Obstruction to Equivariant Ribbon Concordance from Equivariant Khovanov Homology

by

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DISSERTATION ABSTRACT

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In the past decade, there has been renewed interest in studying ribbon concordances, a particular class of smoothly embedded surfaces in the 4-ball with boundary a link in 3-sphere. One reason is that information on these surfaces, provides deeper understanding of topology and geometry of 4 dimensional manifolds. Khovanov homology, a combinatorial link invariant, has proved to be an effective tool to study embedded surfaces in the 4-ball.

The main goal of this thesis is to study such cobordisms which also carry a group action. In particular, we study the equivariant Khovanov homology of links in $\mathbb{R}^3 \backslash z$-axis that are invariant under the action of $\mathbb{Z}_p$ by rotation around the z-axis. We prove that the equivariant Khovanov homology is functorial up to a sign. We use that result to derive an obstruction to equivariant ribbon concordances between p-periodic knots.

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To Aiden

TABLE OF CONTENTS

Chapter Page

LIST OF FIGURES

Figure Page

CHAPTER 1

INTRODUCTION

For a prime integer p, we can consider the action of $\mathbb{Z}_p$ (the cyclic group of order p) on $\mathbb{R}^3$ by rotation by $\frac{2\pi}{p}$ around the z-axis. As this action is proper with fixed-point set the z-axis, it induces an action on $S^3 = \mathbb{R}^3 \cup \infty$ with the fixed points the unknot $U = z\text{-axis} \cup \infty$. Throughout this manuscript, we denote the z-axis in $\mathbb{R}^3$ by $\boldsymbol{z}$ and the origin in $\mathbb{R}^n$ by $\mathbf{0}$.

Definition 1.1. A link $L \subset \mathbb{R}^3 \backslash \boldsymbol{z} = S^3 \backslash U$ is called *p-periodic* if it is invariant under the action of $\mathbb{Z}_p$.

In this work, We view p as fixed. Hence, when it will not cause confusion we drop the p from the notation. Also, we denote this action by θ. The action of $\mathbb{Z}_p$ on S^3 can be extended to $S^3 \times [0, 1]$, if we let the action be trivial on $[0, 1]$. Let us denote the extended action by $\tilde{\theta}$. Then $\tilde{\theta}$ has fixed annulus $\tilde{U} = U \times [0, 1]$.

By a *cobordism* between links L_0 and L_1 we refer to a smoothly embedded surface Σ in $\mathbb{R}^3 \times [0, 1]$ with $\partial\Sigma \subset \mathbb{R}^3 \times \{0, 1\}$ and $\partial\Sigma \cap (\mathbb{R}^3 \times \{i\}) = L_i$ for $i = 0, 1$. Given a smooth, compact cobordism Σ in $S^3 \times [0, 1]$ from L to L', a *movie* of the link cobordism Σ is a finite collection of link diagrams $\{D_i \| i = 0, \ldots, k\}$ for a non-negative integer k, such that the link diagram D_{t_i} is related to $D_{t_{i+1}}$ by either a single birth, a single saddle, a single death, or a Reidemeister move, localized to a disk in $\mathbb{R}^2$.

By work of Carter and Saito [CS98], every smooth cobordism Σ as above has a movie. Additionally, they proved, if two smooth cobordisms $\Sigma, \Sigma' \subset \mathbb{R}^3 \times [0, 1]$ between links L_0 and L_1 are ambient isotopic relative to the boundary, then their movies differ by finitely many *movie moves* (see also [Bar05], and [Kho06]).

Definition 1.2. For periodic links L_0 and L_1 in $\mathbb{R}^3 \backslash \boldsymbol{z}$, a smooth cobordism $\Sigma \subset (\mathbb{R}^3 \backslash \boldsymbol{z}) \times [0, 1]$ is called an *equivariant cobordism* if it is invariant under the extended action $\tilde{\theta}$, and is disjoint from the annulus $\tilde{U}$.

In section 4.2, we show that equivariant cobordisms $\Sigma \subset (\mathbb{R}^3 \backslash \boldsymbol{z}) \times [0, 1]$ have *equivariant movie* presentations. Two equivariant cobordisms Σ and Σ' are called *equivariantly isotopic*, if there is an ambient isotopy of $(\mathbb{R}^3 \backslash \boldsymbol{z}) \times [0, 1]$ relative to boundary such that the isotopy commutes with the action $\tilde{\theta}$. For a smooth equivariant cobordism, we define,

Definition 1.3. An *equivariant movie move* means transforming the movie of a p-equivariant cobordism by Carter Saito movie moves localized to p disjoint disks in $\mathbb{R}^2 \backslash \mathbf{0}$ such that the disks and the movies correspond under the action of $\mathbb{Z}_p$ by rotation on $\mathbb{R}^2 \backslash \mathbf{0}$.

Analogous to the result above, we show

Theorem 1.4. *Fix two equivariantly isotopic, equivariant cobordisms Σ and Σ' from L_0 to L_1 represented by equivariant movies. Then the equivariant movie of Σ differs from the equivariant movie of Σ' by finitely many equivariant movie moves.*

Khovanov homology [Bar05; Kho00; Kho02] is a link invariant that assigns to the link diagram of L a bigraded chain complex of R-modules $C_{\mathrm{Kh}}(L; R)$, for any unital commutative ring R. The graded Euler characteristic of Khovanov homology is the Jones polynomial. Moreover, Khovanov homology assigns a chain map to any elementary string move (birth, saddle, death, or Reidemeister move).

Assume that Σ and Σ' are smooth, ambient isotopic cobordisms in $\mathbb{R}^3 \times [0, 1]$ with boundary $\partial\Sigma = \partial\Sigma' = L_0 \coprod L_1$. Jacobson [Jac04], Bar-Natan [Bar05], and Khovanov [Kho06] proved that the maps induced on Khovanov homology (denoted by $\mathrm{Kh}(L)$ for a link L) by Σ and Σ' are equal up to a sign i.e., $\mathrm{Kh}(\Sigma) = \pm\mathrm{Kh}(\Sigma')$. This property of Khovanov homology is called *functoriality*.

A smooth *concordance* between knots K_0 and $K_1 \subset S^3$ is a smooth embedded annulus $A : S^1 \times [0, 1] \hookrightarrow S^3 \times [0, 1]$ with $K_1 = A(S^1 \times \{1\}) \subset S^3 \times \{1\}$ and $K_0 = A(S^1 \times \{0\}) \subset S^3 \times \{0\}$. We also call the image of the annulus $A(S^1 \times [0, 1]) = F$ a concordance between $K_1 = A(S^1 \times \{1\})$ and $K_0 = A(S^1 \times \{0\})$. If the projection $S^3 \times [0, 1] \to [0, 1]$ is a Morse function when restricted to F with only critical points of index 0 and 1 (so no local maxima), then we say that F is a *ribbon concordance* between K_0 and K_0.

As a consequence of the functoriality of Khovanov homology, one can study Khovanov homology of a ribbon concordance. A framework for studying ribbon concordances is Bar-Natan's dotted cobordism treatment of Khovanov homology [Bar05]. Levine and Zemki [LZ19] showed that if F is a ribbon concordance from L_0 to L_1, then the induced map $\mathrm{Kh}(F) : \mathrm{Kh}(L_0) \to \mathrm{Kh}(L_1)$ is a bigraded split injection.

Given a finite group G and a G-space X, *equivariant (co)homology* (also known in the literature as the Borel equivariant homology or Cartans mixing construction) is effective for studying the (co)homological behavior of both the space and the G-action [Hsi75]. The equivariant (co)homology of X is given by the (co)homology $H_*^G(X) := H_*(E \times_G X)$ (respectively $H^*_G(X) := H^*(E \times_G X)$) where EG is the universal G-bundle.

In the case of having an R-chain complex (C, d) with a G-action, the equivariant (co)homology of C, computes the group (co)homology of an $R[G]$-module M, with coefficients in the $R[G]$-chain complex (C, d) [Bro82]. Hence, the *equivariant Khovanov homology* of a p-periodic link for a choice of unital ring R is defined as follows:

Definition 1.5. The *equivariant Khovanov homology* of the p-periodic link L is defined as

$$\mathrm{EKh}(L, M) = \mathrm{Ext}_{R[\mathbb{Z}_p]}(M, C_{\mathrm{Kh}}(L; R)) \tag{1.1}$$

where M is a $R[\mathbb{Z}_p]$-module, and $C_{\mathrm{Kh}}(L; R)$ is the Khovanov chain complex.

Equivariant Khovanov homology for p-periodic links was recently studied by Borodzik and Politarczyk [BP21]. They also studied equivariant Bar-Natan and Lee homologies for periodic knots. We restate the construction of equivariant Khovanov homology in chapter 3.

In chapter 4, we show that equivariant Khovanov homology is functorial up to a sign. More specifically, we prove that the chain maps induced on the equivariant Khovanov chain complex by equivariantly isotopic equivariant cobordisms are chain homotopic up to a factor of $(\pm 1)^p$.

Theorem 1.6. *Given two equivariantly isotopic equivariant link cobordisms Σ, Σ' from a p-periodic link L_0 to L_1, the induced morphism*

$$\mathrm{EKh}(\Sigma), \mathrm{EKh}(\Sigma') : \mathrm{EKh}(L_0; \mathbb{Z}) \to \mathrm{EKh}(L_1; \mathbb{Z}) \tag{1.2}$$

differ by a factor of $(\pm 1)^p$.

Lastly, in the equivariant case, we show that there is an equivariant analogue of the neck cutting relation that holds in equivariant Khovanov homology (see section 4.3.1). Therefore, we conclude that,

Theorem 1.7. *Fix a smooth equivariant ribbon concordance F between periodic knots K and K'. The map induced by F on equivariant Khovanov homology is a split injection.*

CHAPTER 2

BACKGROUND

In this chapter we will introduce some background and notation that we will use throughout the arguments in this thesis.

2.1 Tangles and diskular tangles

It is often useful to discuss only small "pieces" of a link while disregarding everything else. Tangles may be thought of as small pieces or local pictures of knots or links, and they provide a useful language to describe local manipulations. For example, the Reidemeister moves alter at most 3 crossings in a small surroundings. Our approach to the tangle decomposition of a link is the frameworks introduced in [Kho02; Bar05; LLS21]. We first introduce the tangles from [Kho02]. However, we mainly utilize the setup of *planar arc diagram* introduced in [Bar05, §5] which coincides with *diskular tangles* in [LLS21].

By a *link* in the 3-dimensional sphere we refer to a smooth embedding of finite collection of circles $L : \sqcup_n S^1 \to S^3$. If $n = 1$, it is called a *knot*. Here, $\sqcup$ denotes the disjoint union. By abuse of notation we refer to the image of embedding $L := L(\sqcup_n S^1) \subset S^3$ as the link L. A *link diagram* D is a projection of a link L onto the $\mathbb{R}^2$ (xy-plane) or S^2 such that the only intersections are transverse double points.

For a link L, the link diagram D inherits an orientation from L. When D has the induced orientation, it is referred to as the *oriented* link diagram. If we forget the induce orientation of D, then we call D an *unoriented* link diagram.

Definition 2.1. For positive even integers m, n, by an unoriented (m, n)-*tangle* T we mean proper and smooth embedding of $\frac{m+n}{2}$ arcs and finitely many circles in the cube $[0, 1]^3$ such that

- The boundary of the arcs are on the faces $[0,1]\times\{0,1\}\times[0,1]$ of the cube. More specifically, the boundary of the arcs are the points

$$\{(\frac{k}{n+1}, 0, \frac{1}{2})\}_{k=1}^{n} \bigcup \{(\frac{j}{m+1}, 1, \frac{1}{2})\}_{j=1}^{m}$$

- The arcs are perpendicular to the faces of the cube near their boundary.

An (n, n)-*tangle diagram* is a projection of an (n, n)-tangle to the unit square $[0, 1]^2$ where all singular points are transverse intersections (just as in link diagrams).

The above choices are imposed to make the process of gluing tangles easier. This notion of tangles is equivalent to the classical definition of tangles except our tangles are in a 3-dimensional cube rather than a 3-ball. Unoriented tangles in $[0, 1]^3$ form a category with objects the even numbers and morphisms the tangles (see [Kho02]), which here is denoted by $\mathcal{T}$.

Definition 2.2. For m, n positive even integers, an unoriented (m, n)-*flat tangle* is $\frac{n+m}{2}$ many arcs and finitely many circles smoothly embedded in $[0, 1]^2$ so that the boundaries of the arcs are the points

$$\{(\frac{k}{n+1}, 0)\}_{k=1}^{n} \bigcup \{(\frac{j}{m+1}, 1)\}_{j=1}^{m}$$

Also, the arcs are perpendicular to the top and bottom sides of the square..

Figure 2.1 (a) shows a $(4, 4)$-flat tangle. Following the notation from [Kho02], the set of (m, n)-flat tangles is dented by $\hat{B}^m_n$. For consistency with the literature, we also omit zero in subscript or superscript. That is, we denote $B^m = B^m_0$ and $B_n = B^0_n$. Let B^m_n denote the subset of $\hat{B}^m_n$ consisting of flat tangles with no closed components (figure 2.1 (a)). We can concatenate two flat tangles $a \in B^m_n$ and $b \in B^n_k$ to obtain $ab \in B^m_k$ as follows: We first shrink height of a and b to $[\frac{1}{2}, 1]$ and $[0, \frac{1}{2}]$ respectively. Then glue

a on top of the b. The result will be a flat (m, k)-tangle. Also, for a flat (m, n)-tangle $b \in B^m_n$ we can reflect it about the line $[0, 1] \times \{\frac{1}{2}\}$ and obtain $\hat{b} \in B^n_m$.

For $a_0, a_1 \in B^m_n$, by an *admissible cobordism* between a_0 and a_1 we mean a smooth properly embedded surface $\Sigma \subset [0, 1]^2 \times [0, 1]$ such that $\partial\Sigma \cap ([0, 1]^2 \times \{i\}) = a_i$ for $i = 0, 1$, and Σ is diffeomorphic to product cobordism near all of the faces of cube $[0, 1]^3$. Admissible surfaces have been treated thoroughly in [Kho02, §2.3]. This makes the collection of (m, n)-flat tangles into a 2-category $\mathbb{TL}$ with objects positive even integers, morphism $B^m_n = \mathrm{Hom}(n, m)$, and admissible cobordisms as 2-morphisms.

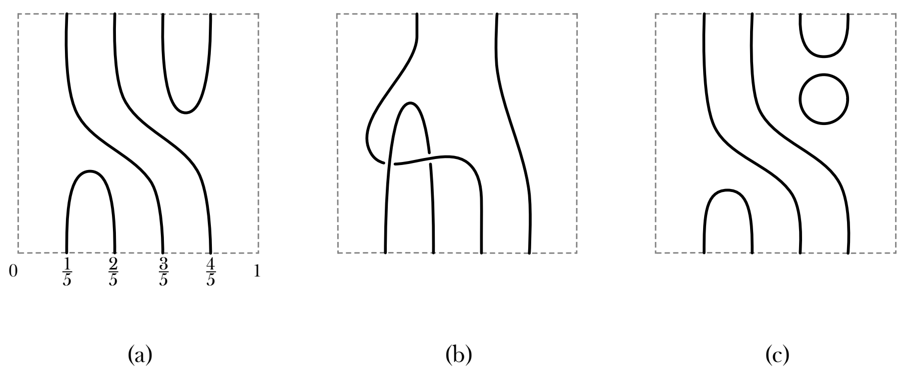


Figure 2.1: All of the figures are examples of a tangle diagrams in the square $[0, 1]^2$. (a) Shows a flat tangle with no closed components. (b) Shows a tangle diagram. (c) Shows a flat tangle with a closed components.

Let $\mathbb{D}$ be the standard closed disk centered at the origin in $\mathbb{C}$ and $\mathbb{D}_i \subseteq \mathbb{D}$ be disjoint open sub-disks for $i = 1, \ldots, k$ such that $\overline{\mathbb{D}}_i \subset \mathring{\mathbb{D}}$. A *$k$-punctured disk $\mathcal{D}$* is

$$\mathcal{D} = \mathbb{D} \backslash \big(\mathbb{D}_1 \cup \mathbb{D}_2 \cup \cdots \cup \mathbb{D}_k\big) \tag{2.1}$$

We can partition the boundary of $\mathcal{D}$ in to two sets: The outer boundary $\partial\mathbb{D} = S^1$ and k inner boundaries $\partial_i\mathcal{D} = \partial\mathbb{D}_i$ for $i = 1, \ldots, k$. Up to scaling and translating, any one of those inner boundaries is the unit circle $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$. Hence, we treat all boundary components as the unit circle. For an arbitrary non-negative integer m, a set of *m-marked points* on a unit circle consists of m points $\{p_j \mid p_j \neq 1, j = 1, \ldots, m\} \subset \partial\mathbb{D}$. For example, we can choose $\{p_1 = e^{\frac{2\pi i}{m+1}}, \ldots, \ p_m = e^{\frac{2\pi i m}{m+1}}\}$ i.e., non-trivial roots of unity as our m-marked points. We use $1 \in S^1 \subset \mathbb{D}$ as the base point. For a choice of boundary components of $\mathcal{D}$ and any positive integer m, we can set m-marked points on that boundary component. The importance of such choices will be more clear in the gluing process.

