An Excision Theorem in Heegaard Floer Theory

by

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DISSERTATION ABSTRACT

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Let Y_1 be a closed, oriented 3-manifold and Σ denote a non-separating closed, orientable surface in Y_1 which consists of two connected components of the same genus. By cutting Y_1 along Σ and re-gluing it using an orientation-preserving diffeomorphism of Σ we obtain another closed, oriented 3-manifold Y_2. When the excision surface Σ is of genus one, we show that twisted Heegaard Floer homology groups of Y_1 and Y_2 (twisted with coefficients in the universal Novikov ring) are isomorphic. We use this excision theorem to demonstrate that certain manifolds are not related by the excision construction on a genus one surface. Additionally, we apply the excision formula to compute twisted Heegaard Floer homology groups of surgery on certain two-component links, including some families of 2-bridge links.

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INTRODUCTION

1.1 Excision Construction

Floer homology is a powerful tool in geometry and topology, arising as the homology of chain complexes generated by geometric data and defined by counting solutions to certain partial differential equations. There are three main flavors: Lagrangian intersection Floer homology, which studies intersections of Lagrangian submanifolds in symplectic manifolds; instanton Floer homology, based on solutions to the anti-self-dual Yang–Mills equations on 3- and 4-manifolds; and monopole Floer homology, which involves solutions to the Seiberg–Witten monopole equations. The first two were introduced by Andreas Floer in the 1980s and have since inspired a broad range of developments across mathematical physics and low-dimensional topology.

Computing Floer homology directly is often extremely difficult. As a result, a major part of the theory focuses on developing structural properties that allow indirect computations. These properties enable us to leverage a few accessible computations to extract rich information about the underlying manifolds.

Excision formulas study the behavior of Floer homologies under certain cutting and gluing of a 3-manifold along a surface. To explain the excision construction more precisely, let Y_1 be a closed, oriented 3-manifold with at most two connect components and Σ_i, $i = 1, 2$, be two disjoint closed, connected, oriented, non-separating surfaces in Y_1 of the same genus. If Y_1 has two components Y_{11} and Y_{12}, we require that Σ_i is a surface in Y_{1i}, $i = 1, 2$. Let $h : \Sigma_1 \to \Sigma_2$ be an orientation-preserving diffeomorphism. Cut Y_1 along $\Sigma_1 \cup \Sigma_2$ and denote the resulting manifold by Y'. Y' is a manifold with four boundary components:

$$\partial Y' = \Sigma_1 \cup -\Sigma_1 \cup \Sigma_2 \cup -\Sigma_2.$$

Glue Σ_1 to $-\Sigma_2$ and $-\Sigma_1$ to Σ_2, through the diffeomorphism h for each case, to obtain a closed manifold Y_2 (see Figure 1.1). We say that Y_2 is obtained from Y_1 by *excision* along the surfaces Σ_1 and Σ_2.

An excision formula which studies the behavior of instanton homology under cutting and gluing of a three manifold along a surface of genus one was introduced by Floer (see [BD95]). Kronheimer and Mrowka proved a similar formula for the monopole and instanton Floer homology where the cutting and gluing is along higher

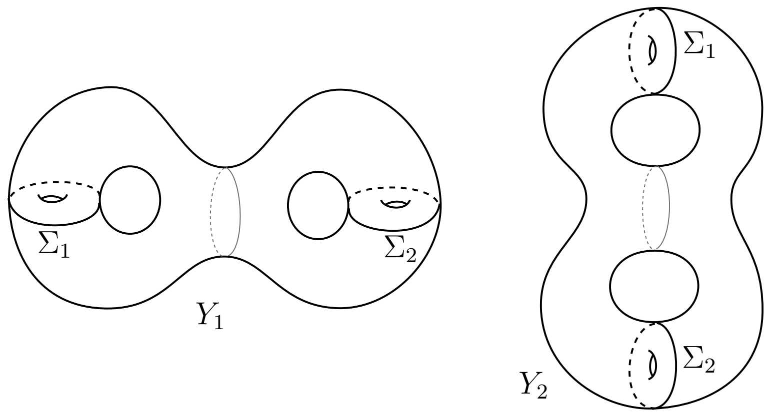

Figure 1.1. The construction of excision when Y_1 is connected.

genus surfaces (see [KM10, Theorem 3.1 and Theorem 7.7]). When this surface is of genus one, there is a version of the excision formula for monopole Floer homology with local coefficients (see [KM10, Theorem 3.2]).

Floer's excision theorem is a crucial tool in constructing monopole/instanton Floer homology for a sutured manifold and using it to define an invariant for a knot in a closed manifold. In fact, to a sutured manifold (M, γ), one can associate a closed three manifold $(Y, \overline{R})$ where $\overline{R}$ is a closed surface in Y. Kronheimer and Mrowka (see [KM10, Section 4]) defined monopole/instanton Floer homology groups for (M, γ) as the monopole/instanton Floer homology groups of $(Y, \overline{R})$, which are well-defined using Floer's excision theorem. They used this construction to define a monopole knot homology group $KHM(Z, K)$ for a knot K in a closed three manifold Z (see [KM11a]), which provided a tool to prove that Khovanov homology detects the unknot (see [KM11a]).

Another tool for studying knots and links in a 3-manifold is singular instanton Floer homology, which was defined for a 3-manifold Y with a link L (see [KM11a, KM11b]). The excision formula is valid in singular instanton Floer homology as long as the link L is either disjoint from the excision surface or intersects it in an odd number of points (see [Str12, XZ19, XZ23]). This singular version of the excision theorem is used in [XZ19] to define an instanton Floer homology for sutured manifolds with tangles which is used later to show that annular Khovanov homology detects the unlink, among other applications.

Ye's excision theorem (see [LY23, Theorem 3.30]) relates the Heegaard Floer homology groups of Y_1 and Y_2 in some specific Spin^c structures. More precisely, let Y be a closed, oriented 3-manifold and F be a closed, oriented surface in Y such that each component of Y has a non-empty intersection with F. Let F_i, $i = 1, \ldots, m$, denote the connected components of F. The *adjunction inequality* places a restriction on $c_1(\mathfrak{s}) \in H^2(Y;\mathbb{Z})$ with respect to F_i. Let $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ be a Spin^c-structure such that $\widehat{HF}(Y,\mathfrak{s}) \neq 0$. Then

$$|\langle c_1(\mathfrak{s}), [F_i]\rangle| \leq 2g(F_i) - 2.$$

Define

$$\mathrm{Spin}^c(Y|F) := \{\mathfrak{s} \in \mathrm{Spin}^c(Y) \mid \langle c_1(\mathfrak{s}), [F_i]\rangle = 2g(F_i) - 2,\ 1 \leq i \leq m\},$$

$$HF^+(Y|F) := \bigoplus_{\mathfrak{s}\in \mathrm{Spin}^c(Y|F)} HF^+(Y,\mathfrak{s}),$$

where $g(F_i)$ denotes the genus of F_i. With this introduction, Ye's excision theorem is as follows.

Theorem 1.1. *[LY23, Theorem 3.30] Let Y_2 be constructed from Y_1 as explained above. Denote the excision surface in Y_1 and its corresponding copy in Y_2 by $F = \Sigma_1 \cup \Sigma_2$ where $g(\Sigma_i) \geq 2$, $i = 1, 2$. Then*

$$HF(Y_1|F) \cong HF(Y_2|F).$$

Moreover, this isomorphism and its inverse are induced by restricted graph cobordisms. Here $HF(Y,\mathfrak{s})$ denotes $HF_{red}(Y,\mathfrak{s}) = HF^+(Y,\mathfrak{s}) \cong \mathbf{HF}^-(Y,\mathfrak{s})$ where $\mathbf{HF}^-$ is the completion of HF^- with respect to the U-power filtration.

Also, when the excision surface is of genus one, Ai and Peters proved a version of the excision formula for a closed oriented 3–manifold which fibers over the circle (see [AP10, Theorem 1.3] and Theorem 4.1 in Section 3). In this thesis, we prove an excision theorem in ω-twisted Heegaard Floer theory for arbitrary genus one surfaces. Our main theorem is:

Theorem 1.2. *Let Y_2 be constructed from Y_1 as explained above and $F = \Sigma_1 \cup \Sigma_2$. Suppose that $\iota_{F,i} : F \hookrightarrow Y_i$ and $\iota_{Y',i} : Y' \hookrightarrow Y_i$ denote the inclusions of F and Y' into*

Y_i. *Let* $\omega_i \in H^2(Y_i;\mathbb{R})$ *be such that* $\iota^*_{F,1}(\omega_1) = \iota^*_{F,2}(\omega_2) \neq 0$ *and* $\iota^*_{Y',1}(\omega_1) = \iota^*_{Y',2}(\omega_2)$. *If* $g(\Sigma_i) = 1$, $i = 1,2$, *then*

$$\underline{HF}(Y_1;\Lambda_{\omega_1}) \cong \underline{HF}(Y_2;\Lambda_{\omega_2}).$$

Moreover, this isomorphism and its inverse are induced by restricted graph cobordisms. Here $\underline{HF}(Y,\mathfrak{s};\Lambda_\omega)$ *denotes*

$$\underline{HF}_{red}(Y,\mathfrak{s};\Lambda_\omega) = \underline{HF}^+(Y,\mathfrak{s};\Lambda_\omega) \cong \underline{HF}^-(Y,\mathfrak{s};\Lambda_\omega).$$

In this thesis, we work over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ unless otherwise stated. In Theorem 1.2, Λ_ω denotes the universal Novikov ring, which is equipped with an $\mathbb{F}_2[H^1(Y;\mathbb{Z})]$-module structure via a 2-dimensional cohomology class $\omega \in H^2(Y;\mathbb{R})$. $\underline{HF}(Y,\mathfrak{s};\Lambda_\omega)$ denotes a version of Heegaard Floer homology groups which is twisted with coefficients in Λ_ω (see Chapter 3 for more details). Note that Theorem 1.2 implies that the choice of the diffeomorphism h in the excision construction does not matter (see Remark 4.9 for a precise statement).

We can use Theorem 1.2 to check if two given 3-manifolds are not related by the excision construction. Let L be a link in S^3 and let $S^3_0(L)$ denote the 3-manifold obtained by performing 0-surgery on the components of L.

Corollary 1.3. *Let* W_n *denote the* n*-twisted Whitehead link (see Figure 5.2 on the right for* $n > 0$*). If* $n, m \neq 0$ *and* $|n| \neq |m|$*, then* $S^3_0(W_n)$ *and* $S^3_0(W_m)$ *are not related by the excision construction on a genus one surface.*

Another application of Theorem 1.2 is in computing twisted Heegaard Floer homology groups of manifolds obtained by performing surgery on certain families of 2-bridge links.

Corollary 1.4. *Let* $C(m,\pm1,n)$ *denote a 2-bridge link shown in Figure 5.7 such that* $m, n \neq 0$*. Then*

$$\underline{HF}^+(S^3_0(C(n,\pm1,-n));\Lambda_\omega) \cong \Lambda^{n^2},$$

and when $|m+n| = 1$*, there is a short exact sequence*

$$0 \to \Lambda^{|mn|} \to \underline{HF}^+(S^3_{0,2m+2n}(C(m,\pm1,n)) \to \Lambda[U^{-1}] \to 0,$$

where $[\omega] \in H^2(S^3_{0,2m+2n}(C(m,\pm1,n));\mathbb{R})$ *is* $\lambda\mathrm{PD}[\eta] \neq 0$, $\lambda \in \mathbb{R}$*. Here* $S^3_{0,2n+2m}(C(m,\pm1,n)$ *is the manifold obtained by performing surgery on the components of* $C(m,\pm1,n)$ *with framings 0 and* $2n+2m$*, and* η *is the meridian of the red component of* $C(m,\pm1,n)$ *(see Figure 5.7 on the bottom right).*

1.3 Organization

In Chapter 2, we review Heegaard Floer theory. In Chapter 3, we discuss twisted Heegaard Floer theory where the Heegaard Floer homology groups are twisted by the Novikov ring. In Chapter 4, we modify the proof of Theorem 1.1 to the twisted case to provide a proof for Theorem 1.2. In Chapter 5, Theorem 1.2 is applied to compute Heegaard Floer homology groups of a 3-manifold obtained by 0-surgery on the n-twisted Whitehead link (see Figure 5.2). As a result of this computation, we provide proofs for Corollary 1.3 and Corollary 1.4.

CHAPTER 2

AN INTRODUCTION TO HEEGAARD FLOER THEORY

In this chapter, we first review the essential background for Heegaard Floer homology, followed by a discussion of the filtered version of the theory induced by knots.

2.1 Background on Heegaard Floer Homology

Associated to a closed, connected, oriented 3-manifold Y, there are Heegaard Floer homology groups denoted $HF^{\circ}(Y)$, $\circ \in \{\pm, \wedge, \infty\}$, introduced by Ozsváth-Szabó (see [OS04d]), which are modules over the ring $\mathbb{Z}[U]$ or $\mathbb{F}_2[U]$. These modules split by Spin^c structures as $\oplus_{\mathfrak{s}\in \mathrm{Spin}^c(Y)} HF^{\circ}(Y, \mathfrak{s})$. The modules are constructed using a pointed Heegaard diagram. In this section, we recall the constructions of these groups. Much of the material in this section is based on [OS04d], [OS06b], and [Hom20].

Topological background

Heegaard diagrams

We begin by recalling the notion of a Heegaard diagram.

Definition 2.1. A *pointed Heegaard diagram* is described as

$$\mathcal{H} = (\Sigma, \boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_g\}, \boldsymbol{\beta} = \{\beta_1, \dots, \beta_g\}, z),$$

where Σ is a closed, orientable surface of genus g, $\{\alpha_i\}_{i=1}^{g}$ (resp. $\{\beta_i\}_{i=1}^{g}$) are simple, closed, and disjoint curves on the surface Σ, and $z \in \Sigma \setminus \boldsymbol{\alpha} \cup \boldsymbol{\beta}$. Furthermore, we assume that $\Sigma \setminus \boldsymbol{\alpha}$ (resp. $\Sigma \setminus \boldsymbol{\beta}$) is connected.

Given a Heegaard diagram, one can construct a closed, oriented 3-manifold. Specifically, consider $\Sigma \times [-1, 1]$, and attach 2-handles along the curves $\alpha \times \{-1\}$ (resp. $\beta \times \{1\}$) in $\Sigma \times \{-1\}$ (resp. $\Sigma \times \{1\}$). This yields a 3-manifold with two spherical boundary components. By gluing 3-balls to these boundary components, we obtain a closed, oriented 3-manifold, which is unique up to homeomorphism. Moreover, every closed, oriented 3-manifold can be described by a Heegaard diagram. The diagram is not unique for a given manifold Y. However, different Heegaard diagrams for the same manifold are related by a sequence of simple modifications known as *Heegaard moves*.

Let $\mathcal{H}$ be a Heegaard diagram for a closed, oriented 3-manifold Y. We use $\mathcal{H}$ to construct the chain complexes $CF^{\circ}(Y)$, $\circ \in \{\pm, \wedge, \infty\}$, such that their chain homotopy types are invariants of Y. Let

$$\mathrm{Sym}^g(\Sigma) = \frac{\Sigma^{\times g}}{S_g}$$

denote the g-fold symmetric product of Σ, where S_g is the symmetric group on g letters, obtained by permuting the factors. One can show that $\mathrm{Sym}^g(\Sigma)$ is a smooth manifold of dimension $2g$. There are two g-dimensional sub-manifolds

$$\mathbb{T}_{\boldsymbol{\alpha}} = \alpha_1 \times \cdots \times \alpha_g, \qquad \mathbb{T}_{\boldsymbol{\beta}} = \beta_1 \times \cdots \times \beta_g$$

of $\mathrm{Sym}^g(\Sigma)$. We assume that curves in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ intersect transversely, which implies that $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$ also intersect transversely. The chain complex $\widehat{CF}(\mathcal{H})$ is freely generated over $\mathbb{F}$ by the intersections of $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$. Let $V_w = \{w\} \times \mathrm{Sym}^{g-1}(\Sigma)$, where $w \in \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$. Choose a generic path of almost complex structures J_s on $\mathrm{Sym}^g(\Sigma)$. Roughly, the differential

$$\partial : \widehat{CF}(\mathcal{H}) \to \widehat{CF}(\mathcal{H})$$

counts J_s-holomorphic Whitney disks in $\mathrm{Sym}^g(\Sigma)$ that connects two generators and avoid the subvariety V_z. We will provide the precise definition of the differential in the following subsection.

Whitney disks

Let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$. Let e_α, respectively e_β, denote the arc in $\partial\mathbb{D}$ with $Re(z) \geq 0$, respectively $Re(z) \leq 0$. A *Whitney disk* from $\mathbf{x}$ to $\mathbf{y}$ is a continuous map $\phi : \mathbb{D} \to \mathrm{Sym}^g(\Sigma)$ such that

- $\phi(-i) = \mathbf{x}$ and $\phi(i) = \mathbf{y}$;
- $\phi(e_\alpha) \subset \mathbb{T}_{\boldsymbol{\alpha}}$ and $\phi(e_\beta) \subset \mathbb{T}_{\boldsymbol{\beta}}$.

Using the Riemann mapping theorem, one can think of a Whitney disk as a map $\phi : [0,1] \times i\mathbb{R} \subset \mathbb{C} \to \mathrm{Sym}^g(\Sigma)$ such that

- $\lim_{t\to-\infty} \phi(s+it) = \mathbf{x}$ and $\lim_{t\to+\infty} \phi(s+it) = \mathbf{y}$;
- $\phi(\{1\} \times \mathbb{R}) \subset \mathbb{T}_{\boldsymbol{\alpha}}$ and $\phi(\{0\} \times \mathbb{R}) \subset \mathbb{T}_{\boldsymbol{\beta}}$.

Let $\pi_2(\mathbf{x}, \mathbf{y})$ denote the set of homotopy classes of Whitney disks from $\mathbf{x}$ to $\mathbf{y}$. Let $w \in \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ and let

$$n_w : \pi_2(x, y) \to \mathbb{Z}$$

denote the algebraic intersection number $n_w(\phi) = \#\phi^{-1}(V_w)$. Let $D_1, \cdots, D_m$ denote the closures of the components of $\Sigma \setminus \boldsymbol{\alpha} \cup \boldsymbol{\beta}$. Given $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, the *domain* associated to ϕ is the formal linear combination of the regions $\{D_i\}_{i=1}^m$

$$D(\phi) = \sum_{i=1}^{m} n_{z_i}(\phi) D_i$$

where the $z_i \in D_i$ are points in the interior of the D_i. If all the coefficients $n_{z_i}(\phi) \geq 0$, then we write $D(\phi) \geq 0$. A 2-chain $\mathcal{P} = \sum_{i=1}^n D_i$ is called a *periodic domain* if $n_z(\mathcal{P}) = 0$ and $\partial\mathcal{P}$ is a sum of α- and β-curves. For each $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, a class $\phi \in \pi_2(\mathbf{x}, \mathbf{x})$ with $n_z(\phi) = 0$ is called a *periodic class*. The domain associated to a periodic class is a periodic domain.

Spin^c structures

Fix a Riemannian metric g over Y. Two unit vector fields v_i, $i = 1, 2$, are called homologous if they are homotopic in the complement of finitely many disjoint 3-balls in Y. Spin$^c(Y)$ is defined as the space of homology classes of unit vector fields over Y. There is a non-canonical one-to-one correspondence between Spin$^c(Y)$ and $H^2(Y; \mathbb{Z})$.

There is a natural map $\mathfrak{s}_z : \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} \to \text{Spin}^c(Y)$ defined as follows. Let f be a Morse function on Y compatible with the Heegaard diagram $\mathcal{H}$. Corresponding to each $\mathbf{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, there is a g-tuple of trajectories for the gradient flow of f connecting the index one critical points to the index two critical points. Similarly, z gives a trajectory connecting the index zero critical point with the index three critical point. The gradient vector field is non-zero outside a tubular neighborhood of these $g + 1$ trajectories. Since each trajectory connects critical points of different parities, the gradient vector field has index 0 on all the boundary spheres of the subset, so it can be extended as a nowhere vanishing vector field over Y . The homology class of the nowhere vanishing vector field obtained in this manner (after renormalizing, to make it a unit vector field) gives the Spin^c structure $\mathfrak{s}_z(\mathbf{x})$. One can show that $\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}_z(\mathbf{y})$ if and only if $\pi_2(\mathbf{x}, \mathbf{y}) \neq \emptyset$ (see [OS04d, Lemma 2.19]).

Moduli spaces

Let $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ and $\mathcal{M}(\phi)$ denote the moduli space of holomorphic representatives of ϕ. $\mathcal{M}(\phi)$ is smooth and its expected dimension is called the *Maslov index*, $\mu(\phi)$. There is an $\mathbb{R}$-action on $\mathcal{M}(\phi)$ coming from vertical translations. Let $\widehat{\mathcal{M}}(\phi) = \frac{\mathcal{M}}{\mathbb{R}}$. If $\mu(\phi) = 1$, then $\widehat{\mathcal{M}}(\phi)$ is a compact manifold of dimension zero.

The differentials

We can partition the intersection points in $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ according to the Spin^c structures. In particular, we have $\widehat{CF}(\mathcal{H}) = \oplus_{\mathfrak{s}\in \mathrm{Spin}^c(Y)} \widehat{CF}(\mathcal{H}, \mathfrak{s})$ where $\widehat{CF}(\mathcal{H}, \mathfrak{s})$ consists of points $\mathbf{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ with $\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}$.

Let $\mathcal{H}$ be a Heegaard diagram for a rational homology sphere. The differential $\partial : \widehat{CF}(\mathcal{H}, \mathfrak{s}) \to \widehat{CF}(\mathcal{H}, \mathfrak{s})$ is defined to be

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}), \\ \mu(\phi)=1, \\ n_z(\phi)=0}} \# \widehat{\mathcal{M}}(\phi) \mathbf{y}$$

Note that the differential respects the splitting $\widehat{CF}(\mathcal{H}) = \oplus_{\mathfrak{s}\in \mathrm{Spin}^c(Y)} \widehat{CF}(\mathcal{H}, \mathfrak{s})$.

$\widehat{CF}(\mathcal{H}, \mathfrak{s})$ is endowed with a relative grading called the Maslov grading:

$$\mathrm{gr}(\mathbf{x}, \mathbf{y}) = \mu(\phi) - 2n_z(\phi)$$

which is well defined ([OS06b, Proposition 5.14 and Lemma 7.1]). From the definition, the differential ∂ lowers the Maslov grading by 1. Also, $\partial^2 = 0$ (see [OS04d]). Let $\widehat{HF}(\mathcal{H}, \mathfrak{s}) = H_*(\widehat{CF}(\mathcal{H}, \mathfrak{s}))$. In [OS04d], it is shown that $\widehat{HF}(\mathcal{H}, \mathfrak{s})$ is independent of the Heegaard diagram, the base point, the complex structure, and other choices in the definition.

Theorem 2.2 ([OS04d])**.** *Let $\mathcal{H}$ and $\mathcal{H}'$ be pointed Heegaard diagrams of Y, and $\mathfrak{s} \in Spin^c(Y)$. Then the homology groups $\widehat{HF}(\mathcal{H}, \mathfrak{s})$ and $\widehat{HF}(\mathcal{H}', \mathfrak{s})$ are isomorphic.*

Then, $\widehat{HF}(Y, \mathfrak{s})$ is defined as $\widehat{HF}(\mathcal{H}, \mathfrak{s})$. In the following, we recall the construction of other variants of Heegaard Floer homology groups. $CF^\infty(\mathcal{H}, \mathfrak{s})$ is a free module generated by the intersection points $\mathbf{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ over $\mathbb{F}[U, U^{-1}]$ with $\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}$. Here

U is a formal variable with $\mathrm{gr}(U) = -2$. The differential is defined as

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \sum_{\substack{\phi \in \pi_2(\mathbf{x},\mathbf{y}),\\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(\phi) U^{n_z(\phi)} \mathbf{y}$$

That is, the Whitney disks are allowed to meet the basepoint and the variable U counts the algebraic intersection number $n_z(\phi)$ of $\phi(\mathbb{D})$ and V_z. Equivalently, $CF^{\infty}(\mathcal{H}, \mathfrak{s})$ can be considered as the free Abelian group generated by pairs $[\mathbf{x}, i]$ where $\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}$, and $i \in \mathbb{Z}$. When viewed this way, the differential is defined by

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \sum_{\substack{\phi \in \pi_2(\mathbf{x},\mathbf{y}),\\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(\phi) [\mathbf{y}, i - n_z(\phi)].$$

The U-action is given by $U[\mathbf{x}, i] = [\mathbf{x}, i - 1]$. For a rational homology sphere, $HF^{\infty}(Y, \mathfrak{s}) \cong \mathbb{Z}[U, U^{-1}]$ (see [OS04c]).

Let $CF^{-}(\mathcal{H}, \mathfrak{s})$ denote the subgroup of $CF^{\infty}(\mathcal{H}, \mathfrak{s})$ which is freely generated by pairs $[\mathbf{x}, i]$, where $i < 0$. Let $CF^{+}(\mathcal{H}, \mathfrak{s})$ denote the quotient group $CF^{\infty}(\mathcal{H}, \mathfrak{s})/CF^{-}(\mathcal{H}, \mathfrak{s})$.

