Rankin–Cohen-Type Operators on Hermitian Modular Forms

by

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DISSERTATION ABSTRACT

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We construct Rankin–Cohen-type differential operators on scalar-valued Hermitian modular forms, giving an explicit construction in signature (n, n). The bilinear differential operators given here specialize to the original Rankin–Cohen operators in the case $n = 1$, and to the operators previously constructed in some small cases in this setting by Böcherer–Das and Martin–Senadheera. Our approach builds on previous work by Eholzer–Ibukiyama in the case of Siegel modular forms, together with results of Kashiwara–Vergne on the representation theory of unitary groups. We then connect the results here to those of Ban in a similar setting.

This dissertation includes previously published material.

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CHAPTER 1

INTRODUCTION

1.1 Overview

Modular forms, or more generally automorphic forms, are central objects in modern number theory. They have a broad range of applications. Early in their development, modular forms were studied as generating functions encoding arithmetic data. More recently, they most famously played a vital role in the proof Fermat's Last Theorem. They also provide a link between algebraic number theory and representation theory in the Langlands program.

In the classical setting, modular forms can be viewed as holomorphic functions on the complex upper half-plane satisfying specific transformation properties under an action of $\mathrm{SL}_2(\mathbb{Z})$. The derivative of a modular form is not in general a modular form itself, since it fails to satisfy the same transformation properties. However, in 1956 R. A. Rankin was able to describe differential operators sending modular forms to modular forms [35]. H. Cohen exhibited a special case in [10], introducing particular bilinear differential operators on the graded ring of modular forms. These operators (or a version of them) originated in the study of invariant theory in the work of Gordan in the 19th century, and have appeared since in a variety of areas including Lie theory and harmonic analysis. A more detailed account of their history can be found in [19, §1].

Within number theory, these Rankin–Cohen operators or Rankin–Cohen brackets are of interest since they provide the unique way of combining derivatives of two modular forms to produce another modular form of a higher weight. That is, for modular forms f and g of weights k and ℓ respectively, for each integer $v > 0$ the Rankin–Cohen bracket indexed by k, ℓ and v is the unique bilinear differential operator (i.e. a bilinear combination of the derivatives of f and g), up to rescaling, which gives a modular form of weight $k + \ell + 2v$. Representation-theoretically, the Rankin–Cohen operators explicitly describe the decomposition of the tensor product of the $\mathfrak{sl}_2$-representations generated by f and g. Furthermore, for $v > 0$ the resulting modular forms are cusp forms, which hold special significance in the literature. In this paper we focus on a higher-dimensional generalization, the case of scalar-valued Hermitian modular forms of signature (n, n), and produce analogous results in this new setting.

1.2 Main Results

We prove the following results for Hermitian modular forms of signature (n, n) under some technical conditions on the weights $k_1, \ldots, k_r$ and v below:

Theorem 1.1 (Imprecise version of Theorem 4.1 and Corollary 4.3.)**.** *Covariant multilinear differential operators sending r scalar-valued Hermitian modular forms of integer weights $k_1, \ldots, k_r$ respectively to a scalar-valued Hermitian modular form of weight $2v + \sum_{i=1}^{r} k_i$ for a fixed $v \geq 0$ correspond to a certain space of homogeneous pluri-harmonic polynomials (as defined in §3.2).*

We also prove a more general version of Theorem 1.1 for scalar-valued Hermitian modular forms of signature (p, q) in Theorem 4.7 and Corollary 4.9.

Theorem 1.2 (Imprecise version of Theorem 5.6.)**.** *In the case $r = 2$, for each $v > 0$ and $n > 1$ there is a unique (up to rescaling) differential operator sending scalar-valued Hermitian modular forms of weights k_1 and k_2 respectively to a Hermitian modular form of weight $k_1 + k_2 + 2v$. We also give integral linear relations satisfied by the coefficients of the differential operators.*

As a consequence we extract that for $v > 0$ the image of these bilinear differential operators is contained in the space of cusp forms in Proposition 5.9. In addition, since the relations between coefficients of the operators are integral, the ring of definition for the modular forms is preserved.

This paper builds on several previous works generalizing Rankin–Cohen-type operators beyond the classical setting. Choie–Eholzer [7] used Jacobi forms to construct Rankin–Cohen-type operators for Siegel modular forms. These results were later generalized by Eholzer–Ibukiyama [14] and Ibukiyama [24] using a different approach involving theory of pluriharmonic polynomials, in particular the work of Kashiwara–Vergne [26]. In the case of Hermitian modular forms, Böcherer–Das [2] used direct computational methods to produce Rankin–Cohen-type operators in signature $(2, 2)$, and one in signature (n, n) raising the total weight by 2 (i.e. from weights k_1 and k_2 to weight $k_1 + k_2 + 2$). Martin–Senadheera [33] used Jacobi forms to produce analogues of Rankin–Cohen brackets for Hermitian modular forms over the field extension $\mathbb{Q}(i)/\mathbb{Q}$ and signature $(2, 2)$. Ban [1] constructed Rankin–Cohen-type operators (non-explicitly) for both Siegel modular forms and certain Hermitian modular forms on $\mathrm{SU}(p, q)$ (although his results in fact apply to $\mathrm{U}(p, q)$, see

Chapter 6) from the perspective of representation theory, again making use of [26]. Note that while Ban proves a version of Theorem 1.1, the methods are very different to those in this paper.

This paper follows a similar approach to [14, 24], again using results of [26], to produce Rankin–Cohen-type operators on Hermitian modular forms in a more general setting than before. We then explain the connection to [1] to give a representation-theoretic interpretation of these results.

1.3 Connections to Other Results

Due to the different areas in which Rankin–Cohen operators appear, and the varied terms they go by, it is difficult to give an exhaustive list of the papers relating to them. Here we give a selection of closely related works, focused within number theory.

In the classical setting, the Rankin–Cohen brackets endow rich algebraic structure on the graded algebra of modular forms as described by Zagier [42], and generalized to a study of "automorphic pseudo-differential operators" by Cohen–Manin–Zagier [11]. Their algebraic and representation-theoretic properties have been the subject of further works, for example by El Gradechi [19]. Also from the perspective of representation theory, Rankin–Cohen brackets are a special example of "symmetry-breaking operators" in the sense of Kobayashi. This viewpoint has been studied by Kobayashi–Pevzner [27, 28, 29] and by Clerc [8, 9] in terms of Jordan algebras.

Rankin–Cohen-type differential operators have been valuable tools to study and prove results concerning automorphic forms. They have found use in settings such as the study of mod p differential operators [15], and generalizing Garrett's "pullback formula" to the Hermitian setting [41]. Furthermore, the image of the Rankin–Cohen operators in the elliptic and Siegel settings has been investigated in, for example, [31] and [25]. Similar investigations could be now be carried out for the operators constructed in this paper.

1.4 Organization

In Chapter 2 we shall give an account of the situation for classical modular forms. This can be viewed as the motivating story which we seek to generalize throughout the rest of this paper. Then, in Chapter 3 we give the relevant defini-

tions for Hermitian modular forms and introduce the particular spaces of polynomials of interest to us. Chapters 4 and 5 contain work from Francis Dunn, *Rankin–Cohen Type Differential Operators on Hermitian Modular Forms*, Res. Number Theory **10** (2024), no. 88. In Chapter 4 we establish the correspondence between differential operators on Hermitian modular forms and certain polynomials, stating and proving the precise version of Theorem 1.1. We prove Theorem 1.2 in Chapter 5 by giving an explicit description of the particular space of polynomials that we are interested in, and finding linear relations that coefficients of the specific polynomials we care about must satisfy. We also consider the effect of our differential operators on Fourier expansions of Hermitian modular forms, and prove that the non-trivial bilinear operators produce cusp forms. Finally, we connect the work presented here to the representation-theoretic viewpoint in Chapter 6. In particular, we deduce that the operators constructed give generators in the decomposition of a certain representation.

CHAPTER 2

CLASSICAL MODULAR FORMS AND RANKIN–COHEN BRACKETS

The aim of this chapter is to provide an overview of the motivating story for classical modular forms, which we shall spend the remainder of this paper generalizing to higher dimensions. The goal is not to give a complete exposition of all the background material involved, nor develop all of the connections that arise. As such, only essential definitions shall be given, and results in this chapter will be stated without proof. The interested reader is invited to consult the references for further details.

2.1 Modular Forms as Holomorphic Functions

First, we shall describe modular forms as certain holomorphic functions. The definitions given here are very standard. We do not follow any one particular source, but there are many good references including, for example, [30, Chapter 3] or [6, Chapter 1].

Definition 2.1. We shall denote by $\mathcal{H}$ the complex upper half-plane, that is

$$\mathcal{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}.$$

Definition 2.2. The special linear group with coefficients in $\mathbb{R}$ for dimension 2 is

$$\mathrm{SL}_2(\mathbb{R}) := \{g \in M_{2,2}(\mathbb{R}) \mid \det(g) = 1\},$$

where $M_{2,2}(\mathbb{R})$ is the space of two-by-two matrices with coefficients in $\mathbb{R}$.

The group $\mathrm{SL}_2(\mathbb{R})$ (or indeed, $\mathrm{SL}_2(\mathbb{C})$) acts on $\mathbb{C} \cup \{\infty\}$ by Möbius transformations. Explicitly, a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ acts on $z \in \mathbb{C}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \begin{cases} \frac{az+b}{cz+d} & cz + d \neq 0 \\ \infty & cz + d = 0 \end{cases},$$

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \infty = \frac{a}{c}$. This restricts to an action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}$ since $\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathcal{H}$. This is a transitive action; for any two

elements $z, w \in \mathcal{H}$ there is a matrix $g \in \mathrm{SL}_2(\mathbb{R})$ such that $g \cdot z = w$. We shall be particularly interested in the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$. Note that upon restricting coefficients to $\mathbb{Z}$, this action is no longer transitive.

Definition 2.3 (Modular Form)**.** A *modular form* (also called a *classical modular form*, or an *elliptic modular form*) is a function $f : \mathcal{H} \to \mathbb{C}$ such that:

(i) The function f is holomorphic.

(ii) The function f is *modular* (with respect to $\mathrm{SL}_2(\mathbb{Z})$). That is, for a non-negative integer k and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we put

$$(f|_k\gamma)(z) := (cz + d)^{-k} f(\gamma \cdot z).$$

Then, we require that

$$f|_k\gamma = f$$

for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. The integer k is called the *weight* of f.

(iii) The function f is *of moderate growth* (or is *holomorphic at infinity*), that is for $x, y \in \mathbb{R}$ and $y > 0$ we have that $f(x + iy)$ is bounded as $y \to \infty$.

We denote by M_k the space of modular forms of weight k. If $f : \mathcal{H} \to \mathbb{C}$ satisfies the second and third properties and is infinitely real-differentiable (but not necessarily holomorphic) we say that f is a *modular function*.

Remark 2.4. The expression $cz + d$ is often called an *automorphy factor*, and we write $j(\gamma, z) := (cz + d)$ with γ as above. So condition (ii) above may be written

$$j(\gamma, z)^{-k} f(\gamma \cdot z) = f(z).$$

Note that for $g_1, g_2 \in \mathrm{SL}_2(\mathbb{R})$, j satisfies

$$j(g_1 g_2, z) = j(g_1, g_2 \cdot z) j(g_2, z).$$

Remark 2.5. The above definition is for "level 1" modular forms, since we require property (ii), modularity, for all of $\mathrm{SL}_2(\mathbb{Z})$. One can in general consider modular forms of level Γ (denoted $M_k(\Gamma)$) for a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, where one only requires modularity holds for matrices inside Γ. Modular forms of levels other than 1 are in general interesting and meaningful objects to study, however for what

is to follow the level actually plays no part (which is perhaps interesting in itself). So, we may refer to other levels for generality, but it suffices to stay in the level 1 case, and often no level will be specified. One may also consider modular forms of non-integral weight; this does affect the situation and so we restrict ourselves to integral weights for other reasons. In the level 1 setting, the integral weights for which there are non-zero modular forms are the even non-negative integers excluding 2, and modular forms in weight 0 are precisely the constant functions.

For a modular form f, the moderate growth property (condition (iii) above) is equivalent to f having a Fourier expansion

$$f(z) = \sum_{n \geq 0} a_n e^{2\pi i n z} \qquad a_n \in \mathbb{C}.$$

This is often written as $\sum_{n\geq 0} a_n q^n$ with $q = e^{2\pi i z}$, and thus commonly called a "q-expansion".

Definition 2.6 (Cusp Form)**.** A modular form f is called a *cusp form* if the constant coefficient a_0 in the Fourier expansion of f is equal to zero. We write S_k (resp. $S_k(\Gamma)$) for the space of cusp forms of weight k and level 1 (resp. level Γ).

Cusp forms hold special significance in the literature. The reasons why are outside the scope of this paper, but this definition is made so that it can be pointed out when they arise.

2.2 Rankin–Cohen Brackets as Differential Operators

Given a modular form f of weight k, since f is holomorphic (and thus infinitely differentiable) one might consider its derivative $f'(z) = \frac{df}{dz}$. This is again a holomorphic function on $\mathcal{H}$ of moderate growth, but is no longer modular. Indeed, some elementary calculus shows us that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$f'(z) = (cz+d)^{-(k+2)} f'(\gamma \cdot z) - kc(cz+d)^{-(k+1)} f(\gamma \cdot z).$$

The first term on the right hand side matches what one would need for f' to be modular of weight $k+2$, but the second "error" term prevents this. There are a couple of ways one could recover modularity. One way (which we shall return to later) is to use a *Maass–Shimura operator* δ_k, defined as

$$\delta_k f(z) := \frac{1}{2\pi i} f'(z) - \frac{k}{4\pi y} f(z), \qquad \delta_k^{r+1} f = \delta_{k+2r} \circ \cdots \circ \delta_{k+2} \circ \delta_k f,$$

where $y = \mathrm{Im}(z)$. The "correction term" $\frac{k}{4\pi y} f$ ensures that $\delta_k f$ is indeed modular of weight $k+2$. However, since this expression involves y, $\delta_k f$ is no longer holomorphic in general. Since δ_k is not an expression purely in $\frac{d}{dz}$, some authors refer to this as a *pseudo-differential operator*. To preserve holomorphicity, one would instead prefer to work with differential operators in this stricter sense, only applying certain combinations of $\frac{d}{dz}$. This is the approach we now take. Rankin [35] was able to describe differential operators sending modular forms to modular forms, and later Cohen [10] explicitly exhibited the special case of such bilinear operators, which we now define.

Definition 2.7 (Rankin–Cohen Brackets)**.** For each pair of weights k_1 and k_2, and non-negative integer v, we define the *Rankin–Cohen bracket* applied to two modular forms f_1 of weight k_1 and f_2 of weight k_2 as

$$[f_1, f_2]_v^{(k_1,k_2)}(z) := [f_1, f_2]_v(z) := \sum_{r=0}^{v} (-1)^r \binom{v+k_1-1}{v-r} \binom{v+k_2-1}{r} f_1^{(r)}(z) f_2^{(v-r)}(z),$$

where by $f^{(r)}(z)$ we mean the renormalized r-th derivative $\left(\frac{1}{2\pi i}\right)^r \frac{d^r f}{dz^r}$. This particular normalization is chosen to preserve the ring of definition for the Fourier coefficients. Note that the operator above does depend on the initial weights k_1 and k_2, but this is often suppressed in the notation.

These Rankin–Cohen brackets satisfy several interesting properties. For now, we conclude this section by highlighting three of them.

1. With f_1 and f_2 as above, the function $[f_1, f_2]_v$ is indeed a modular form, and its weight is $k_1 + k_2 + 2v$.

2. Furthermore, if $v > 0$ then $[f_1, f_2]_v$ is always cusp form.

3. The Rankin–Cohen bracket $[\cdot,\cdot]_v^{(k_1,k_2)} = [\cdot,\cdot]_v$ is the unique (up to rescaling) bilinear differential operator $\mathbf{D}$ which, acting on any pair of holomorphic functions $\mathbb{H} \to \mathbb{C}$, satisfies

$$\mathbf{D}(\cdot|_{k_1}\gamma, \cdot|_{k_2}\gamma) = \mathbf{D}(\cdot,\cdot)|_{k_1+k_2+2v}\gamma \qquad \forall \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

2.3 Modular Forms as Functions on the Group

Viewing modular forms as functions on a Lie group allows us more tools to study them, specifically the representation theory of Lie algebras. One of the early instances of this perspective occurs in the work of Harris [22], from which we take inspiration for the following introduction. A significantly more detailed treatment can be found in [6, Chapter 2].

The group $\mathrm{SL}_2(\mathbb{R})$ is a Lie group, meaning that as well as a group it also has the structure of a manifold (over $\mathbb{R}$), and the group operations are continuous maps on this manifold. We may alternatively view modular forms as functions on the group $\mathrm{SL}_2(\mathbb{R})$ itself, using the fact that the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}$ is transitive. For a function $f : \mathcal{H} \to \mathbb{C}$ and an integer k we can define a corresponding function on $\mathrm{SL}_2(\mathbb{R})$ by

$$\phi_{k,f} : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C} \qquad \phi_{k,f}(g) = j(g,i)^{-k} f(g \cdot i).$$

If f is modular of weight k then a straightforward calculation shows that for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$

$$\phi_{k,f}(\gamma g) = \phi_{k,f}(g).$$

So in this case we may view $\phi_{k,f}$ as a function on the quotient space $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ (or, more generally, $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ if f has level Γ). The stabiliser of $i \in \mathcal{H}$ in $\mathrm{SL}_2(\mathbb{R})$ is a maximal compact subgroup, specifically the group of rotation matrices $\mathrm{SO}_2(\mathbb{R}) = \left\{ r_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}$, and so we must have that

$$\phi_{k,f}(g r_\theta) = j(g,i)^{-k} j(r_\theta, i)^{-k} f(g \cdot i) = e^{-ik\theta} \phi_{k,f}(g). \tag{2.1}$$

Conversely, given a function $\phi : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ such that $\phi(\gamma g) = \phi(g)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\phi(g r_\theta) = e^{-ik\theta}\phi(g)$ for all $r_\theta \in \mathrm{SO}_2(\mathbb{R})$, and some fixed integer k, we can define

$$f_{k,\phi} : \mathcal{H} \to \mathbb{C} \qquad f_{k,\phi}(z) = j(g,i)^k \phi(g),$$

where g is any element of $\mathrm{SO}_2(\mathbb{R})$ such that $g \cdot i = z$. Then $f_{k,\phi}$ is well-defined and satisfies the modularity condition in weight k. When the weight is clear from the context, we shall simply write f_ϕ or ϕ_f. Smooth (meaning infinitely real-differentiable) functions on $\mathcal{H}$ correspond to smooth functions on the manifold $\mathrm{SL}_2(\mathbb{R})$, as well as an equivalence of "moderate growth" conditions. So we have described a correspondence between modular functions and smooth, moderate-growth functions on

$\mathrm{SL}_2(\mathbb{R})$ left-invariant under $\mathrm{SL}_2(\mathbb{Z})$ (or, more generally, Γ) and satisfying a transformation property under right-translation by $\mathrm{SO}_2(\mathbb{R})$. We shall denote the space of such functions with weight k (as in (2.1)) by $C^\infty_{k,\mathrm{mod}}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R}), SO_2(\mathbb{R}))$, or, for the space of such functions of any weight, by $C^\infty_{\mathrm{mod}}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R}), SO_2(\mathbb{R}))$.

Observe that this identification does depend on our choice of basepoint $i \in \mathcal{H}$, but any other would be the same up to conjugation inside $\mathrm{SL}_2(\mathbb{R})$. Note that our picture is not quite complete as we have not yet encoded the holomorphicity of a modular form f as a property of the corresponding function on the group ϕ_f.

2.4 Lie Theory for Modular Forms

Definition 2.8 (Lie Algebra)**.** Abstractly, a *(real or complex) Lie algebra* is a vector space $\mathfrak{g}$ (over $\mathbb{R}$ or $\mathbb{C}$) together with an alternating bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity. That is, for any $X, Y, Z \in \mathfrak{g}$ we have

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

A representation of a Lie algebra is an action of $\mathfrak{g}$ on a vector space V compatible with the Lie bracket $[,]$. That is, a representation is (ρ, V) with

$$\rho : \mathfrak{g} \to \mathrm{End}(V) \qquad \text{such that} \qquad \rho([X,Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X) \quad \forall X, Y \in \mathfrak{g},$$

where $[X,Y]$ is the Lie bracket of X and Y inside $\mathfrak{g}$.

Definition 2.9 (Universal Enveloping Algebra)**.** For a Lie algebra $\mathfrak{g}$ over a base field L, recall that the tensor algebra of $\mathfrak{g}$ is given by $T(\mathfrak{g}) = \bigoplus_{i=0}^\infty \mathfrak{g}^{\otimes i}$, where $\mathfrak{g}^{\otimes 0} = L$. We can extend the Lie bracket on $\mathfrak{g}$ to $T(\mathfrak{g})$ by (recursively) taking

$$[X \otimes Y, Z] = X \otimes [Y,Z] + [X,Z] \otimes Y \qquad \text{and} \qquad [X, Y \otimes Z] = [X,Y] \otimes Z + Y \otimes [X,Z].$$

Then the *universal enveloping algebra* of $\mathfrak{g}$ is

$$\mathbf{U}(\mathfrak{g}) = T(\mathfrak{g})/\sim, \quad \text{where } [X,Y] \sim X \otimes Y - Y \otimes X \quad \forall X, Y \in T(\mathfrak{g}).$$

Typically, when writing elements of $\mathbf{U}(\mathfrak{g})$, we will omit the tensor products and simply write e.g. $XY - YX$ for $X \otimes Y - Y \otimes X$.

Remark 2.10. The role of the universal enveloping algebra for a Lie algebra $\mathfrak{g}$ is, in some ways, analogous to that of the group ring for a finite group. Of particular note for our purposes is that representations of $\mathfrak{g}$ are the same as modules over

$\mathbf{U}(\mathfrak{g})$. In addition, the following well-known theorem gives a straightforward description of elements of $\mathbf{U}(\mathfrak{g})$.

Theorem 2.11 (Poincaré–Birkhoff–Witt (PBW))**.** *If $X_1, \ldots, X_n$ is an ordered basis for a finite-dimensional Lie algebra $\mathfrak{g}$, then the universal enveloping algebra $\boldsymbol{U}(\mathfrak{g})$ has a basis given by*

$$\{X_1^{a_1} \ldots X_n^{a_n} \mid a_j \in \mathbb{Z}_{\geq 0}, j = 1, \ldots, n\}.$$

This is often called a PBW basis for $\boldsymbol{U}(\mathfrak{g})$.

Example 2.12. To any Lie group G we have an associated Lie algebra obtained by taking the tangent space at the identity, $\mathrm{Lie}\, G = T_I G$. Given a Lie group G, we shall often write $\mathfrak{g}$ for $\mathrm{Lie}\, G$. In our current setting we have the (real) Lie group $G = \mathrm{SL}_2(\mathbb{R})$, which has (real) Lie algebra

$$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) = \{X \in M_{2,2}(\mathbb{R}) \mid \mathrm{tr}(X) = 0\}.$$

The Lie bracket in this realization is precisely the commutator of two matrices, consistent with the commonly-used notation for $[g, h]$ as the commutator. For the purposes herein, unless explicitly stated otherwise all Lie groups will be *matrix Lie groups* (Lie groups which are closed subgroups of $\mathrm{GL}(n)$), and all Lie algebras will arise as the tangent spaces of these matrix Lie groups. Throughout this chapter we will mainly be concerned with this specific example of SL_2 and $\mathfrak{sl}_2$ for classical modular forms.

Definition 2.13 (Complexification)**.** For a Lie algebra $\mathfrak{g}$, the *complexification of* $\mathfrak{g}$ is $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. (In our example, if $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$.)

