THE HOOK FORMULA: DETERMINING THE NUMBER OF STANDARD TABLEAUX FOR A PARTITION OF A NATURAL NUMBER

by

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The partitioning of natural numbers leads to the discussion of Ferrers diagrams and standard tableaux. Hooks are useful shapes within such diagrams consisting of a cell and all cells directly below it and directly to its right. The Hook Formula is a simple way to determine the number of standard tableaux for a given partition of n and is given by $n!$ divided by the product of hook lengths. Explorations of the Hook Formula show its possible usefulness in fields such as computer science and contextualize its history in mathematics. A bijection between the set of general tableaux and the set of ordered pairs of standard tableaux and a record keeping array show how the Hook Formula comes about. Lastly presenting an algorithm to create a standard tableau and showing that the probability of generating a specific tableau via this process is the multiplicative inverse of the formula proves the Hook Formula itself.

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SECTION I

INTRODUCTION

In this thesis, I shall explain a theorem called the Hook Formula, which is used for determining the total number of standard Young tableaux of a partition. I shall also present a proof of this theorem that extensively uses probability and an explanation based on a bijection. In addition I will explore the possible applications, uses, and significance of this formula, and of partitions and tableaux in general. My intent is to also explore the context from which this formula was derived, and give a brief history of its discovery and of its several proofs.

The purpose behind this exploration is to make a fairly complex mathematical theorem and its proof more accessible to an audience that does not have an in-depth background in math. Although some of the mathematical structures and processes involved in the Hook Formula can be very complicated, I shall build and explain each step and structure as I use them and define each term as I go, thereby making this theorem and proof more readable and comprehensible to someone who has not studied math extensively. This thesis is not for a completely lay audience and will likely require a basic familiarity with

college algebra to grasp fully the theorem, the concepts that follow, and the steps I will take in explaining and proving it, but this thesis is for the most part "self contained." All unusual structures, definitions, and notations are explained as they are introduced.

I would also like to expose the process of rigorous proof and definition to non-mathematicians, but in an intricate and nontrivial way. In the purest of math, nothing can go unproven or undefined - we must build and verify every tool and structure we use in order for our proof to be truly rigorous, and for the proof to really "prove" something.

I would also like to present a theorem and its proof in a more holistic manner than most math papers do. Most publications in mathematics, including the pre-existing works on the Hook Formula, assume a certain base (and sometimes very advanced) knowledge of math at the onset. This creates a sense of elitism since one has to have studied for years just to read about one formula, and it cuts off many mathematical tools from the general population. Most math papers that focus on specific topics like the Hook Formula also leave their results in isolation and do not relate them to other branches of math or to other disciplines - they give no supporting information to

contextualize their conclusions. In this paper I hope to eliminate the exclusivity of mathematics treatises and put the Hook Formula in a greater mathematical and practical context by presenting some applications to other mathematical structures and to other disciplines.

This thesis, then, is a synthesis of works on the Hook Formula - compiling multiple explanations, applications, and history into one treatise for the first time. We will begin with an introduction to the natural numbers and build each of the pertinent structures as we proceed. We will also see a number of examples of tableaux and of the Hook Formula and explore this formula from various perspectives.

SECTION II

THE BASICS: PARTITIONS, DIAGRAMS, AND TABLEAUX

Let us begin with some definitions, starting with the natural numbers. The set of *natural numbers*, denoted $\mathbf{N}$, is the set $\{1, 2, 3, 4, \ldots\}$, that is, the set of positive integers. It is an infinite, but countable, set. A *partition* of a particular natural number n is a sequence of weakly decreasing natural numbers whose sum is n. That is, if $n \in \mathbf{N}$ (the symbol "$\in$" means "exists in" or "is an element of"), then a partition λ of length k for n is a sequence $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_k$ and $\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_k = n$. The *length* of a partition λ is the number of elements the sequence contains and is denoted $|\lambda|$. For example, given the natural number $n = 5$, one possible partition is $\lambda = (3, 1, 1)$. Note that $3 \geq 1 \geq 1$ and that $3 + 1 + 1 = 5$, hence, this sequence satisfies the definition of a partition. This particular partition of 5 has length $|\lambda| = 3$. Some of the other possible partitions of 5 are the sequence $\mu = (3, 2)$, which has length $|\mu| = 2$ and the sequence $\nu = (2, 1, 1, 1)$ with length $|\nu| = 4$.

One common way to depict a partition of this nature is with a Ferrers diagram*. A *Ferrers diagram* for a partition λ of length k for n is an array of connected, rectangular cells with k "left-justified rows and λ_i cells in row i" [4]. This means that a Ferrers diagram for a particular λ is a picture of n cells all lined up in rows with the number of cells in each row corresponding to the numbers in the partition λ, and all the rows are lined up to the left. For example, the partitions from above could be pictured with Ferrers diagrams as in Figure 1.

$\lambda = (3, 1, 1)$ $\mu = (3, 2)$



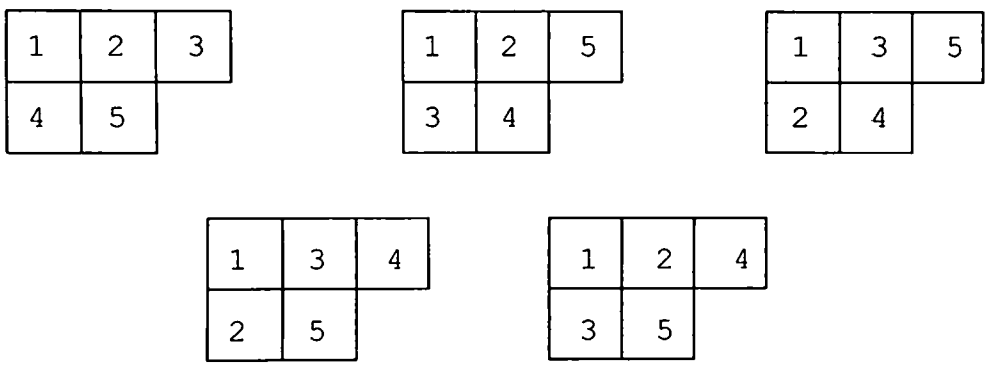
[Figure 1: Two Ferrers diagrams for different partitions of 5]

A *standard Young tableau*†, also often just called a *standard tableau*, for a given partition λ of n is a Ferrers diagram for the partition in which each cell is filled in with a distinct natural number 1 through n such that as you read each row from left to right the numbers increase and

* The name "Ferrers diagram" is in honor of the 19th Century mathematician and physicist Norman Macleod Ferrers (1829-1903).