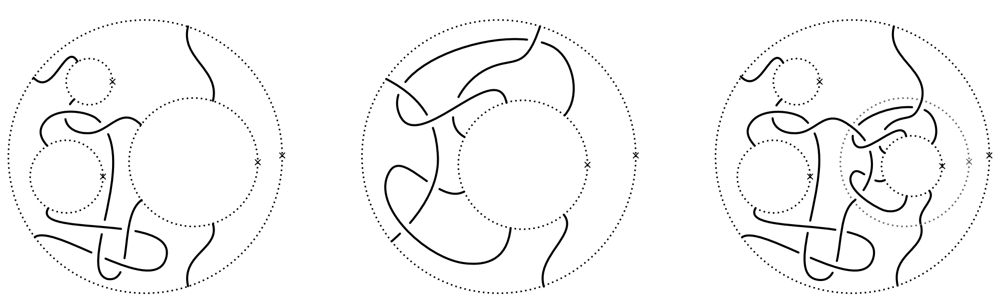
Definition 2.3. An $(m_1, \ldots, m_k; n)$-*diskular tangle* $T = T^{(m_1,\ldots,m_k;n)}$ is a diagram of a tangle in a thickened k-punctured disk $\mathcal{D}$ with the following information:

1. The $n, m_1, \ldots, m_k$ are fixed, positive, even integers.
2. There are m_i marked points on $\partial_i\mathcal{D}$ (ith inner boundary of $\mathcal{D}$) and n marked points on the outer boundary of $\mathcal{D}$.
3. Each boundary component has $1 \in \partial_i\mathcal{D}$ as the base point.
4. The tangle diagram T consist of finitely many immersed circles and $\frac{1}{2}(n + m_1 + \ldots + m_k)$ arcs in $\mathcal{D}$ with the boundary of the arcs on the marked points.
5. The arcs are perpendicular (making the right angle with the tangent vectors to $\partial\mathcal{D}$ at marked points) near the boundary of $\mathcal{D}$.

Figure 2.2 (a) and (b) depicts two diskular tangles. If there are no inner boundaries, and n-marked points on the outer boundary, we denote the diskular tangle by $T^{(;n)}$. By contrast, $T^{(0,\ldots,0;m)}$ would imply that there are inner boundaries with no marked points, and one outer boundary with m-marked points.

Remark 2.4. The choice of $1 \in S^1$ as the base point on the boundaries of $\mathcal{D}$ might appear excessive to the reader as we only allow tangle diagrams to be scaled or translated. This choice of base point is indeed extra here. However, it will be used in section 2.4.2.

Two diskular tangles $S = S^{(l_1,\ldots,l_j;m_i)}$ and $R = R^{(m_1,\ldots,m_i,\ldots,m_k;n)}$ can be composed by gluing S from its outer boundary to the ith inner boundary of R to form a new $(m_1, \ldots, m_{i-1}, l_1, \ldots, l_j, m_{i+1}, \ldots, m_k; n)$-diskular tangle denoted by $R \circ_i S$. The reader should note that here the subscript $\circ_i$ specifies that we glue S into R's ith boundary. In the case of unoriented tangles, to be able to get a diskular tangle from $R \circ_i S$ we need the number of marked points on the ith inner boundary of R, m_i, to be equal to the number of marked points on the outer boundary of S. However, if one works with oriented tangles, then the orientation of arcs in R that meet the $\partial_i R$ must match with the orientation of arcs that start or terminate on the outer boundary of S. Gluing can be made canonical if we identify $\partial_i R$ with $\partial_{\text{out}} S$ by scaling and translating of S and R (Figure 2.2 (c)).



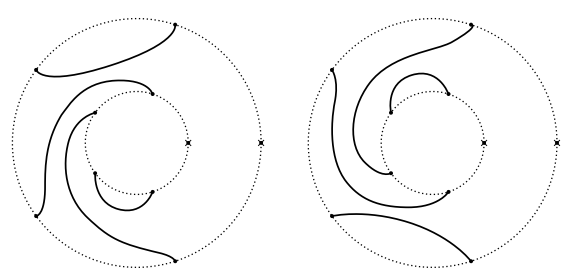
(a) (b) (c)

Figure 2.2: (a) Shows a diskular tangle $T^{(2,2,4;4)}$ and (b) shows a diskular tangle $T^{(6;4)}$. (c) depicts the composition $T^{(2,2,4;4)} \circ_3 T^{(6;4)}$ which is a $(2, 2, 6; 4)$-diskular tangle.

Similarly, the $(m_1, \cdots, m_k; n)$-*flat* diskular tangles are defined similarly to definition 2.3 but we require a smooth and proper embedding of $\frac{1}{2}(n + m_1 + \cdots + m_k)$ arcs and finitely many circles in a k-punctured disk $\mathcal{D}$. We denote by $B^{(m_1,\cdots,m_k;n)}$ the collection of $(m_1, \ldots, m_k; n)$-flat diskular tangles.

For any square (m, n)-tangle diagram, identifying the right and left side of the square $[0, 1]^2$ will transform the square into an annulus. This also provides a one-to-one correspondence between tangle diagrams in $[0, 1]^2$ and diskular tangles $T^{(m;n)}$ in the annulus $A = S^1 \times [0, 1]$ that do not intersect the ray $\{1\} \times [0, 1]$. Given an $(m; n)$-flat diskular tangle $R \in B^{(m;n)}$ we can reflect the arcs radially around the middle circle in the annulus and denote the resulting flat diskular tangle $\widehat{R} \in B^{(m;n)}$ (figure 2.3).

The collection of $(m; n)$-flat diskular tangles in the annulus $A \subset \mathbb{C}$ with no closed components and arcs that avoid the ray $r_1 = \{(0, t) \mid t \in [0, +\infty)\} \cap A$ up to ambient isotopy of $\mathcal{D}$ relative to the boundary is denoted by $\boldsymbol{B}^{(m;n)}$. The base point chosen in definition 2.3 helps with making the identification of flat diskular tangles in $\boldsymbol{B}^{(m;n)}$ with flat tangles in $[0, 1]^2$ more concrete.



(a) (b)

Figure 2.3: (a) Shows a flat (4; 4)-diskular tangle and (b) shows the reflection of same flat diskular tangle $\widehat{T}^{(4;4)}$.

Definition 2.5. Given two $(m_1, \ldots, m_k; n)$-diskular tangle S, T, by an *elementary cobordism* $\mathfrak{s} : S \to T$, we mean any of following:

- A planar isotopy $f_t : \mathcal{D} \to \mathcal{D}$ of k-punctured disks such that f_t restricted to the boundary is the identity for all t.
- Any of the Reidemeister moves away from the boundary.
- Any of the Morse moves (birth, death, and adding a saddle).

Figure 2.4 shows a composition of two saddle addition elementary cobordisms.

In [LLS21, Lemma 4.7], the authors showed that collection of diskular tangles forms a multicategory $\mathbb{T}$ enriched in categories. In this multicategory, an unoriented $(m_1, \ldots, m_k; n)$-diskular tangle $S = S^{(m_1, \cdots, m_k; n)}$ is a multi-morphism between objects $m_1, m_2, \cdots, m_k$ and n i.e., $S \in \mathrm{Hom}_{\mathbb{T}}(m_1, \cdots, m_k; n)$. The identity morphism is the tangle $\mathrm{Rad}_n \in \mathrm{Hom}_{\mathbb{T}}(n; n)$ which is the radial crossingless matching of n-marked points on the outer boundary and the n-marked points on the inner boundary. In this setting, any elementary cobordism (definition 2.5) between $(m_1, \ldots, m_k; n)$-diskular tangles $S, T \in \mathrm{Hom}_{\mathbb{T}}(m_1, \ldots, m_k; n)$, is a 2-morphisms in $\mathbb{T}$.

In light of the categorical refinement of diskular tangles in [LLS21, Section 4], we denote the collection of diskular tangles for fixed integers $m_1, \ldots, m_k, n$ by $\mathbb{T}^{(m_1, \cdots, m_k; n)}$. For instance, instead of writing the $S = S^{(m_1, \cdots, m_k; n)}$ we will write $S \in \mathbb{T}^{(m_1, \cdots, m_k; n)}$ to denote an $(m_1, \cdots, m_k; n)$-diskular tangle.

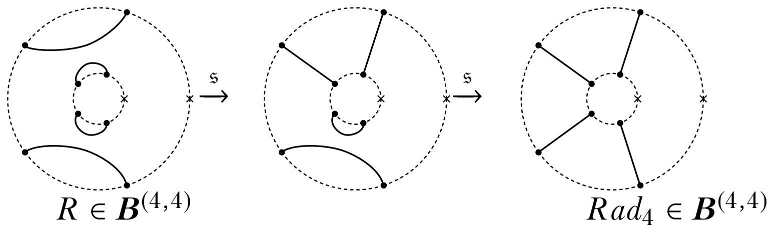


Figure 2.4: Composition of two Morse saddle elementary cobordism between the (4; 4)-flat diskular tangle R on the left, and (4; 4)-diskular tangle Rad_4 on the right.

2.2 Periodic links and tangles

2.2.1 Periodic links

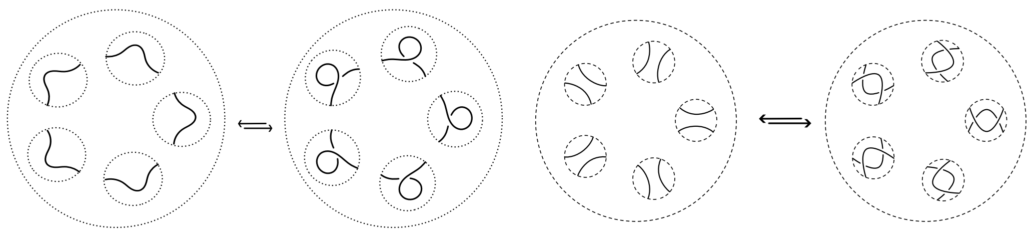
In this section we study the p-periodic links (definition 1.1) in more depth and introduce periodic tangle decompositions for p-periodic links. Recall from chapter 1, that a link $L \subset \mathbb{R}^3 \backslash \boldsymbol{z}$ is called p-periodic if it is invariant under the action of $\mathbb{Z}_p$ by rotation for some positive integer p. For simplicity, we use the term periodic links instead of p-periodic link when it will not cause confusion.

Periodic links have periodic diagrams. That is to say, there is a generic position of the periodic link L such that the image of L by the projection $\mathbb{R}^3 \backslash \boldsymbol{z} \to \mathbb{R}^2 \backslash \mathbf{0}$ denoted by $D \subset \mathbb{R}^2$, is a link diagram that misses the origin, and is taken to itself by $\frac{2\pi}{p}$ rotation around the origin.

Analogous to ordinary links, one can define the equivariant Reidemeister moves as transformations on a diagram of a periodic link.

Definition 2.6. A *p-equivariant Reidemeister move* means applying a Reidemeister move to a periodic link diagram of a periodic link, localized on p disjoint, closed disks in $\mathbb{R}^2 \backslash \mathbf{0}$, so that disks and Reidemeister moves correspond under the action $\mathbb{Z}_p$.

Figure 2.5 depicts an example of a 5-equivariant Reidemeister move. Given a p-periodic link L in $\mathbb{R}^3\backslash\boldsymbol{z}$ (similarly in $S^3\backslash U$, where U is the fixed unknot), the quotient link $\overline{L} = L/\mathbb{Z}_p$ is a link in $\mathbb{R}^3\backslash\boldsymbol{z}$ (respectively in $S^3\backslash U$). Throughout this paper, an overline indicates the quotient of a periodic link by $\mathbb{Z}_p$. Both L and $\overline{L}$ can be considered as annular links by considering them as links in $\mathbb{R}^3\backslash\boldsymbol{z}$. The projection to the xy-plane will provide an annular link diagram of L and $\overline{L}$ in an annulus centered at origin of $\mathbb{R}^2$.



(a) (b)

Figure 2.5: (a) Equivariant type I Reidmeister move and (b) Equivariant type II Reidmeister moves for a 5-periodic link.

Theorem 2.7. *Two periodic links L_0 and L_1 represented by periodic link diagrams D_0 and D_1 are equivariantly isotopic if and only if D_0 is obtained from D_1 by a finite sequence of equivariant Reidemeister moves.*

Proof. Consider the quotient link diagram $\overline{D}_0$. Any annular Reidemeister move on $\overline{D_0}$ can be pulled back to an equivariant Reidemeister move. Therefore, the theorem follows from the similar result in annular links topology [HP89, Theorem 1]. □

The action of $\mathbb{Z}_p$ on $\mathbb{R}^3$ (respectively S^3) can be extended to an action on $\mathbb{R}^3 \times [0, 1]$ (respectively $S^3 \times [0, 1]$) if we let the action be trivial on $[0, 1]$. This extended action has fixed annulus $U \times [0, 1]$. We will use this extended action in section 4.2.

2.2.2 *Periodic Tangles*

We start by introducing a notion of admissible periodic diskular tangles. Recall, in the composition of diskular tangles (definition 2.3) we required gluing to identify the marked points on each boundary component in a canonical way.

Definition 2.8. An $(m_1, \ldots, m_k; n)$-diskular tangle E is an *admissible p-periodic diskular tangle*, if the followings hold.

(a) We have $n = 0$, i.e., there are no arcs in the diagram of E with boundary on $\partial_{out}\mathcal{D}$, the outer boundary of $\mathcal{D}$ for a k punctured disk $\mathcal{D}$ centered at origin in $\mathbb{R}^2$.

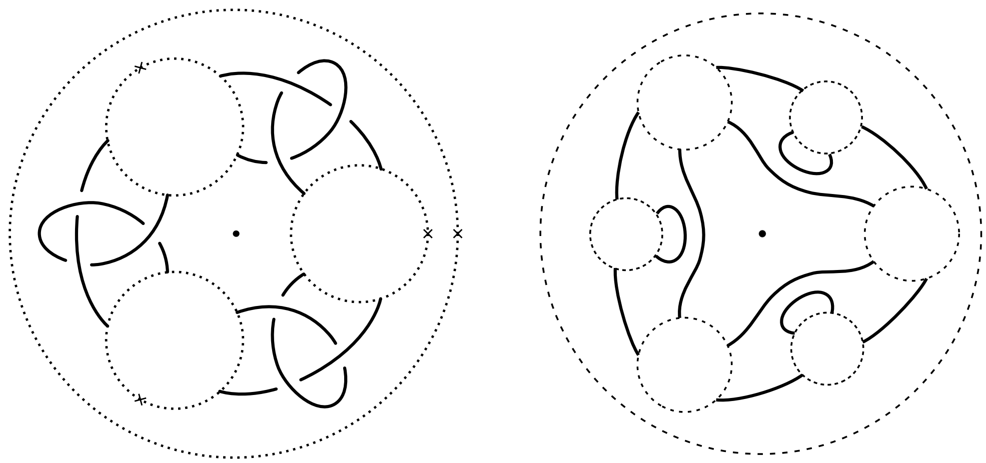
(b) We have $k = p$ and the $\mathbb{Z}_p$ acts on the set of inner boundaries $\partial_{in}\mathcal{D} = \{\partial_i\mathcal{D}\}_{i\in\mathbb{Z}_p}$ by cyclic permutation, i.e., $\theta(\partial_i\mathcal{D}) = \partial_{i+1}\mathcal{D}$ for $i \in \mathbb{Z}_p$.

(c) We relaxed the criteria that $1 \in \partial_i\mathcal{D}$ be the base point in definition 2.3 and for each $i = 1, \ldots, k$ there are $m_i + 1$ points on the inner boundary $\partial_i\mathcal{D}$. One of the $m_i + 1$ points is set to be the base point. Also, the base points on inner boundaries correspond under the action of $\mathbb{Z}_p$.

(d) The arcs and circles in E correspond under the rotation by $\mathbb{Z}_p$.

Remark 2.9. A few notes about the definition 2.8:

1. In (c), the requirement $k = p$ could be generalized to k being divisible by p. Then the inner boundaries $\partial_{in}\mathcal{D} = \{\partial_i\mathcal{D}\}_{i=1}^{k}$ can be partitioned into subsets $\sqcup_{l=1}^{p}\mathcal{I}_l$ that correspond by cyclic permutation by action of $\mathbb{Z}_p$. However, this generalization is not necessary for the proof of the main results in this paper.



(a) (b)

Figure 2.6: (a) Shows an admissible 3-periodic diskular tangle $T^{(4,4,4;)}$. (b) Shows a non admissible flat 3-periodic diskular tangle $T^{(4,4,4,4,4,4;)}$.

2. Condition (c) implies that the number of marked points on inner boundaries are equal, i.e., $m_1 = \ldots = m_k$. Moreover, the action of $\mathbb{Z}_p$ maps the marked points on $\partial_i \mathcal{D}$ bijectively to marked points on $\partial_{i+1}\mathcal{D}$.

3. Marked points on the inner boundaries corresponding under the rotation, would also rotate the base point. To make sure gluing process is canonical in periodic diskular tangles, we relaxed the criteria that $1 \in \partial_i \mathcal{D}$ be the base point. Instead we just required that there be $m_i + 1$ point on the ith inner boundary of equivariant diskular tangle E.

Definition 2.10. By a *p-equivariant diskular tangle*, we refer to p disjoint copies of a diskular tangles $T_0, \ldots, T_{p-1} \in \mathbb{T}^{(m_1,\ldots,m_k;n)}$ in $\mathbb{R}^2 \backslash \mathbf{0}$ such that

1. We have the action $\mathbb{Z}_p$ on $\{T_j\}_{j\in\mathbb{Z}_p}$ with $\theta(T_j) = T_{j+1}$.

We can extended the notion of elementary cobordims (definition 2.5) to periodic diskular tangles.

Definition 2.11. A *p-equivariant elementary cobordisms* $\tilde{\mathfrak{s}}$ between p-periodic diskular tangles E and E', also denoted by $\tilde{s} : E \to E'$, is applying an elementary cobordism to E localized to p disjoint disks away from the center of E such that the elementary cobordism correspond under the under the rotation by $\mathbb{Z}_p$.

2.3 Link cobordisms

In this section, we begin with definitions for embeddings and isotopy of manifolds. Let N be an n-dimensional smooth manifold and $\Sigma \hookrightarrow N$ be a smooth submanifold of N. A pair of embeddings $i_0, i_1 : \Sigma \to N$ are *isotopic* if they are related by a smooth map $f : \Sigma \times [0, 1] \to N$ such that each $f_t = f|_{\Sigma\times\{t\}}$ is a smooth embedding, and $f_0 = i_0$ and $f_1 = i_1$.