Definition 2.14 (Adjoint action)**.** For a matrix Lie group G with Lie algebra $\mathfrak{g}$, we have the Adjoint action of G on $\mathfrak{g}$ given by

$$\mathrm{Ad}(g)(X) := \mathrm{Ad}_g(X) := gXg^{-1} \qquad \text{for } g \in G, X \in \mathfrak{g}.$$

This is well-defined, $gXg^{-1} \in \mathfrak{g}$, and the Adjoint action on $\mathfrak{g}$ extends to one on $\mathfrak{g}_{\mathbb{C}}$.

In our setting, let $K = \mathrm{SO}_2(\mathbb{R})$, a maximal compact subgroup of $\mathrm{SL}_2(\mathbb{R})$ as before. Then K is a Lie subgroup of $\mathrm{SL}_2(\mathbb{R})$, and put $\mathfrak{k}$ for its Lie algebra (which is a Lie subalgebra of $\mathfrak{sl}_2(\mathbb{R})$ realised by the corresponding inclusion of tangent spaces).

Then the Adjoint action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathfrak{sl}_2(\mathbb{C})$ restricts to an action of K on $\mathfrak{sl}_2(\mathbb{C})$, and this decomposes as

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-.$$

Here, $\mathfrak{k}_{\mathbb{C}} = \mathbb{C} \cdot H$, $\mathfrak{p}_+ = \mathbb{C} \cdot E$ (sometimes called *raising operators*) and $\mathfrak{p}_- = \mathbb{C} \cdot F$ (*lowering operators*) where

$$E = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad F = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},$$

and we have that

$$\mathrm{Ad}(r_\theta)(E) = e^{-2i\theta}E, \qquad \mathrm{Ad}(r_\theta)(H) = H, \qquad \mathrm{Ad}(r_\theta)(F) = e^{2i\theta}F.$$

Also, note that

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

This decomposition makes the choice of basis $\{E, H, F\}$ for $\mathfrak{sl}_2(\mathbb{C})$ a useful one which we shall keep referring to. This is conjugate inside $\mathrm{SL}_2(\mathbb{C})$ to the (in some settings) more standard choice of basis for $\mathfrak{sl}_2(\mathbb{C})$ consisting of

$$\hat{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \hat{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{F} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

via the change-of-basis given by $C = -\frac{1+i}{2}\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}$ and $C^{-1} = \frac{1+i}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ (so $\hat{E} = CEC^{-1}$ etc.).

Definition 2.15 (Exponential map)**.** For a real or complex matrix Lie group G with Lie algebra $\mathfrak{g}$, there is an *exponential map*

$$\exp : \mathfrak{g} \to G \qquad \exp(X) = I + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + \cdots = \sum_{j=0}^{\infty} \frac{1}{j!}X^j.$$

This is a continuous mapping which indeed converges (in fact for any $n \times n$ real or complex matrix), and we have that $\exp(\mathrm{Ad}(g)(X)) = \exp(gXg^{-1}) = g\exp(X)g^{-1}$ (see, for example, [21, §2.1]).

Via the exponential map we have an action of $\mathfrak{g}$ on the space of smooth functions $\phi : G \to \mathbb{C}$ given by

$$X * \phi(g) := \frac{d}{dt}\phi(g \exp(tX))|_{t=0} \quad g \in G, X \in \mathfrak{g}.$$

Note this action preserves modularity, since if ϕ is left-invariant by a subgroup $\Gamma \leq G$ then so is $X * \phi$. We also claim that if ϕ is of moderate growth, then so is $X * \phi$. Furthermore, if G and $\mathfrak{g}$ are real, we can extend this map linearly to the complexification $\mathfrak{g}_{\mathbb{C}}$.

Now, put $C_k^\infty(G, K)$ for the set of smooth functions ϕ on $G = \mathrm{SL}_2(\mathbb{R})$ satisfying the transformation property with respect to K as in (2.1). Then, with E, F, H as above and $\phi \in C_k^\infty(G, K)$, short calculations show that

$$E * \phi \in C_{k+2}^\infty(G, K), \quad H * \phi = k\phi \in C_k^\infty(G, K), \quad \text{and } F * \phi \in C_{k-2}^\infty(G, K).$$

Therefore, we can translate the action of these particular elements to operations on modular functions. For a modular function f of weight k we can compute that the action of E on $\phi = \phi_f$ corresponds to

$$E * \phi \leftrightarrow f_{k+2, E*\phi} = 2i\frac{df}{dz} + \frac{k}{y}f,$$

where we write $z = x + iy$ as usual. Note that this is (a rescaling of) the Maass–Shimura operator δ_k mentioned earlier. Similarly, it can be calculated that

$$F * \phi \leftrightarrow f_{k-2, F*\phi} = -2iy^2\frac{df}{d\bar{z}}.$$

We therefore have the following well-known lemma (e.g. [22, Lemma 1] or [19, Prop. 2.20]):

Lemma 2.16. *For a modular function f of weight k and the corresponding function on the group $\phi_{k,f}$, f is holomorphic (and therefore a modular form) if and only if $F * \phi_{k,f} = 0$.*

So, we can now complete our correspondence between modular forms and functions on the group itself; modular forms of weight k and level Γ correspond precisely to functions $\phi \in C_{k,\mathrm{mod}}^\infty(\Gamma\backslash G, K)$ such that $\mathfrak{p}_- * \phi = 0$, ϕ is killed by all the lowering operators in the Lie algebra. Sometimes it is said that such a ϕ is *of holomorphic type*.

Definition 2.17 ($(\mathfrak{g}, K)$-modules)**.** The actions of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$ and $K = \mathrm{SO}_2(\mathbb{R})$ on $C_k^\infty(G, K)$ as above makes $C^\infty(G, K)$ (or, restricting, $C^\infty(\Gamma\backslash G, K)$ or $C^\infty_{\mathrm{mod}}(\Gamma\backslash G, K)$) a $(\mathfrak{g}_{\mathbb{C}}, K)$-*module*. More generally, for any reductive Lie group G with Lie algebra $\mathfrak{g}$ and maximal compact subgroup K, a $(\mathfrak{g}, K)$-module is a vector space V endowed with representations of $\mathfrak{g}$ and K satisfying the following compatibility conditions.

(i) Under the action of K, V must decompose into a direct sum of finite-dimensional invariant subspaces.

(ii) For any $X \in \mathfrak{k} = \mathrm{Lie}\ (K)$ and $f \in V$, $X \cdot f = \frac{d}{dt}(\exp(tX) \cdot f)|_{t=0}$.

(iii) For any $g \in K$, $X \in \mathfrak{g}$, and $f \in V$, $g \cdot X \cdot g^{-1} \cdot f = (\mathrm{Ad}(g)(X)) \cdot f$.

We have not defined what it means for a Lie group to be *reductive*, nor for a $(\mathfrak{g}, K)$-module to be *admissible* and *unitary* (the "correct" subcategory of representations to consider), but in this paper all the Lie groups and all the $(\mathfrak{g}, K)$-modules which appear shall, respectively, be so. More details can be found in, for example, [6, §2.4-2.5].

Remark 2.18 (Weights). Those more familiar with Lie theory may be used to *weights* as eigenvalues of, or linear functionals on, a particular subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. In our current setting, take $\mathfrak{h} = \mathfrak{k}_{\mathbb{C}} = \mathbb{C} \cdot H$ and recall from above that $H * \phi = k\phi$ for $\phi \in C_k^\infty(G, K)$. That is, if ϕ is weight k (using our previous definition) then ϕ is also a weight vector of weight k or $(H \mapsto k)$ in the Lie-theoretic sense, so the two definitions agree here. More generally, the weight of a modular form (or automorphic form) in dimension n will be a representation ρ of $\mathrm{GL}(n, \mathbb{C})$ (we could think of "weight k" above as "weight $\det^k$" as a representation of $\mathrm{GL}(1, \mathbb{C})$). This does not quite match up with the Lie-theoretic definition. However, since ρ will be a weight representation in this paper, the discrepancy is not too large. In what follows, therefore, we simply use "weight" without too much fear of ambiguity. Those less familiar with Lie theory may safely disregard this remark.

2.5 Rankin–Cohen Brackets as Representation-theoretic Operators

Although Rankin–Cohen brackets have appeared implicitly in this context in, for example, [22], to the author's knowledge the particular characterization we now give was first described by El Gradechi [19] (although it may have been known earlier) and soon afterwards appeared in related works such as [32] and [1].

For a non-constant modular form f of weight k and level 1 (so $k \in 2\mathbb{Z}_{\geq 4}$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$), consider the subrepresentation or sub-$(\mathfrak{g}_{\mathbb{C}}, K)$-module inside $C^\infty(G, K)$ (indeed, inside $C^\infty_{\mathrm{mod}}(\Gamma\backslash G, K)$) generated by $\phi = \phi_f \in C^\infty_{k,\mathrm{mod}}(G, K)$. We denote this by ρ_ϕ, and the corresponding (isomorphic) representation inside $C^\infty_{\mathrm{mod}}(\mathbb{H})$ by ρ_f. Since representations of the Lie algebra are the same as modules over the universal enveloping algebra $\rho_\phi = \mathbf{U}(\mathfrak{g}_{\mathbb{C}}) * \phi$, and ρ_f is comprised of the corresponding functions $\{f_{X*\phi} \mid X \in \mathbf{U}(\mathfrak{g}_{\mathbb{C}})\}$. By considering the action of elements of the PBW basis for $\mathbf{U}(\mathfrak{g}_{\mathbb{C}}) = \mathbf{U}(\mathfrak{sl}_2(\mathbb{C}))$ of the form $E^v H^w F^x$ (with E, H, F as before), we can see that ρ_ϕ has a basis given by $\{E^v * \phi \mid v \in \mathbb{Z}_{\geq 0}\}$ since $F * \phi = 0$ and H acts on ϕ by a scalar. As noted previously, the weight of $E^v * \phi$ is $k + 2v$. In fact, ρ_ϕ is an irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-module isomorphic to the so-called *(upwards) Verma module* of lowest weight k, which we shall denote by V_k. Under the isomorphism, the choice of abstract generator ("lowest weight vector") of V_k maps to ϕ. (See, for example, [22, §2] or [6, §2.5].) We shall denote by $\rho(\lambda)$ the piece of the $(\mathfrak{g}, K)$-module ρ with weight λ. So, for example, we have

$$\rho_\phi(\lambda) = \begin{cases} \mathbb{C} \cdot E^v * \phi & \text{if } \lambda = k + 2v, v \in \mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise.} \end{cases}$$

Next, we shall consider the tensor product of two such representations. That is, for $i = 1, 2$, take a non-constant modular form f_i with weight k_i, put $\phi_i = \phi_{k_i, f_i}$ and $\rho_i = \rho_{\phi_i} \cong V_{k_i}$. Now, $\rho_1 \otimes \rho_2$ again lies inside $C^\infty(G, K)$ by realizing the tensor product as (pointwise) function multiplication. Similarly to above, $\rho_1 \otimes \rho_2$ has a basis given by $\{E^{v_1} * \phi_1 \otimes E^{v_2} * \phi_2 \mid v_1, v_2 \in \mathbb{Z}_{\geq 0}\}$ and $E^{v_1} * \phi_1 \otimes E^{v_2} * \phi_2$ has weight $k_1 + 2v_1 + k_2 + 2v_2$, so

$$\dim_{\mathbb{C}}((\rho_1 \otimes \rho_2)(\lambda)) = \begin{cases} v + 1 & \text{if } \lambda = k_1 + k_2 + 2v, v \in \mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.19 ([22, Lemma 2] or [19, Prop. 4.1])**.** *In our current setting, with $k_1, k_2 \in 2\mathbb{Z}_{\geq 4}$, the tensor product of the two Verma modules decomposes into irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-modules as*

$$V_{k_1} \otimes V_{k_2} \cong \bigoplus_{v=0}^{\infty} V_{k_1 + k_2 + 2v}.$$

In particular, inside each non-zero weight space in $\rho_1 \otimes \rho_2$, there is a unique (up to scalar) non-zero function of holomorphic type, which generates the summand

isomorphic to $V_{k_1+k_2+2v}$. In weight k_1+k_2, this must be $\phi_1\phi_2$, corresponding to f_1f_2. In higher weights k_1+k_2+2v (with $v>0$) this is some linear combination of $\{(E^{v_1}*\phi_1)(E^{v_2}*\phi_2) \mid v_1+v_2=v\}$ corresponding to some holomorphic modular form of the same weight. This corresponding modular form (generating the summand of $\rho_{f_1}\otimes\rho_{f_2}$ isomorphic to $V_{k_1+k_2+2v}$) is indeed $[f_1,f_2]_v$. One way to see this is to use the following lemma. This was obtained by Lanphier as a corollary of a stronger result, but can be independently verified by a straightforward computation.

Lemma 2.20 ([32, Corollary 1])**.** *For modular forms f_1 and f_2 of weights k_1 and k_2 respectively, their Rankin–Cohen bracket can be expressed using the Maass–Shimura operator δ. as*

$$[f_1,f_2]_v(z)=\sum_{r=0}^{v}(-1)^r\binom{v+k_1-1}{v-r}\binom{v+k_2-1}{r}(\delta_{k_1}^r f_1)(z)(\delta_{k_2}^{v-r}f_2)(z).$$

That is, in the definition of the Rankin–Cohen bracket, one may replace $\frac{1}{2\pi i}\frac{d}{dz}$ by the appropriate Maass–Shimura operator.

In particular, recall that the action of E corresponds precisely to δ. So we have that

$$[f_1,f_2]_v \leftrightarrow \phi_{[f_1,f_2]_v}=\sum_{r=0}^{v}(-1)^r\binom{v+k_1-1}{v-r}\binom{v+k_2-1}{r}(E^r*\phi_1)(E^{v-r}*\phi_2).$$

That is, $\phi_{[f_1,f_2]_v}$ lies inside $(\rho_1\otimes\rho_2)(k_1+k_2+2v)$ and is of holomorphic type. Therefore, it must be the unique such one, which generates the summand of $\rho_1\otimes\rho_2$ isomorphic to $V_{k_1+k_2+2v}$. Summarizing, we obtain the following characterization of the Rankin–Cohen brackets.

Proposition 2.21. *With f_1,f_2,ϕ_1,ϕ_2 as above, we have that*

$$\rho_{\phi_1}\otimes\rho_{\phi_2}=\bigoplus_{v=0}^{\infty}\rho_{\phi_{[f_1,f_2]_v}},$$

or equivalently

$$\rho_{f_1}\otimes\rho_{f_2}=\bigoplus_{v=0}^{\infty}\rho_{[f_1,f_2]_v}.$$

Note that this is not merely an isomorphism but actual equality. The forms $[f_1,f_2]_v$ are explicit elements inside $\rho_{f_1}\otimes\rho_{f_2}$ which are generators for the irreducible subrepresentations, of course with $\rho_{[f_1,f_2]_v}\cong V_{k_1+k_2+2v}$ for each $v=0,1,2,\ldots$.

CHAPTER 3

HERMITIAN MODULAR FORMS AND PLURIHARMONIC POLYNOMIALS

In this chapter we introduce what will be the central objects in the rest of this paper, giving the key definitions together with some additional context.

3.1 Unitary Groups and Hermitian Modular Forms

Following the account given in [18], one of the early motivations for the study of modular forms was through the perspective of generating functions counting solutions to quadratic forms. For example, writing

$$r_m(n) := \big|\{(a_1, a_2, \ldots, a_m) \in \mathbb{Z}^m \mid a_1^2 + \cdots + a_m^2 = n\}\big|,$$

the function $f(z) = \sum_{n \geq 0} r_4(n) e^{2\pi i n z}$ turns out to be a modular form (of weight 2 and level $\Gamma_0(4)$). One can then recover the celebrated Four Squares Theorem by writing f in terms of a known basis for $M_2(\Gamma_0(4))$ to calculate all of its Fourier coefficients, as outlined in [12, §1.2]. We can view $r_m(n)$ as counting the solutions to $Q_m(x) = n$, where $x = {}^t(x_1, \ldots, x_m)$ and $Q_m(x) = {}^t x 1_m x$. Here, ${}^t A$ denotes the transpose of a matrix or vector A, and 1_m denotes the m-by-m identity matrix. More generally, given a positive definite quadratic form Q in g variables with integral coefficients (corresponding to a half-integral positive definite g-by-g matrix $T = T_Q$), one could consider

$$r_m(Q) := \big|\{A \in M_{m,g}(\mathbb{Z}) \mid {}^t A A = T_Q\}\big|.$$

Here, $M_{m,g}(\mathbb{Z})$ is the set of m-by-g matrices with coefficients in $\mathbb{Z}$. This again counts the number of ways to re-write Q as a sum of m squares of linear terms (take $g = 1$ and $Q = Q_m$ to recover $r_m(Q) = r_m(n)$). This was generalized further to expressing one quadratic form in terms of another. Given Q and $\tilde{Q}$ two positive definite integral quadratic forms in g and m variables respectively, the relevant sets to count are

$$\{A \in M_{m,g}(\mathbb{Z}) \mid Q(x) = \tilde{Q}(Ax)\} = \{A \in M_{m,g}(\mathbb{Z}) \mid {}^t A T_{\tilde{Q}} A = T_Q\}.$$

This version of the problem led to the study of Siegel modular forms, introduced by the eponymous Siegel [37, 38, 39, 40], involving the symplectic groups $\mathrm{Sp}(2g)$. Hel Braun then considered a variant with Q and $\tilde{Q}$ Hermitian forms defined over $\mathcal{O}$ the

ring of integers of some quadratic imaginary extension of $\mathbb{Q}$, counting

$$\{A \in M_{m,g}(\mathcal{O}) \mid Q(x) = \tilde{Q}(Ax)\} = \{A \in M_{m,g}(\mathcal{O}) \mid {}^t\overline{A}T_{\tilde{Q}}A = T_Q\}.$$

This led to the introduction of Hermitian modular forms by Braun [3, 4, 5], the specific kind of modular forms with which we are concerned. More recently, Hermitian modular forms have gained significance for other reasons, some of which could not have reasonably been foreseen at the time of their inception. For instance, unitary groups have associated *Shimura varieties*, i.e. one can view certain quotients of the space $\mathbb{H}_n$ (defined below) as moduli spaces for particular kinds of abelian varieties. Additionally, the representation theory of unitary groups (in particular, of $\mathrm{U}(n,n)$) is closely related to that of general linear groups. In the modern day, this has proven especially useful for the study of Galois representations in the context of the Langlands program. These applications are beyond the scope of this paper, but hopefully provide the reader with some idea of the motivation for the continued study of these objects. Some more detail (and many references) can be found in e.g. [16, §1.1].

Definition 3.1 (Unitary Groups)**.** Throughout this section, without loss of generality we take integers $p \geq q \geq 1$. Let

$$H = \begin{pmatrix} 0 & 0 & -i1_q \\ 0 & 1_{p-q} & 0 \\ i1_q & 0 & 0 \end{pmatrix} \in M_{p+q,p+q},$$

written in $q, p-q, q$-by-$q, p-q, q$ blocks respectively. Then we define

$$\mathrm{U}(p,q) := \{\gamma \in M_{p+q,p+q}(\mathbb{C}) \mid \gamma^* H \gamma = H\}$$

to be the *unitary group of signature* (p,q), where γ^* denotes the conjugate-transpose ${}^t\overline{\gamma}$. Note that allowing $q = 0$, $\mathrm{U}(p,0)$ is the usual unitary group $\mathrm{U}(p)$. We write $\mathrm{U}(p,q)(R)$ to specify that the entries of the matrices lie in a subring R of $\mathbb{C}$.

A collection of generators of $\mathrm{U}(p,q)$ is given by matrices of the following five forms:

(I) $\begin{pmatrix} A & 0 & 0 \\ 0 & 1_{p-q} & 0 \\ 0 & 0 & (A^*)^{-1} \end{pmatrix}$ for any $A \in \mathrm{GL}(q,\mathbb{C})$

(II) $\begin{pmatrix} 1_q & 0 & B \\ 0 & 1_{p-q} & 0 \\ 0 & 0 & 1_q \end{pmatrix}$ where B is any q-by-q Hermitian matrix

(III) $\begin{pmatrix} 0 & 0 & -1_q \\ 0 & 1_{p-q} & 0 \\ 1_q & 0 & 0 \end{pmatrix}$

(IV) $\begin{pmatrix} 1_q & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1_q \end{pmatrix}$ for any $\alpha \in \mathrm{U}(p-q)$

(V) $\begin{pmatrix} 1_q & iu^* & \frac{i}{2}u^*u \\ 0 & 1_{p-q} & u \\ 0 & 0 & 1_q \end{pmatrix}$ for any $u \in M_{p-q,q}(\mathbb{C})$.

In the special case of *even signature* i.e. signature (n,n) where $p = q = n$ we have $H = \begin{pmatrix} 0 & -i1_n \\ i1_n & 0 \end{pmatrix} \in M_{2n,2n}$ and

$$\mathrm{U}(n,n) := \{\gamma \in M_{2n,2n}(\mathbb{C}) \mid \gamma^* H \gamma = H\}.$$

Similarly, a collection of generators of $\mathrm{U}(n,n)$ is given by matrices of the following three forms:

(I) $\begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$ for any $A \in \mathrm{GL}(n,\mathbb{C})$

(II) $\begin{pmatrix} 1_n & B \\ 0 & 1_n \end{pmatrix}$ where B is any n-by-n Hermitian matrix

(III) $\begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$.

Remark 3.2. While we shall give some results for general signature, parts of this paper (the whole of §5, for example) specialize to the important case of signature (n,n). Therefore, we shall continue drawing attention to the definitions in this particular case as well as giving the more general versions.

Definition 3.3 (Hermitian Upper Half-space)**.** We define the *Hermitian upper half-space* as

$$\mathbb{H}_{p,q} := \left\{ Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \in M_{p,q}(\mathbb{C}) \mid \frac{1}{2i}(Z_1 - Z_1^*) - \frac{1}{2} Z_2^* Z_2 > 0 \right\}$$

where $Z_1 \in M_{q,q}(\mathbb{C})$ and $Z_2 \in M_{p-q,q}(\mathbb{C})$, and we write $A > 0$ for a square matrix A to mean that A is positive definite. For even signature, this specializes to

$$\mathbb{H}_n := \mathbb{H}_{n,n} = \left\{ Z \in M_{n,n}(\mathbb{C}) \mid \frac{1}{2i}(Z - Z^*) > 0 \right\}.$$

We have an action of $\mathrm{U}(p,q)$ on $\mathbb{H}_{p,q}$ given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}$$

for $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{U}(p,q)$ ($A \in M_{p,p}$, $B \in M_{p,q}$, $C \in M_{q,p}$, $D \in M_{q,q}$) and $Z \in \mathbb{H}_{p,q}$.

Definition 3.4 (Automorphy Factor)**.** We define the *canonical automorphy factor* $J(g,Z) = (J_1(g,Z), J_2(g,Z)) \in \mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C})$ for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{U}(p,q)$ and $Z \in \mathbb{H}_{p,q}$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1_p & Z \\ 0 & 1_q \end{pmatrix} = \begin{pmatrix} 1_p & g \cdot Z \\ 0 & 1_q \end{pmatrix} \begin{pmatrix} J_1(g,Z) & 0 \\ C & J_2(g,Z) \end{pmatrix}.$$

A straightforward computation gives that $J_i(g_1 g_2, Z) = J_i(g_1, g_2 Z) J_i(g_2, Z)$ for $i = 1, 2$.

