LOWER TAIL PROBABILITIES AND THE GAUSSIAN PROCESS

by

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This paper is devoted to the study of the lower tail probabilities for Gaussian Process. It will begin with a brief summary of continuous probability theory, at least enough to allow an undergraduate math major of approximately the junior level to appreciate the theorem and examples presented. The proof of the so-called Slepian Lemma will follow the discussion of continuous probability theory and lead into a discussion of three applications that arise from the Slepian Lemma. The first example comes from Brownian pursuits, the second emerges from the study of real zeros of random polynomials and the third example results from the first passage time for the Slepian process. Some numerical analysis of the second two examples will conclude.

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Chapter 1

Introduction

Problems surrounding the nature of so-called random events have arisen through the centuries from numerical data and results of observations made by the sciences; the development was also stimulated by the practical requirements of insurance companies and abstract problems associated with games of chance.

Probability as a science found its roots around the seventeenth century with the concept of mathematical expectation and the theorems of addition and multiplication for probabibility. Notable advances to probability theory of the seventeenth century include Pascal, Fermat and Huygens. The rise of the eighteenth century brought notable contributions from Bernolli, de Moivre, Laplace, Gauss, and Poisson which led to many applications in probability theory to that of statistics. The second half of the nineteenth century was dominated by the Russian (St. Petersburg) school, which include such famous mathematicians as Chebyshev, Markov, and Kolmogorov. During this period the laws of probability theory were applied to dependent random variables which as a result allowed it to be applied to the natural sciences (especially to physics). The modern period of probability theory follows from the formulation of axiomatic probability; the demand for axioms stemmed primarily from practical applications, since it was necessary to refine and produce an orderly structure to the basic notions of probability when applying its theories across disciplines.[7]

The developments of the last century have brought about many intriguing problems in the area of probability theory. One such problem arises from determining the rate of lower tail probability of real valued Gaussian random processes. The aforementioned problem can be stated as

$$P(\sup_{0\leq t\leq T} X(t) \leq 0) \text{ as } T \to \infty \tag{1.1}$$

where $X(t)$ is a stationary Gaussian process.

There are many reasons to study (1.1), other than its own importance, three of which include Brownian pursuits, the study of real zeros of random polynomials, and the first passage time for the so-called Slepian process.

This thesis should be accessible to any math major of approximately junior status who has had classes in linear algebra, vector calculus and introduction to analysis; however, to get the most out of this thesis one should be familiar with probability theory. Chapter 2 should cover enough of the key definitions and concepts needed to appreciate the last three chapters. Chapter 3 will cover the main theorem of this paper and its proof (The Comparison Principle and its corollary the Slepian Lemma). Chapter 4 will discuss three examples arising from applications of Brownian pursuits, the study of real zeros of random polynomials, and the first passage time for the so-called Slepian process. Chapter 5 will conclude the paper with some numerical results obtained from simulations run on Mathematica.

Chapter 2

Definitions and Key Concepts

To understand the key theorems and their application it is important to be at least acquainted with probability theory. The key definitions and concepts needed for this acquaintance will be discussed in this chapter as a foundation for the last three chapters which will follow.

Suppose we have an experiment which terminates with an *outcome*, and this experiment is characterized by the fact that we cannot predicted the outcome with certainty prior to the performance of the experiment.

Now suppose we have the aforementioned case, but the experiment is of such a nature that a collection of every possible outcome can be described prior to its performance and if this kind of experiment can be repeated under the same conditions, it is called a *random experiment*, and the collection of every possible outcome is called the *sample space.*

Definition 2.1 (Sample Space) Let Ω denote the set of every possible outcome of a random experiment; that is, Ω is the sample space.

Probability has been formalized with three axioms which create the foundation for all of probability theory. The first axiom is that the probability of an event is between

zero and one; the second axiom is that the probability of the entire space is one; and the last axiom is that probability of number of disjoint events is the addition of their probabilities.

Definition 2.2 (Axiomatic Probability Space) A *probability function* is a real-valued function defined on some class of subsets on Ω. With the following properties:

(1) $0 \leq P(A) \leq 1$

(2) $P(\Omega) = 1$

(3) if $A_1, A_2, \ldots$ is a disjoint sequence of events ($A_i \cap A_j = \emptyset, i \neq j$), then

$$P\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sum_{k=1}^{\infty} P(A_k)$$

A random variable can be thought of as the numeric result of operating a non-deterministic mechanism or performing a non-deterministic experiment to generate a random result. An important note to realize is that unlike the common practice with other mathematical variables, a random variable cannot be assigned a value. A random variable does not describe the actual outcomes of a particular experiment, but instead describes a yet to be determined outcome in terms of the real numbers.

Definition 2.3 (Random Variable) Consider a random experiment with a sample space Ω. A function X, which assigns to each element $c \in \Omega$ one and only one real number $X(c) = x$, is called a *random variable.* The space of X is the set of real numbers $\mathfrak{A} = \{x : x = X(c), c \in \Omega\}$. Random variables will be denoted with capitol letters.

A Cumulative Distribution Function (cdf) completely describes the distribution of a real valued random variable.

Definition 2.4 (Cumulative Distribution Function *cdf*) Let X denote a random variable. The *cumulative distribution function* of X, denoted by $F(x)$, is given by $F(x) = P(X < x)$ for $x \in \Re$. Note that $P(a < X < b) = F(b) - F(a)$.

A continuous random variable is basically self explanatory, it is a random variable that is continuos instead of discrete.

Definition 2.5 (Continuous Random Variable) Let X denote a random variable with a cdf $F(x)$. X is said to be a *continuous* if the cdf $F(x)$ is continuous in $\Re$ and its first derivative exists and is continuous except for, at most, a finite number of points in any finite interval.

The Probability Density Function is the derivative of cdf, which allows one to represent the probability function in terms of integrals.

Definition 2.6 (Probability Density Function *pdf*) Let $F(x)$ be the *cumulative distribution function* for a continuous random variable X. Then $f(x)$, given by

$$f(y) = \frac{d}{dx}F(x) = F'(x)$$

wherever the derivative exists, is called the *probability density function* for the random variable X. Two important properties of the pdf are that $f(x) \geq 0$ for any value of x. and $\int_{-\infty}^{\infty} f(x)dx = 1$.

The joint cdf and pdf are just the generalizations of the single variable cdf and pdf into multiple dimensions.

Definition 2.7 (Joint cdf) Let $X_1, \ldots, X_n$ be continuous random variables defined on Ω. Then the *joint cumulative distribution function* $F(x_1, \ldots, x_n)$ is denoted by

$$F(x_1, \ldots, x_n) = P(X_1 < x_1, \ldots, X_n < x_n), \text{ where } x_1, \ldots, x_n \in \Re$$

Definition 2.8 (Joint pdf) Let $X_1, \ldots, X_n$ be continuous random variables with joint cdf $F(x_1, \ldots, x_n)$. If there exists a nonnegative function $f(x_1, \ldots, x_n)$ such that

$$F(x_1, \ldots, x_n) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} f(t_1, \ldots, t_n)dt_1 \ldots dt_n, \text{ where } x_1, \ldots, x_n \in \Re$$

then $X_1, \ldots, X_n$ are said to be *jointly continuous random variables.* The function $f(x_1, \ldots, x_n)$ is called the *joint probability density function.*

Mathematical Expectation or expected value of a random variable is the sum of the probability of each of the possible outcomes of the experiment multiplied by its pay-off/value. In other words it represents the average amount one expects to win/acquire etc.