The group $CF^{-}(\mathcal{H}, \mathfrak{s})$ is a subcomplex of $CF^{\infty}(\mathcal{H}, \mathfrak{s})$, and there is a short exact sequence of chain complexes:

$$0 \to CF^{-}(\mathcal{H}, \mathfrak{s}) \xrightarrow{\iota} CF^{\infty}(\mathcal{H}, \mathfrak{s}) \xrightarrow{\pi} CF^{+}(\mathcal{H}, \mathfrak{s}) \to 0.$$

Note that U restricts to an endomorphism of $CF^{-}(\mathcal{H}, \mathfrak{s})$ and induces an endomorphism on the quotient $CF^{+}(\mathcal{H}, \mathfrak{s})$. There is a short exact sequence

$$0 \to \widehat{CF}(\mathcal{H}, \mathfrak{s}) \xrightarrow{\iota} CF^{+}(\mathcal{H}, \mathfrak{s}) \xrightarrow{U} CF^{+}(\mathcal{H}, \mathfrak{s}) \to 0,$$

where $\iota(\mathbf{x}) = [\mathbf{x}, 0]$.

Let $HF^{\circ}(\mathcal{H}, \mathfrak{s})$, $\circ \in \{\infty, \pm\}$, denote the homology groups corresponding to the complexes $CF^{\circ}(\mathcal{H}, \mathfrak{s})$. It is proved in [OS04d] that $HF^{\circ}(\mathcal{H}, \mathfrak{s})$, $\circ \in \{\infty, \pm\}$, are topological invariants of Y and $\mathfrak{s}$, in the sense that they are independent of the choices used in their definitions.

When $b_1(Y) > 0$, we should restrict to *admissible* Heegaard diagrams to make sure that the sum in the definition of the differential remains finite. Different variants of the theory require different types of admissibility.

A Heegaard diagram is called *strongly admissible* for the Spin^c structure $\mathfrak{s}$ if for every non-trivial periodic domain D with $\langle c_1(\mathfrak{s}), H(D)\rangle = 2n \geq 0$, D has some coefficient $> n$. Here $H(D)$ denotes the homology class corresponding to the periodic domain D. A pointed Heegaard diagram is called *weakly admissible* for $\mathfrak{s}$ if for each non-trivial periodic domain D with $\langle c_1(\mathfrak{s}), H(D)\rangle = 0$, D has both positive and negative coefficients.

Theorem 2.3. *[OS04d, Theorem 4.15] Let Y be a three-manifold equipped with a Spin^c structure $\mathfrak{s}$.*

- *If $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is strongly $\mathfrak{s}$-admissible, then $CF^\infty(\mathcal{H}, \mathfrak{s})$ is a chain complex, with subcomplex CF^- and quotient complex CF^+.*
- *If $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is weakly $\mathfrak{s}$-admissible, then $CF^+(\mathcal{H}, \mathfrak{s})$ is a chain complex with subcomplex $\widehat{CF}(\mathcal{H}, \mathfrak{s})$.*

Remark 2.4. There is an equivalent cylindrical reformulation of Heegaard Floer homology, suggested by Lipshitz (see [Lip06]), where J-holomorphic surfaces $u : S \to \Sigma \times [0,1] \times \mathbb{R}$ with cylindrical ends are counted (see [Lip06] for more details).

3+1 TQFT

Heegaard Floer homology fits into the framework of a $(3+1)$-dimensional TQFT (see [OS04a] and [OS06a]). Let W be a smooth, oriented four manifold such that $\partial W = -Y_1 \amalg Y_2$ where Y_i, $i = 1, 2$, are closed, oriented, compact and connected 3-manifolds. In this case, we say W is a cobordism from Y_1 to Y_2. This cobordism induces a map

$$F^\circ_{W,\mathfrak{t}} : HF^\circ(Y_1, \mathfrak{s}_1) \to HF^\circ(Y_2, \mathfrak{s}_2),$$

where $\mathfrak{t} \in \mathrm{Spin}^c(W)$, and $\mathfrak{s}_i = \mathfrak{t}|_{Y_i}$ (see [OS06a, Theorem 1.1]). We will review the construction of these maps in the next chapter.

Graph cobordisms

In the following chapters, we will work with cobordisms between disconnected 3-manifolds. For that purpose, we will introduce multi-pointed 3-manifolds.

Definition 2.5. [Zem19, Definition 3.1] A multi-pointed 3-manifold is a pair $(Y, \mathbf{w})$ consisting of a closed, oriented 3-manifold with a finite collection of basepoints $\mathbf{w} \subset Y$ such that each component of Y contains at least one basepoint.

Definition 2.6. [Zem19, Definition 4.1] Suppose $(Y, \mathbf{w})$ is a multi-pointed 3-manifold. A multi-pointed Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w})$ for $(Y, \mathbf{w})$ is a tuple such that $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a standard Heegaard diagram for Y and $\mathbf{w} \subset \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$.

Let $\mathbf{w} = \{w_1, \ldots, w_n\}$. Then

$$\mathbb{F}_2[U_{\mathbf{w}}] := \mathbb{F}_2[U_{w_1}, \ldots, U_{w_n}].$$

Also, $\mathbb{F}_2[U_{\mathbf{w}}, U_{\mathbf{w}}^{-1}]$ denotes the ring obtained by formally inverting each of the variables U_{w_i}. If $\mathbf{k} = (k_1, \ldots, k_n)$ is an n-tuple, we write

$$U_{\mathbf{w}}^{\mathbf{k}} := U_{w_1}^{k_1} \cdots U_{w_n}^{k_n}.$$

We write U_i for U_{w_i}. Also, $k[U_{\mathbf{w}}]$ and $k[U_{\mathbf{w}}, U_{\mathbf{w}}^{-1}]$, where k is a ring, are defined similarly. Similar to the connected case above, multi-pointed 3-manifolds can be described using multi-pointed Heegaard diagrams $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w})$ and there is a map

$$\mathfrak{s}_{\mathbf{w}} : \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} \to \mathrm{Spin}^c(Y).$$

We recall the definition of $CF^\circ(Y)$ when Y is disconnected. Let $(Y, \mathbf{w}) = (Y_1, \mathbf{w}_1) \amalg (Y_2, \mathbf{w}_2)$, where $(Y_i, \mathbf{w}_i)$, $i = 1, 2$ is a connected, multi-pointed 3-manifold. Let $\mathcal{H}_i$, $i = 1, 2$, be weakly admissible multi-pointed Heegaard diagrams for $(Y_i, \mathbf{w}_i)$. Define

$$(CF^\circ(\mathcal{H}_1 \amalg \mathcal{H}_2), \partial_{J_s}) := (CF^\circ(\mathcal{H}_1), \partial_{J_{s_1}}) \otimes_{\mathbb{F}_2} (CF^\circ(\mathcal{H}_2), \partial_{J_{s_2}}).$$

where $J_s = J_{s_1} \amalg J_{s_2}$ is a generic path of almost complex structures in Y.

The graph TQFT of Zemke (see [Zem19]) generalizes to cobordisms with disconnected ends where he defines a notion of *ribbon graph cobordism* (W, Γ) between $(Y_1, \mathbf{w}_1)$ and $(Y_2, \mathbf{w}_2)$. Here Γ is an embedded graph with $\Gamma \cap Y_i = \mathbf{w}_i$, and is decorated with a *formal ribbon structure*. Zemke proves that there are two chain maps

$$F^A_{W,\Gamma,\mathfrak{t}}, F^B_{W,\Gamma,\mathfrak{t}} : CF^\circ(Y_1, \mathbf{w}_1, \mathfrak{s}_1) \to CF^\circ(Y_2, \mathbf{w}_2, \mathfrak{s}_2),$$

which are diffeomorphism invariants of (W, Γ) up to $\mathbb{F}_2[U]$-equivariant chain homotopy. Here $\mathfrak{t} \in \mathrm{Spin}^c(W)$ and $\mathfrak{s}_i = \mathfrak{t}|_{Y_i}$. (See Subsection 3.2 for more details). We denote the map induced on the chain complex by $f^\circ_{W,\Gamma,\mathfrak{t}}$ and the map induced on the homology by $F^\circ_{W,\Gamma,\mathfrak{t}}$.

In the following chapters, following [Zem21, Section 12] and [JZ23, Subsection 2.1], $\doteq$ and $\dot{\simeq}$ indicate equality or chain homotopy of morphisms, respectively, up to some unit. We work with 2-dimensional cohomology classes $[\omega] \in H^2(X; \mathbb{R})$ where X is either a closed 3-manifold or a 4-dimensional cobordism. We consider $k[H^1(Y; \mathbb{Z})]$-modules M where k can be either $\mathbb{F}_2$ or $\mathbb{Z}$.

2.2 Knot Floer Homology

We recall some facts from knot Floer homology theory that are used in this thesis. There are several variants of bi-graded knot Floer homology groups, introduced independently by Ozsváth-Szabó and Rasmussen (see [OS04b, Ras03]).

Definition 2.7. A doubly pointed Heegaard diagram for a knot $K \subset S^3$ is a tuple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$, where $w, z \in \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$, such that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a Heegaard diagram for S^3 splitting it into two handlebodies H_1 and H_2. Then K is the union of arcs a and b where a is an arc in $\Sigma \setminus \boldsymbol{\alpha}$ connecting w to z, pushed slightly into H_1 and b is an arc in $\Sigma \setminus \boldsymbol{\beta}$ connecting z to w, pushed slightly into H_2.

Indeed, given a knot K in S^3, there exists a doubly pointed Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ that represents K (see [OS06b, Proposition 12.1]). The chain complex $CFK^\infty(K)$ is generated by elements in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ as a free $\mathbb{Z}[U, U^{-1}]$-module with the differential given by

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}), \\ \mu(\phi) = 1}} \# \widehat{\mathcal{M}}(\phi) U^{n_w(\phi)} \mathbf{y}.$$

This is a $\mathbb{Z}$-filtered chain complex and is well-defined up to filtered chain homotopy equivalence. Associated to each generator there are two gradings, called the *Maslov grading* M and *Alexander grading* A. One can think of $CFK^\infty(K)$ as freely generated over $\mathbb{Z}$ by triples $[\mathbf{x}, i, j]$ where $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $i, j \in \mathbb{Z}$, and $A(\mathbf{x}) = j - i$. $[\mathbf{x}, i, j]$ corresponds to the generator $U^{-i}x$. $CFK^\infty(K)$ is called the *full knot Floer complex*. Each generator $[\mathbf{x}, i, j]$ is presented by a dot located at the (i, j) coordinates on the plane. The sub-complex generated by $[\mathbf{x}, 0, j']$ induces a filtration on $\widehat{CF}(S^3)$

$$\cdots \subset \mathcal{F}(K, j-1) \subset \mathcal{F}(K, j) \subset \mathcal{F}(K, j+1) \subset \cdots \subset \widehat{CF}(S^3),$$

where $\mathcal{F}(K, j)$ is freely generated over $\mathbb{Z}$ by $[\mathbf{x}, 0, j']$ such that $j' \leq j$. The homology of the associated graded complex $\oplus_j \mathcal{F}(K, j)/\mathcal{F}(K, j-1)$ is denoted by

$$\widehat{HFK}(K) = \bigoplus_{i,s} \widehat{HFK}_i(K, s),$$

where i and s are the Masolv grading and the Alexander grading. $\widehat{HFK}(K)$ categorifies the Alexander polynomial (see [OS04b]) in the sense that

$$\Delta_K(t) = \sum_{i,s} (-1)^i \dim \widehat{HFK}_i(K, s) t^s.$$

For alternating knots, $\widehat{HFK}$ is determined by the Alexander polynomial. Indeed, if $\Delta_K(t) = a_0 + \sum_{s>0} a_s(T^s + T^{-s})$ denotes the symmetrized Alexander polynomial of K, then for the alternating knot K, $\widehat{HFK}(K, s)$ is supported in dimension $s + \frac{\sigma}{2}$ where σ denotes the signature of K and

$$\widehat{HFK}(K, s) \cong \mathbb{Z}^{|a_s|}.$$

One can define a third grading $\delta = A - M$, called the *δ-grading*. A knot K is called *Floer homologically thin*, or *thin* for short, if $\widehat{HFK}(K)$ is supported in a single δ-grading. If the homology is supported on the diagonal $\delta = -\sigma/2$, where σ denotes the knot signature, then we say the knot is σ-thin.

Fixing a field k, the Heegaard Floer homology group $\widehat{HF}(S^3)$ of the three-sphere, with coefficients in k, is isomorphic to k, supported in homological grading zero. One can define the following:

$$\tau(K) = \min\{j \in \mathbb{Z} | i_* : H_*(\mathcal{F}(K, j)) \to H_*(\widehat{CF}(S^3)) \text{ is non-trivial}\}.$$

Another sub-complex of $CFK^\infty(K)$ which is generated by $[\mathbf{x}, i, j]$, $i \leq 0$, is denoted by $CFK^-(K)$ and induces a filtration on $CF^-(S^3)$. The filtered chain homotopy type of $CFK^-(K)$ is a knot invariant. If K is thin, $CFK^-(K)$ is completely determined by $\tau(K)$ and $\Delta_K(t)$, the Alexander polynomial associated to K (see [Pet13, Theorem 4]). Note that for a σ-thin knot $\tau(K) = -\frac{\sigma}{2}$.

CHAPTER 3

TWISTED HEEGAARD FLOER THEORY

In this chapter, we review twisted Heegaard Floer theory where the coefficients are weighted by a 2-dimensional cohomology class ω. This is a special case of Heegaard Floer homology with twisted coefficients (see [OS04c], [JM08], [Zem21]).

3.1 Twisted Heegaard Floer homology groups

We begin by reviewing the Novikov ring.

Novikov ring

Let k be a commutative ring and $\Gamma \subset \mathbb{R}$ be an additive subgroup of $\mathbb{R}$. The *Novikov ring* of Γ, $\mathrm{Nov}(\Gamma)$, is a ring consisting of formal sums $\sum a_{r_i} t^{r_i}$ where $r_1 < r_2 < \cdots$, $r_i \in \Gamma$, and $r_i \to \infty$. Here t is a formal variable. Equivalently,

$$\mathrm{Nov}(\Gamma) = \{\sum_{r\in\Gamma} a_r t^r | a_r \in k, \quad \#\{r | a_r \neq 0,\ r < c\} < \infty,\ \forall c \in \mathbb{R}\}.$$

The multiplication is defined as

$$(\sum_{r\in\Gamma} a_r t^r) \cdot (\sum_{r\in\Gamma} b_r t^r) = \sum_{z\in\Gamma} (\sum_{r+s=z} a_r \cdot b_s) t^z.$$

When k is a field, the Novikov ring is also a field. $\mathrm{Nov}(\mathbb{R})$ is called the universal Novikov ring and is denoted by Λ. In this thesis, $k = \mathbb{F}_2$ unless otherwise stated.

Let Y be a closed, oriented 3-manifold and fix $[\omega] \in H^2(Y;\mathbb{R})$. This cohomology class induces an $\mathbb{F}_2[H^1(Y;\mathbb{Z})]$-module structure on Λ: for each $\eta \in H^1(Y;\mathbb{Z})$, define $t^\eta \cdot t^r := t^{r+\int_Y \eta\cup\omega}$ where $\int_Y \eta \cup \omega = \langle \eta \cup \omega, [Y]\rangle$ and $[Y]$ denotes the fundamental class of Y. Let Λ_ω denote Λ equipped with this $\mathbb{F}_2[H^1(Y;\mathbb{Z})]$-module structure. Let $a, b \in M$ where M is a Λ-module. By $a \doteq b$, we mean that there exists $z \in \mathbb{R}$ such that $a = t^z \cdot b$.

ω-twisted chain complexes

Let $(Y, \mathbf{w})$ denote a multi-pointed 3-manifold. Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w})$ denote a multi-pointed, weakly admissible Heegaard diagram for $(Y, \mathbf{w})$, $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, and J be a generic almost complex structure. Fix a 2–cocycle representative $\omega \in [\omega]$. The

ω-twisted chain complex, $\underline{CF}^{\infty}(\mathcal{H}, \mathbf{w}, \mathfrak{s}; \Lambda_{\omega})$, is a $\Lambda[U_{\mathbf{w}}, U_{\mathbf{w}}^{-1}]$-module which is freely generated by the points $\mathbf{x} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}$ with $\mathfrak{s}_{\mathbf{w}}(\mathbf{x}) = \mathfrak{s}$. Let

$$n_{\mathbf{w}}(\phi) := (n_{w_1}(\phi), \cdots, n_{w_n}(\phi)).$$

The differential is defined as follows:

$$\underline{\partial}\mathbf{x} = \sum_{\mathbf{y} \in T_{\alpha} \cap T_{\beta}} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}), \\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(\phi) U_{\mathbf{w}}^{n_{\mathbf{w}}(\phi)} t^{\omega([\phi])} \, \mathbf{y},$$

where $[\phi]$ (sometimes denoted by $\widetilde{D}(\phi)$) is a 2-chain in Y obtained from $D(\phi)$ (which is the domain associated to ϕ and is a 2-chain in Σ) by coning the α- and β-boundaries of $D(\phi)$ using gradient trajectories. Here $\omega([\phi])$ (sometimes denoted by $\int_{\widetilde{D}(\phi)} \omega$) is the evaluation of ω on $[\phi]$. Note that $\partial \widetilde{D}(\phi)$ depends only on $\mathbf{x}$ and $\mathbf{y}$. Notice also that the isomorphism class of the chain complex only depends on the cohomology class $[\omega]$ and is independent from the choice of the 2-cycle representative $\omega \in [\omega]$ (see [AP10, Section 2.1]). The homology group is denoted by $\underline{HF}^{\infty}(\mathcal{H}, \mathfrak{s}; \Lambda_{\omega})$ which is a $\Lambda[U]$-module. A similar construction works for $\underline{CF}^{-}$, $\underline{CF}^{+}$, $\underline{\widehat{CF}}$.

Definition 3.1. [JZ23, Definition 2.3] Suppose that $\mathcal{C}$ is a category and I is a set. A *transitive system* in $\mathcal{C}$, indexed by I, is a collection of objects $(X_i)_{i \in I}$, as well as a distinguished morphism $\Psi_{i \to j} : X_i \to X_j$ for each $(i, j) \in I \times I$, such that

1. $\Psi_{i \to j} = \Psi_{j \to k} \circ \Psi_{i \to j}$
2. $\Psi_{i \to i} = \mathrm{id}_{X_i}$.

Here, we work with two categories. The first category is the projectivized category of Λ-modules $\mathcal{C} = P(\Lambda - \mathrm{Mod})$ where the objects are Λ-modules and the morphism set $\mathrm{Hom}_{\mathcal{C}}(X_1, X_2)$ is the projectivization of $\mathrm{Hom}_{\Lambda}(X_1, X_2)$ under the action of units of Λ. In this category, given morphisms $f, g \in \mathrm{Hom}_{\Lambda}(X_1, X_2)$, we will use the notation $f \doteq g$ if $f = u \cdot g$ for some unit $u \in \Lambda$. The second category is the projectivized homotopy category $\mathcal{C} = P(K(\Lambda - \mathrm{Mod}))$. The objects of $\mathcal{C}$ are chain complexes over Λ. The morphism set $\mathrm{Hom}_{\mathcal{C}}(X_1, X_2)$ is the projectivization of $H_*(\mathrm{Hom}_{\Lambda}(X_1, X_2))$ under the action of units of Λ. In this category, given chain maps $\phi, \psi \in H_*(\mathrm{Hom}_{\Lambda}(X_1, X_2))$, $\phi \dot{\simeq} \psi$ means $\phi \simeq u \cdot \psi$ for some unit $u \in \Lambda$ and $\simeq$ means chain homotopy. When $\phi \dot{\simeq} \psi$, ϕ and ψ are called *projectively equivalent* (see [JZ23, Subsection 2.1]). A transitive system over one of the above categories is called a *projective transitive* *system*. The symbols $\doteq$ and $\dot{\simeq}$ are defined slightly differently in [JZ23], where they denote equality or chain homotopy equivalence up to a factor of t^z for some $z \in \mathbb{R}$.

$\underline{HF}^\circ(Y, \mathfrak{s}; \Lambda_\omega)$, $\circ \in \{\pm, \infty, \wedge\}$, forms a projective transitive system of Λ-modules indexed by the set of pairs $(\mathcal{H}, J)$, where $\mathcal{H}$ is an $\mathfrak{s}$ admissible diagram of Y, and J is a generic almost complex structure. (see [JZ23, Theorem 3.1] and [Zem21, Remark 12.1]). Note that when ω is the zero cohomology class, we have

$$\underline{CF}^-(Y, \mathfrak{s}; \Lambda_\omega) \cong CF^-(Y, \mathfrak{s}) \otimes_{\mathbb{F}_2} \Lambda.$$

Also when a three manifold is a disjoint union of two connected three manifolds Y and Y', we define

$$\underline{CF}^-(Y \amalg Y', \mathfrak{s} \oplus \mathfrak{s}'; \Lambda_{\omega \oplus \omega'}) := \underline{CF}^-(Y, \mathfrak{s}; \Lambda_\omega) \otimes_\Lambda \underline{CF}^-(Y', \mathfrak{s}'; \Lambda_{\omega'}).$$

Remark 3.2. In this remark the coefficients are taken over $\mathbb{Z}$. There is a universally twisted Heegaard Floer homology, denoted by $\underline{HF}^\circ(Y, \mathfrak{s})$ which is a $\mathbb{Z}[H^1(Y;\mathbb{Z})]$-module (see [OS04c, Section 8]). When M is a $\mathbb{Z}[H^1(Y;\mathbb{Z})]$-module, $\underline{HF}^\circ(Y, \mathfrak{s}; M)$ is defined as the homology group of the chain complex

$$\underline{CF}^\circ(Y, \mathfrak{s}; \mathbb{Z}[H^1(Y;\mathbb{Z})]) \otimes_{\mathbb{Z}[H^1(Y;\mathbb{Z})]} M.$$

If $M = \mathbb{Z}$ (where the elements of $\mathbb{Z}[H^1(Y;\mathbb{Z})]$ act trivially), $\underline{HF}^\circ(Y, \mathfrak{s}; M) = HF^\circ(Y, \mathfrak{s})$. From this point of view, $\underline{HF}^\circ$ is a lift of HF°. When $M = \Lambda_\omega$, $\underline{HF}^\circ(Y, \mathfrak{s}; M)$ is the ω-twisted Heegaard Floer homology group of Y. Note that there is a twisted version, $\underline{HF}(Y, \mathfrak{s}; \omega)$ (see [OS04a, Section 3.1]) which is different from the ω-twisted version defined above. In fact, $\underline{HF}(Y, \mathfrak{s}; \omega)$ is obtained by taking $M = \mathbb{Z}[\mathbb{R}]$ as a $\mathbb{Z}[H^1(Y;\mathbb{Z})]$-module. Therefore, $\underline{HF}(Y, \mathfrak{s}; \Lambda_\omega)$ is the Novikov completion of $\underline{HF}(Y, \mathfrak{s}; \omega)$.

3.2 ω-Twisted graph cobordisms

In the following, ω-twisted graph cobordisms are reviewed. (See [Zem21, JZ23] for more details). Let $(Y_i, \mathbf{w}_i)$, $i = 1, 2$, be two multi-pointed three manifolds.

Definition 3.3. [Zem19, Definition 3.2] A *ribbon graph cobordism* (W, Γ) from $(Y_1, \mathbf{w}_1)$ to $(Y_2, \mathbf{w}_2)$ consists of a cobordism W from Y_1 to Y_2 and an embedded finite graph Γ such that

1. there is no vertex of valence zero in Γ;

2. $\Gamma \cap Y_i = \mathbf{w}_i$, $i = 1, 2$. Furthermore, each point of $\mathbf{w}_i$ has valence 1 in Γ;

3. each edge of Γ is smoothly embedded;

4. each vertex of Γ is decorated with a cyclic ordering of all the edges that meet at it. Such a decoration is called a *formal ribbon structure*.

When W is obtained from $Y \times I$ by attaching 4-dimensional i-handles, $i = 1, 2, 3$, away from the basepoints $\mathbf{w} \subset Y$, and $\Gamma = \mathbf{w} \times I$, (W, Γ) is called a *restricted graph cobordism*. Let $(\mathcal{H}_i, J_i)$, $i = 1, 2$, denote a weakly admissible Heegaard diagram for $(Y_i, \mathbf{w}_i)$. Let $[\omega] \in H^2(W; \mathbb{R})$, and $\omega_i = \omega|_{Y_i}$, $i = 1, 2$. The cobordism (W, Γ) induces a map

$$\underline{f}^-_{W,\Gamma,\mathfrak{t};\Lambda_\omega} : \underline{CF}^-(\mathcal{H}_1, \mathbf{w}_1, \mathfrak{t}|_{Y_1}; \Lambda_{\omega_1}) \to \underline{CF}^-(\mathcal{H}_2, \mathbf{w}_2, \mathfrak{t}|_{Y_2}; \Lambda_{\omega_2})$$

on the chain complexes with ω-twisted coefficient where $\mathfrak{t} \in \mathrm{Spin}^c(W)$ (see [JZ23, Section 3] and [Zem21, Subsection 12.3]). This map is well-defined in the projectivized homotopy category which means that it is independent, up to homotopy, from the choices made in its definition and the following diagram is commutative up to homotopy and up to an overall factor t^z.

$$\begin{array}{ccc}
\underline{CF}^-(\mathcal{H}_1, \mathbf{w}, \mathfrak{t}|_{Y_1}; \Lambda_{\omega_1}) & \xrightarrow{\underline{f}^-_{W,\Gamma,\mathfrak{t};\Lambda_\omega}} & \underline{CF}^-(\mathcal{H}_2, \mathbf{w}, \mathfrak{t}|_{Y_2}; \Lambda_{\omega_2}) \\
\Big\downarrow {\scriptstyle \underline{\Psi}_{(\mathcal{H}_1,J_1)\longrightarrow(\mathcal{H}'_1,J'_1),\mathfrak{t}|_{Y_1}}} & & \Big\downarrow {\scriptstyle \underline{\Psi}_{(\mathcal{H}_2,J_2)\longrightarrow(\mathcal{H}'_2,J'_2),\mathfrak{t}|_{Y_2}}} \\
\underline{CF}^-(\mathcal{H}'_1, \mathbf{w}, \mathfrak{t}|_{Y_1}; \Lambda_{\omega_1}) & \xrightarrow{\underline{f}^-_{W,\Gamma,\mathfrak{t};\Lambda_\omega}} & \underline{CF}^-(\mathcal{H}'_2, \mathbf{w}, \mathfrak{t}|_{Y_2}; \Lambda_{\omega_2})
\end{array}$$

where $\underline{\Psi}_{(\mathcal{H}_i,J_i)\longrightarrow(\mathcal{H}'_i,J'_i),\mathfrak{t}|_{Y_i}}$ denotes a transition map between two pairs of weakly admissible Heegaard diagrams $(\mathcal{H}_i, J_i)$ and $(\mathcal{H}'_i, J'_i)$ of $(Y_i, \mathbf{w}_i, \mathfrak{t}|_{Y_i})$ (see [JZ23, Section 7]).