Assume there exists a smooth map $f : N \times [0, 1] \to N$ such that $f_t = f|_{N\times\{t\}}$ is a diffeomorphism for all $t \in [0, 1]$, $f_0 = id_N$ and $f_1 \circ i_0 = i_1$. Then, we call the embeddings i_0 and i_1 *ambiently isotopic*. If the isotopy f between i_0 and i_1 fixes $\partial\Sigma \times [0, 1]$ set-wise, we call f a *boundary-preserving isotopy*.

In this manuscript, the ambient space is $N = \mathbb{R}^3 \times [0, 1]$, and Σ is an orientable surface with boundary links in $\mathbb{R}^3 \times \{0, 1\}$. Moreover, by isotopy we mean ambient isotopy.

Given a smooth, compact cobordism Σ in $S^3 \times [0, 1]$ from L to L', after possibly a small perturbation, the projection $\pi : \mathbb{R}^3 \times [0, 1] \to [0, 1]$ restricted to Σ − which by abuse of notation we also denote it by π − is a Morse function with distinct critical values. Let ρ denotes the projection $\mathbb{R}^3 \times [0, 1] \to \mathbb{R}^2$ to the xy-plane restricted to Σ. Then $D_t = \rho \circ \pi^{-1}(t)$ for $t \in [0, 1]$ is an one-parameter family of link diagrams except at finitely many singular points $t_1, \ldots, t_k \in [0, 1]$. At those singular levels, the link diagram D_t undergoes a single elementary cobordism, i.e., a planar isotopy, birth, saddle, death, or Reidemeister move (see [CS98]).

Definition 2.12. A *movie* $\boldsymbol{M}_\Sigma = \{D_i\}_{i=0}^k$ of a link cobordism $\Sigma : L_0 \to L_1$ is a finite sequence of link diagrams D_i for $i = 0, \ldots, k$, with successive pairs of diagrams related by an elementary cobordism (planar isotopy, Morse move, or Reidemeister move) localized to a disk in $\mathbb{R}^2$. Individual diagrams in the sequence are often called *frames*.

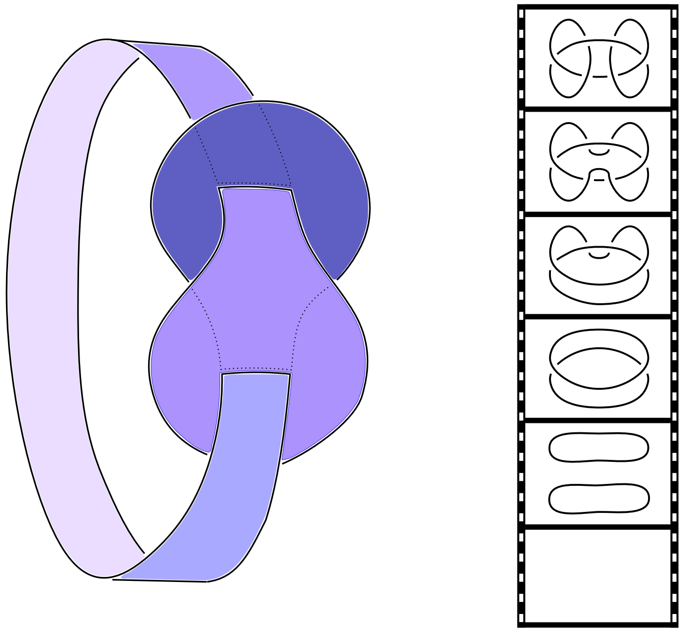
To obtain the frames of a movie, we use the following process. Let $t_1, \ldots, t_k$ be the singular points of Σ, as described above, ordered with respect to the interval factor of $\mathbb{R}^3 \times [0, 1]$. Then for each $i \in \{1, \ldots, k\}$, the point t_i has a sufficiently small neighborhood $[t_i - \varepsilon_i, t_i + \varepsilon_i]$ in which the diagrams $D_{t_i - \varepsilon_i}$ and $D_{t_i + \varepsilon_i}$ are related by a Morse move or Reidemeister move. Additionally, each interval $[t_i + \varepsilon_i, t_{i+1} - \varepsilon_{i+1}]$ describes an isotopy between $D_{t_i+\varepsilon_i}$ and $D_{t_{i+1}-\varepsilon_{i+1}}$. Let $t_0 = 0$ and $t_{k+1} = 1$. Then the desired movie is the sequence of diagrams corresponding to the points $\{t_0, t_1 - \varepsilon_1, t_1 + \varepsilon_1, \ldots, t_{k+1}\}$. Figure 2.7 shows a movie of a surface that bounds a knot, from [Sun22].

Movies associated to isotopic link cobordisms are related by a sequence of *movie moves*, which locally adjust the frames of a movie [CS98]. For a list of movie moves, see figures 5-9 in [Kho06]. The movie moves can be divide in two types:

1. Movies that compose planer isotopy, or commuting isotopy with Morse moves or Reidemeister movies. In [Kho06] these are moves 8-22, 24 and 31.

2. Remaining ones which are a sequence of Morse moves and Reidemeister moves and no isotopy.

For a more detailed treatment of the movie moves, we refer the reader to [Bar05; Kho06; LLS21]. We will revisit the movie moves in section 2.4.3 and 4.2.



(a) (b)

Figure 2.7: (a) Broken surface diagram of ribbon concordance that bounds the Trefoil connected sum to its mirror. (b) A movie of the cobordism in (a).

2.4 Khovanov Invariants

This section explores the necessary background for the computing Khovanov invariant of tangles and the maps on Khovanov homology induced by link cobordisms [Kho06]. We begin by reviewing the construction of Khovanov homology for diskular tangles in section 2.4.2. We then briefly discuss the maps on Khovanov homology induced by link cobordisms. Finally, in section 2.4.4, we study the maps induced by ribbon concordances and their effect on Khovanov homology, following the work of Adam S. Levine and Ian Zemki [LZ19].

2.4.1 Khovanov invariant for links

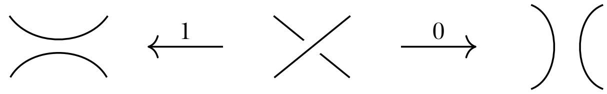
Let us fix $\mathbb{F}$ to be a commutative unital ring.

Definition 2.13. An $(n+1)$-dimensional *TQFT* is a functor $\mathcal{F}$ from the category Cob^{n+1} consisting of closed n-dimensional manifolds as objects and $(n+1)$-dimensional cobordisms between n-manifold as morphisms to the category of finitely generated R-modules, such that

- $\mathcal{F}$ is multiplicative: $\mathcal{F}(Y \coprod Y') = \mathcal{F}(Y) \otimes \mathcal{F}(Y')$.
- $\mathcal{F}$ is involutory: If $\overline{Y}$ is Y with the opposite orientation, then $\mathcal{F}(\overline{Y}) = \mathcal{F}(Y)^*$, where $M^* = \mathrm{Hom}_{\mathbb{F}}(M, \mathbb{F})$ is the dual module.

Being a functor means that, if $C_{01} : Y_0 \to Y_1$ and $C_{12} : Y_1 \to Y_2$ are cobordisms, then $\mathcal{F}(C_{12} \circ C_{01}) = \mathcal{F}(C_{12}) \circ \mathcal{F}(C_{01}) : \mathcal{F}(Y_0) \to \mathcal{F}(Y_2)$. Also, $\mathcal{F}$ sends the identity cobordism $Y \times [0, 1] : Y \to Y$ to the identity map.

We denote by $\underline{2}^n = \{0, 1\}^n$, the n-dimensional cube. For $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_n) \in \underline{2}^n$, if α and β differ at the i^{th} bit with $\alpha_i = 0$ and $\beta_i = 1$, then we define a directed edge $e_{\alpha \to \beta}$ from α to β. To construct the Khovanov invariant for a link L, we need the *cube of resolutions* defined as follows. Let D denote the unoriented link diagram of L. For $\alpha \in \{0, 1\}^n$, the vertex D_α is obtained by the complete resolution of the diagram D where the crossing c_i is smoothed according to the bit α_i according to following picture.



(2.2)

If there is an edge $\alpha \to \beta$, then D_α and D_β differ only in the neighborhood of one crossing. The resolution D_β is obtained from D_α by either

- merging two circles of D_α into one circle in D_β or
- splitting one circle of D_α into two circles in D_β.

So we have the edge $D_\alpha \to D_\beta$ and this edge corresponds to either a merge or split.

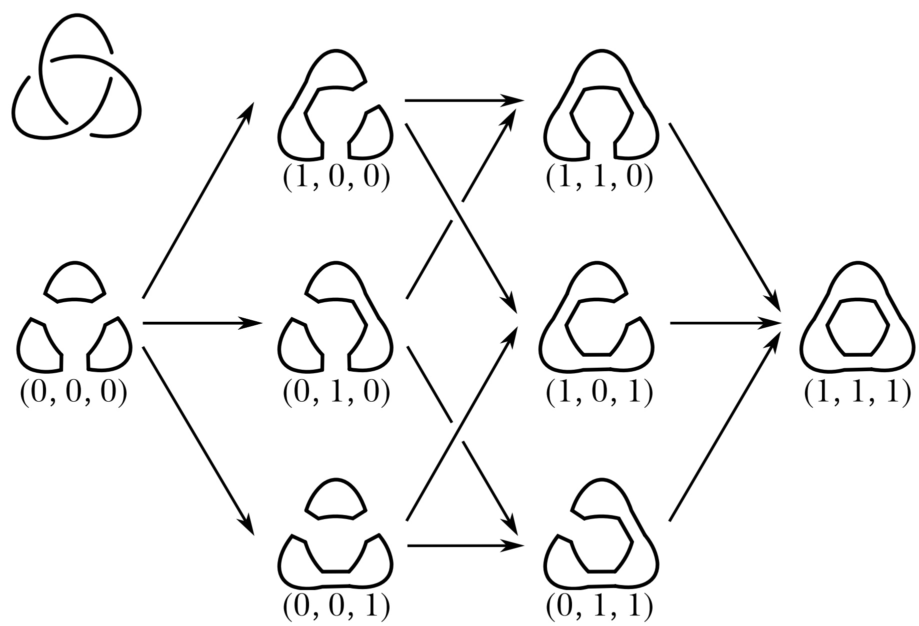


Figure 2.8: Cube of resolutions for the trefoil.

We use a convention similar to [Bar05] for grading shift operations. For a bigraded $\mathbb{F}$-module $A = \bigoplus_{(i,j)\in\mathbb{Z}\oplus\mathbb{Z}} A^{i,j}$, we denote by $[n]$ and $\{m\}$ the following shifts in the bigrading.

$$(A\{m\}[n])^{i,j} = A^{i-n,j-m}. \tag{2.3}$$

Let us denote $\mathcal{A} = \mathbb{F}[X]/(X^2)$, which is a Frobenius algebra if we define a multiplication, comultiplication, unit and counit as follows.

- The multiplication:

$$m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \tag{2.4}$$
$$\mathbf{1} \otimes \mathbf{1} \mapsto \mathbf{1}$$
$$\mathbf{1} \otimes \boldsymbol{X}, \boldsymbol{X} \otimes \mathbf{1} \mapsto \boldsymbol{X}$$
$$\boldsymbol{X} \otimes \boldsymbol{X} \mapsto 0$$

- The co-multiplication:

$$\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \tag{2.5}$$
$$\mathbf{1} \mapsto \mathbf{1} \otimes \boldsymbol{X} + \boldsymbol{X} \otimes \mathbf{1}$$
$$\boldsymbol{X} \mapsto \boldsymbol{X} \otimes \boldsymbol{X}$$

- The unit:

$$\iota : \mathbb{F} \to \mathcal{A} \tag{2.6}$$
$$1 \mapsto \mathbf{1}$$

- The co-unit:

$$\varepsilon : \mathcal{A} \to \mathbb{F} \tag{2.7}$$
$$\mathbf{1} \mapsto 0 \in \mathbb{F}$$
$$\boldsymbol{X} \mapsto 1 \in \mathbb{F}$$

Also, $\mathcal{A}$ can be considered as a graded $\mathbb{F}$-module by choosing $\mathrm{gr}_q(\mathbf{1}) = -1$ and $\mathrm{gr}_q(\boldsymbol{X}) = 1$. In the literature, gr_q is referred to as the *quantum grading* [Kho00; Bar05]. (see [Bar05; Kho00; Kho02]).

Remark 2.14. there are different convention for the choice $\mathrm{gr}_q(\mathbf{1})$ and $\mathrm{gr}_q(\boldsymbol{X})$ of quantum grading. For instance in [Bar05], $\mathrm{gr}_q(\mathbf{1}) = 1$ and $\mathrm{gr}_q(\boldsymbol{X}) = -1$. Additionally, the choice $\mathrm{gr}_q(\mathbf{1}) = 0$ and $\mathrm{gr}_q(\boldsymbol{X}) = 2$ makes $\mathcal{A}$ a graded $\mathbb{F}$-algebra.

To construct the Khovanov chain complex we use the *Khovanov TQFT* denoted by $\mathcal{F} : \mathcal{C}ob^{1+1} \to \mathbb{F} - \text{Mod}$ (see [Kho00]) induced by $\mathcal{A}$. Here, $\mathcal{C}ob^{1+1}$ is the category consisting of closed 1-dimensional manifolds as objects, and surfaces with boundary as morphisms.

Let $\pi_0(D_\alpha)$ denote the set of connected components (circles) in the resolution D_α. The TQFT $\mathcal{F}$ assigns to the vertex D_α of the cube of the resolutions of D the $\mathbb{F}$-module

$$V_\alpha = \mathcal{F}(D_\alpha) = \bigotimes_{\pi_0(D_\alpha)} \mathcal{A} \cong \mathcal{A}^{\otimes|\pi_0(D_\alpha)|} \tag{2.8}$$

where $|\pi_0(D_\alpha)|$ denotes the number of elements in $\pi_0(D_\alpha)$. Hence, as an $\mathbb{F}$-module, V_α is generated by pure tensors $\{\mathbf{1} \otimes \cdots \otimes \mathbf{1}, \mathbf{1} \otimes \cdots \otimes \boldsymbol{X}, \ldots, \boldsymbol{X} \otimes \cdots \otimes \boldsymbol{X}\}$.

Given an edge $e_{\alpha\to\beta}$, we assign the map $\mathcal{F}(e_{\alpha\to\beta}) : V_\alpha \to V_\beta$ as follows.

- If the D_β differs from D_α by a merge, then $\mathcal{F}(D_\alpha \to D_\beta)$ is the multiplication map m (equation 2.4) tensored with the identity on the other components.
- If the D_β differs from D_α by a split, then $\mathcal{F}(D_\alpha \to D_\beta)$ is the comultiplication map Δ (equation 2.5) tensored with the identity on the other components.

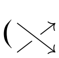
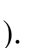
We also need a choice of sign to build a chain complex. For the edge $\alpha \to \beta$, define the sign by

$$\epsilon_{\alpha\to\beta} = (-1)^{\sum_{j=1}^{i-1} \alpha_j} = (-1)^{\sum_{j=1}^{i-1} \beta_j} \tag{2.9}$$

where α and β differ at the i^{th} bit with $\alpha_i = 0$ and $\beta_i = 1$.

Now we can obtain a chain complex as follows. Define (C^i, d^i) by

$$C^i = \bigoplus_{|\alpha|=i} V_\alpha\{|\alpha| - N_+ + 2N_-\}$$

where N_+ and N_- respectively denote the number of positive crossings () and negative crossings (). The differentials $d^i : C^i \to C^{i+1}$ are defined as $\sum_{|\alpha|=i} \epsilon_{\alpha\to\beta}\mathcal{F}(e_{\alpha\to\beta})$.

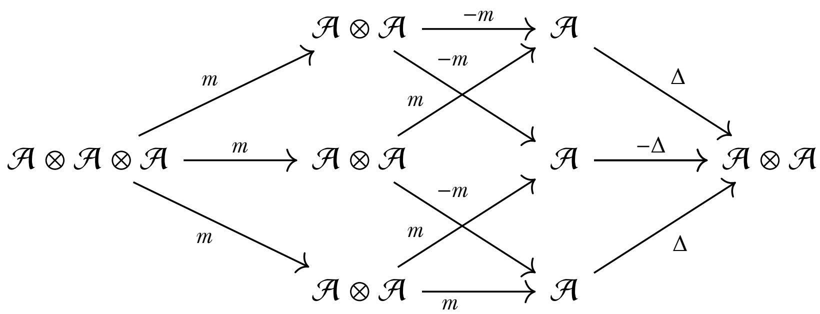


Figure 2.9: Khovanov chain complex of trefoil derived from the cube of resolution in figure 2.8.

To have the classical construction of Khovanov tangle invariant, we need to shift the homological grading by N_-.

Definition 2.15. The *Khovanov chain complex* of a link diagram D is denoted by $(C^i_{\mathrm{Kh}}(D;\mathbb{F}), d^i_{\mathrm{Kh}})$ and defined by

$$C_{\mathrm{Kh}}(D;\mathbb{F}) = C[-N_-].$$

Example 2.16. The chain complex (C, d) for the trefoil in figure 2.8, is given by

$$C^0 = \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \xrightarrow{d_{\mathrm{Kh}}} C^1 = \bigoplus_3 \mathcal{A}\otimes\mathcal{A} \xrightarrow{d_{\mathrm{Kh}}} C^2 = \bigoplus_3 \mathcal{A} \xrightarrow{d_{\mathrm{Kh}}} C^3 = \mathcal{A}\otimes\mathcal{A} \tag{2.10}$$

where the chin complex (C, d) is build by taking the total complex of figure 2.9. The Khovanov chian complex $(C_{\mathrm{Kh}}, d_{\mathrm{Kh}})$ is

$$C^{-3}_{\mathrm{Kh}} = \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \xrightarrow{d_{\mathrm{Kh}}} C^{-2}_{\mathrm{Kh}} = \bigoplus_3 \mathcal{A}\otimes\mathcal{A} \xrightarrow{d_{\mathrm{Kh}}} C^{-1}_{\mathrm{Kh}} = \bigoplus_3 \mathcal{A} \xrightarrow{d_{\mathrm{Kh}}} C^{-3}_{\mathrm{Kh}} = \mathcal{A}\otimes\mathcal{A} \tag{2.11}$$

For the majority of this work, we can choose $\mathbb{F}$ to be $\mathbb{Z}$. Also, we drop the coefficient ring $\mathbb{F}$ from the notation if it is $\mathbb{Z}$.