Remark 3.5. Another common approach in the literature is to follow Shimura's definition of automorphy factors λ and μ [36, §I.3]. There is no large disparity here; in fact, $J_2(g,Z) = CZ + D = \mu(g,Z)$, so this automorphy factor is identical. For the other automorphy factor, in signature (n,n) we have that $J_1(g,Z) = (ZC^* + D^*) = {}^t\lambda(g,Z)^{-1}$, the two are not identical but are isomorphic. In other signatures, Shimura works with a different (but isomorphic) realization of $\mathrm{U}(p,q)$, so a direct comparison is not as clean to write down, although one could translate between the two if they so wished.

Definition 3.6 (Hermitian Modular Forms)**.** Fix a quadratic imaginary extension L of a totally real number field L^+. Let Γ be a congruence subgroup of $\mathrm{U}(p,q)(\mathcal{O}_L)$. Let (ρ, V) be an algebraic representation of $\mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C})$, with V a finite-dimensional vector space over $\mathbb{C}$. A Hermitian modular form of level Γ and weight ρ (or type ρ) is a function $f : \mathbb{H}_{p,q} \to V$ such that:

(i) The function f is holomorphic.

(ii) For any $\gamma \in \Gamma$ we put

$$(f|_\rho \gamma)(Z) := \rho(J(\gamma, Z))^{-1} f(\gamma \cdot Z).$$

Then, for every $\gamma \in \Gamma$ we require that

$$f|_\rho \gamma = f.$$

(iii) If $p = q = 1$ and $L^+ = \mathbb{Q}$, we require that f is holomorphic at every cusp. This is equivalent to condition (iii) of definition 2.3. Note that other than this exceptional case, this condition is satisfied automatically by Koecher's principle.

For much of this paper, we shall focus on scalar-valued forms, i.e. with $V = \mathbb{C}$. In this case, $\rho(J_1, J_2) = \det^{\kappa_1}(J_1) \det^{\kappa_2}(J_2)$ for some (for our purposes) integers κ_1 and κ_2. In this case we say f is weight (κ_1, κ_2), and we say f is weight k for some integer k in the case that $\rho(J_1, J_2) = \det^k(J_2) = \det^k(CZ + D)$, i.e. weight $(0, k)$.

Hermitian modular forms also have Fourier expansions, as stated in [16]. In signature (n, n) and any fixed level Γ, there is a $\mathbb{Z}$-lattice of Hermitian matrices H_Γ such that $\begin{pmatrix} 1_n & h \\ 0 & 1_n \end{pmatrix} \in \Gamma$ for every $h \in H_\Gamma$. Then put $H_\Gamma^\vee$ for the dual of H_Γ with respect to the trace pairing. That is,

$$H_\Gamma^\vee := \{h = h^* \in M_{n,n}(\mathbb{C}) \mid \mathrm{Tr}_{L^+/\mathbb{Q}}(\mathrm{Tr}(hH_\Gamma)) \subset \mathbb{Z}\}.$$

Here, Tr is the usual trace of a square matrix, and $\mathrm{Tr}_{L^+/\mathbb{Q}}$ is the number field trace. Then a Hermitian modular form f of level Γ has a Fourier expansion of the form

$$f(Z) = \sum_{h \in H_\Gamma^\vee} c(h) e^{2\pi i \,\mathrm{Tr}(hZ)}.$$

As in [36, Prop. 5.7], if $L^+ \neq \mathbb{Q}$ or $n \neq 1$ then $c(h) = 0$ unless h is positive semi-definite. Then we say f is a *cusp form* if additionally $c(h) = 0$ except for when h is positive-definite.

Remark 3.7 (Terminology). The term *automorphic form* is commonly used in the literature to describe any higher-dimensional generalization of a modular form. For instance, one could say that Hermitian modular forms are automorphic forms for $\mathrm{U}(n,n)$ or $\mathrm{U}(p,q)$ and similarly that Siegel modular forms are automorphic forms for the symplectic groups $\mathrm{Sp}(2n)$.

In this section we have defined Hermitian modular forms as holomorphic functions, in a higher-dimensional analogue of Definition 2.3. This is the perspective we take throughout Chapters 4 and 5. We shall revisit Hermitian modular forms defined as functions on $\mathrm{U}(p,q)$, more similarly to §2.3, in Chapter 6.

3.2 Pluriharmonic Polynomials

The differential operators we shall construct are described in terms of this specific class of polynomials. Their representation-theoretic properties have been studied extensively by Kashiwara–Vergne [26], some of which we shall exploit for our purposes in this paper. For this section, p and q will be positive integers.

Definition 3.8 (Harmonic and Pluriharmonic)**.** Let $P(X,Y)$ be a polynomial in matrix variables $X = (x_{s,u}) \in M_{q,k}(\mathbb{C})$ and $Y = (y_{t,u}) \in M_{p,k}(\mathbb{C})$. Set

$$\Delta_{s,t}(X,Y) = \sum_{u=1}^{k} \frac{\partial^2}{\partial x_{s,u} \partial y_{t,u}} \qquad (1 \leq s \leq q, 1 \leq t \leq p).$$

Then we say that P is *harmonic* if $\sum_{s=1}^{q} \Delta_{s,s}(X,Y)P = 0$ and *pluriharmonic* if $\Delta_{s,t}(X,Y)P = 0$ for every $1 \leq s \leq q$, $1 \leq t \leq p$.

Definition 3.9 (Homogeneous)**.** There is a group action of $\mathrm{GL}(q,\mathbb{C}) \times \mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(k,\mathbb{C})$ on polynomials $P(X,Y)$ given by

$$(A,B,C) \cdot P(X,Y) = P(AXC^{-1}, BY^tC).$$

For integers $v, v' \geq 0$ we say that P is *homogeneous of degree* (v,v') if, setting $C = 1_k$, for every $(A,B) \in \mathrm{GL}(q,\mathbb{C}) \times \mathrm{GL}(p,\mathbb{C})$ we have

$$(A,B,1_k) \cdot P(X,Y) = P(AX,BY) = \det(A)^v \det(B)^{v'} P(X,Y).$$

In the case $v = v'$, we shall simply say that P is homogeneous of degree v.

Remark 3.10. Note that $P(X, Y)$ is pluriharmonic if and only if $P(AX, BY)$ is harmonic for every $(A, B) \in \mathrm{GL}(q, \mathbb{C}) \times \mathrm{GL}(p, \mathbb{C})$. So a homogeneous polynomial is pluriharmonic if and only if it is harmonic.

Definition 3.11 (Invariance and Association)**.** For positive integers r and $k_1, \ldots, k_r$ with $k_1 + \cdots + k_r = k$, let $K = K(k_1, \ldots, k_r)$ be the (image of the) block-diagonal embedding

$$\mathrm{GL}(k_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(k_r, \mathbb{C}) \hookrightarrow \mathrm{GL}(k, \mathbb{C}).$$

We say $P(X, Y)$ is *K-invariant* if $P(XC^{-1}, Y^t C) = P(X, Y)$ for every matrix C in K. For any such polynomial P, there is a polynomial $Q(R_1, \ldots, R_r)$ with variables $R_1, \ldots, R_r \in M_{q,p}(\mathbb{C})$ such that, writing $X = (X_1, \ldots, X_r)$ and $Y = (Y_1, \ldots, Y_r)$ with $X_i \in M_{q,k_i}$, $Y_i \in M_{p,k_i}$ for $i = 1, \ldots, r$, $P(X, Y) = Q(X_1 {}^t Y_1, \ldots, X_r {}^t Y_r)$. For such a P and Q we say that Q is *associated to* P.

The particular spaces of polynomials we are concerned with are as follows:

Definition 3.12 (Notation)**.** Denote by $\mathcal{P}_{(q,p),(v,v')}(k_1, \ldots, k_r)$ the space of all K-invariant homogeneous polynomials P of degree (v, v'). Let $\mathcal{Q}_{(q,p),(v,v')}(r)$ be the space of all polynomials Q in the coefficients of $q \times p$ matrices $R_1, \ldots, R_r$ such that

$$Q(AR_1 {}^t B, \ldots, AR_r {}^t B) = \det(A)^v \det(B)^{v'} Q(R_1, \ldots, R_r)$$

for all $(A, B) \in \mathrm{GL}(q, \mathbb{C}) \times \mathrm{GL}(p, \mathbb{C})$. Let $\mathcal{H}_{(q,p),(v,v')}(k_1, \ldots, k_r)$ be the subspace of $\mathcal{Q}_{(q,p),(v,v')}(r)$ consisting of the elements $Q \in \mathcal{Q}_{(q,p),(v,v')}(r)$ such that $Q(X_1 {}^t Y_1, \ldots, X_r {}^t Y_r)$ is pluriharmonic considered as a polynomial in the entries of $X = (X_1, \ldots, X_r)$ and $Y = (Y_1, \ldots, Y_r)$, $X_i \in M_{q,k_i}$, $Y_i \in M_{p,k_i}$. For ease of notation, in the case $p = q = n$ we write subscript n instead of (n, n), and similarly in the case $v = v'$ we use the subscript v instead of (v, v). In this special case, we put $\mathcal{Q}_n(r) := \oplus_{v \geq 0} \mathcal{Q}_{n,v}(r)$.

Remark 3.13. When $k_i \geq \min(p, q)$ for each $1 \leq i \leq r$, $\mathcal{H}_{(q,p),(v,v')}(k_1, \ldots, k_r)$ is the space of polynomials Q associated to the degree-(v, v') homogeneous K-invariant pluriharmonic polynomials $P \in \mathcal{P}_{(q,p),(v,v')}(k_1, \ldots, k_r)$ as above.

We shall be concerned with differential operators associated with these pluriharmonic polynomials, with the association made explicit by the following definition.

Definition 3.14 (Differential Operators)**.** For a matrix variable $Z = (z_{t,s}) \in M_{p,q}$ we define the $q \times p$ matrix of derivatives $\frac{\partial}{\partial Z}$ by

$$\left(\frac{\partial}{\partial Z}\right)_{s,t} = \frac{\partial}{\partial z_{t,s}}.$$

Then, given a polynomial $Q(R_1, \ldots, R_r)$ in the entries of $q \times p$ matrices $R_1, \ldots, R_r$, we can define a differential operator on holomorphic functions $M_{p,q}^r \to \mathbb{C}$ in $p \times q$-shaped variables $Z_1, \ldots, Z_r$ by $Q\left(\frac{\partial}{\partial Z_1}, \ldots, \frac{\partial}{\partial Z_r}\right)$, for which we shall often simply write $Q\left(\frac{\partial}{\partial Z}\right)$.

Remark 3.15. It should be noted that here we shall work with $\mathcal{H}_{(q,p),\cdot}(\cdot)$ with $p \geq q$ rather than $\mathcal{H}_{(p,q),\cdot}(\cdot)$ as in [26] or [1]. This is merely a matter of convention arising from the choice of shape of $\frac{\partial}{\partial Z}$. From the perspective of differential operators, it is this author's opinion that the definition given here (with $\frac{\partial}{\partial Z}$ of shape $q \times p$, the shape of ${}^tZ$) is "more natural" whereas from the perspective of representation theory (in particular, for the methods in [1]) choosing $\frac{\partial}{\partial Z}$ to have the same shape as Z is more convenient. For many of the intermediate lemmas or propositions this will make no difference since the definition of the polynomials themselves is the same. When discussing, for example, differential operators of the form $Q\left(\frac{\partial}{\partial Z}\right)$, one can translate between the two perspectives merely by switching X and Y (i.e. take $\tilde{P}(X, Y) = P(Y, X)$) or taking the transpose of the variables R_j (i.e. $\tilde{Q}(R_1, \ldots, R_r) = Q({}^tR_1, \ldots, {}^tR_r)$). We shall endeavour to make it clear to the reader when this translation is necessary when aligning the two perspectives in Chapter 6.

CHAPTER 4

CHARACTERIZATION OF DIFFERENTIAL OPERATORS

The following two chapters are, for the most part, from Francis Dunn, *Rankin–Cohen Type Differential Operators on Hermitian Modular Forms*, Res. Number Theory **10** (2024), no. 88. We adapt the argument of [14, 24] from the setting of Siegel modular forms to that of Hermitian modular forms. Respectively, §4.1 and Chapter 5 in this paper are §3 and §4 from [13]. The differences to the original publication amount to (an expansion of) Remark 4.2, and minor corrections of typographical errors and to Lemma 4.5. Then, we give a subsequent generalization of Theorem 4.1 in §4.2, which did not appear in [13].

4.1 In Even Signature

In this section we give the full, precise statement of Theorem 1.1 and prove the characterization of differential operators on Hermitian modular forms of signature (n,n) in terms of the pluriharmonic polynomials defined above.

Theorem 4.1. *Let L/L^+ be a quadratic imaginary extension of a totally real number field, $k_1,\dots,k_r,v\in\mathbb{Z}_{>0}$ with each $k_i\geq n$ and $\frac{|\mu(L)|}{2}[\mathcal{O}_L^\times:\mu(L)\mathcal{O}_{L^+}^\times]$ dividing v, where $\mu(L)$ is the set of roots of unity contained in L. For a polynomial Q on $M_{n,n}^r$, we have*

$$Q\left(\frac{\partial}{\partial Z}\right)\left(F(\gamma Z_1,\dots,\gamma Z_r)\prod_{i=1}^{r}\det(CZ_i+D)^{-k_i}\right)|_{Z_1=\cdots=Z_r=Z}$$
$$=\det(CZ+D)^{-(k+2v)}Q\left(\frac{\partial}{\partial\gamma Z}\right)(F(\gamma Z,\dots,\gamma Z)) \quad (4.1)$$

for $k=\sum_{j=1}^r k_r$ and every $\gamma=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in U(n,n)(\mathcal{O}_L)$, if and only if $Q\in\mathcal{H}_{n,v}(k_1,\dots,k_r)$.

Remark 4.2. Regarding the condition on the divisibility of v, Ibukiyama remarked to the author that this is essentially the condition of a real Lie group for the unitary group in question. Other authors have dealt with the issue which causes this to arise in similar but slightly differing fashions. Martin–Senadheera [33] introduced

a character on the resulting output, Ban [1] restricted his attention to special unitary groups (i.e. determinant equal to 1), and Böcherer–Das [2] mainly worked with special unitary groups but proposed a "twisted automorphy factor" in this setting. This author notes that one may replace the condition on the degree v in the theorem above by any of these variations and the resulting statement is proved by an almost identical argument. These perspectives can be united by observing that the $J^*(\gamma, Z)$ of Böcherer–Das is the same as Shimura's $\lambda(\gamma, Z)$, which is equal to ${}^tJ_1(\gamma, Z)^{-1}$ of Definition 3.4 (as pointed out in the subsequent remark). Then, in the case of Martin–Senadheera, one recovers their character twist by noting that $\det(\lambda(\gamma, Z)) = \det(\gamma)^{-1}\det(\mu(\gamma, Z))$ (as observed, for example, in [17, §3.1.1, (17)]). In §4.2 we shall re-formulate the operators here as raising the weight by $(-v, v)$ (perhaps closest to the version suggested by Böcherer–Das) which coincides with raising by $2v = (0, 2v)$ upon restricting to $\mathrm{SU}(n, n)$ or under the condition on v given in the theorem above.

Corollary 4.3 (Differential Operators on Hermitian Modular Forms)**.** *If, in the setting of the theorem, we have $F(Z_1, \ldots, Z_r) = f_1(Z_1)\ldots f_r(Z_r)$ with each f_j $(j = 1, \ldots, r)$ a scalar-valued Hermitian modular form of weight k_j of some level Γ then applying the differential operator associated to some polynomial Q gives a Hermitian modular form of weight $2v + k$ and level Γ if $Q \in \mathcal{H}_{n,v}(k_1, \ldots, k_r)$.*

Proof of Corollary 4.3. This follows because, for such an F,

$$\begin{aligned}
&\left(Q\left(\frac{\partial}{\partial Z}\right)(f_1\ldots f_r)\right)(Z_1, \ldots, Z_r)|_{Z_1=\cdots=Z_r=Z} \\
&\quad = Q\left(\frac{\partial}{\partial Z}\right)(f_1(Z_1)\ldots f_r(Z_r))\,|_{Z_1=\cdots=Z_r=Z} \\
&\quad = Q\left(\frac{\partial}{\partial Z}\right)\left(f_1(\gamma Z_1)\ldots f_r(\gamma Z_r)\prod_{i=1}^{r}\det(CZ_i + D)^{-k_i}\right)|_{Z_1=\cdots=Z_r=Z} \\
&\quad = \det(CZ + D)^{-(k+2v)}Q\left(\frac{\partial}{\partial \gamma Z}\right)(f_1(\gamma Z)\ldots f_r(\gamma Z)) \\
&\quad = \det(CZ + D)^{-(k+2v)}\left(Q\left(\frac{\partial}{\partial Z}\right)(f_1\ldots f_r)\right)(\gamma Z, \ldots, \gamma Z),
\end{aligned}$$

where the penultimate equality is the relation (4.1). □

Remark 4.4. The reverse implication for Corollary 4.3 is not quite true. One counterexample, pointed out by T. Ibukiyama, is that if we choose k_1 and Γ such that

the only Hermitian modular form of weight k_1 and level Γ is the zero function, then the differential operator corresponding to any polynomial Q will act on the zero function and return the zero function, which is of course a Hermitian modular form of every weight and level.

Proof of Theorem 4.1. We start by supposing $Q \in \mathcal{H}_{n,v}(k_1, \ldots, k_r)$. It is sufficient to show that the result holds for a collection of generators of $U(n,n)$ such as the one given in Definition 3.1.

Case I We show the relation for $\gamma = \begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$ for $A \in \mathrm{GL}(n, \mathbb{C})$. First, note that

$$\frac{\partial}{\partial Z} = A^* \frac{\partial}{\partial W} A,$$

where $W = \gamma \cdot Z = AZA^*$. Furthermore, note that $A \in \mathrm{GL}(n, \mathcal{O}_L)$ and $\det(A) \in \mathcal{O}_L^\times$. For v as in the statement of the theorem, $\det(A)^v$ and $\det(A^*)^v$ must lie inside $\mathcal{O}_{L^+}^\times \subset \mathbb{R}$ and thus

$$\det(A)^v \det(A^*)^v = \det(A)^{2v} = \det(A^*)^{2v}.$$

Then

$$\begin{aligned}
&Q\left(\frac{\partial}{\partial Z}\right)\left(F(AZ_1A^*, \ldots, AZ_rA^*)\det(A^*)^k\right)|_{Z_1=\cdots=Z_r=Z} \\
&= \det(A^*)^k Q\left(\frac{\partial}{\partial Z}\right)(F(AZA^*, \ldots, AZA^*)) \qquad \text{since } A \text{ and } A^* \text{ do not depend on } Z \\
&= \det(A^*)^k Q\left(A^*\frac{\partial}{\partial W}A\right)(F(AZA^*, \ldots, AZA^*)) \qquad \text{by the equality above} \\
&= \det(A^*)^{k+v}\det(A)^v Q\left(\frac{\partial}{\partial W}\right)(F(W, \ldots, W)) \qquad \text{since } Q \text{ is homogeneous of degree } v \\
&= \det(A^*)^{k+2v} Q\left(\frac{\partial}{\partial W}\right)(F(W, \ldots, W)) \qquad \text{by the observation above,}
\end{aligned}$$

and this is precisely the relation (4.1) in this case.

Case II We consider $\gamma = \begin{pmatrix} 1_n & B \\ 0 & 1_n \end{pmatrix}$ with B Hermitian, $B = B^*$. Then $W = \gamma \cdot Z = Z + B$ and $\frac{\partial}{\partial Z} = \frac{\partial}{\partial W}$.

We have

$$\begin{aligned} & Q\left(\frac{\partial}{\partial Z}\right)\left(F(Z_1+B,\ldots,Z_r+B)\det(1_n)^{-k}\right)|_{Z_1=\cdots=Z_r=Z} \\ &= Q\left(\frac{\partial}{\partial Z}\right)(F(Z+B,\ldots,Z+B)) \\ &= Q\left(\frac{\partial}{\partial W}\right)(F(W,\ldots,W)), \end{aligned}$$

and we are done.

Case III We take $\gamma = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$. So $W = \gamma \cdot Z = -Z^{-1}$ and

$$\frac{\partial}{\partial Z} = (-Z^{-1})\frac{\partial}{\partial W}(-Z^{-1}).$$

We argue in a similar fashion to the analagous Theorem 2 of Ibukiyama [24]. We see that there exist functions $G_M(Z)$, independent of F, such that

$$Q\left(\frac{\partial}{\partial Z}\right)(F(W,\ldots,W)\det(-Z)^{-k}) = \sum_{|M|\leq \deg(Q)} G_M(Z)\left(Q_M\left(\frac{\partial}{\partial Z}\right)F\right)(W),$$

where the sum runs over $n \times n$ matrices $M = (m_{s,t})$ with non-negative integer entries satisfying

$$|M| := \sum_{1\leq s,t,\leq n} m_{s,t} \leq \deg(Q),$$

and Q_M is the monomial associated to M, i.e.

$$Q_M\left(\frac{\partial}{\partial Z}\right) = \prod_{1\leq s,t\leq n}\left(\frac{\partial}{\partial Z}\right)_{s,t}^{m_{s,t}} = \prod_{1\leq s,t\leq n}\frac{\partial^{m_{s,t}}}{\partial z_{t,s}^{m_{s,t}}}.$$

Therefore, if we can verify that the commutation relation holds for a particular holomorphic function F for which $Q_M\left(\frac{\partial}{\partial Z}\right)(F)$ are linearly independent at any point Z (or linearly independent over the ring of holomorphic functions), then the commutation relation holds for all holomorphic functions.

We take $F_0(Z) := \exp(i\operatorname{Tr}(Y^*ZY))$ where Y is an $n \times k$ matrix of independent variables $y_{s,t}$. Then

$$Q_M\left(\frac{\partial}{\partial Z}\right)(F_0)(Z) = i^{|M|}Q_M(YY^*)F_0(Z).$$

Since the $y_{s,t}$ are independent variables, the $Q_M\left(\frac{\partial}{\partial Z}\right)(F)$ are linearly independent for any Z. So, we prove the commutation relation for $\gamma = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$ and F_0. To

do so, we use the following technical lemma, a more general version of which was proved by Kashiwara–Vergne:

Lemma 4.5. *[26, 4.2, p.39]*

Let $P = P(X,Y)$ be a pluriharmonic polynomial, homogeneous of degree v. Then we have the following equality:

$$\int_{M_{n,k}(\mathbb{C})} e^{i\,\mathrm{Tr}\,UY^*} e^{i\,\mathrm{Tr}\,U^*Y} e^{i\,\mathrm{Tr}\,U^*ZU} P(U,\overline{U})|dU|^2 = (-2\pi i)^{nk} \det(-Z)^{-(k+2v)} e^{i\,\mathrm{Tr}\,Y^*(-Z)^{-1}Y} P(Y,\overline{Y}),$$

where $|dU|^2 = 2^k d(ReU_1)d(ImU_1)\ldots d(ReU_k)d(ImU_k)$ with U_j the j-th column of U.

