$RO(C_3)$-GRADED BREDON COHOMOLOGY AND C_P-SURFACES

by

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DISSERTATION ABSTRACT

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Let p be an odd prime, and let C_p denote the cyclic group of order p. We use equivariant surgery methods to classify all closed, connected 2-manifolds with an action of C_p. We then use this classification in the case $p = 3$ to compute the $RO(C_3)$-graded Bredon cohomology of all C_3-surfaces in constant $\underline{\mathbb{Z}/3}$ coefficients as modules over the cohomology of a point. We show that the cohomology of a C_3-surface is completely determined by its genus, number of fixed points, and whether or not its underlying surface is orientable.

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CHAPTER I

INTRODUCTION

Let p be an odd prime, and let C_p denote the cyclic group of order p. We achieve two goals in this paper:

1. Classify all closed and connected 2-manifolds with an action of C_p up to equivariant isomorphism.

2. Compute the $RO(C_3)$-graded Bredon cohomology of all closed, connected 2-manifolds with a C_3-action in constant $\underline{\mathbb{Z}/3}$ coefficients.

More specifically, we define ways of constructing classes of C_p-surfaces using equivariant surgery methods and prove that all C_p-surfaces can be constructed in this way. This geometric classification then lends itself to cohomology computations in $RO(G)$-graded Bredon theory.

Dugger gave a similar classification of C_2-surfaces in [10]. In his paper, Dugger gave a complete list of isomorphism classes of C_2-surfaces and developed a full set of invariants which would determine the isomorphism class of a given surface with involution. We use similar methods to show that all nontrivial, closed, connected C_p-surfaces are in one of six families of isomorphism classes of C_p-surfaces. Various papers have treated aspects of the classification result in Theorems 4.0.2 and 4.0.3, mostly focusing on the orientable case [1, 3, 4, 16, 18, 19]. The new idea presented in this classification and in that of [10] is construction of isomorphism classes via equivariant surgeries.

The second goal of this paper is to compute the $RO(C_3)$-graded Bredon cohomology of all C_3-surfaces given in the classification, in constant $\underline{\mathbb{Z}/3}$

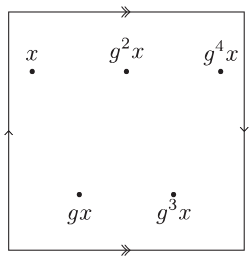
coefficients. This Bredon theory provides a nice analogue for singular cohomology in $\mathbb{Z}/3$ coefficients. Many recent computations have been done in $RO(G)$-graded Bredon theory [6, 9, 8, 11, 12, 13, 15, 17, 20], including a similar computation of the cohomology of equivariant surfaces in the $G = C_2$ case [12].

For a C_3-surface X, let $F(X)$ denote the number of fixed points of X. It is useful to note that when the action is non-trivial, $F(X)$ must be finite. We also let $\beta(X)$ denote the β**-genus** of X. This is defined to be $\dim_{\mathbb{Z}/2} H^1_{\text{sing}}(X; \mathbb{Z}/2)$. We show in this paper that the $RO(C_3)$-graded Bredon cohomology in $\underline{\mathbb{Z}/3}$ coefficients of a non-trivial C_3-surface X is completely determined by $F(X)$, $\beta(X)$, and whether or not X is orientable.

1.1. Classification of C_p-surfaces

The idea behind our classification result is to show that all C_p-surfaces can be described in terms of other simpler C_p-surfaces. Some examples of these "building block" surfaces are $S^{2,1}$ and M_1^{free} which can be described as the 2-sphere and torus (respectively) rotating about the axis passing through each of their centers. Other examples include the non-orientable spaces N_2^{free} and $N_1[1]$ whose C_p-actions are shown in Figure 1 in the case $p = 5$. The final family of spaces needed for our classification is denoted Hex_n for $n \geq 1$. We can think of Hex_1 as a $2p$-gon with opposite edges identified and a rotation action of $e^{2\pi i/p}$. Then Hex_n consists of n copies of Hex_1 glued together in a particular way. These surfaces are described in greater detail in Chapter III, but we can also see this gluing demonstrated in Figure 2 in the case $p = 3$.

Before precisely stating the classification result, let us define some equivariant surgery operations.



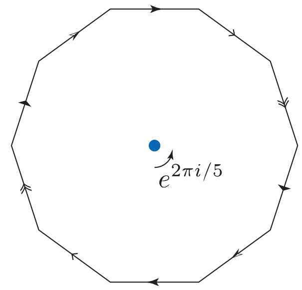
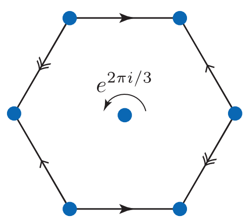


FIGURE 1. The spaces N_2^{free} (left) and $N_1[1]$ (right) in the case $p = 5$.



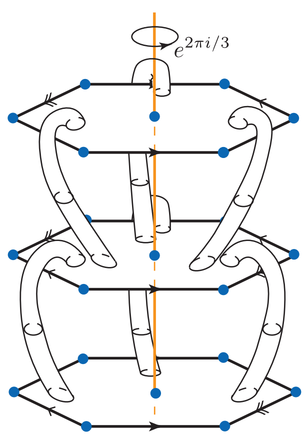


FIGURE 2. The spaces Hex_1 (left) and Hex_3 (right) in the case $p = 3$.

Let Y be a non-equivariant surface and X a surface with a nontrivial order p homeomorphism $\sigma\colon X \to X$. Define $\tilde{Y} := Y \setminus D^2$, and let D be a disk in X so that D is disjoint from each of its conjugates $\sigma^i D$. Let $\tilde{X}$ denote X with each of the $\sigma^i D$ removed. Choose an isomorphism $f\colon \partial\tilde{Y} \to \partial D$. We let $X\#_p Y$ denote the space

$$\left[\tilde{X} \sqcup \coprod_{i=0}^{p-1} \left(\tilde{Y} \times \{i\}\right)\right] / \sim$$

where $(y, i) \sim \sigma^i(f(y))$ for $y \in \partial\tilde{Y}$ and $0 \le i \le p-1$. This process is called equivariant connected sum surgery.

We use R_p to denote the space $S^{2,1}$ with p conjugate disks removed. Let X be a C_p-surface. Choose a disk D_1 in X that is disjoint from each of its conjugates. Then remove D_1 and its conjugates to form the space $\tilde{X}$. Let D be the disk in $S^{2,1}$ which was removed (along with its conjugates) to form R_p. Choose an isomorphism $f\colon \partial D_1 \to \partial D$ and extend this equivariantly to an isomorphism $\tilde{f}\colon \partial\tilde{X} \to \partial R_p$. We then define $X + [R_p]$ to be the space

$$\left(\tilde{X} \sqcup R_p\right) / \sim$$

where $x \sim \tilde{f}(x)$ for $x \in \partial\tilde{X}$. This process is called equivariant ribbon surgery.

In this paper, we prove that up to isomorphism all C_p-surfaces can be constructed by starting with M_1^{free}, $S^{2,1}$, N_2^{free}, $N_1[1]$, or Hex_n (for some n) and performing a series of equivariant connected sum and ribbon surgeries. If X is a surface with order p homeomorphism σ_X and Y is a surface with order p homeomorphism σ_Y, we say that X and Y are isomorphic if there exists a homeomorphism $f\colon X \to Y$ such that $f \circ \sigma_X = \sigma_Y \circ f$.

Theorem 1.1.1. *Let X be a connected, closed, orientable surface with an action of C_p. Then X can be constructed via one of the following surgery procedures, up to $\mathrm{Aut}(C_p)$ actions on each of the pieces.*

1. $M_1^{free} \#_p M_g,\ g \geq 0$
2. $(S^{2,1} + k[R_p])\,\#_p M_g,\ k, g \geq 0$
3. $(\mathrm{Hex}_n + k[R_p])\,\#_p M_g,\ k, g \geq 0,\ n \geq 1$

Let N_r denote the genus $\beta = r$, closed non-orientable surface.

Theorem 1.1.2. *Let X be a connected, closed, non-orientable surface with an action of C_p. Then X can be constructed via one of the following surgery procedures, up to $\mathrm{Aut}(C_p)$ actions on each of the pieces.*

1. $N_2^{free} \#_p N_r,\ r \geq 0$
2. $(S^{2,1} + k[R_p])\,\#_p N_r,\ r \geq 1$
3. $(N_1[1] + k[R_p])\,\#_p N_r,\ k, r \geq 0$

1.2. $RO(C_3)$-graded Bredon cohomology of C_3-surfaces

Up to isomorphism, there are two irreducible real representations of C_3, namely the trivial representation ($\mathbb{R}_{\text{triv}}$) and the two-dimensional representation given by rotation of 120° about the origin ($\mathbb{R}^2_{\text{rot}}$). So any element of $RO(C_3)$ can be represented as $\mathbb{R}_{\text{triv}}^{\oplus p-2q} \oplus (R^2_{\text{rot}})^{\oplus q}$ and is completely determined by the values p and q. As a result, $RO(C_3)$-graded Bredon cohomology can be viewed as a bigraded theory, with the cohomology of a C_3-space X denoted $H^{*,*}(X; \underline{\mathbb{Z}/3})$.

Let $\mathbb{M}_3$ denote the $RO(C_3)$-graded Bredon cohomology of a fixed point in $\underline{\mathbb{Z}/3}$ coefficients. In this paper, we compute the cohomology of all closed, connected,

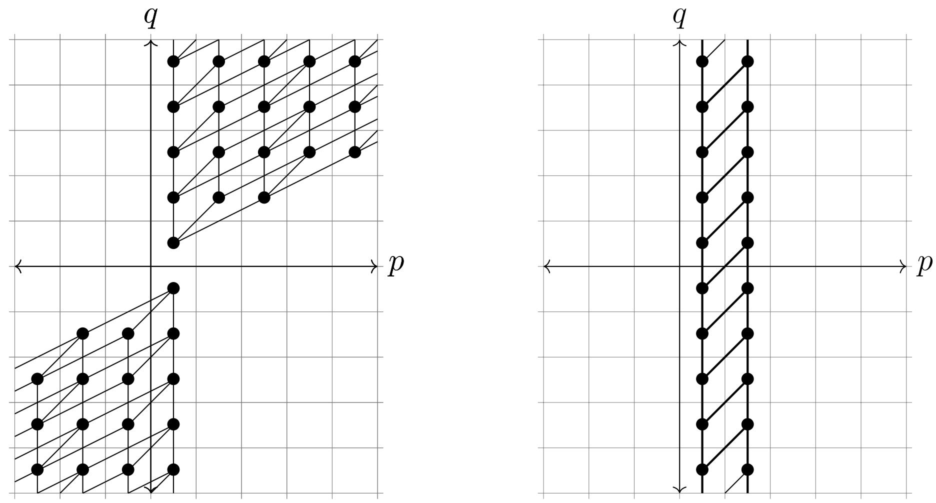


FIGURE 3. The $\mathbb{M}_3$-modules $\mathbb{M}_3$ (left) and $H^{*,*}(S^1_{\text{free}})$ (right).

non-trivial C_3-surfaces as $\mathbb{M}_3$-modules. It turns out there are only a few $\mathbb{M}_3$-modules which show up in the cohomology of C_3-surfaces. These modules are $\mathbb{M}_3$, the cohomology of the freely rotating circle (S^1_{free}), the cohomology of C_3, and a module called $\mathbb{EB}$ which denotes the reduced cohomology of the unreduced suspension of C_3. Since our Bredon theory is bigraded, we can depict each of these modules in the (p, q)-plane, where the (p, q)th cohomology group is depicted above and to the right of the (p, q)th spot on the grid. Figures 3 and 4 give depictions of these $\mathbb{M}_3$-modules in the (p, q)-plane. Each dot in these figures represents a copy of $\mathbb{Z}/3$.

The $\mathbb{M}_3$-module structure of these important pieces are discussed more thoroughly in Chapter II. With this brief overview of the basic pieces, we can now preview the main result on the cohomology of C_3-surfaces:

Theorem 1.2.1. *Let X be a free C_3-surface with genus β.*

1. *If X is orientable, then*

$$H^{*,*}(X) \cong H^{*,*}(S^1_{\mathit{free}}) \oplus \Sigma^{1,0} H^{*,*}(S^1_{\mathit{free}}) \oplus \left(\Sigma^{1,0} H^{*,*}(C_3)\right)^{\oplus \frac{\beta-2}{3}}.$$

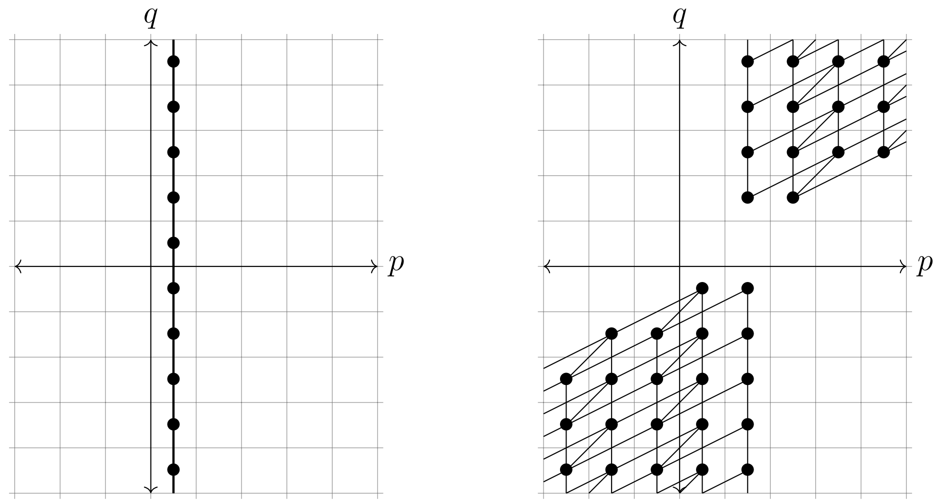


FIGURE 4. The $\mathbb{M}_3$-modules $H^{*,*}(C_3)$ (left) and $\mathbb{EB}$ (right).

2. If X is non-orientable, then

$$H^{*,*}(X) \cong H^{*,*}(S^1_{free}) \oplus \left(\Sigma^{1,0}H^{*,*}(C_3)\right)^{\oplus\frac{\beta-2}{3}}.$$

Theorem 1.2.2. *Let X be a C_3-surface with genus β and $F > 0$ fixed points.*

1. If X is orientable, then

$$H^{*,*}(X) \cong \mathbb{M}_3 \oplus \Sigma^{2,1}\mathbb{M}_3 \oplus \mathbb{EB}^{\oplus F-2} \oplus \left(\Sigma^{1,0}H^{*,*}(C_3)\right)^{\oplus\frac{\beta-2F+4}{3}}$$

2. If X is non-orientable and F is even, then

$$H^{*,*}(X) \cong \mathbb{M}_3 \oplus \mathbb{EB}^{\oplus F-2} \oplus \left(\Sigma^{1,0}H^{*,*}(C_3)\right)^{\oplus\frac{\beta-2F+1}{3}}$$

3. If X is non-orientable and F is odd, then

$$H^{*,*}(X) \cong \mathbb{M}_3 \oplus \mathbb{EB}^{\oplus F-1} \oplus \left(\Sigma^{1,0}H^{*,*}(C_3)\right)^{\oplus\frac{\beta-2F+1}{3}}$$

We can quickly observe from these results that given any C_3-surface X, the Bredon cohomology of X is completely determined by $\beta(X)$, $F(X)$, and whether or not X is orientable.

There is a potential concern that $\frac{\beta-2F+4}{3}$ and $\frac{\beta-2F+1}{3}$ may not be integers. However we prove in Chapter IV that $F(X) \equiv 2 - \beta(X) \pmod 3$ for any space X with a C_3-action, so this is not an issue. Consequently, when the action on X is free, it must be that $\frac{\beta-2}{3} \in \mathbb{Z}$.

1.3. Organization of the Paper

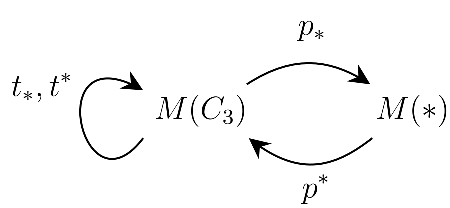
In Chapter II we discuss important properties and computational tools for Bredon cohomology. Equivariant surgery procedures are outlined in Chapter III. Chapter IV contains a statement of the main classification theorem for nontrivial C_p-surfaces. We then compute the Bredon Cohomology of all nontrivial C_3-surfaces in Chapters V and VI. Some important equivariant surgery results are proved in Appendix A. A detailed proof of the main classification theorem from Chapter IV is given in Appendices B and C.

CHAPTER II

PREMILINARIES ON $RO(C_3)$-GRADED BREDON CHOMOLOGY

In this chapter we discuss background knowledge and computational tools for $RO(G)$-graded Bredon Cohomology in the case $G = C_3$. This theory takes coefficients in a Mackey functor, so we begin with a discussion of Mackey functors and define the specific Mackey functor which will be used throughout the paper. We next discuss notation and terminology related to this theory and introduce several computational tools which will be used throughout this paper. This chapter ends with a few small computations which utilize these tools and introduce some of the key methods used in later computations.

Definition 2.0.1. A **Mackey Functor** M for $G = C_3$ is the data of



where $M(C_3)$ and $M(*)$ are abelian groups, and p^*, p_*, t^*, and t_* are homomorphisms that satisfy

i. $(t^*)^3 = id$

ii. $(t_*)^3 = id$

iii. $t^*p^* = p^*$

iv. $p_*t_* = p_*$

v. $t_*t^* = id$

vi. $p^*p_* = 1 + t^* + (t^*)^2$.

In this paper we will be primarily focused on **the constant $\mathbb{Z}/3$ Mackey functor**, which is denoted $\underline{\mathbb{Z}/3}$ and is defined by $M(C_3) = M(*) = \mathbb{Z}/3$, $p^* = t^* = t_* = id$, and $p_* = 0$.

2.1. Bigraded Theory

For a group G, the $RO(G)$-graded Bredon cohomology of a space with a G-action is graded on the Grothendieck group of real, orthogonal, finite-dimensional G-representations. In the case $G = C_3$, there are only two such irreducible G-representations up to isomorphism. These are the 1-dimensional trivial representation $\mathbb{R}_{\text{triv}}$, and the 2-dimensional representation given by rotation of the plane about the origin by 120°. We denote this representation by $\mathbb{R}^2_{\text{rot}}$.

Given a C_3-representation V, we can write $V = (\mathbb{R}_{\text{triv}})^{\oplus p-2q} \oplus (\mathbb{R}_{\text{rot}})^{\oplus q}$ where p represents the total dimension of V and q represents the number of copies of $\mathbb{R}_{\text{rot}}$ in V. Notice that V is completely determined by the values of p and q, so $RO(C_3)$ is a rank 2 free abelian group. In particular, we can write $H^{p,q}_{C_3}(X; M)$ to denote the Vth cohomology group of X in this theory. Note that the subscript of C_3 will be omitted when the context of $G = C_3$ is understood. We also let $\mathbb{R}^{p,q}$ denote the C_3-representation $(\mathbb{R}_{\text{triv}})^{\oplus p-2q} \oplus (\mathbb{R}_{\text{rot}})^{\oplus q}$ and the element $(p-2q)[\mathbb{R}_{\text{triv}}] + q[\mathbb{R}^2_{\text{rot}}]$ of $RO(C_3)$.

Let V be a real G-representation, and consider the space $\hat{V}$ obtained by one-point compactifying V by adding a fixed point at infinity. The space $\hat{V}$ is equivalent to a sphere with a G-action. We call this a **representation sphere** and denote it by S^V.

We can then form the equivariant suspension

$$\Sigma^V := S^V \wedge X$$

where X is a G-space with a fixed base point. If X is a free G-space, we can add a fixed base point to form the space $X_+ := X \sqcup \{*\}$. In general, the notation X_+ represents a G-space X with a disjoint base point which is fixed under the action of G.

For every finite-dimensional, real, orthogonal G-representation V, we get natural isomorphisms

$$\Sigma^V \colon \tilde{H}_G^{\alpha}(-;M) \to \tilde{H}_G^{\alpha+V}\left(\Sigma^V(-);M\right)$$

where coefficients are taken in the Mackey functor M. Given a cofiber sequence of based G-spaces

$$X \xrightarrow{f} Y \to C(f)$$

we get a Puppe sequence

$$X \to Y \to C(f) \to \Sigma^{\mathbf{1}} X \to \Sigma^{\mathbf{1}} Y \to \Sigma^{\mathbf{1}} C(f) \to \cdots$$

where $\mathbf{1}$ represents the 1-dimensional trivial representation of G. We can then use the suspension isomorphism to get a long exact sequence

$$\tilde{H}_G^V(X;M) \leftarrow \tilde{H}_G^V(Y;M) \leftarrow \tilde{H}_G^V(C(f);M) \leftarrow \tilde{H}_G^{V-\mathbf{1}}(X;M) \leftarrow \tilde{H}_G^{V-\mathbf{1}}(Y;M) \leftarrow \cdots$$

for each $V \in RO(G)$.

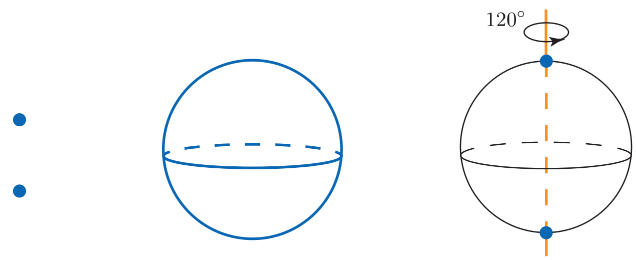


FIGURE 5. The representation spheres $S^{0,0}$, $S^{2,0}$, and $S^{2,1}$, respectively.

In the case $G = C_3$, we know $V \cong \mathbb{R}^{p,q}$ for some p and q. For brevity, we use $S^{p,q}$ to denote the representation sphere $S^{\mathbb{R}^{p,q}}$. Examples of representation spheres in this case can be found in Figure 5. We use blue to denote points which are fixed under the action. We additionally use $\Sigma^{p,q}X$ to denote the Vth suspension of a C_3-space X. This means there are isomorphisms

$$\Sigma^{p,q}\colon \tilde{H}^{a,b}(X;M) \to \tilde{H}^{a+p,b+q}(\Sigma^{p,q}X;M)$$

for all $p, q \geq 0$. Moreover, given a cofiber sequence of based C_3-spaces

$$X \xrightarrow{f} Y \to C(f)$$

we get a long exact sequence

$$\cdots \to \tilde{H}^{p,q}(Y;M) \to \tilde{H}^{p,q}(X;M) \xrightarrow{d^{p,q}} \tilde{H}^{p+1,q}(C(f);M) \to \tilde{H}^{p+1,q}(Y;M) \to \cdots$$

for each $q \in \mathbb{Z}$.

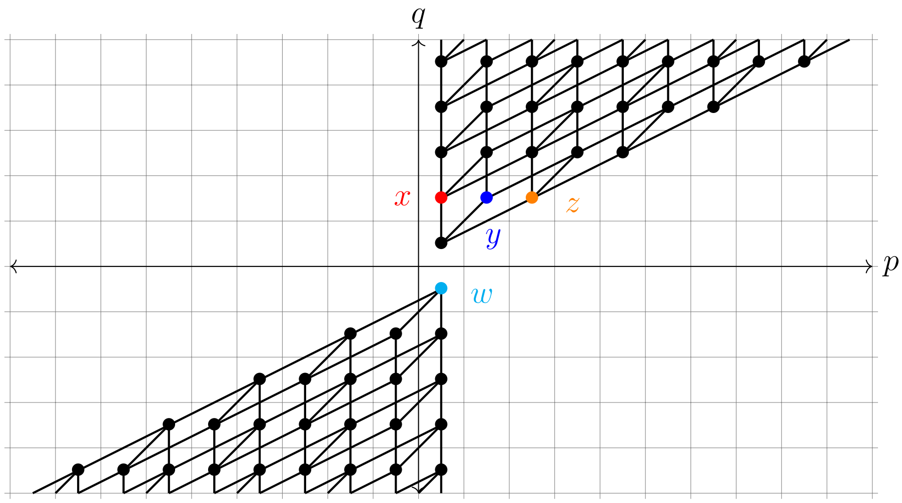


FIGURE 6. The ring $\mathbb{M}_3 = H^{*,*}(pt)$.

2.2. Cohomology of Orbits

Here we give the cohomology of $pt = C_3/C_3$ and the free orbit C_3 in constant $\underline{\mathbb{Z}/3}$ coefficients. These computations have been done in [14], so we just give the ring structure below.

Let $\mathbb{M}_3$ denote the ring $H^{*,*}(\text{pt}; \underline{\mathbb{Z}/3})$ which is depicted in Figure 6. The (p, q) spot on the grid denotes the cohomology group $H^{p,q}(\text{pt}; \underline{\mathbb{Z}/3})$, and each dot represents a copy of $\mathbb{Z}/3$. Solid lines indicate ring structure as we explain below. We use the convention that the (p, q)th entry is plotted above and to the right of the (p, q)th coordinate.

We will refer to the portion above the p-axis as the "top cone" and the portion below as the "bottom cone". The top cone is isomorphic to the polynomial ring $\mathbb{Z}/3[x, y, z]/(y^2)$ where x is a generator of $\mathbb{Z}/3$ in degree $(0, 1)$, y is a generator in degree $(1, 1)$, and z is a generator in degree $(2, 1)$. Multiplication by x is denoted by vertical lines, multiplication by y is denoted by lines of slope 1, and multiplication by z is denoted by lines of slope $1/2$.

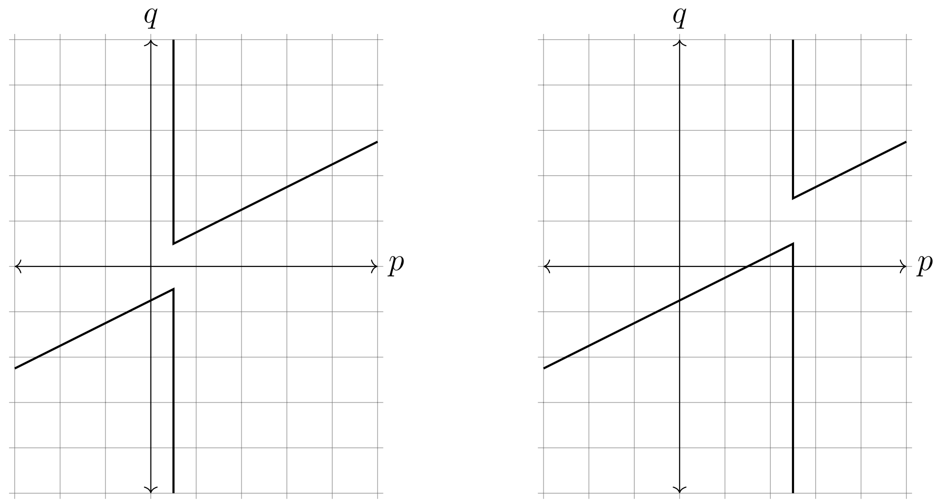


FIGURE 7. Abbreviated representations of $\mathbb{M}_3$ and $\Sigma^{2,1}\mathbb{M}_3$, respectively.

The generator w in degree $(0, -1)$ is infinitely divisible by x and z and is divisible by y. For example, there is an element denoted $\frac{w}{x}$ in degree $(0, -2)$ with the property that $x \cdot \frac{w}{x} = w$. More generally, all nonzero elements of the bottom cone are of the form $\pm\frac{w}{x^k y^i z^\ell}$ for some $k, \ell \in \mathbb{N}$ and $i \in \{0, 1\}$.

Going forward we will use an abbreviated picture for $\mathbb{M}_3$ which we can see in Figure 7. Although this simpler version allows us to keep our diagrams from getting too busy, we are leaving out a lot of information about the ring structure.

Given any C_3-space X, there is an equivariant map $X \rightarrow pt$ sending everything to a fixed point. We then get an induced map $\mathbb{M}_3 \rightarrow H^{*,*}(X; \underline{\mathbb{Z}/3})$. This means that for any C_3-space X, $H^{*,*}(X; \underline{\mathbb{Z}/3})$ can be made into an $\mathbb{M}_3$-module. In this paper, we will utilize this structure and compute the cohomology of all non-trivial C_3-surfaces as modules over $\mathbb{M}_3$.

We next consider the free orbit C_3. As an $\mathbb{M}_3$-module, the cohomology of C_3 is isomorphic to $x^{-1}\mathbb{M}_3/(y, z)$. The module $H^{*,*}(C_3; \underline{\mathbb{Z}/3})$ is given on the left of Figure 8 with an abbreviated picture on the right which we will use in future computations.

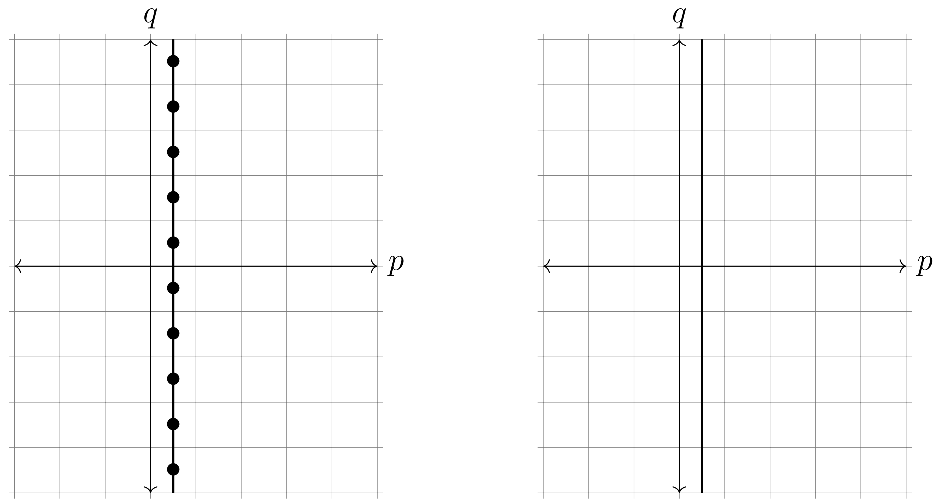


FIGURE 8. The module $H^{*,*}(C_3)$ (left) and an abbreviated representation (right).

2.3. Computational Tools

We now introduce several properties relating $RO(C_3)$-graded Bredon cohomology to singular cohomolgy. These lemmas will be extremely useful in later computations.

Lemma 2.3.1 (The Quotient Lemma)**.** *Let X be a finite C_3-CW complex. We have the following isomorphism for all p:*

$$H^{p,0}(X;\underline{\mathbb{Z}/3}) \cong H^{p,0}(X/C_3;\underline{\mathbb{Z}/3}) \cong H^p_{sing}(X/C_3;\mathbb{Z}/3).$$

A proof for the analogous statement in the $G = C_2$ case is nearly identical to that of the C_3 case and can be found in [12].

Lemma 2.3.2. *Let Y be a non-equivariant space. The cohomology of the free C_3-space $C_3 \times Y$ is given by*

$$H^{*,*}(C_3 \times Y;\underline{\mathbb{Z}/3}) \cong \mathbb{Z}/3[x,x^{-1}] \otimes_{\mathbb{Z}/3} H^*_{sing}(Y;\mathbb{Z}/3)$$

as $\mathbb{M}_3$*-modules.*

Proof. For this proof, all coefficients are understood to be $\underline{\mathbb{Z}/3}$, so we will suppress the notation.

The equivariant map $C_3 \times Y \to C_3$ sending each copy of Y to a single point induces a map $H^{*,*}(C_3) \to H^{*,*}(C_3 \times Y)$. Since $H^{*,*}(C_3) \cong \mathbb{Z}/3[x, x^{-1}]$, we can make $H^{*,*}(C_3 \times Y)$ a graded algebra over $\mathbb{Z}/3[x, x^{-1}]$. This means there exist natural maps

$$\mathbb{Z}/3[x, x^{-1}] \otimes_{\mathbb{Z}/3} H^{*,0}(C_3 \times Y) \to H^{*,*}(C_3 \times Y).$$

Restricting to the qth graded piece gives us a map

$$\left[\mathbb{Z}/3[x, x^{-1}] \otimes_{\mathbb{Z}/3} H^{*,0}(C_3 \times Y)\right]^q \to H^{*,q}(C_3 \times Y)$$

which is natural in Y. In particular, this is a map of cohomology theories. We can quickly see that this map is an isomorphism on both $Y = pt$ and $Y = C_3$, which means it defines an isomorphism of equivariant cohomology theories for each q.

