MENDING MATHEMATICS

by

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A THESIS

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Shlomo Libeskind

The primary goal of my thesis is to attempt to formulate problems in mathematics that are geared more towards developing student interest and multi-purpose reasoning skills rather than repetitive actions. These problems were created using strategies described in various pieces of literature I have read on the subject. Some were created from scratch, others were based on problems found in mathematical texts. The goal is to see if these problems generate student interest in mathematics. These problems were typed into pdf files using Latex Code and sent to high school teachers to present to the students. After attempting to solve the problems, the students took a survey to rate their experience.

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Introduction

It’s no secret that the United States education system struggles with graduating students from high school who are adept at mathematics. Numbers from the National Center for Educational Statistics show that among countries in the OECD (Organization for Economic Co-operation and Development) the United States scores below average with a 481, whereas 494 is average. Furthermore, overall 29 education systems scored above the United States. More shocking than these student scores, however, is the impression that the mathematics education system in the U.S. has had on adults. In a survey of random adults in the United States 71% of participants could not calculate the miles per gallon on a road trip and 58% of them could not calculate a 10% tip on a lunch bill. Although these adults are clearly ill-equipped mathematically for the world, a measly 15% of participants stated that they wish they had learned more in mathematics while they were in school (Willis 2010). Clearly, the United States has issues developing students who can utilize math skills in the future.

So the question remains, how can the system be changed to increase student interest and performance in mathematics? How can we take a system with supposed dry subject matter and turn it into a source of valuable knowledge to carry throughout adult life? These are the questions that I hope my thesis touch upon, starting with a small population.

Problem Formulation Strategy

Let us narrow our scope. One subject in mathematics where students first begin to struggle with math is algebra. Studies have shown that interest in math actually depreciates during adolescence in general, and algebra classes in particular are where it happens. Algebra classes also hit low income students harder than others, failure rates in Algebra 1 are high in general, but even higher for lower income students. Some even suggest that algebra 1 should be scrapped altogether as a class (Howell 2012). However, algebra is a basic principal of mathematics, one that plays a silent but essential role in daily lives. Algebra also opens the door for more advanced concepts in mathematics and is vital for most scientific field. So if we do not remove algebra, what can we do to improve it? Through my search for methods of improving algebra education, I have discovered an extensive list of strategies which may be utilized in the classroom to aide in improving student interest and skill with mathematics. From this list, I derive the framework for the algebra problems to be presented to students. Unfortunately, because I am not a teacher, I do not have access to my own group of mathematics students to utilize many of these strategies. So, eliminating those options I am left with the following:

1. Demonstrate the usefulness of math in day to day scenarios or with some imagination
2. Create problems that are Achievable Challenges
3. Use the video game model for teaching students
4. Reduce negativity from mistakes

5. Limit repetition
6. Use estimation to make answers appear error-less
7. Turn mistakes into learning opportunities
8. Use problems that can be solved using multiple approaches
9. Include multimedia
10. Do not help the student too much, encourage intuition
11. Ask short questions

(Willis 2010, Boaler 2000, Meyer 2010)

This list is what sparked my idea for a proper research project. All these accessible strategies relate not directly to the teacher, but the formulation of the problems themselves. Therefore, if I could make a problem set of my own using these strategies, could there be a significant response of interest from students interacting with the problems? From here, I made my problems set for Algebra 1 students. Now, let us take a more detailed look of what these strategies entail:

Strategy Descriptions:

1. Presenting problems in the context of real life scenarios allows students to better understand what a question is asking because it correlates to concepts they can relate to. It can also act as a method for leveling the playing field between students of different levels of math experience because they can all comprehend what the problem is asking
2. Achievable challenges is a concept created by Judy Willis that involves formulating problems so that they are manageable but not too easy. This way the

problems are not so hard that the student gives up, but not so easy that they get bored with the material.

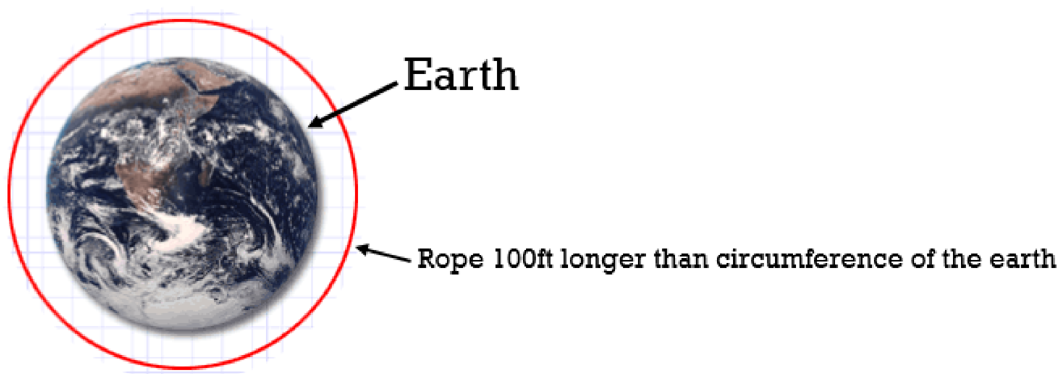
3. The video game model, as described in “Learning to Love Math” is a way of presenting a series of problems so that they increase in difficulty but as the difficulty increases the participant is able to conquer them using concepts they learned in previous problems. This model can apply to creating problems in mathematics because it encourages feelings of accomplishment and progress.
4. Negativity discourages students from participating in problems, so if negative feelings towards failure involving math problems can be minimized, fewer students will dismiss problems as not being worth the effort.
5. The repetition of the same problem can cause students to grow bored of the material. It has also been proven that repeatedly studying the same material can result in “over studying” which can actually diminish the material learned.
6. By allowing students to estimate, Math problems no longer have right or wrong answers they just have more correct and less correct answers. This further limits the negativity that math problems can create.
7. By twisting mistakes and making them seem less like an error and more like an opportunity, students are more likely to learn and improve skills without the barrier of negative feelings.
8. By offering students multiple avenues to solve problems students can be more creative with their approaches to problems. This makes students feel less like they are replicating a process and more like they are creating their own solution.

9. Multimedia, such as pictures, allows students to understand the problem visually, not just in text.
10. Allowing a student to rely on intuition gives them the opportunity to experience the thrill of accomplishment on a task for themselves, and learn how to use their own intuition to grasp concepts
11. Shorter questions are less intimidating to students so they feel more comfortable working out ways to solve it.

Clearly, not all of these strategies can be crammed together into every problem on the worksheet. So, attempts must be made to utilize problems which employ a good number of strategies without overcomplicating them. The following is one example which uses a significant number of the above strategies.

Example Problem with Analysis

Suppose the earth has a rope tied tightly around its equator. Let 100 feet of rope be added to extend the length of the rope that wraps around the earth. Pretend that this rope can now float above the earth so that it is at an equal distance from the earth at all points. What is the distance between the earth and the rope? Could a person walk under it?



