A Jack Approach to the Equivariant Edge and Vertex Measures

by

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DISSERTATION ABSTRACT

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Doctor of Philosophy in Mathematics

Title: A Jack Approach to the Equivariant Edge and Vertex Measures

This work provides a description of several results in the field of Jack symmetric functions. We first provide a combinatorial explanation of a well known geometric concept, the equivariant edge measure of Donaldson–Thomas theory, and examine a combinatorial argument of Haiman which encodes some data about Hilbert schemes. We continue by discussing generalizing some of these notions to the equivariant vertex measure. In particular, we show that away from the Calabi-Yau specialization, the equivariant vertex measure is not a probability measure on plane partitions.

Next we describe a connection between the Pieri rule for Jack functions and some work of Ciocan-Fontanine, Konvalinka, and Pak, as well as ideas about how one might extend such work. Work in this direction has given rise to probability distributions on Young diagrams/tableaux.

This dissertation includes previously published co-authored material.

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For Boba and Chai

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CHAPTER 1

INTRODUCTION

This work explores several combinatorial notions relating to and using Jack symmetric functions. Much of this work is probabilistically flavored. In addition to the new mathematics featured in this thesis, we present some conjectural work and expository content.

The main accomplishments set forth in what follows include a combinatorial explanation of the edge term of the equivariant vertex measure of Donaldson-Thomas theory as well as preliminary results regarding the equivariant vertex measure itself. We describe a connection between a combinatorial notion of Haiman and Jack combinatorics, then use this to expound on the three-dimensional case of the completed problem, describing how our two-dimensional methods can be generalized and for what applications they are functional.

We provide some ostensibly new Jack-theoretic probability distributions on Young objects which we believe correspond to hook walks on Young diagrams. The hook walks are conjectural; the probability distributions themselves are proven via other methods.

1.1 Organization and Contributions

The remainder of this chapter is devoted to an overview of the mathematics discussed in this work.

Chapter 2 provides background information about the concepts and notation used in the remainder of this document. It begins by introducing integer partitions and Young objects, as well as hook lengths and goes on to provide the framework for the Jack context. It also presents structural propositions which we make great use of in proceeding chapters.

Chapter 3 provides a combinatorial description of the equivariant edge measure from Donaldson–Thomas theory described in [Mau+06a] and [Mau+06b] using Jack symmetric functions. We introduce an object we call the *corner polynomial* of a Young diagram which describes the locations on the diagram at which a cell may be added or removed. This chapter's conclusion is that the equivariant vertex measure is (up to conventions) the Jack Plancherel measure. This work was originally written in [PY25] and is joint work with B. Young. A version is now published in

Séminaire Lotharingien de Combinatoire as part of the Proceedings of The Formal Power Series and Algebraic Combinatorics conference proceedings.

Chapter 3 serves as a warm-up of sorts for Chapter 5, which discusses preliminary work on the three-dimensional version of the same problem. In particular, we expound on notions which can be generalized from the 2D case to the 3D case by setting forth a definition of a 3D corner polynomial and using it to concisely describe the ratio of equivariant vertex measures of certain similar plane partitions. With these tools in hand, we show that outside of a certain special case, the sign of equivariant vertex measure with real parameters is not predictable based only on the size of a given plane partition. Appendix B describes the Markov chain Monte Carlo method we used to initially conjecture these ideas.

Between these two stories, Chapter 4 provides an aside in which we discuss another geometric notion which can be described combinatorially. In [Hai98], Haiman walks the reader through an argument showing that the Hilbert scheme of n points in the plane is smooth of dimension $2n$ as an introduction to the following work on Catalan generalization. The objects he introduces are related to Jack Plancherel measure through the generating function for Hilbert schemes. A direct combinatorial connection is desirable, and Chapter 4 fleshes out a such a relationship by demonstrating a bijection and branching rule.

In Chapter 6, we begin by introducing a connection between a specialization of the Pieri rule for Jack functions and a work of Ciocan-Fontanine, Konvalinka, and Pak [CKP10]. The bulk of the chapter is spent presentating a number of Jack flavored probability distributions on Young diagrams and tableaux. These stem from identities detailed in Appendix A.

1.2 Future Directions

In Chapter 5, we show that real-valued parameters generally fail to make the equivariant vertex measure a probability measure. We focused on parameter sets in $\mathbb{R}^3$ so as to demonstrate the lack of an analogy with the edge measure and Jack Plancherel measure connection in Chapter 3. Our work in Chapter 5 provides a $\mathbb{R}^3$-geometric based strategy to understand this problem using linear algebra of (hyper)planes and the angles between them. All of this said, to our knowledge, it is possible that with complex-valued parameters we are able to avoid this issue essentially entirely. The methods we provide herein will not suffice on their own, so we

assigning this as a future avenue to explore.

In addition to the connection illuminated in Chapter 6, a direction for future work is elucidated by describing what a pullback of a Pieri rule generalization and the (WHL) of [CKP10] would look like. Moreover, potential hook walks are outlined as conjectures for alternate proofs for the presented identities.

CHAPTER 2

BASIC NOTIONS

This chapter explains necessary background information and set up to introduce the main results in the next section. The content was furnished from [PY25], a joint work with myself and B. Young. We follow the exceptional exposition of Stanley from [Sta89]; more information about the Jack symmetric functions can be found in [Mac98, Chapter 10].

2.1 Partitions, Young Diagrams, and Tableaux

A *partition* λ is a vector $(\lambda_1, \lambda_2, \lambda_3, \dots)$ with weakly decreasing natural number entries only finitely many of which are nonzero. The *size* of λ is

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots . \tag{2.1}$$

If $|\lambda| = n$ then we say that λ *partitions* n or λ is a partition of n, denoted $\lambda \vdash n$. Each of the λ_i is called a *part* of λ. The number of nonzero parts of λ is said to be the *length* of λ, denoted $\mathrm{len}(\lambda)$. In this thesis, we identify a partition λ with its *Young* (or *Ferrers*) *diagram* in English notation with zero-indexed matrix coordinates, (row, column):

$$\lambda = \{(i, j) \mid 0 \le i \le \mathrm{len}(\lambda) - 1,\ 0 \le j \le \lambda_i - 1\}. \tag{2.2}$$

The Young diagram for the partition $\lambda = (3, 3, 1)$ is shown in Figure 2.1. This diagram has *cells* (or *boxes*): $\{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0)\}$.

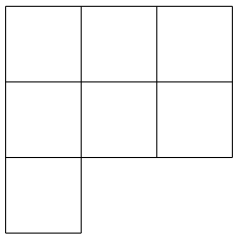


Figure 2.1. The diagram for $\lambda = (3, 3, 1)$.

Given two partitions λ and μ, we write $\mu \subseteq \lambda$ if the $\mu_i \le \lambda_i$ for all i. In other words, $\mu \subseteq \lambda$ if the diagram of μ fits inside the diagram of λ when they are overlaid

with their upper left cells aligned. The partially ordered set defined by the relation $\subseteq$ is called *Young's Lattice*. The *conjugate* λ' of a partition λ is the partition with diagram

$$\lambda = \{(j, i) \mid 0 \leq i \leq \text{len}(\lambda) - 1, 0 \leq j \leq \lambda_i - 1\}, \tag{2.3}$$

i.e., λ' is the partition obtained by reflecting λ across the northwest to southeast diagonal.

A *standard Young tableau* is a filling of each box of a shape λ with the integers in the set $\{1, 2, \ldots, |\lambda|\}$ such that each row increases rightward and each column increases downward. See Figure 2.2.

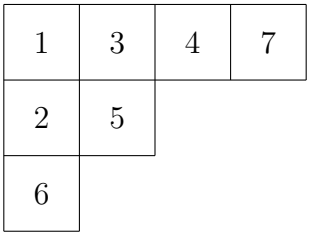


Figure 2.2. A standard Young tableau of shape $\lambda = (4, 2, 1)$.

2.2 Hook Lengths and the Plancherel Measure

With this background, we can now define the *arm*, *leg*, *hook*, and *hook length* of a cell in a Young diagram. Before providing a rigorous definition, we give an intuitive explanation: The hook length of a cell is the number of cells directly to its right (the arm) and below it (the leg) plus one. See Figure 2.3.

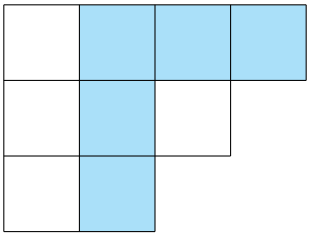


Figure 2.3. The blue boxes denote the hook of the cell $(0, 1)$ in $\lambda = (4, 3, 2)$.

More rigorously, given a cell (i, j) in λ, the *arm* and *leg* of (i, j) are the sets

$$\mathrm{arm}_\lambda(i,j) = \{(i,k) |\ j \le k \le \lambda_i\} \quad \text{and} \quad \mathrm{leg}_\lambda(i,j) = \{(k,j) |\ i \le k \le \lambda'_j\}. \tag{2.4}$$

Moreover, we define the *arm length* and *leg length* of (i,j) to be

$$a_\lambda((i,j)) = |\mathrm{arm}_\lambda(i,j)| = \lambda_i - j, \qquad \ell_\lambda((i,j)) = |\mathrm{leg}_\lambda(i,j)| = \lambda'_j - i \tag{2.5}$$

respectively.

The *hook* of the cell (i,j) in λ is the union of the singleton set $\{(i,j)\}$, the arm of (i,j), and leg of (i,j).

$$\mathrm{hook}_\lambda(i,j) = \{(i,j)\} \cup \{(i,k) |\ j \le k \le \lambda_i\} \cup \{(k,j) |\ i \le k \le \lambda'_j\} \tag{2.6}$$

The *hook length* of a cell $\square \in \lambda$ is

$$h_\lambda(\square) = |\mathrm{hook}_\lambda(\square)| = a_\lambda(\square) + \ell_\lambda(\square) + 1. \tag{2.7}$$

The partition λ may be suppressed from the notation for these notions if it is clear from context.

The product of all hook lengths for a shape λ is denoted H_λ. In other words,

$$H_\lambda = \prod_{\square \in \lambda} h_\lambda(\square). \tag{2.8}$$

Perhaps surprisingly, the number of standard Young tableaux of a given shape λ can be procured via a hook length computation.

Theorem 2.1 (Frame, Robinson, Thrall [FRT54])**.** *The number of standard Young tableaux of shape λ of size n is*

$$f^\lambda = \frac{n!}{H_\lambda}. \tag{2.9}$$

Moreover, the sum of the squares of these counts over all shapes λ of size n is $n!$.

Theorem 2.2 (Frobenius [Fro00], Young [You02])**.**

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n! \tag{2.10}$$

This theorem essentially restates the fact that the sum of the squares of the dimensions of the irreducible representations of a finite group G over an algebraically closed field is the size of G for the special case that G is the symmetric group on n elements, S_n.

Combining Theorem 2.1 and Theorem 2.2 yields a probability distribution on partitions of n.

Theorem 2.3 ([BO05])**.**

$$1 = \sum_{\lambda \vdash n} \frac{n!}{H_\lambda^2} \tag{2.11}$$

This distribution is called the Plancherel measure.

2.3 Symmetric Functions

We follow the conventions of Richard Stanley in [Sta89] and [Sta99] for various bases of Λ, the algebra of symmetric functions. Specifically, the monomial symmetric function corresponding to λ is

$$m_\lambda = \sum_\alpha x^\alpha \tag{2.12}$$

where $\alpha = (\alpha_1, \alpha_2, \dots)$ ranges over all permutations of the vector $\lambda = (\lambda_1, \lambda_2, \dots)$. The power sum symmetric function corresponding to λ is

$$p_\lambda = p_{\lambda_1} \cdot p_{\lambda_2} \cdot \dots \cdot p_{\lambda_{\mathrm{len}(\lambda)}} \tag{2.13}$$

where

$$p_i = m_i. \tag{2.14}$$

Lastly, the Schur symmetric function corresponding to λ is denoted s_λ. There is an involution $\omega : \Lambda \to \Lambda$ on the algebra of symmetric functions which sends s_λ to $s_{\lambda'}$. The Schur symmetric functions and ω are described in detail in [Sta99, Sections 7.6, 7.10, and 7.15].

2.4 The Jack Context

Jack functions are homogeneous symmetric functions in $x_1, x_2, x_3, \dots; t$ indexed by integer partitions. We will use J_λ for the (integral form) Jack polynomial corresponding to the partition λ [Ale24]. The variable t is called the Jack parameter.

The set of Jack symmetric functions form a basis for $\Lambda \otimes \mathbb{Q}(t)$. Moreover, Jack functions "sit between" Macdonald functions and Schur functions in terms of generality. Figure 2.4 shows the relationship between Macdonald functions, Jack functions, and Schur functions. The scalar C is an explicit rational function in t that depends on λ.

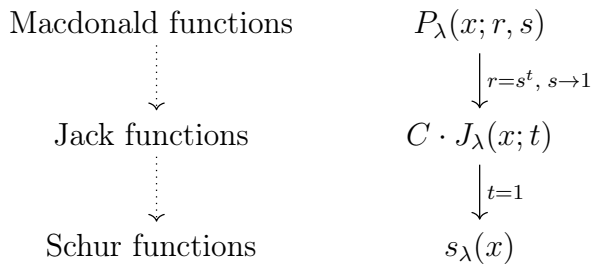


Figure 2.4. Macdonald functions, Jack functions, and Schur functions.

Jack symmetric functions are orthogonal with respect to the scalar product $\langle -, - \rangle$ on $\Lambda \otimes \mathbb{Q}(t)$ given by

$$\langle p_\lambda, p_\mu \rangle_t = \delta_{\lambda,\mu} z_\lambda t^{\mathrm{len}(\lambda)}$$

where

$$z_\lambda = 1^{n_1(\lambda)} \cdot 2^{n_2(\lambda)} \cdot 3^{n_3(\lambda)} \cdot \cdots \cdot n_1(\lambda)! \cdot n_2(\lambda)! \cdot n_3(\lambda)! \cdot \cdots \tag{2.15}$$

and $n_i(\lambda)$ is the number of parts of λ equal to i.

Note that when $t = 1$, this is the standard scalar product on symmetric functions. The notation $j_\lambda := \langle J_\lambda, J_\lambda \rangle_t$ is used for the pairing of a Jack function with itself.

Definition 2.4 ([Sta89], page 77)**.** The Jack symmetric functions are uniquely characterized by the following three conditions.

(1) Orthogonality: $\langle J_\lambda, J_\mu \rangle_t = 0$ when $\lambda \neq \mu$ where the scalar product is defined above.

(2) Triangularity: Select coefficients $v_{\lambda\mu}(t)$ so that

$$J_\lambda = \sum_\mu v_{\lambda\mu}(t) m_\mu.$$

Then $v_{\lambda\mu}(t) = 0$ unless $\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i$ for all i.

(3) Normalization: If $|\lambda| = n$, then the coefficient $v_{\lambda 1^n}$ of $x_1 x_2 x_3 \cdots x_n$ of J_n is $n!$.

There is a Jack generalization of the involution $\omega : \Lambda \to \Lambda$ which we call the Jack involution and write

$$\omega_t : \Lambda \otimes \mathbb{Q}(t) \to \Lambda \otimes \mathbb{Q}(t). \tag{2.16}$$

This involution is defined by

$$\omega_t(p_i) = (-1)^{i-1} t p_i \tag{2.17}$$

for $i \in \mathbb{Z}$ [Mac98, Chapter VI, Section 10].

There are two usual t-generalizations of the notion of hook length.

Definition 2.5 ([Sta89], page 95)**.** The upper and lower hook lengths of a cell $\square \in \lambda$ are given by

$$h^*_\lambda(\square) = t(a(\square) + 1) + \ell(\square),$$
$$h^\lambda_*(\square) = t(a(\square)) + 1 + \ell(\square)$$

respectively. Moreover, the product of all of the upper hook lengths in λ is given by

$$H^*_\lambda := \prod_{\square \in \lambda} h^*_\lambda(\square)$$

and the product of all lower hook lengths in λ is given by

$$H^\lambda_* := \prod_{\square \in \lambda} h^\lambda_*(\square).$$

In a sense, this is just the standard hook length except that arm boxes are worth t instead of 1. The upper and lower hook lengths differ in what weight we assign the pivot box: the upper hook length assigns the pivot box t and the lower hook length assigns the pivot box 1.

Definition 2.6 ([Ker00], [CDM23])**.** The Jack Plancherel measure is a probability measure on partitions of n given by

$$\mathbb{P}(\lambda) = \frac{n! \cdot t^n}{H^*_\lambda H^\lambda_*}. \tag{2.18}$$

Definition 2.7. We write

$$w_{\text{Jack}}(\lambda) := \prod_{\square \in \lambda} \frac{1}{h^*_\lambda(\square) h^\lambda_*(\square)} = \frac{1}{H^*_\lambda H^\lambda_*}, \tag{2.19}$$

a scaled version of the Jack Plancherel measure.

Much like for Schur symmetric functions, there is a Pieri rule for Jack symmetric functions ([Sta89], page 98). For the purposes of this document, it suffices to present just a special case of the Jack Pieri rule.

Corollary 2.8 (Pieri Rule for Jack Symmetric Functions, One Box Version)**.** *Suppose $\lambda \vdash n$, $\mu \vdash (n-1)$, and $\mu \subseteq \lambda$. Then*

$$\langle J_\mu J_1, J_\lambda \rangle_t = t \cdot j_\mu \cdot \prod_{\square \in \text{Row}(\lambda/\mu)} \frac{h^*_\lambda(\square)}{h^*_\mu(\square)} \prod_{\square \in \text{Col}(\lambda/\mu)} \frac{h^\lambda_*(\square)}{h^\mu_*(\square)} \tag{2.20}$$

where $\text{Col}(\lambda/\mu)$ *(resp.* $\text{Row}(\lambda/\mu)$*) is the set of boxes that appear in the same column (resp. row) of λ as the corner box λ/μ excluding the box λ/μ itself.*

CHAPTER 3

THE EQUIVARIANT EDGE MEASURE

The work in this chapter is joint work with Young and appears in [PY25]. We credit Maciej Dolega for supplying the main idea in the proof of Theorem 3.2.

3.1 Introduction

We study a measure on integer partitions called the *equivariant edge measure*: a measure on partitions which arises implicitly in the edge term in the localization computation of the Donaldson–Thomas invariants of a toric threefold. We combinatorially show that the equivariant edge measure is, up to choices of convention, equal to the Jack Plancherel measure. Our work is motivated by a problem in plane partition combinatorics that is well-known to enumerative geometers, but perhaps less well known to combinatorialists: the *equivariant vertex measure*. We first describe this motivating problem combinatorially.

3.1.1 Motivation: The Equivariant Vertex Measure

In their 2006 papers ([Mau+06a; Mau+06b]), Maulik, Nekrasov, Okounkov, and Pandharipande (hereafter, MNOP) compute the Donaldson–Thomas partition function of a threefold with a torus action; they use Atiyah–Bott localization to reduce the computation to the torus fixed loci of the action. One result of their calculations is an exotic measure w on plane partitions, called the *equivariant vertex measure*. This measure has has three parameters u, v, w coming from the torus action. The equvariant vertex measure of a plane partition π is a certain rational function in u, v, w, defined as follows: first, we temporarily choose 3 other formal parameters, r, s, t. We write

$$Q = \sum_{(i,j,k)\in\pi} r^i s^j t^k \qquad \text{and} \qquad \overline{Q} = \sum_{(i,j,k)\in\pi} r^{-i} s^{-j} t^{-k},$$

where both sums are over the boxes in the 3D Young diagram corresponding to π. Next, write

$$F = Q - \frac{\overline{Q}}{rst} + Q\overline{Q}\frac{(1-r)(1-s)(1-t)}{rst} =: \sum_{i,j,k} c_{ijk} r^i s^j t^k.$$

One checks that F is a Laurent polynomial with no constant term, so it makes sense to "swap the roles of addition and multiplication" in this formula. As far as we understand the geometry, the expression for F appears in the computation of the character of the torus action, and the "swap" happens in the process of the localization calculation of the Donaldson–Thomas invariants from this character; however, for the purposes of this note, the swap is a purely formal operation. We may finally define the *equivariant vertex measure*:

$$\mathsf{w}(\pi) = \prod_{i,j,k} (iu + jv + kw)^{-c_{ijk}}. \tag{3.1}$$

One of the results of [Mau+06b] is a closed form for the partition function of the equivariant vertex measure. Namely, let $|\pi|$ denote the number of boxes in the 3D Young diagram for π. Then

$$Z := \sum_{\pi} \mathsf{w}(\pi) q^{|\pi|} = M(q)^{-\frac{(u+v)(v+w)(w+u)}{uvw}}, \qquad \text{where} \qquad M(q) = \prod_{i \geq 1} \left(\frac{1}{1-q^i} \right)^i \tag{3.2}$$

is Macmahon's generating function for plane partitions [Mac01]. Note also that our notation differs from that in [Mau+06a; Mau+06b]: we use different variable names, and we use q in place of $-q$.

Although the definition of $\mathsf{w}(\pi)$ is completely combinatorial, the proof of the formula for Z is intricate and geometric. Such a closed formula as (3.2), in principle, puts w into the class of *exactly solved* models in statistical mechanics. One might expect that $\mathsf{w}(\pi)/Z$ is a *probability* measure on plane partitions π but in Chapter 5 we show that this is almost never true. However, there is a case in which we have significant information.

Definition 3.1. We call the case in which $u + v + w = 0$ the *Calabi-Yau specialization.*

In the Calabi-Yau specialization everything simplifies directly: we have $|\mathsf{w}(\pi)| = 1$ and $Z = M(q)$, recovering Macmahon's formula; random plane partitions under this measure have been extensively studied by many authors (see for instance [KOS06; OR03] for this precise problem, or [Gor21] for a survey).

The remainder of this chapter is dedicated to the study of a warm-up problem of sorts one dimension down.

3.1.2 The Equivariant Edge Measure

Having introduced $\mathsf{w}(\pi)$ and proclaimed it to be hard to study, we now turn to a two-dimensional version of this problem. The subject of this chapter is the analogue of the equivariant vertex measure for ordinary partitions which implicitly arises in [Mau+06a]. More specifically, it arises in the "edge term", coming from a torus-invariant *line* in the localization calculation. Given a partition λ, as in [Mau+06a, Section 4.8] define generating functions

$$Q(\lambda) = \sum_{(i,j)\in\lambda} r^i s^j \qquad \text{and} \qquad \overline{Q}(\lambda) = \sum_{(i,j)\in\lambda} r^{-i} s^{-j} \tag{3.3}$$

where the sums are taken over the coordinates of all cells in λ. Then, following [Mau+06a, Equation (12)], define

$$F(\lambda) = -Q(\lambda) - \frac{\overline{Q}(\lambda)}{rs} + \frac{Q(\lambda)\overline{Q}(\lambda)(1-r)(1-s)}{rs}. \tag{3.4}$$

We then swap the role of addition and multiplication much as before, though for convenience in stating the main result, we include a minus sign on v, and omit one in the exponent (as compared with the definition of w). Given an index set A and a Laurent polynomial $G = \sum_{(i,j)\in A} c_{i,j} r^i s^j$ in the variables r and s with no constant term, we define the swap operation as follows:

$$\operatorname{swap}(G) = \operatorname{swap}\left(\sum_{(i,j)\in A} c_{i,j} r^i s^j\right) = \prod_{(i,j)\in A} (iu - jv)^{c_{i,j}} \tag{3.5}$$

We then define the *equivariant edge measure* w_{MNOP} to be the swap operation applied to $F(\lambda)$:

$$w_{\mathrm{MNOP}}(\lambda) := \operatorname{swap}(F(\lambda)). \tag{3.6}$$

3.1.3 The Jack Plancherel Measure

We now introduce a second, *a priori different*, measure on Young diagrams. The *Jack Plancherel measure* was first studied in [Ker00]; it is a probability measure on partitions of n, defined by

$$w_{\mathrm{Jack}}(\lambda) := \frac{1}{\prod_{(i,j)\in\lambda} h^*(i,j) h_*(i,j)}. \tag{3.7}$$

The upper and lower hook lengths h^* and h_* are in turn defined carefully in Section 2.1. They are two perturbations of the standard hook length of a cell in a Young diagram, and they depend on a single parameter β (which we homogenize using two parameters u, v).

The Jack Plancherel measure is described in [BO05] as a natural perturbation of ordinary Plancherel measure. Okounkov [Oko06] also advanced the viewpoint that the Plancherel measure is a close analogue of the Gausiann Unitary ensemble from random matrix theory and shares many of the same universal asymptotics; in this framework the Jack Plancherel measure should have many of the universal asymptotics of the random-matrix-theoretic β ensembles. Indeed, the bulk asymptotics were later computed in [DF16] and the edge asymptotics in [GH19]. See also [BD23] for a recent interpretation of Jack character values (of which the Jack Plancherel measure is one) in terms of enumerations of maps on surfaces.