2.4.2 Khovanov invariant for tangles

In this section, we review the construction of the Khovanov invariant for tangles (section 2.1).

Definition 2.17. Let $\mathbb{F}$ be as above. For an even positive integer n, we define the *arc algebra* $\mathcal{H}^n$ as follows.

$$\mathcal{H}^n = \bigoplus_{a,b \in \boldsymbol{B}^{(0;n)}} \mathcal{F}(\hat{a} \circ b)\{n/2\} \tag{2.12}$$

where $\boldsymbol{B}^{(0;n)}$ denotes the collection of isotopy classes of flat $(0;n)$-diskular tangles with no closed components (section 2.1), and $\mathcal{F}$ is the Khovanov TQFT functor (section 2.4.1).

Remark 2.18. The condition that arcs in $\boldsymbol{B}^{(0;n)}$ do not intersect the ray $r_1 = \{1\} \times [0, 1]$, is used to conclude the arc algebra $\mathcal{H}^n$ defined above is identical to the ring H^n defined in [Kho02, section 2.4].

The $\mathbb{F}$-module $\mathcal{H}^n$ is an associative ring. Fix elements $x, y \in \mathcal{H}^n$. Then there are elements $a_x, a_y, b_x, b_y \in \boldsymbol{B}^{(0;n)}$ such that $x \in \mathcal{F}(\widehat{a_x} \circ b_x)$ and $y \in \mathcal{F}(\widehat{a_y} \circ b_y)$. To define the product of x and y we consider two cases:

1. If $b_x \neq a_y$, then we define the product $x \cdot y = 0 \in \mathcal{H}^n$.
2. If $c = b_x = a_y$, then we can define the saddle cobordism $\widehat{c} \circ c \xrightarrow{Sad} \mathrm{Rad}_n$ by composition of elementary saddles from an arc in b_x to the reflected image of that arc in $\widehat{b_x}$. See figure 2.4 for a movie of the Sad_4. By gluing the $(n;0)$-diskular tangle $\widehat{a_x}$ to the outer boundary of $\widehat{c} \circ c$, and gluing the b_y to the inner boundary of $\widehat{c} \circ c$, we have the following cobordism:

$$\widehat{a_x} \circ c \circ \widehat{c} \circ b_y \xrightarrow{Sad} \widehat{a_x} \circ \mathrm{Rad}_n \circ b_y = \widehat{a_x} \circ b_y. \tag{2.13}$$

Applying the TQFT functor $\mathcal{F}$ to this cobordism results in a map

$$\mathcal{F}(\widehat{a}_x \circ b_x) \otimes \mathcal{F}(\widehat{a}_y \circ b_y) \xrightarrow{\mathfrak{m}} \mathcal{F}(\widehat{a}_x \circ b_y), \tag{2.14}$$

where $x \cdot y = \mathfrak{m}(x \otimes y)$ is composition of multiplication maps (equation 2.4). Hence, the product is defined by $x \cdot y = \mathfrak{m}(x \otimes y)$.

For more details see [Kho02; LLS21].

Now we can define the Khovanov invariant for a diskular tangle $R \in \mathbb{T}^{(m_1,\cdots,m_k;n)}$. First, to a flat $(m_1, \ldots, m_k; n)$-diskular tangle R, we can assign a graded $\mathbb{F}$-module

$$V(R) = \bigoplus_{a_i \in \boldsymbol{B}^{(0;m_i)}, b \in \boldsymbol{B}^{(0;n)}} \mathcal{F}(\hat{b} \circ R \circ (a_1, \ldots, a_k))\{\frac{n}{2}\}. \tag{2.15}$$

where $\{\cdot\}$ denotes the shift in quantum grading gr_q, and $\mathcal{F}$ is the Khovanov TQFT.

The $V(R)$ defined above, is an $(\mathcal{H}^{m_1} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{H}^{m_k}, \mathcal{H}^n)$-bimodule as follows. By a construction similar to the product in the $\mathcal{H}^n$, for $l = m_1, \ldots, m_k, n$ we can define a product $\mathcal{H}^l \otimes V(R) \to V(R)$. However, in the construction of Khovanov homology (section 2.4.1), the map $\mathcal{F}(\sqcup_k S^1) \to \mathcal{F}(\sqcup_{k\pm 1} S^1)$ is either a merge or a split on one of the circles and the identity on the rest. Hence, we are choosing an order on the circles.

The $V(R)$ can be viewed as an object in **Bim**, the category dg multi-modules [LLS21, Lemma 4.2], where by a $(\mathcal{H}^{m_1}, \cdots, \mathcal{H}^{m_k}; \mathcal{H}^n)$-multimodule M we mean a $(\mathcal{H}^{m_1} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{H}^{m_k}, \mathcal{H}^n)$-bimodule. For the $(\mathcal{H}^{m_1}, \cdots, \mathcal{H}^{m_k}; \mathcal{H}^n)$-multimodule A and the $(\mathcal{H}^{s_1}, \cdots, \mathcal{H}^{s_i}; \mathcal{H}^{m_j})$-multimodule B, we can define the ith partial tensor product to be

$$A \underset{j}{\otimes} B = A \underset{\mathcal{H}^{m_j}}{\otimes} B, \tag{2.16}$$

which is a $(\mathcal{H}^{m_1}, \cdots, \mathcal{H}^{m_{j-1}}, \mathcal{H}^{s_1}, \cdots, \mathcal{H}^{s_i}, \mathcal{H}^{m_{j+1}}, \cdots, \mathcal{H}^{m_k}; \mathcal{H}^n)$-multi module. More generally one can form tensor product a multi-modules B_j over $(\mathcal{H}^{j,1}, \ldots, \mathcal{H}^{j,s_j}; \mathcal{H}^{m_j})$ for $j = 1, \ldots, k$ with a multi-module A over $(\mathcal{H}^{m_1}, \ldots, \mathcal{H}^{m_k}; \mathcal{H}^n)$:

$$(B_1, \ldots, B_k) \otimes_{\mathcal{H}^{m_1}, \ldots, \mathcal{H}^{m_k}} A = (B_1 \otimes_{\mathbb{F}} \ldots \otimes_{\mathbb{F}} B_k) \underset{\mathcal{H}^{m_1} \otimes_{\mathbb{F}} \ldots \otimes_{\mathbb{F}} \mathcal{H}^{m_k}}{\otimes} A. \tag{2.17}$$

Fix an $(m_1, \dots, m_k; n)$-diskular tangle T with N crossings. For each $\alpha \in \underline{2}^N$, let T^α denote the α-resolution of T according to equation (2.2). Therefore, T^α is a flat $(m_1, \dots, m_k; n)$-diskular tangle and by (2.15) we can assign a $(\mathcal{H}^{m_1} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{H}^{m_k}, \mathcal{H}^m)$-bimodule $V(T^\alpha)$ to T^α.

To build a tangle invariant for an $(m_1, \dots, m_k; n)$-diskular tangle R with N crossings, we assign a chain complex $C(R) = (C^i(R), d^i)$ of $(\mathcal{H}^{m_1} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{H}^{m_k}, \mathcal{H}^m)$-bimodules to R as follows

$$C^i(R) = \bigoplus_{\substack{|\alpha|=i \\ \alpha \in \underline{2}^N}} \big(V(R^\alpha)\big)\{|\alpha| + N_+ - 2N_-\} \tag{2.18}$$

where R^α denotes the resolution of R for $\alpha \in \underline{2}^N$, $\mathcal{F}$ is the Khovanov TQFT functor defined in section 2.4.1, and N_+ and N_- denote the number of positive and negative crossings respectively. The differentials for $C_{\mathrm{Kh}}(R)$ are defined similar to section 2.4.1.

Definition 2.19. The Khovanov invariant of the diskular tangle $R \in \mathbb{T}^{(m_1, \cdots, m_k; n)}$ is the chain complex

$$C_{\mathrm{Kh}}(R) = C(R)[N_-], \tag{2.19}$$

where $(C(R), d)$ is defined above.

Gluing process: Fix an $(m_1, \dots, m_k, n_i)$-diskular tangle T and an $(n_1, \dots, n_r, p)$-diskular tangle S, for $1 \leq i \leq r$. The composition (gluing) $T \circ_i S$ corresponds to the following theorem.

Theorem 2.20. *[LLS21, Lemma 4.16] For diskular tangles S and T as above, there is an isomorphism*

$$C_{Kh}(T) \otimes_{\mathcal{H}^{n_i}} C_{Kh}(S) \to C_{Kh}(T \circ_i S) \tag{2.20}$$

as $(\mathcal{H}^{n_1} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{H}^{n_{i-1}} \otimes_{\mathbb{F}} \big(\mathcal{H}^{m_1} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{H}^{m_k}\big) \otimes_{\mathbb{F}} \mathcal{H}^{n_{i+1}} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{H}^{n_r}, \mathcal{H}^p)$-bimodules.

Also, if we compose $(r_{i,1}, \cdots, r_{i,l_i}; m_i)$-diskular tangles S_i for $i = 1, \ldots, k$ and $(m_1, \ldots, m_k; n)$-diskular tangle T to build $(r_{1,1}, \cdots, r_{k,l_k}; n)$-diskular tangle $T \circ (S_1, \ldots, S_k)$, then we have isomorphism of chain complexes:

$$C_{\mathrm{Kh}}(T) \underset{\mathcal{H}^{m_1} \otimes \cdots \otimes \mathcal{H}^{m_k}}{\otimes} (C_{\mathrm{Kh}}(S_1), \ldots, C_{\mathrm{Kh}}(S_k)) \cong C_{\mathrm{Kh}}(T \circ (S_1, \ldots, S_k)), \tag{2.21}$$

as $(\mathcal{H}^{r_{1,1}} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{H}^{r_{k,l_k}}, \mathcal{H}^n)$-bimodules. Hence, we can write

$$C_{\mathrm{Kh}}(T \circ (S_1, \ldots, S_k)) \cong C_{\mathrm{Kh}}(T) \underset{\mathcal{H}^{m_1} \otimes \cdots \otimes \mathcal{H}^{m_k}}{\otimes} (C_{\mathrm{Kh}}(S_1) \otimes \cdots \otimes C_{\mathrm{Kh}}(S_k)) \tag{2.22}$$

Given an elementary cobordism $\mathfrak{s}$ between diskular $(m_1, \cdots, m_k; n)$-tangles R and S, it induces a $(\mathcal{H}^{m_1} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{H}^{m_k}, \mathcal{H}^n)$-bimodules morphism $C_{\mathrm{Kh}}(\mathfrak{s}) : C_{\mathrm{Kh}}(R) \to C_{\mathrm{Kh}}(S)$ with grading shift $(0, \chi(\mathfrak{s}))$.

2.4.3 Functoriality of Khovanov

Khovanov homology is *functorial* up to a sign. That is to say, for an ambient isotopy between cobordism Σ and Σ' in $\mathbb{R}^3 \times [0, 1]$, the maps induced on Khovanov homology are equal up to a sign.

Theorem 2.21. *[Jac04; Bar05; Kho06] Let $\Sigma, \Sigma' \subset \mathbb{R}^3 \times [0, 1]$ be two smooth cobordism from L_0 to L_1. If they are isotopic relative to the boundary, then up to a sign they induce chain homotopic maps on the Khovanov chain complex. Hence, the induced map on homology satisfies*

$$Kh(\Sigma) = \pm Kh(\Sigma') : Kh(L_0) \to Kh(L_1). \tag{2.23}$$

We will not prove theorem 2.21 here and refer the reader to [Jac04; Bar05; Kho06], but we discuss the central ideas involved in the proof. The movies of isotopic cobordisms are related by finite sequence of movie moves. Hence, to prove the theorem, we should compute the chain homotopy between maps induced by movie moves. The result follows by showing the Khovanov chain complex of equivalent movies have only $\pm\mathrm{Id}$ as chain maps.

2.4.4 Obstruction to ribbon concordance

This section is devoted to re-stating the main result form [LZ19]. Their result provides an obstruction to ribbon concordance, from Khovanov homology. First, we recite a few notions related to the study of smooth surfaces using Khovanov homology.

Let $\mathcal{C}ob^2$ denote the cobordism category of links, with objects the links in S^3 and morphism consisting of smooth surfaces embedded in $S^3 \times [0, 1]$. Let $\mathcal{K}ob^2$ denote the pre-additive $\mathbb{Z}$-linear category freely generated by $\mathcal{C}ob^2$. The category $\mathcal{K}ob^2$ has the same objects as $\mathcal{C}ob^2$, and morphisms from L to L' in $\mathcal{K}ob^2$ are finite formal linear combinations of cobordisms from L to L' in $\mathcal{C}ob^2$ and the composition is induced from the composition in $\mathcal{C}ob^2$. Also, let $\mathcal{K}ob^2_\bullet$ denotes the pre-additive category of *dotted* cobordisms [Bar05, Section 11.2]. In this category objects are links in S^3. A morphism is a finite formal sum of dotted cobordisms $\mathcal{S}$, where a dotted cobordism means a properly embedded smooth surface possibly with boundary in $S^3 \times [0, 1]$ and finitely many dots (marked points) on its interior.

Definition 2.22. Let Σ be a cobordism and h be a smoothly embedded 3-dimensional 1-handle $[-1, 1] \times \mathbb{D}^2$ in $\mathbb{R}^3 \times [0, 1]$ such that $\mathfrak{n} = h \cap \Sigma$ is an embedded annulus $[-1, 1] \times S^1 \hookrightarrow \Sigma$ with the movie of $\mathfrak{n}$ is given by figure 2.10 (a). We call $\mathfrak{n}$ a *standard neck* on the cobordism Σ.

Figure 2.10 shows the local movie of a standard neck on a cobordism. It is consists of two band sums such that the number of connected components on the left frame of the movie is equal to the number of the connected components on the right frame. Up to isotopy of a cobordism Σ, we can transform any neck to a standard neck.

Let Σ_+ (respectively Σ_-) be the result of deleting the neck $\mathfrak{n} \subset \Sigma$ from Σ and smoothly gluing $\{-1, 1\} \times \mathbb{D}^2 \times \{0\}$ on the new boundary, and putting a new dot on

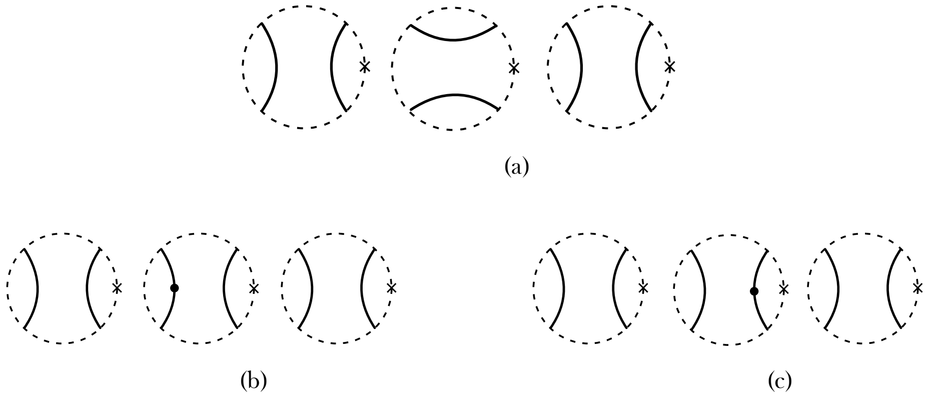


Figure 2.10: (a) Shows the movie of a standard neck. (b) and (c) show the movie of the cobordisms Σ_+ and Σ_- respectively after the standard neck cut.

$\{1\} \times \mathbb{D}^2 \times \{0\}$ (respectively $\{-1\} \times \mathbb{D}^2 \times \{0\}$).

At the Khovanov homology package, a dot on a connected component of a cobordism induces the multiplication by $X \in \mathcal{A}$ on the component with the dot and the identity on the rest of the components. Moreover, the placement of the dots on Σ_+ (respectively Σ_-) does not affect the map $\mathrm{Kh}(\Sigma_+)$ (respectively $\mathrm{Kh}(\Sigma_-)$) induced on Khovanov homology. Hence analogous to the theorem 2.21, we have,

Theorem 2.23. *[BLS17; Sar20] If Σ and Σ' are isotopic dotted cobordisms relative to the boundary in $S^3 \times [0, 1]$, then they are related by a sequence of movie moves and dot isotopies. Moreover, the maps induced by Σ, Σ' are chain homotopic up to a sign.*

We will not prove this theorem here, but we will give some insight to how the proof works. A movie for a dotted cobordism Σ consists of sequence of elementary cobordisms (a sequence of planar isotopies, Reidemeister moves, births, saddles, and deaths), and *dot additions* that are frames in the movie with a dot appearing on an arc. Then, we can show that moving a dot under or over a crossing will induce chain homotopic maps on Khovano chain complex. Hence, the maps induced on Khovanov homology by isotopic dotted cobordisms are equal up to a sign.

The Khovanov homology satisfies the following neck cutting relation.