Please answer this in the following steps:

a) What information do you think you need to solve this problem?

b) Let the radius of the earth be R and let x be the distance between the surface of the earth and the rope. Using the variables listed, write two expressions for the circumference of the circle formed by the rope: one using R and 100, the other using R and x. (Reminder: the circumference of a circle is 2r where r is the radius)

c) Set the expressions you got in part (b) equal to each other and solve for x.

d) What is the answer? Could someone walk under this rope?

e) Would the answer be different if we did the same with the planet Mars? Why or why not?

The solution to this problem, to be precise, is $100/2\pi$. More interesting, however, is the fact that no additional information is needed for this problem than what is provided; the math actually works out nicely to this answer. Furthermore, because the student does not need to know what the radius of the earth is to come to this conclusion, this answer is actually the same no matter what circular object is surrounded by a rope. In other words, if you tied a rope tight around a basketball and then added a hundred feet to that rope, there would still be $100/2\pi$ ft. of distance between the rope and the ball, just like with the earth.

Notice that this problem applies strategies 1 through 6 and 9 to 11. Here is why (each number corresponds to one of the strategies described in the previous section):

1. While the scenario presented is a stretch (Pun intended) it still contains substance from real life that students are familiar with.
2. This problem is semi-challenging because it requires knowledge of algebra and circumferences. While the overall question is hard, the steps leading up to it are not so hard.
3. The problem presents the questions in order of simplest formula to most complicated formula and all the problems lead up to answering a bigger question.
4. Students can do some of the math wrong and still get the overall question right.
5. There is little repetition in this problem since all the steps require different approaches.

6. It is possible for the student to estimate what they think the answer is and still come to the correct conclusion.
7. This strategy is not seen in this particular problem.
8. This strategy is not seen in this particular problem.
9. An image is provided that displays the information from the problem.
10. Students are not spoon-fed information in this problem, they need to surmount the problem themselves.
11. The entire story problem is long, but the questions leading up to the answer are short.

Thus, this is the type of question I proposed to teachers in the hopes that the strategies could spark interest in their students.

Previous Research

There are some examples of research done previously which involve strategies for improving student performance in algebra. For instance, consider the study “Correct vs. Incorrect Examples in Algebra” by Booth, J.L., Lange, K.E., Koedinger, K.R., & Newton, K.J. As the title suggests, this study observed student performance in algebra based on the types of example problems they were given. A group of students were given examples of how to correctly solve problems, while another group was instructed on how NOT to solve algebra problems. Interestingly, this research determined that incorrect approaches to solving example problems, whether given alone or in conjunction with correct examples, are actually especially beneficial for students’ learning. This is related to problem #2 on my problems list in which the student needs to read a problem that is solved incorrectly and find the error. Another study into algebra is entitled “Learning Effects of Examples Applied to College Algebra Student Interests” by Carrie A. Campbell. This study applies directly to the “Making math problems more directly related to their life” strategy for helping students learn. Campbell is able to determine the effectiveness of this strategy. From her study she concludes that the students who were given example problems based on their interests performed significantly better on quizzes than the students who had normal problems. This also supports the learning strategy of presenting math problems in the context of real life scenarios. Although this project does not cater to all student interests, it can still reach out to some. We also see from Thompson and Senk’s study in 2001 entitled “The Effects of Curriculum on Achievement in Second-Year Algebra: The Example of the University of Chicago School Mathematics Project” in which students who are taught algebra in the context of real life applications tend to actually apply algebra in real life scenarios more effectively than students who learn algebra the traditional way. Thus, this further demonstrating the significance of the strategy to present problems in the context of real life scenarios. All this previous research considered, I have been unable to find a research study that presents new and unusual problems to determine whether those problems can spark interest. It is my hope that my modest research can begin to fill this void of knowledge, or, at least point to a potential area for further research.

An Analysis of Current Textbooks

Today in the U.S., The Common Core State Standards Initiative is in effect, setting guidelines for how students from Kindergarten through grade twelve should understand Mathematics and English Language Arts. This initiative was sponsored by the National Governors Association and the Council of Chief State School Officers with the goal of establishing standards for education which are consistent across all states. The expectation is that these standards will allow for students across the country to be prepared for college or entry into the workforce after high school graduation. So, let us consider a common core text book such as "Algebra 1 Common Core" by Pearson Inc., in particular the section on variables and expressions. The goal of this section is to write algebraic expressions. The practice problems in this section demonstrate the typical textbook structure: start with simple problems, increase complexity, and end with word problems. Practice with plain and simple examples such as "Write an algebraic expression for each word phrase: 4 more than p." It is perfectly acceptable to start with simple examples so that students can apply their recently obtained knowledge. Then we see more complicated problems to get the student thinking more critically, such as "9.85 less than the product of 37 and t." This problem forces the student to use more of what they learned than the basics. Then, finally, we expect to end the section with a word problem related to writing expressions. The example from this section is the following: "You and some friends are going to a museum. Each ticket costs $4.50. Part A: If n is the number of tickets purchased, write an expression that gives the total cost of buying n tickets. Part B: Suppose the total cost for n tickets is $36. What is the total cost if one more ticket is purchased?" This problem is essentially repeating the questions of the previous problems in word problem form. But it is not just this word problem in the section, every word problem in the section is just the application of the same principals. There is nothing new or interesting for the student to encounter, nothing to make the student think critically or use more problem solving efforts beyond basic expression writing. Furthermore, the only strategies that this problem utilizes are the use of day to day scenarios and short questions. The previous problem about the rope around the earth, on the other hand, presents the same problems but in the context of a situation that pushes students' imaginations. This allows students to discover an interesting solution. It is the same principle, but with a more desirable ending. And with such an ending a student can feel more accomplished and more engaged with the work, rather than experience the monotony of plugging information into a formula and producing the required solution from the section. Furthermore, the rope around the earth problem utilizes many more of the aforementioned strategies to help encourage student engagement.

But let us look beyond the simpler word problems in this book. This textbook also has a selection of "Challenge Problems" marked with a C to indicate that the problem is meant to test the student further. Could these problems provide more interesting or insightful problems? Here is an example of one Challenge Problem: "Your friend uses the order of operations to find the value of 11(39-3). Would you prefer to use the Distributive Property instead?" (Charles, 2012: 52). Looking at this problem, what would capture the student's interest? Why would they care which method to use? Furthermore, this is hardly a challenge problem. If anything this is a problem about writing an opinion. A real challenge problem should be one worth solving for a student, one where they are legitimately curious about the answer. It should also be hard enough that they learn and feel accomplished when they complete it. Another Challenge Problem which approaches this goal is the following: “A company sells men’s basketballs with a circumference of 29.5 in. They also sell youth basketballs with a circumference of 27.75 in. The company has cube-shaped packaging boxes with edges that are either 8 in, 9 in., or 10 in. long. What is the smallest box in which each ball can be packaged?” (Charles, 2012: 183). This problem uses math in a real life scenario to demonstrate its usefulness, and it involves a subject that many students enjoy. It is slightly more difficult than other problems from this section because it requires students to know how to get a diameter out of a circumference. Unfortunately, however, there is not interesting end to this problem. There is nothing there to really promote critical thinking for the students and the answer is not surprising or interesting.