Now, applying this to the situation at hand:

$$\begin{aligned}
&Q\left(\frac{\partial}{\partial Z}\right)(F_0(-Z^{-1})\det(-Z)^{-k}) \\
&= Q\left(\frac{\partial}{\partial Z}\right)\left((-2\pi i)^{-nk}\int e^{i\,\mathrm{Tr}\,UY^*} e^{i\,\mathrm{Tr}\,U^*Y} e^{i\,\mathrm{Tr}\,U^*ZU}|dU|^2\right) \quad \text{by Lemma 4.5 with } P \text{ trivial} \\
&= (-2\pi i)^{-nk}\int e^{i\,\mathrm{Tr}\,UY^*} e^{i\,\mathrm{Tr}\,U^*Y} Q\left(\frac{\partial}{\partial Z}\right)\left(e^{i\,\mathrm{Tr}\,U^*ZU}\right)|dU|^2 \quad \text{by Leibniz integral rule} \\
&= (-2\pi i)^{-nk}\int e^{i\,\mathrm{Tr}\,UY^*} e^{i\,\mathrm{Tr}\,U^*Y} e^{i\,\mathrm{Tr}\,U^*ZU} i^{\deg(Q)} Q(UU^*)|dU|^2 \\
&= (-2\pi i)^{-nk} i^{\deg(Q)}\int e^{i\,\mathrm{Tr}\,UY^*} e^{i\,\mathrm{Tr}\,U^*Y} e^{i\,\mathrm{Tr}\,U^*ZU} P(U,\overline{U})|dU|^2 \quad \text{where } P \text{ assoc. to } Q \\
&= (-2\pi i)^{-nk} i^{\deg(Q)} (-2\pi i)^{nk} \det(-Z)^{-(k+2v)} e^{i\,\mathrm{Tr}\,Y^*(-Z)^{-1}Y} P(Y,\overline{Y}) \quad \text{by Lemma 4.5} \\
&= \det(-Z)^{-(k+2v)} i^{\deg(Q)} Q(YY^*) e^{i\,\mathrm{Tr}\,Y^*(-Z)^{-1}Y} \\
&= \det(-Z)^{-(k+2v)} Q\left(\frac{\partial}{\partial W}\right)(F_0(W)) \quad \text{and we are done.}
\end{aligned}$$

So, if the polynomial Q is a member of the space $\mathcal{H}_{n,v}(k_1,\ldots,k_r)$ then equation (4.1) is satisfied for every element of our unitary group. Conversely, suppose that Q is a polynomial on $M_{n,n}^r$ satisfying equation (4.1), restated below:

$$\begin{aligned}
Q\left(\frac{\partial}{\partial Z}\right)\left(F(\gamma Z_1,\ldots,\gamma Z_r)\prod_{i=1}^{r}\det(CZ_i+D)^{-k_i}\right)|_{Z_1=\cdots=Z_r=Z} \\
= \det(CZ+D)^{-(k+2v)} Q\left(\frac{\partial}{\partial \gamma Z}\right)(F(\gamma Z,\ldots,\gamma Z)) \qquad (4.1)
\end{aligned}$$

for every matrix $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(n,n)$. We again consider $F_0(Z) = \exp(i \operatorname{Tr} Y^* Z Y)$. We start with the homogeneity property, and consider the relation (4.1) for the polynomial Q in the case of a matrix $\gamma = \begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$ for any $A \in \mathrm{GL}(n, \mathcal{O}_L)$:

$$Q\left(\frac{\partial}{\partial Z}\right)\left(\exp(i \operatorname{Tr} Y^* A Z A^* Y) \det(A^*)^k\right) = \det(A^*)^{k+2v} Q\left(\frac{\partial}{\partial (A Z A^*)}\right)\left(\exp(i \operatorname{Tr} Y^* A Z A^* Y)\right).$$

Computing each side, this becomes

$$\det(A^*)^k Q(A^* Y Y^* A) \exp(i \operatorname{Tr} Y^* A Z A^* Y) = \det(A^*)^{k+2v} Q(Y Y^*) \exp(i \operatorname{Tr} Y^* A Z A^* Y).$$

Since the matrix Y is a matrix of independent variables and this holds for every value of Z and every $A \in \mathrm{GL}(n, \mathcal{O}_L)$, we must have that

$$Q(A^* Y Y^* A) = \det(A^*)^{2v} Q(Y Y^*).$$

In particular, for $A \in \mathrm{GL}(n, \mathcal{O}_L)$ and v as in the statement of Theorem 4.1,

$$Q(A^* Y Y^* A) = \det(A^*)^v \det(A)^v Q(Y Y^*).$$

Since the relationship is for $Q(YY^*)$ a polynomial in the entries of YY^*, we can find a unique polynomial P with $P(Y, \overline{Y}) = Q(YY^*)$, and more generally $P(X,Y) = Q(X^t Y)$. So far we've shown that this polynomial P is homogeneous (of degree v). To show that the original polynomial Q is indeed associated to an harmonic polynomial P, we need the following lemma:

Lemma 4.6. *Given a homogeneous polynomial $P(X,Y)$ in the entries of $X, Y \in M_{n,k}(\mathbb{C})$ which we identify with $\mathbb{C}^N$ ($N = nk$) which satisfies*

$$\int_{\mathbb{C}^N} e^{-i\langle w,z\rangle} e^{-i\langle z,w\rangle} e^{-\langle w,w\rangle} P(w, \overline{w}) |dw|^2 = (2\pi)^N e^{-\langle z,z\rangle} P(-iz, -i\overline{z}) \qquad (4.2)$$

for every $z \in \mathbb{C}^N$, P must be harmonic (in the sense of Definition 3.8). Here, $\langle , \rangle$ is the standard Hermitian form on $\mathbb{C}^N$ (which may also be written $\langle w, z\rangle = \operatorname{Tr}(w z^*)$).

Proof of Lemma 4.6. We identify $\mathbb{C}^N$ with $\mathbb{R}^{2N}$ and use the mean value relationship for harmonic (the usual definition of harmonic) polynomials on $\mathbb{R}^{2N}$

$$\int_{\mathbb{R}^{2N}} e^{-2i(x,y)} e^{-(x,x)} F(x) dx = \pi^N e^{-(y,y)} F(-iy),$$

which holds for every $y \in \mathbb{R}^{2N}$ for a polynomial F if and only if F is harmonic on $\mathbb{R}^{2N}$. Here, $(,)$ is the standard inner product on $\mathbb{R}^{2N}$. Writing $w = x_1 + ix_2$ and $z = y_1 + iy_2$ with $x_1, x_2, y_1, y_2 \in \mathbb{R}^N$, $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^{2N}$, setting $F(x) = P(x_1 + ix_2, x_1 - ix_2)$, and recalling that $|dw|^2 = 2^N dx_1 dx_2 = 2^N dx$ it follows immediately that equation (4.2) holds if and only if $P(x_1+ix_2, x_1-ix_2)$ is harmonic as a function on $\mathbb{R}^{2N}$, i.e. $P(w, \overline{w})$ with $w \in \mathbb{C}^N = M_{n,k}(\mathbb{C})$ satisfies

$$\sum_{r=1}^{n}\sum_{s=1}^{k} \frac{\partial^2}{\partial w_{r,s}\partial \overline{w} r, s} P(w, \overline{w}) = 0. \tag{4.3}$$

Thinking of $P(w, \overline{w}) = P(x_1 + ix_2, x_1 - ix_2)$ as an analytic function on $\mathbb{R}^{2N} \subset \mathbb{C}^N \times \mathbb{C}^N$ (embedded via $(x_1, x_2) \mapsto (x_1+ix_2, x_1-ix_2)$), we have an equality of real analytic functions in equation (4.2) which holds precisely when equation (4.3) holds. We can extend equation (4.2) via analytic continuation to all of $\mathbb{C}^N \times \mathbb{C}^N$. Since the analytic continuation of $P(w, \overline{w})$ is of course $P(X, Y)$, equation (4.3) becomes

$$\sum_{r=1}^{n}\sum_{s=1}^{k} \frac{\partial^2}{\partial x_{r,s}\partial y_{r,s}} P(X, Y) = 0,$$

which is precisely our definition 3.8 for P to be harmonic. □

Returning to the proof of Theorem 4.1. It remains to show that equation (4.2) holds for the polynomial P associated to our differential operator. To do this, we use the relation (4.1) for $\gamma = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$ and the function $F_0(Z) = \exp(i \operatorname{Tr}(Y^* Z Y))$ as above. The commutation relation tells us that

$$Q\left(\frac{\partial}{\partial Z}\right)\left(\det(-Z)^{-k} F_0(-Z^{-1})\right) = \det(-Z)^{-(k+2v)} Q\left(\frac{\partial}{\partial \gamma Z}\right)\left(F_0(-Z^{-1})\right).$$

Just as in Case III above, the left-hand side is equal to

$$(-2\pi i)^{-nk} i^{\deg(Q)} \int_{M_{n,k}(\mathbb{C})} e^{i \operatorname{Tr} UY^*} e^{i \operatorname{Tr} U^* Y} e^{i \operatorname{Tr} U^* ZU} P(U, \overline{U}) |dU|^2.$$

For the right-hand side, using the homogeneity of Q and computing we find

$$\begin{aligned}
&\det(-Z)^{-(k+2v)} Q\left(\frac{\partial}{\partial \gamma Z}\right)\left(F_0(-Z^{-1})\right) \\
&= \det(-Z)^{-k} Q\left(\frac{\partial}{\partial Z}\right)\left(F_0(-Z^{-1})\right) \\
&= \det(-Z)^{-k} i^{\deg(Q)} F_0(-Z^{-1}) Q((-Z^{-1}) Y Y^* (-Z^{-1})).
\end{aligned}$$

So

$$(-2\pi i)^{-nk}\int_{M_{n,k}(\mathbb{C})} e^{i\operatorname{Tr}UY^*}e^{i\operatorname{Tr}U^*Y}e^{i\operatorname{Tr}U^*ZU}P(U,\overline{U})|dU|^2 \\ = \det(-Z)^{-k}e^{i\operatorname{Tr}\left(Y^*(-Z^{-1})Y\right)}Q((-Z^{-1})YY^*(-Z^{-1})).$$

Since this holds for any value of Z, let's take $Z = i\alpha^2$ for a positive definite Hermitian matrix α. Then we can write the above as follows:

$$(-2\pi i)^{-nk}\int_{M_{n,k}(\mathbb{C})} e^{i\operatorname{Tr}UY^*}e^{i\operatorname{Tr}U^*Y}e^{-\operatorname{Tr}((\alpha U)^*(\alpha U))}P(U,\overline{U})|dU|^2 \\ = (-i)^{-nk}\det(-\alpha)^{-2k}e^{-\operatorname{Tr}\left((\alpha^{-1}Y)^*(\alpha^{-1}Y)\right)}Q(i\alpha^{-2}YY^*i\alpha^{-2}).$$

Re-arranging we have

$$\int_{M_{n,k}(\mathbb{C})} e^{i\operatorname{Tr}UY^*}e^{i\operatorname{Tr}U^*Y}e^{-\operatorname{Tr}((\alpha U)^*(\alpha U))}P(U,\overline{U})|dU|^2 \\ = (2\pi)^{nk}\det(-\alpha)^{-2k}e^{-\operatorname{Tr}\left((\alpha^{-1}Y)^*(\alpha^{-1}Y)\right)}P\left(-i\alpha^{-1}(\alpha^{-1}Y), -i\ \overline{\alpha^{-1}(\alpha^{-1}Y)}\right).$$

Finally, we can change variables to $W = -\alpha U$ and $V = \alpha^{-1}Y$, and since $|dU|^2 = \det(-\alpha)^{-2k}|dW|^2$ we have

$$\int_{M_{n,k}(\mathbb{C})} e^{-i\operatorname{Tr}WV^*}e^{-i\operatorname{Tr}W^*V}e^{-\operatorname{Tr}(W^*W)}P(\alpha^{-1}W,\overline{\alpha^{-1}W})|dW|^2 \\ = (2\pi)^{nk}e^{-\operatorname{Tr}(V^*V)}P\left(-i\alpha^{-1}V, -i\ \overline{\alpha^{-1}V}\right).$$

So by Lemma 4.6, $P(\alpha^{-1}X, \alpha^{-1}Y)$ satisfies (4.2) and is thus harmonic. We already showed P was homogeneous of degree v, so as α is Hermitian it follows that $P(X,Y)$ is harmonic, and indeed pluriharmonic by Remark 3.10, concluding the proof of Theorem 4.1. □

4.2 Generalization to Other Signatures

Here, we generalize Theorem 4.1 to other signatures (p,q) (with $p \geq q > 0$) and other scalar weights.

Theorem 4.7. *Let L/L^+ be a quadratic imaginary extension of a totally real number field, and fix a signature (p,q) (with $p \geq q > 0$). Fix non-negative integers v, v' and $\kappa_j^{(1)}, \ldots, \kappa_j^{(r)}$ with $j = 1, 2$ such that $k_i := \kappa_1^{(i)} + \kappa_2^{(i)} \geq q$ for each $i = 1, \ldots, r$. Put $\kappa_j = \sum_{i=1}^r \kappa_j^{(i)}$ and $k = \sum_{i=1}^r k_i$. Then for a polynomial Q on $M_{q,p}^r$, we have*

$$\begin{aligned} &Q\left(\frac{\partial}{\partial Z}\right)\left(F(\gamma Z_1, \ldots, \gamma Z_r)\prod_{i=1}^r \det(J_1(\gamma, Z_i))^{\kappa_1^{(i)}} \det(J_2(\gamma, Z_i))^{-\kappa_2^{(i)}}\right)|_{Z_1=\cdots=Z_r=Z} \\ &\qquad = \det(J_1(\gamma, Z))^{\kappa_1+v'} \det(J_2(\gamma, Z))^{-(\kappa_2+v)} Q\left(\frac{\partial}{\partial \gamma Z}\right)(F(\gamma Z, \ldots, \gamma Z)) \quad (4.4) \end{aligned}$$

for every $\gamma \in U(p,q)(\mathcal{O}_L)$ if and only if $Q \in \mathcal{H}_{(q,p),(v,v')}(k_1, \ldots, k_r)$.

Proof. The proof is almost identical to that of Theorem 4.1, so we give a more concise argument here. First, putting $W = \gamma \cdot Z$ as usual, a straightforward computation gives

$$dW = J_1(\gamma, Z) dZ J_2(\gamma, Z)^{-1},$$

where dZ (resp. dW) is the matrix of 1-forms with the same shape as Z, $(dZ)_{ij} = dZ_{ij}$ (c.f. [36, §I.3, Lemma 3.4]). So, we have that

$$\frac{\partial}{\partial Z} = J_2(\gamma, Z)^{-1} \frac{\partial}{\partial W} J_1(\gamma, Z).$$

Now, if γ is a generator of $\mathrm{U}(p,q)(\mathcal{O}_L)$ of the form (I), (II), (IV) or (V), $J(\gamma, Z)$ does not depend on Z. In these cases, therefore, if $Q \in \mathcal{H}_{(q,p),(v,v')}(k_1, \ldots, k_r)$ then (4.4) is satisfied because

$$\begin{aligned} &Q\left(\frac{\partial}{\partial Z}\right)\left(F(\gamma Z_1, \ldots, \gamma Z_r)\prod_{i=1}^r \det(J_1(\gamma, Z_i))^{\kappa_1^{(i)}} \det(J_2(\gamma, Z_i))^{-\kappa_2^{(i)}}\right)|_{Z_1=\cdots=Z_r=Z} \\ &= \det(J_1(\gamma, Z))^{\kappa_1} \det(J_2(\gamma, Z))^{-\kappa_2} Q\left(\frac{\partial}{\partial Z}\right)(F(\gamma Z_1, \ldots, \gamma Z_r))\,|_{Z_1=\cdots=Z_r=Z} \\ &= \det(J_1(\gamma, Z))^{\kappa_1} \det(J_2(\gamma, Z))^{-\kappa_2} Q\left(J_2(\gamma, Z)^{-1}\frac{\partial}{\partial \gamma Z} J_1(\gamma, Z)\right)(F(\gamma Z_1, \ldots, \gamma Z_r))\,|_{Z_1=\cdots=Z_r=Z} \\ &= \det(J_1(\gamma, Z))^{\kappa_1+v'} \det(J_2(\gamma, Z))^{-(\kappa_2-v)} Q\left(\frac{\partial}{\partial \gamma Z}\right)(F(\gamma Z, \ldots, \gamma Z)), \end{aligned}$$

where in the first equality we use that the automorphy factors are independent of Z to move them past the differential operator, and in the final equality we use the homogeneity of Q. For the action of the final generator, (III) $\begin{pmatrix} 0 & 0 & -1_q \\ 0 & 1_{p-q} & 0 \\ 1_q & 0 & 0 \end{pmatrix}$, we rather show that the relation holds for the inverse, $\gamma = \begin{pmatrix} 0 & 0 & 1_q \\ 0 & 1_{p-q} & 0 \\ -1_q & 0 & 0 \end{pmatrix}$. Note that (temporarily) writing $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \in \mathbb{H}_{p,q}$, we have

$$W = \gamma \cdot Z = \begin{pmatrix} -Z_1^{-1} \\ -Z_2 Z_1^{-1} \end{pmatrix}, \quad J_1(\gamma, Z) = \begin{pmatrix} -Z_1^{-1} & 0 \\ -Z_2 Z_1^{-1} & 1 \end{pmatrix} \quad \text{and} \quad J_2(\gamma, Z) = -Z_1,$$

so $\det(J_1(\gamma, Z)) = \det(J_2(\gamma, Z))^{-1} = \det(-Z_1)^{-1}$. We use a similar lemma to Lemma 4.5.

Lemma 4.8. *Let P be a pluriharmonic polynomial, homogeneous of degree (v, v'). Then, for any $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \in \mathbb{H}_{p,q}$ we have the following equality:*

$$\int_{M_{q,k}(\mathbb{C})} e^{i \operatorname{Tr} UY^*} e^{i \operatorname{Tr} U^* Y} e^{i \operatorname{Tr} U^* Z_1 U} e^{i \operatorname{Tr} V^* Z_2 Y} P\left(U, \overline{\begin{pmatrix} U \\ iV \end{pmatrix}}\right) |dU|^2$$
$$= (-2\pi i)^{qk} \det(-Z_1)^{-(k+v+v')} e^{i \operatorname{Tr} Y^* (-Z_1)^{-1} Y} e^{\operatorname{Tr} V^* (-Z_2 Z_1^{-1}) Y} P\left(Y, \overline{\begin{pmatrix} Y \\ iV \end{pmatrix}}\right).$$

Once again, this can be readily deduced from [26, 4.2, p.39]. Then, we consider the function

$$\begin{aligned} F_1((Z_j)_{j=1}^r) &:= \exp\left(i \sum_{j=1}^r \operatorname{Tr} \begin{pmatrix} Y_j \\ iV_j \end{pmatrix}^* Z_j Y_j\right) \\ &= \exp\left(i \sum_{j=1}^r \operatorname{Tr}\big(Y_j^* (Z_j)_1 Y_j\big)\right) \exp\left(\sum_{j=1}^r \operatorname{Tr}\big(V_j^* (Z_j)_2 Y_j\big)\right), \end{aligned}$$

where for each $j = 1, \ldots, r$, Y_j (resp. V_j) is a $q \times k_i$ (resp. $(p-q) \times k_i$) matrix of independent variables, and $Z_j = \begin{pmatrix} (Z_j)_1 \\ (Z_j)_2 \end{pmatrix} \in \mathbb{H}_{p,q}$. Note that this is completely

analogous to the function F_0 defined in the proof of Theorem 4.1 (of course, V is zero-dimensional in signature (n,n)) but here we try a little harder to avoid any abuse of notation. As before, for any monomial Q_M we have

$$Q_M\left(\left(\frac{\partial}{\partial Z_j}\right)_j\right)(F_1((Z_j)_j)) = i^{|M|}Q_M\left(\left(Y_j\begin{pmatrix}Y_j\\ iV_j\end{pmatrix}^*\right)_j\right)F_1((Z_j)_j).$$

Since each $k_j \geq q$ the components of $Y_j\begin{pmatrix}Y_j\\ iV_j\end{pmatrix}^*$ are algebraically independent, so it suffices to show the result for F_1. But the relation for F_1 follows from Lemma 4.8 by an almost identical argument to that for F_0 using Lemma 4.5. So, if $Q \in \mathcal{H}_{(q,p),(v,v')}(k_1,\ldots,k_r)$ then (4.4) is satisfied. The converse similarly follows. Considering the action of generators of the forms (I,II,IV,V) on F_1 gives homogeneity. Then, harmonicity again follows from Lemma 4.6 by essentially the same reasoning as before, noting that $P\left(Y, \overline{\begin{pmatrix}Y\\ iV\end{pmatrix}}\right)$ satisfies the hypotheses (considered as a polynomial in Y and $\overline{Y}$) by Lemma 4.8. □

Corollary 4.9. *Suppose, in the setting of Theorem 4.7, we have that $F(Z_1,\ldots,Z_r) = f_1(Z_1)\ldots f_r(Z_r)$ with each f_j $(j=1,\ldots,r)$ a scalar-valued Hermitian modular form of weight $(-\kappa_1^{(j)},\kappa_2^{(j)})$ of some level Γ. Then applying the differential operator associated to some polynomial $Q \in \mathcal{H}_{(q,p),(v,v')}(k_1,\ldots,k_r)$ gives a Hermitian modular form of weight $(-(\kappa_1+v'),\kappa_2+v)$ and level Γ.*

Remark 4.10. Firstly, Corollaries 4.3 and 4.9 are not completely unknown in the literature (they are in some sense variants of [1, Theorem 4.2.5], although the statements are slightly different; see Chapter 6 for a direct comparison). However, the perspective and method of proof are new in the case of Hermitian modular forms. Secondly, it is worth noting that we say nothing about the existence of (non-trivial) such operators in this theorem, since for many choices of p,q,v,v' the space of polynomials $\mathcal{H}_{(q,p),(v,v')}(k_1,\ldots,k_r)$ is zero. For example, one can consider the action of $\alpha 1_q$ and $\alpha 1_p$ for any $\alpha \in \mathbb{C}^\times$ on such a polynomial Q. We have that

$$Q(\alpha 1_q Z_1,\ldots,\alpha 1_q Z_r) = Q(\alpha Z_1,\ldots,\alpha Z_r) = Q(Z_1\alpha 1_p,\ldots,Z_r\alpha 1_p).$$

By the homogeneity of Q, the left-hand side is $\alpha^{qv}Q(Z_1,\ldots,Z_r)$ and the right-hand side is $\alpha^{pv'}Q(Z_1,\ldots,Z_r)$. Since this holds for every value of Z and every value of

$\alpha \in \mathbb{C}^\times$, if $qv \neq pv'$ then Q must be identically zero. For instance, in signature (n, n), to have any chance of constructing non-zero operators we must therefore require that $v = v'$, which is precisely the case in Theorem 4.1 (and the case we discuss further in Chapter 5).

CHAPTER 5

EXPLICIT DESCRIPTION OF BILINEAR DIFFERENTIAL OPERATORS

From Francis Dunn, *Rankin–Cohen Type Differential Operators on Hermitian Modular Forms*, Res. Number Theory **10** (2024), no. 88, §4.