Remark 3.4. When Γ is a path (which is a connected graph with two vertices) in W and M is a $\mathbb{Z}[H^1(Y_1; \mathbb{Z})]$-module, there is an induced $\mathbb{Z}[H^1(Y_2; \mathbb{Z})]$-module $M(W)$ (see [OS06a, Subsection 2.7]) and the map

$$\underline{F}^\circ_{W,\mathfrak{t};M} : \underline{HF}^\circ(Y_1, \mathfrak{t}|_{Y_1}; M) \to \underline{HF}^\circ(Y_2, \mathfrak{t}|_{Y_2}; M(W))$$

is uniquely defined up to multiplication by ± 1, left-translation by an element of $H^1(Y_1; \mathbb{Z})$, and right translation by an element of $H^1(Y_2; \mathbb{Z})$ (see [OS06a, Theorem 3.8]). When $[\omega] \in H^2(W; \mathbb{R})$ and $M = \Lambda_{\omega_1}$, where $\omega_1 = \omega|_{Y_1}$, then $M(W) = \Lambda_{\omega_2}$, where $\omega_2 = \omega|_{Y_2}$.

In the following, we briefly review the constructions of the induced cobordism maps. Note that there is a handle decomposition of W obtained by i-handle attachments, $0 \leq i \leq 4$, away from the basepoints:

$$W = W_0 \cup W_1 \cup W_2 \cup W_3 \cup W_4$$

where W_i consists of i-handles, $0 \leq i \leq 4$. This decomposition is induced by a Morse function f. Let v denote a gradient-like vector field for f. We can assume that the graph Γ is disjoint from the critical points of f, descending manifolds of index 1 critical points, ascending manifolds of index 3 critical points, and from both ascending and descending manifolds of index 2 critical points (see [Zem19, Lemma 10.4]). A ribbon graph cobordism (W, Γ) induces two maps

$$F^A_{W,\Gamma,\mathfrak{t}}, F^B_{W,\Gamma,\mathfrak{t}} : CF^-(Y_1, \mathbf{w}_1, \mathfrak{s}_1) \to CF^-(Y_2, \mathbf{w}_2, \mathfrak{s}_2),$$

where $\mathfrak{t} \in \mathrm{Spin}^c(W)$, $\mathfrak{s}_i = \mathfrak{t}|_{Y_i}$, $i = 1, 2$. These maps are diffeomorphism invariants of (W, Γ), up to $\mathbb{F}_2[U]$-equivariant chain homotopy (see [Zem19, Theorem A]). When Γ consists of a collection of paths, each connecting $\mathbf{w}_1$ to $\mathbf{w}_2$, we have $F^A_{W,\Gamma,\mathfrak{t}} \simeq F^B_{W,\Gamma,\mathfrak{t}}$ (see [Zem19, Theorem B]). $F^A_{W,\Gamma,\mathfrak{t}}$ and $F^B_{W,\Gamma,\mathfrak{t}}$ are compositions of maps induced by i-handle attachments, $0 \leq i \leq 4$, and *graph action maps* $\mathfrak{A}_{\mathcal{G}}$ and $\mathfrak{B}_{\mathcal{G}}$, respectively, associated to $(Y, \mathcal{G} = (\Gamma, \mathbf{w}_0, \mathbf{w}_1))$ where $\mathcal{G}$ is an embedding of Γ in Y (see [Zem19] and the following discussion). There is a twisted cobordism map

$$\underline{F}^B_{W,\Gamma,\mathfrak{t};\Lambda_\omega} : \underline{CF}^-(Y_1, \mathbf{w}_1, \mathfrak{s}_1; \Lambda_{\omega_1}) \to \underline{CF}^-(Y_2, \mathbf{w}_2, \mathfrak{s}_2; \Lambda_{\omega_2}),$$

which is described in [Zem21]. We briefly review the construction by describing the maps involved in the definition. We describe the maps that will be used in later sections in more details. We denote the map induced on the chain complex by $\underline{f}^-_{W,\Gamma,\mathfrak{t};\Lambda_\omega}$ and the map induced on the homology by $\underline{F}^-_{W,\Gamma,\mathfrak{t};\Lambda_\omega}$.

1. 4-dimensional 0- and 4-handle attachment are equivalent to adding or removing (S^3, w_0). In this case, the associated map is induced by the isomorphism

$$\begin{aligned}\underline{CF}^-(Y \amalg S^3, \mathbf{w} \cup \{w_0\}, \mathfrak{s} \oplus \mathfrak{s}_0; \Lambda_{\omega \oplus \omega_0}) &\cong \underline{CF}^-(Y, \mathbf{w}; \Lambda_\omega) \otimes_\Lambda \underline{CF}^-(S^3, w_0, \mathfrak{s}_0; \Lambda_{\omega_0}) \\ &\cong \underline{CF}^-(Y, \mathbf{w}; \Lambda_\omega) \otimes_\Lambda \Lambda[U_0].\end{aligned}$$

   Here ω_0 denotes the zero cohomology class. In fact, if $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w})$ is a weakly admissible Heegaard diagram for Y and $\mathcal{H}_0 = (\Sigma_0, \{\alpha_0\}, \{\beta_0\}, w_0)$ is the standard Heegaard diagram for S^3 where $\alpha_0 \cap \beta_0 = \{x_0\}$, the associated map corresponding to a 0-handle attachment is

$$\underline{CF}^-(\mathcal{H}, \mathfrak{s}; \Lambda_\omega) \to \underline{CF}^-(\mathcal{H} \amalg \mathcal{H}_0, \mathfrak{s} \otimes \mathfrak{s}_0; \Lambda_\omega \otimes \Lambda_{\omega_0})$$
$$\mathbf{y} \mapsto \mathbf{y} \otimes x_0,$$

where $\mathbf{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ is a generator of $\underline{CF}^-(\mathcal{H}, \mathfrak{s}; \Lambda_\omega)$.

2. If $(W, \Gamma) : (Y, \mathbf{w}) \to (Y', \mathbf{w})$ is a restricted graph cobordism given by i-handle additions, $i = 1, 2, 3$, the induced maps are define in [OS06a, Section 4] (see [JZ23, Sections 6 and 7], for more details).

For a general ribbon graph cobordism, the graph action map $\mathfrak{A}_\mathcal{G}$ is a composition of maps associated to some *elementary graphs*. For the case of restricted graph cobordisms, we only have one type of these elementary graphs called *translations* for which the embedding $\mathcal{G} = (\Gamma, \mathbf{w}_0, \mathbf{w}_1)$ is such that $|\mathbf{w}_0| = |\mathbf{w}_1|$ and each edge of Γ connects a vertex of $\mathbf{w}_0$ to a vertex of $\mathbf{w}_1$. There are two maps involved in the definition of a twisted $\underline{\mathfrak{A}}_\mathcal{G}$ associated to a translation: *twisted free stabilization maps* $\underline{S}_w^{\pm}$ (which correspond to adding or removing a base point), and *twisted relative homology maps* $\underline{A}_\lambda$ (which correspond to a path λ between two basepoints w_1 and w_2). A twisted graph action map associated to a restricted graph cobordism, where $\mathcal{G} = (\Gamma, \mathbf{w}_0, \mathbf{w}_1)$ corresponds to a translation, is as follows:

$$\underline{\mathfrak{A}}_\mathcal{G} = (\prod_{w \in \mathbf{w}_0} \underline{S}_w^-) \circ (\prod_{e \in E(\Gamma)} \underline{A}_e) \circ (\prod_{w \in \mathbf{w}_1} \underline{S}_w^+).$$

See [Zem19] for more details. Here we recall the definition of twisted free stabilization maps (see [Zem21] for a definition of twisted relative homology maps).

Let $z \in \mathbf{w}$ and $\mathcal{H}_0 = (S^2, \{\alpha\}, \{\beta\}, \{w_0, w_1\})$ denote a Heegaard diagram with $\alpha \cap \beta = \{\theta^+, \theta^-\}$ where θ^+ is the generator with the higher grading (see Figure 3.1). Let $\mathcal{H}' = (\Sigma, \boldsymbol{\alpha} \cup \{\alpha\}, \boldsymbol{\beta} \cup \{\beta\}, \mathbf{w} \cup \{w_0\})$ denote the Heegaard diagram for $(Y, \mathbf{w} \cup \{w_0\})$ obtained by the connected sum of $\mathcal{H}$ and $\mathcal{H}_0$. To perform the connected sum, we remove the interiors of two disks around $z \in \Sigma$ and $w_1 \in S^2$ and identify the boundaries of these disks. We denote the basepoint in this region by z. For appropriate choices of almost complex structure (see [Zem19, Definition 6.2 and Proposition 6.3]), one can define the free stabilization maps

$$\begin{aligned} \underline{S}_{w_0}^+ : \underline{CF}^-(Y, \mathbf{w}, \mathfrak{s}; \Lambda_\omega) &\to \underline{CF}^-(Y, \mathbf{w} \cup \{w_0\}, \mathfrak{s}; \Lambda_\omega) \\ \underline{S}_{w_0}^+(\mathbf{x}) &= \mathbf{x} \times \theta^+, \end{aligned} \tag{3.1}$$

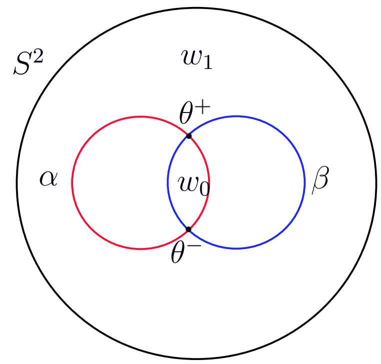

Figure 3.1. The doubly pointed Heegaard diagram $\mathcal{H}_0$ used in the definition of free stabilization maps. The Heegaard surface is a sphere.

and

$$\begin{aligned} &\underline{S}^-_{w_0} : \underline{CF}^-(Y, \mathbf{w} \cup \{w_0\}, \mathfrak{s}; \Lambda_\omega) \to \underline{CF}^-(Y, \mathbf{w}, \mathfrak{s}; \Lambda_\omega) \\ &\underline{S}^-_{w_0}(\mathbf{x} \times \theta^-) = \mathbf{x}, \quad \underline{S}^-_{w_0}(\mathbf{x} \times \theta^+) = 0. \end{aligned} \tag{3.2}$$

The twisted free stabilization maps are well-defined chain maps by the same argument as the twisted 1-handle and 3-handle maps (see [JZ23, Section 7.3] and [Zem19, Section 6] for more details).

Note that when W, Y_1, and Y_2 are non-empty and connected, $\mathbf{w}_i = \{w_i\}$, $i = 1, 2$, and Γ is a path from w_1 to w_2, $\underline{F}^-_{W,\Gamma,\mathfrak{s};\omega}$ coincides with the map defined by Ozsváth and Szabó (see [OS06a]).

3.3 Absolute grading and surgery exact sequence

For this subsection, we work over $\mathbb{Z}[H^1(Y;\mathbb{Z})]$. When $\mathfrak{s}$ is torsion, $HF^\circ(Y, \mathfrak{s})$ is generated by homogeneous elements $\mathfrak{U}$ which are equipped with a relative grading function

$$\mathrm{gr} : \mathfrak{U} \times \mathfrak{U} \to \mathbb{Z}.$$

This relative $\mathbb{Z}$-grading is lifted to an absolute $\mathbb{Q}$-grading

$$\widetilde{\mathrm{gr}} : \mathfrak{U} \to \mathbb{Q},$$

in the sense that if $\eta, \xi \in \mathfrak{U}$,

$$\mathrm{gr}(\eta, \xi) = \widetilde{\mathrm{gr}}(\eta) - \widetilde{\mathrm{gr}}(\xi).$$

See [OS06a, Theorem 7.1] for more details. A map induced by a cobordism W from Y_1 to Y_2, which is endowed with a Spin^c structure $\mathfrak{t}$ whose restriction to Y_i is torsion, shifts the grading by the formula

$$\widetilde{\mathrm{gr}}(F_{W,\mathfrak{t}}(\xi)) - \widetilde{\mathrm{gr}}(\xi) = \frac{c_1(\mathfrak{t})^2 - 2\chi(W) - 3\sigma(W)}{4}, \tag{3.3}$$

where $\sigma(W)$ denotes the signature of the intersection form

$$\begin{aligned} Q :& H_2(W) \otimes H_2(W) \to \mathbb{Z} \\ & (a, b) \mapsto a \cdot b. \end{aligned}$$

Note that every class $a \in H_2(W)$ can be represented by a smoothly embedded closed oriented surface and $a \cdot b$ denotes the signed intersection of the surfaces that represent a and b. To define $c_1(\mathfrak{t})^2$, we recall some facts. From the long exact sequence

$$H^2(W, \partial W; \mathbb{Q}) \to H^2(W; \mathbb{Q}) \to H^2(\partial W; \mathbb{Q})$$

and the assumption that $c_1(\mathfrak{t}|_{\partial W})$ is torsion, or equivalently $c_1(\mathfrak{t}|_{\partial W}) = 0 \in H^2(\partial W; \mathbb{Q})$, there is $x \in H^2(W, \partial W; \mathbb{Q})$ whose image is $c_1(\mathfrak{t})$. From here, $c_1(\mathfrak{t})^2$ is defined as $\langle c_1(\mathfrak{t}) \cup x, [W] \rangle$.

Let K be a knot in an integer homology sphere Y and $Y_i = Y_i(K)$ be obtained from Y by performing an i-surgery along K, $i = 0, 1$. There is a long exact sequence

$$\cdots \to HF^\circ(Y) \xrightarrow{f_1^\circ} HF^\circ(Y_0) \xrightarrow{f_2^\circ} HF^\circ(Y_1) \xrightarrow{f_3^\circ} HF^\circ(Y) \to \dots$$

where the maps f_i°, $i = 1, 2, 3$, are induced from cobordisms W_i which correspond to attaching 2-handles (see [OS03, Theorem 9.1]). Therefore, for each cobordism, we have $\chi(W_i) = 1$. To compute $\sigma(W_i)$, recall that when a 4-manifold W with boundary is obtained by integral surgery on a framed link $\mathcal{L}$ in Y, its intersection form is isomorphic to the linking matrix of $\mathcal{L}$ (when $Y = S^3$, this is Theorem 6.2 of [Sav12] which is true when we replace S^3 with a homology sphere). In the case at study, since W_1 is obtained by attaching a 0-framed 2-handle to $Y \times [0, 1]$, $\sigma(W_1) = 0$. Also W_2 is obtained from $Y_0 \times [0, 1]$ by attaching a single 2-handle where we perform -1-surgery on an unknot U which links K geometrically once. As a result, $W_1 \cup_{Y_0} W_2$ is a 4-manifold obtained by performing integral surgery on the two component link $K \amalg U$ in Y, with linking matrix $\left(\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Therefore, $\sigma(W_1) + \sigma(W_2) = \sigma(W_1 \cup W_2) = 0$ (see [Kir89, Theorem 5.3]). This shows that $\sigma(W_2) = 0$.

Note that f_i°, $i=1,2$, are sums of maps associated to W_i, which means that

$$f_i^\circ = \sum_{\mathfrak{t}\in \mathrm{Spin}^c(W_i)} F^\circ_{W_i,\mathfrak{t}}.$$

Also, Spin^c structures on both W_1 and W_2 are uniquely determined by their restrictions to Y_0. Let $\mathfrak{t}_0$ denote the Spin^c structure over Y_0 with trivial first Chern class. Using Formula (3.3) and the fact that $\sigma(W_i)=0$, $i=1,2$, the component of f_1° that maps into $HF^\circ(Y_0,\mathfrak{t}_0)$ shifts degree by $-1/2$ and the restriction of f_2° to $HF^\circ(Y_0,\mathfrak{t}_0)$ shifts degree by $-1/2$. Also, f_3° is non-increasing on the grading. Indeed, $f_3^\circ = F^\circ_{W_3}$ where W_3 is the reverse of a cobordism $\overline{W_3}$ obtained from $-Y$ by attaching a 2-handle along K with framing 1. Since $\sigma(\overline{W_3})=1$ and the orientation of W_3 is the opposite of the orientation of $\overline{W_3}$, we have $\sigma(W_3)=-1$. Therefore,

$$\widetilde{\mathrm{gr}}(F^\circ_{W_3,\mathfrak{t}}(\xi)) - \widetilde{\mathrm{gr}}(\xi) = \frac{c_1(\mathfrak{t})^2+1}{4}. \tag{3.4}$$

We recall the computation of $c_1(\mathfrak{t})^2$ when W is a cobordism between two homology spheres with the intersection form (p). First recall that $H^2(W)$ acts freely and transitively on $\mathrm{Spin}^c(W)$ and the first Chern class is a map $c_1 : \mathrm{Spin}^c(W)\to H^2(W)$ such that $c_1(h+\mathfrak{t}) = 2h + c_1(\mathfrak{t})$, $h\in H^2(W)$, $\mathfrak{t}\in \mathrm{Spin}^c(W)$. A cohomology class $x\in H^2(W)$ is called a *characteristic element* if $x\cup h \equiv h\cup h \mod 2$, for all $h\in H^2(W,\partial W)$. Since $c_1(\mathfrak{t})\equiv w_2(\mathfrak{t}) \mod 2$ (w_i denotes the i^{th}-Stiefel-Whitney class), the Wu formula shows that $c_1(\mathfrak{t})$ is a characteristic element:

$$w_2 = \sum_{i+j=2} Sq^i(v_j) = v_2 + v_1\cup v_1 = v_2,$$
$$w_2\cup h = v_2\cup h = Sq^2(h) = h\cup h.$$

Here $h\in H^2(W,\partial W;\mathbb{F}_2)$, v_i denotes the relative Wu class and Sq^i is the Steenrod square (see [Ker57, Section 7]). Note that $v_1 = w_1 = 0$.

Let $H^2(W)\cong \mathbb{Z}\langle y\rangle$ and $H^2(W,\partial W)\cong \mathbb{Z}\langle z\rangle$. Therefore, $c_1(\mathfrak{t})\cup z\equiv z\cup z \mod 2$. From the long exact sequence

$$0 = H^1(\partial W)\to H^2(W,\partial W)\to H^2(W)\to H^2(\partial W)\cong \mathbb{Z}_p \to H^3(W,\partial W)\cong H_1(W)=0,$$

the generator z is mapped to $py\in H^2(W)$. Let $c_1(\mathfrak{t}) = ky$, for some $k\in\mathbb{Z}$. Since $c_1(\mathfrak{t})$ is a characteristic element, we have

$$\langle c_1(\mathfrak{t})\cup z, [W]\rangle \equiv \langle z\cup z,[W]\rangle \mod 2,$$
$$\langle z\cup z,[W]\rangle = p,$$
$$\langle y\cup z,[W]\rangle = 1/p\langle z\cup z,[W]\rangle.$$

This shows that $k \equiv p \mod 2$. Let $c_1(\mathfrak{t}_j) = (2j+p)y$. We want to compute $c_1(\mathfrak{t})^2$.

$$c_1(\mathfrak{t}_j)^2 = \frac{(2j+p)^2}{p}\langle y \cup z, [W]\rangle = \frac{(2j+p)^2}{p^2}\langle z \cup z, [W]\rangle = \frac{(2j+p)^2}{p}. \tag{3.5}$$

For the cobordism W_3, we have $p = -1$. If use the formulas in (5.4) and (5.3), we see that f_3° is non-increasing on the grading.

To state the twisted surgery exact sequence, note that we have $\mathbb{Z}[H^1(Y_0)] \cong \mathbb{Z}[t, t^{-1}]$ where t is a generator of $H^1(Y_0;\mathbb{Z})$. If M denotes a $\mathbb{Z}[U]$-module, there is an induced $\mathbb{Z}[U, t, t^{-1}]$-module structure on $M[t, t^{-1}] = M \otimes \mathbb{Z}[t, t^{-1}]$. There is a $\mathbb{Z}[U, t, t^{-1}]$-equivariant long exact sequence

$$\cdots \to HF^+(Y)[t, t^{-1}] \xrightarrow{\underline{f}_1^+} \underline{HF}^+(Y_0) \xrightarrow{\underline{f}_2^+} HF^+(Y_1)[t, t^{-1}] \xrightarrow{\underline{f}_3^+} \ldots$$

(see [OS04c, Theorem 9.21]). There is a similar exact sequence for the hat version.

Note that for a closed oriented, Spin^c 3-manifold $(N, \mathfrak{s})$, there is an $\mathfrak{s}$-grading on $\mathbb{Z}[H^1(N;\mathbb{Z})]$

$$\mathrm{gr}_{\mathfrak{s}}(x) = -\langle c_1(\mathfrak{s}) \cup x, [N]\rangle.$$

where $x \in H^1(N;\mathbb{Z})$, which makes $\mathbb{Z}[H^1(N;\mathbb{Z})]$ into a graded ring (see [JM08, Definition 3.1]). This equips the fully twisted Heegaard Floer homology groups with a relative $\mathbb{Z}$-grading.

Definition 3.5. [JM08, Definition 3.2] Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a pointed Heegaard triple describing the 3-manifold N. Fix a Spin^c structure $\mathfrak{s}$ for N and an additive assignment $\{A_{\mathbf{x},\mathbf{y}}\}$ for the diagram. The relative $\mathbb{Z}$ grading between generators $[\mathbf{x}, i]$ and $[\mathbf{y}, j]$ for $\underline{CF}^\circ(N, \mathfrak{s}; \mathbb{Z}[H^1(N;\mathbb{Z})])$ is defined by

$$\underline{\mathrm{gr}}([\mathbf{x}, i], [\mathbf{y}, j]) = \mu(\phi) + 2(i-j) - 2n_z(\phi) - \langle c_1(\mathfrak{s}) \cup A_{\mathbf{x},\mathbf{y}}(\phi), [N]\rangle,$$

where ϕ is any element of $\pi_2(\mathbf{x}, \mathbf{y})$. More generally, if $r_1, r_2 \in \mathbb{Z}[H^1(N;\mathbb{Z})]$ are homogeneous elements, then we set

$$\underline{\mathrm{gr}}(r_1 \cdot [\mathbf{x}, i], r_2 \cdot [\mathbf{y}, j]) = \underline{\mathrm{gr}}([\mathbf{x}, i], [\mathbf{y}, j]) + \mathrm{gr}_{\mathfrak{s}}(r_1) - \mathrm{gr}_{\mathfrak{s}}(r_2).$$

Here, since Y and Y_1 are homology spheres, $\underline{\mathrm{gr}} = \mathrm{gr}$. Also, for the Spin^c manifold $(Y_0, \mathfrak{t}_0)$, $\underline{\mathrm{gr}} = \mathrm{gr}$. Therefore, the component of $\underline{f}_1^+$ that maps into $\underline{HF}^+(Y_0, \mathfrak{t}_0)$ shifts degree by $-1/2$ and the restriction of $\underline{f}_2^+$ to $\underline{HF}^+(Y_0, \mathfrak{t}_0)$ shifts degree by $-1/2$. Also, $\underline{f}_3^+$ is non-increasing on the grading.

Completion of Heegaard Floer homology groups. We work with the completions of Heegaard Floer homology groups, of which we recall the definition here (see [AM69, Chapter 10] and [MO24, Section 2] for more details). Let G be a topological commutative group. The completion $\boldsymbol{G}$ of G may be defined in the usual sense using Cauchy sequences (see [AM69, Chapter 10] for the definition of Cauchy sequence). Let us consider the case that G has a topology where $0 \in G$ has a fundamental system of neighborhoods consisting of subgroups

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n \supseteq \ldots$$

In this topology, U is a neighborhood of 0 if and only if U contains G_n for some n. Note that the topology of G is uniquely determined by the neighborhoods of 0 in G. From here, one can see that $\boldsymbol{G} \cong \varprojlim G/G_n$. From this viewpoint and further properties of $\boldsymbol{G}$, one can check that taking completion is an exact functor (see [AM69, Proposition 10.2, Corollary 10.3]).