We know from the Quotient Lemma that $H^{*,0}(C_3 \times Y) \cong H^*_{\text{sing}}(Y)$ for all Y, so we have isomorphisms $\mathbb{Z}/3[x, x^{-1}] \otimes_{\mathbb{Z}/3} H^*_{\text{sing}}(Y) \to H^{*,*}(C_3 \times Y)$ for each qth graded piece. Together, these give us an isomorphism of $\mathbb{M}_3$-modules, and the result follows. □

Another useful tool to aid us in computations is the **forgetful map**. Let X be a pointed C_3-space. For every integer q, we have map

$$\tilde{H}^{p,q}(X; \underline{\mathbb{Z}/3}) \xrightarrow{\psi} \tilde{H}^p_{\text{sing}}(X; \mathbb{Z}/3).$$

To understand this map, for each $V \cong \mathbb{R}^{p,q}$, we define $H^{p,q}(X; \underline{\mathbb{Z}/3})$ as maps from X to the equivariant Eilenberg-MacLane space $K(\underline{\mathbb{Z}/3}, p, q)$. Forgetting this equivariant structure leaves us with a map from the underlying topological space X to the Eilenberg-MacLane space $K(\mathbb{Z}/3, p)$.

Example 2.3.3. We will now use these tools to compute the cohomology of the freely rotating sphere S^1_{free}. All coefficients are understood to be $\underline{\mathbb{Z}/3}$, so the coefficient notation will be suppressed. We begin by constructing a cofiber sequence

$$C_{3+} \hookrightarrow S^1_{\text{free}+} \to S^{1,0} \wedge C_{3+}$$

which is illustrated in Figure 9. This cofiber sequence gives rise to a long exact sequence on cohomology:

$$\cdots \to \tilde{H}^{p,q}(S^{1,0} \wedge C_{3+}) \to H^{p,q}(S^1_{\text{free}}) \to H^{p,q}(C_3) \xrightarrow{d^{p,q}} \tilde{H}^{p+1,q}(S^{1,0} \wedge C_{3+}) \to \cdots$$

for each value of q. The total differential of these long exact sequences $d = \bigoplus_{p,q} d^{p,q}$ is an $\mathbb{M}_3$-module map, so we can understand $H^{*,*}(S^1_{\text{free}})$ by computing the total differential and solving the extension

$$0 \to \text{coker}(d) \to H^{*,*}(S^1_{\text{free}}) \to \ker(d) \to 0.$$

Figure 10 shows all possible nonzero differential maps

$$d^{p,q} \colon H^{p,q}(C_3) \to \tilde{H}^{p+1,q}(S^{1,0} \wedge C_{3+}).$$

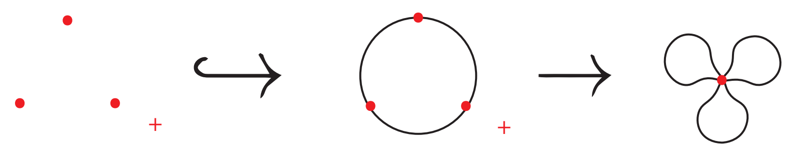


FIGURE 9. The cofiber sequence $C_{3+} \hookrightarrow S^1_{\text{free}+} \to S^{1,0} \wedge C_{3+}$.

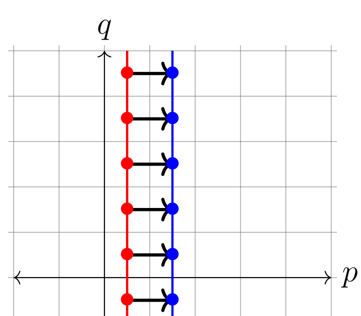


FIGURE 10. The differential $d\colon H^{*,*}(C_3) \to \tilde{H}^{*+1,*}(S^{1,0} \wedge C_{3+})$.

Since $H^{p,q}(C_3) \cong \mathbb{Z}/3[x, x^{-1}]$, we know that $H^{0,q}(C_3)$ and $\tilde{H}^{1,q}(S^{1,0} \wedge C_{3+})$ must be $\mathbb{Z}/3$ for each q. By linearity of the differential, $d^{0,q}$ is either 0 or an isomorphism for all q.

The Quotient Lemma tells us that $H^{1,q}(S^1_{\text{free}}) \cong \mathbb{Z}/3$, so the map $\tilde{H}^{1,q}(S^{1,0} \wedge C_{3+}) \to H^{1,q}(S^1_{\text{free}})$ in the long exact sequence must be an isomorphism when $q = 0$. This implies that $d^{0,0} = 0$, and thus the total differential is 0 by linearity. We can then conclude that $H^{p,q}(S^1_{\text{free}}) = \mathbb{Z}/3$ when $p = 0$ or 1.

There is still a question of whether or not the extension $\text{coker}(d) \to H^{*,*}(S^1_{\text{free}}) \to \ker(d)$ is trivial. In particular, we want to know if ya is nonzero for $a \in H^{0,q}(S^1_{\text{free}})$. To do this, we will instead compute the $\mathbb{M}_3$-module structure on $\tilde{H}^{*,*}(S^{1,0} \wedge (S^1_{\text{free}+})) \cong \tilde{H}^{*-1,*}(S^1_{\text{free}+}) \cong H^{*-1,*}(S^1_{\text{free}})$ using another cofiber sequence.

The space $S^{1,0} \wedge (S^1_{\text{free}+})$ can be constructed as the cofiber of the map $S^{0,0} \to S^{2,1}$. Suspending along the Puppe sequence yields

$$S^{2,1} \to S^{1,0} \wedge (S^1_{\text{free}+}) \to S^{1,0} \wedge S^{0,0}.$$

From here we can follow the same procedure of looking at the long exact sequence on cohomology and computing its differential $d\colon H^{*,*}(S^{2,1}) \to H^{*+1,*}(S^{1,0} \wedge S^{0,0})$ which is shown in the left of Figure 11. We know the group structure of $\tilde{H}^{*,*}\left(S^{1,0} \wedge (S^1_{\text{free}+})\right)$ from our previous computations, so it must be the case that d maps the generator of $\Sigma^{2,1}\mathbb{M}_3$ to z times the generator of $\Sigma^{1,0}\mathbb{M}_3$. The right of Figure 11 shows $\ker(d)$ and $\text{coker}(d)$ in this case. Comparing the information from Figures 10 and 11 (noting that the latter represents a shifted copy of $H^{*,*}(S^1_{\text{free}})$), we can see that

$$H^{*,*}(S^1_{\text{free}}) \cong x^{-1}\mathbb{M}_3/(z).$$

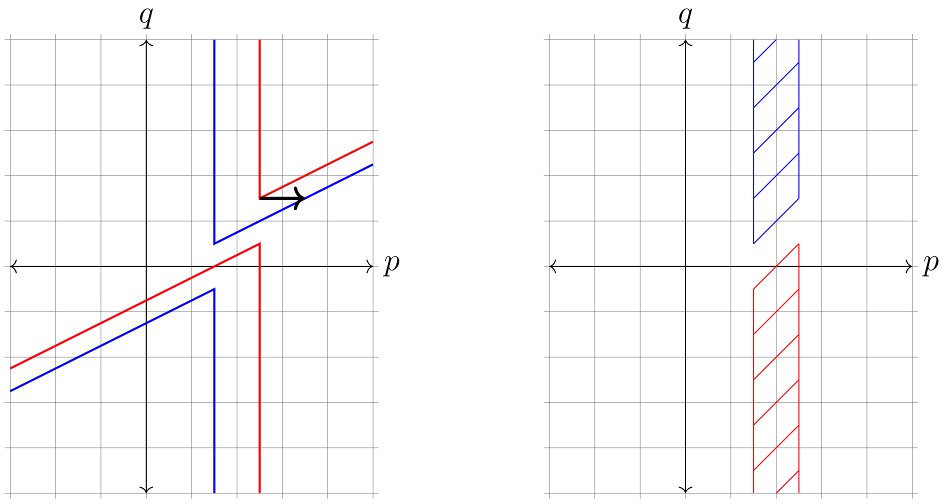


FIGURE 11. The map $d\colon \Sigma^{2,1}\mathbb{M}_3 \to \Sigma^{1,0}\mathbb{M}_3$ (left) and its kernel and cokernel (right).

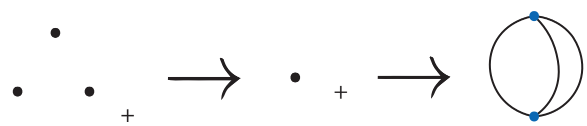


FIGURE 12. The cofiber sequence $C_{3+} \to S^{0,0} \to EB$.

Example 2.3.4. We next compute the reduced cohomology of the "eggbeater" space. The eggbeater, denoted by EB_3 (or EB when the action of C_3 is understood), can be defined as the cofiber of the map $C_{3+} \to S^{0,0}$ which sends all of C_3 to a fixed point. An illustration of this cofiber sequence can be found in Figure 12.

To determine the cohomology of EB, we will instead consider the cofiber sequence

$$EB \hookrightarrow S^{2,1} \to S^{2,0} \wedge C_{3+}$$

which is depicted in Figure 13. We can extend this via the Puppe sequence to get another cofiber sequence

$$S^{1,0} \wedge C_{3+} \to EB \to S^{2,1}.$$

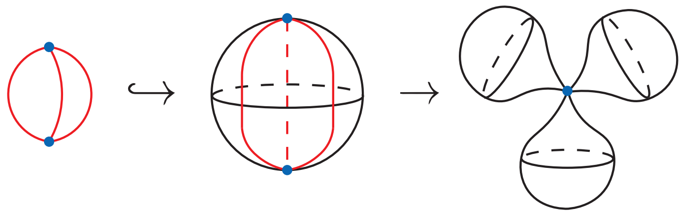


FIGURE 13. The cofiber sequence $EB \hookrightarrow S^{2,1} \to S^{2,0} \wedge C_{3+}$.

Thus we get a long exact sequence on cohomology

$$\cdots \to \tilde{H}^{p,q}(S^{1,0} \wedge C_{3+}) \to \tilde{H}^{p,q}(EB) \to \tilde{H}^{p,q}(S^{2,1}) \xrightarrow{d} H^{p+1,q}(S^{1,0} \wedge C_{3+}) \cdots$$

which can be understood by analyzing its total differential

$$d^{p,q} \colon \tilde{H}^{p,q}(S^{1,0} \wedge C_{3+}) \to \tilde{H}^{p+1,q}(S^{2,1})$$

for all (p, q). The differential $d^{1,0}$ is depicted in Figure 14. Since the total differential $\bigoplus_{p,q} d^{p,q}$ is an $\mathbb{M}_3$-module map and $\tilde{H}^{p,q}(S^{1,0} \wedge C_{3+}) = 0$ when $p \neq 1$, we only need to compute $d^{1,0}$.

Observe that $EB/C_3 \simeq pt$, so using the Quotient Lemma we know that

$$\tilde{H}^{p,0}(EB) \cong \tilde{H}^{p}_{\text{sing}}(EB/C_3) \cong \tilde{H}^{p}_{\text{sing}}(pt).$$

So $\tilde{H}^{p,0}(EB) = 0$ for all p. It then must be the case that $d^{1,0}$ is an isomorphism. By linearity, we have that $d^{1,q}$ is an isomorphism for all $q \leq 0$. We next want to understand the extension problem

$$0 \to \text{coker}(d) \to \tilde{H}^{p,q}(EB) \to \ker(d) \to 0.$$

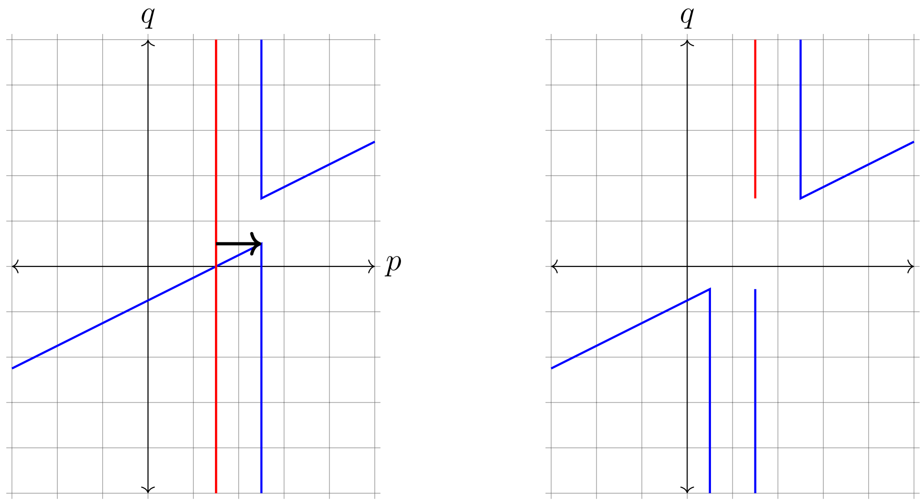


FIGURE 14. The differential $d^{0,0}$ (left), and ker(d) and coker(d) (right).

Since $\text{coker}(d) \subseteq \tilde{H}^{p,q}(EB)$, the module structure of coker(d) is preserved. The extension here is nontrivial which we can see by going through a similar computation with the cofiber sequence $S^{0,0} \to EB \to S^{1,0} \wedge (C_{3+})$. In the end, we can think of the $\mathbb{M}_3$-module structure on $\tilde{H}^{*,*}(EB)$ as generated by elements α in degree $(2,1)$ and β in degree $(1,1)$ with the relations $y\beta = 0$ and $z\beta = y\alpha$. A more complete picture of this module structure is depicted on the left of Figure 15. For brevity, we will use the representation of $\tilde{H}^{*,*}(EB; \underline{\mathbb{Z}/3})$ shown to the right of Figure 15.

Going forward, we will let $\mathbb{EB}$ represent the $\mathbb{M}_3$-module $\tilde{H}^{*,*}(EB; \underline{\mathbb{Z}/3})$.

Example 2.3.5. Let $N_1[1]$ denote the C_3-surface depicted in Figure 16. Note that the underlying topological space is $\mathbb{R}P^2$, sometimes denoted by N_1. To compute the cohomology of this surface, let us consider the cofiber sequence

$$S^1_{\text{free}+} \hookrightarrow N_1[1] \to S^{2,1}$$

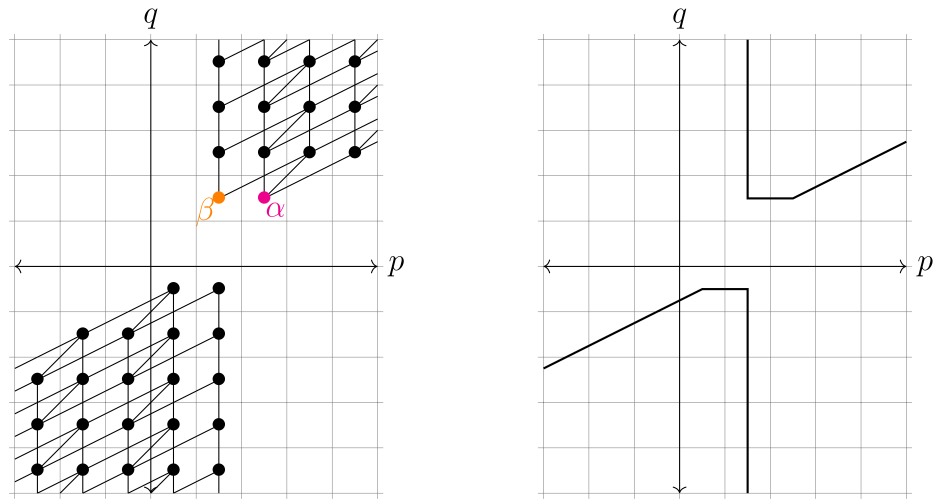


FIGURE 15. The $\mathbb{M}_3$-module $\mathbb{EB}$ (left) with abbreviated representation (right).

which is illustrated in Figure 17. Thus we have the following long exact sequence on cohomology

$$\cdots \to \tilde{H}^{p,q}(S^{2,1}) \to H^{p,q}(N_1[1]) \to H^{p,q}(S^1_{\text{free}}) \xrightarrow{d^{p,q}} \tilde{H}^{p+1,q}(S^{2,1}) \to \cdots$$

As in the previous examples, in order to compute $H^{p,q}(N_1[1])$ we need to understand the differential maps

$$d^{p,q}\colon H^{p,q}(S^1_{\text{free}}) \to \tilde{H}^{p+1,q}(S^{2,1})$$

for each (p,q). We can use the Quotient Lemma to compute $d^{1,0}$ which will then determine the value of all other possible nonzero differential maps. The differential $d^{1,0}$ is depicted in Figure 18.

We can observe that $N_1[1]/C_3 \cong \mathbb{R}P^2$. Recall $H^p_{\text{sing}}(\mathbb{R}P^2;\mathbb{Z}/3) \cong \mathbb{Z}/3$ when $p=0$, and it is 0 for all other values of p. This implies the differential $d^{1,0}$ must be an isomorphism. The $\mathbb{M}_3$-module structure of $H^{*,*}(S^1_{\text{free}})$ then guarantees that $d^{1,q}$

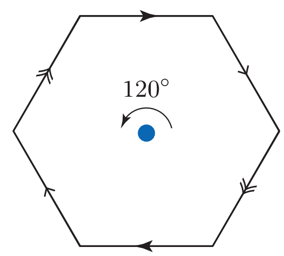


FIGURE 16. Then C_3-surface $N_1[1]$ whose underlying surface is $\mathbb{R}P^2$.

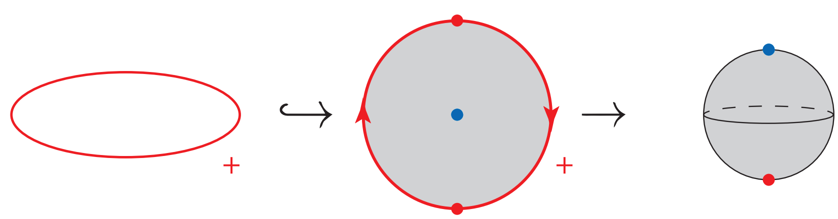


FIGURE 17. The cofiber sequence $S^1_{\text{free}+} \hookrightarrow N_1[1]_+ \to S^{2,1}$.

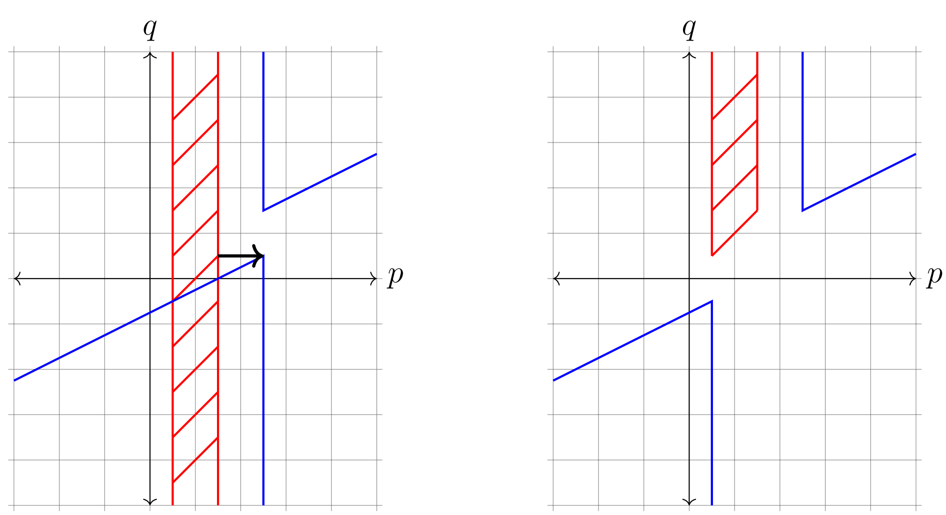


FIGURE 18. The differential $d^{1,0}$ (left), and $\ker(d)$ and $\operatorname{coker}(d)$ (right).

is an isomorphism for all $q \leq 0$. We similarly find that $d^{0,q}$ must be an isomorphism for $q < 0$.

Now we are left to solve the extension problem of $\mathbb{M}_3$-modules

$$0 \to \text{coker}(d) \to H^{*,*}(N_1[1]) \to \ker(d) \to 0$$

However since $\mathbb{M}_3$ must be a submodule of $H^{*,*}(N_1[1])$, we can immediately conclude that $H^{*,*}(N_1[1]) \cong \mathbb{M}_3$ and the extension is nontrivial.

Remark 2.3.6. *In general, given a C_3-space X with at least one fixed point, $\mathbb{M}_3$ is a summand of $H^{*,*}(X; \underline{\mathbb{Z}/3})$. This is because the inclusion $pt \hookrightarrow X$ induces a surjective $\mathbb{M}_3$-module map $H^{*,*}(X) \twoheadrightarrow H^{*,*}(pt) = \mathbb{M}_3$.*

Remark 2.3.7. *Going forward we will use the notation $N_r[F]$ to denote a non-orientable C_3-surface of genus r and F fixed points. There is a concern that this notation will not be well-defined, but it turns out to be the case that genus and the number of fixed points is enough to determine a closed, non-orientable C_3-surface up to isomorphism.*

When discussing non-equivariant surfaces, M_g will always be used to denote the "g-holed torus" (ie. the closed, orientable surface with genus $\beta = 2g$), and N_r will denote the closed, non-orientable surface of genus $\beta = r$.

CHAPTER III

C_P-EQUIVARIANT SURGERIES OF SURFACES

Let p be an odd prime. There are $(p-1)/2$ isomorphism classes of C_p-actions on $\mathbb{R}^2$ corresponding to rotation about the origin by a pth root of unity. Rotation of the plane by ω_i is isomorphic to rotation by ω_j only when $\omega_i = \overline{\omega_j}$. However if we consider such rotations up to an action of $\mathrm{Aut}(C_p)$, then we are left with only one isomorphism class of nontrivial actions on $\mathbb{R}^2$. In this chapter we lay the ground work for a classification of closed surfaces with a nontrivial action of C_p up to an action of $\mathrm{Aut}(C_p)$.

In 2019, Dugger classified all C_2-actions on surfaces using a method called **equivariant surgery** [10]. In this chapter, we define analogues of these equivariant surgery methods for surfaces with an action of C_p where p is any odd prime.

These surgeries can be used to build more interesting C_p-surfaces out of simpler ones. We will eventually see in Chapter IV that up to isomorphism, all C_p-surfaces can be constructed by performing these surgery methods on some specific families of surfaces.

3.1. Equivariant Connected Sums

Definition 3.1.1. Let Y be a non-equivariant surface and X a surface with a nontrivial order p homeomorphism $\sigma\colon X \to X$. Define $\tilde{Y} := Y \setminus D^2$, and let D be a disk in X so that D is disjoint from each of its conjugates $\sigma^i D$. Similarly let $\tilde{X}$ denote X with each of the $\sigma^i D$ removed. Choose an isomorphism $f\colon \partial\tilde{Y} \to \partial D$.

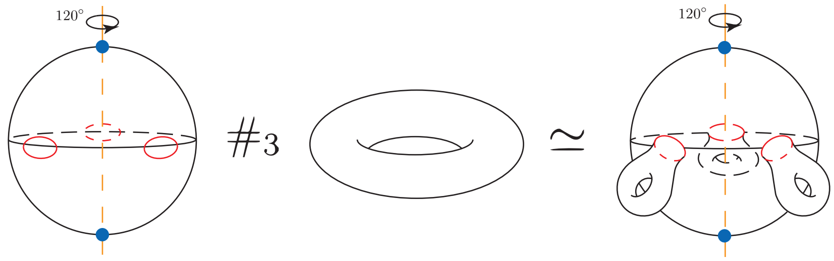


FIGURE 19. We can see above the result of the surgery $S^{2,1}\#_3 M_1$.

We define an **equivariant connected sum** $X\#_p Y$, by

$$\left[\tilde{X} \sqcup \coprod_{i=0}^{p-1} \left(\tilde{Y} \times \{i\}\right)\right] / \sim$$

where $(y, i) \sim \sigma^i(f(y))$ for $y \in \partial\tilde{Y}$ and $0 \leq i \leq p-1$. We can see an example of this surgery in Figure 19.

We will prove in Proposition A.0.3 that the space $X\#_p Y$ is independent of the chosen disk D.

Remark 3.1.2. *Any nontrivial C_p-surface has only a finite number of isolated fixed points since each fixed point must have a neighborhood isomorphic to $\mathbb{R}^2$ with a rotation action.*

For a C_p-space X with F fixed points and β-genus β_1 and a non-equivariant surface Y with β-genus β_2, $X\#_p Y$ has F fixed points and β-genus $\beta_1 + p\beta_2$.

3.2. C_p-equivariant Ribbon Surgeries

Recall that in the case $p = 3$, the representation sphere $S^{2,1}$ is the space S^2 with a homeomorphism $\sigma\colon S^2 \to S^2$ given by rotation by 120°. When p is any odd prime, there are $(p-1)/2$ non-isomorphic actions on S^2 given by rotation by a primitive pth root of unity about the axis passing through its north and south poles. When the prime p is understood, we let $S^{2,1}_{(i)}$ denote this sphere with rotation by $e^{2\pi i/p}$ where $1 \leq i \leq p-1$.

Definition 3.2.1. Let D be a disk in $S^{2,1}_{(i)}$ that is disjoint from each of its conjugate disks. We define a **C_p-equivariant ribbon** as

$$S^{2,1}_{(i)} \setminus \left(\coprod_{j=0}^{p-1} \sigma^j D \right),$$

and we denote this space $R_{p,(i)}$. We can see $R_{p,(1)}$ depicted in Figure 20 in the cases $p = 3$ and $p = 5$. The action of $R_{p,(i)}$ can be described as rotation about the orange axis. There are two fixed points of this action, given by the points in blue where the axis of rotation intersects the surface.

Definition 3.2.2. Let X be a surface with a nontrivial order p homeomorphism $\sigma\colon X \to X$. Choose a disk D_1 in X that is disjoint from $\sigma^j D_1$ for each j. Then remove each of the $\sigma^j D_1$ to form the space $\tilde{X}$. As in Definition 3.2.1, let D be the disk in $S^{2,1}_{(i)}$ which was removed (along with its conjugates) to form $R_{p,(i)}$. Choose an isomorphism $f\colon \partial D_1 \to \partial D$ and extend this equivariantly to an isomorphism $\tilde{f}\colon \partial\tilde{X} \to \partial R_{p,(i)}$. We then define **$C_p$-ribbon surgery** on X to be the space

$$\left(\tilde{X} \sqcup R_{p,(i)} \right) / \sim$$

where $x \sim \tilde{f}(x)$ for $x \in \partial\tilde{X}$. This is a new C_p-surface which we will denote $X + [R_{p,(i)}]$.

Remark 3.2.3. *There is an action of* $\mathrm{Aut}(C_p)$ *on* $S^{2,1}_{(i)}$ *(and thus* $R_{p,(i)}$*) given by* $\sigma S^{2,1}_{(i)} = S^{2,1}_{(\sigma(i))}$ *for* $\sigma \in \mathrm{Aut}(C_p)$*. Our goal is to classify all* C_p*-surfaces using*

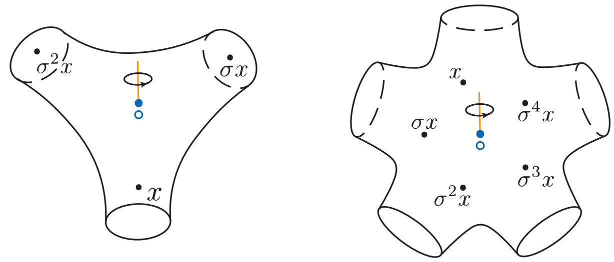


FIGURE 20. The C_p-surface R_p in the cases $p = 3$ (left) and $p = 5$ (right).

equivariant surgery methods up to this action of $\mathrm{Aut}(C_p)$ *on each of the surgery pieces. Going forward, we will use the notation* $X + [R_p]$ *to denote a* C_p*-surface obtained by performing some* C_p*-ribbon surgery on* X*. The notation* $X + [R_p]$ *therefore refers to several distinct isomorphism classes of* C_p*-surfaces which can be obtained from each other by the action of* $\mathrm{Aut}(C_p)$ *on each of the surgery pieces. We similarly let* $S^{2,1}$ *denote the 2-sphere with a rotation action of* C_p*, noting that each of these can be obtained from the standard rotation of* $e^{2\pi/p}$ *by this action of* $\mathrm{Aut}(C_p)$*.*

In the $p = 3$ *case, this action of* $\mathrm{Aut}(C_p)$ *is trivial since* $S^{2,1}_{(1)} \cong S^{2,1}_{(2)}$*. Thus the notation* $X + [R_3]$ *(as well as* $S^{2,1}$*) is well-defined and denotes a single* C_3*-surface up to equivariant isomorphism.*

We will prove in Corollary A.0.2 that the space $X + [R_{p,(i)}]$ is independent of the chosen disk D_1.

For a C_p-surface X with F fixed points and β-genus β, the space $X + [R_{p,(i)}]$ has $F + 2$ fixed points and β-genus $\beta + 2(p - 1)$.

Let $X + k[R_{p,(i)}]$ denote the surface obtained by performing C_p-ribbon surgery k times on X. We will see in Corollary A.0.2 that $+[R_{p,(i)}]$-surgery is independent of the choice of disk D_1. Because of this, C_p-ribbon surgery is associative and commutes with itself, making this notation well-defined.

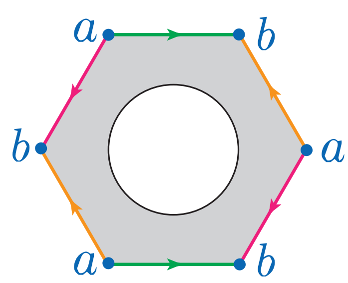


FIGURE 21. The C_3-surface TR_3.

Definition 3.2.4. We next define the C_p-surface $TR_{p,(i)}$ using a gluing diagram. Start with a $2p$-gon with a disk removed from its center. Then identify opposite edges of the $2p$-gon in the same direction to obtain the space $TR_{p,(i)}$. Figure 21 shows this in the case $p = 3$. The action on $TR_{p,(i)}$ is defined by rotation about its center by an angle corresponding to the pth root of unity $e^{2\pi i/p}$ $(1 \leq i \leq (p-1)/2)$. Note that $TR_{p,(i)} \cong TR_{p,(j)}$ only when $j = i$ or $j = p - i$. This surface is orientable with one boundary component.

When $p = 3$, $TR_{3,(1)} \cong TR_{3,(2)}$, so for simplicity of notation we will denote this space by TR_3.

Lemma 3.2.5. *The surface $TR_{p,(i)}$ has two fixed points.*

Proof. Consider the space $TR_{p,(i)}$ with its first p edges labeled $e_1, \ldots, e_p$ as shown in Figure 22. Since opposite edges of the $2p$-gon are identified, all other edges are named accordingly. Let v_1 be the starting vertex of e_1, and let v_2 be the ending vertex of e_1. We first claim that all other vertices of the $2p$-gon representing $TR_{p,(i)}$ must be identified with either v_1 or v_2. Looking at the edge labeled e_2 towards the top of the polygon, we see that e_2 shares a starting vertex with e_1. Now looking at its opposite edge, it is also the case that e_2 shares an ending vertex with e_1. We can

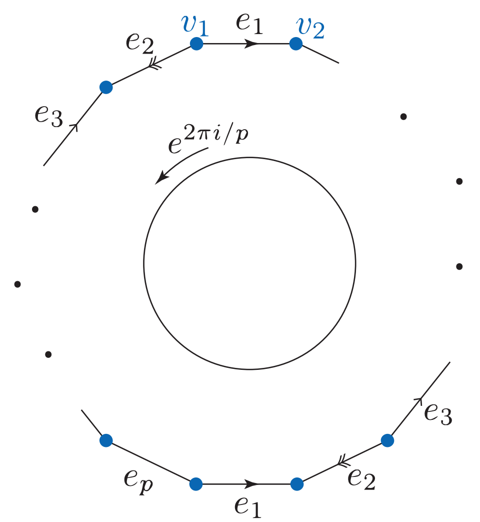


FIGURE 22. The surface $TR_{p,(i)}$.

keep going to see that e_3 must share starting and ending vertices with e_2, and in fact all vertices e_k must have starting vertex v_1 and ending vertex v_2.

Finally observe that since the action of C_p takes e_1 to e_k for some k, the vertices v_1 and v_2 are fixed under the action. Thus, $TR_{p,(i)}$ has two fixed points. $\square$

Definition 3.2.6. Let X be a non-trivial C_p-space with at least one isolated fixed point x. Choose a neighborhood D_x of x that is fixed by the action of σ. We then let $\tilde{X}$ denote $X \setminus D_x$. The action on the boundary of $\tilde{X}$ will be rotation by $e^{2\pi i/p}$ for some i. Fix an isomorphism $f\colon \partial\tilde{X} \to \partial TR_{p,(i)}$. The **$C_p$-twisted ribbon surgery** on X is given by

$$\left(\tilde{X} \sqcup TR_{p,(i)}\right) / \sim$$

where $y \sim f(y)$ for $y \in \partial\tilde{X}$. We denote this new space by $X +_x [TR_p]$.

For a C_p-surface X with F fixed points and β-genus $\beta(X)$, the space $X +_x$ $[TR_p]$ has $F + 1$ fixed points and β-genus $\beta(X) + 2(p-1)$.