Definition 2.9 (Mathematical Expectation) If $\int \int_{\Re} g(x,y)f(x,y)dxdy < \infty$, then the *mathematical expectation* is given by,

$$\mathbb{E}[g(X,Y)] = \int_{\Re^2} g(x,y)f(x,y)dxdy$$

The Mean is the the average as calculated using the mathematical Expectation.

Definition 2.10 (Mean)

$$\mu = \mathbb{E}[X]$$

The variance of a random variable is a measure of its statistical dispersion. The square root of the variance will give one the standard deviation.

Definition 2.11 (Variance)

$$Var[X] = \sigma^2 = \mathbb{E}[X - \mathbb{E}[X]]^2 = \mathbb{E}[X - \mu]^2 = \mathbb{E}[X^2] - \mu^2$$

Covariance can be thought of as a measure of linear dependence between two random variables, although this is not the linear dependence studied in linear algebra.

Definition 2.12 (Covariance)

$$Cov(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mu_X\mu_Y$$

The Gaussian distribution, which is often known as the Normal distribution or even sometimes as the bell curve, is symmetric distribution around the mean with a bell like shape.

Definition 2.13 (Gaussian Distribution) Let $\mu, \sigma \in \Re$, $\sigma > 0$. Then the Gaussian (or Normal) distribution is denoted as $\mathcal{N}(\mu, \sigma^2)$ if the pdf is defined by:

$$f(x) = (2\pi)^{-\frac{1}{2}}\sigma^{-1}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

To show that Gaussian is a pdf we need demonstrate that it integrates to one and that it is greater than or equal to zero; but this is an exponential function so we know $f(x) \geq 0$ by definition. To show that the Normal distribution integrates to one:

$$\left[\int f(x)dx\right]^2 = \iint f(x_1)f(x_2)dx_1dx_2$$

Switch to polar coordinates centered around (μ,μ)

$$x_1 = \rho cos\theta + \mu$$

$$x_2 = \rho sin\theta + \mu$$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial \rho} & \frac{\partial x_1}{\partial \theta} \\ \frac{\partial x_2}{\partial \rho} & \frac{\partial x_2}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -\rho sin\theta \\ sin\theta & \rho cos\theta \end{vmatrix} = \rho$$

$$\Rightarrow (2\pi)^{-1}\sigma^{-2}\int_0^{2\pi}\int_0^{\infty}\exp\left\{-\frac{\rho^2}{2\sigma^2}\right\}\rho d\rho d\theta = 1$$

The Standard Normal Distribution is a Gaussian Distribution with mean equal to zero and variance of one.

Definition 2.14 (Standard Normal Distribution) The standard normal distribu-

tion $\mathcal{N}(0,1)$, then the cdf is denoted by:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{ -\frac{t^2}{2} \right\} dt$$

The Multivariate Normal Distribution is the generalization of the Normal distribution to multidimensional space.

Definition 2.15 (Multivariate Normal Distribution) Let $X \in \Re^n$ be a random vector with mean μ and covariance matrix Σ. X is said to have a multivariate normal distribution if the pdf is given by

$$f(x_1, x_2, \ldots, x_n) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu) \right\}, \text{ where } \Sigma = \mathbb{E}[(X-\mu)(X-\mu)']$$

The Characteristic Function, a very useful tool in probability theory, allows one to show many aspects about a distribution and can also be used to calculate the mean and variance of a random variable.

Definition 2.16 (Characteristic Function, cf) Let $X \in \Re^n$ be random vector, then the characteristic function of x, commonly called the fourier transform, is defined as follows:

$$\varphi(t) = \mathbb{E}[e^{i<t,X>}]$$

Example 1: The Normal Distribution ($\mathcal{N}(\mu, \sigma^2)$) cf is:

$$\varphi(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$$

Example 2: The multivariate normal distribution ($\mathcal{N}(\mu, \Sigma)$) cf is:

$$\varphi(t) = e^{i<\mu,t> - \frac{1}{2}(t-\mu)'\Sigma(t-\mu)}$$

Lemma 2.1 (Relationship between cf and pdf) If $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\varphi(t)| dt_1 \ldots dt_n < \infty$, then the pdf can be written as

$$f(x_1, \ldots, x_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-i<t,x>} \varphi(t) dt_1 \ldots dt_n$$

Definition 2.17 (The Indicator Function) The indicator function of subset A of a set X is a function

$$1_A : X \to \{0, 1\}$$

defined as follows:

$$1_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A \end{cases}$$

Property 2.1 If X is a probability space with probability measure P and A is a measurable set, then 1_A becomes a random variable whose expected value is equal to the probability of A:

$$\mathbb{E}[1_A] = \int_X 1_A(x) dP = \int_A dP = P(A).$$

Chapter 3

The Comparison Principle

In order to study lower tail for the three examples mentioned in Chapter 1, we will need a particular method which will reduce the problem to studying relatively simple random functions. This method is known as the Comparison Principle: "*The less correlated the values of a Gaussian random function are, the larger is the probability that it would deviate from the mean.*"[6] To give a precise meaning to this statement we will use the notation and basic ideas used in MA Lifshits book *Gaussian Random Functions* for the following theorem and corollaries. The goal of this chapter is prove the Slepian Lemma (Corollary 3.3), which we will use in the examples in Chapter 4.

Theorem 3.1 Let $\xi = (\xi_1, ..., \xi_n)$ and $\eta = (\eta_1, ..., \eta_n)$ be independent centered Gaussian vectors. Let $f : \Re^n \to \Re^1$ be a function with bounded second-order derivatives. Then

$$\mathbb{E}[f(\eta)] - \mathbb{E}[f(\xi)] = \frac{1}{2}\int_0^1 \sum_{i,j}(\mathbb{E}[\eta_i\eta_j] - \mathbb{E}[\xi_i\xi_j])\mathbb{E}\frac{\partial^2 f}{\partial x_i \partial x_j}((1-\alpha)^{\frac{1}{2}}\xi + \alpha^{\frac{1}{2}}\eta)d\alpha. \quad (3.1)$$

Proof Let Σ^ξ and Σ^η be the covariance operators of the independent centered Gaussian vectors ξ and η. Given any $\alpha \in (0,1)$, the auxiliary distribution vector $\zeta_\alpha = (1-\alpha)^{\frac{1}{2}}\xi + \alpha^{\frac{1}{2}}\eta$, with the covariance operator $\Sigma = (1-\alpha)\Sigma^\xi + \alpha\Sigma^\eta$, and $x \in \Re^n$ has a pdf of the

form

$$p(\alpha, x) = \frac{1}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}} exp\left\{ -\frac{1}{2}x'\Sigma^{-1}x \right\}$$

Where the following density is the solution to the parabolic partial differential equation.

$$\frac{\partial}{\partial\alpha}p(\alpha, x) = \frac{1}{2}\sum_{i,j}(r_{ij}^{\eta} - r_{ij}^{\xi})\frac{\partial^2}{\partial x_i \partial x_j}p(\alpha, x) \tag{3.2}$$

Verification of (3.2)

$$\frac{\partial}{\partial\alpha}p(\alpha, x) = \sum_{1\le i\le j\le n}\frac{\partial}{\partial(r_{ij})}p(\alpha, x)\frac{\partial(r_{ij})}{\partial\alpha}$$