Let A be a ring and I be an ideal of A. Denote the completion of A with respect to this ideal by $\boldsymbol{A}$. If M is an A-module, the completion $\boldsymbol{M}$ of M is an $\boldsymbol{A}$-module. As an example, when $A = \mathbb{F}_2[U]$ (resp. $A = \mathbb{F}_2[U, U^{-1}]$) and $I = (U)$, then $\boldsymbol{A} = \mathbb{F}_2[\![U]\!]$, the ring of formal power series (resp. $\boldsymbol{A} = \mathbb{F}_2[\![U, U^{-1}]$ ring of semi-infinite Laurent polynomials). $\boldsymbol{CF}^-$ (resp. $\boldsymbol{CF}^\infty$), the completion of the chain complex CF^- (resp. CF^∞), is a chain complex with the same generators as of CF^- (resp. CF^∞) with coefficients in $F_2[\![U]\!]$ (resp. $\mathbb{F}_2[\![U, U^{-1}]$). Since completion is an exact functor, $\boldsymbol{HF}^-$, the homology group of $\boldsymbol{CF}^-$, is the completion of HF^-. Similarly, $\boldsymbol{HF}^\infty$ is the completion of HF^∞. One can define the completions for $\widehat{HF}$ and HF^+ similarly but since the action of U is nilpotent on each generator, the completion matches with the original group. Therefore, we have the following exact sequence for a closed, oriented three manifold Y:

$$\cdots \longrightarrow \boldsymbol{HF}^-(Y) \longrightarrow \boldsymbol{HF}^\infty(Y) \longrightarrow HF^+(Y) \longrightarrow \cdots . \tag{3.6}$$

Let

$$\boldsymbol{HF}_{\text{red}}(Y) := \text{Coker}(\boldsymbol{HF}^\infty(Y) \to HF^+(Y)) \cong \text{Ker}(\boldsymbol{HF}^-(Y) \longrightarrow \boldsymbol{HF}^\infty(Y)).$$

Note that for the definition of HF^- and HF^∞, one needs to restrict to strongly admissible Heegaard diagrams. However, Lemma 4.13 of [OS04d] shows that the differentials for $\boldsymbol{CF}^-$ and $\boldsymbol{CF}^\infty$ are finite even for weakly admissible Heegaard diagrams. Also, note that for any torsion Spin^c structure $\mathfrak{s}$, the groups $HF^-(Y,\mathfrak{s})$ and $HF^\infty(Y,\mathfrak{s})$ are determined by $\boldsymbol{HF}^-$ and $\boldsymbol{HF}^\infty$, respectively, where $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ (see [MO24, Section 2]). But for a non-torsion Spin^c structure $\mathfrak{s}$, as $(1-U^N)HF^\infty(Y,\mathfrak{s}) = 0$ for some $N \geq 1$, and $1-U^N$ is invertible in $\mathbb{Z}[\![U]\!]$, we have $\boldsymbol{HF}^\infty(Y,\mathfrak{s}) = 0$. Therefore, from the long exact sequence in (3.6), we have $\boldsymbol{HF}^-(Y,\mathfrak{s}) \cong HF^+(Y,\mathfrak{s})$, where $\mathfrak{s}$ is non-torsion (see [MO24, Section 2]).

When we consider the completion of Heegaard Floer homology groups with respect to (U) and with coefficients in Λ_ω, we also have the long exact sequence

$$\cdots \longrightarrow \underline{\boldsymbol{HF}}^-(Y,\mathfrak{s};\Lambda_\omega) \longrightarrow \underline{\boldsymbol{HF}}^\infty(Y,\mathfrak{s};\Lambda_\omega) \longrightarrow \underline{HF}^+(Y,\mathfrak{s};\Lambda_\omega) \longrightarrow \cdots .$$

By Corollary 8.7 of [JM08], when $(Y,\mathfrak{s})$ is a Spin^c manifold, $c_1(\mathfrak{s})$ is torsion, and $[\omega] \neq 0$, $\underline{\boldsymbol{HF}}^\infty(Y,\mathfrak{s};\Lambda_\omega) = 0$. This implies that

$$\underline{\boldsymbol{HF}}^-(Y,\mathfrak{s};\Lambda_\omega) \cong \underline{HF}^+(Y,\mathfrak{s};\Lambda_\omega)$$

It is worth mentioning that for torsion Spin^c structures,

$$\underline{HF}^-(Y,\mathfrak{s};\Lambda_\omega) \cong \underline{HF}^+(Y,\mathfrak{s};\Lambda_\omega).$$

This follows from the fact that $\underline{HF}^\infty(Y,\mathfrak{s};\Lambda_\omega) = 0$ when $c_1(\mathfrak{s})$ is torsion and $[\omega] \neq 0$ (see [JM08, Corollary 8.5]). Therefore,

$$\underline{\boldsymbol{HF}}_{\mathrm{red}}(Y,\mathfrak{s};\Lambda_\omega) = \underline{HF}^+(Y,\mathfrak{s};\Lambda_\omega) \cong \underline{\boldsymbol{HF}}^-(Y,\mathfrak{s};\Lambda_\omega),$$

and

$$\underline{HF}_{\mathrm{red}}(Y,\mathfrak{s};\Lambda_\omega) = \underline{HF}^+(Y,\mathfrak{s};\Lambda_\omega) \cong \underline{HF}^-(Y,\mathfrak{s};\Lambda_\omega).$$

Sums and compositions of twisted cobordism maps. Let (W,Γ) be a restricted graph cobordism from $(Y_1,\mathbf{w}_1)$ to $(Y_2,\mathbf{w}_2)$. Let $[\omega] \in H^2(W;\mathbb{R})$ and $\omega_i = \omega|_{Y_i}$, $i = 1,2$. Assume further that $\mathfrak{T}$ is a subset of $\{\mathfrak{t} \in \mathrm{Spin}^c(W) \mid \mathfrak{t}|_{Y_1} = \mathfrak{s}_1, \mathfrak{t}|_{Y_2} = \mathfrak{s}_2\}$. The map

$$\underline{\boldsymbol{f}}^-_{W,\Gamma,\mathfrak{T};\Lambda_\omega} : \underline{\boldsymbol{CF}}^-(\mathcal{H}_1,\mathbf{w}_1,\mathfrak{s}_1;\Lambda_{\omega_1}) \to \underline{\boldsymbol{CF}}^-(\mathcal{H}_2,\mathbf{w}_2,\mathfrak{s}_2;\Lambda_{\omega_2})$$

is defined as $\sum_{\mathfrak{t}\in\mathfrak{T}} \underline{\boldsymbol{f}}^-_{W,\Gamma,\mathfrak{t};\Lambda_\omega}$, where $\mathcal{H}_i$, $i = 1,2$, is a Heegaard diagram for Y_i. Since the ground ring is $\Lambda[\![U]\!]$, there is no need for $\mathfrak{T}$ to be finite. Also note that by [OS06a, Theorem 3.3], this sum is well defined over the power series ring.

Lemma 3.6. *[Zem21, Lemma 12.4] The map* $\underline{\boldsymbol{f}}^-_{W,\Gamma,\mathfrak{T};\Lambda_\omega}$ *is well defined up to an overall factor* t^z, $z \in \mathbb{R}$.

This lemma states that when we change the auxiliary data used to construct the cobordism map, each $\underline{\boldsymbol{f}}^-_{W,\Gamma,\mathfrak{T};\Lambda_\omega}$ changes by the same factor of t^z, where z is independent of $\mathfrak{T}$. In general, we need to require that all the Spin^c structures $\mathfrak{t} \in \mathfrak{T}$ restrict to the same Spin^c structures on Y_1 and Y_2. Indeed, when $\omega \neq 0$, the twisted Heegaard Floer groups $\underline{HF}^\circ(Y, \mathfrak{s}; \Lambda_\omega)$ are well-defined up to an overall factor t^z. Therefore, these groups are natural only when we restrict to one Spin^c structure at a time. In general, one needs to check if the maps

$$\underline{f}^\circ_{W,\Gamma;\Lambda_\omega} = \sum_{\mathfrak{s}\in \mathrm{Spin}^c(W)} \underline{f}^\circ_{W,\Gamma,\mathfrak{s};\Lambda_\omega}$$

are well defined.

We will need the following version of composition law.

Lemma 3.7. *[Zem21, Lemma 12.5] Suppose that*

$$(W_1, \Gamma_1) : (Y, \mathbf{w}) \to (Y', \mathbf{w}') \quad \textit{and} \quad (W_2, \Gamma_2) : (Y', \mathbf{w}') \to (Y'', \mathbf{w}'')$$

are graph cobordisms. Write $(W, \Gamma) = (W_2, \Gamma_2) \circ (W_1, \Gamma_1)$. *If* $\mathfrak{S}_1 \subset \mathrm{Spin}^c(W_1)$ *and* $\mathfrak{S}_2 \subset \mathrm{Spin}^c(W_2)$ *are sets of* Spin^c *structures which all have the same restrictions to* Y, Y', *and* Y'', *write* $\mathfrak{S}(W, \mathfrak{S}_1, \mathfrak{S}_2)$ *for the set of* Spin^c *structures on* W *which restrict to an element of* $\mathfrak{S}_1$ *and an element of* $\mathfrak{S}_2$. *Then*

$$\underline{\boldsymbol{F}}^B_{W,\Gamma,\mathfrak{S};\Lambda_\omega} \dot{\simeq} \underline{\boldsymbol{F}}^B_{W_2,\Gamma_2,\mathfrak{S}_2;\Lambda_{\omega_2}} \circ \underline{\boldsymbol{F}}^B_{W_1,\Gamma_1,\mathfrak{S}_1;\Lambda_{\omega_1}}.$$

where $\omega_i = \omega|_{W_i}$. *If* ω *vanishes on one of the 3-manifolds* Y, Y', *or* Y'', *we may relax the requirement that all elements of* $\mathfrak{S}_1$ *and* $\mathfrak{S}_2$ *have the same restriction to that 3-manifold.*

Remark 3.8. In the following section, we work with 3-manifolds Y such that $\underline{HF}^\circ(Y, \mathfrak{s}; \Lambda_\omega) \cong 0$ where $0 \neq [\omega] \in H^2(Y; \mathbb{R})$ and $c_1(\mathfrak{s}) \neq 0$. Therefore, Lemma 3.6 is still valid when we replace $\underline{\boldsymbol{f}}^-_{W,\Gamma,\mathfrak{T};\Lambda_\omega}$ with $\underline{f}^-_{W,\Gamma,\mathfrak{T};\Lambda_\omega}$ (for such 3-manifolds). Also, Lemma 3.7 is true when we replace $\underline{\boldsymbol{F}}^B_{W,\Gamma,\mathfrak{S};\Lambda_\omega}$ with $\underline{F}^B_{W,\Gamma,\mathfrak{S};\Lambda_\omega}$.

CHAPTER 4

PROOF OF THE MAIN THEOREM

The proof is a slight modification of the proof of Theorem 1.1. We use the following result from [AP10]. In this section, $\mathcal{H}$, $\mathcal{H}_0$, etc., will be used to describe Heegaard diagrams that are possibly different from the same notations used in the previous chapter.

Theorem 4.1 ([AP10])**.** *Suppose Y is a closed, oriented 3-manifold that fibers over the circle with torus fiber $\mathcal{F}$, and $[\omega] \in H^2(Y;\mathbb{R})$ is a cohomology class such that $\omega(\mathcal{F}) \neq 0$. Then we have an isomorphism of Λ-modules*

$$\underline{HF}^+(Y;\Lambda_\omega) \cong \Lambda.$$

Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}_0 \cup \{z\})$ be a multi-pointed Heegaard diagram for a multi-pointed 3-manifold $(Y, \mathbf{w}_0 \cup \{z\})$ and $\mathcal{H}_0 = (S^2, \alpha, \beta, \{w_0, w_1\})$ denote the doubly pointed Heegaard diagram in Figure 3.1 with $\alpha \cap \beta = \{\theta^{\pm}\}$ where $\mathrm{gr}(\theta^+, \theta^-) = 1$. Let

$$\mathcal{H}_1 = (\Sigma \# S^2, \boldsymbol{\alpha} \cup \{\alpha\}, \boldsymbol{\beta} \cup \beta, \mathbf{w}_0 \cup \{w_0, w_1\})$$

be the connected sum of $\mathcal{H}$ and $\mathcal{H}_0$ where the connected sum is formed at the points z and w_1. Let $[\omega] \in H^2(Y;\mathbb{R})$. The following theorem is a twisted version of [OS08, Proposition 6.5].

Theorem 4.2. *Let $\mathcal{H}$, $\mathcal{H}_0$, and $\mathcal{H}_1$ be as above. Then $\underline{CF}^-(\mathcal{H}_1, \mathfrak{s}; \Lambda_\omega)$ is identified with the mapping cone of*

$$\underline{CF}^-(\mathcal{H}, \mathfrak{s}; \Lambda_\omega) \otimes_\Lambda \Lambda[U_{w_0}]\langle \theta^- \rangle \xrightarrow{U_{w_0} - U_z} \underline{CF}^-(\mathcal{H}, \mathfrak{s}; \Lambda_\omega) \otimes_\Lambda \Lambda[U_{w_0}]\langle \theta^+ \rangle.$$

Before stating the proof, we recall the definition of a mapping cone. Let (A, ∂_A) and (B, ∂_B) denote two $\mathbb{F}_2$-graded chain complexes and $f : A \to B$ be a chain map. The mapping cone $M(f)$ is the chain complex with the underlying group $A \oplus B$ endowed with a differential $(a, b) \mapsto (\partial_A(a), f(a) + \partial_B(b))$. There is a short exact sequence of chain maps

$$0 \to B \xrightarrow{\iota} Cone(f) \xrightarrow{\pi} A \to 0, \tag{4.1}$$

such that the connecting homomorphism in the associated long exact sequence is the map on homology induced by f.

Proof. We modify the proof of [OS08, Proposition 6.5] for ω-twisted coefficients. Choose an almost complex structure such that the neck length of the connected sum is very big. We have $\underline{CF}^-(\mathcal{H}_1, \mathfrak{s}; \Lambda_\omega) \cong C_{\theta^+} \oplus C_{\theta^-}$, where $C_{\theta^\pm}$ denote all the generators $\mathbf{x} \times \theta^\pm$, $\mathbf{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$.

Let $\phi \in \pi_2(\mathbf{x} \times \theta^+, \mathbf{y} \times \theta^+)$ or $\phi \in \pi_2(\mathbf{x} \times \theta^-, \mathbf{y} \times \theta^-)$. We have $\phi = \phi_1 \# \phi_2$ where $\phi_1 \in \pi_2(\mathbf{x}, \mathbf{y})$ is a Whitney disk in $\mathcal{H}$ and $\phi_2 \in \pi_2(\theta^+, \theta^+)$ or $\phi_2 \in \pi_2(\theta^-, \theta^-)$ is a Whitney disk in $\mathcal{H}_0$. As discussed in [OS08], the only case where ϕ has a holomorphic representative is when

$$\mu(\phi_2) = 2n_{w_1}(\phi_2), \quad n_{w_0}(\phi_2) = 0.$$

Using [OS08, Theorem 5.1], the $\mathbf{y} \times \theta^\pm$ component of $\underline{\partial}^-(\mathbf{x} \times \theta^\pm)$ is

$$\sum_{\{\phi_1 \in \pi_2(\mathbf{x},\mathbf{y}) | \mu(\phi_1)=1\}} \sum_{u_1 \in \widehat{\mathcal{M}}(\phi_1)} \sum_{\phi_2 \in \pi_2(\theta^\pm, \theta^\pm)} \#\{u_2 \in \mathcal{M}(\phi_2) | \rho_1(u_1) = \rho_2(u_2)\} t^{\omega([\phi_1]) + \omega([\phi_2])}.$$

Here $\rho_i : \mathcal{M}(\phi_i) \to Sym^k(\mathbb{D})$ with $\rho_1(u) = u^{-1}(\{z\} \times Sym^{d_1 - 1}(\Sigma))$, $\rho_2(u) = u^{-1}(\{w_1\})$, d_1 is the number of curves in $\boldsymbol{\beta}$, $k = n_z(\phi_1) = n_{w_1}(\phi_2)$, and $\mathbb{D}$ is the unit disk in $\mathbb{C}$. Note that $\partial D(\phi_2)$ is a union of α- and β-curves in $\mathcal{H}_0$ and $[\phi_2]$, which is obtained by coning the α and β boundaries of $D(\phi_2)$, is a sum of copies of an S^2 which bounds a 3-ball in Y. Therefore, $\omega([\phi_2]) = 0$. This shows that the $\mathbf{y} \times \theta^\pm$ component of $\underline{\partial}^-(\mathbf{x} \times \theta^\pm)$ is identified with the $\mathbf{y}$ component of $\underline{\partial}^-(\mathbf{x})$.

The argument in the proof of Proposition 6.5 of [OS08] shows that if $\mathbf{y} \times \theta^-$ appears in $\underline{\partial}^-(\mathbf{x} \times \theta^+)$, domains of the Whitney disks that connect $\mathbf{y} \times \theta^-$ to $\mathbf{x} \times \theta^+$ consist of the two bigons D_1 and D_2 connecting θ^- to θ^+ away from the basepoints in $\mathcal{H}_0$. The same argument as the previous case shows that $\omega([D_1 - D_2]) = 0$. Therefore the corresponding disks in the differential cancel and the C_{θ^-} component of $\underline{\partial}^-(\mathbf{x} \times \theta^+)$ is trivial.

Finally, as shown in the proof of Proposition 6.5 [OS08], the C_{θ^+} component of $\underline{\partial}(\mathbf{x} \times \theta^-)$ is given by

$$U_z \, t^{\omega([\phi_1])} \, \mathbf{x} \times \theta^+ + U_{w_0} \, t^{\omega([\phi_2])} \, \mathbf{x} \times \theta^+,$$

where ϕ_2 is a disk such that its domain is a bigon connecting θ^- to θ^+ which contains w_0 and ϕ_1 is a disk such that its domain is $\Sigma \# D_2$ where D_2 is a bigon in $\mathcal{H}_0$ connecting θ^- to θ^+ and containing w_1. $\partial D(\phi_2 - \phi_1)$ is a union of α- and β-curves in $\mathcal{H}_1$ and $[\phi_2 - \phi_1]$ is homotopic to the Heegaard surface $\Sigma \# S^2$. Therefore $\omega([\phi_1]) = \omega([\phi_2])$ (as $\Sigma \# S^2$ is the boundary of a handlebody in Y). This completes the proof since the mapping cone of $U_{w_0} - U_z$ is isomorphic to the mapping cone of $t^{\omega([\phi_1])}(U_{w_0} - U_z)$. □

Corollary 4.3. *If $U_{w_0} \neq U_z$ in Theorem 4.2, $\underline{S}^+_{w_0}$ induces isomorphism on $\underline{HF}^\circ$, $\circ \in \{+,-,\infty\}$, and $\underline{S}^-_{w_0}$ induces zero maps.*

Proof. When $\circ \in \{-,\infty\}$, from Theorem 4.2 and the long exact sequence induced from the short exact sequence in (4.1), we have

$$\underline{HF}^-(\mathcal{H}_1,\mathfrak{s};\Lambda_\omega) \cong \frac{\underline{HF}^-(\mathcal{H},\mathfrak{s};\Lambda_\omega)[U_{w_0}]\langle\theta^+\rangle}{(U_{w_0}-U_z)}.$$

Then the result is obtained directly from the definition of the free stabilization maps (see Section 3.2). The case of HF^+ is implied from the fact that the free stabilization maps are compatible with the long exact sequence in (3.6). □

Let $\mathcal{F} = S^1 \times S^1$ and $Y = \mathcal{F} \times S^1$. Assume that W is a cobordism which is obtained from $\mathcal{F} \times D^2$ by removing a 4-ball. More precisely, let $D_0 \subset D^2$ be a small disk and $p \in \mathcal{F}$. Remove a small neighborhood of $\{p\} \times D_0$ from $\mathcal{F} \times D^2$ to obtain the cobordism W from S^3 to Y such that $(\{p\} \times (\overline{D^2 \setminus D_0})) \cap S^3$ is $\{p\} \times \partial D_0$. Suppose $w_0 \in S^3$ and $w \in Y$ are basepoints. Let $\Gamma \subset W$ be any path connecting w_0 to w. Let $\eta = \{p\} \times (S^1 = \partial D^2) \subset Y$ and $[\omega] = d \cdot \mathrm{PD}([\eta]) \in H^2(Y;\mathbb{R})$ where $d \neq 0$. Suppose $[\bar{\omega}] \in H^2(W;\mathbb{R})$ be $d \cdot \mathrm{PD}(\{p\} \times (\overline{D^2 \setminus D_0}))$.

Lemma 4.4. *With the above notation, the map*

$$\underline{F}^-_{W,\Gamma;\Lambda_{\bar{\omega}}} : \Lambda[U_0] \cong \underline{HF}^-(S^3, w_0, \mathfrak{s}_0; \Lambda) \to \underline{HF}^-(Y, w; \Lambda_\omega) \cong \Lambda$$

is well-defined up to an overall factor and is a nonzero map.

Proof. The proof is a straightforward modification of the proof of [LY23, Lemma 3.33]. Let P denote a pair of pants as shown in Figure 4.1. Let Y_i, $i = 1,2,3$, be three copies of Y, $w_i \in Y_i$, $i = 1,2,3$, and $w_3' \in Y_3$ be basepoints. Let $W_1 = Y_1 \times I$, $W_2 = \mathcal{F} \times P$, $\Gamma_1 = \{w_1\} \times I \subset W_1$, and $\Gamma_2 \subset W_2$ denote two paths connecting w_1 to w_3 and w_2 to w_3' (note that we consider W_2 as a cobordism from $Y_1 \amalg Y_2$ to Y_3). Suppose η_i and ω_i, $i = 1,2,3$, are copies of η and ω in Y_i, $[\bar{\omega}_2] = d \cdot \mathrm{PD}([\{p\} \times P]) \in H^2(W_2;\mathbb{R})$, and $[\bar{\omega}_1] = d \cdot \mathrm{PD}([\mu_1 \times I]) \in H^2(W_1;\mathbb{R})$.

Note that the map $\underline{F}^-_{W_2,\Gamma_2;\Lambda_{\bar{\omega}_2}}$ is a sum of maps. By Theorem 4.1, $\underline{HF}^-$ of the 3-manifolds appearing in ∂W_2 are supported in the torsion Spin^c-structure. Therefore, by Remark 3.8, we can apply Lemma 3.6 for the map $\underline{F}^-_{W_2,\Gamma_2;\Lambda_{\bar{\omega}_2}}$, to say that it is well-defined up to an overall factor. Similarly, $\underline{F}^-_{W_1 \sqcup W, \Gamma_1 \cup \Gamma; \Lambda_{\bar{\omega}_1 \otimes \bar{\omega}}}$ and $\underline{F}^-_{W,\Gamma;\Lambda_{\bar{\omega}}}$ are well-defined maps. Let (W', Γ') be the composition of the cobordism W_2 with $W_1 \sqcup W$

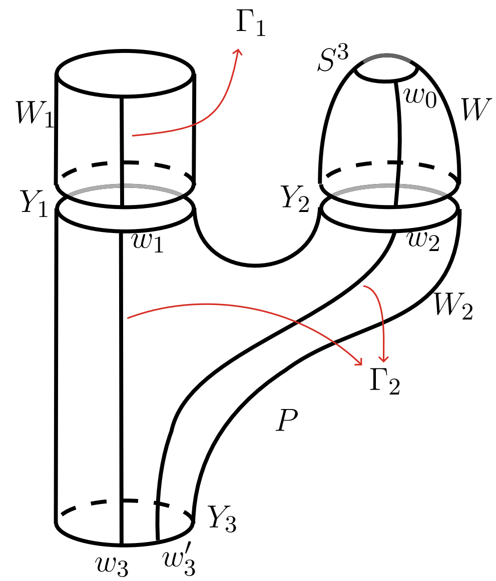

Figure 4.1. This is Figure 7 of [LY23].

then a 0-handle attachment. W' is the product cobordism $Y \times I'$, where I' is a closed connected interval, and Γ' is a two component path. This gives the free stabilization $\underline{S}^+_{w_0}$ (see [LY23, Remark 3.19]) which induces an isomorphism by Corollary 4.3. More precisely, let $[\omega'] \in H^2(W';\mathbb{R})$ be $PD(\eta \times I')$. Then $\omega'|_{W_2} = \overline{\omega}_2$, $\omega'|_{W_1 \amalg W} = \overline{\omega}_1 \oplus \overline{\omega}$, and $\underline{F}^-_{W',\Gamma';\Lambda_{\omega'}} = \underline{S}^+_{w_0}$ (note that Lemma 3.7 explains how the map induced by W' is related to the composition of maps induced by W_2 and $W_1 \amalg W$ for the twisted case). We have

$$\underline{F}^-_{W_1 \sqcup W, \Gamma_1 \cup \Gamma; \Lambda_{\bar{\omega}_1 \oplus \bar{\omega}}} = \underline{F}^-_{W_1,\Gamma_1;\Lambda_{\bar{\omega}_1}} \otimes_\Lambda \underline{F}^-_{W,\Gamma;\Lambda_{\bar{\omega}}} = \mathrm{id}_{Y_1} \otimes_\Lambda \underline{F}^-_{W,\Gamma;\Lambda_{\bar{\omega}}}.$$

Since $\underline{F}^-_{W',\Gamma';\Lambda_{\omega'}} = \underline{S}^+_{w_0}$, $\underline{F}^-_{W,\Gamma;\Lambda_{\bar{\omega}}}$ is nonzero. □

Let $\mathcal{H}_2 = (\Sigma \# S^2, \boldsymbol{\alpha}' \cup \{\alpha'\}, \boldsymbol{\beta}' \cup \{\beta'\}, \mathbf{w}_0 \cup \{w_0, w_1\})$ be the connected sum of $\mathcal{H}$ and $\mathcal{H}_0$ where the connected sum is formed at the points z and w_0 and the basepoint z is replaced with w_0. (Here, $\boldsymbol{\alpha}'$, α', $\boldsymbol{\beta}'$, and β' are small Hamiltonian isotopies of $\boldsymbol{\alpha}$, α, $\boldsymbol{\beta}$, and β.) See Figure 4.2. Recall that $\mathcal{H}_1$ is the connected sum of $\mathcal{H}$ and $\mathcal{H}_0$ where the connected sum is formed at the points z and w_1. Also, $\mathcal{H}_1$ and $\mathcal{H}_2$ are related by a sequence of Heegaard moves.