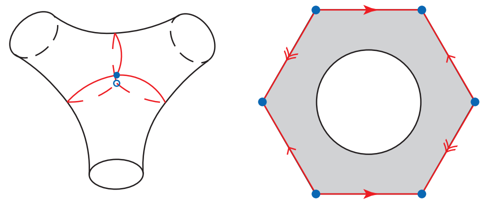


FIGURE 23. The spaces $R_{p,(i)}$ (left) and $TR_{p,(i)}$ (right), each containing EB_p in red.

Remark 3.2.7. *We will see in Corollary A.0.2 that* $+[R_{p,(i)}]$*-surgery does not depend on the initial disks chosen for the surgery, making the notation* $X + [R_{p,(i)}]$ *well defined. Unfortunately, the same is not true of twisted ribbon surgery. To specify our choice of initial fixed point* x*, we will use the notation* $X +_x [TR_p]$*. We will see in Example 3.4.5 a space* X *and choices of fixed points* x *and* y *where* $X +_x [TR_p] \ncong X +_y [TR_p]$*.*

Let X be a C_p-space with two distinct fixed points x and y. By Proposition A.0.1, there exists a simple path α in X from x to y that does not intersect its conjugate paths. Observe that the union of all conjugates of α is isomorphic to EB_p, where EB_p denotes the unreduced suspension of C_p. In particular, given any C_p-space X with at least two isolated fixed points, we can find a copy of EB_p sitting inside X. We know from Lemma C.1.1 that a neighborhood of this copy of EB_p must be isomorphic to $R_{p,(i)}$ or $TR_{p,(i)}$. Given such a space, we can "undo" the corresponding ribbon surgery to construct a new space $X - [R_{p,(i)}]$ (respectively $X - [TR_p]$) which we define below. Figure 23 shows us how $R_{p,(i)}$ and $TR_{p,(i)}$ can be viewed as neighborhoods of EB.

Definition 3.2.8. Let X be a C_p-surface with isolated fixed points a and b, and suppose the corresponding EB_p containing a and b has a neighborhood homeomorphic to $R_{p,(i)}$. Then $\tilde{X} := X \setminus R_{p,(i)}$ has p boundary components, and there is an isomorphism $f\colon \partial\tilde{X} \to \partial\left(D^2 \times C_p\right)$. Define $X - [R_{p,(i)}]$ to be

$$\left(\tilde{X} \sqcup \left(D^2 \times C_p\right)\right) / \sim$$

where $a \sim f(a)$ for $a \in \partial\tilde{X}$.

As a result of this surgery, the space $X - [R_{p,(i)}]$ has 2 fewer fixed points, and its β-genus is reduced by $2(p-1)$ from that of X. Moreover, if X was a connected C_p-surface with at least 3 fixed points, then $X - [R_{p,(i)}]$ is also connected. This does not have to be the case when $F = 2$ however. For example, there exists $EB_p \subseteq S^{2,1}$ such that $(S^{2,1}\#_p M_1) - [R_p] \cong M_1 \times C_p$.

Let $a, b \in X$ be fixed points such that a and b live in some copy of $TR_{p,(i)}$ inside of X. We can similarly define $X -_{a,b} [TR_p]$ to be the result of surgery which removes this copy of $TR_{p,(i)}$ from X and glues in $D^{2,1}$ along the boundary. As one would expect, the space $X -_{a,b} [TR_p]$ has one fewer fixed point and β-genus $p-1$ smaller than X.

Remark 3.2.9. *Although by Corollary A.0.2 we know $+[R_{p,(i)}]$ is independent of the disks chosen, $-[R_{p,(i)}]$ surgery does depend on a choice of EB_p. Two different choices of $R_{p,(i)}$ in a space can result in different spaces once $-[R_{p,(i)}]$ is performed. As a result, the notation $X - [R_{p,(i)}]$ is not well defined. Going forward, we will use the notation $X - [R_p]$ when the choice of $R_{p,(i)}$ is understood. Figure 24 shows this using the example $S^{2,1}\#_3 M_1$. For the choice of EB on the left, $-[R_3]$ surgery results in the space $M_1 \times C_3$. For the choice on the right, $-[R_3]$ surgery results in the space M_1^{free}*

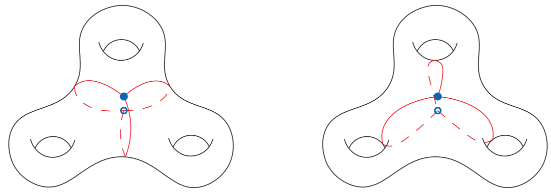


FIGURE 24. Two choices of EB in $S^{2,1}\#_3 M_1$.

Proposition 3.2.10. *Let X be a non-trivial C_p-surface and Y a non-equivariant surface. Then $\left(X + [R_{p,(i)}]\right) \#_p Y \cong (X \#_p Y) + [R_{p,(i)}]$. If X has a fixed point x, it is also true that $(X +_x [TR_p]) \#_p Y \cong (X \#_p Y) +_x [TR_p]$.*

Additionally, if X is a space for which $-[R_p]$ or $-[TR_p]$-surgeries are defined, then $(X - [R_p]) \#_p Y \cong (X \#_p Y) - [R_p]$ (respectively $(X - [TR_p]) \#_p Y \cong (X \#_p Y) - [TR_p]$).

In other words,the equivariant connected sum surgery operation commutes with $\pm[R_{p,(i)}]$ and $\pm[TR_p]$ on all C_p-surfaces X for which these surgeries are defined.

In the case of $-[R_p]$ or $\pm[TR_p]$ surgeries, this is clear because these surgery operations take place in the neighborhood of fixed points while we can choose to perform any equivariant connected sum operation away from these fixed points. The proof that equivariant connected sum surgery commutes with $+[R_{p,(i)}]$-surgery is similar to the argument presented in the proof of Corollary A.0.2 and is left to the reader.

3.3. Mobius Band Surgeries

Definition 3.3.1. Represent the Möbius band as the usual quotient of the unit square where $(0, y) \sim (1, 1-y)$. We define $(p-1)/2$ actions of C_p on the mobius band as follows. For a generator σ of C_p, let $\sigma(x, y) = \left(x + \frac{i}{p}, 1 - y\right)$ for $1 \leq i \leq$

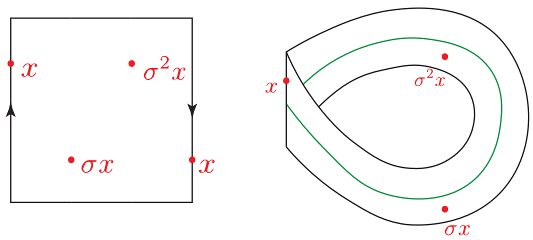


FIGURE 25. The C_3-space MB_3, whose underlying space is the mobius band.

$(p-1)/2$. Denote this space $MB_{p,(i)}$. Figure 25 gives a visual representation of this action in the case $p = 3$.

Note that the action on the boundary of $MB_{p,(i)}$ is the rotation action of S^1 by $e^{-2\pi i/p}$.

When $p = 3$, $MB_{3,(1)} \cong MB_{3,(2)}$, so for simplicity of notation we will denote this space by MB_3.

Definition 3.3.2. Let X be a non-trivial C_p-surface with fixed point x. Choose a neighborhood D_x of x which is fixed under the action of $\sigma \in C_p$. The C_p-space $\tilde{X} := X \setminus D_x$ has a distinguished boundary component isomorphic to S^1 with rotation by some angle $e^{2\pi i/p}$. Fix an equivariant isomorphism $f\colon \partial\tilde{X} \to \partial MB_{p,(i)}$. We can then define a new C_p-space

$$\left(\tilde{X} \sqcup MB_{p,(i)}\right) / \sim$$

where $x \sim f(x)$ for $x \in \partial\tilde{X}$. Denote this new space by $X +_x [FMB_p]$. This process is called **fixed point to mobius band surgery**.

Remark 3.3.3. *Given a C_p space X with F fixed points and β-genus β, the space $X + [FMB_p]$ has $F - 1$ fixed points and genus $\beta + 1$.*

Definition 3.3.4. We can similarly define **mobius band to fixed point surgery** on a C_p space X with $MB_{p,(i)} \subseteq X$. This procedure is the reverse process of $+_x[FMB_p]$ surgery in the sense that it removes MB_p from X and glues in a copy of $D^{2,1}$ along the boundary. The resulting space is denoted $X + [MB_pF]$. This notation will only be used when the choice of mobius band is understood.

3.4. Examples in the $p = 3$ Case

In this section we will highlight some of the surfaces we can now build using equivariant surgery. Although each of the following examples have analogues for higher p, we will focus mainly on the $p = 3$ case.

Example 3.4.1 (Free Torus)**.** There is a free C_3-action on the torus M_1 given by rotation of 120° about its center. Denote this C_3-space by M_1^{free}. From this, we can perform an equivariant connected sum operation with the g-holed torus M_g to construct the space $M_{3g+1}^{\text{free}} := M_1^{\text{free}} \#_3 M_g$. The result is a free C_3-action on the $(3g+1)$-holed torus (ie. the orientable surface with beta genus $\beta = 6g+2$). We will see in the next chapter that up to equivariant isomorphism there is only one free action of C_3 on M_{3g+1}. The space M_{3g+1}^{free} can be seen in Figure 26 in the case $g = 2$.

Example 3.4.2 ($\text{Sph}_g[F]$)**.** The representation sphere $S^{2,1}$ is defined as the 2-sphere with a rotation action of 120° about the axis passing through the north and south poles of the sphere. Since ribbon surgery and connected sum surgery commute with each other, we can consider the space $\text{Sph}_{2k+3g}[2k+2] := (S^{2,1} + k[R_3])\#_3 M_g$ which is constructed by performing ribbon surgery k times on $S^{2,1}$ and then performing connected sum surgery with the orientable surface M_g.

The space $\text{Sph}_{2k+3g}[2k+2]$ has $2k+2$ fixed points and is non-equivariantly isomorphic to M_{2k+3g}.

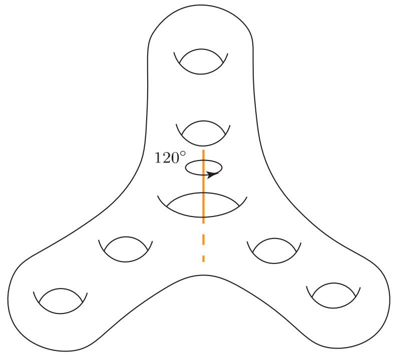


FIGURE 26. The space M_7^{free}.

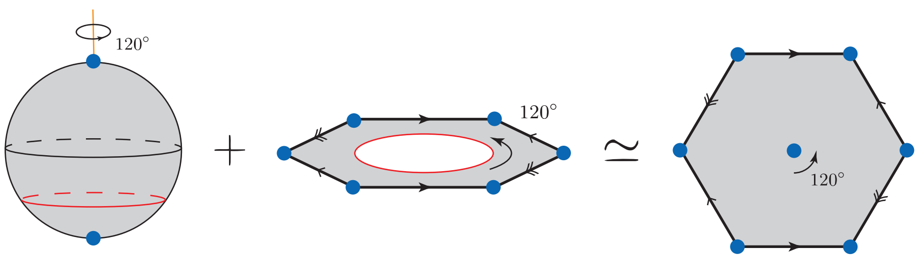


FIGURE 27. The space $\text{Hex}_1 = S^{2,1} +_S [TR_3]$.

Example 3.4.3 (Non-free Torus)**.** Let Hex_1 (Figure 27) denote the space $S^{2,1} +_S [TR_3]$ where S denotes the south pole of $S^{2,1}$. Then Hex_1 has β-genus $\beta = 2$ and 3 fixed points. We can additionally observe that the space $S^{2,1} +_N [TR_3]$ (where N is the north pole this time) is isomorphic to Hex_1.

Example 3.4.4 (Free Klein Bottle)**.** The representation sphere $S^{2,1}$ has two fixed points, so we can consider the space $S^{2,1} + 2[FMB_3] := (S^{2,1} +_N [FMB_3]) +_S [FMB_3]$ where we perform $+[FMB_3]$ surgery on both the north and south poles. The resulting space must be free of β-genus $\beta = 2$. We denote this free Klein Bottle by N_2^{free}.

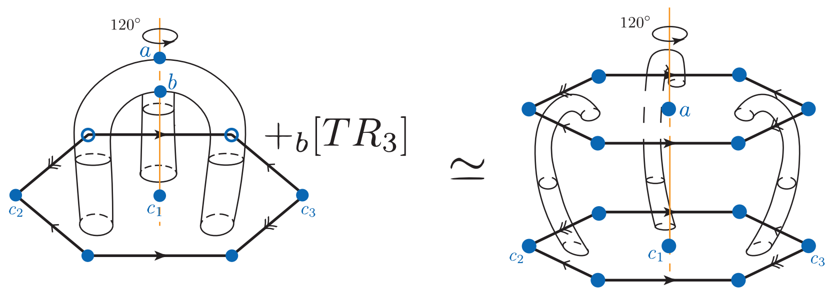


FIGURE 28. Twisted ribbon surgery centered on b yields the space Hex_2.

Other free non-orientable surfaces can be constructed by performing equivariant connected sum surgery on N_2^{free}. We will see in the next chapter that up to isomorphism there is only one free action on N_{2+3r} for each $r \geq 0$, namely

$$N_{2+3r}^{\text{free}} := N_2^{\text{free}} \#_3 N_r.$$

Example 3.4.5 (Hex_n)**.** Consider the surface $\text{Hex}_1 + [R_3]$ with β-genus $\beta = 6$ and $F = 5$ fixed points. Label the fixed points as shown in Figure 28. As a result of Lemma C.1.3, we know that $+_{c_i}[TR_3]$-surgery results in a space isomorphic to $S^{2,1} + 2[R_3]$. One naturally asks the question: Does twisted ribbon surgery yield the same space when centered around the fixed points a or b? As it turns out, we get the same result after performing $+_a[TR_3]$-surgery, but ribbon surgery centered on the point b yields a different C_3-surface. This new surface (which we will call Hex_2) is depicted in Figure 28. Proposition C.1.6 contains the proof of the fact that Hex_2 and $S^{2,1} + 2[R_3]$ are non-isomorphic surfaces.

Now that we have a new C_3-surface Hex_2, we can construct surfaces of the form $\text{Hex}_2 + k[R_3] \#_3 M_g$ for some $k, g \geq 0$. This brings us back to our previous question. What if we performed twisted ribbon surgery on $\text{Hex}_2 + k[R_3] \#_3 M_g$? Does the result depend on the chosen fixed point? Ultimately, the answer depends on k. When $k = 0$, twisted ribbon surgery is independent of the chosen fixed point. This is not true when $k > 0$ however. In this case there are two isomorphism classes of spaces which can be obtained by performing twisted ribbon surgery on $\mathrm{Hex}_2 + k[R_3]\#_3 M_g$. We prove these facts in Appendix C.

For now, let us examine this through the $k = 1$, $g = 0$ case. The space $\mathrm{Hex}_2 + [R_3]$ is shown on the left of Figure 29. Performing twisted ribbon surgery centered on any point other than b results in the space $\mathrm{Hex}_1 + 3[R_3]$. However $+_b[TR_3]$-surgery produces a different space which we will call Hex_3 (the space on the right of Figure 29).

In general, we can inductively define a space Hex_n by starting with the space $\mathrm{Hex}_{n-1} + [R_3]$ and performing twisted ribbon surgery centered on a specific fixed point. Just as Hex_3 is represented in Figure 29 as a tower of three hexagons, the space Hex_n for $n \geq 1$ can be thought of as a tower of n hexagons connected in a similar way.

An analogous collection of C_p-spaces (for $p > 3$) can be defined and will be denoted Hex_n^p when the prime p is not understood. The C_p-space Hex_n^p has $3n$ fixed points and β-genus $\beta = (3n - 2)(p - 2)$.

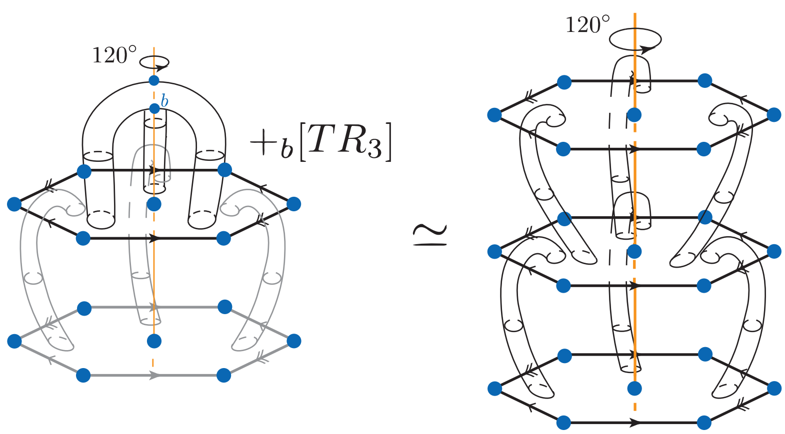


FIGURE 29. Twisted ribbon surgery centered on b yields the space Hex_3.

CHAPTER IV

CLASSIFYING C_P ACTIONS

In this chapter we state the main classification theorem for nontrivial, closed surfaces with an action of C_p for any odd prime p. All surfaces are defined up to an action of $\mathrm{Aut}(C_p)$ on each of the surgery pieces.

The proof of the classification of free C_p-surfaces can be found in Appendix B, while the proof of the non-free case is in Appendix C.

Lemma 4.0.1. *Let X be a surface with beta genus β. If $\sigma\colon X \to X$ is a C_p-action with F fixed points, then $F \equiv 2 - \beta \pmod{p}$.*

Proof. The space $X \setminus X^{C_p}$ is a free C_p-space with Euler characteristic $2 - \beta - F$. Since the action is free, $X \setminus X^{C_p} \to (X \setminus X^{C_p})/C_p$ is a p-fold covering space. In particular, the Euler characteristic of $X \setminus X^{C_p}$ must be a multiple of p. □

Theorem 4.0.2. *Let X be a connected, closed, orientable surface with an action of C_p. Then X can be constructed via one of the following surgery procedures, up to $\mathrm{Aut}(C_p)$ actions on each of the pieces.*

1. $M^{\mathit{free}}_{1+pg} := M^{\mathit{free}}_1 \#_p M_g,\ g \geq 0$
2. $\mathrm{Sph}_{(p-1)k+pg}[2k+2] := (S^{2,1} + k[R_p])\,\#_p M_g,\ k, g \geq 0$
3. $\mathrm{Hex}_{n,(3n-2)(p-1)/2+(p-1)k+pg}[3n+2k] := (\mathrm{Hex}_n + k[R_p])\,\#_p M_g,\ k, g \geq 0,\ n \geq 1$

Theorem 4.0.3. *Let X be a connected, closed, non-orientable surface with an action of C_p. Then X can be constructed via one of the following surgery procedures, up to $\mathrm{Aut}(C_p)$ actions on each of the pieces.*

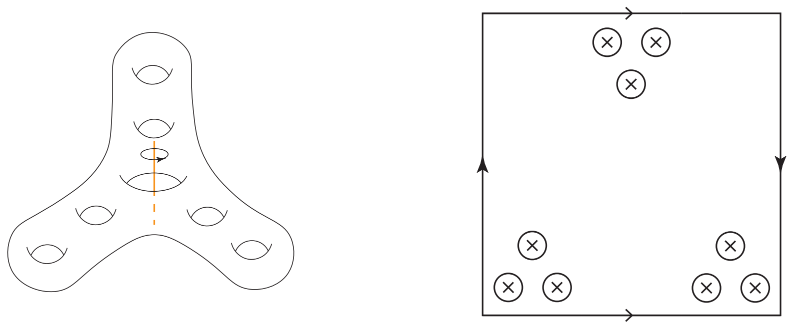


FIGURE 30. The C_3-spaces M_7^{free} (left) and N_{11}^{free} (right).

1. $N_{2+pr}^{free} \cong N_2^{free} \#_p N_r$, $r \geq 0$

2. $N_{2(p-1)k+pr}[2k+2] \cong (S^{2,1} + k[R_p]) \#_p N_r$, $r \geq 1$

3. $N_{1+2(p-1)k+pr}[1+2k] \cong (N_1[1] + k[R_p]) \#_p N_r$, $k, r \geq 0$

Remark 4.0.4. *Although unfortunate, it is important to note that for orientable surfaces, β and F do not provide enough information to distinguish between these classes. For example, when $p = 3$, $\text{Hex}_{2,4}[6]$ and $\text{Sph}_4[6]$ are non-isomorphic orientable surfaces with $\beta = 4$ and $F = 6$. See Proposition C.1.6 for a proof of this fact.*

In the case of non-orientable surfaces, F and β do distinguish between isomorphism classes. In other words, given a non-orientable surface X with specific values for F and β, one can explicitly determine how X was constructed via equivariant surgeries.

Some examples of spaces in each of these classes are shown in the case $p = 3$ in Figures 30, 31, 32.

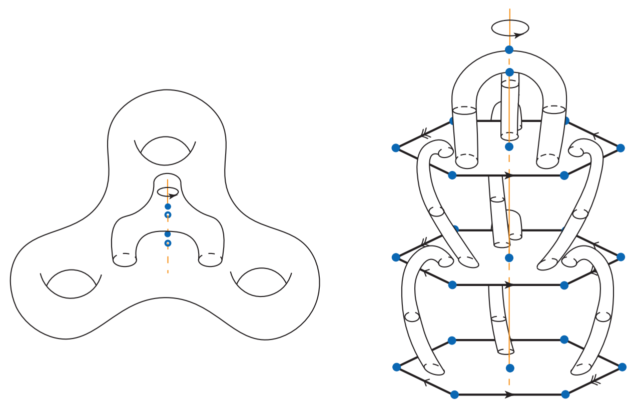


FIGURE 31. The C_3-spaces $S^{2,1} + [R_3]\#_3 M_1$ (left) and $\text{Hex}_3 + [R_3]$ (right).

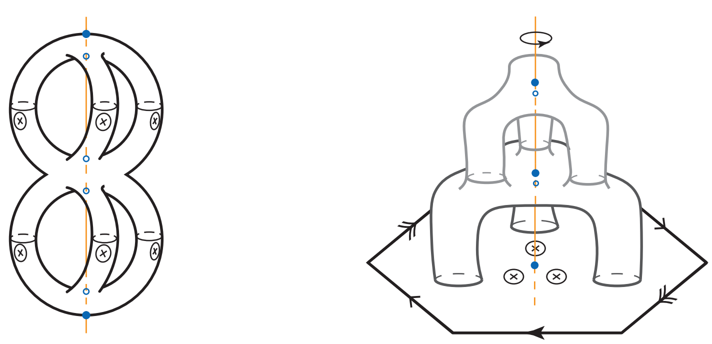


FIGURE 32. The C_3-spaces $S^{2,1} + 2[R_3]\#_3 N_2$ (left) and $N_1[1] + 2[R_3]\#_3 N_1$ (right).

CHAPTER V

COHOMOLOGY COMPUTATIONS OF FREE C_3-SURFACES

In this chapter, we compute the cohomology of all free C_3-surfaces in $\underline{\mathbb{Z}/3}$-coefficients. We start by computing the cohomology of M_1^{free} and N_2^{free}. Then we utilize the equivariant surgery construction of M_{1+3g}^{free} in Example 3.4.1 and N_{2+3r}^{free} in Example 3.4.4 to compute the cohomology of all free C_3-surfaces.

Remark 5.0.1. *In the following chapters, we will use the convention that if X is a non-equivariant surface, then*

$$\tilde{X} := X \setminus D^2.$$

Additionally, all coefficients in this chapter are understood to be $\underline{\mathbb{Z}/3}$.

Theorem 5.0.2. *Let X be a free C_3-surface with genus β.*

1. *If X is orientable, then*

$$H^{*,*}(X) \cong H^{*,*}(S^1_{\mathit{free}}) \oplus \Sigma^{1,0} H^{*,*}(S^1_{\mathit{free}}) \oplus \left(\Sigma^{1,0} H^{*,*}(C_3)\right)^{\oplus \frac{\beta-2}{3}}.$$

2. *If X is non-orientable, then*

$$H^{*,*}(X) \cong H^{*,*}(S^1_{\mathit{free}}) \oplus \left(\Sigma^{1,0} H^{*,*}(C_3)\right)^{\oplus \frac{\beta-2}{3}}.$$

Example 5.0.3. We can define the rotating torus by $M_1^{\text{free}} = S^1_{\text{free}} \times S^{1,0}$. This gives rise to the cofiber sequence

$$S^1_{\text{free}+} \hookrightarrow \left(M_1^{\text{free}}\right)_+ \to S^{1,0} \wedge \left(S^1_{\text{free}+}\right)$$

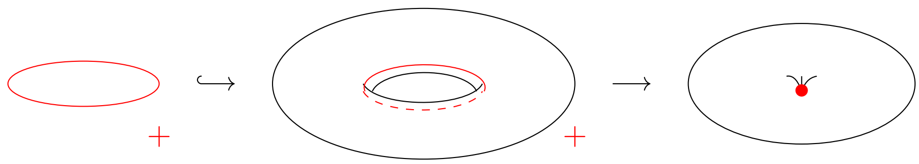


FIGURE 33. The cofiber sequence $S^1_{\text{free}+} \hookrightarrow {M_1^{\text{free}}}_+ \to S^{1,0} \wedge \left(S^1_{\text{free}+}\right)$.

which we can see depicted in Figure 33. For each q we then get a long exact sequence on cohomology

$$\to \tilde{H}^{p,q}(S^{1,0} \wedge (S^1_{\text{free}+})) \to H^{p,q}(M_1^{\text{free}}) \to H^{p,q}(S^1_{\text{free}}) \xrightarrow{d^{p,q}} \tilde{H}^{p+1,q}(S^{1,0} \wedge (S^1_{\text{free}+})) \to .$$

Together these long exact sequences have total differential

$$d := \bigoplus_{p,q} d^{p,q} \colon H^{*,*}\left(S^1_{\text{free}}\right) \to \tilde{H}^{*+1,*}\left(S^{1,0} \wedge \left(S^1_{\text{free}+}\right)\right)$$

which is shown in Figure 34. To compute $H^{*,*}(M_1^{\text{free}})$, we will analyze the total differential and solve the corresponding extension problem

$$0 \to \operatorname{coker}(d) \to H^{*,*}(M_1^{\text{free}}) \to \ker(d) \to 0.$$

We can see from Figure 34 that the only possible nonzero differentials are $d^{0,q}$ and $d^{1,q}$. Since d is an $\mathbb{M}_3$-module map, it suffices to compute $d^{0,0}$ and $d^{1,0}$. The quotient Lemma tells us that

$$\begin{aligned} H^{p,0}\left(M_1^{\text{free}}\right) &\cong H^p_{\text{sing}}\left(M_1^{\text{free}}/C_3\right) \\ &\cong H^p_{\text{sing}}(M_1) \end{aligned}$$

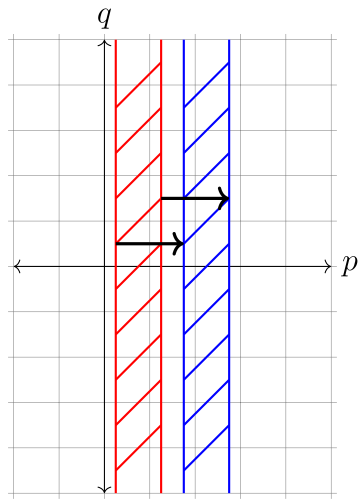


FIGURE 34. The differential $d\colon H^{*,*}\left(S^1_{\text{free}}\right) \to \tilde{H}^{*+1,*}\left(S^{1,0} \wedge \left(S^1_{\text{free}+}\right)\right)$.

which is $\mathbb{Z}/3$ when $p = 0, 2$ and $\mathbb{Z}/3 \oplus \mathbb{Z}/3$ when $p = 1$. So $d^{0,0}$ and $d^{1,0}$ must be the zero map, and thus all differentials are zero by linearity. This leaves us to determine if the following extension is trivial:

$$0 \to \Sigma^{1,0} H^{*,*}(S^1_{\text{free}}) \to H^{*,*}(M_1^{\text{free}}) \to H^{*,*}(S^1_{\text{free}}) \to 0$$

The only other possibility is a non-trivial z-extension from $\ker(d)$ to $\operatorname{coker}(d)$. This begs the question: does there exist $\alpha \in H^{0,q}(M_1^{\text{free}})$ so that $z\alpha \neq 0$?

The following composition is the identity map, implying π_2^* is injective on cohomology.

$$S^1_{\text{free}} \xrightarrow{\cong} \text{pt} \times S^1_{\text{free}} \hookrightarrow M_1^{\text{free}} \xrightarrow{\pi_2} S^1_{\text{free}}$$

Since $H^{0,q}(M_1^{\text{free}})$ and $H^{0,q}(S^1_{\text{free}})$ are both $\mathbb{Z}/3$, it must be that π_2^* is an isomorphism in degrees $(0, q)$. Now let $\alpha \in H^{0,q}(M_1^{\text{free}})$. Then there exists $\beta \in H^{0,q}(S^1_{\text{free}})$ such that $\alpha = \pi_2^*(\beta)$. Then $z\alpha = \pi_2^*(z\beta) = 0$ since $\beta \in H^{*,*}(S^1_{\text{free}}) =$

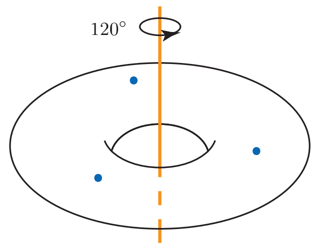


FIGURE 35. The space $\hat{M}_1$ where the blue points are identified to a single point.

$x^{-1}\mathbb{M}_3/(z)$. Thus the extension is trivial and

$$H^{*,*}(M_1^{\text{free}}) \cong H^{*,*}(S^1_{\text{free}}) \oplus \Sigma^{1,0} H^{*,*}(S^1_{\text{free}})$$

We now turn our attention to the general case. Recall that M_{3g+1}^{free} can be constructed via the equivariant connected sum: $M_1^{\text{free}} \#_3 M_g$. This construction suggests a map

$$\left(\tilde{M}_g \times C_3\right)_+ \hookrightarrow M_{3g+1}^{\text{free}}{}_+$$

whose cofiber is the C_3-space depicted in Figure 36. We denote this space by $\hat{M}_1$. The three blue points shown in the figure are all identified, making it a single fixed point under the C_3-action. In order to utilize the corresponding long exact sequence on cohomology, we first need to compute $\tilde{H}^{*,*}(\hat{M}_1)$.

To do this, we use another cofiber sequence

$$C_{3+} \hookrightarrow M_1^{\text{free}}{}_+ \to \hat{M}_1$$

which we can extend to the cofiber sequence

$$M_1^{\text{free}}{}_+ \hookrightarrow \hat{M}_1 \to S^{1,0} \wedge (C_{3+}) \, .$$

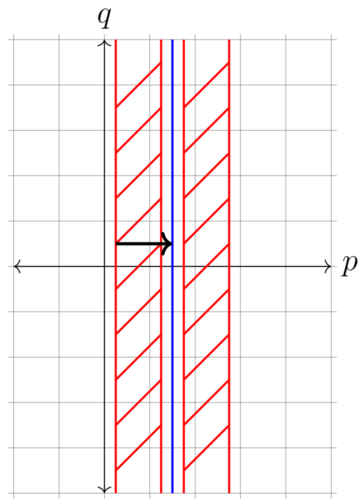


FIGURE 36. The differential $d\colon H^{*,*}(M_1^{\text{free}}) \to \tilde{H}^{*+1,*}(S^{1,0} \wedge C_{3+})$.

We next consider the long exact sequence on cohomology which has total differential $d := \bigoplus_{p,q} d^{p,q}$, where

$$d^{p,q}\colon H^{p,q}(M_1^{\text{free}}) \to \tilde{H}^{p+1,q}(S^{1,0} \wedge C_{3+}).$$

We can see from Figure 36 that it suffices to compute $d^{0,0}$. The Quotient Lemma tells us $\tilde{H}^{0,0}(\hat{M}_1) \cong \tilde{H}^0_{\text{sing}}(\hat{M}_1/C_3) \cong \tilde{H}^0_{\text{sing}}(M_1) = 0$. This means $d^{0,0}$ must be an isomorphism. Thus we conclude $d^{0,q}$ is an isomorphism for all q. So $\text{coker}(d) = 0$ and we have

$$\tilde{H}^{*,*}(\hat{M}_1) \cong \Sigma^{1,0} H^{*,*}(C_3) \oplus \Sigma^{1,0} H^{*,*}(S^1_{\text{free}}).$$

Now that we know the cohomology of $\hat{M}_1$, we can return to the cofiber sequence

$$\left(\tilde{M}_g \times C_3\right)_+ \hookrightarrow M^{\text{free}}_{3g+1_+} \to \hat{M}_1$$

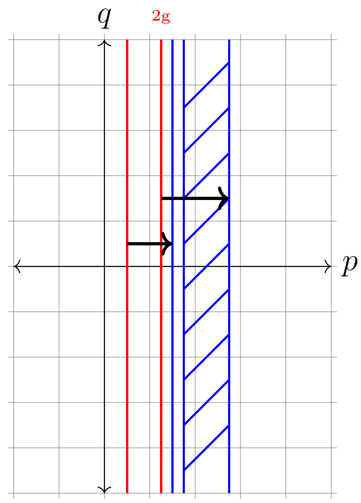


FIGURE 37. The differential $d\colon H^{*,*}(\tilde{M}_g \times C_3) \to \tilde{H}^{*+1,*}(\hat{M}_1)$.

and its corresponding long exact sequence on cohomology. For each q, we get an exact sequence with differential

$$d^{p,q}\colon H^{p,q}(\tilde{M}_g \times C_3) \to \tilde{H}^{p+1,q}(\hat{M}_1)$$

By Lemma 2.3.2, we know $H^{*,*}(\tilde{M}_g \times C_3) \cong \mathbb{Z}/3[x, x^{-1}] \otimes_{\mathbb{Z}/3} H^*_{\text{sing}}(\tilde{M}_g)$. We can see in Figure 37 that we only need to compute the differential when p is 0 or 1.