We need to show that $\frac{\partial}{\partial(r_{ij})}p(\alpha, x) = \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}p(\alpha, x)$. Now, we note that

$$p(\alpha, x) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} e^{-i(t_1x_1+t_2x_2+\ldots+t_nx_n)}e^{-\frac{1}{2}(t_2^2r_{11}+\ldots+t_n^2r_{nn}+\sum_{1\le i<j\le n}2t_it_jr_{ij})}dt_1\ldots dt_n$$

So for $i < j$,

$$\frac{\partial}{\partial r_{ij}}p(\alpha, x) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}(-t_it_j)e^{-i(t_1x_1+t_2x_2+\ldots+t_nx_n)}e^{-\frac{1}{2}(t_2^2r_{11}+\ldots+t_n^2r_{nn}+\sum_{1\le i<j\le n}2t_it_jr_{ij})}dt_1\ldots dt_n$$

$$\frac{\partial}{\partial x_i}p(\alpha, x) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}(-it_i)e^{-i(t_1x_1+t_2x_2+\ldots+t_nx_n)}e^{-\frac{1}{2}(t_2^2r_{11}+\ldots+t_n^2r_{nn}+\sum_{1\le i<j\le n}2t_it_jr_{ij})}dt_1\ldots dt_n$$

$$\frac{\partial^2}{\partial x_i \partial x_j}p(\alpha, x) =$$
$$\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}(i\cdot it_it_j)e^{-i(t_1x_1+t_2x_2+\ldots+t_nx_n)}e^{-\frac{1}{2}(t_2^2r_{11}+\ldots+t_n^2r_{nn}+\sum_{1\le i<j\le n}2t_it_jr_{ij})}dt_1\ldots dt_n$$
$$= \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}(-t_it_j)e^{-i(t_1x_1+t_2x_2+\ldots+t_nx_n)}e^{-\frac{1}{2}(t_2^2r_{11}+\ldots+t_n^2r_{nn}+\sum_{1\le i<j\le n}2t_it_jr_{ij})}dt_1\ldots dt_n$$

Which is what we want, now if we notice that we are only interested in summing over half the interval it becomes clear that for $(i < j)$ $\frac{\partial}{\partial(r_{ij})}p(\alpha, x) = \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}p(\alpha, x)$. now we

need to check $i = j$ case.

$$\frac{\partial}{\partial r_{ii}}p(\alpha,x) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}(-\frac{1}{2}t_i^2)e^{-i(t_1x_1+t_2x_2+\ldots+t_nx_n)}e^{-\frac{1}{2}(t_2^2r_{11}+\ldots+t_n^2r_{nn}+\sum_{1\le i<j\le n}2t_it_jr_{ij})}dt_1\ldots dt_n$$

$$\frac{\partial}{\partial x_i}p(\alpha,x) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}(-it_i)e^{-i(t_1x_1+t_2x_2+\ldots+t_nx_n)}e^{-\frac{1}{2}(t_2^2r_{11}+\ldots+t_n^2r_{nn}+\sum_{1\le i<j\le n}2t_it_jr_{ij})}dt_1\ldots dt_n$$

$$\frac{\partial^2}{\partial x_i\partial x_i}p(\alpha,x) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}(i\cdot it_i^2)e^{-i(t_1x_1+t_2x_2+\ldots+t_nx_n)}e^{-\frac{1}{2}(t_2^2r_{11}+\ldots+t_n^2r_{nn}+\sum_{1\le i<j\le n}2t_it_jr_{ij})}dt_1\ldots dt_n$$

$$= \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}(-t_i^2)e^{-i(t_1x_1+t_2x_2+\ldots+t_nx_n)}e^{-\frac{1}{2}(t_2^2r_{11}+\ldots+t_n^2r_{nn}+\sum_{1\le i<j\le n}2t_it_jr_{ij})}dt_1\ldots dt_n$$

which makes it clear that $\frac{\partial}{\partial(r_{ij})}p(\alpha,x) = \frac{1}{2}\frac{\partial^2}{\partial x_i\partial x_j}p(\alpha,x)$ for $(i=j)$ case. Now it is clear that $\frac{\partial}{\partial\alpha}(r_{ij}) = (r_{ij}^{\eta} - r_{ij}^{\xi})$. thus (3.2) is verified. Now, we want to look at

$$\mathbb{E}[f(\zeta_\alpha)] = \int_{-\infty}^{\infty}f(x)p(\alpha,x)dx_1\ldots dx_n$$

and which we then want to differentiate with respect to α. Thus giving us,

$$\frac{\partial}{\partial\alpha}\mathbb{E}[f(\zeta_\alpha)] = \int_{-\infty}^{\infty}f(x)\frac{\partial}{\partial\alpha}p(\alpha,x)dx_1\ldots dx_n$$

We substitute the pde we verified in equation (3.2).

$$\begin{aligned}
&\int_{-\infty}^{\infty}f(x)\frac{\partial^2}{\partial\alpha}p(\alpha,x)dx_1\ldots dx_n\\
&\quad= \int_{-\infty}^{\infty}f(x)\left(\frac{1}{2}\sum_{ij}(r_{ij}^{\eta}-r_{ij}^{\xi})\frac{\partial^2}{\partial x_i\partial x_j}p(\alpha,x)\right)dx_1\ldots dx_n\\
&\quad= \int_{\infty}^{\infty}-\frac{1}{2}\sum_{ij}(r_{ij}^{\eta}-r_{ij}^{\xi})f(x)\frac{\partial^2}{\partial x_i\partial x_j}p(\alpha,x)dx_1\ldots dx_n\\
&\quad= \frac{1}{2}\sum_{ij}(r_{ij}^{\eta}-r_{ij}^{\xi})\int_{-\infty}^{\infty}f(x)\frac{\partial^2}{\partial x_i\partial x_j}p(\alpha,x)dx_1\ldots dx_n
\end{aligned}$$

So then, we want to integrate the following expression twice by parts.

$$\int_{-\infty}^{\infty} f(x)\frac{\partial^2}{\partial x_i \partial x_j}p(\alpha,x)dx_1\dots dx_n$$

which gives us,

$$\begin{aligned}
&\int_{-\infty}^{\infty} f(x)d\left(\frac{\partial}{\partial x_i}p(\alpha,x)\right)dx_1\dots dx_n \\
&\quad = f(x)\frac{\partial}{\partial x_i}p(\alpha,x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial}{\partial x_i}f(x)p(\alpha,x)dx_1\dots dx_n \\
&\quad = 0 - \int_{-\infty}^{\infty}\frac{\partial}{\partial x_i}f(x)\frac{\partial}{\partial x_i}p(\alpha,x)dx_1\dots dx_n \\
&\quad = -\frac{\partial}{\partial x_i}f(x)p(\alpha,x)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\frac{\partial^2}{\partial x_i\partial x_j}f(x)p(\alpha,x)dx_1\dots dx_n \\
&\quad = 0 + \int_{-\infty}^{\infty}\frac{\partial^2}{\partial x_i\partial x_j}f(x)p(\alpha,x)dx_1\dots dx_n = \int_{-\infty}^{\infty}\frac{\partial^2}{\partial x_i\partial x_j}f(x)p(\alpha,x)dx_1\dots dx_n
\end{aligned}$$

Thus we have,

$$\frac{1}{2}\sum_{ij}(r_{ij}^{\eta}-r_{ij}^{\xi})\int_{-\infty}^{\infty}\frac{\partial^2}{\partial x_i\partial x_j}f(x)p(\alpha,x)dx_1\dots dx_n$$

$$= \frac{1}{2}\sum_{ij}(r_{ij}^{\eta}-r_{ij}^{\xi})\mathbb{E}\frac{\partial^2}{\partial x_i\partial x_j}f(\zeta_\alpha)$$