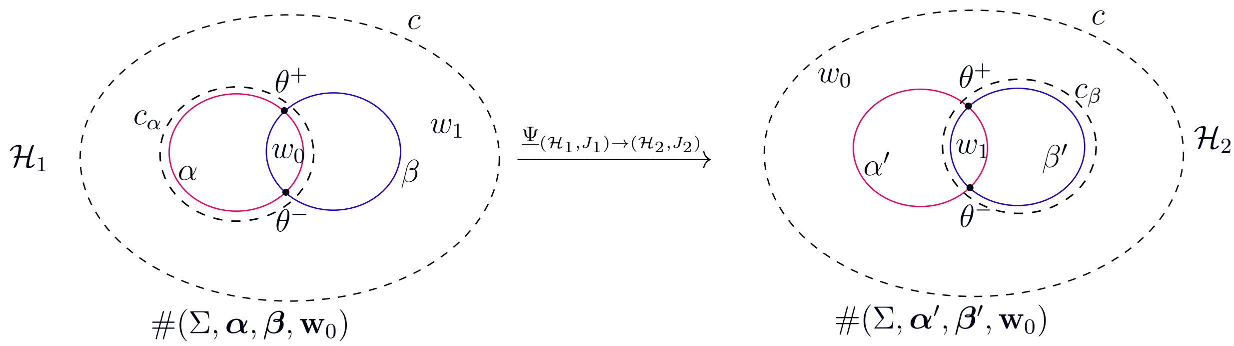

Figure 4.2. This is Figure 14.1 of [Zem19].

Proposition 4.5. *With the above notation, there are choices of almost complex structures J_1 and J_2 on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, such that*

$$\underline{\Psi}_{(\mathcal{H}_1,J_1)\to(\mathcal{H}_2,J_2),\mathfrak{s}} \doteq \begin{bmatrix} * & 0 \\ (\underline{\Phi}^{\boldsymbol{\beta}'}_{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'})^{U_z\to U_{w_0}}_{U_{w_1}} \circ (\sum U^i_{w_0} U^j_{w_1} (\underline{\partial}_{i+j+1})_{U_{w_0},w_1}) \circ (\underline{\Phi}^{\boldsymbol{\beta}\to\boldsymbol{\beta}'}_{\boldsymbol{\alpha}})^{U_z\to U_{w_1}}_{U_{w_0}} & * \end{bmatrix}.$$

Here, $\underline{\Phi}^{\boldsymbol{\beta}\to\boldsymbol{\beta}'}_{\boldsymbol{\alpha}}$ denotes the transition map from $\underline{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}_0 \cup \{z\}, \mathfrak{s}; \Lambda_\omega)$ to $\underline{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}', \mathbf{w}_0 \cup \{z\}, \mathfrak{s}; \Lambda_\omega)$. $\underline{\Phi}^{\boldsymbol{\beta}'}_{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'}$ is defined similarly. If $G : C_1 \to C_2$ is a map of $\Lambda[U_z]$-modules, write $G^{U_z\to U_w}$ for the induced map

$$G^{U_z\to U_w} := G \otimes_\Lambda id_{\Lambda[U_z,U_w]/(U_w-U_z)}.$$

If R is a Λ-algebra and $G : C_1 \to C_2$ is a map of R-modules, write G_{U_w} for the map of $R \otimes_\Lambda \Lambda[U_w]$-modules

$$G_{U_w} := G \otimes id_{\Lambda[U_w]} : C_1 \otimes_\Lambda \Lambda[U_w] \to C_2 \otimes_\Lambda \Lambda[U_w]. \tag{4.2}$$

$\underline{\Psi}_{(\mathcal{H}_1,J_1)\to(\mathcal{H}_2,J_2),\mathfrak{s}}$ is the transition map from the chain complex $\underline{CF}^-(\mathcal{H}_1, \mathfrak{s}; \Lambda_\omega)$ to $\underline{CF}^-(\mathcal{H}_2, \mathfrak{s}; \Lambda_\omega)$. Note that the generators in $\mathcal{H}_i$, $i = 1, 2$, can be written as $C_{\theta^+} \oplus C_{\theta^-}$ and in the matrix presentation for $\underline{\Psi}_{(\mathcal{H}_1,J_1)\to(\mathcal{H}_2,J_2),\mathfrak{s}}$, the first row and column of this matrix correspond to θ^+, and the second row and column to θ^-. We extend this matrix presentation to other maps. If V denotes the 2-dimensional vector space $\langle\theta^+, \theta^-\rangle$ and G is a homomorphism

$$G : C_1 \otimes_\Lambda V \to C_2 \otimes_\Lambda V,$$

then we will write G as a 2×2 block matrix using the decomposition induced by θ^+ and θ^-. Also, we can write the differential on $\underline{CF}^-(\mathcal{H}_i, \mathfrak{s}; \Lambda_\omega)$ as $\underline{\partial}_{\mathcal{H}_i} = \sum_{j=0}^{\infty} \underline{\partial}_j U^j_z$.

This and the definition of G_{U_w} in Equation (4.2) explains the notation $(\underline{\partial}_{i+j+1})_{U_{w_0,w_1}}$ in the statement of Proposition 4.5.

Proposition 4.5 is the twisted version of Theorem 14.1 of [Zem19] and the proof easily modifies to the twisted case. We briefly restate the steps of the proof according to [Zem19].

Let c denote a curve along the connected sum neck of Σ and S^2, c_α be a small isotopy of α, and c_β be a small isotopy of β such that $c_\alpha \cap c = \emptyset$ and $c_\beta \cap c = \emptyset$ (see Figure 4.2). Let $\mathbf{T} = (T_1, T_2)$ and $J_\alpha(\mathbf{T})$ denote an almost complex structure which is stretched along c and c_α with neck-lengths T_1 and T_2 respectively. Lemma 14.2 of [Zem19] proves that when the neck-lengths are large enough, the relative neck-lengths T_1 and T_2 do not matter. More precisely, the twisted version of this lemma is as follows.

Let J_1 and J_2 denote two almost complex structures on $\Sigma \times [0,1] \times \mathbb{R}$. There is an almost complex structure J' on $\Sigma \times [0,1] \times \mathbb{R}$ such that it agrees with J_1 on $\Sigma \times [0,1] \times (-\infty,-1]$ and with J_2 on $\Sigma \times [0,1] \times [1,+\infty)$, and a transition map $\underline{\Psi}_{(\mathcal{H},J_1)\to(\mathcal{H},J_2),\mathfrak{s}}$ is defined by counting J'-holomorphic disks ϕ of zero Maslov index in $\Sigma \times [0,1] \times \mathbb{R}$, weighted by $t^{\omega([\phi])}$, where $[\phi]$ is the associated 2-chain to ϕ obtained by coning off the α- and the β- boundaries of $D(\phi)$. We say that J' interpolates between J_1 and J_2.

Lemma 4.6. *Let $\tilde{\mathcal{H}}$ be one of $\mathcal{H}_i$, $i = 1, 2$, or the diagram $\mathcal{H}_{1.5}$ shown in Figure 4.3. There is a constant N such that if $\mathbf{T}$ and $\mathbf{T}'$ are two pairs of neck lengths, all of whose components are greater than N, then there is a non-cylindrical almost complex structure J' interpolating $J_\alpha(\mathbf{T})$ and $J_\alpha(\mathbf{T}')$, respectively, $J_\beta(\mathbf{T})$ and $J_\beta(\mathbf{T}')$, satisfying*

$$\underline{\Psi}_{J_\alpha(\mathbf{T})\to J_\alpha(\mathbf{T}'),\mathfrak{s}} = \underline{\Psi}_{J',\mathfrak{s}} = \begin{pmatrix} id & 0 \\ 0 & id \end{pmatrix},$$
$$\underline{\Psi}_{J_\beta(\mathbf{T})\to J_\beta(\mathbf{T}'),\mathfrak{s}} = \underline{\Psi}_{J',\mathfrak{s}} = \begin{pmatrix} id & 0 \\ 0 & id \end{pmatrix}.$$

Proof. The proof is almost the same as the proof of [Zem19, Lemma 14.2]. We consider the case $\tilde{\mathcal{H}} = \mathcal{H}_1$. The general plan is to take two sequences $\mathbf{T}_i = (T_{1,i}, T_{2,i})$, $\mathbf{T}'_i = (T'_{1,i}, T'_{2,i})$ of pairs of neck-lengths. Let J'_i denote the almost complex structure which interpolates $J_\alpha(\mathbf{T}_i)$ and $J_\alpha(\mathbf{T}'_i)$ and is non-cylindrical only in a neighborhood of c and c_α. $\underline{\Psi}_{J'_i,\mathfrak{s}}$ counts the number of J'_i-holomorphic disks with Maslov index zero and has a matrix presentation as

$$\begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}.$$

Let $\phi\#\phi_0 \in \pi_2(\mathbf{x}\times x, \mathbf{y}\times y)$. By the index formula ([Zem19, Formula 14.5]),

$$\mu(\phi\#\phi_0) = \mu(\phi) + \mathrm{gr}(x,y) + 2m_2(\phi_0),$$

Classes with $\mathrm{gr}(x,y) = 0$, contribute to A_i or D_i, classes with $\mathrm{gr}(x,y) = -1$ contribute to B_i, and classes with $\mathrm{gr}(x,y) = 1$ contribute to B_i. Here $m_2(\phi_0)$ denotes the coefficient of the disk ϕ_0 in the region containing w_0 (see Figure 4.2 on the left). As the two pairs of neck-lengths approach infinity, for each sequence of J_i'-holomorphic disks u_i, we can take preimage of u_i into subregions of $\Sigma\times[0,1]\times\mathbb{R}$ to construct holomorphic curves with additional boundary circles, u_i^l, u_i^m, and u_i^r such that each converges to curves in punctured manifolds $(S^2\setminus\{p_0\})\times[0,1]\times\mathbb{R}$, in $S^1\times\mathbb{R}\times[0,1]\times\mathbb{R}$, and in $\Sigma\setminus\{p\}\times[0,1]\times\mathbb{R}$, where p_0 and p denote the connected sum points corresponding to the circles c_α and c, respectively. Note that $S^1\times\mathbb{R}$ is a cylinder corresponding to the cylinder $S^1\times[0,1]$ with boundary components c_α and c. Consequently, ϕ and ϕ_0 admit broken homomorphic representatives on $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and (S^2, α, β^l), where the curve β^l is the result of cutting β along its intersection with c_α, and then collapsing the ends to a point. The index formula and the fact that ϕ has a broken holomorphic representative imply that for the disks which contribute to A_i and D_i, ϕ and ϕ_0 are constant and therefore $\phi\#\phi_0$ has a unique $\tilde{J}_i$-holomorphic representative and $\omega([\phi]) = \omega([\phi_0]) = 0$, which means that $A_i = D_i = 1$. The same argument as in the proof of [Zem19, Lemma 14.2] shows that $B_i = C_i = 0$. □

Let $\mathcal{H}_{1.5}$ denote the Heegaard diagram in Figure 4.3. The following lemma is similar to Theorem 4.2 but with different basepoints and different choices of almost complex structures.

Lemma 4.7. *([Zem19, Lemma 14.3]) Let J_α denote an almost complex structure on the Heegaard diagram $\mathcal{H}_{1.5}$ which is stretched along c and c_α (see Figure 4.3). For sufficiently large neck lengths along c and c_α, we have*

$$\underline{\partial}_{\mathcal{H}_{1.5},J_\alpha,\mathfrak{s}} = \begin{pmatrix} (\underline{\partial}_{\mathcal{H}_0})^{U_z\to U_{w_1}} & t^w(U_{w_1}+U_{w_0}) \\ 0 & (\underline{\partial}_{\mathcal{H}_0})^{U_z\to U_{w_1}} \end{pmatrix},$$

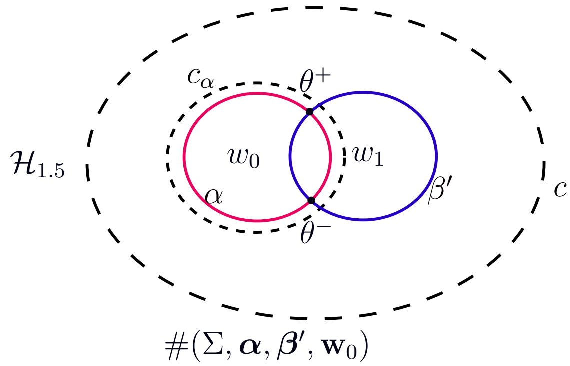

Figure 4.3. This is Figure 14.2 of [Zem19].

for some $w \in \mathbb{R}$. If J_β denotes an analogous almost complex structure stretched sufficiently along c and c_β, then

$$\underline{\partial}_{\mathcal{H}_{1.5},J_\beta,\mathfrak{s}} = \begin{pmatrix} (\underline{\partial}_{\mathcal{H}_0})^{U_z \to U_{w_0}} & t^w(U_{w_1} + U_{w_0}) \\ 0 & (\underline{\partial}_{\mathcal{H}_0})^{U_z \to U_{w_0}} \end{pmatrix},$$

for some $w \in \mathbb{R}$.

Proof. The proof of [Zem19, Lemma 14.3] works here with a slight modification and we briefly restate it. By Lemma 4.6, the relative neck-lengths of c and c_α do not affect the computation when the neck-lengths are sufficiently large. Let

$$\underline{\partial}_{\mathcal{H}_{1.5},J_\alpha,\mathfrak{s}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

For a homotopy class of a disk $\phi \# \phi_0 \in \pi_2(\mathbf{x} \times x, \mathbf{y} \times y)$, we have the index formula

$$\mu(\phi \# \phi_0) = \mu(\phi) - \mathrm{gr}(x, y) + 2m_2(\phi_0).$$

Let m_1, m_2, n_1 and n_2 denote the multiplicities of the regions of $(S^2, \alpha, \beta', \{w_0, w_1\})$ shown in Figure 4.4. Classes with $\mathrm{gr}(x, y) = 1$ contribute to C, classes with $\mathrm{gr}(x, y) = 0$ contribute to A and D, and classes with $\mathrm{gr}(x, y) = -1$ contribute to B. By the index formula and stretching the neck along c, one can check that the only classes with $\mathrm{gr}(x, y) = -1$ are the classes with domains one of the two bigons D_i, $i = 1, 2$, containing the basepoints w_0 and w_1. Let ϕ_i, $i = 1, 2$, denote the classes with $D(\phi_i) = D_i$. As was argued in the proof of Theorem 4.2, $\omega([\phi_1]) = \omega([\phi_2])$. Therefore, $B \doteq U_{w_0} + U_{w_1}$.

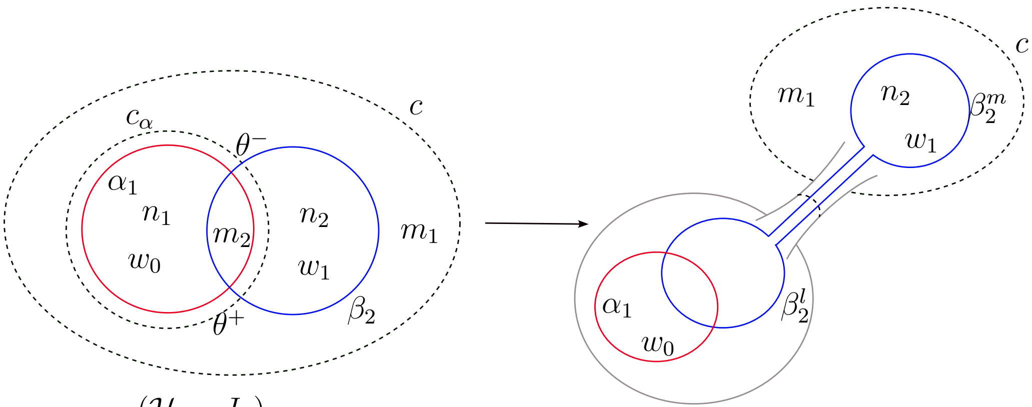

Figure 4.4. Figure 14.5 of [Zem19].

To compute A and D, note that stretching the neck around c and then using the above index formula implies that $\mu(\phi) = 1$ and $m_2(\phi_0) = 0$. For a fixed neck-length around c_α, the argument in the proof of Theorem 4.2 computes the functions A and D without the consideration about the basepoints. This is proved similarly to the untwisted case. In fact, as the neck around c_α approaches infinity, a two-punctured sphere with one β curve and no α curves degenerates (see Figure 4.4). By considering the contributions of the limiting curves on this degenerate diagram, one can prove that $m_1(\phi_0) \leq n_2(\phi_0)$. This observation and the index formula result in $n_1(\phi_0) = 0$ and $n_2(\phi_0) = m_1(\phi_0)$. This implies that any curve which makes non-trivial contribution to A or D is counted with a factor of $U_{w_1}^{m_1(\phi)}$ and no factor of U_{w_0}. This completes the computation of A and D.

For a sufficiently large neck-length around c and using the index formula, one can check that the only possible disks that contribute to C are either a disk $\psi := \phi \# \phi_0$ where ϕ is a constant class and ϕ_0 is a disk where the domain is a bigon with $m_2(\phi_0) = 1$ or a disk $\phi' \# \phi'_0$ with $\mu(\phi') = 2$ and $m_2(\phi'_0) = 0$. For each neck-length $T_i(c_\alpha)$ along c_α, there is a point $d_i(\phi'_0) \in [0,1] \times \mathbb{R}$ associated with the bigon ϕ'_0 in $\mathcal{H}_0$ such that as $T_i(c_\alpha)$ approaches infinity, $d_i(\phi'_0)$ approaches $\{0\} \times \mathbb{R}$. Therefore, using the maximum modulus principle, the domain of each limit curve u is a component of $\Sigma \setminus \boldsymbol{\beta}'$. As c stretches, one can use transversality results to show that u is a representative for ϕ', and we have $\omega([\phi' \# \phi'_0]) = \omega([\psi])$. One can also check that $\# \widehat{\mathcal{M}}(\phi' \# \phi'_0) = \# \widehat{\mathcal{M}}(\psi)$. This proves that $C = 0$. $\square$

Proof of Proposition 4.5. Let J_1 and J_2 be the almost complex structures J_α and J_β

in Lemma 4.7. One can use the computation for the differential in Lemma 4.7 above and Lemma 14.5 of [Zem19] to show that the transition map $\underline{\Psi}_{(\mathcal{H}_{1.5},J_\alpha)\to(\mathcal{H}_{1.5},J_\beta),\mathfrak{s}}$ is given by

$$\begin{pmatrix} id & 0 \\ * & id \end{pmatrix},$$

where $* \doteq \sum_{i,j\geq 0} U_{w_0}^i U_{w_1}^j (\underline{\partial}_{i+j+1})_{U_{w_0},U_{w_1}}$. One can decompose $\underline{\Psi}_{(\mathcal{H}_1,J_\alpha)\to(\mathcal{H}_2,J_\beta),\mathfrak{s}}$ as

$$\underline{\Psi}_{(\mathcal{H}_1,J_\alpha)\to(\mathcal{H}_2,J_\beta),\mathfrak{s}} = \underline{\Psi}_{(\mathcal{H}_{1.5},J_\beta)\to(\mathcal{H}_2,J_\beta),\mathfrak{s}} \circ \underline{\Psi}_{(\mathcal{H}_{1.5},J_\alpha)\to(\mathcal{H}_{1.5},J_\beta),\mathfrak{s}} \circ \underline{\Psi}_{(\mathcal{H}_1,J_\alpha)\to(\mathcal{H}_{1.5},J_\alpha),\mathfrak{s}}$$

The twisted versions of Proposition 14.6 and Proposition 14.8 of [Zem19] hold (with the same proof) and one can use them to compute $\underline{\Psi}_{(\mathcal{H}_1,J_\alpha)\to(\mathcal{H}_{1.5},J_\alpha),\mathfrak{s}}$ and $\underline{\Psi}_{(\mathcal{H}_{1.5},J_\beta)\to(\mathcal{H}_2,J_\beta),\mathfrak{s}}$. In fact, in matrix notation, $\underline{\Psi}_{(\mathcal{H}_1,J_\alpha)\to(\mathcal{H}_2,J_\beta)}$ equals

$$\begin{pmatrix} (\underline{\Phi}^{\boldsymbol{\beta}'}_{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'})^{U_z\to U_{w_0}}_{U_{w_1}} & 0 \\ 0 & (\underline{\Phi}^{\boldsymbol{\beta}'}_{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'})^{U_z\to U_{w_0}}_{U_{w_1}} \end{pmatrix} \begin{pmatrix} id & 0 \\ * & id \end{pmatrix} \begin{pmatrix} (\underline{\Phi}^{\boldsymbol{\beta}\to\boldsymbol{\beta}'}_{\boldsymbol{\alpha}})^{U_z\to U_{w_1}}_{U_{w_0}} & 0 \\ 0 & (\underline{\Phi}^{\boldsymbol{\beta}\to\boldsymbol{\beta}'}_{\boldsymbol{\alpha}})^{U_z\to U_{w_1}}_{U_{w_0}} \end{pmatrix}.$$

This completes the proof. See [Zem19, Section 14] for more details. □

At this step, we can prove the twisted version of Lemma 3.35 of [LY23].

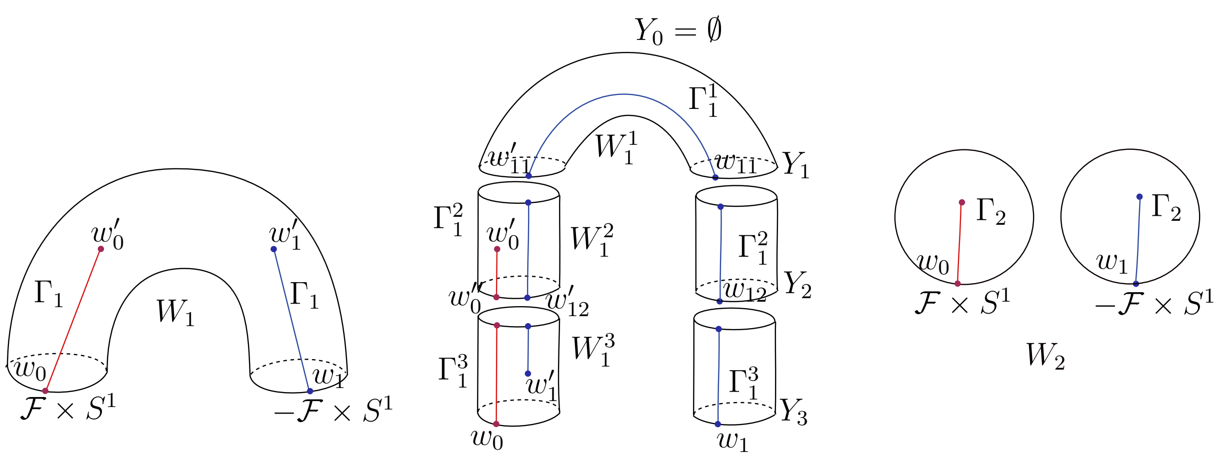

Figure 4.5. This is Figure 8 in [LY23], related to Lemma 4.8

Lemma 4.8. *Let $Y = \mathcal{F} \times S^1$, $[\omega] = d \cdot \mathrm{PD}([\eta]) \in H^2(Y;\mathbb{R})$ where $\mathcal{F} = S^1 \times S^1$ $\eta = \{p\} \times S^1$, $p \in \mathcal{F}$, $d \neq 0$. Let $W_1 = Y \times I$, viewed as a cobordism from $\emptyset$*

to $Y \amalg (-Y)$, *and* $[\overline{\omega}_1] = d \cdot \mathrm{PD}([\eta \times I]) \in H^2(W;\mathbb{R})$. *Let* $w_0 \in Y$, $w_1 \in -Y$, $w_0', w_1' \in W_1$ *and let* $\Gamma_1 \subset W_1$ *denote a two component path that connects* w_i *and* w_i', $i = 0, 1$, *as shown in the left of Figure 4.5. Let* (W_2, Γ_2) *denote a graph cobordism from* $\emptyset$ *to* $Y \amalg (-Y)$ *with* $W_2 \cong (\mathcal{F} \times D^2) \amalg (-\mathcal{F} \times D^2)$ *and* Γ_2 *a two component path such that the components connect two basepoints in* Y *and* $-Y$ *to the basepoints in* W_2 *as shown in the right of Figure 4.5. Let* $[\overline{\omega}_2] \in H^2(W_2;\mathbb{R})$ *be such that whose restriction to each boundary component is* $d \cdot \mathrm{PD}([\eta])$. *Then*

$$\underline{f}^-_{W_1,\Gamma_1;\Lambda_{\overline{\omega}_1}} \doteq \underline{f}^-_{W_2,\Gamma_2;\Lambda_{\overline{\omega}_2}}.$$

Proof. The proof is similar to the proof of [LY23, Lemma 3.35]. We only need to modify it to the twisted case. First note that the two cobordisms W_i, $i = 1, 2$, induce maps from $\underline{CF}^-(\emptyset)$ to $\underline{CF}^-(Y \amalg -Y, \{w_1, w_0\}; \Lambda_\omega)$ (see Subsection 3.4). From

$$\underline{CF}^-(Y \amalg -Y, \{w_1, w_0\}; \Lambda_{\omega \oplus \omega}) = \underline{CF}^-(Y, w_1; \Lambda_\omega) \otimes_\Lambda \underline{CF}^-(-Y, w_0; \Lambda_\omega)$$

and

$$\underline{HF}^-(Y, w_1; \Lambda_\omega) \simeq \bigoplus_{\mathfrak{s} \in Spin^c(Y)} \underline{HF}^-(Y, w_1, \mathfrak{s}; \Lambda_\omega),$$

and the fact that only one of the above summands is nonzero (by Theorem 4.1), by Lemma 3.6 and Remark 3.8, the induced maps $\underline{F}^-_{W_i,\Gamma_1;\Lambda_{\overline{\omega}_i}}$, $i = 1, 2$, are well-defined. Let $\mathcal{R} = \Lambda[\![U_1, U_2]\!]$. We have $\underline{CF}^-(\emptyset) = \mathcal{R}$ (see Remark 3.6 of [LY23]). $\underline{CF}^-(-Y, w_1; \Lambda_\omega)$ is chain isomorphic to $\mathrm{Hom}_{\mathcal{R}}(\underline{CF}^-(Y, w_1; \Lambda_\omega), \mathcal{R})$ ([Zem21, Subsection 12.2]). In other words, there is the following canonical isomorphism

$$\underline{CF}^-(-Y, w_1; \Lambda_\omega) \cong \underline{CF}^-(Y, w_1; \Lambda_\omega)^\vee := \mathrm{Hom}_{\mathcal{R}}(\underline{CF}^-(Y, w_1; \Lambda_\omega), \mathcal{R}).$$

By Theorem 4.1, $\underline{CF}^-(Y, w_1; \Lambda_\omega)$ is chain homotopy equivalent to the chain complex

$$0 \to \Lambda[\![U_1]\!]\langle x \rangle \xrightarrow{U_1} \Lambda[\![U_1]\!]\langle y \rangle \to 0. \tag{4.3}$$

By Lemma 4.4, $\underline{F}^-_{W_2,\Gamma_2;\Lambda_{\overline{\omega}_2}}$ is non-zero. Therefore, up to a unit in Λ and chain homotopy, the map $\underline{f}^-_{W_2,\Gamma_2;\Lambda_{\overline{\omega}_2}}$ (on the chain level) sends the generator of $\underline{CF}^-(\emptyset)$ to $y \otimes x^\vee$ where $x^\vee$ is dual of x. To compute $\underline{f}^-_{W_1,\Gamma_1;\Lambda_{\overline{\omega}_1}}$, we decompose (W_1, Γ_1) into three pieces (W_1^i, Γ_1^i) (see Figure 4.5 in the middle). Let $\overline{\omega}_{1i} = \overline{\omega}_1|_{W_1^i}$.