Again, we know from the Quotient Lemma that

$$\begin{aligned} H^{p,0}(M^{\text{free}}_{3g+1}) &\cong H^p_{\text{sing}}(M^{\text{free}}_{3g+1}/C_3) \\ &\cong H^p_{\text{sing}}(M_{g+1}). \end{aligned}$$

In particular,

$$H^{0,0}(M^{\text{free}}_{3g+1}) = \mathbb{Z}/3$$
$$H^{1,0}(M^{\text{free}}_{3g+1}) = \mathbb{Z}/3^{\oplus 2g+2}$$
$$H^{2,0}(M^{\text{free}}_{3g+1}) = \mathbb{Z}/3$$

So all differentials must be zero. Thus we are left to solve the extension problem

$$0 \to \tilde{H}^{*,*}(\hat{M}_1) \to H^{*,*}(M^{\text{free}}_{3g+1}) \to H^{*+1,*}(\tilde{M}_g \times C_3) \to 0.$$

All elements of the lower cone of $\mathbb{M}_3$ must act trivially on elements which are infinitely divisible by x. So we only need to determine if $y\alpha$ or $z\alpha$ are nonzero for $\alpha \in H^{0,q}(M^{\text{free}}_{3g+1})$. Consider the following map of cofiber sequences:

$$\begin{array}{ccccc} (\tilde{M}_g \times C_3)_+ & \hookrightarrow & {M^{\text{free}}_{3g+1}}_+ & \longrightarrow & \hat{M}_1 \\ \downarrow & & \downarrow \varphi & & \downarrow \\ {C_3}_+ & \hookrightarrow & {M^{\text{free}}_1}_+ & \longrightarrow & \hat{M}_1 \end{array}$$

Recall that the differential for the long exact sequence corresponding to the top cofiber sequence was shown to be zero. Moreover, in a previous computation we showed that the differential in the long exact sequence corresponding to ${M^{\text{free}}_1}_+ \to \hat{M}_1 \to S^{1,0} \wedge {C_3}_+$ was always surjective. This implies the differential in the long exact sequence for the bottom cofiber sequence must be 0. So we have the following commutative diagram where the rows are exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{H}^{p,q}(\hat{M}_1) & \longrightarrow & H^{p,q}(M^{\text{free}}_{3g+1}) & \longrightarrow & H^{p,q}(\tilde{M}_g \times C_3) & \longrightarrow & 0 \\
 & & \uparrow \text{id} & & \uparrow \varphi^* & & \uparrow & & \\
0 & \longrightarrow & \tilde{H}^{p,q}(\hat{M}_1) & \longrightarrow & H^{p,q}(M^{\text{free}}_1) & \longrightarrow & H^{p,q}(C_3) & \longrightarrow & 0
\end{array}$$

Row exactness implies φ^* is injective. In fact, φ^* must be an isomorphism in dimension $(0, q)$ for all q since both the domain and the codomain are $\mathbb{Z}/3$. Let $\alpha \in H^{0,q}(M^{\text{free}}_1)$. Then $\varphi^*(y\alpha) = y\varphi^*(\alpha)$. We know $y\alpha \neq 0$ in $H^{1,q+1}(M^{\text{free}}_1)$, so injectivity implies $y\varphi^*(\alpha) \neq 0$. Surjectivity in degrees $(0, q)$ implies $y\beta \neq 0$ for all nonzero $\beta \in H^{0,q}(M^{\text{free}}_{3g+1})$. Also note that φ^* must be an isomorphism in degrees $(2, q)$ for all q. We know $z\varphi^*(\alpha) = \varphi^*(z\alpha) = 0$ since $z\alpha = 0$ in $H^{2,q+1}(M^{\text{free}}_1)$. So the action of z on $H^{0,q}(M^{\text{free}}_{3g+1})$ must be 0. Putting this together, we conclude

$$H^{*,*}(M^{\text{free}}_{3g+1}) \cong H^{*,*}(S^1_{\text{free}}) \oplus \Sigma^{1,0} H^{*,*}(S^1_{\text{free}}) \oplus \left(\Sigma^{1,0} H^{*,*}(C_3)\right)^{\oplus 2g}$$

Example 5.0.4. We now compute the cohomology of all free non-orientable C_3-surfaces. Recall the construction for the free C_3 action on the Klein Bottle N^{free}_2, as defined in Example 3.4.4.

To compute the cohomology of this space, we start with the cofiber sequence

$$S^1_{\text{free}+} \hookrightarrow N^{\text{free}}_{2\ +} \to N_1[1]$$

which we can see illustrated in Figures 38 and 39. We saw in Example 2.3.5 that $\tilde{H}^{*,*}(N_1[1]) = 0$, so we can immediately conclude

$$H^{*,*}(N^{\text{free}}_2) \cong H^{*,*}(S^1_{\text{free}}).$$

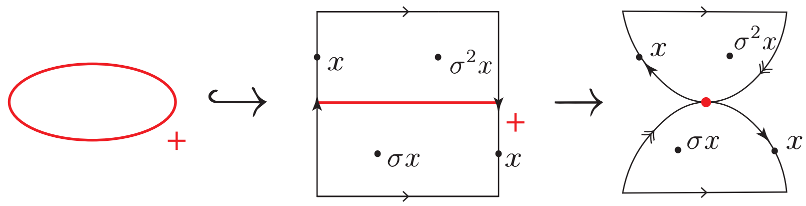


FIGURE 38. The cofiber sequence $S^1_{\text{free}+} \xrightarrow{f} N_2^{\text{free}}{}_+ \to \text{cofib}(f)$.

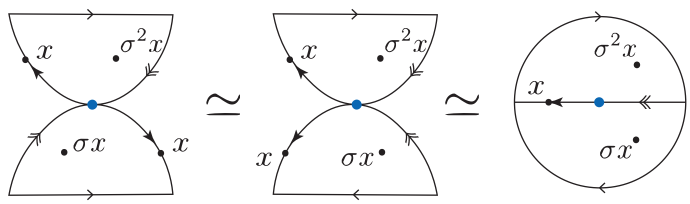


FIGURE 39. The cofiber of the map f above is equivalent to the space $N_1[1]$.

We turn next to the more general case of $N_{2+3r}^{\text{free}} = N_2^{\text{free}} \#_3 N_r$ for $r \geq 1$. For this we consider the cofiber sequence

$$\left(\tilde{N}_r \times C_3\right)_+ \hookrightarrow N_{2+3r+}^{\text{free}} \to \hat{N}_2 \tag{5.0.1}$$

To make use of this, we first must compute the reduced cohomology of the space $\hat{N}_2$.

The space $\hat{N}_2$ can be constructed as the cofiber of the map $C_{3+} \hookrightarrow N_2^{\text{free}}{}_+$. Using the Puppe sequence, we can instead consider the cofiber sequence

$$N_2^{\text{free}}{}_+ \to \hat{N}_2 \to \Sigma^{1,0} C_{3+}$$

and its corresponding long exact sequence on cohomology

$$\cdots \to \tilde{H}^{p,q}(\Sigma^{1,0}C_{3+}) \to \tilde{H}^{p,q}(\hat{N}_2) \to H^{p,q}(N_2^{\text{free}}) \xrightarrow{d} \tilde{H}^{p+1,q}(\Sigma^{1,0}C_{3+}) \to \cdots .$$

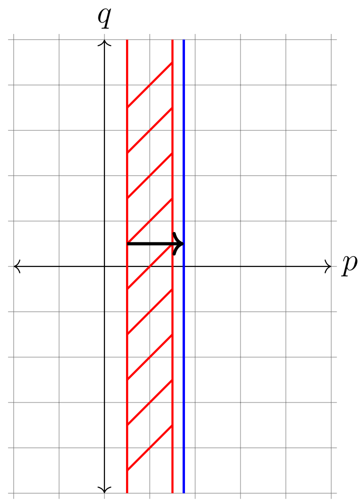


FIGURE 40. The differential $d\colon H^{*,*}(N_2^{\text{free}}) \to \tilde{H}^{*+1,*}(\Sigma^{1,0}C_{3+})$.

Our goal is to compute the differential of this sequence, which can be seen in Figure 40. First notice that $\hat{N}_2/C_3 \simeq N_2$, so by the Quotient Lemma we have $\tilde{H}^{p,0}(\hat{N}_2) \cong \tilde{H}^p_{\text{sing}}(N_2)$ which is $\mathbb{Z}/3$ for $p = 1$ and 0 otherwise. In particular, $\tilde{H}^{0,0}(\hat{N}_2) = 0$ which implies the differential

$$d^{0,q}\colon H^{0,q}(N_2^{\text{free}}) \to \tilde{H}^{1,q}(\Sigma^{1,0}C_{3+})$$

is an isomorphism for $q = 0$. By linearity, we can conclude that this differential is in fact an isomorphism for all q. So $\operatorname{coker}(d) = 0$ and $\tilde{H}^{*,*}(\hat{N}_2) \cong \ker(d)$. In particular,

$$\tilde{H}^{*,*}(\hat{N}_2) \cong \Sigma^{1,0}H^{*,*}(C_3).$$

We can now turn back to our original cofiber sequence (5.0.1) and examine its corresponding long exact sequence on cohomology

$$\cdots \to \tilde{H}^{p,q}(\hat{N}_2) \to H^{p,q}(N_{2+3r}^{\text{free}}) \to H^{p,q}(\tilde{N}_r \times C_3) \xrightarrow{d} \tilde{H}^{p+1,q}(\hat{N}_2) \to \cdots .$$

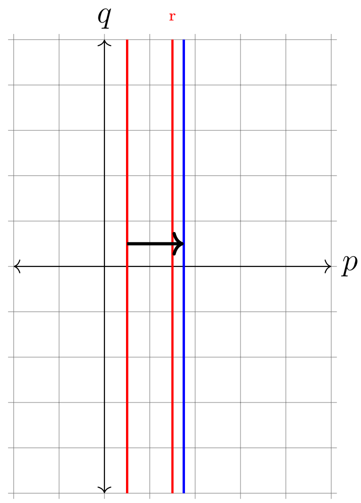


FIGURE 41. The differential to (5.0.1), $d\colon H^{*,*}(\tilde{N}_r \times C_3) \to \tilde{H}^{*+1,*}(\hat{N}_2)$.

As in previous examples, our strategy is to compute the total differential

$$d\colon H^{*,*}(\tilde{N}_r \times C_3) \to \tilde{H}^{*+1,*}(\hat{N}_2)$$

as seen in Figure 41.

Since $N^{\text{free}}_{2+3r}/C_3 \simeq N_{2+r}$, we know by the Quotient Lemma that $H^{p,0}(N^{\text{free}}_{2+3r}) \cong H^p_{\text{sing}}(N_{2+r})$ which is $\mathbb{Z}/3$ for $p = 0$, $(\mathbb{Z}/3)^{r+1}$ when $p = 1$, and 0 otherwise. Linearity of the differential guarantees that this map is zero in all degrees.

All that remains is to solve the extension problem

$$0 \to \tilde{H}^{*,*}(\hat{N}_2) \to H^{*,*}(N^{\text{free}}_{2+3r}) \to H^{*,*}(\tilde{N}_r \times C_3) \to 0.$$

In particular, we need to determine if $y\alpha$ is nonzero for $\alpha \in H^{0,q}(N^{\text{free}}_{2+3r})$. Consider the following map of cofiber sequences:

$$\begin{array}{ccccc}
(\tilde{N} \times C_3)_+ & \hookrightarrow & N^{\text{free}}_{2+3r} & \longrightarrow & \hat{N}_2 \\
\downarrow & & \downarrow q & & \downarrow \\
C_{3+} & \hookrightarrow & N^{\text{free}}_2 & \longrightarrow & \hat{N}_2
\end{array}$$

The differential for each of the corresponding long exact sequences was found in the above computations to be zero. Thus we have the following commutative diagram where the rows are exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{H}^{*,*}(\hat{N}_2) & \longrightarrow & H^{*,*}(N^{\text{free}}_{2+3r}) & \longrightarrow & H^{*,*}(\tilde{N}_r \times C_3) & \longrightarrow & 0 \\
& & \uparrow id & & \uparrow q^* & & \uparrow & & \\
0 & \longrightarrow & \tilde{H}^{*,*}(\hat{N}_2) & \longrightarrow & H^{*,*}(N^{\text{free}}_2) & \longrightarrow & H^{*,*}(C_3) & \longrightarrow & 0
\end{array}$$

Row exactness implies that q^* is injective. Moreover, for nonzero $\beta \in H^{0,q}(N^{\text{free}}_2)$ we know that $y\beta \neq 0$ in $H^{1,q+1}(N^{\text{free}}_2)$. Thus for any nonzero $\alpha \in H^{0,q}(N^{\text{free}}_{2+3r})$, we know $\alpha = q^*(\beta)$ for some nonzero $\beta \in H^{0,q}(N^{\text{free}}_2)$. By the above remarks, it follows that $y\alpha = yq^*(\beta) = q^*(y\beta) \neq 0$. So we can conclude

$$H^{*,*}(N^{\text{free}}_{2+3r}) \cong H^{*,*}(S^1_{\text{free}}) \oplus \left(\Sigma^{1,0} H^{*,*}(C_3)\right)^{\oplus r}.$$

CHAPTER VI

COHOMOLOGY COMPUTATIONS OF NON-FREE C_3-SURFACES

Theorem 6.0.1. *Let X be a C_3-surface with genus β and $F > 0$ fixed points.*

1. *If X is orientable, then*

$$H^{*,*}(X) \cong \mathbb{M}_3 \oplus \Sigma^{2,1}\mathbb{M}_3 \oplus \mathbb{EB}^{\oplus F-2} \oplus \left(\Sigma^{1,0} H^{*,*}(C_3)\right)^{\oplus \frac{\beta-2F+4}{3}}$$

2. *If X is non-orientable and F is even, then*

$$H^{*,*}(X) \cong \mathbb{M}_3 \oplus \mathbb{EB}^{\oplus F-2} \oplus \left(\Sigma^{1,0} H^{*,*}(C_3)\right)^{\oplus \frac{\beta-2F+1}{3}}$$

3. *If X is non-orientable and F is odd, then*

$$H^{*,*}(X) \cong \mathbb{M}_3 \oplus \mathbb{EB}^{\oplus F-1} \oplus \left(\Sigma^{1,0} H^{*,*}(C_3)\right)^{\oplus \frac{\beta-2F+1}{3}}$$

Remark 6.0.2. *Since $H^{*,*}(X)$ is completely determined by β, F, and whether or not X is orientable, it follows from the observations in Remark 4.0.4 that $RO(C_3)$-graded Bredon cohomology in $\underline{\mathbb{Z}/3}$ coefficients is not a complete invariant. It cannot distinguish between isomorphism classes of C_3-spaces.*

This is true in the case of both orientable and non-orientable surfaces. We can again reference Remark 4.0.4 and observe for exmample that $H^{*,*}(\mathrm{Hex}_{2,4}[6]) \cong H^{*,*}(\mathrm{Sph}_4[6])$. We can also find an example of this in the non-orientable surfaces $N_3[2]$ and $N_1[1]$.

We will prove this result by directly computing the cohomology of all non-free C_3-surfaces. These computations will be broken up into four classes of non-free surfaces according to our classification in Theorems 4.0.2 and 4.0.3. We begin by considering the following lemmas which will aid us in these computations.

Lemma 6.0.3. *The group* $\mathrm{Ext}^{1,(0,0)}_{\mathbb{M}_3}(\mathbb{EB}, \mathbb{EB})$ *is trivial. In particular, given a short exact sequence of* $\mathbb{M}_3$*-modules*

$$0 \to \mathbb{EB} \hookrightarrow X \twoheadrightarrow \mathbb{EB} \to 0$$

it must be that $X \cong \mathbb{EB} \oplus \mathbb{EB}$.

Proof. We begin by constructing the first few terms of a free resolution

$$\cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\eta} \mathbb{EB}$$

of $\mathbb{EB}$ over $\mathbb{M}_3$. Recall from Example 2.3.4 that $\mathbb{EB}$ is generated by α in degree $(2,1)$ and β in degree $(1,1)$ with $y\beta = 0$ and $z\beta = y\alpha$.

Define $F_0 = \mathbb{M}_3\langle a_0\rangle \oplus \mathbb{M}_3\langle b_0\rangle$ where a_0 and b_0 are generators of each copy of $\mathbb{M}_3$ in degrees $(2,1)$ and $(1,1)$, respectively. There is a surjection $\eta\colon F_0 \to \mathbb{EB}$ given by $a_0 \mapsto \alpha$, $b_0 \mapsto \beta$. Its kernel is generated by yb_0 and $zb_0 - ya_0$, so we can construct another map $d_1\colon \mathbb{M}_3 langlea_1\rangle \oplus \mathbb{M}_3\langle b_1\rangle \to F_0$ (where $|a_1| = (2,2)$ and $|b_1| = (3,2)$) such that $d_1(a_1) = yb_0$ and $d_1(b_1) = zb_0 - ya_0$. Let F_1 denote the module $\mathbb{M}_3\langle a_1\rangle \oplus \mathbb{M}_3\langle b_1\rangle$.

Notice that $\ker(d_1)$ is generated by ya_1 and $za_1 - yb_1$. For $F_2 := \mathbb{M}_3\langle a_2\rangle \oplus \mathbb{M}_3\langle b_2\rangle$ (with $|a_2| = (3,3)$ and $|b_2| = (4,3)$), we define the map $d_2\colon F_2 \to F_1$ given by $d_2(a_2) = ya_1$ and $d_2(b_2) = za_1 - yb_1$. We can stop here as this is the only part of the free resolution necessary to understand the first Ext group.

Next apply the functor $\mathrm{Hom}_{\mathbb{M}_3}(-, \mathbb{EB})$ of degree preserving maps to our free resolution:

$$\mathrm{Hom}(F_0, \mathbb{EB}) \xrightarrow{d_1^*} \mathrm{Hom}(F_1, \mathbb{EB}) \xrightarrow{d_2^*} \mathrm{Hom}(F_2, \mathbb{EB}) \to \cdots$$

and compute $\ker(d_2^*)/\operatorname{im}(d_1^*)$.

Let's start by computing d_2^*. Let f be an element of $\mathrm{Hom}(F_1, \mathbb{EB}) = \mathrm{Hom}(\mathbb{M}_3\langle a_1\rangle \oplus \mathbb{M}_3\langle b_1\rangle, \mathbb{EB})$. Since f is determined by its values on a_1 and b_1, let us say $f(a_1) = r$ and $f(b_1) = s$ for some $r, s \in \mathbb{EB}$ in degrees $(2,2)$ and $(3,2)$, respectively. Then $d_2^*(f) \in \mathrm{Hom}(F_2, \mathbb{EB}) = \mathrm{Hom}(\mathbb{M}_3\langle a_2\rangle \oplus \mathbb{M}_3\langle b_2\rangle, \mathbb{EB})$ is determined by its values on a_2 and b_2. We have

$$\begin{aligned} d_2^*(f)(a_2) &= f(d_2(a_2)) = f(ya_1) = yf(a_1) = yr \\ d_2^*(f)(b_2) &= f(d_2(b_2)) = f(za_1 - yb_1) = zf(a_1) - yf(b_1) = zr - ys. \end{aligned}$$

So $f \in \ker(d_2^*)$ exactly when $yr = 0$ and $zr - ys = 0$ in $\mathbb{EB}$.

Recall that r must be some element of $\tilde{H}^{2,2}(EB)$, so $yr \neq 0$ unless $r = 0$. So $f \in \ker(d_2^*)$ if and only if $f(a_1) = 0$. Next observe that $ys = 0$ for any $s \in \tilde{H}^{3,2}(EB)$. In particular, there are two nonzero elements of $\ker(d_2^*)$; namely, the maps such that $a_1 \mapsto 0$ and $b_1 \mapsto \pm y\alpha$. Call these maps f_+ and f_-. We will see that both of these maps are in $\operatorname{im}(d_1^*)$, proving that $\mathrm{Ext}^{1,(0,0)}_{\mathbb{M}_3}(\mathbb{EB}, \mathbb{EB}) = 0$.

To show this, we compute d_1^*. Given a map $g \in \mathrm{Hom}(F_0, \mathbb{EB}) = \mathrm{Hom}(\mathbb{M}_3\langle a_0\rangle \oplus \mathbb{M}_3\langle b_0\rangle, \mathbb{EB})$, we know g is determined by its values on a_0 and b_0, so let's suppose $g(a_0) = t$ and $g(b_0) = u$ for some $t \in \tilde{H}^{2,1}(EB)$ and $u \in \tilde{H}^{1,1}(EB)$. Then $d_1^*(g) \in \mathrm{Hom}(F_1, \mathbb{EB}) = \mathrm{Hom}(\mathbb{M}_3\langle a_1\rangle \oplus \mathbb{M}_3\langle b_1\rangle, \mathbb{EB})$ and can be determined by its values on a_1 and b_1. In particular,

$$d_1^*(g)(a_1) = g(d_1(a_1)) = g(yb_0) = yg(b_0) = yu$$
$$d_1^*(g)(b_1) = g(d_1(b_1)) = g(zb_0 - ya_0) = zg(b_0) - yg(a_0) = zu - yt.$$

Then we can see that $u = -\beta$, $t = \alpha$ defines an element of $\mathrm{Hom}(F_1, \mathbb{EB})$ whose image under d_1^* is equal to f_+. Similarly, $u = \beta$, $t = -\alpha$ defines an element of $\mathrm{Hom}(F_1, \mathbb{EB})$ whose image under d_1^* is f_-.

□

Lemma 6.0.4. *The group* $\mathrm{Ext}^{1,(2,1)}_{\mathbb{M}_3}(\mathbb{EB}, \mathbb{M}_3) \cong \mathrm{Ext}^{1,(0,0)}_{\mathbb{M}_3}(\mathbb{EB}, \Sigma^{2,1}\mathbb{M}_3)$ *is trivial.*

Proof. We begin by considering the same free resolution for $\mathbb{EB}$ over $\mathbb{M}_3$ as in the proof of Lemma 6.0.3:

$$\cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\eta} \mathbb{EB}$$

To compute $\mathrm{Ext}^{1,(0,0)}_{\mathbb{M}_3}(\mathbb{EB}, \Sigma^{2,1}\mathbb{M}_3)$, we next apply the functor $\mathrm{Hom}(-, \Sigma^{2,1}\mathbb{M}_3)$ of degree preserving maps to this free resolution. We claim that $\ker(d_2^*)/\operatorname{im}(d_1^*)$ is trivial.

Let $f \in \ker(d_2^*)$. So f is some map $f\colon \mathbb{M}_3\langle a_1\rangle \oplus \mathbb{M}_3\langle b_1\rangle \to \Sigma^{2,1}\mathbb{M}_3$. Suppose $f(a_1) = s$ and $f(b_1) = t$ for some $s, t \in \Sigma^{2,1}\mathbb{M}_3$. Recall from the previous lemma that $|a_1| = (2,2)$ and $|b_1| = (3,2)$. Since f is degree preserving, we have that $|s| = (2,2)$ and $|t| = (3,2)$.

Now, $d_2^*(f)$ is a map $d_2^*(f)\colon \mathbb{M}_3\langle a_2\rangle \oplus \mathbb{M}_3\langle b_2\rangle \to \Sigma^{2,1}\mathbb{M}_3$ given by $d_2^*(f)(a_2) = ys$ and $d_2^*(f)(b_2) = zs - yt$. Since $f \in \ker(d_2^*)$, we know $ys = 0$ and $zs - yt = 0$. We can see from Figure 42 that $ys = 0$ only when $s = 0$. Since $s = 0$, the second relation simplifies to the requirement that $-yt = 0$. This is true for any element of

$\Sigma^{2,1}\mathbb{M}_3$ in degree $(3,2)$. This tells us that any function f in $\ker d_2^*$ must be of the form $a_1 \mapsto 0$, $b_1 \mapsto t$ for any t in degree $(3,2)$.

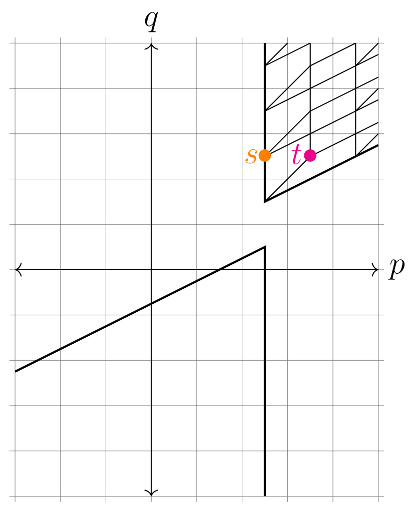


FIGURE 42. The top cone module structure of $\Sigma^{2,1}\mathbb{M}_3$. Note $ys \neq 0$ when $s \neq 0$.

It turns out that any map of this form is also in $\text{im}(d_1^*)$. Let a denote the generator of $\Sigma^{2,1}\mathbb{M}_3$. We want to show the maps $a_1 \mapsto 0$ and $b_1 \mapsto \pm ya$ are in the image of d_1^*. Define the map $g_+ \colon F_0 \to \Sigma^{2,1}\mathbb{M}_3$ given by $a_0 \mapsto a$ and $b_0 \mapsto 0$. Then $d_1^*(g_+) \colon F_1 \to \Sigma^{2,1}\mathbb{M}_3$ is given by:

$$d_1^*(g_+)(a_1) = g_+(d_1(a_1)) = g_+(yb_0) = yg_+(b_0) = 0$$

$$d_1^*(g_+)(b_1) = g_+(d_1(b_1)) = g_+(zb_0 - ya_0) = -ya.$$

A similar computation shows the image of the map $g_- \colon F_0 \to \Sigma^{2,1}\mathbb{M}_3$ given by $a_0 \mapsto -a$ and $b_0 \mapsto 0$ under d_1^* sends a_1 to 0 and b_1 to ya.

So $\ker(d_2^*) = \text{im}(d_1^*)$ and $\text{Ext}^{1,(2,1)}(\mathbb{EB}, \mathbb{M}_3)$ is trivial. $\square$

Together, Lemmas 6.0.3 and 6.0.4 tell us that given a short exact sequence of the form

$$0 \to \Sigma^{2,1}\mathbb{M}_3 \oplus \mathbb{EB}^{\oplus k} \to X \to \mathbb{EB}^{\oplus \ell} \to 0,$$

the $\mathbb{M}_3$-module X must be isomorphic to $\Sigma^{2,1}\mathbb{M}_3 \oplus \mathbb{EB}^{\oplus k+\ell}$. We can even take things one step further to conclude any extension

$$0 \to \Sigma^{2,1}\mathbb{M}_3 \oplus \mathbb{EB}^{\oplus k} \to X \to \mathbb{M}_3 \oplus \mathbb{EB}^{\oplus \ell} \to 0$$

must be trivial by the projectivity of $\mathbb{M}_3$ as an $\mathbb{M}_3$-module.

Lemma 6.0.5. *There are no nontrivial extensions*

$$0 \to \Sigma^{2,1}\mathbb{M}_3 \oplus \left(\Sigma^{1,0}H^{*,*}(C_3)\right)^{\oplus 2g} \to X \to \mathbb{M}_3 \oplus \mathbb{EB} \to 0$$

Proof. Using Lemmas 6.0.3 and 6.0.4 as well as the fact that $\mathbb{M}_3$ is free, we only need to show $\mathrm{Ext}^{1,(0,0)}\left(\mathbb{EB}, \Sigma^{1,0}H^{*,*}(C_3)\right) = 0$. Using the free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to \mathbb{EB}$$

defined in the proof of Lemma 6.0.3, we can see that $\mathrm{Hom}\left(F_1, \Sigma^{1,0}H^{*,*}(C_3)\right)$ must be 0. Recall F_1 is isomorphic to two copies of $\mathbb{M}_3$ generated in degrees $(3,2)$ and $(2,2)$. Since $\Sigma^{1,0}H^{*,*}(C_3)$ is concentrated in degrees $(1,q)$, there are no degree preserving maps $F_1 \to \Sigma^{1,0}H^{*,*}(C_3)$. Thus $\mathrm{Ext}^{1,(0,0)}\left(\mathbb{EB}, \Sigma^{1,0}H^{*,*}(C_3)\right)$ must be zero. □

With these lemmas, we are now ready to begin computations.

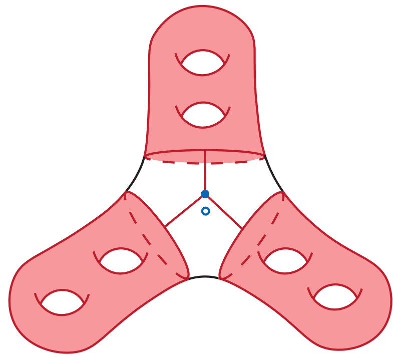


FIGURE 43. The space $Y \subset \mathrm{Sph}_{3g}[2]$ in red with $g = 2$. Note $Y \simeq \bigvee_{2g} S^{1,0} \wedge C_{3+}$.

Class 1 ($\mathrm{Sph}_{2k}\, 3g[2k{+}2] \cong S^{2,1}{+}k[R_3]\#_3 M_g$)**.** Recall that the space $\mathrm{Sph}_{2k+3g}[2k{+}2]$ is orientable with $\beta = 2(2k + 3g)$ and $F = 2k + 2$. In particular, $F - 2 = 2k$ and $\frac{\beta-2F+4}{3} = 2g$. We will show that

$$H^{*,*}(\mathrm{Sph}_{2k+3g}[2k+2]) \cong \mathbb{M}_3 \oplus \Sigma^{2,1}\mathbb{M}_3 \oplus \left(\Sigma^{1,0}H^{*,*}(C_3)\right)^{\oplus 2g} \oplus \mathbb{EB}^{\oplus 2k}$$

by induction on k. The first case to consider is $k = 0$. Recall that $\mathrm{Sph}_{3g}[2] := S^{2,1}\#_3 M_g$.

To begin the computation we will construct a cofiber sequence

$$Y_+ \hookrightarrow \mathrm{Sph}_{3g}[2]_+ \to S^{2,1}$$

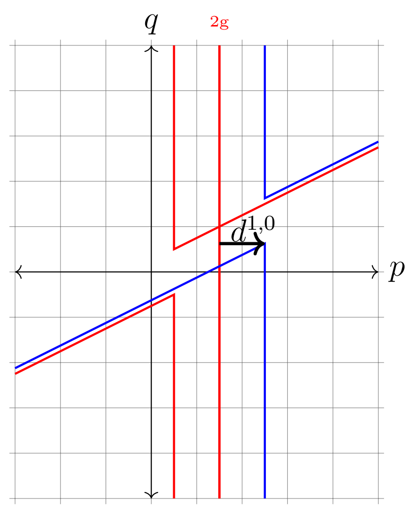
where Y is the space in red depicted in Figure 43. The space Y is homotopy equivalent to $\tilde{M}_g \wedge C_{3+}$ which deformation retracts onto $\left(\bigvee_{2g} S^{1,0}\right) \wedge C_{3+}$. This gives us a long exact sequence on cohomology:

$$\cdots \to \tilde{H}^{p,q}(S^{2,1}) \to H^{p,q}(\mathrm{Sph}_{3g}[2]) \to H^{p,q}(Y) \xrightarrow{d} \tilde{H}^{p+1,q}(S^{2,1}) \to \cdots$$

which can be understood by analyzing its total differential

$$\bigoplus_{p,q} d^{p,q} \colon H^{p,q}(Y) \to \tilde{H}^{p+1,q}(S^{2,1}).$$

We plot the domain and target space of the differential below:



Since the total differential is an $\mathbb{M}_3$-module map, it is completely determined by its values in degrees $(0,0)$ and $(1,0)$ by linearity. It is immediate that $d^{0,0} = 0$ since $\tilde{H}^{1,0}(S^{2,1}) = 0$, and we can use the Quotient Lemma to determine $d^{1,0}$. In particular, $\mathrm{Sph}_{3g}[2]/C_3 \simeq M_g$, and so $H^{2,0}(\mathrm{Sph}_{3g}[2]) \cong \mathbb{Z}/3$. Therefore something in degree $(2,0)$ must be in the cokernel of d. This can only happen if $d^{1,0} = 0$.