† The term "standard Young tableau" is in honor of the mathematician Alfred Young (1873-1940).

as you read each column from top to bottom the numbers also increase. (A side note: There are also *non-standard* or *general tableaux*, which are Ferrers diagrams for the partition that have been filled in with the numbers 1 through n in any order, and we will work with these structures some in Section V.) Figure 2 shows examples of different standard Young tableaux for the above example with $\mu = (3, 2)$.



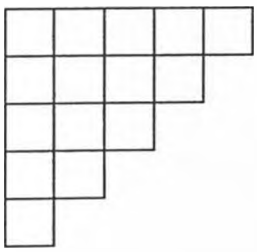
[Figure 2: The 5 different standard Young tableau for the partition $\mu = (3, 2)$ of 5]

Note that in each column the numbers increase as you move down, and in each row the numbers increase as you move to the right. A *corner cell* is a cell that is the farthest to the right in its row and the farthest down in its column. A different but equivalent way to define this is as follows: A corner cell of a Ferrers diagram is a cell with the property that if it is removed, the remaining shape is still a Ferrers diagram (though for a different partition).

In the first example of Figure 2, the corner cells are the two boxes holding the numbers 3 and 5. Note also that the number 1, being the lowest, always must lie in the upper-most left cell and that the number *n* must always lie in a corner.

Clearly for most partitions, there are multiple ways to number these cells to form standard Young tableaux. Check for yourself that there are no standard Young tableaux for the partition (3, 2) of 5 other than those pictured in Figure 2.

In fact, as a partition gets larger, the number of different standard Young tableaux for it gets extremely large very quickly. Try working out the standard Young tableaux for the partition $\lambda = (5, 4, 3, 2, 1)$ of 15 whose Ferrers diagram is pictured in Figure 3.



[Figure 3: The Ferrers diagram for the partition $\lambda = (5, 4, 3, 2, 1)$ of 15]

You will soon discover that there a far more ways to number this partition to form standard Young tableaux than you

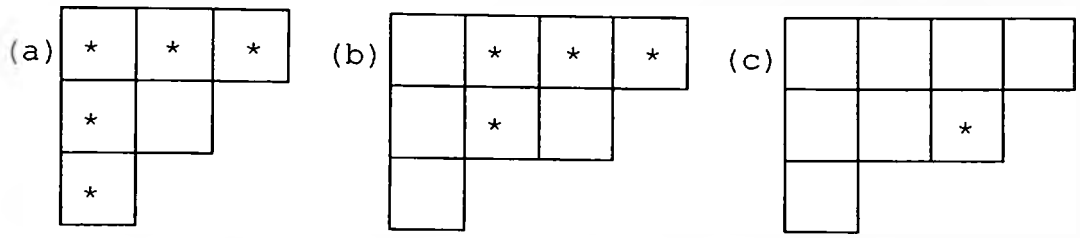
would ever care to figure out. In fact, there are 292,864 different ways to number this diagram to form standard Young tableaux! Obviously, we do not want to figure out every one of these tableaux just to find out how many there are, and we would hope that there is a simple formula to help us. And, magically enough, there is.

SECTION III

THE HOOK FORMULA

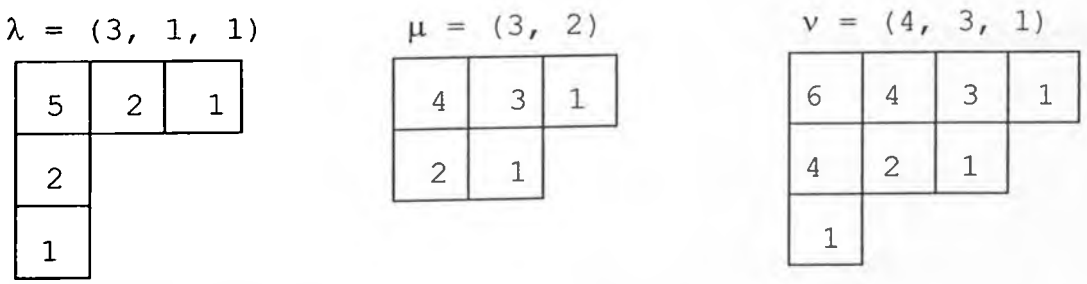
The *Hook Formula* is a formula that allows us to easily calculate the total number of distinct standard Young tableaux for any given partition. But before we can examine this formula, we need to explore the shapes called hooks.

A *hook* is a portion of the Ferrers diagram that includes a given cell, all the cells below it in the same column, and all the cells to its right in the same row. We call a cell in the i^{th} row and the j^{th} column of the Ferrers diagram the cell (i, j). For each cell (i, j), then, we have a hook which we will denote $H_{i,j}$. In set notation, $H_{i,j}$ is the union $\{(i, j')$ such that $j' \geq j\} \cup \{(i', j)$ such that $i' \geq i\}$. This notation is taken from [9] and is merely another way to denote that $H_{i,j}$ is a cell along with all the cells straight below it and straight to its right. For example, Figure 4 shows depictions of several hooks for two different Ferrers diagrams.



[Figure 4: Several Ferrers diagrams with some hooks denoted. (a) Hook $H_{1,1}$ for $\lambda = (3, 2, 1)$; (b) hook $H_{1,2}$ for $\mu = (4, 3, 1)$; (c) hook $H_{2,3}$ for $\mu = (4, 3, 1)$]

So every diagram of a partition λ for a natural number n has exactly n hooks since there is exactly one hook per cell in the tableau for λ. Each hook $H_{i,j}$ has *hook length* $|H_{i,j}| = h_{i,j}$, which is the total number of cells in the hook. From the examples in Figure 4, (a) $H_{1,1}$ has hook length $h_{1,1} = 5$, (b) $H_{1,2}$ has hook length $h_{1,2} = 4$, and (c) $H_{2,3}$ has hook length $h_{2,3} = 1$. Figure 5 shows several Ferrers diagrams in which each cell (i, j) is filled in with its corresponding hook length $h_{i,j}$.