3.1.4 Main Result and Outline

Our main result is that the equivariant edge measure w_{MNOP} is the Jack Plancherel measure.

Theorem 3.2. *The Jack Plancherel measure of a partition λ is the same as the equivariant edge measure of λ up to a sign, i.e.*

$$w_{\mathrm{Jack}}(\lambda) = -w_{\mathrm{MNOP}}(\lambda). \tag{3.8}$$

It is no surprise to see the Jack Plancherel measure arising in enumerative geometry. For instance, Okounkov [Oko06] described how the Jack Plancherel measure appears in Seiberg–Witten theory, and in the study of the Hilbert scheme of points in the plane. The latter relationship is surely closely related to our work, since this Hilbert scheme appears also as a fundamental object in the computation of the Donaldson–Thomas invariants in [Mau+06a; Mau+06b]. Indeed, it would not entirely shock us to learn that our result is known to geometers; however, we were not able to find a clear statement or a proof in the literature. Moreover, as both sides of Theorem 3.2 may be defined completely combinatorially, one would like a combinatorial proof. We provide such a proof here.

In Section 3.2, we provide background information on partitions and hook lengths as well as the necessary components of the work of MNOP in [Mau+06a]

and [Mau+06b] and full definition of our "swap" operation, the equivariant vertex measure, and the Jack Plancherel measure. Our main results appear in Section 3.3, in which we provide ratios of measures for growing a partition by one box in both the Jack Plancherel measure and the equivariant edge measure. We then show that the ratios of measures are equal. Routine induction reveals that the equivariant edge measure is the Jack Plancherel measure up to a sign.

3.2 Background Information and Notation

For a partition λ define the u, v *upper* and *lower hook lengths* as

$$h^*_\lambda(\square) = u \cdot \ell(\square) + v(a(\square) + 1)$$
$$h^\lambda_*(\square) = u(1 + \ell(\square)) + v \cdot a(\square)$$

respectively. Note that both the u, v upper and lower hook lengths specialize to the standard hook length on Young diagrams when u and v are both 1. Indeed, this can be viewed as a homogenization of the Jack hook lengths described in Section 2.4. Note that the Jack parameter t from Chapter 1 is just v/u. For the remainder of this chapter, we use the homogenized hook lengths.

$3v+u$ $2v+2u$	$2v+u$ $v+2u$	v u
$2v$ $v+u$	v u	

Figure 3.1. The upper and lower hook lengths of each box in $\lambda = (3, 2)$.

The Jack Plancherel measure $w_{\text{Jack}}(\mu)$ (as defined in (3.7)) of the partition μ in Figure 3.1 is the product of the inverses of each of the expressions printed in the cells of μ.

3.2.1 Corners

We say a cell (i, j) in a partition λ is an *inside corner* of λ if neither $(i + 1, j)$ nor $(i, j + 1)$ is in λ. We also add two artificial inside corners with coordinates

$(-1, \infty)$ and $(\infty, -1)$. Similarly, we say a cell $(i, j) \notin \lambda$ is an *outside corner* of λ if any of the following hold:

- both of $(i-1, j)$ and $(i, j-1)$ are in λ
- $(i, j) = (1, \lambda_1 + 1)$
- $(i, j) = (\lambda'_1 + 1, 1)$.

Label the inside corners of λ from the bottom left to the top right as

$$(\rho_0, \gamma_0), (\rho_1, \gamma_1), (\rho_2, \gamma_2), \ldots, (\rho_m, \gamma_m), (\rho_{m+1}, \gamma_{m+1}) \tag{3.9}$$

where $\rho_0 = \gamma_{m+1} = \infty$ and $\rho_{m+1} = \gamma_0 = -1$. This means the outside corners are

$$(\rho_1 + 1, 0), (\rho_2 + 1, \gamma_1 + 1), (\rho_3 + 1, \gamma_2 + 1), \ldots, (0, \gamma_m + 1).$$

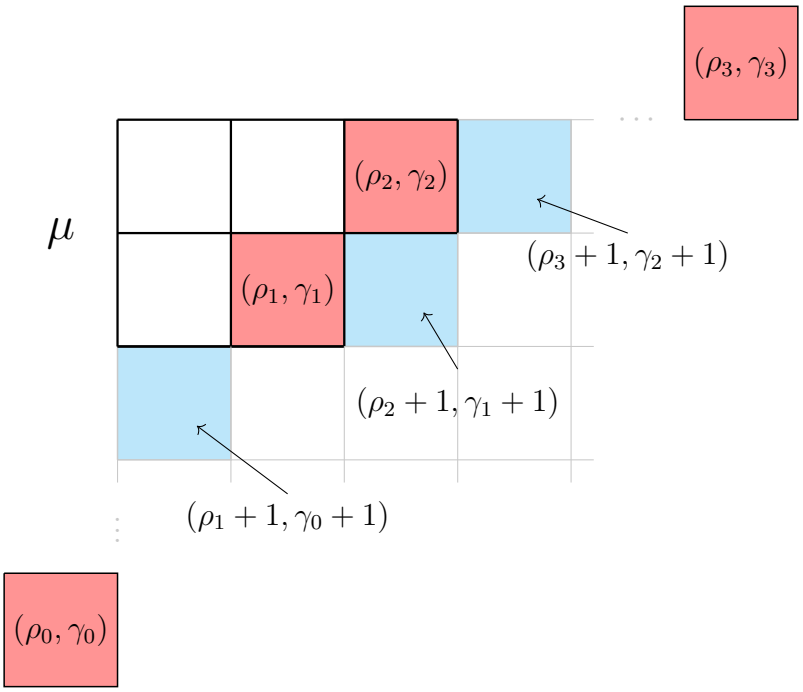


Figure 3.2. Inside and outside corners of $\mu = (3, 2)$.

Figure 3.2 shows the inside (red/darker) and outside (blue/lighter) corners of $\mu = (3, 2)$. Here, $(\rho_0, \rho_1, \rho_2, \rho_3) = (\infty, 1, 0, -1)$ and $(\gamma_0, \gamma_1, \gamma_2, \gamma_3) = (-1, 1, 2, \infty)$.

3.2.2 Equivariant Edge Measure Preliminaries

We now turn to a discussion of the quantities $Q(\lambda)$, $\overline{Q}(\lambda)$, $F(\lambda)$, and $w_{\mathrm{MNOP}}(\lambda)$ defined in Section 3.1.2. Where unambiguous, we write $Q = Q(\lambda)$ and so forth. The following two lemmas concern a generating function which we call the "corner polynomial" and a useful property of the swap operation.

Lemma 3.3. *The "corner polynomial" for a partition λ is*

$$\begin{aligned} C = C(\lambda) &:= Q(1-r)(1-s) \\ &= 1 + \sum_{(i,j)\ \textit{(true) inside corner of } \lambda} r^{i+1}s^{j+1} - \sum_{(i,j)\ \textit{outside corner of } \lambda} r^i s^j \\ &= 1 + \sum_{k=1}^{m} r^{\rho_k+1}s^{\gamma_k+1} - \sum_{k=1}^{m+1} r^{\rho_k+1}s^{\gamma_{k-1}+1}. \end{aligned} \tag{3.10}$$

Proof. Observe that

$$Q(1-r)(1-s) = Q - Qr - Qs + Qrs. \tag{3.11}$$

We consider the contribution of each cell in the plane to C. There are five cases.

Case 1. The top left cell in any diagram λ is only represented in the first term on the RHS of (3.11) meaning that this cell will contribute $r^0s^0 = 1$ to C.

Case 2. The remainder of cells inside the diagram λ have a contribution of zero to C.

- If the cell is in the top row of λ, then is has a cell in λ to its left, but not above it nor northwest of it. Hence it appears once in the first term and once in the third term of the RHS of (3.11) which cancel making its total contribution 0.
- If the cell is in the far left column of λ, we can make an analogous argument.
- If the cell is otherwise in λ, it appears once in each of the four terms of the RHS of (3.11) yielding a total contribution of 0.

Case 3. Note that cells (i, j) which are outside corners of λ either

- have cells of λ to the left, above, and northwest of themselves, hence appearing once in each of the last three terms of the RHS of (3.11),

- have a cell of λ to the left of themselves, hence appearing in the $-Qs$ term of (3.11) only, or
- have a cell of λ above themselves, hence appearing in the $-Qr$ term of (3.11) only.

In any case, each outside corner of λ contributes a total of $-r^i s^j$ to C.

Case 4. Given a (true, not artificial) inside corner (i, j) in λ, the cell $(i + 1, j + 1)$ contributes $r^{i+1}s^{j+1}$ to C since it appears once in the last term of the RHS of (3.11) but not in any other terms.

Case 5. All as yet unmentioned cells are outside of λ and do not have cells of λ directly to the right, above, or northwest of themselves, hence have no effect on C.

□

Define $\overline{C}$ similarly:

$$\overline{C} := \overline{Q}(1 - r^{-1})(1 - s^{-1}) = \frac{\overline{Q}(1 - r)(1 - s)}{rs}.$$

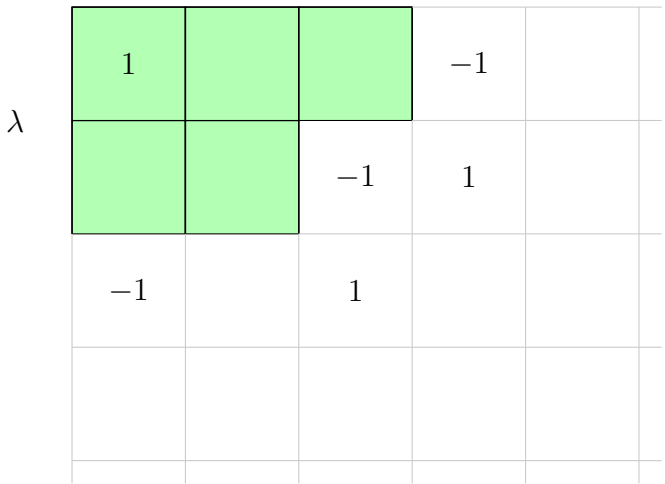


Figure 3.3. Coefficients of the corner polynomial for the shape $\lambda = (3, 2)$.

We provide a visual representation of the corner polynomial in Figure 3.3: Inside every cell in $\lambda = (3, 2)$ is the coefficient of its contribution to C. Empty cells contribute nothing to C. For example, the cell $(1, 2)$ contributes $-1 \cdot r^1 s^2$ to C.

Lemma 3.4. *For Laurent polynomials F and G in the variables r and s with no constant terms, the quotient of swaps of F and G is the swap of the difference of F and G. In other words,*

$$\operatorname{swap}(F-G) = \frac{\operatorname{swap}(F)}{\operatorname{swap}(G)}. \tag{3.12}$$

Proof. Suppose

$$F = \sum_{i,j} c_{i,j} r^i s^j, \qquad\qquad G = \sum_{i,j} d_{i,j} r^i s^j.$$

Applying the swap operation yields

$$\operatorname{swap}(F) = \prod_{i,j} (iu - jv)^{c_{i,j}}, \qquad\qquad \operatorname{swap}(G) = \prod_{i,j} (iu - jv)^{d_{i,j}}.$$

Note also that

$$F - G = \sum_{i,j} (c_{i,j} - d_{i,j}) r^i s^j \qquad \text{and} \qquad \operatorname{swap}(F-G) = \prod_{i,j} (iu - jv)^{c_{i,j} - d_{i,j}}.$$

Taking the quotient of $\operatorname{swap}(F)$ and $\operatorname{swap}(G)$, we have

$$\frac{\operatorname{swap}(F)}{\operatorname{swap}(G)} = \frac{\prod_{i,j}(iu-jv)^{c_{i,j}}}{\prod_{i,j}(iu-jv)^{d_{i,j}}} = \prod_{i,j} (iu-jv)^{c_{i,j}-d_{i,j}} = \operatorname{swap}(F-G).$$

□

3.3 Main Results

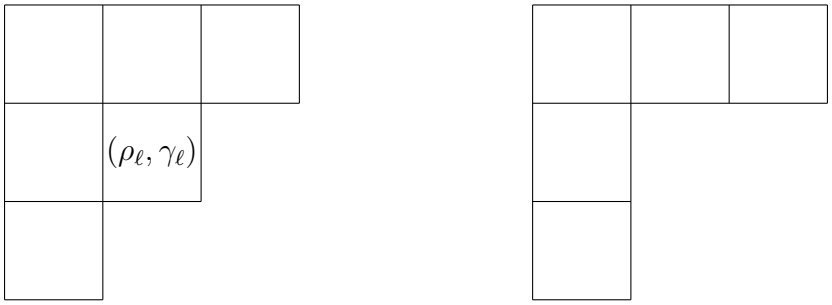
In this section, we compute both the ratio of Jack Plancherel measures and the ratio of equivariant edge measures of two partitions that differ by exactly one corner. Once this is accomplished, we compare these ratios. Then Theorem 3.2 follows by induction on Young's lattice.

Given a partition λ and a distinguished inside corner of λ, (ρ_ℓ, γ_ℓ), define a new partition μ by removing (ρ_ℓ, γ_ℓ) from λ. Note that $\mu \subseteq \lambda$ and $|\lambda| = |\mu| + 1$. Figure 3.4 shows an example of such a pair of Young diagrams.

3.3.1 Ratio of Jack Plancherel Measures

Recall that we defined the Jack Plancherel measure as

$$w_{\text{Jack}}(\lambda) = \frac{1}{\prod_{\square \in \lambda} h^*_\lambda(\square) h^\lambda_*(\square)}$$



(a) $\lambda = (3, 2, 1)$ *(b)* $\mu = (3, 1, 1)$

Figure 3.4. The partition μ is obtained by removing the corner (ρ_ℓ, γ_ℓ) from λ.

in Section 3.1. We now compute the ratio of Jack Plancherel measures of two partitions λ and μ related as described above.

Proposition 3.5. *Let μ and λ be partitions such that $|\mu| + 1 = |\lambda|$ and $\mu \subseteq \lambda$ and let m be the number of non-artificial inside corners of λ. Then*

$$\frac{w_{\text{Jack}}(\lambda)}{w_{\text{Jack}}(\mu)} = \prod_{k=1}^{m} A(k)$$

where

$$A(k) = \begin{cases} \frac{((\rho_\ell - \rho_k - 1)u - (\gamma_\ell - \gamma_k - 1)v) \cdot ((\rho_\ell - \rho_k)u - (\gamma_\ell - \gamma_k)v)}{((\rho_\ell - \rho_k - 1)u - (\gamma_\ell - \gamma_{k-1} - 1)v) \cdot ((\rho_\ell - \rho_k)u - (\gamma_\ell - \gamma_{k-1})v)} & \text{if } k < \ell \\ \frac{((\rho_\ell - \rho_k)u - (\gamma_\ell - \gamma_k)v) \cdot ((\rho_\ell - \rho_k - 1)u - (\gamma_\ell - \gamma_k - 1)v)}{((\rho_\ell - \rho_{k+1})u - (\gamma_\ell - \gamma_k)v) \cdot ((\rho_\ell - \rho_{k+1} - 1)u - (\gamma_\ell - \gamma_k - 1)v)} & \text{if } k > \ell \\ \frac{1}{(\rho_\ell - \rho_{\ell+1})u \cdot ((\rho_\ell - \rho_{\ell+1} - 1)u + v)} \cdot \frac{uv}{(u + (\gamma_\ell - \gamma_{\ell-1} - 1)v) \cdot ((\gamma_\ell - \gamma_{\ell-1})v)} & \text{if } k = \ell. \end{cases} \tag{3.13}$$

Proof. Label the corners of the partition λ as in (3.9). We have

$$\frac{w_{\text{Jack}}(\lambda)}{w_{\text{Jack}}(\mu)} = \frac{\prod_{\square \in \mu} h^*_\mu(\square) h^\mu_*(\square)}{\prod_{\square \in \lambda} h^*_\lambda(\square) h^\lambda_*(\square)} = \frac{1}{uv} \cdot \prod_{\square \in \mu} \frac{h^*_\mu(\square) h^\mu_*(\square)}{h^*_\lambda(\square) h^\lambda_*(\square)}. \tag{3.14}$$

Since λ and μ differ by exactly one cell, namely $(\rho_\ell, \gamma_\ell) \in \lambda$, we note that $h^*_\lambda(\square) = h^*_\mu(\square)$ for all cells in μ except for those in the row or column of (ρ_ℓ, γ_ℓ). The same can be said for the lower hook lengths. Hence significant cancellation occurs in (3.14).

Call the set of cells that share a row with (ρ_ℓ, γ_ℓ) (i.e. directly to the left of (ρ_ℓ, γ_ℓ)) $\mathrm{Row}((\rho_\ell, \gamma_\ell))$ and the set of cells sharing a column with (ρ_ℓ, γ_ℓ) (i.e. directly above (ρ_ℓ, γ_ℓ)) $\mathrm{Col}((\rho_\ell, \gamma_\ell))$.

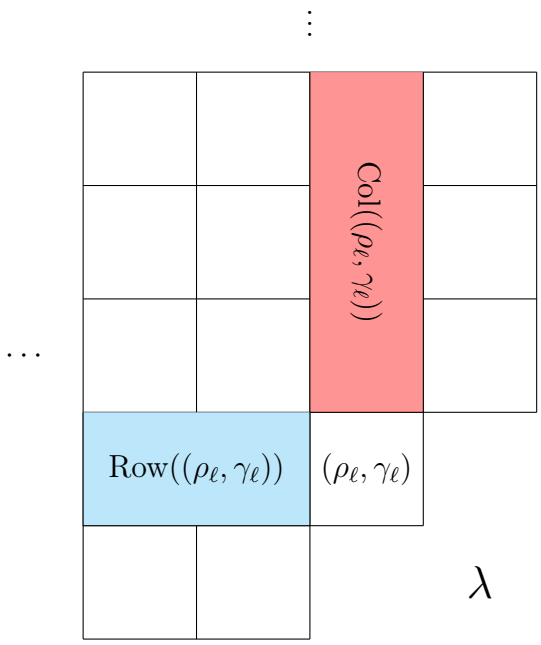


Figure 3.5. $\mathrm{Row}((\rho_\ell, \gamma_\ell))$ and $\mathrm{Col}((\rho_\ell, \gamma_\ell))$.

Now

$$\frac{w_{\mathrm{Jack}}(\lambda)}{w_{\mathrm{Jack}}(\mu)} = \frac{1}{uv} \cdot \prod_{\square \in \mathrm{Row}((\rho_\ell, \gamma_\ell))} \frac{h^*_\mu(\square) h^\mu_*(\square)}{h^*_\lambda(\square) h^\lambda_*(\square)} \cdot \prod_{\square \in \mathrm{Col}((\rho_\ell, \gamma_\ell))} \frac{h^*_\mu(\square) h^\mu_*(\square)}{h^*_\lambda(\square) h^\lambda_*(\square)}.$$

We split these products to viewthe expression as a product over corners of λ.

$$\begin{aligned}\frac{w_{\mathrm{Jack}}(\lambda)}{w_{\mathrm{Jack}}(\mu)} = \frac{1}{uv} \cdot \prod_{k=1}^{\ell-1} \left(\prod_{i=\gamma_{k-1}+1}^{\gamma_k} \frac{h^*_\mu((\rho_\ell, i)) h^\mu_*((\rho_\ell, i))}{h^*_\lambda((\rho_\ell, i)) h^\lambda_*((\rho_\ell, i))} \right) \\ \cdot \prod_{k=\ell+1}^{m} \left(\prod_{i=\rho_{k+1}+1}^{\rho_k} \frac{h^*_\mu((i, \gamma_\ell)) h^\mu_*((i, \gamma_\ell))}{h^*_\lambda((i, \gamma_\ell)) h^\lambda_*((i, \gamma_\ell))} \right) \\ \cdot \prod_{i=\gamma_{\ell-1}+1}^{\gamma_\ell - 1} \frac{h^*_\mu((\rho_\ell, i)) h^\mu_*((\rho_\ell, i))}{h^*_\lambda((\rho_\ell, i)) h^\lambda_*((\rho_\ell, i))} \cdot \prod_{i=\rho_{\ell+1}+1}^{\rho_\ell - 1} \frac{h^*_\mu((i, \gamma_\ell)) h^\mu_*((i, \gamma_\ell))}{h^*_\lambda((i, \gamma_\ell)) h^\lambda_*((i, \gamma_\ell))} \end{aligned} \tag{3.15}$$

How we proceed depends on if the last two products of (3.15) are empty or nonempty. We complete this argument via cases.

Case 1. If the final two products are nonempty each inside product in the first line and both of the last two products telescope, leaving

$$\begin{aligned}\frac{w_{\mathrm{Jack}}(\lambda)}{w_{\mathrm{Jack}}(\mu)} = \frac{1}{uv}\cdot\prod_{k=1}^{\ell-1}\frac{h^*_\mu((\rho_\ell,\gamma_k))h^\mu_*((\rho_\ell,\gamma_k))}{h^*_\lambda((\rho_\ell,\gamma_{k-1}+1))h^\lambda_*((\rho_\ell,\gamma_{k-1}+1))}\\ \cdot\prod_{k=\ell+1}^{m}\frac{h^*_\mu(\rho_k,\gamma_\ell)h^\mu_*(\rho_k,\gamma_\ell)}{h^*_\lambda(\rho_{k+1}+1,\gamma_\ell)h^\lambda_*(\rho_{k+1}+1,\gamma_\ell)}\\ \cdot\frac{h^*_\mu((\rho_\ell,\gamma_\ell-1))h^\mu_*((\rho_\ell,\gamma_\ell-1))}{h^*_\lambda((\rho_\ell,\gamma_{\ell-1}+1))h^\lambda_*((\rho_\ell,\gamma_{\ell-1}+1))}\\ \cdot\frac{h^*_\mu((\rho_\ell-1,\gamma_\ell))h^\mu_*((\rho_\ell-1,\gamma_\ell))}{h^*_\lambda((\rho_{\ell+1}+1,\gamma_\ell))h^\lambda_*((\rho_{\ell+1}+1,\gamma_\ell))}.\end{aligned}$$

Rewriting without the use of hook lengths, we have

$$\begin{aligned}\frac{w_{\mathrm{Jack}}(\lambda)}{w_{\mathrm{Jack}}(\mu)} =\prod_{k=1}^{\ell-1}\frac{((\rho_k-\rho_\ell)u+(\gamma_\ell-\gamma_k)v)((\rho_k-\rho_\ell+1)u+(\gamma_\ell-\gamma_k-1)v)}{((\rho_k-\rho_\ell)u+(\gamma_\ell-\gamma_{k-1})v)((\rho_k-\rho_\ell+1)u+(\gamma_\ell-\gamma_{k-1}-1)v)}\\ \cdot\prod_{k=\ell+1}^{m}\frac{((\rho_\ell-\rho_k-1)u+(\gamma_k-\gamma_\ell+1)v)((\rho_\ell-\rho_k)u+(\gamma_k-\gamma_\ell)v)}{((\rho_\ell-\rho_{k+1}-1)u+(\gamma_k-\gamma_\ell+1)v)((\rho_\ell-\rho_{k+1})u+(\gamma_k-\gamma_\ell)v)}\\ \cdot\frac{1}{uv}\cdot\frac{uv}{((\gamma_\ell-\gamma_{\ell-1})v)(u+(\gamma_\ell-\gamma_{\ell-1}-1)v)}\cdot\frac{uv}{((\rho_\ell-\rho_{\ell+1}-1)u+v)((\rho_\ell-\rho_{\ell+1})u)}.\end{aligned}$$

Simplification gives the desired result.

Case 2. If the last two products are empty we can omit them. Figure 3.5 depicts a situation in which this occurs. Note that in this case, $\rho_{\ell+1}=\rho_\ell-1$ and $\gamma_{\ell-1}=\gamma_\ell-1$. With this in mind, the $k=\ell$ case of the ratio of Jack Plancherel measures (3.13) reduces to $\frac{1}{uv}$.