1. (W_1^1, Γ_1^1) is a cobordism from $\emptyset$ to $(Y_1 \amalg -Y_1, \mathbf{w}_1)$ where Y_1 is a copy of Y, $\mathbf{w}_1$ intersects each of Y and $-Y$ in a single basepoint $\{w_{11}'\}$ and $\{w_{11}\}$, respectively,

and Γ_1^1 is a path that connects the two basepoints $\{w'_{11}\}$ and $\{w_{11}\}$. This is a twisted cotrace map which, up to a unit in Λ, sends the generator of $\underline{CF}^-(\emptyset) = \mathcal{R}$ to $x \otimes x^\vee + y \otimes y^\vee$ where $x^\vee$ and $y^\vee$ are duals of x and y (see [Zem21, Theorem 1.7 and Subsection 12.2]).

2. (W_1^2, Γ_1^2) is a two component cobordism from $(Y_1 \amalg -Y_1, \mathbf{w}_1)$ to $(Y_2 \amalg -Y_2, \mathbf{w}_2)$ where Y_2 is a copy of Y and $\mathbf{w}_2$ intersects Y_2 in two basepoints $\{w''_0, w'_{12}\}$ and intersect $-Y_2$ in a single basepoint $\{w_{12}\}$. Two components of Γ_1^2 connect w'_{11} and w_{11} in Y_1 and $-Y_1$, respectively, to w'_{12} and w_{12} in Y_2 and $-Y_2$, respectively. Also, one component of Γ_1^2 connects w'_0 to $w''_0 \in Y_2$. One component of this cobordism corresponds to identity and the other one is a free stabilization $S^+_{w''_0}$. Let $\mathcal{H}$ and $\mathcal{H}_1$ be Heegaard diagrams for Y_1 and Y_2, respectively, where $\mathcal{H}_1$ (see Figure 4.2 on the left) is a connected sum of $\mathcal{H}$ with $\mathcal{H}_0$ (see Figure 3.1) such that a basepoint in $\mathcal{H}$ is identified with w'_{12}. We have

$$\underline{f}^-_{W_1^2,\Gamma_1^2;\Lambda_{\varpi_{12}}} : \underline{CF}^-(\mathcal{H};\Lambda_\omega) \otimes_\Lambda \underline{CF}^-(-\mathcal{H};\Lambda_\omega) \to \underline{CF}^-(\mathcal{H}_1;\Lambda_\omega) \otimes_\Lambda \underline{CF}^-(-\mathcal{H};\Lambda_\omega).$$

By Theorem 4.2, we can compute $\underline{CF}^-(\mathcal{H}_1;\Lambda_\omega)$ in terms of $\underline{CF}^-(\mathcal{H};\Lambda_\omega)$, where the later is chain homotopy equivalent to the chain complex in (4.3). Therefore, using the definition of the free stabilization map (see Equation (3.1)), the above map sends a generator $u \otimes v$ to $u \otimes v \otimes \theta^+$, up to a unit (see the proof of Lemma 3.35 of [LY23] for more details). Here, $u \in \{x, x^\vee\}$, $v \in \{y, y^\vee\}$, and $-\mathcal{H}$ is a Heegaard diagram for Y_1 obtained from $\mathcal{H}$ by reversing the orientation of the Heegaard surface.

3. (W_1^3, Γ_1^3) is a two component cobordism from $(Y_2 \amalg -Y_2, \mathbf{w}_2)$ to $(Y_3 \amalg -Y_3, \mathbf{w}_3)$ where Y_3 is a copy of Y and $\mathbf{w}_3$ intersects each of Y_3 and $-Y_3$ in w_0 and w_1, respectively. One component corresponds to the identity and the other one is a free stabilization map $S^-_{w'_{12}}$. Let $\mathcal{H}_2$ be a Heegaard diagram for Y_2 which is a connected sum of $\mathcal{H}$ with $\mathcal{H}_0$ where a basepoint in $\mathcal{H}$ is identified with w''_0 (see Figure 4.2 on the right). We have

$$\underline{f}^-_{W_1^3,\Gamma_1^3;\Lambda_{\varpi_{13}}} : \underline{CF}^-(\mathcal{H}_2;\Lambda_\omega) \otimes \underline{CF}^-(-\mathcal{H};\Lambda_\omega) \to \underline{CF}^-(\mathcal{H};\Lambda_\omega) \otimes \underline{CF}^-(-\mathcal{H};\Lambda_\omega).$$

By Theorem 4.2 and the chain complex in (4.3), $\underline{f}^-_{W_1^3,\Gamma_1^3;\Lambda_{\varpi_{13}}}$, up to a unit, sends a generator $u \otimes v \otimes \theta^-$ to $u \otimes v$ and $u \otimes v \otimes \theta^+$ to 0 (see the proof of Lemma 3.35 of [LY23] for more details).

Therefore,

$$\underline{f}^-_{W_1,\Gamma_1;\Lambda_{\varpi_1}} = \underline{f}^-_{W_1^3,\Gamma_1^3;\Lambda_{\varpi_{13}}} \circ \underline{\Psi}_{(\mathcal{H}_1,J_1)\to(\mathcal{H}_2,J_2)} \circ \underline{f}^-_{W_1^2,\Gamma_1^2;\Lambda_{\varpi_{12}}} \circ \underline{f}^-_{W_1^1,\Gamma_1^1;\Lambda_{\varpi_{13}}},$$

where J_i, $i=1,2$, are the almost complex structures in Proposition 4.5. Note that

$$\begin{aligned} A &= \underline{f}^-_{W_1^2,\Gamma_1^2;\Lambda_{\varpi_{12}}} \circ \underline{f}^-_{W_1^1,\Gamma_1^1;\Lambda_{\varpi_{13}}}(1) = \underline{f}^-_{W_1^2,\Gamma_1^2;\Lambda_{\varpi_{12}}}(x\otimes x^\vee + y\otimes y^\vee) \\ &= x\otimes x^\vee \oplus \theta^+ + y\otimes y^\vee \oplus \theta^+ = (x\otimes x^\vee + y\otimes y^\vee, 0), \end{aligned}$$

where 1 denotes the generator of $\mathcal{R}$ and the last equality shows A in matrix notation with $\theta^\pm$ components. Since $\underline{f}^-_{W_1^3,\Gamma_1^3;\Lambda_{\varpi_{13}}}$ sends a generator $u\otimes v\otimes\theta^+$ to zero, by Proposition 4.5, we only need to find the image of the component $x\otimes x^\vee + y\otimes y^\vee$ of A under the entry

$$(\underline{\Phi}^{\boldsymbol{\beta}'}_{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'})^{U_z\to U_{w_0''}}_{U_{w_{12}'}} \circ (\sum U^i_{w_0''} U^j_{w_{12}'}(\underline{\partial}_{i+j+1})_{U_{w_0''},w_{12}'}) \circ (\underline{\Phi}^{\boldsymbol{\beta}\to\boldsymbol{\beta}'}_{\boldsymbol{\alpha}})^{U_z\to U_{w_{12}'}}_{U_{w_0''}} \tag{4.4}$$

of $\underline{\Psi}_{(\mathcal{H}_1,J_1)\to(\mathcal{H}_2,J_2),\mathfrak{s}}$. Note that $x\otimes x^\vee \oplus +y\otimes y^\vee$ has no U-power, therefore, since $(\underline{\Phi}^{\boldsymbol{\beta}'}_{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'})^{U_z\to U_{w_0''}}_{U_{w_{12}'}}$ and $(\underline{\Phi}^{\boldsymbol{\beta}\to\boldsymbol{\beta}'}_{\boldsymbol{\alpha}})^{U_z\to U_{w_{12}'}}_{U_{w_0''}}$ are non-zero maps, they can be regarded as the identity, up to a unit. By Theorem 4.2 and Formula (4.3), $\underline{CF}^-(\mathcal{H}_2,\{z,w_{12}'\};\Lambda_\omega)\otimes \underline{CF}^-(-\mathcal{H},\{z\};\Lambda_\omega)$ with the differential $\underline{\partial} = \sum_{i\geq 1}^{\infty}\underline{\partial}_i U_z^i$ is isomorphic with

$$\begin{array}{ccc} \mathcal{R}\langle x, y^\vee\rangle & \xrightarrow{U_z} & \mathcal{R}\langle x, x^\vee\rangle \\ {\scriptstyle U_{w_{12}'}}\big\downarrow & & \big\downarrow{\scriptstyle U_{w_{12}'}} \\ \mathcal{R}\langle y, y^\vee\rangle & \xrightarrow{U_z} & \mathcal{R}\langle y, x^\vee\rangle. \end{array}$$

Therefore, the map in Formula (4.4), sends $x\otimes x^\vee + y\otimes y^\vee$ to $y\otimes x^\vee$, up to a unit, and we have

$$\underline{f}^-_{W_1^3,\Gamma_1^3;\Lambda_{\varpi_{13}}} \circ \Psi_{(\mathcal{H}_1,J_1)\to(\mathcal{H}_2,J_2)}(A) = \underline{f}^-_{W_1^3,\Gamma_1^3;\Lambda_{\varpi_{13}}}(*\otimes\theta^+ + y\otimes x^\vee\otimes\theta^-) = y\otimes x^\vee.$$

This observation completes the proof. □

Proof of Theorem 1.2. The proof is a slight modification of the proof of Theorem 3.30 of [LY23]. One can construct a cobordism W from Y_1 to Y_2. In fact, let Y' denote a three manifold obtained by cutting Y_1 along $F=\Sigma_1\cup\Sigma_2$ and P be a saddle with boundary and corners (see Figure 4.6 on the left). If we glue $Y'\times I$ to $P\times\Sigma$, where $\Sigma\cong\Sigma_1\cong\Sigma_2$, as in Figure 4.6 on the right, we obtain the cobordism W (see [LY23] for more details). Suppose Y_1 is disconnected (the proof for the connected case is similar).

Let $W' = -W$ be the upside-down cobordism from Y_2 to Y_1 and $W_A = W \cup_{Y_2} W'$ and $W_B = W' \cup_{Y_1} W$. We can obtain W_A and W_B from a product cobordism as follows. Suppose that W''_A (resp. W''_B) is the product cobordism $Y_1 \times [-1,1]$ (resp. $Y_2 \times [-1,1]$). A neighborhood of $F \times \{0\}$ in W''_A (resp. W''_B) can be identified with $F \times [-\epsilon,\epsilon] \times [-\epsilon,\epsilon]$ for some small $\epsilon > 0$. If we remove the interior of $F \times [-\epsilon,\epsilon] \times [-\epsilon,\epsilon]$ and identify its two boundary components with the boundary components of the manifold $\Sigma \times S^1 \times [-\epsilon',\epsilon']$, where $\epsilon' > 0$, we obtain W_A (see Figure 4.7) (resp. W_B). Let $Y_A := \Sigma \times S^1 \subset W_A$ and $N(Y_A)$ denote a neighborhood of Y_A. Similarly, let $Y_B := \Sigma \times S^1 \subset W_B$ and $N(Y_B)$ denote a neighborhood of Y_B.

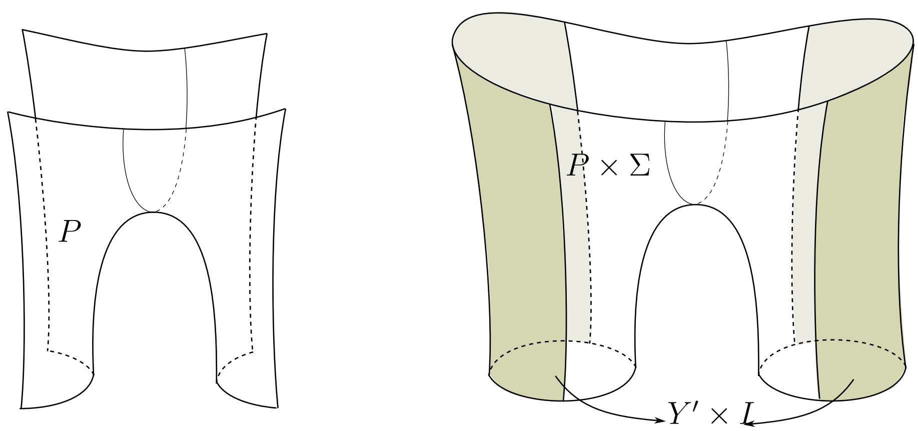

Figure 4.6. Construction of the cobordism W from Y_1 to Y_2. Left: A saddle P. Right: Gluing $Y' \times I$ to $P \times \Sigma$.

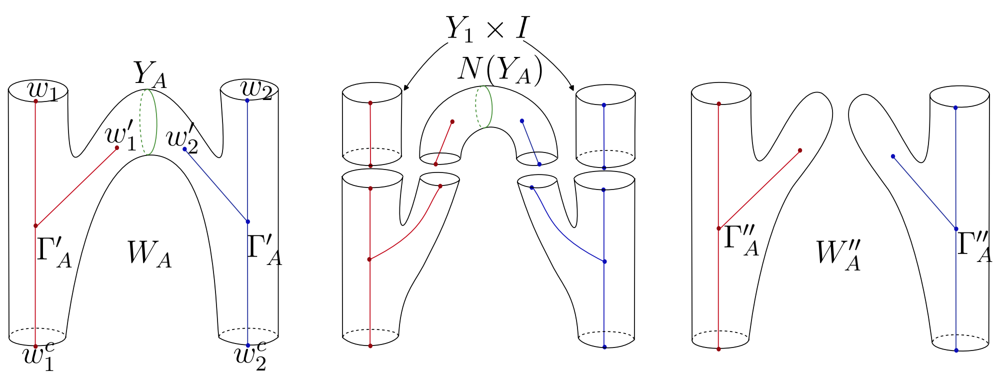

Figure 4.7. Figures 10 and 11 of [LY23]

Let w_i, $i = 1,2$, denote two basepoints, one in each connected component of Y_1. Note that W_A is a cobordism from Y_1 to Y_1. Let w_i^c, $i = 1,2$, denote the corresponding basepoints in another copy of Y_1 and Γ_A be a two component path which connects w_i to w_i^c in the boundaries of W_A. Let z_i, $i=1,2$, denote two basepoints in Y_2. Note that W_B is a cobordism from Y_2 to Y_2. Let z_i^c, $i=1,2$, denote the corresponding basepoints in another copy of Y_2 and Γ_B be a two component path which connects z_i to z_i^c in the two boundaries of W_B. Let w_i', $i=1,2$, be two points in the interior of W_A and Γ_A' be obtained from Γ_A by connecting one of the w_i' by an arc to each component of Γ_A. Each component of Γ_A' has a vertex of valence three (see Figure 4.7 on the left). Fix an ordering for these vertices. From the Mayer-Vietoris sequence

$$\cdots \to H^2(W;\mathbb{R}) \to H^2(Y' \times I) \oplus H^2(\Sigma \times P) \to \cdots$$

and the assumptions on $\omega_i \in H^2(Y_i;\mathbb{R})$, there are $\overline{\omega} \in H^2(W;\mathbb{R})$, $\omega_A \in H^2(W_A;\mathbb{R})$, and $\omega_B \in H^2(W_B;\mathbb{R})$ such that $\overline{\omega}|_{Y_i} = \omega_i$, $\omega_A|_{Y_1} = \omega_1$, $\omega_B|_{Y_2} = \omega_2$, $\omega_A|_{Y_A} \neq 0$, and $\omega_B|_{Y_B} \neq 0$.

One can decompose (W_A, Γ_A') into two cobordisms (see Figure 4.7 in the middle). One cobordism consists of $N(Y_A)$ (as a cobordism from $\emptyset$ to $\Sigma \times S^1 \amalg (-\Sigma \times S^1)$) and two copies of $Y_1 \times I$. If we replace $N(Y_A)$ with a cobordism corresponding to $\Sigma \times D^2 \amalg (-\Sigma \times D^2)$, we will obtain a cobordism (W_A'', Γ_A'') (see Figure 4.7 on the right). Γ_A'' is the graph induced by Γ_A'. Therefore, there are two edges in Γ_A'' with endpoints in the interior of W_A'' (one end point in each component is of valence one). By [Zem19, Lemma 7.15], one can replace Γ_A'' with a two component path which connects w_i to w_i^c in the boundaries of W_A''. We also denote this graph by Γ_A''. Therefore, (W_A'', Γ_A'') is diffeomorphic to the product cobordism. If $\mathcal{H}_1$ denotes a Heegaard diagram for Y_1, the induced map

$$\underline{F}^-_{W_A'', \Gamma_A''; \Lambda_{\omega_A'}} = \bigoplus_i \underline{F}^-_{W_A'', \Gamma_A'', \mathfrak{t}_i; \Lambda_{\omega_A'}}$$

where

$$\underline{F}^-_{W_A'', \Gamma_A'', \mathfrak{t}_i; \Lambda_{\omega_A'}} : \underline{HF}^-(\mathcal{H}_1, \{w_1, w_2\}, \mathfrak{s}_i; \Lambda_{\omega_1}) \to \underline{HF}^-(\mathcal{H}_1, \{w_1, w_2\}, \mathfrak{s}_i; \Lambda_{\omega_1}),$$

is a diagonal matrix with units in Λ on the diagonal. Here, $[\omega_A'] \in H^2(W_A'';\mathbb{R})$ is such that $\omega_A'|_{Y_1} = \omega_1$, and $\mathfrak{t}_i$ denotes the Spin^c structure in $\mathrm{Spin}^c(W_A'')$ whose restriction to the boundary components of W_A'', which are two copies of Y_1, is $\mathfrak{s}_i \in \mathrm{Spin}^c(Y_1)$.

By Lemma 4.8,

$$\underline{F}^-_{W_A, \Gamma_A'; \Lambda_{\omega_A}} \doteq \underline{F}^-_{W_A'', \Gamma_A''; \Lambda_{\omega_A'}}.$$

By [Zem19, Lemma 7.15], (W_A, Γ_A) and (W_A, Γ_A') induce the same cobordism map up to a unit. Therefore, $\underline{F}^-_{W_A, \Gamma_A; \Lambda_{\omega_A}}$ is an invertible diagonal matrix $[\lambda_i]$, where $\lambda_i \in \Lambda$.

If $\mathcal{H}_2$ denotes a Heegaard diagram for Y_2, a similar argument proves that the induced map

$$\underline{F}^-_{W_B,\Gamma_B;\Lambda_{\omega_B}} : \bigoplus_i \underline{HF}^-(\mathcal{H}_2, \{z_1, z_2\}, \mathfrak{s}'_i; \Lambda_{\omega_2}) \to \bigoplus_i \underline{HF}^-(\mathcal{H}_2, \{z_1, z_2\}, \mathfrak{s}'_i; \Lambda_{\omega_2}),$$

is a diagonal matrix $[\gamma_i]$ with units $\gamma_i \in \Lambda$ on the diagonal. Here, $\mathfrak{s}'_i \in \mathrm{Spin}^c(Y_2)$.This proves that

$$\underline{HF}(Y_1; \Lambda_{\omega_1}) \cong \underline{HF}(Y_2; \Lambda_{\omega_2}).$$

This isomorphism is induced by a cobordism. In the following, we show that the map induced by the cobordism is well-defined. Let

$\mathfrak{T}_{ij} = \{\mathfrak{t} \in \mathrm{Spin}^c(W) \mid \mathfrak{t}|_{Y_1} = \mathfrak{s}_i,\ \mathfrak{t}|_{Y_2} = \mathfrak{s}'_j\}$ and $\mathfrak{T}'_{ij} = \{\mathfrak{t} \in \mathrm{Spin}^c(W') \mid \mathfrak{t}|_{Y_2} = \mathfrak{s}'_i,\ \mathfrak{t}|_{Y_1} = \mathfrak{s}_j\}$.

Let $F = [F_{ij}]$ and $G = [G_{ij}]$ where

$$F_{ji} = \sum_{\mathfrak{t} \in \mathfrak{T}_{ij}} \underline{F}_{W,\Gamma,\mathfrak{t};\Lambda_{\varpi}} : \underline{HF}^-(\mathcal{H}_1, \{w_1, w_2\}, \mathfrak{s}_i; \Lambda_{\omega_1}) \to \underline{HF}^-(\mathcal{H}_2, \{z_1, z_2\}, \mathfrak{s}'_j; \Lambda_{\omega_2}),$$

$$G_{ji} = \sum_{\mathfrak{t} \in \mathfrak{T}'_{ij}} \underline{F}_{W',\Gamma,\mathfrak{t};\Lambda_{\varpi}} : \underline{HF}^-(\mathcal{H}_2, \{z_1, z_2\}, \mathfrak{s}'_i; \Lambda_{\omega_2}) \to \underline{HF}^-(\mathcal{H}_1, \{w_1, w_2\}, \mathfrak{s}_j; \Lambda_{\omega_1}),$$

and Γ consists of a pair of paths that connects w_i to z_i, $i = 1, 2$. We have

$$\underline{F}^-_{W',\Gamma;\Lambda_{\varpi}} = \bigoplus_j (\sum_i G_{ij}),$$

and we will show that $\pi_{s_j} \circ \underline{F}^-_{W',\Gamma;\Lambda_{\varpi}}$ is well-defined up to a unit. Indeed, by Lemma 3.6, F_{ij} and G_{ij} are well-defined. Let $\mathcal{H}'_i$, $i = 1, 2$, denote two Heegaard diagrams for Y_i with transition maps

$$\underline{\Psi}_i = \underline{\Psi}_{(\mathcal{H}_1,J_1) \longrightarrow (\mathcal{H}'_1,J'_1),\mathfrak{s}_i}, \quad \underline{\Phi}_i = \underline{\Phi}_{(\mathcal{H}_2,J_2) \longrightarrow (\mathcal{H}'_2,J'_2),\mathfrak{s}'_i}.$$

Let $G' = [G'_{ij}]$ where

$$G'_{ji} : \underline{HF}^-(\mathcal{H}'_2, \{z_1, z_2\}, \mathfrak{s}_i; \Lambda_{\omega_2}) \to \underline{HF}^-(\mathcal{H}'_1, \{w_1, w_2\}, \mathfrak{s}_j; \Lambda_{\omega_1}),$$

is the map induced by the cobordism W'. From the above argument,

$$F \circ G = [\gamma_i], \qquad G \circ F = [\lambda_i]. \tag{4.5}$$

Therefore,

$$G \circ [\gamma_i] = G \circ (F \circ G) = (G \circ F) \circ G = [\lambda_i] \circ G.$$

If $G_{ij} \neq 0$, $\gamma_j = \lambda_i$. Similarly, there are invertible diagonal matrices $[\gamma'_i]$ and $[\lambda'_i]$ such that

$$F \circ \underline{\Psi}^{-1} \circ G' \circ \underline{\Phi} = [\gamma'_i], \qquad \underline{\Psi}^{-1} \circ G' \circ \underline{\Phi} \circ F = [\lambda'_i], \tag{4.6}$$

where $\underline{\Psi}$ and $\underline{\Phi}$ are diagonal matrices with $\underline{\Psi}_{ii} = \underline{\Psi}_i$, $\underline{\Phi}_{ii} = \underline{\Phi}_i$. This shows that $\gamma'_j = \lambda'_i$ if $(\underline{\Psi}^{-1} \circ G' \circ \underline{\Phi})_{ij} \neq 0$. From Equations (4.5) and (4.6), we have

$$[\lambda_i]\underline{\Psi}^{-1} \circ G' \circ \underline{\Phi} = (G \circ F) \circ \underline{\Psi}^{-1} \circ G' \circ \underline{\Phi} = G \circ (F \circ \underline{\Psi}^{-1} \circ G' \circ \underline{\Phi}) = G[\gamma'_i],$$

which shows that $\lambda_i(\underline{\Psi}^{-1} \circ G' \circ \underline{\Phi})_{ij} = G_{ij}\gamma'_j$. Therefore, when $G_{ij} \neq 0$, $\gamma_j(\underline{\Psi} \circ G' \circ \underline{\Phi})_{ij} = G_{ij}\gamma'_j$. From here,

$$\sum_i G_{ij} = \frac{\gamma_j}{\gamma'_j} \sum_i (\underline{\Psi}^{-1} \circ G' \circ \underline{\Phi})_{ij} \doteq \sum_i (\underline{\Psi}^{-1} \circ G' \circ \underline{\Phi})_{ij}.$$

This completes the proof. □

Remark 4.9. Note that Theorem 1.2 implies that we can use two orientation-preserving diffeomorphisms $h, h' : \Sigma_1 \to \Sigma_2$ to identify Σ_1 with $-\Sigma_2$ using h and identify $-\Sigma_1$ with Σ_2 using h'. Indeed, assume that Y_2 is obtained from Y_1 by excision along the surfaces Σ_1 and Σ_2 (using the orientation-preserving diffeomorphism h). Let $Y = Y_2 \cup T_\phi$, where $\phi = h^{-1} \circ h'$ and T_ϕ is the mapping torus of ϕ. By Theorem 1.2 and Theorem 4.1,

$$\underline{HF}(Y_3; \Lambda_{\omega_3}) \cong \underline{HF}(Y_2; \Lambda_{\omega_2}),$$

for a generic choice of $[\omega_3] \in H^2(Y_3; \mathbb{R})$ (as stated in Theorem 1.2). Note that if we cut Y_1 along $\Sigma_1 \cup \Sigma_2$ to obtain the three manifold Y', and identify Σ_1 with $-\Sigma_2$ using h, and $-\Sigma_1$ with Σ_2 using h', we obtain the 3-manifold Y_3.