Furthermore, consider some problems out of the textbook “Core Connections Algebra.” This textbook is heralded as an excellent example of how textbooks should teach. This textbook does appear to sway from the typical textbook formula because it relies more heavily on word problems rather than computing plain math. If we look for interesting problems, however, we run into similar results as the “Algebra 1 Common Core” textbook. For instance, consider the following problem: “After the math contest, Basil noticed that there were four extra-large pizzas that were left untouched. In addition, another three slices of pizza were uneaten. If there were a total of 51 slices of pizza left, how many slices does an extra-large pizza have?” (Dietiker, 2013: 145) Again, this problem does present mathematics in a day-to-day life scenario, but why is it interesting? Why would a student care about how many pizza slices are in each box? And what is so interesting about finding such a solution? Further digging into the book yielded no problems of significant interest in regards to how engaging the presentation of the problem was, nor how thought-provoking the solution was.

Research Process

The results from this research project were obtained from high school students in four Algebra 1 classes from three different high schools in Oregon: Sheldon High School, Springfield High School, and Chiloquin High School. The math problems I designed were given to each high school teacher who presented the problems to their students with student and parental consent. The students then had the option to complete a survey about their experiences with the math problems. Each problem includes a footnote of what Common Core Standard that the problem abides by. The students were provided with the following survey questions:

Please circle how strongly you agree or disagree with each statement

1. I find mathematics difficult

Strongly Disagree Disagree Neutral Agree Strongly Agree

2. I found these problems difficult

Strongly Disagree Disagree Neutral Agree Strongly Agree

3. I found these problems interesting

Strongly Disagree Disagree Neutral Agree Strongly Agree

4. I would like to see more problems like these in my class

Strongly Disagree Disagree Neutral Agree Strongly Agree

5. What did you like/dislike about these problems?

6. Which problems did you find the most interesting? Why?

7. Which problems did you find the least interesting? Why?

8. Which problems did you find the hardest? Why?

9. Which problems did you find the easiest? Why?

Notice that a Likert Scale was used for the first four questions which will likely be the more important questions, while the free response questions are meant to get a better and more specific response from the students as to what they thought about the problems. The results from this survey were the primary resource for my research and will be analyzed in the section to follow. 122 students participated in this survey from four different classes at three different schools.

Data Analysis

First, we address the most important question, how much interest did the students find in these problems? From this survey, we generally get positive results. More students found these problems interesting than those that did not find them interesting. 47 students stated that they either agreed or strongly agreed that the problems were interesting, that is 39% of the respondents. Meanwhile, 25 students or 20% of the students disagreed or strongly disagreed that the problems were interesting, almost half as many. Conversely, the students' response to whether they wanted to see more similar problems in their class was less ecstatic: 23 students wanted to see these problems in their class (19%) compared to 50 students who said that they did not want to see more problems like these problems in their class (41%). When students were asked what they disliked about the problems, their most common response was that the problems were too hard (46 students, or 38%). In contrast, the most common thing that the students enjoyed about the problems was the fact that they forced the students to think more critically (21 students, or 17%). These figures and more can be found in Table 1.

Table 1		
Question	Number who Agree (%)	Number who Disagree (%)
I find math difficult.	37 (30%)	32 (26%)
I found these problems difficult.	66 (54%)	9 (7%)
I found these problems interesting.	47 (39%)	25 (20%)
I want to see more problems like these in my class.	23 (19%)	50 (41%)

Additionally, there are some interesting relationships between the data points from the survey. Each response to the Likert Scale questions were assigned a value: one for strongly disagree, two for disagree, three for neutral, four for agree, and five for strongly agree. There was a fairly strong correlation between difficulty in math for the students and their difficulty with the problems. There is a small correlation between student interest in mathematics and student interest in the math problems. Unexpectedly, there was a huge lack of correlation between finding mathematics difficult and student interest in the problems. Lastly, there was a weak negative correlation between the problem difficulty and student interest. These specific relationships and the correlation coefficients (“r”) are displayed in figures 1 through 4.

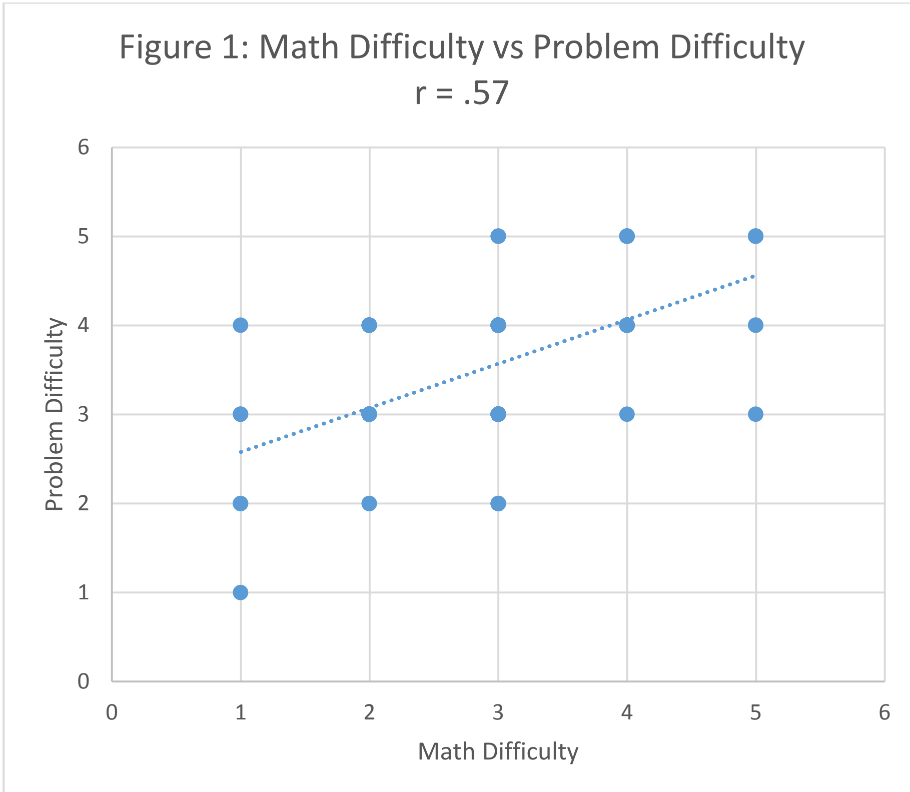


Figure 1: Math Difficulty vs Problem Difficulty
r = .57
Problem Difficulty
Math Difficulty
0
1
2
3
4
5
6