Picking up where we left off, as in Case 1, each inside product in (3.15) telescopes. We are left with

$$\begin{aligned}\frac{w_{\mathrm{Jack}}(\lambda)}{w_{\mathrm{Jack}}(\mu)} = \frac{1}{uv}\cdot\prod_{k=1}^{\ell-1}\frac{h^*_\mu((\rho_\ell,\gamma_k))h^\mu_*((\rho_\ell,\gamma_k))}{h^*_\lambda((\rho_\ell,\gamma_{k-1}+1))h^\lambda_*((\rho_\ell,\gamma_{k-1}+1))}\\ \cdot\prod_{k=\ell+1}^{m}\frac{h^*_\mu(\rho_k,\gamma_\ell)h^\mu_*(\rho_k,\gamma_\ell)}{h^*_\lambda(\rho_{k+1}+1,\gamma_\ell)h^\lambda_*(\rho_{k+1}+1,\gamma_\ell)}.\end{aligned}$$

Rewriting without the use of hook lengths, we have

$$\frac{w_{\mathrm{Jack}}(\lambda)}{w_{\mathrm{Jack}}(\mu)} = \frac{1}{uv}\cdot\prod_{k=1}^{\ell-1}\frac{((\rho_k-\rho_\ell)u+(\gamma_\ell-\gamma_k)v)((\rho_k-\rho_\ell+1)u+(\gamma_\ell-\gamma_k-1)v)}{((\rho_k-\rho_\ell)u+(\gamma_\ell-\gamma_{k-1})v)((\rho_k-\rho_\ell+1)u+(\gamma_\ell-\gamma_{k-1}-1)v)}$$

$$\cdot \prod_{k=\ell+1}^{m} \frac{((\rho_\ell - \rho_k - 1)u + (\gamma_k - \gamma_\ell + 1)v)((\rho_\ell - \rho_k)u + (\gamma_k - \gamma_\ell)v)}{((\rho_\ell - \rho_{k+1} - 1)u + (\gamma_k - \gamma_\ell + 1)v)((\rho_\ell - \rho_{k+1})u + (\gamma_k - \gamma_\ell)v)}.$$

As a product over corners of λ we have

$$\frac{w_{\text{Jack}}(\lambda)}{w_{\text{Jack}}(\mu)} = \prod_{k=1}^{\#\text{ of corners of }\lambda} A(k)$$

where

$$A(k) = \begin{cases} \frac{((\rho_\ell-\rho_k-1)u-(\gamma_\ell-\gamma_k-1)v)\cdot((\rho_\ell-\rho_k)u-(\gamma_\ell-\gamma_k)v)}{((\rho_\ell-\rho_k-1)u-(\gamma_\ell-\gamma_{k-1}-1)v)\cdot((\rho_\ell-\rho_k)u-(\gamma_\ell-\gamma_{k-1})v)} & \text{if } k < \ell \\ \frac{((\rho_\ell-\rho_k)u-(\gamma_\ell-\gamma_k)v)\cdot((\rho_\ell-\rho_k-1)u-(\gamma_\ell-\gamma_k-1)v)}{((\rho_\ell-\rho_{k+1})u-(\gamma_\ell-\gamma_k)v)\cdot((\rho_\ell-\rho_{k+1}-1)u-(\gamma_\ell-\gamma_k-1)v)} & \text{if } k > \ell \\ \frac{1}{uv} & \text{if } k = \ell. \end{cases}$$

as claimed.

□

3.3.2 Ratio of Equivariant Edge Measures

We now compute the ratio of the equivariant edge measures of two partitions that differ by a single box.

Proposition 3.6. *Let μ and λ be partitions such that $|\mu| + 1 = |\lambda|$ and $\mu \subseteq \lambda$. Then*

$$\frac{w_{\text{MNOP}}(\lambda)}{w_{\text{MNOP}}(\mu)} = \prod_{k=1}^{m} B(k)$$

where

$$B(k) = \begin{cases} \frac{((\rho_\ell-\rho_k)u-(\gamma_\ell-\gamma_k)v)((\rho_\ell-\rho_k-1)u-(\gamma_\ell-\gamma_k-1)v)}{((\rho_\ell-\rho_k)u-(\gamma_\ell-\gamma_{k-1})v)((\rho_\ell-\rho_{k+1}-1)u-(\gamma_\ell-\gamma_k-1)v)} & \text{if } k < \ell \\ \frac{((\rho_\ell-\rho_k)u-(\gamma_\ell-\gamma_k)v)((\rho_\ell-\rho_k-1)u-(\gamma_\ell-\gamma_k-1)v)}{((\rho_\ell-\rho_k)u-(\gamma_\ell-\gamma_{k-1})v)((\rho_\ell-\rho_{k+1}-1)u-(\gamma_\ell-\gamma_k-1)v)} & \text{if } k > \ell \\ \frac{-uv}{(\gamma_\ell-\gamma_{\ell-1})v\cdot((\rho_\ell-\rho_{\ell+1}-1)u+v)} \cdot \frac{1}{((\rho_\ell)u-(\gamma_\ell-\gamma_m)v)((\rho_\ell-\rho_1-1)u-(\gamma_\ell-1)v)} & \text{if } k = \ell. \end{cases} \tag{3.16}$$

Proof. Once again, label the corners of the partition λ as in (3.9). By Lemma 3.4 it suffices to compute $\mathrm{swap}(F(\lambda) - F(\mu))$. Note that

$$F(\lambda) - F(\mu) = -Q(\lambda) - \frac{\overline{Q}(\lambda)}{rs} + \frac{Q(\lambda)\overline{Q}(\lambda)(1-r)(1-s)}{rs} - \left(-Q(\mu) - \frac{\overline{Q}(\mu)}{rs} + \frac{Q(\mu)\overline{Q}(\mu)(1-r)(1-s)}{rs}\right). \tag{3.17}$$

Since λ and μ differ by exactly one cell $(\rho_\ell, \gamma_\ell) \in \lambda$, we have

$$Q(\mu) = Q(\lambda) - r^{\rho_\ell} s^{\gamma_\ell}.$$

Substituting,

$$F(\lambda) - F(\mu) = -r^{\rho_\ell} s^{\gamma_\ell} - r^{-\rho_\ell - 1} s^{-\gamma_\ell - 1} + r^{-\rho_\ell - 1} s^{-\gamma_\ell - 1} Q(\lambda)(1-r)(1-s) + r^{\rho_\ell - 1} s^{\gamma_\ell - 1} \overline{Q}(\lambda)(1-r)(1-s) - \frac{(1-r)(1-s)}{rs}.$$

In terms of corner polynomials,

$$F(\lambda) - F(\mu) = -r^{\rho_\ell} s^{\gamma_\ell} - r^{-\rho_\ell - 1} s^{-\gamma_\ell - 1} + r^{-\rho_\ell - 1} s^{-\gamma_\ell - 1} C(\lambda) + r^{\rho_\ell} s^{\gamma_\ell} \overline{C}(\lambda) - \frac{(1-r)(1-s)}{rs}. \tag{3.18}$$

Simplifying using Lemma 3.3 we have

$$F(\lambda) - F(\mu) = \sum_{k=1}^{\ell-1} r^{\rho_k - \rho_\ell} s^{\gamma_k - \gamma_\ell} + \sum_{k=\ell+1}^{m} r^{\rho_k - \rho_\ell} s^{\gamma_k - \gamma_\ell} - \sum_{k=1}^{m+1} r^{\rho_k - \rho_\ell} s^{\gamma_{k-1} - \gamma_\ell} + \sum_{k=1}^{\ell-1} r^{\rho_\ell - \rho_k - 1} s^{\gamma_\ell - \gamma_k - 1} + \sum_{k=\ell+1}^{m} r^{\rho_\ell - \rho_k - 1} s^{\gamma_\ell - \gamma_k - 1} - \sum_{k=0}^{m} r^{\rho_\ell - \rho_{k+1} - 1} s^{\gamma_\ell - \gamma_k - 1}.$$

We can now apply the swap operation:

$$\begin{aligned} &\mathrm{swap}(F(\lambda) - F(\mu)) = \\ &= \prod_{k=1}^{\ell-1} ((\rho_k - \rho_\ell)u - (\gamma_k - \gamma_\ell)v) \cdot ((\rho_\ell - \rho_k - 1)u - (\gamma_\ell - \gamma_k - 1)v) \\ &\cdot \prod_{k=\ell+1}^{m} ((\rho_k - \rho_\ell)u - (\gamma_k - \gamma_\ell)v) \cdot ((\rho_\ell - \rho_k - 1)u - (\gamma_\ell - \gamma_k - 1)v) \\ &\cdot \left(\prod_{k=1}^{m} ((\rho_k - \rho_\ell)u - (\gamma_{k-1} - \gamma_\ell)v) \cdot ((\rho_\ell - \rho_{k+1} - 1)u - (\gamma_\ell - \gamma_k - 1)v)\right)^{-1} \\ &\cdot -uv \cdot ((\rho_{m+1} - \rho_\ell)u - (\gamma_m - \gamma_\ell)v)^{-1} \cdot ((\rho_\ell - \rho_1 - 1)u - (\gamma_\ell - \gamma_0 - 1)v)^{-1}. \end{aligned} \tag{3.19}$$

Splitting this into the three cases $k < \ell$, $k > \ell$, $k = \ell$ yields the desired result. □

3.3.3 Comparison of Ratios

So far, we have expressed ratios of both the Jack Plancherel measure and the equivariant edge measure as products over the corners of λ. It remains to show that these ratios are equal, and then use that information inductively to show that $w_{\text{Jack}}(\lambda) = -w_{\text{MNOP}}(\lambda)$.

Theorem 3.7. *We have*

$$\frac{w_{\text{Jack}}(\lambda)}{w_{\text{Jack}}(\mu)} = \frac{w_{\text{MNOP}}(\lambda)}{w_{\text{MNOP}}(\mu)}. \tag{3.20}$$

Proof. We have

$$\frac{\frac{w_{\text{Jack}}(\lambda)}{w_{\text{Jack}}(\mu)}}{\frac{w_{\text{MNOP}}(\lambda)}{w_{\text{MNOP}}(\mu)}} = \prod_{k=1}^{m} \frac{A(k)}{B(k)}$$

by Proposition 3.5 and Proposition 3.6. We simplify this in three cases: $k < \ell$, $k > \ell$, and $k = \ell$. When $k < \ell$, all but one term of $A(k)$ agrees with $B(k)$. The remaining term in $A(k)$ is

$$T_A(k) := \frac{1}{(\rho_\ell - \rho_k - 1)u - (\gamma_\ell - \gamma_{k-1} - 1)v}$$

and for $B(k)$ is

$$T_B(k) := \frac{1}{(\rho_\ell - \rho_{k+1} - 1)u - (\gamma_\ell - \gamma_k - 1)v}.$$

Note that $T_B(k) = T_A(k+1)$. So via traditional cancellation and telescoping,

$$\prod_{k=1}^{\ell-1} \frac{A(k)}{B(k)} = \prod_{k=1}^{\ell-1} \frac{T_A(k)}{T_B(k)} = \frac{T_A(1)}{T_B(\ell-1)} = \frac{-u - (\gamma_\ell - \gamma_{\ell-1} - 1)v}{(\rho_\ell - \rho_1 - 1)u - (\gamma_\ell - 1)v}.$$

We proceed analogously with the $k > \ell$ case. Define

$$T^A(k) = \frac{1}{(\rho_\ell - \rho_{k+1})u - (\gamma_\ell - \gamma_k)v} \quad \text{and} \quad T^B(k) = \frac{1}{(\rho_\ell - \rho_k)u - (\gamma_\ell - \gamma_{k-1})v}.$$

Then

$$\prod_{k=\ell+1}^{m} \frac{A(k)}{B(k)} = \prod_{k=\ell+1}^{m} \frac{T^A(k)}{T^B(k)} = \frac{T^A(m)}{T^B(\ell+1)} = \frac{(\rho_\ell - \rho_{\ell+1})u}{(\gamma_\ell - \gamma_m)v - (\rho_\ell)u}.$$

Remark 3.8. When λ has zero corners, i.e. when $m = 0$, the products above are empty. When λ has one corner then one can verify that the two expressions above are both one.

Now

$$\frac{\frac{w_{\text{Jack}}(\lambda)}{w_{\text{Jack}}(\mu)}}{\frac{w_{\text{MNOP}}(\lambda)}{w_{\text{MNOP}}(\mu)}} = \prod_{k=1}^{m} \frac{A(k)}{B(k)} = \frac{(-1)((\gamma_\ell - \gamma_{\ell-1} - 1)v + u)}{(\gamma_\ell - 1)v - (\rho_\ell - \rho_1 - 1)u} \cdot \frac{(\rho_\ell - \rho_{\ell+1})u}{(\gamma_\ell - \gamma_m)v - (\rho_\ell)u} \cdot \frac{A(\ell)}{B(\ell)}.$$

We cancel, revealing

$$\frac{\frac{w_{\text{Jack}}(\lambda)}{w_{\text{Jack}}(\mu)}}{\frac{w_{\text{MNOP}}(\lambda)}{w_{\text{MNOP}}(\mu)}} = 1.$$

□

Proof of Theorem 3.2. We proceed by induction on the size of a partition using the partition ν with one box as our base case. Let s be the unique cell in ν. We have

$$w_{\text{Jack}}(\nu) = \frac{1}{h^*(s)h_*(s)} = \frac{1}{uv}$$

and

$$\begin{aligned} w_{\text{MNOP}}(\nu) &= \text{swap}\left(-Q(\nu) - \frac{\overline{Q}(\nu)}{rs} + \frac{Q(\nu)\overline{Q}(\nu)(1-r)(1-s)}{rs}\right) \\ &= \text{swap}\left(-\frac{1}{s} - \frac{1}{r}\right) = -\frac{1}{uv} \end{aligned} \tag{3.21}$$

as we expect. Suppose that $w_{\text{Jack}}(\mu) = -w_{\text{MNOP}}(\mu)$ for all partitions of size $n-1$. By Theorem 3.7, given a partition λ of size n, we know that

$$\frac{w_{\text{Jack}}(\lambda)}{w_{\text{Jack}}(\mu)} = \frac{w_{\text{MNOP}}(\lambda)}{w_{\text{MNOP}}(\mu)}$$

for any $\mu \subseteq \lambda$ of size $n-1$. Multiplying both sides by $w_{\text{Jack}}(\mu) = -w_{\text{MNOP}}(\mu)$ yields

$$w_{\text{Jack}}(\lambda) = -w_{\text{MNOP}}(\lambda).$$

□

We explain why there is no similar result in the three-dimensional case in Chapter 5.

CHAPTER 4

AN ARROW ARGUMENT OF HAIMAN

Before presenting our work on the equivariant vertex measure in Chapter 5, we discuss *Haiman's arrows* and provide an aside concerning a bijection between them and terms in the Jack Plancherel measure. This will provide language and context with which to discuss our work in the following chapter.

4.1 Relationships between Objet Étudiés

In Chapter 3 we showed that the equivariant edge measure from Donaldson–Thomas theory is, in essence, the Jack Plancherel measure. However, the fact that these two notions encode the same data has been known to geometers for some time. Indeed, both are equivalent to the generating function for the Hilbert scheme of points in the plane. The novelty in our work was not in the statement of the main theorem, but in the nature by which we arrived at the result: combinatorially and without any reliance on geometric machinery.

In 1998, Haiman demonstrated another seemingly unrelated manipulation of combinatorial objects that is equivalent to the generating function for the Hilbert scheme of points in the plane [Hai98]. In this work, he describes assigning two arrows to each cell in an integer partition λ of size n. Each arrow then corresponds to a basis element of the Hilbert scheme. Figure 4.2 shows the two arrows u_{hk} and d_{hk} corresponding to the shaded cell (h, k) in the Young diagram. (Note that we preserve as much of Haiman's original notation as possible, save for using English

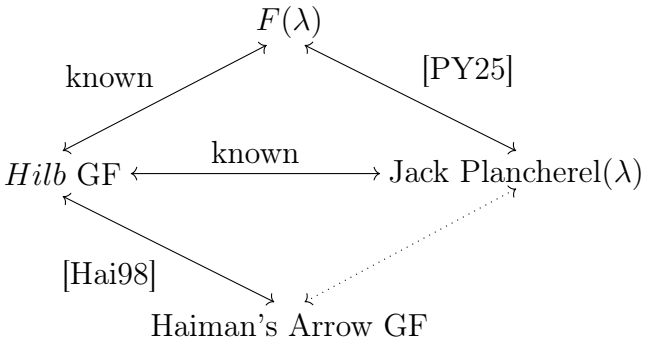


Figure 4.1. The Hilbert scheme of points in the plane and associated objects.

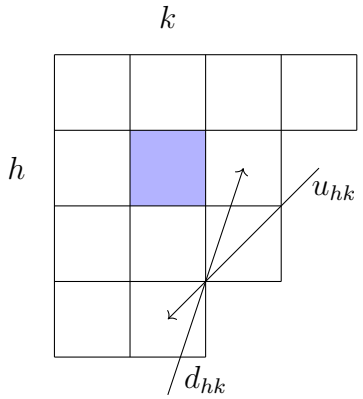


Figure 4.2. The arrows d_{hk} and u_{hk} for (h, k).

notation for Young diagrams rather than the French notation). Via this bijection, Haiman shows that the Hilbert scheme is smooth and irreducible of dimension $2n$.

4.2 A Direct Combinatorial Bijection

This object is equivalent to the Jack Plancherel measure via Hilbert schemes. However, it would behoove us to consider whether or not equivalence may be described by combinatorial bijection only, without appealing to geometric notions.

Note that for a given shape λ of size n, there are $2n$ hook length terms in $w_{\text{Jack}}(\lambda)$ as given in (3.7). There is a natural bijection between terms in this product and the $2n$ arrows which comprise Haiman's arrow basis for λ: the arrows $d_{h,k}$ and $u_{h,k}$ corresponding to the hook based at cell (h, k) in λ are mapped to $h^{\lambda}_{*}((h, k))$ and $h^{*}_{\lambda}((h, k))$ respectively.

$$\begin{aligned} d_{h,k} &\mapsto h^{\lambda}_{*}((h,k)) \\ u_{h,k} &\mapsto h^{*}_{\lambda}((h,k)) \end{aligned} \tag{4.1}$$

The choice of upper and lower is not arbitrary. Indeed, each arrow encodes a hook length in its vertical and horizontal components, so we can explicitly compute $w_{\text{Jack}}(\lambda)$ from the set of $2n$ arrows on λ. Call the vertical and horizontal displacements of an arrow c_{hk}

$$\ell(c_{hk}) \qquad \text{and} \qquad a(c_{hk}) \tag{4.2}$$

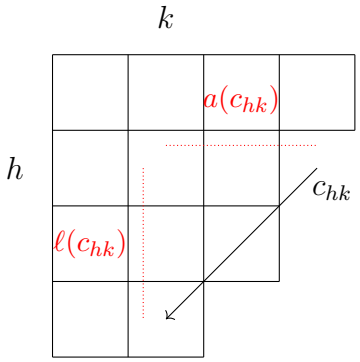


Figure 4.3. Vertical and horizontal displacement of an arrow.

respectively. See Figure 4.3. We have purposely chosen this notation to be suggestive of arm and leg lengths.

The hook length which corresponds to c_{hk} is given explicitly by

$$u \cdot \ell(c_{hk}) + v \cdot a(c_{hk}) \tag{4.3}$$

where u and v are the Jack parameters as in Chapter 3.

To compute $w_{\text{Jack}}(\lambda)$ from the set of arrows on λ take the product over all arrows c_{hk} of the reciprocal of (4.3).

$$\prod_{\text{arrows } d_{hk}} \frac{1}{u \cdot \ell(d_{hk}) + v \cdot a(d_{hk})} \prod_{\text{arrows } u_{hk}} \frac{1}{u \cdot \ell(u_{hk}) + v \cdot a(u_{hk})} = w_{\text{Jack}} \tag{4.4}$$

4.3 Branching Rule

For a given shape λ, call its set of Hilbert scheme basis arrows $A(\lambda)$ and call the set of hook lengths $\text{Hooks}(\lambda)$.

Theorem 4.1. *The bijection β from arrows to hook lengths described in the previous section commutes with the action of adding a box to λ.*

In other words, suppose $\lambda \subset \nu$ and $|\lambda| + 1 = |\nu|$. Then the following diagram commutes.

$$\begin{array}{ccc} A(\lambda) & \xleftrightarrow{\beta_\lambda} & \text{Hooks}(\lambda) \\ {\scriptstyle +\square}\downarrow & & \downarrow{\scriptstyle +\square} \\ A(\nu) & \xleftrightarrow{\beta_\nu} & \text{Hooks}(\nu) \end{array}$$

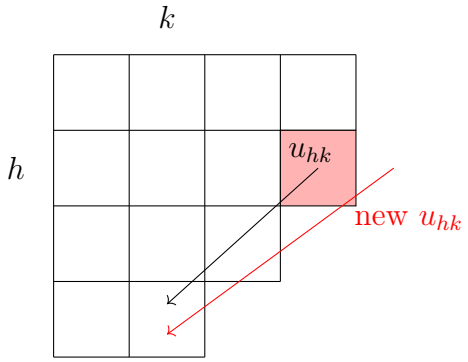


Figure 4.4. Effect of adding a box on arrows.

Proof. We show that adding a box to a shape λ to get a new shape ν has the same effect on both the Hilbert scheme basis arrows and on the set of hook lengths. For arrows, the addition of $\square$ effects only arrows corresponding to cells whose hook contains $\square$, i.e. Row($\square$) and Col($\square$). Indeed, the hook of any other cell is unaffected, so its arrows remain unchanged. For a cell (h, k) in Row($\square$), both the tail of u_{hk} and the tip of d_{hk} are shifted to the right by one cell. This increases $a(u_{hk})$ and $a(d_{hk})$ by one (and does not effect $\ell(u_{hk})$ or $\ell(d_{hk})$). In Figure 4.4, u_{hk} is shown both before and after adding the box at $(1, 3)$.

As for hook lengths, adding a box $\square$ to λ effects only the hook lengths in the row and column of the added box $\square$. See Figure 3.5. The addition of $\square$ increases the arm length of each cell in Row($\square$) by one.

Analogous statements can be made for cells in Col($\square$) for both arrows and hook lengths.

Since arrows and hook lengths are effected the same way by the addition of a cell to a diagram λ, the bijection β commutes with cell addition. $\square$

CHAPTER 5

ON THE EQUIVARIANT VERTEX MEASURE

5.1 Introduction

In this chapter, we set out progress towards combinatorially describing the equivariant vertex measure (3.1). This work is guided by our insights about the edge term investigated in Chapter 3. First, we describe a natural generalization of the corner polynomial defined in Section 3.2.2 in three dimensions. Then we use this object to achieve the main thrust of this chapter: showing that outside of the Calabi-Yau specialization (Definition 3.1), the equivariant edge measure is not a probability measure. Indeed, the sign of the measure on a plane partition (defined below) cannot be predicted by the size of the plane partition alone. We follow our previous conventions and notation from Section 3.1.

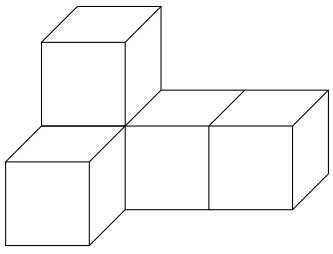
In the three dimensional setting, we use an analogue of Young diagrams which were mentioned but not defined in Section 3.1.

Definition 5.1. A *plane partition* is a two-dimensional array of nonnegative integers which is finitely supported and weakly decreases along both the rows and columns. The *size* of a plane partition π is the sum of all of the array entries.

A plane partition can be understood as a collection of boxes stacked in the corner of a room. The number of boxes in the stack at a given location (i, j) is the (i, j) entry of the array in zero-indexed matrix coordinates.

$$\begin{array}{lll} 2 & 1 & 1 \\ 1 & & \end{array}$$

(a) A plane partition as an array.



(b) A plane partition viewed as boxes.

Figure 5.1. A plane partition of size five shown in two different ways.

We add the parameter t to r and s from Chapter 3 for the added dimension. See Figure 5.2.

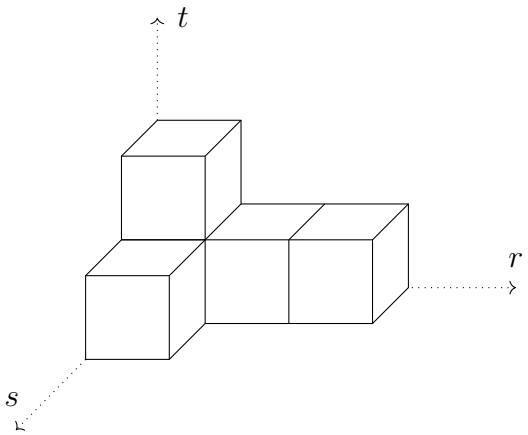


Figure 5.2. The directions r, s, and t.

Much of our work from Chapter 3 carries over to three dimensions. In particular, we have a 3D analog of Lemma 3.4. We begin by defining the swap operation.

Definition 5.2. In three dimensions, the swap operation on Laurent polynomials in r, s and t with no constant term is defined as follows:

$$\operatorname{swap}\left(\sum_{i,j,k} c_{ijk} r^i s^j t^k\right) := \prod_{i,j,k} (iu + jv + kw)^{-c_{ijk}}.$$

Notably, the 3D swap operation differs from the 2D swap operation in the signs between summands and on the exponents. However, the relationship between the swap operation and differences and quotients from Chapter 3 remains.