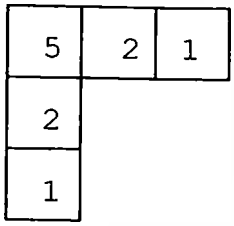
[Figure 5: Three Ferrers diagrams with the appropriate hook length values in each cell]

Now we have enough information to be able to look at the Hook Formula directly. Recall that this formula gives us the total number of standard Young tableaux for a given partition λ of n, denoted f^{λ}. In other words, f^{λ} is the number of different ways that you can place the numbers 1 to n into a Ferrers diagram such that each row increases as you move right and each column increases as you move down. So the Hook Formula is as follows: If n is a natural number and if λ is a partition of n, then the total number of standard Young tableaux for λ is

$$f^{\lambda} = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}} .$$

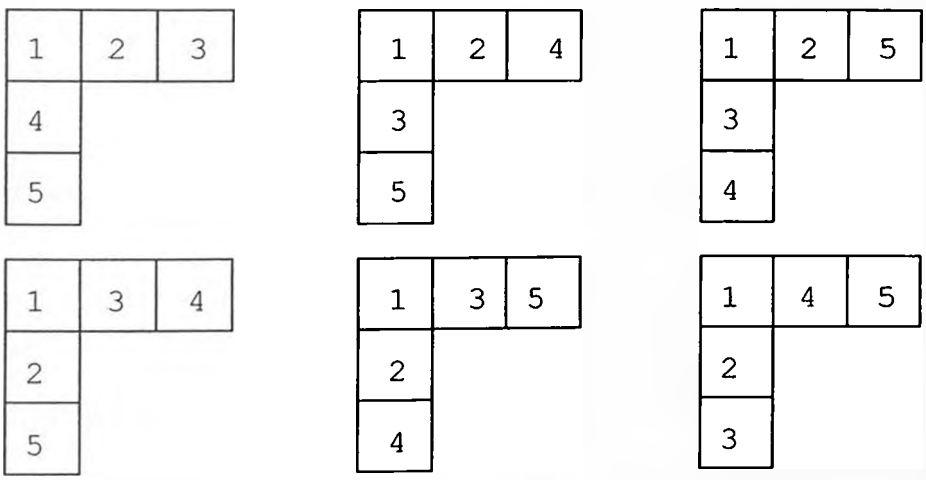
The notation "$n!$" reads "n factorial," and means n multiplied by all of the natural numbers below it, so, $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n - 1) \cdot n$. The notation "$\prod_{(i,j)\in\lambda} h_{i,j}$" is shorthand for "the product of all $h_{i,j}$ such that $(i, j) \in \lambda$", or in other words, the product of all the hook lengths of the partition.

For example given the partition $\lambda = (3, 1, 1)$ of $n = 5$, we have the following hook lengths for each cell:



[Figure 6: The hook lengths for λ = (3, 1, 1) filled in to their appropriate cells]

So according to the Hook Formula, this partition has a total number of $f^{\lambda} = (5!) / (5 \cdot 2 \cdot 2 \cdot 1 \cdot 1)$ standard Young tableaux. That is, $f^{\lambda} = (5 \cdot 4 \cdot 3 \cdot 2) / (5 \cdot 4) = (3 \cdot 2) = 6$. Hence there are six standard Young tableaux for this shape. Let us verify this by listing them, as depicted in Figure 7.



[Figure 7: The six standard Young tableaux for the partition λ = (3, 1, 1)]

For small partitions such as this, it is easy to list all the combinations of numbers to get the standard Young tableaux, but for larger partitions, it is much more

difficult and time consuming, hence the usefulness of the Hook Formula. Hook lengths are quick and easy to determine, so with this formula, we can establish the number of combinations without figuring them all out. For example, given the partition $\lambda = (6, 5, 3, 3, 1)$ of $n = 18$, we have the following hook lengths for each cell:

10	8	7	4	3	1
8	6	5	2	1	
5	3	2			
4	2	1			
1					

[Figure 8: The hook lengths for $\lambda = (6, 5, 3, 3, 1)$ filled in to their appropriate cells]

Therefore f^{λ} for this partition is $(18!) / (10 \cdot 8 \cdot 7 \cdot 4 \cdot 3 \cdot 1 \cdot 8 \cdot 6 \cdot 5 \cdot 2 \cdot 1 \cdot 5 \cdot 3 \cdot 2 \cdot 4 \cdot 2 \cdot 1 \cdot 1) =$ 8,270,262. This partition of 18, thought not very large and with length of only 5, has over 8 million different standard Young tableaux! Clearly finding each of these tableaux would be very time consuming and difficult. For big partitions of large natural numbers, it can even be quite time consuming for computers to find each of the standard tableaux. The partition $\lambda = (50, 20, 10, 5, 5, 5,$

3, 2) of $n = 100$ helps show how even a computer might run into difficulty evaluating every tableaux to determine if it is standard. There are more than 9.3×10^{157} permutations of the numbers 1 through 100 that could fit into this λ-diagram that the computer would have to check, and about 4.5×10^{55} of these are standard tableaux. The Hook Formula, however, allows us to determine exactly how many there are, without figuring out each of them.

SECTION IV

APPLICATIONS, HISTORY, AND CONTEXT

Knowing these numbers of combinations has many useful applications in fields such as information storage and computer science. Computers have limited processing capabilities and must frequently partition large pieces of information into smaller, more manageable, and well-organized chunks, hence the importance of partitioning. Furthermore, shapes and arrays of numbers such as standard Young tableaux are helpful in keeping information organized and accessible. The Hook Formula becomes useful, then, in finding out how much information can be stored in a given partition. We can use the formula to find out which partitions of a number n allow the most standard Young tableaux, and hence, can hold the greatest amount of information. Computer scientists can apply the Hook Formula to their partitioning to maximize the amount and efficiency of information storage given limited space and processing capabilities.

The Hook Formula also has applications in other fields of mathematics, such as in studies of the representations of symmetric groups. It also has applications to topics in classical mathematics, such as the study of polygons. For

example, when determining the number of triangulations of a regular polygon, that is, the number of ways we can draw diagonals of polygons to create touching but non-overlapping triangles, we find that this value is $(2n)! / (n! \cdot (n+1)!)$ for a regular $(n+2)$-sided polygon. This formula is identical to the number of standard Young tableaux as determined by the Hook Formula for the partition $\lambda = (n, n)$ of $2n$.

A mathematician from Michigan State University named J. S. Frame first proposed the Hook Formula and proved the conjecture with another mathematician named G. Robinson in 1953. The formula was also discovered and proved in the same fashion independently by R. M. Thrall in the same year, and the three mathematicians published the first proof of the Hook Formula in 1954 [1]. As an intriguing side note, Thrall not only discovered the same proof simultaneously, but he too was in Michigan, just down the highway at the University of Michigan. There must have been something mathematical in the Michigan air at the time. Frame, Robinson, and Thrall's proof relied heavily on a complicated formula discovered independently by both G. Frobenius [2] and A. Young (restated in [5]) derived from an in-depth study of symmetric groups.