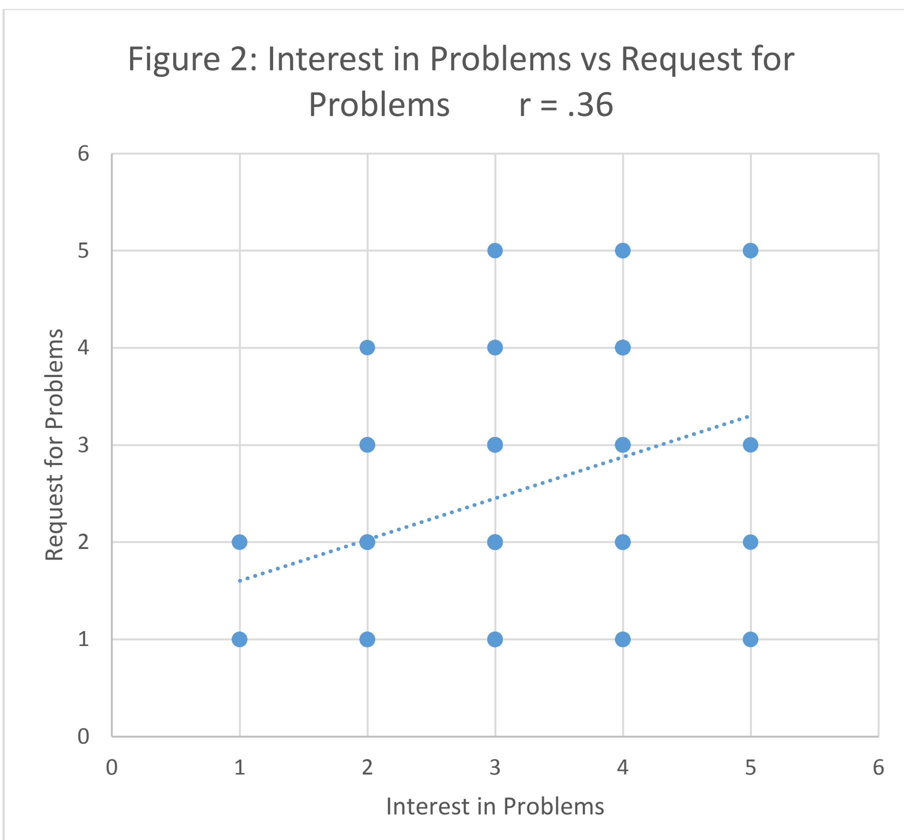


Figure 2: Interest in Problems vs Request for Problems r = .36
Request for Problems
Interest in Problems
0
1
2
3
4
5
6

Figure 3: Math Difficulty vs Problem Interest r = -.004

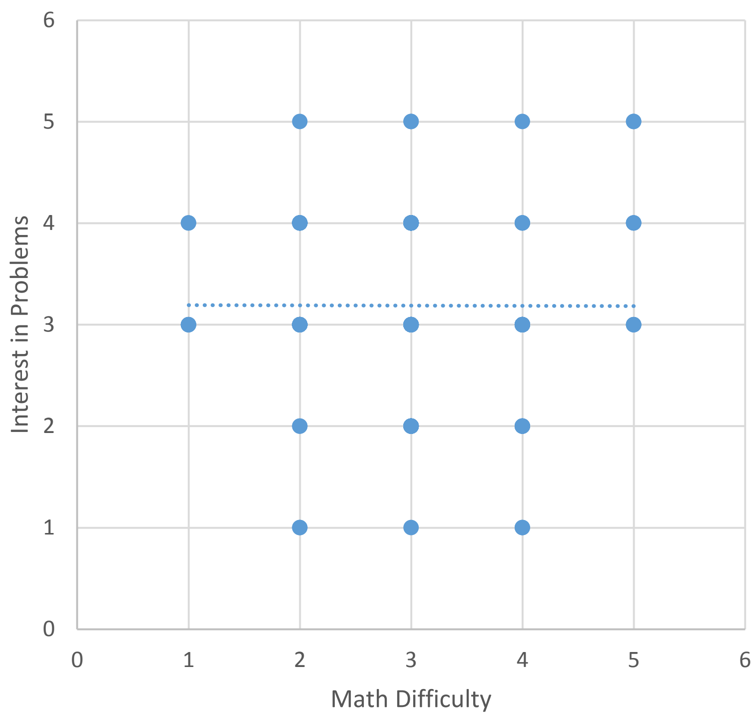
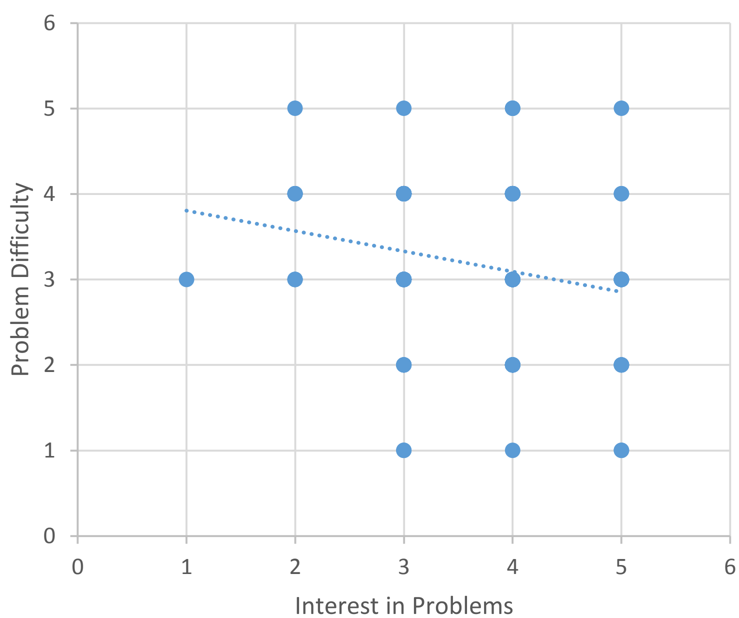