Lemma 5.3. *In three dimensions, the swap of a difference is the same as a quotient of swaps. More explicitly, for Laurent polynomials with integer coefficients*

$$F = \sum_{i,j,k} f_{i,j,k} r^i s^j t^k \quad \text{and} \quad G = \sum_{i,j,k} g_{i,j,k} r^i s^j t^k \tag{5.1}$$

with no constant term,

$$\mathit{swap}(F - G) = \frac{\mathit{swap}(F)}{\mathit{swap}(G)}. \tag{5.2}$$

Proof. We have

$$\operatorname{swap}(F - G) = \operatorname{swap}\left(\sum_{i,j,k} f_{i,j,k} r^i s^j t^k - \sum_{i,j,k} g_{i,j,k} r^i s^j t^k\right)$$

$$
\begin{aligned}
&= \operatorname{swap}\left(\sum_{i,j,k}(f_{i,j,k} - g_{i,j,k})r^i s^j t^k\right) \\
&= \prod_{i,j,k}(iu + jv + kw)^{g_{i,j,k} - f_{i,j,k}} \\
&= \prod_{i,j,k}(iu + jv + kw)^{g_{i,j,k}} \cdot \prod_{i,j,k}(iu + jv + kw)^{-f_{i,j,k}} \\
&= \operatorname{swap}\left(\sum_{i,j,k} g_{i,j,k} r^i s^j t^k\right)^{-1} \cdot \operatorname{swap}\left(\sum_{i,j,k} f_{i,j,k} r^i s^j t^k\right) \\
&= \frac{\operatorname{swap}(F)}{\operatorname{swap}(G)} \qquad (5.3)
\end{aligned}
$$

as claimed. □

5.2 The 3D Corner Polynomial

The most salient 3D parallel from our 2D is the existence of a "corner polynomial".

Definition 5.4. The *corner polynomial* for a plane partition π is

$$C_3(\pi) = Q(\pi) \cdot (1 - r) \cdot (1 - s) \cdot (1 - t). \qquad (5.4)$$

We omit the subscripted 3 when it is clear from context.

Theorem 5.5. *The corner polynomial $C_3(\pi)$ is named as such because it describes the corners and saddle points of the plane partition π. Specifically,*

$$
\begin{aligned}
C_3 &= Q(1 - r)(1 - s)(1 - t) \\
&= 1 + \sum_{(i,j,k)\ \textit{non-monkey saddle point}} r^i s^j t^k + 2 \sum_{(i,j,k)\ \textit{monkey saddle point}} r^i s^j t^k \\
&\quad - \sum_{(i,j,k)\ \textit{outside corner}} r^i s^j t^k - \sum_{(i,j,k)\ \textit{inside corner}} r^{i+1} s^{j+1} t^{k+1}. \qquad (5.5)
\end{aligned}
$$

Figure 5.3 shows a monkey saddle point. Figure 5.4 shows all other types of saddle points. A right divot saddle point is marked in magenta on the face closest to the reader. A right bump saddle is marked in orange farther from the reader. The left (resp. bottom) version of each of these saddles can be achieved by reflecting this image in a mirror held vertically (resp. horizontally) perpendicular to the page.

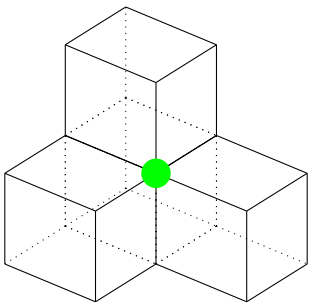


Figure 5.3. A monkey saddle point.

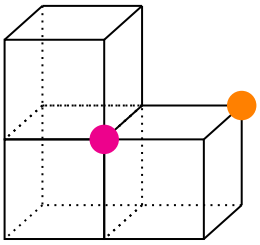


Figure 5.4. Bump and divot saddles.

Proof. For a polynomial P and monomial m, write $[m]P$ for the coefficient of m in P. We view an integral polynomial P in r, s, and t as an assignment of an integer to each lattice point in $\mathbb{Z}^3$ by taking the integer at (i, j, k) to be the coefficient of $r^i s^j t^k$ in P.

In this proof, we say that the lattice point (i, j, k) *has value* $n \in \mathbb{Z}$ *in* P if the coefficient on $r^i s^j t^k$ in P is n.

The reader is invited to imagine embedding the plane partition π into the lattice $\mathbb{Z}^3$ with its base corner at $(0, 0, 0)$. One can verify that the corner polynomial $C_3(\pi)$ of the plane partition π in Figure 5.5 is the sum of the monomial $r^i s^j t^k$ with coefficient

- 1 if there is a red circle at (i, j, k),
- 2 if there is a green circle at (i, j, k) (not pictured)
- -1 if there is a blue circle at (i, j, k), and
- 0 otherwise

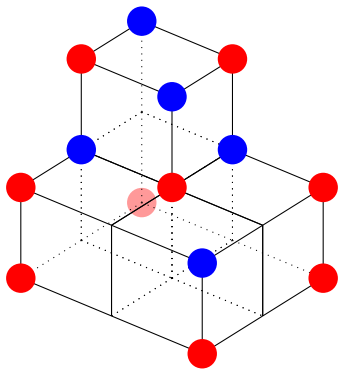


Figure 5.5. The corner polynomial $C_3(\pi)$ of the plane partition π.

for $(i, j, k) \in \mathbb{Z}^3$. We view a plane partition π as a vertical stack of 2D partitions each of which is contained by the partition directly below it. We call the 2D partition at height h in this stack λ^h. Figure 5.6 shows a plane partition π and its stack of 2D partitions $\lambda^1 = (1)$ and $\lambda^0 = (2, 2)$.

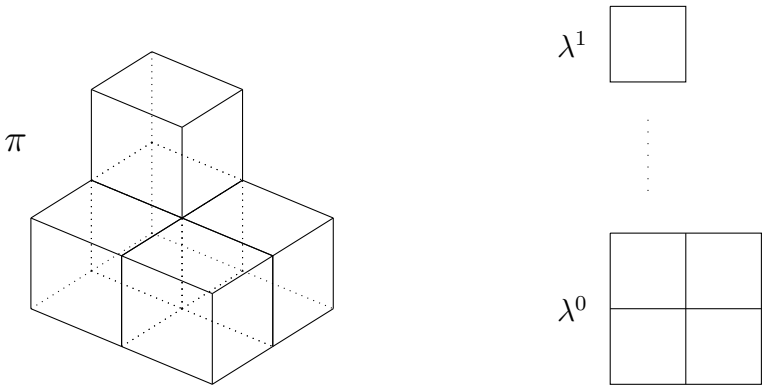


Figure 5.6. The plane partition π as a stack of two 2D partitions.

The 2D corner polynomial C_2^h of λ^h is known. (See Lemma 3.3 for an explicit description of C_2^h.) Analogously to above, for each h view C_2^h as an assignment of integers to the lattice points in $\mathbb{Z}^2$. See Figure 5.7. Note that an inside corner (i, j, k) of a partition λ^k has a zero at its lattice point, but the lattice point $(i + 1, j + 1, k)$ that is shifted by one in each of the r and s directions from (i, j, k) has value 1 in C_2^k. We call $(i + 1, j + 1, k)$ the *inside corner marker* for (i, j, k).

By "stacking" these assignments, we create an assignment of integers to each lattice point in $\mathbb{Z}^3$. This assignment is exactly that of $Q(\pi) \cdot (1 - r) \cdot (1 - s) = \frac{C_3}{(1-t)}$

by definition of C_2^h.

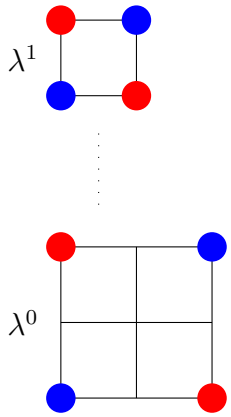


Figure 5.7. The 2D corner polynomials C_2^0 and C_2^1.

With this in mind, the coefficient of $r^i s^j t^k$ in C_3 is $[r^i s^j]C_2^k - [r^i s^j]C_2^{k-1}$. We compute this in cases.

Case 1.

Since $[r^0 s^0]C_2^0$ is 1 and $[r^0 s^0]C_2^{-1} = 0$ (as λ^{-1} is the empty diagram), we have $[r^0 s^0 t^0]C_3 = 1 - 0 = 1$.

Case 2.

Define a lattice point (i, j, k) in $\mathbb{Z}^3$ that is on the interior of the 2D partition λ^k at height k and sits below a lattice point contained in the partition at height $k + 1$ an *unexposed lattice point*. In other words, an unexposed lattice point is one that can't be viewed from any angle outside the plane partition π. All unexposed lattice points besides $(0, 0, 0)$ will contribute a 0 to C_3. Indeed, such a lattice point contributes 0 to C_2^k, and the lattice point below it contributes 0 to C_2^{k-1} as it is interior to λ^{k-1} by containment of λ^k.

It remains to check the contribution of lattice points which are external to π and lattice points which lie on the surface of π.

Case 3.

If a lattice point (i, j, k) and the point below it $(i, j, k - 1)$ both have the same value in C_2^k and C_2^{k-1} respectively, i.e. both are outside corners (so have value -1), both are inside corners shifted by one in both the r and s directions (so have value

1), or both are neither of the previously mentioned situations (so have value 0), in their respective 2D partitions then $[r^i s^j t^k]C_3 = 0$.

The all remaining cases consider a lattice point (i, j, k) which is assigned a different number than the point below it at $(i, j, k-1)$ in the corresponding polynomials C_2^k and C_2^{k-1}.

Case 4.

If $[r^i s^j]C_2^k = 1$ and $[r^i s^j]C_2^{k-1} = -1$ then $[r^i s^j t^k]C_3 = 1 - (-1) = 2$ and (i, j, k) is the position of an inside corner marker of λ^k appearing above an outside corner of λ^{k-1}, in other words, a monkey saddle point in π. See Figure 5.3.

Remark 5.6. It is never the case that an outside corner of λ^k appears above an inside corner marker of λ^{k-1}, as this would not result in a valid plane partition when stacked. Hence there are no monomials with coefficient $-1 - 1 = -2$ in the corner polynomial.

Case 5.

Only in three situations does a coefficient of 0 appear in C_2^k when a -1 appears as a coefficient on the same monomial in C_2^{k-1}. These are

1. at the lattice point where a horizontal edge of λ^k appears above an outside corner of λ^{k-1},

2. at the lattice point where a vertical edge of λ^k appears above an outside corner of λ^{k-1}, and

3. where a lattice point corresponding to an outside corner of λ^{k-1} does not have any lattice points corresponding to boxes from λ^k on top of it.

The first and second situations above represent right and left bump saddle points respectively. The last describes a bottom divot saddle point. See Figure 5.4 for an illustration of such saddles. Each monomial corresponding to one of these features appears with coefficient $0 - (-1) = 1$ to C_3.

Case 6.

Similarly to the previous case, we find that there are only three situations in which a 1 appears as a coefficient on a monomial in C_2^k when a 0 appears as a coefficient on the same monomial in C_2^{k-1}. These are

1. at the lattice point where an inside corner marker of λ^k appears above a horizontal edge of λ^{k-1},

2. at the lattice point where an inside corner marker of λ^k appears above a vertical edge of λ^{k-1}, and

3. at the lattice point where an inside corner marker of λ^k appears above the interior of λ^{k-1}.

The first and second situations above represent right and left divot saddle points respectively. The last describes a bottom bump saddle point. Each monomial corresponding to one of these features appears with coefficient $1 - 0 = 1$ to C_3.

Remark 5.7. The lattice point value arrangements in the previous two cases are in fact the only ways of achieving non-monkey saddle points in π. So together they show that each of the monomials from non-monkey saddle points in a plane partition contribute 1 to C_3.

Case 7.

A lattice point which is external to and not on the surface of a shape λ^k which has value 0 in C_2^k above a lattice point which has value 1 in C_2^{k-1} indicates an outside corner of the plane partition π. The corresponding contribution to C_3 has coefficient $0 - 1 = -1$.
Note: A box which is interior to or on an edge of a shape λ which has value 0 in C_2^k cannot be above a box which has value 1 in C_2^{k-1}, as that would not result in a valid plane partition.

Case 8.

If a lattice point which has value -1 in C_2^k (i.e. an outside corner of λ^k) appears above a lattice point internal to π which has value 0 in C_2^{k-1}, then this represents a place we can add a box to π, i.e. an outside corner of π. Such a monomial will have coefficient $-1 - 0 = -1$ in C_3.
Note: A lattice point which is an inside corner of λ^k, i.e. has value -1 in C_2^k, cannot appear above a lattice point which is external π or on the surface of π with value 0 in $C_{2,h-1}$ or else we will not have a valid plane partition.

Remark 5.8. The lattice point value arrangements in the previous two cases are in fact the only arrangements which produce inside or outside corners of π. Hence

these last two cases show that inside and outside corners of π contribute their corresponding monomial with coefficient -1 to C_3.

□

5.3 Sign Unpredictability

In what follows, we demonstrate that away from the Calabi-Yau specialization, (3.2) is not a probability measure on plane partitions π for any parameter vector $(u, v, w) \in \mathbb{R}^3$. We do this via examining the sign of $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$ for certain plane partitions π and σ which differ by one box.

Select $q \in (-1, 1)$ and $q \neq 0$. If q is positive, then we must have $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$ positive for all plane partitions π and σ, for this is the only way that all summands $\mathsf{w}(\nu)q^{|\nu|}$ in (3.2) to have the same sign. If q is negative, then we must have $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$ negative for all plane partitions π and σ which differ in size by one, for then $\mathsf{w}(\nu)$ and $q^{|\nu|}$ both alternate in sign as $|\pi|$ increases by one. This is the only way that all summands $\mathsf{w}(\nu)q^{|\nu|}$ in (3.2) to have the same sign when q is negative. We demonstrate that neither of these possibilities occur.

Note that in the Calabi-Yau specialization, the ratio of these two vertex measures is exactly -1. See Remark 5.11.

We introduce a specific class of plane partitions which depend on three positive integer parameters. Then we use the 3D corner polynomial to consider the problem in terms of pairs of Haiman arrows. There is significant efficiency in splitting the problem into two cases: one for parameters $(u, v, w) \in \mathbb{R}^3$ in the $+++$ or $---$ octants of $\mathbb{R}^3$ and one for parameters in any of the other six octants of $\mathbb{R}^3$. We then provide criteria for coming up with parameters $(a, b, c) \in \mathbb{Z}^3_{>0}$ for which the sign contribution of each pair can be controlled. Careful consideration of these signs simultaneously yields the theorem.

5.3.1 A Family of Simple Plane Partitions

For positive integers a, b, and c, let σ be the plane partition with the shape of an $a \times b \times c$ rectangular prism. Define π to be the plane partition that is identical to σ besides an extra box in the top left stack. See Figure 5.8.

Definition 5.9. We say a, b and c are the *dimensions* of σ and π.

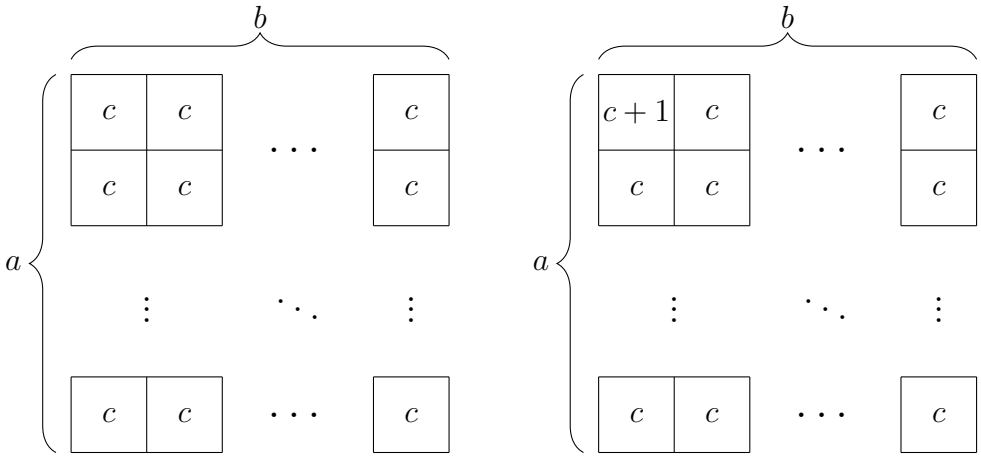


Figure 5.8. Diagrams for plane partitions σ and π

Theorem 5.10 (Sign Unpredictability)**.** *Choose nonzero real-valued parameters u, v, and w such that $u+v+w \neq 0$. We can choose dimensions a, b, and c in $\mathbb{Z}_{>0}$ for σ (and thus π) such that the equivariant vertex measure ratio of these two plane partitions $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$ is positive. Similarly, we can choose dimensions a, b, and c in $\mathbb{Z}_{>0}$ for σ (and thus π) such that the equivariant vertex measure ratio of these two plane partitions $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$ is negative. Hence the sign of the equivariant edge measure is not determined solely by the size of a plane partition.*

The rest of this chapter is devoted to proving this theorem.

5.3.2 Computation of $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$

Presently it will be useful to describe $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$ explicitly. Lemma 5.3 and Theorem 5.5 greatly minimize the necessary work, as σ has exactly three non-monkey saddle points at $(a,b,0)$, $(a,0,c)$, and $(0,b,c)$, three outside corners at $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$, and one inside corner at (a,b,c). Hence the corner polynomial for σ is

$$C(\sigma) = 1 + r^a t^c + r^a s^b + s^b t^c - r^a - s^b - t^c - r^{a+1} s^{b+1} t^{c+1}.$$

Note that

$$\begin{aligned} F(\pi) - F(\sigma) &= t^c - r^{-1}s^{-1}t^{-c-1} + C(\sigma)(r^{-1}s^{-1}t^{-c-1}) - \overline{C}(\sigma)(t^c) - (1-r^{-1})(1-s^{-1})(1-t^{-1}) \\ &= t^c - r^{-1}s^{-1}t^{-c-1} + r^{-1}s^{-1}t^{-c-1} + r^{a-1}s^{-1}t^{-1} + r^{a-1}s^{b-1}t^{-c-1} + r^{-1}s^{b-1}t^{-1} \end{aligned}$$

$$
\begin{aligned}
&- r^{a-1}s^{-1}t^{-c-1} - r^{-1}s^{b-1}t^{-c-1} - r^{-1}s^{-1}t^{-1} - r^a s^b \\
&- (t^c + r^{-a} + r^{-a}s^{-b}t^c + s^{-b} - r^{-1}t^c - s^{-b}t^c - 1 - r^{-a-1}s^{-b-1}t^{-1}) \\
&- (1 - r^{-1})(1 - s^{-1})(1 - t^{-1}).
\end{aligned}
$$

Cancellation yields

$$
\begin{aligned}
F(\pi) - F(\sigma) = {} & r^{a-1}s^{-1}t^{-1} + r^{-1}s^{b-1}t^{-1} + r^{a-1}s^{b-1}t^{-c-1} \\
& - r^{a-1}s^{-1}t^{-c-1} - r^{-1}s^{b-1}t^{-c-1} - r^a s^b - r^{-a} - s^{-b} - r^{-a}s^{-b}t^c \\
& + r^{-a}t^c + s^{-b}t^c + r^{-a-1}s^{-b-1}t^{-1} + r^{-1} + s^{-1} + t^{-1} - r^{-1}t^{-1} - r^{-1}s^{-1} - s^{-1}t^{-1}.
\end{aligned}
$$

We now apply the swap operation to get

$$
\begin{aligned}
\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)} &= \operatorname{swap}\left(\frac{F(\pi)}{F(\sigma)}\right) = \operatorname{swap}\left(F(\pi) - F(\sigma)\right) \\
&= \frac{((a-1)u - v + (-c-1)w)(-u + (b-1)v + (-c-1)w)}{((a-1)u - v - w)(-u + (b-1)v - w)} \\
&\quad \cdot \frac{(au + bv)(au)(bv)}{((a-1)u + (b-1)v + (-c-1)w)(-au + cw)(-bv + cw)} \\
&\quad \cdot \frac{(-au - bv + cw)(-u - v)(-v - w)(-u - w)}{((-a-1)u + (-b-1)v - w)(-u)(-v)(-w)}.
\end{aligned} \tag{5.6}
$$

Writing

$$
\begin{aligned}
x_1 &= (a-1, -1, -c-1) & y_1 &= (-a, 0, c) \\
x_2 &= (-1, b-1, -c-1) & y_2 &= (0, -b, c) \\
x_3 &= (a, b, 0) & y_3 &= (-a-1, -b-1, -1) \\
x_4 &= (-a, -b, c) & y_4 &= (a-1, b-1, -c-1) \\
x_5 &= (-a, 0, 0) & y_5 &= (a-1, -1, -1) \\
x_6 &= (0, -b, 0) & y_6 &= (-1, b-1, -1) \\
x_7 &= (-1, -1, 0) & y_7 &= (0, 0, -1) \\
x_8 &= (0, -1, -1) & y_8 &= (-1, 0, 0) \\
x_9 &= (-1, 0, -1) & y_9 &= (0, -1, 0)
\end{aligned}
$$

we can restate the final expression in (5.6) as a product of vector dot products in $\mathbb{R}^3$:

$$
\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)} = \prod_{1 \le i \le 9} \frac{x_i \cdot (u, v, w)}{y_i \cdot (u, v, w)}. \tag{5.7}
$$

Remark 5.11. Note that for each pair of vectors (x_i, y_i), we have $x_i = -y_i - (1,1,1)$. With this in mind, each (x_i, y_i) can be viewed as a pair of 3D Haiman arrows. See [Rüh18] for more about 3D Haiman arrows. Moreover, in the Calabi-Yau specialization, each ratio $\frac{x_i \cdot (u,v,w)}{y_i \cdot (u,v,w)}$ simplifies to -1, so $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$ is $(-1)^9 = -1$. With this in hand, for the remainder of this chapter we need only consider parameter tuples (u, v, w) which do not sum to zero.

We prove Theorem 5.10 by considering the sign contribution of each vector pair's term $\frac{x_i \cdot (u,v,w)}{y_i \cdot (u,v,w)}$ to the overall product $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$. The sign contribution of the pair (x_i, y_i) can be obtained by considering the relationship between the parameter vector $(u, v, w) \in \mathbb{R}^3$ and the planes whose normal vectors are x_i and y_i. In what follows, we identify a vector pair (x_i, y_i) with the pair of planes for which x_i and y_i are the normal vectors.

We next provide a definition which makes discussing sign contribution easier.

Definition 5.12. We say that $(u, v, w) \in \mathbb{R}^3$ *lies between* a pair of planes of the form $jx + ky + \ell = 0$ and $(-j-1)x = (-k-1)y + (-\ell - 1)z = 0$ in $\mathbb{R}^3$ if one of the following two conditions holds:

$$(u,v,w) \cdot (j,k,\ell) > 0 \text{ and } (u,v,w) \cdot (-j-1, -k-1, -\ell - 1) > 0$$

$$\text{or}$$

$$(u,v,w) \cdot (j,k,\ell) < 0 \text{ and } (u,v,w) \cdot (-j-1, -k-1, -\ell - 1) < 0.$$

In other words, (u, v, w) lies between the two planes if the normal vectors of both planes make acute angles with (u, v, w) or the normal vectors of both planes make obtuse angles with (u, v, w).

With this definition in mind, if a vector (u, v, w) lies between a plane pair (x_i, y_i), the term corresponding to (x_i, y_i) of $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$ is positive. Otherwise, the term is negative.

For $i = 7, 8, 9$, the pair (x_i, y_i) is perpendicular and does not depend on any of a, b, or c. Moreover, none of these six planes properly intersects the $+++$ and $---$ octants of $\mathbb{R}^3$.

Lemma 5.13. *For all (u, v, w) in the $+++$ and $---$ octants of $\mathbb{R}^3$*

$$\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)} = \prod_{7 \le i \le 9} \frac{x_i \cdot (u,v,w)}{y_i \cdot (u,v,w)} > 0.$$

Proof. Note that

$$x_7 \cdot (u, v, w) = -u - v < 0 \quad \text{and} \quad y_7 \cdot (u, v, w) = -w < 0.$$

So

$$\frac{x_7 \cdot (u, v, w)}{y_7 \cdot (u, v, w)} > 0.$$

The same result for pairs (x_8, y_8) and (x_9, y_9) follows from symmetry. □

The proof proceeds identically for $u, v, w < 0$.

Remark 5.14. Every other octant in $\mathbb{R}^3$ contains regions of points (u, v, w) which yield

$$\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)} = \prod_{7 \le i \le 9} \frac{x_i \cdot (u, v, w)}{y_i \cdot (u, v, w)} > 0.$$

However, each of these octants also has regions of points for which

$$\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)} = \prod_{7 \le i \le 9} \frac{x_i \cdot (u, v, w)}{y_i \cdot (u, v, w)} < 0$$

as well.

