,0>.
    - Starting at 0, this would be <0,1,2,3,4,5,6,7,8,9>
- Regarding 10-2*:
    - 3-6* should first be learned as <0, 2, 4>
    - 3-6*(2^{nd}) is <4, 0, 2>
    - 3-6*(2^{nd}) (1➔9) is then <4,5,6,7,8,9,t,e,0,2>
    - Starting at 0, this would be <0,1,2,3,4,5,6,7,8,t>
- Regarding 10-3*:
    - 4-10* should first be learned as <0, 2, 3, 5>
    - 4-10*(3^{rd}) is <5, 0, 2, 3>;
    - 4-10*(3^{rd})(1➔8) is then <5,6,7,8,9,t,e,0,2,3>.
    - Starting at 0, this would be <0,1,2,3,4,5,6,7,9,t>
- Regarding 10-4*,
    - 4-21* should first be learned as <0, 2, 4, 6>;
    - 4-21*(3^{rd}) is <6, 0, 2, 4>;
    - 4-21*(3^{rd})(1➔7, 8➔10) is then <6,7,8,9,t,e,0,2,3,4>.
    - Starting at 0, this would be <0,1,2,3,4,5,6,8,9,t>
- Regarding 10-5*,
    - 4-23* should first be learned as <0, 2, 5, 7>;
    - 4-23*(3^{rd}) is <7, 0, 2, 4>;
    - 4-23*(3^{rd})(1➔6, 7➔10) is then <7,8,9,t,e,0,2,3,4,5>.
    - Starting at 0, this would be <0,1,2,3,4,5,7,8,9,t>
- Regarding 10-6*,
    - 4-25* should first be learned as <0, 2, 6, 8>;
    - 4-25*(3^{rd}) is <8, 0, 2, 6>;
    - 4-25*(3^{rd})(1➔5, 6➔10) is then <8,9,t,e,0,2,3,4,5,6>
    - Starting at 0, this would be <0,1,2,3,4,6,7,8,9,t>.

For two reasons, in future chapters, I will not explicitly provide the process associated with making sense of a set class's prime form's "other name."

1. There would be a ton of extra pages that most would probably not ever read through.
2. There is an advantage to you working through these processes on your own**. One of the biggest skills associated with this curriculum is "rotating set classes" in your head**.
    a. For most, this is not a challenge when there are only 3 elements. For 5 and more pitches, it is most likely a challenge for most.

Nonetheless, should you work extensively through this curriculum, by the end you should be much quicker with rotating all of the set classes; especially, in regards to their prime forms (or the transposition associated with the given "other name"). Furthermore, other knowledge about the properties of specific set classes in question could help.

First note that a comparison can be made between Forte's numbering and the "Dewey Decimal system."

1. Forte ranks set classes by their concentration and distribution of intervals in their associated interval vector.
2. Here is a "Dewey Decimal way" of labeling set classes (treated here as pc sets); describe them as the path traversed from 12-1* to them.
    a. To see this, create a "decimal" as such:
        i. the "10^{th}'s place" refers to the missing pitch of the aggregate that defines the rotation of 11-1* (section 2.2)
        ii. the "100^{th}'s place" refers to the "missing pitch" that connects 11-1* to its 10-note subset of interest.
        iii. the "1000th's place" refers to the missing pitch that connects "100th's place's" 10-note subset to its following "1000th's place" 9-note subset of interest. Etc.

Here is an example: take .8t2.

- 8 refers to the pitch-class removed from 12-1* to create the rotation of 11-1* of interest (rotation 4).
- t refers to the pitch-class removed from 11-1* (rotation 4) to create a 10-note subset of interest (10-2*).
- 2 refers to the pitch-class removed from the 10-note subset of interest to create the 9-note subset of interest (9-8b).

Note that there are multiple other "dewey decimal numbers" that can yield 9-5b from 12-1*: e.g., .864.

Alternatively, using the just given example, you could label the pathway from 12-1* to 8-5b, in terms of the Forte numbers traversed—put parentheses around Forte numbers higher than 9. This would be .148. (11-1*, 10-4*, 9-8)

When you this restrict this "Dewey decimal system" to two digits, it becomes apparent that the pattern, let's call it pattern A that relates the rotations of 11-1* to 12-1*, also relates the rotations of 11-1* to the 10-note set classes. Note that:

1. Just as the prime form of the 11-note chord and its rotations/modes never contain e (pitch class 11), the prime forms of the 10-note set classes never contain t or e;
2. Just as the "missing pitch" decreases as the number of the 11-note scale's rotation/mode increases (refer to chapter 2.2), the additional "missing pitch" of the 10-note set classes decreases as the Forte number of the 10-note set class increases.

However, pattern A breaks down more and more as the set class's cardinality veers towards 6 (from above and below);

1. Different non-linear principles organize the set of set classes of cardinalities 3 ≤ 9.
2. Only when the pathway from 12-1* to your set class of interest, also the chromatic cluster of a given cardinality, can be described as (1*,1*,1*, etc.) does the same relation hold.
   a. This is discussed further below.

As your knowledge of various set classes accumulates, tracing these pathways can be an important means of both deepening your understanding of a particular set class as well as each of those supersets/subsets to which you are comparing it.

<u>Patterns to take note of</u>:

There are often commonalities between Forte numbers whose ordering number (the number after the dash) is the same. The most pronounced is between those set classes whose ordering number is 1*: 10-1*, 9-1*, 8-1* etc.. It's helpful to think of the relationship between 11-1* and the 10-note set classes (10-1* – 10-6*) as the prototype for this particular pattern.

11-1* includes the cluster 10-1* and following set classes, 10-2* – 10-6*; in ascending order, each 10-note set class differs by a single decreasing pitch. This principle can be extended to 10-1* and its maximal subsets, 9-1* - 9-5* (or 9-1* and its maximal subsets etc.) as the Forte number increases, the "missing pitch" decreases.[42] Once you learn the superset's, e.g, 10-1*'s, different chord representations, such as Maj (2:b2,2,#2,4,#4,5,b6), you can predictably derive its subsets' different representations—for instance, 9-1* has Maj (2:b2,2,#2,4,#4,5); 9-2a has Maj (2:b2,2,#2,4,#4,b6); 9-3a has Maj (2:b2,2,#2,4,5,b6); 9-4a has Maj (2:b2,2,#2,#4,5,b6); and 9-5b[43] has Maj (2:b2,2,4,#4,5,b6). Notice, how the missing note decreases: respectively, in 9-1* this is b6; 9-2a, 5; 9-3a, #4; 9-4a, 4; 9-5b, #2. If one were to look at the component scales, 9-2b, 9-3b, 9-4b, and 9-5a, one could find an analogous process, wherein the "missing note" increases.

In general, there is a benefit to considering the set classes associated with the *minimal* supersets (n +1) of a scale that you are interested in. By examining these set classes' component scales'

[42] Mazzola's numbering system described earlier organizes these set classes (actually M5-classes) in the same manner. However, as his numbering system does not begin anew with each cardinality, it is more difficult to see this directly from the labeling.

[43] Why is there a "flipping" in orientation regarding 9-5? Well, it's a result of a very special property of it; best observed by examining 3-5. 3-5 consists of a tritone and P5 that are conjoined — e.g., C-F# and C-G. The tritone and its minimally perturbed P5, evidence multiple symmetries regarding the 12-tone space. 9-5 can be expressed as concatenated inversely-oriented versions of 3-5: C-F-B-E-Bb-Eb etc. C-F-B = 3-5a; F-B-E = 3-5b; B-E-Bb =3-5a etc. It takes all twenty-four component scales (twelve 3-5a + twelve 3-5b) concatenated to return to 3-5 in its original orientation and key. In short, 3-5/9-5 is oriented differently, witnessed through how its iteration maps onto 12-tone space, to all other set classes.

rotations/modes, one can see most,[44] if not all, of the ways in which the set class of interest can be enriched by the addition of an additional note and how this manifests in the context of each rotation accounted for. For instance, in the most straightforward case, any way in which you enrich a 10-note set class by a single note, yields 11-1*. Furthermore, as 11-1* is symmetrical, a symmetry can be found in the way in which the various 10-note subsets' T_n-classes relate to each other. For example, take this rotation/mode of 11-1*, Dom (**1**:b2,2,#2,4,#4,5,b6,6):

- 10-2* has Dom (**10**:[redacted],2,#2,4,#4,5,b6,6) and 10-2* has Dom (**1**:b2,2,#2,4,#4,5, b6,[redacted])
- 10-3* has Dom (**9**:b2,[redacted],#2,4,#4,5,b6,6) and 10-3* has Dom (**1**:b2,2,#2,4,#4,5,[redacted], 6)
- 10-4* has Dom (**8**:b2,2,[redacted],4,#4,5,b6,6) and 10-4* has Dom (**1**:b2,2,#2,4,#4,[redacted],b6,6)
- 10-5* has Min (**7**:b2,2,[redacted],4,#4,5,b6,6) and 10-5* has Dom (**1**:b2,2,#2,4,[redacted],5,b6,6)
- 10-6* has (**1,5**:b2,2,#2,[redacted],#4,5, b6, 6)

Since 11-1* representation was laid out symmetrically, it was easier to observe the manifestation of this symmetry via the ordering of the 10-note set classes. Generally speaking, if a chord, X, has a minimal superset Y* that is symmetrical, then both X and X^{-1} will yield Y* with the addition of a single pitch (oppositely positioned) in relation to it.

Although the above demonstration highlighted the how a simple rule applies to a host of set classes, in general, when learning the set classes, there tends to be certain quirks/exceptions that pop up when seeking to apply some simple rule more generally. This is because the space that judges distances between set classes, in terms of contained pitches (and is of cardinality 3 or less) is not Cartesian. Instead, it's better described via group theoretic principles or on a torus. Furthermore, once you deal with set classes that are greater than 3, you will need more than 3 dimensions to visualize them.[45] Hence, this pedagogy's focus on building up an intuition around

[44] My charts do not always describe every rotation of a scale.
[45] While Mazzola and others have come up with ways to help, or as in the case of Quinn, who taking account of the various symmetries at play, then offers suggestions on how we can project 12-dimensional pitch vectors into 3-dimensional space, these various visualization methods have not been embedded into this approach. I will include helpful maps, but cannot advocate for or against restructuring this pedagogy so as to imagine set classes as figures whose various limbs and joints accommodate higher-order cardinalities. Moreover, truly adopting some of these visualization methods would be so intensive and extensive that a central benefit of this method, its building on familiarity with common musical descriptions, would be lost. However, I would be interested to hear about any attempts to do so.

how various set classes relate, learning helpful "tendencies," or better put, heuristics. Over time, as you learn more set classes in a greater number of ways, you will refine your understanding and knowledge of when and where these "tendencies" break down. In other words, you will enrich your heuristics.

Finally, having just mentioned group theory, this is an opportune time to refer to a couple of Lewin's GISs (generalized interval systems) mentioned earlier. Lewin defines a GIS framework where the differences are proportional rather than fixed and, when dealing with timbres/spectra, he defines another GIS where an element of IVLS is the difference/relation between two spectra. As mentioned before, these two GIS recognize that there are differences between how we perceive musical change, often exponential, and how we understand it analytically.

Analogously, differences will develop between our understanding of various set classes and analytical insights first used to quicken their internalization. Not all pathways between a set class and its minimal superset may arise equally in your musical consciousness/practice. For instance, a major triad's (3-11b's) minimal supersets are 4-14, 4-17*, 4-18, 4-19, 4-20*, 4-22, 4-26*, 4-27, and 4-Z29. Yet, all but 4-17*, 4-18, and 4-19 are diatonic supersets. In other words, if you are exploring how the triad is encapsulated into 4-note subsets of the major scale, you will never run into 4-17*- 4-19. Similarly, if you are exploring how the triad is encapsulated into 4-note subsets of the hexatonic scale, you will only run into 4-17*, 4-19, and 4-20*. If you explore the triad through both scales, you will encounter all of 3-11's minimal supersets, but some will be more reinforced than others. The point is that your "perception" of how 3-11 transforms via the addition of a single note will deepen as you "enrich your heuristic" however, it will never evolve into either the best way or a completely comprehensive way to understand 3-11. Rather, it will just become the most enriched way that you have yet found in regard to your interests. Ultimately, the breadth of this pedagogy facilitates your enriching your understanding of these musical elements to whatever degree or level your curiosity impels you towards.

A:

This section is devoted to intervals. **Memorize all of their Forte*[46] names right away**. As mentioned earlier, intervals have also been imbued with metaphysical import. Hauer, considered "intervals, conceived as colors, [as] gestures and units in the spiritual plane that [they inhabit]." No novel methods are offered towards learning intervals. A few oft-used methods are listed below. They regard:

1. Learning intervals vis-à-vis their typical tonal context:
    a. Compare each note to the tonic (or dominant).
        i. e.g., a Maj 6th is found between $\hat{5}$ and $\hat{3}$ (from I), $\hat{1}$ and $\hat{6}$ (from IV), and $\hat{2}$ and $\hat{7}$ (from V).
2. Learning intervals independent to their tonal context.
    a. Compare each note to the previous.
    b. Associate each interval with a familiar song.
        i. The assumption being that one can transfer their knowledge of an interval in a familiar context to a less familiar, tonal or atonal, one.
3. Teaching intervals in regards to locations on an instrument.
    a. P5s relate the open strings on many instruments. Certain intervals are understood as distances between fingers in a particular register of the instrument.
        i. Horn or woodwind players may conceive of intervals in terms of fingering patterns on their instrument.

There are strengths and weaknesses associated with each of these approaches. Although, I will only compare and contrast the first two approaches. I know of no systematic method associated with the last approach, no. 3.

Approach 1's strength lies in its acknowledging that there is more to defining an interval than a single fixed ratio. As western singers and string players know, as well as musicians ensconced in musical traditions that do not privilege equal temperament, not all Maj 2nds are the same.

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[46] Disregard that Forte retreated from labeling set classes of cardinality ≤ 2 and ≥ 10. For our purposes, as alluded to above, it is helpful to learn all of these set class labels.

Assuming just intonation, refer back to the last section, D:C = 9/8, E:D = $\frac{5/4}{9/8} = 10/9$. As such, in many contexts, tuning inflections must be made to account for these subtle differences. Moreover, singing in tune often requires one to adjust to the entire harmonic field, not just the previous pitch. Approach 2, is pretty much oblivious to these types of distinctions.

On the other hand, approach 2 provides an inroad to non-tonal music that approach 1 does not.[47] As much non-tonal music presumes equal temperament, treating these intervals as fixed entities coincides with the way that many 20th century composers engaged musical pitch in their works. Furthermore, as various works, e.g., serial works, were composed so as to minimize any sense of a tonal center, attempting to interpret any note in terms of a fixed tonal center could be misguided.

However, in any live music performance, the intonation of human musicians using non-fixed pitched instruments will be affected by the room and the harmonic field. It would also be misleading to assume that the composer's envisioning of pitch completely captures the performed pitch. Finally, there are many musics, Western and Non-Western, that are sensitive to intervals smaller than the semitone and distinguish between those inflections, whether they are conceived of as gestures or as a fixed entity.

Intervals have also been used as a scaffold for melodic and harmonic dictation. During dictation, musicians will often be asked to first notate the outer voices and then fill in the inner. This pedagogy extends this approach by making an assumption; that from an aural standpoint, it

[47] In tonal music, there are more usages of a major 2 (or major 6th) than this method accommodates. Specifically, this occurs at the interstices of modulations. If in C major, the melodic pair E-F# arrives, how do we interpret the F#? If we are modulating to D major, it would be F#:E = $10/9$ in a key that is yet to arrive. If we are modulating to E major, it would be F#:E = $9/8$, yet again in a key that is yet to arrive. What if doesn't modulate? If the harmony changes while the E is being held — say from a I in C to a V^9 of D, does one in real time adjust the intonation of the held E? typically not. Let's say that the listener was perceptive enough and well-enough informed to be able to recognize such subtle shift in a held E as foreshadowing a specific modulation—even when the performing musician goes unaccompanied; for instance, a solo violinist not using double-stops; would that necessarily be the point? Sometimes, ambiguity is important. In short, even in tonal music, approach 1 has limits. How does it apply to a tonal piece played on a honkytonk piano? It depends on whether you think of the honkytonk rendition as being a deformation of what it should be, rather than similar in tuning scheme but not of the same fundamental type.

is helpful to think of more complex chords as more subtle expressions of simpler chords. As you accumulate chords/set classes that you are familiar with, you reflect on how they alter the timbre. Of course, many additional components affect timbre in a live setting. However, this pedagogy asserts that a lot of headway can be made teasing out some of the structural determinants. For instance, how do various voicing-related transformations affect our perception of a chord writ large? Well, if the "our" [in the last sentence] is "your", then, on your instrument of choice, over the course of your engaging this pedagogy, you may find a very meaningful answer, regarding your own listening practices, to a question that is intractable, when it's asked more generally.

One of the greatest benefits of learning all of the set classes (or, at least, a good number of them) is that you have an exceptionally large and finely gradated "data set" upon which you can test the bounds of your perception. Put another way, you create an opportunity to ask and test questions that you may care about, such as: Under what conditions am I unable to hear that an additional note has been added? Is this related to a specific voicing, whether the note added doubles the same note in another register, or the consonance/dissonance of the chord itself?

Furthermore, if you fully embrace this method, you will have some familiarity with all of the chords. One hiccup typically occurring in answering questions about one's abilities to perceive musical material relates to a disproportionate amount of exposure to certain chords, such as the major triad, and virtually no exposure (at least as a distinct unit) to most other things.[48] Developing a relatively uniform relationship to so many other set classes, could allow you to ask these questions in a less potentially skewed way. You could start with two (or many), previously unfamiliar, now moderately familiar, set classes and then record yourself playing them (or many) under some manipulation. "Rinse and repeat" with the same set classes and different manipulations. Come back a few months later, can you identify each of the manipulations used? When is this not the case?

Why is this important? Students trained to hear triads only in root position can have difficulty hearing triads in rotation. Typically, to overcome this, we train them to identify each rotation

[48] This may be an apt description: everything is construed as either close to a familiar chord or "completely" different.

separately and in a tonal context. Is this the best fix for learning other chords and their rotations? If so, which other chords and why? In short, how transferable is our perception of rotation in regard to triads and 7th chords to less familiar set classes? When is it more helpful to think a specific rotation as a different mode as opposed to a different rotation? In this instance, does how we think about it inform our ability to perceive it? These are open questions. We will go further into these considerations in future chapters.

Moving forward, when training yourself to hear a new set class, begin with prime form and as you explore the various rotations in closed position, ask yourself, how is this sound shaped by the interval adjoining its outer pitches? How is the total sound "a refinement" of its outer pitches?

10 note chord	**9 note subsets**	**Other name**	**Maj 7**	**Min7**	**MinMaj7**	**Dom 7**	**Dim7**
10-1*	1*,1*	2-3*(1st)(1→10)	(**2**:b2,2,#2,4,#4,5,b6)		(**7**:b2,2,#4,5,b6,6,#6)		(**1**:b2,2,b4,4,5,b6)
	2		(**3**:b2,2,#2,4,#4,5,#6)				
	3		(**4**:b2,2,#2,4,b5,6,#6)				
	4		(**5**:b2,2,#2,4,#5,6,#6)				
	5		(**6**:b2,2,#2,5,b6,6,#6)				
			(**9**:b2,4,#4,5,b6,6,#6)				
			(**10**:#2,4,#4,5,b6,6,#6)				
10 note chord	**9 note subsets**	**Other name**	**Maj 7**	**Min7**	**MinMaj7**	**Dom 7**	**Dim7**
10-2*	1*	3-6*(2nd)(1→9)	(**2**:b2,2,#2,4,#4,5,6)		(**6**:b2,2,4,5,b6,6,#6)	(**1**:b2,2,#2,4,#4,5,b6)	
	2		(**3**:b2,2,#2,4,#4,#5,#6)		(**8**:b2,4,#4,5,b6,6,#6)	(**10**:2,#2,4,#4,5,b6,6)	
	6*,6*		(**4**:b2,2,#2,4,5,6,#6)				
	7		(**5**:b2,2,#2,#4,#5,6,#6)				
	8		(**7**:b2,2,#4,5,b6,6,#6)				
	9*		(**9**:2,4,#4,5,b6,6,#6)				
10 note chord	**9 note subsets**	**Other name**	**Maj 7**	**Min7**	**MinMaj7**	**Dom 7**	**Dim7**
10-3*	2	4-10*(3rd)(1→8)	(**2**:b2,2,#2,4,#4,#5,6)		(**5**:b2,2,4,#4,#5,6,#6)	(**1**:b2,2,#2,4,#4,5,6)	
	3		(**3**:b2,2,#2,4,5,b6,#6)		(**8**:2,4,#4,5,b6,6,#6)	(**9**:b2,#2,4,#4,5,b6,6)	
	7		(**4**:b2,2,#2,#4,5,6,#6)				
	10*,10*		(**6**:b2,2,4,5,b6,6,#6)				
	11		(**7**:b2,#2,#4,5,b6,6,#6)				
			(**10**:2,#2,4,#4,5,b6,6)				
10 note chord	**9 note subsets**	**Other name**	**Maj 7**	**Min7**	**MinMaj7**	**Dom 7**	**Dim7**
10-4*	3	4-21*(3rd)(1→7,8→10)	(**2**:b2,2,#2,4,5,b6,6)		(**4**:b2,2,4,#4,5,6,#6)	(**1**:b2,2,#2,4,b5,b6,6)	
	4		(**3**:b2,2,#2,#4,5,b6,#6)			(**8**:b2,2,4,#4,5,b6,6)	
	6*		(**5**:b2,2,4,#4,#5,6,#6),				
	8		(**6**:b2,#2,4,5,b6,6,#6)				
	11		(**7**:2,#2,#4,5,b6,6,#6)				
	12*		(**9**:b2,#2,4,#4,5,b6,6)				
			(**10**:2,#2,4,#4,5,b6,#6)				
10 note chord	**9 note subsets**	**Other name**	**Maj 7**	**Min7**	**MinMaj7**	**Dom 7**	**Dim7**
10-5*	4	4-23*(3rd)(1→6,7→10)	(**2**:b2,2,#2,#4,5,b6,6)	(**7**:b2,2,4,#4,5,b6,6)	(**3**:b2,2,4,#4,5,b6,#6)	(**1**:b2,2,#2,4,5,b6,6)	
	5		(**4**:b2,2,4,#4,5,6,#6)				
	7		(**5**:b2,#2,4,#4,#5,6,#6)				
	9*		(**6**:2,#2,4,5,b6,6,#6)				
	9*		(**8**:b2,2,4,#4,5,b6,6)				
	11		(**9**:b2,#2,4,#4,5,b6,#6)				
			(**10**:2,#2,4,#4,5,6,#6)				
10 note chord	**9 note subsets**	**Other name**	**Maj 7**	**Min7**	**MinMaj7**	**Dom 7**	**Dim7**
10-6*	5,5	4-25*(1st,3rd)(1→5,6→10)	(**3,8**:b2,2,4,#4,5,b6,#6)		(**2,7**:b2,2,4,#4,5,b6,6)	(**1,5**:b2,2,#2,#4,5,b6,6)	
	8,8		(**4,9**:b2,#2,4,#4,5,6,#6)				
	10*,10*,10*,10*		(**5,10**:2,#2,4,#4,#5,6,#6)				

10-1*[0,1,2,3,4,5,6,7,8,9]	**10-2***[0,1,2,3,4,5,6,7,8,t]	**10-3***[0,1,2,3,4,5,6,7,9,t]	**10-4***[0,1,2,3,4,5,6,8,9,t]	**10-5***[0,1,2,3,4,5,7,8,9,t]	**10-6***[0,1,2,3,4,6,7,8,9,t]

Figure 10 A

10 note chord	9 note subsets	Other name	Maj7	Min7	MinMaj7	Dom7	Dim7
10-1*	1*,1*	2-3*(1st)(1→10)	(**2**:6,#6)		(**7**:(3),4)		(**1**:(b7),(7))
	2*,2*		(**3**:b6,6)				
	3*,3*		(**4**:5,b6)				
	4*,4*		(**5**:#4,5)				
	5*,5*		(**6**:4,b5)				
			(**9**:2,#2)				
			(**10**:b2,2)				

10 note chord	9 note subsets	Other name	Maj7	Min7	MinMaj7	Dom7	Dim7
10-2*	1*,1*	3-6*(2nd)(1→9)	(**2**:b6,#6)		(**6**:(3),#4)	(**1**:6,(7))	
	2		(**3**:5,6)		(**8**:2,(3))	(**10**:b2,(7))	
	6*,6*		(**4**:#4,#5)				
	7		(**5**:4,5)				
	8		(**7**:#2,4)				
	9*		(**9**:b2,#2)				

10 note chord	9 note subsets	Other name	Maj7	Min7	MinMaj7	Dom7	Dim7
10-3*	2	4-10*(3rd)(1→8)	(**2**:5,#6)		(**5**:(3),5)	(**1**:#5,(7))	
	3		(**3**:#4,6)		(**8**:b2,(3))	(**9**:2,(7))	
	7		(**4**:4,#5)				
	10*,10*		(**6**:#2,#4)				
	11		(**7**:2,4)				
			(**10**:b2,#6)				

10 note chord	9 note subsets	Other name	Maj7	Min7	MinMaj7	Dom 7	Dim7
10-4*	3	4-21*(3rd)(1→7,8→10)	(**2**:#4,#6)		(**4**:(3),#5)	(**1**:5,(7))	
	4		(**3**:4,6)			(**8**:#2,(7))	
	6*		(**5**:#2,5)				
	8		(**6**:2,#4)				
	11		(**7**:b2,4)				
	12*		(**9**:2,#6)				
			(**10**:b2,6)				

10 note chord	9 note subsets	Other name	Maj7	Min7	MinMaj7	Dom7	Dim7
10-5*	4	4-23*(3rd)(1→6,7→10)	(**2**:4,#6)	(**7**:(3),(7))	(**3**:(3),6)	(**1**:#4,(7))	
	5		(**4**:#2,#5)				
	7		(**5**:2,5)				
	9*		(**6**:b2,#4)				
	9*		(**8**:#2,#6)				
	11		(**9**:2,6)				
			(**10**:b2,b6)				

10 note chord	9 note subsets	Other name	Maj7	Min7	MinMaj7	Dom7	Dim7
10-6*	5,5	4-25*(1st,3rd)(1→5,6→10)	(**3,8**:#2,6)		(**2,7**:(3),#6)	(**1,5**:4,(7))	
	8,8		(**4,9**:2,#5)				
	10*,10*,10*,10*		(**5,10**:b2,5)				

10-1*[0,1,2,3,4,5,6,7,8,9]	**10-2***[0,1,2,3,4,5,6,7,8,t]	**10-3***[0,1,2,3,4,5,6,7,9,t]	**10-4***[0,1,2,3,4,5,6,8,9,t]	**10-5***[0,1,2,3,4,5,7,8,9,t]	**10-6***[0,1,2,3,4,6,7,8,9,t]

Figure 10 B

Part II, Chapter 11: 2.3

Section (A_3, R_9); $(x+y)^3$.

R:

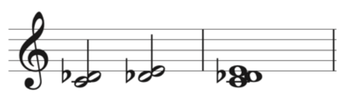
Unlike the 10 and 11-note set classes, I do not recommend learning all of the 9-note set classes outright; at least not in the sense of memorizing them and all of their rotations; they are a bit too big and unwieldy. However, it is imperative to learn the entirety of the 3-note set classes immediately. Subsequent chapters will ask you to build your understanding of the new set classes (of cardinality ≤ 6) on their previously learned subsets. Also, just as we used the literal (as opposed to abstract) complementary dyads to help shape our understanding of the decads, the complementary trichords will be used to help shape our understanding of the nonads.

Mostly, this section's focus will be on both "technical" and "musical" considerations. Technical considerations include: seeing how the just-discussed patterns apply to the nonads; introducing other patterns; comparing these set classes to certain set classes of cardinality 3, 8, and 10, and getting familiar with both "well known" (as much as that is possible) set classes and frequently occurring voicing combinations. Musical considerations include a discussion on the omnipresent tri-fold framings that break musical form down into beginning/middle/last and "motivic approaches" whose smallest presentation is 3 units. Let's begin by learning the 3-note set classes shown below (Figure 2.3.1)!

Set class	Prime Form	Prime Form: Added Intervals	2 note subsets	Other Name	Easy Description
3-1*	(0,1,2)	m2 + m2	1*,1*,2*	2-2*(1→3)	Chromatic cluster
3-2ab	(0,1,3)	m2 + M2	1*,2*,3*	Maj(7,1,2)	Do, Re, Meh or La, Ti, Do or $\hat{1}, \hat{2}, \hat{7}$ of Maj9 or $\hat{1}$, $\hat{6}$, $\hat{7}$ of Maj7
3-3ab	(0,1,4)	m2 + m3	1*,3*,4*	Mm(7,1,3)	$\hat{1}, \hat{3}, \hat{7}$ of MinMaj7 or $\hat{1}, \hat{5}, \hat{7}$ of Maj7^{+5}
3-4ab	(0,1,5)	m2 + M3	1*,4*,5*	Maj(7,1,3)	$\hat{1}, \hat{3}, \hat{7}$ of Maj7 or $\hat{1}$, $\hat{5}$, $\hat{7}$ of Maj7
3-5ab	(0,1,6)	m2 + P4	1*,5*,6*	Maj(7,1,4)	$\hat{1}, \hat{4}, \hat{7}$ of Maj Scale or $\hat{4}$, $\hat{7}$, $\hat{3}$ of Maj Scale
3-6*	(0,2,4)	M2 + M2	2*,2*,4*	Maj(1,2,3)	Do, Re, Mi or alternatively, $\hat{1}, \hat{6}, \hat{7}$ of natural minor
3-7ab	(0,2,5)	M2 + m3	2*,3*,5*	Maj(1,2,4)	$\hat{1}, \hat{3}, \hat{7}$ of min7 or $\hat{1}$, $\hat{5}$, $\hat{7}$ of min7
3-8ab	(0,2,6)	M2 + M3	2*,4*,6*	Maj(4,5,7)	$\hat{1}, \hat{3}, \hat{7}$ of Dom7 or $\hat{1}, \widehat{b5}$, $\hat{7}$ of ø7
3-9*	(0,2,7)	P4 + P4	2*,5*,5*	Maj(1,2,5)	Stacked 5ths or 4ths
3-10*	(0,3,6)	m3 + m3	3*,3*,6*	Maj(7,2,4)	Diminished triad
3-11ab	(0,4,7)	m3 + M3	3*,4*,5*	Mm(1,3,5)	Minor and Major triad
3-12*	(0,4,8)	M3 + M3	4*,4*,4*	Mm(7,3,5)	Augmented triad.

Figure 2.3 1

Below (Figure 2.3.2), following the description given in "Prime Form: Added Intervals," is an example of how 3-3a is constructed.



m2 + m3 = 3-3a

Figure 2.3 2

Note: In the case of the "other name" column and in general, I will always refer to the conventional usage (e.g., starting on tonic) when I use a conventional name (e.g., Major or Melodic minor). When I use set class names, such as 7-32A (harmonic minor) and 4-27A (half-diminished chord), the assumed reference is prime form. For example, if 4-26*(2nd)(2➔4) appears in the other name column (as it does for 5-Z37*), then you are expected to take the second inversion of its prime form <0,3,5,8>, which is <5,8,0,3>; **not** the 2nd inversion of a min 7 chord, which in "C" would be <7,t,0,3>. On the other hand, if you see Maj(2)(6➔3), you are expected to start with a major scale, let's say <0,2,4,5,7,9,e>; **not** the prime form of 7-35*, which is <0,1,3,5,6,8,t>. Why do I alternate between usages? The goal of this pedagogy is to remove barriers from your internalizing these set classes. Not force you, for consistency's sake, to call *ti* $\hat{1}$ and *do* $\hat{2}$. The benefit, consistency's sake, always referring to prime/normal form/order, would be grossly outweighed by its discouraging you to put this new set class in dialogue with your exceedingly enriched understanding of those "foundational" scales/chords. Rather, I'd like the level of your depth of understanding of those "foundational" scales, and their latent harmonic and phrase implications, to be a bar above which you seek to enrich your understanding of other set classes. For that to happen, especially, you should not fetter your (most likely) most salient scalar anchors with overhead. **Use prime form (or "root" position) as a way to get prime forms of set classes you do not know; not to redefine the "root" positions for the set classes that you already know intimately**!

Furthermore, under "other Name" above, I have listed the set classes in regard to Major and Melodic minor (7-35* and 7-34*), even though, all but one of the 3-note set classes, 3-1*, can be found in Melodic minor.[49] I have chosen to use Major and Melodic minor scale references, rather than just Melodic minor, as most readers are probably better acquainted with Major than the Melodic minor and I'd rather have versions of the "other name" that are easier to mentally work into prime form. For instance, take Mm(7,1,2), Mm(2,3,4), Maj(7,1,2), and Maj(3,4,5) as the possible expressions of 3-2's prime form. I chose Maj(7,1,2), as the reference is Major (which is preferable to Melodic Minor) and it requires the smallest transposition to put back into the key of "C"–the "key" of the prime form. Just transpose up by a half-step rather than down by a Major 3rd. I prioritize ease of the later transposition over use of Major vs. Melodic Minor.

Again, I strive to make the introduction of to the set classes as easy as possible. I am not worried about biasing you in doing this; the assumption is that once you learn these set classes, your usage of them will play a much bigger role in your conception of them than this practical introduction ever could. Finally, why use "the other name" at all in this description of 3-note set classes? I'm introducing you to the rationale behind how I describe many of the set classes of larger cardinality; where this column may prove to be the most helpful. By the end of this pedagogy, those "other name" descriptors should be very easy to handle, intuitive, and will complement your even more enriched familiarity with well-utilized set classes.

By looking at the above ordering, a pattern emerges: 3-1*(2 x m2), 3-6*(2 x M2), 3-9*(2 x P4), 3-10* (2 x m3), and 3-12*(2 x M3) consists of double copies of the same interval. With the exception of the "out-of-place" 3-9*, which when stacked surpasses the octave, as the Forte

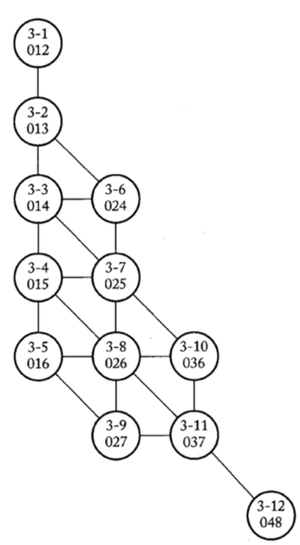
[49] Major (7-35*), on the other hand, contains all but 3 of the 3-note set classes — 3-1*, 3-3, and 3-12*.

number increases so does the repeated interval. In between these repeated interval set classes, the interval juxtaposed with the repeated interval increases; look again at 3-1* through 3-5ab—the repeated interval is (0,1) and the juxtaposed intervals respectively are: (1,2), (1,3), (1,4), (1,5), and (1,6). Why aren't there more numbers, then, between 3-10* and 3-12*, such as a m3 + P4, m3 + A4, m3 + P5, etc.? Well, a closer inspection reveals that we have already listed those set classes before; respectively they are, 3-11b, 3-10*, 3-7a, etc.

Put alternatively, the set class(es)

- 3-1* – 3-5 contain a representation of the form (0, 1, x); (0,1) is fixed and the 3rd component, x, varies.
- 3-6* – 3-9* contain a representation of the form (0, 2, x); where (0, 2) is fixed and the 3rd component, x, varies between 3≤x≤10.
- 3-10* and 3-11 contain a representation of the form (0, 3, x); where (0, 3) is fixed and the 3rd component, x, varies between 6≤x≤9.
- 3-12* is of the form (0, 4, 8).

In his article "Uniformity, Balance, and Smoothness in Atonal Voice Leading," Joseph N. Straus provides this visual counterpart (Figure 2.3.3) to the relationships described above (Straus 2003, 337).



EXAMPLE 22. *Optimal offsets for trichords (as the map of a voice leading space).*

Figure 2.3 3

Therefore, my recommendation is to learn this pattern first—it could take you as little as 5 minutes—then enrich it. First, memorize the "other name." To do so, similar to the method shown in section 2.2, take C Melodic minor, take the scale degrees associated with it — for instance, Mm(7,2,3) is B-D-Eb and then "transpose this to a C bass," yielding C, D#, E/Fb.

You may soon realize that in most of the set classes mentioned above, the scale degree listed first is either a 7 or a 1. When it's a 7, you can do the transposition *up front*; utilizing the same scale degrees in regard to an imagined Db major/minor scale rather than a C major/minor scale. In the one case where the first scale degree listed in the "other name" column is 4 (3-8*), you can first imagine a G major scale. Eventually, putting the transposition on the front, as opposed to back end, will most likely lead to the fastest—near, if not completely, automatic—processing. As the cardinality increases, the vast majority of the "front-end transpositions of the "other names" description, will be in regard to either a Db or D major/minor scale. It will be quite rare for anything else to occur.

Please, **do not take these tips as prescriptions on how one should think about these set classes musically, but just as advice from someone who has learned them and has noticed how certain skills get entrenched**. Again, one of the biggest advantages to endeavoring to learn many set classes is to then put them in dialogue with each other. Learning them all in similar formats will facilitate this. By adopting more formats, one can mitigate the "biasing" effect of a single approach.

Other Formats:

In the above chart, there is no usage of a classification method that dominates the description of the set classes of a higher cardinality—classifying the set class rotations in regard to the Maj7 (3-4a), Dom7(3-8a), min7(3-7a), dim7(4-28*) chords. However, as we are only formally introducing most of those chords now, I did not want to muddy up a nomenclature that you are just getting used to. So, as a substitute, I have introduced the "**Easy Description**" column—a category only used for the 3-note set classes. When the set class is not symmetrical—in that case, there is only one component tri-chord—the first of the two descriptions refers to the "a" component, and the last to the "b" component. **However, as not all are in their most compact rotation, the methods described above for finding prime form may or may not apply.** If one of the descriptions is not in its most compact rotation, such as $\hat{1}, \hat{5}, \hat{7}$ of Maj7, and you want to use it to get the prime form, first rotate it in your head to its most compact form, getting $\hat{5}, \hat{7}, \hat{1}$ of Maj 7.

On to the 9-note set classes!

Charts 9c and 9d express the 9 note set classes in terms of their "missing" chords. From this perspective, they are as easy (or difficult) to learn as the 3-note set classes. Nonetheless, as mentioned before, the aim of this pedagogy is to familiarize you with what they are; not just what they are not. As it turns out, the former is substantially more difficult. However, if your 10* and 11 note set classes are internalized, it is still manageable: if not in one sitting, within a week.[50]

Below, Figure 2.3.4 is a notated example of how 9-3a in chart 9C is to be read. Remember that in the second and third bars all variable pitches except, respectively, #4,6,#6 & 4,b6,6 are presented.

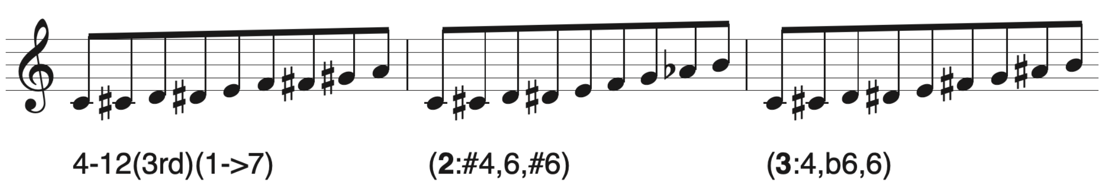


Figure 2.3 4

[50] However, expect automaticity with the 9-note set classes to take substantially longer: their size, 9 notes, makes them hard to handle cognitively; and, their abundance of rotations—e.g., 9-11 has 18 described rotations—means that learning even one 9-note set class is not trivial. Really, becoming automatic with the 9s could be considered as a brass ring associated with "mastering" this pedagogy.

10-1*	**9-1***
Dim7 (**1**:b2,2,b4,4,5,b6)	-----
Maj7 (**2**:b2,2,#2,4,#4,5,b6)	Maj7 (**2**:b2,2,#2,4,#4,5)
Maj7 (**3**:b2,2,#2,4,#4,5,#6)	-----
Maj7 (**4**:b2,2,#2,4,b5,6,#6)	Maj7 (**3**:b2,2,#2,4,b5,#6)
Maj7 (**5**:b2,2,#2,4,#5,6,#6)	Maj7 (**4**:b2,2,#2,4,6,#6)
Maj7 (**6**:b2,2,#2,5,b6,6,#6)	Maj7 (**5**:b2,2,#2,b6,6,#6)
MinMaj7 (**7**:b2,2,#4,5,b6,6,#6)	MinMaj7 (**6**:b2,2,5,b6,6,#6)
Maj7 (**9**:b2,4,#4,5,b6,6,#6)	------
Maj7 (**10**:#2,4,#4,5,b6,6,#6)	Maj7 (**9**:4,#4,5,b6,6,#6)

Figure 2.3 5

Grafting onto Straus's ordering, we could describe 9-1*–9-5 as 10-1*'s various representations minus a single pitch. Let's compare 10-1* to 9-1* (Figure 2.3.5). Notice two things. First, not all of the 10-note or 9-note set classes' rotations are represented; this is simply because those modes/rotations cannot be described by Maj7, Min7, Dom7, MinMaj7, and/or Dim7. While I could have technically labeled the rest with something else, this is where I drew the line; just sticking to these 5 labels, already provides so much and the demands of this pedagogy are more than met. If you are interested in naming the rest, my recommendation is to do that at a much later time; and more to the point, do it for those set classes you care the most about or in seeking to understand how some type of transformation acts on different voicings. I envision it as a great refinement that, down the road, is dictated by specific musical interests. Again, though, you decide what is important to you! And, if you want have those gaps filled in from the beginning, please do; it's your prerogative.

10-1*	**9-2a**
Dim7 (**1**:b2,2,b4,4,5,b6)	-----
Maj7 (**2**:b2,2,#2,4,#4,5,b6)	Maj7 (**2**:b2,2,#2,4,#4,b6)
Maj7 (**3**:b2,2,#2,4,#4,5,#6)	Maj7 (**3**:b2,2,#2,4,5,#6)
Maj7 (**4**:b2,2,#2,4,b5,6,#6)	Maj7 (**4**:b2,2,#2,#4,6,#6)
Maj7 (**5**:b2,2,#2,4,b6,6,#6)	Maj7 (**5**:b2,2,#2,b6,6,#6)
Maj7 (**6**:b2,2,#2,5,b6,6,#6)	Maj7 (**6**:b2,2,5,b6,6,#6)
MinMaj7 (**7**:b2,2,#4,5,b6,6,#6)	MinMaj7 (**7**:b2,#4,5,b6,6,#6)
Maj7 (**9**:b2,4,#4,5,b6,6,#6)	
Maj7 (**10**:#2,4,#4,5,b6,6,#6)	Dom7 (**9**:#2,4,#4,5,b6,6)

Figure 2.3 6

In both cases, you should notice that when comparing the subsets 9-1* and 9-2a with the superset 10-1*, as the rotations proceed, so will the expected shifting of the "missing" element. Why did I not indicate a grey missing pitch for the 10th rotation of 10-1*? It's because that "missing pitch" is the Major 7th, the #6 in 10-1*s Maj(**10**:#2,4,#4,5,b6,6,#6) is the b7 in the Dom(**9**:#2,4,#4,5,b6,6) chord.

This is a lot of material to contend with up front. So let me reiterate here the big "big" picture. Although the ante required may feel substantial, if you practice this, it will begin to feel a lot less abstract; especially, as mentioned earlier, when you start to get familiar with certain number combinations: mentally (as auralized possibilities), aurally, and through your playing. Moreover, remember that this is only the start of the learning process — forming the containers. As you learn subsets (and supersets) of each of these set classes, you are simultaneously forming new containers and filling in previous ones. You may naturally start to ask questions such as, "what set classes contain this specific number combination (itself a set class)? Or similarly, in what circumstance, or approximate rotation number, does it tend to appear? In short, this process of knowledge acquisition entails identifying things by how they relate to other things and in turn, how other things relate to them. In many ways, this is an analogue version of Mazzola's Encyclospace (a continually expanding "encyclopedia" that endeavors to cover all possible musical topics and can be navigated by any documented means of association) and the cognitive models that he pins it on.

As suggested earlier, in regards to Forte's article (Forte 1987), the mere act of naming these various set classes, especially after contextualizing them in terms of their sub(super)sets, can "activate" them in your musical consciousness. They start to take on a life of their own. Moreover, the relatively small and particular ways that rotation impacts "missed notes" of the superset will start to feel standard. As the cardinality of the set classes approaches six, the organization of set classes becomes much harder to describe—remember Quinn's reference to a 12 by 12 cube and orbifolds. Nonetheless, after a while you'll know this stuff so well, having accumulated so much familiarity with the nooks and crannies of those voice-leading spaces. The less trodden paths: you'll either have a *gut feeling* about a lead you can follow in real time; or, something you'll be able to mentally calculate although it may take up to a few seconds; too slow for most improvisation, but blistering fast when weighing your compositional options. The most trodden paths will be instantaneous.

Also, as mentioned earlier, the maximal subsets (cardinality n-1) of fully chromatic set classes (cardinality n), tend to be more straightforwardly related than other maximal subsets of non-clustered set classes are to those same non-fully chromatic set classes. Below (Figure 2.3.7) is an example of the relation between such a non-clustered set class and a maximal subset.

10-4*	**9-8b**
Dom 7 (**1**:b2,2,#2,4,b5,b6,6)	Dom7 (**5**:b2,2,#2,#4,b6,6)
Maj7 (**2**:b2,2,#2,4,5,b6,6)	MinMaj7 (**6**:b2,2,4,5,b6,6)
Maj7 (**3**:b2,2,#2,#4,5,b6,#6)	Maj7 (**7**:b2,2,#4,5,b6,#6)
MinMaj7 (**4**:b2,2,4,#4,5,6,#6)	MinMaj7 (**8**:b2,4,#4,5,6,#6)
Maj7 (**5**:b2,2,4,#4,#5,6,#6),	Maj7 (**9**:2,4,#4,#5,6,#6)
Maj7 (**6**:b2,#2,4,5,b6,6,#6)	-----
Maj7 (**7**:2,#2,#4,5,b6,6,#6)	Dom7 (**1**:2,#2,#4,5,b6,6)
Dom7 (**8**:b2,2,4,#4,5,b6,6)	Dom7 (**2**:b2,2,4,#4,5,b6)
Maj7 (**9**:b2,#2,4,#4,5,b6,6)	Maj7 (**3**:b2,#2,4,#4,5,6)
Maj7 (**10**:2,#2,4,#4,5,b6,#6)	Maj7 (**4**:2,#2,4,#4,#5,#6)

***Figure 2.3** 7*

Like the other comparisons made, the rotations in the maximal subset differ from those in 10-4* by the “missed pitch.” However, notice that there is a tritone shift. The prominence of the tritone in both 3-8 and 9-8 rears its head here. You can notice this by examining any rotations from 9-8b. Let’s take Maj (**7**:b2,2,#4,5,b6,#6). There is a tritone between: $\hat{1}$ and $\widehat{\#4}$; $\widehat{b2}$ and $\hat{5}$; $\hat{2}$ and $\widehat{b6}$; and, $\hat{3}$ and $\widehat{\#6}$. Remember, that Maj refers to $\hat{1}$, $\hat{3}$, and $\hat{7}$. If there is a standard rule about these set classes, it’s that there are no standard rules; the relative simplicity of the relation between a cluster and its subsets is quite exceptional. Still though, in general, expect quirks to appear.

Important observations:

As alluded to earlier, since the set of all set classes of a particular cardinality contain all combinations of numbers expressed in the jazz chord/figured bass, if you substitute one permissible number for another, that combination will appear in the modes/rotations associated with another set class. Again, let’s take 9-8b’s Maj7 (**7**:b2,2,#4,5,b6,#6) and substitute 4 for #4, the result is 9-10*'s Maj7(**3**:b2,2,4,5,b6,#6).[51] In short, two set classes that differ by a single switch of a number are R_p related.[52] To find these R_p-related set classes, simply find the subsets of a given set class's complement.[53] For instance, 9-8's complement is 3-8. Its subsets are 2-2*, 2-4*, and 2-6*. Hence, minus 3-8, the union of 10-2*, 10-4*, and 10-6*'s maximal subsets, yields all of the set classes that are in R_p relationship with 9-8; which, in this case, is every set class of cardinality 9. While for the largest set classes this may seem interesting but not very revealing, it Is very helpful to test yourself in this manner. You will notice, for instance, that for symmetrical set classes, systematic changes of the both the notes and the notes substituted for them, will yield pairs of inversely related components of set classes. For non-symmetrical set classes, this isn't the case, and you will typically find twice as many set classes in Rp relation. The rationale for this was discussed in Part II's chapter 2.2.

Here are a couple of observations to also keep in mind, the higher the Forte number is, prior to where exclusively Z set classes begin, the more likely it is that the following charts will give expression to all of the rotations;[54] especially in the set classes of larger cardinality. Secondly, the task of learning both components of a particular set class can be diminished by realizing that the given expressions are inversely related; in short, if you know the associated expression for the 2nd rotation of cardinality n, you will most likely know a rotation of the other component scale of that same set class. Below is a breakdown of how inverses will manifest in the above expressions. I urge you to memorize this as soon as possible. What this means is that you find the patterns, and then apply them through multiple examples in your head.

Finally, note that Dim7 is a special label. This is how its numbers work: first, Dim always implies fully Dim7 (4-28*, [0,3,6,9]); and second, the scale degrees are the scale degrees associated with a Maj scale.

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[51] Actually, expect similar comparisons with 3-8 to yield the opposite component in the compared set class; again, the tritone rears its head!

[52] Two set classes are R_p related if they share a maximal subset.

[53] There will be an exception, later discussed, regarding the hexachords.

[54] Here is an example: take 9-8a. Amongst 9-note set classes 8 is relatively high, 12 is the highest. It turns out that all of its voicing can be classified. Therefore, 9-8a's voicing 1 is the inverse of 9-8b's voicing 1; 9-8a's voicing 2 is the inverse of 9-8b's voicing 9; 9-8a's voicing 3 is the inverse of 9-8b's voicing 8, 9-8a's voicing 4 is the inverse of 9-8b's voicing 7, etc.

Maj7	Maj7 (~~3~~)	**MinMaj7**	Maj7 (~~3~~)	**Dom7**	Min7 (~~3~~)	**Min7**	Min7	**Dim7**	Dim7
b2	5, #6	b2	#5, #6	b2	b5, 6	b2	5, 6	b2	b6
2	5, 6	2	#5, 6	2	b5, b6	2	5, b6	2	5
#2	5, b6	3	#5/b6	#2	#4,5	3	5	b3	#4/b5
3	5	b4	5, b6	3	#4	b4	#4,5	~~b4~~	~~4~~
~~4~~	~~#4, 5~~	~~4~~	~~b5, b6~~	~~4~~	~~4, b5~~	~~4~~	~~4, 5~~	~~4~~	~~b4~~
~~#4/b5~~	~~4, 5~~	~~#4/b5~~	4, #5	#4/b5	b4, b5	#4/b5	b4	#4/b5	b3
5	3, 5	5	3, #5	5	3, b5	5	3	5/bb6	2
#5/b6	#2, 5	#5/b6	#2, #5	#5/b6	2, b5	#5/b6	2	b6	b2
6	2, 5	6	2, #5	6	b2, b5	6	b2	-----	-----
#6	b2, 5	#6	b2, #5	-----	-----	-----	-----	-----	-----

Figure 2.3 8

Above is a summary of the chord-quality inverse rules (Figure 2.3.8).

1. The inverse of a Maj7 or MinMaj7 chord can be either a Maj7 or MinMaj7 chord. It depends on whether or not there is a 5, a b6, or both. Condition b is the only one that yields a MinMaj7.
   a. If the first chord contains a 5 and no b6 **or** a 5 and a b6, its inverse is Maj7.
   b. If the first chord contains no 5 and a b6, its inverse is MinMaj7.
   c. If the first chord contains no 5 **and** no b6, its inverse is not provided.
2. The inverse of a Dom7 or Min7 chord can be either a Dom7 or a Min7 chord. It depends on whether or not there is a #4/b5, a 5, or both. Condition b is the only one that yields a Min.
   a. If the first chord contains a #4/b5 and a 5 **or** a #4/b5 and no 5, its inverse is Dom7.
   b. If the first chord contains no #4/b5 and a 5, its inverse is Min7.
   c. If the first chord contains no #4/b5 and no 5, its inverse is not provided.
3. The inverse of a Dim7 is a Dim7. There are no other chord types that span the interval of a Dim7.

Finally, when taking the inverse of number, such as 2 in the context of a Dom7, reflect it over the grey axis—2^{nd} from top, 2, goes to 2^{nd} from bottom, #5/b6.

Let's do some practice now; an example of each. While, this may be a little cumbersome to calculate at first; it soon becomes very fast.

- Maj7:
    - Take 9-3a, Maj7 (**2**:b2,2,#2,4,5,b6). Since, Maj7 (**2**:b2,2,#2,4,5,b6) contains 5 and a b6, it meets condition 1a. Its inverse will be in Major: b6 ➔ #2 and 5 ➔ 3. The rest of the flips entail b2 ➔ #6, 2 ➔ 6, #2 ➔ b6, 4 ➔ #4. The result is 9-3b, Maj (**9**:#2,#4,5,b6,6,#6).
- MinMaj7:
    - Take 9-6*, MinMaj7 (**4**:b2,2,4,5,6,#6). Since, (**4**:b2,2,4,5,6,#6) contains 5 and no b6, it meets condition 1a. As such, its inverse is contained in a Maj chord: 5 ➔ 3. The rest of the flips entail b2 ➔ #6, 2 ➔ 6, b3 ➔ #5, 4 ➔ #4, 6 ➔ 2, #6 ➔ b2. The result is 9-6*, Maj (**5**:b2,2,#4,#5,6,#6).
- Dom7:
    - Take 9-11a, Dom7 (**8**:b2,#2,4,b5,b6,6). Since, (**8**:b2,#2,4,b5,b6,6) contains b5 and no 5, it meets condition 1a. As such, its inverse will be in Dom: b5 ➔ 3. The rest of the flips entail b2 ➔ 6, #2 ➔ 5, 3 ➔ b5, 4 ➔ 4, b6 ➔ 2, and b6 ➔ b2. The result is 9-11b, Dom (**1**:b2,2,4,#4,5,6).
- Min7:
    - Take 9-5a, Min7 (**6**:b2,2,#4,5,b6,6). Since (**6**:b2,2,#4,5,b6,6) contains #4 and 5, it meets condition 1a. As such, its inverse will be in Dom: #4 ➔ 3. The rest of the flips entail b2 ➔ 6, 2 ➔ b6, 3 ➔ 5, 5 ➔ #2, b6 ➔ 2, and 6 ➔ b2. The result is 9-5b, Dom (**5**:b2,2,#2,5,b6,6).
- Dim:
    - Take 9-5a, Dim7 (**1**:b2,2,b4,5,b6). The inverse of a Dim must be a Dim; there are no other chord types offered that span a dim 7. Its inverse will contain: b2 ➔ b6, 2 ➔ 5, b4 ➔ 4, 5 ➔ 2, and b6 ➔ b2. The result is 9-5b, Dim7 (**1**:b2,2,4,5,b6).

As you may have noticed, knowing the number of a rotation will not necessarily give you the number of its rotation in the component scale. For example, take 9-3 again, the inverse of Maj7 (**2**:b2,2,#2,4,5,b6) is Maj7 (**9**:#2,#4,5,b6,6,#6). Adding the rotation numbers together, **2** + **9** yields 11. On the other hand, take 9-6*'s MinMaj7 (**4**:b2,2,4,5,6,#6) and its inverse; Maj7

(**5**:b2,2,#4,#5,6,#6). Adding the rotation numbers together, **4** + **5** yields 9. This variation is due to the way prime form and normal order are taken. In short, to get prime form, we seek out the rotation that lists its densest possible representation of pitches first. By definition then, the exact inverse of prime form in the component scale will typically not list its densest representation of pitches first. Moreover, normal order represents the densest representation of pitches for whichever component scale of the set class is offered. In short, our designation of a component scale's normal order is independent of the prime form. As such, no general rule will let us deduce what the rotation will be of its exact inverse in the other component scale. However, once we figure out one, the rest will follow.

Figure 2.3 9

For example, again take 9-3a's Maj (**3**:b2,2,#2,#4,5,#6), it's inverse is 9-3b's Maj7(**8**:b2,4,5,b6,6,#6). Notice that the rotation numbers add together, **3** + **8**, to yield 11; in short, in 9-3, assuming that every rotation fits into a given classification type, inversely-related rotations will always add up to 11.[55] For practice, double check this statement! This is a simple consequence of the definition of symmetry and has been described by numerous authors. Nonetheless, this observation will give you a way to double check your memory and create associations between the inversely-related rotations of component scales; enriching your understanding of the set class being discussed as well as your notion of how symmetry transforms familiar chord shapes.

Finally, let us give homage to Hanson. Here are the 9-note chords that he singled out. With each, I'll show their decomposition into either smaller familiar set classes and outwardly generated intervals about an axis. While he does show the others, he shows them as the end point of a process that involves their superimposing various transpositions of their 3-note complement.

[55] Only those that can be categorized into the chord-type labels provided will be given though!

However, as the process doesn't have to be complete, he'll just show those aggregates that are more familiar—for instance, regarding 9-11b, 5-27a = 3-11a (B♭ + G♭)—it's not clear exactly how the larger set decomposes into conjunct and disjunct transpositions of its 3-note complement. I would like to add that 9-9* can be expressed as 8 stacked 5ths and 9-12* is one of Messiaen's modes of limited transposition.

- 9-1*(Example 46-8b)
  - About D♭ axis = B♭♭-F; C♭-E♭; C-E♭♭; B♭-F♭.
- 9-6* (Chapter 10)
  - Projection of the Major Second beyond the Six-Tone Series
    - C–D–E–F#–G#–Bb; G–A–B
- 9-9* (Example 46-3a, 46-4a)
  - About F# axis – B-C#; E-G#; A-D#; D-A#
  - $P5^2$ + M3 = B–F#–C#; D–A#; D#–A; E–G#
- 9-10* (Example 46-6b)
  - About F# axis – B-C#; D-A#; E#-G; E-G#
  - $m3^2$ + M3 = B–F#–C#; D–A#; E#–G; E–G#
- 9-12* (Example 46-7a)
  - About F# axis – D-A#; D#-A; E#-G; B-C#
  - $M3^2$+ m2 = D–F#–A#; D#–A; E#–G; B–C#

A:

This section and subsequent ones build on a concept mentioned in the last section—intervals as a scaffold for melodic and harmonic dictation; complex chords are aural refinements of the outer intervals. The rotations of 3-note set classes are then an interval with a note in the middle. Mazzola's numbering system, for the 3-note set classes, is a systematic description of all of the 3-note chords, classified into set classes that are similarly presented. When one bundles all of the 3-note chords into sets of intervals that are smaller than an octave, one can get the columns below.

When one bundles all of the trichords into sets of intervals that are greater than an octave, one can get the rows below. For example, Figure 2.3 10 below shows how the interval C-B can be partitioned.

A couple of notes on Figure 2.3 10: with the exception of the last columns, the boundaries of the above half-triangle traverse 3-1*-3-6* and are symmetric about 3-5. One can notice how order of the set classes in the rows above align with the pathways that connect the 3-note set classes in Straus's chart. I recommend tracing this out yourself. Hopefully, your appreciation for Straus's map is growing as we proceed. It's elegant, appealing in its simplicity, and it sheds light on many of the 3-note set class group's structural properties. In general, as your knowledge of set classes grow, continue to enrich your understanding through the investigation of voice-leading maps.

	C-B	**C-Bb**	**C-A**	**C-Ab**	**C-G**	**C-F#**	**C-F**	**C-E**	**C-Eb**	**C-D**
C-Db	C-Db-B (3-1*)	C-Db-Bb (3-2b)	C-Db-A (3-3b)	C-Db-Ab (3-4b)	C-Db-G (3-5b)	C-Db-F# (3-5a)	C-Db-F (3-4a)	C-Db-E (3-3a)	C-Db-Eb (3-2b)	C-Db-D (3-1*)
C-D	C-D-B (3-2a)	C-D-Bb (3-6*)	C-D-A (3-7b)	C-D-Ab (3-8b)	C-D-G (3-9*)	C-D-F# (3-8a)	C-D-F (3-7a)	C-D-E (3-6*)	C-D-Eb (3-2a)	—
C-Eb	C-Eb-B (3-3a)	C-Eb-Bb (3-7a)	C-Eb-A (3-10*)	C-Eb-Ab (3-11b)	C-Eb-G (3-11a)	C-Eb-F# (3-10*)	C-Eb-F (3-7b)	C-Eb-E (3-3b*)	—	—
C-E	C-E-B (3-4a)	C-E-Bb (3-8a)	C-E-A (3-11a)	C-E-Ab (3-12*)	C-E-G (3-11b)	C-E-F# (3-8b)	C-E-F (3-4a)	—	—	—
C-F	C-F-B (3-5a)	C-F-Bb (3-9*)	C-F-A (3-11b)	C-F-Ab (3-11a)	C-F-G (3-9*)	C-F-F# (3-5b)	—	—	—	—
C-F#	C-F#-B (3-5b)	C-F#-Bb (3-8b)	C-F#-A (3-10*)	C-F#-Ab (3-8a)	C-F#-G (3-5a)	—	—	—	—	—
C-G	C-G-B (3-4b)	C-G-Bb (3-7b)	C-G-A (3-7a)	C-G-Ab (3-4a)	—	—	—	—	—	—
C-Ab	C-Ab-B (3-3b)	C-Ab-Bb (3-6*)	C-Ab-A (3-3a)	—	—	—	—	—	—	—
C-A	C-A-B (3-2b)	C-A-Bb (3-2a)	—	—	—	—	—	—	—	—
C-A#	C-A#-B (3-1*)	—	—	—	—	—	—	—	—	—

Figure 2.3 10

To find out the minimum span of any rotation in a given cardinality, subtract one from the cardinality. For instance, the minimum span of 9-1* [0,1,2,3,4,5,6,7,8] is 8 half-steps, a minor 6th. The rotation that has the smallest span in any given cardinality is the prime form of the cluster. As such, all rotations of the 9-note set classes can be aurally approximated, keeping the outer notes the same and providing one pitch in the middle, by the trichords presented in the first four columns.

For the 9-note set classes, all rotations that can be described by either Maj7, MinMaj7, Dom7, and Min7 are approximated by trichords in the first two columns. Furthermore, if you hear something that sounds either like a b2/b9 or a #6, and there is a Major 7, then you know that that 9-note set class contains 3-1*. Or, if you hear something that sounds like a 2/9 and there is a Major 7, then you know that that 9-note set class contains 3-2a, etc..[56] As alluded to earlier, the inverses of the trichords underlying the chord classifications used (let's call them the **basic trichords**)—3-3a, 3-4a, 3-7a, and 3-8a—do not specify the 3rd of the rotation. As such 3-3b can be found in both Maj and MinMaj rotations—it sounds as a #5/b6. Plus, note that 3-3a can also be found in a Maj 7 categorized chord–it sounds as a #2.

From experience though, not all tri-chords figure as prominently into our aural perceptions of a chord. In regards to Maj, I would rate the approximating trichords (Figure 2.3.11), in order from most prominent to least: as 3-5b, 3-5a (both are found in $Dom_{7}^{\#9,13}$ chords), 3-3a, 3-4a, 3-1*, 3-1*, 3-3b, 3-2a, 3-2b, 3-4b. Nonetheless, this rating is personal and no assertion is made about this being the best rating; yet undone experimental research would need to weigh in.[57] Similarly, different trichords may figure in less or more prominently as the classification category (e.g., Dom or Min) changes.[58] However, when working with students or teaching yourself, you can let the students (or yourself) weigh in and then introduce the material in such a way that you learn to identify the most salient approximations first and the most subtle ones later.

[56] Of course, this may not seem particularly significant when accounting for a nonad, all nonads contain all trichords; however, it does give a lot of information regarding the set classes of cardinality 4 and 5.
[57] Morris's suggestion to 'teach the most contrasting pc-sets first; the more similar second' resonates with the list offered. The more "striking" a tri-chord is, the more distant (contrasting) it sounds to the other tri-chords.
[58] Acoustical, room dimensions, and stylistic etc. factors can also figure into this. It's best not to generalize too much—giving the impression that this is less context dependent then it actually is. Rather, find out what is most salient in the musical contexts in which you and/or the class envision these sounds.

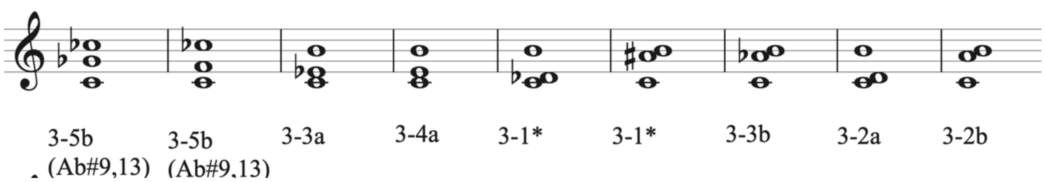

Figure 2.3 11

One advantage of knowing so many of the rotations of a given set class is that as the chapters proceed you can better hear how many of a set class's rotations may approximate the modes/rotations found of their supersets; you'll have multiple manifestations of each set class at your immediate disposal.

One helpful way to practice is to go to the minimal supersets to get an appreciation for a trichord. For instance, let's say that you want to get more familiar with 3-3. First, look at the 8 note maximal **sub**sets of 9-3; these are 8-2, 8-3*, 8-4, 8-7*, 8-12, 8-Z15, 8-17, 8-18, 8-19. This means that 4-2, 4-3*, 4-4, 4-7*, 4-12, 4-Z15, 4-17, 4-18, and 4-19 are the minimal **super**sets of 3-3. How does 3-3 sound in those contexts? Refreshing what was discussed earlier, notice that as 3-3 is not symmetrical, there are up to 9 different ways that one can add a pitch to 3-3a and yield a distinct set class. Conversely, when learning 3-3b, the added pitches will yield, in reverse order, the other component of the associated set class. This activity refines both your aural impression of 3-3, putting it into many more contexts, and enriches your perception of the supersets addressed. In a similar fashion, you can better understand an interval, by looking at how the various trichords shed light on it.

However, at least at first, "limit infinity" by sticking to the **basic expressions** of the tri-chords studied that are found in the first three columns of the above chart. What happens when you add a note to one of those expressions of a trichord in closed position? Have a friend play various tetrachords that contain one of two basic trichords in its respective basic expression—melodically, vertically, or both. Can you tell which one has which basic trichord? Furthermore, I recommend prioritizing recognizing the basic trichords first—it more easily ties in with the charts given, and the most salient, second. A trick question could involve 4-7*, whose rotation <1,4,5,0>, Maj (**2**:#2), has the basic expression of 3-3a and 3-4a.

A reasonable question that may spring to your mind when reading this is, "Isn't this a bit much?" "Hasn't it already been shown that our ability to pick out even the most familiar musical entities, like a major triad, cannot be isolated from the stylistic expectations and our interaction at some level with meaning?" What meaning is attached to one's recognition that 4-7* is a "shade" of the *primary colors* 3-3 and 3-4? Especially, in these sterile, outside of actual music practice, contexts? Furthermore, even if you can find some piece in the wild that accidentally may use one of these concoctions, (in certain repertoires, that's not too hard) it still doesn't seem like that "meaning" is significant to warrant this level of detail? Or, even more to the point, will this appreciation that we are cultivating even benefit the musical interpretation of the chord in question? When reading a comic book, are you supposed to find and then attribute meaning in the slightest shifts in color that accompany random fluctuations in the paper making process?

This pedagogy wants you to ask and then attempt to answer this question. Let it guide how deeply you explore set classes. Good musicians of any stripe develop a sense of what they need to do to get a musical idea across. Often this amounts to filtering out the noise—or at least reducing it. Classical keyboardists do it when they train their hands to project a greater linear independence (e.g., in the realization of a four-part fugue by Bach) than they may be able to keep track of in a live performance; or a singer singing a word with a pronunciation that were it spoken instead would be completely unintelligible. The vastness of this pedagogy makes it a prime site for your testing out many of the parameters that inform how much nuance is too much for your particular message. The mere engagement with this pedagogy could help you understand this in a more personal way and could inform how you negotiate parameters, like timbre, that are often considered obliquely. Can you distinguish between a cluster and almost cluster in the set classes of larger cardinality?

Along these lines, one of the best "solos" that I've heard was an accomplished jazz guitarist's rendition of a melody with his pedals. Here is my impression: there was so much packed into how he played each note; in rhythmic placement and tone. I sensed that that guitarist was presenting us with a glimpse into a well-tended portion of an aural garden that extended far beyond what we could ever get full access to. It unfolded in his imagination and was the result of his shoehorning the "infinity" of his musical life experience into that moment.

Again, reflecting on this bigger question—how does one filter out the noise surrounding their musical idea—is also asking, "what do they find most essential to their musical idea/sound?" This "external" exploration into this question may happen during your practicing of this pedagogy, or when you quickly apply your acquired knowledge at some later stage. You can better understand some still amorphous concept you have by quickly exposing it to an infinite number of conditions. As I mentioned before, people do acquire this "it" factor, knowing just how much to say and how to say it. However, while it may be assumed to emerge spontaneously from the immense talent of those rare performers, in actuality, it often "arises" when students over many years respond well to feedback from their experienced teachers or critical bandmates.

