COST-MINIMIZING NETWORKS AND POLYHEDRAL CONES

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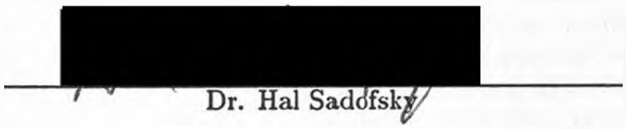
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We model immiscible fluid clusters in space by cost-minimizing polyhedral surfaces, where "cost" is a weighted area. We also discuss planar configurations of immiscible fluid clusters modeled by cost-minimizing networks, where cost is a weighted length. We give necessary and sufficient conditions for local minimization of networks in the plane and cones of planes meeting along a line in space. A cone is minimizing if and only if a certain geometric "point-placing" condition is fulfilled. We extend these results to allow a single additional fluid not present in the original configuration. It is not known whether point-placing and minimization are equivalent for cones of planes meeting at a point, but we conjecture the same point-placing condition and discuss some special examples of such minimizing configurations.

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1 Introduction

A classical question in plane geometry, known as the Steiner problem, asks for the network of line segments with the least total length connecting a set of points. It is well known that in nodes not in the original set of points, segments meet only in threes at 120° angles. The same is true of planar networks of soap films, so they are good models of minimizing networks and surfaces. Jean Taylor [8] proves that soap films in space can meet in two ways: three surfaces can meet along a curve at 120° angles, or six surfaces can meet along four curves which meet at a point at angles of about 109°. Locally, under repeated magnification, such singularities look and behave like three planes meeting along a line at 120° angles and six planes meeting along four lines meeting at a point. These are the only singularities that area minimizing surfaces can have in space.

1.1 Cost-Minimizing Networks in the Plane and Surfaces in Space

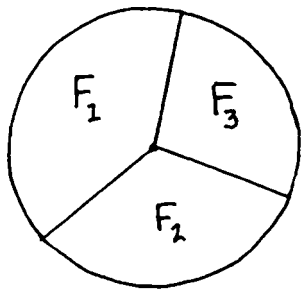


Figure 1: A minimizing configuration of three fluids F_1, F_2, F_3 separated by line segments meeting at a point.

One generalization of the Steiner problem seeks to find the cost-minimizing way to connect points in the plane, where cost is a weighted length. Our model for cost-minimizing networks comes from immiscible fluid clusters. When two immiscible fluids meet in the plane, they form an interface which has an associated energy proportional to length. This energy gives rise to a "cost-constant" for each pair of fluids. When several fluids come together, they form a network separating the fluids. The fluids arrange themselves to minimize the total energy or "cost" of the configuration. For example, a minimizing configuration involving three fluids might consist of three line segments meeting at a point, as in figure 1.

The fluid interpretation of our problem suggests imposing area constraints, which we do not, but both Frank Morgan [7] and Brian White [9], [10] consider this possibility. We will focus on the behavior of such networks in the neighborhood of a node (figure 2). Thus we focus on a local problem in the global configuration. As in the case of Steiner networks, we would like to be able to determine the angles at which segments meet in cost-minimizing networks, depending on the cost-constants. We consider configurations of fluids in the unit disk, with a fixed (possibly empty) arc on the boundary of the disk associated with each fluid. To preserve the global structure of the configuration, we ask that all our networks separate the arcs of the disk the same way. Given that the node is in the center, we characterize which boundary conditions result in a minimizer. It is known (Proposition 4.1) that a cone of segments meeting at a point is minimizing if a certain "point-placing" condition can be satisfied, even among fluids which may not be present in the configuration. A point is placed for each fluid in such a way that the vector between two points is perpendicular to the interface separating the fluids. The distances of the points from one another depend on the cost constants. Proposition 4.5 shows point-placing is necessary for minimization among present fluids.