This remark motivates our splitting of the problem into two cases: tuples of parameters $(u, v, w) \in \mathbb{R}^3$ in the $+++$ or $---$ octants, and parameter values in one of the other six octants.

5.3.3 Parameter Tuples in the $+++$ and $---$ Octants of $\mathbb{R}^3$

Since both planes in each pair pass through the origin, a parameter vector (u, v, w) in a given octant of $\mathbb{R}^3$ will have the same relationship with the plane pair (x_i, y_i) as the vector $(-u, -v, -w)$ does in the octant obtained by reversing all of the signs. Hence our conclusions about the parameter vectors in the $+++$ octant which follow can be analogously applied to the $---$ octant.

For parameter tuples (u, v, w) in the $+++$ octant of $\mathbb{R}^3$, we begin by eliminating the first case presented in Theorem 5.10, i.e. we show that for any $(u, v, w) \in \mathbb{R}^3$ with $u, v, w \neq 0$, we can select $(a, b, c) \in \mathbb{Z}^3_{>0}$ such that $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)} > 0$. By Lemma 5.13, the contribution from each of the last three vector pairs (x_7, y_7), (x_8, y_8), (x_9, y_9) is

positive. It remains to show that we can pick dimensions a, b and c so that the contribution from the other six vector pairs is positive overall as well. The following proposition shows just this: that for most any $(u, v, w) \in \mathbb{R}^3$, we can find a tuple $(a, b, c) \in \mathbb{Z}^3_{>0}$ such that for $1 \leq i \leq 6$, the term corresponding to the pair (x_i, y_i) in the product representation of $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$ is negative. Hence with this choice of (a, b, c) the first six pairs of vectors above make an overall positive contribution to the product $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$. Note that this proposition is much more general than necessary, as in the $+++$ octant, u, v, and w will all be positive by definition. The excess generality will be helpful for our work in Section 5.3.4 below.

Proposition 5.15. *Suppose $(u, v, w) \in \mathbb{R}^3$ such that $u, v, w \neq 0$. Then there exists $m \in \mathbb{Z}$ such that if $(a, b, c) = (m, m^2, 1)$, then for $1 \leq i \leq 6$, $x_i \cdot (u, v, w)$ and $y_i \cdot (u, v, w)$ have opposite signs.*

Proof. Suppose $(u, v, w) \in \mathbb{R}^3$ such that $u, v, w \neq 0$. Suppose further that $a = m$, $b = m^2$ and $c = 1$. Then $(a, b, c) = (m, m^2, 1) \in \mathbb{Z}^3_{>0}$ depends only on a choice of m. In this setting, note that

$$\begin{aligned} x_3 \cdot (u, v, w) &= m^2 v + mu \\ y_3 \cdot (u, v, w) &= (-m - 1)u + (-m^2 - 1)v - w \\ &= -m^2 v - mu - u - v - w. \end{aligned}$$

These are both quadratics in m and, regardless of the sign of y, have leading terms with opposite signs. Hence for $m = m_3$ sufficiently large, $x_3 \cdot (u, v, w)$ and $y_3 \cdot (u, v, w)$ have opposite signs. We make a similar argument for the pair v_4 and u_4. Note that

$$\begin{aligned} x_4 \cdot (u, v, w) &= -m^2 v - mu + w \\ y_4 \cdot (u, v, w) &= (m - 1)u + (m^2 - 1)v - 2w \\ &= m^2 v + mu - u - v - 2w. \end{aligned}$$

So for $m = m_4$ sufficiently large, $x_4 \cdot (u, v, w)$ and $y_4 \cdot (u, v, w)$ have opposite signs. Select $m \in \mathbb{Z}_{>0}$ such that

$$m > \max\left(m_3, m_4, \frac{v + 2w}{u} + 1, \frac{w}{u}, \frac{u + v + w}{u}\right)$$

and

$$m^2 > \max\left(\frac{u+2w}{v}+1, \frac{w}{v}, \frac{u+v+w}{v}\right).$$

By our work above, each of the arrow pairs x_3, y_3 and for v_4, u_4 contribute negatively to the product $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$. Observe that

$$\begin{aligned} x_1 \cdot (u,v,w) &= (m-1)u - v - 2w \\ y_1 \cdot (u,v,w) &= -mu + w. \end{aligned}$$

Since

$$m > \max\left(\frac{v+2w}{u}+1, \frac{w}{u}\right),$$

we have the following inequalities. If $u > 0$, then $m > \frac{w}{u}$ implies $(-a,0,c) \cdot (u,v,w) = y_1 \cdot (u,v,w) < 0$, and $m > \frac{v+2w}{u}+1$ implies $(a-1,-1,-c-1)\cdot(u,v,w) = x_1 \cdot (u,v,w) > 0$. If $u < 0$, then $m > \frac{v+2w}{u}+1$ implies $(a-1,-1,-c-1)\cdot(u,v,w) = x_1 \cdot (u,v,w) < 0$, and $m > \frac{w}{u}$ implies $(-a,0,c)\cdot(u,v,w) = y_1 \cdot (u,v,w) > 0$. We proceed similarly with the pair x_2, y_2. Observe that

$$\begin{aligned} x_2 \cdot (u,v,w) &= (m^2-1)v - u - 2w \\ y_2 \cdot (u,v,w) &= -m^2 v + w. \end{aligned}$$

Since

$$m^2 > \max\left(\frac{u+2w}{v}+1, \frac{w}{v}\right),$$

we have the following inequalities. If $v > 0$, then $m^2 > \frac{w}{v}$ implies $-m^2v + w < 0$ and $m^2 > \frac{u+2w}{v}+1$ implies $(m^2-1)v - u - 2w > 0$. If $v < 0$, then $m^2 > \frac{u+2w}{v}+1$ means that $(m^2-1)v - u - 2w < 0$ and $m^2 > \frac{w}{v}4$ means that $-m^2v + w > 0$.

It remains to check the last two pairs x_5, y_5 and x_6, y_6. First,

$$\begin{aligned} x_5 \cdot (u,v,w) &= -mu \\ y_5 \cdot (u,v,w) &= (m-1)u - v - w \\ &= mu - u - v - w. \end{aligned}$$

Since $m > \frac{u+v+w}{u}$, if $u > 0$, $mu > u+v+w$, so $y_5 \cdot (u,v,w) > 0$ and $x_5 \cdot (u,v,w) = -mu < 0$. Similarly, if $u < 0$, we have $mu < u, v, w$ so $y_5 \cdot (u,v,w) < 0$ and $x_5 \cdot (u,v,w) = -mu > 0$. Lastly,

$$x_6 \cdot (u,v,w) = -m^2 v$$

$$\begin{aligned} y_6 \cdot (u,v,w) &= (m^2-1)v - u - w \\ &= m^2 v - u - v - w. \end{aligned}$$

Since $m^2 > \frac{u+v+w}{v}$, if $v > 0$, then $m^2 v > u + v + w$ so $y_6 \cdot (u,v,w) > 0$ and $x_6 \cdot (u,v,w) = -m^2 v < 0$. Similarly, if $v < 0$, we have $m^2 v < u + v + w$ meaning $y_6 \cdot (u,v,w) < 0$ and $x_6 \cdot (u,v,w) = -m^2 v > 0$. So both of the final two pairs x_5, y_5 and x_6, y_6 contribute negatively to the product $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$. □

Using a, b, and c as determined above, the contributions from each of the first six vector pairs (x_i, y_i) is negative and the contribution from each of the last three is positive. So overall, $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)} > 0$ for (u,v,w) in the $+++$ octant of $\mathbb{R}^3$.

It remains to show that the overall sign on $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$ can be negative for vectors (u,v,w) in the $+++$ octant. As the last three pairs of vectors are guaranteed to each give positive contributions to $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$, we consider the first six pairs of vectors. Via the lemmas below, we show that we can find parameter values a, b, and c such that

- there exist a, b, and c such that (u,v,w) lies between the plane pair (x_2, y_2), and
- none of the other plane pairs (x_i, y_i) for $1 \leq i \leq 6$.

We begin with the first bulleted statement.

Lemma 5.16. *Suppose p and q are positive real numbers. Further, suppose that α, β, and M are positive integers. If p and q are $\mathbb{Q}$-linearly independent, $\{\alpha p - \beta q\}$ is dense in $\mathbb{R}$.*

Proof. Note that $\frac{p}{q} \notin \mathbb{Q}$ and for any $\beta \in \mathbb{Z}_{>0}$, $\beta\left(-\frac{q}{p}\right) < 0$. Let $\varepsilon > 0$. By Weyl's Equidistribution Theorem [Wey16], $\left\{\left\{\beta\left(-\frac{q}{p}\right)\right\} : \beta \in \mathbb{Z}_{>0}\right\}$ is dense in $[0,1]$. With this in mind, for $n \in \mathbb{R}$, we can pick

$$\beta = \left\lceil 1 + \max\left(M, -\frac{p}{q}(n - M - 1)\right)\right\rceil$$

so that $\left|\{n\} - \left\{\beta\left(-\frac{q}{p}\right)\right\}\right| < \varepsilon$. Pick α to be the integer closest to $n + \beta\left(\frac{q}{p}\right)$. By design, $\alpha > M$. Then

$$\alpha - \beta\left(\frac{q}{p}\right) = k$$

is such that $|k-n| < \varepsilon$. So $\{\alpha - \beta\left(\frac{q}{p}\right) : \alpha, \beta \in \mathbb{Z}_{>0}\}$ is dense in $\mathbb{R}$. Multiplying a set by a real number will not affect density, meaning

$$\{\alpha p - \beta q : \alpha, \beta \in \mathbb{Z}_{>0}\}$$

is dense in $\mathbb{R}$. □

Corollary 5.17. *Suppose p, q, and r are positive real numbers such that $p+q+r > 0$. For any $M \in \mathbb{Z}_{>0}$ there exist infinitely many $\alpha, \beta \in \mathbb{Z}_{>0}$, where $\alpha, \beta > M$ so that $0 < \alpha p - \beta q < p+q+r$.*

Proof. If p and q are $\mathbb{Q}$-linearly independent, this follows directly from Lemma 5.16. Suppose p and q are $\mathbb{Q}$-linearly dependent, i.e. $p, q \in \mathbb{R}$ such that $p = mq$ for some $m \in \mathbb{Q}$. Consider the sequence $\{q_k\}$ where $q_k = q \cdot 2^{\frac{1}{k}}$. This sequence converges to q and since $2^{\frac{1}{k}}$ is irrational for all k, p and q_k are $\mathbb{Q}$-linearly independent. By Lemma 5.16, there exist sequences $\{\alpha_k\}$ and $\{\beta_k\}$ corresponding to $\{q_k\}$ such that for all k,

$$0 < \alpha_k p - \beta_k q_k < p + q_k + r.$$

Since these new sequences $\{\alpha_k\}$ and $\{\beta_k\}$ consist only of positive integers, as k increases and q_k approaches q the sequences $\{\alpha_k\}$ and $\{\beta_k\}$ must stabilize. The stable values α_s and β_s then satisfy

$$0 < \alpha_s p - \beta_s q < p + q + r.$$

□

Remark 5.18. This can be done analogously for $p+q+r < 0$. We omit the proof here.

Proposition 5.19 (Lying Between Pair (x_2, y_2))**.** *Any (u, v, w) in the $+++$ octant of $\mathbb{R}^3$ lies between the planes*

$$(-1)\,x \;+\; (b-1)\,y \;+(-c-1)\,z \;= 0 \text{ and } (-b)\,y \;+(c)\,z \;= 0$$

in $\mathbb{R}^3$, i.e. either

$$bv - cw > u + v + w \text{ and } -bv + cw > 0 \tag{5.8}$$

or

$$bv - cw < u + v + w \text{ and } -bv + cw < 0 \tag{5.9}$$

for some values of b and c in $\mathbb{Z}_{>0}$ with $b > \frac{u+v+w}{v}$ and $c > \frac{au}{w}$.

Remark 5.20. It may not be clear to the reader why we insist that $b > \frac{u+v+w}{v}$ and $c > \frac{au}{w}$ in the statement of this proposition now; however, these two conditions will allow us to avoid other plane pairs in Lemma 5.23 and Lemma 5.24 below.

Proof. In the case of (5.8), finding $b, c \in \mathbb{Z}_{>0}$ so that (u, v, w) lies between these two planes is equivalent to finding b and c so that

$$0 < bv - cw < u + v + w.$$

This is true by Corollary 5.17 with $v = p$, $w = q$, and $M = \max\left(\frac{u+v+w}{v}, \frac{au}{w}\right)$. □

The following lemmas show that a parameter set a, b, and c as selected above to guarantee (u, v, w) lies between the plane pair (x_2, y_2) can be constructed to avoid the other five plane pairs (x_i, y_i) for $1 \leq i \leq 6$.

Lemma 5.21 (Avoiding Plane Pair 3)**.** *Any vector (u, v, w) in the $+++$ octant of $\mathbb{R}^3$ will not lie between the plane pair (x_3, y_3).*

Proof. Any vector (u, v, w) in the $+++$ octant of $\mathbb{R}^3$ will have

$$au + bv > 0 \text{ and } (-a-1)\,u \;+\; (-b-1)\,v \;+(-1)\,w \;< 0.$$

So no vector in this octant can lie between the (x_3, y_3) plane pair. □

Lemma 5.22 (Avoiding Plane Pair 5)**.** *Suppose (u, v, w) is in the $+++$ octant of $\mathbb{R}^3$. For $a > \frac{u+v+w}{u}$, (u, v, w) will not lie between the plane pair (x_5, y_5).*

Proof. A vector (u, v, w) in $\mathbb{R}^3$ being between the planes

$$-ax = 0 \quad \text{and} \quad (a-1)x - y - z = 0$$

means

$$0 < au < u + v + w$$

since $u + v + w > 0$. For $a > \frac{u+v+w}{u}$ this will not occur. □

Lemma 5.23 (Avoiding Plane Pair 6)**.** *Any vector (u, v, w) in the $+++$ octant of $\mathbb{R}_3$ with $b > \frac{u+v+w}{v}$ will not lie between the plane pair (x_6, y_6).*

Proof. A vector (u, v, w) in $\mathbb{R}^3$ being between the planes

$$-bx = 0 \quad \text{and} \quad -x + (b-1)y - z = 0$$

means

$$0 < bv < u + v + w.$$

For $b > \frac{u+v+w}{v}$ this will not occur. □

Lemma 5.24 (Avoiding Plane Pair 4)**.** *Let (u, v, w) be a vector in the $+++$ octant of $\mathbb{R}^3$ and let $a > \frac{u+v+w}{u}$. If (u, v, w) lies between the pair (x_2, y_2), it will not also lie between the plane pair (x_4, y_4).*

Proof. Suppose (u, v, w) lies between the plane pair (x_2, y_2). Since $u + v + w > 0$ we have $0 < bv - cw < u + v + w$. So

$$au + bv - cw > au > \frac{u+v+w}{u}u = u + v + w > 0,$$

i.e. $au + bv - cw > u + v + w > 0$. □

Lemma 5.25 (Avoiding Plane Pair 1)**.** *For $c > \frac{au}{w}$, the vector (u, v, w) will not lie between the plane pair (x_1, y_1).*

Proof. Pick $c > \frac{au}{w}$. Then

$$au - cw < au - \frac{au}{w}w = 0.$$

so $au - cw < 0 < u + v + w$. □

Proposition 5.26. *For any (u, v, w) in the $+++$ octant of $\mathbb{R}^3$, there exist $a, b, c \in \mathbb{Z}^3_{>0}$ so that (u, v, w) lies between the plane pair (x_2, y_2) and no other pair (x_i, y_i) for $1 \leq i \leq 6$.*

Proof. Choose b, and c as in Lemma 5.19 with $a > \frac{u+v+w}{u}$. This assures that (u, v, w) lies between the plane pair (x_2, y_2) and misses the plane pairs (x_1, y_1), (x_3, y_3), (x_4, y_4), (x_5, y_5), (x_6, y_6). □

Proposition 5.26 shows that for vectors (u, v, w) in the $+++$ and $---$ octants of $\mathbb{R}^3$, there are parameters $a, b, c \in \mathbb{Z}^3_{>0}$ for which the product

$$\prod_{1 \leq i \leq 6} \frac{x_i \cdot (u, v, w)}{y_i \cdot (u, v, w)} < 0.$$

So overall,

$$\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)} = \prod_{1 \leq i \leq 9} \frac{x_i \cdot (u,v,w)}{y_i \cdot (u,v,w)} < 0.$$

5.3.4 Parameter Tuples in the Remaining Octants of $\mathbb{R}^3$

The octants of $\mathbb{R}^3$ which we haven't yet analyzed are the $++-$, $+-+$, $+--$, $--+$, $-+-$, and $-++$ octants. We can reduce to considering only the $+-+$ octant by symmetry. Indeed, the plane partition π was constructed by adding a box to σ in one of σ's three outside corners arbitrarily. Adding a box in one of the other two outside corners would provide an analogous set of planes; however, the configuration would be rotated 120° about the vector $(1,1,1) \in \mathbb{R}^3$. So constructions relating to the $+-+$ octant can be reproduced in the $-++$ and $++-$ octants. Furthermore, as described in Section 5.3.3, for (u,v,w) in a given octant of $\mathbb{R}^3$, the tuple $(-u,-v,-w)$ in the opposite octant has the same relationship with all plane pairs. So the $-+-$, $-++$, $--+$ octants are also covered by looking at the $+-+$ octant.

We show that for a given tuple of nonzero parameters (u,v,w) in $\mathbb{R}^3$, there exist dimensions which make $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$ positive and dimensions which make $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$ negative.

Proposition 5.27. *Suppose* $(u,v,w) \in \mathbb{R}^3$ *is such that* $u,v,w \neq 0$. *Then there exist* $a,b,c \in \mathbb{Z}_{>0}$ *such that*

$$\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)} = \prod_{1 \leq i \leq 9} \frac{x_i \cdot (u,v,w)}{y_i \cdot (u,v,w)} > 0$$

and $a',b',c' \in \mathbb{Z}_{>0}$ *such that*

$$\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)} = \prod_{1 \leq i \leq 9} \frac{x_i \cdot (u,v,w)}{y_i \cdot (u,v,w)} < 0.$$

Note that the final three terms in the product do not depend on the dimensions a, b, and c. So in what follows we guarantee that we can find dimensions a, b, and c which make the product of the first six terms

- positive so that $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$ has the same sign overall as the product of its last three terms, or
- negative so that $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$ has the opposite sign overall as the product of its last three terms.

We have already done sufficient work towards the former in Proposition 5.15, so we focus on the latter.

In order to show that we can find dimensions which make the overall sign of $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$ disagree with the sign of the product of the last three terms, we show that

- there exist a, b, and c such that (u, v, w) lies between the plane pair (x_3, y_3), and
- none of the other plane pairs (x_i, y_i) for $1 \leq i \leq 6$.

We begin with the first bulleted statement.

Proposition 5.28 (Lying Between Pair (x_3, y_3))**.** *Any (u, v, w) in the $+-+$ octant of $\mathbb{R}^3$ lies between the planes*

$$ax + by = 0 \text{ and } (-a-1)x + (-b-1)y - z = 0$$

i.e.

$$au + bv > 0 \text{ and } (-a-1)u + (-b-1)v - w > 0 \tag{5.10}$$

or

$$au + bv < 0 \text{ and } (-a-1)u + (-b-1)v - w < 0 \tag{5.11}$$

for some values of a and b in $\mathbb{Z}_{>0}$ with $a > \frac{u+v+w}{u}$ and $b > \frac{u+v+w}{v}$.

Remark 5.29. It may not be clear to the reader why we insist that $a > \frac{u+v+w}{u}$ and $b > \frac{u+v+w}{v}$ in the statement of this proposition now; however, these two conditions will allow us to avoid other plane pairs in Lemma 5.31 and Lemma 5.32 below.

Proof. In the case of (5.10), finding $a, b \in \mathbb{Z}_{>0}$ so that (u, v, w) lies between these two planes is equivalent to finding a and b so that

$$0 < au + bv < u + v + w.$$

This is true by Corollary 5.17 with $x = p$, $y = -q$, and $M = \max\left(\frac{u+v+w}{u}, \frac{u+v+w}{v}\right)$. In the case of (5.11), finding $a, b \in \mathbb{Z}_{>0}$ so that (u, v, w) lies between these two planes is equivalent to finding a and b so that

$$0 > au + bv > u + v + w.$$

This is true by the analogue to Corollary 5.17 also with $x = p$, $y = -q$, and $M = \max\left\{\frac{u+v+w}{u}, \frac{u+v+w}{v}\right\}$. □

The following lemmas show that a parameter set a, b, and c as selected above to guarantee (u, v, w) lies between the plane pair (x_3, y_3) can be constructed to avoid the other five plane pairs (x_i, y_i) for $1 \leq i \leq 6$. Note that this work is similar to what appears in Section 5.3.3 above, but requires slightly more thought, as for (u, v, w) in the $+ - +$ octant of $\mathbb{R}^3$, it is possible that $u + v + w < 0$. This was not possible for (u, v, w) in the $+ + +$ octant.

Lemma 5.30 (Avoiding Plane Pair 2)**.** *Any vector (u, v, w) in the $+ - +$ octant of $\mathbb{R}^3$ will not lie between the plane pair (x_2, y_2).*

Proof. Any vector (u, v, w) in the $+ - +$ octant of $\mathbb{R}^3$ will have

$$(-1)\,u \;+\; (b-1)\,v \;+(-c-1)\,w < 0 \text{ and } (-b)\,v \;+(c)\,w \;> 0.$$

So no such vector can lie between this plane pair. □

Lemma 5.31 (Avoiding Plane Pair 5)**.** *Suppose (u, v, w) is in the $+ - +$ octant of $\mathbb{R}^3$. For $a > \frac{u+v+w}{u}$, (u, v, w) will not lie between the plane pair (x_5, y_5).*

Proof. A vector (u, v, w) in $\mathbb{R}^3$ being between the planes

$$-ax = 0 \quad \text{and} \quad (a-1)x - y - z = 0$$

means

$$0 < au < u + v + w$$

if $u + v + w > 0$. For $a > \frac{u+v+w}{u}$ this will not occur. If $u + v + w < 0$, being between these planes means

$$0 > au > u + v + w.$$

For $a \geq 1$, this is not possible, since $a, u > 0$. □

Lemma 5.32 (Avoiding Plane Pair 6)**.** *If $b > \frac{u+v+w}{v}$, the vector (u, v, w) will not lie between the plane pair (x_6, y_6).*

Proof. A vector (u, v, w) in $\mathbb{R}^3$ being between the planes

$$-bx = 0 \quad \text{and} \quad -x + (b-1)y - z = 0$$

means

$$0 > bv > u + v + w$$

if $u + v + w < 0$. For $b > \frac{u+v+w}{v}$ this will not occur. If $u + v + w > 0$, being between these planes means

$$0 < bv < u + v + w.$$

For $b \geq 1$, this is not possible, since $v < 0$ and $b > 0$. □

Lemma 5.33 (Avoiding Plane Pair 4)**.** *For $c > \frac{|u+v+w|}{w}$, any vector (u, v, w) in the $+ - +$ octant of $\mathbb{R}^3$ which lies between the pair (x_3, y_3) will not lie between the plane pair (x_4, y_4).*

Proof. Suppose (u, v, w) lies between the plane pair (x_3, y_3). If $u + v + w > 0$, then because (u, v, w) lies between the pair (x_3, y_3), $0 < au + bv < u + v + w$. We have,

$$\begin{aligned} au + bv - cw < au + bv - \frac{|u + v + w|}{w} w &= au + bv - |u + v + w| \\ &< u + v + w - |u + v + w| \leq 0 \end{aligned}$$

so $au + bv - cw < 0 < u + v + w$.
If $u + v + w < 0$, then because (u, v, w) lies between the pair (x_3, y_3), $0 > au + bv > u + v + w$. In particular, $au + bv - u - v - w > 0$. Since $c > \frac{-u-v-w}{w}$,

$$\begin{aligned} au + bv - cw < au + bv - \frac{-u - v - w}{w} w &= au + bv + u + v + w \\ &< 0 + u + v + w = u + v + w \end{aligned}$$

so $au + bv - cw < u + v + w < 0$. □

Lemma 5.34 (Avoiding Plane Pair 1)**.** *For $c > \max\left(\frac{(a-1)u-v-w}{w}, \frac{au}{w}\right)$, the vector (u, v, w) will not lie between the plane pair (x_1, y_1).*

Proof.

$$au - cw > u + v + w \quad au - cw < 0$$

Suppose (u, v, w) lies between the plane pair (x_1, y_1). There are two cases: $u + v + w > 0$ and $u + v + w < 0$. If $u + v + w > 0$, pick $c > \frac{au}{w}$. Then

$$au - cw < au - \frac{au}{w} w = 0,$$

so $au - cw < 0 < u + v + w$.
If $x + y + z < 0$ pick $c > \frac{(a-1)u-v-w}{w}$,

$$au - cw < au - \frac{(a-1)u - v - w}{w}w = au - au + u + v + w = u + v + w$$

so $au - cw < u + v + w < 0$. □

Proposition 5.35. *For any (u, v, w) in the $+ - +$ octant of $\mathbb{R}^3$, there exist $a, b, c \in \mathbb{Z}^3_{>0}$ so that (u, v, w) lies between the plane pair (x_3, y_3) and no other pair (x_i, y_i) for $1 \leq i \leq 6$.*

Proof. Choose a and b as in Lemma 5.28. Choosing $c > \max\left\{\frac{|x+y+z|}{z}, \frac{(a-1)x-y-z}{z}, \frac{ax}{z}\right\}$ assures that (u, v, w) lies between the plane pair (x_3, y_3) and misses the plane pairs (x_1, y_1), (x_2, y_2), (x_4, y_4), (x_5, y_5), (x_6, y_6) (by the previous five lemmas). □

Proposition 5.35 shows that for vectors (u, v, w) in the $+ - +$ octant of $\mathbb{R}^3$, there are parameters $a, b, c \in \mathbb{Z}^3_{>0}$ for which the product

$$\prod_{1 \leq i \leq 6} \frac{x_i \cdot (u, v, w)}{y_i \cdot (u, v, w)} < 0.$$

For all parameter tuples (u, v, w) in the $+ - +$ octant of $\mathbb{R}^3$, we have now guaranteed existence of a, b, and c which determine the sign of $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$ to be whatever we like.