We are able to determine by linearity that the total differential $\bigoplus_{a,b} d^{a,b}$ must be zero everywhere. This leaves us to solve the extension problem

$$\Sigma^{2,1}\mathbb{M}_3 \hookrightarrow H^{*,*}(\mathrm{Sph}_{3g}[2]) \twoheadrightarrow \mathbb{M}_3 \oplus \left(\Sigma^{1,0} H^{*,*}(C_3)\right)^{\oplus 2g}.$$

By the observations in Remark 2.3.6, we know $\mathbb{M}_3$ splits off as a summand of $H^{*,*}(\mathrm{Sph}_{3g}[2])$. Moreover, the submodule $\left(\Sigma^{1,0}(\mathbb{Z}/3[x,x^{-1}])\right)^{\oplus 2g} \subseteq \ker(d)$ also splits off. To see this, let a be a nonzero element of $H^{1,0}(\mathrm{Sph}_{3g}[2])$. We know $H^{3,1}(\mathrm{Sph}_{3g}[2]) = 0$, so $z \cdot a = 0$. Since $y^2 = 0$ and for all nonzero b in degree $(2,1)$, $y \cdot b \neq 0$, it must be the case that $y \cdot a = 0$. Finally, any lower cone element must act trivially on a since it is infinitely divisible by x. By linearity, we conclude that there cannot be any nonzero y, z, or lower cone extensions coming from $x^\ell a$ for any $\ell \in \mathbb{Z}$.

Thus we can conclude the extension is trivial, and

$$H^{*,*}(\mathrm{Sph}_{3g}[2]) \cong \mathbb{M}_3 \oplus \Sigma^{2,1}\mathbb{M}_3 \oplus \left(\Sigma^{1,0} H^{*,*}(C_3)\right)^{\oplus 2g}.$$

We next proceed to the inductive step, assuming that

$$H^{*,*}(\mathrm{Sph}_{2k+3g}[2k+2]) \cong \mathbb{M}_3 \oplus \Sigma^{2,1}\mathbb{M}_3 \oplus \left(\Sigma^{1,0} H^{*,*}(C_3)\right)^{\oplus 2g} \oplus \mathbb{EB}^{\oplus 2k}$$

for some $k \geq 0$. Let's now use this assumption to compute the cohomology of $\mathrm{Sph}_{2(k+1)+3g}[2(k+1)+2]$.

We proceed by considering the cofiber of a map

$$EB_+ \to \mathrm{Sph}_{2(k+1)+3g}[2(k+1)+2]_+ \tag{6.0.1}$$

which we define below. The cofiber will be homotopy equivalent to $\mathrm{Sph}_{2k+3g}[2k+2] \vee EB$. To see this, first notice that $\mathrm{Sph}_{2k+3g}[2k+2]$ has at least 2 fixed points for any $k \geq 0$. Construct $\mathrm{Sph}_{2(k+1)+3g}[2(k+1)+2]$ by performing C_3-ribbon surgery on $\mathrm{Sph}_{2k+3g}[2k+2]$ in a neighborhood of one of these fixed points. Then construct the map $EB \to \mathrm{Sph}_{2(k+1)+3g}[2(k+1)+2]$ by sending EB into this copy of R_3 used

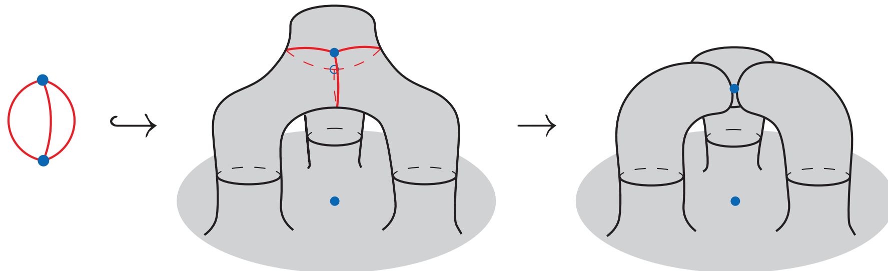


FIGURE 44. The cofiber sequence corresponding to (6.0.1).

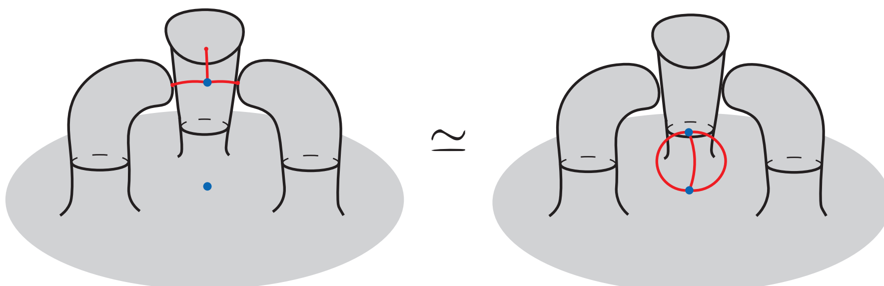


FIGURE 45. Up to homotopy, the cofiber of (6.0.1) is $\mathrm{Sph}_{2(k+1)+3g}[2(k+1)+2] \vee EB$.

to construct $\mathrm{Sph}_{2(k+1)+3g}[2(k+1)+2]$ from $\mathrm{Sph}_{2k+3g}[2k+2]$. Figure 44 shows the cofiber of such a map.

Next notice that this cofiber is homotopy equivalent to the space shown in Figure 45 which is homotopy equivalent to $\mathrm{Sph}_{2k+3g}[2k+2] \vee EB$.

This cofiber sequence gives us a long exact sequence on cohomology given by

$$\to H^{p,q}(\mathrm{Sph}_{2(k+1)+3g}[2(k+1)+2]) \to H^{p,q}(EB) \xrightarrow{d} \tilde{H}^{p+1,q}(\mathrm{Sph}_{2k+3g}[2k+2] \vee EB) \to$$

As in previous examples, we can understand $\tilde{H}^{*,*}(\mathrm{Sph}_{2(k+1)+3g}[2(k+1)+2])$ by computing the total differential

$$d\colon H^{*,*}(EB) \to \tilde{H}^{*+1,*}(\mathrm{Sph}_{2k+3g}[2k+2] \vee EB).$$

The domain and target space of this differential is shown on the (p, q)-axis in Figure 46.

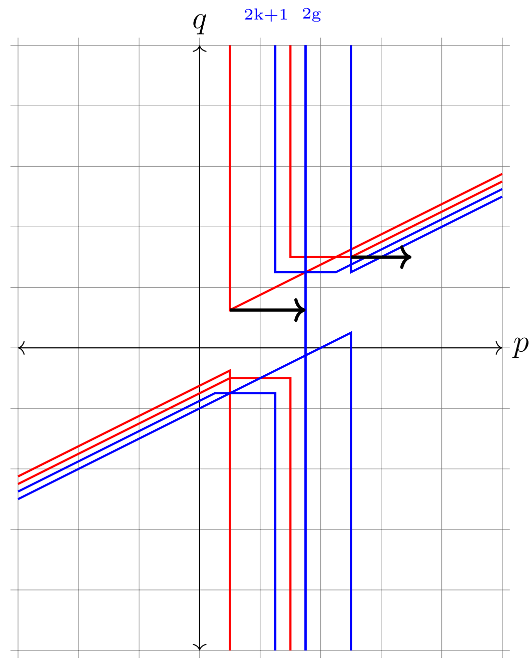


FIGURE 46. The differential for the long exact sequence corresponding to (6.0.1).

To compute this differential, it suffices to determine its value in degrees $(0,0)$, $(1,1)$, and $(2,2)$.

First observe that $\mathrm{Sph}_{2(k+1)+3g}[2(k+1)+2]/C_3 \simeq M_g$. This follows from the fact that for any C_3-space X and non-equivariant space Y, $(X + k[R_3]\#_3 Y)/C_3 \cong (X/C_3)\#Y$. In this case, we have $\mathrm{Sph}_{2(k+1)+3g}[2(k+1)+2] \cong S^{2,1} + k[R_3]\#_3 M_g$, so $\mathrm{Sph}_{2(k+1)+3g}[2(k+1)+2]/C_3 \simeq M_g$. The Quotient Lemma then tells us that $H^{1,0}(\mathrm{Sph}_{2(k+1)+3g}[2(k+1)+2]) \cong (\mathbb{Z}/3)^{\oplus 2g}$. In particular, $d^{0,0}$ must be 0.

We saw in Example 2.3.4 that $\mathbb{EB} \cong \mathbb{M}_3\langle \alpha, \beta\rangle/(y\beta, y\alpha - z\beta)$ where α is in degree $(2,1)$ and β is in degree $(1,1)$. There is nothing for $d_1^{2,1}$ to hit, so $d(\alpha) = 0$. Moreover, $0 = yd(\alpha) = d(y\alpha) = d(z\beta)$. If $d(\beta) \neq 0$, then linearity of d would imply $d(z\beta) \neq 0$. This tells us the total differential must be 0.

This leaves us to solve the extension problem

$$\left(\Sigma^{1,0} H^{*,*}(C_3)\right)^{\oplus 2g} \oplus \mathbb{EB}^{\oplus 2k+1} \oplus \Sigma^{2,1}\mathbb{M}_3 \hookrightarrow H^{*,*}(\mathrm{Sph}_{2(k+1)+3g}[2(k+1)+2]) \twoheadrightarrow \mathbb{M}_3 \oplus \mathbb{EB}.$$

We can then use Lemmas 6.0.3, 6.0.4, and 6.0.5 to determine that there can be no non-trivial extensions. Thus finally we have that

$$H^{*,*}(\mathrm{Sph}_{2(k+1)+3g}[2(k+1)+2]) \cong \mathbb{M}_3 \oplus \Sigma^{2,1}\mathbb{M}_3 \oplus \left(H^{*,*}(C_3)\right)^{\oplus 2g} \oplus \mathbb{EB}^{\oplus 2(k+1)}$$

and the result holds by induction.

Class 2 ($\mathrm{Hex}_{n,3n-2+2k+3g}[3n+2k]$)**.** Our next goal will be to compute the cohomology of the space $\mathrm{Hex}_{n,3n-2+2k+3g}[3n+2k]$ for all $n \geq 1$ and $k, g \geq 0$. First recall that $\mathrm{Hex}_{n,3n-2+2k+3g}[3n+2k] \cong (\mathrm{Hex}_n + k[R_3]) \#_3 M_g$ with $\beta = 2(3n-2+2k+3g)$ and $F = 3n+2k$. Therefore $F - 2 = 3n + 2k - 2$ and $(\beta - 2F + 4)/3 = 2g$. So we will work towards proving

$$\mathrm{Hex}_{n,3n-2+2k+3g}[3n+2k] \cong \mathbb{M}_3 \oplus \Sigma^{2,1}\mathbb{M}_3 \oplus \mathbb{EB}^{\oplus(3n-2+2k)} \oplus \left(\Sigma^{1,0} H^{*,*}(C_3)\right)^{\oplus 2g}.$$

This will be done in several steps. First we consider the case $g = k = 0$ and $n = 1$. Then we confirm that the result holds for $n = 1$, $g = 0$, and $k \geq 0$. The next step will be to induct on n and compute cohomology in the case $n > 1$, $g = 0$, and $k \geq 0$. The final step will be to consider the $g > 0$ case.

As previously stated, we begin with the computation of $H^{*,*}(\mathrm{Hex}_1)$. There is a cofiber sequence

$$EB \hookrightarrow \mathrm{Hex}_1 \to S^{2,1} \tag{6.0.2}$$

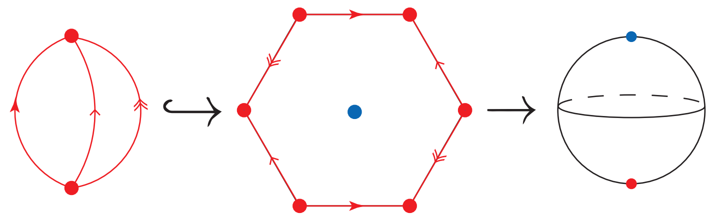


FIGURE 47. The cofiber sequence $EB \hookrightarrow \mathrm{Hex}_1 \to S^{2,1}$.

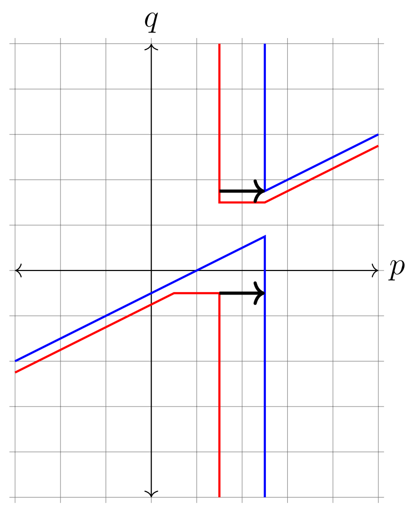


FIGURE 48. The differential for the long exact sequence corresponding to (6.0.2).

which we can see depicted in Figure 47. This gives a long exact sequence on cohomology

$$\cdots \to \tilde{H}^{p,q}(S^{2,1}) \to \tilde{H}^{p,q}(\mathrm{Hex}_1) \to \tilde{H}^{p,q}(EB) \xrightarrow{d_1^{p,q}} \tilde{H}^{p+1,q}(S^{2,1}) \to \cdots$$

which can be understood by computing its total differential

$$\bigoplus_{p,q} d_1^{p,q} \colon \tilde{H}^{*+1,*}(EB) \to \tilde{H}^{*,*}(S^{2,1})$$

shown in Figure 48.

Similar reasoning to that of the last example tells us that the total differential to this cofiber sequence must be zero. In particular, we can use the module structure $\mathbb{EB} \cong \mathbb{M}_3\langle \alpha, \beta\rangle/(y\beta, y\alpha - z\beta)$ and the fact that $d^{2,1}(\alpha) = 0$ to determine that $d^{p,q}$ must be zero for all (p, q).

Since the differential is identically zero, we know $\ker(d) = \mathbb{EB}$ and $\operatorname{coker}(d) = \Sigma^{2,1}\mathbb{M}_3$. We now have to solve the extension problem

$$0 \to \Sigma^{2,1}\mathbb{M}_3 \to \tilde{H}^{*,*}(\mathrm{Hex}_1) \to \mathbb{EB} \to 0.$$

The above sequence is split as a consequence of Lemmas 6.0.3 and 6.0.4, and we have

$$\tilde{H}^{*,*}(\mathrm{Hex}_1) \cong \Sigma^{2,1}\mathbb{M}_3 \oplus \mathbb{EB}.$$

As per the obervations in Remark 2.3.6, it follows that

$$H^{*,*}(\mathrm{Hex}_1) \cong \mathbb{M}_3 \oplus \Sigma^{2,1}\mathbb{M}_3 \oplus \mathbb{EB}.$$

Next assume that for some $k \geq 0$ and $g = 0$, the cohomology of $\mathrm{Hex}_{1,1+2k+3g}[2k + 3]$ is as stated in Theorem 6.0.1, and we will show that it holds true for $\mathrm{Hex}_{1,1+2(k+1)}[2(k + 1) + 3]$.

Consider the cofiber sequence

$$EB \hookrightarrow \mathrm{Hex}_{1,1+2(k+1)}[2(k + 1) + 3] \to \mathrm{Hex}_{1,1+2k}[2k + 3] \vee EB$$

whose corresponding long exact sequence on cohomology has differential

$$d\colon \mathbb{EB} \to \tilde{H}^{*+1,*}(\mathrm{Hex}_{1,1+2k}[3 + 2k] \vee EB).$$

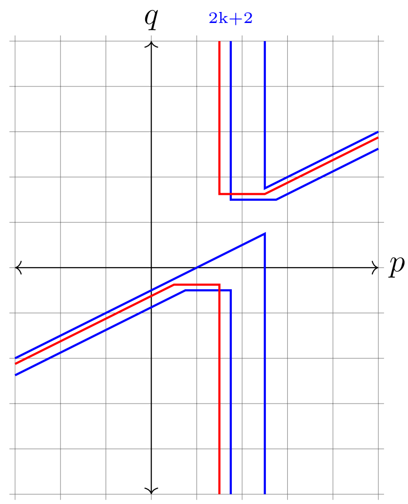


FIGURE 49. The differential $d^{p,q}\colon \mathbb{EB}^{p,q} \to \tilde{H}^{p+1,q}(\mathrm{Hex}_{1,1+2k}[2k+3] \vee EB)$.

To compute this differential, we consult Figure 49 and observe that $\alpha \in \mathbb{EB}$ in degree $(2,1)$ must map to 0 as there is nothing in degree $(3,1)$. However using a similar argument to that in the base case, it must be that $d(\beta) = 0$ by linearity. In particular, the total differential is zero.

Moreover, all extensions are trivial as a consequence of Lemmas 6.0.3 and 6.0.4. From this, we can easily see what $\tilde{H}^{*,*}(\mathrm{Hex}_{1,1+2(k+1)}[2(k+1)+3])$ must be. Thus we have

$$H^{*,*}(\mathrm{Hex}_{1,1+2(k+1)}[2(k+1)+3]) \cong \mathbb{M}_3 \oplus \Sigma^{2,1}\mathbb{M}_3 \oplus \mathbb{EB}^{\oplus 2k+3}$$

as desired.

Our next goal is to consider $H^{*,*}(\mathrm{Hex}_{n,3n-2+2k}[3n+2k])$, which will be done by inducting on n. The base case has been completed with the computation of $H^{*,*}(\mathrm{Hex}_{1,1+2k}[3+2k])$ in the previous step.

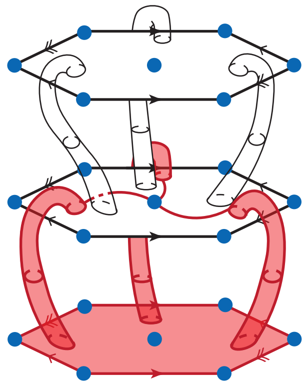


FIGURE 50. The space $Y \simeq EB \vee EB \vee EB$ is shown in red in the case $n = 2$, $k = 0$.

Assume for some $n \geq 1$ that

$$H^{*,*}(\mathrm{Hex}_{n,3n-2+2k}[3n+2k]) \cong \mathbb{M}_3 \oplus \Sigma^{2,1}\mathbb{M}_3 \oplus \mathbb{EB}^{\oplus(3n-2+2k)}.$$

There is a cofiber sequence

$$Y_+ \hookrightarrow \mathrm{Hex}_{n+1,3(n+1)-2+2k}[3(n+1)+2k]_+ \to \mathrm{Hex}_{n,3n-2+2k}[3n+2k]$$

where Y is the space depicted in Figure 50. This space is homotopy equivalent to $EB \vee EB \vee EB$. As usual, we want to consider the differential in the corresponding long exact sequence on cohomology:

$$d\colon H^{*,*}(Y) \to \tilde{H}^{*,*}(\mathrm{Hex}_{n,3n-2+2k}[3n+2k])$$

The spaces $H^{*,*}(Y)$ and $\tilde{H}^{*,*}(\mathrm{Hex}_{n,3n-2+2k}[3n+2k])$ are shown in Figure 51.

Since d is an $\mathbb{M}_3$-module map, we only need to consider the value of d in degrees $(0,0)$, $(1,1)$, and $(2,1)$. The Quotient Lemma guarantees that $d^{0,0} = 0$.

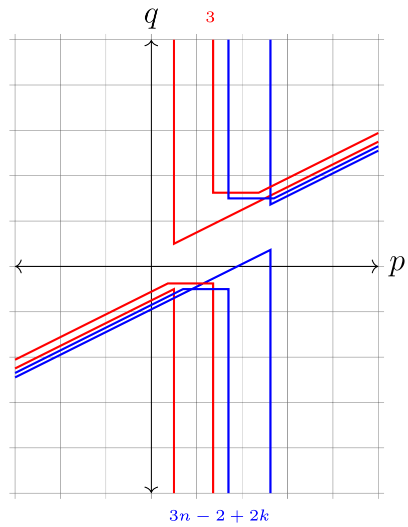


FIGURE 51. The spaces $H^{*,*}(Y)$ and $\tilde{H}^{*,*}(\mathrm{Hex}_{n,3n-2+2k}[3n+2k])$.

Since there is nothing in degree $(3,1)$, it also must be the case that $d^{2,1} = 0$. A similar strategy from previous examples utilizing the module structure of $\mathbb{EB}$ guarantees that in fact all differentials must be 0. This leaves us with the extension problem

$$\Sigma^{2,1}\mathbb{M}_3 \oplus \mathbb{EB}^{\oplus(3n-2+2k)} \hookrightarrow H^{*,*}(\mathrm{Hex}_{n+1,3(n+1)-2+2k+3g}[3(n+1)+2k]) \twoheadrightarrow \mathbb{M}_3 \oplus \mathbb{EB}^{\oplus 3} \to 0.$$

We know from Lemmas 6.0.3 and 6.0.4 that this extension is trivial. Thus we have

$$H^{*,*}(\mathrm{Hex}_{n,3n-2+2k}[3n+2k]) \cong \mathbb{M}_3 \oplus \Sigma^{2,1}\mathbb{M}_3 \oplus \mathbb{EB}^{\oplus 3n-2+2k}.$$

Finally, we will compute the cohomology of $\mathrm{Hex}_{n,3n-2+2k+3g}[3n+2k]$ when $g > 0$. For this we construct the cofiber sequence

$$Y_+ \hookrightarrow \mathrm{Hex}_{n,3n-2+2k+3g}[3n+2k]_+ \to \mathrm{Hex}_{n,3n-2+2k}[3n+2k]$$

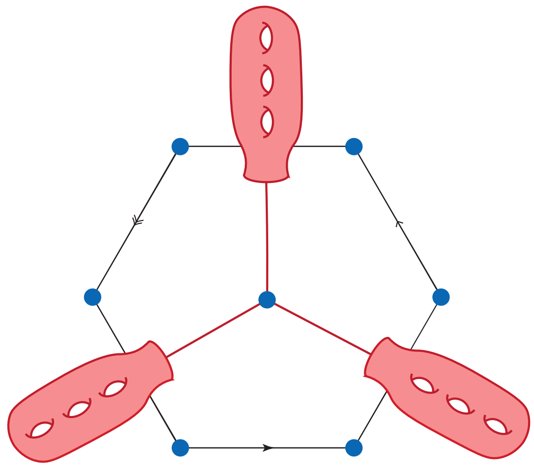


FIGURE 52. The space $Y \simeq \bigvee_{2g} S^{1,0} \wedge C_{3+}$, shown in red for $n = 1, k = 0, g = 3$.

where Y is the space in Figure 52. Recall that $Y \simeq \left(\bigvee_{2g} S^{1,0}\right) \wedge C_{3+}$. So the long exact sequence corresponding to this cofiber sequence has differential

$$d^{p,q}\colon H^{p,q}(Y) \to \tilde{H}^{*,*}(\mathrm{Hex}_{n,3n-2+2k}[3n+2k]).$$

This differential can be see in Figure 53.

Since there is nothing for it to hit, we can easily observe that $d^{0,0} = 0$. The Quotient Lemma additionally allows us to conclude $d^{1,0} = 0$, and thus $d^{1,q} = 0$ by linearity. In particular, the total differential is zero.

We then turn to solve the extension problem

$$0 \to \mathrm{coker}(d) \to H^{*,*}(\mathrm{Hex}_{n,3n-2+2k+3g}[3n+2k]) \to \ker(d) \to 0$$

where $\mathrm{coker}(d) = \mathbb{EB}^{\oplus 3n-2+2k} \oplus \Sigma^{2,1}\mathbb{M}_3$ and $\ker(d) = \mathbb{M}_3 \oplus (\Sigma^{1,0}H^{*,*}(C_3))^{\oplus 2g}$. We again recall Remark 2.3.6 and observe that $\mathbb{M}_3 \subseteq \ker(d)$ must split off as a

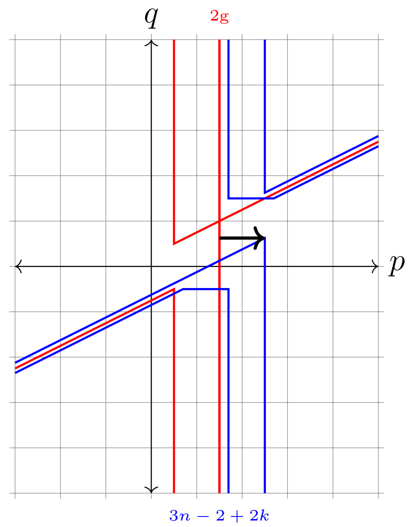


FIGURE 53. The differential $d\colon H^{*,*}(Y) \to \tilde{H}^{*,*}(\mathrm{Hex}_{n,3n-2+2k}[3n+2k])$.

summand of $H^{*,*}(\mathrm{Hex}_{n,3n-2+2k+3g}[3n+2k])$. A similar argument to that of the base case in Class 1 guarantees that $(\Sigma^{1,0}H^{*,*}(C_3))^{\oplus 2g}$ in $\ker(d)$ must split off as well.

Thus we can conclude the extension is trivial, and

$$H^{*,*}(\mathrm{Hex}_{n,3n-2+2k+3g}[3n+2k]) \cong \mathbb{M}_3 \oplus \Sigma^{2,1}\mathbb{M}_3 \oplus \mathbb{EB}^{\oplus 3n-2+2k} \oplus \left(\Sigma^{1,0}H^{*,*}(C_3)\right)^{\oplus 2g}.$$

Class 3 ($N_{4k+3r}[2k+2] \cong S^{2,1} + k[R_3]\#_3 N_r$)**.** Recall that the space $N_{4k+3r}[2k+2]$ is non-orientable when $r \geq 1$ with $\beta = 4k+3r$ and $F = 2k+2$. Then $F-2 = 2k$ and $(\beta - 2F + 1)/3 = r - 1$. So our goal is to show

$$H^{*,*}(N_{4k+3r}[2k+2]) \cong \mathbb{M}_3 \oplus \mathbb{EB}^{\oplus 2k} \oplus \left(\Sigma^{1,0}H^{*,*}(C_3)\right)^{\oplus r-1}$$

We begin with the cofiber sequence

$$\left(\tilde{N}_r \times C_3\right)_+ \hookrightarrow N_{4k+3r}[2k+2]_+ \to \mathrm{Sph}_{2k}[2k+2] \vee EB.$$

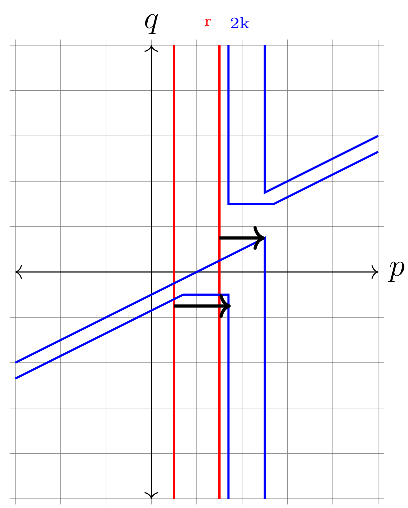
This gives us the following long exact sequence on cohomology

$$\cdots \to H^{p,q}(N_{4k+3r}[2k+2]) \to H^{p,q}(\tilde{N}_r \times C_3) \xrightarrow{d} \tilde{H}^{p+1,q}(\mathrm{Sph}_{2k}[2k+2] \vee EB) \to \cdots$$

with differential

$$d^{p,q} \colon H^{p,q}(\tilde{N}_r \times C_3) \to \tilde{H}^{p+1,q}(\mathrm{Sph}_{2k}[2k+2] \vee EB)$$

as shown below:



To determine if this differential is nonzero, we start with the Quotient Lemma. Observe that $N_{4k+3r}[2k+2]/C_3 \simeq N_r$, and we have

$$\tilde{H}^p_{\mathrm{sing}}(N_r; \mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3 & \text{for } p = 0 \\ (\mathbb{Z}/3)^{r-1} & \text{for } p = 1 \\ 0 & \text{else} \end{cases}$$

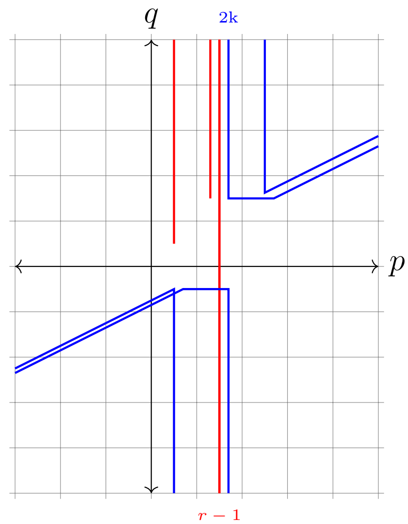


FIGURE 54. The modules ker(d) and coker(d).

Thus it must be the case that $d^{1,0}$ is nonzero. Otherwise $H^{2,0}(N_{4k+3r}[2k+2]) \neq 0$ and we would contradict the results of the Quotient Lemma. By linearity, we get that $d^{1,q}$ is nonzero for all $q \leq 0$ and is zero when $q > 0$.

We next turn to $d^{0,q}$. Using a similar argument from the base case of Class 1, we can observe that since $\mathbb{M}_3$ and coker(d) are submodules of $H^{*,*}(N_{4k+3r}[2k+2])$, $d^{0,q}$ cannot be zero.

We are left to determine if the extension

$$\operatorname{coker}(d) \hookrightarrow H^{*,*}(N_{4k+3r}[2k+2]) \twoheadrightarrow \ker d$$

is nontrivial, where coker(d) and ker(d) are depicted in Figure 54. We already know that this extension is nontrivial since $\mathbb{M}_3 \subseteq H^{*,*}(N_{4k+3r}[2k+2])$. As in the computation for Class 1, we get that $(\Sigma^{1,0}H^{*,*}(C_3))^{\oplus r-1} \subseteq \ker d$ must split off as a summand of $H^{*,*}(N_{4k+3r}[2k+2])$ since no possible nontrivial extensions from this module can exist in this case.

Finally, we conclude that

$$H^{*,*}(N_{4k+3r}[2k+2]) \cong \mathbb{M}_3 \oplus \left(\Sigma^{1,0} H^{*,*}(C_3)\right)^{\oplus r-1} \oplus \mathbb{EB}^{\oplus 2k}.$$

Class 4 ($N_{1+4k+3r}[2k+1] \cong N_1[1] + k[R_3] \#_3 N_r$)**.** We next compute the cohomology of $N_{1+4k+3r}[2k+1]$ for $k, r \geq 0$. Recall that the case for $k = r = 0$ was done in Example 2.3.5. Our more general computation will be done in two steps. First we will restrict to when $r = 0$. Then we will allow $r \geq 0$ and compute the cohomology in the general case.

The space $N_{1+4k+3r}[2k+1]$ is non-orientable with $\beta = 1 + 4k + 3r$ and $F = 2k+1$. In particular, $F - 1 = 2k$ and $(\beta - 2F + 1)/3 = r$. So our goal is to show

$$N_{1+4k+3r}[2k+1] \cong \mathbb{M}_3 \oplus \mathbb{EB}^{\oplus 2k} \oplus \left(\Sigma^{1,0} H^{*,*}(C_3)\right)^{\oplus r}$$

Recall that if $r = 0$, then $N_{1+4k+3r}[2k+1] \cong N_1[1] + k[R_3]$. We start with a cofiber sequence

$$S^1_{\text{free}+} \hookrightarrow N_1[1] + k[R_3]_+ \to \text{Sph}_{2k}[2k+2]$$

with long exact sequence

$$\to \tilde{H}^{p,q}(\text{Sph}_{2k}[2k+2]) \to H^{p,q}(N_{1+4k}[2k+1]) \to H^{p,q}(S^1_{\text{free}}) \xrightarrow{d} \tilde{H}^{p+1,q}(\text{Sph}_{2k}[2k+2]) \to$$

on cohomology. We once again try to determine the total differential $\bigoplus_{p,q} d^{p,q}$ which is highlighted on the left of Figure 55.