Now take into account that $r_{ij}^{\eta} = \mathbb{E}[\eta_i\eta_j]$ and $r_{ij}^{\xi} = \mathbb{E}[\xi_i\xi_j]$. Now we integrate the expression with respect to *alpha* over the interval 0 to 1.

$$\mathbb{E}[f(\zeta_\alpha)] = \int_0^1 \frac{1}{2}\sum_{ij}(\mathbb{E}[\eta_i\eta_j]-\mathbb{E}[\xi_i\xi_j])\mathbb{E}\frac{\partial^2}{\partial x_i\partial x_j}f(\zeta_\alpha)d\alpha$$

Corollary 3.2 Let ξ, η, and f be the same as given by the assumptions in Theorem 3.1. Assume that the set of indices $[0,n]\times[0,n]$ may be split into two parts, Π_1 and Π_2, so that the equality $\mathbb{E}[\xi_i\xi_j] = \mathbb{E}[\eta_i\eta_j]$ holds for $(i,j)\in\Pi_1$, and the equality $\mathbb{E}[\xi_i\xi_j] \le \mathbb{E}[\eta_i\eta_j]$ holds for $(i,j)\in\Pi_2$, and, for all $x\in\Re^n$, we have $\frac{\partial^2}{\partial x_i\partial x_j}f(x)\ge 0$. Then

$\mathbb{E}[f(\xi)] \le \mathbb{E}[f(\eta)]$.

Observe that the assumptions on smoothness of the function f can be relaxed. If function f may be approximated by smooth function f_m in such a way that

$\mathbb{E}[f_m(\xi)] \to \mathbb{E}[f(\xi)]$ and $\mathbb{E}[f_m(\eta)] \to \mathbb{E}[f(\eta)]$, and $\frac{\partial^2}{\partial x_i \partial x_j} f(\cdot) \ge 0$ for $(i,j) \in \Pi_2$, then, by taking the limit in the sequence of inequalities $\mathbb{E}[f_m(\xi)] \le \mathbb{E}[f_m(\eta)]$, we again obtain $\mathbb{E}[f(\xi)] \le \mathbb{E}[f(\eta)]$.

Corollary 3.3 (The Slepian-Schläfli form of the Comparison Principle) Let $\xi = (\xi_1, \ldots, \xi_n)$ and $\eta = (\eta_1, \ldots, \eta_n)$ be centered Gaussian random vectors with $\mathrm{Var}[\xi_i] = \mathrm{Var}[\eta_i]$ for all $i \le n$, and $\mathbb{E}[\xi_i \xi_j] \le \mathbb{E}[\eta_i \eta_j]$ for all $i \ne j$. Then for any $r \in \Re^1$

$$P\left\{\sup_i \xi_i > r\right\} \ge P\left\{\sup_i \eta_i > r\right\}. \tag{3.3}$$

Proof. Assume $z = -r$ and define a function $f : \Re^n \to \Re^1$ by the formula

$$f \equiv 1_{\{\inf(x_i) \ge z\}} = \begin{cases} 1 & \text{if } \inf\{x_i\} \ge z \\ 0 & \text{if } \inf\{x_i\} < z \end{cases}$$

Let us approximate f by the functions

$$f_m(x) \equiv \prod_{q=1}^{n} \Phi(m(x_q - z)),$$

then we have for $i \ne j$

$$\frac{\partial^2}{\partial x_i \partial x_j} f(x) = \frac{1}{2\pi} m^2 \exp\left\{\frac{-m^2((x_i - z)^2 + (x_j - z)^2)}{2}\right\} \prod_{q=1 q \ne i q \ne j}^{n} \Phi(m(x_q - z)) \ge 0$$

It is obvious that $f(x) = \lim_m f_m(x)$ on the set $\{x \in \Re^n | x_q \neq z, 1 \leq q \leq n\}$ and, given that min $\mathrm{Var}[\xi_i] = \min \mathrm{Var}[\eta_i] > 0$ or $z \neq 0$, this implies the equalities

$$\mathbb{E}[f(\xi)] = \lim_m \mathbb{E}[f_m(\xi)] \text{ and } \mathbb{E}[f(\eta)] = \lim_m \mathbb{E}[f_m(\eta)] \tag{3.4}$$

We may consider the vectors ξ and η independent. We will now apply Corollary 2 to ξ, η, f_m, and the sets of indices

$$\Pi_1 \equiv \{(i,i), 1 \leq i \leq n\} \text{ and } \Pi_2 \equiv \{(i,j), i \neq j\},$$

and then take the limit in (3.4) to obtain the inequality

$$P\left\{\inf_i \xi_i > z\right\} = \mathbb{E}[f(\xi)] \leq \mathbb{E}[f(\eta)] = P\left\{\inf_i \eta_i > z\right\}.$$

One should notice that $\inf_i \xi_i = -\sup_i(-\xi_i)$ and that ξ and $-\xi$ are identically distributed and the same can be said for η. Thus we can rewrite the above inequality as

$$P\left\{-\sup_i \xi_i > -r\right\} \leq P\left\{-\sup_i \eta_i > -r\right\},$$

which is equivalent to (3.3).

Chapter 4

Three Examples

Our first example comes from Brownian pursuits. Let $W_0, W_1, \ldots, W_n$ be independent standard Brownian motion, starting at 0, and define the stopping time

$$\tau_n = \inf\{t > 0 : W_i(t) - 1 = W_0(t) \text{ for some } 1 \leq i \leq n\}.$$

Then τ_n can be viewed as a capture time in a random pursuit setting. Assume that a Brownian prisoner escapes, running along the path of W_0. In his pursuit, there are n independent Brownian policemen. These policemen run along the paths of $W_1, \ldots, W_n$, respectively. The prisoner begins one unit ahead of the policemen. τ_n represents the capture time when the fastest of the policemen apprehends the prisoner. In Wenbo V. Li and Qi-Man Shao (2004) they tackle the question for which n is $\mathbb{E}\tau_n < \infty$.

It is known that

$$P(\tau_n > t) \sim ct^{-\gamma_n} \text{ as } t \to \infty \tag{4.1}$$

where γ_n is determined by the first eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator on a subset of the unit sphere S^n in R^{n+1}. Of course, $\gamma_1 = \frac{1}{2}$ by the reflection principle, and from Li and Shao (2004) we know that $\gamma_2 = \frac{3}{4}$, $\gamma_3 < 1$.