CHAPTER 5

APPLICATIONS OF EXCISION FORMULA

Let K be a genus one knot, and $Y_p = S^3_p(K)$ be the 3-manifold obtained by performing p-surgery on K. Therefore, Y_0 contains a non-separating torus. In this section, first we compute $\underline{HF}^+(Y_0;\Lambda)$ when K is a *twist knot* which is, by definition, an n-twisted Whitehead double of the unknot (see Figure 5.2). Then we use Theorem 1.2 to compute Heegaard Floer homology groups for the manifold obtained by cutting Y_0 along the non-separating torus and regluing it using a Dehn twist. We use this computation to prove Corollary 1.3 and Corollary 1.4.

5.1 Examples of 3-Manifolds Related by the Excision Construction

The negative n-twisted Whitehead double of the unknot, denoted $D_-(U,n)$ where U is the unknot, is shown in Figure 5.2 on the left (the " $-$ " indicates the parity of the clasp). Let $D_+(U,n)$ denote the positive n-twisted Whitehead double of the unknot, where the clasp is positive. Note that $D_-(U,n)$ is an alternating knot with the Alexander polynomial $\Delta_{D_-(U,n)}(t) = n(t+t^{-1})+(-2n+1)$. Since alternating knots are σ-thin, the argument in Section 2.2 determines $CFK^-(D_-(U,n))$ and therefore its full knot complex (see Figure 5.1).

Lemma 5.1. *Let F be a genus one surface obtained by capping off the Seifert surface of $D_-(U,n)$ (resp. $D_+(U,-n)$), $n>0$, in $S^3_0(D_-(U,n))$ (resp. $S^3_0(D_+(U,-n))$) and $[\omega]\in H^2(S^3_0(D_\pm(U,\mp n));\mathbb{R})$ be a cohomology class such that $\omega([F])\neq 0$. There is an isomorphism of Λ-modules*

$$\underline{HF}(S^3_0(D_\pm(U,\mp n));\Lambda_\omega)\cong \Lambda^n,$$

supported in a single grading.

Proof. We use the idea in the proof of Lemma 8.6 of [OS03] to compute $\underline{HF}^+(S^3_0(D_-(U,n)))$. We find it easier to compute $\widehat{HF}(Y_1)$ first, where $Y_1 = S^3_1(D_-(U,n))$. The surgery formula (see [OS04b, Theorem 4.4]) relates $HF^\circ(S^3_p(K))$, $\circ\in\{\wedge,\pm,\infty\}$, to the homology of certain sub-complexes of $CFK^\infty(K)$. In fact, $\widehat{HF}(S^3_p(K),[s])\cong H_*(C\{\max(i,j-s)=0\})$ and $HF^+(S^3_p(K),[s])\cong H_*(C\{\max(i,j-s)\geq 0\})$ when $p\geq 2g(K)-1$ and $|s|\leq p/2$. Here $[s]\in\mathbb{Z}_p$ is the corresponding Spin^c structure on $S^3_p(K)$. Therefore, $\widehat{HF}(Y_1,0)\cong \mathbb{Z}^{n-1}_{(-1)}\oplus\mathbb{Z}^n_{(-2)}$ generated by w_i, y_i, and a_3 where $M(w_i)=-1$ and $M(y_i)=M(a_3)=-2$, $i=1,\dots,n-1$ (see Figure 5.1).

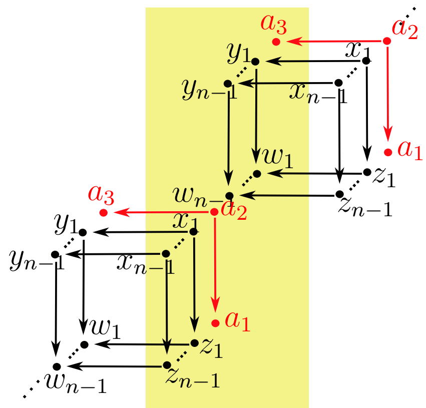

Figure 5.1. This picture shows the knot complex for $D_-(U,n)$ where each generator $[x,i,j]$ is shown as x. To distinguish the generators, the U^0 column which contains the generators with $[x,0,j]$ is highlighted. In this column, $A(w_i)=A(x_i)=A(a_2)=0$, $A(a_3)=A(y_i)=1$, $A(a_1)=A(z_i)=-1$, $i=1,\cdots,n-1$, and $M(a_3)=0$. With this information, the generator a_3, for instance, which is not in the U_0 column represents $[a_3,-2,0]$.

Let t be a generator for $H^1(Y_0;\mathbb{Z})$. Then we can think of $\mathbb{Z}[H^1(Y_0;\mathbb{Z})]$ as $L(t):=\mathbb{Z}[t,t^{-1}]$. For any $\mathbb{Z}[U]$-module M, let $M[t,t^{-1}]$ denote the induced module over $\mathbb{Z}[U,t,t^{-1}]$. There is a $\mathbb{Z}[U,t,t^{-1}]$-equivariant long exact sequence (see [OS04c, Theorem 9.21])

$$\cdots \to \widehat{HF}(S^3)[t,t^{-1}] \xrightarrow{\widehat{f}_1} \underline{\widehat{HF}}(Y_0) \xrightarrow{\widehat{f}_2} \widehat{HF}(Y_1)[t,t^{-1}] \xrightarrow{\widehat{f}_3} \dots \tag{5.1}$$

where the component of $\widehat{f}_1$ mapping into $\widehat{HF}(Y_0,\mathfrak{t}_0)$ (now thought of as absolutely $\mathbb{Q}$-graded) has degree $-1/2$, the restriction of $\widehat{f}_2$ to $\widehat{HF}(Y_0,\mathfrak{t}_0)$ has degree $-1/2$, while $\widehat{f}_3$ is non-increasing in grading (see Subsection 3.3). Since $\widehat{HF}(Y_1)[t,t^{-1}]$ is supported in degrees -1 and -2 while $\widehat{HF}(S^3)[t,t^{-1}]$ is supported in degree 0, and $\hat{f}_3$ is non-increasing, $\widehat{f}_3$ vanishes identically. Also, by the adjunction inequality (see [OS04c, Theorem 7.1]), $\underline{\widehat{HF}}(Y_0,\mathfrak{t})=0$ for $\mathfrak{t}\neq\mathfrak{t}_0$. Therefore, $\underline{\widehat{HF}}(Y_0)\cong L(t)^n\oplus L(t)^n$, generated in dimensions $-\frac{1}{2}$ and $-\frac{3}{2}$. Theorem 7.1 of [OS04c] is stated for HF^+, but the argument, as explained in [OS04c], applies to the case of $\underline{\widehat{HF}}$.

In general, if $\underline{\widehat{HF}}_k(Y)=0$, $k\geq m$, the long exact sequence

$$\cdots \xrightarrow{\delta} \underline{\widehat{HF}}_{i+1}(Y) \xrightarrow{i_*} \underline{HF}^+_{i+1}(Y) \xrightarrow{U} \underline{HF}^+_{i-1}(Y) \xrightarrow{\delta} \underline{\widehat{HF}}_i(Y) \xrightarrow{i_*} \cdots, \tag{5.2}$$

implies that $U:\underline{HF}^+_{i+1}(Y)\to\underline{HF}^+_{i-1}(Y)$, $i\geq m$, is an isomorphism. In this sequence, δ is the connecting map. Therefore, $\underline{HF}^\infty_i(Y)\cong\underline{HF}^+_i(Y)$, $i\geq m-1$. Indeed, if

$x \in \underline{HF}_i^+(Y)$, $i \geq m-1$, for large n, there is a $y_n \in \underline{HF}^+(Y)$ such that $x = U^n y_n$. Therefore, $\delta(x) = U^n \delta(y_n) = 0$ since $\delta(y_n) = 0$ for large n where δ is the connecting map in the long exact sequence corresponding to the short exact sequence

$$0 \to \underline{CF}_i^-(Y) \to \underline{CF}_i^\infty(Y) \to \underline{CF}_i^+(Y) \to 0.$$

This induces the following short exact sequence

$$0 \to \underline{HF}_i^-(Y) \to \underline{HF}_i^\infty(Y) \to \underline{HF}_i^+(Y) \to 0,$$

$i \geq m-1$. For large i, since $\underline{HF}_i^-(Y) = 0$, we have $\underline{HF}_i^\infty(Y) \cong \underline{HF}_i^+(Y)$. From here and the isomorphism $U : \underline{HF}_{i+1}^+(Y) \to \underline{HF}_{i-1}^+(Y)$, $i \geq m$, we have $\underline{HF}_i^\infty(Y) \cong \underline{HF}_i^+(Y)$, $i \geq m-1$. The computation of $\underline{HF}^\infty(Y)$ (see Theorem 10.12 of [OS04c]) shows that there is a $\mathbb{Z}[U, U^{-1}] \otimes \mathbb{Z}[t, t^{-1}]$-module isomorphism $\underline{HF}^\infty(Y_0, \mathfrak{t}_0) \cong \mathbb{Z}[U, U^{-1}]$ where here the latter group is endowed with a trivial action by $\mathbb{Z}[t, t^{-1}]$ (meaning that t acts by 1). To determine the homological grading, note that using the long exact sequence that relates $\widehat{\underline{HF}}(Y_0)$ and $\underline{HF}^+(Y_0)$, we have

$$\begin{aligned} \underline{HF}_{-i}^+(Y_0) &\cong \underline{HF}_{-\frac{5}{2}}^+(Y_0), \quad \text{if } i \equiv -5/2 \mod 2 \text{ and } i \geq -5/2, \\ \underline{HF}_{-i}^+(Y_0) &\cong \underline{HF}_{-\frac{7}{2}}^+(Y_0), \quad \text{if } i \equiv -7/2 \mod 2 \text{ and } i \geq -7/2. \end{aligned}$$

Since $\underline{HF}_{-i}^+(Y_0) = 0$, for large i, we have

$$\begin{aligned} \underline{HF}_{-i}^+(Y_0) &= 0, \quad \text{if } i \equiv -5/2 \mod 2 \text{ and } i \geq -5/2, \\ \underline{HF}_{-i}^+(Y_0) &= 0, \quad \text{if } i \equiv -7/2 \mod 2 \text{ and } i \geq -7/2. \end{aligned}$$

A similar argument shows that $\underline{HF}_i^+(Y_0) = 0$, when $i \not\equiv (2j+1)/2 \mod 2$, where $j \in \mathbb{Z}$. From here and the long exact sequence in (5.2), we have

$$0 \to \widehat{\underline{HF}}_{-3/2}(Y_0) \cong L(t)^n \to \underline{HF}_{-3/2}^+(Y_0) \to 0.$$

Therefore, $\underline{HF}_{-3/2}^+(Y_0) \cong L(t)^n$. Since $U : \underline{HF}_{i+1}^+(Y_0) \to \underline{HF}_{i-1}^+(Y_0)$, $i \geq 1/2$, is an isomorphism, we only need to determine $\underline{HF}_{1/2}^+(Y_0)$ and $\underline{HF}_{-1/2}^+(Y_0)$. Since $\underline{HF}_i^\infty(Y_0) \cong \underline{HF}_i^+(Y_0)$, $i \geq -1/2$ and $\underline{HF}^\infty(Y_0, \mathfrak{t}_0) \cong \mathbb{Z}[U, U^{-1}]$, we have one of the following possibilities:

$$\underline{HF}_{1/2}^+(Y_0) = 0, \text{ and } \underline{HF}_{-1/2}^+(Y_0) \cong \mathbb{Z}, \tag{5.3}$$

or,

$$\underline{HF}_{1/2}^+(Y_0) \cong \mathbb{Z}, \text{ and } \underline{HF}_{-1/2}^+(Y_0) = 0. \tag{5.4}$$

Note that by applying the short exact sequence (5.2), we see that Equation (5.4) cannot occur. Otherwise, we would have:

$$\underline{\widehat{HF}}(Y_0)_{1/2} \to \underline{HF}^+(Y_0)_{1/2} \to \underline{HF}^+(Y_0)_{-3/2} \to \underline{\widehat{HF}}(Y_0)_{-1/2} \to \underline{HF}^+(Y_0)_{-1/2},$$

or equivalently,

$$0 \to \mathbb{Z} \to \mathbb{Z}[t,t^{-1}]^n \to \mathbb{Z}[t,t^{-1}]^n \to 0.$$

This is impossible, since $\mathbb{Z}$ cannot inject in $\mathbb{Z}[t,t^{-1}]^n$ as a $\mathbb{Z}[t,t^{-1}]$-module. This completes the computation

$$\underline{HF}_i^+(Y_0) \cong \begin{cases} \mathbb{Z}, & \text{if } i \equiv -1/2 \mod 2 \text{ and } i \geq -1/2, \\ L(t)^n & \text{if } i = -3/2, \\ 0 & \text{otherwise.} \end{cases}$$

To compute $\underline{HF}^+(Y_0;\Lambda_\omega)$, we follow the computation in [AP10, Proof of Theorem 1.3]. First note that the above computation is true with coefficients in $\mathbb{F}_2$

$$\underline{HF}_i^+(Y_0;\mathbb{F}_2[t,t^{-1}]) \cong \begin{cases} \mathbb{F}_2, & \text{if } i \equiv -1/2 \mod 2 \text{ and } i \geq -1/2, \\ \mathbb{F}_2[t,t^{-1}]^n & \text{if } i = -3/2, \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\underline{CF}^+(Y_0;\Lambda_\omega) \cong \underline{CF}^+(Y_0;\mathbb{Z}[t,t^{-1}]) \otimes_{\mathbb{Z}[t,t^{-1}]} \Lambda_\omega$$

and $\mathbb{F}_2[t,t^{-1}]$ is a PID, by the universal coefficient theorem, there is the following exact sequence

$$0 \to \underline{HF}^+(Y_0;\mathbb{F}_2[t,t^{-1}]) \otimes_{\mathbb{F}_2[t,t^{-1}]} \Lambda_\omega \to \underline{HF}^+(Y_0;\Lambda_\omega) \to \operatorname{Tor}_1^{\mathbb{F}_2[t,t^{-1}]}(\underline{HF}^+(Y_0;\mathbb{F}_2[t,t^{-1}]),\Lambda) \to 0.$$

As shown in [AP10], $\operatorname{Tor}_1^{\mathbb{F}_2[t,t^{-1}]}(\underline{HF}^+(Y_0;\mathbb{F}_2[t,t^{-1}]),\Lambda) = 0$ and the result follows for $S_0^3(D_-(U,n))$. Note that $D_+(U,-n)$ is the mirror image of $D_-(U,n)$. Therefore, $S_0^3(D_+(U,-n))$ is homeomorphic to $-S_0^3(D_-(U,n)) = -Y_0$. As discussed in [Zem21, Subsetion 12.2], the complex $\underline{CF}^-(-Y_0,\mathfrak{s};\Lambda_\omega)$ is chain homotopic to $\operatorname{Hom}_{\Lambda[U]}(\underline{CF}^-(Y_0,\mathfrak{s};\Lambda_\omega);\Lambda[U])$. By the Universal Coefficient Theorem, there is a split short exact sequence

$$0 \to \operatorname{Ext}^1_{\Lambda[U]}(\underline{HF}^-(Y_0;\Lambda_\omega),\Lambda[U]) \to \underline{HF}^-(-Y_0;\Lambda_\omega) \to \operatorname{Hom}_{\Lambda[U]}(\underline{HF}^-(Y_0;\Lambda_\omega),\Lambda[U]) \to 0.$$

We have

$$\mathrm{Hom}_{\Lambda[U]}(\underline{HF}^-(Y_0;\Lambda_\omega),\Lambda[U]) \cong \mathrm{Hom}_{\Lambda[U]}(\Lambda^n,\Lambda[U]) = 0.$$

If we use the free resolution

$$0 \to \Lambda[U]^n \xrightarrow{U} \Lambda[U]^n \to 0$$

and apply $\mathrm{Hom}_{\Lambda[U]}(-,\Lambda[U])$,

$$0 \leftarrow \Lambda[U]^n \xleftarrow{U} \Lambda[U]^n \leftarrow 0,$$

then $\mathrm{Ext}^1_{\Lambda[U]}(\underline{HF}^-(Y_0;\Lambda_\omega),\Lambda[U])$ is the cohomology at position 1, which is isomorphic to Λ^n. This completes the proof. $\square$

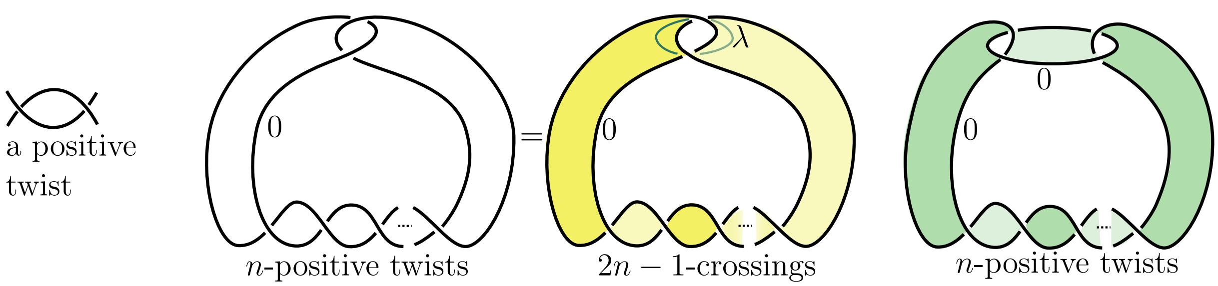

Figure 5.2. Left: the three manifold Y_0 obtained by performing 0-surgery on $D_-(U,n)$. λ is a non-separating curve on the non-separating torus in Y_0. Right: the three manifold obtained by performing 0-surgery on the n-twisted Whitehead link W_n.

Figure 5.2, in the middle, shows a non-separating torus $T \subset Y_0$ which is obtained by capping off the genus one Seifert surface for $D_-(U,n)$, shown in yellow. The zero-framing on λ differs from the framing induced by the Seifert surface by 1. So, performing 0-surgery on lambda has the effect of composing the identification with a Dehn twist on lambda. This results in a manifold $S^3_0(W_n)$ obtained by performing 0-surgery on the n-twisted Whitehead link W_n, $n > 0$, shown in Figure 5.2 on the right. Note that the Kirby diagram for $S^3_0(W_n)$, is obtained by a Kirby move from the Kirby diagram shown in the middle of Figure 5.2 where λ has framing 0.

Corollary 5.2. *Let $S^3_0(W_n)$ be a closed, orientable three manifold obtained by performing 0-surgery on the n-twisted Whitehead link W_n, $n \neq 0$. Suppose that T is the non-separating torus in $S^3_0(W_n)$ obtained by capping off the genus 1 Seifert surface of*

W_n, which is shown in green on the right of Figure 5.3. Let $[\omega]$ be a 2-dimensional cohomology class in $H^2(S^3_0(W_n);\mathbb{R})$ such that $\omega([T]) \neq 0$. Then

$$\underline{HF}(S^3_0(W_n);\Lambda_\omega) \cong \Lambda^{|n|}.$$

Proof. Use Lemma 5.1 and Theorem 1.2. □

Proof of Corollary 1.3. Assume that $S^3_0(W_n)$ and $S^3_0(W_m)$ are related by the excision construction, and $[\omega_i] \in H^2(S^3_0(W_i);\mathbb{R})$, $i = n, m$, are 2-dimensional cohomology classes that satisfy the conditions of Theorem 1.2. Furthermore, we can assume that $\omega_i([T_i]) \neq 0$ where T_i is the non-separating torus in $S^3_0(W_i)$ obtained by capping off the genus 1 Seifert surface of W_i. Now Corollary 5.2 contradicts the result of Theorem 1.2. □

Remark 5.3. That $\underline{HF}(S^3_0(W_n);\Lambda_\omega) \cong \Lambda^{|n|}$ can also be obtained using the techniques used in the proof of Theorem 4.1. See Corollary 2.3 of [AN09].

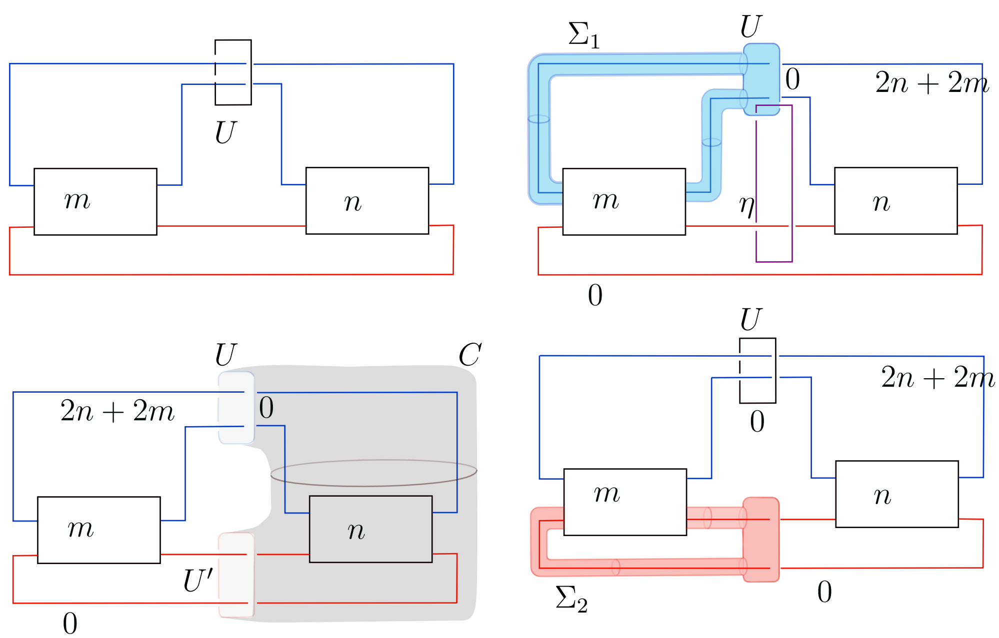

Figure 5.3. Top left: (m,n)-twisted Borromean rings $B(m,n)$. Here, the boxes labeled m and n denote m and n full twists, respectively. Bottom left: a cylinder C which bounds U and U'. Top right and bottom right: two non-separating tori Σ_1 and Σ_2 in $S^3_{0,0,2n+2m}(B(m,n))$. Top right: the curve η intersects Σ_i, $i = 1, 2$, geometrically once and is a meridian for the red curve.

Let $B(m,n)$ denote (m,n)-*twisted Borromean rings* as shown in Figure 5.3 on the top left. $B(m,n)$ is a three component link such that each component is an unknot. In Figure 5.3, the boxes labeled m and n denote m and n full twists, respectively. Therefore, the blue and red components are twisted $m+n$ times. Let U denote the component shown in black in Figure 5.3 on the top left. Note that $B(1,-1)$ represents the Borromean rings. Let $S^3_{0,0,2n+2m}(B(m,n))$ be a manifold obtained by performing 0-surgeries on U and the red component and $2n+2m$-surgery on the blue component of $B(m,n)$. There are two non-separating tori Σ_1 and Σ_2 in $S^3_{0,0,2n+2m}(B(m,n))$ shown in Figure 5.3 on the top right and on the bottom right. Indeed, the unknot U with framing 0 specifies a 2-sphere S. The blue component intersects S geometrically twice with opposite orientations. If we remove a neighborhood of two intersection points in S and connect the two circle boundaries with a tubular neighborhood of the blue arc that connects them, we obtain Σ_1. Let U' be the unknot as shown in Figure 5.3 on the bottom left. There is cylinder C in $S^3_{0,0,2n+2m}(B(m,n))$ such that $\partial C = U \cup U'$ (see Figure 5.3 on the bottom left). Therefore, U' specifies a 2-sphere which can be used to construct the surface Σ_2. Let η be the curve shown in Figure 5.3 on the top left. The curve η intersects Σ_i, $i = 1,2$, geometrically once and is the meridian for the red component.