We shall return to the setting of Theorem 4.1 (or equivalently, of Theorem 4.7 with $p = q = n$ and $v = v'$, if one prefers that formulation). The following work appeared as [13, §4]. In this section we shall state and prove Theorem 1.2. We shall explicitly describe differential operators in the case $r = 2$, i.e. the space $\mathcal{H}_{n,v}(k_1, k_2)$, for a fixed CM field L/L^+ and k_1, k_2, n, v as in the statement of Theorem 4.1. That is, we describe the space of polynomials $P(X, Y)$ on entries $X, Y \in M_{n,k}$ (where $k = k_1 + k_2$) satisfying the following three conditions:

$$(*) \qquad P(AX, BY) = \det(A)^v \det(B)^v P(X, Y) \quad \forall A, B \in \mathrm{GL}(n, \mathbb{C}),$$

$$(**) \qquad P(XC^{-1}, Y^t C) = P(X, Y) \quad \forall C \in \mathrm{GL}(k_1, \mathbb{C}) \oplus \mathrm{GL}(k_2, \mathbb{C}) \subset \mathrm{GL}(k, \mathbb{C}),$$

where $\mathrm{GL}(k_1, \mathbb{C}) \oplus \mathrm{GL}(k_2, \mathbb{C})$ sits inside $\mathrm{GL}(k, \mathbb{C})$ as the block-diagonal embedding, and the pluriharmonic condition

$$(***) \qquad \Delta_{s,t} P(X, Y) = \sum_{u=1}^{k} \frac{\partial^2}{\partial x_{s,u} \partial y_{t,u}} P(X, Y) = 0 \quad \forall 1 \leq s, t \leq n.$$

Note that condition $(**)$ is equivalent to being able to write $P(X, Y)$ as a polynomial $Q({X_1}^t Y_1, {X_2}^t Y_2)$ in the entries of two $n \times n$ matrices, where X_1 and Y_1 are the first k_1 columns of X and Y respectively, X_2 and Y_2 the last k_2 columns of X and Y respectively. In this case, writing W and Z for the $n \times n$ variables of the polynomial Q, condition $(*)$ can be reformulated as

$$(*) \qquad Q(AW^t B, AZ^t B) = \det(A)^v \det(B)^v Q(W, Z).$$

Up to isomorphism (so to match up with Shimura's description in [36]) this is equivalent to:

$$(*) \qquad Q({}^t AWB, {}^t AZB) = \det(A)^v \det(B)^v Q(W, Z).$$

As in Definition 3.12 we denote by $\mathcal{H}_{n,v}(k_1, k_2)$ the space of such polynomials Q corresponding to polynomials P satisfying properties $(*)$, $(**)$ and $(***)$. This sits inside the space $\mathcal{Q}_{n,v}(2)$ of polynomials Q satisfying $(*)$ (and $(**)$ by virtue of

being polynomials in $X_1{}^tY_1, X_2{}^tY_2$). Let $\mathcal{Q}_n(2) = \oplus_{v\geq 0}\mathcal{Q}_{n,v}(2)$. Note that this is a graded ring.

The notation used for monomials here is $X^\alpha Y^\beta$ or $W^l Z^m$ where α, β, l and m are "index matrices" with entries in $\mathbb{Z}_{\geq 0}$. More precisely, for a matrix variable $X \in M_{n,k}$ and $\alpha \in M_{n,k}(\mathbb{Z}_{\geq 0})$

$$X^\alpha = \prod_{i,j} x_{ij}^{\alpha_{ij}},$$

and similarly for Y^β, W^ℓ and Z^m.

Recall that

$$\det(W) = \sum_{\sigma\in S_n} \mathrm{sign}(\sigma) W^\sigma,$$

where by an abuse of notation we write σ for both the permutation in S_n and the corresponding permutation matrix, i.e. $\sigma_{ij} = 1$ if $\sigma(i) = j$, and $\sigma_{ij} = 0$ otherwise.

Define polynomials $Q_a(W, Z)$ for $a = 0, \ldots, n$ by

$$\det(W + \lambda Z) = \sum_{a=0}^{n} Q_a(W, Z)\lambda^a.$$

The goal is to eventually show that the space $\mathcal{Q}_n(2)$ is generated by the polynomials $\{Q_0, \ldots, Q_n\}$ as a $\mathbb{C}$-algebra. We begin with the following initial result:

Lemma 5.1. *The polynomials $Q_0, \ldots, Q_n$ are a $\mathbb{C}$-basis for $\mathcal{Q}_{n,1}(2)$.*

Proof. First, the polynomials $Q_0, \ldots, Q_n$ must be linearly independent over $\mathbb{C}$ since they share no monomials. To prove that they span $\mathcal{Q}_{n,1}(2)$, we show that any non-zero $Q \in \mathcal{Q}_{n,1}(2)$ has some multiple of one of the polynomials $Q_0, \ldots, Q_n$ as a summand. For any such Q, take a monomial with a non-zero coefficient, $C_{\ell,m}W^\ell Z^m$. Since $Q \in \mathcal{Q}_{n,1}(2)$, $\ell + m$ must be a permutation matrix in order for Q to satisfy the homogeneity condition $(*)$ under the action of diagonal matrices. Let a be the number of non-zero entries of m (so $0 \leq a \leq n$). Also by the homogeneity condition considered for permutation matrices, we know that Q in fact has a summand

$$\sum_{\sigma\in S_n}\sum_{\tau\in S_n} \frac{C_{\ell,m}}{a!(n-a)!}\mathrm{sign}(\sigma)\mathrm{sign}(\tau)W^{\sigma\ell\tau}Z^{\sigma m\tau}.$$

Observing that we may write

$$Q_a(W, Z) = \sum_{\sigma,\tau\in S_n} \frac{1}{a!(n-a)!}\mathrm{sign}(\sigma)\mathrm{sign}(\tau)W^{\sigma^{-1}(I_{n-a}\oplus 0_a)\tau}Z^{\sigma^{-1}(0_{n-a}\oplus I_a)\tau},$$

we see that the summand of Q we found above is (a re-ordering of) the polynomial $C_{\ell,m}Q_a$.

□

Since all of the polynomials $Q_0, \ldots, Q_n \in \mathcal{Q}_{n,1}(2)$, it follows that for any $v \geq 1$, any product of v of $Q_0, \ldots, Q_n$ lies inside $\mathcal{Q}_{n,v}(2)$, that is $\mathbb{C}[Q_0, \ldots, Q_n]_v \subseteq \mathcal{Q}_{n,v}(2)$ and $\mathbb{C}[Q_0, \ldots, Q_n] \subseteq \mathcal{Q}_n(2)$. To show that these containments are in fact equalities, we show that the v-products of $Q_0, \ldots, Q_n$ are a basis for $\mathcal{Q}_{n,v}(2)$ for each v. Their linear independence is a consequence of the next lemma, then we show that $\mathcal{Q}_{n,v}(2)$ is of the correct dimension, specifically $\binom{n+v}{v}$.

Lemma 5.2. *The polynomials $Q_0(W,Z), Q_1(W,Z), \ldots, Q_n(W,Z)$ are algebraically independent in $\mathbb{C}[W,Z]$.*

Proof. Recall that we defined the polynomials Q_a (for each $a = 0, \ldots, n$) by

$$\det(W + \lambda Z) = \sum_{a=0}^{n} Q_a(W,Z)\lambda^a.$$

In particular, evaluating at $Z = 1_n$ we have that

$$\det(W + \lambda 1_n) = \sum_{a=0}^{n} Q_a(W, 1_n)\lambda^a$$

and therefore $Q_a(W, 1_n)$ is the $(n-a)$th elementary symmetric polynomial in the eigenvalues of W (counted with multiplicity). Explicitly, if W is diagonal then we have

$$\sum_{a=0}^{n} s_{n-a}(w_{1,1}, w_{2,2}, \ldots, w_{n,n})\lambda^a = \prod_{i=1}^{n}(w_{i,i} + \lambda) = \det(W + \lambda 1_n) = \sum_{a=0}^{n} Q_a(W, 1_n)\lambda^a,$$

where s_{n-a} is the $(n-a)$th elementary symmetric polynomial on n arguments (and s_0 is the constant polynomial 1). So any non-trivial algebraic relation on $Q_0, \ldots, Q_n$ restricts to an algebraic relation on $s_0, \ldots, s_n$ inside the ring of polynomials in n variables (given by the eigenvalues of W). But the elementary symmetric polynomials are algebraically independent, so the polynomials $Q_0, \ldots, Q_n$ must be too. □

Proposition 5.3. *The dimension of $\mathcal{Q}_{n,v}(2)$ as a vector space over $\mathbb{C}$ is $\binom{n+v}{v}$ for each integer $v > 0$. (The same holds for $v = 0$ trivially.)*

Proof. First, we have that $\mathbb{C}[W, Z] \cong \mathbb{C}[W] \otimes \mathbb{C}[Z]$ not only as $\mathbb{C}$-algebras but also as $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$ representations. Now, $\mathcal{Q}_{n,v}(2) \subset \mathbb{C}[W, Z]_{nv}$ since, by considering the action of diagonal matrices, for example, or by previous arguments, each polynomial $Q \in \mathcal{Q}_{n,v}(2)$ is homogeneous of total degree nv in W and Z. The standard decomposition of tensor products is

$$\mathbb{C}[W, Z]_{nv} \cong \bigoplus_{a=0}^{nv} \mathbb{C}[W]_a \otimes \mathbb{C}[Z]_{nv-a}.$$

Now, the space $\mathbb{C}[W]_a$ together with its $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$-action $(A, B)f(W) = f({}^tAWB)$ is precisely the representation τ^a of Shimura [36, §12], and similarly $\mathbb{C}[Z]_{nv-a}$ with the same action is Shimura's τ^{nv-a}. Since $\mathcal{Q}_{n,v}(2)$ is precisely the summand of $\det^v \boxtimes \det^v$ inside $\mathbb{C}[W, Z]_{nv}$, we just need to find the summands (or, for the dimension, the multiplicity of) the representation $\det^v \boxtimes \det^v$ in all of the summands $\mathbb{C}[W]_a \otimes \mathbb{C}[Z]_{nv-a} = \tau^a \otimes \tau^{nv-a}$ $(a = 0, \ldots, nv)$.

By [36, Theorem 12.7] (attributed to L.-K. Hua), for any irreducible representations ρ_1 and ρ_2 of $\mathrm{GL}(n, \mathbb{C})$, the representation $\rho_1 \boxtimes \rho_2$ occurs as a summand in the representation τ^a with multiplicity one if and only if ρ_1 and ρ_2 correspond to the same partition λ of a (with at most n parts), $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ and $\lambda_1 + \cdots + \lambda_n = a$, and does not occur otherwise. For representations ρ_1 and ρ_2 both corresponding to the same n-part partition λ of a, and representations σ_1 and σ_2 both corresponding to the same n-part partition μ of $nv - a$, the multiplicity of $\det^v \boxtimes \det^v$ inside $(\rho_1 \boxtimes \rho_2) \otimes (\sigma_1 \boxtimes \sigma_2)$ is the product of the multiplicity of $\det^v$ in $\rho_1 \otimes \sigma_1$ and $\rho_2 \otimes \sigma_2$ respectively.

Since this multiplicity depends only on the partitions λ and μ, the representation $\det^v$ occurs with the same multiplicity within $\rho_1 \otimes \sigma_1$ and $\rho_2 \otimes \sigma_2$. This multiplicity is precisely given by the Littlewood–Richardson coefficient $\mathrm{LR}_{\lambda\mu}^{(v^n)}$, where (v^n) denotes the partition $(v, v, \ldots, v)$ of nv. Either by application of the Littlewood–Richardson rule (as stated in [20, Chapter 5] for example), or as a consequence of a result of Okada (stated as a remark following Theorem 2.1 of [34]) the coefficient

$\mathrm{LR}_{\lambda\mu}^{(v^n)} = 1$ if and only if $\lambda_1 \leq v$ and $\mu_i + \lambda_{n+1-i} = v$ for each $i = 1, \ldots, n$, and $\mathrm{LR}_{\lambda\mu}^{(v^n)} = 0$ otherwise.

So, for each $a = 0, \ldots, nv$ and each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of a with $\lambda_1 \leq v$ there is exactly one copy of $\det^v \boxtimes \det^v$ inside $\tau^a \otimes \tau^{nv-a}$. That is, the multiplicity of $\det^v \boxtimes \det^v$ inside $\tau^a \otimes \tau^{nv-a}$ is equal to the number of Young diagrams with a boxes fitting inside $(v)^n$, the $v \times n$ rectangle. So the total multiplicity of $\det^v \otimes \det^v$ in $\mathbb{C}[W, Z]_{nv}$ is equal to the number of Young diagrams of any size inside $(v)^n$. By drawing a lattice path along the lower-right edge of any such Young diagram, we see that this is equal to the number of lattice paths from $(0, 0)$ to (v, n). This is equal to $\binom{n+v}{v}$ since the path is $n + v$ steps long, and we just need to choose v of those steps at which to travel right or equivalently n of the steps at which to travel upwards.

□

Corollary 5.4. *We have*

$$\mathcal{Q}_n(2) = \mathbb{C}[Q_0, \ldots, Q_n] \qquad \textit{and} \qquad \mathcal{Q}_{n,v}(2) = \mathbb{C}[Q_0, \ldots, Q_n]_v.$$

Proof. By Lemma 5.1, the polynomials $Q_0, \ldots, Q_n$ are a basis for $\mathcal{Q}_{n,1}(2)$ as a vector space over $\mathbb{C}$. Any v-product of the polynomials $Q_0, \ldots, Q_n$ is an element of $\mathcal{Q}_{n,v}(2)$. By Lemma 5.2, the polynomials $Q_0, \ldots, Q_n$ are algebraically independent and therefore all the distinct v-products of $Q_0, \ldots, Q_n$ are linearly independent over $\mathbb{C}$. There are $\binom{n+v}{v}$ such products, and by Proposition 5.3 this is the dimension of $\mathcal{Q}_{n,v}(2)$, so the set of v-products of the polynomials $Q_0, \ldots, Q_n$ is a basis for $\mathcal{Q}_{n,v}(2)$ for every degree v.

□

Remark 5.5. A priori, it is unclear that the space of polynomials inside $\mathcal{Q}_{n,v}(2)$ which arise as polynomials of the form $P(X, Y)$ is necessarily all of $\mathcal{Q}_{n,v}(2)$. That is, not every polynomial $Q \in \mathcal{Q}_{n,v}(2)$ might correspond to a polynomial $P(X, Y)$. However, having shown that $Q_0, \ldots, Q_n$ generate $\mathcal{Q}_n(2)$ then we see that this is indeed the case since, setting $W = X_1{}^t Y_1$ and $Z = X_2{}^t Y_2$ as usual,

$$\sum_{m=0}^{n} Q_m(W, Z)\lambda^m = \det(W + \lambda Z) = \det\big(X_1{}^t Y_1 + \lambda X_2{}^t Y_2\big) = \sum_{m=0}^{n} P_m(X, Y)\lambda^m,$$

for some polynomials $P_m(X, Y)$. By comparing coefficients of λ^m, we must have that $P_m(X, Y) = Q_m(X_1{}^t Y_1, X_2{}^t Y_2)$.

Theorem 5.6. *For each fixed n and v, writing a polynomial $Q \in \mathcal{Q}_{n,v}(2)$ as*

$$Q = \sum_{\alpha} C(\alpha) \prod_{j=0}^{n} Q_j^{\alpha_j},$$

where the sum ranges over index tuples $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ with $\sum_{j=0}^{n} \alpha_j = v$, Q is pluriharmonic i.e. $Q \in \mathcal{H}_{n,v}(k_1, k_2)$ if the coefficients $C(\alpha)$ with $\alpha \neq (0, \ldots, 0, v)$ satisfy the following linear relations:

$$\begin{aligned} 0 =& C(\alpha)\alpha_m(k_1 + 1 - n + m) + C(\alpha - 1_m + 1_{m+1})(\alpha - 1_m + 1_{m+1})_{m+1}(k_2 - m) \\ &+ C(\alpha)\alpha_m(\alpha_m - 1) \\ &+ \sum_{\substack{m < \ell \leq \ell' \leq n \\ \ell + \ell' - m - 1 \leq n}} C(\tilde{\alpha}(m, \ell, \ell'))\tilde{\alpha}(m, \ell, \ell')_\ell(\tilde{\alpha}(m, \ell, \ell')_{\ell'} - \delta_{\ell,\ell'})(2 - \delta_{\ell,\ell'}) \\ &- \sum_{\substack{m < \ell \leq \ell' \leq n \\ \ell + \ell' - m \leq n}} C(\alpha(m, \ell, \ell'))\alpha(m, \ell, \ell')_\ell(\alpha(m, \ell, \ell')_{\ell'} - \delta_{\ell,\ell'})(2 - \delta_{\ell,\ell'}). \end{aligned}$$

Here $0 \leq m < n$ is the least integer such that $\alpha_m > 0$ and we write $\alpha(m, \ell, \ell') := \alpha - 1_m + 1_\ell + 1_{\ell'} - 1_{\ell+\ell'-m}$ and $\tilde{\alpha}(m, \ell, \ell') := \alpha - 1_m + 1_\ell + 1_{\ell'} - 1_{\ell+\ell'-m-1}$. In particular, every coefficient $C(\alpha)$ is determined by the value of $C(0, \ldots, 0, v)$ so there is a unique (up to scaling) pluriharmonic polynomial inside $\mathcal{Q}_{n,v}(2)$, $\mathcal{H}_{n,v}(k_1, k_2)$ is one-dimensional.

Proof. We begin by rephrasing $(***)$, restated below, in terms of the polynomial $Q(W, Z)$ associated to $P(X, Y)$.

$$(***) \qquad \Delta_{i,j} P(X, Y) = \sum_{u=1}^{k} \frac{\partial^2}{\partial x_{i,u} \partial y_{j,u}} P(X, Y) = 0 \quad \forall 1 \leq i, j \leq n.$$

First we re-write the operator $\Delta_{i,j}$ (in terms of X, Y acting on $P(X, Y)$) as a differential operator in terms of $W = X_1^t Y_1$ and $Z = X_2^t Y_2$ acting on the polynomial $Q(W, Z)$ associated to P (so $P(X, Y) = Q(W, Z)$). We have

$$w_{i,j} = \sum_{u=1}^{k_1} x_{i,u} y_{j,u} \qquad z_{i,j} = \sum_{u=k_1+1}^{k} x_{i,u} y_{j,u},$$

$$\sum_{u=1}^{k} \frac{\partial}{\partial y_{j,u}} = \sum_{s=1}^{n} \left(\sum_{u=1}^{k_1} x_{s,u} \frac{\partial}{\partial w_{s,j}} + \sum_{u=k_1+1}^{k} x_{s,u} \frac{\partial}{\partial z_{s,j}} \right),$$

and then

$$\begin{aligned}
&\sum_{u=1}^{k} \frac{\partial}{\partial x_{i,u}} \frac{\partial}{\partial y_{j,u}} \\
&= k_1 \frac{\partial}{\partial w_{i,j}} + \sum_{s,t=1}^{n} \sum_{u=1}^{k_1} x_{s,u} y_{t,u} \frac{\partial}{\partial w_{s,j}} \frac{\partial}{\partial w_{i,t}} + k_2 \frac{\partial}{\partial z_{i,j}} + \sum_{s,t=1}^{n} \sum_{u=k_1+1}^{k} x_{s,u} y_{t,u} \frac{\partial}{\partial z_{s,j}} \frac{\partial}{\partial z_{i,t}} \\
&= k_1 \frac{\partial}{\partial w_{i,j}} + \sum_{s,t=1}^{n} w_{s,t} \frac{\partial}{\partial w_{s,j}} \frac{\partial}{\partial w_{i,t}} + k_2 \frac{\partial}{\partial z_{i,j}} + \sum_{s,t=1}^{n} z_{s,t} \frac{\partial}{\partial z_{s,j}} \frac{\partial}{\partial z_{i,t}}.
\end{aligned}$$

We write

$$L_{i,j}^{(k_1)} := k_1 \frac{\partial}{\partial w_{i,j}} + \sum_{s,t=1}^{n} w_{s,t} \frac{\partial}{\partial w_{s,j}} \frac{\partial}{\partial w_{i,t}} \quad \text{and} \quad L_{i,j}^{\prime(k_2)} := k_2 \frac{\partial}{\partial z_{i,j}} + \sum_{s,t=1}^{n} z_{s,t} \frac{\partial}{\partial z_{s,j}} \frac{\partial}{\partial z_{i,t}}.$$

So for a polynomial P satisfying $(**)$, $\Delta_{i,j}(P(X,Y)) = 0$ if and only if $(L_{i,j}^{(k_1)} + L_{i,j}^{\prime(k_2)})(Q(W,Z)) = 0$ where Q is the polynomial associated to P. Recall that for a polynomial P (and associated Q) satisfying $(*)$ and $(**)$, $(***)$ holds if and only if $\sum_{i=1}^{n} \Delta_{i,i}(P(X,Y)) = 0$.

Therefore, to find polynomials in $\mathcal{H}_{n,v}(k_1,k_2)$ we should calculate the action of $L_{i,i}^{(k_1)}$ and $L_{i,i}^{\prime(k_2)}$ on $Q_0,\ldots,Q_n$ and on products. First, for $Q,Q' \in \mathcal{Q}_n(2)$ we have

$$L_{i,i}^{(k_1)}(QQ') = L_{i,i}^{(k_1)}(Q)\cdot Q' + Q\cdot L_{i,i}^{(k_1)}(Q') + \sum_{s,t=1}^{n} w_{s,t} \left(\frac{\partial}{\partial w_{s,i}} Q \cdot \frac{\partial}{\partial w_{i,t}} Q' + \frac{\partial}{\partial w_{i,t}} Q \cdot \frac{\partial}{\partial w_{s,i}} Q' \right),$$

and

$$L_{i,i}^{\prime(k_2)}(QQ') = L_{i,i}^{\prime(k_2)}(Q)\cdot Q' + Q\cdot L_{i,i}^{\prime(k_2)}(Q') + \sum_{s,t=1}^{n} z_{s,t} \left(\frac{\partial}{\partial z_{s,i}} Q \cdot \frac{\partial}{\partial z_{i,t}} Q' + \frac{\partial}{\partial z_{i,t}} Q \cdot \frac{\partial}{\partial z_{s,i}} Q' \right).$$

For ease of notation, we use the following:

$$(Q,Q')_{i,W} := \sum_{s,t=1}^{n} w_{s,t} \left(\frac{\partial}{\partial w_{s,i}} Q \cdot \frac{\partial}{\partial w_{i,t}} Q' + \frac{\partial}{\partial w_{i,t}} Q \cdot \frac{\partial}{\partial w_{s,i}} Q' \right),$$

and

$$(Q,Q')_{i,Z} := \sum_{s,t=1}^{n} z_{s,t} \left(\frac{\partial}{\partial z_{s,i}} Q \cdot \frac{\partial}{\partial z_{i,t}} Q' + \frac{\partial}{\partial z_{i,t}} Q \cdot \frac{\partial}{\partial z_{s,i}} Q' \right).$$

So, re-writing the above, we obtain:

$$L_{i,i}^{(k_1)}(QQ') = L_{i,i}^{(k_1)}(Q)\cdot Q' + Q\cdot L_{i,i}^{(k_1)}(Q') + (Q,Q')_{i,W},$$

and

$$L'^{(k_2)}_{i,i}(QQ') = L'^{(k_2)}_{i,i}(Q) \cdot Q' + Q \cdot L'^{(k_2)}_{i,i}(Q') + (Q, Q')_{i,Z}.$$

Next, we calculate $L^{(k_1)}_{i,i}(Q_a)$ and $L'^{(k_2)}_{i,i}(Q_a)$ for $a = 0, \ldots, n$. There are two easy cases: recall that $Q_0(W, Z) = \det(W)$ and $Q_n(W, Z) = \det(Z)$ so

$$L^{(k_1)}_{i,i}(Q_n) = L'^{(k_2)}_{i,i}(Q_0) = 0.$$

For the other cases, we compute $L^{(k_1)}_{i,i}(\det(W + \lambda Z))$ and $L'^{(k_2)}_{i,i}(\det(W + \lambda Z))$ respectively, and consider the coefficients of λ^a.