Proof. (of Proposition 5.27) For a given tuple of parameters (u, v, w) in the $+ - +$ octant of $\mathbb{R}^3$ the last three terms of the product

$$\prod_{1 \leq i \leq 9} \frac{x_i \cdot (u, v, w)}{y_i \cdot (u, v, w)} \tag{5.12}$$

are determined by the relationship of the vector (u, v, w) with the last three plane pairs, which do not depend on the dimensions a, b and c. Proposition 5.15 shows that selecting good values of the plane partition dimensions a, b, and c guarantees that the product of first six terms can be made to be positive, i.e. the overall sign of (5.12) will match that of the product of its final three terms. Proposition 5.35 shows that selecting good values of the plane partition dimensions a, b, and c guarantees that the product of first six terms can be made to be negative, i.e. the overall sign of (5.12) will oppose that of the product of its final three terms. □

We can finally complete what we set out to do.

Proof. (of Theorem 5.10) Suppose a parameter tuple (u, v, w) in $\mathbb{R}^3$ where $u, v, w \neq 0$ lies in either of the $+++$ octant or the $---$ octant of $\mathbb{R}^3$. Proposition 5.15 and Proposition 5.26 show that we can pick dimensions a, b, and c so that $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$ can be positive or negative as we wish. Suppose a parameter tuple (u, v, w) in $\mathbb{R}^3$ where $u, v, w \neq 0$ lies in any other octant of $\mathbb{R}^3$. Proposition 5.27 shows that we can select dimensions a, b, and c so that $\frac{\mathsf{w}(\pi)}{\mathsf{w}(\sigma)}$ can be positive or negative, as we wish. $\square$

CHAPTER 6

PROBABILITY DISTRIBUTIONS

6.1 The one box Pieri rule and the CKP identity

There is a curious connection between Jack functions and the *weighted branching rule for hook lengths* (WHL) of Ciocan-Fontanine, Konvalinka, and Pak.

Theorem 6.1 ([CKP10], page 2, The Weighted Branching Rule for Hook Lengths)**.**

$$\sum_{\text{corners } (r,s)\in\lambda} x_r y_s \prod_{i=1}^{r-1}\left(1+\frac{x_i}{x_{i+1}+\cdots+x_r+y_{s+1}+\cdots+y_{\lambda_i}}\right) \cdot\prod_{j=1}^{s-1}\left(1+\frac{y_j}{x_{r+1}+\cdots+x_{\lambda'_j}+y_{j+1}+\cdots+y_s}\right)=\sum_{(i,j)\in\lambda} x_i y_j \qquad \text{(WHL)}$$

The authors of [CKP10] make no mention of Jack functions in their description and exploration of this identity. However, there is a connection.

Proposition 6.2 (P., 2023)**.** *Specializing $x_k = t$ and $y_k = 1$ in* (WHL) *for all k implies the one-box version of the Jack Pieri rule.*

Recall that the one-box version of the Jack Pieri rule is given in (2.20). We are interested in generalizing (WHL). In particular, we would like to find a generalization of the one box version of the Jack Pieri rule that yields a pullback of (WHL).

There are several options to consider as the top right entry of Figure 6.1 since there are several ways to generalize the idea of "adding one box to a diagram." For example, in terms of Schur symmetric functions, both the Pieri rule and the Murnaghan-Nakayama rule specialize to the one-box Pieri rule. There are likely other possibilities worth considering in the Jack case.

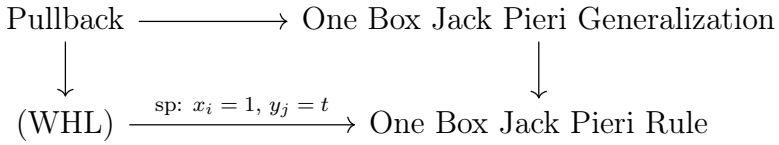


Figure 6.1. The relationship between (WHL) and the one box Pieri rule.

This work yielded three interesting probability distributions on Young diagrams and tableaux.

6.2 Downward Induction

The first probability distribution on Young diagrams below gives rise to the following probability distribution on Young tableaux.

6.2.1 Probability Distribution on Young Diagrams

The following lemma provides a probability distribution on partitions μ of an integer $n-1$ for each partition λ of n.

Lemma 6.3 (P.-Young, 2023)**.** *Suppose the size of λ is n. Then*

$$\sum_{\mu \vdash n-1} \frac{\langle J_\mu J_1, J_\lambda \rangle_t}{j_\mu \cdot n \cdot t} = 1. \tag{6.1}$$

The idea of the following proof is courtesy of Valentin Féray.

Proof. Equation (10.29) on page 382 of [Mac98] states that if

$$J_\lambda = \sum_\mu c_{\lambda,\mu} p_\mu$$

then $c_{\lambda,1^n} = 1$. So for any $\lambda \vdash n$ we have

$$\begin{aligned} \langle p_{1^n}, J_\lambda \rangle_t = \langle p_{1^n}, \sum_\mu c_{\lambda,\mu} p_\mu \rangle_t &= \sum_\mu c_{\lambda,\mu} \langle p_{1^n}, p_\mu \rangle_t \\ &= c_{\lambda,1^n} \langle p_{1^n}, p_{1^n} \rangle_t = 1 \cdot t^{\ell(1^n)} \cdot z_{1^n} = n! \cdot t^n. \end{aligned}$$

Write $p_{1^n} = p_1^n$ as a sum over Jack symmetric functions.

$$p_1^n = n! \cdot t^n \cdot \sum_{\lambda \vdash n} \frac{J_\lambda}{j_\lambda}$$

We can use this to simplify the expression we are interested in. Note that $p_1 = J_1$.

$$\begin{aligned} \sum_{\mu \vdash n-1} j_\mu^{-1} \langle J_\mu J_1, J_\lambda \rangle_t &= \left\langle \frac{1}{(n-1)! \cdot t^{n-1}} \cdot p_1^{n-1} \cdot J_1, J_\lambda \right\rangle_t \\ &= \frac{1}{(n-1)! \cdot t^{n-1}} \left\langle p_1^{n-1} \cdot J_1, J_\lambda \right\rangle_t \end{aligned}$$

$$= \frac{1}{(n-1)! \cdot t^{n-1}} \left\langle (J_1)^{n-1} \cdot J_1, J_\lambda \right\rangle_t$$
$$= \frac{1}{(n-1)! \cdot t^{n-1}} \left\langle (J_1)^{n}, J_\lambda \right\rangle_t$$
$$= \frac{n! \cdot t^n}{(n-1)! \cdot t^{n-1}} = nt \tag{6.2}$$

The penultimate equality follows from Proposition 2.3 in [Sta89]. This is what we wanted. □

6.2.2 Probability Distribution on Young Tableaux

From Lemma 6.3, we have derived a probability distribution on Young tableaux by inducting on the size of the partition μ.

Theorem 6.4 (P., 2024)**.** *Let $\lambda \vdash n$ and let $\varnothing = \lambda^0 \subseteq \lambda^1 \subseteq \lambda^2 \subseteq \cdots \subseteq \lambda^{n-1} \subseteq \lambda$ be a saturated chain in Young's lattice beginning at the empty partition $\varnothing$ and ending at λ. (In other words, $\lambda^i \lessdot \lambda^{i+1}$.) Then*

$$1 = \sum_{\varnothing = \lambda^0 \subseteq \lambda^1 \subseteq \lambda^2 \subseteq \cdots \subseteq \lambda^{n-1} \subseteq \lambda} \frac{1}{n! t^n} \prod_{i=0}^{n-1} \frac{\langle J_{\lambda^i} J_1, J_{\lambda^{i+1}} \rangle_t}{j_{\lambda^i}}. \tag{6.3}$$

When t is positive, the summand corresponding to $\varnothing = \lambda^0 \subseteq \lambda^1 \subseteq \lambda^2 \subseteq \cdots \subseteq \lambda^{n-1} \subseteq \lambda$ on the right hand side in (6.3) can be viewed as the probability in this distribution of selecting the standard Young tableau generated by putting i into the box $\lambda^i / \lambda^{i-1}$.

Proof. Multiplying both sides of (6.1) by J_λ yields

$$\sum_{\mu \vdash n-1} \frac{\langle J_\mu J_1, J_\lambda \rangle_t J_\lambda}{j_\mu \cdot n \cdot t} = J_\lambda.$$

Note that the only terms contributing nontrivially to this sum are those corresponding to $\mu \subseteq \lambda$. We can write μ in this form as well.

$$\sum_{\nu \vdash n-2} \frac{\langle J_\nu J_1, J_\mu \rangle_t J_\mu}{j_\nu \cdot (n-1) \cdot t} = J_\mu.$$

Substituting for J_μ yields

$$\sum_{\mu \vdash n-1} \frac{\left\langle \left(\sum_{\nu \vdash n-2} \frac{\langle J_\nu J_1, J_\mu \rangle_t J_\mu}{j_\nu \cdot (n-1) \cdot t} \right) J_1, J_\lambda \right\rangle_t J_\lambda}{j_\mu \cdot n \cdot t} = J_\lambda.$$

Now we cancel a J_λ from both sides and simplify via bilinearity.

$$\sum_{\mu\vdash n-1}\sum_{\nu\vdash n-2}\frac{\langle J_\nu J_1, J_\mu\rangle_t}{j_\nu\cdot(n-1)\cdot t}\frac{\langle J_\mu J_1, J_\lambda\rangle_t}{j_\mu\cdot n\cdot t}=1$$

Repeat this substitution $n-2$ times more until we reach the empty partition. Then

$$\frac{1}{n!t^n}\sum_{\varnothing\subseteq\lambda^0\subseteq\lambda^1\subseteq\lambda^2\subseteq\cdots\subseteq\lambda^{n-1}\subseteq\lambda}\prod_{i=0}^{n-1}\frac{\langle J_{\lambda^i}J_1, J_{\lambda^{i+1}}\rangle_t}{j_{\lambda^i}}=1$$

where $\lambda^i\vdash i$. □

Our future work entails developing a hook walk that yields this distribution of tableaux in the style of [GNW79] and [GNW84].

6.3 Upward Induction

We have also discovered a related lemma that yields a similar probability distribution. However, this distribution is over partitions $\lambda\vdash n$ and the distribution depends on choice of a *smaller* partition $\mu\vdash n-1$.

Lemma 6.5 (P.-Young, 2024)**.** *Fix a partition $\mu\vdash n-1$. Then*

$$\sum_{\lambda:\mu\nearrow\lambda}\frac{\langle J_\mu J_1, J_\lambda\rangle_t}{j_\lambda}=1. \tag{6.4}$$

Our initial experiments suggest that the resulting distribution can be achieved combinatorially by the most straightforward Jack generalization of the outside hook walk presented in [GNW84]: Start at a cell (i,j) outside of λ such that $i\geq\lambda'_1$ and $j\geq\lambda_1$. Then move to a given arm cell with probability $\frac{t}{ta(\square)+\ell(\square)}$ and move to a given leg cell with probability $\frac{1}{ta(\square)+\ell(\square)}$. This is illustrated in Figure 6.2. Repeat this until an outside corner of the diagram is reached. Add this cell to the diagram and then repeat this entire process with the enlarged diagram.

This being said, our current proof does not use such methods.

Proof. Recall that $J_1=p_1$ and $(p_1)^n=p_{1^n}$. By Prop 2.3 from [Sta89], we have

$$\begin{aligned}\sum_{\lambda:\mu\nearrow\lambda}\frac{\langle J_\mu J_1, J_\lambda\rangle_t}{j_\lambda}&=\left\langle J_\mu J_1, \sum_{\lambda:\mu\nearrow\lambda}\frac{J_\lambda}{j_\lambda}\right\rangle_t=\frac{1}{n!\cdot t^n}\langle J_\mu J_1,(J_1)^n\rangle_t\\&=\frac{1}{n!\cdot t^n}\langle J_\mu\cdot p_1,(p_1)^n\rangle_t=\frac{1}{n!\cdot t^n}\langle J_\mu\cdot p_1,p_{1^n}\rangle_t.\end{aligned}$$

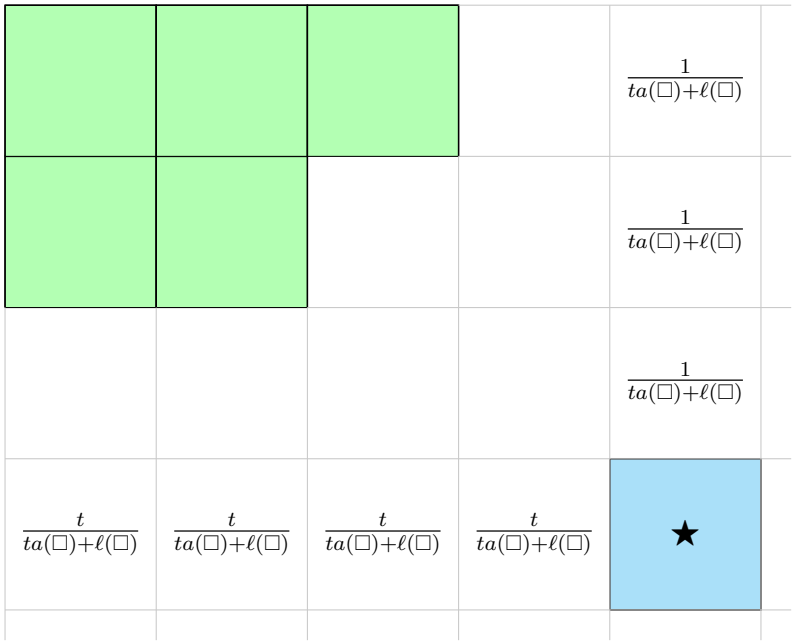


Figure 6.2. First step probabilities for a Lemma 6.5 hook walk.

Suppose that we write $J_\mu = \sum_{\nu \vdash n-1} c^\mu_\nu p_\nu$. So we can write

$$\frac{1}{n! \cdot t^n} \langle J_\mu \cdot p_1, p_{1^n} \rangle_t = \frac{1}{n! \cdot t^n} \left\langle \left(\sum_{\nu \vdash n-1} c^\mu_\nu p_\nu \right) p_1, p_{1^n} \right\rangle_t = \frac{1}{n! \cdot t^n} \sum_{\nu \vdash n-1} c^\mu_\nu \langle p_\nu p_1, p_{1^n} \rangle_t .$$

Since the power sum symmetric functions are orthogonal with respect to this scalar product the only nontrivial summand is the one corresponding to $\nu = 1^{n-1}$, as $p_{1^{n-1}} \cdot p_1 = p_{1^n}$. With that in mind, we have

$$\begin{aligned} \frac{1}{n! \cdot t^n} \sum_{\nu \vdash n-1} c^\mu_\nu \langle p_\nu p_1, p_{1^n} \rangle_t &= \frac{c^\mu_{1^{n-1}}}{n! \cdot t^n} \langle p_{1^n}, p_{1^n} \rangle_t \\ &= \frac{c^\mu_{1^{n-1}}}{n! \cdot t^n} \cdot z_{1^n} \cdot t^{\ell(1^n)} \\ &= \frac{c^\mu_{1^{n-1}}}{n! \cdot t^n} \cdot n! \cdot t^n \\ &= c^\mu_{1^{n-1}}. \end{aligned} \tag{6.5}$$

It remains to show that $c^\mu_{1^{n-1}} = 1$. We use Corollary 4.3.1 from [HW18]. We need not bother with the sum as there is a unique graph on n vertices with n connected

singleton components. Hence

$$c^{\mu}_{1^{n-1}} = J^{(t)}_{\mu'}(x)|_{p_{1^n}} = (-1)^0 \prod_{\{u,v\}\in H\setminus G_\mu} -\text{hook}^{(t)}_{\mu}(u). \tag{6.6}$$

The product is the trivial since there aren't any edges in H. So $c^{\mu}_{1^{n-1}} = 1$. □

Future work on this project involves finding a tableaux probability distribution analogue to Lemma 6.5 much as there was in Section 6.2. We expect this distribution to be over *skew tableaux* of shape λ/μ where λ and μ are fixed.

Definition 6.6. A *skew shape* λ/μ is the collection of cells which belong to λ, but not to μ, i.e.

$$\lambda/\mu = \{(i,j) \in \lambda | (i,j) \notin \mu\}.$$

A *skew tableau* of shape λ/μ is a filling of the cells of the shape λ/μ with the set $\{1, 2, \ldots, |\lambda| - |\mu|\}$ increasing rightward and downward.

Figure 6.3 shows an example of such a tableaux.

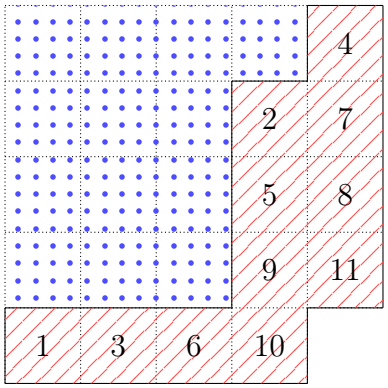


Figure 6.3. The striped zone is a skew tableaux of shape $(5,5,5,5,4)/(4,3,3,3)$.

APPENDIX A

COMPUTATIONS OF JACK SCALAR PRODUCTS

Throughout the process of the work described herein, we have established several evaluations of scalar products of Jack functions. In keeping with our goals in Chapter 6, many of these ideas arise from considering generalizations of the one box version of the Jack Pieri rule.

A.1 Single Row Pairings

Proposition A.1. *Let λ be the one-row partition of size n. Then*

$$\langle (J_1)^{n-2} J_2, J_\lambda \rangle_t = n! \cdot t^n \cdot (t+1).$$

Proof. By Proposition 2.3 from [Sta89], we know that

$$(J_1)^{n-2} = (x_1 + x_2 + \cdots)^{n-2} = t^{n-2}(n-2)! \sum_{\nu \vdash (n-2)} J_\nu j_\nu^{-1}.$$

This means that we can write

$$\langle (J_1)^{n-2} J_2, J_\lambda \rangle_t = t^{n-2}(n-2)! \sum_{\nu \vdash (n-2)} j_\nu^{-1} \langle J_\nu J_2, J_\lambda \rangle_t.$$

Note that all summands except for one will be zero. (See Proposition 5.3 in [Sta89].) Indeed, the only partition of $n-2$ that permits an addition of a horizontal 2-strip to form λ is the one-row partition of size $n-2$. Henceforth we will call this partition μ. We can now do away with the sum.

$$\langle (J_1)^{n-2} J_2, J_\lambda \rangle_t = t^{n-2} \cdot (n-2)! \cdot j_\mu^{-1} \cdot \langle J_\mu J_2, J_\lambda \rangle_t \tag{A.1}$$

We can use Pieri's rule to evaluate the scalar product $\langle J_\mu J_2, J_\lambda \rangle_t$. We get

$$\langle J_\mu J_2, J_\lambda \rangle_t = \prod_{s \in \mu} h_*^\mu(s) \cdot 2! \cdot t^2 \cdot \prod_{s \in \mu} h_\lambda^*(s) \cdot \prod_{s \in \lambda/\mu} h_*^\lambda(s). \tag{A.2}$$

We now examine the last product on the right. There are exactly two boxes in λ/μ. The lower lambda hook length of these boxes is written inside each of them in Figure A.1.

This means we can rewrite (A.2) as

$$\langle J_\mu J_2, J_\lambda \rangle_t = 2t^2 \cdot (1+t) \cdot \prod_{s \in \mu} h_*^\mu(s) h_\lambda^*(s).$$

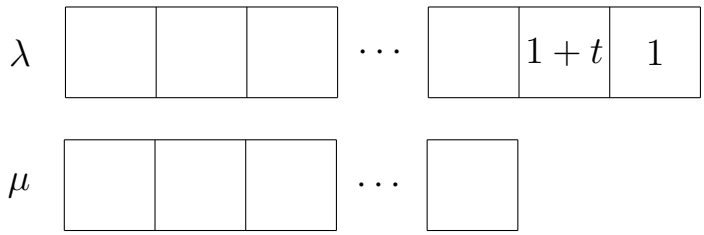


Figure A.1. The partitions λ and μ as described in (A.2)

Now we can plug this into (A.1).

$$\begin{aligned}\langle (J_1)^{n-2} J_2, J_\lambda \rangle_t &= t^{n-2} \cdot (n-2)! \cdot j_\mu^{-1} \cdot \left(2t^2 \cdot (1+t) \cdot \prod_{s \in \mu} h_*^\mu(s) h_\lambda^*(s) \right) \\ &= 2t^n \cdot (n-2)! \cdot (1+t) \cdot j_\mu^{-1} \cdot \prod_{s \in \mu} h_*^\mu(s) h_\lambda^*(s)\end{aligned}$$

We next replace j_μ with the explicit value in terms of products of hook lengths given by Theorem 5.8 in [Sta89] and combine it with the existing product.

$$\langle (J_1)^{n-2} J_2, J_\lambda \rangle_t = 2t^n \cdot (n-2)! \cdot (1+t) \cdot \prod_{s \in \mu} \frac{h_*^\mu(s) h_\lambda^*(s)}{h_*^\mu(s) h_\mu^*(s)}$$

After cancellation, this yields

$$\langle (J_1)^{n-2} J_2, J_\lambda \rangle_t = 2t^n \cdot (n-2)! \cdot (1+t) \cdot \prod_{s \in \mu} \frac{h_\lambda^*(s)}{h_\mu^*(s)}.$$

This product telescopes! The last $n-4$ terms of the numerator directly cancel with the first $n-4$ terms in the denominator. Additionally, all factors of t are eliminated since there are the same number of terms in the numerator and the denominator and each is linear in t. After cancellation, we are left with

$$\langle (J_1)^{n-2} J_2, J_\lambda \rangle_t = 2t^n \cdot (n-2)! \cdot (1+t) \cdot \frac{n \cdot (n-1)}{2}.$$

Simplifying yields

$$\langle (J_1)^{n-2} J_2, J_\lambda \rangle_t = t^n \cdot n! \cdot (1+t).$$

□

We can say much more than Proposition A.1. There's nothing special about adding a two-strip; we can add a strip of any length.