Proposition 2.24 (Neck Cutting Relation). *[Bar05, section 11.2] Assume* $\mathfrak{n}$ *is a standard neck on the cobordism* Σ*, and* Σ_+ *and* Σ_- *are the result of cutting the neck* $\mathfrak{n}$*. Then,*

$$Kh(\Sigma) = \pm Kh(\Sigma_+) \pm Kh(\Sigma_-) \tag{2.24}$$

At the level of the Khovanov chain complex, the induced chain map $C_{\text{Kh}}(\Sigma) : C_{\text{Kh}}(L_0) \to C_{\text{Kh}}(L_1)$ is up to a sign chain homotopic (more strongly, equal) to the chain map $C_{\text{Kh}}(\Sigma_+)+C_{\text{Kh}}(\Sigma_-) : C_{\text{Kh}}(L_0) \to C_{\text{Kh}}(L_1)$.

Lastly we have,

Theorem 2.25. *[LZ19] If* F *is a smooth ribbon concordance from* L_0 *to* L_1 *, then the map induced on Khovanov homology is a bi-graded split injection.*

CHAPTER 3

EQUIVARIANT KHOVANOV HOMOLOGY

Section 3.1 is devoted to studying the equivariant cohomology of chain complexes with a group action. We will direct our focus on Khovanov chain complex of periodic links in section 3.2. Also, we fix the choice of our commutative ring $\mathbb{F}$ to be the integers $\mathbb{Z}$.

3.1 Equivariant cohomology

Let G be a finite cyclic group. We will denote by EG the universal G-bundle. More precisely, EG is a contractible G-CW complex such that the action of G is free. The orbit space EG/G is the classifying space BG.

Example 3.1. For the cyclic group $G = \mathbb{Z}_2$, $E\mathbb{Z}_2 = S^\infty$, with the antipodal action of $\mathbb{Z}_2$. Just as the quotient of the antipodal action on S^n gives $\mathbb{R}P^n$, the quotient space is $B\mathbb{Z}/2 = S^\infty/\mathbb{Z}_2 = \mathbb{R}P^\infty$.

Example 3.2. For $G = \mathbb{Z}_p$ with p an odd prime, we have a similar construction. Just as we can consider S^{2n-1} as the unit sphere in $\mathbb{C}^n$, we can similarly consider S^∞ as the unit sphere in $\mathbb{C}^\infty$. Then $\mathbb{Z}_p$ acts by multiplication by $e^{2\pi i/p}$. This is a free action of $\mathbb{Z}_p$ on S^∞. So we conclude that $E\mathbb{Z}_p \cong S^\infty$ and $B\mathbb{Z}_p \cong S^\infty/\mathbb{Z}_p$. The space $B\mathbb{Z}_p$ is an infinite-dimensional lens space.

Given X a topological G-space, the G-*equivariant cohomology* (also referred to as *Borel equivariant cohomology*) of X is defined as

$$H^*_G(X;\mathbb{Z}) := H^*(EG \times X/G;\mathbb{Z}), \tag{3.1}$$

where $(EG \times X)/G$ is the quotient of $EG \times X$ by the diagonal action of G. For more treatment of this topic, see [Wei94; Bro82; BP21].

For computations, the cellular cochain complex $C^*_{\text{cell}}(EG;\mathbb{Z})$ can be considered as a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module where G acts on $\mathbb{Z}$ trivially. Additionally, the action of G on X makes the cochain complex $C^*_{\text{cell}}(X;\mathbb{Z})$ a $\mathbb{Z}[G]$-module. Hence,

$$H^*_G(X;\mathbb{Z}) = \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, C^*_{\text{cell}}(X;\mathbb{Z})). \tag{3.2}$$

The G-equivariant cohomology of X is equivalent to computing the group homology of $\mathbb{Z}$ with coefficients in $C^*_{\text{cell}}(X;\mathbb{Z})$ [Bro82, Chapter VII]. Instead of $\mathbb{Z}$, we could use any $\mathbb{Z}[G]$-module M.

With abuse of notation we denote the cellular chain complex $C^{\text{cell}}_\bullet(EG,\mathbb{Z})$ by EG. That is, we have a chain complex

$$EG : \cdots \xrightarrow{1-\theta} EG^2 = \mathbb{Z}[G] \xrightarrow{1+\cdots+\theta^{p-1}} EG^1 = \mathbb{Z}[G] \xrightarrow{1-\theta} EG^0 = \mathbb{Z}[G] \longrightarrow 0 \tag{3.3}$$

where $\mathbb{Z}[G] = \mathbb{Z}[\theta]/(1-\theta^p)$, and p is the order of the group G.

Proposition 3.3. *Let $EG^* = \mathrm{Hom}_{\mathbb{Z}[G]}(EG, \mathbb{Z}[G])$, for a bounded below cochain complex C over $\mathbb{Z}[G]$, there is a natural isomorphism*

$$C \otimes_{\mathbb{Z}[G]} EG^* \to \mathrm{Hom}_{\mathbb{Z}[G]}(EG, C) \tag{3.4}$$

of $\mathbb{Z}$-modules give by $c \otimes x \mapsto \phi_{c\otimes x} : EG \to C$ that is defined by $\phi_{c\otimes x}(a) = cx(a)$ for $c \in C$, $x \in EG^*$, and $a \in EG$.

Proof. It is immediate from the definition of equivariant cohomology. □

Hence for computations we could use the following definition of the equivariant homology/cohomology.

Corollary 3.4. *For a G-space X, the G-equivariant homology of X with coefficients in $\mathbb{Z}$ is the homology of the chain complex*

$$C^G_*(X;\mathbb{Z}) = EG \otimes_{\mathbb{Z}[G]} C^{CW}_*(X;\mathbb{Z}). \tag{3.5}$$

Hence

$$H_*^G(X;\mathbb{Z}) = \mathrm{Tor}_{\mathbb{Z}[G]}(C_*^{CW}(X;\mathbb{Z}), \mathbb{Z}). \tag{3.6}$$

Similarly the G-equivariant cohomology of X with coefficients in $\mathbb{Z}$, is the homology of the cochain complex

$$C_G^*(X;\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}[G]}(EG, C_{CW}^*(X;\mathbb{Z})). \tag{3.7}$$

Hence

$$H_G^*(X;\mathbb{Z}) = \mathrm{Ext}_{\mathbb{Z}[G]}(\mathbb{Z}, C_{CW}^*(X;\mathbb{Z})). \tag{3.8}$$

3.2 Equivariant Khovanov homology

Let $\mathrm{Kh}(L)$ and $C_{\mathrm{Kh}}(L)$ respectively denote the Khovanov homology and the Khovanov chain complex of a link L, with coefficients in $\mathbb{Z}$ (section 2.4.2). Also, to avoid introducing new notation, we denote by θ the generator of the group ring $\mathbb{Z}[\mathbb{Z}_p] = \mathbb{Z}[\theta]/(\theta^p - 1)$.

Definition 3.5. The *equivariant Khovanov homology* for a p-periodic link L with coefficients in $\mathbb{Z}[\mathbb{Z}_p]$-module M is defined as

$$\mathrm{EKh}(L, M) = \mathrm{Ext}_{\mathbb{Z}[\mathbb{Z}_p]}(M, C_{\mathrm{Kh}}(L;\mathbb{Z})). \tag{3.9}$$

Note that in literature despite the fact that Khovanov invariant is defined as a cochain complex, it has referred to as Khovanov homology. We also adhere to this inconsistency here, and refer to the equivariant Khovanov invariant defined above by equivariant Khovanov homology instead of equivariant Khovanov cohomology.

Hence, for computations using the Khovanov chain complex,

$$C_{\mathrm{EKh}}^k(L, M) = \bigoplus_{i+j=k} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_p]}(P_i, C_{\mathrm{Kh}}^j(L)) \tag{3.10}$$

where $(P_\bullet, \delta_\bullet)$ is a projective resolution of M as a $\mathbb{Z}[\mathbb{Z}_p]$-modules. Differentials for this complex are given by $d = \delta + d_{\mathrm{Kh}}$.

Since C_{Kh} is a finitely generated free abelian group, if we fix $M = \mathbb{Z}$, we have,

$$C^k_{\mathrm{EKh}}(L; \mathbb{Z}) = \bigoplus_{i+j=k} \mathbb{Z}[\mathbb{Z}_p]^* \underset{\mathbb{Z}[\mathbb{Z}_p]}{\otimes} C^i_{\mathrm{Kh}}(L) \cong \bigoplus_{j\in\mathbb{Z}} C^{k-j}_{\mathrm{Kh}}(L). \tag{3.11}$$

Computing the equivariant Khovanov homology using periodic diskular tangles is done by decomposing the periodic diagram D of the periodic link L, as

$$D = T \circ (S_1, \ldots, S_p) \tag{3.12}$$

where $T = T^{(n,\cdots,n;0)}$, $S_1 = \cdots = S_p = T^{(;n)}$, and $\mathbb{Z}_p$ maps T to itself, and acts by permutation on $(S_1, \cdots, S_p)$. Hence,

$$C_{\mathrm{EKh}}(D) = E\mathbb{Z}_p \bigotimes_{\mathbb{Z}[\mathbb{Z}_p]} \left(C_{\mathrm{Kh}}(T) \underset{\mathcal{H}^n \otimes \cdots \otimes \mathcal{H}^n}{\otimes} C_{\mathrm{Kh}}(S_1, \ldots, S_p) \right) \tag{3.13}$$

$$\tag{3.14}$$

where the tensor product on the right hand side is by the gluing lemma 2.20.

3.3 Skew group ring

In this section we are going to study the algebraic structure of the Khovanov invariant of an admissible p-periodic diskular tangle. In summary, we show the Khovanov chain complex associated to a p-equivariant diskular tangle is projective over a Khovanov arc algebra (definition 2.17) equipped with the $\mathbb{Z}_p$ action. Here unless otherwise stated, tensor product $\otimes$ denotes the tensor product over $\mathbb{Z}$.

Definition 3.6. Let R be a unital ring, G a finite group and $\theta : G \to \mathrm{Aut}_{\mathrm{Ring}}(R)$ a group homomorphism. The *skew group ring* of G over R induced by θ as a left R-module is given by

$$R_\theta[G] = \bigoplus_{g\in G} R\{g\}, \tag{3.15}$$

where $g \in G$ is considered as a formal variable and $\mathrm{Aut}_{\mathrm{Ring}}(R)$ denotes the group of ring automorphisms of R. The addition in $R_\theta[G]$ is component-wise and multiplication is given by $rg \cdot sh = r\theta(g)(s)gh \in R\{gh\}$ for $rg \in R\{g\}$ and $sh \in R\{h\}$.

To simplify the notation, we denote $R_g = R\{g\}$, and we will write $\theta_g(b)$ to denote the image of $b \in R$ by the ring isomorphism $\theta(g) : R \to R$. Also, we can see elements of $R_\theta[G]$ as formal sums

$$R_\theta[G] = \{\sum_{g\in G} r_g g | r_g \in R_g\}. \tag{3.16}$$

Proposition 3.7. *The skew group ring $R_\theta[G]$ defined above is a unital associative ring with multiplicative identity given by* $1 = 1_R 1_G \in R_{1_G}$.

The proof is straightforward using the associativity of R and the fact that θ is a group homomorphism.

Example 3.8. Assume $R = \mathbb{Z}$ and $G = \mathbb{Z}_p$ for p a positive integer. Let $\theta : \mathbb{Z}_p \to \mathrm{Aut}_{\mathrm{Ring}}(\mathbb{Z})$ be defined by

$$\theta(g) = id_{\mathbb{Z}} \quad \text{for all } g \in \mathbb{Z}_p. \tag{3.17}$$

Then $\mathbb{Z}_\theta[\mathbb{Z}_p]$ is the usual group ring $\mathbb{Z}[\mathbb{Z}_p]$.

Another example of a skew group ring is the following.

Example 3.9. Let A be a unital ring and assume $G \leqslant S_p$ is a subgroup of the pth symmetric group for a positive integer p. We have an action of S_p on $A^{\otimes p} = A\otimes \cdots \otimes A$ by permutation of factors. Therefore, G also acts on $A^{\otimes p}$. We can define a multiplication for the A-module $A^{\otimes p}\otimes\mathbb{Z}[G]$ by

$$(a \otimes g)(b \otimes h) = a \cdot g(b) \otimes gh \tag{3.18}$$

for $a, b \in A^{\otimes p}$ and $g, h \in G$. The resulting $\mathbb{Z}$-algebra $A^{\otimes p} \otimes \mathbb{Z}[G]$ is call the *wreath product* of A with G, and we denote it by $A \wr G$.

It follows from definition 3.6 and proposition 3.7 that:

Proposition 3.10. *For a skew group ring $R_\theta[G]$, the following are true.*

(a) The R-module R_{1_G} is a subring of $R_\theta[G]$ containing the identity 1.

(b) For $g, h \in G$, $R_g \cdot R_h = R_{gh}$

The skew group ring belongs to a more general class of rings known as *strongly group graded rings*. We encourage the reader to look at [Dad80; CK96; BG00] for more details on strongly group graded ring.

Let us fix an $R_\theta[G]$-module M and $\mathbb{Z}[G]$-module V. Then $M \otimes V$ is naturally a left $R_\theta[G] \otimes \mathbb{Z}[G]$-module. Additionally, one can make $V \otimes M$ a left $R_\theta[G]$-module as follows. The *semi-diagonal action* of $R_\theta[G]$ on $M \otimes V$ is defined by

$$rg \cdot (v \otimes m) = gv \otimes r \cdot g(m) \tag{3.19}$$

This $R_\theta[G]$ action makes $M \otimes V$ a left $R_\theta[G]$-module. Again the proof of this statement is left as an exercise.

Now we can compare the projective modules over $R_\theta[G]$ with R-modules.

Lemma 3.11. *[BG00, lemma 4.1] Let V be a left $\mathbb{Z}[G]$-module, and M a left $R_\theta[G]$-module. Assume that V is free as a $\mathbb{Z}$-module. If M is projective as an $R_\theta[G]$-module, then so is $V \otimes M$ as a left $R_\theta[G]$-module with the semi-diagonal action.*

Lemma 3.12. *[BG00, lemma 4.3] Let M be an $R_\theta[G]$-module. If M is projective as a left R-module, then $\mathbb{Z}[G] \otimes M$ with semi-diagonal action is projective as a left $R_\theta[G]$-module.*

Both lemma 3.11 and 3.12 are proven for strongly group graded rings in [BG00] of which skew group rings are a special case.

Now we will focus our attention to the special cases $R = \mathcal{H}^n$ (see definition 2.17), and $G = \mathbb{Z}_p$.

Lemma 3.13. *[Kho02, Proposition 3] Let $S \in \mathbb{T}^{(;n)}$ be an $(;n)$-diskular tangle. The $\mathcal{H}^n$-module $C_{Kh}(S)$ is a finitely generated and projective $\mathcal{H}^n$-module.*

Proof. By the correspondence between the tangles diagrams in $[0, 1]^2$ and diskular tangles (section 2.1), we can consider the $(;n)$-diskular tangle S as a tangle diagram in $[0, 1]^2$. Hence, the lemma follows from [Kho02, Proposition 3]. □

For p-equivariant diskular tangle $(S_1, \ldots, S_p)$ with $S_1 = \cdots = S_p \in T^{(;n)}$ (see definition 2.10) we have,

Corollary 3.14. *The Khovanov chain complex of $C_{Kh}(S_1, \ldots, S_p) = C_{Kh}(S_1) \otimes \cdots \otimes C_{Kh}(S_p)$ is projective as an $\mathcal{H}^n \otimes \cdots \otimes \mathcal{H}^n$-module.*

Proof. This is immediate from lemma 3.13. □

The cyclic group $\mathbb{Z}_p$ also acts on both $\mathcal{H}^n \otimes \cdots \otimes \mathcal{H}^n$ and $C_{\mathrm{Kh}}(S_1, \ldots, S_p) = C_{\mathrm{Kh}}(S_1) \otimes \cdots \otimes C_{\mathrm{Kh}}(S_p)$ by cyclic permutation, denoted by θ. Hence, we can consider $C_{\mathrm{Kh}}(S_1) \otimes \cdots \otimes C_{\mathrm{Kh}}(S_p)$ as a module over $\mathcal{R}^n_\theta = (\mathcal{H}^n \otimes \cdots \otimes \mathcal{H}^n)_\theta[\mathbb{Z}_p]$.

Theorem 3.15. *Given a p-equivariant $(;n)$-diskular tangle $(S_1, \ldots, S_p)$, the $\mathcal{R}^n_\theta$-module $\mathbb{Z}[\mathbb{Z}_p] \otimes C_{Kh}(S, \ldots, S)$ with the semi-diagonal action is a projective $\mathcal{R}^n_\theta$-module.*

Proof. Proof follows immediately from corollary 3.14 and lemma 3.12. □

CHAPTER 4

OBSTRUCTION TO EQUIVARIANT RIBBON CONCORDANCE

In this chapter, we study the maps induced on equivariant Khovanov homology by equivariant cobordisms. In section 4.2, we prove equivariant Khovanov homology is functorial up to a factor of $(\pm 1)^p$ where p is the order of the group. Using that result, we study the map induced by a ribbon concordance in section 4.3.

4.1 Equivariant movie moves

Here, we introduce movie presentations for equivariant cobordisms between periodic links. Also, we prove that for equivariantly isotopic cobordisms, their movies are related by equivariant movie moves. In what follows, p is a fixed prime integer, and is the order of the group. In an overview, we use techniques similar to section 2.4.3.

Definition 4.1. A *p-equivariant movie* of an equivariant link cobordism $\Sigma : L_0 \to L_1$, is a finite sequence of periodic link diagrams $\boldsymbol{EM}_\Sigma = \{E_i\}_{i=0}^k$, with successive pairs of diagrams related by a p-equivariant elementary cobordism (definition 2.11) localized to p disjoint closed disks in $\mathbb{R}^2\backslash\mathbf{0}$ such that the elementary cobordisms correspond under the action of $\mathbb{Z}$. Individual diagrams E_i in the sequence are called *equivariant frames*.