As usual we start with the Quotient Lemma. Observe that $(N_1[1] + k[R_3])/C_3 \simeq \mathbb{R}P^2$, so it must be that $H^{p,0}(N_{1+4k}[2k+1]) = 0$ for $p \neq 0$. In particular, $d^{1,0}$ must be an isomorphism, and by linearity we can determine the

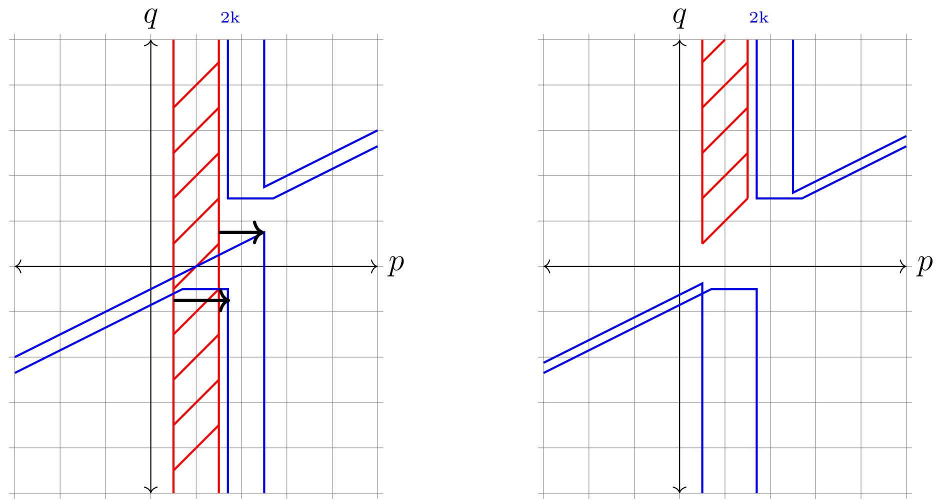


FIGURE 55. The differential d (left) and its kernel and cokernel (right).

behavior of the differential in all other degrees. In particular, $d^{p,q}$ is 0 when $(p,q) = (0,0)$ or $q \geq 1$. Othwerwise $d^{p,q} \neq 0$ with image in $\Sigma^{2,1}\mathbb{M}_3 \subseteq \tilde{H}^{*,*}(\mathrm{Sph}_{2k}[2k+2])$.

Now that we know the value of the differential, we can find $\ker(d)$ and $\mathrm{coker}(d)$. These modules are depicted on the right of Figure 55. We are left to solve the extension problem

$$0 \to \mathrm{coker}(d) \to H^{*,*}(N_{1+4k}[2k+1]) \to \ker(d) \to 0.$$

Since $\mathbb{M}_3 \subseteq H^{*,*}(N_{1+4k}[2k+1])$, we can immediately see that there must be a nontrivial extension. Knowing that $\mathbb{M}_3$ is a summand of $H^{*,*}(N_{1+4k}[2k+1])$, there is only one possible solution:

$$H^{*,*}(N_{1+4k}[2k+1]) \cong \mathbb{M}_3 \oplus \mathbb{EB}^{\oplus 2k}.$$

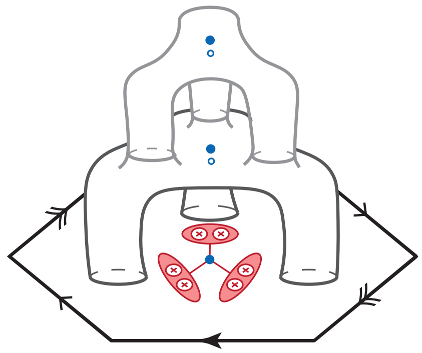


FIGURE 56. The space Y in $N_{1+2k+3r}[2k{+}1]$ is shown in red in the case $r = k = 2$.

We finally turn to the general case with $r \geq 0$ and start by constructing the cofiber sequence

$$Y_+ \hookrightarrow N_{1+4k+3r}[2k+1]_+ \to N_{1+4k}[2k+1]$$

where Y is the space shown in red in Figure 56. Notice that Y is homotopy equivalent to $(\bigvee_r S^{1,0}) \wedge C_{3+}$.

From here we can examine the differential $d\colon H^{*,*}(Y) \to \tilde{H}^{*+1,*}(N_{1+4k}[2k+1])$ of the corresponding long exact sequence on cohomology. The left diagram of Figure 57 shows the $\mathbb{M}_3$-modules $H^{*,*}(Y)$ and $\tilde{H}^{*,*}(N_{1+4k}[2k+1])$. Since $\tilde{H}^{p+1,0}(N_{1+4k}[2k+1]) = 0$ for all p, we immediately see that $d^{p,0}$ must be 0. By linearity, this guarantees the differential $d^{p,q}$ must be the zero map for all (p,q).

Since the total differential is 0, $\ker(d) = H^{*,*}(Y) \cong \mathbb{M}_3 \oplus (\Sigma^{1,0} H^{*,*}(C_3))^{\oplus r}$ and $\operatorname{coker}(d) = \tilde{H}^{*,*}(N_{1+4k}[2k+1]) \cong \mathbb{EB}^{\oplus 2k}$. We now must solve the final extension problem

$$0 \to \mathbb{EB}^{\oplus 2k} \to H^{*,*}(N_{1+4k+3r}[2k+1]) \to \mathbb{M}_3 \oplus \left(\Sigma^{1,0} H^{*,*}(C_3)\right)^{\oplus r} \to 0.$$

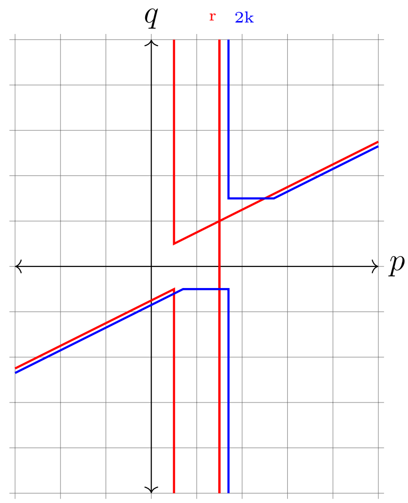


FIGURE 57. The spaces $H^{*,*}(Y) = \ker(d)$ and $\tilde{H}^{*,*}(N_{1+4k}[2k+1]) = \operatorname{coker}(d)$.

We know that $\mathbb{M}_3$ must split off as a summand of $H^{*,*}(N_{1+4k+3r}[2k+1])$, and we have seen before in the computation for Class 1 that there can be no nontrivial extensions from $(\Sigma^{1,0}H^{*,*}(C_3))^{\oplus r}$ to $\mathbb{EB}$. So our extension must be trivial, and we get the desired result of

$$H^{*,*}(N_{1+4k+3r}[2k+1]) \cong \mathbb{M}_3 \oplus \left(\Sigma^{1,0}(H^{*,*}(C_3))\right)^{\oplus r} \oplus \mathbb{EB}^{\oplus 2k}.$$

APPENDIX A

SURGERY INVARIANCE RESULTS

Let p be an odd prime. This appendix contains proofs for some of the basic surgery invariance results outlined in Chapter III.

Proposition A.0.1. *Let X be a closed, connected 2-manifold with a map $\sigma\colon X \to X$ such that $\sigma^p = 1$. Let $a, b \in X \setminus X^{C_p}$ such that $a \neq \sigma^k b$ for any k. Then there exists a simple path α in X from a to $\sigma^k b$ for some k such that α does not intersect any of its conjugate paths. In other words, $\alpha(s) \neq \sigma^k \alpha(t)$ for all k, s, and t ($k \neq 0$ if $s = t$).*

Proof. Choose a smooth, simple path α in $X \setminus X^{C_p}$ from a to b, and suppose α intersects its conjugate paths. Let z be a point in the intersection of α with one of its conjugates. Since $z \in X \setminus X^{C_p}$, it must be distinct from $\sigma^k z$ for all k. We can therefore take a Euclidean neighborhood of z small enough so that it is disjoint from each of its conjugate neighborhoods. Then we may alter α in this neighborhood so that z is an isolated point in the intersection. We may also alter α in such a way that this intersection is transverse and α intersects with only one of its conjugates at the point z. Proceeding in this way, we can assume that every point of intersection is transverse—and so there are only finitely many such points.

Let $z \in \alpha$ be the first point of intersection of α with one of its conjugates. We know that α intersects $\sigma^k \alpha$ for some k at this point, but the proof is similar in each value of k. We will assume $z = \sigma x$ for some $x \in \alpha$. Figure 58 shows α, $\sigma\alpha$, and $\sigma^2\alpha$ around z, σz, and $\sigma^2 z$ for the case $p = 3$. Choose a small "sidepath" from α to $\sigma\alpha$ as shown in red in Figure 58 that avoids z. Define a new path α' from a to σb as follows.

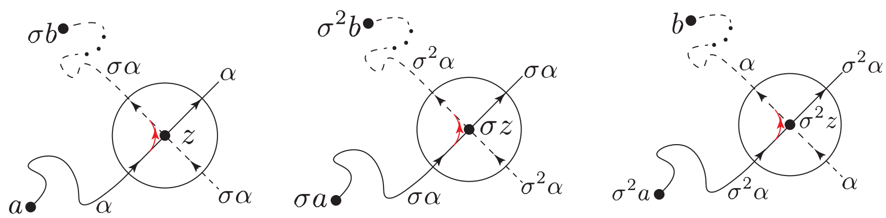


FIGURE 58. The paths α, $\sigma\alpha$, and $\sigma^2\alpha$ and an alternative path from $\sigma^i a$ to $\sigma^j b$.

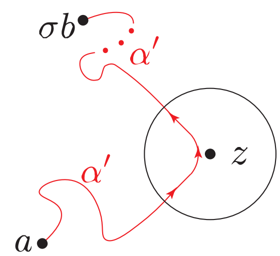


FIGURE 59. The path α'.

1. Start at a and follow α until just before getting to z.

2. Take the chosen sidepath to $\sigma\alpha$.

3. Follow $\sigma\alpha$ until reaching σb.

Now α' (shown in Figure 59) is a path which goes from a to σb and intersects its conjugates in strictly less points than α. Next move on to the next intersection of α' with one of its conjugates and perform the same procedure. Continuing in this way, one will eventually construct a path from a to $\sigma^k b$ which does not intersect its conjugate paths. □

Corollary A.0.2. *Let X be a path-connected, closed 2-manifold with a C_p action. Let Y_1 be obtained from X by removing disjoint conjugate disks embedded in $X \setminus X^{C_p}$ and sewing in a C_p-ribbon. Let Y_2 be similarly obtained from X, but using a different set of conjugate embedded disks. Then $Y_1 \cong Y_2$.*

Proof. Let $D_i, \sigma D_i, \ldots, \sigma^{p-1} D_i$ be the names of the disjoint disks removed to make Y_i from X. Let a_i denote the center of D_i. Then by Proposition A.0.1 there is a path α from a_1 to $\sigma^k a_2$ for some k that does not intersect its conjugate paths. From here, we can obtain an equivariant homeomorphism $X \to Y$ by following a nearly identical procedure to the proof of Corollary A.3 in [10]. $\square$

Proposition A.0.3. *Let X be a path-connected, closed 2-manifold with a C_p action, and let M be a non-equivariant connected surface. The equivariant isomorphism type of $X \#_p M$ is independent of the choice of disks used in the construction.*

The proof of this proposition is nearly identical to that of Corollary A.0.2.

Proposition A.0.4. *Let X and Y be equivariant 2-manifolds that both contain a C_p-ribbon. If $X - [R_p] \cong Y - [R_p]$, then $X \cong Y$.*

An analogous statement and proof of this fact for the $p = 2$ case can be found in Proposition 3.11 of [10].

APPENDIX B

FREE CLASSIFICATION PROOF

In order to prove Theorems 4.0.2 and 4.0.3, we will induct on the number of fixed points of a given C_p-surface. In this appendix, we prove the base case for this argument. In other words, we will prove in this appendix that every closed surface with a free C_p action is either isomorphic to M^{free}_{1+pg} for some g or N^{free}_{2+pr} for some r.

Let X be a path-connected non-equivariant space. Let $\mathcal{S}_p(X)$ denote the set of isomorphism classes of free C_p-spaces Y that are path-connected and have the property that $Y/C_p \cong X$.

Proposition B.0.1. *There is a bijection between $\mathcal{S}_p(X)$ and the set of nonzero orbits in $H^1_{sing}(X;\mathbb{Z}/p)/\operatorname{Aut}(X)$.*

An analogous proof of this fact for the $p = 2$ case is provided in [10], but we will summarize the main idea here. Given an element Y of $\mathcal{S}(X)$, we get a principal $\mathbb{Z}/p$ bundle $Y \to X$ by choosing an isomorphism $Y/C_p \to X$. This then corresponds to an element of $H^1(X;\mathbb{Z}/p)$ via its characteristic class. To make this association well-defined, we must quotient out by the automorphisms of X.

With this proposition, our goal is now to understand the action of $\operatorname{Aut}(X)$ on $H^1_{\text{sing}}(X;\mathbb{Z}/p)$. This is given by a group homomorphism

$$\operatorname{Aut}(X) \to \operatorname{Aut}(H^1_{\text{sing}}(X;\mathbb{Z}/p)).$$

Recall that the **full mapping class group** $\mathcal{M}(X)$ of a space X is defined to be $\mathcal{M}(X) = \operatorname{Aut}(X)/\mathcal{I}(X)$ where $\mathcal{I}(X)$ is the subgroup of automorphisms that are isotopic to the identity. Since $\mathcal{I}(X)$ acts trivially on $H^1_{\text{sing}}(X;\mathbb{Z}/p)$, our action

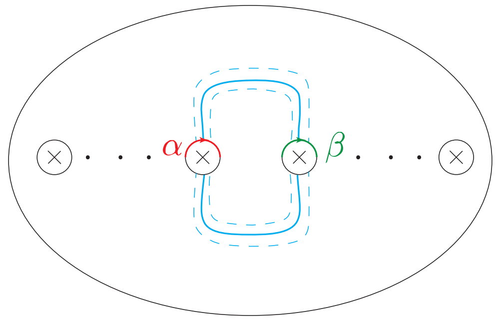


FIGURE 60. The circles α (red) and β (green). We consider $T_{\alpha,\beta}$ about the blue curve.

$\mathrm{Aut}(X) \to \mathrm{Aut}(H^1_{\text{sing}}(X;\mathbb{Z}/p))$ descends to a map

$$\mathcal{M}(X) \to \mathrm{Aut}(H^1_{\text{sing}}(X;\mathbb{Z}/p)).$$

B.1. The Mapping Class Group

We begin with some preliminaries on the mapping class group of non-orientable surfaces. It is shown in [5] that the mapping class group of a closed, non-orientable surface is generated by Dehn twists and crosscap slide maps. In this section we will demonstrate how these types of maps act on $H^1_{\text{sing}}(N_r;\mathbb{Z}/p)$. Since $H^1_{\text{sing}}(N_r;\mathbb{Z}/p)$ and $H_1^{\text{sing}}(N_r;\mathbb{Z}/p)$ are dual in $\mathbb{Z}/p$ coefficients, we can instead show how these generators act on homology.

For the remainder of this section, we will let $H^1(N_r)$ and $H_1(N_r)$ denote the singular cohomology and homology of N_r in $\mathbb{Z}/p$ coefficients.

Example B.1.1 (Dehn Twists)**.** We can represent N_r as a sphere with r crosscaps. Let α and β represent the center circles of two adjacent crosscaps as shown in Figure 60

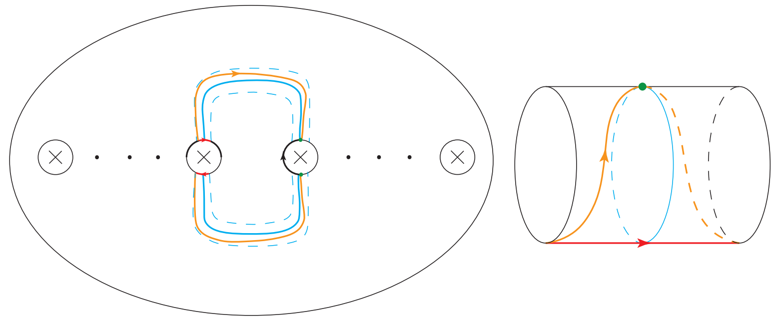


FIGURE 61. The image of the red path under the Dehn twist $T_{\alpha,\beta}$ is shown in orange.

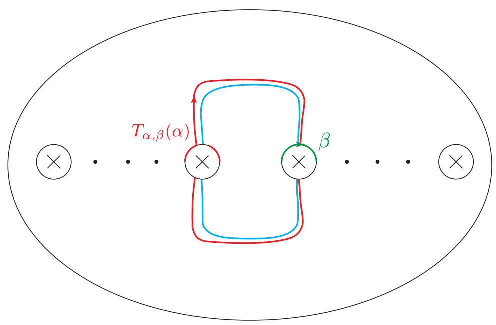


FIGURE 62. The curve $T_{\alpha,\beta}(\alpha)$ in red and β in green.

Recall that the Dehn Twist about the blue curve (which we will denote $T_{\alpha,\beta}$) will fix everything in N_r outside of some neighborhood of the curve. Figure 61 depicts the image under $T_{\alpha,\beta}$ on the portion of α lying in the neighborhood.

Figure 62 depicts the image of α under $T_{\alpha,\beta}$ in red.

To understand $T_{\alpha,\beta}(\alpha)$ as an element of $H_1(N_r)$, we will first introduce some notation. See Figure 63.

We will make several observations:

1. $\alpha = \alpha_1 + \alpha_5 + \alpha_4$ in $H_1(N_r)$.

2. $T_{\alpha,\beta}(\alpha) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ in $H_1(N_r)$.

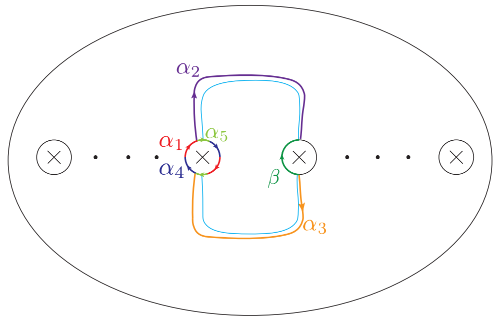


FIGURE 63. Together, the paths α_2, $-\beta$, α_3, $-\alpha_5$, $-\alpha_4$, $-\alpha_1$, and $-\alpha_5$ bound a disc.

3. The homology class $\alpha_2 - \beta + \alpha_3 - \alpha_5 - \alpha_1 - \alpha_4 - \alpha_5$ is zero since these curves bound a disk.

We can then see that

$$\begin{aligned} 0 &= \alpha_2 - \beta + \alpha_3 - \alpha_5 - \alpha_1 - \alpha_4 - \alpha_5 \\ &= \alpha_2 + \alpha_3 - \beta - \alpha - \alpha_5 \\ &= \alpha_2 + \alpha_3 - \beta - \alpha + \alpha_1 + \alpha_4 - \alpha \\ &= T_{\alpha,\beta}(\alpha) - \beta - 2\alpha. \end{aligned}$$

So $T_{\alpha,\beta}(\alpha)$ is homologous to $2\alpha + \beta$.

Next we will understand $T_{\alpha,\beta}(\beta)$ in $H_1(N_r)$. Again, consider α and β depicted in Figure 64, and let $T_{\alpha,\beta}$ denote the Dehn Twist about the blue curve.

Observe that the image of β under $T_{\alpha,\beta}$ can be represented by the green curve depicted in Figure 65.

Additionally notice that together $T_{\alpha,\beta}(\beta)$ and α bound a disk in N_r. So $T_{\alpha,\beta}(\beta) = -\alpha = (p-1)\alpha$ in $H_1(N_r)$. This proves the following proposition.

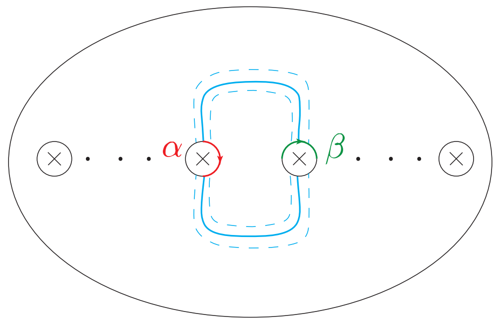


FIGURE 64. The circles α (in red) and β (in green).

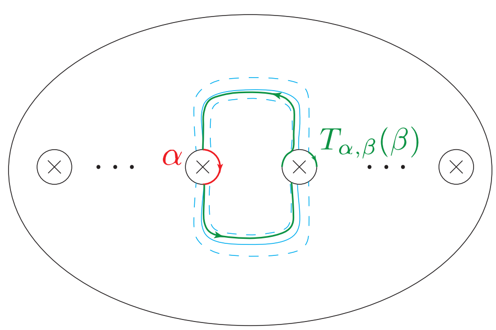


FIGURE 65. The curve $T_{\alpha,\beta}(\beta)$ in green.

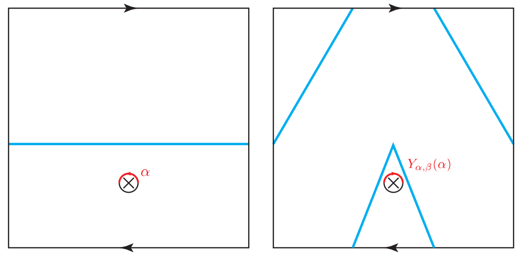


FIGURE 66. The result of the map $Y_{\alpha,\beta}$ which slides the circle α along β.

Proposition B.1.2. *Choose a representation of N_r as S^2 with r crosscaps whose center circles are denoted $\alpha_1, \ldots, \alpha_r$. Then for distinct i, j, and k, we have*

$$T_{\alpha_i,\alpha_j}(\alpha_k) = \begin{cases} 2\alpha_i + \alpha_j & \textit{if } k = i \\ -\alpha_i & \textit{if } k = j \\ \alpha_k & \textit{for } k \neq i, j \end{cases}$$

in $H_1^{sing}(N_r; \mathbb{Z}/p)$.

Example B.1.3 (Crosscap Slides)**.** Let $r \geq 2$ and represent N_r as a sphere with r crosscaps. We will describe the crosscap slide map $Y_{\alpha,\beta}$, where α and β are the center circles of two of the crosscaps of N_r.

Consider an open neighborhood of β containing α. We can think of this as a Möbius band (whose center circle is β) containing the crosscap with center circle α. The map $Y_{\alpha,\beta}$ is described as "sliding" the crosscap containing α along the center circle of the Möbius band until it's back to its original position. The action of this map on the neighborhood of β is depicted in Figure 66. Notice that this action sends α to $-\alpha$ and fixes all boundary points. We can extend this to a map on all of N_r by allowing $Y_{\alpha,\beta}$ to act as the identity everywhere outside of this neighborhood.

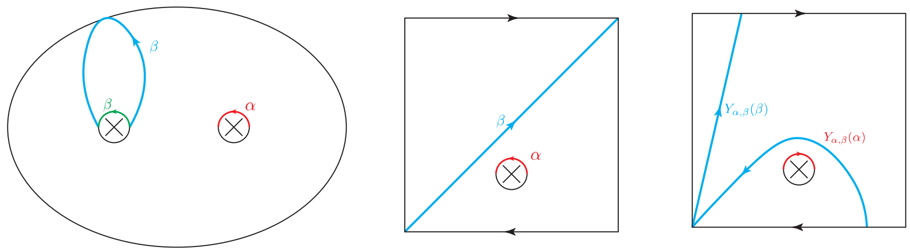


FIGURE 67. The crosscap slide $Y_{\alpha,\beta}(\alpha)$.

Our next goal is to understand $Y_{\alpha,\beta}(\beta)$ in $H_1(N_r)$. The left picture of Figure 67 shows α in red and β in green. Observe that the blue circle depicted in the left and center pictures of Figure 67 is homologous to β, so for this computation we will use the blue circle in place of β.

The right picture in Figure 67 shows the images of α and β under $Y_{\alpha,\beta}$.

Figure 68 depicts α (red), β (orange), $Y_{\alpha,\beta}(\beta)$ (blue), and a path γ (green) which bound the gray disk. The picture on the right more clearly illustrates this disk after making the appropriate identifications.

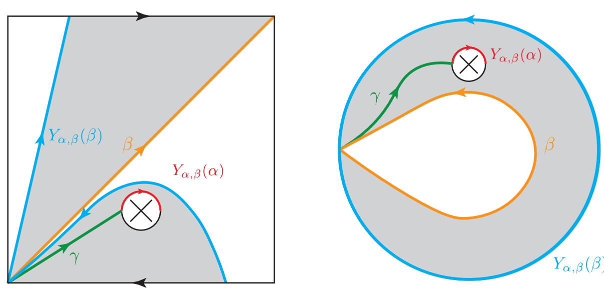


FIGURE 68. The gray region is a disk bounded by $Y_{\alpha,\beta}(\beta) + \gamma + 2Y_{\alpha,\beta}(\alpha) - \gamma - \beta$.

This gives us that

$$\begin{aligned} 0 &= Y_{\alpha,\beta}(\beta) + \gamma - \alpha - \alpha - \gamma - \beta \\ &= Y_{\alpha,\beta}(\beta) - 2\alpha - \beta. \end{aligned}$$

In general, we have

$$\begin{aligned} Y_{\alpha,\beta}(\alpha) &= -\alpha \\ Y_{\alpha,\beta}(\beta) &= 2\alpha + \beta \end{aligned}$$

and if δ is the center of a crosscap in N_r with $\delta \neq \alpha, \beta$, then $Y_{\alpha,\beta}(\delta) = \delta$.

B.2. Non-orientable Case

Proposition B.2.1. *Let X be a closed, connected, non-orientable surface of genus $r \geq 3$. There is only one nonzero orbit in $H^1(X; \mathbb{Z}/p)/\operatorname{Aut}(X)$.*

Proof. We can choose to represent X as a sphere with r crosscaps $\alpha_1, \ldots, \alpha_r$ as in Figure 69. Then we can choose $\alpha_1, \ldots, \alpha_{r-1}$ as generators for $H_1(X; \mathbb{Z}/p)$. Since homology and cohomology are dual in $\mathbb{Z}/p$ coefficients, it is sufficient to check that the action of $\operatorname{Aut}(X)$ on $H_1(X; \mathbb{Z}/p)$ has a single non-zero orbit.

The action of $\operatorname{Aut}(X)$ factors through an action of the mapping class group $\mathcal{M}(X)$. Earlier in the chapter, we discussed generators for $\mathcal{M}(X)$. To prove this proposition, we will determine how these generators for $\mathcal{M}(X)$ act on the α_i. For ease of notation, we define $T_{i,j} := T_{\alpha_i,\alpha_j}$ and $Y_{i,j} = Y_{\alpha_i,\alpha_j}$.

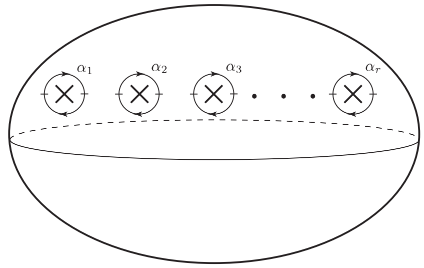


FIGURE 69. The space N_r represented as a sphere with r crosscaps.

Let us start by considering the case $r = 3$. We first claim that $T_{1,2}^{\ell}(\alpha_1) = (\ell+1)\alpha_1 + \ell\alpha_2$. This can be quickly verified using induction. The $\ell = 0$ case is immediate, and

$$\begin{aligned} T_{1,2}^{\ell+1}((\ell+1)\alpha_1 + \ell\alpha_2) &= (\ell+1)(2\alpha_1 + \alpha_2) - \ell\alpha_1 \\ &= (2\ell+2)\alpha_1 + (\ell+1)\alpha_2 - \ell\alpha_1 \\ &= (\ell+2)\alpha_1 + (\ell+1)\alpha_2. \end{aligned}$$

For simplicity, we let the tuple (c_1, c_2) represent the element $c_1\alpha_1 + c_2\alpha_2 \in H_1(X; \mathbb{Z}/p)$. Now for each $1 \leq k \leq p-1$, let S_k be the set

$$\begin{aligned} S_k &= \{(k,0), (k\cdot 2, k), (k\cdot 3, k\cdot 2), \ldots, (k\cdot(p-1), k\cdot(p-2)), (0, k\cdot(p-1))\} \\ &= \left\{T_{1,2}^{\ell}(k,0)\right\} \end{aligned}$$

and let $\tilde{S}_k$ be the singleton set containing (k,k). Observe that $(c_1, c_2) \in S_k$ if and only if $c_1 - c_2 = k$. Thus every nonzero element of $H_1(X; \mathbb{Z}/p)$ is in at least one of the S_k or $\tilde{S}_k$. One can also check that the map $T_{1,2}$ fixes all elements of the form (k,k).

Next we'll consider the action of $Y_{1,3}$ on the S_k and $\tilde{S}_k$. Since $Y_{1,3}(\alpha_1) = -\alpha_1 = (p-1)\alpha_1$, $(1,0)$ maps to $(p-1,0)$. So these elements are in the same orbit, and it must be that $S_1 \cup S_{p-1}$ is contained in a single orbit.

Similarly, we have

$$Y_{1,3}(2,1) = (p-2,1) \in S_{p-3}.$$

This implies the elements of S_1 and S_{p-3} are in the same orbit. Therefore, $S_1 \cup S_{p-1} \cup S_{p-3}$ is contained in a single orbit. Continuing in this way, we can see that in general

$$Y_{1,3}(s,s-1) = (p-s,s-1) \in S_{p-(2s-1)}.$$

for all $1 \leq s \leq p-1$. Moreover, as s ranges from 1 to $p-1$, $S_{p-(2s-1)}$ ranges over all the S_k. This tells us that $\bigcup_{k=1}^{p-1} S_k$ is contained in a single orbit.

Finally, we can check that (k,k) must also be in this orbit for each k. We have

$$Y_{1,3}(k,k) = (p-k,k) \in S_{p-2k}.$$

So $\tilde{S}_k \cup S_{p-2k}$ is contained in the same orbit for each k. Since every nonzero element of $H_1(X;\mathbb{Z}/p)$ is in S_k or $\tilde{S}_k$ for some k, there must be a single nonzero orbit in $H_1(X;\mathbb{Z}/p)/\mathrm{Aut}(X)$.

Let us now turn to the more general $r > 3$ case. For ease of notation, we will denote elements of $H_1(X;\mathbb{Z}/p)$ by an $(r-1)$-tuple. We will show that every nonzero element is in the same orbit as $(1,0,\ldots,0)$ under the action of Dehn twists and crosscap slides. Let $(c_1,c_2,\ldots,c_{r-1}) \in H_1(X;\mathbb{Z}/p)$ be nonzero, and let c_i be the rightmost nonzero coordinate of the tuple. First suppose $i=1$. We know from the $r=3$ case that there exist compositions of $T_{1,2}$ and $Y_{1,3}$ which take $(c_1,0)$ to $(1,0)$.

Since the maps $T_{j,k}$ and $Y_{j,k}$ fix all coordinates other than j and k of any given tuple, we can use $T_{1,2}$ and $Y_{1,r}$ to take $(c_1, 0, \ldots, 0)$ to $(1, 0, \ldots, 0)$ in the $r > 3$ case.

For $i > 1$, our tuple is of the form $(c_1, c_2, \ldots, c_{i-1}, c_i, 0, \ldots, 0)$. We again know from the $r = 3$ case that there is a composition of $T_{1,2}$ and $Y_{1,3}$ which takes (c_{i-1}, c_i) to $(1, 0)$. We can use the same compositions (replacing $Y_{1,3}$ with $Y_{1,r}$) in the $r > 3$ case to take $(c_1, \ldots, c_i, 0, \ldots, 0)$ to $(c_1, \ldots, c_{i-2}, 1, 0, \ldots, 0)$. Now we have a new nonzero tuple in the same orbit as the original tuple whose rightmost nonzero coordinate is in the $(i-1)$st position. We can repeat the above process until we get that the tuple $(c_1, \ldots, c_i, 0, \ldots, 0)$ is in the same orbit as $(1, 0, \ldots, 0)$. Since every nonzero element is in the same orbit as $(1, 0, \ldots, 0)$, it must be that there is a single nonzero orbit in $H_1(X; \mathbb{Z}/p)/\mathrm{Aut}(X)$. □

We next consider the genus $r < 3$ case. Since $H^1(N_1; \mathbb{Z}/p) = 0$, there can be no nonzero orbits in $H^1(N_1; \mathbb{Z}/p)/\operatorname{Aut}(N_1)$. Thus the only additional case to check is $r = 2$.

Proposition B.2.2. *There are $(p-1)/2$ nonzero orbits in $H^1(N_2; \mathbb{Z}/p)/\operatorname{Aut}(N_2)$.*

Proof. As in the $r \geq 3$ case, we can choose to represent N_2 as a sphere with 2 crosscaps α and β. Then $H_1(N_2; \mathbb{Z}/p) \cong \mathbb{Z}/p$ with generator α. Additionally, $\alpha = -\beta$ in $H_1(N_2; \mathbb{Z}/p)$.

Let us first check how Dehn twists act on α. We know $T_{\alpha,\beta}(\alpha) = 2\alpha + \beta = \alpha$ and $T_{\beta,\alpha}(\alpha) = -\beta = \alpha$. So Dehn twists act trivially on $H_1(N_2; \mathbb{Z}/p)$.

We also know that $Y_{\alpha,\beta}(\alpha) = -\alpha$ and $Y_{\beta,\alpha}(\alpha) = 2\beta + \alpha = -\alpha$.

This gives us $(p-1)/2$ nonzero orbits, each containing $k\alpha$ and $-k\alpha$ for each $1 \leq k \leq (p-1)/2$. □

B.3. Orientable Case

When X is an orientable surface, $\mathrm{Aut}(X)$ preserves the symplectic form given by the cup product. So the map $\mathrm{Aut}(X) \to \mathrm{Aut}(H^1(X))$ factors through the symplectic group $\mathrm{Sp}(2g, \mathbb{Z}/p)$. We again reference [10] for similar details in the $p = 2$ case.