Further, M. Bramson and d. Griffeath (1991) conjecture that $\gamma_4 > 1$ as their simulation suggests that $\gamma_4 \approx 1.032$. From Li and Shao (2004) we know that $\gamma_5 > 1$. Early, using closely related independent stationary OrnsteinUhlenbeck processes and the theory of large deviations, from Li and Shao (2004) we know Kesten showed that

$$0 < \liminf_{n\to\infty} \frac{\gamma_n}{\log n} \leq \limsup_{n\to\infty} \frac{\gamma_n}{\log n} \leq \frac{1}{4} \tag{4.2}$$

and conjectured the existence of $\lim_{n\to\infty} \frac{\gamma_n}{\log n}$. From Li and Shao (2004) we know that $\lim_{n\to\infty} \frac{\gamma_n}{\log n} = \frac{1}{4}$. To see the connection with (1.1) from the point of view of the theory of Gaussian processes, we note that estimating the tail of τ_n, that is, $P(\tau_n > t)$ as $t \to \infty$, is the same as estimating the lower tail probability for the Gaussian process $X(k,s) = W_k(s) - W_0(s)$ indexed by $(k,s) \in \{1,...,n\} \times [0,1]$, that is, $P(\max_{1\leq k\leq n} \sup_{0\leq s\leq 1}(W_k(s) - W_0(s)) \leq x)$ as $x \to 0$. In fact, for any $t > 0$, by the Brownian scaling

$$P(\tau_n > t) = P\left(\max_{1\leq k\leq n} \sup_{0\leq s\leq t} (W_k(s) - W_0(s)) < 1\right)$$

$$= P\left(\max_{1\leq l\leq n} \sup_{0\leq s\leq 1} (W_k(s) - W_0(s)) < t^{-\frac{1}{2}}\right)[5]$$

In particular we can look at the capture time of the fractional Brownian motion pursuit. Let $\{B_{k,\alpha}(t); t \geq 0\}$ $(k = 0, 1, 2, \ldots, n)$ denote independent fractional Brownian motions of order $\alpha = 1$ all starting from 0. let

$$\tau_n := \tau_{n,\alpha} = \inf\left\{t > 0 : \max_{1\leq k\leq n} B_{k,\alpha}(t) = B_{0,\alpha}(t) + 1\right\}$$

Li and Shao (2004) raise the question of when is $\mathbb{E}(\tau_n)$ finite? Also from Li and Shao (2004) we know that this question is the same as estimating the lower tail probabilities of $\max_{1\leq k\leq n} \sup_{0\leq t\leq 1}(B_{k,\alpha}(t) - B_{0,\alpha}(t))$. For any $s > 0$, by the fractional

Brownian scaling,

$$P(\tau_n > s) = P\Big(\max_{1\le k\le n}\sup_{0\le t\le s}(B_{k,\alpha}(t) - B_{0,\alpha}(t)) < 1\Big)$$

$$= P\Big(\max_{1\le k\le n}\sup_{0\le t\le s}(B_{k,\alpha}(t) - B_{0,\alpha}(t)) < s^{-\frac{\alpha}{2}}\Big).$$

Now let

$$X_{k,\alpha}(t) = e^{-\frac{t\alpha}{2}}B_{k,\alpha}(t) \text{ for } k = 0,\dots,n.$$

and the following limit exists by subadditivity,

$$c_{n,\alpha} := -\lim_{T\to\infty}\frac{1}{T}\log P\Big(\sup_{0\le t\le T}\max_{1\le k\le n}(X_{k,\alpha}(t) - X_{0,\alpha}(t)) \le 0\Big) \tag{4.3}$$

which allows for,

Theorem 4.1

$$P\Big(\max_{1\le k\le n}\sup_{0\le t\le s}(B_{k,\alpha}(t) - B_{0,\alpha}(t)) < x\Big) = x^{\frac{2c_{n,\alpha}}{\alpha+o(1)}} \text{ as } x \to \infty \tag{4.4}$$

Proof

In this paper we will only deal with $\alpha = 1$, It will suffices to show that

$$\liminf_{x\downarrow 0}\frac{\log P\Big(\max_{1\le k\le n}\sup_{0\le t\le s}(B_{k,\alpha}(t) - B_{0,\alpha}(t)) \le x\Big)}{\log(\frac{1}{x})} \ge -\frac{2c_\alpha}{\alpha} \tag{4.5}$$

and

$$\liminf_{x\downarrow 0}\frac{\log P\Big(\max_{1\le k\le n}\sup_{0\le t\le s}(B_{k,\alpha}(t) - B_{0,\alpha}(t)) \le x\Big)}{\log(\frac{1}{x})} \le -\frac{2c_\alpha}{\alpha} \tag{4.6}$$

By the scaling property of the fractional Brownian motion, for any $0 < x < \frac{1}{2}$

$$P\Big(\max_{1\le k\le n}\sup_{0\le t\le(\frac{2}{\alpha}\log(\frac{1}{x})}(X_{k,\alpha}(t) - X_{0,\alpha}(t)) \le 0\Big)$$

$$= P\Big(\max_{1\le k\le n} \sup_{x^{\frac{2}{\alpha}}\le t\le 1} (B_{k,\alpha}(t) - B_{0,\alpha}(t)) \le 0 \Big)$$

$$\le P\Big(\max_{1\le k\le n} \sup_{x^{\frac{2}{\alpha}}\le t\le 1} (B_{k,\alpha}(t) - B_{0,\alpha}(t)) \le x \Big) \tag{4.7}$$

$$\le \frac{P\Big(\max_{1\le k\le n} \sup_{0\le t\le 1}(B_{k,\alpha}(t) - B_{0,\alpha}(t)) \le x \Big)}{P\Big(\max_{1\le k\le n} \sup_{0\le t\le x^{\frac{2}{\alpha}}}(B_{k,\alpha}(t) - B_{0,\alpha}(t)) \le x \Big)}$$

$$= \frac{P\Big(\max_{1\le k\le n} \sup_{0\le t\le 1}(B_{k,\alpha}(t) - B_{0,\alpha}(t)) \le x \Big)}{P\Big(\max_{1\le k\le n} \sup_{0\le t\le 1}(B_{k,\alpha}(t) - B_{0,\alpha}(t)) \le 1 \Big)}$$

where we have used the Slepian Lemma (Corollary 3.3) in the last inequality. Now (4.5) follows from (4.3) and (4.7).

If we were doing this in general we would need to break this into multiple cases, but we are only dealing with the case where $\alpha = 1$.

For $\alpha = 1$ it is easy to see that, for $t \ge a \ge 0$,

$$\mathbb{E}\Big\{ [(B_{k,\alpha}(t) - B_{0,\alpha}(t)) - (B_{k,\alpha}(a) - B_{0,\alpha}(a))](B_{k,\alpha}(a) - B_{0,\alpha}(a)] \Big\} = 0 \tag{4.8}$$

Therefore by the Slepian Lemma (Corollary 3.3),

$$P\Big(\max_{1\le k\le n} \sup_{0\le t\le (\frac{2}{\alpha})\log(\frac{1}{x})} X_{k,\alpha}(t) - X_{0,\alpha}(t) \le 0 \Big)$$

$$= \Big(\max_{1\le k\le n} \sup_{x^{\frac{2}{\alpha}}\le t\le 1} B_{k,\alpha}(t) - B_{0,\alpha}(t) \le 0 \Big) \tag{4.9}$$

$$\ge P\Big(\max_{1\le k\le n} \sup_{x^{\frac{2}{\alpha}}\le t\le 1} [(B_{k,\alpha}(t) - B_{0,\alpha}(t)) - (B_{k,\alpha}(x^{\frac{2}{\alpha}}) - B_{0,\alpha}(x^{\frac{2}{\alpha}})) \le 0)] \le x, (B_{k,\alpha}(x^{\frac{2}{\alpha}}) - B_{0,\alpha}(x^{\frac{2}{\alpha}})) \le -1$$

$$\ge P\Big(\max_{1\le k\le n} \sup_{x^{\frac{2}{\alpha}}\le t\le 1} [(B_{k,\alpha}(t) - B_{0,\alpha}(t)) - (B_{k,\alpha}(x^{\frac{2}{\alpha}}) - B_{0,\alpha}(x^{\frac{2}{\alpha}})) \le 0)] \le x \Big) P((B_{k,\alpha}(x^{\frac{2}{\alpha}}) - B_{0,\alpha}(x^{\frac{2}{\alpha}}$$

$$\geq P\Big(\max_{1\leq k\leq n}\sup_{0\leq t\leq 1}(B_{k,\alpha}(t)-B_{0,\alpha}(t))\leq x\Big)P((B_{k,\alpha}(1)-B_{0,\alpha}(1))\leq -1).$$

Hence (4.6) holds by (4.3).