Figure 4: Problem Difficulty vs Interest r = -.22



Conclusion

This research demonstrates mixed results about the problems as to how much interest they generate. Although more students found the problems interesting than those who did not (39% vs 20%), the students who wanted to see more problems like these in their classes are a minority (17%). This begs the question: where is that discrepancy? Why do the students find the problems interesting but do not want to see more in their classes. A multitude of factors play into this result. First, the difficulty of the problems is likely a key factor. The difficulty of the problems was the number one complaint that students had in the survey. Furthermore, the education system probably influenced the students' responses. The students were interested in the problems at the time, but probably felt that if they were given these problems for class as an assignment or test question that they could not confidently say they would be able to complete it. This is concerning to them because their grades would suffer. It is also important to look at the overall interest. The fact that there were almost twice as many students who found these problems interesting as those that found it uninteresting demonstrates that there is an audience of students that are attracted to these types of problems. Although many of these students may not want to see these problems assigned in class, these students could greatly benefit from a low pressure environment where they could attempt the problems without risk of it affecting their grade. Furthermore, this research demonstrates the importance of the teaching strategy "Achievable Challenges." These problems were likely too difficult for the Algebra 1 level. This is displayed by the 66 students (54%) who found the problems difficult as well as the common response from students disliking the level of difficulty (38%). The small negative correlation between student interest and problem difficulty in Figure 4 may also support this. A similar research project in which these problems are given to higher level students, perhaps algebra 2 or even pre-calculus, could give a better representation of student interest. Unfortunately, these problems appear too difficult for students who already find math challenging. As evidenced by Figure 1, students that already find math hard tended to find these problems hard as well. Fascinatingly, however, there is no correlation between students finding math difficult and finding these problems interesting. This exhibits how a set of similar problems given with the appropriate level of difficulty to an algebra class could actually be engaging for students who may find mathematics difficult in general. Altogether, while these problems did not generate as much engagement or interest as one would hope, it displays how there is a cluster of students who could appreciate such problems. It also shows that students who find math difficult can still be drawn to math problems that require more critical thinking.

Algebra 1 Problems

Name:

Instructions: Complete the following problems as instructed, then, complete the survey at the end (if you would like).

1. Think of a number between 1 and 10. Then, do the following:

- Add the next integer after your number to your number
- Add 9 to the number you just got
- Divide the number you now have by 2
- Subtract the number you started with from your current number

The number in your head right now is 5

Now, answer the following question: How did you end up with 5 without this problem knowing the number you thought of? Use some algebra. (Hint: Choose x to be the number you started with, then try to work through the steps)

CCSS.Math.Content.HSA.REI.A.1: Creating Equations: Understand solving equations as a process of reasoning and explain the reasoning

2. This next question will mistakenly prove that 2 = 1. You must find the error.

(a) Let $a = b$

(b) Then, multiplying both sides by b gives us: $ab = b^2$

(c) Subtracting a from both sides gives us: $ab - a^2 = b^2 - a^2$

(d) So we factor out $(b - a)$ from both sides and get: $a(b - a) = (b + a)(b - a)$

(e) Now we divide both sides by $(b - a)$ and get: $a = b + a$

(f) But, $a = b$, so that is the same as: $a = 2a$

(g) So now we divide by a and at last we get: 1 = 2

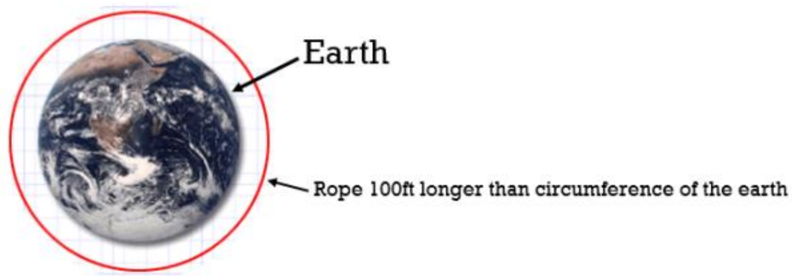
When you find the error, circle it and explain why it does not work.

CCSS.Math.Content.HSA.REI.A.1 Understand solving equations as a process of reasoning and explain the reasoning

3. A book costs $1 plus half its price. How much does it cost?

CCSS.Math.Content.HSA.REI.A.1: Understand solving equations as a process of reasoning and explain the reasoning

4. Suppose the earth has a rope tied tot around its equator. Let 100 feet of rope be added to extend the length of the rope that wraps around the earth. Pretend that this rope can now float above the earth so that it is at an equal distance from the earth at all points. What is the distance between the earth and the rope? Could a person walk under it? Please answer this in the following steps:



(a) What information do you think you need to solve this problem?

(b) Let the radius of the earth be R and let x be the distance between the surface of the earth and the rope. Using the variables listed, write two expressions for the circumference of the circle formed by the rope: one using R and 100, the other using R and x. (Reminder: the circumference of a circle is $2\pi r$ where r is the radius)

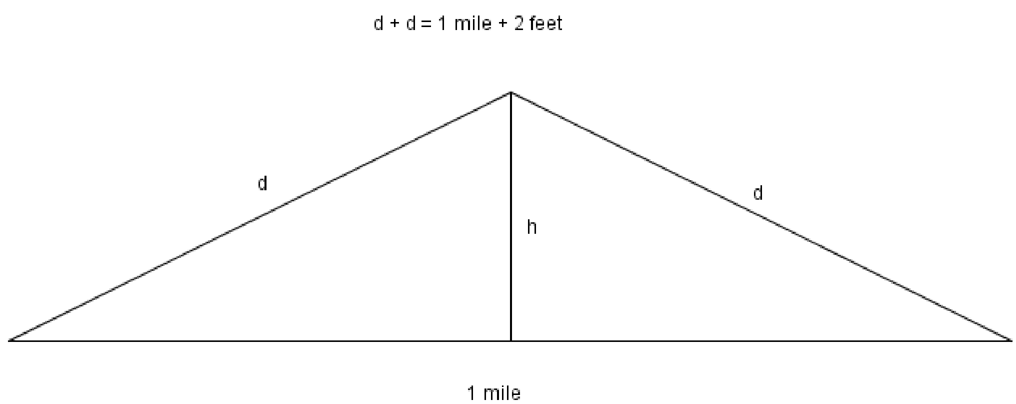
(c) Set the expressions you got in part (b) equal to each other and solve for x.

(d) What is the answer? Could someone walk under this rope?

(e) Would the answer be different if we did the same with the planet Mars? Why or why not

CCSS.Math.Content.HSA.CED.A.2: Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales

5. Rails on a train track are typically restrained in order to avoid a "sun kink." A sun kink is when the rails lengthen due to hot weather or shorten due to cold weather. Suppose a one mile stretch of rail is not restrained and will expand 2 feet. What will be the height (h) of the buckle? Is it a dangerous amount? (Reminder: a mile is 5280 ft., and the Pythagorean Theorem says that in a right triangle $a^2 + b^2 = c^2$ where c is the hypotenuse). Use the diagram below and assume that the buckle forms an isosceles triangle.



CCSS.Math.Content.HSA.REI.A.1: Understand solving equations as a process of reasoning and explain the reasoning

6. Suppose you travel directly between home and school at a rate of 60 miles per hour and return directly home from school at a rate of 40 miles per hour. You take the same route both times. What is your average rate for the entire round trip? The answer is NOT the arithmetic average (60+ 40)/2 = 50. Hint: use the fact that $d = rt$ (d = distance, r = rate, t = time). The average rate is the entire distance divided by time.