Yet, both of these methods, if you can even call the first a method, are very indirect. I submit then that using this pedagogy whets, cognitively speaking, this "knowing what not to play" ability. You are primed to see a thousand different takes on your material and have gained experience, through intentioned activity, at separating the wheat from the chaff. Let me be clear though, I'm not claiming that "knowing what not to play" is something simple like a formula that can be applied across various genres; that is absolute nonsense. Rather, I'm just suggesting that over time, getting used to asking that question and then making a habit of actively putting it into dialogue with your musical endeavors may help you get quicker at finding a context-specific acceptable answer to it—in whatever terms that may be.

9 note chord	8 note subsets	Other name	Maj7	Min7
9-1*	1*,1*,2,4,5,6*	2-4*(1st)(1→9)	(**2**:b2,2,#2,4,#4,5),(**3**:b2,2,#2,4,b5,#6),(**4**:b2,2,#2,4,6,#6),(**5**:b2,2,#2,#5,6,#6),(**9**:4,#4,5,b6,6,#6)	
9-2	1*,2,3,10,11,12,13,14,Z29	3-7(2nd)(1→8)	(**2**:b2,2,#2,4,#4,#5),(**3**:b2,2,#2,4,5,#6),(**4**:b2,2,#2,#4,6,#6),(**6**:b2,2,5,b6,6,#6)	
9-2B		3-7B(1st)(2→9)	(**3**:b2,2,#2,4,b5,6),(**4**:b2,2,#2,4,#5,#6),(**5**:b2,2,#2,5,6,#6),(**8**:b2,#4,5,b6,6,#6)	
9-3	2,3*,4,7*,12,Z15,17,18,19	4-12(3rd)(1→7)	(**2**:b2,2,#2,4,5,b6),(**3**:b2,2,#2,#4,5,#6),(**5**:b2,2,4,#5,6,#6),(**6**:b2,#2,5,b6,6,#6),(**9**:#2,4,#4,5,b6,6)	
9-3B		4-12B(1st)(3→9)	(**2**:2,#2,4,#4,5,b6),(**4**:b2,2,#2,4,#5,6),(**5**:b2,2,#2,5,b6,#6),(**8**:b2,4,5,b6,6,#6),(**9**:#2,#4,5,b6,6,#6)	
9-4	4,5,7*,8*,11,14,16,19,20*	4-22(3rd)(1→6,7→9)	(**2**:b2,2,#2,#4,5,b6),(**4**:b2,2,4,b5,6,#6),(**5**:b2,#2,4,#5,6,6),(**6**:2,#2,5,b6,6,#6),(**8**:b2,4,#4,5,b6,6), (**9**:#2,4,#4,5,b6,#6)	
9-4B		4-22B(1st)(1→3,4→9)	(**2**:b2,#2,4,#4,5,b6),(**3**:2,#2,4,#4,5,#6),(**5**:b2,2,#2,5,b6,6),(**8**:b2,4,#4,#5,6,#6),(**9**:#2,4,5,b6,6,#6)	
9-5	5,6*,8*,9*,13,Z15,16,18,Z29	4-27(3rd)(1→5,6→9)	(**3**:b2,2,4,#4,5,#6),(**4**:b2,#2,4,b5,6,#6),(**5**:2,#2,4,#5,6,#6),(**8**:b2,4,#4,5,b6,#6),(**9**:#2,4,#4,5,6,#6)	(**6**:b2,2,#4,5,b6,6)
9-5B		4-27B(1st)(1→4,5→9)	(**2**:b2,2,4,#4,5,b6),(**3**:b2,#2,4,#4,5,#6),(**4**:2,#2,4,b5,6,#6),(**8**:b2,4,#4,5,6,#6),(**9**:#2,4,#4,#5,6,#6)	
9-6*	2,11,21*,21*,22,24*	4-21*(3rd)(1→7)	(**2**:b2,2,#2,4,5,6),(**3**:b2,2,#2,#4,#5,#6),(**5**:b2,2,#4,#5,6,#6),(**7**:2,#4,5,b6,6,#6)	
9-7	4,10*,11,13,Z15,22,23*,26*,27	5-23(4th)(1→6)	(**2**:b2,2,#2,#4,#5,6),(**4**:b2,2,#4,5,6,#6),(**6**:2,4,5,b6,6,#6),(**9**:2,#2,4,#4,5,6)	(**7**:2,4,#4,5,b6,6)
9-7B		5-23B(1st)(4→9)	(**2**:2,4,#4,5,b6,6),(**5**:b2,2,#2,#4,5,6),(**7**:b2,2,4,5,6,#6),(**8**:b2,#2,#4,#5,6,#6)	(**1**:b2,4,#4,5,b6,6)
9-8	5,12,Z15,16,21*,24*,25*,27,Z29	5-33*(4th)(1→5,6→8)	(**3**:b2,2,4,#4,#5,#6),(**4**:b2,#2,4,5,6,#6),(**5**:2,#2,#4,#5,6,#6),(**8**:2,4,#4,5,b6,#6)	
9-8B		5-33*(1st)(2→4,5→9)	(**3**:b2,#2,4,#4,5,6),(**4**:2,#2,4,#4,#5,#6)(**7**:b2,2,#4,5,b6,#6),(**9**:2,4,#4,#5,6,#6)	
9-9*	6*,14,16,22,23*,23*	5-35*(2nd)(1→4,5→8)	(**2**:b2,2,4,#4,5,6),(**3**:b2,#2,4,#4,#5,#6),(**4**:2,#2,4,5,6,#6), (**6**:b2,2,#4,5,b6,6),(**8**:2,4,#4,5,6,#6)	(**1**:b2,2,4,#4,5,b6),(**5**:b2,2,4,5,b6,6)
9-10*	12,13,18,27,28*	6-Z23*(5th)(1→5)	(**3**:b2,2,4,5,b6,#6),(**4**:b2,#2,#4,5,6,#6),(**9**:2,#2,4,#4,#5,6)	
9-11	14,17*,18,19,20*,22,26*,27,Z29	6-33B(5th)(1→4,5→7)	(**2**:b2,2,4,#4,#5,6) (**3**:b2,#2,4,5,b6,#6),(**4**:2,#2,#4,5,6,#6),(**6**:b2,#2,#4,5,b6,6),(**9**:2,#2,4,5,b6,6)	(**1**:b2,2,4,#4,5,6)
9-11B		6-33(3rd)(1→3,4→6)	(**2**:b2,#2,4,#4,#5,6),(**3**:2,#2,4,5,b6,#6),(**5**:b2,2,4,5,b6,6),(**6**:b2,#2,#4,5,b6,#6),(**9**:2,#2,#4,5,b6,6)	(**4**:b2,2,4,b5,b6,6)
9-12*	19,19,19,24*,24*,24*	6-35*(1→3,4→6,7→9)	(**2,5,8**:b2,#2,4,5,b6,6),(**3,6,9**:2,#2,#4,5,b6,#6)	

9-1*	9-2a	9-2b	9-3a	9-3b	9-4a	9-4b	9-5a	9-5b	9-6*
[0,1,2,3,4,5,6,7,8]	[0,1,2,3,4,5,6,7,9]	[0,2,3,4,5,6,7,8,9]	[0,1,2,3,4,5,6,8,9]	[0,1,3,4,5,6,7,8,9]	[0,1,2,3,4,5,7,8,9]	[0,1,2,4,5,6,7,8,9]	[0,1,2,3,4,6,7,8,9]	[0,1,2,3,5,6,7,8,9]	[0,1,2,3,4,5,6,8,t]
9-7a	9-7b **	9-8a	9-8b **	9-9*	9-10*	9-11a	9-11b **	9-12*	
[0,1,2,3,4,5,7,8,t]	[0,1,2,3,4,5,7,9,t]	[0,1,2,3,4,6,7,8,t]	[0,1,2,3,4,6,8,9,t]	[0,1,2,3,5,6,7,8,t]	[0,1,2,3,4,6,7,9,t]	[0,1,2,3,5,6,7,9,t]	[0,1,2,3,5,6,8,9,t]	[0,1,2,4,5,6,8,9,t]	

Figure 9 A

9 note chord	Other name	MinMaj7	Dom7	Dim7
9-1*	2-4*(1st)(1→9)	(**6**:b2,2,5,b6,6,#6)		
9-2	3-7(2nd)(1→8)	(**5**:b2,2,4,#5,6,#6),(**7**:b2,#4,5,b6,6,#6)	(**9**:#2,4,#4,5,b6,6)	
9-2B	3-7B(1st)(2→9)	(**6**:b2,2,#4,#5,6,#6),(**9**:4,#4,5,b6,6,#6)	(**2**:b2,2,#2,4,#4,5)	
9-3	4-12(3rd)(1→7)	(**4**:b2,2,4,b5,6,#6),(**7**:2,#4,5,b6,6,#6)	(**8**:b2,4,#4,5,b6,6)	
9-3B	4-12B(1st)(3→9)	(**6**:b2,2,#4,5,6,#6)	(**3**:b2,2,#2,4,b5,6)	
9-4	4-22(3rd)(1→6,7→9)	(**3**:b2,2,4,#4,5,#6)		
9-4B	4-22B(1st)(1→3,4→9)	(**6**:b2,2,#4,5,b6,#6)	(**4**:b2,2,#2,4,b6,6)	
9-5	4-27(3rd)(1→5,6→9)	(**2**:b2,2,4,#4,5,b6)		(**1**:b2,2,b4,5,b6)
9-5B	4-27B(1st)(1→4,5→9)	(**6**:b2,2,#4,5,b6,6)	(**5**:b2,2,#2,5,b6,6)	(**1**:b2,2,4,5,b6)
9-6*	4-21*(3rd)(1→7)	(**4**:b2,2,4,5,6,#6),(**6**:b2,4,5,b6,6,#6)	(**1**:b2,2,#2,4,b5,b6)(**8**:2,4,#4,5,b6,6),(**9**:2,#2,4,#4,5,b6)	
9-7	5-23(4th)(1→6)	(**3**:b2,2,4,5,b6,#6),(**5**:b2,4,#4,#5,6,#6)	(**1**:b2,2,#2,4,5,6),(**8**:b2,#2,4,#4,5,b6)	
9-7B	5-23B(1st)(4→9)	(**6**:b2,2,4,#4,#5,#6),(**9**:2,4,5,b6,6,#6)	(**3**:2,#2,4,#4,5,6),(**4**:b2,2,#2,4,5,b6)	
9-8	5-33*(4th)(1→5,6→8)	(**2**:b2,2,4,#4,5,6),(**7**:b2,4,#4,5,b6,6)	(**1**:b2,2,#2,#4,5,b6)(**6**:b2,2,#4,5,b6,6),(**9**:2,#2,4,b5,b6,6)	
9-8B	5-33*(1st)(2→4,5→9)	(**6**:b2,2,4,5,b6,6),(**8**:b2,4,#4,5,6,#6)	(**1**:2,#2,#4,5,b6,6),(**2**:b2,2,4,#4,5,b6),(**5**:b2,2,#2,#4,b6,6)	
9-9*	5-35*(2nd)(1→4,5→8)	(**7**:b2,4,#4,5,b6,#6)	(**9**:2,#2,4,5,b6,6)	
9-10*	6-Z23*(5th)(1→5)	(**2**:b2,2,4,#4,#5,6),(**5**:2,4,#4,#5,6,#6),(**7**:2,4,#4,5,b6,6)	(**1**:b2,2,#2,#4,5,6)(**6**:b2,#2,#4,5,b6,6),(**8**:b2,#2,4,#4,5,6)	
9-11	6-33B(5th)(1→4,5→7)	(**7**:2,4,#4,5,b6,#6)	(**5**:b2,2,4,5,b6,6),(**8**:b2,#2,4,b5,b6,6)	
9-11B	6-33(3rd)(1→3,4→6)	(**7**:2,4,#4,5,6,#6)	(**1**:b2,2,4,#4,5,6),(**8**:b2,#2,4,5,b6,6)	
9-12*	6-35*(1→3,4→6,7→9)		(**1,4,7**:b2,2,4,b5,b6,6)	

9-1*	9-2a	9-2b	9-3a	9-3b	9-4a	9-4b	9-5a	9-5b	9-6*
[0,1,2,3,4,5,6,7,8]	[0,1,2,3,4,5,6,7,9]	[0,2,3,4,5,6,7,8,9]	[0,1,2,3,4,5,6,8,9]	[0,1,3,4,5,6,7,8,9]	[0,1,2,3,4,5,7,8,9]	[0,1,2,4,5,6,7,8,9]	[0,1,2,3,4,6,7,8,9]	[0,1,2,3,5,6,7,8,9]	[0,1,2,3,4,5,6,8,t]
9-7a	9-7b **	9-8a	9-8b **	9-9*	9-10*	9-11a	9-11b **	9-12*	
[0,1,2,3,4,5,7,8,t]	[0,1,2,3,4,5,7,9,t]	[0,1,2,3,4,6,7,8,t]	[0,1,2,3,4,6,8,9,t]	[0,1,2,3,5,6,7,8,t]	[0,1,2,3,4,6,7,9,t]	[0,1,2,3,5,6,7,9,t]	[0,1,2,3,5,6,8,9,t]	[0,1,2,4,5,6,8,9,t]	

Figure 9 B

9 note chord	8 note subsets	Other name	Maj7 (minus listed elements)	Min7 (minus listed elements)
9-1*	1*,1*,2,4,5,6*	2-4*(1st)(1→9)	(**2**:b6,6,#6),(**3**:5,#5,6),(**4**:#4,5,b6),(**5**:4,#4,5),(**9**:b2,2,#2)	
9-2	1*,2,3,10,11,12,13,14,Z29	3-7(2nd)(1→8)	(**2**:5,6,#6),(**3**:#4,#5,6),(**4**:4,5,b6),(**6**:#2,4,#4)	
9-2B		3-7B(1st)(2→9)	(**3**:5,b6,#6),(**4**:#4,5,6),(**5**:4,b5,b6),(**8**:2,#2,4)	
9-3	2,3*,4,7*,12,Z15,17,18,19	4-12(3rd)(1→7)	(**2**:#4,6,#6),(**3**:4,b6,6),(**5**:#2,#4,5),(**6**:2,4,#4),(**9**:b2,2,#6)	
9-3B		4-12B(1st)(3→9)	(**2**:b2,6,#6),(**4**:#4,5,#6),(**5**:4,#4,6),(**8**:2,#2,#4),(**9**:b2,2,4)	
9-4	4,5,7*,8*,11,14,16,19,20*	4-22(3rd)(1→6,7→9)	(**2**:4,6,#6),(**4**:3,5,b6),(**5**:2,#4,5),(**6**:b2,4,#4),(**8**:2,#2,#6),(**9**:b2,2,6)	
9-4B		4-22B(1st)(1→3,4→9)	(**2**:2,6,#6),(**3**:b2,b6,6),(**5**:4,#4,#6),(**8**:2,#2,5),(**9**:b2,2,#4)	
9-5	5,6*,8*,9*,13,Z15,16,18,Z29	4-27(3rd)(1→5,6→9)	(**3**:#2,b6,6),(**4**:2,5,b6),(**5**:b2,#4,5),(**8**:2,#2,6),(**9**:b2,2,b6)	(**6**:(3),4,(7))
9-5B		4-27B(1st)(1→4,5→9)	(**2**:#2,6,#6),(**3**:2,b6,6),(**4**:b2,5,b6),(**8**:2,#2,b6),(**9**:b2,2,5)	
9-6*	2,11,21*,21*,22,24*	4-21*(3rd)(1→7)	(**2**:#4,#5,#6),(**3**:4,5,6),(**5**:#2,4,5),(**7**:b2,#2,4)	
9-7	4,10*,11,13,Z15,22,23*,26*,27	5-23(4th)(1→6)	(**2**:4,5,#6),(**4**:#2,4,b6),(**6**:b2,#2,#4),(**9**:b2,b6,#6)	(**7**:b2,(3),(7))
9-7B		5-23B(1st)(4→9)	(**2**:b2,#2,#6)(**5**:4,b6,#6),(**7**:#2,#4,b6),(**8**:2,4,5)	(**1**:2,(3),(7))
9-8	5,12,Z15,16,21*,24*,25*,27,Z29	5-33*(4th)(1→5,6→8)	(**3**:#2,5,6),(**4**:2,b5,b6),(**5**:b2,4,5),(**8**:b2,#2,6)	
9-8B		5-33*(1st)(2→4,5→9)	(**3**:2,b6,#6),(**4**:b2,5,6),(**7**:#2,4,6),(**9**:b2,#2,5)	
9-9*	6*,14,16,22,23*,23*	5-35*(2nd)(1→4,5→8)	(**2**:#2,#5,#6),(**3**:2,5,6),(**4**:b2,#4,#5),(**6**:#2,4,#6),(**8**:b2,#2,b6)	(**1**:(3),6,(7)), (**5**:(3),#4,(7))
9-10*	12,13,18,27,28*	6-Z23*(5th)(1→5)	(**3**:#2,#4,6),(**4**:2,4,b6),(**9**:b2,5,#6)	
9-11	14,17*,18,19,20*,22,26*,27,Z29	6-33B(5th)(1→4,5→7)	(**2**:#2,5,#,6),(**3**:2,#4,6),(**4**:b2,4,b6),(**6**:2,4,#6),(**9**:b2,#4,#6)	(**1**:4,#5,(7))
9-11B		6-33(3rd)(1→3,4→6)	(**2**:2,5,#6),(**3**:b2,#4,6), (**5**:#2,#4,#6) (**6**:2,4,6),(**9**:b2,4,#6)	(**4**:(3),5,(7))
9-12*	19,19,19,24*,24*,24*	6-35*(1→3,4→6,7→9)	(**2,5,8**:2,#4,#6),(**3,6,9**:b2,4,6)	

9-1*	9-2a	9-2b	9-3a	9-3b	9-4a	9-4b	9-5a	9-5b	9-6*
[0,1,2,3,4,5,6,7,8]	[0,1,2,3,4,5,6,7,9]	[0,2,3,4,5,6,7,8,9]	[0,1,2,3,4,5,6,8,9]	[0,1,3,4,5,6,7,8,9]	[0,1,2,3,4,5,7,8,9]	[0,1,2,4,5,6,7,8,9]	[0,1,2,3,4,6,7,8,9]	[0,1,2,3,5,6,7,8,9]	[0,1,2,3,4,5,6,8,t]
9-7a	9-7b **	9-8a	9-8b **	9-9*	9-10*	9-11a	9-11b **	9-12*	
[0,1,2,3,4,5,7,8,t]	[0,1,2,3,4,5,7,9,t]	[0,1,2,3,4,6,7,8,t]	[0,1,2,3,4,6,8,9,t]	[0,1,2,3,5,6,7,8,t]	[0,1,2,3,4,6,7,9,t]	[0,1,2,3,5,6,7,9,t]	[0,1,2,3,5,6,8,9,t]	[0,1,2,4,5,6,8,9,t]	

Figure 9 C

9 note chord	Other name	MinMaj7(minus listed elements)	Dom 7(minus listed elements)	Dim(minus listed elements)
9-1*	2-4*(1st)(1→9)	(**6**:(3),4,#4)		
9-2	3-7(2nd)(1→8)	(**5**:(3),#4,5),(**7**:2,(3),4)	(**9**:b2,2,(7))	
9-2B	3-7B(1st)(2→9)	(**6**:(3),4,5),(**9**:b2,2,(3))	(**2**:#5,6,(7))	
9-3	4-12(3rd)(1→7)	(**4**:(3),5,b6),(**7**:b2,(3),4)	(**8**:2,#2,(7))	
9-3B	4-12B(1st)(3→9)	(**6**:(3),4,b6)	(**3**:5,b6,(7))	
9-4	4-22(3rd)(1→6,7→9)	(**3**:(3),b6,6)		
9-4B	4-22B(1st)(1→3,4→9)	(**6**:(3),4,6)	(**4**:4,#4,(7))	
9-5	4-27(3rd)(1→5,6→9)	(**2**:(3),6,#6)		(**1**:4,(#6),(7))
9-5B	4-27B(1st)(1→4,5→9)	(**6**:(3),4,#6)	(**5**:4,#4,(7))	(**1**:3,(#6),(7))
9-6*	4-21*(3rd)(1→7)	(**4**:(3),#4,b6),(**6**:2,(3),#4)	(**1**:5,6,(7)),(**8**:b2,#2,(7)),(**9**:b2,6,(7))	
9-7	5-23(4th)(1→6)	(**3**:(3),#4,6),(**5**:2,(3),5)	(**1**:#4,b6,(7)),(**8**:2,6,(7))	
9-7B	5-23B(1st)(4→9)	(**6**:(3),5,6),(**9**:b2,3,#4)	(**3**:b2,#5,(7)),(**4**:#4,6,(7))	
9-8	5-33*(4th)(1→5,6→8)	(**2**:(3),b6,#6),(**7**:2,(3),#6)	(**1**:4,6,(7)),(**6**:#2,4,(7)),(**9**:b2,5,(7))	
9-8B	5-33*(1st)(2→4,5→9)	(**6**:(3),#4,#6),(**8**:2,(3),#5)	(**1**:b2,4,(7)),(**2**:#2,6,(7)),(**5**:4,5,(7))	
9-9*	5-35*(2nd)(1→4,5→8)	(**7**:2,(3),6)	(**9**:b2,#4,(7))	
9-10*	6-Z23*(5th)(1→5)	(**2**:(3),5,#6)(**5**:b2,(3),5),(**7**:b2,(3),#6)	(**1**:4,b6,(7)),(**6**:2,4,(7)),(**8**:2,#5,(7))	
9-11	6-33B(5th)(1→4,5→7)	(**7**:b2,(3),6)	(**5**:#2,#4,(7)),(**8**:2,5,(7))	
9-11B	6-33(3rd)(1→3,4→6)	(**7**:b2,(3),b6)	(**1**:#2,#6,(7)),(**8**:2,#4,(7))	
9-12*	6-35*(1→3,4→6,7→9)		(**1,4,7**:#2,5,(7))	

9-1*	9-2a	9-2b	9-3a	9-3b	9-4a	9-4b	9-5a	9-5b	9-6*
[0,1,2,3,4,5,6,7,8]	[0,1,2,3,4,5,6,7,9]	[0,2,3,4,5,6,7,8,9]	[0,1,2,3,4,5,6,8,9]	[0,1,3,4,5,6,7,8,9]	[0,1,2,3,4,5,7,8,9]	[0,1,2,4,5,6,7,8,9]	[0,1,2,3,4,6,7,8,9]	[0,1,2,3,5,6,7,8,9]	[0,1,2,3,4,5,6,8,t]
9-7a	9-7b **	9-8a	9-8b **	9-9*	9-10*	9-11a	9-11b **	9-12*	
[0,1,2,3,4,5,7,8,t]	[0,1,2,3,4,5,7,9,t]	[0,1,2,3,4,6,7,8,t]	[0,1,2,3,4,6,8,9,t]	[0,1,2,3,5,6,7,8,t]	[0,1,2,3,4,6,7,9,t]	[0,1,2,3,5,6,7,9,t]	[0,1,2,3,5,6,8,9,t]	[0,1,2,4,5,6,8,9,t]	

Figure 9 D

Part II, Chapter 12: 2.4

Section (A_4, R_8); $(x+y)^4$.

R:

With the 8-note set classes, we have finally arrived with set classes that may feel either familiar, like the octatonic scale, or at a minimum, less abstract. They tend to be more manageable than the nonads and decads. However, the straightforward organizational principles found earlier in the higher set classes apply less here. You can find them in "pockets" of the 8-note set classes, but not through their entirety. While this was addressed in more general terms in Part I, I want to present it again; now that we have more tools with which to extract our understanding of this concept with.

Let's examine this observation: "where there is a parsimonious[59](or not), non-repeating way to connect all of the set classes of cardinality ≤ 3, I have yet to find one for cardinality 4 (the complement of 8)." The easiest explanation for this difficulty may lie in the high number of symmetrical set classes (15 out of 29!!)—a symptom of cardinalities 4 and 8 being even and not relatively prime to 12.[60] Typically, symmetrical sets are parsimoniously connected to half as many set classes as their non-symmetrical counterparts.[61] Furthermore, a couple of those set classes—4-9* and 4-28*—have so many axes of symmetry that they can only be connected parsimoniously to one other set class—respectively, 4-16 and 4-27.

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[59] Again, in this pedagogy, two set classes are considered *parsimonious* if there exists a representation of each that differ by a single half-step. E.g., 3-2 and 3-3 are parsimonious as C#,D,E (3-2a) differs from C#,D,F (3-3a) by a single half-step.
[60] I have not yet proved that this is the case; however, from my preliminary investigations, I suspect it. Nonetheless, even if there proves to be a parsimonious and non-repeated way to connect all of the set classes, as shown by the following commentary, my observation still stands.
[61] This is an immediate consequence of their being symmetrical. To see this, express said set class symmetrically about an axis. Raising one pitch that is distance x from the axis by y half-steps yields some scale Z. Conversely, lowering the inversely related pitch that is distance -x from the axis by y half-steps yields Z^1. Z and Z^1 are component scales of the same set class.

How does this discussion of parsimony impact you? Structurally speaking, the two “most” straight-forward ways of relating sets—straight-forward in the sense that a musician may stumble upon them with no explicit instruction—is by shared subsets and/or a superset/subset relationship. Parsimony is an example of shared subsets; when you change one scale of cardinality n by a single half-step, the new scale and the old scale share a subset of cardinality n-1—the Rp relation discussed in the last chapter. Learning the different seventh chords that are associated with the major scale and/or isolating out the horn parts in a funk band are examples of the superset/subset relation. Other, less structurally obvious ways to relate sets include: span/register/timbre, which are less susceptible to systematic inquiry; and being Z-related, which, bar the hexachords, only applies to a few set classes.

Now, how does this discussion of parsimony tie back into the original point? It highlights the fact that there aren’t any simple organizational principles that underlie how the various 4 note set classes relate parsimoniously. As you explore these set classes then, your initial intuition about how two sets may or may not relate parsimoniously may lead you astray. For example, there is only one way to change parsimoniously from one transposition of 4-27a into 4-28* and yet there are 4 different ways to change from one transposition of 4-28* into 4-27a (at 4 different transposition levels). From a group theoretical standpoint, this may “make sense.” However, from the perspective of changing one note at a time, as soloists typically do, this is wonky. 4-28* has a very different gravity to 4-27. One can’t assume that the same type of motions when situated in one region of the 4-note set class space will yield the same type of responses when situated in another region. Plus, the difference is not minimal; it’s drastic!!

Finally, how does one respond to this? This pedagogy attributes a benefit to learning how to navigate the set classes in terms of the shared subsets and the superset/subset considerations just discussed. Yet, it also acknowledges that there are no simple ways that uniformly apply to all set classes in these two cardinalities.

While this pedagogy can give guidance on some easy-to-notice pockets[62] — for instance, 4-1*/8-1* through 4-6*/8-6*, it would be misleading to imply that there is somehow a segmentation of these cardinalities' set class space that would neatly distinguish all pockets from each other. Rather, many of these pockets can overlap. For instance, look at 4-25*. It is in a 4-9* (0,1,6,7) – 4-25* (0,2,6,8) – 4-28*(0,3,6,9) pocket (those symmetrical 4-note set classes that fold about the tritone) as well as in a 3-6* (0,2,t,0) – 4-21*(3,5,7,9) – 4-25* (2,4,8,t)[63] pocket (pairs of major 2nds that are separated by some multiplication of a whole step). Which "way of operating" is more essential to 4-25*? I would submit the answer, neither. However, in terms of your interest and musical inclinations, one may be more essential.

Along those lines, set class maps akin to the Straus 3-note map given earlier are less insightful into 4-note space. Again, this is not a fault of the maps, rather it's a result, alluded to above, of the space being sufficiently complicated that a single 2D rendering of this now 4D space (as opposed to 3D) can't be as complete. It seems that there was a bigger incentive in the theoretical literature to find the best mapping of the set class spaces of cardinalities 4-6 rather than a "collection of helpful" mappings. Unfortunately then, while the accomplishments have been impressive, they can actually mislead the reader into thinking that the maps of the larger cardinalities are as authoritative as those maps pertaining to the smaller cardinalities. Rather, we need more maps. Maps that are convincing and maximally dissimilar from each other; in whatever way that may be defined. From my perspective, the best question is not what is the best map of cardinality 4, but what is the smallest number of maps that we can get that will give us comparable insight into cardinality 4 to what we have into cardinality 3.

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[62] I'm defining *pockets* in two senses. The more general sense, given in this paragraph, is as any collection of set classes that you find relatable. The more specific senses, typically used later, are as collections of set classes of the same cardinality that: one, are related parsimoniously; and two, either:

1. Express this via a shared classification type. E.g., {Dom 7(b2), Dom 7(2), Dom 7(#2), Dom 7(4), Dom 7(#4), Dom 7(5), Dom 7(b6), and Dom 7(6)} or {4-12a, 4-21*, 4-Z15b, 4-16b, 4-25*, 4-27b, 4-24*, 4-Z29a}. Or,
2. Express this via shared scale degrees. E.g., {Maj (2), Dom (2), MinMaj (2), Min(2)} or {4-11a, 4-21*, 4-3*, 4-11b}

[63] Even though this pedagogy focuses on set class space, many musical lines of inquiry, such as this one here, may not only distinguish between component scales, but also distinguish between octaves. So, the 3-6* as a set class label still applies, but this *pocket*, includes set classes of smaller cardinality with doubled pitches. As discussed earlier, Quinn and Tymoczko have grappled with the conflict arising when a certain set of voice leading paths leads to doublings in one context and not in another. Particularly around the question of how does one create a terminology and calculus for voice leading transformation that doesn't fold when doublings occur?

Ultimately though, even those optimal set of maps would have a bias. As one main objective of this pedagogy is to help you reflect on and then "productively cultivate" your bias (this is an art, not a science!), I recommend creating your own maps. Towards this end, the numerous listings of the set class's rotations are given. For instance, as I learned them, certain descriptions of chords seemed more salient to me than others. When working with the 8-note set classes, I found certain pockets of them that shared some memorable combination of scale degrees, let's say #4, 5, and 6. However, my impression was further subdivided than this. One particularly salient subset of this pocket included those 8-note set classes whose other two notes were the three different ways one could select two scale degrees from {b2, 2, and #2}. Another meaningful subset included those set classes that shared the same five scale degrees, #4, 5, 6 plus two others, but fell under the different chord types. I also became aware of the inversely-related patterns. For instance, fixing 2, 4, and 5 in Maj (the inverse of #4,5, and 6), and then selecting the other two scales degrees from {b6, 6, and #6}.

Just in going through these examples, you may notice a biasing affect built into my classification by chord type. You are correct. However, no matter how I presented the information, it would lead to some biasing affect. I stand by my earlier claim that once these set classes sufficiently get a "life of their own" in your consciousness, they will "break through" any limits suggested by my classification system. However, in my estimation, considering historical factors, such as how many institutions in the U.S. have taught music theory and performance, the relative simplicity of this proposed classification system—in large part due to inherited familiarity (figured bass and Jazz notation)—offers too much pedagogical benefit to scuttle due to already-known biases — such as preferencing tonal music. Of course, I'm always open to other suggestions.

In the set classes of cardinality 5 through 7, this statement still holds. However, due to the different symmetries involved, the composition and behavior of many of the pockets (in the specific senses) are very different. However, there are some that are analogous too.

Notable sets of set classes that may be worth starting your exploration of the octads with are:

- Those that are modes of limited transposition:

        - mode 2, 8-28*; mode 6, 8-25*; and mode 4, 8-9*.
    - 8-28* is the octatonic scale and, according to Hanson, a projection of the m3; it consists of h-w-h (half-whole-half) repeated at the tritone; 8-25* consists of h-h-w repeated at the tritone; and 8-9*, h-h-h repeated at the tritone. The former has 8-fold symmetry; the latter two, 4-fold.
        - This can be easily seen when examining their maximal subsets; 8-28* has one (two scales); the others have four (Respectively, 3 set classes and 2 set classes).
        - This can also be seen in the duplications of their chords' representation: e.g., 8-25* contains Maj(**3,7**:2,4,#4,#5,#6) and 8-28* contains MinMaj(**2,4,6,8**:2,4,#4,#5,6).
- Those that are rife with P5s.
        - Primarily, 8-23*, 8-9*, and 8-6*
        - Secondarily, 8-26* and 8-22.
        - Thirdly, 8-14.
    - 8-23* can be expressed exclusively as a stack of P5s; 8-9*, discussed above, can be expressed as two disjunct 4-23*s (itself 3 stacked P5s) separated by a tritone; 8-6*, can be expressed as two disjunct 4-23*s separated by a m2.
    - 8-23*, 8-26*, and 8-22 are all minimal supersets of 7-35*, the Major scale and the single 7-note set class that can be expressed exclusively in P5s.
        - 8-23* can be expressed as two representations of 7-35* separated by a P5—e.g., C major + G major
        - 8-26* can be expressed as representations of 7-35* and 7-32 that respectively are separated by a m3—e.g., C major + A harmonic minor, or share the same key—C major + C harmonic major.
        - 8-22 can be expressed as representations of 7-35* and 7-34* that either share the same key—C major + C melodic minor, or are separated by a M2—e.g., C Major + D Melodic minor.
    - 8-14 contains all but one of the minimal supersets of 6-32*, the single 6-note set class that consists of 5 stacked P5s. 7-35* is the missing minimal superset.
- Other combinations of familiar scales:

        - 8-27, 8-18, and 8-17*
    - 8-27 can be expressed as 3 different combinations of familiar scales.
        - 8-27 can be expressed as a representation of 7-34* and 7-32 that either share the same key (8-27a)—e.g., C Melodic minor + C Harmonic minor, or are separated by a P5 (8-27b)—e.g., C Melodic minor + G Harmonic major.
        - 8-27 can be expressed as a representation of 7-34* and 7-31, which is the single maximal subset of 8-28*, that either are separated by a m6 (8-27a)—e.g., C melodic minor + Ab 7-31a (h-W), or are separated by a M6 (8-27b)—e.g., Db Melodic minor + Bb 7-31b (W-h).
        - 8-27 can be expressed as a representation of 7-32 and 7-31 that either are separated by a m6 (8-27a)—e.g., C Harmonic minor + Ab 7-31a (h-W), or are separated by a M2 (8-27b)—e.g., Ab Harmonic Major + Bb 7-31b (W-h).
    - 8-18 can be expressed as a representation of 7-32 and 7-31 that either are separated by a M7 (8-18a)—e.g., Db Harmonic major + C 7-31b (W-h), or are separated by a M7 (8-18b)—e.g., G Harmonic minor + F# 7-31a (h-W).
    - 8-17* can be expressed as two inversely related component scales of 7-32 that share the same key—e.g., C Harmonic minor + C Harmonic major
- The minimal supersets of 7-33* (0,1,2,4,6,8,t)—the whole-tone scale plus 1.
    - 8-21*, 8-24*, 8-25*
        - Hanson discusses 8-21* under the projection of the M2. Quinn expounds on this (Hanson 1960 p.91; Quinn 2006 p.129).
    - 8-21* can be expressed as a representation of 7-33* and 7-34*[64] that either are separated by a m2—e.g., C 7-33* + Db Melodic minor, or are separated by a M7—e.g., D 7-33* + C# Melodic minor.

[64] Note the similarity between the first rotation of 7-33* and the non-rotated melodic minor. The former has a b2, the latter a natural 2. This reflects a disadvantage of following the typical naming convention for set classes that treats the "prime" form of 7-34* as melodic minor rather than as an "alt chord", Dom(**1**:b2,#2,#4,#5); it obscures their similarity. You have to look at different rotations in each set class to see this. Keep this is mind as you work with these most common scales. Had I used 7-34*s typical representation, 8-21* would have been expressed as C 7-33* + C 7-34* and D 7-33* + C 7-34*. As you learn this, think about which representation is easier for you. Ultimately, stick to whichever representations work best for you.

- 8-24*'s prime form is very similar to 8-21*s; the difference is in where the chromatic clusters appear—in the former, (0,1,2) and (2,3,4); in the latter, (0,1,2) and (4,5,6). Since they share a whole tone scale, the distribution of chord types is also very similar.
- 8-25*'s prime form is also very similar to 8-21*s (and 8-24*s); the difference is in where the chromatic clusters appear—in the former, (0,1,2) and (2,3,4); in the latter, (0,1,2) and (6,7,8). Also, since they share a whole tone scale, the distribution of chords types is very similar.

What you may come to notice is that between these 3 set classes, you have all of 8-note set classes that are either Dom7 and contain (2,#4,5,b6), "Aug 9,#4,5," or its inverse, Dom7(2,#2, #4, b6), "Aug7(9,#9, #4)."[65] As mentioned earlier, noticing when certain set classes contain all chords of a certain type that fix some configuration of numbers, can be extremely helpful.

Furthermore, remember, one should not expect to learn the 8-note set classes quickly. Focus on absorbing the various pockets that fascinate you. When they have settled enough, they will anchor other chords. There is just TOO much information for it not to all become mush in one sitting or even ten; especially for these closer-to-maximally-even set-chords—there are so many rotations listed. The more you can tether these set classes to musical (or non-musical) experiences that you care about, the better. Some of those experiences could consist you spending an afternoon doodling around with even one set class. Another could be your fixing a specific voicing of Dom7(2,#4,5,b6), 6-21b, and then playing around with dyads foreign to it; exploring how the shift of these dyads effects the sound. Do this while fixing other voicings of 6-21b etc.

Other groupings (pockets) worth exploring are listed below. While I'm not giving them the "notable" status, they are of course all compelling. Working through the list below will give you a substantial "in" to all of the 8-note set classes. If you take the inverse of the following fixed chords and the referred-to configurations of numbers, you will be exposed to all but the diminished rotations of the 8-note set classes.

[65] Note that these 4-note figurations can be associated with the set classes 4-5b, and 4-Z15a.

8-note pockets (Figure 2.4.1): Fixing some combination of Maj (~4,~5,~6) and Dom (~4,~5,~6), select the other two notes from {b2, 2, and #2}.

Maj (b2,2)	**MinMaj (b2,2)**	**Min (b2,2)**	**Maj (b2,#2)**	**Maj (2,#2)**	**Dom (b2,2)**	**Dom (b2,#2)**	**Dom (2,#2)**	**(~4,~5,~6)**	**Fixed Maj**	**Fixed MinMaj**	**Fixed Dom**	**Fixed Min**
8-2a	8-1*	NA	8-4a	8-5a	NA	NA	NA	(b6,6,#6)	6-Z37*	6-Z36a	NA	NA
8-10*	8-2b	NA	8-12b	8-14b	NA	NA	NA	(5,6,6#)	6-Z40b	6-Z39a	NA	NA
8-12a	8-4b	NA	8-17*	8-19b	NA	NA	NA	(5,b6,#6)	6-15b	6-14a	NA	NA
8-14a	8-5b	8-6*	8-19a	8-20*	8-Z29a	8-18a	8-16a	(5,b6,6)	6-14b	6-15a	6-Z39b	6-Z40a
8-11a	8-3*	NA	8-13a	8-Z29a	NA	NA	NA	(#4,6,#6)	6-Z41b	6-Z42*	NA	NA
8-21*	8-11b	NA	8-22a	8-24*	NA	NA	NA	(#4,#5,#6)	6-22b	6-Z46b	NA	NA
8-22b	8-13b	8-Z29b	8-26*	8-27b	8-24*	8-27a	8-25*	(#4,#5,6)	6-Z24b	6-27b	6-21b	6-Z45*
8-Z15a	8-7*	NA	8-18a	8-19a	NA	NA	NA	(#4,5,#6)	6-Z43b	6-Z44b	NA	NA
8-23*	8-Z15b	8-18b	8-27a	8-26*	8-27b	8-28*	8-27a	(#4,5,6)	6-Z25b	6-Z28*	6-Z23*	6-27a
8-16a	8-8*	8-16b	8-20*	8-19b	8-25*	8-27b	8-24*	(#4,5,b6)	6-16b	6-Z44a	6-21a	6-Z46a
8-4a	8-2a	NA	8-5a	8-6*	NA	NA	NA	(4,6,#6)	6-Z38*	6-Z41a	NA	NA
8-12b	8-10*	NA	8-14b	8-Z29b	NA	NA	NA	(4,#5,#6)	6-Z17b	6-Z47b	NA	NA
8-17*	8-12a	8-14a	8-19b	8-18b	8-19a	8-17*	8-12b	(4,#5,6)	6-Z19b	6-Z49*	6-16a	6-Z47a
8-13a	8-11a	NA	8-Z29a	8-14a	NA	NA	NA	(4,5,#6)	6-18b	6-Z48*	NA	NA
8-22a	8-21*	8-26*	8-24*	8-22b	8-26*	8-27b	8-23*	(4,5,6)	6-Z26*	6-34b	6-Z25a	6-33b
8-18a	8-Z15a	8-27a	8-19a	8-17*	8-27a	8-26*	8-22a	(4,5,b6)	6-Z19a	6-31b	6-Z24a	6-32*
8-5a	8-4a	NA	8-6*	8-5b	NA	NA	NA	(4,b5,#6)	6-7*	6-18a	NA	NA
8-14b	8-12b	8-19b	8-Z29b	8-13b	8-19b	8-18b	8-Z15b	(4,b5,6)	6-18a	6-30b	6-Z43a	6-Z29*
8-Z29a	8-13a	8-24*	8-14a	8-12a	8-24*	8-22b	8-21*	(4,b5,b6)	6-Z17a	6-Z50*	6-22a	6-33a
8-6*	8-5a	8-14b	8-5b	8-4b	8-Z29b	8-13b	8-11b	(4,#4,5)	6-Z38*	6-Z17b	6-Z41b	6-Z47b

Figure 2.4 1

More 8-note pockets (Figure 2.4.2): similarly, fixing some combination of MinMaj (~4,~5, ~5 ~6) and Min (~4,~5, ~5 ~6), select the other note from {b2, 2}.

Maj(b2)	MinMaj(b2)	Maj(2)	MinMaj (2)	Maj (#2)	Dom (b2)	Min (b2)	Dom (2)	Min (2)	Dom (#2)	(~4,~5,~5,~6)	Fixed Maj	Fixed MinMaj	Fixed Dom	Fixed Min
8-3*	8-2a	8-11a	8-4a	8-7*	NA	NA	NA	NA	NA	**(5,b6,6,#6)**	7-3b	7-3a	NA	NA
8-11b	8-10*	8-21*	8-12b	8-Z15b	NA	NA	NA	NA	NA	(#4,#5,6,#6)	7-9b	7-10b	NA	NA
8-7*	8-11a	8-Z15a	8-13a	8-8*	NA	NA	NA	NA	NA	(4,#5,6,#6)	7-6b	7-Z12*	NA	NA
8-13b	8-12a	8-22b	8-17*	8-18b	NA	NA	NA	NA	NA	**(#4,5,6,6#)**	7-Z36b	7-16b	NA	NA
8-Z15b	8-21*	8-23*	8-22a	8-16b	NA	NA	NA	NA	NA	(4,5,6,6#)	7-14b	7-24b	NA	NA
8-8*	8-Z15a	8-16a	8-18a	8-9*	NA	NA	NA	NA	NA	(4,b5,6,6#)	7-7b	7-19a	NA	NA
8-Z29b	8-14a	8-24*	8-19a	8-19b	NA	NA	NA	NA	NA	**(#4,5,b6,#6)**	7-13b	7-Z17*	NA	NA
8-18b	8-22b	8-27b	8-26*	8-20*	NA	NA	NA	NA	NA	(4,5,b6,#6)	7-Z38b	7-27b	NA	NA
8-16b	8-23*	8-25*	8-27a	8-16a	NA	NA	NA	NA	NA	(4,#4,#5,#6)	7-15*	7-29b	NA	NA
8-14b	8-Z29a	8-22a	8-18a	8-17*	8-12b	8-13a	8-21*	8-Z15a	8-12a	**(#4,5,b6,6)**	7-11b	7-16a	7-8*	7-10a
8-19b	8-24*	8-26*	8-27a	8-19a	8-17*	8-22a	8-22b	8-23*	8-14a	(4,5,b6,6)	7-Z37*	7-10a	7-11a	7-23b
8-20*	8-27b	8-27a	8-28*	8-18a	8-19a	8-26*	8-24*	8-27b	8-Z29a	(4,#4,#5,6)	7-Z38a	7-31b	7-13a	7-25b
8-9*	8-16a	8-16b	8-20*	8-8*	NA	NA	NA	NA	NA	**(4,#4,5,#6)**	7-7a	7-20b	NA	NA
8-16a	8-25*	8-23*	8-27b	8-Z15a	8-18a	8-27a	8-22a	8-26*	8-13a	(4,#4,5,6)	7-14a	7-28b	7-Z36a	7-25a
8-8*	8-16b	8-Z15b	8-18b	8-7*	8-Z15a	8-23*	8-21*	8-22b	8-11a	(4,#4,5,b6)	7-6a	7-Z18b	7-9a	7-23a

Figure 2.4 2

Finally, the last "pockets" that I recommend exploring are:

- {8-1*, 8-2, 8-3*, 8-4, 8-5, and 8-6*} and its subset {8-1*, 8-2, 8-4, 8-5}
    - These obey the nice ordering typified in the set classes of cardinality 2 and 3.
    - Greater familiarity with the comparable group of set classes in cardinalities 4-7, especially 7, will make this a relatively easy addition.
    - Furthermore, as these chords are the nearest chords/set classes to complete clusters, they exhibit the "extreme" and typically unfamiliar chords; as such, their typical fixed combinations stand out. For instance, many have either {b2,2,#2} or {b6, 6, #6} or some contiguous combination of both such as {b2,2,#2,6,#6}. Furthermore, due to the high volume of half-steps, the most likely chord types are Maj and MinMaj.
- {8-7*, 8-8*, and 8-9*}
    - They all have rotations that pivot around the number configuration Maj {4,#4,5}, 6-Z38* and some combination of one selection from {b2,#2} and {b6,#6}.

A:

As mentioned earlier, the smaller, in terms of cardinality, of two complementary set classes is substantially easier to learn; especially in regard to this system of classifying set classes by chord type. In large part, the difficulty of learning (memorizing) the larger set classes has to do with the larger number of rotations that can fit into one of the given chord types; Take 8-22 and 8-27 which have 16 distinct classifiable rotations associated with each (9-8 and 9-11 have 18!). Musically speaking this is a wonderful thing; not only is 9-11 rich in terms of being the complement of the major/minor triad, meaning it can easily be decomposed in multiple iterations of major and/or minor triads, but there are just SO many different categorically recognizable rotations associated with it. This potential for exploration may be inspiring to you; it presents so many ways to recontextualize and deepen your understanding of what you may already be intimately familiar with.

On the other hand, while the 4-note set classes are more graspable in terms of size and ease of manipulation, the vast majority of their representations are not expressible in terms of the discussed chord classifications. In this this limited sense, through this pedagogy, learning them may seem trivial next to their counterpart — there are fewer rotations and most are not classified/identified. However, even if easier learning them well is also not a trivial endeavor. Even though I will not focus on labeling many of these tetrads' rotations, for instance, 4-2a's third rotation {D,E,C,C#}, this still provides a meaningful lens through which to compare: many Major and MinMaj chords — such as{D,E,F#,C,C#}, {D,E,F,C,C#}; other Maj or MinMaj supersets in which it is contained; or even, other non-classifiable (in this system) rotations/modes.

Also, perhaps most importantly, musically speaking, that "non-classifiable" rotation may be very important to you! In terms of good or bad, there is no value judgment placed on the inclusion/exclusion of certain rotations in this classification scheme. Again, let me reiterate that this omission is due to efficacy, not taste. Limiting the classification types, simplifies the already difficult task of getting a good working frame; these set classes are complex! Of course, if one wants to augment the already given list of set class rotations/modes, one should. A more ambitious task could be to "classify" those yet unclassified rotations/modes. Yet, as I consider that an ancillary pursuit that, save the following comment, this paper will not pursue this any further. If I were to pursue this further, I would weigh these two competing concerns:

- Classify by set class.
- Classify by some combination of pitch distribution — number of pitches, span, and/or the salience of the trichord (the tetrad's outer pitches and some middle pitch) that you deem best approximate it.

However, by the time you were familiar with the bulk of the other rotations/modes, this may be a moot point; a refined sense of a 9-note set class would likely reflect an aural sensitivity and mental agility that would make those not-classified voicings easier to onboard.
We'll now discuss another skill that would aid such a future endeavor: a better understanding of how changes in voicing interact with your perception of a chord. As you engage this material, you will see that you have a veritably limitless workshop through which you can test out the impact of particular voicing transformations — of which rotation/mode is but one — on your

understanding of a set class/chord. Can you learn to identify specific voicing transformations, such as going from 2nd to 5th rotation/mode, independent of the scale used? How do different set classes' distribution of chord type amongst rotations/modes impact your understanding of chord type? For instance, 4-2a contains Maj (b2) and MinMaj (#6). This dialectical process should enrich your understanding of, and ability to audiate, the set classes in focus, regardless of whether the elusive task of recognizing the voicing transformation independent of set class being transformed is attainable.

First though, here are the 4-notes set classes and their rotations (Figures 2.4.3a and 2.4.3b):

4 note chord	Prime/Normal	3 note subsets	Other Name	Maj7	Min7	MinMaj7
4-1*	[0,1,2,3]	1*,2	2-3 (1→4)			
4-2	[0,1,2,4]	1*,2,3,6*	3-6*(1→3)			(**2**:b2)
4-2B	[0,2,3,4]		3-6* (2→4)			
4-3*	[0,1,3,4]	2,3	Octatonic(1→			(**2**:2)
4-4	[0,1,2,5]	1*,3,4,7	3-7(1→3)	(**2**:b2)		(**3**:#6)
4-4B	[0,3,4,5]		3-7B(2→4)			
4-5	[0,1,2,6]	1*,4,5,8	3-8(1→3)	(**3**:#6)		
4-5B	[0,4,5,6]		3-8B (2→4)			
4-6*	[0,1,2,7]	1*,5,9*	3-9*(1→3)			
4-7*	[0,1,4,5]	2,3	Hm(5→1)	(**2**:#2)		
4-8*	[0,1,5,6]	3,4	Maj (7,1,3,4)	(**2**:4)		
4-9*	[0,1,6,7]	5,5	Octatonic			
4-10*	[0,2,3,5]	2,7	Maj (6,7,1,2)		(**2**:b2	
4-11	[0,1,3,5]	2,4,6*,7	Maj (1→4)	(**2**:2**)**		
4-11B	[0,2,4,5]		Maj (3→6)		(**2**:2)	
4-12	[0,2,3,6]	2,3,8,10*	Mm(7,1,2,4)			(**3**:6)
4-12B	[0,3,4,6]		Mm (7,2,3,4)			
4-13	[0,1,3,6]	2,5,7,10*	Maj (7,1,2,4)		(**3**:6)	
4-13B	[0,3,5,6]		Maj (7,2,3,4)			
4-14	[0,2,3,7]	2,4,9*,11	Maj (6,7,1,3)	(**3**:6)		
4-14B	[0,4,5,7]		Maj (1,3,4,5)			
4-Z15	[0,1,4,6]	3,5,7,8	Mm (7,1,3,4)			(**2**:4)
4-Z15B	[0,2,5,6]		Mm (6,7,2,3)			
4-16	[0,1,5,7]	5,7,8,9*	Maj (3,4,6,7)	(**2**:#4)		
4-16B	[0,2,6,7]		Maj (4,5,7,1)			
4-17*	[0,3,4,7]	3,11	Hm (6,7,1,3)			(**2**:#5)
4-18	[0,1,4,7]	3,5,10*,11	Octatonic			(**2**:#4)
4-18B	[0,3,6,7]		Octatonic			
4-19	[0,1,4,8]	3,7,11,12*	Mm (7,1,3,5)			(**2**:5)
4-19B	[0,3,4,8]		Mm (7,2,3,5)	**(4**:#5)		
4-20*	[0,1,5,8]	4,11	Maj (3,5,7,1)	(**4**:5)		
4-21*	[0,2,4,6]	6*,8	Whole-Tone			
4-22	[0,2,4,7]	6*,7,9*,11	Maj (1,2,3,5)		(**3**:b6	
4-22B	[0,3,5,7]		Maj (6,1,2,3)			
4-23*	[0,2,5,7]	7,9*	Maj (1,2,4,5),		(**2**:4)	
4-24*	[0,2,4,8]	6*,8,12*	Mm (3,4,5,7)			
4-25*	[0,2,6,8]	8,8	Mm (3,4,6,7)			
4-26*	[0,3,5,8]	7,11	Min 7 (2nd)		(**2**:5)	
4-27	[0,2,5,8]	7,8,10*,11	Half-Dim(3rd)		**(2:b5**	
4-27B	[0,3,6,8]		Dom 7 (1st)			
4-28*	[0,3,6,9]	10*	Dim. 7			
4-Z29	[0,1,3,7])	3,5,10*,11	Maj (3,4,5,7)			
4-Z29B	[0,4,6,7]		Maj (4,6,7,1)			

4 note chord	Prime/Normal	3 note subsets	Other Name	Dom 7	Dim7
4-1*	[0,1,2,3]	1*,2	2-3 (1→4)		
4-2	[0,1,2,4]	1*,2,3,6*	3-6*(1→3)		
4-2B	[0,2,3,4]		3-6* (2→4)		
4-3*	[0,1,3,4]	2,3	Octatonic(1→4), HW		
4-4	[0,1,2,5]	1*,3,4,7	3-7(1→3)		
4-4B	[0,3,4,5]		3-7B(2→4)		
4-5	[0,1,2,6]	1*,4,5,8	3-8(1→3)		
4-5B	[0,4,5,6]		3-8B (2→4)		
4-6*	[0,1,2,7]	1*,5,9*	3-9*(1→3)		
4-7*	[0,1,4,5]	2,3	Hm(5→1)		
4-8*	[0,1,5,6]	3,4	Maj (7,1,3,4)		
4-9*	[0,1,6,7]	5,5	Octatonic (1,2,6,7)		
4-10*	[0,2,3,5]	2,7	Maj (6,7,1,2)		
4-11	[0,1,3,5]	2,4,6*,7	Maj (1→4)		
4-11B	[0,2,4,5]		Maj (3→6)		
4-12	[0,2,3,6]	2,3,8,10*	Mm(7,1,2,4)	(**2**:b2)	
4-12B	[0,3,4,6]		Mm (7,2,3,4)		
4-13	[0,1,3,6]	2,5,7,10*	Maj (7,1,2,4)		
4-13B	[0,3,5,6]		Maj (7,2,3,4)		
4-14	[0,2,3,7]	2,4,9*,11	Maj (6,7,1,3)		
4-14B	[0,4,5,7]		Maj (1,3,4,5)		
4-Z15	[0,1,4,6]	3,5,7,8	Mm (7,1,3,4)		
4-Z15B	[0,2,5,6]		Mm (6,7,2,3)	(**2**:#2)	
4-16	[0,1,5,7]	5,7,8,9*	Maj (3,4,6,7)		
4-16B	[0,2,6,7]		Maj (4,5,7,1)	(**2**:4)	
4-17*	[0,3,4,7]	3,11	Hm (6,7,1,3)		
4-18	[0,1,4,7]	3,5,10*,11	Octatonic (1,2,4,6)		
4-18B	[0,3,6,7]		Octatonic (1,3,5,6)		
4-19	[0,1,4,8]	3,7,11,12*	Mm (7,1,3,5)		
4-19B	[0,3,4,8]		Mm (7,2,3,5)		
4-20*	[0,1,5,8]	4,11	Maj (3,5,7,1)		
4-21*	[0,2,4,6]	6*,8	Whole-Tone	(**2**:2)	
4-22	[0,2,4,7]	6*,7,9*,11	Maj (1,2,3,5)		
4-22B	[0,3,5,7]		Maj (6,1,2,3)		
4-23*	[0,2,5,7]	7,9*	Maj (1,2,4,5), P4s		
4-24*	[0,2,4,8]	6*,8,12*	Mm (3,4,5,7)	(**3**:b6)	
4-25*	[0,2,6,8]	8,8	Mm (3,4,6,7)	(**2,4**:#4)	
4-26*	[0,3,5,8]	7,11	Min 7 (2^{nd})		
4-27	[0,2,5,8]	7,8,10*,11	Half-Dim(3^{rd})		
4-27B	[0,3,6,8]		Dom 7 (1^{st})	(**4**:5)	
4-28*	[0,3,6,9]	10*	Dim. 7		(**1,2,3,4**:⦸)
4-Z29	[0,1,3,7])	3,5,10*,11	Maj (3,4,5,7)	(**3**:6)	
4-Z29B	[0,4,6,7]		Maj (4,6,7,1)		

Figure 2.4 3(a-b)

Here is a table giving the pockets (Figure 2.4.4):

	b2	2	#2	4	#4/b5	5	#5/b6	6	#6
Maj	4-4a	4-11a	4-7*	4-8*	4-16a	4-20*	4-19b	4-14a	4-5a
MinMaj	4-2a	4-3*	NA	4-Z15a	4-18a	4-19a	4-17*	4-12a	4-4a
Dom	4-12a	4-21*	4-Z15b	4-16b	4-25*	4-27b	4-24*	4-Z29a	NA
Min	4-10*	4-11b	NA	4-23*	4-27a	4-26*	4-22a	4-13a	NA

Figure 2.4 4

In the Jazz Piano Book, Mark Levine bundles the 24 possible permutations, sans octave displacement, of four distinct pitches into 6 categories.[66] Closed, "Drop 2", "Drop 3", "Drop 24", "Drop 23", "Drop 23 flip." The closed category never spans more than an octave; the "Drop 23 flip" never less than two octaves and the rest always between one and two. "Drop" always refers to the closed voicing and can be understood as "take the closed voicing and drop the __ (& __) pitch(es) from the top down an octave. Therefore, the "drop 2" voicing of 4-12a's Dom (2:b2), e.g., <C, Db, E, Bb> , is <E, C, Db, Bb>; the "drop 23 flip" is <E, Db, C, Bb>. "Drop 1" would just be rotation. Note that I am listing these ordered pitches in ascending order.[67]

How can we systematically explore the impact of particular voice leading transformations while accumulating familiarity with the set classes? One way is to practice sequences of set classes (scales/chords) that are set in the same permutation category. Below is a set of set class loops that collectively contain the entirety of the 4-note set classes.[68] Should you choose to learn these set classes at the piano or another chordal instrument, you will find that parsimony does not

[66] Without octave placement means that for any 4 ascending pitches, such as <D, C, B, F>, if D is D_4, then: C can only be C_5 (**not** C_6); B, only B_5; and F, F_6.
[67] Tymozcko also discusses these voicing classifications in his recent article, "Approximate Set classes"(Tymoczko 2023).
[68] Finding a parsimonious map which connects each tetrad (set class component) exactly once has eluded me. While I have not formally proved its impossibility, from my investigation, discovering such a map seems unlikely. The issue regards a few factors: no two symmetrical chords are parsimonious; there are more symmetrical than non-symmetrical set classes (15:14); not all non-symmetrical set classes parsimoniously connect to a symmetrical class, and the paucity of non-symmetrical set classes that have a lot of connectivity. The longest chain that I have found, parsimoniously connects 42 of 43 set classes.

undermine voicing category. In other words, should you choose to begin a loop with a drop-2 voicing of a set class, each chord then traversed will also be in a drop-2 voicing.

As a pianist, I found drop 2 voicings very helpful to practice; I can often play them in the right hand. Plus, if I cannot, I can incorporate the lowest note into the left hand's functioning — whether it is playing a bass line or some other accompanimental figuration. In a jazz ensemble, drop-2 and even drop-3 can be great for comping. Closed voicings are often used when the hands perform different functions; often in solo piano music. Due to awkwardness of playing and spread in register, I have thought of the other categories as better for arranging and composition. However, consider them as you will. In short, the main take-away that I would like to impart is that there is a benefit to learning these 4-note set classes in a least a couple of different voicing categories. It enriches your understanding of them—perhaps even adjusting your perception of these set classes to being melodic rather than harmonic can also add to your sense of how voicing transformations: impact your perception of a set class; increase your chances of recognizing these set classes quickly "in the wild;" and, quickly improve many performance-related skill sets.

Similarly, as mentioned earlier, we can learn the four-note set classes as a refinement of the 3-note set classes. We have already done that for 3-3a, 3-4a, 3-7a, and 3-8a above — they are built in to the proposed classification scheme. Factors to consider are the span of the trichord, which is being kept within an octave, and salience of the inner voice. Generally speaking, the closer the middle pitch in the trichord is to the center of the span, the easier it is to think of the following tetrad as a refinement. Below (Figure 2.4.5) are "refinements" of trichords that are not listed in the above "pockets table" (Figure 2.4.4). As they are easily deduced from this same table, I will not include 3-3b, 3-4b, 3-7b, and 3-8b. Furthermore, I will not look at rotations of trichords that span less than a P5.

	b2	2	#2/b3	3/b4	4	#4/b5	5	#5/b6	6	#6/b7	7
3-1* $[\hat{1},\widehat{b2},\hat{7}]$	Any tetrad with either a M7 and a b2 or a M7 and a #6										
3-2ab $[\hat{1},\hat{2}/\hat{6},\hat{7}]$	Any tetrad with either a M7 and a 2 or a M7 and a 6										
3-5a $[\hat{1},\hat{4},\hat{7}]$	4-5a	4-13a	4-Z15a	4-8*	NA	4-9*	NA	4-18b	4-Z29b	4-6*	NA
3-5b $[\hat{1},\#\hat{4},\hat{7}]$	4-6*	4-Z29a	4-18a	4-16a	4-9*	NA	4-8*	4-Z15b	4-13b	4-5b	NA
3-6* $[\hat{1},\hat{2},\widehat{b7}]$	4-2b	NA	4-11b	4-21*	4-22a	4-24*	4-22b	4-21*	4-11a	NA	-----
3-9* $[\hat{1},\hat{4},\widehat{b7}]$	4-14a	4-22a	4-23*	4-16b	NA	4-16a	NA	4-22b	4-14b	NA	-----
3-10* $[\hat{1},\hat{4},bb\hat{7}]$	4-12b	4-13b	NA	4-18b	4-27b	4-28	4-27a	4-18a	NA	-----	-----
3-11a $[\hat{1},\widehat{b3},\hat{5}]$	4-Z29	4-14a	NA	4-17*	4-22b	4-18b	NA	-----	-----	-----	-----
3-11b $[\hat{1},\hat{3},\hat{5}]$	4-18a	4-22a	4-17*	NA	4-14b	4-Z29b	NA	-----	-----	-----	-----
3-12* $[\hat{1},\hat{3},\widehat{b6}]$	4-19a	4-24*	4-19b	NA	4-19a	4-24*	4-19b	NA	-----	-----	-----

Figure 2.4 5

Looking at the above chart also gives insight into the 8-note set classes. Revisit the earlier part of this chapter with this in mind. For instance, look at the 3-10* row. All of the 8 note complements of these set classes have a strong diminished component (just look in the diminished column 2_5_8B and 8D). Similarly, just as 3-4, 3-7, and 3-9 contain one or more P5s, the 8 complements associated with them are rife with P5s. Likewise, the complements of the minimal supersets of 3-6* and 3-8*(Dom) are often supersets of the whole-tone scale; the latter actually contains all of the 8-note supersets of 6-35*, the whole tone scale.

If you have not already done so, it’s important to start learning the prime/normal forms (transposition 0) associated with each of these 4-note set classes. As mentioned earlier, they can be found in the “other forms” column. These will become important anchors. At this point, it is less critical to do this for the 8-note set classes; return to that task after having learned the hexads, if not the septads. However, if there are certain 8-note set classes that you are already particularly interested in; by all means learn those—as needed, either skip ahead or look at Forte’s list of set classes.