The second example comes from the study of real zeros of random polynomials. Let $\{Z_i, i \geq 0\}$ be independent standard normal random variables. In their study on the probability that the random polynomial $\sum_{i=0}^{n} Z_i x^i$ does not have real root in $\Re$, from Dembo, Poonen, Shao and Zeitouni (2002) we know,

$P\Big(\sum_{i=0}^{n} Z_i x^i \leq 0 \ \forall\ x \in \Re^1\Big) = n^{-b+o(1)}$ as n $\to \infty$ through even integers, where

$$b = -4\lim_{T\to\infty}\frac{1}{T}\log P\Big(\sup_{0\leq t\leq T} X_t \leq 0\Big) \tag{4.10}$$

and X_t is a centered stationary Gaussian process with

$$\mathbb{E}[X_s X_t] = \frac{2e^{-\frac{|t-s|}{2}}}{1+e^{-|t-s|}} \text{ for } s,t \geq 0.[5] \tag{4.11}$$

Furthermore, they proved that $0.4 < b \leq 2$. Here we find a very closed relation between (4.10) and the lower tail probability for a connected dual Gaussian process. [5]

Our third example comes from an old problem of the first passage time for the so-called Slepian process. Let $S(t), t \geq 0$, be the Slepian process, which is the Gaussian process with mean zero and covariance $\mathbb{E}[S(t)S(s)] = (1-|t-s|)1_{\{|t-s|\leq 1\}}$. It is easy to see that $S(t)$ can be represented in terms of the standard Wiener process W(t) by $S(t) = W(t) - W(t+1), t \geq 0$. The first passage probability

$$Q_a(T) = P\Big(\sup_{0\leq t\leq T} S(t) \leq a\Big)$$

was studied by many authors. In particular, Slepian (1961) found a simple expression when $T \leq 1$, L.A. Shepp (1971) gave an explicit but very hard to evaluate formula in terms of a T-fold integral for an integer T and a $(2[T]+2)$-fold integral for a non-integer

T . The question of finding bounds on

$$\lim_{n\to\infty} n^{-1} \log P\Big(\sup_{0\le t\le n} S(t) \le a \Big) \tag{4.12}$$

(assuming the limit exists) was asked in L.A. Shepp's (1971) paper since the formula of $Q_a(T)$ is apparently not suited for either numerical calculation or asymptotic estimation. From Li and Shao (2004), we have that the limit in (4.12) exists. [5]

Chapter 5

Numerical Analysis

The simulations I ran for the problem in estimating the probability in the study of real zeros of random polynomials is based on the research done by Li and Shao (2004). [5] The following basic explanation of the algorithm used will follow, and for further information I have included the program used to do the simulations at the end of the chapter. To estimate the probabilities I used a frequency distribution:

$$\frac{1}{m}\sum_{k=1}^{m} 1_{\{\max_{1\le i\le n} X_{k,i}\le a\}} \approx P(\max_{1\le i\le n} X_{k,i} \le a)$$

with $m = 20,000$, $h = .05$, $n = 150$, $a = \{0, .01, .02, \ldots, 3\}$ To find the lower tail probabilities we will need to take the logarithm of this probability and divide it by n times h.

$$\frac{\ln\{P(\max_{1\le i\le n} X_{k,i} \le a)\}}{nh}$$

The random variable $X_{k,i}$ was derived as follows: $B = (b_{ij}, 1 \le i, j \le n)$ where B is an n by n Matrix, in this case, $n = 150$.

$$b_{ij} = \mathbb{E}X_iX_j = \frac{2e^{-\frac{h|j-i|}{2}}}{1+e^{-h|j-i|}}$$

A is lower triangular matrix with $i \le j$ defined as follows:

$$A = (a_{ij}, 1 \leq i, j \leq n)$$

where $a_{11} = b_{11}^{\frac{1}{2}}$,

$$a_{i1} = \frac{b_{i1}}{a_{11}} \ , i = 2, \ldots, n$$

$i \neq j$

$$a_{ij} = \frac{1}{a_{jj}}(b_{ij} - \sum_{k=1}^{j-1} a_{ik}a_{jk}), j = 1, \ldots, i-1$$

and for $i = j$

$$a_{ii} = (b_{ii} - \sum_{k=1}^{i-1} a_{ik}^2)^{\frac{1}{2}}$$

I then constructed the random vector as follows:

$$X = AY \text{ where } Y \sim i.i.d.N(0,1)$$

The simulation produced the following results. A graphical representation can be found at the end of the chapter under (5.1) and (5.2):

a	$P(\max_{1\le i\le n} X_{k,i} \le a)$	$\frac{\ln\{P(\max_{1\le i\le n} X_{k,i}\le a)\}}{nh}$
0	0.2574	-0.18095
0.05	0.26135	-0.178919
0.1	0.26465	-0.177246
0.15	0.268	-0.175569
0.2	0.27085	-0.174159
0.25	0.27425	-0.172495
0.3	0.27715	-0.171093
0.35	0.2792	-0.17011
0.4	0.28205	-0.168756
0.45	0.28435	-0.167673
0.5	0.28645	-0.166692
0.55	0.289	-0.16551
0.6	0.2915	-0.164362
0.65	0.2937	-0.16336
0.7	0.2962	-0.162229
0.75	0.2989	-0.161019
0.8	0.3013	-0.159953
0.9	0.305	-0.158326
0.95	0.3067	-0.157585
1	0.3089	-0.156632
1.05	0.31105	-0.155707
1.1	0.313	-0.154874
1.15	0.31535	-0.153876
1.2	0.3175	-0.15297
1.25	0.3198	-0.152008
1.3	0.32165	-0.151239
1.35	0.32435	-0.150124
1.4	0.32695	-0.14906
1.45	0.32905	-0.148206

a	$P(\max_{1\le i\le n} X_{k,i} \le a)$	$\frac{\ln\{P(\max_{1\le i\le n} X_{k,i}\le a)\}}{nh}$
1.5	0.33095	-0.147438
1.55	0.33285	-0.146675
1.6	0.3351	-0.145777
1.65	0.33825	-0.144529
1.7	0.3406	-0.143606
1.75	0.34375	-0.142379
1.8	0.3457	-0.141625
1.85	0.3484	-0.140587
1.9	0.3509	-0.139634
1.95	0.35275	-0.138933
2	0.3549	-0.138123
2.05	0.35715	-0.13728
2.1	0.3601	-0.136183
2.15	0.36235	-0.135353
2.2	0.3648	-0.134454
2.25	0.36725	-0.133562
2.3	0.3692	-0.132856
2.35	0.3721	-0.131812
2.4	0.37445	-0.130973
2.45	0.3769	-0.130103
2.5	0.3794	-0.129222
2.55	0.3817	-0.128416
2.6	0.38425	-0.127528
2.65	0.38685	-0.126629
2.7	0.38905	-0.125873
2.75	0.3911	-0.125172
2.8	0.39295	-0.124543
2.85	0.3947	-0.123951
2.9	0.3971	-0.123142
2.95	0.39955	-0.122322

To simulate the slepian process I again used a frequency distribution, but with $h = .05$, $m = 1,000,000$, $n = 100$, and $k = 1,000$, where Z_i is an i.i.d random variable from a N(0,1) distribution. The following algorithm was employed:

$$X(ih) = \frac{1}{\sqrt{k}} \sum_{j=1+kih}^{k+kih} Z_j \text{ with } i = 1, \ldots, n$$

we again then looked at the frequency distribution as follows:

$$\frac{1}{m} \sum_{k=1}^{m} 1_{\{\max_{1 \le i \le n} X_{k,i} \le a\}} \approx P(\max_{1 \le i \le n} X_{k,i} \le a)$$

and to find the lower tail probabilities,

$$\frac{\ln \{P(\max_{1 \le i \le n} X_{k,i} \le a)\}}{nh}$$

Unfortunately I was never able to acquire any useful data; the general theory implied that I would need m to be around $1,000,000$ to receive good data and due to time constraints this was not possible. However I have included the Mathematica program at the end of the chapter that I attempted to run.