Today there are several different ways by which one can prove the Hook Formula. First I shall present a bijective explanation for the Hook Formula. This bijection was developed by I. M. Pak and A. V. Stoyanovskii [7] in the early 1990s and the complete proof was presented with the help of J. C. Novelli [6] in 1997. I shall not present the complete proof here, but instead merely outline and explain the bijection, for the purpose of better understanding the Hook Formula. Then I will present a complete proof based on a probabilistic method adapted from B. E. Sagan [8], and originally developed by Greene, Nijenhuis, and Wilf [3].

SECTION V

A BIJECTIVE EXPLANATION

What follows in this section is a bijective explanation of the Hook Formula. It is not a complete proof (which follows in Section VI), but it does help explain *why* such a simple formula for the number of standard Young tableaux for a given partition exists. It seems rather mysterious that the total number of standard tableaux can be determined so concisely with the Hook Formula. This bijective explanation, however, helps illuminate why such a formula exists.

We need a few definitions before we can proceed. A *function* f: $X \rightarrow Y$, read "from set X to set Y," is a mapping scheme that assigns some element $y \in Y$ to each element $x \in X$. When a function f assigns each unique $x \in X$ to a unique $y \in Y$, then f is said to be *one-to-one*. When every element $y \in Y$ is assigned to some element $x \in X$ by the function f, then f is said to be *onto* (Grammatical note: In this context, "onto" is used as an adjective). So with a one-to-one function, no two different values of $x \in X$ will yield the same value of $y \in Y$, and with an onto function, every element in Y is assigned to an element in X. A

bijective function f: $X \to Y$ is a function from X to Y that is both one-to-one and onto. Bijective functions are also called *bijections*. Since bijections are both one-to-one and onto, every distinct element $y \in Y$ is assigned to some unique $x \in X$, and thus the sets X and Y have the exact same number of elements. Furthermore, bijections have the property that they can map sets in two directions, that is, they can map from set X to set Y and from set Y to set X, and bijections are sometimes written like this: $X \leftrightarrow Y$.

Now we can use a bijection to help explain the reasoning behind to Hook Formula. Once again, the Hook Formula is

$$f^{\lambda} = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}$$

where f^{λ} is the number of standard tableaux for a partition λ and $\prod_{(i,j)\in\lambda} h_{i,j}$ is the product of the hook lengths of λ. We can rewrite this equation as

$$n! = f^{\lambda} \cdot \prod_{(i,j)\in\lambda} h_{i,j}.$$

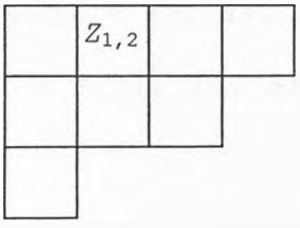
Next let us examine three Ferrers diagrams of shape λ. Let X be a general or non-standard tableau of shape λ, that is, X might not be standard, and the numbers 1 through n have been filled into the cells in any order. Let Y be a

standard tableau of shape λ. And finally let Z be a λ-shaped array of numbers for which the number of choices for each cell $Z_{i,j}$ is $h_{i,j}$, that is, the number of choices for each cell in Z is the same as the length of the corresponding hook for that cell. Since order does not matter in the numbering of X, then the total number of general tableaux X is $n!$. The total possible number of standard Young tableaux Y is, of course, f^{λ}. Since each cell $Z_{i,j}$ of Z has $h_{i,j}$ possibilities, then the total number of Z is $\prod_{(i,j)\in\lambda} h_{i,j}$. The set (Y, Z) is the set of ordered pairs of all Y and all Z and, thus, has size equal to the size of Y times the size of Z. So, finding a bijection $X \leftrightarrow (Y, Z)$ is the same as showing that $n! = f^{\lambda} \cdot \prod_{(i,j)\in\lambda} h_{i,j}$, and, as luck would have it, there is just such a bijection.

With this bijection we shall take any general tableau X and transform it into a standard tableau Y while keeping a record of the transformation in the form of array Z.

Let the specific possible entries $Z_{i,j}$ for the cells of Z be the integers between the arm length and the negative of the leg length. The *arm* of a cell (i, j) is defined as all of the cells in the same row to the right of (i, j), and the *leg* of a cell (i, j) is defined as all of the cells in the same column below (i, j). We write the arm length

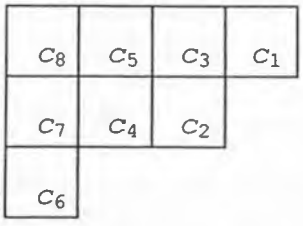
of cell (i, j) as $al_{i,j}$ and the leg length as $ll_{i,j}$. So let $Z_{i,j}$ be an integer and $-ll_{i,j} \leq Z_{i,j} \leq al_{i,j}$. Note that the hook length $h_{i,j}$ for cell (i, j) is $al_{i,j} + ll_{i,j} + 1$ since hook length is the sum of all cells directly to the right and directly below plus one for the cell (i, j) itself. Also note that, including zero, there are $h_{i,j}$ integers between $-ll_{i,j}$ and $al_{i,j}$. For example, in the partition $\lambda = (4, 3, 1)$, whose Ferrers diagram is pictured in Figure 9, the leg length of cell (1, 2) is 1, the arm length is 2, and the hook length is 4, and thus the four numeric options for cell $Z_{1,2}$ are -1, 0, 1, and 2.



[Figure 9: The Ferrers diagram for $\lambda = (4, 3, 1)$ with cell (1, 2) marked]

Next let us define an ordering and general naming system for the cells of our generic λ-shaped diagrams. Define (i, j) as less than or equal to (i', j'), written $(i, j) \leq (i', j')$, exactly when $j > j'$ or when $j = j'$ and $i \geq i'$ (this ordering and notational system is adapted from [9]). Now let us label the cells of our λ-shaped diagrams

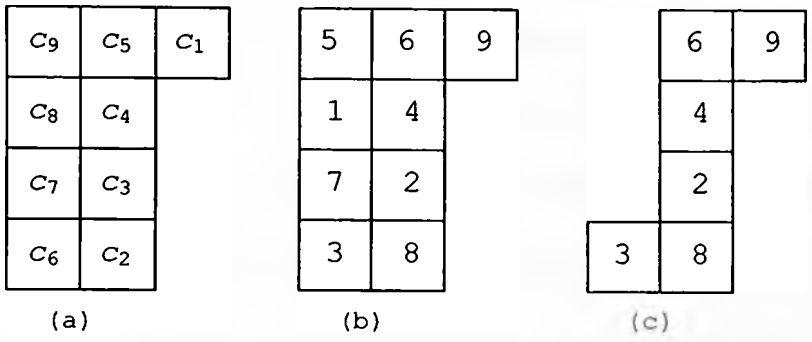
as c_1, c_2, …, c_n and order them such that $c_1 < c_2 < \ldots < c_n$. Figure 10 shows this numbering and ordering system applied to the partition $\lambda = (4, 3, 1)$.