8 note chord	7 note subsets	Other Name	Maj	Min
8-1*	1*,1*,2,4,5	2-5*(1st)(1→8)	(**2**:b2,2,#2,4,b5),(**3**:b2,2,#2,4,#6),(**4**:b2,2,#2,6,#6)	
8-2	1*,2,3,8*,9,11,13,Z36	3-8(2nd)(2→8)	(**2**:b2,2,#2,4,5),(**3**:b2,2,#2,#4,#6),(**5**:b2,2,#5,6,#6)	
8-2B		3-8B(1st)(2→8)	(**3**:b2,2,#2,4,6),(**4**:b2,2,#2,b6,#6),(**8**:#4,5,#5,6,#6)	
8-3*	1*,3,10,16,Z17*	3-10*(2nd)(1→6)	(**2**:b2,2,#2,4,b6),(**3**:b2,2,#2,5,#6),(**6**:b2,5,b6,6,#6)	
8-4	2,3,4,6,11,14,Z37*,Z38	4-14(3rd)(1→6)	(**2**:b2,2,#2,#4,5),(**4**:b2,2,4,6,#6),(**5**:b2,#2,#5,6,#6)(**8**:4,#4,5,b6,6)	
8-4B		4-14B(1st)(3→8)	(**2**:2,#2,4,#4,5),(**4**:b2,2,#2,b6,6),(**8**:4,5,#5,6,#6)	
8-5	4,5,6,7,9,13,15*,Z38	4-24*(3rd)(1→5,6→8)	(**3**:b2,2,4,b5,#6)(**4**:b2,#2,4,6,#6),(**5**:2,#2,#5,6,#6)(**8**:4,#4,5,b6,#6)	
8-5B		4-24*(1→3,4→8)	(**2**:b2,#2,4,#4,5),(**3**:2,#2,4,#4,#6),(**8**:4,#4,#5,6,#6)	
8-6*	5,7,14,Z36	4-26*(1→4,5→8)	(**2**:b2,2,4,#4,5),(**3**:b2,#2,4,b5,#6)(**4**:2,#2,4,6,#6),(**8**:4,#4,5,6,#6)	(**5**:b2,2,5,#5,6)
8-7*	3,6,Z18,21	4-17*(3rd)(1→6)	(**2**:b2,2,#2,5,b6),(**5**:b2,4,#5,6,#6),(**6**:#2,5,#5,6,#6), (**8**:#2,4,#4,5,b6)	
8-8*	6,7,Z12*,20,22*	4-26*(3rd)(1→5,6→8)	(**4**:b2,4,b5,6,#6)(**5**:#2,4,#5,6,#6),(**7**:b2,4,#4,5,b6),(**8**:#2,4,#4,5,#6)	
8-9*	7,7,19,19	4-28*(1→4,5→8)	(**2,7**:b2,4,#4,5,#6),(**3,8**:#2,4,b5,6,#6)	
8-10*	2,10,23,25	4-23*(2nd)(2→7)	(**3**:b2,2,#2,#4,6),(**5**:b2,2,5,6,#6)	(**8**:4,#4,5,b6,6)
8-11	2,3,9,Z12*,23,24,26,27	4-22(3rd)(1→6)	(**2**:b2,2,#2,#4,#5),(**4**:b2,2,#4,6,#6),(**6**:2,5,b6,6,#6)	
8-11B		4-22B(1st)(4→8)	(**4**:b2,2,#2,5,6),(**7**:b2,#4,#5,6,#6)	
8-12	4,8*,10,16,Z18,26,28,31	5-28B(2nd)(3→7)	(**2**:2,#2,4,b5,b6),(**5**:b2,2,5,b6,#6)	
8-12B		5-28(3rd)(2→6)	(**4**:b2,2,4,#5,#6),(**5**:b2,#2,5,6,#6)	
8-13	4,10,Z12*,19,25,29,31,Z36	5-25(4th)(1→5)	(**3**:b2,2,4,5,#6),(**4**:b2,#2,#4,6,#6)	(**6**:b2,#4,5,b6,6)
8-13B		5-25B(1st)(3→8)	(**3**:2,#2,4,b5,6),(**7**:b2,#4,5,6,#6)	
8-14	5,11,Z17*,Z18,20,23,27,29	5-35*(1→3,4→7)	(**2**:b2,#2,4,b5,b6),(**3**:2,#2,4,5,#6),(**5**:b2,2,5,b6,6)	(**4**:b2,2,4,b6,6)
8-14B		5-35*(3rd)(2→5,6→8)	(**3**:b2,2,4,b5,6),(**4**:b2,#2,4,#5,#6),(**5**:2,#2,5,6,#6)(**7**:b2,#4,5,b6,6)	(**2**:b2,2,4,#4,5)

8-1* [0,1,2,3,4,5,6,7]	**8-2a**[0,1,2,3,4,5,6,8]	**8-2b**[0,2,3,4,5,6,7,8]	**8-3***[0,1,2,3,4,5,6,9]	**8-4a**[0,1,2,3,4,5,7,8]	**8-4b**[0,1,3,4,5,6,7,8]
8-5a[0,1,2,3,4,6,7,8]	**8-5b**[0,1,2,4,5,6,7,8]	**8-6***[0,1,2,3,5,6,7,8]	**8-7***[0,1,2,3,4,5,8,9]	**8-8***[0,1,2,3,4,7,8,9]	**8-9***[0,1,2,3,6,7,8,9]
8-10*[0,2,3,4,5,6,7,9]	**8-11a**[0,1,2,3,4,5,7,9]	**8-11b**[0,2,4,5,6,7,8,9]	**8-12a**[0,1,3,4,5,6,7,9]	**8-12b**[0,2,3,4,5,6,8,9]	**8-13a**[0,1,2,3,4,6,7,9]
8-13b[0,2,3,5,6,7,8,9]	**8-14a**[0,1,2,4,5,6,7,9]	**8-14b**[0,2,3,4,5,7,8,9]	**8-Z15a**[0,1,2,3,4,6,8,9]	**8-Z15b**[0,1,3,5,6,7,8,9]	**8-16a**[0,1,2,3,5,7,8,9]
8-16b[0,1,2,4,6,7,8,9]	**8-17***[0,1,3,4,5,6,8,9]	**8-18**a[0,1,2,3,5,6,8,9]	**8-18b**[0,1,3,4,6,7,8,9]	**8-19a**[0,1,2,4,5,6,8,9]	**8-19b**[0,1,3,4,5,7,8,9]
8-20*[0,1,2,4,5,7,8,9]	**8-21***[0,1,2,3,4,6,8,t]	**8-22a**[0,1,2,3,5,6,8,t]	**8-22b**[0,1,2,3,5,7,9,t]**	**8-23***[0,1,2,3,5,7,8,t]	**8-24***[0,1,2,4,5,6,8,t]
8-25*[0,1,2,4,6,7,8,t]	**8-26***[0,1,2,4,5,7,9,t]	**8-27a**[0,1,2,4,5,7,8,t]	**8-27b**[0,1,2,4,6,7,9,t]**	**8-28***[0,1,3,4,6,7,9,t]	**8-Z29a**[0,1,2,3,5,6,7,9]
8-Z29b[0,2,3,4,6,7,8,9]					

Figure 8 A

8 note chord	7 note subsets	Other Name	MinMaj 7	Dom7	Dim7
8-1*	1*,1*,2,4,5	2-5*(1st)(1→8)	(**5**:b2,2,#5,6,#6)		
8-2	1*,2,3,8*,9,11,13,Z36	3-8(2nd)(2→8)	(**4**:b2,2,4,6,#6),(**6**:b2,5,b6,6,#6)	(**8**:4,#4,5,b6,6)	
8-2B		3-8B(1st)(2→8)	(**5**:b2,2,5,6,#6)	(**2**:b2,2,#2,4,b5)	
8-3*	1*,3,10,16,Z17*	3-10*(2nd)(1→6)	(**4**:b2,2,#4,6,#6),(**7**:#4,5,b6,6,#6)		(**1**:b2,2,b4,4),(**8**:b4,4,5,b6)
8-4	2,3,4,6,11,14,Z37*,Z38	4-14(3rd)(1→6)	(**3**:b2,2,4,b5,#6),(**6**:2,5,#5,6,#6)		
8-4B		4-14B(1st)(3→8)	(**5**:b2,2,5,b6,#6)	(**3**:b2,2,#2,4,6)	
8-5	4,5,6,7,9,13,15*,Z38	4-24*(3rd)(1→5,6→8)	(**2**:b2,2,4,#4,5)		
8-5B		4-24*(1→3,4→8)	(**6**:b2,2,5,b6,6)	(**5**:b2,2,#2,b6,6)	
8-6*	5,7,14,Z36	4-26*(1→4,5→8)			
8-7*	3,6,Z18,21	4-17*(3rd)(1→6)	(**3**:b2,2,#4,5,#6)		
8-8*	6,7,Z12*,20,22*	4-26*(3rd)(1→5,6→8)	(**2**:b2,2,#4,5,b6)		
8-9*	7,7,19,19	4-28*(1→4,5→8)			(**1,5**:b2,2,5,b6)
8-10*	2,10,23,25	4-23*(2nd)(2→7)	(**4**:b2,2,4,#5,#6),(**6**:b2,#4,#5,6,#6)	(**2**:b2,2,#2,4,5)	(**1**:2,b4,4,5)
8-11	2,3,9,Z12,23,24,26,27	4-22(3rd)(1→6)	(**3**:b2,2,4,5,#6),(**5**:b2,4,#5,6,#6)	(**8**:#2,4,#4,5,b6)	
8-11B		4-22B(1st)(4→8)	(**5**:b2,2,#4,#5,#6),(**8**:4,5,b6,6,#6)	(**2**:2,#2,4,#4,5),(**3**:b2,2,#2,4,b6)	
8-12	4,8*,10,16,Z18,26,28,31	5-28B(2nd)(3→7)	(**4**:b2,2,4,#5,6),(**6**:b2,#4,5,6,#6)	(**3**:b2,2,#2,#4,6)(**8**:#2,#4,5,b6,6)	(**1**:b2,b4,4,5)
8-12B		5-28(3rd)(2→6)	(**3**:b2,2,4,b5,6),(**6**:2,#4,#5,6,#6)(**8**:4,#4,5,b6,6)	(**2**:b2,2,#2,#4,5),(**7**:b2,#4,5,b6,6)	(**1**:2,b4,4,b6)
8-13	4,10,Z12,19,25,29,31,Z36	5-25(4th)(1→5)	(**2**:b2,2,4,b5,b6),(**5**:2,4,#5,6,#6)	(**8**:#2,4,#4,5,6)	(**1**:b2,2,b4,5)
8-13B		5-25B(1st)(3→8)	(**5**:b2,2,#4,#5,6),(**6**:b2,4,5,b6,#6)(**8**:4,#4,#5,6,#6)	(**2**:b2,#2,4,#4,5),(**4**:b2,2,#2,5,6)	(**1**:2,4,5,b6)
8-14	5,11,Z17*,Z18,20,23,27,29	5-35*(1→3,4→7)	(**6**:b2,#4,5,b6,#6)	(**8**:#2,4,5,b6,6)	
8-14B		5-35*(3rd)(2→5,6→8)	(**8**:4,#4,5,b6,#6)		

8-1* [0,1,2,3,4,5,6,7]	**8-2a**[0,1,2,3,4,5,6,8]	**8-2b**[0,2,3,4,5,6,7,8]	**8-3***[0,1,2,3,4,5,6,9]	**8-4a**[0,1,2,3,4,5,7,8]	**8-4b**[0,1,3,4,5,6,7,8]
8-5a[0,1,2,3,4,6,7,8]	**8-5b**[0,1,2,4,5,6,7,8]	**8-6***[0,1,2,3,5,6,7,8]	**8-7***[0,1,2,3,4,5,8,9]	**8-8***[0,1,2,3,4,7,8,9]	**8-9***[0,1,2,3,6,7,8,9]
8-10*[0,2,3,4,5,6,7,9]	**8-11a**[0,1,2,3,4,5,7,9]	**8-11b**[0,2,4,5,6,7,8,9]	**8-12a**[0,1,3,4,5,6,7,9]	**8-12b**[0,2,3,4,5,6,8,9]	**8-13a**[0,1,2,3,4,6,7,9]
8-13b[0,2,3,5,6,7,8,9]	**8-14a**[0,1,2,4,5,6,7,9]	**8-14b**[0,2,3,4,5,7,8,9]	**8-Z15a**[0,1,2,3,4,6,8,9]	**8-Z15b**[0,1,3,5,6,7,8,9]	**8-16a**[0,1,2,3,5,7,8,9]
8-16b[0,1,2,4,6,7,8,9]	**8-17***[0,1,3,4,5,6,8,9]	**8-18**a[0,1,2,3,5,6,8,9]	**8-18b**[0,1,3,4,6,7,8,9]	**8-19a**[0,1,2,4,5,6,8,9]	**8-19b**[0,1,3,4,5,7,8,9]
8-20*[0,1,2,4,5,7,8,9]	**8-21***[0,1,2,3,4,6,8,t]	**8-22a**[0,1,2,3,5,6,8,t]	**8-22b**[0,1,2,3,5,7,9,t]**	**8-23***[0,1,2,3,5,7,8,t]	**8-24***[0,1,2,4,5,6,8,t]
8-25*[0,1,2,4,6,7,8,t]	**8-26***[0,1,2,4,5,7,9,t]	**8-27a**[0,1,2,4,5,7,8,t]	**8-27b**[0,1,2,4,6,7,9,t]**	**8-28***[0,1,3,4,6,7,9,t]	**8-Z29a**[0,1,2,3,5,6,7,9]
8-Z29b[0,2,3,4,6,7,8,9]					

Figure 8 B

8 note chord	7 note subsets	Other Name	Maj7	Min7
8-Z15	6,9,10,14,19,28,30,32	5-26(4th)(1→5)	(3:b2,2,#4,5,#6),(5:2,4,#5,6,#6),(8:#2,4,#4,5,6)	(6:2,#4,5,b6,6)
8-Z15B		5-26B(1st)(4→7)	(2:2,4,#4,5,b6),(7:b2,4,5,6,#6),(8:#2,#4,#5,6,#6)	
8-16	7,14,15,Z18,20,24,29,30	5-34*(4th)(1→4,6→8)	(2:b2,2,#4,5,b6),(4:2,4,b5,6,#6),(7:b2,4,#4,5,6)(8:#2,4,#4,#5,#6)	
8-16B		5-34*(1→3,5→8)	(3:2,4,#4,5,#6),(7:b2,4,#4,#5,#6),(8:#2,4,5,6,#6)	(5:b2,2,#4,5,b6)
8-17*	11,16,21,32	6-29*(3→6)	(2:2,#2,4,5,b6),(4:b2,2,4,#5,6),(5:b2,#2,5,b6,#6),(8:#2,#4,5,b6,6)	(3:b2,2,4,b5,6)
8-18	16,Z18,19,22*,31,32,Z36,Z38	6-27B(5th)(1→4)	(2:b2,2,4,5,b6),(3:b2,#2,#4,5,#6),(8:#2,4,#4,#5,6)	
8-18B		6-27(5→8)	(4:2,#2,4,#5,6),(7:b2,4,5,b6,#6),(8:#2,#4,5,6,#6)	(5:b2,2,#4,5,6)
8-19	13,Z17*,21,21,22*,26,30,Z37*	6-34B(1→3,4→6)	(2:b2,#2,4,5,b6),(3:2,#2,#4,5,#6),(5:b2,#2,5,b6,6),(8:#2,4,5,b6,6)	
8-19B		6-34(3→5,7→8)	(2:2,#2,#4,5,b6),(4:b2,#2,4,#5,6)(5:2,#2,5,b6,#6),(7:b2,4,5,b6,6),(8:#2,#4,5,b6,#6)	
8-20*	20,21,27,Z38*	6-32*(1→3,6→8)	(2:b2,#2,#4,5,b6),(5:2,#2,5,b6,6),(7:b2,4,#4,#5,6)(8:#2,4,5,b6,#6)	
8-21*	8*,9,24,33*,33*,34*	5-33*(4th)(1→5)	(3:b2,2,#4,#5,#6),(5:2,#4,#5,6,#6)	
8-22	11,23,24,27,30,34*,35*,Z36	6-33(5th)(1→4)	(2:b2,2,4,5,6),(3:b2,#2,#4,#5,#6),(6:2,#4,5,b6,6)	(1:b2,2,4,b5,b6), (5:b2,4,5,b6,6)
8-22B		6-33B(1st)(5→8)	(4:2,#2,4,5,6), (6:b2,2,#4,#5,6), (8:2,#4,5,6,#6)	(2:2,4,#4,5,b6), (5:b2,2,4,5,6)
8-23*	14,23,29,35*,35*	6-32*(5th)(1→4)	(2:b2,2,#4,5,6),(4:2,4,5,6,#6),(7:2,4,#4,5,6)	(1:b2,2,4,5,b6),(5:2,4,5,b6,6),(6:2,4,#4,5,b6)
8-24*	13,26,30,33*,33*	6-35*(1→3,4→6)	(2:b2,#2,4,5,6),(3:2,#2,#4,#5,#6),(6:2,#4,5,b6,#6)	
8-25*	15*,15*,28,28,33*,33*	6-35*(rt,3rd)(1→3,5→7)	(3,7:2,4,#4,#5,#6)	
8-26*	25,27,32,35*,Z37*	Mixolydian(1→3)	(2:b2,#2,#4,#5,6),(5:2,4,5,b6,6), (8:2,#2,#4,5,6)	(4:b2,4,b5,b6,6),(6:2,4,#4,5,6)
8-27	25,26,28,29,31,32,34*,Z38*	Mixolydian(b6)(1→3)	(2:b2,#2,#4,5,6),(7:2,4,#4,#5,6)	(6:b2,4,#4,5,6)
8-27B		Locrian(natural 2)(6→8)	(5:2,#2,#4,#5,6),(8:2,4,5,b6,#6)	(1:2,4,#4,#5,6)
8-28*	31,31,31,31	Octatonic		
8-Z29	5,13,16,19,20,24,25,28	5-34*(4th)(1→4,5→7)	(2:b2,2,4,b5,b6),(3:b2,#2,4,5,#6),(4:2,#2,#4,6,#6)	
8-Z29B		5-34*(2→4,5→8)	(3:b2,#2,4,b5,6),(4:2,#2,4,#5,#6),(7:b2,#4,5,b6,#6)	(5:b2,2,#4,b6,6)

8-1* [0,1,2,3,4,5,6,7]	8-2a[0,1,2,3,4,5,6,8]	8-2b[0,2,3,4,5,6,7,8]	8-3*[0,1,2,3,4,5,6,9]	8-4a[0,1,2,3,4,5,7,8]	8-4b[0,1,3,4,5,6,7,8]
8-5a[0,1,2,3,4,6,7,8]	8-5b[0,1,2,4,5,6,7,8]	8-6*[0,1,2,3,5,6,7,8]	8-7*[0,1,2,3,4,5,8,9]	8-8*[0,1,2,3,4,7,8,9]	8-9*[0,1,2,3,6,7,8,9]
8-10*[0,2,3,4,5,6,7,9]	8-11a[0,1,2,3,4,5,7,9]	8-11b[0,2,4,5,6,7,8,9]	8-12a[0,1,3,4,5,6,7,9]	8-12b[0,2,3,4,5,6,8,9]	8-13a[0,1,2,3,4,6,7,9]
8-13b[0,2,3,5,6,7,8,9]	8-14a[0,1,2,4,5,6,7,9]	8-14b[0,2,3,4,5,7,8,9]	8-Z15a[0,1,2,3,4,6,8,9]	8-Z15b[0,1,3,5,6,7,8,9]	8-16a[0,1,2,3,5,7,8,9]
8-16b[0,1,2,4,6,7,8,9]	8-17*[0,1,3,4,5,6,8,9]	8-18a[0,1,2,3,5,6,8,9]	8-18b[0,1,3,4,6,7,8,9]	8-19a[0,1,2,4,5,6,8,9]	8-19b[0,1,3,4,5,7,8,9]
8-20*[0,1,2,4,5,7,8,9]	8-21*[0,1,2,3,4,6,8,t]	8-22a[0,1,2,3,5,6,8,t]	8-22b[0,1,2,3,5,7,9,t]**	8-23*[0,1,2,3,5,7,8,t]	8-24*[0,1,2,4,5,6,8,t]
8-25*[0,1,2,4,6,7,8,t]	8-26*[0,1,2,4,5,7,9,t]	8-27a[0,1,2,4,5,7,8,t]	8-27b[0,1,2,4,6,7,9,t]**	8-28*[0,1,3,4,6,7,9,t]	8-Z29a[0,1,2,3,5,6,7,9]
8-Z29b[0,2,3,4,6,7,8,9]					

Figure 8 C

8 note chord	7 note subsets	Other Name	MinMaj7	Dom7	Dim7
8-Z15	6,9,10,14,19,28,30,32*	5-26(4th)(1→5)	(**2**:b2,2,4,5,b6),(**4**:b2,4,b5,6,#6)	(**7**:b2,4,#4,5,b6)	(**1**:b2,2,b4,b6)
8-Z15B		5-26B(1st)(4→7)	(**5**:b2,2,#4,5,6)	(**3**:2,#2,4,b5,6),(**4**:b2,2,#2,5,b6)	(**1**:b2,4,5,b6)
8-16	7,14,15,Z18,20,24,29,30	5-34*(4th)(1→4,6→8)	(**3**:b2,4,#4,5,#6)	(**5**:2,#2,5,#5,6)	
8-16B		5-34*(1→3,5→8)	(**2**:b2,4,#4,5,b6)	(**5**:2,#2,4,#5,6)	
8-17*	11,16,21,32	6-29*(3→6)	(**6**:2,#4,5,6,#6)	(**5**:b2,4,5,b6,6)	(**1**:b2,b4,4,b6)
8-18	16,Z18,19,22*,31,32,Z36,Z38	6-27B(5th)(1→4)	(**4**:2,4,b5,6,#6) (**6**:2,#4,5,b6,6)	(**5**:b2,#2,5,b6,6),(**7**:b2,4,#4,5,6)	(**1**:b2,2,4,b6)
8-18B		6-27(5→8)	(**2**:2,4,#4,5,b6)	(**3**:b2,#2,4,b5,6)	(**1**:b2,b4,5,b6)
8-19	13,Z17*,21,21,22*,26,30,Z37*	6-34B(1→3,4→6)	(**6**:2,#4,5,b6,#6)	(**4**:b2,2,4,#5,6),(**7**:b2,4,b5,b6,6)	
8-19B		6-34(3→5,7→8)		(**2**:b2,2,4,b5,6)	
8-20*	20,21,27,Z38*	6-32*(1→3,6→8)	(**3**:2,4,#4,5,#6)	(**4**:b2,#2,4,b6,6)	
8-21*	8*,9,24,33*,33*,34*	5-33*(4th)(1→5)	(**2**:b2, 2, 4, 5, 6) (**4**:b2,4,5,6,#6)	(**1**:b2,2,#2,#4,b6),(**6**:2,#4,5,b6,6),(**7**:2,4,#4,5,b6),(**8**:2,#2,4,b5,b6)	
8-22	11,23,24,27,30,34*,35*,Z36	6-33(5th)(1→4)	(**6**:2,4,5,6,#6)	(**7**:2,4,#4,5,6),(**8**:2,#2,4,5,b6)	
8-22B		6-33B(1st)(5→8)	(**7**:b2,4,5,b6,#6)	(**1**:2,4,5,b6,6),(**3**:b2,#2,4,b5,b6)	
8-23*	14,23,29,35*,35*	6-32*(5th)(1→4)	(**3**:b2,4,#4,#5,#6)	(**8**:2,#2,4,5,6)	
8-24*	13,26,30,33*,33*	6-35*(1→3,4→6)	(**5**:b2,4,5,b6,6)	(**1**:b2,2,4,b5,b6),(**4**:b2,2,#4,b6,6),(**7**:2,4,b5,b6,6),(**8**:2,#2,#4,5,b6)	
8-25*	15*,15*,28,28,33*,33*	6-35*(rt,3rd)(1→3,5→7)	(**2,6**:b2,4,#4,5,6)	(**1,5**:b2,2,#4,5,b6),(**4,8**:2,#2,#4,b6,6)	
8-26*	25,27,32,35*,Z37*	Mixolydian(1→3)	(**3**:2,4,5,b6,#6)	(**1**:b2,2,4,5,6), (**7**:b2,#2,4,5,b6)	
8-27	25,26,28,29,31,32,34*,Z38*	Mixolydian(b6)(1→3)	(**3**:2,4,#4,#5,#6),(**5**:2,4,5,b6,6)	(**1**:b2,2,4,5,b6),(**4**:b2,#2,#4,b6,6),(**8**:2,#2,#4,5,6)	
8-27B		Locrian(natural 2)(6→8)	(**3**:2,4,#4,5,6), (**7**:b2,4,#4,#5,6)	(**2**:b2,#2,#4,5,b6),(**4**:b2,#2,4,5,6), (**6**:b2,2,#4,5,6),	
8-28*	31,31,31,31	Octatonic	(**2,4,6,8**:2,4,#4,#5,6)	(**1,3,5,7**:b2,#2,#4,5,6)	
8-Z29	5,13,16,19,20,24,25,28	5-34*(4th)(1→4,5→7)	(**6**:b2,#4,5,b6,6)	(**5**:b2,2,5,b6,6),(**8**:#2,4,b5,b6,6)	(**1**:b2,2,4,5)
8-Z29B		5-34*(2→4,5→8)	(**8**:4,#4,5,6,#6)	(**2**:b2,2,4,#4,5)	(**1**:2,b4,5,b6)

8-1* [0,1,2,3,4,5,6,7]	**8-2a**[0,1,2,3,4,5,6,8]	**8-2b**[0,2,3,4,5,6,7,8]	**8-3***[0,1,2,3,4,5,6,9]	**8-4a**[0,1,2,3,4,5,7,8]	**8-4b**[0,1,3,4,5,6,7,8]
8-5a[0,1,2,3,4,6,7,8]	**8-5b**[0,1,2,4,5,6,7,8]	**8-6***[0,1,2,3,5,6,7,8]	**8-7***[0,1,2,3,4,5,8,9]	**8-8***[0,1,2,3,4,7,8,9]	**8-9***[0,1,2,3,6,7,8,9]
8-10*[0,2,3,4,5,6,7,9]	**8-11a**[0,1,2,3,4,5,7,9]	**8-11b**[0,2,4,5,6,7,8,9]	**8-12a**[0,1,3,4,5,6,7,9]	**8-12b**[0,2,3,4,5,6,8,9]	**8-13a**[0,1,2,3,4,6,7,9]
8-13b[0,2,3,5,6,7,8,9]	**8-14a**[0,1,2,4,5,6,7,9]	**8-14b**[0,2,3,4,5,7,8,9]	**8-Z15a**[0,1,2,3,4,6,8,9]	**8-Z15b**[0,1,3,5,6,7,8,9]	**8-16a**[0,1,2,3,5,7,8,9]
8-16b[0,1,2,4,6,7,8,9]	**8-17***[0,1,3,4,5,6,8,9]	**8-18**a[0,1,2,3,5,6,8,9]	**8-18b**[0,1,3,4,6,7,8,9]	**8-19a**[0,1,2,4,5,6,8,9]	**8-19b**[0,1,3,4,5,7,8,9]
8-20*[0,1,2,4,5,7,8,9]	**8-21***[0,1,2,3,4,6,8,t]	**8-22a**[0,1,2,3,5,6,8,t]	**8-22b**[0,1,2,3,5,7,9,t]**	**8-23***[0,1,2,3,5,7,8,t]	**8-24***[0,1,2,4,5,6,8,t]
8-25*[0,1,2,4,6,7,8,t]	**8-26***[0,1,2,4,5,7,9,t]	**8-27a**[0,1,2,4,5,7,8,t]	**8-27b**[0,1,2,4,6,7,9,t]**	**8-28***[0,1,3,4,6,7,9,t]	**8-Z29a**[0,1,2,3,5,6,7,9]
8-Z29b[0,2,3,4,6,7,8,9]					

Figure 8 D

Part II, Chapter 13: 2.5

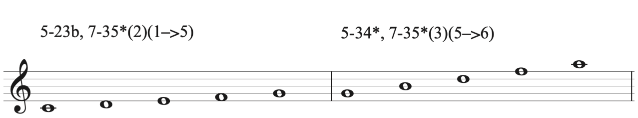
Section (A_5, R_7); $(x+y)^5$.

R:

Unlike the previous sections, I will introduce both the 7-note and the 5-note set classes in this first section.

Beginning with the 5-note set classes: the “other chords” column of my chart is coded a little differently. Since the vast majority of the pentads are subsets of the just-mentioned familiar 7-note set classes, I have opted to describe them in terms of those supersets. In the relevant cases, strictly when dealing with supersets, the arrow does not refer to chromatic clusters that span from the starting to ending numbers, where both are scale degrees. Instead, the arrow indicates a string of consecutive pitches, wherein the starting and ending number indicate scale degrees. The (2) indicates that the underlying scale is expressed in its typical ordering, consecutive 2nds. For instance, 5-23b, 7-35*(2)(1➔5), spans the first five scale degrees of a major scale; in C major, this is C-D-E-F-G. On the other hand, 5-34*, 7-35*(3)(5➔6): starts on scale degree 5, spans five consecutive pitches of a major scale expressed as stacked/consecutive thirds, and ends on scale degree 6; in C major, this is G-B-D-F-A[69].

As mentioned above, the chart 5A defaults to expressing the 5-note set classes in terms of the most familiar 7-note set classes: 7-35* (Maj), 7-34*(Mm), 7-32ab (Hm & HM), and 7-31ab (the octatonic subsets HW & WH). When there are multiple candidates for familiar supersets, I default to the one with the highest Forte number. For instance, even though 5-34* is contained in 7-34*(Mm) and 7-35*(Maj), I only reference 7-35* as a superset.



69

Maj7:

Scale Degree	**(b2)**	**(2)**	**(#2)**	**(4)**	**(#4)**	**(5)**	**(#5/b6)**	**(6)**	**(#6)**	**Fixed Maj**
(b2)	**NA**	5-2a	5-3a	5-6a	5-14a	5-Z38a	5-Z37*	5-11a	5-4a	4-4a
(2)		**NA**	5-3b	5-Z12*	5-24a	5-27a	5-26b	5-23a	5-9a	4-11b
(#2)			**NA**	5-6b	5-Z18a	5-21a	5-21b	5-Z18b	5-6a	4-7*
(4)				**NA**	5-7b	5-20a	5-22*	5-20b	5-7a	4-8*
(#4)					**NA**	5-20b	5-30b	5-29b	5-15*	4-16a
(5)						**NA**	5-21b	5-27b	5-Z38b	4-20*
(#5/b6)							**NA**	5-Z17*	5-13b	4-19b
(6)								**NA**	5-5a	4-14a
(#6)									**NA**	4-5a

MinMaj7:

Scale Degree	**(b2)**	**(2)**	**(4)**	**(#4/b5)**	**(5)**	**(#5/b6)**	**(6)**	**(#6)**	**Fixed MinMaj**
(b2)	NA	5-1*	5-9a	5-Z36	5-13a	5-11b	5-8*	5-2a	4-2a
(2)		NA	5-10a	5-16a	5-Z17*	5-16b	5-10b	5-3a	4-3*
(4)			NA	5-19b	5-30a	5-32a	5-28a	5-14a	4-Z15a
(#4)				NA	5-22*	5-32b	5-31b	5-Z38a	4-18a
(5)					NA	5-21a	5-26a	5-Z37*	4-19a
(#5/b6)						NA	5-16a	5-11a	4-17*
(6)							NA	5-4a	4-3*
(#6)								NA	4-2a

Dom7:

Scale Degree	**(b2)**	**(2)**	**(#2)**	**(4)**	**(#4/b5)**	**(5)**	**(#5/b6)**	**(6)**	**Fixed Dom7**
(b2)	NA	5-8*	5-10b	5-Z18b	5-28a	5-31b	5-26a	5-16a	4-12a
(2)		NA	5-9b	5-24b	5-33*	5-34*	5-33*	5-24a	4-21*
(#2)			NA	5-14b	5-28b	5-32b	5-30b	5-19a	4-Z15b
(4)				NA	5-15*	5-29a	5-30a	5-20a	4-16b
(#4)					NA	5-28a	5-33*	5-28b	4-25*
(5)						NA	5-26b	5-25b	4-27b
(#5/b6)							NA	5-13a	4-24*
(6)								NA	4-13a

Min7:

Scale Degree	(b2)	(2)	(4)	(#4/b5)	(5)	(#5/b6)	(6)	Fixed Min7
(b2)	NA	5-2b	5-23a	5-25a	5-25b	5-23b	5-10a	4-10*
(2)		NA	5-23b	5-26a	5-27b	5-24b	5-Z12*	4-11b
(4)			NA	5-29b	5-35*	5-35*	5-29a	4-23*
(#4)				NA	5-32a	5-34*	5-31a	4-27a
(5)					NA	5-27a	5-25a	4-26*
(#5/b6)						NA	5-Z36a	4-22a
(6)							NA	4-13a

Dim 7:

Scale Deg.	(b2)	(2)	(b4)	(4)	(5)	(#5/b6)	Fixed Dim7
	5-31a	5-31b	5-31a	5-31b	5-31a	5-31b	4-28*

Figure 2.5 1(a-e) - The 5-note pockets

Alternatively, you can think of the 5-note set classes as minimal supersets of the 4-note set classes. One way of encapsulating this is to learn a row in the above table; each 5-note set class in a given row fixes the same 4-note set class component. Also, remember, if you take the inverse of the fixed 4-note chord, the inverses of the associated 5-note set class components will likewise be associated with it. However, it may wreak havoc on the chord type classification given. There is no implication that the intervallic inversions yielded will subscribe to the same chord type; plus, the "added scale degree" will also have to be inversed. Check Figure 2.3 8 for details on how to do this. However, notice that not all of the 4-note set classes are represented. Missing are 4-1*, 4-6*, 4-9*, 4-14, and 4-Z29. Of course, this does not mean that those 4-note set classes are not contained in the listed 5-note set classes, but just that the above charts do not classify those 5-note set classes in regard to the "missing 4-note set classes."

Below (Figure 2.5.2), I have aggregated those set classes most used in Jazz. Here I have expressed them as a root note plus a 4-note component of a set class. In a jazz ensemble, the pianist often focuses on root-less voicings of a chord. The assumption is that the bass player would double the root note. For example, take the transposition 4-14b, C-E-F-G. If the bass note is Db, you get Maj #2, #4 and the associated 5-note set class is 5-Z18; with bass note Ab, Maj #5, 6 and 5-Z17* etc.

A couple of notes: In Jazz pedagogy, it's often recommended that one does not play the natural 4 with a Maj 3; particularly in the context of a Maj 7 chord. As such, I've highlighted all of those "dissonant" chords in light grey. However, I didn't exclude them as my assumption is that anyone curious enough to pursue this pedagogy, may, at minimum, not be bothered by the extra information, and, at maximum, interested in those "dissonant" chords. What I did exclude were set classes of cardinality 4 or 5 that either included clusters (e.g., 4-1*) or didn't fall into one of the two following categories: some type of 7th (e.g., Dim 7, b9) or 6,9 chord (Maj #2,5,6) with no Maj 7). Furthermore, all of the 5-note set classes used were subsets of all but one of the familiar 7 and 8-note set classes typically used in "traditional" Jazz:[70] 8-28*, 7-35*, 7-34*, 7-32, and 7-31. The two exceptions being: chords only found in Harmonic Major and/or chords that include clusters and can be derived from the bebop scale (8-23*).

Using this chart below (Figure 2.5.2), one can see how much mileage can be attained through knowing one 4-note set class well; especially, if you've also learned it in either its drop 2 or drop 3 forms. This chart provides a prime example of how one can build one's understanding of a set class in terms of its maximal subsets. Again, I recommend using the parsimonious maps introduced in the last chapter to greatly facilitate learning these. When I sought to memorize/internalize them myself without the maps, I learned about one every two weeks. After I changed the order in which I learned new ones, exclusively taking ones that were parsimonious to ones that I had already learned, my learning rate increased to about 2 (all in drop 2 form) every 3 days. After internalizing these forms though, one would still have to practice "conjuring them up" in a performance context.

[70] I am not referring to Free, Avant-Garde, or Experimental Jazz. In those genres anything can appear. I'm referring to those styles often taught in the first few years of a Jazz program—Swing, Bebop, 1950's Miles Davis, etc.

Set classes	Maj	MinMaj7	Dom7	Min	Sus	Ø	Associated 5-note Set classes
4-8*				Min $^{2, 5, b6}$, Min $^{9, 13}$			5-20b, 5-Z12*
4-9*			Dom $^{\#9, 13}$ (2x)				5-19a
4-12a			Dom $^{\#4,5}$				5-28a
4-12b		MinMaj ^{2,4}	Dom $^{5, b6}$	Min $^{\#4, 5, 6}$			5-10a, 5-26b, 5-31a
4-13b	Maj $^{2, 4}$		Dom $^{5, 13}$				5-Z12*, 5-25b
4-14a	Maj $^{\#4, 5}$					Ø ^{11}	5-20b, 5-29b
4-14b	Maj $^{\#2, \#4}$, Maj $^{\#5,6}$			Min $^{5, b13}$, Min $^{9, 11}$			5-Z18a, 5-Z17*, 5-27a, 5-23b
4-Z15a			Aug ^{11}	Min $^{b9, 13}$			5-30a, 5-10a
4-Z15b		MinMaj ^{2,6}	Dom $^{\#11, 13}$				5-10b, 5-28b
4-16a	Maj ^{4,6}		Dom $^{9, 13}$, Aug $^{\#9}$, Dom ^{11,13}	Min $^{2, 5, 6}$			5-20b, 5-24a, 5-30b, 5-20a
4-16b	Maj $^{\#2,6}$			Min $^{11, 13}$, Min $^{9, b13}$			5-Z18b, 5-29a, 5-24b
4-17*	Maj $^{5, b6}$	MinMaj $^{2, \#4}$		Min $^{5, \#11}$			5-21b, 5-16a, 5-32a
4-18a	Maj $^{4, \#5}$		Dom $^{5, \#9}$, Dom $^{b9, 13}$				5-22*, 5-32b, 5-16a
4-18b		MinMaj $^{2, \#5}$	Dom $^{b9, 11}$			Ø ^{13}	5-16b, 5-Z18b, 5-31a
4-19a	Maj $^{\#2, 5}$	MinMaj $^{5, b6}$			Sus $^{b9, 13}$	Ø ^{9}	5-21a, 5-21a, 5-Z17*, 5-26a
4-19b		MinMaj ^{2,5}, MinMaj $^{\#4, 5}$					5-Z17*, 5-22*
4-20*	Maj $^{\#2, \#5}$			Min $^{5, 9}$	Sus $^{9, 13}$		5-21b, 5-27b, 5-27a
4-21*			Aug $^{\#4}$				5-33*
4-22a	Maj $^{\#4, \#5}$			Min $^{4, 5}$			5-30b, 5-35*
4-22b	Maj $^{2, \#4}$, Maj $^{5, 6}$			Min $^{b9, 11}$		Ø ^{b13}	5-24a, 5-27b, 5-23a, 5-34*
4-23*	Maj $^{5, 6, 9}$, Maj $^{2, 6}$, Maj $^{\#4, 6}$			Min $^{b2, b6}$, Min $^{4, b6}$			5-35*, 5-23a, 5-29b, 5-23b, 5-35*
4-24*		MinMaj ^{5,6}, MinMaj ^{4,5}	Dom. $^{9, \#11}$				5-26a, 5-30a, 5-33*
4-25*		MinMaj $^{4, 6}$ (2x)	Dom. $^{9, b13}$ (2x)				5-28a, 5-33*
4-26*	Maj $^{2, 5}$	MinMaj $^{\#4, \#5}$			Sus ^{5,9}	Ø ^{b9}	5-27a, 5-32b, 5-35*, 5-25a
4-27a		MinMaj $^{4, \#5}$	Dom $^{5, 9}$, Dom $^{b9, b13}$		Sus $^{5, b9}$		5-32a, 5-34*, 5-26a, 5-29b
4-27b	Maj $^{2, \#5}$	MinMaj $^{\#4,6}$	Dom $^{b9, \#11}$	Min $^{5, b9}$	Sus $^{9, b13}$		5-26b, 5-31b, 5-28a, 5-25b, 5-34*
4-28*			Dom $^{5, b9}$ (4x)				5-31b
4-Z29a	Maj $^{4, 5}$	MinMaj $^{\#5,6}$	Dom $^{\#9, \#11}$				5-20a, 5-16a, 5-28b
4-Z29b		MinMaj $^{4, b5}$	Dom $^{9, 11}$	Min $^{5, 13}$			5-19b, 5-24b, 5-25a

Figure 2.5 2

Due to the above charts collectively representing most of the 5-note set classes in regard to their familiar supersets, maximal subsets, and parsimonious relationships, I believe that there is enough here for you to not only start your exploration of these set classes, but to get very far as well. The learning of the 'cluster' set classes could also benefit from practicing them in the following groupings: 5-1*, 5-2, and 5-3, which conform to a nice ordering principle, and the already-listed first rows (b2) of the Maj and MinMaj parsimony charts.

On the other hand, even with my providing the following guidance, the 7-note set classes may still take quite a while to feel well-acquainted with. Arguably, they, in their totality, are the most difficult to learn. While there are fewer septads than hexads, the added complexity takes its toll.[71]

As with the 5-note set classes, the 7-note parsimony charts are presented below. As before, to get all of the voicings, one would need to create the parsimony charts in regard to the inverse of the fixed hexachords. However, as mentioned before, these would cut across the classification types already given. Essentially, I do not see this as a big concern. Your increased familiarity with these, your learning of Figure 6 A, and facile work figuring out complements, will adequately prepare you; they really do represent the vast majority. However, if you are going to invert one row, I recommend inverting the first row of each of the 7-note classification charts given above.

[71] Throughout this comment, I'm assessing the level of difficulty/challenge in terms of: one, how numerous the associated classifiable chords are; two, whether or not they are symmetric; three, whether or not they conform to straightforward principles (e.g., the first set classes of any cardinality), and four, relatability. Technically speaking, certain 9-note set classes—9-7, 9-8, and 9-11 are the most difficult individual set classes to learn; they can be considered difficult in all three of the above criteria. While the 10 and 11-note set classes house more classifiable chords, they are not considered difficult by the second and third criteria. Finally, as shown in the previous chapter, understanding the 8-note set classes is greatly aided by: the great number of symmetries, the relative simplicity of their complement, and the ability to describe the set classes with the most classifiable rotations in terms of already familiar 7-note scales—Major, Harmonic and Melodic minor, 7-31 ("diminished"), and 7-33* ("whole tone + 1). The 8-note set classes that meet all 4 criteria of difficulty are 8-11 – 8-16 and 8-Z29—although 8-12, 8-13, and 8-Z15 have a strong diminished aspect to them. However, due to the relative paucity of these set classes, their only being bigger than the 7-note classes by a single-note and, relative to their cardinality, their only having a medium number of classifiable rotations, the 8-note set classes fall short of wresting the award of "the most difficult set classes, in terms of cardinality, to learn" from the 7-note set classes. Let me know if you feel differently about this!!

Maj 7:

(b2)	(2)	(#2)	(4)	(#4/b5)	(5)	(~4,~5,~6)	Fixed Maj
7-3a	7-9a	7-6a	7-6b	7-9b	7-3b	**(b6,6,#6)**	6-Z37*
7-10b	7-23a	7-Z18b	7-14b	7-Z36b		**(5,6,6#)**	6-Z40b
7-16b	7-26a	7-21b	7-Z38b	7-13b		**(5,b6,#6)**	6-15b
7-Z17*	7-27a	7-21a	7-Z37*	7-11b		**(5,b6,6)**	6-14b
7-Z12*	7-24a	7-19a	7-7b			**(#4,6,#6)**	6-Z41b
7-24b	7-33a	7-30b	7-15a			**(#4,#5,#6)**	6-22b
7-27b	7-34*	7-32b	7-Z38a			**(#4,#5,6)**	6-Z24b
7-19a	7-30a	7-22a	7-7a			**(#4,5,#6)**	6-Z43b
7-29b	7-35*	7-32a	7-14a			**(#4,5,6)**	6-Z25b
7-20b	7-30b	7-21b	7-6a			**(#4,5,b6)**	6-16b
7-6a	7-14a	7-7a				**(4,6,#6)**	6-Z38a
7-Z18b	7-28b	7-20b				**(4,#5,#6)**	6-Z17b
7-21b	7-32a	7-22*				**(4,#5,6)**	6-Z19b
7-19a	7-29a	7-20a				**(4,5,#6)**	6-18b
7-30b	7-35*	7-26b				**(4,5,6)**	6-Z26*
7-22*	7-32b	7-21b				**(4,5,b6)**	6-Z19a
7-7a	7-15*	7-7b				**(4,b5,#6)**	6-7*
7-20b	7-29b	7-19b				**(4,b5,6)**	6-18a
7-20a	7-28a	7-Z18a				**(4,b5,b6)**	6-Z17a
7-7b	7-14b	7-6b				**(4,#4,5)**	6-Z38*

(b2,2)	(b2,#2)	(2,#2)	(~5,~6)	Fixed Maj
7-2a	7-4a	7-5a	**(6,#6)**	5-5a
7-8*	7-11b	7-13b	**(b6,6#)**	5-13b
7-10a	7-16a	7-Z17*	**(5,#6)**	5-Z38b
7-9a	7-Z36a	7-13a	**(#4,#6)**	5-15*
7-4a	7-5a	7-5b	**(4,#6)**	5-7a
7-11a	7-Z37*	7-Z38b	**(#5,6)**	5-Z17*
7-23b	7-26b	7-27b	**(5,6)**	5-27b
7-23a	7-25a	7-25a	**(#4,6)**	5-29b
7-11b	7-13b	7-Z36b	**(4,6)**	5-20b
7-Z18a	7-21a	7-21b	**(5,b6)**	5-21b
7-24a	7-27a	7-26a	**(#4,#5)**	5-30b
7-16a	7-Z17*	7-16b	**(4,#5)**	5-22*
7-14a	7-Z38a	7-Z37*	**(#4,5)**	5-20b
7-Z36a	7-13a	7-11a	**(4,5)**	5-20a
7-5a	7-5b	7-4b	**(4,b5)**	5-7b

MinMaj7:

(b2)	(2)	(4)	(#4/b5)	(5)	(~4,~5,~6)	Fixed MinMaj
7-2a	7-4a	7-Z12*	7-10b	7-3a	**(b6,6,#6)**	6-Z36a
7-8*	7-11b	7-24b	7-16b		**(5,6,6#)**	6-Z39*
7-11a	7-Z37*	7-27b	7-Z17*		**(5,b6,#6)**	6-14a
7-13a	7-Z38a	7-26b	7-16a		**(5,b6,6)**	6-15a
7-10a	7-16a	7-19a			**(#4,6,#6)**	6-Z42*
7-23b	7-26b	7-29b			**(#4,#5,#6)**	6-Z46b
7-25b	7-31b	7-31b			**(#4,#5,6)**	6-27b
7-Z18a	7-21a	7-20b			**(#4,5,#6)**	6-Z44b
7-28a	7-32b	7-28b			**(#4,5,6)**	6-Z28*
7-20a	7-22*	7-Z18b			**(#4,5,b6)**	6-Z44a
7-9a	7-Z36a				**(4,6,#6)**	6-Z41a
7-23a	7-25a				**(4,#5,#6)**	6-Z47b
7-26a	7-31a				**(4,#5,6)**	6-Z49*
7-24a	7-27a				**(4,5,#6)**	6-Z48*
7-33*	7-34*				**(4,5,6)**	6-34b
7-30a	7-32a				**(4,5,b6)**	6-31b
7-14a	7-Z38a				**(4,b5,#6)**	6-18a
7-28b	7-31b				**(4,b5,6)**	6-30b
7-29a	7-31a				**(4,b5,b6)**	6-Z50*
7-15*	7-Z38b				**(4,#4,5)**	6-Z17b

(b2,2)	(~5,~6)	Fixed MinMaj
7-1*	**(6,#6)**	5-4a
7-2b	**(b6,6#)**	5-11a
7-3b	**(5,#6)**	5-Z37*
7-3a	**(#4,#6)**	5-Z38
7-2a	**(4,#6)**	5-14a
7-4b	**(#5,6)**	5-16a
7-9b	**(5,6)**	5-26a
7-10b	**(#4,6)**	5-31b
7-8*	**(4,6)**	5-28a
7-6b	**(5,b6)**	5-21a
7-Z12*	**(#4,#5)**	5-32b
7-10a	**(4,#5)**	5-32a
7-6a	**(#4,5)**	5-22*
7-9a	**(4,5)**	5-30a
7-4a	**(4,b5)**	5-19b

Dom 7:

(b2)	(2)	(#2)	(4)	(#4/b5)	(~4,~5,~6)	Fixed Dom
7-16	7-24a	7-Z18a	7-11a	7-8*	**(5,b6,6)**	6-Z39b
7-26b	7-33a	7-28a	7-13a		**(#4,#5,6)**	6-21b
7-31b	7-34*	7-31a	7-Z36a		**(#4,5,6)**	6-Z23*
7-28b	7-33*	7-26a	7-9a		**(#4,5,b6)**	6-21a
7-21a	7-30a	7-20a			**(4,#5,6)**	6-16a
7-32b	7-35*	7-29a			**(4,5,6)**	6-Z25a
7-32a	7-34*	7-27a			**(4,5,b6)**	6-Z24a
7-22*	7-30b	7-19a			**(4,b5,6)**	6-Z43a
7-30a	7-33*	7-24a			**(4,b5,b6)**	6-22a
7-19a	7-24b	7-Z12*			**(4,#4,5)**	6-Z41a

(b2,2)	(b2,#2)	(2,#2)	(~5,~6)	Fixed Dom
7-13a	7-Z38a	7-15*	**(#5,6)**	5-13a
7-25b	7-31b	7-29b	**(5,6)**	5-25b
7-26a	7-31a	7-28a	**(#4,6)**	5-28b
7-Z37*	7-Z38b	7-14b	**(4,6)**	5-20a
7-28a	7-32b	7-30b	**(5,b6)**	5-26b
7-33*	7-34*	7-33*	**(#4,#5)**	5-33*
7-26b	7-27b	7-24b	**(4,#5)**	5-30a
7-28b	7-31b	7-26b	**(#4,5)**	5-28a
7-25a	7-25b	7-23b	**(4,5)**	5-29a
7-13b	7-Z36b	7-9b	**(4,b5)**	5-15*

Min 7:

(b2)	(2)	(4)	(#4/b5)	(~4,~5,~6)	Fixed Min
7-Z36a	7-14a	7-23b	7-10a	**(5,b6,6)**	6-Z40a
7-25a	7-28b	7-25b		**(#4,#5,6)**	6-Z45*
7-31a	7-32a	7-25a		**(#4,5,6)**	6-27a
7-29a	7-30a	7-23a		**(#4,5,b6)**	6-Z46a
7-27a	7-29a			**(4,#5,6)**	6-Z47a
7-34*	7-35*			**(4,5,6)**	6-33b
7-35*	7-35*			**(4,5,b6)**	6-32*
7-32a	7-32b			**(4,b5,6)**	6-Z29*
7-35*	7-34*			**(4,b5,b6)**	6-33a
7-29b	7-27b			**(4,#4,5)**	6-Z47b

(b2,2)	(~5,~6)	Fixed Min
7-5b	**(#5,6)**	5-Z36a
7-Z36b	**(5,6)**	5-25a
7-16b	**(#4,6)**	5-31a
7-11a	**(4,6)**	5-29a
7-14b	**(5,b6)**	5-27a
7-24b	**(#4,#5)**	5-34*
7-23b	**(4,b6)**	5-35*
7-Z18b	**(#4,5)**	5-32a
7-23a	**(4,5)**	5-35*
7-11b	**(4,b5)**	5-29b

Dim 7:

(b2)	(2)	(~5,~6)	Fixed Dim
7-19a	7-19b	**(5,b6)**	6-Z42*
7-32b	7-31b	**(4,b6)**	6-27b
7-32a	7-28b	**(b4,b6)**	6-Z28*
7-28a	7-25b	**(4,5)**	6-Z45*
7-31a	7-25a	**(b4,5)**	6-27a
7-16b	7-10b	**(b4,4)**	6-Z42*

Figure 2.5 3(a-e) – 7-note pockets

Nonetheless, if you can learn all the set classes of this cardinality well; it is mostly downhill afterwards. Outside of 7-30–7-35* and perhaps 7-22*, all of the 7-note set classes are likely unfamiliar. Furthermore, I would only put 7-1* – 7-5, into the "organized nicely" group. As such, two significant types of help will be less effective here. Also, as we have not covered the hexads directly yet, you may not see them as an aid. Therefore, due to these septads' difficulty and in the current absence of all of the helping information, I would recommend waiting until after having read the following section on hexads before even attempting to seriously dive into them.

For instance, one great preliminary approach would be to focus on groups that consist of the minimal supersets of familiar hexads, such as 6-Z47b, the blues scale. While this can be facilitated by the 7-note parsimonious charts above—e.g., regarding 6-Z47b, look at MinMaj 7 (4,#5,#6) and Min 7 (4,#4,5), it doesn't have to be; moreover you may also want to explore the minimal supersets of its inverse, 6-Z47a, and complement/Z-relation 6-Z25. It may be that in a short order, if you take 7-1* – 7-5, you will have encountered most of them. These set classes consist mostly of the most highly clustered hexads; 6-1* – 6-8* and 6-Z36 – 6-Z39*. Yet, as you may notice, 7-1* – 7-5 are, predominantly, the "straightforwardly organized" group. So, you have an in.

Another good start entails exploring the 7-note complements of the 5-note parsimony groupings given above. For instance, here is the first row (Figure 2.5.4), (b2), of the 5-note parsimony grouping.

Scale Degree	**(b2)**	**(2)**	**(#2)**	**(4)**	**(#4)**	**(5)**	**(#5/b6)**	**(6)**	**(#6)**	**Fixed MinMaj**
(b2)	5-2a	5-3a	**NA**	5-6a	5-14a	5-Z38a	5-Z37*	5-11a	5-4a	4-4a

Figure 2.5 4

As such, consider beginning your 7-note parsimonious explorations with 7-2a, 7-3a, 7-6a, 7-14a, 7-Z38a, 7-Z37a, 7-11a, and 7-4a. The relative simplicity of the 5-note, as opposed to the 7-note, set classes, may come across as a more manageable introductory approach; it may be less overwhelming. Remember though, this is not easy. You could reasonably expect the time needed for mastery of the seven note-set classes to be best measured in months, if not years.

However, even just learning these already mentioned charts and relationships is not enough to get a "full sense" of the 7-note set classes. In the following portion of this chapter, we will discuss this more fully. After that though, you will be exposed to the majority of techniques/perspectives that this pedagogy will ask you to engage in regard to internalizing/exploring these set classes. In short, the more "fully enriched" your understanding of the 7-note set classes is, the stronger your "core" grasp of this approach is. I would even go so far as to say that your depth of understanding of the 7-note classes as a whole, may be the best measure of how great your mastery of this pedagogy is.

A:

If there is ambiguity around whether it makes more sense to default to thinking of a hexad as a scale or a chord, with the 7-note set classes, there is less ambiguity in this regard. I submit that, generally speaking, it makes more sense to treat a 7-note set class component as a scale, a resource for smaller chords, rather than as a chord in of itself. Plus, learning the septads' complements, the pentads, is both: significantly more difficult than learning the tetrad complements of the octads; and, less helpful to use, due to the septads' own complexity, when seeking to characterize the septads.

However, it is true that as all but one 7-note scale, 7-Z12*, contains its complement; and as such, you may assert that, generally speaking, the 7-note set classes are only their 5-note complements plus two notes. Nonetheless, even with this information, there are still up to 21, $^{(7 \times 6)}/_{2}$, different set classes that could contain that pentad. As such, a complementary pentad subset is not informative enough to immediately distinguish the 7-note set class in question from other ones, and as a mere descriptor it is harder to grasp than the 4-note set classes.

Furthermore, as there are more rotations in the pentads vs. the tetrads, it is more likely that very different sounding rotations of that same pentad could be manifest within the same rotation of the 7-note complement. Theoretically, this could be greater evidence for connecting the complements; yet, in real time, from an aural perspective, it may be difficult to know what to do with that information. As cognitive units that we imprint on our understanding of a larger set class, the relative simplicity of the tetrads can go a long way. Even if they only distinguish each septad very partially, they are quick to use when seeking to make sense of a larger set class.

For practicing though, it is extremely helpful to explore exactly how a given seven-note scale embeds its complement (and other subsets)! Moreover, this portion of the chapter will do that, offering many resources that are of this nature. At this point though, I officially submit that the maximal subsets are the "best" way to characterize the larger set classes. Instead, use the pentads indirectly: first use the pentads to characterize the hexads and then the hexads to characterize the septads.

That being said, amongst other things, this chapter will include two groups of "charts:" one that shows the different ways one can select the pentad subsets and another that consists of examples of musical pieces that systematically describe each 7-note scale in terms of its tetrad subsets. Even if the pentads are awkward soundbites of their 7-note supersets, in the context of a more in-depth exploration into a 7-note set class of interest they can be wonderful guides. To do this, I will indirectly build on the earlier voicing discussions, by examining systematic ways to extract subsets from a superset; this is akin to classifying particular pentad subsets by the given superset's generic pentad subset/type.

I label the three categories of 5-note subset extraction from a 7-note scale as 12345, 12346, and 12356. In other words, any 5-note subsets of a 7-note scale can be rearranged into one of these forms. Although, to do so may require "diatonic transposition or inversion." E.g., 12345 = 23456 = 56712 = 21765 etc.. Below are the extractions for 7-31 - 7-35*. By applying extractions in the given forms to all of the modes of both components of a set class, we take account of all "diatonic transpositions and inversions" and a total account of all of the 5-note subsets of the particular 7-note set class is given. For visual clarity, I did not mark the A component of a set class. For example, 7-31's Mode 2's 12345 is {C, Db, Eb, Fb, Gb} = 5-10A; even though just 5-10 is written. Or, 7-32B's Mode 4's 13527 can be reduced to 71235, Mode 3's 12346, which is equal to {C, Db, Eb, Fb, Ab} = 5-Z17*.

<u>So, what is so significant about the following chart (Figure 2.5 5)</u>?

Well, there are two main things: First, it shows you the "pentad vector" of a given set class. This is an unbiased accounting of the distribution of the pentads in relation to a 7-note superset. Fundamentally, if we assign a set of pitches as a scale rather than a chord than we are treating it as resource for potential chords and melodic configurations, more so than as a discrete entity. In that sense, the above chart (and others that show the distribution of its subsets) are necessary for "understanding it." While this chart may seem less helpful for the familiar sets given[72]—you may already feel that you understand them well enough—for the unfamiliar 7-note supersets it can provide a lot of information quickly. Especially, as you accrue a refined sense of what these 5-note set class's components sound like, how they are manifested under various transformations (e.g., drop 3 voicing and inversion), and how they behave in various contexts (e.g., 4-27b, a dominant 7th chord, suggests one transposition of 7-35*—G^7 implies C Major, but two of 7-34*—F^7 implies C melodic minor and Bb melodic minor). In turn then, as a composer, it may give you insight into potential ambiguities (such as one chord pointing to two different transpositions of a superset) that can be exploited.

[72] Due to space limitations, I have only provided these few familiar set classes. The benefit of their familiarity is that it is easier to test your understanding of the concept in relation to them. While one could argue that, an additional 6 pages is not bad. The follow up would be, should I include another 7 pages in regards to the 4s and then do something similar for the 8-note (6 and 5-note subsets) and 9-note (7 and 6-note subsets) supersets? In the future, it would be better to create a website wherein these things are easily accessed.

7-31	**1**	**#2**	**3**	**#4**	**5**	**6**	**b7**	**12345**	**12346**	**12356**
mode 1	C	D#	E	F#	G	A	Bb	5-16B	5-31(4th)	5-32
mode 2	C	Db	Eb	Fb	Gb	Abb	Bbb	5-10	5-16	5-19
mode 3	C	D	Eb	F	Gb	Ab	B	5-10B	5-25	5-28
mode 4	C	Db	Eb	Fb	Gb	A	Bb	5-10	5-16B(1st)	5-31
mode 5	C	D	Eb	F	G#	A	B	5-25	5-25B	5-19B(2nd)
mode 6	C	Db	Eb	F#	G	A	Bb	5-19	5-31	5-28B
mode 7	C	D	E#	F#	G#	A	B	5-28B	5-32B(3rd)	5-32
7-31B	**1**	**2**	**b3**	**4**	**b5**	**b6**	**bb7**	**12345**	**12346**	**12356**
mode 1	C	D	Eb	F	Gb	Ab	Bbb	5-10B	5-25	5-28
mode 2	C	Db	Eb	Fb	Gb	Abb	Bb	5-10	5-16	5-19
mode 3	C	D	Eb	F	Gb	A	B	5-10B	5-25B	5-31B
mode 4	C	Db	Eb	Fb	G	A	Bb	5-16	5-16B(1st)	5-28B
mode 5	C	D	Eb	F#	G#	A	B	5-28	5-31B	5-19B(2nd)
mode 6	C	Db	E	F#	G	A	Bb	5-19B	5-32	5-32B
mode 7	C	D#	E#	F#	G#	A	B	5-25B	5-31B(4th)	5-32B
7-32	**1**	**2**	**b3**	**4**	**5**	**b6**	**7**	**12345**	**12346**	**12356**
mode 1	C	D	Eb	F	G	Ab	B	5-23	5-25	5-20B
mode 2	C	Db	Eb	F	Gb	A	Bb	5-Z12*	5-26B	5-31
mode 3	C	D	E	F	G#	A	B	5-26	5-27B	5-30
mode 4	C	D	Eb	F#	G	A	Bb	5-Z18B	5-31B	5-29B
mode 5	C	Db	E	F	G	Ab	Bb	5-Z18	5-21	5-22*
mode 6	C	D#	E	F#	G	A	B	5-16B	5-31(4th)	5-32
mode 7	C	Db	Eb	Fb	Gb	Ab	Bbb	5-10	5-Z17*	5-29
7-32B	**1**	**2**	**3**	**4**	**5**	**b6**	**7**	**12345**	**12346**	**12356**
mode 1	C	D	E	F	G	Ab	B	5-23B	5-26	5-30B
mode 2	C	D	Eb	F	Gb	A	Bb	5-10B	5-25B	5-31B
mode 3	C	Db	Eb	Fb	G	Ab	Bb	5-16	5-Z17*	5-20
mode 4	C	D	Eb	F#	G	A	B	5-Z18B	5-31B	5-29B
mode 5	C	Db	E	F	G	A	Bb	5-Z18	5-21B	5-32B
mode 6	C	D#	E	F#	G#	A	B	5-26B	5-31(4th)	5-22*(2nd)
mode 7	C	Db	Eb	F	Gb	Ab	Bbb	5-Z12*	5-27B	5-29
7-33*	**1**	**b2**	**b3**	**4**	**5**	**6**	**7**	**12345**	**12346**	**12356**
mode 1	C	Db	Eb	F	G	A	B	5-24	5-26B	5-28B
mode 2	C	D	E	F#	G#	A#	B	5-33*	5-33*	5-33*(2nd)
mode 3	C	D	E	F#	G#	A	Bb	5-33*	5-34*	5-30
mode 4	C	D	E	F#	G	Ab	Bb	5-24B	5-33*	5-30B
mode 5	C	D	E	F	Gb	Ab	Bb	5-9B	5-26	5-33*
mode 6	C	D	Eb	Fb	Gb	Ab	Bb	5-8*	5-13B	5-28
mode 7	C	Db	Ebb	Fb	Gb	Ab	Bb	5-9	5-13	5-15*
7-34*	**1**	**2**	**b3**	**4**	**5**	**6**	**7**	**12345**	**12346**	**12356**
mode 1	C	D	Eb	F	G	A	B	5-23	5-25B	5-29B
mode 2	C	Db	Eb	F	G	A	Bb	5-24	5-26B	5-28B
mode 3	C	D	E	F#	G#	A	B	5-33*	5-34*	5-30
mode 4	C	D	E	F#	G	A	Bb	5-24B	5-34*	5-35*
mode 5	C	D	E	F	G	Ab	Bb	5-23B	5-26	5-30B
mode 6	C	D	Eb	F	Gb	Ab	Bb	5-10B	5-25	5-28
mode 7	C	Db	Eb	Fb	Gb	Ab	Bb	5-10	5-Z17*	5-29
7-35*	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**12345**	**12346**	**12356**
mode 1	C	D	E	F	G	A	B	5-23B	5-27B	5-35*
mode 2	C	D	Eb	F	G	A	Bb	5-23	5-25B	5-29B
mode 3	C	Db	Eb	F	G	Ab	Bb	5-24	5-27	5-20
mode 4	C	D	E	F#	G	A	B	5-24B	5-34*	5-35*
mode 5	C	D	E	F	G	A	Bb	5-23B	5-27B	5-35*
mode 6	C	D	Eb	F	G	Ab	Bb	5-23	5-25	5-20B
mode 7	C	Db	Eb	F	Gb	Ab	Bb	5-Z12*	5-27	5-29

Figure 2.5 5

Of course, a disclaimer must be made, not all collections of pitches cohere as a scale equally well in our consciousness. Determining the criteria for perceiving a set of pitches (not even pitch-classes) as a scale is a main topic of Scale Theory. I submit that, as with everything else, what we can hear is an interaction between our training (signification), the adopted terms used (signifiers), and "what it is" (signified). Through your practice, you can assess for yourself how much increased training and terminology can mediate your perception of a scale. Once, you've explored 7-7B [4-27B(2nd)(1➔3)(4➔7)], {0,1,2,6,7,8,9} in terms of its subsets and have developed a good sense of how it could be projected in terms of a harmonic texture, do you then conceptualize it as just two disjunct clusters or something that has the potential to also project 4-7*(2x), 4-8*(3x), 4-12, 4-13, 4-14(2x), 4-Z15, 4-16(5x), 4-18, 4-23*, 4-25*, 4-27, and 4-29(2x)? Furthermore, if you were to engage multiple transpositions of 7-7b or spread it over multiple voice parts, how completely could you obscure its clusters? Would that impact whether or not you still perceive it as one scale or two unrelated clusters[73] when it comes up "in the wild"?

This leads to my second point, the realization that all pentads arising in the form of 12346 in regard to a 7-note superset can be expressed as stacked 3rds, and 12356, as stacked 4ths. One significant consequence of this pertains to voicings. When possible, there is a preference for stacking chords in 3rds or 4ths. These charts quickly show you which pentads, in regard to a particular 7-note superset subscribe to that description. Notice that we are discussing general 3rds and 4ths (dclens), not just those of respectively {3 or 4} or {5 or 6} half-steps; this is only the case for 7-35*—a result of diatonic set theory (Clough and Douthett 1991). As the Forte number lowers (or exceeds 35*), the spans of these dclens can vary significantly; so much so, that one also confronts questions regarding "what is the point of labeling something as stacked in

[73] Of course, such an example from the wild would have to be a *reasonable* representation of 7-7b! In other words, no minute long silences between clusters or extreme differences between the orchestration (or dynamics) assigned to each cluster. Similarly, the segmentation would have to be clear; preferably, in one register/octave. Speed, accentuation, and the degree which its treated as a motive (in either one or two parts) could also impact its recognizability as well as how it's set in relief to other harmonic entities; does it appear as a melodic embellishment in one voice of a harmony that is presented in another? etc.

As alluded to before though, the goal of this line of inquiry, in the context of this pedagogy, is not to get a general answer to this cognitive question, but just to provide another forum for you to test how your acquisition of these set classes as cognitive units mediates your relationship to music—as you imagine/audiate and/or perceive/identify it.

Reflecting on these and other processes over long stretches of time can give you great feedback on "your musical voice," helping you tease out the myriad stream of influences that factor into your compositional and improvisational "ear." That being said, if large groups of students were to study aspects of this pedagogy in a single institutional setting, one could imagine a pretest/posttest research design that could elicit evidence pro (or con) for a more general understanding of this cognitive question.

4ths, if few (or none) of those 4ths sound like the categories of 4th that we have been trained to hear? Well, at minimum, if you are working with a scale consisting of mostly clusters, at least this chart would point you to the "most spread out pentads possible" sans octave displacement; then subjecting these pentads to the most spread-out categories of voicing transformations discussed in chapter 4, would exaggerate the spread to the utmost.

Moving on, there are twenty-four 5-note voicing transformation categories. One would expect drop 432 to yield the biggest spreads.

- No Drop: Normal
- Drop: 2, 3, and 4
- Drop: 13, 14, 15; 21, 23, 24, 25; 32, 34, 35; 41, 42, 43; 51.
- Drop: 234, 243, 324, 342, 423, and 432

However, twenty-four voicing transformation categories is a lot![74] In this case, it could be helpful to extend the line of reasoning that led to the last section's expression of the common-to-

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[74] Plus … I'll stick my foot out and make a proclamation, even though it goes against the general ethos of this pedagogy, "seeking to internalize 120 hexachord voicing transformation categories [I won't even list them] is probably not only a waste of time but has no practical benefit." 6 categories are not only hard enough to learn, but upon learning them, one can reasonably expect to be able to engage them in compositional practice (without deferring to a pencil and/or computer) and while improvising; using your understanding of those 6 categorizations' impact in tandem with other musical considerations. In general, aim for "a good enough" impression of how transformations impact the harmonic fields that you care about; not an encyclopedic capacity to list but, in turn, not actually process all potential relationships—is there even one person in the world that has the working memory to tackle such a feat quickly. And … if it were even possible (again this is a big if) to relegate it to some automatic outgrowth of something learned procedurally (through the motor system), I again would caution against it — as it would still be a big waste of time. There are diminishing returns on such obsessive pursuits of nuance; isn't the proscribed bounds of this pedagogy already big enough.

More to the point though, this specific aside may only be needed for a non-existent reader of this text! Any real reader would probably ignore any such request this (or any other author) would make touting the benefits of internalizing 120 voicing categories. Moreso, I am using this over-the-top example as a way to again hammer in a core message of this pedagogy; "Ultimately, the only way to make this pedagogy manageable and worth it, is to connect it to your musical interests and then in a dialectical fashion seek to better understand those same (and expanded) musical interests while exploring various facets of the suggested curriculum."

Don't go overboard! My aim is to minimize the time it will take you to go as deep into it as you desire. However, it will only work if you are actively steering the ship, employing common sense, and aligning it with things that you care about! It's important to repeat this now, as for some readers, they may be starting to finally realize the extreme depth of this ocean surrounding them. Nonetheless, if your continued exploration is tethered to your interest, you will eventually be able to navigate those bottomless waters that are meaningful to you to an amazing level of depth.

Take this metaphor, imagine crossing a busy street with neither an intention to get to the other side or a want to parse out those objects in the landscape that are most meaningful to your health—fast moving cars. How long would you last? Even if you had the most amazing luck and could unconsciously avoid all cars, how long would it take to

Jazz 5-note set classes; conceive of a 5-note set class's component's transformation as a 4-note voicing transformation over some fixed bass. Similarly, you could conceive of a voicing transformation on a hexad (should you even want to think of it as a chord) as a 4-note voicing transformation wherein the bass and soprano are fixed; musically speaking, they are typically fixed beforehand—an example of weighing considerations foreign to just the "variations in the ordering of the inner voices." Various partimento treatises, including C.P.E. Bach's (C. P. E. Bach 1949) and even more recently, Gerre Hancock's (Hancock 1994), recommend anchoring your learning of proper voice leading in tonal figurations through fixed outer voices. Nonetheless, it is also interesting to observe that all 5-note subsets of a 7-note superset can be expressed as the uniform expansion of some generic interval in that superset. This knowledge informed the "other names" column of the 5-note set classes.

I am going to conclude this chapter with a demonstration of both approaches listed above for building your understanding of an unfamiliar scale on the distribution of its subsets.

I created 462 "pedagogical etudes." The goal of these etudes was to give a quick musical snapshot of each of the 7-note set classes. Simply by listening to them and engaging collections of them one gets a (relatively!) "unbiased" impression of the set classes examined, how certain common transformations, in the rhythmic and pitch domain—namely symmetry and augmentation—impact general chords writ large. Can we get a better sense of what a symmetrical transformation sounds like writ large? Furthermore, digging into these etudes, or creating ones on your own, is a forum for getting at questions such as how perceivable is 7-7B as a unit. Is it more perceivable if it expresses bigger uniform generic intervals?

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aimlessly wander over to the other side? Then if you did survive, who do you think would have a better impression of that road? Someone who crossed that street with purpose, a will to survive, and a want to go somewhere; or, alternatively, someone who stumbled across without any such intentions? In short, this pedagogy is too big to believe that immersion or rote work will be enough to get you there. You must be very intentional with every step and see it as just a backdrop for gaining greater insight into what's most important to you. In short, to "master this material" your primary aim can't be the mastery of this material. Instead, it has to be about better understanding your musical interests.

As it took a decent span of time to write all of these etudes, of which I'll only include one here,[75] it may not come as a surprise that not all were constructed via the same method. Namely, the first 60 or so were done creatively, and the last 400 or so were done through the use of an algorithm. The example that I am including is based on the algorithm. At first, I was interested in "making them more exciting/novel." However, I soon changed my mind. While a particular mode of a 7-note scale may be made more memorable through a creative rendition of it, that same creativity can obscure a potential function of this pedagogy, gaining a better understanding of how abstract concepts, like inversion, can impact our understanding of material; which could be better understood by applying these transformations in uniform ways across all of the set classes. Furthermore, there already is a substantial body of music literature out there that can be curated in such a way as to introduce and make salient, many of these set classes. Multiple authors, such as Hanson and Forte, have already done just that (Hanson 1960a; Forte 1973).[76]

However, for this "other potential function" of this pedagogy, nothing even remotely like this (as far as I know!) exists. Individually, these "etudes" offer insight into a specific mode of a set class; as a set of "etudes" associated with a particular set class, they offer insight into the overall distribution of 4-note chords within that set class; as an entire collection, they offer insight into how audible certain common transformations actually are—can we even learn to recognize these transformations outside of the accumulation of specific examples, like the reversed last movement of Berg's lyric suite? How does tempo effect perception? With software, you can play and or listen to them in any key and tempo. Hence, I decided upon sticking to an algorithm.

While this algorithm did restrict my musical choices, it did not eradicate them. Over time, I started to develop, in the context of this algorithm, a sense of what starting rhythms (measured in how evenly they are distributed) best aligned with various set class modes (which can also be measured in how evenly they are distributed). In many cases, the results were decent; of course, a good performance can elevate even the "less successful" miniatures. However, in other cases

[75] They need their own dedicated space; a space wherein musical considerations, such as tempo, can be discussed. Curating them and then maximizing their utility in regards to this pedagogy, could even be another dissertation in of itself!

[76] Although, their goal typically was just to demonstrate an instance of a set class — not as in the case of this pedagogy, to meditate for substantial periods of time on individual set classes — such as introducing them in all of their modes.

the results were magical; and represented a sonic landscape that I could never have stumbled upon via my previous creative efforts. For the purpose of the “etude” however, it’s okay to have “less successful” miniatures; independent of “aesthetic quality,” they are all short enough to weather and their value should be judged, at least partially, on how they add to your general impression of the set as a whole. Along these lines, variety was prized; as such, I utilized many different time signatures and no two starting rhythms were the same.