Here we'll introduce some notation. For an $n \times n$ matrix A, we write $A^{\left[\begin{smallmatrix} i_1, \ldots, i_\ell \\ j_1, \ldots, j_\ell \end{smallmatrix}\right]}$ for the $(n - \ell) \times (n - \ell)$ matrix obtained from A by deleting rows indexed $i_1, \ldots, i_\ell$ and columns indexed $j_1, \ldots, j_\ell$. Then, we write

$$\det\left((W + \lambda Z)^{\left[\begin{smallmatrix} i_1, \ldots, i_\ell \\ j_1, \ldots, j_\ell \end{smallmatrix}\right]}\right) = \sum_{m=0}^{n-\ell} Q_m^{\left[\begin{smallmatrix} i_1, \ldots, i_\ell \\ j_1, \ldots, j_\ell \end{smallmatrix}\right]}(W, Z)\lambda^m.$$

That is, $Q_m^{\left[\begin{smallmatrix} i_1, \ldots, i_\ell \\ j_1, \ldots, j_\ell \end{smallmatrix}\right]}(W, Z)$ is the coefficient of λ^m in $\det(W + \lambda Z)^{\left[\begin{smallmatrix} i_1, \ldots, i_\ell \\ j_1, \ldots, j_\ell \end{smallmatrix}\right]}$ (and we take $Q_m^{\left[\begin{smallmatrix} i_1, \ldots, i_\ell \\ j_1, \ldots, j_\ell \end{smallmatrix}\right]}(W, Z) = 0$ if $m < 0$ or $m > n - \ell$). Now:

$$\begin{aligned}
L^{(k_1)}_{i,i}(\det(W + \lambda Z)) &= k_1 \frac{\partial}{\partial w_{i,i}} \det(W + \lambda Z) + \sum_{s=1}^{n}\sum_{t=1}^{n} w_{s,t} \frac{\partial}{\partial w_{s,i}} \frac{\partial}{\partial w_{i,t}} \det(W + \lambda Z) \\
&= k_1 \det(W + \lambda Z)^{\left[\begin{smallmatrix} i \\ i \end{smallmatrix}\right]} + w_{i,i} \frac{\partial}{\partial w_{i,i}} \det(W + \lambda Z)^{\left[\begin{smallmatrix} i \\ i \end{smallmatrix}\right]} \\
&+ \sum_{\substack{s=1 \\ s \neq i}}^{n} w_{s,i} \frac{\partial}{\partial w_{s,i}} \det(W + \lambda Z)^{\left[\begin{smallmatrix} i \\ i \end{smallmatrix}\right]} + \sum_{\substack{t=1 \\ t \neq i}}^{n} w_{i,t} \frac{\partial}{\partial w_{i,i}} (-1)^{i+t} \det(W + \lambda Z)^{\left[\begin{smallmatrix} i \\ t \end{smallmatrix}\right]} \\
&+ \sum_{\substack{s=1 \\ s \neq i}}^{n}\sum_{\substack{t=1 \\ t \neq i}}^{n} w_{s,t} \frac{\partial}{\partial w_{s,i}} (-1)^{i+t} \det(W + \lambda Z)^{\left[\begin{smallmatrix} i \\ t \end{smallmatrix}\right]}.
\end{aligned}$$

Notice that in the middle three terms in the sum above we deleted the ith row or the ith column, so those partial derivatives of determinants are all zero. So

$$\begin{aligned}
L^{(k_1)}_{i,i}(\det(W + \lambda Z)) &= k_1 \det(W + \lambda Z)^{\left[\begin{smallmatrix} i \\ i \end{smallmatrix}\right]} + \sum_{\substack{s=1 \\ s \neq i}}^{n}\sum_{\substack{t=1 \\ t \neq i}}^{n} w_{s,t} \frac{\partial}{\partial w_{s,i}} (-1)^{i+t} \det(W + \lambda Z)^{\left[\begin{smallmatrix} i \\ t \end{smallmatrix}\right]} \\
&= k_1 \det(W + \lambda Z)^{\left[\begin{smallmatrix} i \\ i \end{smallmatrix}\right]} - \sum_{\substack{s=1 \\ s \neq i}}^{n}\sum_{\substack{t=1 \\ t \neq i}}^{n} w_{s,t} (-1)^{s+t+\delta_{s>i}+\delta_{t>i}} \det(W + \lambda Z)^{\left[\begin{smallmatrix} s,i \\ t,i \end{smallmatrix}\right]},
\end{aligned}$$

where $\delta_{s>i} = 1$ if $s > i$ and 0 otherwise, and above we use that for $i \neq t$ we have $\delta_{i>t} + \delta_{t>i} = 1$. We compute:

$$\begin{aligned}
&\sum_{\substack{s=1\\s\neq i}}^{n}\sum_{\substack{t=1\\t\neq i}}^{n} w_{s,t}(-1)^{s+t+\delta_{s>i}+\delta_{t>i}} \det(W+\lambda Z)^{\left[\begin{smallmatrix}s,i\\t,i\end{smallmatrix}\right]} \\
&\qquad = \sum_{\substack{s=1\\s\neq i}}^{n}\sum_{\substack{t=1\\t\neq i}}^{n} (w_{s,t}+\lambda z_{s,t})(-1)^{s+t+\delta_{s>i}+\delta_{t>i}} \det(W+\lambda Z)^{\left[\begin{smallmatrix}s,i\\t,i\end{smallmatrix}\right]} \\
&\qquad\quad - \sum_{\substack{s=1\\s\neq i}}^{n}\sum_{\substack{t=1\\t\neq i}}^{n} \lambda z_{s,t}(-1)^{s+t+\delta_{s>i}+\delta_{t>i}} \det(W+\lambda Z)^{\left[\begin{smallmatrix}s,i\\t,i\end{smallmatrix}\right]} \\
&\qquad = (n-1)\det(W+\lambda Z)^{\left[\begin{smallmatrix}i\\i\end{smallmatrix}\right]} - \lambda\frac{d}{d\lambda}\left(\det(W+\lambda Z)^{\left[\begin{smallmatrix}i\\i\end{smallmatrix}\right]}\right).
\end{aligned}$$

That is,

$$L_{i,i}^{(k_1)}(\det(W+\lambda Z)) = k_1\det(W+\lambda Z)^{\left[\begin{smallmatrix}i\\i\end{smallmatrix}\right]} - (n-1)\det(W+\lambda Z)^{\left[\begin{smallmatrix}i\\i\end{smallmatrix}\right]} + \lambda\frac{d}{d\lambda}\left(\det(W+\lambda Z)^{\left[\begin{smallmatrix}i\\i\end{smallmatrix}\right]}\right).$$

Comparing coefficients of λ^a (for $a = 0, \ldots, n-1$) we find

$$L_{i,i}^{(k_1)}(Q_a(W,Z)) = (k_1+1-n+a)Q_a^{\left[\begin{smallmatrix}i\\i\end{smallmatrix}\right]}(W,Z). \tag{5.1}$$

Similarly, $L_{i,i}^{\prime(k_2)}(\det(W+\lambda Z))$ is equal to the following:

$$\begin{aligned}
&k_2\frac{\partial}{\partial z_{i,i}}\det(W+\lambda Z) + \sum_{s=1}^{n}\sum_{t=1}^{n} z_{s,t}\frac{\partial}{\partial z_{s,i}}\frac{\partial}{\partial z_{i,t}}\det(W+\lambda Z) \\
&= k_2\lambda\det(W+\lambda Z)^{\left[\begin{smallmatrix}i\\i\end{smallmatrix}\right]} - \sum_{\substack{s=1\\s\neq i}}^{n}\sum_{\substack{t=1\\t\neq i}}^{n} z_{s,t}(-1)^{s+t+\delta_{s>i}+\delta_{t>i}}\lambda^2\det(W+\lambda Z)^{\left[\begin{smallmatrix}s,i\\t,i\end{smallmatrix}\right]} \\
&= k_2\lambda\det(W+\lambda Z)^{\left[\begin{smallmatrix}i\\i\end{smallmatrix}\right]} - \lambda^2\frac{d}{d\lambda}\left(\det(W+\lambda Z)^{\left[\begin{smallmatrix}i\\i\end{smallmatrix}\right]}\right).
\end{aligned}$$

Again by comparing coefficients of λ^a (for $a = 1, \ldots, n$) we get that

$$L_{i,i}^{\prime(k_2)}(Q_a(W,Z)) = (k_2+1-a)Q_{a-1}^{\left[\begin{smallmatrix}i\\i\end{smallmatrix}\right]}(W,Z). \tag{5.2}$$

Next, for $0 \leq a, b \leq n$, we compute:

$$\begin{aligned}
(Q_a,Q_b)_{i,W} &= \sum_{s,t=1}^{n} w_{s,t}\left((-1)^{s+i}Q_a^{\left[\begin{smallmatrix}s\\i\end{smallmatrix}\right]}(-1)^{i+t}Q_b^{\left[\begin{smallmatrix}i\\t\end{smallmatrix}\right]} + (-1)^{i+t}Q_a^{\left[\begin{smallmatrix}i\\t\end{smallmatrix}\right]}(-1)^{s+i}Q_b^{\left[\begin{smallmatrix}s\\i\end{smallmatrix}\right]}\right) \\
&= \sum_{s,t=1}^{n}(-1)^{s+t}w_{s,t}\left(Q_a^{\left[\begin{smallmatrix}s\\i\end{smallmatrix}\right]}Q_b^{\left[\begin{smallmatrix}i\\t\end{smallmatrix}\right]} + Q_a^{\left[\begin{smallmatrix}i\\t\end{smallmatrix}\right]}Q_b^{\left[\begin{smallmatrix}s\\i\end{smallmatrix}\right]}\right),
\end{aligned} \tag{5.3}$$

and

$$(Q_a, Q_b)_{i,Z} = \sum_{s,t=1}^{n} z_{s,t} \left((-1)^{s+i} Q_{a-1}^{[s]}{}_{[i]} (-1)^{i+t} Q_{b-1}^{[i]}{}_{[t]} + (-1)^{i+t} Q_{a-1}^{[i]}{}_{[t]} (-1)^{s+i} Q_{b-1}^{[s]}{}_{[i]} \right)$$
$$= \sum_{s,t=1}^{n} (-1)^{s+t} z_{s,t} \left(Q_{a-1}^{[s]}{}_{[i]} Q_{b-1}^{[i]}{}_{[t]} + Q_{a-1}^{[i]}{}_{[t]} Q_{b-1}^{[s]}{}_{[i]} \right). \tag{5.4}$$

Notice that

$$\sum_{s=1}^{n} (-1)^{s+t} (w_{s,t} + \lambda z_{s,t}) \det(W + \lambda Z)^{[s]}_{[i]} = \delta_{i,t} (-1)^{i+t} \det(W + \lambda Z) = \delta_{i,t} \det(W + \lambda Z). \tag{5.5}$$

Here $\delta_{i,t}$ is the usual Kronecker delta. This equality holds since the left hand side is (up to sign) the determinant of the matrix obtained from $W + \lambda Z$ by copying the entries of the t-th column to the i-th column, which is 0 if $i \neq t$ and the usual determinant of $W + \lambda Z$ otherwise. Comparing coefficients of λ^a on each side of (5.5) we find

$$\sum_{s=1}^{n} (-1)^{s+t} w_{s,t} Q_a^{[s]}{}_{[i]} + \sum_{s=1}^{n} (-1)^{s+t} z_{s,t} Q_{a-1}^{[s]}{}_{[i]} = \delta_{i,t} Q_a,$$

and

$$\sum_{t=1}^{n} (-1)^{s+t} w_{s,t} Q_a^{[i]}{}_{[t]} + \sum_{t=1}^{n} (-1)^{s+t} z_{s,t} Q_{a-1}^{[i]}{}_{[t]} = \delta_{s,i} Q_a.$$

In particular,

$$\sum_{t=1}^{n} \sum_{s=1}^{n} (-1)^{s+t} w_{s,t} Q_a^{[s]}{}_{[i]} Q_b^{[i]}{}_{[t]} + \sum_{t=1}^{n} \sum_{s=1}^{n} (-1)^{s+t} z_{s,t} Q_{a-1}^{[s]}{}_{[i]} Q_b^{[i]}{}_{[t]} = Q_a Q_b^{[i]}{}_{[i]},$$

and

$$\sum_{s=1}^{n} \sum_{t=1}^{n} (-1)^{s+t} w_{s,t} Q_a^{[i]}{}_{[t]} Q_b^{[s]}{}_{[i]} + \sum_{s=1}^{n} \sum_{t=1}^{n} (-1)^{s+t} z_{s,t} Q_{a-1}^{[i]}{}_{[t]} Q_b^{[s]}{}_{[i]} = Q_a Q_b^{[i]}{}_{[i]}.$$

Together with (5.3) and (5.4) we extract the following:

$$(Q_a, Q_b)_{i,W} + (Q_a, Q_{b+1})_{i,Z} = 2 Q_a Q_b^{[i]}{}_{[i]} \quad \text{and} \quad (Q_a, Q_b)_{i,W} + (Q_{a+1}, Q_b)_{i,Z} = 2 Q_a^{[i]}{}_{[i]} Q_b,$$

so

$$(Q_a, Q_b)_{i,W} = 2 Q_a Q_b^{[i]}{}_{[i]} - 2 Q_{a-1}^{[i]}{}_{[i]} Q_{b+1} + (Q_{a-1}, Q_{b+1})_{i,W}, \tag{5.6}$$

and

$$(Q_a, Q_b)_{i,Z} = 2 Q_{a-1}^{[i]}{}_{[i]} Q_b - 2 Q_{a-1} Q_b^{[i]}{}_{[i]} + (Q_{a-1}, Q_{b+1})_{i,Z}. \tag{5.7}$$

To proceed, we need the following lemma:

Lemma 5.7. *For any $n \geq 1$, the polynomials*

$$\sum_{i=1}^{n} Q_a^{\begin{bmatrix} i \\ i \end{bmatrix}} \qquad 0 \leq a \leq n-1$$

are linearly independent over $\mathcal{Q}_n(2)$.

Proof of Lemma 5.7. We use induction on n. For $n = 1$, the statement is trivially true since the only polynomial $Q_0^{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ is the determinant of the empty matrix i.e. the constant 1. Suppose the statement holds up to $n-1$, and suppose there exist polynomials R_a in $Q_0, \ldots, Q_n$ for $0 \leq a \leq n-1$ such that

$$\sum_{a=0}^{n-1} R_a(Q_0, \ldots, Q_n) \left(\sum_{i=1}^{n} Q_a^{\begin{bmatrix} i \\ i \end{bmatrix}} \right) = 0.$$

We show that $R_a = 0$ for every $0 \leq a \leq n-1$. Consider the restriction to $z_{j,n} = z_{n,j} = 0$ for every $1 \leq j \leq n$ and $w_{j,n} = w_{n,j} = 0$ for every $1 \leq j \leq n-1$. In this case we have $Q_n = 0$, $Q_m = w_{n,n} Q_m^{\begin{bmatrix} n \\ n \end{bmatrix}}$ for $0 \leq m \leq n-1$, $Q_a^{\begin{bmatrix} n \\ n \end{bmatrix}}$ is unchanged for every a, and for $0 \leq i \leq n-1$ $Q_a^{\begin{bmatrix} i \\ i \end{bmatrix}}$ becomes $w_{n,n} Q_a^{\begin{bmatrix} i,n \\ i,n \end{bmatrix}}$. So our equation above restricts to

$$\sum_{a=0}^{n-1} R_a(w_{n,n} Q_0^{\begin{bmatrix} n \\ n \end{bmatrix}}, \ldots, w_{n,n} Q_{n-1}^{\begin{bmatrix} n \\ n \end{bmatrix}}, 0) \left(\sum_{i=1}^{n-1} w_{n,n} Q_a^{\begin{bmatrix} i,n \\ i,n \end{bmatrix}} + Q_a^{\begin{bmatrix} n \\ n \end{bmatrix}} \right) = 0.$$

Comparing terms containing $w_{n,n}$ we have

$$\sum_{a=0}^{n-1} R_a(w_{n,n} Q_0^{\begin{bmatrix} n \\ n \end{bmatrix}}, \ldots, w_{n,n} Q_{n-1}^{\begin{bmatrix} n \\ n \end{bmatrix}}, 0) \left(\sum_{i=1}^{n-1} w_{n,n} Q_a^{\begin{bmatrix} i,n \\ i,n \end{bmatrix}} \right) = 0.$$

Further restricting to $w_{n,n} = 1$, note that we obtain exactly the relation for the case $n-1$ with the top left $(n-1) \times (n-1)$ minors of W and Z, $W^{\begin{bmatrix} n \\ n \end{bmatrix}}$ and $Z^{\begin{bmatrix} n \\ n \end{bmatrix}}$. So we must have that $R_a(Q_0, \ldots, Q_{n-1}, 0) = 0$ for every a, and thus $R_a(Q_0, \ldots, Q_{n-1}, Q_n)$ is divisible by Q_n. But we can factor out a suitable power of Q_n and repeat the same argument, showing that for each $0 \leq a \leq n-1$, $R_a(Q_0, \ldots, Q_n)$ is divisible by arbitrarily large powers of Q_n and thus must be zero.

□

Returning to the proof of Theorem 5.6. We consider the action of the Laplacian operator $\sum_{i=1}^{n} \Delta_{i,i} = \sum_{i=1}^{n} L_{i,i}^{(k_1)} + L_{i,i}^{\prime(k_2)}$ on a polynomial $R \in \mathcal{Q}_{n,v}(2)$. By Corollary

5.4 we may write

$$R = \sum_{\alpha} C(\alpha) \prod_{m=0}^{n} Q_m^{\alpha_m} = \sum_{\alpha} C(\alpha) Q^{\alpha},$$

where we sum over $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ with $|\alpha| := \alpha_0 + \alpha_1 + \cdots + \alpha_n = v$, and coefficients $C(\alpha) \in \mathbb{C}$. As a consequence of Lemma 5.7 we can obtain relations among the coefficients $C(\alpha)$ by the vanishing of the coefficient of $\sum_{i=1}^{n} Q_a^{[i]_i}$ in $\sum_{i=1}^{n} L_{i,i}^{(k_1)}(R) + L_{i,i}^{\prime(k_2)}(R)$. Using (5.1), (5.2), (5.6) and (5.7) we find:

$$\begin{aligned}
&(L_{i,i}^{(k_1)} + L_{i,i}^{\prime(k_2)})(R) \\
&= \sum_{\alpha} C(\alpha) \Bigg(\sum_{m=0}^{n} \alpha_m \frac{Q^{\alpha}}{Q_m} (L_{i,i}^{(k_1)} + L_{i,i}^{\prime(k_2)})(Q_m) \\
&\qquad + \sum_{0 \leq m \leq \ell \leq n} \alpha_m (\alpha_\ell - \delta_{m,\ell}) (1 - \frac{\delta_{m,\ell}}{2}) \frac{Q^{\alpha}}{Q_m Q_\ell} \left((Q_m, Q_\ell)_{i,W} + (Q_m, Q_\ell)_{i,Z} \right) \Bigg) \\
&= \sum_{\alpha} C(\alpha) \Bigg(\sum_{m=0}^{n} \alpha_m \frac{Q^{\alpha}}{Q_m} \left((k_1 + 1 - n + m) Q_m^{[i]_i} + (k_2 + 1 - m) Q_{m-1}^{[i]_i} \right) \\
&\qquad + \sum_{0 \leq m \leq \ell \leq n} \alpha_m (\alpha_\ell - \delta_{m,\ell}) (1 - \frac{\delta_{m,\ell}}{2}) \frac{Q^{\alpha}}{Q_m Q_\ell} \Big(2 Q_m Q_\ell^{[i]_i} - 2 Q_{m-1}^{[i]_i} Q_{\ell+1} \\
&\qquad\qquad + (Q_{m-1}, Q_{\ell+1})_{i,W} + 2 Q_{m-1}^{[i]_i} Q_\ell - 2 Q_{m-1} Q_\ell^{[i]_i} + (Q_{m-1}, Q_{\ell+1})_{i,Z} \Big) \Bigg).
\end{aligned}$$

For each $m = 0, \ldots, n$, if we sum over $i = 1, 2, \ldots, n$, the "coefficient" of $\sum_{i=1}^{n} Q_m^{[i]_i}$ is some homogeneous polynomial R_m of degree $v - 1$ with respect to the polynomials $Q_0, \ldots, Q_n$, that is, some element of $\mathcal{Q}_{n,v-1}(2)$. If $\sum_{i=1}^{n} L_{i,i}^{(k_1)}(R) + L_{i,i}^{\prime(k_2)}(R)$ is equal to zero then by Lemma 5.7, each R_m must be the zero polynomial. Moreover, by Lemma 5.2 the scalar coefficient of each monomial (in the polynomials $Q_0, \ldots, Q_n$) of R_m must in fact be zero. Therefore we extract the coeffient of the

term $\frac{Q^\alpha}{Q_m}\sum_{i=1}^n Q_m^{\left[\begin{smallmatrix} i \\ i \end{smallmatrix}\right]}$ for each fixed $m = 0, \ldots, n-1$ and α:

$$\begin{aligned}
0 =& C(\alpha)\alpha_m(k_1+1-n+m) + C(\alpha - 1_m + 1_{m+1})(\alpha - 1_m + 1_{m+1})_{m+1}(k_2 - m) \\
&+ \sum_{\substack{0\le \ell \le \ell' \le m \\ \ell+\ell'-m \ge 0}} C(\alpha(m,\ell,\ell'))\alpha(m,\ell,\ell')_\ell(\alpha(m,\ell,\ell')_{\ell'} - \delta_{\ell,\ell'})(2-\delta_{\ell,\ell'}) \\
&- \sum_{\substack{0\le \ell \le \ell' \le m \\ \ell+\ell'-m-1 \ge 0}} C(\tilde{\alpha}(m,\ell,\ell'))\tilde{\alpha}(m,\ell,\ell')_\ell(\tilde{\alpha}(m,\ell,\ell')_{\ell'} - \delta_{\ell,\ell'})(2-\delta_{\ell,\ell'}) \\
&+ \sum_{\substack{m< \ell \le \ell' \le n \\ \ell+\ell'-m-1 \le n}} C(\tilde{\alpha}(m,\ell,\ell'))\tilde{\alpha}(m,\ell,\ell')_\ell(\tilde{\alpha}(m,\ell,\ell')_{\ell'} - \delta_{\ell,\ell'})(2-\delta_{\ell,\ell'}) \\
&- \sum_{\substack{m< \ell \le \ell' \le n \\ \ell+\ell'-m \le n}} C(\alpha(m,\ell,\ell'))\alpha(m,\ell,\ell')_\ell(\alpha(m,\ell,\ell')_{\ell'} - \delta_{\ell,\ell'})(2-\delta_{\ell,\ell'}).
\end{aligned}$$

In the relations above, we use the notation $\alpha(m,\ell,\ell') := \alpha - 1_m + 1_\ell + 1_{\ell'} - 1_{\ell+\ell'-m}$ and $\tilde{\alpha}(m,\ell,\ell') := \alpha - 1_m + 1_\ell + 1_{\ell'} - 1_{\ell+\ell'-m-1}$. We also take $C(\beta)$ to be zero if $\beta_j < 0$ for any $0 \le j \le n$. We fix the lexicographic order on monomials i.e. on tuples α. That is, since all monomials concerned have the same degree v,

$$\alpha > \beta \Leftrightarrow \exists j \text{ s.t. } \alpha_i = \beta_i \quad \forall 0 \le i < j \text{ and } \alpha_j > \beta_j.$$

Now, fix $\alpha \neq (0, \ldots, 0, v)$ (so $\alpha > (0, \ldots, 0, v)$). Choose m to be the smallest $0 \le m \le n-1$ such that $\alpha_m > 0$. The relation for this choice of α and m becomes that given in the statement of Theorem 5.6:

$$\begin{aligned}
0 =& C(\alpha)\alpha_m(k_1+1-n+m) + C(\alpha - 1_m + 1_{m+1})(\alpha - 1_m + 1_{m+1})_{m+1}(k_2 - m) \\
&+ C(\alpha)\alpha_m(\alpha_m - 1) \\
&- 0 \\
&+ \sum_{\substack{m< \ell \le \ell' \le n \\ \ell+\ell'-m-1 \le n}} C(\tilde{\alpha}(m,\ell,\ell'))\tilde{\alpha}(m,\ell,\ell')_\ell(\tilde{\alpha}(m,\ell,\ell')_{\ell'} - \delta_{\ell,\ell'})(2-\delta_{\ell,\ell'}) \\
&- \sum_{\substack{m< \ell \le \ell' \le n \\ \ell+\ell'-m \le n}} C(\alpha(m,\ell,\ell'))\alpha(m,\ell,\ell')_\ell(\alpha(m,\ell,\ell')_{\ell'} - \delta_{\ell,\ell'})(2-\delta_{\ell,\ell'}).
\end{aligned}$$

In particular, for each index $\alpha > (0, \ldots, 0, v)$ we have a linear relation determining the coefficient $C(\alpha)$ in terms of the values of coefficients $C(\beta)$ with $\beta < \alpha$. Therefore, we can write each coefficient $C(\alpha)$ in terms of $C((0, \ldots, 0, v))$ and so the

polynomial R is unique up to scaling. That is, the space of polynomials $\mathcal{H}_{n,v}(k_1, k_2)$ is one-dimensional and therefore by Corollary 4.3 the corresponding differential operator on Hermitian modular forms is unique up to scaling. □

Remark 5.8. The coefficients of the original Rankin–Cohen brackets as given in [42], for example, satisfy the corresponding linear relations given in Theorem 5.6 in the case $n = 1$. Additionally, once re-written in terms of the polynomials Q_0 ,Q_1 and Q_2 the differential operators on Hermitian modular forms described by Martin–Senadheera [33] have coefficients which satisfy the linear relations in Theorem 5.6 for $n = 2$, and the respective conditions on the degree v agree in the case studied *loc. cit.*.