CCSS.Math.Content.HSA.REI.A.1: Understand solving equations as a process of reasoning and explain the reasoning

7. A student was asked to subtract 3 from a given number and then divide the answer by 9. Instead the student subtracted 9 from a number and then divided by 3, getting an answer of 43. If the student had worked out the problem correctly, what would be the answer?

CCSS.Math.Content.HSA.REI.A.1: Understand solving equations as a process of reasoning and explain the reasoning

8. (Non-algebra) Use exactly three 3s and any numerical mathematical symbols to write mathematical expressions.

Example: $8 = 3! \times \frac{3!}{3}$

Try to make two of these expressions that are equal to 9 in different ways.

CSS.Math.Content.HSA.APR.D.6: Rewrite rational expressions

9. Pick a number. Triple it. Add 12. Divide the new sum by three. Subtract four. What do you get? Use variables to show why this occurs.

CCSS.Math.Content.HSA.REI.A.1: Understand solving equations as a process of reasoning and explain the reasoning

10. (Non-algebra) Would you make more money by getting 10,000 dollars a day every day for a month, or, by getting a penny the first day and then have your total amount of money double each day? Write an explanation of how you arrived at your answer.

CCSS.Math.Content.HSA.REI.A.1: Understand solving equations as a process of reasoning and explain the reasoning

11. Notice that if you add 3 to the numerator and the denominator of 1/3, you get a new fraction that is double the original fraction.

(a) Find a fraction that will triple when its denominator is added to both its numerator and its denominator.

(b) Find a fraction that will quadruple when its denominator is added to both its numerator and its denominator.

12. Solve each of the following systems of two equations with two unknowns and answer the questions that follow.

(i) $x + 2y = 3$

$4x + 5y = 6$

(ii) $2x + 3y = 4$

$5x + 6y = 7$

(iii) $-2x - y = 0$

$x + 2y = 3$

(a) What are your solutions?

(b) Write another system that has the same solution as what you got in part (a)

(c) Consider the following system:

$nx + (n + 1)y = n + 2$

$(n + 3)x + (n + 4)y = n + 5$

Show that for all numbers *n* that this system has the same solution as those in part (a)

(d) Does n have to be an integer? Why or why not?

(e) Show that the system

$33x + 40y = 47$

$50x + 53y = 56$

Has the same solutions as in part (a)

Make a conjecture about what systems that differ from the ones in (c) will have the same solutions as in (a)

CCSS.MATH.CONTENT.HSA.REI.C.6: Solve systems of linear equations exactly and approximately

13. Consider a cube with an edge of length a. Which of the following will make a cube with a bigger volume:

(a) A cube whose volume is 20 times the volume of the cube with edge a

(b) Another cube whose edge is 3 times the length of the edge of the original cube (that is, its edge length is $3a$)

Why does the cube you have chosen have a larger volume? (Show with algebra) Hint: the volume of a cube is x^3 where x is the length of one edge.

CCSS.Math.Content.HSA.APR.C.4: Use polynomial identities to solve problems

14. Jeff has a nickel and a dime on his person. Sally asks him to put one coin in his left hand and the other in his right. She says to double the value of the coin in his right hand and triple the value of the coin in his left hand and then add the values together. Jeff only told her the digit in the ones place, but, Sally was immediately able to figure out which hand each coin was in. How is this possible? (Hint, think evens and odds)

CCSS.Math.Content.HSA.SSE.B.3: Write expressions in equivalent forms to solve problems

15. A boy was idly standing by his house pondering ways to make money. A strange man comes up to him and tells him: I can get you some money! Go cross that bridge and walk back, and when you get back I will double your money! But, because I am being so generous, I will take 24 dollars back from you each time. The boy went twice and the man kept his word, but upon the third time he returned from the bridge he discovered that he had 24 dollars left and so he had to give it all to the man. He had been fooled.

(a) How much money did he start with?

(b) How much money would the boy need to start with to make listening to the stranger profitable?

CCSS.Math.Content.HSA.REI.A.1: Understand solving equations as a process of reasoning and explain the reasoning

16. The following will prove that the weight of an elephant is the same as the weight of a mosquito. Suppose x is the weight of the elephant and y is the weight of the mosquito. Suppose their sum is $2v$. Then,

$x + y = 2v$, so $x - 2v = -y$, thus, $x = -y + 2v$.

Now since $x - 2v = -y$, we can multiply the left by $x - 2v$ and the right by $-y$ to get: $x^2 - 2vx = y^2 - 2vy$.

If we add v^2 to both sides we get $x^2 - 2vx + v^2 = y^2 - 2vy + v^2$ or $(x - v)^2 = (y - v)^2$.

Taking square root gives us $xv = yv$, so, $x = y$.
What mistake was made here?

CCSS.Math.Content.HSA.REI.A.1: Understand solving equations as a process of reasoning and explain the reasoning

17. (Non-algebra) Look at the following columns of numbers:

123456789	1
12345678	21
1234567	321
123456	4321
12345	54321
1234	654321
123	7654321
12	87654321
1	987654321

The numbers on the right are the same as the numbers on the right only reversed. Which column has a larger total if you add all the numbers up?

(Try to answer this without actually adding them up. Then, actually add them up to see if you were right)

CCSS.Math.Content.HSA.SSE.B.3: Write expressions in equivalent forms to solve problems

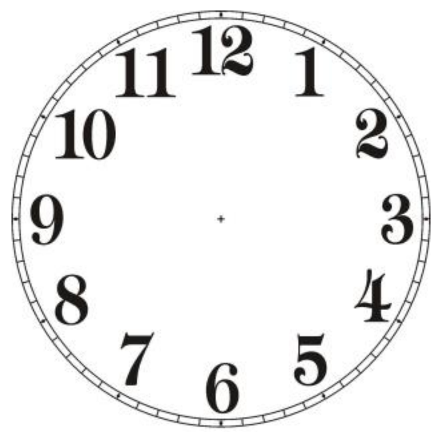
18. (Non-algebra) A cab driver noticed that his odometer read with the number 15951 miles. He realized that this is a palindromic number: it reads the same backward as forward.

"Curious," the driver said to himself. "It will be a long time before that happens again." But 2 hours later the odometer showed a new palindromic number.

How fast was the car traveling for those two hours?

CCSS.Math.Content.HSA.SSE.B.3: Write expressions in equivalent forms to solve problems

19. Take any regular watch face. Can you divide the face with 2 straight lines that do not cross so that the sums of each part are equal? (First, you may want to find out what the sum of each part must be)



CCSS.Math.Content.HSA.REI.A.1: Understand solving equations as a process of reasoning and explain the reasoning

20. Find two different numbers between 0 and 100 such that each original number is three times the product of its digits.