Similar to section 2.3, $\pi : (\mathbb{R}^3\backslash\boldsymbol{z}) \times [0, 1] \to \mathbb{R}^2\backslash\mathbf{0}$ denotes the projection to the xy-plane. Fix a smooth equivariant cobordism $\Sigma : L_0 \to L_1$, between periodic links L_0 and L_1. We call Σ *generic*, if the projection $\rho : (\mathbb{R}^3\backslash\boldsymbol{z}) \times [0, 1] \to [0, 1]$ restricted to Σ is a Morse function.

Lemma 4.2. *Any equivariant link cobordism Σ can be represented by an equivariant movie.*

Proof. Let $\overline{\Sigma}$ be the quotient cobordism in $(\mathbb{R}^3\backslash\boldsymbol{z}) \times [0, 1]$. After small perturbations of $\overline{\Sigma}$ away from the $\boldsymbol{z}\times[0, 1]$, $\overline{\Sigma}$ is a generic cobordism. By [GLW18, Lemma 15], the link cobordism $\overline{\Sigma} \subset \mathbb{R}^3 \times [0, 1]$ can be presented by an annular movie $\boldsymbol{M}_{\overline{\Sigma}} = \{\overline{E}_{t_i}\}$ (A movie in which every frame avoids the origin). Pulling back the movie $\boldsymbol{M}_{\overline{\Sigma}}$ with the quotient map induces an equivariant movie $\boldsymbol{M}_{\Sigma}$ as desired. ◻

An *equivariant isotopy* refers to a 1-parameter family of diffeomorphisms f_s of $(\mathbb{R}^3 \backslash \boldsymbol{z}) \times [0, 1]$ for $s \in [0, 1]$, such that $f_0 = id$ and f_s is the identity on the boundary for all s, and equivariant under the action $\tilde{\theta}$. Hence, for all s, the image of f_s restricted to Σ is a p-equivariant link cobordism.

Fix periodic links L_0 and L_1 and equivariant cobordisms $\Sigma, \Sigma' : L_0 \to L_1$. We call Σ and Σ' *equivariantly isotopic*, if there is an equivariant isotopy f_s such that $f_1(\Sigma) = \Sigma'$.

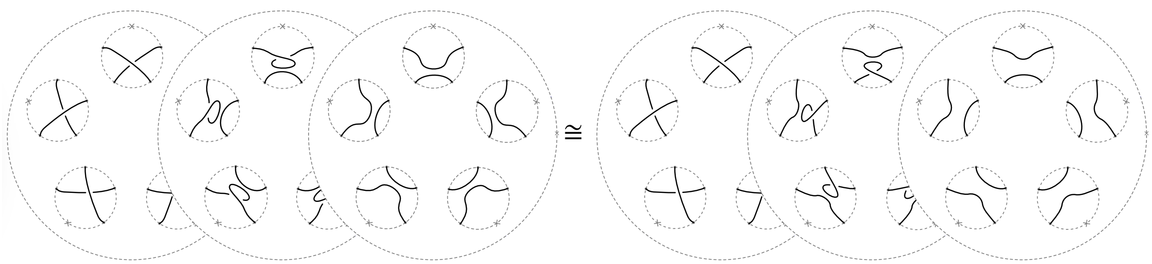


Figure 4.1: This figure shows an equivariant movie move. The equivariant movie on the left is equivalent to the equivariant movie on the right. These two movies differ by commutativity of a saddle move and Reidemeister move of type I.

Now we can prove the theorem 1.4.

Theorem (1.4)**.** *Suppose two equivariantly isotopic, equivariant link cobordisms Σ, Σ' are specified by equivariant movies $\boldsymbol{EM}_{\Sigma}$ and $\boldsymbol{EM}_{\Sigma'}$ respectively. Then there exists a finite sequence of equivariant movie moves that transforms $\boldsymbol{EM}_{\Sigma}$ to $\boldsymbol{EM}_{\Sigma'}$.*

Proof of theorem 1.4. By definition 1.2, if Σ is an equivariant link cobordism from L_0 to L_1, its quotient $\overline{\Sigma}$ by the extended action $\tilde{\theta}$ on $(\mathbb{R}^3 \backslash \boldsymbol{z}) \times [0, 1]$ is an annular link cobordism

$\overline{\Sigma} \subset (\mathbb{R}^3 \backslash \boldsymbol{z}) \times [0, 1]$. By assumption Σ and Σ' are two equivariant cobordisms. Their quotient cobordisms $\overline{\Sigma}$ and $\overline{\Sigma'}$ are isotopic annular link cobordisms in $(\mathbb{R}^3 \backslash \boldsymbol{z}) \times [0, 1]$. By [GLW18, Lemma A.2], $\overline{\Sigma}$ and $\overline{\Sigma'}$ are related by a finite sequence of movie moves localized to a disks away from the origin in $\mathbb{R}^2$. The pull back of those movie moves are a sequence of equivariant movie moves from $\boldsymbol{EM}_{\Sigma}$ to $\boldsymbol{EM}_{\Sigma'}$. □

4.2 Functoriality of Equivariant Khovanov Homology

In this section, we prove theorem 1.6. First we mention a few abstract homological algebra facts. In what follows, unless otherwise stated, we use $\mathbb{Z}$ as our coefficient ring.

Lemma 4.3. *Suppose C, D are chain complexes, $f_0, f_1 : C \to D$ are chain homotopic maps by a chain homotopy h. This information is equivalent to a chain map $H : I \otimes C \to D$, where I is the chain complex*

$$\begin{aligned} I^1 = \mathbb{Z}\{s\} &\longrightarrow I^0 = \mathbb{Z}\{s_0, s_1\} \\ s &\longmapsto s_1 - s_0, \end{aligned} \tag{4.1}$$

such that $H(s_i \otimes x) = f_i(x)$, for $i = 0, 1$, moreover, $H(s \otimes x) = h(x)$.

The proof of the lemma above is trivial and left as an exercise for the reader. Given f_0, f_1 as above, then $f_0^{\otimes p}, f_1^{\otimes p} : C^{\otimes p} \to D^{\otimes p}$ are also chain homotopic with the chain homotopy given by

$$k = \sum_{i=0}^{p-1} \pm f_1^{\otimes i} \otimes h \otimes f_0^{\otimes p-1-i} : C^{\otimes p} \to D^{\otimes p}[1] \tag{4.2}$$

where the sign comes from the Koszul sign convention for tensor product of chain complexes.

Similarly, we introduce a chain map using the chain map H (lemma 4.6) as follows.

$$K : (I \otimes C)^{\otimes p} \longrightarrow D^{\otimes p} \tag{4.3}$$

$$K(t_1 \otimes x_1 \otimes \cdots \otimes t_p \otimes x_p) = H(t_1 \otimes x_1) \otimes \cdots \otimes H(t_p \otimes x_p)$$

Equivalently, we can rewrite the equation (4.3) as $K : I^{\otimes p} \bigotimes C^{\otimes p} \longrightarrow D^{\otimes p}$.

Lemma 4.4. *Assume (C, d) and (C', d') are chain complexes of R-modules and n is an integer. Let $\{g_i : C_i \to C'_i\}$ for $i \leq n$ be a family of group homomorphisms such that $g_i \circ d = d' \circ g_i$. If for $i \geq n$, the C_i are projective and $H_i(C') = 0$, then we can extended the g_i to a chain map $g : C \to C'$. Moreover, g is unique up to chain homotopy.*

Proof. By induction, we can assume for $k \geq n$ we have extended the g_i for $i \leq k$. For $k + 1$ we have a commutative diagram

$$\begin{array}{ccccc} C_{k+1} & \xrightarrow{d} & C_k & \xrightarrow{d} & C_{k-1} \\ \Big\downarrow{\scriptstyle g_{k+1}} & & \Big\downarrow{\scriptstyle g_k} & & \Big\downarrow{\scriptstyle g_{k-1}} \\ C'_{k+1} & \xrightarrow{d'} & C'_k & \xrightarrow{d'} & C'_{k-1} \end{array} \tag{4.4}$$

where $d' \circ g_k \circ d = g_{k-1} \circ d \circ d = 0$. As C_{k+1} is projective and the bottom row is exact, the map g_{k+1} (dashed arrow) exists. Suppose now that $\mathfrak{g} : C \to C'$ is a second extension of $\{g_i\}_{i \leq n}$. Assume by induction the $\phi_i : C_i \to C'_{i+1}$ have been defined for $i \leq k$, where $k \geq n$, and $d'h_i + h_{i-1}d = g_i - \mathfrak{g}_i$.

$$\begin{array}{ccccccc} & & C_{k+1} & \xrightarrow{d} & C_k & \xrightarrow{d} & C_{k-1} \\ & \swarrow{\scriptstyle \phi_{k+1}} & \Big\downarrow{\scriptstyle g_k - \mathfrak{g}_k} & \swarrow{\scriptstyle \phi_k} & \Big\downarrow{\scriptstyle g_{k-1} - \mathfrak{g}_{k-1}} & \swarrow{\scriptstyle \phi_k} & \\ C'_{k+2} & \xrightarrow{d'} & C'_{k+1} & \xrightarrow{d'} & C'_k & & \end{array} \tag{4.5}$$

By assumptions, we have

$$d' \circ \phi_k \circ d = (g_k - \mathfrak{g}_k - \phi_{k-1} \circ d) \circ d$$

$$= g_k \circ d - \mathfrak{g}_k \circ d - \phi_{k-1} \circ d \circ d$$
$$= d' \circ g_k - d' \circ \mathfrak{g}_k = d' \circ (g_k - \mathfrak{g}_k)$$

As $H_i(C') = 0$ for $i \geq n$, the bottom row in diagram 4.5 is exact. Also, C_k is projective, and this implies the desired map ϕ_{k+1} exists. □

We define an action of $\mathbb{Z}_p$ on $C^{\otimes p}$ by cyclic permutation of factors i.e., $\mathbb{Z}_p$ acts by $\theta \cdot (x_1 \otimes \cdots \otimes x_p) = (-1)^{|x_p|(|x_1|+\cdots|x_{p-1}|)} x_p \otimes x_1 \otimes \cdots \otimes x_{p-1}$ where $|x_i| = \mathrm{gr}_h(x_i)$ denotes the homological degree of the $x_i \in C$. Note that the homotopy in equation (4.2) is not equivariant.

For a ring R, consider the category $\mathbf{Ch}_R$ of chain complexes of R-modules. The morphisms of $\mathbf{Ch}_R$ are chain maps. The homotopy category $\mathcal{K}$ of $\mathbf{Ch}_R$ is defined as follows: The objects of $\mathcal{K}$ are chain complexes (the same as the objects of $\mathbf{Ch}_R$) and the morphisms of $\mathcal{K}$ are the equivalence classes of chain maps in $\mathbf{Ch}_R$ up to chain homotopy. One can check that $\mathcal{K}$ is well defined as a category. By $\mathcal{D}$, we denote the *derived category* of $\mathcal{K}$ defined to be the localization $Q^{-1}\mathcal{K}$ at the collection Q of quasi-isomorphisms.

Lemma 4.5. *[Wei94, Corollary 10.4.7] For a bounded above chain complex M of projective R-modules, there is an isomorphism*

$$Hom_{\mathcal{K}}(M, N) \cong Hom_{\mathcal{D}}(M, N) \tag{4.6}$$

for every R-module N.

A chain complexes (C, d) with $d_i : C_i \to C_{i-1}$ is *bounded above* if there exists an $n \in \mathbb{Z}$ such that $C_i = 0$ for all $i < n$.

To construct an equivariant chain homotopy between chain maps $f_0^{\otimes p}$ and $f_1^{\otimes p}$, we need to define a chain map

$$\widehat{H} : E\mathbb{Z}_p \otimes I \longrightarrow I^{\otimes p} \tag{4.7}$$

satisfying

$$\widehat{H}_k(\alpha \otimes w) = \alpha \cdot \widehat{H}_k(1 \otimes w)$$

for $w \in (I^{\otimes p})_k$, $\alpha \in \mathbb{Z}[\mathbb{Z}_p]$, and $k \geq 0$. First, we define the following morphism at homological degree $k = 0, 1$.

$$\widehat{H}_0(1 \otimes s_j) = s_j \otimes \cdots \otimes s_j \qquad (\text{for } j = 0, 1)$$

$$\widehat{H}_1(1 \otimes s_j) = 0 \qquad (\text{for } j = 0, 1)$$

$$\widehat{H}_1(1 \otimes s) = \sum_{i=1}^{p} s_1^{\otimes p-i-1} \otimes s \otimes s_0^{\otimes i}$$

$$\widehat{H}_k(\theta^l \otimes w) = \theta^l \cdot \widehat{H}_k(1 \otimes w) \quad (\forall w \in I,\ k = 0, 1,\ \text{and } l = 0, \ldots, p-1)$$

where θ, acts by cyclic permutation. We use the partial information at $k = 0, 1$ to extend $\widehat{H}$ to every homological degree by lemma 4.4.

Lemma 4.6. *The map $\widehat{H}$ defined above extends to a chain map.*

Proof. At homological degree $k = 0, 1$, $\widehat{H}$ commutes with the differentials.

$$d \circ \widehat{H}_1(1 \otimes s_j) = 0 = \widehat{H}_0 \circ d(1 \otimes s_j) \qquad (\text{for } j = 0, 1)$$

$$d \circ \widehat{H}_1(\theta^i \otimes s_j) = 0 = \widehat{H}_0 \circ d(\theta^i \otimes s_j) \qquad (\text{for } j = 0, 1)$$

$$\begin{aligned} d \circ \widehat{H}_1(1 \otimes s) &= d(\sum_{i=0}^{p-1} s_1^{\otimes p-i-1} \otimes s \otimes s_0^{\otimes i}) = \sum_{i=0}^{p-1} s_1^{\otimes p-i-1} \otimes (s_1 - s_0) \otimes s_0^{\otimes i} \\ &= s_1^{\otimes p} - s_0^{\otimes p} \\ &= \widehat{H}_0(1 \otimes s_1 - 1 \otimes s_0) = \widehat{H}_0 \circ d(1 \otimes s) \end{aligned}$$

for $i = 1, \cdots p-1$. The same relation will be hold for $\widehat{H}_1(\theta^i \otimes s)$ for $i = 1, \cdots, p-1$.

By definition, $(E\mathbb{Z}_p \otimes I)_k$ is a projective $\mathbb{Z}[\mathbb{Z}_p]$-module for $k \geq 0$, and $H_k(I^{\otimes p}) = 0$ for $k \geq 1$. Hence by lemma 4.4, we can extend the maps $\widehat{H}_1$ and $\widehat{H}_0$ to a chain map $\widehat{H} = \{\widehat{H}_k\}$ inductively. □

Example 4.7. As an example, we have written down the chain map $\widehat{H}$ for $p = 2, 3$.

- For $p = 2$, we can define $\widehat{H}$ as follows.

$$\begin{aligned}
\widehat{H} :& E\mathbb{Z}_2 \otimes I \to I^{\otimes 2} \\
&\widehat{H}_0(a \otimes s_j) = s_j \otimes s_j \\
&\widehat{H}_1(a \otimes s_j) = 0 \\
&\widehat{H}_1(1 \otimes s) = s_1 \otimes s + s \otimes s_0 \\
&\widehat{H}_1(\theta \otimes s) = \theta \cdot \widehat{H}_1(1 \otimes s) = s \otimes s_1 + s_0 \otimes s \\
&\widehat{H}_2(a \otimes s_j) = 0 \\
&\widehat{H}_2(1 \otimes s) = s \otimes s \\
&\widehat{H}_2(\theta \otimes s) = \theta \cdot \widehat{H}_2(1 \otimes s) = -s \otimes s
\end{aligned}$$

  for $j = 0, 1$ and $a = 1, \theta$. Moreover, we define $\widehat{H}_k = 0$ for $k > 2$.

- For $p = 3$, we can define $\widehat{H}$ as follows.

$$\begin{aligned}
\widehat{H} :& E\mathbb{Z}_3 \otimes I \to I^{\otimes 3} \\
&\widehat{H}_0(a \otimes s_j) = s_j \otimes s_j \otimes s_j \\
&\widehat{H}_1(a \otimes s_j) = 0 \\
&\widehat{H}_1(1 \otimes s) = s_1 \otimes s_1 \otimes s + s_1 \otimes s \otimes s_0 + s \otimes s_0 \otimes s_0 \\
&\widehat{H}_1(\theta \otimes s) = \theta \cdot H(1 \otimes s) = s_0 \otimes s_1 \otimes s + s_0 \otimes s \otimes s_0 + s \otimes s_1 \otimes s_1 \\
&\widehat{H}_1(\theta^2 \otimes s) = \theta^2 \cdot H(1 \otimes s) = s_0 \otimes s_0 \otimes s + s_1 \otimes s \otimes s_1 + s \otimes s_0 \otimes s_1 \\
&\widehat{H}_2(a \otimes s_j) = 0
\end{aligned}$$

$$\widehat{H}_2(1 \otimes s) = s \otimes s_1 \otimes s + s \otimes s \otimes s_0$$

$$\widehat{H}_2(\theta \otimes s) = \theta \cdot \widehat{H}_2(1 \otimes s) = s_0 \otimes s \otimes s - s \otimes s \otimes s_1$$

$$\widehat{H}_2(\theta_2 \otimes s) = \theta^2 \cdot \widehat{H}_2(1 \otimes s) = -s_1 \otimes s \otimes s - s \otimes s_0 \otimes s$$

$$\widehat{H}_3(a \otimes s_j) = 0$$

$$\widehat{H}_3(1 \otimes s) = \widehat{H}_3(\theta \otimes s) = \widehat{H}_3(\theta^2 \otimes s) = -s \otimes s \otimes s$$

for $j = 0, 1$ and $a = 1, \theta, \theta^2$. Moreover, we define $\widehat{H}_k = 0$ for $k > 3$.