Proposition B.3.1. *Let X be a closed, connected, orientable surface of genus $g \geq 1$. There is only one nonzero orbit in $H_1(X; \mathbb{Z}/p)/\mathrm{Sp}(2g, \mathbb{Z}/p)$.*

Proof. We first show there is one nonzero orbit in the case $g = 1$. One can easily check that the matrices A and B given by

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

are in $\mathrm{Sp}(2, \mathbb{Z}/p)$. For each nonzero $k \in \mathbb{Z}/p$, the elements of the set

$$S_k = \left\{ \begin{pmatrix} k \\ 0 \end{pmatrix}, \begin{pmatrix} k \\ k \end{pmatrix}, \begin{pmatrix} k \\ 2k \end{pmatrix}, \ldots, \begin{pmatrix} k \\ (p-1)k \end{pmatrix} \right\}$$

are in the same orbit since $A \begin{pmatrix} k \\ nk \end{pmatrix} = \begin{pmatrix} k \\ (n+1)k \end{pmatrix}$. Similarly, for each k the elements of the set

$$T_k = \left\{ \begin{pmatrix} 0 \\ k \end{pmatrix}, \begin{pmatrix} k \\ k \end{pmatrix}, \begin{pmatrix} 2k \\ k \end{pmatrix}, \ldots, \begin{pmatrix} (p-1)k \\ k \end{pmatrix} \right\}$$

are in the same orbit since $B\begin{pmatrix} nk \\ k \end{pmatrix} = \begin{pmatrix} (n+1)k \\ k \end{pmatrix}$. Thus we can see that the orbit containing $\begin{pmatrix} k \\ k \end{pmatrix}$ must also contain all elements of S_k and T_k. In particular, $S_k \cup T_k$ is contained in a single orbit for each k.

For each nonzero $k \in \mathbb{Z}/p$, we can find its multiplicative inverse k^{-1}. Then $\begin{pmatrix} k \\ k^{-1}k \end{pmatrix} = \begin{pmatrix} 1 \cdot k \\ 1 \end{pmatrix}$ is in both S_k and T_1. So for each k, the elements of S_k (and thus T_k) are in the same orbit as T_1. Finally, observe that every nonzero element of $(\mathbb{Z}/p)^2$ is in S_k or T_k for some k. Thus, all nonzero elements are in the same orbit under the action of $\mathrm{Sp}(2, \mathbb{Z}/p)$.

Now suppose $g \geq 2$. Choose a symplectic basis $\{e_1, f_1, \ldots, e_g, f_g\}$ so that $\langle e_i, f_i \rangle = 1$, $\langle f_i, e_i \rangle = -1$, and all other pairings are 0. Denote $v \in (\mathbb{Z}/p)^{2g}$ by $v = [B_1, \ldots, B_g]$ where each $B_i \in (\mathbb{Z}/p)^2$ and

$$v = (B_1)_1 e_1 + (B_1)_2 f_1 + \cdots + (B_g)_1 e_g + (B_g)_2 f_g.$$

Consider the evident homomorphism

$$\mathrm{Sp}(2, \mathbb{Z}/p) \times \cdots \times \mathrm{Sp}(2, \mathbb{Z}/p) \to \mathrm{Sp}(2g, \mathbb{Z}/p).$$

This allows us to represent orbits by vectors $[B_1, \ldots, B_g]$ with $B_i \in \{[0,0], [1,0]\}$ by the $g = 1$ case. Now consider the 4×4 symplectic matrix

$$A = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

where I_2 is the identity matrix. Since A is symplectic, so is $A' = I_{2k} \oplus A \oplus I_{2g-2k-4}$ for any $0 \leq k \leq g-2$. Multiplying a vector $v = [B_1, \ldots, B_g]$ by A' allows us to permute its $(k+1)$st and $(k+2)$nd blocks with the price of a sign. We can then multiply by the appropriate element of $\mathrm{Sp}(2, \mathbb{Z}/p) \times \cdots \times \mathrm{Sp}(2, \mathbb{Z}/p)$ to reduce all coefficients to 1 or 0.

Thus, there are at most $g+1$ orbits of the action of $\mathrm{Sp}(2, \mathbb{Z}/p)$ on $(\mathbb{Z}/p)^{2g}$. These orbits can be represented by the vectors

$$[O, O, \ldots, O] \qquad [T, O, \ldots, O] \qquad [T, T, O, \ldots, O] \qquad \cdots \qquad [T, T, \ldots, T]$$

where $O = [0, 0]$ and $T = [1, 0]$.

Let B be the symplectic matrix

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and observe that when $g \geq 2$, $B \oplus I_{2g-4}$ sends $[T, O, \ldots, O]$ to $[T, T, O, \ldots, O]$. In particular, these two representatives are actually in the same orbit. Moreover, for $0 \leq k \leq g-2$, $I_{2k} \oplus B \oplus I_{2g-2k-4}$ takes $[T, T, \ldots, T, O, \ldots, O]$ (with T in the first $k+1$ entries) to the vector with T in the first $k+2$ entries. Thus, all nonzero vectors in $(\mathbb{Z}/p)^{2g}$ are in the same orbit under the action of the symplectic group. $\square$

Now let Y be an orientable surface with a free C_p-action. We can see from Lemma 4.0.1 that the genus of Y must be $1 + pg$ for some g. This implies Y/C_p is a closed, connected orientable surface of genus $1 + g$. Proposition B.3.1 then implies that there is only one isomorphism class of C_p-spaces whose quotient by C_p is M_{1+g}. So Y must be isomorphic to M^{free}_{1+pg}.

APPENDIX C

NON-FREE CLASSIFICATION PROOF

This appendix contains proofs for Theorems 4.0.2 and 4.0.3. In each case, we begin by establishing several lemmas describing relationships between surfaces constructed using differing equivariant surgery methods. The classification theorems are then proven using induction on the number of fixed points.

C.1. Proof of Classification for Orientable Surfaces

Let us start with the orientable case.

Lemma C.1.1. *Let X be a closed, connected C_p-surface with distinct fixed points x and y. Then there exists $EB_{p,(i)} \subset X$ with $x, y \in EB_{p,(i)}$. Moreover, nbd$(EB_{p,(i)}) \subset X$ must be isomorphic to $R_{p,(i)}$ or $TR_{p,(i)}$.*

Proof. This reduces to a question of how we can glue together the surfaces in Figure 70 (showing the $p = 3$ case) along the red lines using equivariant maps. Any such map is completely determined by how we attach a single edge, and up to isomorphism there are only two choices. One of these produces $R_{p,(i)}$ and the other $TR_{p,(i)}$. □

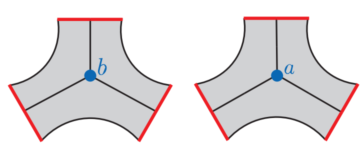


FIGURE 70. Gluing the red edges using an equivariant map results in R_3 or TR_3.

Lemma C.1.2. *There is an equivariant automorphism f on $TR_{p,(i)}$ with distinct fixed points x and y so that $f(x) = y$ and $f(y) = x$ and $f|_{\partial TR_{p,(i)}} = id$.*

Proof. Recall the polygon representation of $TR_{p,(i)}$ as shown in Figure 21. The action of C_p on $TR_{p,(i)}$ corresponds to a rotation action by $e^{2\pi i/p}$ on the polygon.

Let A represent the annulus of width $\epsilon > 0$ inside $TR_{p,(i)}$ so that $\partial TR_{p,(i)}$ is a boundary component of A. Define f so that $f|_A$ is the Dehn twist with $f|_{\partial TR_{p,(i)}} =$ id and f restricted to the other boundary component of A is given by 180° rotation. Then let $f|_{TR_{p,(i)} \setminus A}$ act as rotation by 180°. Notice that f respects the C_p-action of $TR_{p,(i)}$ and swaps x and y as desired. □

Lemma C.1.3. *If $x, y \in \mathrm{Hex}_1$ are distinct fixed points, then $\mathrm{Hex}_1 +_x [TR_p] \cong \mathrm{Hex}_1 +_y [TR_p]$. Moreover,*

$$\mathrm{Hex}_1 +_x [TR_p] \cong \mathrm{Sph}_{p-1}[4].$$

Proof. Given any two distinct fixed points $x, y \in \mathrm{Hex}_1$, there is a copy of TR_p containing them. By Lemma C.1.2, there is an automorphism $\tilde{\varphi}$ of $TR_p \subset \mathrm{Hex}_1$ swapping x and y. This can be extended to an automorphism φ of Hex_1 by defining φ to be $\tilde{\varphi}$ on TR_p and the identity everwhere else. Thus we can define an isomorphism $\mathrm{Hex}_1 +_x [TR_p] \to \mathrm{Hex}_1 +_y [TR_p]$ given by φ everywhere outside of the added copy of TR_p.

Observe that $\mathrm{Hex}_1 +_x [TR_p]$ can be obtained by taking two copies of TR_p and identifying their boundaries. Figure 71 shows how this gives us $\mathrm{Sph}_{p-1}[4]$ in the $p = 3$ case. Choose one copy of TR_p to be a neighborhood of the red EB_p. It's complement in $\mathrm{Sph}_{p-1}[4]$ is another copy of TR_p containing the purple EB_p. □

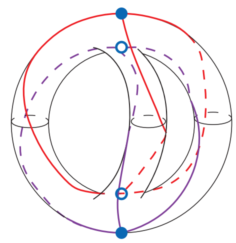


FIGURE 71. Two copies of TR_3 inside $\mathrm{Sph}_2[4]$.

Lemma C.1.4. *If $x, y \in \mathrm{Hex}_n$ ($n \geq 2$) are distinct fixed points, then $\mathrm{Hex}_n +_x [TR_p] \cong \mathrm{Hex}_n +_y [TR_p]$. In other words, twisted ribbon surgery on Hex_n is independent of the fixed point chosen.*

The proof is almost identical to that of Lemma C.1.3. The idea is that any two fixed points in Hex_n are contained in a copy of TR_p. More specifically, this argument shows that $\mathrm{Hex}_n +_x [TR_p] \cong (\mathrm{Hex}_{n-1} + (k+2)[R_p]) \#_p M_g$.

Lemma C.1.5. *If x and y are distinct fixed points in $\mathrm{Sph}_{(p-1)k+pg}[2k+2]$ for some $k, g \geq 0$, then $\mathrm{Sph}_{(p-1)k+pg}[2k+2] +_x [TR_p] \cong \mathrm{Sph}_{(p-1)k+pg}[2k+2] +_y [TR_p]$. In other words, twisted ribbon surgery on $\mathrm{Sph}_{(p-q)k+pg}[2k+2]$ is independent of the chosen fixed point.*

Proof. We can choose to represent $\mathrm{Sph}_{(p-1)k+pg}[2k+2]$ in the following way:

1. Start with $S^{2,1}$.

2. Choose $k+1$ disks $D_1, \ldots D_{k+1}$ centered at the equator of $S^{2,1}$ so that $\sigma^s D_i \cap \sigma^{s'} D_j = \emptyset$ for all i, j, s, s'.

3. Perform $\#_p M_g$-surgery using D_{k+1} and its conjugates.

4. Remove $D_1, \ldots, D_k$ and their conjugates to perform $+[R_p]$-surgery k times. Let R_{p_i} denote the copy of R_p glued to the boundary of $D_i \cup \sigma D_i \cup \cdots \cup \sigma^{p-1} D_i$.

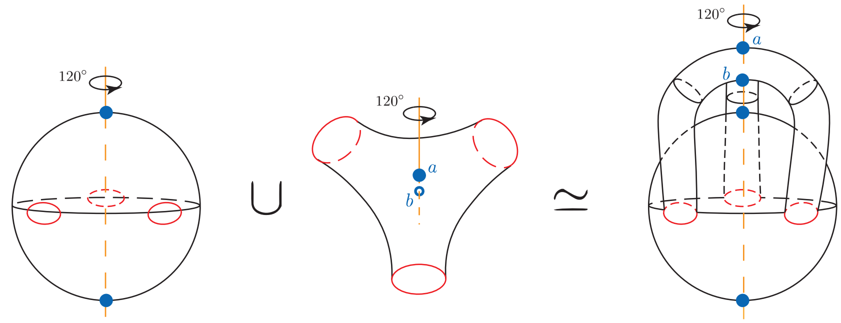


FIGURE 72. A representation of $\mathrm{Sph}_2[4]$ using $+[R_3]$-surgery on $S^{2,1}$.

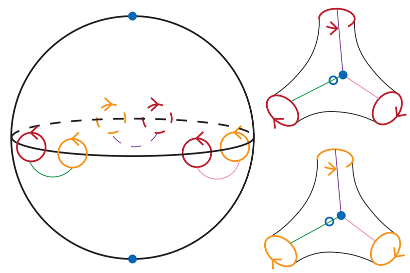


FIGURE 73. A neighborhood of the copy of EB shown here is isomorphic to TR_3.

Suppose each copy of R_p is glued onto $S^{2,1}$ as shown in Figure 72. We call a the "north pole" of R_p and b the "south pole". Figures 73 and 74 depict a path α (in green) from the north pole of R_{p_i} for some i to the north pole of $S^{2,1}$ or R_{p_j} for some j. This figure only shows the path in the case where $k = 2$ and $g = 0$, but in all other cases a similar path can be chosen. Observe that a neighborhood of $\alpha \cup \sigma\alpha \cup \cdots \cup \sigma^{p-1}\alpha$ is isomorphic to TR_p. This can be verified by checking that this neighborhood has only a single boundary component. In this case, we know there exists an automorphism of $\mathrm{Sph}_{(p-1)k+pg}[2k+2]$ swapping the two north poles. Similarly, if given two south poles we can find a copy of TR_p containing them. Thus if x and y are both north poles (respectively south poles), then $\mathrm{Sph}_{(p-1)k+pg}[2k+2] +_x [TR_p] \cong \mathrm{Sph}_{(p-1)k+pg}[2k+2] +_y [TR_p]$.

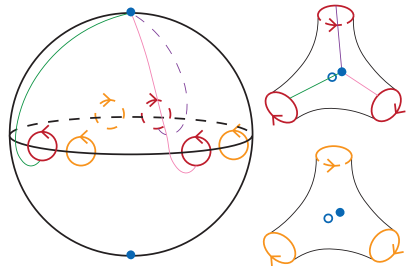


FIGURE 74. A neighborhood of the copy of EB shown here is isomorphic to TR_3.

It remains to show that if x is a north pole and y is a south pole, then $\mathrm{Sph}_{(p-1)k+pg}[2k+2] +_x [TR_p] \cong \mathrm{Sph}_{(p-1)k+pg}[2k+2] +_y [TR_p]$. We will show this by considering the case $x = a$ and $y = b'$ as depicted in Figures 75 and 76. The argument for cases when $k > 1$ or $g > 0$ are similar. If we can show the isomorphism in this case, then for any north pole x' and any south pole and y', we have

$$\begin{aligned}\mathrm{Sph}_{(p-1)k+pg}[2k+2] +_{x'} [TR_p] &\cong \mathrm{Sph}_{(p-1)k+pg}[2k+2] +_x [TR_p]\\ &\cong \mathrm{Sph}_{(p-1)k+pg}[2k+2] +_y [TR_p]\\ &\cong \mathrm{Sph}_{(p-1)k+pg}[2k+2] +_{y'} [TR_p].\end{aligned}$$

Figure 75 depicts the result of $+_a[TR_3]$-surgery on $M_2[4]$, and Figure 76 shows $M_2[4] +_{b'} [TR_3]$. We can construct an isomorphism between these spaces as reflection through the plane of the hexagon.

□

So far we have proven that $X +_x [TR_p]$ is independent of x when X is of the form $\mathrm{Hex}_n \#_p M_g$ or $S^{2,1} + [R_p] \#_p M_g$. We will now spend some time understanding when twisted ribbon surgery fails to be independent of its chosen fixed point.

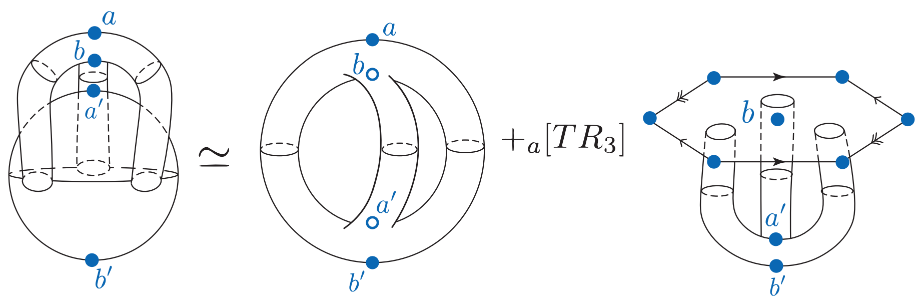


FIGURE 75. The result of $+_a[TR_3]$-surgery on $\mathrm{Sph}_{2k+3g}[2k+2]$ for $k = 1$, $g = 0$.

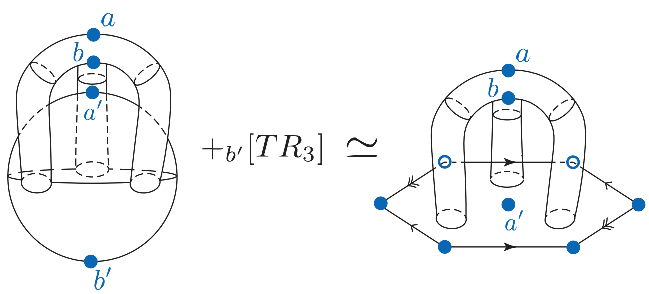


FIGURE 76. The result of $+_{b'}[TR_3]$-surgery on $\mathrm{Sph}_{2k+3g}[2k+2]$ for $k = 1$, $g = 0$.

Proposition C.1.6. *There does not exist an equivariant isomorphism between the C_p-spaces $\mathrm{Sph}_{2(p-1)}[6]$ and Hex_2 (even up to the action of $\mathrm{Aut}(C_p)$).*

Proof. Let X be a nontrivial, orientable C_p-space, and let X^{C_p} denote the fixed set of X. We start by defining a map $X^{C_p} \to C_p$. Fix an orientation for X, and consider the induced orientation on X/C_p. For each fixed point $x \in X^{C_p}$, let $\bar{x}$ represent the image of x in X/C_p. Choose a small loop going around $\bar{x}$ in the direction of the chosen orientation. We can then lift this loop to a path in X going from a point y to gy for some $g \in C_p$. Note that the element g is independent of choice for y. In this way, we can define the map $X^{C_p} \to C_p$ given by $x \mapsto g$.

Theorem 1.1 of [7] states that this map determines the C_p-space X up to isomorphism.

Let us now turn our attention to Hex_2 and $\mathrm{Sph}_{2(p-1)}[6]$. We will show by direct computation that the maps $\mathrm{Hex}_2^{C_p} \to C_p$ and $\mathrm{Sph}_{2(p-1)}[6]^{C_p} \to C_p$ as defined above must be distinct.

We focus our attention on the $p = 3$ case since the argument can be extended to all odd primes. Let g be the generator of C_3 corresponding to counter-clockwise rotation of Hex_2 by 120° about the axis passing through the center of the hexagons. Figure 77 demonstrates that for any fixed point $x \in \mathrm{Hex}_2$, the image of x under the above map is g.

To see this, start by labeling the six fixed points of Hex_2 as $x_1, \ldots, x_6$. Next choose an orientation for Hex_2 and consider the induced orientation on $\mathrm{Hex}_2 / C_3 \simeq S^2$. Figure 77 depicts Hex_2 (left) and S^2 (right) with the chosen orientation in gray. We can then choose a loop in the direction of the orientation about the image of each x_i in S^2. Each of these loops can be lifted to some path in Hex_2. Let $\tilde{x}_i$ ($1 \leq i \leq 6$) denote the starting point of the path lifted from the ith loop. Figure 77 demonstrates that for each i, we get a path from $\tilde{x}_i$ to $g\tilde{x}_i$.

For example, the green loop on the right of Figure 77 goes about the fixed point x_1. We can lift it to the green path in Hex_2. This path starts at the point labeled $\tilde{x}_1$ and ends at the image of $\tilde{x}_1$ under the action of g. So our map in this case sends x_1 to g. Since $\mathrm{Hex}_2^{C_3}$ just consists of the fixed points $x_1, x_2, \ldots, x_6$, we can describe the above map as the tuple (g, g, g, g, g, g).

Let us now choose to represent the space $\mathrm{Sph}_4[6]$ as depicted on the left of Figure 78. Let g represent counter-clockwise rotation of $\mathrm{Sph}_4[6]$ by 120° about the axis passing through the center of the hexagons, and label the six fixed points as $x_1, x_2, \ldots, x_6$. We can then fix an orientation for $\mathrm{Sph}_4[6]$ and choose oriented paths about the image of x_i in $\mathrm{Sph}_4[6]/C_3 \simeq S^2$ for each i. As before, we lift each of these

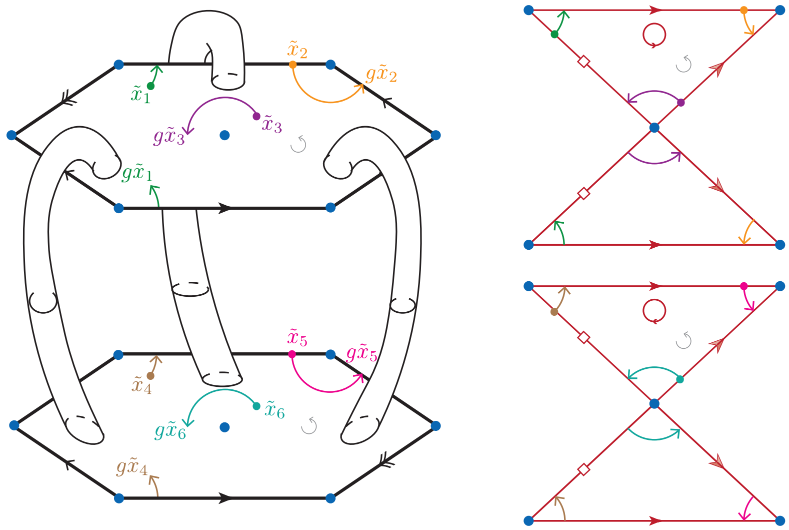


FIGURE 77. The map $\mathrm{Hex}_2^{C_3} \to C_3$, described by the tuple (g, g, g, g, g, g).

loops to a path starting at the point $\tilde{x}_i$, and we look at the endpoint of each lifted path. Figure 78 demonstrates that these endpoints are $g\tilde{x}_1$, $g\tilde{x}_2$, $g\tilde{x}_3$, $g^2\tilde{x}_4$, $g^2\tilde{x}_5$, and $g^2\tilde{x}_6$. Another way to represent this map $\mathrm{Sph}_4[6]^{C_3} \to C_3$ is with the tuple (g, g, g, g^2, g^2, g^2).

Even up to a relabeling of the fixed points and an action of $\mathrm{Aut}(C_3)$, the maps described by (g, g, g, g, g, g) and (g, g, g, g^2, g^2, g^2) must be distinct. In other words, it cannot be the case that $\mathrm{Sph}_4[6]$ and Hex_2 are isomorphic. $\square$

More generally, the same argument shows that $\mathrm{Hex}_2 + k[R_p]\#_p M_g$ is not isomorphic to $\mathrm{Sph}_4[6] + k[R_p]\#_p M_g$ for any k,g.

Remark C.1.7. *The same methods can also be used to show* $\mathrm{Hex}_{n_1} + k_1[R_p]\#_p M_{g_1}$, $\mathrm{Hex}_{n_2} + k_2[R_p]\#_p M_{g_2}$, *and* $S^{2,1} + k_3[R_p]\#_p M_{g_3}$ *are always in distinct isomorphism classes (unless of course* $n_1 = n_2$, $k_1 = k_2$, *and* $g_1 = g_2$*).*

Lemma C.1.8. *When* $k \geq 1$, *there are two isomorphism classes of* C_p*-spaces of the form* $\mathrm{Hex}_{1,(p-1)/2+(p-1)k+pg}[3 + 2k] +_x [TR_p]$ *which depend on the choice of fixed*

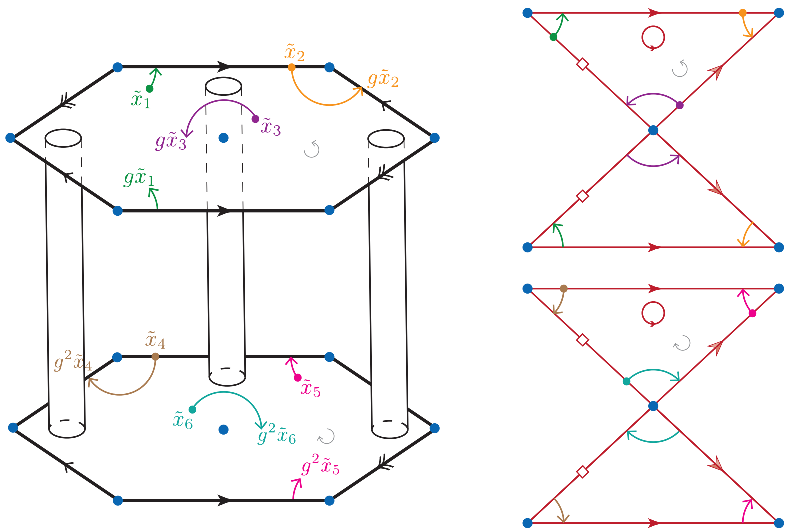


FIGURE 78. The map $\mathrm{Sph}_4[6]^{C_3} \to C_3$, described by the tuple (g, g, g, g^2, g^2, g^2).

point x. In particular, given a fixed point x, $\mathrm{Hex}_{1,(p-1)/2+(p-1)k+pg}[3+2k] +_x [TR_p]$ *is isomorphic to one of the following:*

1. $\mathrm{Sph}_{(p-1)(k+1)+pg}[2+2(k+1)]$

2. $(\mathrm{Hex}_2 + (k-1)[R_p]) \#_p M_g$

Proof. We can represent $\mathrm{Hex}_{1,(p-1)/2+(p-1)k+pg}[3+2k]$ by choosing $k+1$ disks $D_1, \dots D_{k+1}$ on Hex_1 so that $\sigma^s D_i \cap \sigma^{s'} D_j = \emptyset$ for all i, j, s, s'. Remove each $\sigma^j D_i$. Then attach a copy of R_p (denoted R_{p_i}) to $\partial D_i \cup \partial(\sigma D_i) \cup \dots \cup \partial(\sigma^{p-1} D_i)$ for each $i = 1, \dots, k$. Then attach a copy of $C_p \times (M_g \setminus D^2)$ to $\partial D_{k+1} \cup \partial(\sigma D_{k+1}) \cup \dots \cup \partial(\sigma^{p-1} D_{k+1})$. For simplicity of notation, we will let X denote the space $\mathrm{Hex}_{1,(p-1)/2+(p-1)k+pg}[3+2k]$ for the remainder of the proof.

A similar argument as in the previous case shows that if x and y are the north poles (respectively south poles) of R_{p_i} and R_{p_j} for some i, j, then we can find a copy of TR_p containing x and y. This implies that $X +_x [TR_p] \cong X +_y [TR_p]$ for all such x and y.

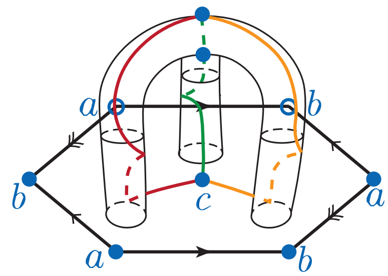


FIGURE 79. A copy of EB_3 whose neighborhood is isomorphic to TR_3.

Let a, b, c be the fixed points originating from the copy of Hex_1 as depicted in Figure 79. This figure depicts a copy of EB_p in X containing the north pole of $(R_p)_1$ and c with a neighborhood isomorphic to TR_p. Figure 79 depicts the case $k = 1, g = 0$, but one could construct a similar copy of EB_p in all other cases. Recall additionally from Lemma C.1.3 that there is a copy of TR_p containing a and c as well as a copy containing b and c. So we have that $X +_x [TR_p] \cong X +_y [TR_p]$ when $x, y \in \{\text{north pole of } (R_p)_i \mid 1 \leq i \leq k\} \cup \{a, b, c\}$. This also holds if $x, y \in \{\text{south pole of } (R_p)_i \mid 1 \leq i \leq k\}$.

At this point we have demonstrated there are at most two isomorphism classes of $\text{Hex}_{1,(p-1)/2+(p-1)k+pg}[3 + 2k] +_x [TR_p]$. We know from Lemma C.1.3 that $X +_c [TR_p] \cong \text{Sph}_{p-1}[4] + k[R_p]\#_p M_g$. By construction in Example 3.4.5, we also know that $X +_x [TR_p] \cong \text{Hex}_2 +(k-1)[R_p]\#_p M_g$ when $x \in \{\text{south pole of } (R_p)_i \mid 1 \leq i \leq k\}$.

We know from Proposition C.1.6 and subsequent remarks that these spaces are not isomorphic. So there must be exactly two isomorphism classes of spaces of the form $\left(\text{Hex}_{1,(p-1)/2+(p-1)k+pg}[3 + 2k]\right) +_? [TR_p]$.

$\square$

Corollary C.1.9. *For $n \geq 2$ and $k \geq 1$, there are two isomorphism classes of C_p-spaces of the form $(\mathrm{Hex}_n + k[R_p]) \#_p M_g +_x [TR_p]$ which depend on the choice of fixed point x. Specifically, given a fixed point x, $(\mathrm{Hex}_n + k[R_p]) \#_p M_g +_x [TR_p]$ is isomorphic to one of the following:*

1. $(\mathrm{Hex}_{n+1} + (k-1)[R_p]) \#_p M_g$
2. $(\mathrm{Hex}_{n-1} + (k+2)[R_p]) \#_p M_g$

The same ideas presented in the proof of Lemma C.1.8 can be extended to this more general case.

Finally, we present a lemma which will help prove the inductive step of our main classification theorem.

Lemma C.1.10. ?? *Let X be a connected C_p-surface for which $X - [R_p]$ is defined. If $F(X) \geq 3$, then $X - [R_p]$ must also be connected.*

Proof. Fix a copy of $R_p \subset X$ on which we will perform $-[R_p]$ surgery. Since $F(X) \geq 3$, there exists at least one additional fixed point $x \in X$ such that $x \notin R_p$. In order to show that $X - [R_p]$ is connected, it suffices to show that $X \setminus R_p$ (the space obtained by removing R_p from X but before gluing in the p conjugate disks) is connected.

We first claim that given any point y in the boundary of $X \setminus R_p$, there is a path from the fixed point x to y. First note that the connected component of $X \setminus R_p$ containing x must have at least one boundary component (which we will call C). Otherwise, X could not have been connected. Thus there is a path from x to any point on C. A conjugate to any such path would be a path from x to $\sigma^i C$. Thus, x must be in the same connected component as each boundary component of $X \setminus R_p$.

Since X is connected, every point $z \in X \setminus R_p$ must be in the same connected component as at least one boundary component. Thus every point in $X \setminus R_p$ must lie in a single boundary component. □

We are now ready to revisit Theorem 4.0.2 and provide a proof of the result.

Theorem C.1.11. *Let X be a connected, closed, orientable surface with an action of C_p. Then X can be constructed via one of the following surgery procedures, up to* $\mathrm{Aut}(C_p)$ *actions on each of the pieces.*

1. $M_{1+pg}^{free} := M_1^{free} \#_p M_g,\ g \geq 0$
2. $\mathrm{Sph}_{(p-1)k+pg}[2k+2] := (S^{2,1} + k[R_p])\, \#_p M_g,\ k, g \geq 0$
3. $\mathrm{Hex}_{n,(3n-2)(p-1)/2+(p-1)k+pg}[3n+2k] := (\mathrm{Hex}_n + k[R_p])\, \#_p M_g,\ k, g \geq 0,\ n \geq 1$

Proof. We induct on the number of fixed points F.

First let X be a free orientable space. By the classification of free C_p-spaces done in Appendix B, $X \cong M_{1+pg}^{\mathrm{free}}$ for some $g \geq 0$.

The case where X is orientable and $F = 1$ does not occur. A proof of this fact can be found in Example 3.3 of [3] or Theorem 7.1 of [2]. Let us move on to the case $F = 2$. Let $x, y \in X$ be distinct fixed points. By Lemma C.1.1, there exists $R_p \subset X$ or $TR_p \subset X$ containing x and y. The latter case is not possible since $X - [TR_p]$ would be a closed, orientable C_p-surface with a single fixed point. So x and y are contained in some R_p in X. Then by the $F = 0$ case, $X - [R_p] \cong M_{1+pg}^{\mathrm{free}}$ or $X - [R_p] \cong M_g \times C_p$. We know from Figure 24 that $+[R_p]$ surgery on either of these spaces results in $S^{2,1} \#_p M_{g'}$ for some $g' \geq 0$. Thus X must be isomorphic to $\mathrm{Sph}_{pg}[2]$.