To the nuts and bolts of the algorithm (in regard to set class 7-1* mode 1):

1. I started with the distribution of 4-note set classes. Here (Figure 2.6.6) they are presented as disjunct tetrads.

<table>
<tr><td>7-note set class</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td></tr>
<tr><td>7-1* mode 1</td><td>C</td><td>Db</td><td>Ebb</td><td>Fbb</td><td>Gbbb</td><td>Abbbb</td><td>Bbbbbb</td></tr>
<tr><td colspan="2">1234/5678</td><td colspan="2">1235/4678</td><td colspan="2">1357/2468</td><td colspan="2">1473/6251 (drop 2)</td></tr>
<tr><td colspan="2">(1,rt)4-1*, (8,1st)4-5B</td><td colspan="2">(1,rt)4-2, (8,1st)4-13B</td><td colspan="2">(1,rt)4-21*, (8,1st)4-11</td><td colspan="2">(1,2nd) 4-12, (8,1st) 4-7*</td></tr>
</table>

Figure 2.5 6

A couple of notes on the new terminology: I paired “inversely-related” extraction forms with each other. Hence, in **1234/5678**, **1234** is the inverse of **8765**. Secondly, I indicated both the inversion and the scale degree associated with the “root” of the rotated (or not) tetrad.

- For instance, in “C,” 7-1* mode 1 is comprised of two tetrads. C-C#-D-Eb and E-F-F#-C. The first tetrad, <0,1,2,3>, is in prime form (although it’s ordered) and its “root” is on the first scale degree—as such, the designation (1,rt) is used. On the other hand, E-F-F#-C, <0,4,5,6>, is also in normal order; however, its “root,” conforming to the labeling provided in for the extraction category, is the 8th degree of the scale.

2. I assigned a different voice part with each extraction category: the soprano played category **1234/5678**, “the scale;” the alto, category **1235/4678**, “nearly a scale;” the tenor, category **1357/2468**, “stacked 3rds;” and the bass, category **1473/6251**, “stacked fourths.”

Collectively, these extractions account for all of the 4-note subsets' possible extraction categories.[77]

3. I chose a time signature and a rhythm to repeat. In this case, I chose $\frac{7}{4}$ as the time signature and sets of rhythmic durations (each is measured in 8th notes): set A is [7,9,6,6] and set B is [10,7,6,5]. Here are the four ascending and descending rhythmic phrases of:
    a. the soprano, ABBA and $[ABBA]^{-1}$;
    b. the alto, $[ABBA]^{-1}$and ABBA;
    c. the tenor, BAAB and $[BAAB]^{-1}$;
    d. the bass, $[BAAB]^{-1}$ and BAAB.

Note that other miniatures also deployed sets of rhythmic durations in symmetrical ways. However, in other miniatures it was often done differently from this example.

4. I played through this miniature at the piano (Figure 2.6.7), present below, and discovered that it works quite well at a very slow tempo; such as quarter note @ 40 bpm.

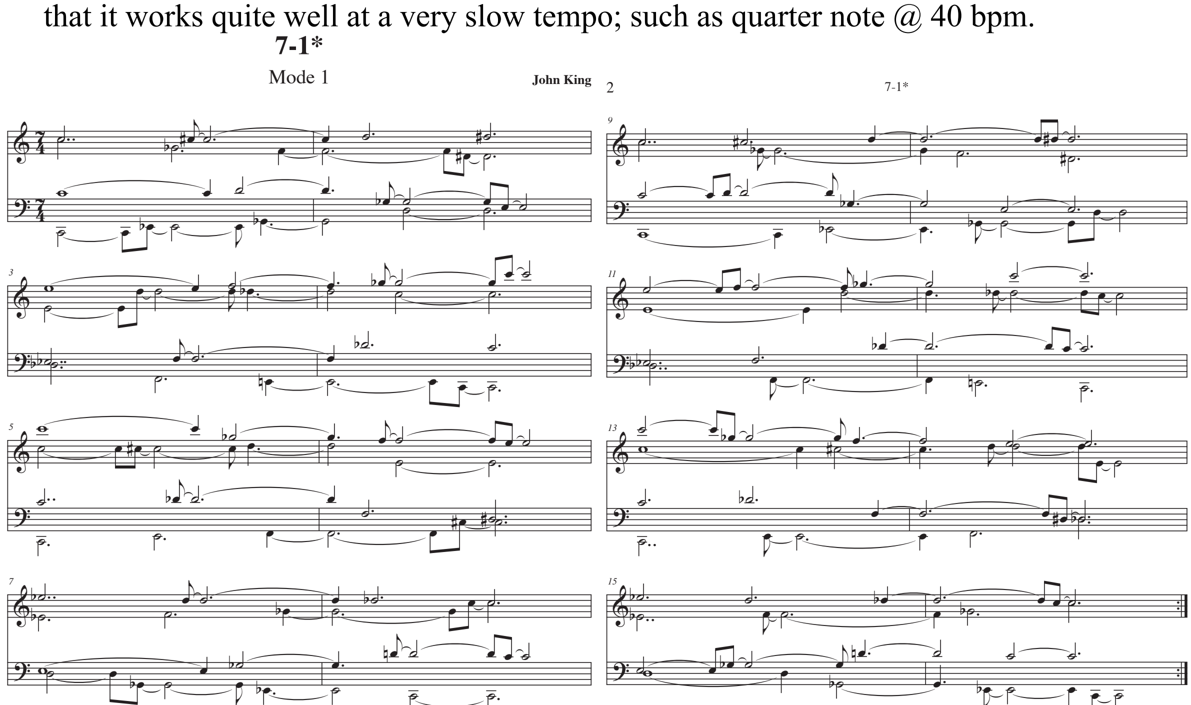


Figure 2.5 7

[77] Due to the systematic exploration of the entirety of this set class's associated modes **and** across extraction category, uniform break down of the scales into disjunct tetrads, the overall distribution of the tetrads is preserved, even if the order of their presentation changes or each is presented an additional time. A distribution is invariant to scale and ordering.

Nonetheless, no matter how effective these various miniatures may be in isolation, I would not recommend listening to four hundred plus back-to-back; no matter how well they are performed. They each create an ambience that is worth settling into. Hearing too many in succession may even have the opposite of the intended affect—instead of savoring the small differences, the strangeness of these stable but sometimes very foreign sound worlds, your brain instead may just "turn off," only noticing the great similarity in the many etudes' construction.

One is enough to digest in any sitting. There really is a lot to explore in each sound bite and all of those layers of symmetry seem to have a calming effect. However, as sonic "moments" that anchor your forays into one set class at a time, they are fantastic. Over time, if you get quite familiar with many, or even some) of them (of my design or yours!), the benefits of their comparison will start to accrue. Again, just please, don't aim to do it all in one listen. Anyone aiming to perform a set of them, could consider using each mode as a "palate cleanser" between other pieces in the set. If not, they could consider using arrangements other than piano; such as a string, vocal, brass, or woodwind quartet. Finally, should one choose to play them back on the piano in a single uninterrupted setting: one should weigh in musical considerations VERY carefully; consider playing them out of order, vary the length of pauses between each, and play up all contrasts (such as in tempos, tone, and even key). Remember, these were meant to be "experiential/pedagogical etudes" not performance pieces. Therefore, should you perform them (or your own, if they are written similarly), you must make up the difference towards selling them as a set. Please, if you find it helpful, add dynamics and even tempo fluctuations, like ritardandos.

Alternatively, if you are interested in exploring the coherency of some scale at different tempo markings, please do so. These "experiential" etudes are not meant to be finished products; rather, they are open ended. As you continue to engage them and every other aspect of this pedagogy, "finish them" in line with whatever your personal objectives are.

5 note chord	4 note subsets	Other Chords	Maj 7	MinMaj7	Dom 7	Min 7
5-1*	1*, 2, 3*	2-4 (1→5)		(**1**:b2,2)		
5-2	1*, 2, 4, 10*, 11	3-7B (1→4)	(**2**:b2,2)	(**3**:b2,#6)		
5-2B		3-7 (2→5)				(**2**:b2,2)
5-3	2, 3*, 4, 7*, 11	4-11B (1→3)	(**2**:b2,#2)	(**3**:2,#6)		
5-3B		4-11 (3→5)	(**2**:2,#2)			
5-4	1*, 4, 5, 12, 13	3-10* (1→4)	(**3**:b2,#6)	(**4**:6,#6)		
5-4B		3-10* (2→5)				
5-5	1*, 5, 6*, 14, Z29	3-11 (1→4)	(**4**:6,#6)			
5-5B		3-11B (2→5)				
5-6	4, 5, 7*, 8*, Z15	4-Z15B (1→3)	(**2**:b2,4),(**3**:#2,#6)			
5-6B		4-Z15 (3→5)	(**2**:#2,4)			
5-7	5, 6*, 8*, 9*, 16	4-16B(1→3)	(**3**:4,#6)			
5-7B		4-16 (3→5)	(**2**:4,b5)			
5-8*	2, 12, 21*	4-21*(2→4)		(**3**:b2,6)	(**2**:b2,2)	
5-9	2, 5, 11, Z15, 21*	4-21*(1→3)	(**3**:2,#6)	(**2**:b2,4)		
5-9B		4-21*(3→5)			(**2**:2,#2)	
5-10	3*, 10, 12, 13, Z15	Mm(2)(7→4)		(**2**:2,4)		(**3**:b2,6)
5-10B		Mm(2)(6→3)		(**3**:2,6)	(**2**:b2,#2)	
5-11	2, 4, 14, 17*, 22	4-22 (2→4)	(**3**:b2,6)	(**4**:#5,#6)		
5-11B		4-22B (2→4)		(**3**:b2,#5)		
5-Z12*	8*, 11, 13	Maj(2)(7→4)	(**2**:2,4)			(**3**:2,6)
5-13	2, 5, 19, 24*, Z29	4-24* (1→3)		(**2**:b2,5)	(**4**:#5,6)	
5-13B		4-24* (2→4)	(**4**:#5,#6)			
5-14	4, 6*, Z15, 16, 23*	4-23* (1→3)	(**2**:b2,#4)	(**3**:4,#6)		
5-14B		4-23* (3→5)			(**2**:#2,4)	
5-15*	5, 16, 25*	4-25* (1→3)	(**3**:#4,#6)		(**5**:4,b5)	
5-16	3*, 12, 17*, 18, Z29	HM(2)(3→7)		(**2**:2,#4),(**4**:#5,6)	(**3**:b2,6)	
5-16B		Hm(2)(6→3)		(**3**:2,#5)		
5-Z17*	3*, 14, 19	Mm(3)(1→2)	(**4**:#5,6)	(**2**:2,5)		
5-Z18	7*, 12, 14, 16, 18	Hm(2)(5→2)	(**2**:#2,#4)			
5-Z18B		Hm(2)(4→1)	(**3**:#2,6)		(**2**:b2,4)	
5-19	9*, 13, Z15, 18, Z29	7-31(2)(5→2)			(**3**:#2,6)	
5-19B		7-31B(2)(6→3)		(**2**:4,b5)		

5-1*[0,1,2,3,4]	**5-2a**[0,1,2,3,5]	**5-2b**[0,2,3,4,5]	**5-3a**[0,1,2,4,5]	**5-3b**[0,1,3,4,5]	**5-4a**[0,1,2,3,6]
5-4b[0,3,4,5,6]	**5-5a**[0,1,2,3,7]	**5-5b**[0,4,5,6,7]	**5-6a**[0,1,2,5,6]	**5-6b**[0,1,4,5,6]	**5-7a**[0,1,2,6,7]
5-7b[0,1,5,6,7]	**5-8***[0,2,3,4,6]	**5-9a**[0,1,2,4,6]	**5-9b**[0,2,4,5,6]	**5-10a**[0,1,3,4,6]	**5-10b**[0,2,3,5,6]
5-11a[0,2,3,4,7]	**5-11b**[0,3,4,5,7]	**5-Z12***[0,1,3,5,6]	**5-13a**[0,1,2,4,8]	**5-13b**[0,2,3,4,8]	**5-14a**[0,1,2,5,7]
5-14b[0,2,5,6,7]	**5-15***[0,1,2,6,8]	**5-16a**[0,1,3,4,7]	**5-16b**[0,3,4,6,7]	**5-Z17***[0,1,3,4,8]	**5-Z18a**[0,1,4,5,7]
5-Z18b*[0,2,3,6,7]	**5-19a**[0,1,3,6,7]	**5-19b**[0,1,4,6,7]			

Figure 5 A

5 note chord	4 note subsets	Other Chords	Maj7	MinMaj7	Dom7	Min7	Dim 7
5-20	8*, 14, 16, 20*, Z29	Maj(4)(5→3)	(**4**:4,5)		(**2**:4,6)		
5-20B		Maj(4)(1→6)	(**2**:#4,5), (**4**:4,6)				
5-21	7*, 17*, 19, 19, 20*	Hm(3)(6→7)	(**2**:#2,5)	(**4**:5,b6)			
5-21B		HM(3)(6→7)	(**2**:#2,#5),(**4**:5,b6)				
5-22*	8*, 18, 19	Hm(4)(7→5)	(**4**:4,#5)	(**2**:#4,5)			
5-23	10*, 11, 14, 22, 23*	Maj(2)(6→3)	(**3**:2,6)			(**2**:b2,4)	
5-23B		Maj(2)(1→5)				(**2**:2,4),(**3**:b2,b6)	
5-24	11, 16, 21*, 22, Z29	Maj(2)(3→7)	(**2**:2,#4)		(**3**:2,6)		
5-24B		Maj(2)(4→1)			(**2**:2,4)	(**2**:2,b6)	
5-25	10*, 13, 26*, 27, Z29	Maj(3)(7→1)				(**2**:b2,b5),(**4**:5,6)	
5-25B		Maj(3)(3→4)			(**4**:5,6)	(**2**:b2,5)	
5-26	11, 12, 19, 24*, 27	Mm(3)(6→7)		(**4**:5,6)	(**3**:b2,#5)	(**2**:2,b5)	
5-26B		Mm(3)(3→4)	(**2**:2,#5)		(**4**:5,b6)		
5-27	11, 14, 20*, 22, 26*	Maj(3)(1→2)	(**2**:2,5)			(**4**:5,b6)	
5-27B		Maj(3)(6→7)	(**4**:5,6)			(**2**:2,5)	
5-28	12, Z15, 25*, 27, Z29	Mm(4)(1→6)		(**3**:4,6)	(**2**:b2,#4),(**5**:#4,5)		
5-28B		Mm(4)(4→2)			(**3**:#4,6), (**5**:#2,#4)		
5-29	13, 14, 16, 23*, 27	Maj(4)(2→7)			(**5**:4,5)	(**3**:4,6)	
5-29B		Maj(4)(4→2)	(**3**:#4,6)			(**5**:4,b5)	
5-30	Z15, 16, 19, 22, 24*	Mm(4)(5→3)$^{+}$		(**2**:4,5)	(**5**:4,#5)		
5-30B		Mm(4)(7→5)	(**5**:#4,#5)		(**3**:#2,#5)		
5-31	12, 13, 18, 27, 28*	7-31(3)(1→2)				(**2**:b5,6)	(**1**:b2), (**4**:5), (**5**:b4)
5-31B		7-31B(3)(1→2)		(**3**:#4,6)	(**2**:b2,5)		(**1**:2), (**4**:b6), (**5**:4)
5-32	Z15, 17*, 18, 26*, 27	Hm(4)(1→6)		(**3**:4,#5)		(**5**:#4,5)	
5-32B		HM(4)(7→5)		(**2**:#4,#5)	(**5**:#2,5)		
5-33*	21*, 24*, 25*	WT(2)(1→5)			(**2**:2,#4),(**3**:2,b6),(**5**:#4,#5)		
5-34*	21*, 22, 27	Maj(3)(5→6)			(**2**:2,5)	(**4**:b5,b6)	
5-35*	22*, 23*, 26*	Major Pentatonic				(**3**:4,b6), (**5**:4,5)	
5-Z36	2, 6*, 13, 18, 22	4-22 (1→3)		(**2**:b2,#4)		(**4**:b6,6)	
5-Z36B		4-22B (3→5)					
5-Z37*	4, 19, 26*	4-26*(2→4)	(**3**:b2,#5)	(**4**:5,#6)			
5-Z38	4, 5, 18, 20*, 27	4-27(1→3)	(**2**:b2,5)	(**3**:#4,#6)			
5-Z38B		4-27B(3→5)	(**5**:5,#6)				

5-20a[0,1,3,7,8]	**5-20b**[0,1,5,7,8]	**5-21a**[0,1,4,5,8]			
5-21b[0,3,4,7,8]	**5-22***[0,1,4,7,8]	**5-23a**[0,2,3,5,7]	**5-23b**[0,2,4,5,7]	**5-24a**[0,1,3,5,7]	**5-24b**[0,2,4,6,7]
5-25a[0,2,3,5,8]	**5-25b**[0,3,5,6,8]	**5-26a**[0,2,4,5,8]	**5-26b***[0,3,4,6,8]	**5-27a**[0,1,3,5,8]	**5-27b**[0,3,5,7,8]
5-28a[0,2,3,6,8]	**5-28b**[0,2,5,6,8]	**5-29a**[0,1,3,6,8]	**5-29b**[0,2,5,7,8]	**5-30a**[0,1,4,6,8]	**5-30b**[0,2,4,7,8]
5-31a[0,1,3,6,9]	**5-31b**[0,2,3,6,9]	**5-32a**[0,1,4,6,9]	**5-32b**[0,1,4,7,9]	**5-33***[0,2,4,6,8]	**5-34***[0,2,4,6,9]
5-35*[0,2,4,7,9]	**5-Z36a**[0,1,2,4,7]	**5-Z36b**[0,3,5,6,7]	**5-Z37***[0,3,4,5,8]	**5-Z38a**[0,1,2,5,8]	**5-Z38b***[0,3,6,7,8]

Figure 5 B

7 note chord	6 note subsets	Other Name	Maj 7	Min 7
7-1*	1*,1*,2,3,Z4*	2-6*(1→7)	(**2**:b2,2,#2,4),(**3**:b2,2,#2,#6)	
7-2	1*, 2, 8* 9, 10*, 11, Z36	3-9*(2nd)(1→6)	(**2**:b2,2,#2,#4),(**4**:b2,2,6,#6)	
7-2B		3-9*(2→7)	(**3**:b2,2,#2,6)	
7-3	1*,14,15,Z36,Z37,Z39,Z40	3-11(2nd)(1→6)	(**2**:b2,2,#2,5),(**5**:b2,#5,6,#6)	
7-3B		3-11B(1st)(2→7)	(**3**:b2,2,#2,#5),(**7**:5,#5,6,#6)	
7-4	2,Z3,5,Z10,Z12,Z13,Z36	4-Z29B(1→5)	(**3**:b2,2,4,#6), (**4**:b2,#2,6,#6)	
7-4B		4-Z29(3→7)	(**2**:2,#2,4,b5)	
7-5	Z3,Z4*,5,Z6*,9,Z11,Z12	4-22B(1→4,5→7)	(**2**:b2,2,4,b5),(**3**:b2,#2,4,#6),(**4**:2,#2,6,#6)	
7-5B		4-22(1→3,4→7)	(**2**:b2,#2,4,b5),(**3**:2,#2,4,#6)	(**4**:b2,2,b6,6)
7-6	5,16,Z17,Z19,Z36,Z37*,Z38*	4-19B(3rd)(1→5)	(**4**:b2,4,6,#6), (**5**:#2,#5,6,#6), (**7**:4,#4,5,b6)	
7-6B		4-19(3→7)	(**2**:#2,4,#4,5),(**7**:4,#5,6,#6)	
7-7	5, Z6*, 7*,18,Z38*,41,Z43	4-27B(1→4)(5→7)	(**3**:b2,4,b5,#6),(**4**:#2,4,6,#6),(**7**:4,#4,5,#6)	
7-7B		4-27(1→3)(4→7)	(**2**:b2,4,#4,5),(**3**:#2,4,b5,#6),(**7**:4,b5,6,#6)	
7-8*	2,21*,21*, Z23*,Z39	4-25*(rt,2nd)(2→5)	(**4**:b2,2,#5,#6)	
7-9	2, 9, 21,22,Z24,Z37*,41	4-24*(3rd)(1→5)	(**3**:b2,2,#4,#6),(**5**:2,#5,6,#6)	
7-9B		4-24*(3→7)	(**7**:#4,#5,6,#6)	
7-10	2,27,Z36,Z40,Z42*,Z45*,Z46	4-27(3rd)(1→5)	(**3**:b2,2,5,#6)	(**6**:#4,5,#5,6)
7-10B		4-27B(2nd)(2→6)	(**3**:b2,5,6,#6)	
7-11	Z3,8*,14, 16, Z24,Z25,Z39	5-29 (3→6)	(**2**:2,#2,4,5),(**4**:b2,2,#5,6)	(**3**:b2,2,4,6)
7-11B		5-29B(2→5)	(**3**:b2,2,4,6), (**4**:b2,#2,#5,#6),(**7**:#4,5,b6,6)	(**2**:b2,2,4,b5)
7-Z12*	Z36,41,Z47,Z49*	4-26*(3rd)(1→5)	(**4**:b2,#4,6,#6)	
7-13	Z4*,Z10,15,16,21,22,2,Z43	5-33* (1→3,4→6)	(**2**:b2,#2,4,5),(**3**:2,#2,#4,#6)	
7-13B		5-33* (2→4,5→7)	(**3**:b2,#2,4,6),(**4**:2,#2,#5,#6),(**7**:#4,5,b6,#6)	
7-14	9, Z12,18,Z25,Z26*,Z38*,Z40	5-27B(1→4)	(**2**:b2,2,#4,5),(**4**:2,4,6,#6),(**7**:4,#4,5,6)	(**5**:2,5,b6,6)
7-14B		5-27 (4→7)	(**2**:2,4,#4,5),(**7**:4,5,6,#6)	(**4**:b2,2,5,b6)
7-15*	7*,Z12,Z17,22	5-33* (1→3)(5→7)	(**3**:2,4,b5,#6),(**7**:4,#4,#5,#6)	(**2**:b2,4,#4,5)
7-16	Z3*, 15, 27, Z28*, Z39,Z42*,Z44	5-31B(4th)(1→4)	(**2**:b2,2,4,#5),(**3**:b2,#2,5,#6)	
7-16B		5-31(4→7)	(**3**:2,#2,4,#5),(**6**:b2,5,b6,#6)	(**4**:b2,2,b5,6)
7-Z17*	Z4*, 14, Z44,Z46	5-34* (1→3,4→6)	(**2**:b2,#2,4,#5),(**3**:2,#2,5,#6),(**5**:b2,5,b6,6)	
7-Z18	5,17,31,Z39,Z40,Z44,Z50*	5-32B(2nd)(3→6)	(**2**:#2,4,#4,#5), (**4**:b2,2,5,b6)	
7-Z18B		5-32(2→5)	(**3**:b2,4,#5,#6),(**4**:#2,5,6,#6)	(**2**:b2,2,#4,5)
7-19	5,18,Z29*,30,41,Z42*,Z43	5-31(3rd)(1→4)	(**3**:b2,4,5,#6),(**4**:#2,#4,6,#6)	
7-19B		5-31B(4→7)	(**6**:b2,#4,5,#6),(**3**:#2,4,b5,6)	

7-1*[0,1,2,3,4,5,6]	**7-2a**[0,1,2,3,4,5,7]	**7-2b**[0,2,3,4,5,6,7]	**7-3a**[0,1,2,3,4,5,8]	**7-3b**[0,3,4,5,6,7,8]	**7-4a**[0,1,2,3,4,6,7]
7-4b[0,1,3,4,5,6,7]	**7-5a**[0,1,2,3,5,6,7]	**7-5b**[0,1,2,4,5,6,7]	**7-6a**[0,1,2,3,4,7,8]	**7-6b**[0,1,4,5,6,7,8]	**7-7a**[0,1,2,3,6,7,8]
7-7b[0,1,2,5,6,7,8]	**7-8***[0,2,3,4,5,6,8]	**7-9a**[0,1,2,3,4,6,8]	**7-9b**[0,2,4,5,6,7,8]	**7-10a**[0,1,2,3,4,6,9]	**7-10b**[0,2,3,4,5,6,9]
7-11a[0,1,3,4,5,6,8]	**7-11b**[0,2,3,4,5,7,8]	**7-Z12***[0,1,2,3,4,7,9]	**7-13a**[0,1,2,4,5,6,8]	**7-13b**[0,2,3,4,6,7,8]	**7-14a**[0,1,2,3,5,7,8]
7-14b[0,1,3,5,6,7,8]	**7-15***[0,1,2,4,6,7,8]	**7-16a**[0,1,2,3,5,6,9]	**7-16b**[0,1,3,4,5,6,9]	**7-Z17***[0,1,2,4,5,6,9]	**7-Z18a**[0,1,2,3,5,8,9]
7-Z18b[0,1,4,6,7,8,9]	**7-19a**[0,1,2,3,6,7,9]	**7-19b**[0,1,2,3,6,8,9] **			

Figure 7 A

7 note chord	6 note subsets	Other Name	MinMaj7	Dom 7	Dim 7
7-1*	1*,1*,2,3,Z4*	2-6*(1→7)	(**4**:b2,2,6,#6)		
7-2	1*, 2, 8* 9, 10*, 11, Z36	3-9*(2nd)(1→6)	(**3**:b2,2,4,#6),(**5**:b2,#5,6,#6)		
7-2B		3-9*(2→7)	(**4**:b2,2,#5,#6)	(**2**:b2,2,#2,4)	
7-3	1*,14,15,Z36,Z37,Z39,Z40	3-11(2nd)(1→6)	(**3**:b2,2,#4,#6),(**6**:5,#5,6,#6)		
7-3B		3-11B(1st)(2→7)	(**4**:b2,2,5,#6)		
7-4	2,Z3,5,Z10,Z12,Z13,Z36	4-Z29B(1→5)	(**2**:b2,2,4,b5),(**5**:2,#5,6,#6)		
7-4B		4-Z29(3→7)	(**4**:b2,2,#5,6)	(**3**:b2,2,#2,6)	
7-5	Z3,Z4*,5,Z6*,9,Z11,Z12	4-22B(1→4,5→7)			
7-5B		4-22(1→3,4→7)			
7-6	5,16,Z17,Z19,Z36,Z37*,Z38*	4-19B(3rd)(1→5)	(**2**:b2,2,#4,5)		
7-6B		4-19(3→7)	(**4**:b2,2,5,b6)		
7-7	5, Z6*, 7*,18,Z38*,41,Z43	4-27B(1→4)(5→7)			
7-7B		4-27(1→3)(4→7)			
7-8*	2,21*,21*, Z23*,Z39	4-25*(rt,2nd)(2→5)	(**3**:b2,2,4,6),(**4**:b2,5,6,#6),	(**2**:b2,2,#2,#4),(**7**:#4,5,b6,6)	
7-9	2, 9, 21,22,Z24,Z37*,41	4-24*(3rd)(1→5)	(**2**:b2,2,4,5),(**4**:b2,4,6,#6)	(**7**:4,#4,5,b6)	
7-9B		4-24*(3→7)	(**4**:b2,2,5,6)	(**2**:2,#2,4,b5),(**3**:b2,2,#2,b6)	
7-10	2,27,Z36,Z40,Z42*,Z45*,Z46	4-27(3rd)(1→5)	(**2**:b2,2,4,#5),(**4**:b2,#4,6,#6)		(**7**:b4,4,5)
7-10B		4-27B(2nd)(2→6)	(**3**:b2,2,#4,6),(**6**:#4,#5,6,#6)	(**2**:b2,2,#2,5)	(**7**:4,5,b6)
7-11	Z3,8*,14, 16, Z24,Z25,Z39	5-29 (3→6)	(**5**:b2,5,b6,#6)	(**7**:4,5,b6,6)	
7-11B		5-29B(2→5)	(**5**:2,5,6,#6)		
7-Z12*	Z36,41,Z47,Z49*	4-26*(3rd)(1→5)	(**2**:b2,2,#4,#5),(**5**:4,#5,6,#6)	(**7**:#2,4,#4,5)	
7-13	Z4*,Z10,15,16,21,22,2,Z43	5-33* (1→3,4→6)	(**5**:b2,5,#5,6)	(**4**:b2,2,b6,6),(**7**:4,b5,b6,6)	
7-13B		5-33* (2→4,5→7)		(**2**:b2,2,4,b5)	
7-14	9,Z12,18,Z25,Z26*,Z38*,Z40	5-27B(1→4)	(**3**:b2,4,b5,#6)		
7-14B		5-27 (4→7)		(**3**:2,#2,4,6)	
7-15*	7*,Z12,Z17,22	5-33* (1→3)(5→7)		(**4**:2,#2,b6,6)	
7-16	Z3*, 15, 27, Z28*, Z39,Z42*,Z44	5-31B(4th)(1→4)	(**4**:2,#4,6,#6),(**6**:#4,5,b6,6)	(**5**:b2,5,b6,6)	(**7**:b4,4,b6)
7-16B		5-31(4→7)	(**7**:#4,5,6,#6)		(**7**:b4,5,b6)
7-Z17*	Z4*, 14, Z44,Z46	5-34* (1→3,4→6)	(**6**:#4,5,b6,#6)		
7-Z18	5,17,31,Z39,Z40,Z44,Z50*	5-32B(2nd)(3→6)	(**5**:b2,#4,5,#6)	(**7**:#2,5,b6,6)	
7-Z18B		5-32(4→7)	(**7**:4,#4,5,b6)		
7-19	5,18,Z29*,30,41,Z42*,Z43	5-31(3rd)(1→4)		(**7**:#2,4,b5,6)	(**5**:b2,5,b6)
7-19B		5-31B(4→7)	(**7**:4,b5,6,#6)	(**2**:b2,4,#4,5)	(**1**:2,5,b6)

7-1*[0,1,2,3,4,5,6]	**7-2a**[0,1,2,3,4,5,7]	**7-2b**[0,2,3,4,5,6,7]	**7-3a**[0,1,2,3,4,5,8]	**7-3b**[0,3,4,5,6,7,8]	**7-4a**[0,1,2,3,4,6,7]
7-4b[0,1,3,4,5,6,7]	**7-5a**[0,1,2,3,5,6,7]	**7-5b**[0,1,2,4,5,6,7]	**7-6a**[0,1,2,3,4,7,8]	**7-6b**[0,1,4,5,6,7,8]	**7-7a**[0,1,2,3,6,7,8]
7-7b[0,1,2,5,6,7,8]	**7-8***[0,2,3,4,5,6,8]	**7-9a**[0,1,2,3,4,6,8]	**7-9b**[0,2,4,5,6,7,8]	**7-10a**[0,1,2,3,4,6,9]	**7-10b**[0,2,3,4,5,6,9]
7-11a[0,1,3,4,5,6,8]	**7-11b**[0,2,3,4,5,7,8]	**7-Z12***[0,1,2,3,4,7,9]	**7-13a**[0,1,2,4,5,6,8]	**7-13b**[0,2,3,4,6,7,8]	**7-14a**[0,1,2,3,5,7,8]
7-14b[0,1,3,5,6,7,8]	**7-15***[0,1,2,4,6,7,8]	**7-16a**[0,1,2,3,5,6,9]	**7-16b**[0,1,3,4,5,6,9]	**7-Z17***[0,1,2,4,5,6,9]	**7-Z18a**[0,1,2,3,5,8,9]
7-Z18b[0,1,4,6,7,8,9]	**7-19a**[0,1,2,3,6,7,9]	**7-19b**[0,1,2,3,6,8,9] **			

Figure 7 B

7 note chord	6 note subsets	Other Name	Maj7	Min7
7-20	Z6*,16,Z17,18,Z44,Z47,Z48	5-35*(1→3,5→7)	(**6**:b2,4,b5,b6),(**7**:#2,4,5,#6)	
7-20B		5-35*(3rd)(1→3,5→7)	(**6**:b2,4,b5,6),(**7**:#2,4,#5,#6),(**2**:b2,#4,5,b6)	
7-21	14,15,16,Z19,20*,31,Z44	6-31B(1→3)	(**2**:b2,#2,5,b6),(**5**:#2,5,#5,6),(**7**:#2,4,5,b6)	
7-21B		6-31(5→7)	(**2**:#2,#4,5,b6),(**4**:2,#2,5,b6),(**6**:b2,4,#5,6),(**7**:#2,5,b6,#6)	
7-22*	Z19,Z43,Z44,Z49*	6-Z49*(5th)(1→3)	(**2**:b2,4,5,b6),(**3**:#2,#4,5,#6),(**7**:#2,4,#5,6)	
7-23	8*,9,32*,33,Z40,Z46,Z47	5-35*(3rd)(2→5)	(**3**:b2,2,#4,6),(**5**:2,5,6,#6)	(**2**:b2,2,4,5),(**7**:4,#4,5,b6)
7-23B		5-35*(3→6)	(**4**:b2,2,5,6)	(**3**:b2,2,4,b6),(**7**:4,5,b6,6)
7-24	9,22,33,34,Z39,41,Z48*	5-34*(4th)(1→4)	(**2**:b2,2,#4,#5),(**4**:2,#4,6,#6)	
7-24B		5-34*(4→7)	(**6**:b2,#4,#5,#6)	(**4**:b2,2,b5,b6)
7-25	Z10*,Z11,27,Z29*,33,Z45*,Z47	6-33B(2→4)	(**3**:b2,#2,#4,6)	(**5**:b2,b5,b6,6),(**7**:4,#4,5,6)
7-25B		6-33(4→6)	(**3**:2,#2,#4,6)	(**7**:4,b5,b6,6)
7-26	Z10,Z15,21,31,Z34,Z46,Z49*	6-34(3→5)	(**2**:2,#2,#4,#5),(**5**:2,5,b6,#6)	
7-26B		6-34B(3→5)	(**4**:b2,#2,5,6)	
7-27	Z11,14,Z24,31,32*,Z47,Z48*	6-32*(1→3)	(**2**:b2,#2,#4,#5),(**5**:2,5,b6,6)	(**4**:b2,4,b6,6)
7-27B		6-32*(5→7)	(**4**:2,#2,5,6),(**6**:b2,#4,#5,6)	(**2**:2,4,#4,5)
7-28	Z12, Z17,21,Z28*,30,34,Z45*	6-34(4→6)	(**2**:2,4,b5,b6)	
7-28B		6-34B(2→4)	(**4**:2,4,#5,#6)	(**5**:2,b5,b6,6)
7-29	Z12,18,Z25,33,Z46,Z47	6-33B(1→3)	(**3**:2,4,5,#6)	(**4**:2,4,b6,6),(**5**:b2,#4,5,b6)
7-29B		6-33(5→7)	(**3**:2,4,b5,6),(**6**:b2,#4,5,6)	(**2**:b2,4,#4,5)
7-30	16,22,Z26*,31,34,Z43,Z46	6-34B(1→3)	(**3**:2,#4,5,#6),(**7**:#2,4,5,6)	(**5**:2,#4,5,b6)
7-30B		6-34(5→7)	(**2**:2,#4,5,b6),(**6**:b2,4,5,6),(**7**:#2,#4,#5,#6)	
7-31	Z13*,Z23*,27,30,Z49*,Z50*	HW		(**5**:b2,#4,5,6)
7-31B		WH		
7-32	Z19,Z24,Z25,27,Z28*,Z29*,31	Harmonic Minor	(**3**:2,4,#5,6),(**6**:#2,#4,5,6)	(**2**:b2,4,b5,6),(**4**:2,#4,5,6)
7-32B		Harmonic Major	(**1**:2,4,5,b6),(**6**:#2,#4,#5,6)	(**2**:2,4,b5,6)
7-33*	21, 22, 34,35*	6-35*(1→3), WT +1	(**3**:2,#4,#5,#6)	
7-34*	Z23*,Z24,33,34	Melodic Minor	(**3**:2,#4,#5,6)	(**2**:b2,4,5,6),(**6**:2,4,b5,b6)
7-35*	Z25,26*,32*, 33*	Major	(**1**:2,4,5,6),(**4**:2,#4,5,6)	(**2**:2,4,5,6),(**3**:b2,4,5,b6),(**6**:2,4,5,b6),(**7**:b2,4,b5,b6)
7-Z36	3,Z11,Z23*,Z25,Z40,41,Z43	5-25B(1→4)	(**2**:b2,2,4,5),(**3**:b2,#2,#4,#6)	(**5**:b2,5,b6,6)
7-Z36B		5-25(4→7)	(**3**:2,#2,4,6),(**7**:#4,5,6,#6)	(**4**:b2,2,5,6)
7-Z37*	Z10,14,Z19,Z26*	6-Z26*(3→5)	(**2**:2,#2,#4,5),(**4**:b2,#2,#5,6),(**7**:4,5,b6,6)	
7-Z38	Z11,Z13*,15,Z17,18,Z19,Z24	6-Z24B(1→3)	(**2**:b2,#2,#4,5),(**7**:4,#4,#5,6)	
7-Z38B		6-Z24(5→7)	(**4**:2,#2,#5,6),(**7**:4,5,b6,#6)	

7-20a[0,1,2,4,7,8,9]	**7-20b**[0,1,2,5,7,8,9]	**7-21a** [0,1,2,4,5,8,9]			
7-21b[0,1,3,4,5,8,9]	**7-22***[0,1,2,5,6,8,9]	**7-23a**[0,2,3,4,5,7,9]	**7-23b**[0,2,4,5,6,7,9]	**7-24a**[0,1,2,3,5,7,9]	**7-24b**[0,2,4,6,7,8,9]
7-25a[0,2,3,4,6,7,9]	**7-25b**[0,2,3,5,6,7,9]	**7-26a**[0,1,3,4,5,7,9]	**7-26b**[0,2,4,5,6,8,9]	**7-27a**[0,1,2,4,5,7,9]	**7-27b**[0,2,4,5,7,8,9]
7-28a[0,1,3,5,6,7,9]	**7-28b**[0,2,3,4,6,8,9]	**7-29a**[0,1,2,4,6,7,9]	**7-29b**[0,2,3,5,7,8,9]	**7-30a**[0,1,2,4,6,8,9]	**7-30b**[0,1,3,5,7,87,9]
7-31a[0,1,3,4,6,7,9]	**7-31b**[0,2,3,5,6,8,9]	**7-32a**[0,1,3,4,6,8,9]	**7-32b**[0,1,3,5,6,8,9]	**7-33***[0,1,2,4,6,8,t]	**7-34***[0,1,3,4,6,8,t]
7-35*[0,1,3,5,6,8,t]	**7-Z36a**[0,1,2,3,5,6,8]	**7-Z36b**[0,2,3,5,6,7,8]	**7-Z37***[0,1,3,4,5,7,8]	**7-Z38a**[0,1,2,4,5,7,8]	**7-Z38b**[0,1,3,4,6,7,8]

Figure 7 C

7 note chord	6 note subsets	Other Name	MinMaj7	Dom7	Dim 7
7-20	Z6*,16,Z17,18,Z44,Z47,Z48	5-35*(1→3,5→7)	(**2**:b2,#4,5,b6)	(**3**:#2,4,b6,6)	
7-20B		5-35*(3rd)(1→3,5→7)	(**3**:4,#4,5,#6)		
7-21	14,15,16,Z19,20*,31,Z44	6-31B(1→3)	(**3**:2,#4,5,#6)	(**4**:b2,4,b6,6)	
7-21B		6-31(5→7)			
7-22*	Z19,Z43,Z44,Z49*	6-Z49*(5th)(1→3)	(**5**:2,#4,5,b6)	(**6**:b2,4,b5,6)	
7-23	8*,9,32*,33,Z40,Z46,Z47	5-35*(3rd)(2→5)	(**4**:b2,4,#5,#6)		
7-23B		5-35*(3→6)	(**5**:b2,#4,#5,#6)	(**2**:2,#2,4,5)	
7-24	9,22,33,34,Z39,41,Z48*	5-34*(4th)(1→4)	(**3**:b2,4,5,#6)	(**5**:2,5,b6,6),(**7**:#2,4,b5,b6)	
7-24B		5-34*(4→7)	(**7**:4,5,6,#6)	(**2**:2,4,#4,5),(**3**:2,#2,4,b6)	
7-25	Z10*,Z11,27,Z29*,33,Z45*,Z47	6-33B(2→4)	(**4**:2,4,#5,#6)	(**2**:b2,2,4,5)	
7-25B		6-33(4→6)	(**5**:b2,#4,#5,6)	(**2**:b2,#2,4,5),(**4**:b2,2,5,6)	
7-26	Z10,Z15,21,31,Z34,Z46,Z49*	6-34(3→5)	(**4**:b2,4,#5,6)	(**3**:b2,2,#4,6),(**7**:#2,#4,5,b6)	
7-26B		6-34B(3→5)	(**5**:2,#4,#5,#6),(**7**:4,5,b6,6)	(**2**:2,#2,#4,5), (**3**:b2,2,4,b6), (**6**:#2,#4,b6,6)	
7-27	Z11,14,Z24,31,32*,Z47,Z48*	6-32*(1→3)	(**3**:2,4,5,#6)	(**7**:#2,4,5,b6)	
7-27B		6-32*(5→7)	(**7**:4,5,b6,#6)	(**3**:b2,#2,4,b6)	
7-28	Z12, Z17,21,Z28*,30,34,Z45*	6-34(4→6)	(**5**:b2,#4,5,6)	(**3**:2,#2,#4,6),(**4**:b2,2,5,b6), (**7**:#2,#4,b6,6)	
7-28B		6-34B(2→4)	(**3**:b2,4,b5,6),(**7**:4,#4,5,6)	(**2**:b2,2,#4,5),(**6**:b2,#4,5,b6)	
7-29	Z12,18,Z25,33,Z46,Z47	6-33B(1→3)	(**2**:b2,4,b5,b6)	(**7**:#2,4,5,6)	
7-29B		6-33(5→7)	(**7**:4,#4,#5,#6)	(**4**:2,#2,5,6)	
7-30	16,22,Z26*,31,34,Z43,Z46	6-34B(1→3)	(**2**:b2,4,5,b6)	(**4**:2,4,b6,6), (**6**:b2,4,b5,b6)	
7-30B		6-34(5→7)		(**3**:2,4,b5,6),(**4**:2,#2,5,b6)	
7-31	Z13*,Z23*,27,30,Z49*,Z50*	HW	(**2**:2,4,b5,b6),(**4**:2,4,#5,6)	(**3**:b2,#2,#4,6), (**7**:#2,#4,5,6)	(**1**:b2,b4,5)
7-31B		WH	(**3**:2,4,b5,6),(**5**:2,#4,#5,6), (**7**:4,#4,#5,6)	(**2**:b2,#2,#4,5),(**4**:b2,#2,5,6),(**6**:b2,#4,5,6)	(**1**:2,4,b6)
7-32	Z19,Z24,Z25,27,Z28*,Z29*,31	Harmonic Minor	(1:2,4,5,b6)	(**5**:b2,4,5,b6)	(**7**:b2,b4,b6)
7-32B		Harmonic Major	(**4**:2,#4,5,6)	(**3**:b2,#2,5,b6),(**5**:b2,4,5,6)	(**7**:b2,4,b6)
7-33*	21, 22, 34,35*	6-35*(1→3), WT +1	(**2**:b2,4,5,6)	(**1**:b2,2,#4,#5)(**4**:2,#4,b6,6),(**5**:2,#4,5,b6), (**6**:2,4,b5,b6),(**7**:2,#2,#4,#5)	
7-34*	Z23*,Z24,33,34	Melodic Minor	(**1**:2,4,5,6)	(**4**:2,#4,5,6),(**5**:2,4,5,b6),(**7**:b2,#2,#4,#5)	
7-35*	Z25,26*,32*, 33*	Major		(**5**:2,4,5,6)	
7-Z36	3,Z11,Z23*,Z25,Z40,41,Z43	5-25B(1→4)	(**4**:2,4,6,#6)	(**7**:4,#4,5,6)	
7-Z36B		5-25(4→7)		(**2**:b2,#2,4,b5)	
7-Z37*	Z10,14,Z19,Z26*	6-Z26*(3→5)	(**5**:2,5,#5,#6)	(**3**:b2,2,4,6)	
7-Z38	Z11,Z13*,15,Z17,18,Z19,Z24	6-Z24B(1→3)	(**3**:2,4,b5,#6),(**5**:2,5,b6,6)	(**4**:b2,#2,b6,6)	
7-Z38B		6-Z24(5→7)	(**2**:2,4,#4,5)	(**3**:b2,#2,4,6)	

7-20a[0,1,2,4,7,8,9]	**7-20b**[0,1,2,5,7,8,9]	**7-21a** [0,1,2,4,5,8,9]			
7-21b[0,1,3,4,5,8,9]	**7-22***[0,1,2,5,6,8,9]	**7-23a**[0,2,3,4,5,7,9]	**7-23b**[0,2,4,5,6,7,9]	**7-24a**[0,1,2,3,5,7,9]	**7-24b**[0,2,4,6,7,8,9]
7-25a[0,2,3,4,6,7,9]	**7-25b**[0,2,3,5,6,7,9]	**7-26a**[0,1,3,4,5,7,9]	**7-26b**[0,2,4,5,6,8,9]	**7-27a**[0,1,2,4,5,7,9]	**7-27b**[0,2,4,5,7,8,9]
7-28a[0,1,3,5,6,7,9]	**7-28b**[0,2,3,4,6,8,9]	**7-29a**[0,1,2,4,6,7,9]	**7-29b**[0,2,3,5,7,8,9]	**7-30a**[0,1,2,4,6,8,9]	**7-30b**[0,1,3,5,7,87,9]
7-31a[0,1,3,4,6,7,9]	**7-31b**[0,2,3,5,6,8,9]	**7-32a**[0,1,3,4,6,8,9]	**7-32b**[0,1,3,5,6,8,9]	**7-33***[0,1,2,4,6,8,t]	**7-34***[0,1,3,4,6,8,t]
7-35*[0,1,3,5,6,8,t]	**7-Z36a**[0,1,2,3,5,6,8]	**7-Z36b**[0,2,3,5,6,7,8]	**7-Z37***[0,1,3,4,5,7,8]	**7-Z38a**[0,1,2,4,5,7,8]	**7-Z38b**[0,1,3,4,6,7,8]

Figure 7 D

Part II, Chapter 14: 2.6

Section (A_6, R_6); $(x+y)^6$.

R:

In this last chapter of Part II, there are no new 'big' concepts to introduce. Instead, we are going to focus on the hexads' peculiar traits, point out familiar ones, relate them to their well-known subsets and supersets, and provide more help in the form of charts.

This chapter will deal with hexads predominantly as a chord in the first section and as a scale in the last section. In other words, when first learning to name them by sight, we'll focus on their "pockets," how they are complements of each other, and offer help in identifying their various rotations. In the aural section, we will focus more on how they can act as a resource for their respective subsets. Likewise, the aid offered will look both at how they are comprised by their subsets and in turn how to best direct previous efforts towards their mastery, such as learning different 4-note categories of voicings. Nonetheless, as no portion of the hexads contains other hexads, the sense of what it means to learn one hexad in terms of another is fraught. As such, the division of the chapter into two distinct sections is also less clear; sometimes, when thinking of the hexad as chord it helps to understand it as scale as well, by looking at the superset/subsets of it and the complementary hexad's supersets/subsets; especially when a hexad is its own complement.

Hopefully, the primary philosophy underlying this pedagogy is starting to congeal for you: learning these set classes is a such a substantial effort that one should simultaneously use a two-pronged attack—one top-down, the other, bottom-up. You are both using "the bigger picture" to identify and assign significance to the multitude of set classes and simultaneously building your understanding of "the bigger picture" through the enrichment of your knowledge of set classes' subsets, rotations, associated voicings, particular traits, etc. — and, in turn, those supersets that contain them. Similarly, learning what something is not, and how it is not, is just as important of a vehicle for learning something as learning "what something is." Or put alternatively, emanating from Mazzola and Yoneda's lemma, no perspective is not situated. To "understand" some set class, we need to know how it relates to other set classes that we do know or just heard. In the context of a tonal piece, this could mean: to ascertain the function of one triad (such as ii minor), you have to relate it to another triad (for instance, I major). In short, this is another sense of our use of the word "enrich." We can enrich our understanding of a set class by aggregating our perspectives on them—each situated from a different set class.

This task is slow, and in many senses never ending. However, if one wishes to "get a feel" for some arbitrary set class to a similar extent as one already has for 4-27 and 3-11, it is the only route I can see.[78] This is something to keep in mind as you plumb the depth and vastness of these hexads. Realize though that in many senses, you will now, finally, be reaching the ocean's floor. Temporarily, your visibility will be limited and the overwhelming pressure from above may obscure that you have at last reached solid footing. Moreover, the ocean is still as vast. Nonetheless, over time you will get better acclimated and, hopefully, feel more comfortable navigating it, while keeping in mind the different experiences gained from traversing different levels of depth. In the penultimate part of this dissertation, Part III, we will even dig into how

[78] Of course, the popularity of certain set classes, like 7-16, will likely never even approach the popularity of 7-35*, as such there will not be the same level of external signification that their usage accrues. If anything, like this pedagogy does, they will first (hopefully, not always!) be understood in comparison to their diatonic compatriots. However, over time, as you and perhaps others wrestle with them in your collective imaginations, armed with deeply enriched understandings of them, they will start to get more autonomy, be less auxiliary, in our general musical consciousness.

Let me be clear, I don't advocate for this because I assign some special significance to one set class or another, but rather because I advocate for a more enriched musical consciousness, period; whether this be parsing out multiple interlocking (or not) rhythms, acquiring a sensitivity to timbre that can lead you to name how it is produced in the voice, through synthesis, through various orchestrations etc. In its ability to move our emotions and communicate, music has few parallels; the more enriched our vocabulary is, the better off we are. However, that doesn't mean we have to create/communicate with one-millionth of the terms that we acquire. The wise chief is wise in part because she both knows the million things to say and the "million minus one" things not to say in a given moment. Analogously, in terms of experience, I assert that blu-rays, with increased resolution, do often offer a significant improvement over DVDs.

When are the gains acquired from increases in resolution offset by increases in cost of production? Do we need 16k? Well, it depends. However, if many of us moved the bar, in terms of "harmonic resolution," a little bit through the learning of some, even minute, portion of this pedagogy, I think we'd be better off for it. Of course, this would have to be done in coordination with creative artists and musicians, who inspired by this increase in resolution, would in turn inspire others with said resolution. If there were no movies whose video release benefitted from a 4k treatment, then I don't think it would be worth it. However, I am glad the gaming companies have created better hardware for successive generation of consoles. There was a trust that, over time, the game makers could and would be better able to exploit the additional resources. There are phenomenal/first-rate games released now that could have also been released on 90's consoles. However, there are many others that could not have.

this initially “harmonic” vocabulary can be successfully used as a frame that can simplify many other non-harmonic musical pursuits and inquiries.

I recommend starting with the most familiar 6-note set classes, subsets of the familiar 7-note set classes, and their complements. The most famous 6-note set class is likely 6-35* {0,2,4,6,8,A}, the whole-tone scale. A not-too-distant second could be 6-32*{0,2,4,5,7,9} the “C” (natural) and “G” (hard) hexachord associated with the Guidonian hand. Let’s start with sc 6-35*: relatively speaking, due to its even 6-fold division of the octave, it has very few subsets; just set classes 5-33*, 4-21*, 4-24*, sc 4-25*, 3-6*, 3-8, 3-12*, 2-2*, 2-4*, 2-6*, and 1-1*. Furthermore, as all but 3-8 are symmetrical, this only amounts to 12 chords/dyads/pitches. This set class is also special as its patterning can also be used to describe an equal temperament of size 6 (as opposed to 12!). If you wish to “experience” the connection between the counting principles underlying the polynomial expansion shown earlier, e.g., Pascal’s triangle, see just how and in how many ways you can select these 12 elements from the whole tone scale (1, 6, 15, 20, 15, 6, 1). Ask how they reflect a count of some dummy variable like xy2. What does that dummy variable refer to?

6-32* can be expressed as stacked P5ths/4ths. Its interval vector <1,4,3,2,5,0> exemplifies the deep scale property and hence is described by the common-tone theorem—both are results of diatonic set-theory; this interval vector can also be used to describe how many common tones occur between an instance of 6-32* and its transposition by the specified amount. For instance, as “C” 6-32* and “F” 6-32* are separated by a P5/P4, they will have 5 tones in common. In this case, “C”, “D”, “F”, “G”, and “A.”

Below, I will categorize the 6-note set classes into tiers relating to their proximity to a familiar hexachord/7-note set class. This a good way to start getting acquainted with both the 7 and 6-note set classes; you are examining how the 6-note set classes are bundled together in 7-note set classes of varying familiarity.

Tier 1: Familiar 7-note set classes

Order Numbers	**7-35***	**7-34***	**7-33***	**7-32a**	**7-32b**	**7-31a**	**7-31b**
1-6	**6-32***	6-33a	**6-22a**	6-Z26*	6-Z24b	**6-Z13***	6-Z23*
2-7	**6-33a**	**6-34a**	6-34a	**6-Z28***	**6-27b**	6-Z23*	6-Z13*
3-1	**6-Z26***	**6-34b**	6-35*	**6-31b**	**6-Z19a**	6-27a	6-27b
4-2	**6-33b**	6-33b	6-34b	**6-Z29***	6-Z29*	**6-Z50***	6-Z49*
5-3	6-32*	**6-Z24b**	**6-22b**	**6-Z19b**	**6-31a**	**6-30a**	**6-30b**
6-4	**6-Z25b**	**6-Z23***	**6-21b**	**6-27a**	6-Z28*	**6-Z49***	6-Z49*
7-5	**6-Z25a**	**6-Z24a**	**6-21a**	6-Z24a	6-Z26*	6-27a	6-27b

Set classes	Z-relation
6-Z13*	**6-Z42***
6-Z19a	**6-Z44b**
6-Z23*	**6-Z45***
6-Z24a	**6-Z46b**
6-Z25a	**6-Z47b**
6-Z26*	**6-Z48***
6-Z28*	6-Z49*
6-Z29*	6-Z50*

Tier 2: Other important 7-note set classes

Order Numbers	**7-1***	**7-7a**	**7-7b**	**7-21a**	**7-21b**	**7-22***
1-6	**6-1***	**6-5a**	6-Z6*	**6-15a**	6-Z19b	6-Z19b
2-7	6-1*	**6-Z6***	6-5b	6-Z19a	**6-15b**	6-Z49*
3-1	**6-2b**	**6-Z43b**	**6-Z41b**	6-31b	**6-14a**	6-Z19a
4-2	**6-Z3b**	**6-18b**	6-Z38*	**6-20***	**6-16b**	6-Z43b
5-3	**6-Z4***	**6-7***	6-7*	**6-Z44b**	**6-Z44a**	6-Z44b
6-4	**6-Z3a**	**6-Z38***	**6-18a**	**6-16a**	6-20*	6-Z44a
7-5	**6-2a**	**6-Z41a**	**6-Z43a**	**6-14b**	6-31a	6-Z43a

Set classes	Z-relation
6-Z3a	**6-Z36**
6-Z4*	**6-Z37***
6-Z6*	**6-Z38***
6-Z12a	6-Z41
6-Z17a	6-Z43
6-Z26*	6-Z48*

Tier 3: Subsets of near-familiar 7-note scales

Order Numbers	**7-23a**	**7-23b**	**7-25a**	**7-25b**	**7-27a**	**7-27b**	**7-29a**	**7-29b**
1-6	**6-8***	**6-9b**	**6-Z10b**	**6-Z11b**	6-Z11a	6-Z24b	6-Z12a	6-Z25b
2-7	**6-9a**	6-8*	**6-Z11a**	**6-Z10a**	6-Z24a	6-Z11b	6-Z25a	6-Z12b
3-1	6-Z46a	**6-Z40a**	6-27a	6-Z45*	6-32*	6-14a	6-33b	6-Z46b
4-2	6-32*	6-Z47a	6-33b	6-Z47a	6-31a	6-Z47b	6-Z50*	6-Z47b
5-3	6-33a	6-33b	6-Z29*	6-Z29*	6-Z48*	6-Z48*	6-18b	6-18a
6-4	6-Z47b	6-32*	6-Z47b	6-33a	6-Z47a	6-31b	6-Z47a	6-50*
7-5	**6-Z40b**	6-Z46b	6-Z45*	6-27b	6-14b	6-32*	6-Z46a	6-33a

Order Numbers	7-24a	7-24b	7-26a	7-26b	7-28a	7-28b	7-30a	7-30b
1-6	6-9a	6-22b	6-Z10a	6-21b	6-Z12b	6-21a	6-Z26*	6-Z26*
2-7	6-22a	6-9b	6-21a	6-Z10b	6-21b	6-Z12b	6-34b	6-22b
3-1	6-33a	**6-Z39a**	6-Z46a	6-15a	6-Z45*	6-28*	6-31b	6-Z46b
4-2	6-34a	**6-Z41a**	6-31a	6-Z49*	6-Z17a	6-34b	6-Z43b	6-16b
5-3	6-Z48*	6-Z48*	6-34a	6-34b	6-30a	6-30b	6-16a	6-Z43a
6-4	**6-Z41b**	6-34b	6-Z49*	6-31b	6-34a	6-Z17b	6-Z46a	6-31a
7-5	**6-Z39b**	6-33b	6-15b	6-Z46b	6-Z28*	6-Z45*	6-22a	6-34a

Figure 2.6 1(tiers 1-3)

How to deal with the Z-relations given? First, observe them — memorizing hexad complement pairs will prove very helpful. Second, recognize that each of these hexads contain a 5-note complement of a 7-note superset of that particular hexad's complement; it's Z relation. Therefore, 6-Z42* is a superset of 5-31ba, which is the complement of 7-31ab, the superset of 6-Z42*'s complement 6-Z13*. As the familiar 7-note set classes' 5-note complements could also be very familiar by this point, you could treat them as a good starting point; for example, take 5-31 (7-31's complement) and interject a half-step somewhere and you get 6-Z42*. This can be deduced also from the "other name" for 6-Z42*, 4-28*(1➔4). 5-31a is 6-Z42* minus the 2nd pitch in 6-Z42's cluster "(1➔4)." In general, use your knowledge of those unfamiliar complements' larger subsets to quickly acclimate yourself to them and their distributions of smaller set classes.

First though, let's investigate the last calculation given. After that, we will explore the above insight in more depth — how to (relatively) quickly use larger subsets of a superset-in-question to help us approximate said distribution; a critical heuristic, but one that we couldn't fully implement until hexad complements were discussed and we knew how to relate subsets of cardinality < 6 to their complements that are > 6.

If the above complementary hexad calculation seems bewildering, you are not alone in feeling this way; this author finds it bewildering too — not in concept as much as application. However, it is what it is; it's a part of the theory that we have to grapple with. Let me state this clearly now, if the cardinality of a subset (A) is < 6 and the cardinality of its complementary superset (Ac) is > 6, then when trying to ascertain whether the former (A) is contained in a superset (B) of the same cardinality as the latter's complement ($|A^c| = |B|$), we must examine whether the subset-in-question (A) is contained in any of the complements of the superset-in-question's 6-note subsets (6-note subsets of B)^c. What a mouthful!!!! In short, to ascertain whether, A ∈ B, where |A| < 6, you must ask if A ∈ (6-note subsets of B)^c.

Why didn't we confront this calculation before? Well, when dealing with anything other than complementary pairs of set classes that are of cardinality 5 and 7, there are workarounds: for instance, one can typically assume that the subset is contained in its (exact) complementary superset.[79] How about determining whether a subset is contained in a set class of the same cardinality as that same subset's complement? E.g., is 4-8* in 8-16? While there are a number of exceptions with the 4/8-pairs—or instance, the 4-note "cluster" set classes are not contained in 8-28* — this is more of an exception than the rule. You would be well-served by employing the default assumption that any 8-note set class contains at least one copy of a 4-note set class.

Furthermore, due to our naming system, and the greater ease with which one can learn the 4-note set classes, it's easier to ask if there are any 4-note set classes that are made explicit by 8-28*'s listed rotations. Moreover, manually finding the 4-note subsets of the (5-note subsets of the (complements of the 6-note subsets of (its 7-note maximal subsets)) is too labor-intensive for all but savants to quickly do in their heads.

How can we use the rotations of 8-28*, MinMaj (2,4,6,8:2,4,#4,#5,6) and Dom (1,3,5,7:b2,#2,#4,5,6), to determine whether or not 8-28* contains 4-28*? First off, neither of the given rotations are expressed as some type of Dim7 chord. However, that doesn't mean that these rotations don't contain it. To contain a Dim7 chord, the expression must have a 0 (which both do), a #2/b3 (which both do), a #4 (which both do), and a 6 (which again, both do). So, the answer is yes!

What other 4-note subsets does it contain? The 4-note subsets that 8-28*'s rotations make immediately clear are (Figure 2.6.2):

[79] There is only one non-hexad exception to this, 5-Z12 is not contained in 7-Z12.

MinMaj7	4-note SC	Dom7	4-note SC
2	4-3*	b2	4-12a
4	4-Z15a	#2	4-Z15b
#4	4-18a	#4	4-25*
#5	4-17*	5	4-27b
6	4-12a	6	4-13a

Figure 2.6 2

With a little more deductive effort, you may realize that since 8-28* is symmetrical, if it contains one component of a set class, than it contains the other as well. Therefore, we have relatively quickly found that 8-28* also contains: 4-3*, 4-12, 4-13, 4-Z15, 4-17*, 4-18, 4-25*, 4-27.

To find the rest, requires a more systematic effort. However, I can't imagine why someone would need to apply such an effort in real time (without at minimum, paper assistance). Again, if you wanted to know whether a particular other 4-note set class is included, such as 4-10*, you could, with minimal mental effort, apply a similar process as just given for 4-28*. 4-10* can be expressed as Min (2:b2). Since Dom(1,3,5,7:b2,#2,#4,5,6) contains a 0, b2, #2, and b7, it contains 4-10*.

We have now, relatively easily, compiled a list of 8-28*'s 4-note subsets that is 10 set classes (or 14 chords) long.[80] So, using the names of these modes'/rotations' expressions we quickly got most of 8-28*'s subsets and very quickly determined if a particular 4-note set class was contained in it. Accordingly, a similar process can be used for the 3/9 and 2/10 complementary cardinality pairs. However, both are a trivially easy task. Each 10 and 9-note set class respectively contains all of the 2- and 3-note set classes. What changes is the distribution of those subsets, or the number of each that is contained within it. However, when your interest extends to grappling with a larger set class with that level of detail, one would benefit from utilizing the many aids found between this chapter and the chapter that first introduced that larger set class: e.g., what are that larger set class's maximal subsets; what are those subsets parsimonious to; what are the complements of its maximal subsets and their parsimoniously-related sets etc.

[80] The total list also contains 4-9*, 4-26*, and 4-Z29.

For instance, if you want to know how many 3-12*s 9-11 contains, look at 9-11's complement 3-11. When asking what 3-11 is a maximal subset of, 4-14, 4-17*, 4-18, 4-19, 4-22, and 4-Z29 etc. may relatively quickly come to mind. You may quickly notice that since this list contains 4-17*, 8-17* is a maximal subset of 9-11. Furthermore, you may also remember that 8-17* contains two copies of 7-32a and 7-32b. This would immediately raise my suspicion that 9-11 contains 6-20*, two augmented triads that are separated by a half-step. Since 8-17* also contains 7-21—at some point in your study this may also be a quick observation—it must contain 6-20*. 7-21 is the only 7-note superset of 6-20*. We then know for certain that 9-11 contains two distinct augmented triads (3-12*s) that are separated by a half-step.

Does it contain 3? No, because if it did, two of those augmented triads would have to be separated by a whole step; again, remember there are only 4 distinct augmented triads. That would mean that there would be an appearance of 6-35*, the whole tone scale. However, by this point in your learning you would just automatically know that the supersets of the whole tone are 7-33*, 8-21*, 8-24*, and 8-25* and that none of their complements, 4-21*, 4-24*, and 4-25*, 5-33*, contain a major or minor triad.

While the above reasoning may seem like a lot of mental calculation now and anything but obvious, I assure you that when some of these other set classes become significantly more familiar, it will become more obvious. More to the point though, it doesn't matter whether you can deduce the exact number of any particular subset in a superset, just by recognizing a few of the superset-in-question's bigger subsets (especially if they are as special as 6-35*!), you will quickly gain a better feel for how many of a certain smaller subset the larger superset-in-question contains. Those bigger subsets can quickly orient you towards understanding the distribution of smaller subsets-in-question. In short, the better you understand those bigger subsets—also capable of being approximated by their own "bigger subsets"—the quicker you will have a good feel for the measured-in-smaller-subsets landscape of the larger superset-in-question.

The next step is to examine the "pockets" (Figure 2.6.3(a-e)). One may also benefit from studying the set classes in the following groups as well:

- 6-1 – 6-5;
- 6-Z6 – 6-7*, 6-Z38*;
- 6-8 – 6-Z12;
- 6-5, 6-Z6, 6-Z12 & 6-Z13;
- 6-14, 6-15, 6-39, 6-40;
- 6-16 – 6-19, 6-43, and 6-44;
- 6-19, 6-31;
- 6-20*- 6-22, 6-35*;
- 6-Z13*, 6-Z23*, 6-27, 6-29 – 6-30, 6-Z49 – 6-Z50*;
- 6-Z25 – 6Z26*, 6-32 – 6-33;
- 6-Z23 – 6-Z24, 6-34 – 6-35*;
- 6-Z36 – 6-Z38
- 6-Z41 – 6-Z42*;
- 6-Z47 – 6–Z48.

Maj 7:

(b2)	**(2)**	**(#2)**	**(4)**	**(#4/b5)**	**(5)**	**(~5,~6)**	**Fixed Maj7**
6-Z326a	6-9a	6-5a	6-Z38*	6-Z41b	6-Z40b	**(6,#6)**	5-5a
6-Z39a	6-21a	6-16b	6-Z17b	6-22b	6-15b	**(#5,6#)**	5-13b
6-Z42*	6-Z46a	6-Z44a	6-18b	6-Z43b		**(5,#6)**	5-Z38b
6-Z41a	6-22a	6-Z43a	6-7*			**(#4,#6)**	5-15*
6-5a	6-Z12a	6-Z6*				**(4,#6)**	5-7a
6-14a	6-Z24a	6-Z44a	6-Z44b	6-Z24b	6-14b	**(#5,6)**	5-Z17*
6-Z46b	6-32*	6-31b	6-Z26*	6-Z25b		**(5,6)**	5-27b
6-Z47b	6-33a	6-Z29*	6-18a			**(#4,6)**	5-29b
6-16b	6-Z25b	6-Z43b				**(4,6)**	5-20b
6-Z44b	6-31a	6-20*	6-Z19a	6-16b		**(5,b6)**	5-21b
6-Z48*	6-34a	6-31a	6-Z17a			**(#4,#5)**	5-30b
6-Z44a	6-Z28*	6-Z44b				**(4,#5)**	5-22*
6-18a	6-Z26*	6-Z19b	6-Z38*			**(#4,5)**	5-20b
6-Z43a	6-Z25a	6-16a				**(4,5)**	5-20a
6-Z6*	6-Z12b	6-5b				**(4,b5)**	5-7b

(b2,2)	**(b2,#2)**	**(2,#2)**	**(~6)**	**Fixed Maj7**
6-2a	6-Z3a	6-Z4*	**(#6)**	4-5a
6-8*	6-Z10b	6-Z11b	**(6)**	4-14a
6-Z39b	6-14b	6-15b	**(b6)**	4-19b
6-Z40a	6-15a	6-14a	**(5)**	4-20*
6-9a	6-Z11a	6-Z10a	**(#4)**	4-16a
6-Z3a	6-Z4*	6-Z3b	**(4)**	4-8*

MinMaj7:

(b2)	(2)	(4)	(#4/b5)	(5)	(~5,~6)	Fixed MinMaj7
6-2a	6-Z3a	6-Z41a	6-Z42*	6-Z39a	**(6,#6)**	5-4a
6-8*	6-Z10b	6-Z47b	6-Z46b	6-14a	**(b6,6#)**	5-11a
6-Z39b	6-14b	6-Z48*	6-Z44b		**(5,#6)**	5-Z37*
6-Z40a	6-15a	6-18a			**(#4,#6)**	5-Z38*
6-9a	6-Z11a				**(4,#6)**	5-14a
6-Z10a	6-Z13*	6-Z49*	6-27b	6-15a	**(#5,6)**	5-16a
6-21b	6-Z24b	6-34b	6-Z28*		**(5,6)**	5-26a
6-Z45*	6-27b	6-Z29*			**(#4,6)**	5-31b
6-21a	6-Z23*				**(4,6)**	5-28b
6-16a	6-Z19b	6-31b	6-Z44a		**(5,b6)**	5-21a
6-Z47a	6-Z49*	6-Z50*			**(#4,#5)**	5-32b
6-Z46a	6-27a				**(4,#5)**	5-32a
6-Z17a	6-Z19a	6-Z17b			**(#4,5)**	5-22*
6-22a	6-Z24a				**(4,5)**	5-30a
6-Z12a	6-Z13*				**(4,b5)**	5-19b

(b2,2)	(~6)	Fixed MinMaj7
6-1*	**(#6)**	4-4a
6-2b	**(6)**	4-12b
6-Z36b	**(#5)**	4-17*
6-Z37*	**(5)**	4-19a
6-Z36a	**(#4)**	4-18a
6-2a	**(4)**	4-Z15a

Dom 7:

(b2)	(2)	(#2)	(4)	(#4/b5)	(5)	(~5,~6)	Fixed Dom
6-15a	6-22a	6-Z17a	6-16a	6-21b	6-Z39b	**(b6,6)**	5-13a
6-27b	6-33a	6-Z50*	6-Z25a	6-Z23*		**(5,6)**	5-25b
6-Z49*	6-34a	6-30a	6-Z43a			**(#4,6)**	5-28b
6-Z19a	6-Z26*	6-18b				**(4,6)**	5-20a
6-Z28*	6-34a	6-31a				**(5,b6)**	5-26b
6-34b	6-35*	6-34a				**(#4,#5)**	5-33*
6-31b	6-34b	6-Z48*				**(4,#5)**	5-30a
6-30b	6-34b	6-Z49*				**(#4,5)**	5-28a
6-Z29*	6-33b	6-Z47a				**(4,5)**	5-29a
6-Z43b	6-22b	6-Z41b				**(4,b5)**	5-15*

(b2,2)	(b2,#2)	(2,#2)	(~6)	Fixed Dom
6-Z10a	6-Z13*	6-Z12b	**(6)**	4-Z29a
6-21b	6-Z24b	6-22b	**(b6)**	4-24*
6-Z45*	6-27b	6-Z46b	**(5)**	4-27b
6-21a	6-Z23*	6-21b	**(#4)**	4-25*
6-Z10b	6-Z11b	6-9b	**(4)**	4-16b

Min 7:

(b2)	(2)	(4)	(#4/b5)	(5)	(~5,~6)	Fixed Min
6-Z11a	6-Z12a	6-Z47a	6-Z45*	6-Z40a	**(#5,6)**	5-Z36
6-Z23*	6-Z25b	6-33b	6-27a		**(5,6)**	5-25a
6-27a	6-Z28*	6-Z29*			**(#4,6)**	5-31a
6-Z24a	6-Z25a				**(4,6)**	5-29a
6-Z25a	6-Z26*	6-32*	6-Z46a		**(5,b6)**	5-27a
6-33b	6-34b	6-33a			**(b5,b6)**	5-34*
6-32*	6-33b				**(4,b6)**	5-35*
6-Z50*	6-31b	6-Z47b			**(#4,5)**	5-32a
6-33a	6-32*				**(4,5)**	5-35*
6-Z25b	6-Z24b				**(4,b5)**	5-29b

(b2,2)	(~6)	Fixed Min
6-Z3b	**(6)**	4-13a
6-9b	**(b6)**	4-22a
6-Z40b	**(5)**	4-26*
6-Z39a	**(#4)**	4-27a
6-8*	**(4)**	4-23*

Dim 7: 4-28* + dyad

	(b2)	(2)	(b4)	(4)	(5)	(b6)
(b2)	NA	6-Z42*	6-27a	6-Z28*	6-30a	6-Z29*
(2)		NA	6-Z45*	6-27b	6-Z29*	6-30b
(b4)			NA	6-Z42*	6-27a	6-Z28*
(4)				NA	6-Z45*	6-27b
(5)					NA	6-Z42*
(b6)						NA

Figure 2.6 3(a-e)

Finally, as a finish to this section, I want to implore you, if you have not begun already, to start memorizing the prime forms of all of the set classes. Only now, after:

- Your having been introduced to all of the set classes,
- Your potentially being somewhat conversant in the set classes of cardinality 5 and under—especially the more familiar ones, and
- Your having some experience mentally manipulating the smaller set classes, are you fully equipped to work with the "other names" column. When mentally exploring a set class,

my default go-to is the prime form/ "other name." Furthermore, all voicing classifications have the prime form as the reference point.