Proposition A.2. *Let λ be the one-row partition of size n. Then*

$$\langle (J_1)^{n-k} J_k, J_\lambda \rangle_t = n! \cdot t^n \cdot \prod_{0<i<k} (it+1). \tag{A.3}$$

Proof. This proof proceeds similarly to above. As before, we can write

$$\langle (J_1)^{n-k} J_k, J_\lambda \rangle_t = t^{n-k}(n-k)! \sum_{\nu \vdash (n-k)} j_\nu^{-1} \langle J_\nu J_k, J_\lambda \rangle_t.$$

Note that all summands except for one will be zero. The only nonzero summand will be the one corresponding to μ, the one-row partition of size $n-k$. We can now do away with the sum.

$$\langle (J_1)^{n-k} J_k, J_\lambda \rangle_t = t^{n-k} \cdot (n-k)! \cdot j_\mu^{-1} \cdot \langle J_\mu J_k, J_\lambda \rangle_t$$

We use Pieri's rule to evaluate the scalar product $\langle J_\mu J_k, J_\lambda \rangle_t$. We get

$$\langle J_\mu J_k, J_\lambda \rangle_t = \prod_{s \in \mu} h_*^\mu(s) \cdot k! \cdot t^k \cdot \prod_{s \in \mu} h_\lambda^*(s) \cdot \prod_{s \in \lambda/\mu} h_*^\lambda(s).$$

Next we write out j_μ^{-1} explicitly and cancel.

$$\langle (J_1)^{n-k} J_k, J_\lambda \rangle_t = t^{n-k} \cdot (n-k)! \cdot \prod_{s \in \mu} \frac{h_*^\mu(s) h_\lambda^*(s)}{h_*^\mu(s) h_\mu^*(s)} \cdot \prod_{s \in \lambda/\mu} h_*^\lambda(s) \cdot k! \cdot t^k$$

$$\langle (J_1)^{n-k} J_k, J_\lambda \rangle_t = t^n \cdot (n-k)! \cdot k! \cdot \prod_{s \in \mu} \frac{h_\lambda^*(s)}{h_\mu^*(s)} \cdot \prod_{s \in \lambda/\mu} h_*^\lambda(s)$$

The first product yields $\binom{n}{k}$. The second product can be rewritten as

$$\prod_{0<i<k} (it+1).$$

Now

$$\langle (J_1)^{n-k} J_k, J_\lambda \rangle_t = n! \cdot t^n \cdot \prod_{0<i<k} (it+1),$$

as claimed. □

A.2 Pairings with Simple Hooks

The following conjecture is closely related to the previous proposition. Indeed, Proposition A.2 can be recovered by setting $r = n$. However, the proof method of Proposition A.2 fails since the sum in the first step does not immediately simplify to one term.

Conjecture A.3. *Suppose that λ is a hook-shaped partition such that $|\lambda| = n$ and the first row of λ has length r. Then*

$$\langle (J_1)^{n-2} J_2, J_\lambda \rangle_t = (r-1)(n-2)!(rt + 2n - r)t^n.$$

At least one other case of the above conjecture is proven: the case when $r = n-1$.

Proposition A.4. *Suppose that λ is a hook-shaped partition such that $|\lambda| = n$ and the first row of λ has length $n-1$. In other words, suppose λ looks like the diagram below with first row of arbitrary length.*

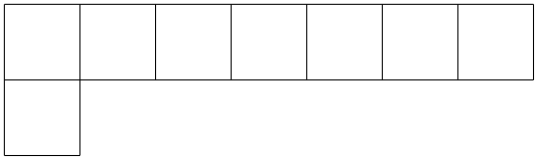


Figure A.2. A hook shaped partition as in Proposition A.4.

Then

$$\langle (J_1)^{n-2} J_2, J_\lambda \rangle_t = (n-2)(n-2)!((n-1)t + n + 1)t^n.$$

Proof. From Stanley's Proposition 2.3 [Sta89], we know that

$$\langle (J_1)^{n-2} J_2, J_\lambda \rangle_t = t^{n-2}(n-2)! \sum_{\mu \vdash (n-2)} j_\mu^{-1} \langle J_\mu J_2, J_\lambda \rangle_t.$$

There are only two terms in the sum that are nonzero, namely, the terms resulting from the partitions onto which a horizontal 2-strip may be added to obtain λ. We will call these partitions μ_1 and μ_2. See Figure A.3.

With this in mind, we now can write

$$\langle (J_1)^{n-2} J_2, J_\lambda \rangle_t = t^{n-2}(n-2)! \left(j_{\mu_1}^{-1} \langle J_{\mu_1} J_2, J_\lambda \rangle_t + j_{\mu_2}^{-1} \langle J_{\mu_2} J_2, J_\lambda \rangle_t \right).$$

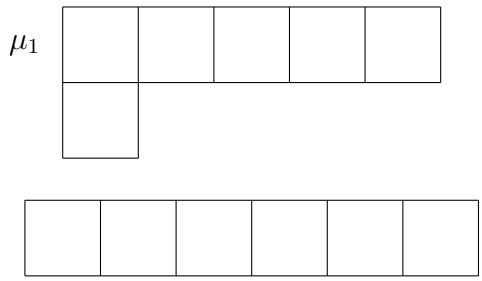


Figure A.3. The two partitions μ_1 and μ_2 .

Using the Pieri Rule to expand in terms of Jack hook lengths, we have the following.

$$\begin{aligned}
&\langle (J_1)^{n-2} J_2, J_\lambda \rangle_t \\
&= t^{n-2}(n-2)! \left(\frac{1}{\prod_{\square \in \mu_1} h^{\mu_1}_*(\square) h^*_{\mu_1}(\square)} \cdot \prod_{\square \in \mu_1} h^{\mu_1}_*(\square) \cdot 2 \cdot t^2 \cdot \prod_{\square \in \mu_1} h^*_\lambda(\square) \cdot \prod_{\square \in \lambda/\mu_1} h^\lambda_*(\square) \right. \\
&\left. + \frac{1}{\prod_{\square \in \mu_2} h^{\mu_2}_*(\square) h^*_{\mu_2}(\square)} \prod_{\square \in \mu_2} h^{\mu_2}_*(\square) \cdot \frac{(n-2)t}{1+(n-3)t} \cdot 2 \cdot t^2 \cdot \prod_{\square \in \mu_2} h^*_\lambda(\square) \cdot \frac{2+(n-2)t}{(n-1)t+1} \right)
\end{aligned}$$

Cancellation yields

$$\begin{aligned}
\langle (J_1)^{n-2} J_2, J_\lambda \rangle_t &= t^n \cdot 2 \cdot (n-2)! \\
&\cdot \left(\prod_{\square \in \mu_1} \frac{h^*_\lambda(\square)}{h^*_{\mu_1}(\square)} \cdot (1+t) + \prod_{\square \in \mu_2} \frac{h^*_\lambda(\square)}{h^*_{\mu_2}(\square)} \cdot \frac{(n-2)t}{1+(n-3)t} \cdot \frac{2+(n-2)t}{(n-1)t+1} \right).
\end{aligned}$$

We can further cancel from the products of fractions of hook lengths.

$$\begin{aligned}
\langle (J_1)^{n-2} J_2, J_\lambda \rangle_t &= t^n \cdot 2 \cdot (n-2)! \left(\frac{(n-1)t+1}{(n-3)t+1} \cdot \frac{(n-2)t \cdot (n-3)t \cdot \cdots \cdot 3t}{(n-4)t \cdot (n-5)t \cdot \cdots \cdot t} \cdot (1+t) \right. \\
&\left. + \frac{(n-1)t+1}{(n-2)t} \cdot \frac{(n-2)t \cdot (n-3)t \cdot \cdots \cdot 2t}{(n-3)t \cdot (n-4)t \cdot \cdots \cdot t} \cdot \frac{(n-2)t}{1+(n-3)t} \cdot \frac{2+(n-2)t}{(n-1)t+1} \right) \\
&= t^n \cdot (n-2)! \left(\frac{(n-1)t+1}{(n-3)t+1} \cdot (n-2) \cdot (n-3) \cdot (1+t) + \frac{2+(n-2)t}{1+(n-3)t} \cdot \frac{2(n-2)}{1} \right) \\
&= t^n \cdot (n-2)! \cdot \left(\frac{((n-1)t+1)(n-2)(n-3)(1+t) + 2(2+(n-2)t)(n-2)}{1+(n-3)t} \right)
\end{aligned}$$

From here, Sage verifies that the last expression above is

$$(n-2)(n-2)!((n-1)t+n+1)t^n$$

as claimed. □

A.3 Single Column Pairings

Proposition A.5. *Let $\lambda \vdash n$ and write $\lambda = (q_1, q_2, \ldots, q_k)$. Then*

$$\langle J_{[q_1]} J_{[q_2]} \cdots J_{[q_k]}, (J_1)^n \rangle_t = n! \cdot t^n.$$

Proof. We begin by using Proposition 2.2 from [Sta89] to write Jack functions in terms of power sum symmetric functions.

$$\begin{aligned}
&\langle J_{[q_1]} J_{[q_2]} \cdots J_{[q_k]}, (J_1)^n \rangle_t \\
&= \left\langle \prod_{1 \le i \le k} \sum_{\nu \vdash q_i} t^{q_i - \ell(\nu)} \cdot n! \cdot z_\nu^{-1} \cdot p_\nu, \left(\sum_{\nu \vdash 1} t^{1-\ell(\nu)} \cdot 1! \cdot z_\nu^{-1} \cdot p_\nu \right)^n \right\rangle_t \\
&= \left\langle \prod_{1 \le i \le k} \sum_{\nu \vdash q_i} t^{q_i - \ell(\nu)} \cdot n! \cdot z_\nu^{-1} \cdot p_\nu, (p_1)^n \right\rangle_t = \left\langle \prod_{1 \le i \le k} \sum_{\nu \vdash q_i} t^{q_i - \ell(\nu)} \cdot n! \cdot z_\nu^{-1} \cdot p_\nu, p_{1^n} \right\rangle_t
\end{aligned}$$

Since the power sum symmetric functions are orthogonal with respect this scalar product, it suffices to find the coefficient of p_{1^n} in the first entry. The only way for two power sum symmetric functions p_λ and p_μ to multiply together and yield some nonzero coefficient on p_{1^n} is if $p_\lambda = p_{1^k}$ and $p_\mu = p_{1^{n-k}}$. Indeed, $p_{1^k} \cdot p_{1^{n-k}} = p_{1^n}$ on the nose. Hence the sum is superfluous, as only the term corresponding to $p_{(1^{q_i})}$ effects the computation.

$$\begin{aligned}
&\langle J_{[q_1]} J_{[q_2]} \cdots J_{[q_k]}, (J_1)^n \rangle_t \\
&\qquad = \left\langle \prod_{1 \le i \le k} t^{q_i - q_i} \cdot q_i! \cdot z^{-1}_{(1^{q_i})} \cdot p_{1^{q_i}}, p_{1^n} \right\rangle_t = \left\langle \prod_{1 \le i \le k} 1 \cdot q_i! \cdot \frac{1}{q_i!} \cdot p_{1^{q_i}}, p_{1^n} \right\rangle_t
\end{aligned}$$

Simplifying, we have

$$\begin{aligned}
\langle J_{[q_1]} J_{[q_2]} \cdots J_{[q_k]}, (J_1)^n \rangle_t &= \langle p_{1^{q_1}} \cdot p_{1^{q_2}} \cdot \cdots \cdot p_{1^{q_k}}, p_{1^n} \rangle_t \\
&= \langle p_{1^n}, p_{1^n} \rangle_t = z_{1^n} \cdot t^{\ell(1^n)} = n! \cdot t^n.
\end{aligned}$$

□

Proposition A.6. *Suppose that*

$$J_{1^n} = \sum_{\mu \vdash n} c_\mu^{1^n} p_\mu$$

where p_μ is the power-sum symmetric function corresponding to μ and $\lambda \vdash n$. Then

$$c_\lambda^{1^n} = (-1)^{n-\ell(\lambda)} \cdot n! \cdot z_\lambda^{-1}.$$

Proof. Recall that $\langle p_\lambda, p_\mu \rangle_t = \delta_{\lambda\mu} \cdot z_\lambda \cdot t^{\ell(\lambda)}$. So

$$c_\lambda^{1^n} = \frac{\langle J_{1^n}, p_\lambda \rangle_t}{t^{\ell(\lambda)} \cdot z_\lambda}.$$

Recall that if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$, then $p_\lambda = p_{\lambda_1} \cdot p_{\lambda_2} \cdot p_{\lambda_3} \cdot \cdots \cdot p_{\lambda_{\ell(\lambda)}}$. We can rewrite the scalar product from above as follows.

$$c_\lambda^{1^n} = \frac{\langle J_{1^n}, p_\lambda \rangle_t}{t^{\ell(\lambda)} \cdot z_\lambda} = \frac{\langle J_{1^n}, p_{\lambda_1} \cdot p_{\lambda_2} \cdot p_{\lambda_3} \cdot \cdots \cdot p_{\lambda_{\ell(\lambda)}} \rangle_t}{t^{\ell(\lambda)} \cdot z_\lambda} \tag{A.4}$$

We now use the endomorphism ω_t described on page 378 of [Mac98]. (When $t = 1$, this is the involution ω on symmetric functions introduced on page 296 of [Sta99].) The map ω_t is defined by

$$w_t(p_r) = (-1)^{r-1} \cdot t \cdot p_r.$$

This means that

$$\omega_{t^{-1}}(p_r) = (-1)^{r-1} \cdot t^{-1} \cdot p_r.$$

So

$$p_r = \omega_t((-1)^{r-1} \cdot t^{-1} \cdot p_r).$$

Replacing p_{λ_i} with $\omega_t((-1)^{\lambda_i - 1} \cdot t^{-1} \cdot p_{\lambda_i})$ in (A.4) yields

$$c_\lambda^{1^n} = \frac{\langle J_{1^n}, \omega_t((-1)^{\lambda_1 - 1} \cdot t^{-1} \cdot p_{\lambda_1}) \cdot \omega_t((-1)^{\lambda_2 - 1} \cdot t^{-1} \cdot p_{\lambda_2}) \cdot \cdots \cdot \omega_t((-1)^{\lambda_{\ell(\lambda)} - 1} \cdot t^{-1} \cdot p_{\lambda_{\ell(\lambda)}}) \rangle_t}{t^{\ell(\lambda)} \cdot z_\lambda}.$$

Since ω_t is an algebra endomorphism,

$$\begin{aligned} c_\lambda^{1^n} &= \frac{\langle J_{1^n}, \omega_t((-1)^{\lambda_1 - 1} \cdot t^{-1} \cdot p_{\lambda_1} \cdot (-1)^{\lambda_2 - 1} \cdot t^{-1} \cdot p_{\lambda_2} \cdot \cdots \cdot (-1)^{\lambda_{\ell(\lambda)} - 1} \cdot t^{-1} \cdot p_{\lambda_{\ell(\lambda)}}) \rangle_t}{t^{\ell(\lambda)} \cdot z_\lambda} \\ &= \frac{\left\langle J_{1^n}, \omega_t((-1)^{\lambda_1 - 1 + \lambda_2 - 1 + \cdots + \lambda_{\ell(\lambda)} - 1} \cdot (t^{-1})^{\ell(\lambda)} \cdot p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{\ell(\lambda)}}) \right\rangle_t}{t^{\ell(\lambda)} \cdot z_\lambda} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\left\langle J_{1^n}, \omega_t((-1)^{n-\ell(\lambda)} \cdot (t^{-1})^{\ell(\lambda)} \cdot p_\lambda) \right\rangle_t}{t^{\ell(\lambda)} \cdot z_\lambda} \\
&= \frac{\left\langle J_{1^n}, (-1)^{n-\ell(\lambda)} \cdot (t^{-1})^{\ell(\lambda)} \cdot \omega_t(p_\lambda) \right\rangle_t}{t^{\ell(\lambda)} \cdot z_\lambda} \\
&= \left\langle J_{1^n}, \omega_t(p_\lambda) \right\rangle_t \cdot \frac{(-1)^{n-\ell(\lambda)} \cdot (t^{-1})^{\ell(\lambda)}}{t^{\ell(\lambda)} \cdot z_\lambda} \\
&= \left\langle J_{1^n}, \omega_t(p_\lambda) \right\rangle_t \cdot \frac{(-1)^{n-\ell(\lambda)}}{t^{2\ell(\lambda)} \cdot z_\lambda}. \qquad \text{(A.5)}
\end{aligned}
$$

Moreover, since ω_t is self-adjoint,

$$
c_\lambda^{1^n} = \left\langle \omega_t(J_{1^n}), p_\lambda \right\rangle_t \cdot \frac{(-1)^{n-\ell(\lambda)}}{t^{2\ell(\lambda)} \cdot z_\lambda}.
$$

A second basis for the Jack symmetric functions which we call the P *basis* is defined from the J basis as follows:

$$
J_\lambda = H_*^\lambda P_\lambda.
$$

Rewriting in terms of the P basis, we have

$$
\begin{aligned}
c_\lambda^{1^n} &= \left\langle \omega_t(H_*^{1^n} P_{1^n}), p_\lambda \right\rangle_t \cdot \frac{(-1)^{n-\ell(\lambda)}}{t^{2\ell(\lambda)} \cdot z_\lambda} \\
&= \left\langle H_*^{1^n} \omega_t(P_{1^n}), p_\lambda \right\rangle_t \cdot \frac{(-1)^{n-\ell(\lambda)}}{t^{2\ell(\lambda)} \cdot z_\lambda} \\
&= H_*^{1^n} \cdot \left\langle \omega_t(P_{1^n}), p_\lambda \right\rangle_t \cdot \frac{(-1)^{n-\ell(\lambda)}}{t^{2\ell(\lambda)} \cdot z_\lambda} \\
&= n! \cdot \left\langle \omega_t(P_{1^n}), p_\lambda \right\rangle_t \cdot \frac{(-1)^{n-\ell(\lambda)}}{t^{2\ell(\lambda)} \cdot z_\lambda} \\
&= \left\langle \omega_t(P_{1^n}), p_\lambda \right\rangle_t \cdot \frac{(-1)^{n-\ell(\lambda)} \cdot n!}{t^{2\ell(\lambda)} \cdot z_\lambda}.
\end{aligned}
$$

By equation (10.17) on page 380 in [Mac98],

$$
c_\lambda^{1^n} = \left\langle Q_n^{(1/t)}, p_\lambda \right\rangle_t \cdot \frac{(-1)^{n-\ell(\lambda)} \cdot n!}{t^{2\ell(\lambda)} \cdot z_\lambda}
$$

where Q_μ is an element of the basis dual to the P basis. We can easily write elements of the Q basis in terms of the J basis:

$$
J_\mu = H_\mu^* Q_\mu.
$$

Hence

$$c_\lambda^{1^n} = \left\langle \frac{1}{H_n^{*(1/t)}} J_n^{(1/t)}, p_\lambda \right\rangle_t \cdot \frac{(-1)^{n-\ell(\lambda)} \cdot n!}{t^{2\ell(\lambda)} \cdot z_\lambda}.$$

Simplifying,

$$\begin{aligned} c_\lambda^{1^n} &= \frac{1}{H_n^{*(1/t)}} \cdot \left\langle J_n^{(1/t)}, p_\lambda \right\rangle_t \cdot \frac{(-1)^{n-\ell(\lambda)} \cdot n!}{t^{2\ell(\lambda)} \cdot z_\lambda} \\ &= \frac{1}{n! \cdot \left(\frac{1}{t}\right)^n} \cdot \left\langle J_n^{(1/t)}, p_\lambda \right\rangle_t \cdot \frac{(-1)^{n-\ell(\lambda)} \cdot n!}{t^{2\ell(\lambda)} \cdot z_\lambda} \\ &= \frac{t^n}{n!} \cdot \left\langle J_n^{(1/t)}, p_\lambda \right\rangle_t \cdot \frac{(-1)^{n-\ell(\lambda)} \cdot n!}{t^{2\ell(\lambda)} \cdot z_\lambda} \\ &= \left\langle J_n^{(1/t)}, p_\lambda \right\rangle_t \cdot \frac{(-1)^{n-\ell(\lambda)} \cdot t^{n-2\ell(\lambda)}}{z_\lambda}. \end{aligned}$$

Proposition 2.2 (b) in [Sta89] allows us to do away with the scalar product. Recall that the power sum basis for symmetric functions is orthogonal (but not orthonormal) with respect to $\langle -, - \rangle_t$. (See [Mac98] page 377 or [Sta89] page 77.)

$$\begin{aligned} c_\lambda^{1^n} &= \left\langle \sum_{\mu \vdash n} \left(\left(\frac{1}{t}\right)^{n-\ell(\mu)} \cdot n! \cdot z_\mu^{-1} \cdot p_\mu \right), p_\lambda \right\rangle_t \cdot \frac{(-1)^{n-\ell(\lambda)} \cdot t^{n-2\ell(\lambda)}}{z_\lambda} \\ &= \left\langle \left(\frac{1}{t}\right)^{n-\ell(\lambda)} \cdot n! \cdot z_\lambda^{-1} \cdot p_\lambda, p_\lambda \right\rangle_t \cdot \frac{(-1)^{n-\ell(\lambda)} \cdot t^{n-2\ell(\lambda)}}{z_\lambda} \\ &= \left(\frac{1}{t}\right)^{n-\ell(\lambda)} \cdot n! \cdot z_\lambda^{-1} \cdot \langle p_\lambda, p_\lambda \rangle_t \cdot \frac{(-1)^{n-\ell(\lambda)} \cdot t^{n-2\ell(\lambda)}}{z_\lambda} \\ &= \left(\frac{1}{t}\right)^{n-\ell(\lambda)} \cdot n! \cdot z_\lambda^{-1} \cdot z_\lambda \cdot t^{\ell(\lambda)} \cdot \frac{(-1)^{n-\ell(\lambda)} \cdot t^{n-2\ell(\lambda)}}{z_\lambda} \end{aligned}$$

Simplifying finally yields

$$c_\lambda^{1^n} = (-1)^{n-\ell(\lambda)} \cdot n! \cdot z_\lambda^{-1}$$

as claimed. □

A.4 Pairings with $(J_2)^n$

Proposition A.7.

$$\langle (J_2)^n, J_{2n} \rangle_t = (2n)! \cdot (t+1)^n \cdot t^{2n} \tag{A.6}$$

Proof. We write each entry in this scalar product in terms of the power sum symmetric functions p_λ. First, we can write J_2 as

$$J_2 = \sum_{\mu \vdash 2} t^{2-\ell(\mu)} \cdot 2! \cdot z_\mu^{-1} p_\mu = tp_2 + p_{1,1}$$

by Proposition 2.2 (b) from [Sta89]. The binomial theorem yields

$$(J_2)^n = (tp_2 + p_{1,1})^n = \sum_{0 \leq k \leq n} \binom{n}{k} (tp_2)^k (p_{1,1})^{n-k}.$$

We also have

$$J_{2n} = \sum_{\mu \vdash 2n} t^{2n-\ell(\mu)} \cdot (2n)! \cdot z_\mu^{-1} \cdot p_\mu.$$

With this in mind,

$$\begin{aligned}
\langle (J_2)^n, J_{2n} \rangle_t &= \left\langle \sum_{0 \leq k \leq n} \binom{n}{k} (tp_2)^k (p_{1,1})^{n-k}, \sum_{\mu \vdash 2n} t^{2n-\ell(\mu)} \cdot (2n)! \cdot z_\mu^{-1} \cdot p_\mu \right\rangle_t \\
&= \sum_{0 \leq k \leq n} \binom{n}{k} t^k \left\langle (p_2)^k (p_{1,1})^{n-k}, \sum_{\mu \vdash 2n} t^{2n-\ell(\mu)} \cdot (2n)! \cdot z_\mu^{-1} \cdot p_\mu \right\rangle_t \\
&= \sum_{0 \leq k \leq n} \binom{n}{k} t^k \left\langle p_{[2^k, 1^{2n-2k}]}, \sum_{\mu \vdash 2n} t^{2n-\ell(\mu)} \cdot (2n)! \cdot z_\mu^{-1} \cdot p_\mu \right\rangle_t .
\end{aligned}$$

For each term in the outside sum, there is only one nontrivial summand in the second entry of the scalar product. Namely, this nontrivial summand will be the one corresponding to $\mu = [2^k, 1^{2n-2k}]$.

$$\begin{aligned}
&\langle (J_2)^n, J_{2n} \rangle_t \\
&\quad = \sum_{0 \leq k \leq n} \binom{n}{k} t^k \left\langle p_{[2^k, 1^{2n-2k}]}, t^{2n-\ell([2^k, 1^{2n-2k}])} \cdot (2n)! \cdot z^{-1}_{[2^k, 1^{2n-2k}]} \cdot p_{[2^k, 1^{2n-2k}]} \right\rangle_t \\
&\quad = \sum_{0 \leq k \leq n} \binom{n}{k} \cdot t^k \cdot t^{2n-\ell([2^k, 1^{2n-2k}])} \cdot (2n)! \cdot z^{-1}_{[2^k, 1^{2n-2k}]} \cdot \left\langle p_{[2^k, 1^{2n-2k}]}, p_{[2^k, 1^{2n-2k}]} \right\rangle_t \\
&\quad = \sum_{0 \leq k \leq n} \binom{n}{k} \cdot t^k \cdot t^{2n-(2n-k)} \cdot (2n)! \cdot z^{-1}_{[2^k, 1^{2n-2k}]} \cdot t^{\ell([2^k, 1^{2n-2k}])} \cdot z_{[2^k, 1^{2n-2k}]} \\
&\quad = \sum_{0 \leq k \leq n} \binom{n}{k} \cdot t^k \cdot t^{2n-(2n-k)} \cdot (2n)! \cdot t^{2n-k}
\end{aligned}$$

$$
\begin{aligned}
&= (2n)! \sum_{0 \le k \le n} \binom{n}{k} t^{2n+k} \\
&= (2n)! \cdot t^{2n} \cdot \sum_{0 \le k \le n} \binom{n}{k} t^{k} \\
&= (2n)! \cdot t^{2n} \cdot (t+1)^n
\end{aligned} \tag{A.7}
$$

□

The data from Sage which suggested the above proposition lead to a conjecture about the other "extremal case": $\langle (J_2)^n, J_{2^n} \rangle_t$. Note that J_{2^n} is the "thinnest" partition such that pairing with $(J_2)^n$ is nontrivial. Here "thinnest" means the partition for which λ_1 is minimal.