We additionally observe in the case $F = 2$ that since $X \cong S^{2,1} \#_p M_{g'}$, there is an equivariant automorphism of X swapping the fixed points x and y. This map φ can be defined as a reflection through the plane perpendicular to the axis of rotation which bisects X. We can thus define an isomorphism $X +_y [TR_p] \to X +_x [TR_p]$ given by φ everywhere outside of the added copy of TR_p.

Next assume $F = 3$. Again, we can find distinct fixed points x and y in X which are contained in $R_p \subset X$ or $TR_p \subset X$. The former is impossible since $X - [R_p]$ would be a closed, orientable C_p-surface with one fixed point. Thus, $x, y \in TR_p$ in X. So $X -_{x,y} [TR_p] \cong S^{2,1} \#_p M_g$ for some g by the previous $F = 2$ case. Finally we can observe that $(S^{2,1} \#_p M_g) + [TR_p] \cong \mathrm{Hex}_1 \#_p M_g$. Since $S^{2,1} \#_p M_g +_? [TR_p]$ is independent of the chosen fixed point, we can conclude that $X \cong \mathrm{Hex}_1 \#_p M_g$.

Since $X \cong \mathrm{Hex}_1 \#_p M_g$, all three fixed points of X live in a copy of $\mathrm{Hex}_1 \setminus (D_2 \times C_p)$. So given any two fixed points in X, there exists $TR_p \subset X$ containing them. By Lemma C.1.2 we can construct an equivariant automorphism of X swapping any two of its fixed points. Therefore $+[TR_p]$ surgery on X is invariant of the choice of fixed point.

For the inductive hypothesis, let $3 < \ell$. For any ℓ' with $3 \leq \ell' < \ell$, suppose that (1) if Z is a connected, closed, orientable C_p-surface with $F = \ell'$, then Z is isomorphic to $\mathrm{Sph}_{(p-1)k+pg}[2k+2]$ or $\mathrm{Hex}_{n,(3n-2)(p-1)/2+(p-1)k}[3n+2k]$ for some $k, g \geq 0$ and $n \geq 1$, and (2) if x and y in Z are distinct fixed points, then $Z +_x [TR_p] \cong Z +_y [TR_p]$. Now let X be a closed, orientable C_p-surface with $F = \ell$. Let $x, y \in X$ be distinct fixed points. By Lemma C.1.1, there exists $R_p \subset X$ or $TR_p \subset X$ containing x and y.

Suppose first that x and y are contained in $R_p \subset X$. Then $X - [R_p]$ has $\ell - 2 \geq 2$ fixed points. Since X was connected and $X - [R_p]$ has at least one fixed point, $X - [R_p]$ must also be connected by Lemma **??**. So we can invoke the inductive hypothesis to conclude that $X - [R_p]$ is isomorphic to one of the following:

1. $\mathrm{Sph}_{(p-1)k+pg}[2k+2] \cong (S^{2,1} + k[R_p]) \#_p M_g$

2. $(\mathrm{Hex}_n + k[R_p]) \#_p M_g$.

In the first case, we can conclude

$$X \cong \left(S^{2,1} + (k+1)[R_p]\right) \#_p M_g \cong \mathrm{Sph}_{(p-1)(k+1)+pg}[2(k+1)+2].$$

In the second case, it follows that

$$X \cong (\mathrm{Hex}_n + (k+1)[R_p]) \#_p M_g.$$

If x and y are contained in $TR_p \subset X$, then $X -_{x,y} [TR_p]$ has $\ell - 1 \geq 3$ fixed points. By the inductive hypothesis, $X -_{x,y} [TR_p]$ is isomorphic to one of the following:

1. $\mathrm{Sph}_{(p-1)k+pg}[2k+2] \cong (S^{2,1} + k[R_p]) \#_p M_g$ for some $k \geq 1$ and $g \geq 0$

2. $(\mathrm{Hex}_n + k[R_p]) \#_p M_g$ for some $n \geq 1$ and $k, g \geq 0$.

We know from Lemma C.1.5 that $+_?[TR_p]$-surgery on $\mathrm{Sph}_{(p-1)k+pg}[2k+2]$ is independent of the chosen fixed point. So if $X -_{x,y} [TR_p] \cong \mathrm{Sph}_{(p-1)k+pg}[2k+2]$, then

$$X \cong (\mathrm{Hex}_1 + k[R_p]) \#_p M_g.$$

Next suppose $X -_{x,y} [TR_p] \cong (\mathrm{Hex}_n + k[R_p]) \#_p M_g$ for some $n \geq 1$ and $k, g \geq 0$. Again, we know from Corollary C.1.9 that if $k \geq 1$ there are two isomorphism classes of spaces for $((\mathrm{Hex}_n + k[R_p]) \#_p M_g) +_a [TR_p]$, depending on the choice of fixed point a. In one case we have

$$X \cong (\text{Hex}_{n-1} + (k+2)[R_p]) \,\#_p M_g.$$

This is also the result of $+_a[TR_p]$-surgery on X when $k = 0$. Assuming $k \geq 1$, it is also possible that

$$X \cong (\text{Hex}_{n+1} + (k-1)[R_p]) \,\#_p M_g.$$

□

For the remainder of this section, we use $\tilde{N}_n$ to denote the space $N_n \setminus D^2$.

C.2. Free Actions on Non-orientable Surfaces with Boundary

Our next goal is to prove the classification theorem for non-orientable C_p-surfaces. We saw that there were no orientable C_p-surfaces with a single fixed point, but this is not the case for non-orientable surfaces. In order to prove the $F = 1$ case, we need to lay a bit of ground work. In this section, we prove that up to an action of $\text{Aut}(C_p)$, there is a single isomorphism class of free C_p-actions on $\tilde{N}_{pn+1}$ for all $n \geq 0$.

As with the orientable case, we will prove Theorem 4.0.3 by induction on F. This result on free actions of $\tilde{N}_{pn+1}$ will help us with the base case $F = 1$.

Proposition C.2.1. *Up to the action of* $\text{Aut}(C_p)$ *there is a single isomorphism class of free* C_p *actions on* $\tilde{N}_{pn+1}$ *for all* $n \geq 0$.

More precisely, there are $(p-1)/2$ non-isomorphic actions on $\tilde{N}_{pn+1}$. These are the $\text{Aut}(C_p)$-conjugates of $MB_p \#_p N_n$ where MB_p is defined in Chapter III.

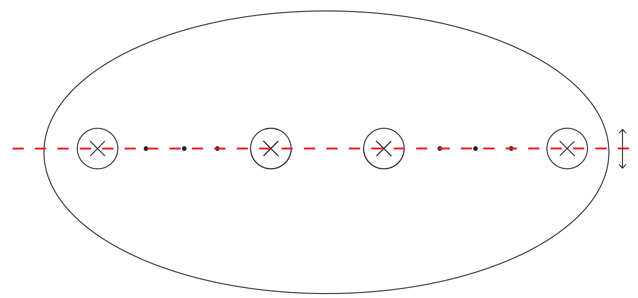


FIGURE 80. Reflection about the orange line sends every α_i to $-\alpha_i$.

Proof. Given a free C_p action on $\tilde{N}_{pn+1}$, the quotient $\tilde{N}_{pn+1}/C_p$ must be a non-orientable surface with a single boundary component and Euler characteristic $\frac{1}{p}(1-(pn+1)) = -n$. The only such space is $\tilde{N}_{n+1}$.

Recall from Chapter III that $\mathcal{S}(\tilde{N}_{n+1})$ denotes the set of isomorphism classes of path-connected, free C_p spaces X so that $X/C_p \cong \tilde{N}_{n+1}$. There is a bijection between $\mathcal{S}(\tilde{N}_{n+1})$ and the set of nonzero orbits of $H^1(\tilde{N}_{n+1};\mathbb{Z}/p)$ under the action of $\mathrm{Aut}(\tilde{N}_{n+1})$.

To prove Proposition C.2.1, we first show that $H_1(\tilde{N}_{n+1};\mathbb{Z}/p)$ has at most $p-1$ nonzero orbits under the action of $\mathrm{Aut}(\tilde{N}_{n+1})$. This will guarantee that the corresponding action on cohomology has at most $p-1$ nonzero orbits, and thus $|\mathcal{S}(\tilde{N}_{n+1})| \leq p-1$. Then we will construct a space $Y \ncong \tilde{N}_{pn+1}$ for $n \geq 1$ with $\frac{p-1}{2}$ non-isomorphic free actions of C_p whose quotient by each of these actions is $\tilde{N}_{n+1}$. This will guarantee that the $\frac{p-1}{2}$ conjugate actions of C_p on $\tilde{N}_{pn+1}$ coming from $MB_p\#_p N_n$ can be the only such actions.

We can choose to represent $\tilde{N}_{n+1}$ as a disk with $n+1$ crosscaps. Pick a basis $\{\alpha_1,\ldots,\alpha_{n+1}\}$ for $H_1(\tilde{N}_{n+1};\mathbb{Z}/p) = (\mathbb{Z}/p)^{n+1}$ given by the center circles of the crosscaps.

Let ψ denote the reflection as shown in Figure 80, and recall our notation $T_{i,j}$ for the Dehn Twist about the curve passing through the ith and jth crosscaps. Similarly $Y_{i,j}$ denotes the crosscap slide which slides the ith crosscap through the jth.

When $n = 0$, the reflection ψ is the only element of the mapping class group which acts nontrivially on homology. It sends an element $k \in \mathbb{Z}/p$ to $p - k$. There are precisely $\frac{p-1}{2}$ nonzero orbits in $H_1(\tilde{N}_1; \mathbb{Z}/p)/\operatorname{Aut}(\tilde{N}_1)$. These orbits are of the form $\{k, p-k\}$ for $1 \leq k \leq \frac{p-1}{2}$.

Let $n \geq 1$ and $\mathbf{c} = (c_1, \ldots, c_{n+1})$ be a nonzero element of $H_1(\tilde{N}_{n+1}; \mathbb{Z}/p)$. We will show that is in the same orbit as either $(k, 0, \ldots, 0)$ for some $1 \leq k \leq \frac{p-1}{2}$ or $(\ell, \ell, \ldots, \ell)$ for some $1 \leq \ell \leq \frac{p-1}{2}$. Let c_i be the rightmost nonzero entry with the property that $c_i \neq c_{i-1}$. We first claim there exists some power of $T_{i-1,i}$ so that

$$(c_1, \ldots, c_{i-1}, c_i, \ldots, c_{n+1}) \sim (c_1, \ldots, c_{i-1} - c_i, 0, c_{i+1}, \ldots, c_{n+1}).$$

We showed in the proof of Proposition B.2.1 that applying $T_{i-1,i}$ to the tuple s times produces the tuple whose $(i-1)$st coordinate is $(s+1)c_{i-1} - sc_i$ and whose ith coordinate is , $sc_i - (s-1)c_{i-1}$. Since $c_{i-1} \neq c_i$, there exists some positive integer s so that $sc_{i-1} - (s-1)c_i \equiv 0 \pmod{p}$. For such an s, it is therefore also true that $(s+1)c_{i-1} - sc_i \equiv c_{i-1} - c_i \pmod{p}$. So applying $T^s_{i-1,i}$ to the tuple $(c_1, \ldots, c_{n+1})$ produces

$$(c_1, \ldots, c_{i-1} - c_i, 0, c_{i+1}, \ldots, c_{n+1})$$

as desired. Notice that applying the appropriate power of $T_{i-1,i}$ either increases the number of zeros in the tuple (in the case that $c_{i-1} \neq 0$) or shifts an existing zero to the right one position (in the case $c_{i-1} = 0$). Repeat this process to obtain the orbit representative $(c_1, \ldots, c_{n+1}) \sim (k, k, \ldots, k, 0, \ldots, 0)$ with $k \neq 0$ and $1 \leq \ell \leq n+1$ nonzero entries.

When $\ell < n+1$,

$$Y_{\ell,n+1}(k, k, \ldots, k, 0, \ldots, 0) = (k, \ldots, k, p-k, 0, \ldots, 0)$$

Since the ℓth entry is not equal to the $(\ell + 1)$st entry, we can repeat the steps outlined in the previous paragraph until we obtain

$$(c_1, \ldots, c_{n+1}) \sim (k', \ldots, k', 0, \ldots, 0)$$

with $k' \neq 0$ and $\ell' < \ell$ nonzero entries. Since k is nonzero and $k \neq p - k$, we know the number of zeros will strictly increase with this process. Therefore we can repeat it until

$$(c_1, \ldots, c_n) \sim (k, 0, \ldots, 0)$$

for some nonzero k.

In the case that $\ell = n+1$ we have $c_i = c_1$ for all i. So

$$(c_1, \ldots, c_{n+1}) = (c_1, c_1, \ldots, c_1).$$

Finally, note that the action of ψ puts $(k, 0, \ldots, 0)$ in the same orbit as $(p - k, 0, \ldots, 0)$ and similarly puts $(k, k, \ldots, k)$ in the same orbit as $(p-k, p-k, \ldots, p-k)$, giving us at most $p-1$ nonzero orbits.

All that is left is to define the space $Y \ncong \tilde{N}_{pn+1}$ and its $(p-1)/2$ actions of C_p. Suppose $n \geq 1$. We have already described in Chapter III the $(p-1)/2$ non-isomorphic actions on both the Möbius band $\tilde{N}_1$ and R_p. Define Y to be the space obtained by removing p conjugate disks from $N_2^{\text{free}}\#_p N_{n-1}$. Recall from Proposition B.2.2 that there are $(p-1)/2$ non-isomorphic actions of C_p on N_2^{free}. This means there must be $(p-1)/2$ non-isomorphic actions of C_p on Y. Note that Y has p boundary components and thus $Y \ncong \tilde{N}_{pn+1}$. Moreover, for any of its $(p-1)/2$ C_p-actions, the quotient of Y by this action is in fact the non-equivariant surface $\tilde{N}_{n+1}$. □

C.3. Proof of Classification for Non-orientable Surfaces

Lemma C.3.1. *There is an equivariant isomorphism*

$$\text{Hex}_1 \#_p N_1 \cong N_1[1] + [R_p]$$

Proof. We will prove this result for the case $p = 3$, noting that $p > 3$ is similar. Figure 81 shows us how $(\text{Hex}_1 \#_3 N_1) - [R_3] \cong N_1[1]$. To begin, we represent $\text{Hex}_1 \#_3 N_1$ as our usual hexagon picture with fixed points a, b, and c as well as 3 crosscaps. A copy of EB containing a and b can be seen in red in the figure on the left. One can check that a tubular neighborhood of this EB has three boundary components and thus must be isomorphic to R_3. The middle of Figure 81 shows the result of removing this copy of R_3. To complete $-[R_3]$ surgery, we glue in the orange, pink, and green disks along the resulting boundary. To more easily see these identifications, we can first perform the intermediate step of "flipping" the red regions and identifying the yellow edges, then having the red change back to grey. The third picture on the right shows the result of the completed $-[R_3]$ surgery. The resulting space is isomorphic to $N_1[1]$. The original statement then follows from Lemma A.0.4. □

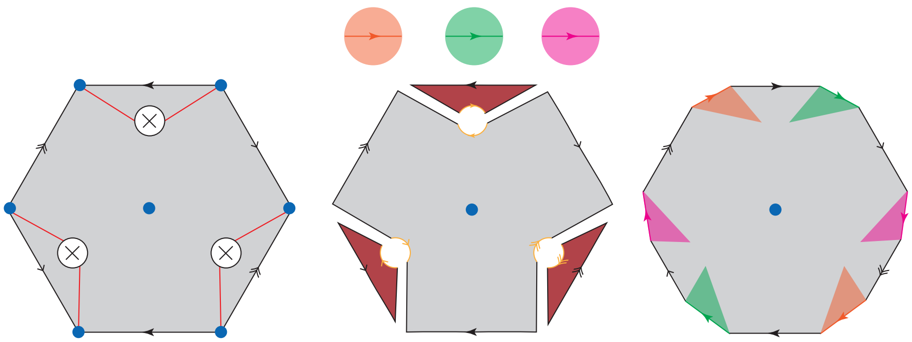


FIGURE 81. The procedure $(\text{Hex}_1 \#_p N_1) - [R_3]$.

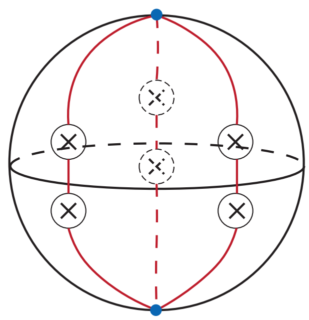


FIGURE 82. A copy of EB whose neighborhood is R_3.

Lemma C.3.2. *There is an equivariant isomorphism*

$$N_2^{free} + [R_p] \cong S^{2,1} \#_p N_2$$

Proof. If we perform $-[R_p]$ surgery on a neighborhood of the copy of EB from $S^{2,1}\#_p N_2$ shown in Figure 82, the result is a connected, non-orientable surface with a free C_p-action. Since $-[R_p]$-surgery reduces β-genus by $2(p-1)$, this surface must have genus $\beta = 2$. In particular, it must be N_2^{free} by our classification of free C_p spaces. It follows from Lemma A.0.4 that $N_2^{\text{free}} + [R_p] \cong S^{2,1}\#_p N_2$. □

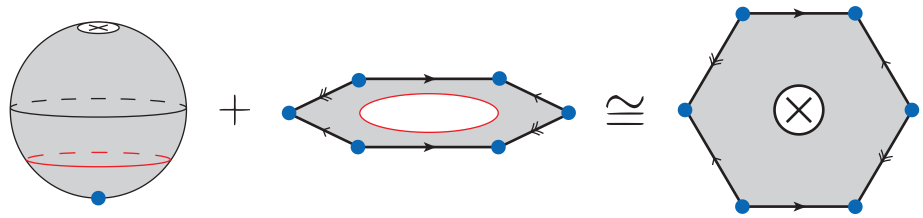


FIGURE 83. The equivariant surgery procedure $N_1[1]+[TR_3]$.

Lemma C.3.3. *There is an equivariant isomorphism*

$$N_1[1]+[TR_p]\cong S^{2,1}\#_p N_1$$

Proof. The C_p-space $\mathrm{Hex}_1+[FMB_p]$ can be constructed in two ways. In addition to performing $+[FMB_p]$ surgery on Hex_1, we could start by constructing $N_1[1]$ as $S^{2,1}+[FMB_p]$. We can then build $N_1[1]+[TR_p]$ by performing the $+[TR_p]$ surgery on the remaining fixed point. These two constructions are demonstrated in Figure 83.

Since both of these constructions yield the same space, it follows that $N_1[1]+[TR_p]\cong \mathrm{Hex}_1+[FMB_p]$. If we next remove a copy of R_p from $\mathrm{Hex}_1+[FMB_p]$ as shown in Figure 84, the result is $C_p\times MB$ where MB denotes the Möbius band. Thus, when we finish the $-[R_p]$ surgery on $\mathrm{Hex}_1+[FMB_p]$ by gluing in p disks on the boundary components, this leaves us with $C_p\times N_1$. Since $C_p\times N_1\cong(S^{2,1}\#_pN_1)-[R_p]$, we get that $\mathrm{Hex}_1+[FMB_p]\cong S^{2,1}\#_pN_1$ by Lemma A.0.4. □

We are now ready to restate and prove Theorem 32 for the classification of non-orientable C_p-surfaces.

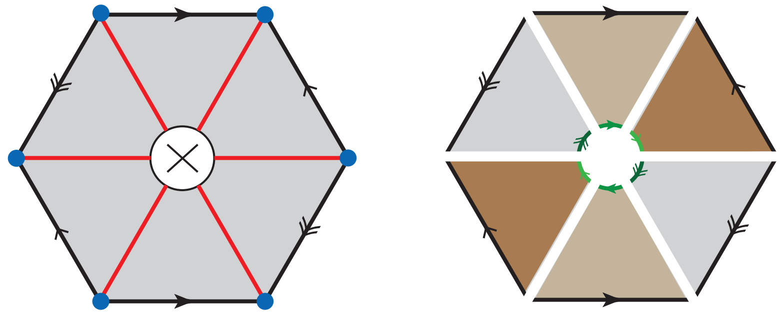


FIGURE 84. Removing R_3 from Hex_1 results in $MB \times C_3$.

Theorem C.3.4. *Let X be a connected, closed, non-orientable surface with an action of C_p. Then X can be constructed via one of the following surgery procedures, up to $\text{Aut}(C_p)$ actions on each of the pieces.*

1. $N_{2+pr}^{free} \cong N_2^{free} \#_p N_r,\ r \geq 0$
2. $N_{2(p-1)k+pr}[2k+2] \cong (S^{2,1} + k[R_p])\, \#_p N_r,\ r \geq 1$
3. $N_{1+2(p-1)k+pr}[1+2k] \cong (N_1[1] + k[R_p])\, \#_p N_r,\ k, r \geq 0$

Moreover, the space X is determined by F and β, with the condition that $F \equiv 2 - \beta \pmod{p}$.

Proof. We induct on the number of fixed points F of X.

First let X be a free non-orientable space. By the classification of free C_p-spaces, $X \cong N_{2+pr}^{\text{free}}$ for some $r \geq 0$.

Let X be a connected, closed, non-orientable C_p-surface with $F = 1$. Then X must have genus $pr + 1$ for some $r \geq 0$ by Lemma 4.0.1. Suppose Y is another closed, connected, genus $pr + 1$ non-orientable C_p-surface with a single fixed point. Let $\tilde{X}$ (respectively $\tilde{Y}$) denote the C_p-space $X \setminus D^{2,1}$ (respectively $Y \setminus D^{2,1}$) where $D^{2,1}$ is a neighborhood of the fixed point of X (respectively Y). Recall that

$\tilde{N}_{pr+1}$ has $(p-1)/2$ non-trivial, non-isomorphic C_p actions up to isomorphism by Proposition C.2.1. After altering the action on Y by $\mathrm{Aut}(C_p)$, we can make the action on $\partial\tilde{Y}$ match that on $\partial\tilde{X}$. Then $\tilde{X} \cong \tilde{Y}$, which extends to an equivariant isomorphism $X \to Y$. Thus there is only one non-orientable C_p-surface of genus $pr+1$ with $F=1$, so it must be isomorphic to $N_1[1]\#_p N_r$.

Suppose $F=2$. Let x and y be the two distinct fixed points of X. By Lemma C.1.1, there exists $R_p \subset X$ or $TR_p \subset X$ containing x and y. If there exists $R_p \subset X$ containing x and y, then $X-[R_p]$ is a free, non-orientable C_p-space. If $X - [R_p]$ is connected, then $X-[R_p] \cong N^{\mathrm{free}}_{2+pr}$ for some $r \geq 0$. So $X \cong N^{\mathrm{free}}_{2+pr} + [R_p] \cong S^{2,1}\#_p N_{r+2}$ by Lemma C.3.2. If $X-[R_p]$ is not connected, then it must be isomorphic to $N_{r'} \times C_p$ for some $r' \geq 1$. In this case, we can see that $X \cong S^{2,1}\#_p N_{r'}$.

Suppose instead we find that x and y are contained in some $TR_p \subset X$. Then $X -_{x,y} [TR_p]$ is a closed, connected, non-orientable C_p-surface with 1 fixed point. In particular, $X -_{x,y} [TR_p] \cong N_1[1]\#_p N_r$ for some r by what we already showed. Recall that equivariant connected sum surgery commutes with all types of C_p-ribbon surgeries. Since X is the result of $+[TR_p]$-surgery on $N_1[1]\#_p N_r$, Lemma C.3.3 tells us that

$$X \cong (N_1[1] + [TR_p])\,\#_p N_r \cong \left(S^{2,1}\#_p N_1\right)\#_p N_r \cong S^{2,1}\#_p N_{r+1}.$$

We next claim that for a closed, non-orientable C_p-surface with $F=2$, there exists a path α between the two fixed points so that a neighborhood of $\alpha \cup \sigma\alpha \cup \cdots \cup \sigma^{p-1}\alpha$ is isomorphic to TR_p. We just showed that $X \cong S^{2,1}\#_p N_r$ for some $r \geq 1$, so we can represent X by a copy of $S^{2,1}$ with pr crosscaps at the equator. Figure 85 shows a path α on X with the desired property in the case when $r=2$ and $p=3$. By Lemma C.1.2, there exists an automorphism of X swapping its fixed

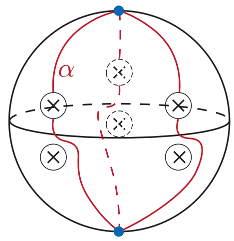


FIGURE 85. A choice of α whose conjugates have a neighborhood isomorphic to TR_3.

points. As in previous cases, this allows us to conclude that $+[TR_p]$ surgery on X is independent of the chosen fixed point.

For the inductive hypothesis, let $2 < \ell$. For any ℓ' with $2 \leq \ell' < \ell$, suppose that (1) if A is a connected, closed, non-orientable C_p-surface with $F = \ell'$, then Z is isomorphic to $N_{2(p-1)k+pr}[2k+2]$ or $N_{1+2(p-1)k+pr}[1+2k]$ for some k, r, and (2) if x and y in Z are distinct fixed points, then $Z +_x [TR_p] \cong Z +_y [TR_p]$. Now let X be a closed, non-orientable C_p-surface with $F = \ell$. Let $x, y \in X$ be distinct fixed points. By Lemma C.1.1, there exists $R_p \subset X$ or $TR_p \subset X$ containing x and y.

Suppose first that x and y are contained in $R_p \subset X$. Then $X - [R_p]$ has $\ell - 2 \geq 1$ fixed points and is thus connected by Lemma **??**. By the inductive hypothesis, $X - [R_p]$ is isomorphic to one of the following:

1. $N_{2(p-1)k+pr}[2k+2] \cong (S^{2,1} + k[R_p]) \#_p N_r$
2. $N_{1+2(p-1)k+pr}[1+2k] \cong (N_1[1] + k[R_p]) \#_p N_r$.

In the first case, we can conclude

$$X \cong \left(S^{2,1} + (k+1)[R_p]\right) \#_p N_r \cong N_{2(p-1)(k+1)+pr}[2(k+1)+2].$$

In the second case, we have

$$X \cong (N_1[1] + (k+1)[R_p]) \#_p N_r \cong N_{1+2(p-1)(k+1)+pr}[1+2(k+1)].$$

If x and y are contained in $TR_p \subset X$, then $X - [TR_p]$ has $\ell - 1 \geq 2$ fixed points. By the inductive hypothesis, $X - [TR_p]$ is isomorphic to one of the following:

1. $N_{2(p-1)k+pr}[2k+2] \cong (S^{2,1} + k[R_p]) \#_p N_r$ $(r \geq 1)$

2. $N_{1+2(p-1)k+pr}[1+2k] \cong (N_1[1] + k[R_p]) \#_p N_r$.

We also know from the inductive assumption that $+[TR_p]$-surgery on $X - [TR_p]$ is independent of the chosen fixed point, so $(X - [TR_p]) + [TR_p] \cong X$. Thus in the first case, we can choose to center our $+[TR_p]$ surgery on the north pole of $S^{2,1}$. Since $r \geq 1$, we have

$$\begin{aligned} X &\cong ((\mathrm{Hex}_1 \#_p N_1) + k[R_p]) \#_p N_{r-1} \\ &\cong (N_1[1] + (k+1)[R_p]) \#_p N_{r-1} \\ &\cong N_{1+2(p-1)(k+1)+p(r-1)}[1+2(k+1)] \end{aligned}$$

where the second isomorphism is by Lemma C.3.1 and the first isomorphism follows from the commutativity of $+[R_p]$-surgery and equivariant connected sum surgery. In the second case, we can choose to center our $+[TR_p]$ surgery on the fixed point originating from the copy of $N_1[1]$. By Lemma C.3.3, we get

$$X \cong \left(S^{2,1} + k[R_p]\right) \#_p N_{r+1} \cong N_{2(p-1)k+p(r+1)}[2k+2].$$

Next we will show that if x and y are distinct fixed points in X, then $X +_x [TR_p] \cong X +_y [TR_p]$. The case where $X \cong N_{2(p-1)k+pr}[2k+2]$ is nearly identical to the orientable case $\mathrm{Sph}_{(p-1)k+pg}[2k+2]$, so we will provide the proof of $+[TR_p]$ invariance only for $X \cong N_{1+2(p-1)k+pr}[1+2k]$.

We represent $N_{1+2(p-1)k+pr}[1+2k]$ by first choosing a disk D in $N_1[1]$ that does not intersect its conjugates. Then choose a representation of $N_{2(p-1)(k-1)+pr}[2(k-1)+2]$ using the same construction as for $\mathrm{Sph}_{(p-1)k+pg}[2k+2]$ in Lemma C.1.5. Next remove p disjoint conjugate disks $D', \sigma D', \ldots, \sigma^{p-1}D'$ from the equator of the sphere $S^{2,1}$ used to construct $N_{2(p-1)(k-1)+pr}[2(k-1)+2]$. Remove D and its conjugates from $N_1[1]$ and identify $\partial\sigma^i D$ with $\partial\sigma^i D'$ (renaming D' if necessary). Let c denote the fixed point in $N_1[1]$. We will show that for any other fixed point x there exists an equivariant automorphism of $N_{1+2(p-1)k+pr}[1+2k]$ which exchanges x and c. If we can show this, then composition of these automorphisms allows us to swap any two fixed points in $N_{2(p-1)(k-1)+pr}[2(k-1)+2]$.

Let $x \neq c$ be a fixed point in $N_{1+2(p-1)k+pr}[1+2k]$. Then x is either contained in the copy of $S^{2,1}$ or $(R_p)_i$ for some i. In any case, there exists a path α from x to c with a neighborhood of $\alpha \cup \sigma\alpha \cup \cdots \cup \sigma^{p-1}\alpha$ isomorphic to TR_p. Figure 86 shows how to construct such a path α when $x \in S^{2,1}$. Note that this figure does not show the $(R_p)_i$, but α can be constructed so that it does not intersect any of the $(R_p)_i$. Similarly, Figure 87 shows how to construct α when $x \in (R_p)_i$ for some i. The choice of α is similar for all i. Again note that α can be constructed so that it does not intersect $(R_p)_j$ when $j \neq i$. One can check that the paths depicted in these figures have a neighborhood isomorphic to TR_p by checking that the chosen neighborhood has a single boundary component. Since x and c are contained in a copy of $TR_p \subset N_{1+2(p-1)k+pr}[1+2k]$, we know from Lemma C.1.2 that there exists

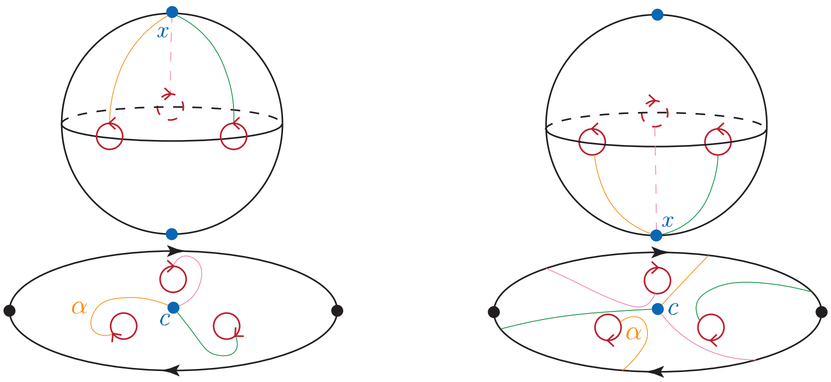


FIGURE 86. Choices of α whose conjugates have a neighborhood isomorphic to TR_3.

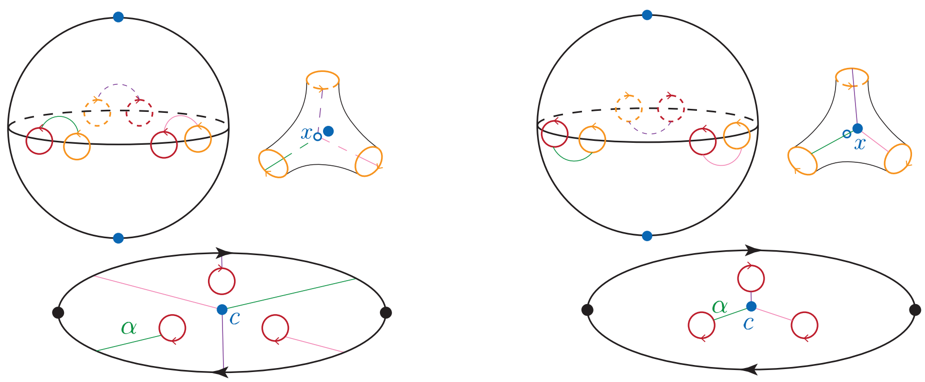


FIGURE 87. Choices of α whose conjugates have a neighborhood isomorphic to TR_3.

an automorphism of $N_{1+2(p-1)k+pr}[1+2k]$ swapping x and c. The result then follows from induction.

$\square$

Corollary C.3.5. *If X and Y are closed, connected, non-orientable C_p-surfaces with $X - [TR_p] \cong Y - [TR_p]$, then $X \cong Y$. In particular, $X +_x [TR_p]$ is independent of the choice of x.*

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