Perhaps most importantly though, when dealing with the vast majority of musicians/academics, if they even know of set classes, they will typically expect you to preface any set class you mention with its prime form. This is where the "other names" earns its badge. While you can mention that 6-33b is [0,2,4,6,7,9], you may get more traction with your audience by either saying, "It's the 1st six degrees of a Lydian scale" or "It's degrees 4 – 2 of a major scale." They will most likely both more quickly get a fuller impression of that set class and not have to:

- First, translate the prime form numbers into pitch names, which may take a second,
- Second, then repeat back to you, something like, "Oh, it's just the 1st six degrees of a lydian scale" or "It's just degrees 4 – 2 of a major scale."

Outside of very specific contexts, few may reward you for knowing the set classes by their Forte numbers. Instead, most, if they will reward you, will do it for your making those set classes relatable and knowing the prime form. Nonetheless, I hope this pedagogy has already amply demonstrated many "internal" rewards for learning to identifying the set classes by their Forte number; two notable rewards are:

- Their facilitation of internalizing these set classes deeply—how much more space would the charts below take up if I used prime form!—and,
- They are not explicitly being linked to a single expression whose bass is "C".

Sometimes, important relationships between two set classes are not always easy to eyeball when comparing prime forms. Exclusively working with prime forms can overemphasize how much insight they can give into a set class. Yes, they show all of the pitch-classes contained in one expression of it; however, they don't make explicit those aspects of that set class that may be most important to you. I actually believe that prime forms typically obfuscate the multiplicity of a set class's representations—giving an impression that they are the "best" representation of a set class. On the other hand, Forte numbers, as an "emptier container", can more easily be molded, through one's conscious exploration, into the best bundle of associations that one wants to ascribe to each set class!

I also "highly recommend" learning all of the maximal subsets associated with the set classes; not because you can't quickly figure it out mentally—especially for set classes of cardinality 3, 4 and even 5; but rather, because it cuts down processing time and mental effort—each worth their weight in gold. After a while, just seeing the Forte name of one studied will quickly conjure up many musical associations that you have bundled into it. Immediately associating a set class of interest with its maximal subsets may "remind" you of aspects of that set class that you have just overlooked; remember, the bigger set classes are so big and have so many associations that the focus should not be on being able to "access everything about them automatically," (an impossible task) but rather, to feel comfortable eliciting—intuitively or consciously—what you, in that moment, may find most relevant about them. In this regard, I found automatically knowing the Forte names of the maximal subsets more helpful than almost anything else.

A:

Again, in this section, there may be some ambiguity around whether we're treating these set classes as chords, sound objects, or scales, resources. After presenting some portion of helpful charts—again we run into the problem of pagination and explanation—we will end this chapter by more openly discussing the challenges of identifying hexads aurally and more generally, handling the cognitive challenges of working with such complex objects: turning around a trichord in your head may be instantaneous, trivially easy, after a while; on the other hand, no matter how well you learn the hexads, doing the same with them may quickly feel exhausting.

The same could be said of the experience of relating hexads to each other vs. relating trichords to each other. This studied material may directly cause you to confront limits in your processing capacity; it did for me. I see this as one of the major perks of this pedagogy though; we get better acquainted with each of our own cognitive limits. Of course, that is just the beginning though as we 1) may soon realize that our limits extend further than we previously thought; and 2) may become more strategic, more quickly assessing how best to achieve whatever "musical goals" —

not abstract goals like "who can mentally "lift" the most numbers at once" — we may have. This often entails finding out what one needs specifically to automate for their own specific musical objective.

The distribution of 4-note set classes amongst the hexads is shown below. Taking a cue from the previous chapter, the hexads are described as "outer" dyads plus inner "tetrads." Each chart fixes the outer dyads and, in each row, a different inner tetrad is composed with it. Included is any combination that yields a hexad whose voicing both:

1. Fits into one of this pedagogy's proposed chord types (or types often found in Jazz/Popular chord notation); and,
2. does not contain two consecutive half-steps.

As the tetrads can successfully combine with the dyads at more than one transposition level, there are often multiple hexads given. The column describing the associated 6-note set classes correspond exactly (both are left to right) with the chord types presented prior in the row. For instance, take the chart that fixes the Maj 7th dyad, the first chord type presented in the 4-3* column is Maj #2,#4,5 and the corresponding first set class given in the "Associated 6-note set classes" column is 6-Z19b.

Again, I am not including all of the potential dyad tetrad combinations; only the M7, m7, M6, and m6 charts. This is for one main reason: presenting twelve charts at once is overkill—not because it's indecipherable but because sometimes "being confronted with too much" information discourages rather than encourages exploration. That is a challenge of a pedagogy of this scope. How does one encourage a broad exploration of set classes without getting overwhelmed? In a future manifestation of this dissertation—for instance, if it became a book, many of these charts would be encapsulated within accompanying software that the reader could then adjust so as to only show them, in any given moment, what they were specifically interested in. Between the scores and charts, there are nearly two thousand pages of charts that could be deemed relevant.[81]

Plus, not all of these charts would speak equally to different readers. Some may find some charts inspiring and others not at all. Furthermore, there is nothing wrong with certain readers, interested in exploring certain components of this pedagogy more thoroughly, "finishing those charts" (or collections of charts) that are only presented partially here (Figures 2.6.4(a-d)). This pedagogy was just meant to provide the legwork needed to offer a navigable path—in a reasonable time (a year or two)—through what many may have seen (or do see) as either an impossible or not worthwhile pursuit. For those in the latter camp, I wonder if some would change their mind were they to see it as actually manageable and as having many of the side benefits that Part III will explore in more depth.

[81] I also have other, not included, practice aids; for instance, audio recordings of all of the set classes' rotations being spoken out loud; the intent being for the listener to listen and then talk along with the recordings. While that may help some, including it in this dissertation would just be too much!

Set classes + M7	Maj	MinMaj	Associated 6-note Set classes
4-3*	Maj$^{\#2,\#4,5}$, Maj^{4,5,b6}	MinMaj^{2,4,b5}	6-Z19b, 6-Z19a, 6-Z13*,
4-7*	Maj$^{\#2,5,b6}$, Maj$^{4,\#5,6}$	MinMaj$^{2,\#4,5}$	6-20*, 6-Z19b, 6-Z19a
4-8*	Maj$^{\#2,\#5,6}$	MinMaj^{2,5,b6}	6-Z19a, 6-Z19b
4-9*		MinMaj$^{2,\#5,6}$(2x)	6-Z13*
4-10*	Maj^{2,4,5}, Maj$^{\#4,5,6}$	MinMaj^{4,b5,b6}	6-Z25a, 6-Z25b, 6-Z50*
4-11a	Maj$^{\#2,\#4,\#5}$, Maj^{4,5,6}	MinMaj^{2,4,5}	6-31a, 6-Z26*, 6-Z24a
4-11b	Maj$^{2,\#4,\#5}$, Maj$^{\#4,\#5,6}$	MinMaj^{4,5,b6}	6-Z26*, 6-Z24b, 6-31b
4-12a	Maj$^{2,4,\#5}$	MinMaj^{4,b5,6}	6-Z28*, 6-30b
4-12b		MinMaj$^{\#4,5,6}$	6-Z28*
4-13a	Maj$^{\#2,\#4,6}$	MinMaj^{2,4,b6}	6-Z29*, 6-27a
4-13b		MinMaj$^{\#4,\#5,6}$	6-Z27b
4-14a	Maj^{2,4,6}		6-Z25b
4-Z15a	Maj$^{\#2,5,6}$	MinMaj$^{2,\#4,\#5}$	6-31b, 6-Z49*
4-Z15b	Maj^{2,5,b6}	MinMaj$^{4,\#5,6}$	6-31a, 6-Z49*
4-16a		MinMaj^{2,5,6}	6-Z24b
4-16b	Maj$^{2,\#5,6}$		6-Z24a
4-18a		MinMaj$^{2,\#4,6}$	6-27b
4-21*	Maj$^{2,\#4,\#5}$	MinMaj^{4,5,6}	6-34a, 6-34b
4-22a	Maj$^{2,\#4,6}$		6-33a
4-23*	Maj^{2,5,6}	MinMaj$^{4,\#5,\#6}$	6-32*, 6-Z47b
4-26*	Maj$^{b2,\#4,6}$		6-Z47b
4-Z29a		MinMaj^{2,4,6}	6-Z23*

Set classes + m7	Dom	Min	Associated 6-note Set classes
4-3*	Dom$^{\#2,\#4,5}$, Dom^{4,5,b6}	Min^{2,4,b5}	6-Z49*, 6-Z24a, 6-Z24b
4-7*	Dom$^{\#2,5,b6}$	Min$^{2,\#4,5}$	6-31a, 6-31b
4-8*		Min^{2,5,b6}	6-Z26*
4-10*	Dom$^{b2,\#2,\#4}$, Dom^{2,4,5}, Dom$^{\#4,5,6}$	Min^{4,b5,b6}	6-Z23*, 6-33b, 6-Z23*, 6-33a
4-11a	Aug$^{\#2,\#4}$, Dom^{4,5,6}	Min^{2,4,5}	6-34a, 6-Z25a, 6-32*
4-11b	Dom$^{b2,\#4,5}$	Min^{b2,4,b5}, Min^{4,5,b6}	6-Z25b, 6-34b, 6-32*
4-12a	Dom$^{b2,\#2,5}$, Aug^{2,4}, Dom^{4,b5,6}		6-27b, 6-34b, 6-Z29*
4-12b	Dom^{b2,4,5}	Min$^{\#4,5,6}$	6-Z29*, 6-27a
4-13a	Dom$^{\#2,\#4,6}$	Min^{2,4,b6}	6-30a, 6-33b
4-13b	Dom$^{b2,\#4,5}$		6-30b
4-14a	Aug$^{b2,\#2}$, Dom^{2,4,6}		6-Z24b, 6-Z26*
4-Z15a	Dom$^{\#2,5,6}$	Min^{2,b5,b6}	6-Z50*, 6-34b
4-Z15b	Dom^{2,5,b6}	Min$^{b2,\#4,5}$	6-34a, 6-Z50*
4-16a		Min^{2,5,6}	6-Z25b
4-16b		Min^{b2,5,b6}	6-Z25a
4-17*	Aug^{b2,4}		6-31b
4-18a		Min$^{2,\#4,6}$	6-Z28*
4-18b	Dom^{b2,5,b6}		6-Z28*
4-19b	Dom^{b2,4,6}		6-Z19a
4-21*	Aug$^{2,\#4}$	Min^{b2,4,5}, Min^{4,5,6}	6-35*, 6-33a, 6-33b
4-22a	Dom$^{2,\#4,6}$	Min^{b2,4,b6}	6-34a, 6-32*
4-22b	Aug$^{b2,\#4}$		6-34b
4-23*	Dom^{2,5,6}	Min^{b2,b5,b6}	6-33a, 6-33b
4-24*		Min^{b2,4,6}	6-Z24a
4-25*		Min^{b2,5,6}(2x)	6-Z23*
4-26*	Dom$^{b2,\#4,6}$		6-Z49*
4-27a	MinMaj$^{4,\#5,\#6}$	Min$^{b2,\#4,6}$	6-Z47b, 6-27a
4-27b	Dom^{b2,5,6}		6-27b
4-Z29a		Min^{2,4,6}	6-Z25a
4-Z29b	Dom$^{b2,\#2,6}$		6-Z13*

Set classes + M6	Maj (6,9)	Maj	MinMaj7	Dom	Min	Dim	Associated 6-note Set classes
4-3*						Dim ^{2,4}, Dim ^{b4,5}	6-27b, 6-27a
4-7*						Dim ^{2,5}	6-Z29*
4-10*	Maj ^{4,5}					Dim ^{b2,b4}, Dim ^{4,b6}	6-32*, 6-27a,6-27b
4-11a					Min ^{2,4,5,6}	Dim ^{b4,b6}	6-33a, 6-Z28*
4-11b	Maj $^{\#4,5}$					Dim ^{b2,4}	6-33b, 6-Z28*
4-12a	Maj $^{b2,\#2,5,6}$, Maj $^{4,\#5}$			Dom $^{\#4,5,6}$			6-Z49*, 6-31b, 6-Z23*
4-12b	Maj ^{b2,4,5,6}		MinMaj ^{2,4,6}				6-31a, 6-Z23*
4-13a				Dom ^{4,5,6}	Min $^{2,4,\#5,6}$		6-Z25a, 6-Z50*
4-13b	Maj $^{b2,\#4,5,6}$	Maj ^{2,4,6}		Dom $^{b2,\#2,6}$			6-Z50*, 6-Z25b, 6-Z13*
4-14a	Maj $^{b2,\#2,\#5,6}$	Maj $^{\#4,5,6}$				Dim ^{4,6}	6-Z19b, 6-Z25b, 6-Z29*
4-14b		Maj $^{\#2,\#4,6}$			Min ^{2,4,6}		6-Z29*, 6-Z25b
4-Z15a						Dim ^{2,b6}	6-30b
4-Z15b						Dim ^{b2,5}	6-30a
4-17*	Maj $^{b2,4,\#5,6}$		MinMaj $^{2,\#4,6}$		Min $^{\#4,5,6}$		6-20*, 6-27b, 6-27a
4-18a		Maj $^{4,\#5,6}$		Dom $^{\#2,5,6}$			6-Z19b, 6-Z50*
4-18b			MinMaj $^{2,\#5,6}$	Dom ^{b2,4,6}			6-Z13*, 6-Z19a
4-19a		Maj $^{\#2,5,6}$				Dim ^{2,6}	6-31b, 6-Z28*
4-19b			MinMaj ^{2,5,6}, MinMaj $^{\#4,5,6}$				6-Z28*, 6-Z24b
4-20*		Maj $^{\#2,\#5,6}$			Min ^{2,5,6}		6-Z19a, 6-Z25b
4-21*	Maj $^{\#4,\#5}$				Min ^{b2,4,5,6}		6-34b, 6-34a
4-22a		Maj $^{\#4,\#5,6}$			Min ^{b2,4,b6}, Min ^{4,5,6}		6-Z24b, 6-31a, 6-33b
4-22b	Maj $^{b2,\#4,\#5,6}$	Maj $^{2,\#4,6}$			Min ^{b2,4,6}		6-31b, 6-33a, 6-Z24a
4-23*		Maj $^{b2,\#4,6}$				Dim ^{b2,b6}	6-Z47b, 6-Z29*
4-24*		Maj $^{2,\#4,6}$	MinMaj ^{4,5,6}				6-34a, 6-34b
4-26*		Maj ^{2,5,6}	MinMaj $^{\#4,\#5,6}$		Min $^{b2,\#4,6}$		6-32*, 6-27b, 6-27a
4-27a			MinMaj $^{4,\#5,6}$	Dom ^{2,5,6}			6-Z49*, 6-33a
4-27b	Maj $^{2,\#5,6}$			Dom $^{b2,\#4,6}$	Min ^{b2,5,6}		6-Z23*, 6-Z49*, 6-Z24
4-28*				Dom ^{b2,5,6} (4x)			6-27b
4-Z29a	Maj ^{4,5,6}			Dom $^{\#2,\#4,6}$			6-Z26*, 6-30a
4-Z29b			MinMaj ^{4,b5,6}	Dom ^{2,4,6}			6-30b, 6-Z26*

Set classes + m6	Maj	MinMaj	Dom	Min	Sus	Dim	Associated 6-note Set classes
4-12a						Dim ^{4,b6}	6-27b
4-12b		MinMaj ^{2,4,b6}					6-27a
4-13a			Dom ^{4,5,b6}			Dim ^{b4,b6}	6-Z24a, 6-Z28*
4-13b	Maj ^{2,4,b6}		Dom $^{b2,\#2,b6}$				6-Z28*, 6-Z24b
4-14a				Min ^{4,b5,b6}			6-33a
4-14b	Maj $^{\#2,\#4,\#5}$			Min ^{2,4,b6}			6-31a, 6-33b
4-Z15b		MinMaj $^{2,\#5,6}$					6-Z13*
4-16a	Maj $^{4,\#5,6}$	MinMaj $^{4,\#5,\#6}$					6-Z19b, 6-Z47b
4-16b	Maj $^{\#2\#5,6}$						6-Z19a
4-17*		MinMaj $^{2,\#4,\#5}$					6-Z49*
4-18a			Dom $^{\#2,5,b6}$			Dim ^{2,b6}	6-31a, 6-30b
4-18b			Dom ^{b2,4,b6}				6-31b
4-19a	Maj $^{\#2,5,b6}$			Min ^{2,b5b6}			6-20*, 6-34b
4-19b		MinMaj ^{2,5,b6}					6-Z19b
4-20*				Min ^{2,5b6}			6-Z26*
4-22a				Min ^{4,5b6}			6-32*
4-22b	Maj $^{2,\#4,\#5}$			Min ^{b2,4,b6}			6-34a, 6-32*
4-23*	Maj $^{2,\#5,6}$, Maj $^{\#4,\#5,6}$						6-Z24a, 6-Z24b
4-24*		MinMaj ^{4,5,b6}	Aug $^{2,\#4}$				6-31b, 6-35*
4-25*		MinMaj $^{4,\#5,6}$					6-Z49*
4-26*	Maj $^{b2,\#4,\#5}$, Maj ^{2,5,b6}			Min ^{b2,b5,b6}	Sus ^{2,5,b6}		6-31b, 6-31a, 6-33b, 6-33a
4-27a			Dom ^{2,5,b6}		Sus ^{b2,5,b6}	Dim ^{b2,b6}	6-34a, 6-Z25b, 6-Z29*
4-27b		MinMaj $^{\#4,\#5,6}$	Aug $^{b2,\#4}$	Min ^{b2,5,b6}			6-27b, 6-34b, 6-Z25a
4-28*			Dom ^{b2,5,b6} (4x)				6-Z28*
4-Z29a	Maj ^{4,5,b6}		Aug $^{\#2,\#4}$				6-Z19a, 6-34a
4-Z29b		MinMaj $^{4,\#4,\#5}$	Aug ^{2,4}				6-Z50*, 6-34b

Figure 2.6 4(a-d)

However, this collection of charts is not only incomplete in the sense of leaving out eight other bass-soprano pairs, but also in the sense that they do not explicitly account for limitations in the execution of these charts. Two important things to consider:

1. The actual distance between the soprano and bass pair can both limit the potential voicings of the inner tetrad and in some cases, if the bass and soprano are sufficiently close in register (namely under an octave) may preclude that tetrad from even being realized; and
2. Depending on the voicing, some of those described chords may not actually sound like the chords described. The main culprit of the latter problems regard chords whose outer voicings comprise either a Maj or Min 3rd; if a chord contains both a M3/d4 and a A2/m3,

the order in which each interval appears above the bass impacts whether or not that is heard as either Maj7/MinMaj7 or Dom7/Min7. In short then, you have to consider both how you would interpret/classify chords whose outer voicings span either a M3 or m3.

a. Moreover, the actual prescribed voicing of some of the given "familiar" hexads, may make them particularly dissonant, better classified differently, and perhaps, like a major-with-a-4 chord, well suited to also being labeled in red.

Now, to follow up on the aural practice suggestions first introduced when discussing trichords as audible units: "conceive of larger audible units such as tetrads and pentads" as audible refinements of a trichord—defined by an outer dyad and a salient middle pitch. In terms of learning the pentads, I did not present a chart that separated the outer from the inner dyads; instead, the chart only fixed the bass note. This was due to that chapter's focus on learning the different tetrads in their six voicing types. However, a chart could also have been given that did as above, split the outer dyad from the inner trichords. In exploring this topic, it may be of benefit to also consider and/or create such a chart. The looming question, is "at what point does the clutter between the outer dyads become so great (such as a tetrad), or the chord's "color" so refined, that distinguishing similar chords aurally becomes exceedingly difficult, if not impossible.

This is a question that deserves an empirical investigation; one unfortunately, as mentioned earlier, that is beyond the scope of this dissertation. However, here again, I urge you the reader to ask yourself this question as you investigate this material. Even without finding evidence supporting a definitive answer that could apply to some general population, you will discover more about what is perceptible to you and in turn how you best want to shape your own musical development with that in mind. Explorations such as recording yourself fixing the outer pitches and then testing yourself on how well you can discern the inner tetrad may prove helpful. Do you not hear the inner tetrad distinctly, but rather just default to comparing each note of the resultant hexad to the bass? If you do hear it distinctly, can you actually name its voicing classification? Are there certain conditions in which you do hear it distinctly and others in which you don't? In your own "experimental setting," factors such as the specific outer dyad, the chosen range of that outer dyad, instrument used, and specific inner tetrad may also mediate the audibility of the inner tetrad. Furthermore, does the tetrad's distance away from prime form impact its intelligibility? For instance, is it easier to identify an inversion in closed position than an inversion in a specific open position? Either way, unless someone else curates the inner tetrads for you, you will most likely know which of a few tetrads may appear in your test. As such, you should be cautioned against interpreting any findings from doing this as capable of shedding light on more general situations; situations in which either you are not picking between at most a few different voicings, or where the results could apply to anyone but you.

Although, in "non-experimental settings" this is a very different beast. So many factors can influence a hexad's audibility. Factors can include: the speed of its presentation; whether the striking of its pitches are staggered; the specific timbres employed; how symmetrical is its presentation, the general familiarity that you may have with that hexad, how that hexad has been used (or not) in the music up to that point—for instance, in the same voicing or in different voicings; how much the listener expects to identify a chord of that complexity; whether it's clearly segmented in the music; its salience in cognitive and structural terms—e.g., is it extremely loud and the first sound that we hear at the beginning of a work, whether the hall it's being performed in obscures (or even cancels out) some of the partials that are needed to clearly identify the inner voices, etc. Another thing to consider is that, even if you hear a chord clearly and know how to classify chords relatively quickly, achieving this in real time may still require more than a person can immediately process. Upon reflection, they may, if the aural resolution (and memory) is sufficiently good (and intact), be able to name an unexpected chord; but still flounder during the first listen. Especially as their working memory is flooded by following chords that obscure the aural impression of the chord in question.

From my experience, I do not believe that people, generally speaking, can achieve such a feat unless they already have great familiarity with a particular hexad (in a particular voicing as well), can automatically name it (due to previous practice), and on some level expect that that chord may appear. Even then, contingencies related to how it may be presented outside of an experimental setting may make it still "impossible" to decipher. Of course, I would welcome research that could better describe how the various factors interact.

Another equally helpful approach to this aural line of inquiry, pertains to reflections over “how saturated in pitches a chord needs to actually be for one to get their “idea” across. A default assumption of this pedagogy is that the perfect understanding should not be the main objective; instead, a good enough impression should be. As such, it may be helpful to really compare the set classes of different cardinality, measured in outer dyads and inner dyads (or trichords or tetrads), and ask under what conditions does that extra note matter. If you just want a fuller texture, when does just adding a doubling, as opposed to a new pitch class, suffice or even offer an improvement. Of course, the calculation is very different if one is conceiving of these hexads as hexachords!

It can also be helpful to examine the portion of CPE Bach’s treatise on “The True Art of Keyboard Playing” that pertains to realizing figured bass. Extreme care is given to the stylistic and aesthetic issues involved with adding something as innocuous as an extra pitch, that in a 20th century Art Music context may not typically be as closely considered: 4 notes can sound much more cluttered than 3, and depending on the particular realization of the figured bass, doublings and or a 5th, can render proper voice leading impractical or even impossible. While we are not bound to the same stylistic considerations as CPE Bach, realize that the depth of his observations could most likely only have been possible in the context of his writing, researching, such a systematic and exhaustive pedagogy as his or as this one. For instance, in various chapters, like that on the 6th chords, he showed all possible, correctly realized, available progressions/algorithms, that alternated between 6th chords. In addition, throughout the treatise, he engaged the formulas and extensive case studies, in an extremely musical manner—not only commenting on the structural concerns but also larger 18th century musical and cultural concerns. Due to the vastness of this 21st century pedagogy and its pointedly not being based in one of a few genres, if I tried to do something akin to CPE Bach, I would be excluding readers who thought of music and musical practice very differently. Rather, it’s up to you to create your “C.P.E Bach” version of this, one that attends to your aesthetic considerations as well as your larger thoughts on how music that you care abouts works or should work. How systematic, exhaustive, and open-ended this pedagogy is could be a perfect platform from which you can explore these types of finer points.

Nonetheless, returning to the earlier point, while there may be limits on what one can infer from a single listening, there are less limits on what one can aurally imagine. A main strength of pursuing this pedagogy along these “experimental” lines could perhaps be summarized as this, “not only may it help you “aurally imagine” more, more clearly, but it may also one: help you develop a better (not perfect!!) sense of under what conditions what you may imagine could be perceptible to the audience—at least in terms of consciously recognized musical units; and two: provide a platform from which to dig deeper into the inner workings of genres or structural considerations that you are partial to.

6 note chord	5 note subsets	Other Repr./Prime Form	Maj7	Min7	MinMaj7	Dom7	Dim7
6-Z25	Z12*,20,23,25,27,29	Major Scale (7→5)	(**2**:2,4,5)	(**3**:2,4,6),(**4**:b2,5,b6)		(**6**:4,5,6)	
6-Z25B		Major Scale (6→4)	(**3**:2,4,6),(**6**:#4,5,6)	(**2**:b2,4,b5)(**4**:2,5,6)			
6-Z26*	20,24,27	Major Scale (3→1)	(**2**:2,#4,5),(**6**:4,5,6)	(**4**:2,5,b6)		(**3**:2,4,6)	
6-27	10,16,25,31,31,32	Harmonic Minor (6→4)(1st)		(**3**:b2,#4,6),(**5**:#4,5,6)	(**2**:2,4,#5)		(**1**:b2,b4), (**6**:b4,5)
6-27B		Harmonic Major (2→7)			(**3**:2,#4,6),(**5**:#4,#5,6)	(**2**:b2,#2,5),(**4**:b2,5,6)	(**1**:2,4),(**6**:4,#5)
6-Z28*	Z12*,22*,26,31	Harmonic Minor (2→7), Harmonic Major (6→4)(1st)	(**2**:2,4,#5)	(**3**:2,#4,6)	(**5**:#4,5,6)	(**4**:b2,5,b6)	(**1**:b2,4),(**6**:b4,b6)
6-Z29*	18,29,31	Harmonic Minor (4→2)(3rd)	(**6**:#2,#4,6)	(**3**:4,b5,6)		(**5**:b2,4,5)	(**1**:b2,b6),(**4**:2,5)
6-30	19,19,28,28,31,31	**i** ✕ **#iv**(2nd)				(**3,6**:#2,#4,6)	(**1,4**:b2,5)
6-30B		**I**(1st) ✕ **#IV**			(**3,6**:4,b5,6)	(**2,5**:b2,#4,5)	(**1,4**:2,b6)
6-31	Z18,21,26,27,30,32	Harmonic Major (5→3)(2nd)	(**2**:2,5,b6), (**7**:#2,#4,#5)			(**4**:#2,5,b6)	
6-31B		Harmonic Minor (3→1)(4th)	(**6**:#2,5,6)	(**4**:2,#4,5)	(**2**:4,5,b6)	(**5**:b2,4,b6)	
6-32*	23,27,35*	Major Scale (1→6)	(**4**:2,5,6)	(**2**:2,4,5),(**3**:b2,4,b6),(**6**:4,5,b6)			
6-33	23,24,25,29,34*,35*	Major Scale (2→7)	(**3**:2,#4,6)	(**2**:b2,4,5),(**6**:4,b5,b6)		(**4**:2,5,6)	
6-33B		Major Scale (4→2)		(**3**:2,4,b6),(**4**:b2,b5,b6),(**6**:4,5,6)		(**2**:2,4,5)	
6-34	24,26,28,30,33*,34*	Melodic Minor (2→7)	(**2**:2,#4,#5)			(**3**:2,#4,6),(**4**:2,5,b6),(**6**:#2,#4,#5)	
6-34B		Melodic Minor (3→1)		(**4**:2,b5,b6)	(**6**:4,5,6)	(**2**:2,#4,5),(**3**:2,4,b6),(**5**:b2,#4,#5)	
6-35*	33*,33*,33*	Whole Tone	(**1,2,3,4,5,6**:2,#4,#5)				
6-Z36	1*,4,5,11,16,Z36	3-11B(1→5)	(**4**:b2,6,#6)		(**2**:b2,2,#4),(**5**:#5,6,#6)		
6-Z36B		3-11(2→6)			(**3**:b2,2,#5)		
6-Z37*	1*,5,13,17*	3-12*(1→5)	(**5**:#5,6,#6)		(**2**:b2,2,5)		
6-Z38*	5,7,20	4-20*(2nd)(1→4)	(**4**:4,6,#6),(**6**:4,#4,5)				
6-Z39	2,4,13,25,26,Z37*	4-27(2→5)	(**4**:b2,#5,#6)	(**2**:b2,2,b5)	(**5**:5,6,#6)		
6-Z39B		4-27B(2→5)	(**3**:b2,2,#5)		(**4**:b2,5,#6)	(**6**:5,b6,6)	
6-Z40	2,5,25,27,Z36,Z38	4-26*(1→4)	(**2**:b2,2,5)	(**5**:5,b6,6)	(**3**:b2,#4,#6)		
6-Z40B		4-26*(3→6)	(**6**:5,6,#6)	(**3**:b2,2,5)			
6-41	4,5,14,15*,28,29	4-27B(1→4)	(**3**:b2,#4,#6)		(**4**:4,6,#6)	(**6**:4,#4,5)	
6-41B		4-27(3→6)	(**6**:#4,6,#6)			(**2**:#2,4,b5)	
6-Z42*	4,31,Z38	4-28*(1→4)	(**3**:b2,5,#6)		(**4**:#4,6,#6)		(**1**:b2,2),(**5**:5,b6),(**6**:b4,4)
6-Z43	6,15*,Z18,20,28,Z38	5-28B(1→3)	(**2**:b2,4,5),(**3**:#2,#4,#6)			(**6**:4,b5,6)	
6-Z43B		5-28(4→6)	(**3**:#2,4,6),(**6**:#4,5,#6)			(**2**:b2,4,b5)	
6-Z44	6,21,22*,32,Z37*,Z38	5-32B(3rd)(1→3)	(**2**:b2,4,#5),(**3**:#2,5,#6)		(**5**:#4,5,b6)		
6-Z44B		5-32(4th)(4→6)	(**3**:#2,4,#5), (**5**:b2,5,b6)		(**6**:#4,5,#6)		
6-Z45*	8*,31,34*,Z36	5-34*(2→4)		(**5**:b5,b6,6)	(**3**:b2,#4,6)	(**2**:b2,2,5)	(**1**:2,b4),(**6**:4,5)
6-Z46	9,11,27,32,34*,Z38	5-34*(1→3)	(**3**:2,5,#6)	(**5**:#4,5,b6)	(**2**:b2,4,#5)		
6-Z46B		5-34*(3→5)	(**4**:b2,5,6)		(**5**:#4,#5,#6)	(**2**:2,#2,5)	
6-Z47	11,14,29,32,35*,Z36	5-35*(1→3)		(**4**:4,b6,6)	(**2**:b2,#4,#5)	(**6**:#2,4,5)	
6-Z47B		5-35*(2→4)	(**3**:b2,#4,6)	(**6**:4,#4,5)	(**4**:4,#5,#6)		
6-Z48*	14,30,35*,Z37*	5-35*(3rd)(1→3)	(**2**:b2,#4,#5)		(**3**:4,5,#6)	(**6**:#2,4,b6)	
6-Z49*	16,28,32	**i** ✕ **VI**(1st)			(**2**:2,#4,#5),(**4**:4,#5,6)	(**3**:b2,#4,6),(**6**:#2,#4,5)	
6-Z50*	19,25,32	**I** ✕ #iv(2nd)		(**4**:b2,#4,5)	(**2**:4,b5,b6)	(**6**:#2,5,6)	

6-Z25a[0,1,3,5,6,8]	**6-Z25b**[0,2,3,5,7,8]	**6-Z26***[0,1,3,5,7,8]	**6-27a**[0,1,3,4,6,9]	**6-27b**[0,2,3,5,6,9]	**6-Z28***[0,1,3,5,6,9]	**6-Z29***[0,1,3,6,8,9]	**6-30a**[0,1,3,6,7,9]
6-30b[0,2,3,6,8,9]	**6-31a**[0,1,3,5,8,9]	**6-31b**[0,1,4,6,8,9]	**6-32***[0,2,4,5,7,9]	**6-33a**[0,2,3,5,7,9]	**6-33b**[0,2,4,6,7,9]	**6-34a**[0,1,3,5,7,9]	**6-34b**[0,2,4,6,8,9]
6-35*[0,2,4,6,8,t]	**6-Z36a**[0,1,2,3,4,7]	**6-Z36b**[0,3,4,5,6,7]	**6-Z37***[0,1,2,3,4,8]	**6-Z38***[0,1,2,3,7,8]	**6-Z39a**[0,2,3,4,5,8]	**6-Z39b**[0,3,4,5,6,8]	**6-Z40a**[0,1,2,3,5,8]
6-Z40b[0,3,5,6,7,8]	**6-Z41a**[0,1,2,3,6,8]	**6-Z41b**[0,2,5,6,7,8]	**6-Z42***[0,1,2,3,6,9]	**6-Z43a**[0,1,2,5,6,8]	**6-Z43b**[0,2,3,6,7,8]	**6-Z44a**[0,1,2,5,6,9]	**6-Z44b**[0,1,2,5,8,9]**
6-Z45*[0,2,3,4,6,9]	**6-Z46a**[0,1,2,4,6,9]	**6-Z46b**[0,2,4,5,6,9]	**6-Z47a**[0,1,2,4,7,9]	**6-Z47b**[0,2,3,4,7,9]	**6-Z48***[0,1,2,5,7,9]	**6-Z49***[0,1,3,4,7,9]	**6-Z50***[0,1,4,6,7,9]

Figure 6 A

6 note chord	5 note subsets	Other Repr./Prime Form	Maj7	Min7	MinMaj7	Dom7	Dim7
6-Z25	Z12*,20,23,25,27,29	Major Scale (7→5)	(2:2,4,5)	(3:2,4,6),(4:b2,5,b6)		(6:4,5,6)	
6-Z25B		Major Scale (6→4)	(3:2,4,6),(6:#4,5,6)	(2:b2,4,b5)(4:2,5,6)			
6-Z26*	20,24,27	Major Scale (3→1)	(2:2,#4,5),(6:4,5,6)	(4:2,5,b6)		(3:2,4,6)	
6-27	10,16,25,31,31,32	Harmonic Minor (6→4)(1st)		(3:b2,#4,6),(5:#4,5,6)	(2:2,4,#5)		(1:b2,b4), (6:b4,5)
6-27B		Harmonic Major (2→7)			(3:2,#4,6),(5:#4,#5,6)	(2:b2,#2,5),(4:b2,5,6)	(1:2,4),(6:4,#5)
6-Z28*	Z12*,22*,26,31	Harmonic Minor (2→7), Harmonic Major (6→4)(1st)	(2:2,4,#5)	(3:2,#4,6)	(5:#4,5,6)	(4:b2,5,b6)	(1:b2,4),(6:b4,b6)
6-Z29*	18,29,31	Harmonic Minor (4→2)(3rd)	(6:#2,#4,6)	(3:4,b5,6)		(5:b2,4,5)	(1:b2,b6),(4:2,5)
6-30	19,19,28,28,31,31	i × #iv(2nd)				(3,6:#2,#4,6)	(1,4:b2,5)
6-30B		I(1st) × #IV			(3,6:4,b5,6)	(2,5:b2,#4,5)	(1,4:2,b6)
6-31	Z18,21,26,27,30,32	Harmonic Major (5→3)(2nd)	(2:2,5,b6), (7:#2,#4,#5)			(4:#2,5,b6)	
6-31B		Harmonic Minor (3→1)(4th)	(6:#2,5,6)	(4:2,#4,5)	(2:4,5,b6)	(5:b2,4,b6)	
6-32*	23,27,35*	Major Scale (1→6)	(4:2,5,6)	(2:2,4,5),(3:b2,4,b6),(6:4,5,b6)			
6-33	23,24,25,29,34*,35*	Major Scale (2→7)	(3:2,#4,6)	(2:b2,4,5),(6:4,b5,b6)		(4:2,5,6)	
6-33B		Major Scale (4→2)		(3:2,4,b6),(4:b2,b5,b6),(6:4,5,6)		(2:2,4,5)	
6-34	24,26,28,30,33*,34*	Melodic Minor (2→7)	(2:2,#4,#5)			(3:2,#4,6),(4:2,5,b6),(6:#2,#4,#5)	
6-34B		Melodic Minor (3→1)		(4:2,b5,b6)	(6:4,5,6)	(2:2,#4,5),(3:2,4,b6),(5:b2,#4,#5)	
6-35*	33*,33*,33*	Whole Tone	(1,2,3,4,5,6:2,#4,#5)				
6-Z36	1*,4,5,11,16,Z36	3-11B(1→5)	(4:b2,6,#6)		(2:b2,2,#4),(5:#5,6,#6)		
6-Z36B		3-11(2→6)			(3:b2,2,#5)		
6-Z37*	1*,5,13,17*	3-12*(1→5)	(5:#5,6,#6)		(2:b2,2,5)		
6-Z38*	5,7,20	4-20*(2nd)(1→4)	(4:4,6,#6),(6:4,#4,5)				
6-Z39	2,4,13,25,26,Z37*	4-27(2→5)	(4:b2,#5,#6)	(2:b2,2,b5)	(5:5,6,#6)		
6-Z39B		4-27B(2→5)	(3:b2,2,#5)		(4:b2,5,#6)	(6:5,b6,6)	
6-Z40	2,5,25,27,Z36,Z38	4-26*(1→4)	(2:b2,2,5)	(5:5,b6,6)	(3:b2,#4,#6)		
6-Z40B		4-26*(3→6)	(6:5,6,#6)	(3:b2,2,5)			
6-41	4,5,14,15*,28,29	4-27B(1→4)	(3:b2,#4,#6)		(4:4,6,#6)	(6:4,#4,5)	
6-41B		4-27(3→6)	(6:#4,6,#6)			(2:#2,4,b5)	
6-Z42*	4,31,Z38	4-28*(1→4)	(3:b2,5,#6)		(4:#4,6,#6)		(1:b2,2),(5:5,b6),(6:b4,4)
6-Z43	6,15*,Z18,20,28,Z38	5-28B(1→3)	(2:b2,4,5),(3:#2,#4,#6)			(6:4,b5,6)	
6-Z43B		5-28(4→6)	(3:#2,4,6),(6:#4,5,#6)			(2:b2,4,b5)	
6-Z44	6,21,22*,32,Z37*,Z38	5-32B(3rd)(1→3)	(2:b2,4,#5),(3:#2,5,#6)		(5:#4,5,b6)		
6-Z44B		5-32(4th)(4→6)	(3:#2,4,#5), (5:b2,5,b6)		(6:#4,5,#6)		
6-Z45*	8*,31,34*,Z36	5-34*(2→4)		(5:b5,b6,6)	(3:b2,#4,6)	(2:b2,2,5)	(1:2,b4),(6:4,5)
6-Z46	9,11,27,32,34*,Z38	5-34*(1→3)	(3:2,5,#6)	(5:#4,5,b6)	(2:b2,4,#5)		
6-Z46B		5-34*(3→5)	(4:b2,5,6)		(5:#4,#5,#6)	(2:2,#2,5)	
6-Z47	11,14,29,32,35*,Z36	5-35*(1→3)		(4:4,b6,6)	(2:b2,#4,#5)	(6:#2,4,5)	
6-Z47B		5-35*(2→4)	(3:b2,#4,6)	(6:4,#4,5)	(4:4,#5,#6)		
6-Z48*	14,30,35*,Z37*	5-35*(3rd)(1→3)	(2:b2,#4,#5)		(3:4,5,#6)	(6:#2,4,b6)	
6-Z49*	16,28,32	i × VI(1st)			(2:2,#4,#5),(4:4,#5,6)	(3:b2,#4,6),(6:#2,#4,5)	
6-Z50*	19,25,32	I × #iv(2nd)		(4:b2,#4,5)	(2:4,b5,b6)	(6:#2,5,6)	

6-Z25a[0,1,3,5,6,8]	**6-Z25b**[0,2,3,5,7,8]	**6-Z26***[0,1,3,5,7,8]	**6-27a**[0,1,3,4,6,9]	**6-27b**[0,2,3,5,6,9]	**6-Z28***[0,1,3,5,6,9]	**6-Z29***[0,1,3,6,8,9]	**6-30a**[0,1,3,6,7,9]
6-30b[0,2,3,6,8,9]	**6-31a**[0,1,3,5,8,9]	**6-31b**[0,1,4,6,8,9]	**6-32***[0,2,4,5,7,9]	**6-33a**[0,2,3,5,7,9]	**6-33b**[0,2,4,6,7,9]	**6-34a**[0,1,3,5,7,9]	**6-34b**[0,2,4,6,8,9]
6-35*[0,2,4,6,8,t]	**6-Z36a**[0,1,2,3,4,7]	**6-Z36b**[0,3,4,5,6,7]	**6-Z37***[0,1,2,3,4,8]	**6-Z38***[0,1,2,3,7,8]	**6-Z39a**[0,2,3,4,5,8]	**6-Z39b**[0,3,4,5,6,8]	**6-Z40a**[0,1,2,3,5,8]
6-Z40b[0,3,5,6,7,8]	**6-Z41a**[0,1,2,3,6,8]	**6-Z41b**[0,2,5,6,7,8]	**6-Z42***[0,1,2,3,6,9]	**6-Z43a**[0,1,2,5,6,8]	**6-Z43b**[0,2,3,6,7,8]	**6-Z44a**[0,1,2,5,6,9]	**6-Z44b**[0,1,2,5,8,9]**
6-Z45*[0,2,3,4,6,9]	**6-Z46a**[0,1,2,4,6,9]	**6-Z46b**[0,2,4,5,6,9]	**6-Z47a**[0,1,2,4,7,9]	**6-Z47b**[0,2,3,4,7,9]	**6-Z48***[0,1,2,5,7,9]	**6-Z49***[0,1,3,4,7,9]	**6-Z50***[0,1,4,6,7,9]

Figure 6 B

PART III: CASE STUDIES

Part III, Chapter 15: Building your own Maps

Part III leverages the skills and knowledge acquired through part II; simplifying the learning of various musical concepts and quickening the performance/learning of associated tasks. For instance, 6-35*, the whole-tone scale, can refer to a scale, a function that shifts pitch-classes by a whole step, and its prime form [0,2,4,6,8,t], or the range of a function, whose domain is pitch-classes, and range, 12-tone pitch-classes multiplied by two.

This chapter will explore this last consideration, using set classes as maps toward 1) understanding how various set classes map to one another under different transformations; and 2) as an aid towards understanding how set classes interrelate as a whole. In line with this pedagogy's overall philosophy, the goal is not to achieve a definitive definition of how 12-tone space is organized but instead a working definition; one that can inform a musician about "best available options" in a scenario they individually might typically face rather than an objective, but not immediately relatable, understanding of musical space.

In a similar fashion, Chapter 20 is geared towards acquiring a better "working" understanding of musical space; however, this time, space is construed as networks of patterns, which are conceptualized as either notes or fingerings/hand positions. However, chapters 21 and 22 change gear. Instead, they focus on using set classes as a way to relate similar concepts that are encountered in different musical domains. Chapter 21 focuses on various theorists' usages of numbers and brackets to describe rhythmic patterns and chapter 22 generalizes a learning strategy that I associate with a work of Monteverdi and the writings of Biancheri. Chapter 23 provides a conclusion to this dissertation.

What are some effective ways to build maps of musical space that are tailored to your musical interests?

Start with your favorite chords.

For some, this may be easy to do; one may already have preferred chords that have become familiar through repeated usage when composing and/or improvising. Alternatively, there may be a chord, such as the set class associated with the mystic chord, 6-34*, that for whatever reason is considered intriguing; subsequently prompting the reviewer to explore it in greater depth. Whether familiarity, your appreciation of how it's typically used — let's say as a signifier of mystery, or how pretty you deem it is, matters not. What does matter is that you care enough about it to want to do a few things: one, internalize it in its various transpositions, intervallic inversions, useful voicings, etc.; two, consider doing the same with some of its subsets/supersets; and three, relate it to other chords that you care about.

For example, let's start with 3-11; the major and minor triad.

- Its subsets are 2-3*, 2-4*, and 2-5*.
- The composition of major triads with major triads yields:
  - 3-11b
  - 6-Z19b
  - 6-33b
  - 5-32b
  - 5-21b
  - 5-27a
  - 6-30b
- The composition of minor triads with minor triads yields:
  - 3-11a
  - 6-Z19a
  - 6-33a
  - 5-32a
  - 5-21a
  - 5-27b
  - 6-30a
- The composition of major and minor triads yields:
  - 4-17*

- 6-Z26*
- 6-Z29*
- 4-26*
- 6-20*
- 5-34*
- 6-Z50*
- 5-Z17*
- 4-20*
- 6-Z49*
- 6-32*
- 5-22*

At this initial stage of analysis, I would look at the familiar supersets that contain most of these subsets.

8-28* contains: 6-Z50, 6-Z49, 6-30, 4-26*
7-35* contains: 6-33, 6-32*, 6-Z26*, 5-34*, 5-27, 4-26*, 4-20*.
7-32 contains: 6-Z50, 6-Z29, 6-Z19, 5-32, 5-22*, 4-26*, 4-20*.
6-20* contains: 5-21, 4-20*, 4-17*

However, 3-11 is so familiar and widely used that I would complement the above search with a literature search. Neo-Riemannian theory is a good first stop. It examines how major and minor triads (and less so, other chords) are related via parsimonious voice leading motion — namely L, P, and R. In the context of Neo-Riemannian theory, parsimony can refer to a single voice moving by half- or whole-step.

- Under L (*Leittonwechsel)*, when the starting triad is in Major, the root of the triad moves down by half-step. E.g., C major -> E minor. On the other hand, under L, when the starting triad is in minor, the 5th of the triad moves up by half-step. E.g., C minor -> Ab major.

- Under P (*Parallel*), when the starting triad is in Major, the 3rd of the triad moves down by half-step. On the other hand, under P, when the starting triad is in minor, the 3rd of the triad moves up by half-step.
- Under R (*Relative*), when the starting triad is in Major, the 5th of the triad moves up by whole-step. On the other hand, under R, when the starting triad is in minor, the root of the triad moves down by whole-step.

The transformations can also be visualized on a *Tonnetz* shown below. On a *Tonnetz*, each triad is represented by either an upward, major, or downward, minor, -facing, triangle. Each vertex of the triangle is a pitch in the triad and each side of the triangle, a dyad of a certain interval, respectively: a minor 3rd (L) a P5th (P), and a major 3rd I.

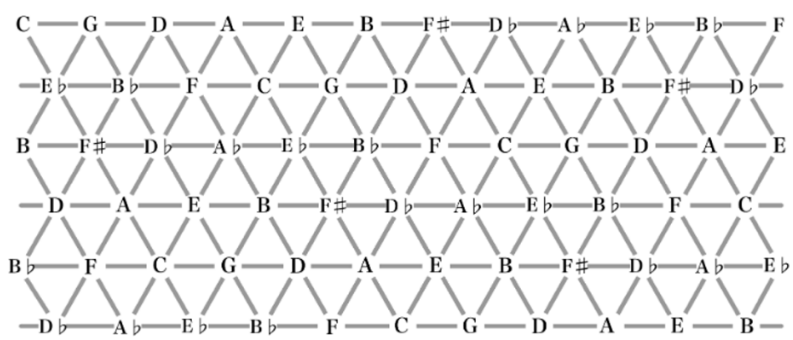
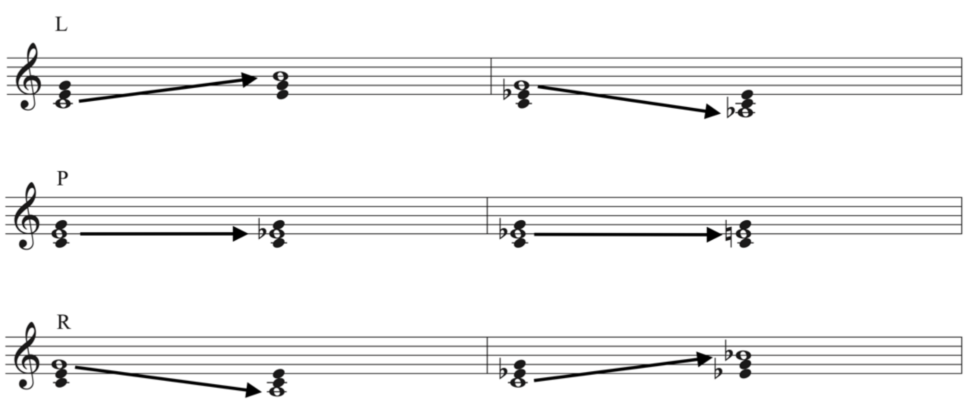


Figure 3.1 1 — a representation of the Tonnetz

- Under L
  - In major, L fixes the minor 3rd dyad (the 3rd and 5th of the starting triad), and then reflects the root over it.
    - For instance, when starting with a C major triad, reflecting C, the root, over the fixed minor 3rd dyad, E and G, yields E minor, E-G-B.
  - In minor, L fixes the minor 3rd dyad (the root and 3rd of the starting triad), and then reflects the 5th over it.
    - For instance, when starting with a C minor triad, reflecting G, the 5th, over the fixed minor 3rd dyad, C and Eb, yields Ab major, Ab-C-Eb.

- Under P
  - In major, P fixes the P5th dyad (the root and 5th of the starting triad), and then reflects the major 3rd over it.
    - For instance, when starting with a C major triad, reflecting E, the 3rd, over the fixed P5th dyad, C and G, yields C minor, C-Eb-G.
  - In minor, P fixes the P5 dyad (the root and 5th of the starting triad), and then reflects the minor 3rd over it.
    - For instance, when starting with a C minor triad, reflecting Eb, the minor 3rd, over the fixed P5 dyad, C and G, yields C major, C-E-G.
- Under R
  - In major, R fixes the major 3rd dyad (the root and 3rd of the starting triad), and then reflects the 5th over it.
    - For instance, when starting with a C major triad, reflecting G, the 5th, over the fixed major 3rd dyad, C and E, yields A minor, A-C-E.
  - In minor, R fixes the major 3rd dyad (the 3rd and 5th of the starting triad), and then reflects the root over it.
    - For instance, when starting with a C minor triad, reflecting C, the root, over the fixed major 3rd dyad, Eb and G, yields Eb major, Eb-G-Bb.



The filled in noteheads refer to pitches that are reflected about. The unfilled noteheads refer to the reflected pitches.

Figure 3.1 2

For our purposes, there is a greater advantage to using the latter geometric interpretation of the L P R transformations. Cohn has classified all potential combinations of L, P, and R. Take LP, (read left to right), one such potential combination of LPR, it transforms a major triad under L first and then under P. Applying LP to a C major triad yields E major.

In short, the total set of pitch-class sets resulting from repeated iterations of combinations of LPR—such as (LP)(LP) etc.—eventually yield symmetrical scales (and sometimes maximally even) scales. For instance, the combination of: C major, LP(C major) = E major, and LP (E major) = {C,E,G,E,G#,B,Ab,C,Eb} or 6-20*. There just needs to be enough repetitions. Interpreted in the preceding manner, combining the results of 3 iterations of $(RP)^0$, $(RP)^1$, $(RP)^2$, and $(RP)^3$ yields 8-28*, and 12 iterations of (LR) = 12-1*.

While the last may seem trivial, the first two cycle-generated sets, 6-20* and 8-28*, are not only prominently referred to in many areas of NR literature (Hexatonic and Octatonic cycles, etc.), but their derivation through Neo-Riemannian theory sheds additional light on their appearance in the list of 4 supersets (and their subsets) given above that arise from different combinations of major and minor triads. In short, components of the major and minor triads, under repetition (the minor 3rd, major 3rd, and P5th) yield maximally even scales (respectively, the augmented triad, the diminished 7th chord and the aggregate). 6-20* and 8-28* are just combinations of these maximally even set classes (3-12* and 4-28*), separated by half-step. As detailed earlier, Hanson's work was based on deriving all possible set classes from combination of smaller set classes. His results would also be a good source to turn to here.

Regarding 7-35*, scalar theory extending back to Vogler and Weber may also serve as a guide. Looking past their grounding of their constructions of these triads in the overtone series and the tonal analytical implications, you find that they associate a triad with all but the 7th degree of the major scale. Built on scale degrees 1, 4, and 5 you have a major triad; scale degrees 2,3, and 6, a minor triad. With such a density of triads, the most of any 7-note scale, it is reasonable that it would be a container for so many combinations of major and minor triads. 7-34* and 7-32 are other expected containers for many combinations of major and minor triads.

Then, due to their being containers for so many combinations of major and minor triads, and/or their ability to shed light on the properties of combinations of major/minor triads, I would consider the above supersets as optimal maps for better understanding triads, how they can combine and transform into each. The *Tonnetz* itself could just be seen as a 3D projection of combinations of a triad's constituent intervals, 3-12*, 4-28*, and 12-1* onto a 2-dimensional plane.

- The P5 axis is horizontal;
- The min 3 SW/NE;
  - In major, you reflect over the SW/NE axis to the right;
  - In minor, the SW/NE axis to the left.
- The major 3rd axis is NW/SE;
  - In major, you reflect over the NW/SE axis to the left;
  - In minor, the NW/SE axis to the right.

Will just practicing combining triads allow you to gain all of the insight forged by many theorists over centuries? Most likely no. How these chords combine and their signification is bigger than understanding any one map of them. Furthermore, the musical compositions themselves are a great source for getting more insight into the significance of these combinations. However, once you realize that there are larger maps that can help you understand the interaction of one or more pitch-class sets, you can use them to your advantage. Remember, you can never have a complete understanding of any chord and its signification musically (and extra musically). However, you can have a greatly enriched understanding; as this demonstration is intended to point out.

So, what are some examples of ways you can use one of these maps to facilitate your learning of the relations involving set class 3-11?

1.) If you are interested in projecting tonal centers and 18th century harmonic sensibilities, consider applying CPE Bach's entire section on triads in *Versuch* (pg.199-233) to bass lines associated with 7-35*. Depending on how open you are to moving between different tonal centers, consider preserving the figure CPE Bach discusses while using the other 3 set classes as material for your bass line.

a. In these chapters, CPE Bach urges his reader to memorize all 24 major and minor triads. He then offers advice on: best doubling practices, preferable (technically and aesthetically) ways to realize figured bass figures, sequences of two triad progressions that preserve the figuring but whose actual triads changed pitch-wise as different portions of the major scale are traversed (these may not work on bass lines derived from other set classes!); and finally, the proper usage of triads, in their various inversions, around cadences.

2.) Refer to improvisation or instrumental manuals that ask you to practice triads in the form of interlocking arpeggios such as C-E-G (ascending), A-E-C# (descending), D-F-A (ascending), B-F#-D# (descending) etc. In the following sequence of triads, the roots of the ascending triads typically outline a scale in focus. At least hundreds, if not thousands, of these types of arpeggios abound in manuals focused on Jazz and Classical repertoire. For instance, for those choosing to outline an octatonic scale, they may take a nod from Neo Riemannian literature and perform a sequence of the transformation RP: C-E-G (ascending), C-A-E, (descending), A-C#-E (ascending), A-F#-C# (descending), F#-A#-C# (ascending), Gb-Eb-Bb (descending), Eb-G-Bb (ascending), Eb-C-G (descending), and then back to C-E-G.

Again though, if a comprehensive practice of all possible combinations is your goal, you may never get to the end of it. Instead, I recommend just practicing these musical exercises in a manner that best helps you accomplish specific musical goals (not just general knowledge acquisition) that you may have.

Why did I choose 3-11 rather than a less familiar set class to demonstrate the power of maps? It's because there is a lot of writing on major/minor triads and they are more relatable. While triads do exhibit special properties that are shared by no other set classes—part of this has to do with their being parsimonious, by half step to both the augmented triad and diminished 7^{th} chord, the biggest reason simply had to do with their familiarity. Had I chosen another set class (or set of set classes) the scope of using such maps may have been harder to grasp; there are fewer musical examples to draw from and many components of those maps may also be unfamiliar.

However, a working knowledge of set classes and how to navigate them subsists of more than just focusing on their internal structure (how they compose with themselves, as we considered above); it consists of understanding on how they relate to other set classes. Nonetheless, this is, once again, where the floodgates truly open as 3-11; while it only relates to 5 chords parsimoniously (other than itself), those 5 chords also relate to numerous chords parsimoniously; some of those are related to 3-11 parsimoniously, others are not. While this list may still feel manageable when dealing with the trichords, since there are only 19 different trichords (12 different set classes), it quickly becomes unmanageable as the cardinality of the set classes starts to increase. Here is where the map building becomes less exact, and building an intuitive sense around how to navigate them becomes more important.

Accordingly, at this next stage, I would look at the set classes that are parsimonious to 3-11: 3-8, 3-9*, 3-10*, 3-11 and 3-12*.[82] In many tonal genres, only 3-11 is considered stable. Depending on your thoughts on this though, you could either explore the “unstable” parsimonious pathways between the stable 3-11 and itself, or you could let other chords into the “stable” club and diminish the number of “unstable” parsimonious pathways. Here are a couple of examples of such pathways:

1. A short one: 3-11b — 3-12* — 3-11a
2. A longer one: 3-11a — 3-8b —3-9* — 3-8a — 3-10* — 3-11b

You could then look at 3-11’s minimal supersets: 4-14, 4-17*, 4-18, 4-19, 4-20*, 4-22, 4-26*, 4-27, and 4-29. Again, all but 4-17* and 4-19 come from 7-35*. 4-17* and 4-19 also come from 6-20* and 8-28*.

You can derive the minimal supersets by keeping the triad fixed and adding a note.

1. C major + C = 3-11b

[82] In essence, this is just looking at the “pockets” of the trichords. However, as we do not yet have enough notes to bundle the trichords into Maj, MinMaj, Dom, and Min categories put at least one note, I am not using the term *pockets* here.

2. C major + Db = 4-18a
3. C major + D = 4-22a
4. C major + Eb = 4-17*
5. C major + E = 3-11b
6. C major + F = 4-14b
7. C major + F# = 4-Z29b
8. C major + G = 3-11b
9. C major + Ab = 4-19b
10. C major + A = 4-26*
11. C major + Bb = 4-27b
12. C major + B = 4-20*

Depending on your interest, you could:

- Explore how the tetrads interact parsimoniously in the context of a superset such as 7-35*.
- Focus on just those tetrads that are typically used in tonal music: 4-20* (Major 7th), 4-26* (Minor 7th), 4-27 (Dominant and Half-Dim 7th), and 4-28* (Diminished 7th).
- Explore the parsimonious pathways between and the pockets associated with your focus on tetrads.

However, there are more ways to relate set classes than through parsimony; we can also adopt one or more similarity measures to aid us in this regard. Again, this opens up an even greater Pandora's box of possibility. As discussed in Part 1, similarity measures typically either quantify how similar two set classes are (e.g., Lewin's EMB, INJ, REL, REL_2)—often requiring that the compared sets are of the same cardinality (e.g., Morris's SIM), or, instead, categorically make such decisions (Rp) (either the two tetrads share a subset of cardinality n-1 or they don't). Furthermore, as Schuijer pointed out, many of the similarity measures regard either similarity in interval vector or in shared subsets.

For instances, REL relates the similarity between two set classes on a scale between 0 and 1—0 is minimally similar. Depending on the measure adopted, you could create pathways between set classes that are similar to at least some degree. In many of these cases, the pathways may not be parsimonious. Take C, Db, and E (3-3a) and 3-10*; they both share a minor 3rd. Hence, they are Rp related. However, there is no transposition of 3-10* that is parsimonious (by semitone motion) to 3-3a. Again, there are too many candidates to exhaustively explore everything. However, coming to your own conclusions about the benefits of different similarity measures, instantiated in various interconnected set class pathways, can provide insight into how your focus on a trichord (and or its substantiation in tetrads) relates more generally to other chords. It can enrich your understanding of that trichord immensely, and expand your notion of similarity.

Ultimately, engaging various similarity measures in your practice may also encourage you to fully embrace the notion that, if there were an exhaustive map (of all maps) of 3-11, it would be a fuzzy map. Greater detail in such maps could only be acquired through unique choices made around how to draw them; preferably, those choices align with your more general interests. Fortunately, however, the burden of learning how changes in similarity measures can change your map's layout does lessen as you learn more set classes and continue to apply these measures. You build specific knowledge around how similar certain set classes are under certain types of measurements and a continually more refined notion of how they are related in regard to similar set classes that you might focus on.

Finally, I will briefly mention that you can build an intuition around set class maps through an analogous building of maps in non-pitch domains; remember, that numbers and brackets can refer to many types of objects. Pitches are only one of them. I'll return to this topic in more depth in the following few chapters.

Part III, Chapter 16: Coordinating Maps of your Instrument with Maps of Set Class Space

This chapter develops an idea assumed by many instrumental manuals, such as Hanon's book of piano exercises and Istvan Thoman's 6-volume compendium of practice patterns, that mastering those manuals will prepare you for any technical difficulties that you may encounter in the related literature. Often times these patterns are derived through systematic measures. For instance, demonstrating all potential fingerings that can enable a trill. On the piano, this could mean first alternating, in sixteenth notes, between fingers 1 and 2, then 1 and 3, 1 and 4, 1 and 5, 2 and 3 etc. At other times, they are distillations of famous tricky passages encountered in the canon. Depending on the focus of the set of exercises, e.g., pedal work or bow work, these pieces can look more like etudes or at least passages of music than simple finger patterns.

This chapter challenges assumptions that equates understanding something with a more formal understanding of that thing—it's an instance of more general principles. Here is an interesting parallel consideration today.

Upon command, the most recent versions of ChatGPT can render poems, whose structure and seeming inventiveness, seem to be at a higher level than what professional teachers can elicit (using general principles as guides) from their students. Even if professional wordsmiths could craft poems as well written as some of those created by ChatGPT, they definitely could not do it in the same amount of time—near instantaneously. Does ChatGPT understand what it does? Definitely not as a human would understand it. However, ChatGPT has passed the Turing test—"a test of a machine's ability to exhibit intelligent behaviour equivalent to, or indistinguishable from, that of a human.[83]"

In an analogous way, one could compare students' mastery of some of these more extensive compendiums as "training" on the technical (and even musical) building blocks—filtered by instrument studied or Genre. The hypothesis being that, for those who successfully finish the set of exercises, scores previously thought impenetrable could then be easy to learn and quickly

[83].“Turing Test,” Wikipedia.2023. Last modified August 3. 2023.https://en.wikipedia.org/wiki/Turing_test.

realized. Where ChatGPT is trained on zillions of data points, here the human learner is trained on data that been selected by experts and is intended to be representative of all (or at least most of the challenges) to be faced. Can mastery of these technical exercises be considered a reasonable substitute for mastery of more general principles?

While I can't imagine anyone unreservedly answering yes to this question, unreservedly answering no also is problematic. There is a tension in teaching between focusing on technique vs. focusing on general principles. Two questions follow:

1. To what degree can concerns around physical technique even be separated from concerns around interpretation? And,
2. If "technical" mastery is not considered a complete substitute for teaching arising out of general musically-informed principles, how much can it be?

Point 2 could be mediated by starting skill level. If some person has some amount of contextual background, could the difference between where they are at and mastery be made up through technical work?

Sets/Manuals of technical exercises differ in how tight their organization is, how thoroughly they cover problems arising in the extant literature, the skill level of the intended reader, and how much they center a long list of technical challenges around certain basic principles —infusing these principles into every aspect of the organization and technical challenges presented. Yet, how successful were these various technical manuals? This is both a music theoretical and musicological question. Nonetheless, answering this question in the context of a single manual is beyond the scope of this pedagogy. Plus, as far as I know, there are not any generally accepted frameworks for making such a nebulous evaluation.

For instance, an interesting outstanding question is, could you train a computer on one or more of these manuals and then see if it could predict reasonable fingerings for contemporaneous scores? Fingerings offered in various printed editions of the same work could be used as points of reference.

Furthermore, different manuals can be organized around different or conflicting principles. Is finger strength or the coordination of larger muscles the core concern underlying successful execution? Does it make sense to develop a universal technique that can be easily adapted to virtually all situations? Or, should the specifics of individual pieces of music determine the technical approach, even when these specifics are at odds with a more comfortable execution? An extreme version of this manner of learning is seen when musicians attempt to adopt the exact body positions of their favorite performers in the execution of the same piece that they are studying. The assumption here is that to achieve the best sound/interpretations you must adopt the specific techniques of those who you presume can best perform the piece. While this is reasonably attemptable for many instrumentalists; this is a much taller order for singers. Lastly, some manuals even focus on compositional technique in addition to physical technique. Is there something fundamentally different about how one should approach the playing of the same passage work in the context of a fugue as opposed to a fantasia?

While answering these questions again are beyond the purview of this dissertation, it's important to raise these issues more generally for two main reasons: one, it's a reminder that even if this pedagogy is highly technical, it must be engaged in a holistic manner—there should be a running dialogue between your more general musical wants, how you engage the material, and what you expect to gain from it; and two, to emphasize that there are limitations in this approach—even though it's a very helpful strategy for many, I am not an advocate of selling unabashed rote work as a solitary means to greater understanding; no matter how well that rote work is curated.

Lastly, before offering suggestions on how to make use of this approach, I would like to briefly compare Istvan Thoman's six-volume piano-technical manual with his student Erno Dohnanyi's short one-volume piano-technical work. The four following points will act as a guide towards evaluating practice strategies that are later suggested.

1. The former presents each exercise in all twelve transpositions; the latter in just those transpositions whose landscape on the piano are sufficiently different to warrant additional attention;

    - The former, due to its presenting each exercise in all twelve transpositions, centers the importance of recognizing this passagework in all twelve transpositions; the latter just centers the physical challenge of the specific task.
    - The former doesn't emphasize any difference in the execution of the same passage work in different transpositions; the latter does.
2. The former, due to its length, can accommodate students who need time for their bodies to adjust to the rising technical difficulties of the exercises; the latter is so short and intense that without explicit guidance or extensive experience as a professional it seems more likely to result in injuries.
3. The former, due to its exceedingly long length, projects thoroughness and completeness—indeed it is a potpourri of many different technical collections — from Czerny's many volumes and Liszt's grab bag of exercises to Clementi's *Gradus et Parnassum*; the latter projects a greater discernment around the single most important set of exercises.
4. The former assorts the various exercises by the larger principles of closed hand position (5 finger exercises), scales (closed hand position and lateral motion), arpeggios (open hand position and lateral movement), octaves, double thirds and sixths, and arpeggios; the latter treats the exercises as embodying the core technical concern rather than just being one of many examples of concerns that extend beyond that exercise.