The Mathematica programs (written by Rodrigo Guzman and Zack Almquist) for estimating the probability of the study of real zeros of random polynomials and the slepian process follow after figures (5.1) and (5.2)on the following pages, respectively.

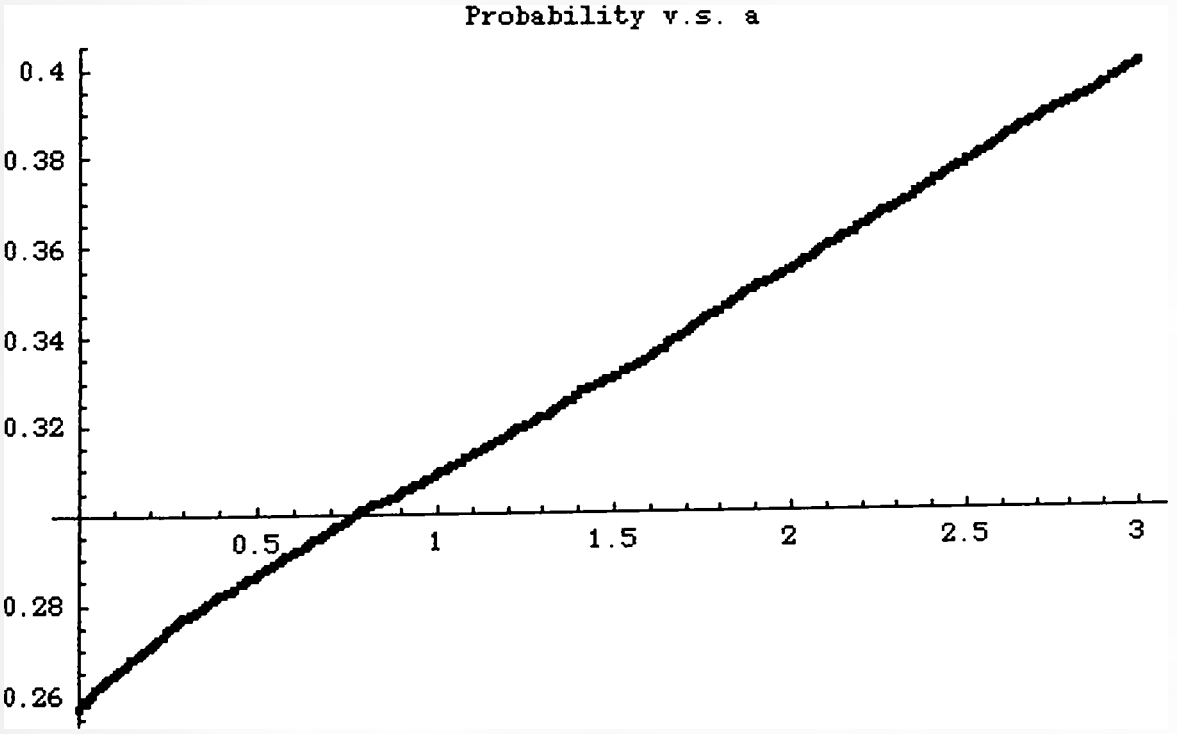

Figure 5.1: A Graph of the probabilities in the study of real zeros of random polynomials.