[Figure 10: A naming and ordering of cells for $\lambda = (4, 3, 1)$]

For example, note that in Figure 10 cell c_2, with coordinates $(i, j) = (2, 3)$, is less than cell c_6, with coordinates $(i', j') = (3, 1)$, since $j = 3 > 1 = j'$; cell c_6, with coordinates $(i, j) = (3, 1)$, is less than cell c_8, with coordinates $(i', j') = (1, 1)$, since $j = 1 = j'$ and $i = 3 > 1 = i'$.

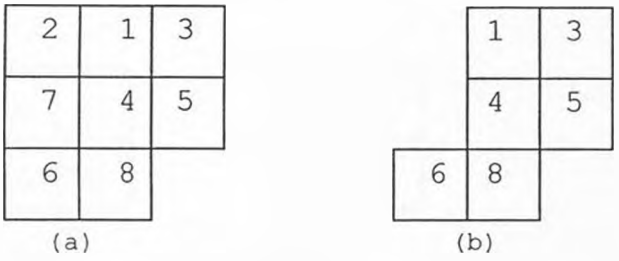
We just need one more definition before delving straight into this bijection. When X is a λ-shaped tableau, not necessarily standard, and c_k is cell in the tableau with $1 \leq k \leq n$, say that $X^{\leq c_k}$ is the tableau X with all cells c_{k+1}, …, c_n removed. Likewise, $X^{<c_k}$ is the tableau X with all cells c_k, c_{k+1}, …, c_n removed. Figure 11 gives an example of this and shows a diagram for the partition $\lambda = (3, 2, 2, 2)$ with the cell naming system as described above, a general (non-standard, that is) tableau X for this λ, and $X^{\leq c_6}$.



[Figure 11: (a) The partition $\lambda = (3, 2, 2, 2)$ with cell designation and ordering; (b) a non-standard tableau X for λ; (c) $X^{\leq c_6}$, the non-standard tableau X with cells c_7, c_8, and c_9 removed]

So our bijection is between an arbitrary non-standard tableau X with $n!$ possibilities for numbering and the pair (Y, Z), which is a standard tableau Y with f^{λ} possibilities for numbering and a λ-shaped array Z with $\Pi_{(i,j)\in\lambda} h_{i,j}$ possibilities for numbering. The bijection involves repeatedly applying an algorithm that moves X towards standardization. We shall complete the algorithm n times, each time moving the tableau X towards a more standardized state, and we shall denote each of these new Xs as X_1, X_2, …, X_n, and for the k^{th} time through the algorithm, we shall use X_{k-1} as our starting point. Define $X = X_0$. The tableau X_n, as we shall soon discover, will be standard. Each time completing the algorithm shall consist of choosing a cell c_k, starting with $k = 1$ and working up, and making $X^{\leq c_k}$ standard. $X^{\leq c_k}$ being standard means that if we look at only

the cells that are less than or equal to c_k, the numbers in these cells will be strictly increasing as we move down and strictly increasing as we move to the right. Figure 12 shows an example of a non-standard X for the partition $\lambda = (3, 3, 2)$ in which the array $X^{\leq c_6}$ is standard.

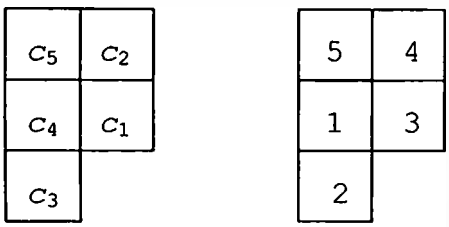


[Figure 12: (a) A non-standard tableau X for $\lambda = (3, 3, 2)$; (b) the array $X^{\leq c_6}$ is standard]

Now the algorithm itself is as follows: From the tableau X_{k-1}, choose the cell c_k and call it c. By our construction, $X^{<c}$ is standard as can be seen from the example that follows. If $X^{\leq c}$ is standard, then define $X_{k-1} = X_k$ and we are done with the k^{th} time through the algorithm. If $X^{\leq c}$ is not standard and c has coordinates (i, j), then say that c' is the cell immediately below or immediately to the right of c that has the minimum value in it. Switch c and c' and redefine c as c'. Recheck $X^{\leq c}$ for standardization with this new c, and repeat the exchanging

process until $X^{\leq c}$ is standard, at which point define $X_{k-1} = X_k$ and we are done with the k^{th} time through the algorithm.

Let us examine this standardization process with an example. Say we have a partition $\lambda = (2, 2, 1)$ of the number 5. X is a non-standard tableau as shown in Figure 13 alongside the cell designation for λ as described above.



[Figure 13: (a) The cell designation and ordering for $\lambda = (2, 2, 1)$; (b) a non-standard tableau X]

First let $k = 1$ and let us work through our algorithm. Using $X_0 = X$ as our beginning tableau, let us isolate cell c_1 and re-designate it as c. The array $X^{\leq c}$, then, is standard since it consists of only the cell containing the integer 3, thus each row and column of this one cell are strictly increasing. X_1, then, is the same as $X_0 = X$. Next, let $k = 2$. X_1 is now our starting tableau and we will isolate the cell c_2 and rename it c. $X^{<c}$ is just the single cell containing 3, which we have already determined is standard. And as we can see, by this construction, as we increase k incrementally, each $X^{<c}$ will be standard. But

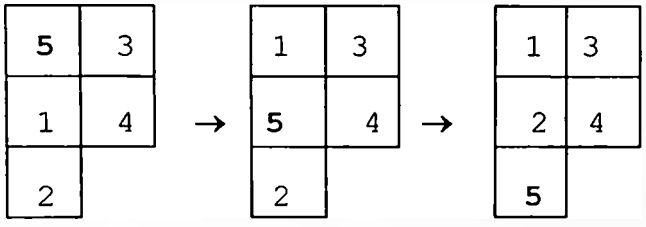
for $k = 2$, $X^{\leq c}$ is not standard since 4 is greater than 3. So next we must exchange this cell with the lower of the cells immediately below and immediately to the right. There are no cells to the right of c_2, thus the cell directly below is our c', and we must exchange c and c' so that c' now contains a 4 and c contains a 3. We then redefine c' as c and check if $X^{\leq c}$ is standard, which it is, and we finish this stage of the algorithm by naming this new tableau X_2. This new tableau is pictured in Figure 14.