Whereas my last chapter focused on building maps through greater knowledge of set classes and how they relate, this chapter focuses on building maps through the acquisition of patterns, often conceived in terms of fingerings, that can then be applied to a wide range of scenarios. The two main approaches that drive this chapter pertain to: one, voicing classifications—first introduced in regard to tetrads and then expanded to account for pentad subsets of septads and finally, hexads; and two, focusing on those supersets whose acquisition best facilitates the learning of the desired set classes of a lower cardinality. As a note, Clough and Douthett's distinction between d-clens and c-clens (generic and specific intervals) proves to be particularly relevant in this chapter, which discusses different musical objectives first and then related approaches second.

Gaining greater familiarity with a given piece's language.

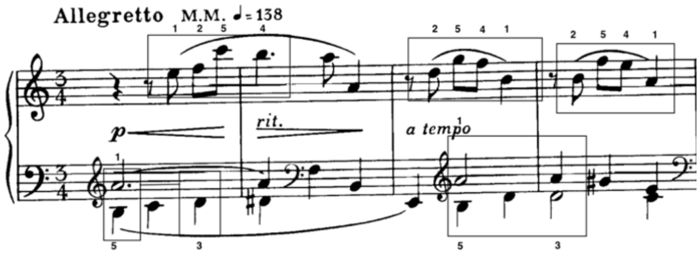


Figure 3.2 1

In the example above (Figure 3.2.1), I have highlighted two repeated finger patterns presented in the opening of Scriabin's 2^{nd} prelude opus 11. This piece is in A minor. In this prelude, as in most others from this set, Scriabin repeats the opening figure at different transposition levels. The first time this occurs is a P5 higher in bar 5; there is a return to the original key at bar 17. At bar 21, this melodic fragment appears down a half-step. The melody appears again in the original at bar 49. The last, a P4 above the original key, occurs at bar 53.

The following two excerpts Figure 3.2.2(a-b) from the RH (in treble clef) at bars 38-39 and 40-41 show how this fingering can anchor a melodic fragment, providing a frame more so than just an exact instance.

Figure 3.2 2(a-b)

So, what would I classify as an important shape to understand in this work: this fingering 1245, where the distance between the first and 5^{th} finger is a diatonic sixth, the 2^{nd} finger can fall a 2^{nd} or 3^{rd} above the 1^{st} and the 4^{th} finger is a step below the 5^{th}. Sometimes the ordering of the 2^{nd} and 1^{st} finger are reversed. When there is greater elaboration, the 3^{rd} finger may act as a bridge between the 2^{nd} and 4^{th}.

While this fingering cannot solely explain any passage,[84] especially as the contrapuntal texture gets denser, learning how it may be instantiated in various transpositions could have aided learning the opening passages. Below (Figure 3.2.3) is an example where the pattern is shifted in two ways: the exterior interval encompasses a 7th and the filled in 3rd note assumes a more prominent position in this adapted fingering/hand shape. This passage acts as the retransition: it's both florid, relatively developmental, and it surreptitiously reintroduces the opening hand shape.

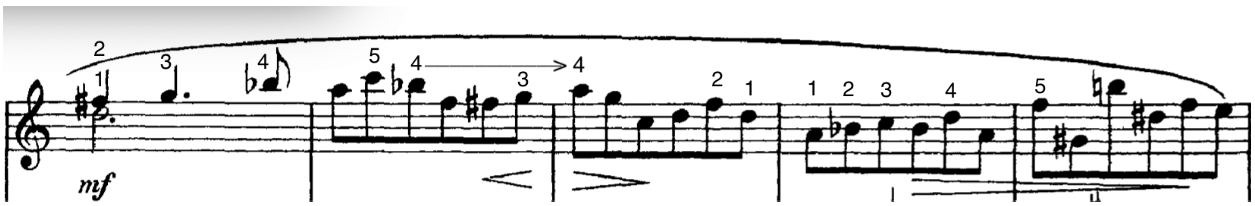


Figure 3.2 3

Even though this may be just as well described in terms of motivic development, I am arguing that a performer could conceive the "motive" being developed as a hand shape and fingering rather than in uniquely melodic and rhythmic terms. Furthermore, if the aim was to familiarize oneself with the language of a piece, it may be profitable to also prioritize the learning of certain patterns conceived in diatonic rather than chromatic steps. Accordingly, practicing these patterns, utilizing a general pianistic technique (wherein the same fingering is meant to be applied independent of considerations of key—the effect being that the performer adjusts the larger rather than the smaller motions) can illuminate many of the potential pathways that this piece may realize.

[84] For many readers, this hand-shape "fingering," especially in bars 2 and 3, may seem both impractical and awkward—for instance, it would necessitate a thumb slide between the "c" and "d" eighth notes on the 2nd beat of bar 3. What it does do is capture important musical considerations in the handshape itself; in bars 2-3, there is a descending scalar line that anchors the passage, C_6(5)-Bb$_5$→A_5(4)-G_5(3)-F_5(2)-D_5(1). The other notes could be thought of as filler. For those who have fully adopted a "universal" piano technique—wherein fingering matters less than handshape and differences are made up through larger body motions (upper arm, shoulder, lower back)—the "awkwardness" of the given fingering (especially at this tempo) may be of trivial concern. For others though, this single handshape may still be considered a poor solution; if they still wanted the musical molding endemic to that handshape, they would instead seek to make up the difference by adopting a friendlier fingering and then adjusting the dynamics of individual notes to affect the wanted phrasing. However, I submit that in the latter approach, there is still a cost incurred, in that the passage becomes harder to remember; there is no longer a single handshape that anchors it.

One great source for working on this particular fingering pattern is the Hanon. Its first 40 exercises are distinguished by their slightly open hand position that has the exterior interval span a 6th and progresses up and then down the diatonic scale. Examples are shown below (Figure 3.2.4). Sometimes the 2nd finger is a step above the first, sometimes it is not. Playing the Hanon exercises in various transpositions and starting on notes other than the tonic, could then prepare the student for some of the specific challenges of Scriabin's piece.

If one is more specific in their intents, they could take each of the highlighted fingerings in the opening example and then run it through various transpositions —varying between whether subsequent transpositions are diatonic or chromatic.

Figure 3.2 4

Regarding point 1 above (learning a language vs. learning a core technique):

- If the goal is to build a language-like fluency in a derivative of Scriabin's style, it's imperative that one learn these patterns in all twelve transpositions.
- The first 40 Hanon exercises are exclusively in C. There is assumption that the core technique covered can be adequately addressed without changing key.

Regarding point 2 above (How much consideration is given to the time needed for the body to adjust to the demands given):

- This approach, building a language-like fluency in a derivative of Scriabin's style, takes as long as you choose — in terms of the material you choose to focus on and how deeply you dig into that material. As such, the pace is either determined by: one, your processing speed—how quickly you can build fluency through improvisation (e.g., how facile you are at transposing or expounding upon ingrained patterns and/or hand shapes); or two, your reading ability—your writing out an exceedingly large amount of material upfront to practice methodically. However, should you then practice that exceedingly large amount of material and not pay attention to variations in physical difficulty, you could definitely run into physical issues (injuries).

- The Hanon is seemingly so simple that it invites playing at speeds faster than indicated and those rushing further into it than their body may be prepared for. Nonetheless, Hanon does acknowledge this potential issue through his design; he makes his second set of 20 exercises double the length of the first 20—the biggest difference is in stamina required, not difficulty of finger movement or speed. One needs the stamina acquired in the learning of the first section to accommodate the challenges of the second section.

Regarding point 3 above (how much is a sense of "getting the big picture" affected):

- Again, if the goal is to build a language-like fluency in a derivative of Scriabin's style, this approach, deeply familiarizing yourself with his motives, is quite thorough; it takes as many patterns/hand shapes as you desire and then puts them in all twelve transpositions. However, its claim is more limited than most pedagogies; it just aims to help you gain fluency in the language of one piece.
- The Hanon seems deceptively straightforward; especially as the skill focused on, finger strength, is put forward as being independent of musical and even anatomical considerations—each finger differs in weight, length, and mobility; equally training them does not lead to equal performance from each. Hanon actually states that *"If all five fingers of the hand were absolutely equally well trained, they would be ready to execute anything written for the instrument, and the only question remaining would be that of fingering, which could be readily solved.*" Perhaps, not by accident, there have been many injuries associated with the learning of the Hanon.

Regarding point 4 above: (How much is the pedagogy rooted in either a single or a set of core principle(s)?)

- When making these suggestions around learning the style of this piece, there is neither an appeal to general principles nor a promise around the applicability of your discovery to other contexts—not even the explication of other portions of that same piece.

- The Hanon makes a nod to covering everything when he includes 20 exercises pertaining to scales, arpeggios, and trills. However, his approach is not integrated. Nonetheless, in the hands of a good teacher, Hanon may be expounded in such a way that considerations of good piano technique extend beyond finger strength and there is more cohesiveness, viewed in terms of exercises offered, to the pedagogy as a whole. Nonetheless, that would be supplementary as the Hanon itself is quite barebone.

Embellishing a pattern or schema — filling in hand shapes:

In this portion of the chapter, we will introduce a topic that will be returned to in chapter 3_4. Since the 9th century, composition and vocal manuals have focused on what constitutes a good phrase—its opening consonance, overall contour, and its cadence. In earlier (and recent) times, vocal phrases have typically set either spoken clauses or words. Similarly, certain melodic figurations have also been associated with different parts of a musical phrase; most notably at the cadences: either the bass descends or ascends by step from 5 to 1. This learning how to set cadences, operationalized in terms of descending pentachords and ascending tetrads, has been a cornerstone of instrumental/compositional pedagogy since the 17th century; and, as just discussed, of vocal pedagogy even much earlier.

This portion of the chapter is similar to the last in that it starts with fingering patterns. There are two big differences though: one, we are exploring these finger patterns in the context of a schema and/or scale fragments—we will discuss the “subset” patterns mentioned in 2_6, and two, we will mention different voicing classifications (such as drop 2 and drop 3).

Again, depending on your interests, there may already be resources available for you—N.B. if you can avoid reinventing the wheel, do, life is too short. If you are interested in the construction of tonal phrases over stock bass lines (although these lines could be in the inner parts), you could again return to the treatises of CPE Bach and even Zarlino, collating and then examining how they treat cadences. Plus, collections of J.S. Bach’s harmonized chorales provide near countless examples of effectively and musically realized basses.

Figure 3.2 5

For instance, in Figure 3.2.5, there are three pentachords (bracketed above) that are set in the bass in BWV 4: in bar 1, the harmony oscillates between F# major and B minor; in bar 5, between E minor and A minor; and in bars 7 and 8, the rising pentachord from E to B in the bass, sets up a tonicization of B minor. All three pentachords appear in the opening part of the phrase, where a given harmony is prolonged. From even this small sample, we will notice some patterns. As is typical in Bach chorales: chords change on each beat, consonant chords come on strong beats, and in the opener (opening part of each phrase)—prolongation of the tonic typically occurs.

Simply practicing alternating between major and minor triads a P5 apart would provide much of the legwork. Depending on your interests, you could start with four voices or three. Your practice could also vary around: which parts had stepwise motion, whether you included 7th chords and or (properly resolved) non-harmonic tones, whether you oscillated between rotations or voicing type (drop 2 vs drop 3). Over time you could do as Bach did in the 3rd example and graft a prolongation and cadence together under the auspices of a single-ascending/descending line (Figure 3.2.6).

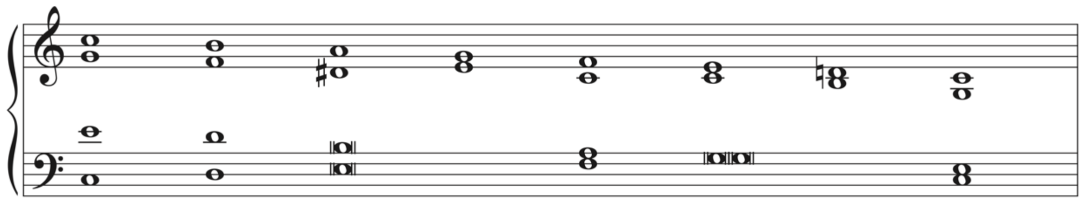


Figure 3.2 6

After working through these enough, you may consider other typical bass motions, such as those that either outline a triad (typically a prolongation) or those that are of the form fa-sol-do or re-sol-do (typically a cadence).

How are these translated into fingerings?

In the case of the bass line pentads (typically the set classes 5-23 and 5-24), they are often played (at least on the piano) in a single hand position and utilize either 1-2-3-4-5 or 5-4-3-2-1. Depending on the key, your hand may open more or less and your fingers are expected to adjust to nearby, chromatically speaking placements.

How are they realized as hand shapes?

Different types of voicings sit very differently in the hands. Compare closed position, C_4-E_4-G_4-C_5 to drop 3, E_3-C_4-G_4-C_5. Many can play closed voicings in one hand; on the other hand, few can play drop 3 voicings in one hand. Furthermore, greater space between voices in a particular voicing indicate more possible melodic figurations that can both encircle each voice and avoid voice crossings.

Below is: the first phrase of the chorale; a table listing the different voicing classifications and associated rotations; and a key that classifies the numbered chords in the excerpt.

A reminder about drop nomenclature: the lower the number is, the higher the voice is in the initial chord. For instance, Drop 2 is translated as "dropping the second to highest pitch in the starting chord below the lowest pitch in that chord. To simplify the descriptions, I will eliminate the use of the word switch; e.g., "Drop 23 switch," which would normally be "dropping 2nd and 3rd to bottom pitches and then switching them (now respectively positions 4 and 3 of the new chord) and replace it with "Drop 32 "; there is an assumption that drop numbers will be listed in numerical order; if not, this will indicate a switch. Drop 1 is just a 1st inversion (rotation) of the starting pitches. As such, it does not change classification type; on the other hand, drop 2, drop 3, drop 13, drop 23, and drop 32 do.

As a help, I am including a listing of the 24 potential voicings (and their rotations) by classification type.[85] If the voicing transformation is a 0th rotation of a drop 2 voicing it will be labeled as just drop 2. If it is a 2nd rotation of drop 2 voicing, it will be labeled as Drop 2 _ 2, etc. Finally, I presume that the starting chord is the most compact version of the chord that bounds the triad with the doubled note. As such, the following table applies to when the root, 3rd, or 5th of the triad is doubled. However, if the 3rd is doubled, then the numbers 1 and 8 refer to the 3rd of the triad rather than the root (the 5 is now called "3" and the 8, "5"). For the 7th chord, treat the number "8" in the following chart as a "7"—no doubling occurs. Lastly, the numbers in the answer that are in bold, refer to non-V (in this context, non-F#) triads.

[85] Finally, a brief comment should be made about the difference between the above approach and figured bass. There are a few main differences:

1. Figured bass focuses on relationships to a bass; the other on transformations from an "ideal" representation of a specific set class.
    a. As such, in figured bass, many different set classes can share the same figure; there is no specifically ordered idealized chord that is being referenced. In other words, one can realize a figured bass figuration correctly and not even know how to identify the quality of the associated chord/set class.
2. The numbers given in the drop table above actually refer to the placement of voicings in an ordered chord. As I was referring to triads/7th chords, I used, for simplicity's sake, 1, 3, 5, and 8. Many readers, especially those with a figured bass background, may find this more relatable than 1, 2, 3, 4. However, there is no implication that the 3, in 1, 3, 5, 8, is actually some type of third above the 1—it is just the 2nd pitch above the root. For the purposes of this chapter, I saw no need to promote a more generalizable terminology.
3. There is no reduction of the different bass configurations into 6 different voicing categories.
4. There is no onboarding of a key signature needed to interpret the numbers.

Rotation	0	1	2	3		0	1	2	3
Drop 0	8	1	3	5	Drop 13	8	1	5	3
	5	8	1	3		3	8	1	5
	3	5	8	1		5	3	8	1
	1	3	5	8		1	5	3	8
Drop 2	8	5	1	3	Drop 23	8	3	5	1
	3	8	5	1		1	8	3	5
	1	3	8	5		5	1	8	3
	5	1	3	8		3	5	1	8
Drop 3	8	3	1	5	Drop 32	8	5	3	1
	5	8	3	1		1	8	5	3
	1	5	8	3		3	1	8	5
	3	1	5	8		5	3	1	8

***Figure 3.2** 7*

1. **Drop 0_3 or Drop 23_2**
2. Drop 2_3 or Drop 3_1 (Drop 2_3 for the adjacent 7th chord)
3. **Drop 0_3 or Drop 23_2**
4. Drop 13_0 or Drop 32_3
5. **Drop 13_3 or Drop 32_2**
6. Drop 23
7. **Drop 13_3 or Drop 32_2**
8. Drop 2_1 or Drop 3_3 (Drop 2_1 for the adjacent 7th chord)
9. **Drop 13_0 or Drop 32_3**

Below (Figure 3.2.8), cases 1 and 4 are written out in notation.

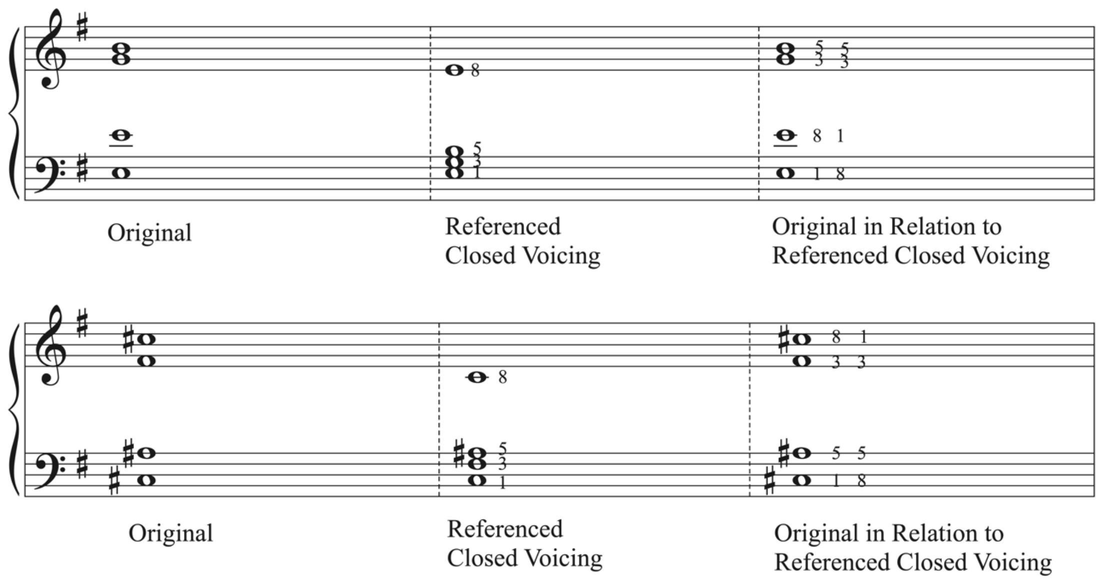


Figure 3.2 8

Even in this short opening section, there appears to be much less variety in the voicing classification of non-V triads than in V triads/chords: 3 out of 5 different forms (2 out of 5 if rotation isn't taken into account) for the former, 4 out of 4 (3 out of 4 if rotation is not taken into account) for the latter. Nonetheless, the point of this is example is not to make any general claims about voicings in Bach; but just to show that even in this short passage, voicing classification may impact the experience of playing the passage.

In chapter 16, it was discussed how, without octave displacement drop 0 spanned an octave at most and drop 32 spanned at minimum 2 octaves. Plus, due to the nature of vocal ranges and rules regarding vocal spread, it would be unexpected for there to be too much octave displacement[86] between voices. The other four voicing classifications are limited (without octave displacement) to between 1 and 2 octaves. Even if the overall range of the chords for those classification types do not change, their distribution (mediated by rotation) may feel quite different—requiring different hand shapes and defaulting towards either putting 2 voices in each

[86] The biggest exception to this occurs between the bass and tenor part, where in more than an octave distance is permitted; however, even though that occurs often, it's rarely the dominant practice within a piece. It happens only once in this chorale — at the opening of bar 7.

hand or 3 in one hand and 1 in the other. Of course, the specific "filler" notes and associated rhythm impact specific fingerings. However, the more general one's technique is—using effective pedaling, keeping the same fingering in different transpositions, playing from the larger motions first, and widespread insight into how to best project the phrasing within a passage (where to breathe, which fingers in a given voicing to lend more weight to—again, mediated by the particular hall and instrument being used, etc.), the less specific filler voices in between may alter the pianist's default reaction to hand shapes associated with different voicing classifications.[87]

Discussing this voicing classification pedagogical approach in line with the four points discussed earlier:

Regarding point 1 above:

- The method above presumes that the performer, barring size-of-hand issues, would—independent of key—apply the same fingering to triads (or 7th chords) of the same voicing classification.

Regarding point 2 above:

- No specification is given regarding the pace of learning or order in which to learn these different classification types.

Regarding point 3 above:

- For any set class you are interested in learning, the voicings of this method project an exhaustive understanding of its potential voicing configurations.

Regarding point 4:

- There is an appeal to a general principle, voicing classification.

[87] While these appeals to "general/universal" technique may seem exceptional, this reflects a 20th century shift in values. Teachers associated with Liszt prepared the student for improvisation. In the 20th century, a well-documented shift in professional values also occurred—a want to align one's interpretation as closely as possible with the composer's intention; put alternatively as, "centering the score and obfuscating the role the performer played behind claims of authority." A technological and marketing change also likely fueled this as did the advent of recordings; it's easier to sell the definitive recording of something over "a" recording of something.

<u>Developing aspects of one's musical style at one's instrument</u>.

As above, this section's examples will pertain to the piano. However, there are analogues for other non-vocal pitched instruments.

First off, every pianist gets greater familiarity with certain played voicings either consciously or unconsciously. It's not uncommon to find a "less-trained" amateur enthusiast playing the same pieces, memorized in childhood, over the ensuing decades. Even with a small amount of formal training, future unscripted sessions at the piano often retain: a basic knowledge of hand positions, a vocabulary of hand shapes (as arpeggios or chords)—first learned in those childhood pieces that are now crystallized; and an enhanced familiarity with how these handshapes are incorporated into improvised pieces that stick mostly to the white keys.

Of course, depending on the individual and their individual interests, this "crystallized hand shape approach" may be significantly more developed. For instance, professional jazz saxophonists, whose primary instrument is not piano (and they may have had only limited formal training), often show great RH dexterity over the piano—quickly being able to realize melodic lines similar to those that they render on their primary instrument. Really, there is infinite variation to how musicians, over the years, will piece together, into a coherent "sound," what they like to hear/play with some learned theory and automated motions. No matter what degree of formal training they may have had, curiosity and good practice habits can lead some very far.

Should this even be done more consciously?

That is a decision best left up to the individual musician. Many different reasons converge at why musicians may doodle around or more directedly explore at their instrument. I've met piano players who feel a cathartic release through playing the piano; in those cases, it seemed, foremost, that those musicians saw the act of playing as a means to work through emotions rather than achieve some purely musical objective. In short, how one relates to an instrument can be an extremely personal thing. Moreover, one could conceive of players who may doodle around on the piano, to maintain a mood or drum up inspiration. They then may go and focus that inspiration into another artistic domain such as painting. In general, the line connecting how an artist inhabits the world, their working methods, and their creations is often neither straight nor easy to pick out.

Depending on your interests, should you choose to do this, the following discussion could be helpful:

Firstly, let's look at Slonimsky's thesaurus of musical scales and melodic patterns.

Most of the book is dedicated to melodic patterns anchored in uniform sequences of intervals — e.g., interspersing ascending and descending sequences of maj 3rds, min 3rds, etc. with min 2nds (e.g., the hexatonic scale, 6-20*, shown below (Figure 3.2.9). Although the chapter titles indicate octaves spanned and number of equal divisions (anchors), the melodic patterns may not share the same span. Furthermore, Slonimsky differentiates between whether the interspersed note(s) lies below the first of the two anchors (infra), between the two anchors (inter), or above the second anchor (ultra).

Figure 3.2 9

Furthermore, even though the number of patterns presented is vast, it is not exhaustive. The interspersed notes (when more than two) tend to be either symmetrical (e.g., 3-1*) or subsets of the major and/or minor scales—in particular: the dominant 7^{th} chords, the minor 7th and their subsets (really, all of the trichords though). Sometimes, the melodic patterns are selected for being symmetrical—for the a^{th} note in the pattern, the $(n-a)^{th}$ note is treated in the opposite manner. It also turns out that many of these opening 1330 melodic patterns are just permutations of a relatively few symmetrical scales—namely 6-20*, 8-28*, and 9-12*. Finally, each pattern is typically accompanied by a harmonic realization, wherein every note is associated with a major or minor triad.

The remainder of Slonimsky's book is a potpourri of various topics ranging from symmetrical rows to pandiatonic patterns and superimposed major scales to double note scales—a re-envisioning of many of the opening patterns, typically 6-20*, 8-28*, and 9-12*, as 3 or 4 dyads rather than a strictly melodic pattern.

If one were to internalize the many patterns offered in Slonimsky's book, one would have an inroad into challenges faced in Bartók's and Messiaen's music, amongst others. Below, Figure 3.2.10 are five passages: the first is Slonimsky's 394th exercise and the second gives the second piano part (the clefs are LH: Bass and RH: Treble) in Bartók's Sonata for Two Pianos and Percussion—one may notice that they are the same but in reverse order; the 3rd and 4th come from, respectively, Slonimsky's exercises 1026 and 982, and the last shows how learning those last two exercises could help with Messiaen's *Canteyodjaya's* bar 6.

Figure 3.2 10

The relation between the Slonimsky and the Bartók passage given is more direct than the relation between the Slonimsky and the Messiaen. This brings up an important point, no book with perhaps less than a million exercises could explicitly account for many of the permutations of the aforementioned symmetrical scales that are often found in the literature. A further restriction of the Slonimsky is that the given patterns repeat until the anchoring interval occurs enough times that the tonic returns. In written music, even if an anchoring interval is repeated numerous times, there is no guarantee that each will be embellished in a similar manner (with the same infra-, inter-, ultra-polated notes).

Nonetheless, regarding jazz improvisation, these extended patterns may better prove their general helpfulness. For instance, for extensive improvisations over a dominant pedal, it is common to improvise over an associated (half-whole) diminished scale. At the appropriate transposition level, many of the figurations in the tritone and *sesquitone* progressions (chapters 1 and 3) would work admirably.

Regarding point 1 above:

- The method does not explicitly ask the performer to learn these melodic patterns in all 12 transpositions. Like with other thesauruses, it seems that the user is expected to pick those exercises, permutations, alternate symmetrical rows etc., that could better substitute for relatively common musical situations—instances wherein a symmetrical scale, or row, may be used.

Regarding point 2 above:

- No specification is given regarding the pace of learning. However, due to the book's organization, there is an implicitly suggested order to follow in navigating it. Nonetheless, there is no explicit mention that it should be followed in any order, nor should it even be completed. Rather, it is a grab bag of important topics of the day (playing symmetrical scales, etc.) that are introduced in a semi-systematic manner.

Regarding point 3 above:

- If the goal is to become familiar with the many manifestations of the symmetrical set classes that may present themselves in the literature, this compendium is not exhaustive. However, without the introduction of more general concepts, no list of exercises could be.

Regarding point 4:

- There is an appeal to a general principle, classification by anchoring interval.

What does the Slonimsky suggest about your own curation of exercises for wide ranging purposes?

1. Without an appeal to more general concepts, no set of exercises has any shot at being exhaustive.
    a. Nonetheless, techniques are shown, for instance octave displacement in Slominsky's pan-diatonic section (p. 192), that if applied to the earlier exercises, could yield substantially more patterns.
2. Similarly, one should differentiate between patterns to internalize and techniques to master. Certain patterns can function as an aid in those different senses.
    a. For instance, a melodic pattern may be a cue for a melody in the soprano or a bass line that is realized a la ragtime—a bass note followed by a related triad that is one or two octaves above.
3. Even such a limited topic as embellished uniform interval successions that collectively spell symmetrical set classes, is, under closer inspection, actually quite unwieldy.
    a. The takeaway, the better that you can delineate exactly what you aim to learn and then tailor those exercises to that, the better.
4. Consider reviewing the first part of this chapter: grounding your exercises in a composer's specific language.

Part 3, Chapter 17: Multi-Modal Transformations

The previous two chapters focused on transferring your growing knowledge of set classes towards a better understanding of some musical landscape of interest; whether it was something as abstract as how all set classes relate, to something more specific, like the musical language of a given composer. In the following two chapters, we will look at how knowledge of set classes can be used to simplify tasks often associated with learning/performing music. This chapter focuses on how set class notation, brackets and numbers, can meaningfully relate to non-harmonic material.

This concept is not new. Schillinger grounds his magnum opus, The Schillinger System of Musical Composition, in his discussion of rhythms—tying all aspects of harmonization, arrangement, and form back to the terminology discussed. Morris explicitly realizes number and bracket terminology in various musical domains. Lewin's *Generalized Musical Intervals and Transformations* explores how the properties associated with intervals, often construed in the harmonic domain, extend to other non-harmonic distances. This chapter, then, will survey the above approaches (there are others) and discuss how working with set classes could supplement them.

Let's start with Schillinger's treatise. The three main concepts that shape Schillinger's treatise are waves, geometric expansion, and permutations. He derives much of the source material from waves' interference/interaction. Permutations and geometric expansion (multiplication and the canonic operators) are the means by which he then generates musical passagework from that source material.

Regarding Rhythm:

Schillinger identifies each wave with a pulse. There is wave interference when two pulses synchronize. For instance, take two pulses, let's say a and b, that are in 3:2 ratio. This means that when they start together, there are three a for every two b and that after either three a or two b, they once again coincide. Schillinger's description reflects the many different ways that this interference can be heard/inferred as:

- An underlying *denominator* pulse that both share; in this case one sixth the overall time span.
- Each pulse separately.
- The *resultant*: the composite rhythm. In this case, it is quarter, two eighths, quarter.

Notably, this description offers an explanation for why certain durations in the composite rhythm may be more accented than others; they are reinforced by more of the different ways of hearing this interference. In the example of 3:2, the moment when the 3 and 2 pulse coincide, is the most accented moment. Here 3 is the major generator (a) and 2 is the minor generator (b).

To account for greater variation in the accenting of rhythms, he introduces the notion of fractioning. Put simply, he introduces a smaller major "generator" whose length is a fraction, b/a, of the major generator. In the case of 3:2, the pulse of the major generator is 1/3 (the minor generator is ½), the pulse of the "smaller major generator" is 2/9 (1/3 ÷ 3/2); and, the overall cycle is 9/9 (as opposed to 6/6). However, three pulses of the smaller major "generator," 2/9 spans only $6/9^{ths}$ of the cycle. Accounting for the complete cycle then requires overlapping 3-groupings of the "smaller major generator," whose starting points are shifted.

Finally, he shows how to generate material through (higher-ordered) permutations of (combinations of permutations of combinations of etc.) unordered sequences of durations and by raising a sequence of durations (a + b) to a particular power (n). Note that permutations can be general (rearrangements of unordered material) or circular (rotations of ordered material).

Regarding Melody, Harmony, and Arrangement:

Other than surveying (and then giving his take on) current ideas in the field, such as voice leading between triads, and making claims about how evolved different nations' usages of the material is (e.g., Schillinger lambasts the West for its reduced rhythmic palette), these chapters are focused on applying (higher-ordered) permutations to the canonic transformations (P, R, I, RI), and multiplication to either unordered melodic (pitches, intervals, motives, phrases) or harmonic (chords) units. Interestingly, he also shows how transformations can simultaneously impact different parameters of a melody or texture—for instance, the first 4 notes in a phrase may cycle through its circular permutations at the same time that the rhythm of the entire phrase cycles through is circular permutations. They coincide at the least common multiple (LCM) of the length of the first four notes (4) and of the entire phrase. Length is determined by number of notes/durations and the overall time elapsed. Finally, he looks at how harmonic and melodic units can be distributed (and then permutations of that distribution) across all of the instruments in a score.

In commonality with this pedagogy then, the bulk of Schillinger's treatise shows how a similar framing can apply to multiple modalities of music: rhythm, melody, harmony, and even arrangement. Namely, he uses the metaphor of a wave and well-known mathematical formulae (like the Fibonacci sequence) to generate material. Then out of that material he generates more material through adopting higher- and higher-order permutations of said material and applying the canonic transformations. Lastly, he then applies his findings broadly.

In this dissertation's methodologies, the "material" is set classes. There is an assumption that there are enough relationships present there that those who know them could then hobble together a quick inroad into many of the tasks that they set their mind to. How helpful this is of course depends on one's musical interests and familiarity with the relevant terminology.

- For instance, express a rhythm, <3,5,8,t>, either as ordered durations (dotted quarter–half-note tied to eighth note—whole note—whole note tied to quarter note) or as time points a la Morris[88] (striking on the 3rd, 5th, 8th, and 10th, eighth note in a bar of 12/8). You could then encode this (either of the two interpretations) as 4-23*. In the first interpretation, rotation yields new material; in the second, it does not. Nonetheless, if you find the mental rotation of pitches easier than numbers—Eb, F, Ab, Bb goes to F, Ab, Bb,

[88] This is discussed more at the end of the chapter.

Eb rather than <3,5,8,t> to <5,8,t,3>, it's great; only a mental translation back to numbers <5,8,t,3> is needed at the end.

  - Here, the translation to pitches may seem unnecessary. However, for more complicated transformations (e.g., inversion or Drop 2_3) or larger set classes, it could be substantially more helpful.

- How about when a set class can't describe the rhythm ? For instance, when the difference between two durations exceeds 12 of whatever the common unit is.
  - You could differentiate between octaves—converting your set classes into ordered pitches (rather than pitch-classes). Or, approximate the rhythm, manipulate that approximation, and then convert that approximation back to the original rhythm at the end.
- How could you accommodate higher-order permutations — pitches or rhythms?
  - If you are familiar with the different voicing classification categories in a particular cardinality, you may remember each permutation more easily. Higher-order permutations could then be described as a sequence of voicing transformations. Such as "apply Drop 0_3 to Drop 13_2 to Drop 3_1."
    - For instance, take an ordered set of durations <3,1,7,4> and conceive of it as a pc-set 4-12a; it has twenty-four possible permutations as demonstrated below. A higher-ordered combination of <3,1,7,4> just combines different permutations of it. Note that <3,1,7,4> is itself a permutation of its most compact form <3,4,7,1>, Drop 32_2.[89]
    - However, while this may be manageable for smaller numbers of elements; it could get less so as the size increases.

[89] Notice, that I am applying the permutation to the compact form rather than the given permutation. This is for two main reasons: it's easier—you don't have to negotiate a relabeling of the given voicing classification system, and it makes the point that all permutations are available. If you are still intent on describing permutations of Drop 32_0 rather than permutations of the most compact form, so "Drop 0_3" refers to Drop 0_3[Drop 32_0] rather than Drop 0_3 of the most compact form, you can; it's just a lot more work—e.g., Drop 0_3[Drop 32_0] <3,4,7,1> sends <3,4,7,1> (<1,3,5,8> in chapter 3-2's table) to <7,3,4,1> (<5,3,1,8>) to <1,4,7,3>(<8,5,1,3>). One approach to doing this more efficiently is to work out and then memorize a table show how voicing transformations compound.

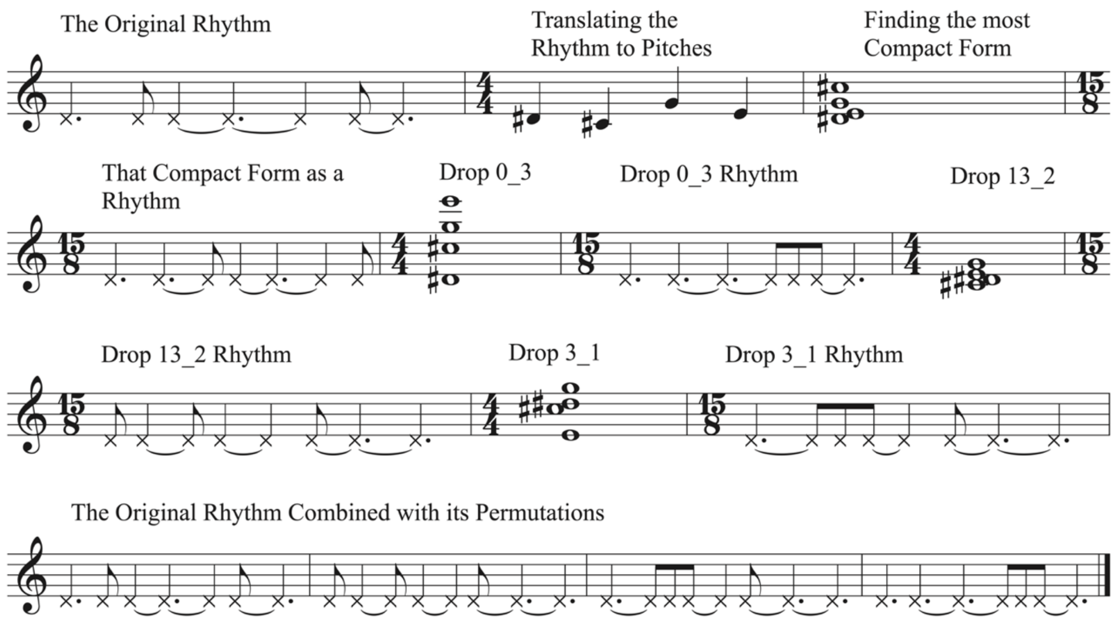


Figure 3.3 1 — Examples of multi-modal transformations

- Of course, arranging is typically done at a computer, manuscript paper, and/or at a keyboard workstation.[90]
    - In the case of a computer, cut and paste (or writing a "permutating music" program) could bypass most of the mental effort described above.
    - In the case of manuscript paper, running through things quickly in your heard beforehand may be a good way to quickly vet a lot of potential options before applying anything to paper. Writing out loads of potential permutations and then picking between them (through playing or reading) can be a tedious process.
    - In the case of a keyboard workstation, how much arranging you do before the performance could mediate how much you could benefit from mental vs computational approaches.

As discussed earlier, Lewin's GMIT used "intervals" to describe distances across a number of musical domains—from harmonic units, to musical events related proportionally, to timbre—as filters, the difference/relation between two spectra, to rhythms and more. By the end of the book,

[90] A keyboard workstation typically includes synthesis, sampling, sequencing of sounds, recording, combination sounds, and effects processing.

he discusses interval systems that can account for fuzzier notions of distance, such as "nearer" or "farther.[91]" To put this to use here though, we must look at an overlapping field of music theory, namely similarity relations—which tries to measure how dis(similar) two particular set classes are. The appendices looked at Derfler's and Schuijer's summaries of the field, as well as a more detailed discussion of Forte's, Tymoczko's, Callender's, Straus's and finally, Quinn's work.

Moreover, as was suggested, in Part 3, chapter 1—for building maps, there could be a great benefit to evaluating the effectiveness of various measurement systems in light of a more hands-on understanding of the set classes. How well do different measures capture your own intuitive notion of the similarity between two set classes? Does one set of approaches, such as comparing interval vectors (e.g., Forte's R1 and R2, Morris's ASIM), or does the number of shared subsets (Forte's Rp, Lewin's EMB) fare better? Or alternatively, when working with Quinn, is it better to compare set classes by how much they exude 2-6*, 3-12*, 4-28*, 6-35*, and 12-1* (the circle of P5/m2)? Finally, Tymozcko and Callender compare set classes in Euclidean space, size 12—comparing those representations of each set class that are the closest in terms of voice leading (e.g., C major and C Aug as opposed to C Major and Db Aug). After determining which type of measurements speak to you best, you could then create a chart that maps out how close all the set classes of a given cardinality are to one another. It's possible then, that by focusing on the regions of this chart/table that describes set classes within a certain proximity to each other, you could refine your sense/perception of how close, in a given measurement system, certain set classes are.

Just as pitches and many rhythms can be described in terms of numbers and brackets, one could then explore whether that just-built chart makes sense when evaluating other modalities of music. For instance, are two rhythms more similar because they have more subsets in common (e.g., 3+3+2+3 & 2+1+3+3) or because their contour is more similar (e.g., 3+3+2+3 & 4+4+3+4)? As rhythms are different than pitch-collections, what one may construe as similar in

[91] As a refresher on Lewin, "in any representative GIS (generalized interval system), there is an underlying set S, a group of Intervals/distances (actions) on the set, and the function (a binary operation) that can both compose intervals with each other as well as with members of the underlying set. The only stipulations are that each interval in IVLS (the set of intervals) applies to every element of S, yielding exactly one other element of S, and that distances accumulate; if there is a distance between s_1 and s_2, and between s_2 and s_3, then there is a distance defined between s_1 and s_3."

one domain could be construed as vastly different in another; there should be no expectation that the same measures of similarity will chart equally well across different musical domains—e.g., perceptibility of relations between rhythms may be more impacted (or at least differently) by changes of tempo and length of the rhythm, than harmonic fields. Yet, you may see this dissonance between the same measures impact in two different domains as a source of inspiration or curiosity worth exploring more deeply. How much of your sense of a rhythm is preserved when a particular rhythm's durations are rearranged or portions of them repeated? Do you find that invocations of the overtone series translate as well into the rhythmic domain as the harmonic? Set class notation and found or newly-built maps of set class/pc-set/pc/pitch space could help you answer some of these questions and gain additional insight into what musical material you value most.

Moreover, by set class terminology, I do not mean that you have to onboard all suggested equivalencies implied by your use of that set class. Since you may already know that set class in a hands-on way (meaning that you have already gained a lot of experience working with it in a variety of ways—from composition, to improvisation, arrangement, and analysis) the terminology is just a pointer for you. You may think even start to think of a particular rhythm as an ordered representation of a certain set class (thought of now in its 'a' or 'b' form). Again, this set class terminology is meant as an anchor/jumping-off-point for you to enrich your perception of these musical things/ideas, not as a definitive description of them.

However, as with other topics in this pedagogy, the point is not for you to find the definitive answer to anything (if that is even possible in most cases), but through the process of exploring these ideas, gain a better personal understanding of how you gauge distance between the musical units that you use. As discussed above, sometimes these notions can be restricted to structural considerations; sometimes, there are other factors—as Lewin gets at through his introduction of networks and fuzzier relations that should also be considered. In the context of a piece, multiple considerations (ranging from timbre, pitch, and placement in the form) can problematize distance measurements that are related to a single domain. With INJ, Lewin discusses similarity in terms of shared subsets and "acceptable" transformations of subsets. This is a powerful starting point for enriching your understanding of what leads you to understand two musical objects as being similar—STRANS is the more general form of IVLS. Again, if interpreted for your specific purposes, set class notation could help you focus in on what makes those objects (structurally, and in terms of context) that you care about more relatable.

Finally, Morris builds off many of Lewin's concepts, including networks mentioned above. Morris makes explicit how to account for rhythms; he labels them as time points in the context of a meter (R. Morris 1987, 299–305). However, he also concretizes time points by making a metronome marking explicit. These time points are not our conception of the distance between two beats in a meter, which may not be uniform.

The subsection of this portion of "orri"'s chapter is actually entitled "Time and Pitch Space Isomorphisms." If there is isomorphism between two different sets (and some set of operations on them that apply to both sets), it means that in regard to those operations, the two sets are equivalent. What does that mean here? In regards to the canonic transformations, Morris shows how they can apply to sets of pitches in pitch space (not pitch class space!!) in an analogous way to rhythms (sets of time points—not beat-classes!) in his time point space—the structure of the two spaces are equivalent. Isomorphisms can also be found between specific pitch-class spaces and modulo time spaces; e.g., 12-tet and 12/8 (without the grouping into dotted quarters!).

More generally though, composition itself is a multi-modal activity. Either (un)consciously or through compositional design or improvisation, one takes some musical idea (or ideas) or material and realizes it (them) through the rhythm, melody, lyrics, performer's body movements, etc. that are bound to any realization of a piece. Put alternatively, whether the germ of the idea begins with a motive, word, feeling, or movement, its final shape manifests in many different ways. Morris's book Composition with Set classes and article "Compositional Spaces and Other Territories," discuss compositional design (pre-compositional plans) and spaces (the material used)—a referendum on creating (through writing and improvising) non-typical musical forms whose coherence is tied to either the repeated use of certain material or some core technique employed. The article also proposes that there can be an intermediate stage of compositional design; one wherein the composer is weighing various options. In the book, much of this discussion is focused on arrays. As mentioned earlier, "arrays are 2D grids of various dimensions. Each box of the grid may (or may not) house certain pitch elements (and/or other embedded arrays). The purpose of the array sets the number and type of restriction on each box—for instance, whether they all contain the same number of elements, or share certain elements". A matrix is one type of array. Arrays then, sans a computer, can be a sophisticated means by which to put in dialogue numbers, whose overloaded meanings can signify multiple types of musical material, and networks, map(s) indicating how that material is coordinated. I recommend working through many of the examples in his book and following whichever leads speak most to you. Your knowledge of set classes can enrich your understanding of the networks that are built and enable you to more quickly internalize those arrays that intrigue you the most.

Finally, Morris, in Composition with Set classes, also introduces the concept of contour space. In contour space, the distance between pitches is undefined—only specifying whether one pitch (cp) is higher or lower than another. While contours are associated with pitch, Morris's contour space can be seen as getting at the notion that our perception of melody, pitch, and rhythm may not be as distinct as standard notation may indicate. When I hear a melody or motive for the first time, I often cannot easily separate my impression of it easily into distinct parts—either distinguishing pitches clearly from rhythm and/or specific pitches from the arpeggio or direction of movement (up, down, encircling, etc.) in which they are ensconced. Instead, I receive an impression of a shape that on additional listens, or mental focus later (if the music sufficiently sticks) I later parse out. Put alternatively, csegs describe general templates in which, a multi-modal manner, the listener or composer may ground their understanding of a melody. In short, using your knowledge of set class terminology—in particular, the first set class of each cardinality (3-1*,4-1*, 5-1* etc.}, can quickly give you insight into a range of contours/templates that can be associated with a given set of pitches.

Part III, Chapter 18: Monteverdi and Biancheri: Embellishing Progressions

Shortly before his 20th birthday, Monteverdi dedicated his first book of Madrigals to the Veronese Count, Marco Verità. Yet, even though this publication came early in his career, 27 January 1587, it demonstrated a compositional skill level that belied his age. What were the pedagogical and conceptual frameworks (contexts) that enabled his early expertise? In short, they were a solid foundation in counterpoint; a great familiarity with his musical material—how to best mine and combine it; and a virtuosic handling of the vihuela—at that time, a composer's aid as well as a musical instrument. This chapter will abstract the latter two background skills; seeking to use set class dexterity to facilitate a similar but adapted approach.

Lex Eisenhardt and Jessie Ann Owens link 16th-century composition and intabulation. Eisenhardt begins his accounting of renaissance guitar practice with a mid-16th century change in guitar tuning; there was a growing trend to strum the guitar/vihuela rather than apportion part of it as a drone (bourdon) for monophonic singing. He attributes this to the rising interest in homophonic vocal genres such as chansons and romances. However, as he points out, in regards to the vihuela, this explanation falls somewhat short—only a few chords on the vihuela could take full advantage of multiple open strings; open strings are louder and more resonant—a boon for performance. This observation prompts Eisenhardt to then suggest that vihuelas, like their cousin the lute,[92] could have been used for the intabulation of vocal polyphonic music onto the guitar (Eisenhardt 2015, 12–14).

Furthermore, in her chapter on 16th-century and earlier composers, Owens provides complementary sources that attest to the effectiveness of intabulation as a pedagogical and compositional aid. Through various methods including music store receipts, visual images, and letters, Owens builds a case that *cartella*, typically-portable erasable tablets, were 258ndispensable for 16th-century composers. Along these lines, she cites Schonsleder, an early

[92] An example of such an intabulation is given in *Composers at Works* (Owens 1997, 151–54).

17th-century writer, who categorized these *cartelle*, *palimpsesti*, into three varieties: a ten-line staff, score,[93] and tablature. Finally, Owens repeats Diruta's keyboard intabulation claim that "once a student had thoroughly mastered the method for making an intabulation from a score, he would be able to make one without needing to write out the parts first in score (Owens 1997, 95)."

The vihuela's tuning, conjoined pairs of P4ths adjoined by a M3, provides further circumstantial evidence; the different voice ranges, SATB, sit about a P4th apart. And unlike the keyboard, or organ, it was relatively easy to tune the vihuela; and, in a time before widespread fixed pitch standards, the vihuela would have been retuned, preserving the intervals between strings, to accommodate whoever was singing along with it. Similarly, its transcription onto a score would accommodate the range of a vocal ensemble, rather than fix the exact pitches that Monteverdi may have strummed on his vihuela.

In addition, as mentioned above, the vihuela was more than a compositional aid, it was a full-blooded instrument. It could entertain; it could dazzle. As such, its performer, in this case Monteverdi, would become familiar with patterns and/or progressions to play on it. Such patterns, namely Sabbatini's (1625) scale harmonization, (Figure 3.4.1, taken from Christensen 2007)— though it probably arose much earlier, and the Romanesca, undergird 'Ardo, si ma non t'amo.' Moreover, it would not be to outlandish to describe certain portions of Monteverdi's work as settings and embellishments of these patterns as much as setting and embellishments of his lyrics.

Example 4: Galeazzo Sabbatini's "scale triads".

Figure 3.4 1

Sabbatini's "scale triads", reproduced in "Regle d'Octave" (T. Christensen 1992, 97)

[93] Mostly likely the score format was less and less common the further back in the16th century that one looks.

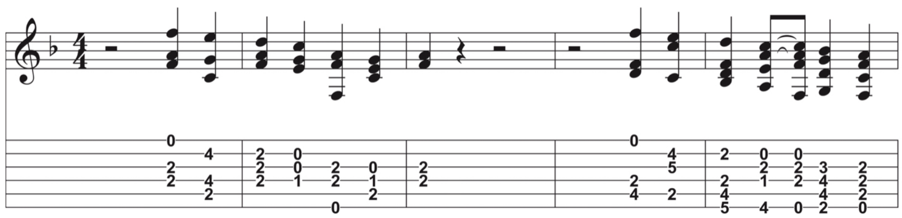
Above are Sabbatini's scale triads, culled from Christensen's chapter on rules of the octave. Notice that he builds root position triads on each scale degree with the exception of scale degrees 3 and 7, upon which he builds first inversion triads. Due to their spacing, as probably was its intent, the scale triads were well-suited to the vihuela.

Now look specifically at "Ardo, si ma non t'amo." Below is an example of one of many passages that can be reduced to such a scale harmonization. In figure 3.4.2, the upper staff reduces the voice parts onto two staves, and the lower system shows how the Romanesca (I-V-vi-iii) underlies the passage. In figure 3.4.3, the same passage is transcribed onto one stave — showing that its range coincides with that of a vihuela. Accompanying this stave is the tablature that would have been associated with said passage; the main point being how easily this passage fits onto the guitar.

Monteverdi's "Ardo, si ma non t'amo," measures 17-22:
A grand staff transcription and reduction
Figure 3.4 2

Monteverdi's "Ardo, si ma non t'amo," measures 17-22:
A treble clef alone and tablature transcription
Figure 3.4 3

The ending passage in 'Ardo, si ma non t'amo,' again sets the Romanesca as well as Sabbatini's scale triads. Shown below is this passage (Figure 3.4.4). Observe by looking at the tablature, how much more difficult (though still playable) it is to render Sabbatini's harmonization of the scale. As the rendering of more difficult/complicated passages can effect a more virtuosic and exciting performance, perhaps Monteverdi's inspiration for this passage arose from how he engaged his physical experience of implementing his frame as much as from any motivic imperative.



Monteverdi's "Ardo, si ma non t'amo," measures 25-7 & 27- 28:
A treble clef alone and tablature transcription[94]
Figure 3.4 4

So, how do we incorporate Monteverdi's approach into the pedagogy of this dissertation?

Below are three approaches. All three explicitly use either pc or set class notation to encode some type of process. How, when, and whether you would use any of these approaches would be determined by the creative task at hand. Therefore, which of the following working methods best suits you, should be seen as a creative and practical choice.

My pragmatic, not nuanced case-specific, response to the musico-philosophical query, "can the meaning of a work be divorced from its method of production?" is a simple, straightforward, no. Put simply, how we practice, informs what we do. How the creative work is envisioned and then made manifest is inextricably linked to that work—just like someone can compose for violin without being a violin maker, the meaning of a work is not tied to the construction of each component of that artistic process; nonetheless, it IS tied to at least one critical component—the component that has been used to distinguish it as an artistic product in some other context. Furthermore, who does the pointing to something as art in one context is often not the same person who "created" the piece being discussed. In other words, a specific deejay's working

[94] Note: to make each subphrase easier to read, an additional bar of rest has been added.

method may reflect more on some dancer's experience of "a fifth of Beethoven" in a night club, than either Beethoven's Partimento or even Walter Murphy's conservatory training.

The first approach is the most straightforward: "memorize a phone number."

If you want to memorize a progression of triads, try encoding them with a string of numbers. For instance, encode the progression of major triads: C, A, Ab, F, E, Eb as <0,9,8,5,4,1>. Familiarity with this sequence of pitches, 6-20* [conceptually and through working knowledge], may then help you quickly transpose this progression; e.g., "in D," this progression is D, B, Bb, G, F#, E♭. In general, a structural understanding of the "telephone number"—in this case, the highly symmetrical, 6-20*—can only add speed and flexibility to your usage of the progression. For example, when making phrasing decisions, the performer's grouping choices may call attention to two embedded symmetries: <0,9,8,5,4,1>'s even order numbers outline one augmented triad – the odd order numbers another; and, I_1[<0,9,8>] = <5,4,1>.[95]

What if some of the triads in this succession are rotated and you want to memorize that feature as well? There are many options:

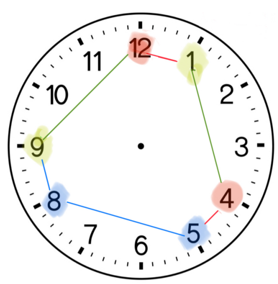


Figure 3.4 5

A visuo-spatial representation of a "phone number."
Rotations are color-coded

[95] The latter symmetry also appears with the rotations of <0,1,4,5,8,9>: I_9[<9,8,5>] = <4,1,0>, and I_5[<8,5,4>] = <1,0,9>

If you have a strong visual memory, perhaps color coding would work. Upon seeing the image of the number in your mind, let's say red (**bold and underline**) is root position, blue (**bold**) is first rotation, green (no bold) is second rotation, you could sear something like this <**0**,**9**,8,5,**4**,**1**> into your memory (Figure 3.4 5).

Or, you could also append letters to the <0r,9s,8f,5f,4r,1s>. If this is too unwieldy still, you can use syllable short hands. The following table gives you such. The syllables were chosen practically. So, if this type of aid were helpful and relevant to you, you could memorize these syllables or make something else up that fits you better. Using the following table, one could remember <0r,9s,8f,5f,4r,1s> by saying <zahr, ness, ayeff, fiveff, fahr, 'wess'> (Figure 6). If you wanted to cast these all as minor chords, you could attach the syllable "em." So you'd say <zahrem, nessem, ayeffem, fiveffem, fahrem, wessem>

	Rt = + ahr	**1st = + eff**	**2nd = + ess**
0 = z +	zahr	zeff	zess
1 = w +	wahr	weff	wess
2 = tw +	twahr	tweff	twess
3 = thr +	thrahr	threff	thress
4 = f +	fahr	feff	fess
5 = fiv +	fivahr	fiveff	fivess
6 = s +	sahr	seff	sess
7 = sev +	sevahr	seveff	sevess
8 = ay +	ayahr	ayeff	ayess
9 = n +	nahr	neff	ness
10 = ten +	tenahr	teneff	teness
11 = lev +	levahr	leveff	levess

A verbal/phonological key for encoding progressions
Figure 3.4 6

While this may seem cumbersome to learn for most, it could enable someone to get extremely fast at memorizing and then quickly executing long series of chords—chords that aren't necessarily connected to phrase-structural expectations. Furthermore, if this way of encoding/decoding becomes second nature, it may greatly improve how quickly you memorize tonal passages. 7th chords could have 'om' on the end; and the corresponding 3rd inversion postfix, 'erd.'

Finally, how would one encode the transposition of a sequence of different chords?

For instance, take C major followed by F#7? One solution is that you could add yet another syllable at the end to the above chart—such as 'om' for dominant; so, <zahr, sahrom>. Or alternatively, you could find the set class that contains C major and F#, 6-30b, and wed your encoding to some particular voicing and inversion of that set class. So, in that case <zero 6-30b> may work. However, this type of encoding is probably not generalizable. Which representation is chosen would depend on each musician's own familiarity with various set classes. Similarly, <zero 6-30b> may only work as a cue for a solitary musician—the one who crafted that association.

Super-common progressions could get their own label. Such as choosing, pun intended, "oof", for a ii-V-I. For, instance, Coltrane's Giant steps would become <e, 2, 7 (sev), t, 3**;** sevoof, t, 3, 6, e**;** throof**,** sevoof**;** levoof, throof**;** 6>. Yes, this is a mouthful. But assuming that this was a useful road of inquiry for you, you'd get proficient at saying and retaining these codes.

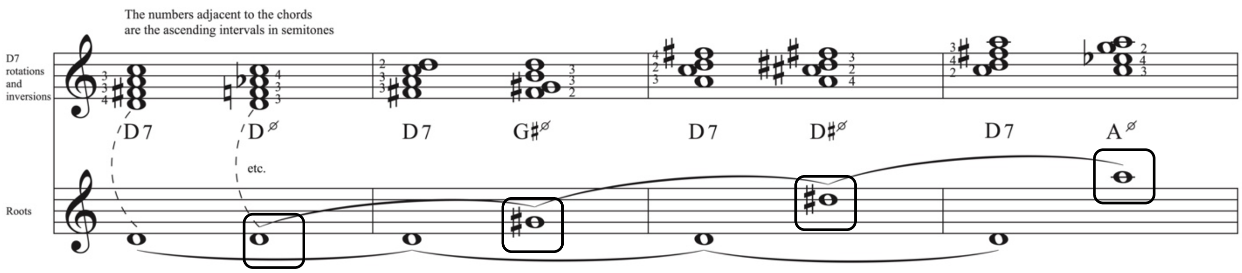
The second approach is similar to the first, in the sense that you have a string of numbers to remember. However, instead of encoding which chords are used in a particular passage, it encodes where a single chord may go under a particular transformation.

Below is what I call finding the *trace* of a chord. Now, instead of the "phone number" representing a series of chords, it represents a single chord under a particular transformation; in this case its inversion. The advantage of a *trace* (rather than a list of chords) is akin to the advantage of GPS over a list of relevant landmarks. No matter where you are, GPS will guide you to the desired movie theater. On the other hand, a list of relevant landmarks only describes a path to a single movie theater.

The following example relates 4-9*, e.g. {D, D#, G#, A}, to the normal order of 4-27; it is a *trace* of how 4-27A can be mapped onto its inverse 4-27B (and vice versa as well).

This is demonstrated below in regards to D^7 (Figure 3.4.7).

- Root Position D^7, inverted around its outer pitches, becomes $D^{\varnothing}$
- First inversion D^7, inverted around its outer pitches becomes $G\#^{\varnothing}$
- Second inversion D^7, inverted around its outer pitches becomes $D\#^{\varnothing}$



The roots of the inverted D7 are D, G#, D#, A = <2,3,8,9> => (0,1,6,7) = 4-9*

A *trace* of 4-27

Figure 3.4 7

Or vice versa {C#, D, G, G#},

- Root Position $D^{\varnothing}$, inverted around its outer pitches, becomes D^7
- First inversion $D^{\varnothing}$, inverted around its outer pitches becomes G^7
- Second inversion $D^{\varnothing}$, inverted around its outer pitches becomes $C\#^7$
- Third inversion $D^{\varnothing}$, inverted around its outer pitches becomes $G\#^7$

The following algorithm is quick way to compute that *trace*[96] for 4-note set classes. Keep in mind that the result applies to ordered sets of pcs, **not** unordered set classes. In other words, the *trace* of two different orderings of D^7 above would yield different results.
The algorithm is:

1. Express the ordered pitch set with integers.
2. Take the sum of each consecutive pair of pitches.

[96] While I discovered this algorithm for 4-note set classes, the inspiration for the proof came from Morris himself.

3. Identify the set class which is comprised of those sums. Or more precisely, the pitch class equivalents of those sums.

E.g.,

1. D-F#-A-C becomes 2-6-9-0
2. 2-6-9-0 becomes 2+6, 6+9, 9+0, and 0+2.
3. {8, 3, 9, 2} equals {0, 1, 6, 7}, which equals 4-9*.

Here is the proof:

In Morris's *Composition with Pitch Classes*, p. 44, TH 2.3.4.2, the following property was proven: $\{\forall pcsa, b | if TnIa = b, then\ a + b = n\}$. The above algorithm extends this property to any ordered set of pitches.

Regarding an ordered set of pcs under inversion, the *trace* accounts for each rotation's associated T_n. For instance, let A = <a_0, a_1, a_2, …. a_{n-1}> and B = <b_0, b_1, b_2, …. b_{n-1}> be inversely-related ordered sets of pcs. By definition, $a_0 + b_{n-1} = a_1 + b_{n-2} \ldots = a_{n-1} + b_0$. Accordingly, the sum associated with the outer pair of pitches, $a_0 + b_{n-1}$ and $a_{n-1} + b_0$, is identical to the sum associated with each of the other aforementioned pairs. Hence, the above property — comparing inversely-related (I_a) pitches under transposition (T_n), is applicable to A and B collectively. T_n is the representative sum. The same rationale applies to each rotation of A and the specific B that is associated with it.

Let r_0 be shorthand for rotation 0, r_1 rotation 1 etc.. Let Tr_1A be the transposition associated with the first rotation of A. The *trace* of A is then <Tr_0, Tr_1 …, Tr_{n-1}>A. Viewing the list of transpositions (directed intervals) as pcs, allows the *trace* to be identified as a set class.

However, you may wish to have other types of traces of your important chord at your disposal — not just inversion. In his *Class Notes* (2001), Morris provides equalities (and their proofs — not included here) that can reduce strings of concatenated M_5, inversions, and transpositions into their simplest form.

- **Composition**
  - Theorem 4.3.1a: $T_xT_y = T_{x+y}$; $M_xM_y = M_{xy}$
  - Theorem 4.3.1b: $M_mT_n = T_{nm}$
- **Inverse**
  - Theorem 4.3.3: $T_{-nm}M_m$ is the inverse of MT_{nm}
- Example:
  - $T_7IM_9IT_{-4}M_5T_3 = T_7IM_5IT_{-4}(M_5T_{5x3}) = T_7IM_9IT_{-4}(T_3M_5) = T_7IM_9IT_{-1}M_5 = T_7IM_9(IT_{-1}M_5) = T_7IM_9(T_{-5}M_5) = T_7IM_9(T_{11x5}M_5) = T_7IM_9(M_5T_{11}) = T_7IM_9(M_5T_{11}) = T_7I(M_9M_5)T_{11} = T_7IM_9T_{11} = T_7I(M_9T_{11}) = T_7I(T_{99}M_9) = T_7(IT_{11}M_9) = T_7(T_{-11x9}M_9) = T_7(T_3M_9) = (T_7T_3)M_9 = T_{10}M_9$
  - So, in short: $T_7IM_9IT_{-4}M_5T_3 = T_{10}M_9$.

As the above steps in the calculation testify, working out the reduction of strings of canonical transformations may be too complicated, generally speaking, to do in one's head; although it would be easy to write a program that automated this calculation. The big takeaway is that, no matter what your fancy is, ALL strings of these concatenated canonical transformations can be reduced to a single pair of T and M. From that vantage point, learning and encoding all or many of those T&M pairs (and their results) for a particular set class IS manageable.

Due to the various symmetries, there are 66 T & M pairs in total. Note that since $T_xM_y = T_{-x}M_{-y}$, when taking the overall tally, complementary subscripts of M cancel each other out. So, there are 12 distinct ($T_{0thru11}$ x $M_{1,11}$) rather than 24. The same reasoning applies to: 12 ($T_{0thru11}$ x $M_{0,12}$), 12($T_{0thru11}$ x $M_{2,10}$), 12($T_{0thru11}$ x $M_{3,9}$), 12($T_{0thru11}$ x $M_{4,8}$), 12($T_{0thru11}$ x $M_{5,7}$); and by extension 6 ($T_{0thru11}$ x $M_{6,6}$). Nonetheless, this yields 78 transformations rather than 66. It turns out that when speaking about the set as a whole, {($T_{0thru11}$ x $M_{1,11}$)}= {($T_{0thru11}$ x $M_{5,7}$)} yield the same results; $M_{1,11}$ and $M_{5,7}$ are counterparts under M_5.

Nonetheless, even if we can get these transformations to look simple on paper, they manifest very differently when applied to various set classes. $M_{1,5,7,11}$ will take any set class (or its M_5 counterpart) through all 12 transpositions. However, when that particular set class has multiple symmetries, there will be a fraction of transpositionally distinct representations of that set class. On the other hand, the other subscripts of M partition the space in such a way that repeated applications of them will never yield all the available transpositions. $M_3 \times M_3 = M_6$; $M_6 \times M_3 = M_9$; $M_9 \times M_3 = M_1$; $M_1 \times M_3 = M_3$. The important takeaway is that if you have a few set classes that you love, you could encode some portion of these 66 possible transformations into an easy-to-remember trace.

For instance, take 6-Z13* under T_3M_4; there are only 3 distinct representations:

- 6-Z13* (0,1,3,4,6,7): $T_3M_4$6-Z13 = T_3(0,4,0,4,0,4) = T_3(0,4) = (3, 7)
- 6-Z13* (1,2,4,5,7,8): $T_3M_4$6-Z13 = T_3(4,8,4,8,4,8) = T_3(4,8) = (7, 11)
- 6-Z13* (2,3,5,6,8,9): $T_3M_4$6-Z13 = T_3(8,0,8,0,8,0) = T_3(8,0) = (11, 3)

- Each diminished triad gets its own pitch.
- This could be encoded as "take 2 from" {3,7,11}, T_3 [3-12*][2-4], or a variety of other expressions. It just depends on how specific you want to be. Also, if you have a familiarity with M_4, its mapping each pitch of a diminished 7th chord to a single pitch, and 6-Z13*'s consisting of two disjunct diminished triads, then this result may have been obvious.

Any more explicit instructions about how to handle (37e) may be a bit of a headache, though. It requires multiple types of encoding. Those explicit instructions could be: first, remember (37e); second, take the remainder of 6-Z13*'s root and 3, third, use this remainder to pick the appropriate dyad: remainder 0, (37); remainder 1, (7e); remainder 2, (e3). Personally speaking, if I were improvising, "take 2 from {3,7,11}" would be enough to spark my imagination and lead me to feel that I was still forging a coherent link. Again, though, all of the separate encoding and longer instructions are easy enough to internalize if you find that set class compelling enough to put the necessary time in.[97]

[97] To aid with *traces* that use inversion and transposition, an asterisk (*) could indicate inversion. For example, take the opening to *Romanesca* a la Monteverdi, [F-C-d-a]. Written as a concatenation of canonic transformations it's T_7 (IT_2)T_7 [3-11b]. Written as a phone number, it's 72*7. Lastly, written as a melody (if you prefer to avoid the asterisk), it's G_4-D_4-G_4, <727>. When the *trace* melody ascends, it's transposition [the implied starting point is C_4]; when it descends, it's transposition and inversion.

In spite of its complexity, this last method can still be seen as an extension of methods that Monteverdi employed in “Ardo, si ma non, t’amo.” Just as Monteverdi anchored much of his composition in the Romanesca (a concatenation of canonic transformations on 3-11), the student can use traces to help them anchor their own improvisations/compositions in other concatenations of canonic transformations on a particular set class.

Finally, the third type of encoding involves using a familiar set class to guide your efforts. As there is less general guidance that can be given on this, this section is quite short. Remember, any given 7-note set class has 60,480 (12 x 7!) representations, how and why one person may find one or a few of its representations memorable could be as individually determined as the individuals who adopt this approach. For instance, Jack Boss thinks of 3-2a as linear motion around the tonic (ti-do-re). 3-6* could be Schenker’s 3-line and 5-23*, his 5-line. Furthermore, set class 4-19, which could be understood as an augmented triad with a half-step against one of its members, may remind you in what important way 3-11 (the major/minor triad) is similar to 3-12 (the augmented triad). The consequences of this observation drives Cohn’s Hyper-Hexatonic System; <0148> = <048> + <148>. The point here, though, is not to assert that memorizing 5-23 may be easier to remember than 5-4-3-2-1. Rather, it just demonstrates how a familiar set class representation, which you’ll acquire more of, could be co-opted to help learn, reinforce, and better utilize some musical material that you are focusing on.