Proposition 4.8. *Assume C, D, f_0, and f_1 are as above. We can define a chain map as follows.*

$$\widehat{K} : E\mathbb{Z}_p \otimes I \otimes C^{\otimes p} \xrightarrow{\widehat{H} \otimes id} (I \otimes C)^{\otimes p} \xrightarrow{K} D^{\otimes p} \tag{4.8}$$

where K is defined in 4.3. Moreover, K is an morphism of $\mathbb{Z}[\mathbb{Z}_p]$-modules.

Proof. This statement follows from the constructions above. □

4.2.1 Proof of theorem 1.6

In this section, we prove the theorem 1.6, which is restated below. The functoriality of Khovanov homology is stated in section 2.4.3. In this section and following sections by $\simeq$, we mean chain homotopy up to a $\pm$ sign.

Theorem (1.6). *For equivariantly isotopic, equivariant link cobordisms Σ and Σ' between p-periodic link L_0 and L_1, the maps induced on equivariant Khovanov homology,*

$$\mathrm{EKh}(\Sigma), \mathrm{EKh}(\Sigma') : \mathrm{EKh}(L_0; \mathbb{Z}) \to \mathrm{EKh}(L_1; \mathbb{Z})$$

are equal up to a factor of $(\pm 1)^p$.

Proof of theorem 1.6. From theorem 1.4, given two equivariant cobordisms $\mathcal{E}, \mathcal{E}' : L_0 \to L_1$ that are equivariantly isotopic relative to the boundary, they are equivalent up to finitely many equivariant movie moves (definition 1.3). Hence, we have to prove that the maps induced by two equivariant movies that are related by an equivariant movie move, are chain homotopic up to a $\pm$ sign. Without loss of generality, we can assume that $\mathcal{E}$ and $\mathcal{E}'$ differ only by one equivariant movie move. Denote by $\boldsymbol{EM}_{\mathcal{E}} = \{E_i\}_{i=0}^k$ and $\boldsymbol{EM}_{\mathcal{E}'} = \{E_i'\}_{i=0}^k$ the equivariant movie of $\mathcal{E}$ and $\mathcal{E}'$ respectively. Hence, $E_0 = E_0'$, and we have a diskular decomposition $E_0 = R \circ (T_0, \dots, T_{p-1})$ for admissible diskular tangles $R \in \mathbb{T}^{(n,\cdots,n;0)}$ and $T_1, \dots, T_p \in \mathbb{T}^{(;n)}$. Also, $E_k = E_k'$ and we have $E_k = R \circ (S_0, \dots, S_{p-1})$ for admissible diskular tangles $R \in \mathbb{T}^{(n,\cdots,n;0)}$ and $S_1, \dots, S_p \in \mathbb{T}^{(;n)}$. Moreover, $(T_0, \dots, T_{p-1})$ and $(S_0, \dots, S_{p-1})$ are p-equivariant $(;n)$-diskular tangles ($\theta(S_i) = S_{i+1}$ and $\theta(T_j) = T_{j+1}$ for $i, j \in \mathbb{Z}_p = \{0, \dots, p-1\}$).

Let us denote $C_i = C_{\text{Kh}}(T_i)$ and $D_i = C_{\text{Kh}}(S_i)$. Both $(T_0, \dots, T_{p-1})$ and $(S_0, \dots, S_{p-1})$ being p-equivariant $(;n)$-diskular tangles implies that $C_0 \cong \dots \cong C_{p_1}$ and $D_0 \cong \dots \cong D_{p_1}$. Therefore, we denote $C = C_0 = \dots = C_{p-1}$ (respectively $D = D_0 = \dots = D_{p-1}$) and we can write the equivariant Khovanov chain complexes for E_0 and E_k by

$$C_{\text{EKh}}(E_0) = E\mathbb{Z}_p^* \underset{\mathbb{Z}[\mathbb{Z}_p]}{\otimes} \big(C_{\text{Kh}}(R \circ (T_0, \dots, T_0)\big) = E\mathbb{Z}_p^* \underset{\mathbb{Z}[\mathbb{Z}_p]}{\otimes} \big(C_{\text{Kh}}(R) \underset{\mathcal{H}^n \otimes \cdots \otimes \mathcal{H}^n}{\otimes} C^{\otimes p}\big) \quad (4.9)$$

and

$$C_{\text{EKh}}(E_k) = E\mathbb{Z}_p^* \underset{\mathbb{Z}[\mathbb{Z}_p]}{\otimes} \big(C_{\text{Kh}}(R \circ (S_0, \dots, S_{p-1})\big) = E\mathbb{Z}_p^* \underset{\mathbb{Z}[\mathbb{Z}_p]}{\otimes} \big(C_{\text{Kh}}(R) \underset{\mathcal{H}^n \otimes \cdots \otimes \mathcal{H}^n}{\otimes} D^{\otimes p}\big) \quad (4.10)$$

Given that $E\mathbb{Z}_p^*$ is isomorphic to $E\mathbb{Z}_p$ as $\mathbb{Z}[\mathbb{Z}_p]$-module, we will drop the dual sign $(\cdot^*)$ from the notation. The chain maps induced by the $\mathcal{E}$ and $\mathcal{E}'$ respectively are given by

$$C_{\text{EKh}}(\mathcal{E}) = id_{C_{\text{Kh}}(R)} \otimes C_{\text{Kh}}(\mathfrak{f}, \dots, \mathfrak{f}) \text{ and } C_{\text{EKh}}(\mathcal{E}') = id_{C_{\text{Kh}}(R)} \otimes C_{\text{Kh}}(\mathfrak{f}', \dots, \mathfrak{f}'), \quad (4.11)$$

where $\mathfrak{f}, \mathfrak{f}' : T_i \to S_i$ for $i \in \mathbb{Z}_p$ denote the composition of elementary cobordisms localized to diskular tangles S_i and T_i in the frames of the movie of $\mathcal{E}$ and $\mathcal{E}'$ respectively.

Also let us donate $f_0 = C_{\mathrm{Kh}}(\mathfrak{f})$ and $f_1 = C_{\mathrm{Kh}}(\mathfrak{f}')$. Our goal is to show the induced maps $C_{\mathrm{EKh}}(\mathcal{E})$ and $C_{\mathrm{EKh}}(\mathcal{E}')$ are equivariantly chain homotopic, possibly up to a sign.

The chain complexes $C^{\otimes p}$ and $D^{\otimes p}$ are $\mathcal{R}^n_\theta$-modules (see section 3.3). The action of θ commutes with $f_0^{\otimes p}$ and $f_1^{\otimes p}$. Hence, $f_0^{\otimes p}$ and $f_1^{\otimes p}$ are also $\mathcal{R}^n_\theta$-module morphisms. Therefore, we need to construct a $\mathcal{R}^n_\theta$-chain homotopy between $f_0^{\otimes p}$ and $f_1^{\otimes p}$.

We have the following commutative diagram in $\mathbf{Ch}_{\mathcal{R}^n_\theta}$, the category of equivariant chain complexes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{K}}(E\mathbb{Z}_p \otimes C^{\otimes p}, D^{\otimes p}) & \longleftarrow & \mathrm{Hom}_{\mathcal{K}}(E\mathbb{Z}_p \otimes C^{\otimes p}, E\mathbb{Z}_p \otimes D^{\otimes p}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{D}}(E\mathbb{Z}_p \otimes C^{\otimes p}, D^{\otimes p}) & \longleftarrow & \mathrm{Hom}_{\mathcal{D}}(E\mathbb{Z}_p \otimes C^{\otimes p}, E\mathbb{Z}_p \otimes D^{\otimes p}) \end{array} \tag{4.12}$$

Here, $\mathcal{K}$ (respectively $\mathcal{D}$) denotes the homotopy category (respectively derived category) of R^n_θ-modules.

Since $E\mathbb{Z}_p$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}_p]$-modules, there is a chain map $E\mathbb{Z}_p \to \mathbb{Z}$. If we give $E\mathbb{Z}_p \otimes D^{\otimes p}$ and $\mathbb{Z} \otimes D^{\otimes p}$ the semi-diagonal action by $\mathcal{R}^n_\theta$ (from section 3.3), there is an induced homomorphism of $\mathcal{R}^n_\theta$-modules $E\mathbb{Z}_p \otimes D^{\otimes p} \to \mathbb{Z} \otimes D^{\otimes p} \cong D^{\otimes p}$. The top and bottom horizontal arrows are induced by post-composing with this map.

By theorem 3.15, $E\mathbb{Z}_p \otimes C^{\otimes p}$ is a projective, bounded above $\mathcal{R}^n_\theta$-chain complex. Hence, lemma 4.5 implies that both of the vertical maps are isomorphisms. Moreover, $E\mathbb{Z}_p$ is a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\mathbb{Z}_p]$-module. Therefore, $E\mathbb{Z}_p \otimes D^{\otimes p}$ is quasi-isomorphic to $D^{\otimes p}$ as a $\mathcal{R}^n_\theta$-chain complex, so the bottom horizontal map is also an isomorphism. Therefore, the top horizontal map is also an isomorphism and the map $\widehat{K}$ defined in proposition 4.8, corresponds to an $\mathcal{R}^n_\theta$-chain homotopy $\tilde{k}$ between chain maps $f_0^{\otimes p}, f_1^{\otimes p} : E\mathbb{Z}_p \otimes C^{\otimes p} \to E\mathbb{Z}_p \otimes D^{\otimes p}$.

Now taking the tensor product with $C_{\mathrm{Kh}}(R)$ as a right $\mathcal{R}^n_\theta$-module, and extending the chain maps $f_0^{\otimes p}$ and $f_1^{\otimes p}$ by the identity on $C_{\mathrm{Kh}}(R)$, we have

$$\tilde{f}_i = id_{C_{\mathrm{Kh}}(R)} \underset{\mathcal{R}_\theta^n}{\otimes} f_i^{\otimes p} : C_{\mathrm{Kh}}(R) \underset{\mathcal{R}_\theta^n}{\otimes} (E\mathbb{Z}_p \otimes C^{\otimes p}) \to C_{\mathrm{Kh}}(R) \underset{\mathcal{R}_\theta^n}{\otimes} (E\mathbb{Z}_p \otimes D^{\otimes p}), \tag{4.13}$$

for $i = 0, 1$. To complete the proof we need to re-write the modules in (4.13) similar to the equations (4.9) and (4.10).

Claim. *As chain complex of Abelian groups, we have,*

$$C_{Kh}(R) \underset{\mathcal{R}_\theta^n}{\otimes} (E\mathbb{Z}_p \otimes C^{\otimes p}) \cong E\mathbb{Z}_p \underset{\mathbb{Z}[\mathbb{Z}_p]}{\otimes} \left(C_{Kh}(R) \underset{(\mathcal{H}^n)^{\otimes p}}{\otimes} C^{\otimes p}\right). \tag{4.14}$$

The same is also true for $D^{\otimes p}$ replaced with $C^{\otimes p}$ in the equation above.

Proof of claim. We can define a bijection by $n \otimes v \otimes m \mapsto v \otimes n \otimes m$. We have to check that this bijection respects the multi-linear relations on the left and right hand side of equation 4.14. Let us fix $r \in (\mathcal{H}^n)^{\otimes p}$, $m \in C^{\otimes p}$, $n \in C_{\mathrm{Kh}}(R)$, and a generator $\theta^i \in \mathbb{Z}[\mathbb{Z}_p]$ for $i \in \{0, \dots, p-1\}$. On the right hand side of (4.14) we have relations,

$$v \otimes n \otimes rm = v \otimes nr \otimes m, \tag{4.15}$$

and

$$\begin{aligned} v \cdot \theta^i \otimes n \otimes m &= v \otimes \theta^i \cdot (n \otimes m) \\ &= v \otimes \theta^i \cdot n \otimes \theta^i \cdot m. \end{aligned} \tag{4.16}$$

On the left hand side of (4.14), we have,

$$n \cdot (r\theta^i) \otimes v \otimes m = n \otimes (r\theta^i) \cdot (v \otimes m) = n \otimes \theta^i \cdot v \otimes r\theta^i \cdot m, \tag{4.17}$$

where the second equality in the above, is by the semi-diagonal $\mathcal{R}_\theta^n$-module structure on $E\mathbb{Z}_p \otimes C^{\otimes p}$. Moreover, $C_{\mathrm{Kh}}(R)$ is a right $\mathcal{R}_\theta^n$-module and $C^{\otimes p}$ is a left $\mathcal{R}_\theta^n$-module but a left $\mathbb{Z}[\mathbb{Z}_p]$-module and the $\mathcal{R}_\theta^n$ action is defined by

$$n \cdot (r\theta^i) = \theta^{-i} \cdot (nr) \tag{4.18}$$

Hence, for the left term in the equation (4.17), we have

$$\begin{aligned}\theta^{-i} \cdot (nr) \otimes v \otimes m \mapsto v \otimes \theta^{-i} \cdot (nr) \otimes m \\ &= v \otimes \theta^{-i} \cdot (nr) \otimes \theta^{-i} \cdot (\theta^{i} \cdot m) \\ &= v \otimes \theta^{-i} \cdot (nr \otimes \theta^{i} \cdot m) \\ &= v \cdot \theta^{-i} \otimes nr \otimes \theta^{i} \cdot m \\ &= \theta^{i} \cdot v \otimes n \otimes r\theta^{i} \cdot m\end{aligned}$$

On the other hand, for the term on the right in the equation (4.17), we have

$$n \otimes \theta^{i} \cdot v \otimes r\theta^{i} \cdot m \mapsto \theta^{i} \cdot v \otimes n \otimes r\theta^{i} \cdot m$$

Therefore, the result follows. □

Now by the claim above, we can write the chain complexes as follows.

$$\tilde{f}_i : E\mathbb{Z}_p \otimes (V \otimes_{(\mathcal{H}^n)^{\otimes p}} C^{\otimes p}) \to E\mathbb{Z}_p \otimes (V \otimes_{(\mathcal{H}^n)^{\otimes p}} D^{\otimes p}). \tag{4.19}$$

The maps f_0 and f_1 that are induced by the before and the after movie of a movie move are chain homotopic up to a sign [Kho06; Bar05; Jac04]. Hence $f_0^{\otimes p}$ and $f_1^{\otimes p}$ are equivariant chain homotopic up to a factor of $(-1)^p$. In conclusion,

$$C_{\mathrm{EKh}}(\Sigma) = id_{C_{\mathrm{Kh}}(R)} \otimes C_{\mathrm{Kh}}(\mathfrak{f}, \ldots, \mathfrak{f}) = id_{C_{\mathrm{Kh}}(R)} \otimes f_0^{\otimes p} \tag{4.20}$$

and

$$C_{\mathrm{EKh}}(\Sigma') = id_{C_{\mathrm{EKh}}(R)} \otimes C_{\mathrm{Kh}}(\mathfrak{f}', \ldots, \mathfrak{f}') = id_{C_{\mathrm{Kh}}(R)} \otimes f_1^{\otimes p} \tag{4.21}$$

are equivariantly chain homotopic up to $(-1)^p$. □

4.3 Obstruction to equivariant concordance

This section is devoted to proving theorem 1.7. Some of the notations are similar to the section 2.4.4.

4.3.1 Equivariant neck cutting relation

Let $\mathrm{K}ob^2_{p,\bullet} \subset \mathcal{K}ob^2_\bullet$ denote the $\mathbb{Z}$-linearized category with objects the collection of p-periodic links. The morphisms of $\mathrm{K}ob^2_{p,\bullet}$ consist of finite formal sum of cobordisms (surfaces) in $\mathbb{R}^3\times[0,1]$ with a choice of finitely many dots on the cobordisms away from their boundaries. These cobordisms are not a priori equivariant under the extended action of $\tilde{\theta}$ but $\mathbb{Z}_p$ acts on formal sums of cobordism by $\tilde{\theta}$ extended linearly. We denote by $(\mathrm{K}ob^2_{p,\bullet})^{\mathbb{Z}_p}$ the fixed set of the action $\tilde{\theta}$.

Additionally, let $\mathbb{Z}_p\mathcal{K}ob^2$ denote the $\mathbb{Z}$-linear category with objects the p-periodic links in $\mathbb{R}^3\backslash\boldsymbol{z}$. Morphisms are the formal linear combination of p-equivariant cobordisms in $(\mathbb{R}^3\backslash\boldsymbol{z})\times[0,1]$ (definition 1.2).

Definition 4.9. Assume $\mathcal{E}$ is a p-equivariant cobordism, and h_i are smooth, embedded 1-handles $[-1,1]\times\mathbb{D}^2 \to \mathbb{R}^3\times[0,1]$. By a *standard p-equivariant neck* $\tilde{\mathfrak{n}} = (\mathfrak{n}_1, \dots, \mathfrak{n}_p)$ on $\mathcal{E}$, we are referring to a collection of p disjoint standard necks $\mathfrak{n}_i = h_i \cap \mathcal{E}$ (see definition 2.22), such that

(EN1) The h_i are permuted by the action $\tilde{\theta}$

(EN2) The h_i are disjoint from the $\boldsymbol{z}\times[0,1]$.

We can define *equivariant neck cutting* as follows. First, for an equivariant cobordism $\Sigma \in \mathbb{Z}_p\mathcal{K}ob^2$, we remove the necks $\mathfrak{n}_i = [-1,1]\times S^1_i \hookrightarrow \Sigma$ for $i = 1, \dots, p$. Then for $i = 1, \dots, p$, we glue disks $\{-1,1\}\times\mathbb{D}^2_i$ to the cutout boundaries obtained from cutting

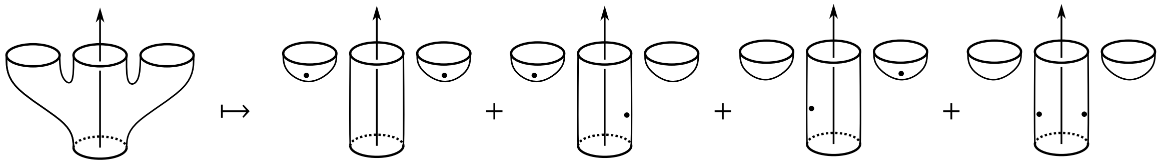


Figure 4.2: An example of the 2-Equivariant neck cutting. The dotted cobordism depicted on the right hand side is $\mathcal{E}_{(+,+)} + \mathcal{E}_{(+,-)} + \mathcal{E}_{(-,+)} + \mathcal{E}_{(-,-)}$.