5	3
1	4
2	

[Figure 14: The tableau X_2, which is X_1 with the 3 and 4 exchanged]

Now for $k = 3$ we use X_2 as our starting tableau and look at $X^{\leq c_3}$. Since $X^{\leq c_3}$ is standard, we just define X_3 as equal to X_2. For $k = 4$ we start with X_3 and look at $X^{\leq c_4}$, which is also standard, so $X_4 = X_3$. For $k = 5$, our starting tableau is X_4, c_5 is c, and we consider $X^{\leq c}$. $X^{\leq c}$ is not standard, so we must exchange the cell containing 5 with the cell just to its right or just below it that contains the lower number. Immediately below cell c is a 1 and immediately to the right is a 3. One is, of course, lower than three, so the cell with the 1 in it we shall call c' and we shall

switch cells c and c'. Now the cell at coordinates (2, 1) contains the number 5 and we will re-designate this cell as c and look at $X^{\leq c}$, which is still not standard. Now the cell immediately below c has a 2 in it and the cell just to the right has a 4 in it, thus, we switch the 5 with the 2 from the cell below, and the 5 ends up in cell (3, 1). This tableau is now X_5. Figure 15 shows this exchanging process for $k = 5$.



[Figure 15: The transformation from X_4 to X_5 showing the movement of the entry of our initial cell c_5]

Notice that in each column of X_5 the cell entries increase as you move down and in each row, the entries increase as you move to the right. X_5, then, is a standard Young tableau and we have transformed a non-standard tableau into a standard one, which we shall now call Y.

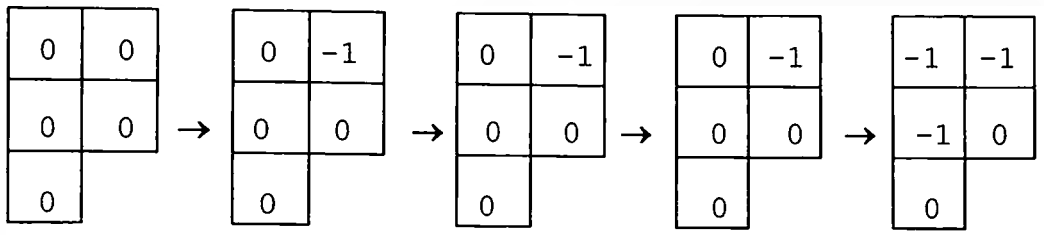
As said above, Z is a "record keeping" λ-shaped array. The changes that the above algorithm forced onto X will be encoded in Z. The entries of Z change as we transform X into the standard Y. Each cell $Z_{i,j}$ has $h_{i,j}$ possible entries, that is, each cell of Z has the same number of

possible entries as the hook for that cell is long. Specifically, $-ll_{i,j} \leq Z_{i,j} \leq al_{i,j}$ with $ll_{i,j}$ and $al_{i,j}$ representing the leg and arm lengths, respectively. Just as we created $X_n = Y$ from a sequence of X_ks with $1 \leq k \leq n$, so will we create $Z_n = Z$ from a sequence of Z_ks with $1 \leq k \leq n$. Z_1 will record the changes that take place going from $X = X_0$ to X_1, Z_2 records the changes going from X_1 to X_2, and so on. $Z_n = Z$ will be the final record keeping array in transforming a non-standard X into a standard Y. Define Z_0 as equal to the λ-shaped array that has a zero in each cell. This is our starting array. For the k^{th} time through the algorithm, we shall say that the entry in cell c_k starts at coordinates (i, j) and ends at coordinates (i', j'). Define Z_k recursively by $Z_k = Z_{k-1}$ except for the entries given below (adapted from [9]):

$$(Z_k)_{m,j} = (Z_{k-1})_{m+1,j} - 1 \text{ when } i \leq m < i' \text{ and}$$

$$(Z_k)_{m,j} = j' - j \text{ when } m = i'.$$

Notice that only the entries of the j^{th} column might change from Z_{k-1} to Z_k. Figure 16 shows Z_1 through Z_5 for the example given above. Notice in Figure 16 that when X_k does not change from X_{k-1}, neither does Z_k from Z_{k-1}.



[Figure 16: The arrays Z_1 through Z_5, in order from left to right, corresponding to the standardization of tableaux X_1 through X_5 in the example above]

In general, the numbers in the cells of Z represent how entries of cells in X move. Entries of zero show no movement, while entries of negative numbers show movement down in a column, and entries of positive numbers show movement to the right in a row.

So we can pair each non-standard tableau X with the standard tableau Y and the array Z showing the changes made to standardize X. The standardization algorithm described earlier can be applied to any λ-shaped tableau X, standard or not. Since there are many more general tableaux for a partition λ than there are standard tableaux, multiple tableaux X will have to map to each standard tableau Y, but each of these different Xs will have different paths towards the standardized form, and thus will have different arrays Z. So each X will correspond to a unique pairing of Y and Z. Thus this pairing is the bijection $X \leftrightarrow (Y, Z)$. The set of general tableaux X has n! elements, the set of standard tableaux Y has f^{λ} elements by definition, and the

set of arrays Z has $\Pi_{(i,j)\in\lambda}\ h_{i,j}$ elements by construction. Therefore, n! = $f^{\lambda} \cdot \Pi_{(i,j)\in\lambda}\ h_{i,j}$ and can then derive the Hook Formula:

$$f^{\lambda} = \frac{n!}{\Pi_{(i,j)\in\lambda}\ h_{i,j}}.$$

It should be noted here that in this preceding bijective explanation of the Hook Formula, we have not fully proved that the mapping $X \leftrightarrow (Y, Z)$ is one-to-one, onto, and reversible and, thus, a true bijection. But hopefully we have explored these structures enough to see that this is so and to see that the sets X and (Y, Z) are of the same size. The complete rigorous proof of the bijection is longer and more complicated than would fit within the scope of this thesis. The bijection between the set of general tableau X and the set (Y, Z), however, helps explain *why* the Hook Formula is what it is. These hook shapes and lengths play a role less random and mysterious than at first it seems. They come in to play when we record the movements of numbers in X in order to transform it into a standardized form Y. Numbers can move down in columns and to the right in rows, thus, hook length is essential and highly informative.