5.1 Application to Fourier Expansions and Cusp Forms

Recall from §3.1 that a $\mathbb{C}$-valued Hermitian modular form f has a Fourier expansion indexed by a lattice H of Hermitian matrices of the form

$$f(Z) = \sum_{h \in H} c(h) e^{2\pi i \operatorname{Tr}(hZ)},$$

for some coefficients $c(h) \in \mathbb{C}$. (This H is the $H_\Gamma^\vee$ of §3.1.) Again, by [36, Prop. 5.7], if $L^+ \neq \mathbb{Q}$ or $n \neq 1$ then $c(h) = 0$ unless h is positive semi-definite. We assume this to be the case. For $\mathbb{C}$-valued Hermitian modular forms $f_1, \ldots, f_r$ of weights $k_1, \ldots, k_r$ respectively let the Fourier series be given by

$$f_j(Z) = \sum_{\substack{h \in H \\ h \geq 0}} c_j(h) e^{2\pi i \operatorname{Tr}(hZ)}$$

for each $1 \leq j \leq r$. For $Q \in \mathcal{H}_{n,v}(k_1, \ldots, k_r)$ denote

$$G(Z) := Q\left(\frac{\partial}{\partial Z}\right) \left(f_1(Z_1) \ldots f_r(Z_r)\right) \big|_{Z_1 = \cdots = Z_r = Z}.$$

After a straightforward calculation, we can write the Fourier series for G in terms of the Fourier series for $f_1, \ldots, f_r$ as

$$G(Z) = \sum_{\substack{h_1, \ldots, h_r \in H \\ h_1, \ldots, h_r \geq 0}} c_1(h_1) \ldots c_r(h_r) Q(h_1, \ldots, h_r) e^{2\pi i \sum_{j=1}^r \operatorname{Tr}(h_j Z)}.$$

Recall that we say that a Hermitian modular form is a cusp form if the Fourier coefficients $c(h)$ are only non-zero at positive definite matrices h.

Proposition 5.9. *If $L^+ \neq \mathbb{Q}$ or $n \neq 1$, then for any $\mathbb{C}$-valued Hermitian modular forms f_1 and f_2 on $U(n,n)$ of level Γ and weights k_1 and k_2 respectively and for any $Q \in \mathcal{H}_{n,v}(k_1,k_2)$ with $v > 0$ satisfying the hypotheses of Theorem 4.1, we have that*

$$G(Z) := Q\left(\frac{\partial}{\partial Z}\right)(f_1(Z_1)f_2(Z_2))\,\big|_{Z_1=Z_2=Z}$$

is a cusp form of weight k_1+k_2+2v and level Γ.

Proof. By the result of Theorem 4.1, we know that G is indeed a Hermitian modular form of the correct weight and level. It remains to show that G is a cusp form. As before, for each f_j $(j=1,2)$ write the Fourier expansion of f_j as

$$f_j(Z) = \sum_{\substack{h\in H\\ h\geq 0}} c_j(h)e^{2\pi i\,\mathrm{Tr}(hZ)}.$$

So, as above, the Fourier expansion of G is given by

$$G(Z) = \sum_{\substack{h_1,h_2\in H\\ h_1,h_2\geq 0}} c_1(h_1)c_2(h_2)Q(h_1,h_2)e^{2\pi i\,\mathrm{Tr}((h_1+h_2)Z)}.$$

Since G is indeed a Hermitian modular form, we know that the coefficient

$$c_1(h_1)c_2(h_2)Q(h_1,h_2)$$

is equal to zero whenever h_1+h_2 is not positive semi-definite. It remains to show that this Fourier coefficient is also zero if h_1+h_2 is positive semi-definite but not positive definite. We shall show that in this case $Q(h_1,h_2)=0$. Indeed, if h_1+h_2 is not positive definite then by definition there is a vector w such that

$$w^*(h_1+h_2)w = 0.$$

Since each of h_1 and h_2 is positive semi-definite, it must be the case that $w^*h_1w = w^*h_2w = 0$. Therefore

$$w^*(h_1+\lambda h_2)w = 0$$

for any constant λ. In particular, $\det(h_1+\lambda h_2)$ is identically zero as a function of λ. With notation as in Lemma 5.1, we have

$$\det(h_1+\lambda h_2) = \sum_{m=0}^{n} Q_m(h_1,h_2)\lambda^m,$$

and so it must be that $Q_m(h_1, h_2) = 0$ for each $m = 0, \ldots, n$. By the description of elements of $\mathcal{H}_{n,v}(k_1, k_2)$ in Theorem 5.6 we know that $Q(h_1, h_2)$ is some homogeneous degree-v polynomial in the values of $Q_0(h_1, h_2), \ldots, Q_n(h_1, h_2)$, and therefore must also be zero. □

We conclude this chapter by noting that we have now shown the analogues of each of the three properties of the original Rankin–Cohen brackets highlighted in 2.7 for these differential operators in this new setting, respectively given by Corollary 4.3, Theorem 5.6, and Proposition 5.9.

CHAPTER 6

FROM THE PERSPECTIVE OF REPRESENTATION THEORY

We now take a perspective more analogous to that of §2.3. Specifically, we follow the approach of Ban [1] in re-proving some of Ibukiyama's results for Rankin–Cohen operators in a more general setting. Taking a viewpoint similar to that of El Gradechi [19] in the classical setting, Ban characterized higher-dimensional Rankin–Cohen operators (or perhaps Rankin–Cohen–Ibukiyama operators) in the language of representation theory. In particular, for automorphic forms $f_1, \ldots, f_d$ of (half-)integral weight, Ban studied the decomposition into irreducibles of the tensor product of the associated representations, i.e. of $\bigotimes_{s=1}^{d} \rho_{f_s}$ (in the notation of §2.3). Ban was then able to prove a characterization result for the generators of (some of) these irreducible subrepresentations. Ban makes use of Howe duality [23], as well as a stronger utilisation of the same work of Kashiwara–Vergne [26] as employed in [24, 14], and in Chapter 4 of this paper (and the corresponding section of [13]). Note that Ban (similarly to Kashiwara–Vergne) treats both the case of symplectic (even, metaplectic) groups $\mathrm{Sp}_{2n}(\mathbb{R})$ and $\mathrm{Mp}_{2n}(\mathbb{R})$ and the case of (special) unitary groups $\mathrm{SU}(p,q)$ and $\mathrm{U}(p,q)$. The results pertain to Siegel modular forms in the first case, corresponding to Ibukiyama's work, and to Hermitian modular forms in the second, which we connect to the results of the previous two chapters. For the purposes of this paper, we shall only describe the second case here.

6.1 The Work of Ban

In this section we aim to give a reasonable account of the results given in [1] without showing all of the details. Note that Ban presents results for $\mathrm{SU}(p,q)$ and half-integral weights. We restrict ourselves here to integral weights, both for simplicity and to make the comparison with results from earlier chapters. More importantly, since Ban's results for $\mathrm{SU}(p,q)$ arise by restricting representations for $\mathrm{U}(p,q)$, we may state his results without further alteration (see Prop. 3.1.5 *loc cit.*). Throughout this section, we shall often write G for the group $\mathrm{U}(p,q)$, $\mathfrak{g}$ for its Lie algebra $\mathfrak{u}(p,q)$, and $\mathfrak{g}_{\mathbb{C}}$ for the complexification of $\mathfrak{g}$. The maximal compact subgroup shall again be denoted K, which is the stabilizer of $\mathbf{i} := \begin{pmatrix} i1_q \\ 0 \end{pmatrix} \in \mathbb{H}_{p,q}$.

Note that $K \cong \mathrm{U}(p) \times \mathrm{U}(q)$ via

$$\mathrm{U}(p) \times \mathrm{U}(q) \ni (k_1, k_2) \leftrightarrow \mathfrak{c} \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \mathfrak{c}^{-1} \in K,$$

where, writing in $(q, p-q, q)$-by-$(q, p-q, q)$ blocks as usual,

$$\mathfrak{c} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i \\ 0 & 2 & 0 \\ i & 0 & 1 \end{pmatrix}.$$

We often use this map to identify K with $\mathrm{U}(p) \times \mathrm{U}(q)$. Note that for $k \in K$ with $k \leftrightarrow (k_1, k_2) \in \mathrm{U}(p) \times \mathrm{U}(q)$, the automorphy factor (as in Definition 3.4) satisfies

$$J(k, \mathbf{i}) = (k_1, k_2).$$

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition, where $\mathfrak{k}$ is the Lie algebra of K. We choose the basis of $\mathfrak{g}_{\mathbb{C}}$ given by

$$\kappa_{i,j} = \mathfrak{c} \begin{pmatrix} e_{i,j} & 0 \\ 0 & 0 \end{pmatrix} \mathfrak{c}^{-1} \quad (1 \le i, j \le p), \qquad \kappa'_{i,j} = \mathfrak{c} \begin{pmatrix} 0 & 0 \\ 0 & e_{i,j} \end{pmatrix} \mathfrak{c}^{-1} \quad (1 \le i, j \le q),$$

$$\pi^+_{i,j} = \mathfrak{c} \begin{pmatrix} 0 & e_{i,j} \\ 0 & 0 \end{pmatrix} \mathfrak{c}^{-1} \ (1 \le i \le q, 1 \le j \le p), \quad \pi^-_{i,j} = \mathfrak{c} \begin{pmatrix} 0 & 0 \\ e_{i,j} & 0 \end{pmatrix} \mathfrak{c}^{-1} \ (1 \le i \le p, 1 \le j \le q),$$

where $e_{i,j}$ is the matrix with the (i,j)-th entry equal to one, and zero elsewhere. Then $\{\kappa_{i,j}\} \cup \{\kappa'_{i,j}\}$ is a basis for $\mathfrak{k}_{\mathbb{C}}$, and $\{\pi^+_{i,j}\} \cup \{\pi^-_{i,j}\}$ is a basis for $\mathfrak{p}_{\mathbb{C}}$. Denote by $\mathfrak{p}^{\pm}$ the $\mathbb{C}$-span of $\{\pi^{\pm}_{i,j}\}$ in $\mathfrak{g}_{\mathbb{C}}$, so $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ + \mathfrak{p}^-$.

Ban considers automorphic forms defined as functions on the group G itself as follows.

Definition 6.1 ([1, Definition 4.1.1])**.** For an irreducible unitary representation (τ, U_τ) of K and a discrete subgroup Γ of G, a holomorphic automorphic form of type τ for Γ is a U^*_τ-valued C^∞ function on G, ϕ, such that

(i) $\phi(gk) = \tau^*(k)^{-1}\phi(g)$ for $k \in K$,

(ii) $\phi(\gamma g) = \phi(g)$ for $\gamma \in \Gamma$,

(iii) ϕ is annihilated by $\mathfrak{p}^-$ through right derivation,

(iv) ϕ is of moderate growth (redundant if $pq > 1$).

Here, for a representation τ, τ^* denotes the dual representation, and similarly for a vector space V, V^* denotes the dual vector space. Note that this definition corresponds to Definition 3.6 in the same way as described in §2.3 for modular forms on $\mathrm{SL}(2,\mathbb{R})$. To make this correspondence completely explicit (between "type τ" and "weight ρ"), for ρ as in Definition 3.6 and τ as above, $\tau^*(k) = \rho(J(k,\mathbf{i})) = \rho(k_1,k_2)$ where $k \leftrightarrow (k_1,k_2)$. For example, in the case of scalar-valued forms of interest to us, weight $(-\kappa_1,\kappa_2) = \det^{-\kappa_1}(J_1)\det^{\kappa_2}(J_2)$ corresponds to type τ where τ is given by $(\kappa_1\mathbf{1}_p; -\kappa_2\mathbf{1}_q)$ (where $\mathbf{1}_p$ is the length-p vector $(1,\dots,1)$). Equivalently, we can view an automorphic form ϕ as (associated to) an intertwining operator from $L(\tau)$ to $C^\infty_{\mathrm{mod}}(\Gamma\backslash G)$, i.e. an element of $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(L(\tau), C^\infty_{\mathrm{mod}}(\Gamma\backslash G))$. Here, $L(\tau)$ denotes the unitary lowest-weight $(\mathfrak{g}_\mathbb{C},K)$-module for G with lowest K-type (or lowest weight) τ. In particular, given such an automorphic form ϕ, the corresponding homomorphism I_ϕ sends the lowest weight vector of $L(\tau)$ to ϕ. Of course, under the action of $(\mathfrak{g}_\mathbb{C},K)$, ϕ generates a copy of $L(\tau)$ inside $C^\infty_{\mathrm{mod}}(\Gamma\backslash G)$ (which is the image of I_ϕ). This equivalence is made precise by the following proposition.

Proposition 6.2 ([1, Prop. 4.1.2])**.** *We have an isomorphism*

$$Hom_{\mathfrak{g}_\mathbb{C},K}(L(\tau), C^\infty_{mod}(\Gamma\backslash G)) \cong [C^\infty_{mod}(\Gamma\backslash G)\otimes U^*_\tau]^{K,\nabla^-=0}.$$

One direction is explained above. For the other direction, given a $(\mathfrak{g}_\mathbb{C},K)$-morphism I, restricting I to the lowest K-type, i.e. to U_τ, gives an element ϕ_I of $\mathrm{Hom}_K(U_\tau, C^\infty_{\mathrm{mod}}(\Gamma\backslash G)) \cong [C^\infty_{\mathrm{mod}}(\Gamma\backslash G)\otimes U^*_\tau]^K$. Note that this K-invariance captures property (i) of Definition 6.1. Then, since $\mathfrak{p}^-$ acts by 0 on $U_\tau \subset L(\tau)$, we must have that $\mathfrak{p}^-$ also acts by zero on ϕ_I since I is a $(\mathfrak{g}_\mathbb{C},K)$-homomorphism. (In Ban's notation, the action of $\mathfrak{p}^-$ by right-derivation on ϕ_I is denoted ∇^-.)

Now, Ban views Rankin–Cohen operators in the sense of the characterization given in Proposition 2.21. That is, take holomorphic automorphic forms $\phi_1,\dots,\phi_d$ of weights $\tau_1,\dots,\tau_d$ and write $I_{\phi_1},\dots,I_{\phi_d}$ for the corresponding intertwining operators as in Proposition 6.2. Then, for each $(\mathfrak{g}_\mathbb{C},K)$-homomorphism $L(\tau) \to \bigotimes_{s=1}^d L(\tau_s)$ we have a resulting $I \in \mathrm{Hom}_{\mathfrak{g}_C,K}(L(\tau), C^\infty_{\mathrm{mod}}(\Gamma\backslash G))$ given by the composition

$$\begin{array}{ccc} L(\tau) & \xrightarrow{\quad I \quad} & C^\infty_{\mathrm{mod}}(\Gamma\backslash G) \\ \downarrow & & \uparrow{\scriptstyle \Delta^*} \\ \bigotimes_{s=1}^d L(\tau_s) & \xrightarrow{\bigotimes_{s=1}^d I_{\phi_s}} & \bigotimes_{s=1}^d C^\infty_{\mathrm{mod}}(\Gamma\backslash G) \end{array}$$

where Δ^* is the pullback of the diagonal embedding $\Delta : G \to \prod_{s=1}^d G$. The assignment of $(\phi_1, \ldots, \phi_d) \mapsto \phi = \phi_I$ is a Rankin–Cohen operator from this perspective. Of course, for an irreducible representation τ, there is a non-zero $(\mathfrak{g}_{\mathbb{C}}, K)$-homomorphism $L(\tau) \to \bigotimes_{s=1}^d L(\tau_s)$ precisely when $L(\tau)$ appears in the decomposition of $\bigotimes_{s=1}^d L(\tau_s)$. When $\phi_1, \ldots, \phi_d$ are of scalar weight, Ban uses Howe duality, together with the work of Kashiwara–Vergne (in particular, [26, Theorem III.6.3]) to prove the decomposition of $\otimes_{s=1}^d L(\tau_s)$, and then gives a description of the automorphic forms ϕ arising this way, which we now follow.

Definition 6.3. The key $(\mathfrak{g}_{\mathbb{C}}, K)$ module will be denoted (l_k, L_k) (or simply L_k) and is defined as follows. We write L_k for $\mathbb{C}[M_{p,k} \times M_{q,k}]$, the space of polynomials in the entries of matrix variables $X = (X_{i,j})$ (p-by-k) and $Y = (Y_{i,j})$ (q-by-k). The representation l_k of $\mathfrak{g}_{\mathbb{C}}$ on L_k is given by:

$$l_k(\kappa_{i,j}) = \sum_{t=1}^{k} X_{i,t} \frac{\partial}{\partial X_{j,t}} + \frac{k}{2}\delta_{i,j}$$

$$l_k(\kappa'_{i,j}) = -\sum_{t=1}^{k} Y_{j,t} \frac{\partial}{\partial Y_{i,t}} - \frac{k}{2}\delta_{i,j}$$

$$l_k(\pi^+_{i,j}) = \sqrt{-1} \sum_{t=1}^{k} X_{i,t} Y_{j,t}$$

$$l_k(\pi^-_{i,j}) = \sqrt{-1} \sum_{t=1}^{k} \frac{\partial^2}{\partial X_{j,t} \partial Y_{i,t}},$$

and for $K \cong \mathrm{U}(p) \times \mathrm{U}(q) \ni c = (c_1, c_2)$ by

$$l_k(c) f(X, Y) = \det(c_1)^k f({}^t c_1 X, c_2^{-1} Y).$$

Also, for $G' = \mathrm{U}(k)$ we define an action on L_k by

$$c \cdot f(X, Y) = f(Xc, Y^t c^{-1}).$$

This is indeed a well-defined $(\mathfrak{g}_{\mathbb{C}}, K)$-module, and the actions of $(\mathfrak{g}_{\mathbb{C}}, K)$ and G' commute. Now, for $(\lambda, V_\lambda) \in \hat{G'}$, $L_k(\lambda) := \mathrm{Hom}_{G'}(V_\lambda, L_k)$ is a $(\mathfrak{g}_{\mathbb{C}}, K)$-module via the action on L_k. So, we have a decomposition as $(\mathfrak{g}_{\mathbb{C}}, K) \times G'$ representations

$$L_k \cong \bigoplus_{\lambda \in \hat{G'}} L_k(\lambda) \boxtimes V_\lambda.$$

Here, $\hat{G'}$ is the *unitary dual of* G', i.e. the set of isomorphism classes of irreducible unitary representations of G'.

Proposition 6.4 ([1, Prop. 3.1.5]). *By Howe duality, the decomposition of L_k above satisfies the following:*

(i) If $L_k(\lambda) \neq 0$, then $L_k(\lambda)$ is a unitary lowest-weight module.

(ii) For any $\lambda, \lambda' \in \hat{G'}$, if $L_k(\lambda) \cong L_k(\lambda') \neq 0$ then $\lambda \cong \lambda'$.

To give the decomposition, we make the following definition.

Definition 6.5. Let Δ_k be the set of pairs of Young diagrams $D = (D_1, D_2)$, writing $D_j = (m_1^{(j)}, \ldots, m_{\ell(D_j)}^{(j)})$ for $j = 1, 2$, such that $\ell(D_1) \leq p$, $\ell(D_2) \leq q$ and $\ell(D_1) + \ell(D_2) \leq k$. For $D \in \Delta_k$ define a pair of row vectors of length p and q respectively, τ_D, and a row vector of length k, λ_D, by

$$\tau_D = (\tau_{D_1}; \tau_{D_2}) = \left(m_1^{(1)}, \ldots, m_{\ell(D_1)}^{(1)}, 0, \ldots, 0; 0, \ldots, 0, -m_{\ell(D_2)}^{(2)}, \ldots, -m_1^{(2)}\right),$$

$$\lambda_D = (m_1^{(2)}, \ldots, m_{\ell(D_2)}^{(2)}, 0, \ldots, 0, -m_{\ell(D_1)}^{(1)}, \ldots, -m_1^{(1)}).$$

Attached to each $D \in \Delta_k$ we have corresponding representations of $K = \mathrm{U}(p) \times \mathrm{U}(q)$ (defined by τ_D) and of $G' = \mathrm{U}(k)$ (defined by λ_D), given in the following way. The representation of $K \cong \mathrm{U}(p) \times \mathrm{U}(q)$ is the irreducible representaion of highest weight $(\tau_{D_1} + k\mathbf{1}_p; \tau_{D_2})$. For brevity, we shall often write $\tau_D + k\mathbf{1}_p$. For $G' = \mathrm{U}(k)$ the representation $(\lambda_D, V_{\lambda_D})$ is the representation of $U(k)$ of highest weight λ_D. Note that the trivial representation of $\mathrm{U}(k)$ occurs by taking D to be the pair of empty Young diagrams (which is always inside Δ_k).

We write $\mathscr{H}_k$ for the space of pluriharmonic polynomials inside L_k. Note that these are precisely the polynomials annihilated by every element of $\mathfrak{p}^-$ (c.f. Definition 3.8), and the space $\mathscr{H}_k$ is preserved by the action of K and G'.

Proposition 6.6 ([1, Prop. 3.1.10]). *We have the following:*

(i) $L_k = \mathscr{H}_k \cdot L_k^{G'}$ [26, Lemma III.5.3].

(ii) $L_k^{G'} = l_k(\mathbf{S}(\mathfrak{p}^+)) \cdot 1$.

(iii) By [26, Theorem III.6.3], under the action of $K \times G'$ we have the decomposition

$$\mathscr{H}_k \cong \bigoplus_{D \in \Delta_k} U_{\tau_D + k\mathbf{1}_p} \boxtimes V_{\lambda_D}.$$

Here, $\mathbf{S}(\mathfrak{p}^+)$ denotes the symmetric algebra of $\mathfrak{p}^+$ (as defined above). This is well-defined since in $\mathfrak{u}(p,q)_{\mathbb{C}}$, $[\mathfrak{p}^+,\mathfrak{p}^+]=0$. Together, these give the full decomposition of L_k.

Proposition 6.7 ([1, Prop. 3.1.7]). *In the decomposition of L_k, the summand $L_k(\lambda)\neq 0$ if and only if $\lambda=\lambda_D$ for some $D\in\Delta_k$. If so, the lowest K-type of $L_k(\lambda)=L_k(\lambda_D)$ is $\tau_D+k\mathbf{1}_p$. Therefore*

$$L_k\cong\bigoplus_{D\in\Delta_k}L(\tau_D+k\mathbf{1}_p)\boxtimes V_{\lambda_D}.$$

Ban then uses the decomposition of L_k to give the decomposition of interest for the purpose of Rankin–Cohen operators.