$\mathfrak{n}_i$. Lastly, for each $1 \leq i \leq p$, we either place a dot on $\{1\} \times \mathbb{D}^2_i$ or on $\{-1\} \times \mathbb{D}^2_i$ and denote the resulting dotted cobordism by $\Sigma_{(\iota_1,\ldots,\iota_{i-1},+,\iota_{i+1},\ldots,\iota_p)}$ if the dot is on $\{1\} \times \mathbb{D}^2_i$ or respectively by $\Sigma_{(\iota_1,\ldots,\iota_{i-1},-,\iota_{i+1},\ldots,\iota_p)}$ if the dot is placed on $\{-1\}\times\mathbb{D}^2_i$. Here $\iota_1, \ldots, \iota_p \in \{+,-\}$. To wit, the result of equivariant neck cutting on an standard equivariant neck $\tilde{\mathfrak{n}}$ on equivariant cobordism Σ can be understood as a map that sends $\Sigma \in \mathbb{Z}_p\mathcal{K}ob^2$ to the formal sum

$$\sum_{(\iota_1,\ldots,\iota_p)\in\{+,-\}^p} \Sigma_{(\iota_1,\ldots,\iota_p)} \in (\mathrm{K}ob^2_{p,\bullet})^{\mathbb{Z}_p}. \tag{4.22}$$

For instance, in the notation above, the $\Sigma_{(+,\cdots,+)}$ indicates that after all of the necks $\mathfrak{n}_i$ were removed and we placed a dot on $\{+1\} \times \mathbb{D}^2$ for each $1 \leq i \leq p$. More generally, after equivariant neck cuts on a p-equivariant cobordism Σ, we have 2^p choices for placements of dots on the glued disks $(\{\pm1\} \times \mathbb{D}^2)^p_{i=1}$. Figure 4.2 shows the equivariant neck cutting on a 2-equivariant cobordism Σ.

Proposition 4.10. *Assume Σ is an equivariant cobordism and $\mathcal{E}$ denotes the equivariant neck cutting of Σ (4.22). Then, equivariant Khovanov homology satisfies the equivariant neck cutting relation i.e.,*

$$C_{EKh}(\Sigma) \simeq \sum_{[\iota]\in\{+,-\}/\mathbb{Z}_p} C_{EKh}\Big(\sum_{(\iota_1,\ldots,\iota_p)\in[\iota]} \Sigma_{(\iota_1,\ldots,\iota_p)}\Big) \tag{4.23}$$

where $\simeq$ chain homotopy up to a sign as was mentioned earlier in this chapter, and $[\iota]$ denotes the class of $\iota \in \{+,-\}^p$ modulo the action of $\mathbb{Z}_p$ by cyclic permutation on $\{+,-\}^p$.

Proof. For simplicity, we can assume that Σ consists of only one standard equivariant neck $\tilde{\mathfrak{n}} = (\mathfrak{n}_i)_{i=1}^{p}$ such that the equivariant movie of Σ is given by considering p copies of the movie in figure 2.10 (a). The general case follows from the simple case inductively. Let $E_i = R \circ (T_{1,i}, \cdots, T_{p,i})$ for $i = 0, 1, 2$ denotes the equivariant movie of the Σ, where R is a $(n, \cdots, n; 0)$-diskular tangle and for $i = 1, \ldots, p$, diskular tangles $T_{0,i}, T_{1,i}$ and $T_{0,i}$ are the $(; 4)$-diskular tangle in figure 2.10 (a) from left to right respectively. The elementary cobordism $\mathfrak{f}_i : T_{0,i} \to T_{2,i}$ (definition 2.5) for $i = 1, \ldots, p$ consists of two elementary saddle moves.

There are induced chain maps $C_{\text{Kh}}(id_R) = id_{C_{\text{Kh}}(R)}$, and

$$\bigotimes_{i=1}^{p} C_{\text{Kh}}(\mathfrak{f}_i) : C_{\text{Kh}}(T_{0,1}) \otimes \ldots \otimes C_{\text{Kh}}(T_{0,p}) \to C_{\text{Kh}}(T_{2,1}) \otimes \ldots \otimes C_{\text{Kh}}(T_{2,p}),$$

where $C_{\text{Kh}}(R)$ and $C_{\text{Kh}}(T_{j,i}, \ldots, T_{j,i})$ are $\mathcal{R}_\theta^n$-modules. Because Khovanov homology satisfies the neck cutting relation we can write

$$\begin{aligned} C_{\text{Kh}}(\Sigma) &= \big(C_{\text{Kh}}(\Sigma_{1,+}) + C_{\text{Kh}}(\Sigma_{1,-})\big) \otimes \cdots \otimes \big(C_{\text{Kh}}(\Sigma_{p,+}) + C_{\text{Kh}}(\Sigma_{p,-})\big) \\ &= \sum_{(\iota_1,\ldots,\iota_p)\in\{+,-\}^p} \bigotimes_{j=1}^{p} C_{\text{Kh}}(\Sigma_{\iota_j}) \\ &= \sum_{(\iota_1,\ldots,\iota_p)\in\{+,-\}^p} C_{\text{Kh}}(\Sigma_{(\iota_1,\ldots,\iota_p)}) \end{aligned} \tag{4.24}$$

where $C_{\text{Kh}}(\Sigma_{i,+}) + C_{\text{Kh}}(\Sigma_{i,-})$ denotes the result of neck cutting on the neck $\mathfrak{n}_i$ for $i = 1, \ldots, p$. Also, the notation Σ_{ι_j} denotes Σ after the all of the p necks have replaced by dotted disks according to $\iota_j \in \{+, -\}^p$. By the action of $\mathbb{Z}_p$ on $\{+, -\}^p$, for each $[\iota'] \in \{+, -\}^p/\mathbb{Z}_p$, the map induced by $\sum_{\iota\in[\iota']} C_{\text{Kh}}(\Sigma_\iota)$ is equivariant. Therefore, we can re-write the sum above over the equivalence classes $\{+, -\}^p/\mathbb{Z}_p$ and we get the equation (4.23), as desired. □

4.3.2 Map induced by equivariant ribbon concordance

Now we can prove theorem 1.7. Our proof mirrors that of [LZ19].

Definition 4.11. An *equivariant ribbon concordance* from a p-periodic knot K_0 to a p-periodic knot K_1 is a smoothly embedded annulus $F \subset (\mathbb{R}^3 \backslash z) \times [0, 1]$ such that the projection $\rho : (\mathbb{R}^3 \backslash z) \times [0, 1] \to [0, 1]$ restricted to F is a Morse function. Moreover,

1. It is invariant under the extended action $\tilde{\theta}$ on $S^3 \times [0, 1]$.
2. It has only index 0 and 1 critical points.

Theorem (1.7)**.** *Fix an equivariant ribbon concordance F between periodic knots K and K'. The map induced by F on equivariant Khovanov homology is a split injection.*

Proof of theorem 1.7. By theorem 1.6, we can equivariantly isotope $F : K_0 \to K_1$ such that the movie of F has the following order. The index 0 critical points (births) appear first, followed by a sequence of equivariant planer isotopy and equivariant Reidemeister moves. Lastly, F has index 1 critical points (saddles), where the saddles can be viewed as attaching unknotted equivariant bands between two strands of the link diagram.

By F^{op} we denote the ribbon concordance F with the opposite orientation. Gluing F along K_1 to F^{op} on K_1 results in the cobordism $F^{\text{op}} \circ F$ from K_0 to itself. Both F and F^{op} are equivariant. Hence, $\Sigma = F^{\text{op}} \circ F$ is an equivariant cobordism. Then Σ can be structured as a finite collection of equivariant unknotted 2-spheres $\{S^2_j\}$ (obtained from index 0 and 2 critical points of Σ) tubed to annulus $K_0 \times [0, 1]$ by a finite collection of standard equivariant necks (obtained from the index 1 critical points of F and F^{op})

For simplicity of notation, we assume that F has only one equivariant index 0 critical point (which consists of p distinct index 0 critical points that correspond by the action $\tilde{\theta}$), and only one equivariant index 1 critical point. The general case would follow by applying the same argument inductively to a general collection of equivariant necks and spheres.

We can perform equivariant neck cutting to eliminate the equivariant neck. Let $\mathcal{E}$ denote the resulting cobordism, which viewed as an element of $(\mathcal{K}ob^2_{\bullet})^{\mathbb{Z}_p}$ is a finite formal sum of cobordisms consisting of finitely many equivariant embedded, unknotted 2-spheres with dots $(S^2, \cdots, S^2)$, and an embedded cylinder C' with dots and not linked with the $(S^2, \cdots, S^2)$ above.

$$\mathcal{E} = \sum_{(\iota_1,\ldots,\iota_p)\in\{\bullet,\circ\}^p} C'_{(\bar{\iota}_1,\ldots,\bar{\iota}_p)} \cup (S^2_{\iota_1}, \ldots, S^2_{\iota_p}) \tag{4.25}$$

where $\bar{\iota}_j = \bullet$ if $\iota_j = \circ$ and, $\bar{\iota}_j = \circ$ if $\iota_j = \bullet$. Also, $S^2_{\iota_j}$ is a dotted sphere if $\iota_j = \bullet$ and is a sphere with no dots if $\iota_j = \circ$.

Not all of the terms in the equation 4.25 are individually equivariant dotted cobordisms. In the formal sum above, the cobordism $C'_{\bullet,\cdots,\bullet} \cup (S^2_\circ, \cdots, S^2_\circ)$ which has all of the dots on C' is an equivariant dotted cobordism. Similarly, the cobordism $C'_{\circ,\cdots,\circ} \cup (S^2_\bullet, \cdots, S^2_\bullet)$ which has all of the dots on the spheres is also an equivariant dotted cobordism. The other terms are not taken to themselves by the action of $\tilde{\theta}$. Moreover, the extended action $\tilde{\theta}$ acts on $(S^2_{\iota_1}, \ldots, S^2_{\iota_n})$ by cyclic permutation of the factors. Let $[\iota] \in \{\bullet, \circ\}^p/\mathbb{Z}_p$ denote the orbits of the action of $\mathbb{Z}_p$ on $\{\bullet, \circ\}^p$ by cyclic permutation. Then, for each $[\iota]$,

$$\mathcal{E}_{[\iota]} = \sum_{(\iota_1,\ldots,\iota_p)\in[\iota]} C'_{(\bar{\iota}_1,\ldots,\bar{\iota}_p)} \cup (S^2_{\iota_1}, \ldots, S^2_{\iota_p}) \tag{4.26}$$

is fixed by $\tilde{\theta}$, so is an equivariant cobordism.

The dotted cobordism $C'_{\bullet,\cdots,\bullet} \cup (S^2_\circ, \cdots, S^2_\circ)$ (respecively $C'_{\circ,\cdots,\circ} \cup (S^2_\bullet, \cdots, S^2_\bullet)$) is equivariantly isotopic to $C_{\bullet,\cdots,\bullet} \cup (S^2_\circ, \cdots, S^2_\circ)$ (respectively $C_{\circ,\cdots,\circ} \cup (S^2_\bullet, \cdots, S^2_\bullet)$), where $C_{\bullet,\cdots,\bullet}$ (respectively $C_{\circ,\cdots,\circ}$) denotes the standard cylinder $K_0 \times [0, 1]$ with p dots (respectively with no dots) which is split from the spheres $(S^2_\circ, \cdots, S^2_\circ)$ (respectively $(S^2_\bullet, \cdots, S^2_\bullet)$). We have

$$\begin{aligned} C_{\mathrm{EKh}}(C'_{\bullet,\cdots,\bullet} \cup (S^2_\circ, \cdots, S^2_\circ)) &\simeq C_{\mathrm{EKh}}(C_{\bullet,\cdots,\bullet} \cup (S^2_\circ, \cdots, S^2_\circ)) \\ &= id_{E\mathbb{Z}_p} \otimes_{\mathbb{Z}[\mathbb{Z}_p]} C_{\mathrm{Kh}}(C_{\bullet,\cdots,\bullet} \sqcup (S^2_\circ, \ldots, S^2_\circ)) \\ &= 0, \end{aligned} \tag{4.27}$$

where the first chain homotopy is by theorem 1.6. Note that the induced map $C_{\mathrm{Kh}}(S^2_\circ) : \mathbb{Z} \to \mathbb{Z}$ by a sphere with no dots is zero, and the map $C_{\mathrm{Kh}}(C_{\bullet,\cdots,\bullet})$ induced by a cobordism with more than one dot is null homotopic. Also,

$$\begin{aligned} C_{\mathrm{EKh}}(C'_{\circ,\cdots,\circ} \cup (S^2_\bullet, \cdots, S^2_\bullet)) &\simeq C_{\mathrm{EKh}}(C_{\circ,\cdots,\circ} \cup (S^2_\bullet, \cdots, S^2_\bullet)). \\ &= id_{E\mathbb{Z}_p} \otimes_{\mathbb{Z}[\mathbb{Z}_p]} C_{\mathrm{Kh}}(C_{\circ,\cdots,\circ} \sqcup (S^2_\bullet, \ldots, S^2_\bullet)). \\ &= id_{E\mathbb{Z}_p} \otimes_{\mathbb{Z}[\mathbb{Z}_p]} \Big(id_{C_{\mathrm{Kh}}(K_0)} \otimes (C_{\mathrm{Kh}}(S^2_\bullet) \otimes \ldots \otimes C_{\mathrm{Kh}}(S^2_\bullet)) \Big). \\ &= id_{C_{\mathrm{EKh}}(K_0)}. \end{aligned} \tag{4.28}$$

This follows because the map $C_{\mathrm{Kh}}(S^2_\bullet) : \mathbb{Z} \to \mathbb{Z}$ induced by a sphere with one dot is multiplication by $1 \in \mathbb{Z}$.

For a fixed $[\eta] \in \{\bullet, \circ\}^p/\mathbb{Z}_p - \{[\bullet, \cdots, \bullet], [\circ, \cdots, \circ]\}$ and $\iota \in [\eta]$, let f_s for $s \in [0, 1]$ denote the isotopy that transforms $C'_{(\bar{\iota}_1,\ldots,\bar{\iota}_p)} \cup (S^2_{\iota_1}, \ldots, S^2_{\iota_p})$ to an ι-dotted cobordism $C_{(\bar{\iota}_1,\ldots,\bar{\iota}_p)} \cup (S^2_{\iota_1}, \ldots, S^2_{\iota_p})$ with C the standard cylinder $K_0 \times [0, 1]$ and disjoint from $(S^2_{\iota_1}, \ldots, S^2_{\iota_p})$. We can equivariantly isotope the equivariant cobordisms (4.26) by $\tilde{f}_s = \sum_{\theta\in\mathbb{Z}_p} \hat{\theta}^{-1} f_s \hat{\theta}$. The $\tilde{f}_s : \mathcal{E}_{[\iota]} \to \mathcal{E}_{[\iota]}$ is an equivariant map with respect to the extended action $\tilde{\theta}$ for all $s \in [0, 1]$. Therefore, $\tilde{f}_s$ induces a $\mathbb{Z}[\mathbb{Z}_p]$-chain homotopy

$$C_{\mathrm{EKh}}(\mathcal{E}_{[\iota]}) \simeq C_{\mathrm{EKh}}(\mathfrak{E}_{[\iota]}), \tag{4.29}$$

where

$$\mathfrak{E}_{[\eta]} = \sum_{(\iota_1,\ldots,\iota_p)\in[\eta]} C_{(\bar{\iota}_1,\ldots,\bar{\iota}_p)} \cup (S^2_{\iota_1}, \ldots, S^2_{\iota_p}). \tag{4.30}$$

Because the chain maps induced on the Khovanov chain complex by a cobordism consisting of more than one dot on a connected component, and the cobordism consisting of a 2-sphere with no dots as a connected component are nullhomotopic, we have

$$\begin{aligned} C_{\mathrm{EKh}}(\mathcal{E}_{[\iota]}) &\simeq C_{\mathrm{EKh}}(\mathfrak{E}_{[\iota]}) \\ &= C_{\mathrm{EKh}}(\sum_{(\iota_1,\ldots,\iota_p)\in[\eta]} C_{(\bar{\iota}_1,\ldots,\bar{\iota}_p)} \cup (S^2_{\iota_1},\ldots,S^2_{\iota_p})) \\ &\simeq 0 \end{aligned}$$

for all $\eta \neq (\bullet, \cdots, \bullet), (\circ, \ldots, \circ)$. Hence we have

$$\begin{aligned} C_{\mathrm{EKh}}(F^{op} \circ F) &\simeq C_{\mathrm{EKh}}\big(\sum_{(\iota_1,\ldots,\iota_p)\in\{\bullet,\circ\}^p} C'_{(\bar{\iota}_1,\ldots,\bar{\iota}_p)} \cup (S^2_{\iota_1},\ldots,S^2_{\iota_p})\big) \\ &\simeq \sum_{[\iota]\in\{\bullet,\circ\}/\mathbb{Z}_p} C_{\mathrm{EKh}}\big(\sum_{(\iota_1,\ldots,\iota_p)\in[\iota]} C_{(\bar{\iota}_1,\ldots,\bar{\iota}_p)} \cup (S^2_{\iota_1},\ldots,S^2_{\iota_p})\big) \\ &= C_{\mathrm{EKh}}(C_{\circ,\cdots,\circ} \cup (S^2_{\bullet},\cdots,S^2_{\bullet})) = id_{C_{\mathrm{EKh}}(L_0)} \end{aligned}$$

This finishes the proof. □

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