SECTION VI

A PROBABILISTIC PROOF

This proof utilizes a process that generates a standard Young tableau of a given shape λ, where λ is a partition of n. It will show that the probability of getting a particular standard λ-tableau P is

$$\frac{\Pi_{(i,j)\in\lambda}\ h_{i,j}}{n!}.$$

Hence the total number of tableaux is the multiplicative inverse of this, and the Hook Formula will be proved. Before we begin, note that if one is to randomly choose an item from n elements, the *probability* of getting a particular element is $1/n$. Also, a comment on notation: $\Pi_{i=1}{}^{m}\ (f_i)$, with $m \in \mathbf{N}$, means the product of all f_is as i is evaluated from 1 to m, that is, $\Pi_{i=1}{}^{m}\ (f_i) = f_1 \cdot f_2 \cdot f_3 \cdot \ldots \cdot f_m$.

The process for generating a standard tableau is to first pick any cell in λ and call it $v_{1,1}$*. Clearly the probability for this first cell is $1/n$ since there are n

* The subscript here is not a cell coordinate designation as is used earlier with hooks and hook lengths, but is instead an indication of algorithmic placement. The notation $v_{p,m}$ means the p^{th} trial and the m^{th} cell choice for that trial.

cells in λ. If $v_{1,1}$ is not a corner cell, choose a $v_{1,2} \in H_{v_{1,1}} - \{v_{1,1}\}$, that is, choose a $v_{1,2}$ in the hook for cell $v_{1,1}$ with the cell $v_{1,1}$ itself removed. This choice, then, has a probability of $1/(h_{v_{1,1}}-1)$. If $v_{1,2}$ is again not a corner cell, then choose a $v_{1,3} \in H_{v_{1,2}} - \{v_{1,2}\}$ (with probability $1/(h_{v_{1,2}}-1)$) and repeat this process until a corner has been reached. Once $v_{1,m}$ for some $m \in \mathbf{N}$ is a corner cell, label it n. Next redefine the λ-diagram to be $\lambda - \{v_{1,m}\}$, that is, λ with the cell $v_{1,m}$ removed and choose a new $v_{2,1} \in \lambda$. The probability for $v_{2,1}$ is now $1/(n-1)$. Repeat the above process for each $v_{2,k}$ until it is in a corner and label it n-1, and so on with $v_{3,k}$, $v_{4,k}$, …, $v_{n,k}$ until all the cells have been labeled with the numbers 1, …, n. Note that since n is placed in a corner of λ, and n-1 is placed in a corner of λ without the cell containing n, and n-2 is placed a corner of λ without the cells containing n and n-1, an so on, then the tableau we have created by this process, call it P, has entries which are strictly increasing as we move right in each row and as we move down in each column. Thus we have just generated a standard Young tableau P. The process of choosing a $v_{p,1}$, then a $v_{p,2}$, and so on up to a $v_{p,m}$, where $v_{p,m}$ is a corner cell and $1 \leq p \leq n$ is called a *trial*. Each trial produces a corner (first of diagram of λ,

then of the diagram of $\lambda - \{v_{1,m}\}$, and so on), and one must complete n trials to create the standard Young tableau P.

So the crux of this proof is showing that the probability of getting any particular tableau P from this process is

$$\text{prob}(P) = \frac{\Pi_{(i,j)\in\lambda}\ h_{i,j}}{n!} .$$

Let us say that ω is the cell containing n and it has coordinates (α, β). Then, $\text{prob}(P) = \text{prob}(\omega) \cdot \text{prob}(P')$, where P' is P with ω removed. Or, $\text{prob}(\omega) = \text{prob}(P) / \text{prob}(P')$, we will denote the partition for P' by λ'. So if we can show that

$$\text{prob}(\omega) = \frac{(\Pi_{v\in\lambda}\ h_v)\ /\ (n!)}{(\Pi_{v'\in\lambda'}\ h_{v'})\ /\ ((n-1)!)} ,$$

then we can induce the desired $\text{prob}(P)$ from above.

By cancellation,

$$\frac{(\Pi_{v\in\lambda}\ h_v)\ /\ (n!)}{(\Pi_{v'\in\lambda'}\ h_{v'})\ /\ ((n-1)!)}$$

simplifies to $(1/n) \cdot \Pi_{v\in W}\ (h_v\ /\ (h_v-1))$, where W is a subset of λ consisting of all $v \neq \omega$ with v in the same column or row as ω. (W, then, is a sort of "reverse hook" for the cell ω with ω itself removed.) One can then write

$(1/n) \cdot \Pi_{v \in W} (h_v / (h_v-1)) = (1/n) \cdot \Pi_{v \in W} (1 + (1 / (h_v-1)))$. We can then split the vertical and horizontal components of W up into two products and write

$$(1/n) \cdot \Pi_{v \in W} (1 + (1 / (h_v-1)))$$

$$= (1/n) \cdot \Pi_{i=1}^{\alpha-1} (1 + (1 / (h_{i,\beta} - 1))) \cdot \Pi_{j=1}^{\beta-1} (1 + (1 / (h_{\alpha,j} - 1))).$$

To simplify the notation in the above equation, let $a_i = h_{i,\beta} - 1$ and $b_i = h_{\alpha,j} - 1$. Then we have

$$(1/n) \cdot \Pi_{v \in W} (h_v / (h_v-1))$$

$$= (1/n) \cdot \Pi_{i=1}^{\alpha-1} (1 + (1 / a_i)) \cdot \Pi_{j=1}^{\beta-1} (1 + (1 / b_j)).$$

Now for the p^{th} trial, which ends at the corner cell $\omega = (\alpha, \beta)$, let R be the set of all $i \neq \alpha$ such that $v_{p,k} = (i, j)$ for some k on the given trial. In other words, R is the set of all rows used during the trial to get to (α, β) with the exception of α itself. Call R a *row projection of the trial*. Likewise let C be the set of all $j \neq \beta$ such that $v_{p,k} = (i, j)$ for some k on the given trial. In other words, C is the set of all columns used during the trial to get to (α, β) with the exception of β itself. Call C a *column projection of the trial*. Furthermore, let $\text{prob}_{R,C}(\omega) = \text{prob}_{R,C}(\alpha, \beta)$ be the sum the probabilities for all trials ending at $\omega = (\alpha, \beta)$ with the given row and column

projections R and C. Although we can get the projections R and C from one trial, there can be several trials with the same projections R and C, so, $\text{prob}_{R,C}(\alpha, \beta)$ is the sum of the probabilities for each of these trials. So the probability of getting a specific $\omega = (\alpha, \beta)$ as our corner cell is the sum of $\text{prob}_{R,C}(\alpha, \beta)$ for all possible row and column projections R and C.