Theorem 6.8 ([1, Theorem 3.2.1]). *Fix positive integers $k_1,\dots,k_d$ with $k=\sum_{s=1}^d k_s$, and $D_s\in\Delta_{k_s}$ for each $s=1,\dots,d$. Put $G'_s=U(k_s)$, and identify $G'_1\times\cdots\times G'_d$ with the block-diagonal embedding inside $G'=U(k)$. Then we have the decomposition of $(\mathfrak{g}_{\mathbb{C}},K)$-modules*

$$\bigotimes_{s=1}^d L(\tau_{D_s}+k_s\mathbf{1}_p)\cong\bigoplus_{D\in\Delta_k}L(\tau_D+k\mathbf{1}_p)\otimes Hom_{G'_1\times\cdots\times G'_d}(V_{\lambda_{D_1}}\boxtimes\cdots\boxtimes V_{\lambda_{D_d}},V_{\lambda_D}).$$

Choosing each D_s to be trivial, since the trivial representation inside V_{λ_D} is precisely the fixed vectors we obtain the following:

Corollary 6.9 ([1, Corollary 3.2.2]).

$$\bigotimes_{s=1}^d L((k_s\mathbf{1}_p,0\mathbf{1}_q))\cong\bigoplus_{D\in\Delta_k}L(\tau_D+k\mathbf{1}_p)\otimes V_{\lambda_D}^{G'_1\times\cdots\times G'_d}.$$

Therefore, there exists a non-zero homomorphism $L(\tau)\to\otimes_{s=1}^d L((k_s\mathbf{1}_p,0\mathbf{1}_q))$ if and only if $\tau=\tau_D+(k\mathbf{1}_p,0\mathbf{1}_q)$ for some $D\in\Delta_k$ such that $V_{\lambda_D}^{G'_1\times\cdots\times G'_d}\neq 0$.

The Rankin–Cohen operators themselves are again described in terms of pluriharmonic polynomials, since the components of $\mathscr{H}_k$ are precisely the lowest-weight subspaces in the decomposition of L_k. For $D\in\Delta_k$, write $\mathscr{H}_k(D)$ for the image of $U_{\tau_D+k\mathbf{1}_p}\boxtimes V_{\lambda_D}$ inside $\mathscr{H}_k$ under the isomorphism given above. Of course, $l_k(\mathbf{S}(\mathfrak{p}^+))\cdot\mathscr{H}_k(D)$ is precisely the summand of L_k isomorphic to $L(\tau_D+k\mathbf{1}_p)\boxtimes V_{\lambda_D}$.

Now, consider $\otimes_{s=1}^d L_{k_s}$, and identify L_{k_s} with the subspace $1\otimes\cdots\otimes L_{k_s}\otimes 1\cdots\otimes 1$. Put $X^{(s)}$ and $Y^{(s)}$ for the variables of L_{k_s}. For $k=\sum_{s=1}^d k_s$, fix the isomorphism

of $\otimes_{s=1}^{d} L_{k_s} \cong L_k$ by adjoining the matrix indeterminates together horizontally i.e. $X = (X^{(1)} \mid X^{(2)} \mid \cdots \mid X^{(d)})$ (and similarly for Y). This is an isomorphism of $(\mathfrak{g}_{\mathbb{C}}, K)$-modules, and is equivariant under the embedding $G'_1 \times \cdots \times G'_d \to G'$ (embedded block-diagonally). So, we identify $\otimes_{s=1}^{d}(l_{k_s}, L_{k_s})$ with (l_k, L_k) (as $(\mathfrak{g}_{\mathbb{C}}, K)$-modules) under this isomorphism. As before, classical invariant theory tells us that the subspace of the invariants $L_{k_s}^{G'_s}$ is the space $\mathbb{C}[Z^{(s)}]$ of polynomials in the entries of $Z^{(s)} = X^{(s)} \cdot {}^t Y^{(s)}$. Therefore we have an injective homomorphism

$$\mathscr{H}_k(D)^{G'_1 \times \cdots \times G'_d} \otimes U^*_{\tau_D + k\mathbf{1}_p} \to L_k^{G'_1 \times \cdots \times G'_d} \otimes U^*_{\tau_D + k\mathbf{1}_p} \cong \bigotimes_{s=1}^{d} L_{k_s}^{G'_s} \otimes U^*_{\tau_D + k\mathbf{1}_p} \cong \bigotimes_{s=1}^{d} \mathbb{C}[Z^{(s)}] \otimes U^*_{\tau_D + k\mathbf{1}_p},$$

where the first map is simply the inclusion of (the direct summand of) pluriharmonic polynomials into (the direct summand of) L_k, as the lowest weight space. Under this chain of maps, write $\Phi_h = \Phi_h(Z^{(1)}, \ldots, Z^{(d)})$ for the image of $h \in \mathscr{H}_k(D)^{G'_1 \times \cdots \times G'_d} \otimes U^*_{\tau_D + k\mathbf{1}_p}$.

Theorem 6.10 ([1, Theorem 4.1.3])**.** *For $s = 1, \ldots, d$ let ϕ_s be a holomorphic automorphic form of type $(k_s\mathbf{1}_p, 0\mathbf{1}_q)$ for Γ. Since $U^*_{(k_s\mathbf{1}_p, 0\mathbf{1}_q)} \cong \mathbb{C}$ we consider each ϕ_s to be valued in $\mathbb{C}$. Now, put $\pi^+ = (\pi^+_{i,j})$, and denote by Δ^* the pullback of functions on $\prod_{s=1}^{d} G$ along the diagonal $\Delta : G \mapsto \prod_{s=1}^{d} G$. Then, for any $h \in \left[\mathscr{H}_k(D)^{G'_1 \times \cdots \times G'_d} \otimes U^*_{\tau_D + k\mathbf{1}_p}\right]^K$, the $U^*_{\tau_D + k\mathbf{1}_p}$-valued function*

$$\Delta^*\left(\Phi_h(\pi^+, \ldots, \pi^+)(\phi_1 \otimes \cdots \otimes \phi_d)\right)$$

is a holomorphic automorphic form of type $\tau_D + k\mathbf{1}_p$ for Γ.

The step of "evaluating" $Z^{(s)}_{i,j}$ at $\pi^+_{i,j}$ defines a $\mathbb{C}$-algebra isomorphism $\mathbb{C}[Z^{(s)}] \to \mathbf{S}(\mathfrak{p}^+)$, and thus gives isomorphisms $\bigotimes_{s=1}^{d} \mathbb{C}[Z^{(s)}] \to \bigotimes_{s=1}^{d} \mathbf{S}(\mathfrak{p}^+)$ and $\bigotimes_{s=1}^{d} \mathbb{C}[Z^{(s)}] \otimes U^*_{\tau_D + k\mathbf{1}_p} \to \bigotimes_{s=1}^{d} \mathbf{S}(\mathfrak{p}^+) \otimes U^*_{\tau_D + k\mathbf{1}_p}$. So, via the action of $\bigotimes_{s=1}^{d} \mathbf{U}(\mathfrak{g}_{\mathbb{C}})$ on $\bigotimes_{s=1}^{d} C^\infty_{\mathrm{mod}}(\Gamma\backslash G)$, $\Phi_h(\pi^+, \ldots, \pi^+)$ gives a homomorphism

$$\bigotimes_{s=1}^{d} C^\infty_{\mathrm{mod}}(\Gamma\backslash G) \to \bigotimes_{s=1}^{d} C^\infty_{\mathrm{mod}}(\Gamma\backslash G) \otimes U^*_{\tau_D + k\mathbf{1}_p}.$$

The key point is that certain pluriharmonic polynomials (of the kind specified in the theorem) give the correct combination of raising operators to apply to $\phi_1, \ldots, \phi_d$ in order to pick out lowest weight vectors in the decomposition of the tensor product in Corollary 6.9.

Finally, Ban converts Theorem 6.10 into a statement for holomorphic automorphic forms (viewed as functions on $\mathbb{H}_{p,q}$). We have the isomorphism

$$M_\tau(\Gamma) \cong [C^\infty_{\mathrm{mod}}(\Gamma\backslash G) \otimes U^*_\tau]^{K,\nabla^-=0}.$$

The correspondence here is the same as that given in §2.3. Recall that for Ban, "type τ" means that f is of weight ρ (as in our choice of definition) where $\tau^*(k) = \rho(J(k,\mathbf{i})) = \rho(k_1,k_2)$ where $k \leftrightarrow (k_1,k_2)$. Then, a holomorphic automorphic form f of type τ corresponds to the function on the group given by

$$\phi_f(g) := \rho(J(g,\mathbf{i}))^{-1} f(g\cdot\mathbf{i}).$$

By Proposition 6.2, we therefore also have that

$$M_\tau(\Gamma) \cong \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(L(\tau), C^\infty_{\mathrm{mod}}(\Gamma\backslash G)),$$

to each f we can associate an intertwining operator, and vice versa. In his own words, Ban "pulls back Theorem 6.10 along this isomorphism" to give the following statement.

Theorem 6.11 ([1, Theorem 4.2.5])**.** *For $s = 1,\dots,d$ let $f_s(z)$ be a holomorphic automorphic form of type $(k\mathbf{1}_p, 0\mathbf{1}_q)$ for Γ, where $z \in \mathbb{H}_{p,q}$. Again, since $U^*_{(k\mathbf{1}_p,0\mathbf{1}_q)} \cong \mathbb{C}$ we consider each f_s as a $\mathbb{C}$-valued function on $\mathbb{H}_{p,q}$. Put*

$$\partial = \left(\frac{\partial}{\partial z_{ij}}\right)_{ij}.$$

Denote by Δ^* the pullback of functions on $\prod_{s=1}^d \mathbb{H}_{p,q}$ along the diagonal embedding $\Delta : \mathbb{H}_{p,q} \to \prod_{s=1}^d \mathbb{H}_{p,q}$. Then, for any $h \in \left[\mathscr{H}_k(D)^{G'_1\times\cdots\times G'_d} \otimes U^*_{\tau_D+k\mathbf{1}_p}\right]^K$, the $U^*_{\tau_D+k\mathbf{1}_p}$-valued function

$$\Delta^*\left(\Phi_h(\partial,\dots,\partial)(f_1\otimes\cdots\otimes f_d)\right)$$

is a holomorphic automorphic form of type $\tau_D + k\mathbf{1}_p$ for Γ.

One of the main points of note in Theorems 6.10 and 6.11 is that the polynomials h and Φ_h are the same in both instances, and give rise to the same automorphic form. That is, if $\phi_s \leftrightarrow f_s$ for each $s = 1,\dots,d$ then

$$\Delta^*\left(\Phi_h(\pi^+,\dots,\pi^+)(\phi_1\otimes\cdots\otimes\phi_d)\right) \leftrightarrow \Delta^*\left(\Phi_h(\partial,\dots,\partial)(f_1\otimes\cdots\otimes f_d)\right).$$

So, together these give a (less explicit) version of Proposition 2.21 for Hermitian modular forms of type $(k_s\mathbf{1}_p, 0\mathbf{1}_q)$ (or weight $(-k_s,0)$).

6.2 Connection to Previous Chapters

In Chapters 4 and 5, we were concerned with the situation in which the resulting automorphic form was also scalar-valued (as well as the original $f_1, \dots, f_d$). Of course, Theorems 6.10 and 6.11 give the entire decomposition, including spaces of forms valued in representations of dimension greater than 1. The scalar-valued forms correspond precisely to the $D = (D_1, D_2) \in \Delta_k$ which have the form $D_1 = v'\mathbf{1}_p$ and $D_2 = v\mathbf{1}_q$ for some integers v and v'. Note that these D are indeed inside Δ_k if and only if $p + q \leq k$, and so we restrict ourselves to this case for the remainder of this discussion. Then, automorphic forms of type $\tau = \tau_D$ are those of weight $(-v', v)$ (in the sense of Definition 3.6); type $\tau_D + k\mathbf{1}_p$ is the same as weight $(-k - v', v)$. Next, recall that the action of $K \cong \mathrm{U}(p) \times \mathrm{U}(q)$ on $h \in \mathscr{H}_k \subset L_k$ is given by

$$l_k(c)h(X, Y) = \det(c_1)^k h({}^t c_1 X, c_2^{-1} Y),$$

where $c \leftrightarrow (c_1, c_2) \in \mathrm{U}(p) \times \mathrm{U}(q)$. But K acts on $h \in \mathscr{H}_k(D)$ as $\tau_D + k\mathbf{1}_p$, so

$$l_k(c)h(X, Y) = \det(c_1)^{k+v'} \det(c_2)^{-v} h(X, Y).$$

This holds for every $(c_1, c_2) \in \mathrm{U}(p) \times \mathrm{U}(q)$, and so any $h \in \mathscr{H}_k(D)$ must be homogeneous of degree (v', v) (in the sense of Definition 3.9). Conversely, if $h \in \mathscr{H}_k$ is homogeneous of degree (v', v) then it must lie in $\mathscr{H}_k(D)$ by the same equality. Fixing an isomorphism $U^*_{\tau_D + k\mathbf{1}_p} \cong \mathbb{C}$ and taking $G'_1 \times \cdots \times G'_d$-invariants, the resulting polynomials $\Phi_h \in \otimes_{s=1}^d \mathbb{C}[Z^{(s)}] \cong \mathbb{C}[Z^{(1)}, \dots, Z^{(d)}]$ are therefore precisely elements of $\mathcal{Q}_{(p,q),(v',v)}(k_1, \dots, k_d)$. Next, we take into account the transposition discrepancy between ∂ of Theorem 6.11 and $\frac{\partial}{\partial Z}$ of Definition 3.14 (used in Chapters 4 and 5), as noted in Remark 3.15. We write $Q_h \in \mathcal{Q}_{(q,p),(v,v')}(k_1, \dots, k_d)$ for the polynomial corresponding to $\Phi_h \in \mathcal{Q}_{(p,q),(v',v)}(k_1, \dots, k_d)$, that is

$$\Phi_h(Z^{(1)}, \dots, Z^{(d)}) = Q_h({}^t Z^{(1)}, \dots, {}^t Z^{(d)}).$$

Of course, then, the operators $\Delta^* \Phi_h(\partial, \dots, \partial)$ are exactly the same operators given by $Q_h\left(\frac{\partial}{\partial Z_1}, \dots, \frac{\partial}{\partial Z_d}\right)|_{Z_1 = \cdots = Z_d}$ as described in Theorem 4.7 in the case that $\kappa_1^{(s)} = k_s$, $\kappa_2^{(s)} = 0$ for each s.

For other initial weights $(-\kappa_1^{(s)}, \kappa_2^{(s)})$ (with $\kappa_1^{(s)} + \kappa_2^{(s)} = k_s$, as in the statement of Theorem 4.7), we can recover the corresponding statement from Ban's work by

simply tensoring by a suitable determinant. Explicitly, we could define representations $(l_{(\kappa_1,-\kappa_2)}, L_{(\kappa_1,-\kappa_2)})$ (with $\kappa_1+\kappa_2=k$) of $(\mathfrak{g}_{\mathbb{C}},K)\times G'$ in the same way as L_k except that for $c\in K$ (corresponding to $(c_1,c_2)\in \mathrm{U}(p)\times\mathrm{U}(q)$ as before) we put

$$l_{(\kappa_1,-\kappa_2)}(c)=\det(c_1)^{\kappa_1}\det(c_2)^{-\kappa_2}f({}^tc_1X,c_2^{-1}Y).$$

So $L_{(\kappa_1,-\kappa_2)}\cong \det{}^{-\kappa_2}\otimes L_k$ as representations of K (since $\det(c)=\det(c_1)\det(c_2)$). Then, as we are just tensoring by a character, the decompositions of $\mathscr{H}_{(\kappa_1,-\kappa_2)}$ (the pluriharmonic polynomials inside $L_{(\kappa_1,-\kappa_2)}$) and of $L_{(\kappa_1,-\kappa_2)}$ are simply obtained from those of $\mathscr{H}_k$ and L_k by tensoring by the same character. For example, we may write a decomposition of $L_{(\kappa_1,-\kappa_2)}$ as $(\mathfrak{g}_{\mathbb{C}},K)\times G'$ representations

$$L_{(\kappa_1,-\kappa_2)}\cong\bigoplus_{\lambda\in\hat{G'}}L_{(\kappa_1,-\kappa_2)}(\lambda)\boxtimes V_\lambda,$$

where $L_{(\kappa_1,-\kappa_2)}(\lambda)=\mathrm{Hom}_{G'}(V_\lambda,L_{(\kappa_1,-\kappa_2)})$. But we can also write this decomposition as

$$L_{(\kappa_1,-\kappa_2)}\cong\det{}^{-\kappa_2}\otimes L_k\cong\bigoplus_{\lambda\in\hat{G'}}\left(\det{}^{-\kappa_2}\otimes L_k(\lambda)\right)\boxtimes V_\lambda.$$

So, using Propositions 6.6(iii) and 6.7 respectively, with Δ_k as above we can write

$$\mathscr{H}_{(\kappa_1,-\kappa_2)}\cong\bigoplus_{D\in\Delta_k}U_{\tau_D+(\kappa_1\mathbf{1}_p,-\kappa_2\mathbf{1}_q)}\boxtimes V_{\lambda_D},$$

$$L_{(\kappa_1,-\kappa_2)}\cong\bigoplus_{D\in\Delta_k}L(\tau_D+(\kappa_1\mathbf{1}_p,-\kappa_2\mathbf{1}_q))\boxtimes V_{\lambda_D}.$$

The subsequent results hold similarly with $(\kappa_1\mathbf{1}_p,-\kappa_2\mathbf{1}_q)$ in place of $(k\mathbf{1}_p,0\mathbf{1}_q)$ and $(\kappa_1^{(s)}\mathbf{1}_p,-\kappa_2^{(s)}\mathbf{1}_q)$ in place of $(k_s\mathbf{1}_p,0\mathbf{1}_q)$. So Corollary 6.9 becomes

$$\bigotimes_{s=1}^{d}L((\kappa_1^{(s)}\mathbf{1}_p,-\kappa_2^{(s)}\mathbf{1}_q))\cong\bigoplus_{D\in\Delta_k}L(\tau_D+(\kappa_1^{(s)}\mathbf{1}_p,-\kappa_2^{(s)}\mathbf{1}_q))\otimes V_{\lambda_D}^{G_1'\times\cdots\times G_d'}.$$

The resulting operators arise as $\Phi_h(\pi^+)$ and $\Phi_h(\partial)$ as in Theorems 6.10 and 6.11 respectively for $h\in\left[\mathscr{H}_{(\kappa_1,-\kappa_2)}(D)^{G_1'\times\cdots\times G_d'}\otimes U^*_{\tau_D+(\kappa_1\mathbf{1}_p,-\kappa_2\mathbf{1}_q)}\right]^K$. Note that, just as in Theorem 4.7 and Corollary 4.9, the polynomials themselves depend only on D and $k_1,\ldots,k_d$, not on the choice of $\kappa_1^{(s)}$ and $\kappa_2^{(s)}$. That is, as sets,

$$\mathscr{H}_k(D)=\mathscr{H}_{(\kappa_1,-\kappa_2)}(D)\qquad\text{and}\qquad\mathscr{H}_k(D)^{G_1'\times\cdots\times G_d'}=\mathscr{H}_{(\kappa_1,-\kappa_2)}(D)^{G_1'\times\cdots\times G_d'},$$

for each choice of $D\in\Delta_k$ and non-negative integers κ_1,κ_2 with $\kappa_1+\kappa_2=k$. Thus, considering only those D corresponding to one-dimensional representations of K as before, we recover Corollary 4.9 (when $p+q\le k_s$ for each s).

We finish by stating a corollary to summarise the significance of the results of Chapter 5 from this perspective. Recall that the work therein concerned the ring of polynomials $\mathcal{Q}_n(2)$, and specifically describing the subspaces $\mathcal{H}_{n,v}(k_1,k_2)$ (given by linear relations on the coefficients in Theorem 5.6). In Ban's notation, note that

$$\mathscr{H}_k((v\mathbf{1}_n, v\mathbf{1}_n))^{G_1'\times G_2',K} \ni h \leftrightarrow \Phi_h \in \mathcal{H}_{n,v}(k_1,k_2),$$

where as usual $k = k_1 + k_2$, $G_1' \times G_2' = \mathrm{U}(k_1) \times \mathrm{U}(k_2) \subset \mathrm{U}(k)$, and $K \cong \mathrm{U}(n) \times \mathrm{U}(n)$ is our choice of maximal compact subgroup of $\mathrm{U}(n,n)$. As such, we can say the following concerning scalar-valued Hermitian modular forms (where we always identify the one-dimensional target space with $\mathbb{C}$).

Corollary 6.12. *We fix the signature to be (n,n) for a positive integer n and fix a level Γ. Take Hermitian modular forms $f_s : \mathbb{H}_n \to \mathbb{C}$ of weights $(-\kappa_1^{(s)}, \kappa_2^{(s)})$ where $\kappa_1^{(s)}, \kappa_2^{(s)}$ are non-negative integers with $k_s := \kappa_1^{(s)} + \kappa_2^{(s)}$ such that $k_s \geq 2n$ for $s = 1,2$. As before, we put ϕ_s for the corresponding function on the group, $\phi_s : G \to \mathbb{C}$. Write ρ_{ϕ_s} (resp. ρ_{f_s}) for the irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-module generated by ϕ_s (resp. f_s) inside $C^\infty_{mod}(G)$ (resp. $C^\infty_{mod}(\mathbb{H}_n)$). Write Q for the unique (up to scaling) polynomial in $\mathcal{H}_{n,v}(k_1,k_2)$ given by the linear relations in Theorem 5.6 with an integer $v \geq 1$ (since the case $v = 0$ is somewhat trivial). Then:*

(i) The polynomial Q gives the unique (up to re-scaling) polynomial in raising operators (via "evaluation" at $\pi^+_{i,j}$) which, when applied to generators of $L((\kappa_1^{(1)}\mathbf{1}_n, -\kappa_2^{(1)}\mathbf{1}_n))$ and $L((\kappa_1^{(2)}\mathbf{1}_n, -\kappa_2^{(2)}\mathbf{1}_n))$ gives a generator of the summand

$$L(((\kappa_1^{(1)}+\kappa_1^{(2)}+v)\mathbf{1}_n, -(\kappa_2^{(1)}+\kappa_2^{(2)}+v)\mathbf{1}_n)) \subset L((\kappa_1^{(1)}\mathbf{1}_n, -\kappa_2^{(1)}\mathbf{1}_n)) \otimes L((\kappa_1^{(2)}\mathbf{1}_n, -\kappa_2^{(2)}\mathbf{1}_n)),$$

as in the decomposition given in Corollary 6.9.

(ii) Via the isomorphisms $L((\kappa_1^{(s)}\mathbf{1}_n, -\kappa_2^{(s)}\mathbf{1}_n)) \cong \rho_{\phi_s} \cong \rho_{f_s}$, $Q(\pi^+)$ therefore gives the unique (up to scaling) holomorphic Hermitian modular form of weight $(-(\kappa_1^{(1)} + \kappa_1^{(2)} + v), (\kappa_2^{(1)} + \kappa_2^{(2)} + v))$ inside $\rho_{\phi_1} \otimes \rho_{\phi_2}$ (resp. $\rho_{f_1} \otimes \rho_{f_2}$) when applied to ϕ_1 and ϕ_2 (resp. f_1 and f_2).

(iii) In the case of holomorphic functions on $\mathbb{H}_n$, the same generator of weight $(-(\kappa_1^{(1)} + \kappa_1^{(2)} + v), (\kappa_2^{(1)} + \kappa_2^{(2)} + v))$ inside $\rho_{f_1} \otimes \rho_{f_2}$ can be obtained by applying the differential operator given by Q to f_1 and f_2, which is a cusp form by Proposition 5.9.

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