PART IV: WRAPPING UP

Part IV, Chapter 19: Conclusion

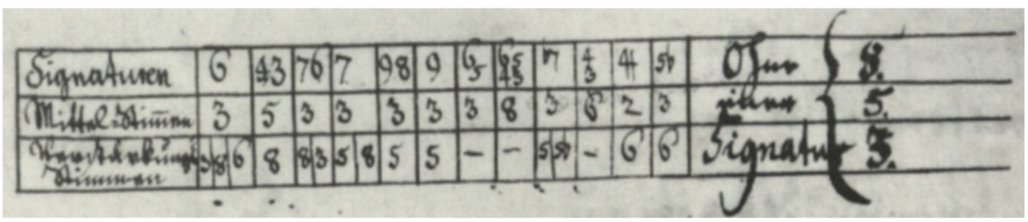
In *The Design of Everyday Things*, Donald Norman asserts that humans are okay with *complexity,* just not with *complicated*. In his lingo, *complicated* implies that an object has a confusing design; how it's presented obscures what it does. However, humans ARE amenable to engaging very complex objects (e.g. an automobile). The two main provisos are that: one, the design illuminates how it works (e.g. a door with a knob signals that the knob should be grasped and pulled); and that, two, this object can be learned in the context of performing some task. Following Norman's advice, this dissertation provides students with the means to elegantly design tasks that help them each achieve their unique, and potentially very complex, musical goals; particularly, those goals that are structural in nature. Furthermore, in Norman's chapter on conceptual models, he shows that the best conceptual model facilitates our understanding of why a product works the way that it does; this is not always synonymous with the model that is the most accurate. Ultimately, the best conceptual model for a product is the one that best engenders the desired behavior.

Humans are not machines; in general, we cannot reliably perform complex calculations accurately – and according to Norman, nor should we be expected to.[98] In other words, any pedagogy that relied strictly on the student's real-time non-computer-assisted calculations of how various pc-sets relate would be doomed to failure. The vast majority of us do not have the mental resources for such an endeavor. Instead, this pedagogy seeks to overcome limits in humans' working memory by assisting the student in building a robust intuition (heuristics) and a good (though not perfect) conceptual/working model for quickly assessing how the pc-sets that they care about relate. One famous example of how an incomplete conceptual model can still yield far reaching insights is J.S. Bach's helpful aid on preferred doublings (Figure 8); his table encourages the execution of good counterpoint (J. S. Bach and Poulin 1994, 61). Furthermore, as the student's set class interests grow, their optimal conceptual model(s) would be flexible enough

[98] Although, as mentioned in the introduction, to what extent we can reliably train ourselves to accurately perform real time calculations with our body, has not been thoroughly examined. Knowing more about this, however, would be very helpful to the student designing their course and in the overall framing adopted by this pedagogy. Over the course of this dissertation, I intend to do more research on this.

to accommodate added information. Nonetheless, finding the most usable inaccurate/incomplete conceptual/working model requires either an accurate conceptual model to begin with, a bucket load of experience to draw from; or preferably, both. At maximum, this dissertation will fully provide that information; at minimum, it will distill the most pertinent take-aways and point the student to sources more dedicated to that line of inquiry. There is just not enough space to derive everything fully.

Lastly, I hope that this dissertation will be a source of inspiration and aid to a plethora of music-pedagogical objectives; with some foreplanning, the student of theory, composition, and performance will be able to *improvise* (executing creative pedagogical tasks) their way into a very powerful understanding of music's structural components. In the meantime, hopefully, they will also enjoy doing it.



This table is from a facsimile of J.S. Bach's preferred doubling:
The top line lists the figuration;
the bottom line, what can be doubled.
Figure 4.1 1

APPENDICES

Appendix A: Music/Math Texts Overview: Neo-Riemannian Theory

Primarily, this Appendix will introduce Neo-Riemannian theory through two texts. The first is Lewin's seminal work, Generalized Musical Intervals and Transformations (GMIT); the second is Derfler's Single-Voice Transformations: a model for parsimonious voice leading. The former book gives insight into the "mathematical soul and disposition" that first breathed life into Neo-Riemannian theory (and many other topics that this dissertation later addresses!!); the latter offers a good survey of the field.

Finally, this and the other appendices are meant as resources and general background for the dissertation's following text. As such, not all ideas presented shortly are directly referenced later. Instead, this overview is just meant to: one, give greater form to a general perspective, articulated by Lewin (and many other authors later referenced) that deeply influenced this dissertation; and two, provide a starting point for your own potential follow-up research later.

Perhaps Klumpenhouwer's (Klumpenhouwer 2000, 157) description of Neo-Riemannian theory as 'more than a set of topics explaining how the non-functional voice-leading properties of triads interact with tonal music,' encapsulates the general spirit of this interaction between mathematics and music inquiry. Neo-Riemannian theory is actually a process by which: one notices patterns that could benefit from a group-theoretic treatment; one rigorously defines the studied objects and how they interact; one explores the implications of this new group-theoretic model; and finally, one applies this new model to the music literature. Bearing this spirit in mind we begin this section with Lewin, an overview of his text, Generalized Musical Intervals and Transformations (GMIT), a background for his introduction of Neo-Riemannian theory, and a definition of group theory.

GMIT

GMIT rigorously provides an introduction of mathematical group theory to music. Ostensibly, group theory looks at mathematical structures called groups.

One type of group structure is a semigroup;[99] it is a set plus a binary operation, BIN, that is closed under said operation, contains an identity element, e, and each element has an inverse (I will confine myself to left inverses). In other words, for any two elements, x & y, of the set, BIN (x,y) is contained in the set and there exists inverses, I_x and I_y such that $I_x x = e = I_y y$.

The following example (Figure A.1)[100] demonstrates a type of group structure; it's called a group action. The group structure describes the actions themselves. In this example, the actions are rotations by multiples of 120º, $\begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}$ and the underlying set is the vertices of an equilateral triangle — $\{(\frac{-1}{2\sqrt{3}}, \frac{1}{2}), (\frac{1}{\sqrt{3}}, 0), (\frac{-1}{2\sqrt{3}}, \frac{-1}{2})\}$; $I_{120} = 240º$, $I_{240} = 120º$ and 0º, 360º = e.

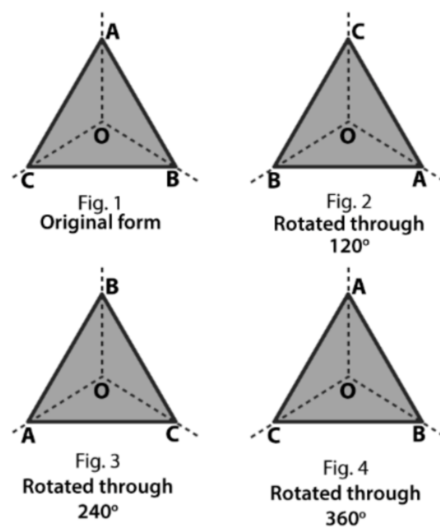


Figure A 1

[99] One example of a semigroup is the chromatic scale (the set) plus T_5 (the binary operation).
[100] "How Many Times Will an Equilateral Triangle Look the Same If Rotated through a Full Rotation?" byju. Accessed August 16, 2023. https://byjus.com/question-answer/how-many-times-will-an-equilateral-triangle-look-the-same-if-rotated-through-a-full/.

As Klumpenhouwer indicated, group structures are abundant; from million-faced crystals, as the example above may have hinted at, to set classes under transposition. In his book, Lewin examines a myriad of group structures that arise in various conceptions of our musical space (timbral, harmonic, durational, etc..): specifically, structures that pertain to measurements of distance in said various spaces.

Lewin defines his GIS (Lewin 2007, 26), the Generalized Interval System, "as an ordered triple (S, IVLS, int), where S, the *space* of the GIS, is a family of elements {a set}, IVLS, the *group* of intervals for the GIS, is a mathematical group, and *int* is a function mapping S x S into IVLS, all subject to the two conditions (A) and (B) following.

(A): For all r,s, and t in S, int(r,s)int(s,t) = int(r,t).
(B): For every s in S and every i in IVLS, there is a unique t in S which lies the interval i from s, that is a unique t which satisfies the equation int(s,t) = i.

In short, in any representative GIS, there is an underlying set S, a group of Intervals/distances (actions) on the set, and the function (a binary operation) that can both compose intervals with each other as well as with members of the underlying set. The only stipulations are that each interval in IVLS applies to every element of S, yielding exactly one other element of S, and that distances accumulate; if there is a distance between s1 and s2, and between s2 and s3, then there is a defined distance between s1 and s3.

Important definitions discussed include equivalence relations and congruence classes. Essentially, in the context of a particular operation, two things are deemed equivalent (under some equivalence relation) when how they "behave" under said operation are indistinguishable. Or put formally, under a relation R, the elements a and b are deemed equivalent if:

1. a ~ a (the reflexive property)
2. a ~ b implies that b ~ a (symmetric property)
3. a ~ b and b ~ c implies that a ~ c.

- An example of an equivalence relation: under "transposition to a pitch whose letter name is a half-step above," pitch and pitch-class space are equivalent. Regardless of register,
    1. $C_4 \sim C_4$: $T_1C_4(D_b) \cong T_1C_4(D_b)$
    2. $C_4 \sim C_5$ & $C_5 \sim C_4$: $T_1C_4(D_b) \cong T_1C_5(D_b)$
    3. Just $C_4 \sim C_5$ & $C_5 \sim C_6$, $C_4 \sim C_6$: $T_1C_4(D_b) \cong T_1C_6(D_b)$
- An example of a congruency class:

Under the above equivalence relation, "transposition to a pitch whose letter name is a half-step above," the set of all octave-related pitches in pitch space yield a congruency class.

An additional helpful term is "surjective homomorphism." For example, there is a surjective homomorphism between pitch space and pitch-class space; in other words, every pitch in pitch space can be mapped to exactly one pitch-class in pitch-class space.

Relatedly, Lewin also broaches discussions around reference units; measurements cannot be discussed independently of the units that they are measured in. Along these lines, he offers a more intuitive reference/unit of intervallic-inversion than previously used: take the intervallic-inverse in regard to a chord/scale's outer pitches rather than the constant that the inversely-related scales' pitches (presented in opposite order) add up to.

In short, Lewin pays significant attention to:

- Grounding many of our musical conceptions/intuitions—such as "passage x seems twice as long as passage y" in mathematical spaces; and,
- Through the introduction of his generalized interval, enabling our comparison of objects in said spaces.

Two things to note:

1. The actions in various spaces are not always commutative; meaning that a * b ≠ b * a.
    a. For instance, the composition of T_3 and M_4 is not commutative—e.g., $(T_3I)M_4$ {C,E,G} = {F#, D} ≠ {C,G#} = $M_4(T_3I)$ {C, E, G}. Morris explores this extensively in his Class Notes for Advanced Atonal Theory (R. Morris 2001, 83–111).

2. Differences in perceptual units are not always compatible with differences in discrete units (such as pitch level) — e.g., one is exponential and the other linear.
    a. Accordingly, Lewin offers a GIS framework accommodating differences that are proportional rather than fixed — the next duration is evaluated as x times longer rather than x units longer than the previous duration.
    b. Similarly, Lewin offers a GIS framework for timbres, formulated as spectra—aggregates of the fundamental frequency's partials at various amplitudes; in this GIS, intervals are the difference/relation between two spectra.

Lewin then looks at conceptions of musical measurement more generally. In Chapter 5, he:

1. Reformulates Forte's interval vector as a particular case of his own function IFUNC—IFUNC tallies the number and type of intervals between two elements in the underlying set of a GIS; these could be pitches, rhythms, etc.
    a. Forte's interval vector then is IFUNC applied to a pitch set and itself.
2. Recasts IFUNC as more than just a way to count up intervals; it is offered as an analogue to our probabilistic listening practices.
    a. The specific number and types of intervals laying between two chords, can predispose us — to greater and lesser degrees — towards hearing those same intervals in the resultant chords.
3. Generalizes IFUNC even further, yielding the embedding (EMB) function—a function that tallies the number of appearances of one chord (or any other element s of the underlying set S) in another.
    a. Note that how we classify a chord[101] impacts the number of instantiations of a chord that we find in another;
    b. Note that EMB's can also be understood in a probabilistic way; as a ratio between:
        i. The number of ways a particular chord, a, is embedded in another chord, b, and
        ii. The total number of possible chord types in b.

[101] For instance, by CANON, those transformations that preserve intervals—such as transposition and inversion. However, others are possible.

Nonetheless, this abstraction is not yet complete. In Chap 6, he:

1. Explores the injection function (INJ) and builds up machinery towards the analysis of serial works. The big differences between EMB (X,Y) and the INJ(X,Y) function are:
    a. One,
        i. EMB counts how many of X **are** in Y,
        ii. INJ counts how much f(X) is in Y, although there is no requirement that this "function" is either 1-1 or onto.
    b. Two,
        i. In EMB, there is an assumption that the specific compositional processes that relate X to Y does not systematically change X;
        ii. In INJ, there is not. INJ then counts the number of occurrences of the modulated form of X in Y.
2. Two functions relevant to atonal analysis are also introduced, W_p and PROT.
    a. W_p, a wedge, describes and/or classifies a pitch via its proximity to the pitch p;
    b. PROT is a set of protocol pairs, pairs of pitches—let's say (p,q);
        i. i.e., in a particular row, L is the subset of PROT, wherein p precedes q.
        ii. i.e., INJ and PROT can interact when discussing how Schoenberg transforms an underlying row to yield specific protocol pairs and, relatedly, how a motive unfolds over a row's span.
3. Light is shed on a problem often posed in atonal music analysis, how can a section of a serial piece sound harmonically homogenous even though the harmonic complexity of its musical surface seemingly indicates otherwise.
    a. Similarly, by comparing how INJ and some modulating function is applied:
        i. *externally*: between two sets—X and Y, and
        ii. *internally*: within each set—X to X and Y to Y,

    the degree to which a given transformation of X is *progressively* or *dispersively* embedded in Y can be assessed.
4. Finally, Lewin develops this machinery further to investigate topics such as:
    a. "Under what circumstances does a certain transformation not change the relation between X and Y?" and
    b. Forte's K and Kh analytical approaches.

Chapter 7 introduces STRANS. STRANS (a simply transitive group of operations on the underlying set S):

1. Expands Lewin's construction of GIS by admitting intervals in IVLS that are not proscribed by a single type of measurement (let's say distance in half steps) but by a host of transformations that apply to each element of IVLS.

Recasts IVLS as an "indexing" function that one-to-one maps STRANS to an ordered list.

This expanded GIS model can accommodate such concerns as:

1. Perceived distance does not always equal literal distance.
    a. "Distance" can be treated as a gesture, e.g., very far or close, as opposed to an absolute amount. This "transformational attitude is much less Cartesian (Lewin 2007, 159)."

This gesture, in turn, can then be considered as an abstract template which can be realized in various ways. Along these lines, Lewin uses network-graphs as the "middle-ground" of a particular transformational gesture.

Chapter 8 digs in to network-graphs associated with Neo–Riemannian transformations; the potential trajectories of interval-preserving (transposition and inversion) functions on triads.

Finally, in chapter 9 to the end, Lewin formally defines and then discusses a(n):

- **Transformation graph** as an ordered quadruple (NODES, ARROW, SGP, TRANSIT) such that
    - the nodes and arrows combine to create a more general "network-graph"
        - two nodes on the graph need not be connected,
    - SGP is a semigroup,
    - TRANSIT is a function mapping ARROW into SGP, and
    - the semigroup product of any two paths (the combined "distance") sharing start and endpoints is the same.

- **Operation graph** as a transformation graph in which SGP is a group.
- **Transformation Network** is a Transformation graph with CONTENT; meaning that a transformation graph is presented with its context — the specific underlying set to which this "gestural template" is applied.

The rest of the book is devoted to examples.

SVT (Single Voice Transformation)

Where Lewin's text offered a deep framing of the field overall, the following discussion, using Derfler's book as a starting point, outlines Neo-Riemannian theory's various subfields. Why is Derfler's 2010 book so comprehensive in its overview of the field?

1. Essentially, it is a published dissertation.
2. Derfler's new metric (and label) for measuring voice-leading distance, single-semitone transformations (SST) and single-fifth transformations (SFT)[102] proves useful in all of these mentioned subfields.

Furthermore, as so much of this dissertation's proposed pedagogy refers to parsimony, construed as SSTs (although, that label is not taken up), SST in of itself is an important topic to broach here.

SST and SFT

Background: According to Derfler, the most common way to compare (well-chosen representations of) set classes, each expressed as ordered vectors of the same cardinality, involves tallying up the distances between their various components. These distances can also be expressed as a transposition, $\mathbf{T}^n$, between respective components of the two pc-sets. In this way, a new type of vector can be introduced, not one that lists elements of a musical object, but rather a "transformational" vector that describes an operation, in this case T_n, whose domain and range are pitch class vectors.

[102] Here 'fifth' should be understood as 'perfect fifth.'

Derfler's SST and SFT are then just such 'transformational vectors,' wherein all but one of their components is the unison transposition T^0.

- In SST_i, i equals the order number of the non-identity transformation; the non-unison transposition is strictly a semitone.
    1. Successive applications of SST can yield any SVT.
    2. SFT, single fifth transformation, substitutes the P5 for the semitone.
    3. Combinations of "successive applications" of SST can yield an SST
    4. To be called parsimonious, the number of "successive applications" of SST required should not be greater than half of the related pc-set's cardinality.
- To describe voice leading between pc-sets of different cardinalities, he introduces:
    1. a $SPLIT_i$ function that transforms a pc-set, expressed as vector, by:
        - Duplicating its i^{th} element,
        - Shifting all ordinal positions greater than i one unit, and then
        - Placing this duplicate in the (i+1) position.
    2. A FUSE function is the inverse of the split, $SPLIT_i^{-1}$.'

SST-succession (and SPLIT-succession) classes acknowledge that, without T_n/T_nI as an equivalence relation, there are many shortest distances between two **set classes** (not pitch-sets).

The three general topics discussed are:

1. The benefits and disadvantages of various framings of voice-leading motion. The theorists discussed are:
    a. Klumpenhouwer and K-nets
    b. Straus's "simply charting the various voice leading paths."
    c. Lewin's recasting voice-leading paths as set relations rather than functions
    d. Callender's looking at continuous transformations between sets.
    e. Tymozcko's representation of chords "as coordinates in n-dimensional geometrical spaces.

    Note that all of these authors/topics, are further discussed in other portions of the mathematical section of this dissertation.

2. The benefit and disadvantages of various visualizations of voice-leading space. The three topics focused on are:
    a. Visualizations of voice-leading space that are rooted in the *Tonnetz*.
    b. Graphs of Optimal Offset between SC Types of the Same Cardinality.
    c. Graphs of Optimal Offset between SC Types of Different Cardinalities.
3. The effectiveness of various labels that describe voice-leading motion in set class space.

Topic 1

a. **K-nets**.

Klumpenhouwer's 1991 dissertation introduces K-nets.[103] K-nets describe the voice-leading between two pitch sets as:

1. Transpositional (T_n) and inversional (T_nI) operations on pitches.
2. A rearranging of parsimoniously-related pitches.
    a. For instance, an A in one voice part, let's say the tenor, goes to a Bb in another, let's say the alto.

The disadvantages (according to Derfler):

1. No voice crossings are permitted in this model.
    - Its visual presentation does not facilitate that level of nuance.
2. A quotient GIS is created that "makes all transformations with the same permutation cycle congruent, regardless of the TTO employed" (Derfler 2010, 11).
    - It can obfuscate the actual musical processes that it is intended to model.
3. A further limitation involves difficulty with relating certain pairs of pc-sets.
    - It's not a universally applicable descriptor.
4. Finally, k-nets are not intuitive enough; models building on T_n/T_nI are preferable.
    - Along those lines, Derfler also criticizes voice-leading approaches (Callahan and Jurkowski's) that are either rooted in K-nets and/or their underlying principles/machinery—such as Perle's cyclic arrays.

[103] One can argue that K-nets are just slivers of George Perle's later-discussed cyclic arrays.

b. "**Simply charting the various voice leading paths**." **(Straus)**

Straus removes much of Klumpenhouwer's overhead by:

1. Simply charting the various voice leading paths and
2. Taking account of the overall voice-leading motion in musical sections of various lengths.

Related topics covered:

1. O'Donell's dissertation that acknowledges that listener's perception of voice leading may not just be related to contiguous chords; i.e., register, but also other factors as well.
    a. As such, successful voice-leading analyses should ally with the (ideal) listener's perceptions.
2. Straus, relaxing the requirement that a single transformation type T_n/T_nI must describe the voice-leading path between two pitch sets, introduces split transformations (a Wedge).
3. Straus's "fuzzy transpositions/inversions" rate the similarity between two pitch sets by measuring the semitonal "offset" of the latter from the nearest T_n/T_nI transformation of the former.
    a. ""**Optimal offset**" is the closest possible distance between any two set classes of the same cardinality, measured in terms of total displacement (Derfler 2010, 27)."
    b. The measure of parsimony (*smoothness*) maintained can be at odds with this measure of similarity.

Disadvantages of Straus:

1. Relating pc-sets of different cardinalities is problematic due to:
    a. The mathematical considerations that Lewin considers below.
    b. Straus's notation leaves unspecified exactly which voices, when the offset is optimal, move where.
    c. One solution: Roeder's notation just compares two ordered pitch vectors (one for each pitch set) and then labels the T_n operation between corresponding component

pitches (or intervals); there is no requirement to assess the latter chord as a transformation of the first chord in its entirety.

c. **Voice-leading paths as functions**.

As voice leading paths do not have unique inverses—there are multiple ways to reconstruct a prior chord from voice-leading paths anchored in the latter, Lewin grapples with defining these paths as functions. Instead, he describes them as set relations.

- These difficulties with a function-based definition become more prominent when addressing music from the late 19th and early 20th century— as common era tonal practices were further extended, there arose greater ambiguity involving the interpretation of harmonic ambiguities.

d. **Continuous transformations between sets**.

Callender looks at continuous transformations between musical sets. These sets are represented as ordered vectors /A/ and can refer to beats, pitches, or harmony that are equivalent under transposition, inversion, permutation, and octave displacement. Distances are the minimal distance found between any two members of the compared set classes $p(/A/, /B/) = \min(p(A', B'))$ for all $A' \in /A/$ and $B' \in /B/$(Derfler 2010, 37). He displays these distances in spaces that are bounded by set classes of a given cardinality. For instance, for cardinality three and its nineteen trichords (Π^3), Callender presents a Euclidean space that is bounded by 3 intersecting lines that together form a right triangle.

e. **N-dimensional geometrical spaces**.

Tymoczko represents chords "as coordinates in n-dimensional geometrical spaces called orbifolds." $\mathbf{T}^n/S_n$ contains all unordered n-note chords of pitch classes, where $\mathbf{T}^n$ represents the quotient space(n-torus) $(\mathbf{R}/12\mathbf{Z})^n$ and S_n is the symmetric group of n. Minimal voice leading between (in which a single chord voice moves by a single semitone) is shown by lines connecting plotted chords in the orbifold (Derfler 2010, 37).” Tymoczko’s description of voice-leading accommodates duplicated chord components.

Topic 2 —Different visualizations for motion in voice-leading space.

The Tonnetz

The *Tonnetz*, a lattice of pitch classes (or pitches), can be referenced to indicate how a triad (represented as a triangle) parsimoniously relates to other instantiations of it. Adjacent instantiations share one or two pitches. To “transform” a triad of three vertices connected by two lines (sides) into one of its neighbors is described by flipping one instantiation over one of its sides (or single pitches). These bi-directional flips over a side are called L, P, R.

- L (*Leittonwechsel*) maps a C major triad to an E minor triad;
- R (relative) maps a C major triad to an A minor triad; and
- P maps a C major triad to a C minor triad.

These relations hold at any pitch level: L maps Db Major to F minor; R map E minor to G major (or G major to E minor) etc.

The flips over a single pitch yield: C Major to Db minor, F minor, and G minor.

Notice, however, that the Tonnetz only addresses voice-leading relations between triads. Derfler then surveys other charts that connect various set classes that are not always of the same cardinality.

The other charts sometimes connect:

- A handful of tetrachords
    - For instance, 4-27, which contains the dominant seventh and half-diminished chords
- Just the triad
    - For instance, Cohn’s “Cube Dance” (Cohn 1998)

In most occasions, these charts are 2D; however, some are in 3-D (Gollin). All illuminate trajectories that late 19th/early 20th century composers (partially or fully) navigated harmonically (consciously or unconsciously). Lastly, these charts vary in terms of how much they specify the specific-voice leading paths.

"**<u>Graphs of Optimal Offset between SC Types of the Same Cardinality</u>** (Derfler 2010, 76)."

These graphs, which are extremely pertinent to later portions of this dissertation, visually connect "strongly parsimonious" (a single SST-succession path) set classes. Many of these charts, mostly applying to tetrachords and triads, are:

- Rendered in 3D,
- Were created by either Straus or Derfler, whose SST labels now specify the voice-leading paths.[104]

Derfler, importantly, points out a core concern pertaining to voicing leading maps in set class space of cardinality 4; whether or not one should duplicate set class 4-19. Without duplication, the geometry of the map is distorted; this is tied into 4-19's maximally even subset's (3-12*) bifurcation of pitch class space. Presumably, duplication impacts:

1. The interpretability of voice-leading distance writ large in the offset maps.
2. The maps compatibility with other maps — "it does not result in a toroidal lattice similar to the one we can obtain from bending the Tonnetz (Derfler 2010, 87)."
3. These maps ability to "clearly model the role that specific set classes play in the Trichordal/Tetrachordal voice-leading system (Derfler 2010, 102)."

<u>"Graphs of Optimal Offset between SC Types of Differing Cardinality</u> (Derfler 2010, 91)"

[104] N.B. Lewin's *Cohn function* is a function with arguments mod N than can be Cohn-flipped in two different ways." In turn, "a mod N function is *Cohn flipped* when a n exchange of different … values at adjacent arguments gives rise to a rotated retrograde of the original function (Derfler 2010, 81)." The "input" of Cohn-functions and flipping are pitch-set classes that are expressed as ordered sequences of intervals rather than pitches; i.e., a major triad is (4,3,5).

Derfler offers a graph that embeds a tetrachordal offset map into Straus's 2003 trichordal offset map. The SPLIT^{-1} function provides the connection.

- This graph's multiple layers allow strongly parsimonious tetrachords to exist on the same layer or between adjacent layers;
- The trichords are the "backbone" that supports the overall structure.
- The more tetrachords with which a tetrachord is parsimonious to, the lower its layer is.

In 2005, Straus offers another map that puts the trichord and tetrachord chords into dialogue. However, according to Derfler, Straus's privileging conformity in the geometry of the map makes many concessions regarding set class duplications.

Topic 3 - The effectiveness of various labels that describe voice-leading motion in set class space

Finally, Derfler compares how various theorists describe parsimonious relations between two members of the same set class and two members of differing ones. Considerations include:

- Degree of specificity regarding the behavior of individual voices,
- Whether dualist language is used,
- The languages dependence on tonal convention (such as identifying the root),
- The set classes covered,
- Whether they describe the transformation of a set class or its individual pitches,
- The conditions in which the proposed terminology may be vague,
- The generalizability of the terminology to set classes other than those focused on,
- The reasoning behind the naming (aural similarity vs distance metric),
- The underlying definitions of parsimony (is it okay to move three voices by a half-step?), and both
- How systematic the reasoning is behind the labels and how exhaustively those labels are applied.

While there are some contenders for alternative labeling systems regarding specific concerns, overall, Derfler demonstrates that his SST (SFT) labeling system, performs well regarding each metric in regard to the tonal and post-tonal literatures. To wrap up this chapter of his book, he creates a few charts that compare how various labelings accommodate the same set class relations.

Finally, on this topic, in his chapter on Dualism in the Oxford Handbook of Neo-Riemannian Theories, Tymozcko rebuts Riemann's attempt to describe chord progressions dualistically and extends a concept that he attributes to Cohn, "dualistic terminology has a natural application to questions about voice leading (Gollin and Rehding 2014, 256)." In short, efficient voice leading (*evl*) can be classified dualistically; meaning that chord A *evl* B can be grouped alongside A^{-1} evl^{-1} B'^{-1}; where A and B are chords and B' is a transposition of B. **This dualist classification of voice leading halves the number of *evl* transformations one would need to learn should one wish to internalize interactions between two inversely-related chords at various transposition levels. I appreciate this pragmatic sentiment that finds ways to reduce the overhead needed for mastering how important chords relate**.

Again, while my dissertation will not explore directly all of the leads provided in this chapter, I recommend working through these topics that speak to you most. Furthermore, Derfler's list of considerations regarding the benefit of various voice-leading labels is absolutely golden. Consider referencing this list later if you (formally or informally) adapt set class labels (discussed extensively in Part III) to reflect and accommodate your specific concerns.

Appendix B: Music/Math Texts Overview: Similarity Relations

This chapter introduces Similarity Relations through four seminal texts: Schuijer's Analyzing Atonal Music; Quinn's three-part article "General Equal-Tempered Harmony," Callender's "Continuous Harmonic Spaces;" and finally, Tymoczko's "Set class Similarity, Voice Leading, and the Fourier Transform."

Schuijer

Analyzing Atonal Music provides an introduction to and overview of set class theory; notably, it contextualizes its development historically. Set class theory, as most other fields, is an outgrowth of communal debates over theoretical concerns of interest. Relevant to this section, Schuijer focuses in on those "debates' regarding "what is the best way to classify pitches?" and "what is the best way to compare and/or measure distances between set classes?" However, determining said best way is not a neutral concern; one significant factor is:

- The specific music being analyzed.
    - For instance, when the musical terrain is dense and unfamiliar, Forte's set class labels may have an advantage; they can quickly relay meaningful information about the harmonic landscape; at minimum, they can be a great first step.
    - However, when common chords are still given premium treatment, chord rotation is deemed worth specifying, and grading dissonance is a desirable trait Krenek's and Hindemith's less systematic classification systems (Schuijer 2008, 138–44) may be a better fit.

Below is an account of those similarity measures Schuijer introduces and then evaluates. He divides similarity measures into three camps; approaches rooted in the:

- Interval-vector
- Subset inclusion, and
- "Absolute" distance.

I'll first look at the interval-vector approaches.

Interval-Vector approaches:

Teitelbaum, circa 1965, compared pitch-class sets' interval vectors by their interval (**APIC**) content (132). He calculated his similarity index by taking the square root of the sums of the squared differences between the two interval vectors' components. It's shown below (Teitelbaum 1965, 88).

$$\sqrt{\sum_{j=i}^{6} (\text{IVEC (I)}_j - \text{IVEC (L)}_j)^2} = \text{Similarity Index}$$

Figure B 1

Morris introduced **SIM**, which, amongst similar set classes was intended to "insure predictable degrees of aural similitude (Schuijer 2008, 134)." Morris calculated the difference vector between compared interval vectors by summing the differences between the corresponding interval vectors components. Note that both this method and Teitelbaum's return a single number that is insensitive to the cardinality of the distance vector.

Forte, comparing set classes of the same cardinality, offered three interval-vector based measures of similarity—pertaining to sets of the same cardinality. Rp pertains to the next subsection.

- R_0 (minimal similarity),
  - When there are no interval components in common
- R_1 (first-order maximal similarity),
  - First-order maximally similar requires that four components be identical and the the remaining two components, in opposite order, be the same.
- R_2 (second-order maximal similarity),
  - Second-order maximally similar requires that four components be identical and does not require that the remaining two components, in opposite order, be the same.
- R_p (sharing a common subset of cardinality N-1).

Schuijer's Forte related criticisms:

- Forte's $R_{0\text{-}2}$ measures do not address "medium degrees of similarity" between the interval vectors.
- Regarding $R_{1\text{-}2}$, different switched-two-interval-vector-components can differ greatly in how they impact a listener's perception of similarity — e.g., if the two "remaining" components are i4 and i5.

<u>Shared Subsets approaches</u>:

Regener introduced the:

- **Common-note vector** (of length 12) whose components indicate the number of shared pitches between the two pc-sets as the transposition-level of one of those pc-sets is shifted 12 times.
    - **Degree of inclusion**: the highest number in a "common-note" vector is called the degree of inclusion.
    - If the sets are of the same cardinality and the degree of inclusion equals cardinality n-1, there is an R_p relation
- **Partition-vector** tallies up how many times 3 common tones, 2 common-tones etc., are found in a common-note vector.

Schuijer's Regener related criticisms:

- It does not take into account the transpositions of the inversion (Lewin later adapts his method in such a way).
    - As such, shared intervallic content cannot be assessed as not every subset with the same intervallic content is taken into account.
- As the cardinality of the compared pitch-class sets rise, the interpretability of the partition vector wanes.
    - A later attempt by Regener to counteract this difficulty, this time by comparing partition vectors, still had shortcomings. An advantage of common-note vectors

was lost. Common-note vectors indirectly indicate the interval content of the compared set classes; partition vectors do not.

<u>Absolute Approaches</u>:

"Absolute approaches" responded to an increasing concern around the difficulty of comparing various similarity measures. For instance,

Morris's **ASIM** "standardizes" his measurement **SIM**. In short, Morris controls for set class size, operationalized as a set class's total number of intervals. Now he can give a measure that ranges from 0, when the compared vectors are the same, to 1, when they are maximally different—they have no intervals in common.

Schuijer's criticisms:

- Only set classes of cardinality 5 or less can be maximally different. As the cardinality of the compared sets increase, the measure's potential range shrinks; as such the significance of SIM, decreases.

Building on Lewin's EMB function, Rahn introduces **TMEMB** that counts the number of appearances of shared subsets of cardinality 3 or greater. "Thus, he obtained a value for the "total mutual embedding in A and B of subsets of all sizes greater than one (Schuijer 2008, 166)." Unlike Regener, all of the canonical transformations are now taken into account. As the cardinality of the compared set classes grew, so does the number returned by TMEMB.

Schuijer's criticisms:

- By a factor of two, Rahn's function overcounts shared subsets, tallying one for both of its instantiations in the compared set classes; this overinflating is further exacerbated when tallying up the many shared intervals.

Like Morris's **ASIM**, Rahn's **ATMEMB** normalizes **TMEMB**'s by placing the results between 0 and 1. "He divided TMEMB by the total number of relevant subsets in A and B (Schuijer 2008, 167)." Z-related sets got a score of 0 in Morris, but not in Rahn.

Lewin introduces REL[105] whose standardized scores also range from 0 to 1. Here, 0 is essentially Forte's R_0 (minimal similarity) and 1, Forte's R_1 (maximal similarity). Nonetheless, Lewin's measure, according to Schuijer, proves more versatile than Forte's as it can be applied to sets of different sizes. Plus, in using multiplication rather than addition (e.g., **ATMEMB**), the REL's highest scores reflect compared interval vectors whose components are in a direct proportion to one another. As such, REL was more sensitive to how exactly how the distribution of sets shared between A and B may affect the listener (Schuijer 2008, 169).

Schuijer's criticisms:

- Isaacson noticed that in REL_2, the full range of [0,1] only resulted when chords of the same cardinality were compared; extreme reductions in range occurred as the cardinalities of the compared sets diverged—so much so that the interpretability of the measure waned.

Isaacson's **IcVSIM** ("Interval-class Vector Similarity"), "which he called "a scaled version of Teitelbaum's similarity index (Schuijer 2008, 174)" mitigated against this last concern by adopting a statistical tool, standard deviation. By examining interval vectors' differences in regard to a normal distribution, the cardinality-related restriction in ranges would (hopefully) cease to obscure the significance (the likeliness or unlikeliness) of the various scores. Now maximal similarity could be achieved between sets of different sizes.
Schuijer's criticism:

IcVSIM's being intended to work with **PIC** rather than **APIC** meant that complementary sets were not maximally similar—as Schuijer points out, it has to do with the difference in how i6 is counted.

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105 Similarly, Isaacson introduces Rel_2. Here, the embedded "chords" are restricted to intervals.

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Thanks in large part to Quinn's and Callenders' soon to be discussed 2007-8 articles, the field of Similarity Measures is now relatively settled. Nonetheless, its concerns are still central to this dissertation's pedagogy. Well-designed maps, with a single glance, can maximally impact the readers' understanding of the described content and how to best navigate it. Straus and others also offered compelling maps — they are addressed in the Neo-Riemannian subsection of this chapter.

In particular, Quinn's article is striking in that it not only offers an intuitive, easy to use, and widely relevant similarity measure, but he arrives at it through a synthesis of theorists' perspectives dating back to the subfield's inception. As such, it's harder to imagine a better candidate for introducing you to the field. Furthermore, his article is also very creative and models how far one can travel—from similarity relations to set class-pedagogy and large taxonomic classification systems like Forte's genera, when seeking answers to bigger questions that one may have. As such, I think it beautifully exemplifies the spirit that I hope will accompany your open-ended embracing of this pedagogy.
Quinn's 2007 three-part article (published in two separate installments) "General Equal-Tempered Harmony" is built off of Lewin's characteristic function[106] (Lewin 1959, 301) and expounds upon the later coined **Intervallic Half-Truth** that was first discussed in his (Quinn's) 2001 article, "Listening to Similarity Relations." In short, the **Intervallic Half-Truth** relates to Quinn's observation that the tetrachords' very highly transitive clusters,[107] were differentiated by a prominent interval; i_1 through i_6. In other words, maybe, "half of the reason" for chords sounding very similar pertains to their all displaying a shared interval prominently.

In Part 1 of his 2006 article, Quinn shows "that higher-order taxonomic categories or **genera** are organized around privileged, highly symmetric chord species (prototypes), which are quite distant from one another in quality space. The taxonomic categories are structured by virtue of

[106] Lewin's characteristic function is the Euclidean distance, $d_p(q,r) = p\sqrt{\sum_i |r_i - q_i|^p}$, between two pitch sets, expressed as 1x12 vectors, a Cartesian coordinate in pitch space, <p_0, p_1 …., p_{11}> — where each entry is either a 0 or 1, depending on whether that pitch (p_0 = 0, p_1 = 1 etc.).
[107] "If a is related to b, and b is related to c than a is, to different degrees, likely related to c."

chords being qualitatively close ("similar") to the prototypes in quality space (Quinn 2006, 121).” In part 2, he uses Clough & Douthett (C&D) to rectify and expand on his **intervallic half-truth** (not one but two generators–one is the complement of the other—are associated with an ME set). Now, he uses qualitative genera. In Part 3, Quinn builds a “model of quality space on the basis of Lewin's 1959 technique, which is in turn based on the discrete Fourier transform (Quinn 2006, 121).[108]”

Part 1—Comparing Higher-Order Taxonomic Categories.

This part examines strategies for classifying chords. Towards this end Quinn adapts Hanson’s algorithm for generating prototypes (of a given cardinality):

1. Take a pitch
2. Add a pitch that is an interval *i* above it.
3. Repeat, until you either get a previous pitch or stop.
4. When you get a previous pitch, raise it by a half-step,
5. Repeat, until all 12 pitches are exhausted or you’re satisfied.

From this, Quinn then extracts 5 properties that collectively, or in part, shaped the development of other taxonomic categories.[109] This includes Eriksson’s maxpoints[110]—prototypes that have maximal inclusion of a particular interval.[111] Buchler’s “contextualiz[ing] the interval vector “by means of "tools that take account of what is minimally and maximally possible [*degree of*

[108] While Quinn coins his works as the Natural Kinds hypothesis, he disavows any claim that his methodology espouses a “truer” or the **most** natural way to relate set classes. Each of the previous similarity measures can position itself as a better adjudicator of similarity in particular analytical scenarios.
[109] The Unique-Prototype Property (UPP): Per cardinality, the six genera have exactly one prototypical species.
1. The Unique-Genus Property (UGP): One prototype cannot represent two different genera.
2. The Intrageneric Inclusion Property (IIP): One prototype of a genera includes all of smaller prototypes of the same genera
3. The Prototype Complementation Property (PCP): the complement of a generic prototype is another prototype of the same genus.
4. The Prototype Familiarity Property (PFP): Most, if not all, of these prototypes are rampant in the literature.

[110] It should be noted that under M_5 transformation these maxpoints’ status as prototypes and ranking under various similarity measures (those based on interval vectors) are invariant.
[111] Unfortunately, these maxpoints, and others that seek a similar definition, do not meet the UPP (6-35* and 6-20* have the same number of ic 4) and UPG (5-33 arises in the 2 and 4 genera) condition.

saturation] in a given cardinality (Quinn 2006, 134)" and thus facilitating interpretable comparisons, *intrageneric affinity*,[112] between chords associated with a given genus.

Quinn then compares the various inclusion-based classification systems such as Hanson's (not explicitly expressed one) and Forte's genera, supergenera, and K and Kh complexes. In short, he concludes that "inclusional similarity relations end up producing largely the same results, [same list of pc-sets, and invariance amongst interval vectors under an M_5 transformation] as intervallic similarity relations (Quinn 2006, 143)." For instance, (super) genera deal with tri-chords that he labels as augmented, diminished, etc.

Finally, Quinn looks at Morris's algebraic genera[113] and Cohn's cyclic approach.[114]

Lastly, Cohn's approach develops his five set-group systems of his own; each is based on a CYCLE homomorphism, a kind of nonstandard multiplicative operator. In essence, it recasts the sets in modulo x; reducing certain maximally even sets into smaller universes.

<u>Part 2</u>

In part 2, Quinn, referring to prototypes—and both intergeneric and intrageneric affinities, crafts "the infrastructure of a theory of chord quality that unifies the distance- and taxonomy-oriented modes of theorizing (Quinn 2007, 4)." To do so, he introduces and then builds off of Clough and

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[112] *Intergeneric affinity* refers to how M_5 related chords can be of different genera but receive the same similarity rating in regards to another

[113]Morris's algebraic genera

1. "Under SG(v), two chords belong to the same species if and only if they have the same interval content.
2. Under SG(1), two chords belong to the same species if and only if they can be transformed into one another by pc transposition. [pc-sets]
3. Under SG(2), two chords belong to the same species if and only if they can be transformed into one another by pc transposition and/or pc inversion. [set classes]
4. Under SG{3), two chords belong to the same species if and only if they can be transformed into one another by pc transposition and/or pc inversion and/or the circle-of-fifths transform (Quinn 2006, 144–54)." [M_5-sets]

[114] Cohn cylic approach:

1. "M_2"; CYCLE-6: |02468A|02468A|; |012345|012345|
2. "M_3"; CYCLE-4: |0369|0369|0369|0369|; |0123|0123|0123|0123|
3. "M_4"; CYCLE-3: |048|048|048|048|; |012|012|012|012|
4. "M_6"; CYCLE-2: |06|06|06|06|06|06|;|01|01|01|01|01|01|
5. "M_{12}"; CYCLE-12: |0|0|0|0|0|0|0|0|0|0|0|0|;|0|0|0|0|0|0|0|0|0|0|0|0|

Douthett's maximally even (ME) theory. In short, his theory of chord quality entails making a one-to-one correspondence between non-trivial **ME** subgenera[115] and qualitative genera (Quinn 2007, 6)." He also seeks a theory that takes account of the various perspectives offered by different similarity measures.

While this very technical second part is compelling, its focus: generated, non-generated, degenerate, and non-degenerate ME sets, in equal temperaments of any size, whose inner and outer generating intervals and or inclusion in larger ME sets class classify them into various prototypes, it may just be a bridge too far for many delving into this pedagogy. Nonetheless, there is a benefit to exploring how set classes to varying degrees manifest generative cycles. What set classes in 12-tet can be classified as either a primary, secondary, or tertiary prototype? Plus, as these terms[116] are revisited/reimagined in part 3, it's helpful to learn them here.

[115] Given a particular *c*, A **ME** subgenus, **M**(c,d), is all of the non-trivial (trivial is *d* = 0, 1, *c*-1, or *c*) ME (M(c,d)) sets and their abstract complements.
[116] Some important terms:

- **G** for generated ME sets (e.g., diatonic); **TINV**: Non-G ME sets. (e.g., octatonic);
- **Index** (in regards to TINV only): *c*/gcf(c,d), the size of the smallest sub-universe into which M(c,d) can be projected —e.g, the index of M(12,8) is 3.
- **Class I** ME sets are TINV and non-degenerate; **Class IIa** (where $d \neq c/2$) ME sets are degenerate; e.g., 4-28*; **Class IIb** ME sets are complement-generated (e.g., 8-28*); **Class III** is neither generated nor complement-generated but is still degenerate; this doesn't occur in 12-tet—one example: 10-tet's M(10,4) and M(10,6) complements, are invariant under T_5 and both decompose the space into (in the spirit of Cohn) two CYCLE-5s. Whereas 10-tet's M(10,2) and M(10,4) are not complements and are both generated by ic_5.
- **g_{out}**: the outer generator (Hanson's shifting by a +1)—in M(10,4), $g_{out} = 2$; **g_{inner}**: the inner generator, the generating interval; e.g., ic_5 in a diatonic scale—in M(10,4), $g_{in} = 5$; **sig**(c,d): (TINV) the index; the size of each partition; **sog**(c,d): the smallest positive integer s such that $s * {}^{d}/_{gcf(c,d)} \equiv \pm 1 \; mod \; sig(c,d)$— The smallest positive integer that generates the partition; in sig(12,8), $[0,1]_3$, this is 1.
- **qualitative genus**: $F(c,q)$ where $0 < q \leq c/2$: a; Its prototypes are the **characteristic ME subgenus** (which include the semi-trivial *d*). A species is a **primary prototype** of F(c,q), irrespective of the genus's class, if and only if **DG** — it is doubly generated by g_{in} = sig(c,q) and g_{out} = sog(c,q) and **CC** — its cardinality is a multiple of gcf(c,q); meaning that all inner cycles of the generation process are "complete." A species is a **secondary prototype** of F(c,q) if it has DG but not necessarily CC (this covers the remaining Hanson projections, which are problematic in many non 12-tet universes — look at 10-tet, discussed above). All secondary prototypes are Kh-related to each other. Meaning that they include each other and exhibit PCP (one of Hanson's properties). A species is a **tertiary prototype** of F(c,q) if and only if it is Kh-related exclusively to all of the genus's primary (but not necessarily secondary) prototypes and is sandwiched (SW) between the next smaller and next larger primary prototypes. For example: F(12,4) has gcf(c,q) = 4; meaning that all primary prototypes have multiples of four notes. The following chords contain 4-28* and are contained by 8-28*, 6-27[013469], an SW, has DG but not CC and 6-30[013679], an SW, has neither DG nor CC. **3**: The tertiary prototypes that are not also secondary prototypes. All secondary prototypes are also tertiary prototypes. F(10,5) is a **3**.

Part 3

Finally in Part 3, Quinn offers another framework for relating chords. Essentially, it is a set of five diagrams that can be used to determine how much of a certain quality, FOURPROP,[117] that a set class has. Each diagram is associated with a divisor, n, of twelve. The diagrams consist of a center point which is encircled by n equally-spaced pans—the pans can also be described as circles that contain pitch-classes. These diagrams are capable of relating the degree to which chords manifest that property.

FOURPROPS should be considered as 1) the characteristic ME species (look at footnote 12) of a qualitative genus, and 2) not having the property by which it is named. It's better to think of a chord as **being balanced** in regard to the property by which it is named; not as equaling that property. A [02468A] does not balance FOURPROP(6), [0145] does. Accordingly, transposition and inversion do not affect the balance. "The content and arrangement of the pans of a Fourier Balance F(c,f) is determined by the same inner and outer generators that determine the structure of the core prototypes of the qualitative genus F(c,f) (Quinn 2007, 40)." Again, many of the terms introduced in this section[118] are updates of those introduced in section 2.

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- FOURPROP(6), the whole tone property — a chord has this property if it "has the same number of notes in one whole-tone set, as it has in the other." Think of two pans, 180º apart; one contains [02468A], the other [13579B].
- FOURPROP(4), the diminished-seventh-chord property — a chord has this property if it "has the same number of notes in common with each of the three diminished-seventh chord sets." It is represented as three pans, 120º apart; one contains [0369], [147A], and [258B].
- FOURPROP(3), the augmented-triad property — a chord has this property if "for any augmented-triad set A, [it] has the same number of notes in common with T_6(A), as it has in common with A." It is represented as 4 pans, 90º apart; now, opposite sides can balance each other. [048], [159], [26A], [37B]. These are called **annihilating pairs**. [134567] is such a chord. 15 balances 37 and 4 balance 6.
- FOURPROP(2), the tritone property, a chord has this property if "for any (0167)-set K, [it] has the same number of notes in common with T_3(K), as it has in common with K." This implies either: It is represented as 3 separate balances whose pans are 180º apart: [0167]-[349A]; [1278]-[45AB]; and [2389]-[56B0], It is represented as 6 pans, 60º apart: [06], [17], [28], [39], [4A], and [5B]. Again, opposite sides can balance each other, such as [0639] as can evenly distributed groups of 3, [02468A]. These are called **annihilating triples.**
- FOURPROP(1), the exceptional property. A chord has this property if "can be expressed as a disjoint union of tritone sets and/or augmented-triad sets." Quinn rewords this as being able to be partitioned into annihilating pairs and triples. It is represented as a clock-face. 12 pans, 30º apart: [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [A], [B].

118 Terminology/Notes (updating chap 2's terms in regard to the new "pans" metaphor):

- **F(c,f)** is "Fourier Balance f in the c-pc universe; its associated qualitative genus (class I, II, III); **M**(c,f) is its characteristic ME subgenus; **M**(c, gcf(c,f)) is its **Pan Species**, which is always class I and a primary prototype of the genus; **sig(c,f)** determines within-pan relationships—its inner generator; and **sog**(c,f)

Note that FOURPROP(0) is one big pan, with all pcs on it 0 and FOURPROP(5) is the same as FOURPROP(1). Also, fuzziness—"since generic prototypicality is maximal Fourier-balance imbalance, prototypicality is the limit case of a phenomenon that comes in degrees."

Below is yet another metaphor that Quinn introduces. The advantage of this last physics metaphor, is that it allows us to quantify to what degree a chord's/set class's construction exhibits the various FOURPROP properties; upon doing so, they can then be compared—as shown in the very last Quinn section. It also provides a segue to the topics covered by the last two authors presented in this chapter.

Physics metaphor:

One significant term that Quinn (re)introduces is **Lw** (Lewins): vectors/arrows that can be added together; represented by length and force. 1 **Lw** is the unit of tipping force exerted by a single pitch class on a Fourier Balance. In this context, the zero vector represents sets of vectors that collectively annihilate each other. For instance, when the set is of size 2, the vectors that annihilate each other are oppositely-oriented and of the same magnitude.

Imagine each FOURPROP as a template wherein the vectors, outwardly directed from the center of each diagram, point to each pan. Being balanced on FOURPROP means that the aggregate of the arrows (each representing a pc in a pc-set) combine so as to form a loop.
Whereas being Imbalanced on FOURPROP is the degree to which a chord is im-balanced on the Fourier Balance; this is proportional to the length of the arrow resulting from the addition of the

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determines between-pan relationships—its outer generator. Except for f = 5, with c = 12, sog(c,f) always equals 1. **Multiplication property:** multiplying a pitch class (pc) p by *n* gives you the pc's pan number.

- Notes on interpretation: the interval class separating adjacent pans is not given directly by f, but by sog(c,f)—in F(11,5): sog(c,f) = -2.5 means that each number must be multiplied by 5 to find its pan. 0 is in pan 0, 1 is in pan 5, 2 is in pan 10 etc. Also, every Class III Fourier Balance has between-pan relationships that cannot be characterized by ic1.
- To find exemplars of (necessary and sufficient conditions): **M**(c,f) is the characteristic ME subgenus—moving clockwise, take all of the numbers in a single pan. Stop before p + 1; **Primary prototype**—do the same as with **M**(c,f); just do more, applying the same procedure to multiple pans; **Tertiary Prototypes—** take arbitrary pcs one at a time for any pan; once a pan has been exhausted (and only then), proceed to the next pan clock-wise and continue. Stop at any point.

arrows associated with the constituent pcs of that chord. The more that a pc-set's pitches refer to the same arrow (especially when they aren't counterbalanced elsewhere) the larger the imbalance. It measures intrageneric affinity—distance from a prototype. Transposition, inversion, and complementation affect orientation, **not** magnitude.

Finally, note that the degree of prototypicality of any chord with respect to any Fourier genus is invariant under transposition and inversion (and complementation) and that no chord of cardinality d, can have a greater force than d Lw.

<u>The Space for Set class Comparison— the harmonic "cube" of dimension 12</u>

Here Quinn introduces us to the space in which these quantified chord qualities/properties can be compared.

Each pc-set is mapped somewhere within the harmonic "cube" of dimension 12 and the vector associated with a set class is $<p_0, p_1, \ldots, p_{11}>$. **The Discrete Fourier Transform (DFT)** is also a 12-tuple, an isomorphism and linear transformation of the chord's coordinates in 12-dimensional harmonic space.

However, due to there being five FOURPROPs, "Fourier Balances" yield only seven vectors of tipping forces. They reduce the two-dimensional (magnitude and orientation) aspect of the tipping force to one, just magnitude. Then Quinn provides great visual insight into how the unseeable 7-dimensional chord manifests in this space. Imagine taking a slice in the plane that stops at the center balance and then folds it on itself. To transform harmonic chord space into quality space simply fold up all of the planes corresponding to two-dimensional Fourier Balances in this manner. The five dimensions that disappear contain chord-specific information, leaving only the information that pertains to the quality of the species; and, again, the intervallic Half-Truth rears its head. In sum, chords near each other in harmonic chord space (e.g., sharing many common tones), must be close to each other in quality space.

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According to Callender, in his "Continuous harmonic Spaces," he "generalizes Ian Quinn's … harmonic characterization of pitch-class sets in equal tempered spaces to chords drawn from continuous pitch and pitch-class spaces. Using the Fourier transform, chords of any real-valued pitches or pitch classes are represented by their *spectra* and located in a harmonic space of all possible chord *spectra* (Callender 2007, 277)."

Where Quinn described a pitch as a discrete entity, Callender[119] casts it as a frequency f_a, which can take on any value in R (the real numbers). Callender then contextualizes this frequency by comparing it to the peaks of an infinitely-extended (in both directions) cos(ine) wave, whose peaks are multiples of another frequency, f_2. For instance, one can compare $f_1 = 5$ to a f_2-cos wave whose peaks are -10, -5, 0, 5, 10…. etc. and valleys are -7.5,-2.5, 0, 2.5, 7.5. In this case, as f_1 aligns with a peak of an f_2-cos wave, it's given a magnitude of 1; if f_1 were -2.5, it'd be given a magnitude of -1. Chords then can be evaluated as compounded magnitudes of its component pitches (in regards to some f_2-cos wave).

Callender also provides other visualizations: a helpful 2D chart that assigns f_2 to the x-axis and magnitude to the y-axis—it can clarify which f_2 yields the greatest response in magnitude to a particular chord (set of f_a); and iconic 3D spaces in which to compare set classes—here he evaluates various chords magnitudes (z) in regard to a particular f_2-cos wave—the set classes are represented as points (x,y) on the underlying plane and integrals are used to measure these spaces. Remember from Quinn that magnitude and orientation are invariant under both transposition and intervallic inversion and that complementation changes the orientation by 180°. He also recasts this f_2-cos wave as a unit circle wherein the d previous peaks (associated with a particular c) are separated by $2\pi/d$ radians.

[119] Although set classes are discrete and Callender's space is continuous, an examination and reflection on his work may still prove useful and/or inspiring in the pursuit of learning set classes and engaging this pedagogy. An important tenet of this dissertation is "Especially when improvising, don't aim for perfection in your processing and utilization of set classes towards some effect, but for good enough (heuristics) handling of them." In short, your notion of what a manipulation of a set class may yield may be fuzzy (in "continuous" space), even if the result of that manipulation is discrete, a set class. Furthermore, certain ideas may actually be continuous (like a gradually getting larger melody, wherein each successive interval is larger than the preceding), yet, it too, can only be realized in discrete terms.

Over the course of the article, Callender builds up his machinery and like Quinn reconciles various intuitive and established measures of set class similarity in regard to his methodology. Furthermore, also deferring to Euclidean measurements, he relies on physics metaphors of vector addition and subtraction when evaluating a set class in regard to a particular F(c,f). He offers a chart that shows the "spectra" of different set classes; its magnitudes in regard to the first 6 f_2-cos waves, F_6, whose peaks are respectively at:

- Harmonic 1: 1-fold division of the octave; unison
- Harmonic 2: 2-fold division of the octave; tritone
- Harmonic 3: 3-fold division of the octave; augmented triad
- Harmonic 4: 4-fold division of the octave; diminished chord
- Harmonic 5: 5-fold division of the octave; none in 12-tet
- Harmonic 6: 6-fold division of the octave; whole-tone

A list of the properties of chord spectra:

1. Chords that share the same interval vector, share the same spectra.
2. Multiplying a chord dilates its spectrum. This is Quinn's multiplication principle.
3. Given a pitch set p, if every member has maximal magnitude in regards to a f_2-cos wave (*l*-cycle) then the spectrum of p is periodic and $1/f_2$.
4. If a pitch-class set belongs to some k-tone equal tempered system, then the spectrum of the pitch-class set is periodic with a period of k/12.
5. Chord spectra are symmetric about z = 0
6. If the spectrum of a pitch-class set is periodic with a period of $1/f_2$ then the entire spectrum of a pitch or pitch-class set is periodic with a period of $1/f_2$.

In order to compare, different spectra, he settles on $d_{pow}|P^2-Q^2|$, rather than |P-Q|, as he asserts that it's better to have the greater magnitudes bear more weight in the calculation. Here, P and Q are vectors in F^6. "||" implies the Euclidean Measurement. Finally, two similarity measures that he relies heavily on are: Scott and Isaacson's (1998) Angle, "which measures the angle between vectors ins a six-dimensional vector space" and Roger's (1999) cos Θ which measures the cos of this same angle (Callender 2007, 304).

***

Finally, I will mention Tymoczko's "Set class Similarity, Voice Leading, and the Fourier Transform," which also builds on Quinn's article. What Tymoczko shows is that Quinn's findings are compatible with a "voice-leading"/contrapuntal perspective.

He too uses the Euclidean distance as a metric, many of the same striking visualizations as Callender, and defines the *distance between any two set classes* as "the size of the minimal voice leading between any of their transpositions or inversions (Tymoczko 2008, 255)." As, with Callender, he treats pitch as a continuous variable. Here is the algorithm to find the nearest compared chord y:

1. Seek those transpositions/inversions of y that share the same pitch sum as x—very often these nearest transpositions have fractional pitches (like $2\,{}^{1}/_{3}$).
2. When the exact voicing of the chords are not specified, seek the rotation of y that yields the minimum voice leading distance.

These various steps are formalized through adoption of the unit circle visualization offered by Callender, the use of vectors (as in Quinn), and through simplifying techniques such as the reduction of pitch-class space to 12/n, where n equals the size of the harmonic spectra being investigated.

The voice-leading distance between specific chords representing harmonic n — for instance, amongst triads, harmonic 3, is represented by (048), (04), and (08) — is treated analogously to Callender's harmonic distance between chords in his continuous harmonic space. The further in voice-leading terms a chord is from the compared-to harmonic-n-set class-representations, the lower its magnitude in regard to the related harmonic-n-spectrum. However, according to Tymoczko, the correlation is not perfect (it's above 95% though).[120] Finally, Tymoczko makes a

[120] Note that voice-leading models typically refer to points on a circle rather than vectors.

case for this approach along these grounds: whereas harmonic space is hard to understand, voice-leading space is much more interpretable; it's easier to quantify what the numbers mean.

Let me embolden an idea that Callender presented in the above article's introduction and that Tymoczko expounded upon in his discussion; **harmonic spaces and voice leading spaces do not conform to the same metric.** Two chords, a and b, that sound very similar may have greater voice leading distance between them than two chords a, again, and c that are sound very different. This is a good time then to re-examine the theoretical cousin to the above theories, similarity relations; the group of theories that focus on voice-leading and voice-leading space; namely Neo-Riemannian theories and appeals to the *Tonnetz*—this ancient tug of war between harmonic and voice-leading approaches to pitch material, dating back to Rameau, is alive and well.

Appendix C: Music/Math Texts Overview: Diatonic Set Theory

Another way to understand musical relationships using mathematical models is provided by research in Diatonic Set Theory. Below is a survey of seminal articles in the field. Collectively, they should be able to give you some indication of the field's breadth. Again, this overview (as other overviews in the dissertation) is not meant to be exhaustive. This topic, as the others, can and have received entire books dedicated towards those ends. Instead, I aim to present you with enough of a background that should you choose to follow in one of the many possible directions implied by this research, you have a good starting off point. Moreover, just as importantly, I encourage you to explore those lines of inquiry when armed with a greater number of set classes at your disposal. For instance, which set classes best approximate *g-generated* subsets of *well-tempered* scales? Or, how does it sound to set maximally-even scales with Euclidean rhythms? In what musical contexts is the notion of generic triads meaningful? Is it meaningful to classify set classes by how dissimilar they are to maximally-even scales of the same cardinality? Etc.

Clough and Douthett's (C&D) 1991 article, Maximally Even Sets, extended and fully grounded concepts that Clough had been publishing on for over a decade. As mentioned before, it relied primarily on reasonings and findings associated with number theory.

In Maximally Even Sets, C&D define and explore maximally even sets (MES). Well known MES are such familiar harmonic resources as the Major, Octatonic, and Whole Tone scales. Briefly, given an equal temperament of size *c* (standing for *chromatic*), typically 12, and a smaller positive integer *d* (standing for *diatonic*), the MES ($M_{c,d}$), where c = x, and d = y ($M_{x,y}$), is the most even distribution of y pitches in an equal temperament of size x.

- $M_{12,2}$ equals the tritone
- $M_{12,3}$ equals the augmented triad;
- $M_{12,4}$ equals the diminished 7th chord
- $M_{12,5}$ equals the pentatonic scale
- $M_{12,6}$ equals the whole tone scale

- $M_{12,7}$ equals the major scale
- $M_{12,8}$ equals the octatonic

Notice that some maximally even sets are symmetrical while others are not.

While C&D give multiple ways to derive $M_{c,d}$, the most utilized is their J-function. The *J-function* with its given parameters—m,c,d—is expressed as $J_{c,d}^{m}\ (N)$. Effectively, *m* is the transposition level. When *m* = 0 and (12(N)+ 0)/7 ranges over N, from 0 to 6, the resultant scale is Db major:

$\{[\frac{12(0)+m}{7}], [\frac{12(1)+m}{7}], [\frac{12(2)+m}{7}], [\frac{12(3)+m}{7}], [\frac{12(4)+m}{7}], [\frac{12(5)+m}{7}], [\frac{12(6)+m}{7}])$.

- The remainder of (12(0) + 0) over 7; **0**,
- The remainder of (12(1) + 0) over 7, **5**;
- The remainder of (12(2) + 0) over 7, **3**;
- The remainder of (12(3) + 0) over 7, **1**;
- The remainder of (12(4) + 0) over 7, **6**;
- The remainder of (12(5) + 0) over 7, **4**;
- The remainder of (12(6) + 0) over 7, **2.**

When m = 1, the major scale is Eb major etc.

Important terms include:

- **Clens:** the distance between two scale degrees in units of the underlying temperament.
    - E.g., The clen between scale degrees 2 and 4 of a major scale is 3.
- **Dlens** (generic interval)**:** the distance between two scale degrees in scale degree units
    - E.g., The dlen (generic interval) between scale degrees 2 and 4 (or 3 and 5, 4 and 6 etc.) of any scale is 2.
- **Spectrum:** all of the clens associated with a scale's generic interval of a particular size.
    - E.g., In a major scale, the generic interval "a 2^{nd}" is associated with a half-step and a whole step.

- **Spec($M_{c,d}$)** a list of all of the spectra (remember, each spectrum is itself a list) that are associated with a particular $M_{c,d}$.
- **CV,** cardinality equals variety: the condition that no two modes of the same scale are identical.
  - This condition is met in regards to the diatonic scale, but not the whole tone scale.
- **Myhill's property**: none of a scale's spectra have more than two clens.
  - The harmonic minor scale does not have this property; its 2^{nd} includes 1, 2, and 3 half-steps.
- **Rounded**: the spectrum of a scale's *step*, is a set of two consecutive integers.
  - As shown above, the diatonic scale has this property
- **CP** (consecutive property)**:** a scale's spectrum consists solely of consecutive integers.
  - All *diatonic* and *pentatonic* sets have the CP.
- A ***g-generated*** maximally even set:
  - One can arrange a g-generated $M_{c,d}$ set so that at least all-but-1 of the clens between successive pitches are identical.
  - E.g., C major can be expressed as B-E-A-D-G-C-F-(B). Except for F-B, all pairs of adjacent pitches are 7 semitones apart. Hence it is a **g-generated** maximally even set.

Perhaps as the above terms indicate, C&D isolate those qualities that make "diatonic" and "pentatonic" scales special. In the first part of the article, they elaborate on what constitutes $M_{c,d}$: Most notably, they answer the following queries:

- Can there be two different $M_{c,d}$ that share the same c and d?
  - The answer is no;
- Prove that the J function is synonymous with $M_{c,d}$;
- Deduce that $M_{c,d}$ are equivalent under transposition and inversion; and
- Offer an alternative classification system for scales and/or shared spectra;
- (Important for part 2) Show that if the cardinality of the scale is greater than ½ of the cardinality of c, then the spectrum of $M_{c,d}$ contains at least one of every type of chromatic interval in C.

However, their inquiries are not restricted to 12-tet; in Part 2, they deduce what conditions are required to have comparable diatonic scales in equal temperaments of different cardinalities. For instance, all "pentatonic" scales' spectra do not contain all of the clen; on the other hand, "diatonic" scales do. Both types are generated (meaning that a single interval can generate them — in the case, of major and pentatonic, this is P5). They then reconceptualize $M_{c,d}$s of various cardinalities by investigation into their properties as an interaction with one or more "tritones" (pseudo-tritones appear when *c* is odd). In Part 3, the focus shifts to Pentatonic sets. Finally, C&D end by discussing 2nd order $M_{c,d}$ — $M_{d,d1}$ (where $0 < d1 < d$). For instance, 3-11* ($M_{7,3}$) is a 2nd order maximally even set in $M_{12,7}$.[121]

However, C&D's herculean effort was not alone; it was preceded and continued by others. Relatively briefly, I will account for some of the other approaches.
In "The Group—Theoretic description of 12-Fold and Microtonal Systems (1980)," Balzano looks for group structures that are isomorphic to Z_{12}. Accordingly, he starts by considering generated sets.[122] Also interested in other temperament analogues to diatonic material and processes, he notes another group structure isomorphic to Z_{12}, Z_3 x Z_4; a framing later used extensively by Mazzola. Finally, he asserts that equal temperaments of size N, a.k.a. cyclic groups (groups generated by a single element), that can be decomposed into two smaller groups of respective sizes, Z_k x Z_{k+1}, are capable of manifesting the special properties of diatonic material and processes. C&D fleshed this and many other things out in the latter part of their 1991 article.

In "Aspects of Well-Formed Scales," Carey and Clampitt, also seeking to better characterize diatonic materials, ingeniously wed a Pythagorean-tuning construction of *well-formed* scales (such as the diatonic, pentatonic, and Danielou's 53-note scale) of size N, to the first n

[121]Put alternatively, again when c = 12, d = 7, and d1 = 3, the unit of the MES of cardinality 7 (the diatonic scale) is a semitone, and the unit of the 2nd order MES of cardinality 3 (the triad) is the diatonic scales' generic 2nd (distance between adjacent scale degrees). As such, this 2nd order MES is the set of triads that can be culled from 3 consecutive odd (or even) scale degrees of the diatonic scale; e.g., $\hat{1}, \hat{3}, \hat{5}$; $\hat{2}, \hat{4}, \hat{6}$; $\hat{3}, \hat{5}, \hat{7}$, etc.
[122] Note that this differs from g-generated maximally even sets; here the entire set of size n must be capable of being generated by a single interval. In g-generated maximally even sets, only n-1 elements of the set are required to be generated by a single interval.

convergences (and semi-convergences)[123] of continued fractions that approximate (over an infinite number of steps) $\log_2 \mu$. In the case of diatonic scales, μ = 3/2 (a P5). All of these scales, in the weakened sense of C&D, are g-generated. Note that in *well-formed* scales, unlike in set class descriptions, order is important. *Degenerate well-formed* scales are *g-generated* subsets of *well-tempered* scales wherein the generating interval and the return-to-origin interval are equal (Clough, 1 999, 79); in non-degenerate scales, this opposite is true (Quinn 2007, 7).

In their later article, "Self-Similar Pitch Structures," Carey and Clampitt observe that "the diatonic scale is self-similar in the following respect: the distribution of semitones [the distribution coefficient] within any diatonic interval is approximately equal to the overall distribution of semitones within the octave, namely two in seven (Carey and Clampitt 1996, 65)." In other words, the diatonic scale "possess[es] a synecdochic property: the part reflects the organization of the whole with a minimal, but inevitable degree of distortion (Carey and Clampitt 1996, 66)." They conclude and prove that a scale has Myhill's property "if and only if it is nondegenerate well-formed, and that it has self-similarity if and only if it has Myhill's property. Also, noticing that since well-formed scales are g-generated, they describe dual well-formed scales that are g^{-1}-generated and then derive the dual's self-similarity measure; the distribution coefficient. In both cases, the two numbers implied by the numerator of the coefficient (the denominator is N)—the frequency of the two clens, α, β, associated with a given dlen (Myhill's property) add up to N, $\alpha = N - \beta$, and their product is 1 Mod_N. Finally, the article focuses on infinite sets, namely those infinite lists of terms that, in the context of a continued fraction, approximate to an irrational number (e.g., the golden ratio, or $\log_2 \mu$). In this context, α, β are reinterpreted as differences between successive terms. Nonetheless, the patterns exhibited by these well-formed binary distributions of α, β are self-similar; or put alternatively, behave like a fractal.

[123] Alternating with successive terms (of a potentially infinite sequence) continued fractions over and under-estimate the rational (or irrational) number they are meant to approximate.

- The n^{th} convergent term is absolutely closer to the approximated term than any of the previous n-terms.
- A semi-convergent term is closer to the approximated term than any of the previous n-terms that, depending on the status of the semi-convergent term (an over- or underestimate), either over- or underestimated the approximated term.