Thus if $\text{prob}_{R,C}(\alpha, \beta) = (1/n) \cdot \Pi_{i \in R} (1 / a_i) \cdot \Pi_{j \in C} (1 / b_j)$, then the sum of this probability for all possible R and C, i.e. $\text{prob}(\omega)$, is $(1/n) \cdot \Pi_{i=1}^{\alpha-1} (1 + (1 / a_i)) \cdot \Pi_{j=1}^{\beta-1} (1 + (1 / b_j))$ which as we saw before is the same as $(1/n) \cdot \Pi_{i=1}^{\alpha-1} (1 + (1 / (h_{i,\beta} - 1))) \cdot \Pi_{j=1}^{\beta-1} (1 + (1 / (h_{\alpha,j} - 1)))$ which equals $(1/n) \cdot \Pi_{v \in W} (1 + (1 / (h_v - 1))) = (1/n) \cdot \Pi_{v \in W} (h_v / (h_v - 1))$ which is

$$\frac{(\Pi_{v \in \lambda} h_v) / (n!)}{(\Pi_{v' \in \lambda'} h_{v'}) / ((n-1)!)}$$

as desired, and therefore the probability of getting a specific standard Young tableau P is $(\Pi_{v \in \lambda} h_v) / (n!)$, and thus the total number of standard Young tableau f^{λ} is $1 / ((\Pi_{v \in \lambda} h_v) / (n!))$, or $(n!) / (\Pi_{v \in \lambda} h_v)$ and the Hook Formula holds. But this chain of implications rests on the

hypothesis that $\text{prob}_{R,C}(\alpha, \beta) = (1/n) \cdot \Pi_{i \in R} (1 / a_i) \cdot \Pi_{j \in C} (1 / b_j)$, so we must now verify this.

Say we have (α, β) as the end point of our trial and row and column projections $R = \{i_1, i_2, i_3, \ldots\}$ and $C = \{j_1, j_2, j_3, \ldots\}$ respectively. If R and C are both empty, then the first and only point of our trial is $v_{1,1} = (\alpha, \beta)$, which has a probability of $1/n$, hence the hypothesis holds. If only one of R and C is empty, say R is empty (it does not matter which one is empty by symmetry), then the only cells we could have in our trial are $v_{1,1} = (\alpha, j_1)$, $v_{1,2} = (\alpha, j_2)$, ... Therefore this has a probability of $(1/n) \cdot (1 / (h_{\alpha, j_1}-1)) \cdot (1 / (h_{\alpha, j_2}-1)) \cdot \ldots = (1/n) \cdot (1 / b_1) \cdot (1 / b_2) \cdot \ldots = (1/n) \cdot \Pi_{j \in C} (1 / b_j)$ by our definition of b_j and by the process we defined for determining our corner cell (α, β). So the hypothesis holds. Finally if R and C are both nonempty, then $v_{1,1}$ must be (i_1, j_1) and there are two choices for $v_{1,2}$: (i_2, j_1) or (i_1, j_2). Define R' as $R - \{i_1\}$ and C' as $C - \{j_1\}$. Then inductively

$$\text{prob}_{R,C}(\alpha, \beta) = (1 / (h_{i_1, j_1}-1)) \cdot (\text{prob}_{R',C}(\alpha, \beta) + \text{prob}_{R,C'}(\alpha, \beta))$$

$$= (1 / (h_{i_1, j_1}-1)) \cdot (1 / (n \cdot a_{i_1}' \cdot a_{i_2} \cdot \ldots \cdot b_{j_1} \cdot b_{j_2} \cdot \ldots) + 1 / (n \cdot a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot b_{j_1}' \cdot b_{j_2} \cdot \ldots))$$

(where a_{i_1}' and b_{j_1}' mean those terms are deleted)

$$= (1 / (h_{i_1,j_1}-1)) \cdot (a_{i_1} / (n \cdot a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot b_{j_1} \cdot b_{j_2} \cdot \ldots) + b_{j_1} / (n \cdot a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot b_{j_1} \cdot b_{j_2} \cdot \ldots))$$

$$= ((a_{i_1} + b_{j_1}) / (h_{i_1,j_1}-1)) \cdot (1 / (n \cdot a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot b_{j_1} \cdot b_{j_2} \cdot \ldots))$$

$= 1 / (n \cdot a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot b_{j_1} \cdot b_{j_2} \cdot \ldots)$ since $a_{i_1} + b_{j_1} = h_{i_1,\beta} - 1 + h_{\alpha,j_1} - 1 = h_{i_1,j_1}-1$ by definition

$$= (1/n) \cdot \Pi_{i \in R} (1 / a_i) \cdot \Pi_{j \in C} (1 / b_j).$$

Therefore the hypothesis holds, and thus the Hook Formula is proven.

In summary, we were trying to show that the total number of standard Young tableaux for a given partition of n is $(n!) / (\Pi_{v \in \lambda} h_v)$. To do this we created a process for generating such a tableau and showed that the probability of getting a specific standard tableau P via this process is $(\Pi_{v \in \lambda} h_v) / (n!)$. Therefore since P has uniform likelihood, that is, each P is equally likely to be generated, then the total number of possibilities is $(n!) / (\Pi_{v \in \lambda} h_v)$.

SECTION VII

CONCLUSION

From this synthesis, we are now familiar with partitions of natural numbers and their depictions in the form of Ferrers diagrams. We have seen the structures of tableaux and hooks, and the special properties of standard Young tableaux. We have also discovered that there is a simple and easy formula for finding out how many standard Young tableaux there are for a partition using hooks. Numerous examples have illuminated this Hook Formula and its applications. Furthermore, a bijection has helped us see how the Hook Formula comes about and the formula has been proven via a probabilistic method.

Clearly the Hook Formula proofs are complex and multifaceted, yet the formula itself is quite simple and elegant. This is often how theorems in mathematics are – we seek simple and powerful formulas and theorems that are easy to use and to apply but that are often exceedingly difficult to prove. This process of proof, however, is utterly essential to the field of mathematics to verify that formulas and theorems we find and use are valid, justified tools.

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