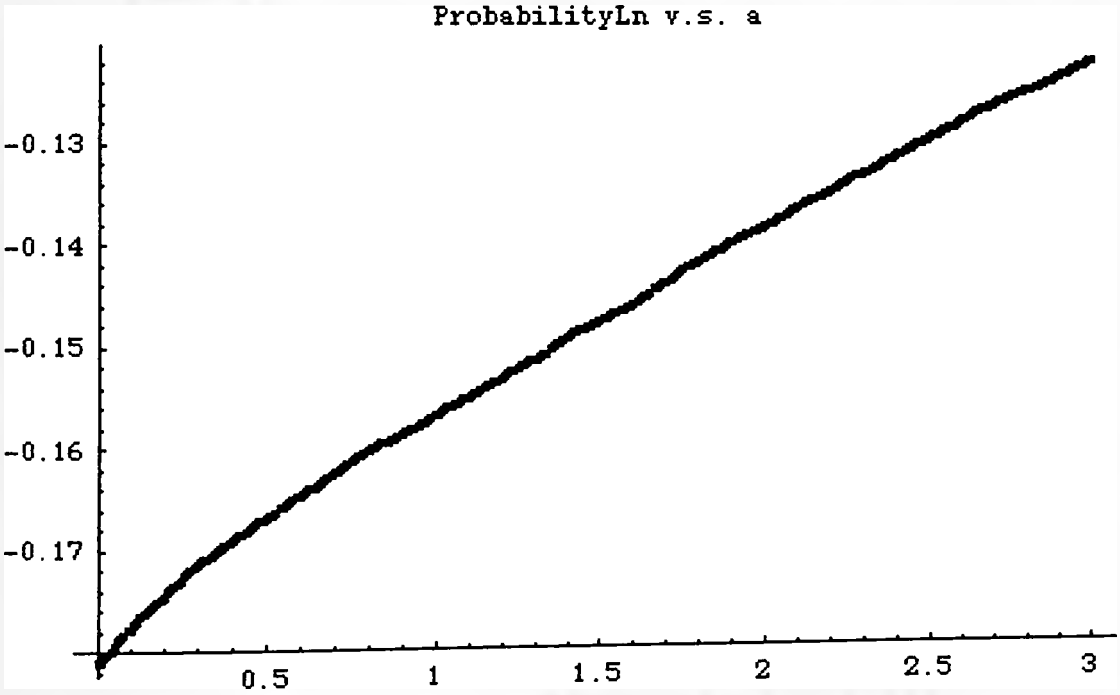

Figure 5.2: A Graph of the logarithmic probabilities in the study of real zeros of random polynomials.

```
<< Statistics`NormalDistribution`
size = 150;
h = 0.05;
theMatrix = {};
Date[]
(*j is column number*)
(*i is row number *)
For[i = 1, i ≤ size, i++,
  row = {};
  For[j = 1, j ≤ size, j++,
   row = Append[row, N[2 Exp[(-h Abs[j-i])/2] / (1 + Exp[-h Abs[j - i]])]];
   If[Mod[i, size / 4] == 0 && j == size,
    Print["bMatrix ", i, ",", j, " ", Date[]]];
  ];
  theMatrix = Append[theMatrix, row];

 ];
Print["Done generating b matrix!", Date[]];

(*generate aMatrix*)
c1 = {theMatrix[[1]][[1]]};
c1 =
  Append[c1, Table[N[theMatrix[[i]][[1]] / c1[[1]]],
     {i, 2, size}]] // Flatten;
c21 = 0;
c22 = N[Sqrt[theMatrix[[2]][[2]] - c1[[2]]^2]];
c2 = {c21, c22};
For[i = 3, i ≤ size, i++,
  c2 = Append[c2,
    1/c22 (theMatrix[[i]][[2]] -
       c1[[i]] * c1[[2]]  )];
 ];
aMatrix = {c1, c2};
bz = False;
For[j = 3, j ≤ size, j++,
```

```
    col = Append[col, Table[0, {k, j, size}]] //
      Flatten;
    ];
    If[Abs[Im[diag]] > 0,
     bz = True;
     diag = 0;
     col = Append[col, Table[0, {k, j, size}]] //
       Flatten;
    ];
    If[!bz, col = Append[col, diag]];
    For[i = j + 1, i ≤ size && !bz, i++,
     entry =
       1
      ------
      diag
       (theMatrix[[i]][[j]] -
         Sum[aMatrix[[k]][[i]] aMatrix[[k]][[j]] ,
          {k, 1, j - 1}]);
     col = Append[col, entry];
     If[Mod[j, size / 5] == 0 && i == size,
      Print["aMatrix: ", i, " ", j, " ", diag,
       " ", entry, " ", Date[]]];
    ];
    aMatrix = Append[aMatrix, col];
   ];
  aMatrix = Transpose[aMatrix];
  Print["Done generating a matrix!", Date[]];

  (*trials*)
  max = 20000;
  cList = {};
  For[i = 1, i ≤ max, i++,
    randomVarVector =
     RandomArray[NormalDistribution[0, 1], size];
    randomVarVector = randomVarVector;
    vector1 = aMatrix.randomVarVector;
    maxVector1 = Max[vector1];
    If[Mod[i, max / 10] == 0, Print["trial #: ", i]];
    nList = {};
    maxK = 3;
    For[k = 0, k ≤ maxK, k = k + .01,
     n = If[maxVector1 ≤ k, 1, 0];
```

```
    kList = {};
    For[i = 1, i ≤ Length[cList], i++,
     kList = Append[kList, cList[[i]][[k]]];
    ];
    fList = Append[fList, kList];
  ];
(*fList[[i]] is list of counts for k =
   0.01*i *)
"Out:" >> randompoly.txt;
Print[
   "============================================="];
probList = {};
probLnList = {};
For[i = 1, i ≤ Length[fList], i++,
   count = Count[fList[[i]], 1];
   prob = N[count / Length[fList[[i]]  ]];
   probList = Append[probList,
     {i*0.01 - 0.01 , prob}  ];
   probLnList = Append[probLnList,
     {i*0.01 - 0.01 , probLn}];
   probLn = N[Log[prob] / (h*size)];
   "k = " <> ToString[ i*0.01 - 0.01] <>
     " prob = " <> ToString[prob] <> " probLn " <>
     ToString[probLn] >>> randompoly.txt;
 ];
```

```
<< Statistics`NormalDistribution`
h = 0.05;
size = 100;
m = 1000;
p = Floor[m + h m size];

(*trials*)
max = 1000000;
cList = {};
For[i = 1, i ≤ max, i++,
  randomVarVector =
   RandomArray[NormalDistribution[0, 1], p];
  vector1 = {};
  For[k = 1, k ≤ size, k++,
   vector1 = Append[vector1,
       1
     ------------ *
     Sqrt[m]
      (Sum[randomVarVector[[j]],
        {j, 1 + h*m*k, m + h*m*k}])];
  ];
  maxVector1 = Max[vector1];
  If[Mod[i, max / 10] == 0, Print["trial #: ", i]];
  nList = {};
  maxK = 3;
  For[k = 0, k ≤ maxK, k = k + .01,
   n = If[maxVector1 ≤ k, 1, 0];
   nList = Append[nList, n];
  ];
  cList = Append[cList, nList];
 ];
fList = {};
For[k = 1, k ≤ Length[cList[[1]]], k++,
  kList = {};
  For[i = 1, i ≤ Length[cList], i++,
   kList = Append[kList, cList[[i]][[k]]];
  ];
  fList = Append[fList, kList];
 ];
(*fList[[i]] is list of counts for k = 0.01*i *)
```

```
count = Count[fList[[i]], 1];
prob = N[count / Length[fList[[i]]  ]];
probList = Append[probList,
  {i * 0.01 - 0.01 , prob}  ];
probLnList = Append[probLnList,
  {i * 0.01 - 0.01 , probLn}];
probLn = N[ Log[prob] / (h * size) ];
"k  =  " <> ToString[ i * 0.01 - 0.01] <>
  "  prob  =  " <> ToString[prob] <> "  probLn  " <>
  ToString[probLn] >>> slepianout.txt;
];
```

Glossary

Axiomatic Probability:
Probability has been formalized with three axioms which create the foundation for all of probability theory. The first axiom is that the probability of an event is between zero and one; the second axiom is that the probability of the entire space is one; and the last axiom is that probability of number of disjoint events is the addition of their probabilities.

Brownian Motion:
Brownian motion was named in honor botanist Robert Brown and for our purposes refers to the mathematical models used to describe the physical phenomenon of random particle motion.

Characteristic Function:
The Characteristic Function, a very useful tool in probability theory, allows one to show many aspects about a distribution and can also be used to calculate the mean and variance of a random variable.

Continuous Random Variable:
A continuous random variable is basically self explanatory, it is a random variable that is continuos instead of discrete.

Covariance:
Covariance can be thought of as a measure of linear dependence between two random variables, although this is not the linear dependence studied in linear algebra.

Cumulative Distribution Function (CDF):
A Cumulative Distribution Function (cdf) completely describes the distribution of a real valued random variable.

Gaussian Distribution:
The Gaussian distribution, which is often known as the Normal distribution or even sometimes as the bell curve, is symmetric distribution around the mean with a bell like shape.

Indicator Function:
See definition in Chapter 2.

Joint CDF:
The joint cdf and pdf are just the generalizations of the single variable cdf and pdf into multiple dimensions.

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Mathematical Expectation:
Mathematical Expectation or expected value of a random variable is the sum of the probability of each of the possible outcomes of the experiment multiplied by its pay-out/value. In other words it represents the average amount one expects to win/acquire etc.

Mean:
The Mean is the the average as calculated using the mathematical Expectation.

Multivariate Normal Distribution:
The Multivariate Normal Distribution is the generalization of the Normal distribution to multidimensional space.

Normal Distribution:
See the Gaussian distribution.

Probability Density Function (PDF):
The Probability Density Function is the derivative of cdf, which allows one to represent the probability function in terms of integrals.

Random Variable:
A random variable can be thought of as the numeric result of operating a non-deterministic mechanism or performing a non-deterministic experiment to generate a random result. An important note to realize is that unlike the common practice with other mathematical variables, a random variable cannot be assigned a value. A random variable does not describe the actual outcomes of a particular experiment, but instead describes a yet to be determined outcome in terms of the real numbers.

Sample Space:
Suppose we have an experiment which terminates with an outcome, and this experiment is characterized by the fact that we cannot predict the outcome with certainty prior to the performance of the experiment. Suppose we have the aforementioned case, but now the experiment is of such a nature that the collection of every possible outcome can be described prior to its performance and if this kind of experiment can be repeated under the same conditions, it is called a random experiment, and the collection of every possible outcome is called the sample space.

Standard Normal Distribution:

The Standard Normal Distribution is a Gaussian Distribution with mean equal to and variance of one.

Variance:
The variance of a random variable is a measure of its statistical dispersion. The square root of the variance will give one the standard deviation.

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