CCSS.Math.Content.HSA.CED.A.2: Create equations in two or more variables to represent relationships between quantities

21. Tyme Lee leaves home for work at exactly 8 a.m. daily. When driving 40 miles per hour (mph), he arrives at work three minutes late. When driving 60 mph, Tyme arrives at work three minutes early. Based on this information, Tyme decides to drive 50mph to arrive at work on time, but to his surprise he arrives early. Find the speed Tyme needs to drive at to arrive on time.

CCSS.Math.Content.HSA.REI.A.1: Understand solving equations as a process of reasoning and explain the reasoning

22. There are 100 coins scattered in a dark room. 90 have heads facing up and 10 have tails up. You cannot distinguish (by feel, etc.) which coins are which. How do you sort the coins into two piles that contain the same number of tails coins? Note: you can flip the coins. Abby answered the question as follows, use algebra to show that her solution is correct:

Choose any 10 coins to make one pile, then turn over all 10 coins in that pile. The number of tails in this pile is now equal to the number of tails in the other pile of 90 coins.

CCSS.Math.Content.HSA.REI.A.1: Understand solving equations as a process of reasoning and explain the reasoning

23. A mule and a donkey were walking along laden with bags of corn. The mule said to the donkey, "If you gave me one of your bags, I would be carrying twice as much as you. But if I gave you one, we would both be carrying equal burdens. How many bags of corn were they each carrying?

CCSS.MATH.CONTENT.HSA.REI.C.6: Solve systems of linear equations exactly and approximately

24. At a meeting, someone counted a total of 66 handshakes. If everyone shook each other's hand, how many people were there?

CCSS.MATH.CONTENT.HSA.APR.B.2: Understand the relationship between zeros and factors of polynomials

Solutions:

1. Let x be the number chosen between one and ten

Add the next integer, which must be $x + 1$, giving us: $x + x + 1 = 2x + 1$

Add 9: $2x + 1 + 9 = 2x + 10$

Divide by 2: $2x+10 = x + 5$

And lastly, subtract the original number: $x + 5 - x = 5$

Thus, your final answer is 5 no matter what your starting number was.

2. The mistake can be spotted at step (e). The reason why this is an error is because $a = b$, so $b - a = 0$. Therefore, dividing by $b - a$ is dividing by 0, which is not possible.

3. Let the price of the book be x. Then you have x = 1/2x + 1, subtracting 1/2x gives us: 1/2x=1, multiplying by 2 gives us: x = 2, so the price is 2 dollars.

4. (a) One interesting thing about this problem is that you do not need to know the radius of the earth to solve this problem.

(b) Circumference formula: $2\pi r$. Our radius for the earth: R. Distance added to rope: 100. Our final formula: $2\pi R + 100$.

Circumference formula: $2\pi r$. The radius of the circle formed by the rope: $R + x$. Our formula for the circumference of the circle formed by the rope: $2\pi(R + x)$.

(c) $2\pi R + 100 = 2\pi(R + x)$. Then we divide both sides by 2π:

$R + 100/2\pi = R + x$. Then we subtract R from both sides:

$100/2\pi = x$

(d) $100/2\pi = 15.915$ feet. How tall is the tallest man in the world?

Almost 9 feet tall. So yes, anyone could walk under the rope. Most shockingly, no matter how big of a circle or sphere is used, if 100 feet is added to the rope, the distance between the rope and that figure will always be $100/2\pi$ ft.

5. 1 mile = 5280 ft. So, half of one rail unexpanded is 2640 ft. The expanded version is then half of 5282 ft. which is 2641 ft. Using the Pythagorean Theorem we get the following:

$2640^2 + h^2 = 2640^2$ $h^2 = 2641^2 - 2640^2$

$h^2 = (2641 - 2640)(2641 + 2640)$ (since $a^2 - b^2 = (a + b)(b + a)$)

$h^2 = 5281$

h = 72.67 ft.
Thus, surprisingly, a mere 2 foot expansion over an entire mile turns into an increase in height of over 70 feet. Clearly a very dangerous obstacle for a train.

6. By definition, average rate = total distance divided by total time. If the distance from home to school was d, the rate was d/60. Similarly, the rate from school to home was d/40. Hence, the average rate must be:

$$\frac{d+d}{\frac{d}{60}+\frac{d}{40}} = \frac{2d}{\frac{40d+60d}{60\times 40}} = \frac{4800d}{d(40+60)} = \frac{4800}{100} = 48.$$

Initially this might appear incorrect, but remember that one rate is greater than the other. At the greater rate, the travel takes less time. Thus, the arithmetic average of the two rates will not work here.

7. You reverse the actions that the student took and then apply the correct one: 43 x 3 = 129, 129 + 9 = 138, 138 – 3 = 135, 135/9 = 15. Thus, the answer is 15.

8. The simpler answer is 3 + 3 + 3 = 9. The slightly more complicated answer is $3^3/3$ = 27/3 = 9

9. Suppose your number is x. Then, you triple it to get 3x, add 12 to get 3x + 12, and divide by 3 to get x + 4, finally, subtract 4 and you have x again. Therefore, through this process you will always get the same number.

10. We can represent the doubling amount as an exponential function, and the $10,000 a day as a linear function. The money doubling each day can be represented by $(.01)2^t$, where t is the number of days. Since you start off with a penny on the first day, and then it doubles for the other 30, your total for the month is $(.01)2^{30}$ which is 21,474,936.48 dollars. The $10,000 a day can be represented by 10,000t, for the number of days t. With this, your total during the month will be $30,000. Surprisingly, you get much more if you take the deal that only starts you off with a penny. This shows how much quicker exponential functions can increase.

11. Suppose the correct fraction is $^a/_b$ .Then, if we add the denominator to the top and bottom we get: $\frac{a+b}{b+b} = \frac{a+b}{2b}$

Then we need the answer to be triple the original. Thus we want: $\frac{a+b}{b+b} = \frac{3a}{b}$.

Multiplying both sides by b gives: $\frac{a+b}{2} = 3a$

Multiplying by 2 gives us $a + b = 6a$, and lastly we subtract a from both sides to get $b = 5a$. Hence, our answer is any fraction where the denominator is 5 times the numerator, so this fraction will always reduce to reduce to $^1/_5$. Repeat the same process for a quadrupled fraction and you will always find your answer to be $^1/_7$

12. (a) The answers for all three problems is $(-1, 2)$

(b) Any system of equations where both of the equations are sequential, i.e. of the form shown in part (c)

(c) Let $x = -1, y = 2$. Then, on the left side of the first equation we have $n(-1) + 2(n+1) = -n + 2n + 2 = n + 2$

For the second equation we have: $(n + 3)(-1) + 2(n+4) = -n - 3 + 2n + 8 = n + 5$

Therefore, $(-1, 2)$ is a solution to this system of equations as well

(d) Yes, the solution $x = -1, y = 2$ will work no matter what n is based on what we did in part (c)

(e) Let $x = -1, y = 2$. Then in the first equation we have $33(-1) + 40(2) = 80 - 43 = 47$. In the second equation we have: $50(-1) + 53(2) = 106 - 50 = 56$. Therefore, $(-1, 2)$ is a solution to this system of equations.