Similarly, in the "Distance Geometry of Music," Demaine et al. apply many of Diatonic Set Theory's concepts to Euclidean Rhythms. Concepts related to C&D's maximally even sets are at home in describing rhythms as well as scales. Euclidean rhythms are a type of distribution of x durations in a meter whose numerator, y, is relatively prime to x. For instance, Euclid (3,8) is the tresillo. Many of the world's rhythms are Euclidean. An algorithm is provided that specifies the length of each of the x durations and by dint, distinguishes Euclidean from non-Euclidean rhythms. They utilize the term deep rhythms (akin to CV, and used by Gamer) and then show the connection between deep rhythms and (maximal evenness). These concepts are also broached in the following article, Clough's et al.'s "Scales, Sets, and Interval Cycles: A Taxonomy."

In their extremely thorough article, Clough et al collate and then compares the many measures that have been proposed to describe diatonic properties. Those that call on irrational and rational generators are: G (g-generated), Well-formed (WF), MP (Myhill property), DE (distributionally even — it's dlen have either one or two elements). Those that call on exclusively rational generators are: ME (maximally even), DP (CV), DT (a *diatonic* set has one tritone, one ambiguous clen — two different dlens that have the same clens, $c \equiv 0 \pmod 4$, d is odd, and $c = 2d + 1$), and finally BZ (Balzano). They then demonstrate which descriptors are interrelated; meaning that you cannot have one without the other. Afterwards, they grade various sets (of different cardinalities and even different temperaments) that have been broached in the Diatonic Set Theory literature. Unsurprisingly, the only set class that exhibits all 8 properties is the diatonic scale. Building on this finding, they then ask which combination of descriptors apply to certain scales/chords. Finally, they present algorithms designed to help theorists find those plausible "F-set" scales (from various cardinalities and temperaments); there are 13, that exhibit various combinations of those 8 descriptors.

In Formal Diatonic Intervallic Notation, Douthett and Hook seek a way to notate intervals that both preserves our notions of addition and subtraction (a M2 + M2 = M3, but $2 + 2 \neq 3$) and allows for needed variety in naming (A, d, M, m). Their solution builds on Brinkman and Agmon's PD notation — (# of half-steps, # of scale degrees), which, according to them, does not make the interval quality apparent. For example, while (5, 3) can be interpreted by a musician as the interval spanning 3 scale degrees which consists of 5 half-steps (an A3), the musician has to supply the extra information about its quality (Aug — greater than the expected); the quality is not contained in the notation. To solve this, they

1. “Modify” PD notation.
    a. The first term of PD notation now signifies deviation from the average distance implied by a 7-fold division of the octave
2. Develop a way to “add” and “subtract” that preserves our numeric sensibilities (2 + 2 = 4). To do so, they
    a. Systematically “counterbalance” deviations in their notation that would result in faulty addition, and
    b. Build on C&D’s J-formula and in doing so make their solution compatible with accidentals.
3. They refer to Group Theory throughout and construct a method that is definitely not meant to be done by hand.

Nonetheless, here is an example of the full machinery of transformational and Diatonic Set Theory being brought to bear on a critical but “negative” aspect of Diatonic scales; the interpretability, in terms of interval names, of the other pitches in the temperament — those that are not contained in the diatonic scale, but are defined by their relation to it.

Appendix D: Music/Math Texts Overview: Guerino Mazzola

While Mazzola's theoretical apparatus and aims ostensibly differ to this pedagogy's, a central concern is shared; help the student handle musical resource in a more intuitive manner — towards either the end of musical practice (composing/improvising) or theoretical (analysis). Furthermore, are approaches both are multi-disciplinary and situate practical musical questions in more general discussions. In fact, bar none, Mazzola's work demonstrates the awesome potential of multi-disciplinary work. Correspondingly though, where I was able to give relatively thorough and technical treatments of some of the other theorists work in this chapter, that is just not possible with Mazzola—the variety of math expected and range of topics covered are simply too great.

Mazzola

There are few topics that Mazzola does not address in his Topos of Music. Notably, he applies various recent Mathematical and Computer Science framings to a litany of music-theoretical topics ranging from Form, to Harmony, Timbre, etc. The core insight fueling his approach, which he roots in Yoneda's lemma, is that the more perspectives that you bring to bear on a topic, the better understanding you can accumulate on that topic. Towards such an end, he created a software that enabled various topics of interest (e.g., a chord, motive, harmony etc.) to be understood from various vantage points; for instance, an F major triad can be described:

- As a T_nI transformation of another triad
- As a non-T_nI transformation of another chord.
- In relation to various chord progressions
- In relation to all available chords
- In relation to harmonic systems that regulate how all available chords interact etc.

However, this software's benefits can also apply to more elusive concepts such as form, timbre, and perception of a melody—concepts that are inherently relational. How significantly must a melody or timbre be altered before it is perceived as something new? Alternatively, how much of a melody do we need to identify it? Relatedly, when assessing Form, is there, structurally speaking, a best starting procedure? Massive portions of his book were devoted to formulating these questions in such a way that would enable, via related coding, his software users to then broach them during the creative act. Stepping even further back, his software is responsive to the fact that even terms that are as straightforward as 'melody' may be defined differently in different regions or historical contexts.

How does this software enable such deep flexibility in so many aspects of idea construction? Well, it lies in the bones of its underlying category-theoretical architecture and view of knowledge that is both hierarchical and associative. Category theory (which I will describe very loosely here) focuses on the underlying schema that organize our understanding of potentially vastly different types of things/ideas. Using non-mathematical examples, hero myths share many commonalities throughout time and place (Campbell 2017); we could label those shared narrative arcs and features as a *category*. How similar are the *category* of hero myths and the *category* 1960's tv coming-of-age stories? Does our familiarity with the *category* of Disney adventure stories inform our responses to and interpretation of the *category* of a bully's behavior in the playground? The last two questions point to, again loosely defined, a *functor'*s critical role in Mazzola's text, as a way to relate different (again, loosely used) *categories*; or put alternatively, account for how a previous mental framing (*category*/schema) can frame a new mental framing (*category*/schema).

To accommodate the hierarchical and associative aspects of his theory, the software's user can add (not erase) associations and hierarchical relations to most terms (there is an indebtedness to JAVA!). Future users can then access the terms of interest from any of those associations. In the domain of language, if enough cat lovers added to the term *cat*, it's possible that a future user could then access the term *cat* from a paw (by two steps maybe from a dog's paw), Tom and Jerry, Siamese cat, feline etc. *Paw* and *feline* together form an example of a hierarchical relation: a paw is contained in the concept of cat and a cat is a type of feline. "Tom and Jerry" is an associative relation with cat.

If the world, moving forward, were to adopt his software for musical discussion and creative play, we could perhaps get closer to capturing an infinite number of quantitatively realized perspectives on a given topic; and, with the aid of computational methods, render new insight into those age-old musical dilemmas that still stump us.

Perhaps, though it’s easiest to view Mazzola’s work through one of its primary applications—as an intuitive computer aided tool for composition, improvisation, analysis, and more generally, musical understanding. As his software gets more traction, it could hopefully not only respond to the composer’s most specific requests but, if sufficiently fitted to their unique perspective, it could outsource the bulk of processing needed to realize generally formed demands — for instance, enabling the composer in real time to compose a score or transform in a one-off specific way certain material via how they move about physical space; ultimately, removing a barrier to improvisation.

It's unfortunate that what goes on under the hood of designing something more accessible often requires a level of technical expertise and specialization that belies the simplicity and the benefit of the final goals. The reception of Mazzola has been unduly hampered by this — to be precise and meet the demands of a theoretical exposition, he has had to write in a way that can come across as indecipherable to those individuals without substantial backgrounds in math, computer science, and music. While he has already made inroads into having more beginner-friendly introduction to his texts, I would love for the theory community at large to put their best foot forward and also initiate the making of more inroads into his text. Let’s not wait until to the 22nd century before his inspired work receives its fair due.

No text can answer everything, but its incorporation into our teaching practices could provide a way forward in addressing some of the deeper issues plaguing the theoretical community. How do we diversify the curriculum, cut down the number of class hours needed to graduate, keep performance standards high, and improve technological proficiency? Giving a student his software, a glove, and the ability to relate their movement to shifts in modulation and other processes, could really help. Furthermore, each student is encouraged to document, through updating the coding language, how they find it important to relate any terms we throw at them. We foreground that all of us are always collectively defining our shared terms.

Appendix E: Cognitive Psychology and Learning Overview: Various Topics

The following chapter is an introduction to various topics in the cognitive psychological literature. Again, this survey makes no claims at being exhaustive. Instead, this chapter is merely intended to introduce a "scientific disposition," discuss helpful terms/concepts, and then discuss a debate that raises many questions. Ultimately, getting the most out of this pedagogy entails more than learning about some portion of its contents; instead, it entails learning more about how you learn. In such a way, you will be best served if you view your foray into it as a series of sample-size one experiments that over time accumulate evidence regarding the learning strategies that best suit you.

What does it mean to be a good scientist?

In chapter 1.2 of Psychological Science, the authors assert that

"one of the hallmarks of a good scientist—or a savvy consumer of scientific research—is *amiable skepticism*. This trait combines openness and wariness. An amiable skeptic remains open to new ideas but is wary of new 'scientific findings' when good evidence and sound reasoning do not seem to support them. An amiable skeptic develops the habit of carefully weighing the facts when deciding what to believe (Phelps, Gazzaniga, and Berkman 2022)."

How does this translate into being a good student of this pedagogy? First, question untested assumptions about your best learning practices; next, design practice regimens that can test those assumptions; finally, refrain from over-claiming.

Regarding point one: Steve Coleman, a prominent saxophonist and composer, taught me this lesson—the value of testing your assumptions—when I took one of his classes as an undergrad. He simply demonstrated musical feats that I had assumed were impossible; one such feat was listening to a recording of Charlie Parker once and then playing back with it so well that the recording and his playing were indistinguishable; another was his immediate coordination of class-chosen rhythms that had different cycle lengths. In that class, a couple of my main take aways were that: "everything" is practicable—even creativity; and more is achievable than one may currently believe. Coleman admitted that, like us, he did not start with any advantages (e.g., perfect pitch).

In short, preexisting assumptions about what you can and can't learn may play a bigger role in what you can and can't learn then nearly any degree of difficulty. View the vastness of this pedagogy as a great site for testing the plausibility of various vast goals that you may have.

Regarding point two: How to design practice regimens that contribute to those facts that you carefully weigh when "deciding what to believe." In the course of researching this pedagogy, I wrote nearly 500 pieces. One purpose of that endeavor was to test whether or not symmetry in formal construction was audible. The pieces varied in harmonic and rhythmic content as well as in the number of ways in which symmetry was manifested—one of those pieces is included in part II of this dissertation. Were you to write your own pieces (or work through/listen to mine that are unpublished), you could also test that assumption.

Ultimately, the sky is the limit in terms of possible test designs and/or possible assumptions that you may look to scrutinize. Nonetheless, consider that, while knowledge of musical specifics may fade—especially as your opportunity to apply them may quickly change, knowledge accumulated about your learning process/or general abilities may remain applicable for a lot longer.

Regarding point 3: Don't overclaim! One critical difference between the science that I am asking you to practice in this application of this pedagogy and the science that is typically practiced as a discipline is that there is **no** want here to infer population-level claims from your sample-size of 1's findings. The end-goal is deeper insight into your own best practices and the accrued benefits of engaging those best practices. Isn't that enough? Focus on whether you can learn to identify symmetrical forms, not whether anyone can.

When seeking to make population-level claims, the practice of sampling methodology is exceedingly complicated.[124] Essentially, you're crushed under the weight of potential confounds—unaccounted-for influences—that may undermine the validity of your population-level claim. To counteract this, scientists continue to refine their methodologies and collaborate as much as possible—we don't yet know all that we don't yet know! These confounds range from biases in our selection of participants, to the order in which we do our investigations (such as doing analyses prior to the results' full collection), to even succumbing to pressures (consciously or not) to publish. Fortunately, statistical methods can control for the influence of unwanted variables and, in doing so, help address (at least, theoretically) potential confounds. Nevertheless, skill and nuance required in both applying these methodologies and interpreting their findings, is significant enough that it is typically developed through formal training.

Furthermore, not all questions worth asking are easily amenable to an experimental design; opportunity costs are real. For instance, regarding selection bias, it is well known that many psychological experimental papers pull applicants strictly from those freshmen and sophomores that are enrolled in a first-year semester long Psychology course. Similarly, many performance objectives, such as putting on a recital, can take up to two years of preparation and study. Could it ever be feasible to employ 1000 people of similar enough backgrounds to follow a music practice routine daily for two years? Even if it were, would the results even be generalizable to students of different backgrounds, who are engaging in different musical literatures?
The typical workaround is to ask if there is an hour-long observable proxy explaining the success/failure of 2 years of practice that one would feel comfortable generalizing from. These "proxies" are typically constructs (e.g., intelligence, rate of processing, memory etc.) that can reasonably be tested in the context of an experimental session and whose results (e.g., SAT scores) correlate with desired performance goals (e.g., College GPA). In short, recognize that many of the ideas introduced in this chapter have been the result of a large accumulation of information on constructs found to correspond with desired performance goals. Also note that the (often ingenious) experiments that provide evidence for (or against) those constructs very rarely

[124] Plus, there is more at stake when you make unsubstantiated claims about others than when you make them about yourself!

engage the specific conditions that you find yourself in when practicing. Plus, every body, mind, and performance goal/piece is different.

So, as you craft your routines, consider the three points above. Should you do so and work with this pedagogy long term, you could receive great feedback on your own best learning practices and the potential limits of your abilities.

Background Concepts:

Part 1—Memory

Memory Models:

The model of memory being presented here is an amalgamation of: one, Atkinson and Shiffrin's Modal Model (Atkinson and Shiffrin 1968); two, Baddeley and Hitch's model of working memory (Baddeley 2008); and three, Cowan's Embedded-Processes model (Cowan 2008).
One, the modal model positions short term memory (STM) as the channel by which our sensory and long-term memories (LTM) enter our consciousness. The STM is also the site for manipulating, retrieving, coding, and rehearsing information.

Two, Baddeley and Hitch's 1974 model STM had been replaced by a system, which, in 1986, they labeled as working memory (WM). It "had three components. It was run by an attentional system, the central executive [which determines where the information goes and is], assisted by two temporary memory stores, one for verbal and sound-based information, the phonological loop, and the other, the visuospatial sketchpad, for visual and spatial information (Baddeley 2008)." Later, Baddeley introduced the episodic buffer, "a limited capacity temporary storage system that is capable of integrating information from a variety of sources" that is controlled by the central executive and that feeds information into and retrieves information from LTM (Baddeley 2000, 421).

Three, Cowan's model emphasizes the interaction of attention and memory and de-emphasizes the distinction between different memory stores. Those memories/contents that are in a relatively inactive state constitute the LTM. Those contents that are above some threshold of activation but outside the focus of attention constitute the STM— even though unconscious, these contents can still influence processing. Those items that are in highest state of activation are those that are in the focus of attention—they are subject to conscious processing. WM is construed as the contents of STM, contents of focus, plus the controlled attention processes of the central executive.

Memory Stores and Concepts

Executive control processes allow one to actively manipulate information (such as adding in your head) and execute retrieval strategies that retrieve memories from long term memory (LTM), maintain them, and then manipulate them. These processes are associated with the ventrolateral and dorsolateral prefrontal cortex (VL- and DLPFC). The VLPFC retrieves memories that the DLPFC then manipulates (Petrides 2000). The prefrontal cortex (PFC) is also associated with inhibitory control—the inhibition of irrelevant information, responses to off-topic stimuli, and unhelpful behavior.

Sensory memory reflects a range of extremely brief memories of just-experienced stimuli. It is associated with our sensory systems, which continually convert sensory energy (e.g., photons, sound waves) into neural representations. Sensory systems have a brief "storage space" for these representations. How we identify these sensory memories depends on the modality—such as iconic memory for visual sensory memories and Echoic memory for audio. E.g., our perception of successive frames in a movie as having motion is due to our sensory memory store sustaining until the next frame.

Here, **short term memory** (STM) is posited as transient storage space for information that is above some level of activation, is readily available, and your attention may not be focused on it.

Where there is a limit on how much and for how long one's STM can hold information, there are no known limits to LTM. Ranges suggested for STM have varied from 4±1 items to 7±2 items.

Rehearsal is a major vehicle by which items are moved from STM to LTM (long term memory). Typically, this involves repeated speaking or imagining of words being rehearsed.

On the other hand, **long term memory** (LTM) is considered a long-term storage space. Typically, it is divided into declarative and non-declarative memory.

Declarative are those memories that you are consciously aware of and can talk about.

- **Semantic memory** is the store for general knowledge about the world. It is associated with medial temporal lobe (MTL) and in the ascribing of meaning, the anterior temporal lobes. E.g., knowing that Jupiter is a planet.
- **Episodic memory** is the store for specific episodes and events. E.g., a memory of walking down the aisle.

How do episodic memories form?

Standard Consolidation Theory, articulated in Squire and Alvarez 1995, asserts that memory traces—formed in the various parts of the brain that are associated with the various senses (hearing, vision, etc.) are transformed, through repeated activation, into a durable coordinated representation that is stored outside of the medial temporal lobe (MTL). The hippocampus is the coordinating brain region for the formation of these multi-modal memory traces. However, there is great debate over both how long consolidation takes and how to best effect it. In the related **multiple-trace theory**, only semantic information is coordinated into durable coordinated representations; episodic memories are not independent of the MTL.

On the other hand, **non-Declarative** memories are those which appear more in terms of your performance and/or behavior; you do not always have conscious access to them.

- **Conditioning** is a learned association between different stimuli or between behaviors and specific rewards and/or punishments. It is associated with neural circuits, such as the interposition nucleus of the cerebellum (eye blink conditioning) and the amygdala (fear conditioning) that are distributed throughout the pre-frontal cortex, amygdala, and the hippocampus. E.g., when training a dog to shake hands by giving it treats.

- **Skill-learning** reflects the learning of and demonstration of a skill. Motor-skill learning is associated with the basil ganglia and is not dependent on the hippocampus. E.g., knowing how to ride a bike.
- **Priming** is when exposure to a stimulus influences your response. It is associated with the various cortical areas that are affiliated with the different modalities primed. E.g., fast food businesses utilizing furniture whose design and coloring discourage longer stays.

Part 2—types of learning:

Implicit vs. Explicit Learning:

Learning is often divided into two categories: Implicit and Explicit. Moreover, studies have shown (Carol A. Seger, 1997) that implicit and explicit learning are facilitated by separate systems.

Implicit Learning:

Implicit learning is an umbrella term for any learning that takes place without focus. **Conditioning** and **priming** are two examples.

As an example of **conditioning**, studies have indicated that humans (or animals) can unconsciously learn to associate certain stimuli with certain rewards (or punishments). When newly presented stimuli of the same ilk are provided, changes in bodily reaction, such as increased salivation, may automatically occur. Along those lines, other studies have shown that that same learned behavior can later be unconsciously unlearned (extinction), when the association between those stimuli and a reward is again altered (Shechner et al. 2014). According to Reber,' implicit Learning has three main features:"(a) Implicit learning produces a tacit knowledge base that is abstract and representative of the structure of the environment; (b) such knowledge is optimally acquired independently of conscious effort to learn; and (c) it can be used implicitly to solve problems and make accurate decisions about novel stimulus circumstances" (p.219) (Cleeremans 1993, 14).

What does Reber mean by “abstract”?

To get a sense of this, let’s examine artificial grammar tasks; a task that Reber employed.

- In these tasks’ learning phase, participants are asked to memorize sequences of letters and, unbeknownst to them, those sequences of letters followed a set of rules.
- In these tasks’ testing phase, the participants were then asked to identify whether or not new sequences of letters were correctly formed.

The results: Reber found evidence that prior exposure to a grammar was associated with reduced recall time—even when the letters used changed. Here, learning is operationalized as quicker response times.

In short, one form of tacit “abstract” knowledge is demonstrated by an increased sensitivity to this grammar.

Sequential pattern recognition is another research design that investigates tacit abstract knowledge. In these tasks, there are regularities in how various stimuli are presented. Over time the participants’ response time to those various patterned stimuli decreased.

Related experiments looked at how participants, as young as 2-3 years old, learned artificial speech (near 2 to 3 minutes of an undifferentiated stream of syllables). “Statistical structure was covertly embedded by constraining the transitional probabilities between syllables in a manner similar to natural speech.” Over repeated and related experiments, “it was established that these very young infants were essentially computing the transitional conditional probabilities among phonemes (Cleeremans, Allakhverdov, and Kuvaldina 2019, 23). Furthermore, in his overview of implicit learning literature, Reber also indicated that the ability to extract statistical structure [is] present across sensory modalities (P. J. Reber, 2013).

What is a good take away—Implicit Learning?

Familiarization is a viable construct. Through exposure we can learn things about certain material without even being aware that we are actually doing so. Furthermore, if there are rules that inform the structure of the material examined, we may even become sensitive to them without either being explicitly aware of them or being able to articulate those rules. Even though how this happens is still only partially understood, evidence suggests that we have an innate (and often unconscious) ability to compute probabilities. Moreover, this still occurs even when we consciously engage with the material in a different manner; Berry and Broadbent found a disassociation between knowledge acquired implicitly and the ability to verbalize it (Cleeremans 1993). As such, trust that there is no "wasted practice." We are always learning something reflecting what and how we study something. Instead, focus on creating practice environments that support, even without conscious attention/manipulation, your long term goals.

Explicit Learning:

On the other hand, **explicit learning** is intentional; it occurs through directed attention.[125]

In Unsworth's article, "Individual differences in working memory capacity and learning," (Unsworth and Engle 2005) the results indicate that the strength of one's working memory significantly impacts explicit learning, but only marginally (if at all) implicit learning. In a series of experiments, Unsworth demonstrated a divergence in performance between high and low working memory groups on explicit learning tasks only. His results also indicated that attention (focus) was not essential to all learning, but just explicit learning.

What is a good take away—explicit learning?

[125] However, there is still disagreement over the role of attention in implicit learning (Carol A. Seger 1997, 110). In addition, Seger documents a split in the literature over how heterogenous implicit learning is—from "all forms of implicit knowledge are seen as essentially similar at their deepest levels" to "implicit learning … as a group of related but dissociable processes or memory systems that may or may not behave similarly in different situations (Carol Augart Seger 1994, 174).

First off, notice that explicit learning can engage more than just semantic learning stores—its definition only pertains to attention. For instance, common advice for practicing the giving of speeches is to imagine, while one is doing so, that one is in front of a hall of people. In doing so, previous episodic memories may be called upon.

That being said, typically-promoted good learning strategies include: making flash cards; taking good notes; synthesizing your understanding of a lecture immediately afterwards; designing your own practice tests; while practicing, systematically changing the rhythm associated with a difficult figuration, etc. All of these are examples of explicit learning strategies; consciously attending to the learning of a task. Of course, depending on the person and task, the successfulness of any given strategy may vary. Consider adopting any or many of the learning strategies just mentioned. Just recognize that in any learning experience, they are not solely responsible for all of the learning that goes on.

For instance, research has indicated that consistency between learning and testing site can impact test performance (Abernethy 1940). Similarly, the manner in which you learn/contextualize the information—for instance, whether you generate the retrieval clues,[126] or find a way make it personally relevant (Rogers, Kuiper, and Kirker 1977)—can similarly impact your performance.

<u>How do implicit learning and explicit learning strategies interact</u>?

In the first chapter of <u>Mechanisms of Implicit Learning: connectionist models of sequence processing</u>, Cleeremans discusses experiments relating to tasks seeking to assess the "relationship between performance and verbalizable knowledge. In their tasks, subjects control the output of a simulated system (e.g., a sugar production plant or a mass transportation system) by determining the value of one or several input variables on each trial … Berry and Broadbent found that performance on the "system control task" was disassociated from performance on the questionnaire … and [later] that adequate capacity to answer questions about the workings of a

[126] In "Optimizing Cue Effectiveness, Recall of 500 and 600 incidentally learned words," provides compelling evidence regarding the benefit of cue generation (Mantyla 1986).

simulated economic system can go along with inability to control the same system …
[Similarly:]

- Cleeremans indicated that inducing subjects to adopt explicit control strategies resulted in decreased task performance (Cleeremans 1993, 10)."

Later research showed that this is not always the case:

- Berry and Broadbent (1988) found an interaction between type of strategy employed (implicit vs. explicit) and type of system relationship, salient vs. non-salient (does the pattern exhibited by the salient stimuli well reflect the underlying pattern?).
    - Implying that explicit strategies work best when the underlying pattern is easy to detect; otherwise, explicit learning can get in the way (Cleeremans 1993, 10).
- Finally, Marascaux found evidence that those who performed the control task best gave the most general and useful rules; those who didn't could answer questions about specific situations that they encountered … [although,] subjects who failed to formulate any rule about the workings of the system … were nevertheless able to answer questions about it, as long as the question used situation that they had previously encountered … … thus, successful control task performance appears to depend both on memory of specific instances and on the use of more general explicit heuristics (Cleeremans 1993, 10)."

In other words, the specifics of the "control learning task" influences best (explicit vs. implicit) learning strategies.

<u>What is a good take away from this—the interaction of implicit and explicit learning strategies</u>?

If we know what we seek to learn, we can curate the information studied such that "what is salient" does not obscure the underlying patterns that we wish to better familiarize ourselves with. There are also individual differences in the extent to which people can generalize from the specific situations that they encounter. It is up to you to better understand what you need to best position yourself towards learning new material. This pedagogy offers material, like learning the different voicings of 4-note chords, that are multi-step and, as such, no single formula or completely "implicit learning approach" can help you generate them quickly (not just recognize them). Choices need to be made and multiple strategies employed to accomplish some aspect of that goal, which is most important to you.

Let's take familiarizing yourself with the voicings of a chord. Learning to quickly recognize or generate a trichord in one key, with neither doublings nor octave displacement allowed (there are only 6 potential voicings) is very different to familiarizing yourself with one tetrad in all keys (there are 691,488 potential voicings[127]), now with octave displacement but still with no doublings allowed. Moreover, one cannot assume that recognizing these differences visually is analogous to learning these differences aurally. Ultimately, multiple strategies (many are offered later) and strategic limiting of your choices, such as in octave displacement and number of doublings permitted, may be needed to ensure a positive result.[128]

Part 3—Learning Strategies:

Relevant concepts and terms to familiarize oneself with.

Heuristics:

Various cognitive biases influence our decision-making processes. Viewed most critically, these biases impair our abilities to make the best decisions; viewed more positively, they are decision-making strategies that take into account the cost of deliberation; in terms of mental effort, the availability, or lack thereof, of pertinent information, and time costs—when a lion charges you, there is not much time to deliberate over exactly the best way to flee or stand your ground. There are a range of cognitive biases ranging from:

- **hindsight bias**—labeling the pros and cons of a decision only after the outcome is known, to
- **gambler's fallacy**—thinking that "random" behavior conforms to the projected over some small number of trials,
- **as-if heuristic**—treating all cues as of equal importance as it takes less effort to do so,

[127] 12 keys * 28[4x7 octaves] * 21[3x7 octaves] * 14[2x7 octaves] * 7[1x7 octaves] = 691,488
[128] We'll revisit this discussion at the end of the chapter.

- **salience bias**—weighing the most-pronounced cue most heavily in your judgement,
- **availability heuristic**—how easily one can conjure up instances that support a hypothesis,
- **anchoring heuristic**—as additional information comes in, an undue bias towards one's initial hypothesis, and
- **confirmation bias**—a blindsight to any information that does not confirm one's held hypothesis.

<u>What is a good take away—heuristics</u>?

When designing your practice routine, you have an opportunity to learn/promote/craft heuristics that will best help you employ that material in later contexts. Examples of "heuristics" (processing shortcuts that enable decision making) often employed are:

- "If you start with a triad (in three voices), want to maintain good voice leading, and are in doubt about what to do next, change a single pitch in the smallest amount possible." Or,
- "If you have a repeated figuration in the bass that ascends diatonically, such as $\hat{3}$-$\hat{4}$-$\hat{2}$-$\hat{3}$, $\hat{4}$-$\hat{5}$-$\hat{3}$-$\hat{4}$, $\hat{5}$-$\hat{6}$-$\hat{4}$-$\hat{5}$, etc., default towards creating a diatonic sequence; i.e., I_6-IV-vii_6-iii, ii_6-V-I_6-IV, iii_6-vi-ii_6-V, etc.."

What processing shortcuts do you think will help you most in a particular future musical endeavor? Are there tools that you can develop that can allow you to quickly and adequately answer complex and infinitely big problems—such as, "what do I do next in my composition," in a relatively short amount of time? Responses could range from developing a questionnaire that helps you focus your efforts, to creating lists of instructions dictating how you should respond if particular musical circumstances arise.

<u>Interference</u>:

Modality-based interference:

Research suggests that memories of different modalities do not compete for the same processing resources. For example, "the verbal/sequential characteristics of verbal working memory are more disrupted by concurrent verbal tasks than by concurrent spatial tasks and that spatial working memory is more disrupted by concurrent spatial than verbal tasks." Similarly, "the central executive is more disrupted by concurrent task activities of higher general demands; tasks that are performed using controlled, rather than more automated processes (Wickens et al. 2013, 200)."

Time-related interference:

Proactive interference occurs when activity engaged in before encoding impacts retrieval. Alternatively, **retroactive interference** occurs when newly learned material interferes with previously acquired material. Moreover, "like concurrent interference, retroactive interference can be reduced or eliminated if the two sources of information are coded to used different working memory components."

<u>What is a good take away—interference</u>?

Take advantage as many parts of this pedagogy as much as you can. The more that you treat the learning of set classes as the accruing of episodic **and** semantic experiences, the better.[129] As no two experiences are identical, in calling upon your enriched experiences, less interference may occur; even if aspects of two memories are coded the same, other aspects are not. Furthermore, if your rendering of certain set classes is automatic[130]—e.g., you can access it by just "feeling" how it fits in your hand, there will be more processing power left for making adjustments in its execution: ranging from rhythmic to pitch-based modulations.

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[129] The **dual-coding principle** addresses this. It asserts that "material is better retained (and more likely to be retrieved) if it has multiple different representations in the brain (Wickens et al. 2013, 231)."

[130] The speed of performance will continue to increase at a rate proportional to the logarithm of the number of trials (Wickens et al. 2013, 232) and the attention or resource demand will continue to decline, allowing the skill to be performed in an automated fashion. Also, overlearning will also decrease the rate of forgetting the skill…).

In all chapters of Part II, a top down (developing appropriate schema) approach is counterpointed with a more bottom-up, aural approach. Accordingly, as you practice an activity like arpeggiating a pc set, try not to frame your own experience to a singular thing. At a minimum, it is a demonstration of your: overcoming technical challenges specific to your instrument, knowledge about a (collection of) set class(es), an aural experience, and an index of every other time that you have practiced arpeggios.

Terms associated with expertise:

A **mental model** "has been defined as a mental structure that reflects the user's understanding of a system … A mental model may be created spontaneously by the user or carefully formed and structured through training (Wickens et al. 2013, 236)." Those developed through experience are typically incomplete and there can be multiple mental models utilized for the same task. Experts tend to have more refined models, greater flexibility in their models, and can switch between models better under stress (Wickens et al. 2013, 236).

Intrinsic tasks: Intrinsic tasks are task that are critical to the domain of expertise; such as, playing the piano is a task intrinsic to being a classical pianist.

Contrived tasks: Those tasks that are not critical to the domain of expertise, but improve due to greater skill in the domain of expertise. For instance, booking gigs is not critical to being a jazz singer; nonetheless, working jazz singers may get a lot of practice booking their own gigs.
Chunking: According to Wickens, a chunk "can be defined as a set of adjacent stimulus units that are tied together by associations in the subject's long-term memory." In turn, Gobet and Clarkson (Wickens et al. 2013, 208) introduced template theory positing that frequently encountered chunks can become part of larger hierarchical structures called templates. Further Wickens, citing Barfield (2004), discussed an experiment suggesting that experts would more quickly commit things that were in their domain of expertise to memory than beginners—"expert programmers could encode more lines of a program per glance and hence recalled more lines of organized code if it had been in order or in random chunks (but not random lines)."

Long-term working memory (LT-WM) is long term memory that is accessed through 'temporary retrieval cues" in working memory; this is positioned as a way that experts can retain and work with more information than WM alone would allow. This can free up mental processing when faced with new situations; they have the bandwidth to register and react to more information in their environment (Wickens et al. 2013, 216). **Scaffolding**, a teaching strategy, augments this. The student is initially given help to ensure correct performance; later, they are weaned off of it. However, it is important to consider that in some contexts it is better to make some errors rather than no errors so that error detection and redressing can be learned (Wickens et al. 2013, 229).

<u>What is a good take away—Expertise</u>?

In short, expertise is manifest in refined mental models of the task, an ability to process the relevant material in larger and larger chunks/templates, and the ability to bring more to bear during a performance through better retrieval cues (LT-WM). Furthermore, greater expertise not only implies greater facility with certain tasks but many auxiliary tasks as well.

As you begin designing some practice routine, consider reverse engineering what it means to be an expert in the to-be-mastered task. While certain skills or mental improvements, such as better chunking, can only be improved over time and repetition, adopting a less rough mental model earlier can save time and promote better all-around learning.

In other words, since being an expert reflects more than proficiency at a single task, treat gaining proficiency in a given task as opportunity to become an expert rather than a call to become proficient at a single task. Design your lesson plans accordingly.

<u>Training strategies—terms</u>:

Adaptive training is another approach. Start with a simplified version of the task. In doing so, the mental workload associated with the task is reduced and more mental resources can be allocated to understanding critical components of the task. Over time, the difficulty of the task increases as it is better learned and become more automatic.

Part-task training decomposes a task into parts and then practices each part separately.

- **Fractionation** decomposes into subsidiary task (for instance, practicing the right and left hand of a piano piece separately).
- **Segmentation** decomposes the task by time (for instance, practicing the first phrase and then the second phrase).

According to Wickens, fractionation generally produces negative transfer—meaning that a percentage of those participants who employ **fractionation** fare worse than those who practiced the task as is.

Variable priority training sees all of the parts practiced together but different emphasis is placed on different aspects of the task (Wickens et al. 2013, 230).

<u>Training strategies—things to watch out for</u>:

According to Wickens, "it is important to note that making errors is a much more salient symptom of learning than is the minor increase of speed (following a logarithmic trend) or reduced attention demand." Hence the following warning: when learners have "complete control over when they may terminate learning or study," … overconfidence that a skill is fully mastered [is invited], when [, actually,] this self-evaluation is heavily dominated by the high salience of error-free performance: "Hey I got it perfect. I'm done (Wickens et al. 2013, 232)."

Similarly, as Bjork points out, training strategy merits should be based upon transfer and not training performance. In particular, Bjork has noted that people intuitively evaluated the ease of learning, training, and practice as a proxy for the quality and effectiveness of that learning: They erroneously think that if learning is easy, it is effective and memory for what is learned will therefore be strong. This is an illusion. People using this heuristic (ease of learning = quality of learning) will often study material less than they should, or choose an inappropriate easy training technique: indicating an overconfidence in their knowledge and skill gain (Wickens et al. 2013, 234).”

What are some good take aways—training strategies?

I believe that Bjork’s point is worth reinforcing; “training strategy merits should be based upon transfer and not training performance.” Whichever strategy you employ, take into consideration the specifics of the when, where, and how of your end goal; such as fluency with set classes a, b, and c in the context of a specific piece, in a specific hall, with specific players, at some already determined time.

For instance, I had a piano teacher who would regularly have his students practice their pieces in the hall of their performance with the video recorder recording them placed between them and where the audience would sit. Later, the teacher would give feedback on the recording. Regardless of how much of the recording that the teacher listened to: the video recorder still became a proxy for his oversight, an audience, and a feeling that every performed note mattered. In such a way, the more abstract practicing of piece, became a much more tangible practicing of the performance of the piece. While this manner of recording of one’s practice may not be the most effective strategy during all phases of one’s learning of a piece, it is, nonetheless, still an example of how important it can be to weigh “more than purely technical” considerations into your strategizing; no matter how flawlessly you may be able to render difficult passages in your living room is no guarantee that you will be able to do the same when battling performance anxiety in a crowded concert hall.

The Prototype/Exemplar Debate:

Finally, this last section focuses on a single debate, stemming for Eleanor Rosch’s work on prototypes and exemplars (Rosch and Lloyd 1978); namely, what part does prototypes versus exemplars play in our accrual of knowledge? After introducing this debate through reviewing two articles, I will talk about some of the implications of the debate regarding learning in general.

The following two articles on classifying visual stimuli bring the prototype exemplar debate to the fore, elucidating current considerations regarding how participants classify objects. The shapes in this experiment are classified by color proximity; the degree to which a circle of a particular color is presented next to another of the same color. Do the participants pick up on a color-independent prototype—a sequence of circles (differentiated by either color's proximity)? Or, do they start to recognize (and perhaps memorize) the various exemplars; with no essentialization occurring?

In "Similarity-Scaling Studies of Dot Pattern Classification and Recognition (JungShin, Nosofsky 1992)", Jung Shin and Nosofsky use their own models to better understand if and when either prototype or exemplar based classification schemes are used. At the heart of their multi-dimensional scaling models, are calculations of the overall distance of the distortions from either the exemplars or a 'best-fit' prototype. Their models accommodate both prototype and exemplar approaches—providing predictions of participant accuracy in recognizing various distortions of dot patterns of various sizes. In all cases, their exemplar model had the smallest SSE (sum of squared estimate of errors). The first experiment tested the participants immediately. The 2nd experiment put in a delayed condition (creating an optimal condition for a prototype effect). The 3rd experiment repeated various exemplars with greater frequency. In all cases, the exemplar predictions were better overall. They even made a case against the contributions of prototype classification schemes to combined exemplar and prototype approaches.

In "False prototype enhancement effects in dot pattern categorization" (Zaki and Nosofsky 2004), Zaki and Nosofsky take further aim at prototype theory; specifically rebutting Posner and Keele's 1968 experiment, which showed that participants tended to endorse "previously unseen prototypes at levels equal to and sometimes higher than the old training items and other new category members"(Zaki and Nosofsky 2004, 390) and Knowlton and Squire's related dot test that asked participants to compare various categories: a prototype, various levels of distortions, and random patterns. The result was that (Zaki and Nosofsky 2004, 391) the participants endorsed the prototype with the greatest probability. Ultimately, they argued that the observed prototype enhancement could be an artifact of either the salience of the particular categorized patterns or due to issues regarding internal validity — that there had been unaccounted-for additional learning of a "prototype."

We have a bias that prefers a root and closed position triad as its best representation.[131] If for no other reason, this arises from how triad identification is taught—compare everything to root position. Implicitly then, many feel that the closed root position form of the triad is its 'ideal' version. This post-16th/17th century notion has also been enhanced through theory that derives the triad from the overtone series.

Perhaps we should better acknowledge, above and beyond inversion, the particular "experience of each instantiation of triad." Maybe, this enrichment of our notion of a triad, in offering us more examples for comparison, will better position us to recognize a random triad than ceasing the studying after a root position triad has been introduced, assuming that it's the best approximation.

Doing so, can also lead us to broach (and re-evaluate) important questions that we may not have thought to ask. For instance:

- What makes one chord more (or less) difficult than another to learn?
    - Is familiarity or structure the unit in which difficulty is primarily measured?
- Independent of a chord's initial voicing, can we be taught to hear specific non-transposition voicing transformations?
- Along those lines, is intervallic inversion, or other forms of rhythmic and pitch-based symmetry-based transformations, something that one can be taught to discern aurally?

What is a main take-away?

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[131] For evidence supporting this claim, examine David Huron's work on musical imagery (Huron 2006, 67–68). When prompting his participants to imagine any sounded chord, 94% of them imagined a major chord and that root position was two to three times more likely than its other forms.

This dissertation's pedagogy will expose you to both a large number of set classes and a just as large number of ways to interact with them. Donning your scientist hat, be *amiably skeptical* about inherited assumptions. Even something as benign as a bias towards thinking of a closed root position as the best representation of that triad can have a profound impact on what you learn and even think you can learn. Then, if so inclined, design practice regimens that are capable of providing evidence, for or against, those assumptions. Finally, use that knowledge to your benefit but refrain from making larger claims that what seems to work for you must work for others.

BIBLIOGRAPHY

Abernethy, Ethel Mary. 1940. "The Effect of Changed Environmental Conditions Upon the Results of College Examinations." *The Journal of Psychology* 10 (2): 293–301. https://doi.org/10.1080/00223980.1940.9917005.

Agmon, Eytan. 1989. "A Mathematical Model of the Diatonic System." *Journal of Music Theory* 33 (1): 1–25.

———. 1996. "Coherent Tone-Systems: A Study in the Theory of Diatonicism." *Journal of Music Theory* 40 (1): 39–59.

Anonymous IV. 1985. *The Music Treatise of Anonymous IV : A New Translation*. Translated by Jeremy Yudkin. Neuhausen-Stuttgart: Neuhausen-Stuttgart : American Institute of Musicology : Hänssler-Verlag.

Atkinson, R.C, and Richard Martin Shiffrin. 1968. *Human Memory: A Proposed System and Its Control Processes*. New York: Academic Press.

Bach, Carl Philipp Emanuel. 1787. *Versuch über die wahre Art das Clavier zu spielen*. 3rd (expanded) edition. Leipzig: Schwickert. https://imslp.org/wiki/Versuch_über_die_wahre_Art_das_Clavier_zu_spielen,_H.868,_870_(Bach,_Carl_Philipp_Emanuel).

Bach, Carl Philipp Emanuel, and William J Mitchell. 2000. *Essay on the True Art of Playing Keyboard Instruments*. New York: Norton.

Bach, Johann Sebastian, and Pamela L. Poulin. 1994. *J. S. Bach's Precepts and Principles for Playing the Thorough-Bass or Accompanying in Four Parts: Leipzig, 1738*. Early Music Series 16. Oxford: Clarendon Press.

Baddeley, Alan. 2000. "The Episodic Buffer: A New Component of Working Memory?" *Trends in Cognitive Sciences* 4 (11): 417–23. https://doi.org/10.1016/S1364-6613(00)01538-2.

———. 2008. "What's New in Working Memory? Research into Working Memory Has Not Stood Still since Baddeley and Hitch Proposed It in 1974. Here, Alan Baddeley Describes What Has Happened since Then." *Psychology Review* 13 (3): 2.

Baker, James M. 1986. *The Music of Alexander Scriabin*. Composers of the Twentieth Century. New Haven: Yale University Press.

Baker, James M, David Beach, and Jonathan W. éditeur Bernard. 2009. *Music Theory in Concept and Practice*. Vol. 8. Eastman Studies in Music. Woodbridge: University of Rochester Press.

Balzano, Gerald J. 1980. "The Group-Theoretic Description of 12-Fold and Microtonal Pitch Systems." *Computer Music Journal* 4 (4): 66–84.

Berry, Michael. 2009. "A New Approach to Post-Tonal Ear Training." *Indiana Theory Review* 27 (1): 23–44.

Block, Steven, and Jack Douthett. 1994. "Vector Products and Intervallic Weighting." *Journal of Music Theory* 38 (1): 21–41.

Boss, Jack Forrest. 1985. "'The Emergence of the Chordal Concept.'" *Unpublished Paper for Claude Palisca's Renaissance Theory and Aesthetics Class*.

———. 2014. *Schoenberg's Twelve-Tone Music: Symmetry and the Musical Idea*. Cambridge, United Kingdom ; New York: Cambridge University Press.

Callahan, Michael. 2008. "Mapping Sum-and-Difference Space: Parallels Between Perle and Lewin." *Theory and Practice* 33: 181–217.

Callender, Clifton. 2007. "Continuous Harmonic Spaces." *Journal of Music Theory* 51 (2): 277–332.

Campbell, Joseph. 2017. *The Hero with a Thousand Faces*. First India printing. Mumbai, India: Yogi Impressions.

Caplin, William Earl. 1998. *Classical Form : A Theory of Formal Functions for the Instrumental Music of Haydn, Mozart, and Beethoven*. New York: New York : Oxford University Press.

Carey, Norman. 2007. "Coherence and Sameness in Well-Formed and Pairwise Well-Formed Scales." *Journal of Mathematics and Music* 1 (July): 79–98. https://doi.org/10.1080/17459730701376743.

Carey, Norman, and David Clampitt. 1989. "Aspects of Well-Formed Scales." *Music Theory Spectrum* 11 (2): 187–206.

———. 1996. "Self-Similar Pitch Structures, Their Duals, and Rhythmic Analogues." *Perspectives of New Music* 34 (2): 62–87.

Carter, Elliott, Nicholas Hopkins, and John F. Link. 2002. *Harmony Book*. Edited by Nicholas S. Hopkins. 1st publ. New York, NY: Carl Fischer.

Chew, Elaine, Adrian Childs, and Ching-Hua Chuan, eds. 2009. *Mathematics and Computation in Music Second International Conference, MCM 2009 John Clough Memorial Conference New Haven, CT, USA, June 2009, Proceedings*. Berlin Heidelberg New York: Springer.

Christensen, Thomas. 1992a. "The 'Règle de l'Octave' in Thorough-Bass Theory and Practice." *Acta Musicologica* 64 (2): 91–117. https://doi.org/10.2307/932911.

———. 1992b. "The Spanish Baroque Guitar and Seventeenth-Century Triadic Theory." *Journal of Music Theory* 36 (1): 1–42. https://doi.org/10.2307/843908.

Christensen, Thomas Street, ed. 2002. *The Cambridge History of Western Music Theory*. 7. print. The Cambridge History of Music. Cambridge: Cambridge Univ. Press.

Cleeremans, Axel. 1993. *Mechanisms of Implicit Learning : Connectionist Models of Sequence Processing*. Neural Network Modeling and Connectionism. Cambridge, Mass: MIT Press.

Cleeremans, Axel, V. M Allakhverdov, and Maria Kuvaldina. 2019. *Implicit Learning: 50 Years On*. London: Routledge, Taylor & Francis Group. https://doi.org/10.4324/9781315628905.

Clementi, Muzio. 7714. "Gradus Ad Parnassum, or l'art de Jouer Le Pianoforte, Démontré Par Des Exercices Dans Le Style Sèvère et Dans Le Style Élégant." *Gradus Ad Parnassum*.

Clifton Callender. 1998. "Voice-Leading Parsimony in the Music of Alexander Scriabin." *Journal of Music Theory* 42 (2): 219–33. https://doi.org/10.2307/843875.

Clough, John. 1979. "Aspects of Diatonic Sets." *Journal of Music Theory* 23 (1): 45–61. https://doi.org/10.2307/843693.

———. 2008. "Flip-Flop Circles and Their Groups." In *Music Theory and Mathematics Chords, Collections, and Transformations*, 23–48. Rochester, NY. USA: University of Rochester Press.

Clough, John, John Cuciurean, and Jack Douthett. 1997. "Hyperscales and the Generalized Tetrachord." *Journal of Music Theory* 41 (1): 67–100.

Clough, John, and Jack Douthett. 1991. "Maximally Even Sets." *Journal of Music Theory* 35 (1/2): 93–173.

Clough, John, Nora Engerbretsen, and Jonathan Kochavi. 1999. "Scales, Sets, and Interval Cycles: A Taxonomy." *Music Theory Spectrum* 21 (1): 74–104.

Clough, John, and Gerald Myerson. 1985. "Variety and Multiplicity in Diatonic Systems." *Journal of Music Theory* 29 (2): 249–70.

———. 1986. "Musical Scales and the Generalized Circle of Fifths." *The American Mathematical Monthly* 93 (9): 695–701.

Cohn, R. 1996. "Maximally Smooth Cycles, Hexatonic Systems, and the Analysis of Late-Romantic Triadic Progressions." *Music Analysis* 15 (1): 9–40. https://doi.org/10.2307/854168.

———. 1997. "Neo-Riemannian Operations, Parsimonious Trichords, and Their 'Tonnetz' Representations." *Journal of Music Theory* 41 (1): 1–66. https://doi.org/10.2307/843761.

Cohn, Richard. 2008. "Pitch-Time Analogies and Transformations in Bartok's Sonata for Two Pianos and Percussion." In *Music Theory and Mathematics Chords, Collections, and Transformations*, 49–71. Rochester, NY. USA: University of Rochester Press.

Cowan, Nelson. 2008. "What Are the Differences between Long-Term, Short-Term, and Working Memory?" *Progress in Brain Research*, Progress in Brain Research, 169: 323–38. https://doi.org/10.1016/S0079-6123(07)00020-9.

Demaine, Erik D., Francisco Gomez-Martin, Hank Meijer, David Rappaport, Perouz Taslakian, Godfried T. Toussaint, Terry Winograd, and David R. Wood. 2009. "The Distance Geometry of Music." *Computational Geometry* 42 (5): 429–54.

Derfler, Brandon Joel. 2010. *Single-Voice Transformations: A Model for Parsimonious Voice Leading*. Newcastle upon Tyne: Cambridge Scholars Publishing.

Diruta, Girolamo, Murray C. Bradshaw, and Edward J. Soehnlen. 1984. *The Transylvanian =: Il Transilvano*. 2 vols. Musicological Studies ; Wissenschaftliche Abhandlungen, v. 38 = Bd. 38. Henryville: Institute of Mediæval Music.

Douthett, Jack. 1999a. "The Theory of Maximally and Minimally Even Sets, the One-Dimensional Antiferromagnetic Ising Model, and the Continued Fraction Compromise of Musical Scales." Mathematics, United States — New Mexico: University of New Mexico.

———. 1999b. "The Theory of Maximally and Minimally Even Sets, the One- Dimensional Antiferromagnetic Ising Model, and the Continued Fraction Compromise of Musical Scales." ResearchGate. https://www.researchgate.net/publication/234262980.

———. 2008. "Filtered Point-Symmetry and Dynamical Voice-Leading." In *Music Theory and Mathematics Chords, Collections, and Transformations*, 72–106. Rochester, NY. USA: University of Rochester Press.

Douthett, Jack, and Julian Hook. 2009. "Formal Diatonic Intervallic Notation." In *Mathematics and Computation in Music Second International Conference, MCM 2009 John Clough Memorial Conference New Haven, CT, USA, June 2009, Proceedings*, 1st ed., 104–14. Berlin Heidelberg New York: Springer.

Douthett, Jack, Martha M. Hyde, and Charles J. Smith, eds. 2008. *Music Theory and Mathematics Chords, Collections, and Transformations*. 1st ed. Rochester, NY. USA: University of Rochester Press.

Douthett, Jack, and Richard J. Krantz. 2008. “Dinner Tables and Concentric Circles: A Harmony of Mathematics, Music, and Physics.” *The College Mathematics Journal* 39 (3): 203–11.

Eisenhardt, Lex. 2015. *Italian Guitar Music of the Seventeenth Century: Battuto and Pizzicato*. Eastman Studies in Music, v. 130. Rochester, NY: University of Rochester Press.

Engebretsen, Nora. 2008. “The ‘Over-Determined’ Triad as a Source of Discord Nascent Groups and the Emergen Chromatic Tonality in Nineteenth-Century German Harmonic Theory.” In *Music Theory and Mathematics Chords, Collections, and Transformations*, 1st ed., 107–36. Rochester, NY. USA: University of Rochester Press.

Ernst, Michael. 2014. “The Category-Theoretical Imperative.” Philosophy, United States — California: University of California — Irvine.

Fischer, Simon. 1997. *Basics: 300 Exercises and Practice Routines for the Violin*. London: Edition Peters.

Forte, Allen. 1963. “Context and Continuity in an Atonal Work: A Set-Theoretic Approach.” *Perspectives of New Music* 1 (2): 72–82. https://doi.org/10.2307/832105.

———. 1964. “A Theory of Set-Complexes for Music.” *Journal of Music Theory* 8 (2): 136–83. https://doi.org/10.2307/843079.

———. 1973. *The Structure of Atonal Music*. 5th ed. Westford, Mass.: The Murray Printing Co.,.

Friedmann, Michael L. 1990. *Ear Training for Twentieth-Century Music*. New Haven: Yale University Press.

Gamer, Carlton. 1967. “Some Combinational Resources of Equal-Tempered Systems.” *Journal of Music Theory* 11 (1): 32–59.

Gauldin, Robert. 1983. “The Cycle-7 Complex: Relations of Diatonic Set Theory to the Evolution of Ancient TonalSystems.” *Music Theory Spectrum* 5: 39–55.

Gjerdingen, Robert O. 2020. *Music in the Galant Style*.

Gollin, Edward. 1998. “Some Aspects of Three-Dimensional ‘Tonnetze.’” *Journal of Music Theory* 42 (2): 195–206. https://doi.org/10.2307/843873.

Gollin, Edward, and Alexander Rehding, eds. 2014. *The Oxford Handbook of Neo-Riemannian Music Theories*. First issued as an Oxford University Press paperback. Oxford New York Auckland Cape Town Dar es Salaam: Oxford University Press.

Gottlieb, G. L, D. M Corcos, S JARIC, and G. C AGARWAL. 1988. “Practice Improves Even the Simplest Movements.” *Experimental Brain Research* 73 (2): 436–40. https://doi.org/10.1007/BF00248235.

Gould, Mark. 2000. “Balzano and Zweifel: Another Look at Generalized Diatonic Scales.” *Perspectives of New Music* 38 (2): 88–105.

Goyette, Jeremiah. 2012. “The Z-Relation in Theory and Practice.” Music Theory, United States — New York: University of Rochester.

Hancock, Gerre. 1994. *Improvising: How to Master the Art*. New York: Oxford University Press.

Hanson, Howard. 1960a. *Harmonic Materials of Modern Music*. New York: New York, Appleton-Century-Crofts.

———. 1960b. *Harmonic Materials of Modern Music; Resources of the Tempered Scale.* New York: Appleton-Century-Crofts.

Hauptmann, Moritz. 1991. *The Nature of Harmony and Metre*. New York: New York : Da Capo Press.

Hepokoski, James A. (James Arnold). 2006. *Elements of Sonata Theory : Norms, Types, and Deformations in the Late Eighteenth-Century Sonata*. Oxford: Oxford.

Hoffman, Justin. 2008. "On Pitch-Class Set Cartography: Relations between Voice-Leading Spaces and Fourier Spaces." *Journal of Music Theory* 52 (2): 219–49.

Hook, Julian. 2007a. "Why Are There Twenty-Nine Tetrachords? A Tutorial on Combinatorics and Enumeration in Music Theory." *Music Theory Online* 13 (4). https://doi.org/10.30535/mto.13.4.1.

———. 2007b. "Why Are There Twenty-Nine Tetrachords? A Tutorial on Combinatorics and Enumeration in Music Theory." *Music Theory Online* 13 (4). https://doi.org/10.30535/mto.13.4.1.

———. 2008. "Signature Transformations." In *Music Theory and Mathematics Chords, Collections, and Transformations*, 137–60. Rochester, NY. USA: University of Rochester Press.

———. 2023. *Exploring Musical Spaces: A Synthesis of Mathematical Approaches*. Oxford Studies in Music Theory. New York, NY: Oxford University Press.

Jensen, Byron William. 1987. *William Schuman's Symphonies Numbers Seven, Eight, Nine, and Ten: An Introduction and Analysis*. Master's Thesis. Manhattan, Kansas: Kansas State University.

Johnson, Mark. 2009. *The Body in the Mind: The Bodily Basis of Meaning, Imagination, and Reason*. Paperback ed., [9. print.]. Chicago: University of Chicago Press.

Johnson, Timothy A. 2003. *Foundations of Diatonic Theory: A Mathematically Based Approach to Music Fundamentals*. Mathematics across the Curriculum. Emeryville, CA: Key College Pub.

———. 2008. "Some Pedagogical Implications of Diatonic and Neo-Riemannian Theory." In *Music Theory and Mathematics Chords, Collections, and Transformations*, 161–73. Rochester, NY. USA: University of Rochester Press.

Juno, Julien, Pierre Audétat, Carlos Agon, and Moreno Andreatta. 2009. "A Generalisation of Diatonicism and the Discrete Fourier Transform as a Mean for Classifying and Characterising Musical Scales." In *Mathematics and Computation in Music Second International Conference, MCM 2009 John Clough Memorial Conference New Haven, CT, USA, June 2009, Proceedings*, 1st ed., 166–79. Berlin Heidelberg New York: Springer.

Justus, Timothy C., and Jamshed J. Bharucha. 2002. "Music Perception and Cognition." In *Stevens' Handbook of Experimental Psychology*, edited by Hal Pashler, pas0111. Hoboken, NJ, USA: John Wiley & Sons, Inc. https://doi.org/10.1002/0471214426.pas0111.

Karpinski, Gary S. 2000. *Aural Skills Acquisition: The Development of Listening, Reading, and Performing Skills in College-Level Musicians*. New York: Oxford University Press.

Kendi, Ibram X. 2017. *Stamped from the Beginning : The Definitive History of Racist Ideas in America*. First trade paperback edition. Definitive History of Racist Ideas in America. New York, NY: New York, NY : Bold Type Books.

Kochavi, Jonathan. 2008. "A Parsimony Metric for Diatonic Sequences." In *Music Theory and Mathematics Chords, Collections, and Transformations*, 174–96. Rochester, NY. USA: University of Rochester Press.

Kottke, F J, D Halpern, J K Easton, A T Ozel, and C A Burrill. 1978. "The Training of Coordination." *Archives of Physical Medicine and Rehabilitation* 59 (12): 567–72.

Laitz, Steven G. (Steven Geoffrey). 2016. *The Complete Musician : An Integrated Approach to Theory, Analysis and Listening*. Fourth edition. Oxford: Oxford.

Lawlor, Robert. 2003. *Sacred Geometry: Philosophy and Practice*. Repr. London: Thames and Hudson.

Lerdahl, Fred, and Ray Jackendoff. 1983. *A Generative Theory of Tonal Music*. Cambridge, Mass.: Cambridge, Mass. : MIT Press.

Lester, Joel. 1992. *Compositional Theory in the Eighteenth Century*. Cambridge, Mass: Harvard University Press.

Levine, Mark. 1989. *The Jazz Piano Book*. Petaluma, CA: Sher Music Co.

Lewin, David. 1959. "THEORY FORUM: Intervallic Relations between Two Collections of Notes." *Journal of Music Theory* 3 (2): 298–298.

———. 2007. *Generalized Musical Intervals and Transformations*. New York: Oxford University Press.

———. 2008. "Transformational Considerations in Schoenberg's Opus 23, Number 3." In *Music Theory and Mathematics Chords, Collections, and Transformations*, 197–221. Rochester, NY. USA: University of Rochester Press.

Liebman, Dave. 1991. *A Chromatic Approach to Jazz Harmony and Melody*. Rottenburg N., Germany: Advance Music.

Mantyla, T. 1986. "Optimizing Cue Effectiveness: Recall of 500 and 600 Incidentally Learned Words." *Journal of Experimental Psychology. Learning, Memory, and Cognition* 12 (1): 66–71. https://doi.org/10.1037/0278-7393.12.1.66.

Mathieu, W. A. 1997. *Harmonic Experience: Tonal Harmony from Its Natural Origins to Its Modern Expression*. Rochester, Vt: Inner Traditions International.

Mazzola, Guerino. 2002. *The Topos of Music*. 1st ed. Basel - Boston - Berlin: Birkhäuser Verlag.

Mazzola, Guerino, Maria Mannone, and Yan Pang. 2016. *Cool Math for Hot Music*. New York, NY: Springer Berlin Heidelberg.

Mckinley, Lily. 1977. *Stylistic Developments in Selected Symphonies of William Schuman: A Comparison of Symphonies Numbers Three and Nine*. Ph.D Dissertation. New York University.

Messiaen, Olivier, and John Satterfield. 1956. *The Technique of My Musical Language*. Bibliotheque-Leduc ; 831, 759. Paris: Alphonse Leduc.

Monteverdi, Claudio. 2011. *Complete Madrigals*. Edited by Andrea Bornstein and Michelangelo Gabbrielli. Score. Vol. 1. 10 vols. Bologna Italy: Ut Orpheus.

Morris, Robert. 1987a. *Composition with Pitch-Classes : A Theory of Compositional Design*. New Haven: New Haven : Yale University Press.

———. 1987b. *Composition with Pitch-Classes: A Theory of Compositional Design*. New Haven: Yale University Press.

———. 2001. *Class Notes for Advanced Atonal Music Theory*. 2 vols. Lebanon NH 03766: Frog Peak Music.

———. 2007. "Mathematics and the Twelve-Tone System: Past, Present, and Future." *Perspectives of New Music* 45 (2): 76–107.

———. 2010. "Some Musical Applications of Minimal Graph Cycles." *Theory and Practice* 35: 95–117.

Morris, Robert D. 1994. “Recommendations for Atonal Music Pedagogy in General: Recognizing and Hearing Set-Classes in Particular.” *Journal of Music Theory Pedagogy* 8: 75–134.
Morris, Robert D. 1998. “Voice-Leading Spaces.” *Music Theory Spectrum* 20 (2): 175–208. https://doi.org/10.2307/746047.
Moseley, Brian Christopher. 2013. “Twelve-Tone Cartography: Space, Chains, and Intimations of ‘Tonal’ Form in Anton Webern’s Twelve-Tone Music.” Music Theory, United States — New York: City University of New York.
Music Theory in Concept and Practice. 1997. Rochester, NY: Rochester, NY : University of Rochester Press.
Mutch, Caleb. 2020. “The Triad in Dispute: Johannes Lippius, His Audiences, and the Disputatio Genre.” *Music Theory Spectrum* 42 (2): 247–59. https://doi.org/10.1093/mts/mtaa008.
Norman, Donald A. 2013. *The Design of Everyday Things*. Revised and Expanded edition. New York, New York: New York, New York : Basic Books.
Owens, Jessie Ann. 1997. *Composers at Work: The Craft of Musical Composition 1450-1600*. New York: Oxford University Press.
Petrides, M. 2000. “The Role of the Mid-Dorsolateral Prefrontal Cortex in Working Memory.” *Experimental Brain Research* 133 (1): 44–54. https://doi.org/10.1007/s002210000399.
Phelps, Elizabeth A., Michael S. Gazzaniga, and Elliot T. Berkman. 2022. *Psychological Science*. 7th edition. New York, NY: W. W. Norton & Company.
Plato, and Donald J. Zeyl. 2000. *Timaeus*. Indianapolis: Hackett Pub. Co.
Popper, Karl R. 1989. *Conjectures and Refutations: The Growth of Scientific Knowledge*. 5th ed. (rev.). London ; New York: Routledge.
Quick, Donya. 2014. “Kulitta: A Framework for Automated Music Composition.” Philosophy, United States — Connecticut: Yale.
Quinn, Ian. 2001. “Listening to Similarity Relations.” *Perspectives of New Music* 39 (2): 108–58.
———. 2006. “General Equal-Tempered Harmony (Introduction and Part I).” *Perspectives of New Music* 44 (2): 114–58.
———. 2007. “General Equal-Tempered Harmony: Parts 2 and 3.” *Perspectives of New Music* 45 (1): 4–63.
Rahn, John. 1987. *Basic Atonal Theory*. Reprint. New York: Schirmer [u.a.].
Rameau, Jean-Philippe. 1971. *Treatise on Harmony*. New York: Dover Publications.
Reber, Arthur S. 1989. “Implicit Learning and Tacit Knowledge.” *Journal of Experimental Psychology. General* 118 (3): 219–35. https://doi.org/10.1037/0096-3445.118.3.219.
Reber, Paul J. 2013. “The Neural Basis of Implicit Learning and Memory: A Review of Neuropsychological and Neuroimaging Research.” *Neuropsychologia* 51 (10): 2026–42. https://doi.org/10.1016/j.neuropsychologia.2013.06.019.
Reimann, Hugo, and William C. Mickelsen. 1977. *Hugo Riemann’s Theory of Harmony: A Study and History of Music Theory, Book III*. Edited and translated by William C. Mickelsen. Lincoln: University of Nebraska Press.
Reina, Rafael. 2015. *Applying Karnatic Rhythmical Techniques to Western Music*. Burlington, VT: Ashgate.
Rogers, T. B, N. A Kuiper, and W. S Kirker. 1977. “Self-Reference and the Encoding of Personal Information.” *Journal of Personality and Social Psychology* 35 (9): 677–88. https://doi.org/10.1037/0022-3514.35.9.677.

Rosenhaus, Steven L. 2008. “An Introduction to George Perle.” *Theory and Practice* 33: 65–71.
Russell, George. 1961. *The Lydian Chromatic Concept of Tonal Organization for Improvisation*. New York: New York : Concept Pub. Co.
Santa, Matthew. 2000. “Analysing Post-Tonal Diatonic Music: A Modulo 7 Perspective.” *Music Analysis* 19 (2): 167–201.
Schenker, Heinrich, and Heinrich Schenker. 2001. *Free Composition*. New Musical Theories and Fantasies 3. Hillsdale, NY: Pendragon Press.
Schillinger, Joseph. 1946. *The Schillinger System of Musical Composition.* New York: New York, C. Fischer.
———. 2003. *Kaleidophone: New Resources of Melody and Harmony : Pitch Scales in Relation to Chord Structures : An Aid to Composers, Performers, Arrangers, Teachers, Song-Writers, Students, Conductors, Critics and All Who Work with Music*. Harwich Port [Mass.: Clock & Rose Press.
Schönberg, Arnold. 1983. *Theory of Harmony*. Translated by Roy E. Carter. 3. [ed.], 1. paperback print. Berkeley Los Angeles: Univ. of California Press.
Schuijer, Michiel. 2008. *Analyzing Atonal Music: Pitch-Class Set Theory and Its Contexts*. Eastman Studies in Music. Rochester, NY: University of Rochester Press.
Sites, William. 2020. *Sun Ra’s Chicago: Afrofuturism and the City*. Historical Studies of Urban America. Chicago: University of Chicago Press.
Skinner, B. F. (Burrhus Frederic). 1938. *The Behavior of Organisms; an Experimental Analysis*. The Century Psychology Series. New York, London: D. Appleton-Century Company, Incorporated.
Slonimsky, Nicolas. 1975. “Thesaurus of Scales and Melodic Patterns.”
Soderberg, Stephen. 2008. “Transformational Etudes Basic Principles and Applications of Interval String Theory.” In *Music Theory and Mathematics Chords, Collections, and Transformations*, 222–44. Rochester, NY. USA: University of Rochester Press.
Solomon, Larry. 2012. “Table of Pitch Class Sets.” November 27, 2012. https://www.scribd.com/document/237597049/Table-of-Pitch-Class-Sets.
Squire, Larry R, and Pablo Alvarez. 1995. “Retrograde Amnesia and Memory Consolidation: A Neurobiological Perspective.” *Current Opinion in Neurobiology* 5 (2): 169–77. https://doi.org/10.1016/0959-4388(95)80023-9.
Straus, Joseph N. 2003. “Uniformity, Balance, and Smoothness in Atonal Voice Leading.” *Music Theory Spectrum* 25 (2): 305–52.
STRAUS, JOSEPH N. 2003. “Uniformity, Balance, and Smoothness in Atonal Voice Leading.” *Music Theory Spectrum* 25 (2): 305–52. https://doi.org/10.1525/mts.2003.25.2.305.
Straus, Joseph Nathan. 2016a. *Introduction to Post-Tonal Theory*. 4th edition. New York: W.W. Norton & Company.
———. 2016b. *Introduction to Post-Tonal Theory*. 4th edition. New York: W.W. Norton & Company.
Taneev, Sergeĭ Ivanovich. 1962. *Convertible Counterpoint in the Strict Style*. Boston: Boston : B. Humphries.
Teitelbaum, Richard. 1965. “Intervallic Relations in Atonal Music.” *Journal of Music Theory* 9 (1): 72–127. https://doi.org/10.2307/843150.

Tymoczko, Dmitri. 2008. "Set-Class Similarity, Voice Leading, and the Fourier Transform." *Journal of Music Theory* 52 (2): 251–72. https://doi.org/10.1215/00222909-2009-017.

———. 2009. "Three Conceptions of Musical Distance." In *Mathematics and Computation in Music Second International Conference, MCM 2009 John Clough Memorial Conference New Haven, CT, USA, June 2009, Proceedings*, 1st ed., 258–72. Berlin Heidelberg New York: Springer.

———. 2023. "Approximate Set Theory." *Journal of Music Theory* 67 (1): 1–70. https://doi.org/10.1215/00222909-10232057.

Ullman, Michael T. 2016. "The Declarative/Procedural Model." In *Neurobiology of Language*, 953–68. Elsevier. https://doi.org/10.1016/B978-0-12-407794-2.00076-6.

Vieru, Anatol. 1985. "Modalism-A 'Third World.'" *Perspectives of New Music* 24 (1): 62–71. https://doi.org/10.2307/832757.

Wickens, Christopher D., Justin G. Hollands, Simon Banbury, and R. Parasuraman. 2013. *Engineering Psychology and Human Performance*. Fourth edition. Boston: Pearson.

Yamaguchi, Masaya. 2006. *The Complete Thesaurus of Musical Scales*. Revised edition. New York: Masaya Music Services.

Žabka, Marek. 2013. "Toward the Reconciliation of Mathematical Theories of Voice Leading." *Journal of Mathematics and Music* 7 (2): 81–88. https://doi.org/10.1080/17459737.2013.824143.

Zarlino, Gioseffo. 1968. *The Art of Counterpoint (Le Istitutioni Harmoniche, Part III)*. Edited by Claude V. Palisca. Translated by Guy Marco. New Haven: Yale University Press.

———. 1983. *On the Modes: Part Four of Le Istitutioni Harmoniche, 1558*. Edited by Claude V. Palisca. Translated by Vered Cohen. Music Theory Translation Series. New Haven: Yale University Press.