Lemma 5.4. *$S^3_0(W_n) \amalg S^3_0(W_m)$ is obtained from $S^3_{0,0,2n+2m}(B(m,n))$ by excision along the surfaces $\Sigma_1 \cup \Sigma_2$ shown in Figure 5.3.*

To prove this lemma, we need two auxiliary lemmas. Let K be a knot in a 3-manifold Y and $\nu(K) \cong S^1 \times D^2$ denote a tubular neighborhood of K in Y. An essential simple closed curve $\lambda \subset \partial\nu(K)$ is called a framing for K. In fact, a framing of K determines a new manifold $Y_\lambda(K)$ obtained by gluing a disk to λ in $\partial(\nu(K))$. We say that λ is a Morse framing if $[\lambda] \in H_1(\nu(K))$ is a generator.

Lemma 5.5. *[HSZ24, Lemma 2.7] Let (Y_0, K_0) and (Y_1, K_1) be knots with Morse framings λ_0 and λ_1 respectively. Let*

$$\phi : \partial(\nu(K_0)) \to \partial(\nu(K_1))$$

be a gluing map which identifies the meridian μ_0 with μ_1, and which maps λ_0 to $-\lambda_1$. Then $(Y_0 \setminus \nu(K_0)) \cup_\phi (Y_1 \setminus \nu(K_1))$ is equal to $(Y_0 \# Y_1)_{\lambda_0+\lambda_1}(K_0 \# K_1)$.

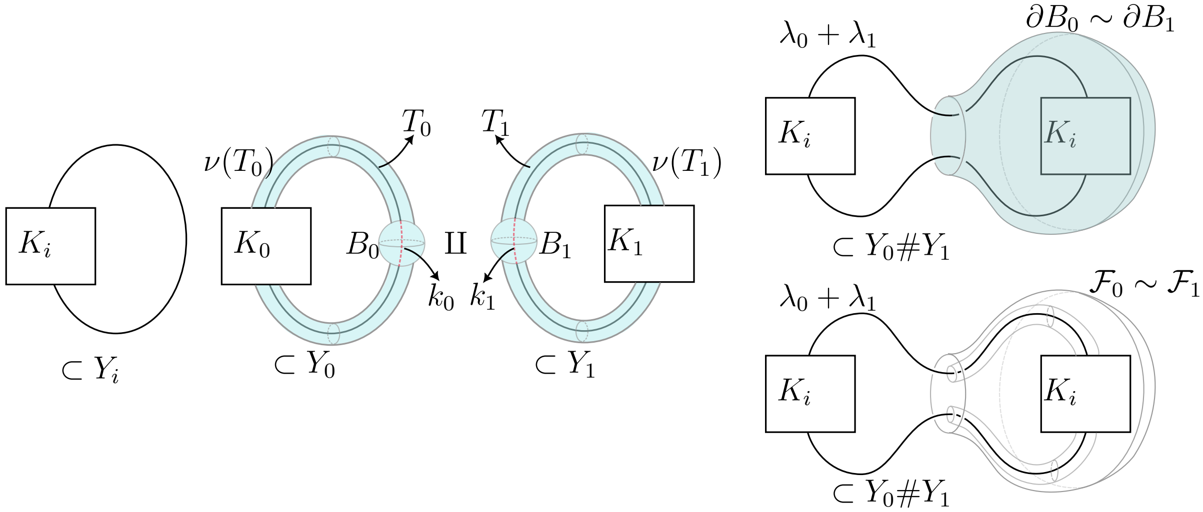

Figure 5.4. Left: a knot K_i in Y_i, $i = 0, 1$. Middle: $\nu(K_i) = B_i \cup \nu(T_i)$. Right-top: $\lambda_0 + \lambda_1$-surgery on $K_0 \# K_1$ in $Y_0 \# Y_1$. Right-bottom: a genus one surface obtained by identifying the surfaces $\mathcal{F}_i = \partial(\nu(T_1) \cup B_i) \subset Y_i$, $i = 0, 1$, by ϕ.

In this lemma, we fix an orientation on λ_i, $i = 0, 1$, and $-\lambda_1$ denotes the curve λ_1 with the orientation reversed. This lemma is proved in [Gor83, Lemma 7.1] and [FM88, Lemma 6.1]. Here, we present a slightly different proof.

Proof. Let $B_i \subset Y_i$, $i = 0, 1$, denote two three balls such that $B_i \cap K_i$, resp. $B_i \cap \lambda_i$ consists of an arc k_i, resp. k_i', $i = 0, 1$. Let $T_i = K_i - k_i$ and $\lambda_i' = \lambda_i - k_i'$. We can write $\nu(K_i) = B_i \cup \nu(T_i)$ (see Figure 5.4 in the middle). We can form $(Y_0 \setminus \nu(K_0)) \cup_\phi (Y_1 \setminus \nu(K_1))$ in two steps. First, remove the interiors of B_i, $i = 0, 1$, and identify the boundary components (this identifies μ_0 with μ_1) then identify λ_0' with $-\lambda_1'$. Then glue the remaining disks on $\partial(\nu(K_i)) \setminus (\mu_i \cup \lambda_i)$, $i = 0, 1$. The first step results in the connected sum $(Y_0 \# Y_1, K_0 \# K_1)$. The second step glues a disk to $\lambda_0 + \lambda_1$. This means performing a $\lambda_0 + \lambda_1$-surgery on $K_0 \# K_1$ in $Y_0 \# Y_1$ (see Figure 5.4 on the right). $\square$

A genus one surface in $(Y_0 \# Y_1)_{\lambda_1 + \lambda_2}(K_0 \# K_1)$ which is obtained by the identification of $\mathcal{F}_0 = \partial(\nu(K_0))$ with $\mathcal{F}_1 = \partial(\nu(K_1))$ via ϕ is shown in Figure 5.4 on the right bottom.

Let K_0 and K_1 be two knots in a 3-manifold Y. We also denote the copies of these knots in $Y \# (S^1 \times S^2)$ by K_0 and K_1. Remove a small arc k_i, with endpoints a_i and b_i, from K_i, then identify a_0 with a_1 and b_0 with b_1 to form a knot K. Let $a = a_0 \sim a_1$ and $b = b_0 \sim b_1$. We form K such that it intersects $\{p\} \times S^2 \subset S^1 \times S^2$ ($p \in S^1$) in a and b with opposite signs. In fact, let k be an arc whose interior is disjoint from K_i, $i = 0, 1$, and has one leg on K_0 and one leg on K_1 such that the arc k intersects $\{p\} \times S^2$ exactly once. Then K is a band sum of K_0 and K_1 using the arc k. Note that by the classical light-bulb theorem, K is independent of the choice of k. We denote K with $K_0 \natural K_1$ (see Figure 5.5 on the right-top). If λ_i is a framing for K_i, there is an induced framing on $K_0 \natural K_1$ denoted by $\lambda_0 \natural \lambda_1$.

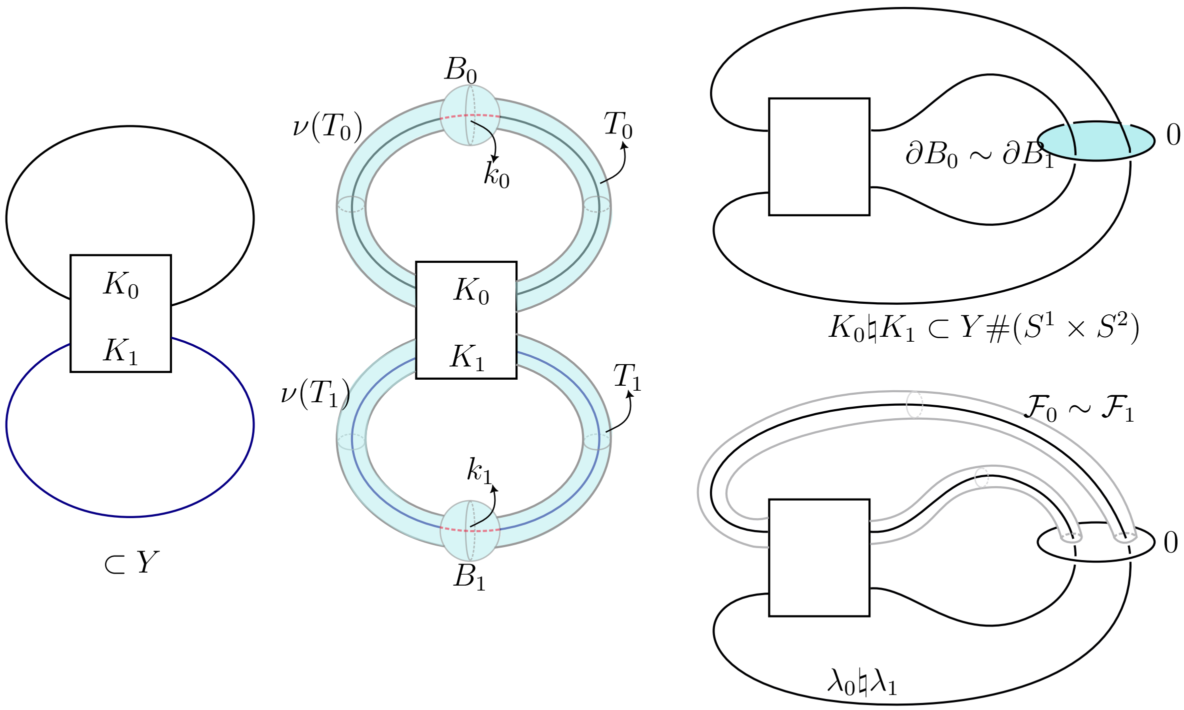

Figure 5.5. Left: two knots K_i, $i = 0, 1$, in Y. Middle: $\nu(K_i) = B_i \cup \nu(T_i)$. Right-top: $K_0 \natural K_1$ in $Y\#S^1 \times S^2$. Right-bottom: a genus one surface in $(Y\#S^1 \times S^2)_{\lambda_0 \natural \lambda_1}(K_0 \natural K_1)$ obtained by identifying the surfaces $\mathcal{F}_i = \partial(\nu(T_1) \cup B_i) \subset Y$, $i = 0, 1$ by ϕ.

Lemma 5.6. *Let $K_0, K_1 \subset Y$ be knots with Morse framings λ_0 and λ_1 respectively. Let*

$$\phi : \partial(\nu(K_0)) \to \partial(\nu(K_1))$$

be a gluing map which identifies the meridian μ_0 with μ_1, and which maps λ_0 to $-\lambda_1$. Then

$$\frac{Y \setminus (\nu(K_0) \cup \nu(K_1))}{x \sim \phi(x)}$$

is equal to $(Y\#S^1 \times S^2)_{\lambda_0 \natural \lambda_1}(K_0 \natural K_1)$.

Proof. Let $B_i \subset Y$, $i = 0, 1$, denote two three balls such that $B_i \cap K_i$, resp. $B_i \cap \lambda_i$ consists of the arc k_i, resp. an arc k_i', $i = 0, 1$. Let $T_i = K_i - k_i$ and $\lambda_i' = \lambda_i - k_i'$. We can write $\nu(K_i) = B_i \cup \nu(T_i)$ (see Figure 5.5 in the middle). We can form

$$\frac{Y \setminus (\nu(K_0) \cup \nu(K_1))}{x \sim \phi(x)}$$

in two steps. First, remove the interiors of B_i, $i = 0, 1$, and identify the boundary components (this identifies μ_0 with μ_1) then identify λ_0' with $-\lambda_1'$. The first step results in $(Y \# S^1 \times S^2, K_0 \natural K_1)$. The second step glues a disk to $\lambda_0 \natural \lambda_1$. This means performing a $\lambda_0 \natural \lambda_1$-surgery on $K_0 \natural K_1$ in $Y \# S^1 \times S^2$ (see Figure 5.5 on the right). □

A genus one surface in $(Y \# S^1 \times S^2)_{\lambda_0 \natural \lambda_1}(K_0 \natural K_1)$ which is obtained by the identification of $\mathcal{F}_0 = \partial(\nu(K_0))$ with $\mathcal{F}_1 = \partial(\nu(K_1))$ via ϕ is shown in Figure 5.5 on the right bottom.

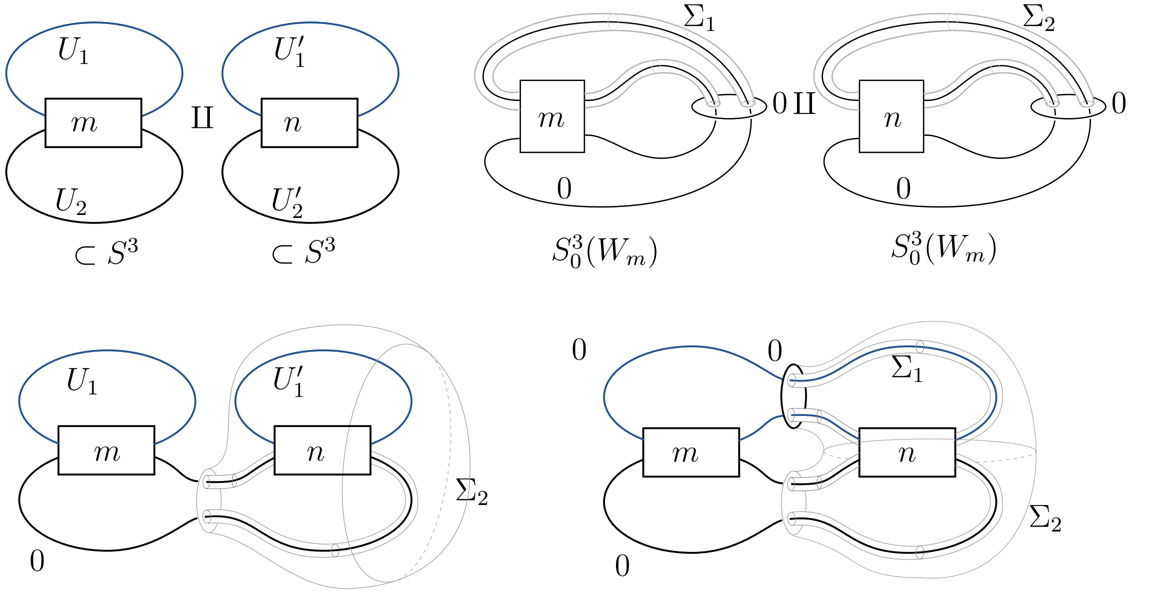

Figure 5.6. Top left: torus links T_m and T_n in two copies of S^3. Here, the boxes labeled m and n denote m and n full twists, respectively. Top right: $S^3_0(W_m) \amalg S^3_0(W_n)$ obtained by applying Lemma 5.6 on the torus links T_m and T_n. Bottom left: the manifold obtained by applying Lemma 5.5 for (S^3, U_2) and (S^3, U_2'). Bottom right: $S^3_{0,0,2n+2m}(B(m, n))$ obtained by applying Lemma 5.6 to the manifold shown on the bottom left.

Proof of Lemma 5.4. Let T_n and T_m denote $(2, 2n)$ and $(2, 2m)$ torus links in two copies of S^3, with components U_i and U_i', $i = 1, 2$, respectively (see Figure 5.6 on the top-left). Let μ_i and γ_i denote the meridian and Seifert longitude of U_i. Consider the Morse framings $\lambda_1 = \gamma_1 + 2m\mu_1$, $\lambda_2 = \gamma_2$ for U_1 and U_2, respectively. Similarly, let μ'_i and γ'_i denote the meridian and Seifert longitude of U'_i. Consider the Morse framings $\lambda'_1 = \gamma'_1 + 2n\mu'_1$, $\lambda'_2 = \gamma'_2$ for U'_1 and U'_2, respectively. If we apply Lemma 5.6 for the components of the torus link T_n (resp. T_m) with the framings λ_i (resp. λ'_i), $i = 1, 2$, we obtain $S^3_0(W_n)$ with the surface Σ_1 (resp. $S^3_0(W_m)$ with the surface Σ_2) (see Figure 5.6 on the top-right). Note that $\lambda_1 \natural \lambda_2$ and $\lambda'_1 \natural \lambda'_2$ correspond to 0-surgeries.

On the other hand, if we apply Lemma 5.5 for (S^3, U_2) and (S^3, U'_2), we obtain the manifold shown in Figure 5.6 on the bottom left with the surface Σ_2. Let U_1 and U'_1 also denote the knots induced by U_1 and U'_1 in the resulting manifold. Finally, if we apply Lemma 5.6 for U_1 and U'_1 in the manifold obtained in the last step, we will obtain $S^3_{0,0,2n+2m}(B(m,n))$ with the surfaces Σ_1 and Σ_2 (see Figure 5.6 on the bottom right).

Therefore, if we cut $S^3_{0,0,2n+2m}(B(m,n))$ along the surfaces Σ_1 and Σ_2, we obtain $(S^3 \setminus N(T_n)) \amalg (S^3 \setminus N(T_m))$ where $N(T_n)$ denotes a neighborhood of T_n. If we reglue the boundary components of $(S^3 \setminus N(T_n)) \amalg (S^3 \setminus N(T_m))$, as explained above, we obtain $S^3_0(W_n) \amalg S^3_0(W_m)$. This shows that $S^3_0(W_n) \amalg S^3_0(W_m)$ is obtained from $S^3_{0,0,2n+2m}(B(m,n))$ by excision along the surfaces Σ_1 and Σ_2. □

Corollary 5.7. *Let* $[\omega] \in H^2(S^3_{0,0,2n+2m}(B(m,n)); \mathbb{R})$ *be* $\lambda \mathrm{PD}[\eta]$, *where* η *is the curve in Figure 5.3 on the top right and* $\lambda \in \mathbb{R} \setminus \{0\}$. *Then*

$$\underline{HF}(S^3_{0,0,2n+2m}(B(m,n)); \Lambda_\omega) \cong \Lambda^{|mn|}.$$

Proof. Let Σ_i, $i = 1, 2$, denote the surface in $S^3_{0,0,2n+2m}(B(m,n))$ shown in Figure 5.3 on the top right and on the bottom right. It is clear that $[\Sigma_i] \neq 0$, $i = 1, 2$, and $F = \Sigma_1 \cup \Sigma_2$ is separating in $S^3_{0,0,2n+2m}(B(m,n))$. By Lemma 5.4, if we perform excision in $S^3_{0,0,2n+2m}(B(m,n))$ along F, we obtain $S^3_0(W_m) \amalg S^3_0(W_n)$. By Theorem 1.2,

$$\underline{HF}(S^3_{0,0,2n+2m}(B(m,n)); \Lambda_\omega) \cong \underline{HF}(S^3_0(W_m) \amalg S^3_0(W_n); \Lambda_{\omega'}), \tag{5.5}$$

where $[\omega'] \in H^2(S^3_0(W_m) \amalg S^3_0(W_n); \mathbb{R})$ satisfies the conditions of Theorem 1.2. Note that

$$\underline{CF}^-(S^3_0(W_m) \amalg S^3_0(W_n); \Lambda_{\omega'}) = \underline{CF}^-(S^3_0(W_m); \Lambda_{\omega'_1}) \otimes_\Lambda \underline{CF}^-(S^3_0(W_n); \Lambda_{\omega'_2}),$$

where $\omega' = \omega'_1 \oplus \omega'_2$, $\omega'_1 \in H^2(S^3_0(W_m); \mathbb{R})$, $\omega'_2 \in H^2(S^3_0(W_n); \mathbb{R})$. Since Λ is a field,

$$\underline{HF}^-(S^3_0(W_m) \amalg S^3_0(W_n); \Lambda_{\omega'}) = \underline{HF}^-(S^3_0(W_m); \Lambda_{\omega'_1}) \otimes_\Lambda \underline{HF}^-(S^3_0(W_n); \Lambda_{\omega'_2}). \tag{5.6}$$

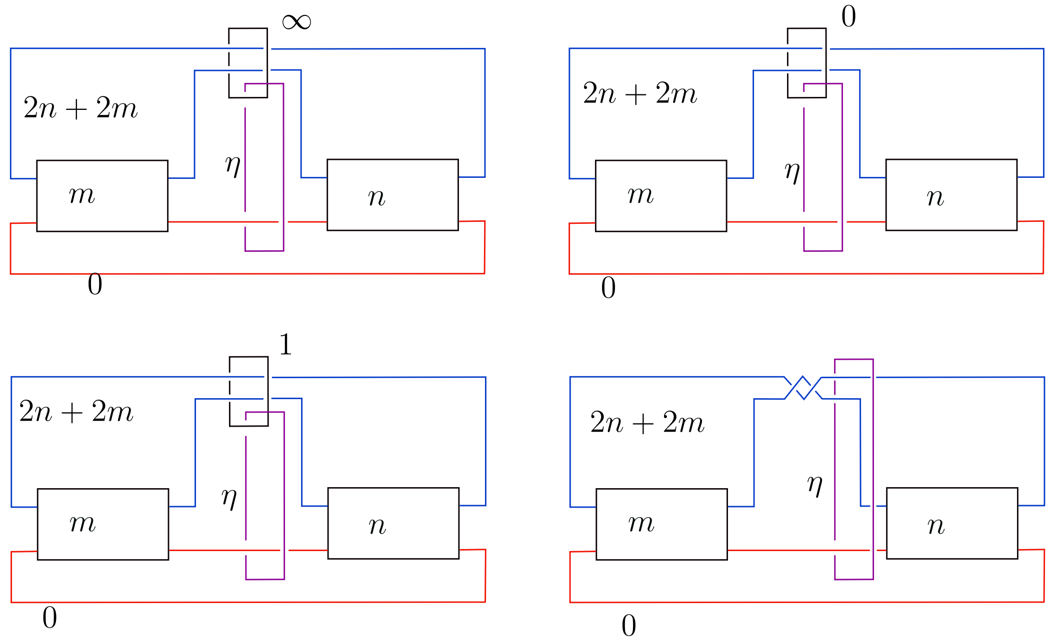

Figure 5.7. Top right: $S^3_{0,0,2n+2m}(B(m,n))$. Bottom: $S^3_{0,2n+2m}(C(m,1,n))$. The boxes labeled m and n denote m and n full twists, respectively.

The result follows from (5.5), (5.6), Corollary 5.2, and the fact that $\underline{HF}(Y,\mathfrak{s};\Lambda_\omega) \cong \underline{HF}^-(Y,\mathfrak{s};\Lambda_\omega)$, when $c_1(\mathfrak{s})$ is torsion and $[\omega] \neq 0$. $\square$

5.2 0-Surgery on 2-bridge Links

Let K be a framed knot in a 3–manifold Y with framing λ and meridian μ. Given an integer r, let $Y_r(K)$ denote the 3–manifold obtained from Y by performing Dehn surgery along the knot K with framing $\lambda + r\mu$. Let $N(K)$ denote a small tubular neighborhood of the knot K and $\eta \subset Y - N(K)$ be a closed curve in the knot complement. Then for any integer r, $\eta \subset Y - N(K) \subset Y_r(K)$ is a closed curve in the surgered manifold $Y_r(K)$. We denote its Poincare dual by $[\omega_r] \in H^2(Y_r;\mathbb{R})$. There is a surgery exact sequence for ω-twisted Floer homology [AP10, Section 3]

$$\cdots \to \underline{HF}^+(Y;\Lambda_\omega) \to \underline{HF}^+(Y_0;\Lambda_{\omega_0}) \to \underline{HF}^+(Y_1;\Lambda_{\omega_1}) \to \cdots, \tag{5.7}$$

where $[\omega] = \mathrm{PD}[\eta]$.

Proof of Corollary 1.4. Let Y be the manifold obtained by performing 0-surgery on the red component and $2n+2m$-surgery on the blue component of $B(m,n)$ (see Figure

5.3 on the top). Let K denote the knot induced from U in Y. Note that

$$Y = \begin{cases} \#^2(S^1 \times S^2) & \text{if } m = -n, \\ S^3 & \text{if } |m+n| = 1, \end{cases}$$

$Y_0 = Y_0(K) = S^3_{0,0,2n+2m}(B(m,n))$ and $Y_1 = Y_1(K) = S^3_{0,2n+2m}(C(m,\pm 1,n))$ (for appropriate choices of the framing). Note that the induced curve η in Y_0 is such that $\omega_0 = \lambda \mathrm{PD}[\eta]$ satisfies the condition in Corollary 5.7 (see Figure 5.7 on the top left). When $m = -n$, let S denote a 2-sphere in Y obtained by capping off the Seifert disk of the 0-framed component in red. In this case, the induced 2-dimensional cohomology class $[\omega']$ in Y, is such that $\omega'([S]) \neq 0$ (see Figure 5.7 on the top left). Therefore, $\underline{HF}(Y;\Lambda_{\omega'}) = 0$ (see [AP10, Subsection 2.2]). Using Corollary 5.7 and the long exact sequence in (5.7), the result follows.

When $|m+n| = 1$, it follows from algebra that the map

$$\Lambda[U^{-1}] \cong \underline{HF}^+(Y;\Lambda_\omega) \to \underline{HF}^+(Y_0;\Lambda_{\omega_0}) \cong \Lambda^{|mn|}$$

in (5.7) is zero and this proves the result. □

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