(f) The conjecture would be that $(-1, 2)$ works for any system of equations where the coefficients of each equation increase sequentially by the same number

13. The first cube has an area of a^3, then, when we multiply the volume of the cube by 20 we get $20a^3$. For the second cube, we start off with an edge length a, then triple it to get a length $3a$. Then, if we make a cube with edge length of $3a$ we get $(3a)^3 = 27a^3$. Clearly, $27a^3 > 20a^3$.

14. For this trick, notice that a nickel is an odd amount, and a dime is an even amount. So, if the dime is in the left hand and the nickel is in the right hand then the result is double the nickel plus triple the dime. The answer here will be even because the odd amount (nickel) is doubled and the even amount (dime) is tripled, so the digit in the one's place will be even. Otherwise, if the nickel is in the left and the dime is in the right the result will be triple the nickel (which will still be odd) plus double the dime (which will still be even) so then the sum will be an odd number. Hence, the last digit will be odd. So, Sally knows that if the digit that Jeff gives her is odd then the nickel is in the left hand, or, if it is even then the nickel is in the right hand.

15. Suppose the idler started out with x dollars. His money was doubled and then he had 24 taken away, resulting in $2x - 24$. Then, repeating the process gives us $2(2x - 24) - 24$. Then, the money gets doubled one more time and he is left with 24: $2(2(2x - 24) - 24) = 24$. Divide by 2: $2(2x - 24) - 24 = 12$, add 24: $2(2x - 24) = 36$, divide by 2: $2x - 24 = 18$, add 24: $2x = 42$, divide by 2: $x = 21$. In order to profit the boy must start off with more than 24 dollars. That way, each time his money doubles he will be making more than the 24 dollars that is taken away.

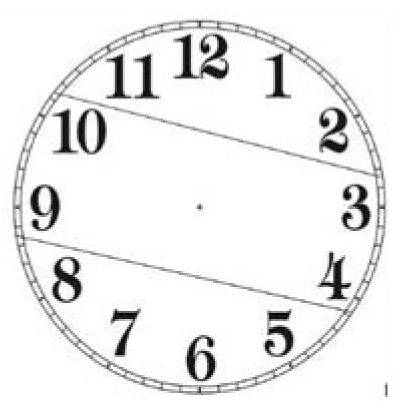
16. In this problem, we stated that the square root of $(y - v)^2$ was $y - v$, but in the context of this problem it should be $-(y - v)$.

$x - v = -(y - v)$; $x + y = 2v$. Note that $x - v$ is positive but $y - v$ is negative. If these numbers were used it would be like saying the following: 25 = 25 implies that 5 = –5. Nonsense.

17. It may not appear so, but these numbers do have the same sum. Look at each column individually. Notice that the 9 ones make up one 9, and two 8's is equivalent to eight 2's and so on. Then, once you add them up, you'll see that they are equal.

18. The first digit could not change in only two hours, so the first and last digits must be one. The five in the ten's place is going to change, so then the second and fourth digits will become six. Then it is up to the middle digit. The distance the driver traveled in 2 hours will depend on this digit, for instance if it is 0, 1, or 2, the result will be 110, 210, or 310 miles. Since the driver must be obeying normal speed limits, the best choice would be 0. Therefore, our number is 16061 and the driver went 110 miles in 2 hours so he must be moving at 55 miles per hour.

19. The approach: the sums of all then numbers on a clock is 78, which divided by 3 gives us 26. Therefore, we must find two lines to draw which create three groups of 26. Since we have the numbers one through twelve we need to pair up the larger numbers with the smaller numbers to make equal groups. Notice that if you add numbers across from each other you get 13, i.e. 12 + 1 = 13, 11 + 2 = 13, and so on. With that knowledge, you can come to this solution:



20. The two solutions are 15 and 24. Let's say our number is x. Then, $x = 10a + b$, where a is the digit in the ten's place, and b is the digit in the one's place.

Also, the problem wants x to be triple the product of the digits, so $x = 3ab$. Put those together you get $10a + b = 3ab$. You can guess and check from here. OR you can go further and say that $10a = 3ab - b$, so $10a = b(3a - 1)$. Therefore we need b or $3a - 1$ to be a multiple of 5. Since we are working with digits, one of them must be equal to 5. So either $3a - 1 = 5$ and then you get 24. Or you get $b = 5$ and you get 15.

21. Let t be the time required for Tyme to get to work on time. Three minutes = 0.05 hours. Since distance is rate multiplied by time, we can say d = (40mph)(t + 0.05). Also, distance = (60 mph)(t – 0.05) Since distance traveled is the same for both rates, 40(t+0.05) = 60(t–0.05). Hence, t +0.05 = 1.5(t–.05) which gives us: $2t + 0.1 = 3t - 0.15$. So, $t = 0.25$ hours. Furthermore, distance = 40(0.25 + 0.05) = (40)(0.03) = 12 miles. That means 12 = r (0.25). So r = 48 mph

OR 40(t + 0.05) = 60(t 0.05) 40t + 2 = 60t3 5 = 20t 0.25 = t. The distance from home to work is 40(0.25 + 0.05), or 12 miles. The average speed will be 12/.25 = 48 mph.

22. Note that the piles do not need to be the same size!
Suppose there were x tails in the 10 coin pile. Since there were initially a total of 10 tails, there must be $10 - x$ tails in the pile of 90 coins. When you flip over all 10 coins in the smaller pile, the x tails in that pile become heads and the remaining coins, which must be $10 - x$ heads, become tails. That matches the larger pile.

23. Suppose the mule carried x bags and the donkey carried y bags. Then the mule's statements can be written as: $x + 1 = 2(y - 1)$ and $x - 1 = y + 1$ from solving this system of equations, we get the answer: $x = 7$, $y = 5$

24. Suppose each of the x people shook each of the $x - 1$ persons, this makes the total number of handshakes $x(x - 1)$. But remember, when Jeff shakes Charlie's hand, Charlie also shakes Jeff's hand. Since we count that as one handshake, we must divide our answer by 2. Hence we have:

$\frac{x(x-1)}{2} = 66$, simplification gives us: $x^2 - x - 132 = 0$. Then, factoring gives us $(x - 12)(x + 11) = 0$, so our answer is either $x = 12$ or $x = -11$. Thus, our answer must be 12.

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