Conjecture A.8.

$$\langle (J_2)^n, J_{2^n} \rangle_t = (2n)!! \cdot \left(\prod_{1 \le k \le n} (t+k) \right) \cdot t^{2n}$$

We can't use the same tools as the previous proposition for this, since J_{2^n} is not easily written in terms of power sums.

We have one further conjecture regarding pairing with $(J_2)^n$.

Conjecture A.9. *Let $n > 1$. Then*

$$
\begin{aligned}
\langle (J_2)^n, J_{[2n-2,2]} \rangle_t &= t^{2n} \cdot (t+1)^{n-2} \\
&\quad \cdot (2n)! \cdot \frac{2n-1}{(2n-2)^2} \left(t^2 + \frac{4n-5}{2n-3} t + \frac{2(2n^2-2n+1)}{2n-3} \right).
\end{aligned} \tag{A.8}
$$

We can say more about pairing with $(J_2)^n$.

Proposition A.10.

$$\langle (J_2)^n, J_{[2n-1,1]} \rangle_t = \sum_{k=0}^{n} \binom{n}{k} t^{2n+k} \left((2n)! - (2n-2)! \left(\frac{1}{t} + 2n - 1 \right) (2k) \right) \tag{A.9}$$

Proof. We'll use the two facts from [Sta89] above plus the information about how to travel between the m and p bases for symmetric functions from [ER91] in order to compute $\langle (J_2)^n, J_{[2n-1,1]} \rangle_t$. By Prop 7.2 from [Sta89],

$$J_{2^1,1^{2n-2}} = (2n)! m_{1^{2n}} + (t+2n-1)(2n-2)! m_{2,1^{2n-2}}.$$

From [ER91], we have

$$m_{1^{2n}} = \sum_{\mu \vdash 2n} M(p,m)_{\mu,1^{2n}} p_\mu = \sum_{\mu} (-1)^{2n-\ell(\mu)} w(B_{1^{2n},\mu}) z_\mu^{-1} p_\mu = \sum_{\mu} (-1)^{2n-\ell(\mu)} z_\mu^{-1} p_\mu$$

$$m_{2,1^{2n-2}} = \sum_{\mu \vdash 2n} M(p,m)_{\mu,[2,1^{2n-2}]} p_\mu = \sum_{\mu} (-1)^{2n-1-\ell(\mu)} w(B_{[2,1^{2n-2}],\mu}) z_\mu^{-1} p_\mu.$$

We need to do a little bit of work to find $w(B_{[2,1^{2n-2}],\mu})$. Note that there are

$$\sum_{i=1}^{\ell(\mu)} \mu_i - 1 = |\mu| - \ell(\mu)$$

$[2,1^{2n-2}]$-brick tabloids of shape μ since each tabloid is a choice of where to put the unique 2-block. All brick tabloids have weight 1 besides the $\ell(\mu) - m_1(\mu)$ of them in which the 2-block appears at the end of a row. These have weight 2. This means that

$$\begin{aligned} w(B_{[2,1^{2n-2}],\mu}) &= \sum_{T \in B_{[2,1^{2n-2}],\mu}} w(T) \\ &= 1 \cdot (\text{total number of tabloids}) \\ &\qquad - 1 \cdot (\text{number of tabloids with weight two}) \\ &\qquad + 2 \cdot (\text{number of tabloids with weight two}) \\ &= (\text{total number of tabloids}) + (\text{number of tabloids with weight two}) \\ &= (|\mu| - \ell(\mu)) + (\ell(\mu) - m_1(\mu)) = |\mu| - m_1(\mu). \end{aligned}$$

To finish up computing $m_{2,1^{2n-2}}$ in terms of power sums, we have

$$\begin{aligned} m_{2,1^{2n-2}} &= \sum_{\mu} (-1)^{2n-1-\ell(\mu)} w(B_{[2,1^{2n-2}],\mu}) z_\mu^{-1} p_\mu \\ &= \sum_{\mu} (-1)^{2n-1-\ell(\mu)} (|\mu| - m_1(\mu)) z_\mu^{-1} p_\mu. \end{aligned} \tag{A.10}$$

Putting this all together,

$$\begin{aligned} &J_{2,1^{2n-2}} \\ &\quad = \sum_{\mu \vdash 2n} \Big((-1)^{2n-\ell(\mu)} (2n)! z_\mu^{-1} \\ &\qquad\qquad + (-1)^{2n-1-\ell(\mu)} (t+2n-1)(2n-2)!(2n-m_1(\mu)) z_\mu^{-1} \Big) p_\mu \\ &\quad = \sum_{\mu \vdash 2n} \left((-1)^{2n-\ell(\mu)} z_\mu^{-1} \left[(2n)! - (2n-2)!(t+2n-1)(2n-m_1(\mu)) \right] \right) p_\mu. \end{aligned}$$

Using Cor 3.5 from [Sta89],

$$J_{2n-1,1} = \sum_{\mu \vdash 2n} \Big((-t)^{2n-\ell(\mu)} \\ \left((-1)^{2n-\ell(\mu)} z_\mu^{-1} \left[(2n)! - (2n-2)! \left(\frac{1}{t} + 2n - 1 \right) (2n - m_1(\mu)) \right] \right) p_\mu \Big).$$

Finally, we can pair with $(J_2)^n$. Recall

$$(J_2)^n = \sum_{k=0}^{n} \binom{n}{k} t^k (p_2)^k (p_{11})^{n-k}.$$

So

$$\begin{aligned}
&\langle (J_2)^n, J_{2n-1,1} \rangle_t \\
&\quad = \left\langle \sum_{k=0}^{n} \binom{n}{k} t^k (p_2)^k (p_{11})^{n-k}, \sum_{\mu \vdash 2n} (-t)^{2n-\ell(\mu)} \left((-1)^{2n-\ell(\mu)} z_\mu^{-1} \right. \right. \\
&\qquad \left. \left. \left[(2n)! - (2n-2)! \left(\frac{1}{t} + 2n - 1 \right) (2n - m_1(\mu)) \right] \right) p_\mu \right\rangle_t \\
&\quad = \sum_{k=0}^{n} \binom{n}{k} t^k \left\langle p_{[2^k, 1^{2n-2k}]}, (-t)^k \left((-1)^k z^{-1}_{[2^k, 1^{2n-2k}]} \right. \right. \\
&\qquad \left. \left. \left[(2n)! - (2n-2)! \left(\frac{1}{t} + 2n - 1 \right) (2k) \right] \right) p_{[2^k, 1^{2n-2k}]} \right\rangle_t \\
&\quad = \sum_{k=0}^{n} \binom{n}{k} t^k (-t)^k \\
&\qquad \left((-1)^k z^{-1}_{[2^k, 1^{2n-2k}]} \left[(2n)! - (2n-2)! \left(\frac{1}{t} + 2n - 1 \right) (2k) \right] \right) \\
&\qquad\qquad \cdot \left\langle p_{[2^k, 1^{2n-2k}]}, p_{[2^k, 1^{2n-2k}]} \right\rangle_t \\
&\quad = \sum_{k=0}^{n} \binom{n}{k} t^k (-t)^k \\
&\qquad \left((-1)^k z^{-1}_{[2^k, 1^{2n-2k}]} \left[(2n)! - (2n-2)! \left(\frac{1}{t} + 2n - 1 \right) (2k) \right] \right) \\
&\qquad\qquad \cdot z_{[2^k, 1^{2n-2k}]} t^{2n-k} \\
&\quad = \sum_{k=0}^{n} \binom{n}{k} t^{2n+k} \left((2n)! - (2n-2)! \left(\frac{1}{t} + 2n - 1 \right) (2k) \right). \qquad \text{(A.11)}
\end{aligned}$$

□

In fact, we can say something more general using the same tools.

A.5 A Fact About Length Two Partitions

Proposition A.11. *If λ is a partition of length 2, say $\lambda = (i+j, i)$, then*

$$J_\lambda = \sum_{\mu \vdash 2i+j} \Bigg((-t)^{2i+j-\ell(\mu)} \cdot \left(\sum_{r=0}^{i} (i)_r ((1/t)+i+j)_r (2(i-r)+j)! (-1)^{2i-r+j-\ell(\mu)} \cdot w(B_{2^r 1^{2(i-r)+j}, \mu}) \cdot z_\mu^{-1} \right) p_\mu \Bigg)$$

where $(q)_r$ is the Pochhammer symbol (or "falling factorial"). In other words,

$$\langle J_\lambda, p_\mu \rangle_t = t^{2i+j} \cdot \sum_{r=0}^{i} (i)_r ((1/t)+i+j)_r (2(i-r)+j)! (-1)^r \cdot w(B_{2^r 1^{2(i-r)+j}, \mu}).$$

Proof. By Proposition 7.2 from [Sta89],

$$J'_\lambda = J_{2^i 1^j} = \sum_{r=0}^{i} (i)_r (t+i+j)_r (2(i-r)+j)! m_{2^r 1^{2(i-r)+j}}.$$

We use [ER91] to write $m_{2^r 1^{2(i-r)+j}}$ in terms of power sum symmetric functions.

$$m_{2^r 1^{2(i-r)+j}} = \sum_{\mu \vdash 2i+j} (-1)^{2i-r+j-\ell(\mu)} \cdot w(B_{2^r 1^{2(i-r)+j}, \mu}) \cdot z_\mu^{-1} \cdot p_\mu$$

Combining these ideas,

$$J'_\lambda = \sum_{r=0}^{i} \left((i)_r (t+i+j)_r (2(i-r)+j)! \sum_{\mu \vdash 2i+j} (-1)^{2i-r+j-\ell(\mu)} \cdot w(B_{2^r 1^{2(i-r)+j}, \mu}) \cdot z_\mu^{-1} \cdot p_\mu \right)$$

$$= \sum_{\mu \vdash 2i+j} \left(\sum_{r=0}^{i} (i)_r (t+i+j)_r (2(i-r)+j)! (-1)^{2i-r+j-\ell(\mu)} \cdot w(B_{2^r 1^{2(i-r)+j}, \mu}) \cdot z_\mu^{-1} \cdot p_\mu \right).$$

Finally we apply Corollary 3.5 to get the following.

$$J_\lambda = J''_\lambda = \sum_{\mu \vdash 2i+j} \Big((-t)^{2i+j-\ell(\mu)} \cdot \left(\sum_{r=0}^{i} (i)_r ((1/t)+i+j)_r (2(i-r)+j)! (-1)^{2i-r+j-\ell(\mu)} \cdot w(B_{2^r 1^{2(i-r)+j}, \mu}) \cdot z_\mu^{-1} \right) p_\mu \Bigg) \tag{A.12}$$

□

APPENDIX B

THE METROPOLIS–HASTINGS ALGORITHM

This appendix describes a Markov chain Monte Carlo (MCMC) method called the Metropolis–Hastings algorithm and describes an implementation of the technique to generate a uniformly random large boxed plane partitions. MCMC methods provide a way to sample from a distribution by performing a random walk between states within the distribution. Such techniques are especially useful when the distribution is difficult to sample from directly, as one need not know much about the distribution in order to effect an MCMC algorithm.

The Metropolis–Hastings algorithm was first proposed in 1953 by Nicholas Metropolis, Arianna W. Rosenbluth, Marshall Rosenbluth, Augusta H. Teller and Edward Teller [Met+53]. In 1970, W.K. Hastings generalized this idea [Has70]. The credit for the original 1953 work is disputed, with M. Rosenbluth and E. Teller each giving different accounts of what the others contributed [Gub05].

In Section B.1, we state the Metropolis–Hastings in generality. Section B.2 describes using this algorithm to generate a large uniformly random boxed plane partition.

B.1 The General Algorithm

The Metropolis–Hastings algorithm is characterized by its reliance on an "acceptance probability" of taking each step. The algorithm proceeds as follows.

1. Choose a *proposal function* $q(a \to b)$ which gives a probability of walking to each state b from the state a.

2. Pick an arbitrary starting state x. Allowing for sufficient mixing time will negate any bias in choice of this starting point.

3. Use $q(x \to y)$ to propose a new state y which we can step to from x.

4. Perform the step to y from x with probability

$$\alpha = \min\left(1, \frac{\pi(y) \cdot q(y \to x)}{\pi(x) \cdot q(x \to y)}\right) \tag{B.1}$$

   where $\pi(x)$ is the probability of selecting x from the stationary distribution.

5. Repeat from step 3 until sufficient mixing has occurred.

This algorithm satisfies the *detailed balance* equation.

Theorem B.1. *For any two states x and y, the Metropolis–Hastings algorithm satisfies detailed balance:*

$$\pi(x)P(x \to y) = \pi(y)P(y \to x) \tag{B.2}$$

where $P(x \to y) = q(x \to y)\alpha(x \to y)$, i.e. $P(x \to y)$ is the probability of moving to the state y given the current state is x in the algorithm.

Proof. Note that

$$\begin{aligned}
\pi(x)P(x \to y) &= \pi(x)q(x \to y)\alpha(x \to y) \\
&= \pi(x)q(x \to y)\min\left(1, \frac{\pi(y)\cdot q(y \to x)}{\pi(x)\cdot q(x \to y)}\right) \\
&= \min(\pi(x)q(x \to y), \pi(y)q(y \to x)) \\
&= \pi(y)q(y \to x)\min\left(1, \frac{\pi(x)\cdot q(x \to y)}{\pi(y)\cdot q(y \to x)}\right) \\
&= \pi(y)q(y \to x)\alpha(y \to x) \\
&= \pi(y)P(y \to x)
\end{aligned} \tag{B.3}$$

as claimed. □

B.2 Generating Uniformly Random Boxed Plane Partitions

As an aside from our ultimate goal, we implemented the Metropolis-Hastings algorithm in SageMath to generate a uniformly random boxed plane partition. A step is taken to be from the current boxed plane partition to a new boxed plane partition which is adjacent to it in the lattice of plane partitions ordered by containment.

1. First, set the proposal function $q(\pi \to \sigma)$ to be

$$q(\pi \to \sigma) = \begin{cases} \frac{1}{\#\text{ of plane partitions adjacent to } \pi} & \text{if } \sigma \text{ is a legal step from } \pi, \\ 0 & \text{otherwise.} \end{cases} \tag{B.4}$$

2. Select the empty plane partition $\varepsilon = \sigma$ as our starting state.

3. Create a list of all of the boxed plane partitions which are adjacent to our current state σ in the (finite) lattice of boxed plane partitions ordered by containment.

4. Select a plane partition π uniformly randomly from this list.

5. Accept this step with probability

$$\alpha(\sigma \to \pi) = \min\left(1, \frac{q(\pi \to \sigma)}{q(\sigma \to \pi)}\right) \tag{B.5}$$

where σ is the current state, π is the proposed adjacent state and

$$q(\sigma \to \pi) = \begin{cases} \frac{1}{\#\text{ of plane partitions adjacent to } \sigma} & \text{if } \pi \text{ is a legal step from } \sigma, \\ 0 & \text{otherwise.} \end{cases} \tag{B.6}$$

6. Repeat from step 3

   a) with π as the current state if you accepted the proposal in step 5.,

   b) or with σ as the current state if you rejected the proposal in step 5.

Our acceptance probability is chosen carefully so that after sufficient "burn in", this yields a uniformly random boxed plane partition. Indeed, by detailed balance, given two boxed plane partitions ρ and σ,

$$\begin{aligned} \frac{\pi(\rho)}{\pi(\sigma)} &= \frac{P(\sigma \to \rho)}{P(\rho \to \sigma)} \\ &= \frac{q(\sigma \to \rho)\alpha(\sigma \to \rho)}{q(\rho \to \sigma)\alpha(\rho \to \sigma)} \\ &= \frac{q(\sigma \to \rho)\min\left(1, \frac{q(\rho \to \sigma)}{q(\sigma \to \rho)}\right)}{q(\rho \to \sigma)\min\left(1, \frac{q(\sigma \to \rho)}{q(\rho \to \sigma)}\right)} \\ &= \frac{\min\left(q(\sigma \to \rho), q(\rho \to \sigma)\right)}{\min\left(q(\rho \to \sigma), q(\sigma \to \rho)\right)} \\ &= 1. \end{aligned} \tag{B.7}$$

Our implementation of this algorithm in SageMath is publicly available on Github [Poh25]. Figure B.1 shows a boxed plane partition in a $10 \times 10 \times 10$ cube generated by this code with $100,000$ steps. Interestingly, the arctic circle phenomenon described in [CLP02] can be observed even in plane partitions of this relatively small size.

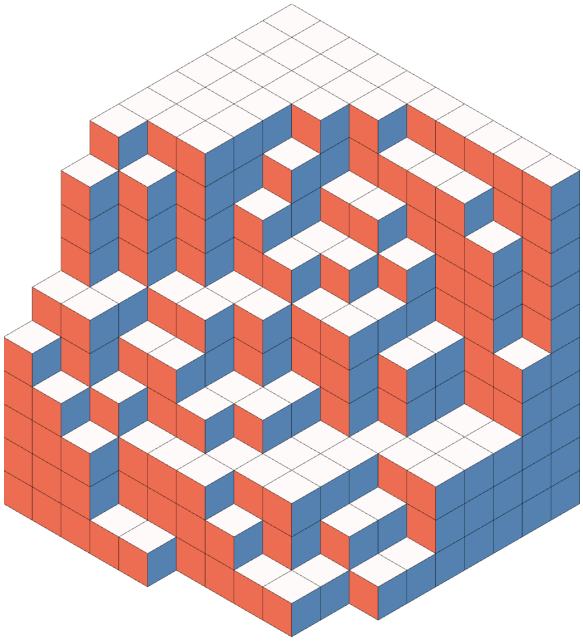


Figure B.1. A boxed plane partition.

REFERENCES CITED

[Ale24] P. Alexandersson, *Jack Polynomials*, URL: https://www.symmetricfunctions.com/jack.htm#jackP (visited on 08/09/2025), 2024.

[BD23] H. Ben Dali and M. Dołęga, *Positive formula for Jack polynomials, Jack characters and proof of Lassalle's conjecture*, arXiv: 2305.07966 [math.CO], 2023.

[BO05] A. Borodin and G. Olshanski, *Z-measures on partitions and their scaling limits*, European J. Combin. **26** (2005), no. 6, 795–834.

[CDM23] C. Cuenca, M. Dołęga, and A. Moll, *Universality of global asymptotics of Jack-deformed random Young diagrams at varying temperatures*, arXiv: 2304.04089 [math.PR], 2023.

[CKP10] I. Ciocan-Fontanine, M. Konvalinka, and I. Pak, *The weighted hook length formula*, arXiv: 1006.0043 [math.CO], 2010.

[CLP02] H. Cohn, M. Larsen, and J. Propp, *The shape of a typical boxed plane partition*, arXiv: math/9801059 [math.CO], 2002.

[DF16] M. Dołęga and V. Féray, *Gaussian fluctuations of Young diagrams and structure constants of Jack characters*, Duke Math. J. **165** (2016), no. 7, DOI: 10.1215/00127094-3449566.

[Die+18] J. Dietz et al., *Automorphism Orbits of Some Metacyclic Groups*, Pi Mu Epsilon J. **14** (2018), no. 9, 557–564.

[ER91] Ö. Eğecioğlu and J. B. Remmel, *Brick tabloids and the connection matrices between bases of symmetric functions*, Discrete Appl. Math. **34** (1991), 107–120, DOI: 10.1016/0166-218X(91)90071-F.

[Fro00] G. Frobenius, *Über die Charaktere der symmetrischen Gruppe*, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften (1900), 516–534.

[FRT54] J. S. Frame, G. d. Robinson, and R. M. Thrall, *The Hook Graphs of the Symmetric Group*, Canadian Journal of Mathematics **6** (1954), 316–324, DOI: 10.4153/CJM-1954-030-1.

[GH19] A. Guionnet and J. Huang, *Rigidity and Edge Universality of Discrete β-Ensembles*, Communications on pure and applied mathematics **72** (2019), no. 9, 1875–1982.

[GKP20] R. Gaybullaev, A. Khudoyberdiyev, and K. Pohl, *Classification of solvable Leibniz algebras with abelian nilradical and $k-1$ dimensional extension*, Comm. Algebra **48** (2020), no. 7, 3061–3078, DOI: 10.1080/00927872.2019.1647439, arXiv: 1808.06171 [math.RA].

[GNW79] C. Greene, A. Nijenhuis, and H. S. Wilf, *A probabilistic proof of a formula for the number of Young tableaux of a given shape*, Adv. Math. **31** (1979), 104–109.

[GNW84] ———, *Another probabilistic method in the theory of Young tableaux*, Journal of Combinatorial Theory, Series A **37** (1984), no. 2, 127–135, DOI: 10.1016/0097-3165(84)90065-7.

[Gor21] V. Gorin, *Lectures on random lozenge tilings*, *Vol. 193*, Cambridge University Press, 2021.

[Gub05] J. E. Gubernatis, *Marshall Rosenbluth and the Metropolis algorithm*, Physics of Plasmas **12** (2005), no. 5, 057303, DOI: 10.1063/1.1887186.

[Hai98] M. Haiman, *t,q-Catalan numbers and the Hilbert scheme*, Discrete Math. **193** (1998), no. 1-3, 201–224, DOI: 10.1016/S0012-365X(98)00141-1.

[Has70] W. K. Hastings, *Monte Carlo Sampling Methods Using Markov Chains and Their Applications*, Biometrika **57** (1970), no. 1, 97–109, DOI: 10.1093/biomet/57.1.97.

[HW18] J. Haglund and A. T. Wilson, *Macdonald Polynomials and Chromatic Quasisymmetric Functions*, arXiv: 1701.05622v2 [math.CO], URL: https://arxiv.org/pdf/1701.05622v2, arXiv preprint, 2018.

[Ker00] S. V. Kerov, *Anisotropic Young diagrams and Jack symmetric functions*, Functional Analysis and Its Applications **34** (2000), 41–51, arXiv: math/9712267 [math.CO].

[KOS06] R. Kenyon, A. Okounkov, and S. Sheffield, *Dimers and amoebae*, Annals of mathematics (2006), 1019–1056.

[Mac01] P. A. MacMahon, *Combinatory analysis, volumes I and II*, *Vol. 137*, American Mathematical Society, 2001.

[Mac98] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford university press, 1998.

[Mau+06a] D. Maulik et al., *Gromov–witten theory and donaldson–thomas theory, i*, Compos. Math. **142** (2006), no. 5, 1263–1285.

[Mau+06b] ———, *Gromov–witten theory and donaldson–thomas theory, ii*, Compos. Math. **142** (2006), no. 5, 1286–1304.

[Met+53] N. Metropolis et al., *Equation of State Calculations by Fast Computing Machines*, Journal of Chemical Physics **21** (1953), no. 6, 1087–1092, DOI: 10.1063/1.1699114.

[Oko06] A. Okounkov, *The uses of random partitions*, XIVth International Congress on Mathematical Physics, World Scientific, 2006, 379–403.

[OR03] A. Okounkov and N. Reshetikhin, *Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram*, J. Amer. Math. Soc. **16** (2003), no. 3, 581–603.

[Poh25] K. Pohl, *Metropolis–Hastings-Plane-Partition-Generation*, URL: https://github.com/kyla-pohl/Metropolis--Hastings-Plane-Partition-Generation (visited on 08/09/2025), GitHub repository, with Kayla Wright and Ben Young, 2025.

[PY25] K. Pohl and B. Young, *Jack Combinatorics of the Equivariant Edge Measure*, Article #7, 12 pages; Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2025), Sapporo, 2025.

[Rüh18] N. Rühling, *The Hilbert Scheme of Points*, Bachelor's thesis, Supervised by Martijn Kool, 2018.

[Sta89] R. P. Stanley, *Some combinatorial properties of Jack symmetric functions*, Adv. Math. **77** (1989), no. 1, 76–115.

[Sta99] ———, *Enumerative Combinatorics. Volume 2*, *Vol. 62*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1999.

[Wey16] H. Weyl, *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77** (1916), no. 3, 313–352, DOI: 10.1007/BF01475864, MR1511862.

[You02] A. Young, *Quantitative substitutional analysis II*, Proceedings of the London Mathematical Society, Series 1 **35** (1902), 361–397.