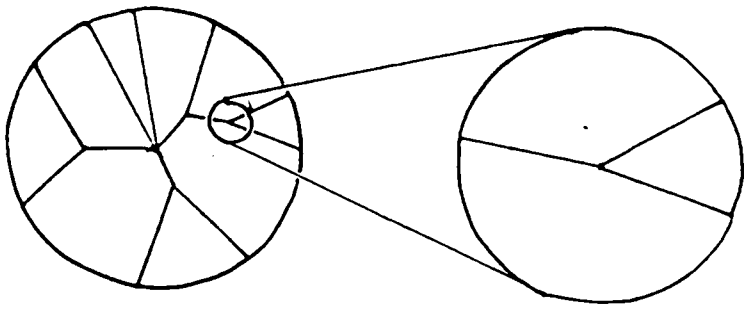


Figure 2: We focus on the singularities in the neighborhood of a node.

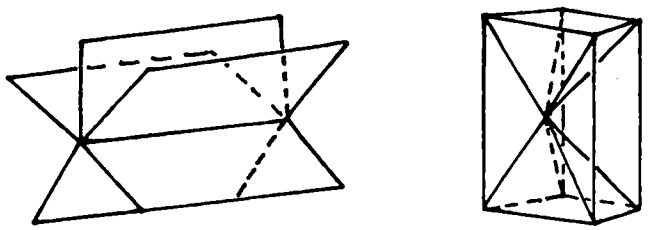


Figure 3: Cost-Minimizing cones of planes meet along lines and at points in space.

The case of greatest physical interest is the structure of singularities in cost-minimizing surfaces among immiscible fluids in space. We focus attention on what happens when several planes meet along a line, and when several planes meet along lines meeting at a point, which Brian White [10] proves are the only types of singularities for planar surfaces (figure 3). We call such surfaces *polyhedral cones* since they are invariant under magnification and because they consist of pieces of planes. As Jean Taylor [8] classified the singularities of soap-films in space, so we would like to classify the singularities of cost-minimizing polyhedral surfaces in space. Again our model is immiscible fluids without volume constraints, which minimize a total cost proportional to area. We work inside the closed unit ball, and fix several regions on the boundary of the ball associated with the various fluids. Again we insist that our surfaces separate the boundary of the unit ball the same way. It is known that a similar point-placing condition as given for networks in the plane is sufficient for minimization of cones even among fluids which may not be present. We prove that point-placing is necessary for minimization for such cones in Theorem 5.2 by reducing it to the case of networks in the plane. It is not known whether point-placing is necessary for minimization for cones of planes meeting at a point, which we discuss in greater detail in sections 6 and 6.2.

1.2 Allowing Additional Fluid

Another open question asks whether point-placing is necessary and sufficient if we allow regions of additional fluids not present in the original configuration. Point placing is known to be sufficient, but is not believed to be necessary, a problem studied in [2]. The authors of that paper propose a counterexample allowing two new fluids in the presence of four. We give their theorem which shows

that point-placing and minimization are equivalent allowing one additional fluid in Theorem 4.10. We generalize their work to cones of planes in space in Theorem 5.2.

One idea developed in [2] while attempting to extend Theorem 4.6 to allow extra fluids is known as supercalibration. We present and utilize this idea in section 4.3, and give a special case of it applied to a different proof of the equivalence of point-placing and minimization in the case of no additional fluids.

1.3 Cones of Planes Meeting at a Point

As we mentioned, it is unknown whether point-placing and minimization are equivalent for polyhedral cones of planes meeting at a point. Point-placing is always sufficient for minimization, and in dealing with the converse we assume points cannot be placed and seek a comparison surface cheaper than the cone. Our calculations in sections 6.2, 6.3, and 6.4 suggest that it is not sufficient to consider nearby planar surfaces. A better understanding of curved surfaces is necessary for proving the necessity of point-placing. We extend a result of [1] in this direction and discuss the structure of planar comparisons and their inability to accurately describe the subtle geometry of such cones.

2 Preliminaries

2.1 Surfaces in Space

We consider the configurations of several immiscible fluids $F_1, \ldots, F_n$ which meet inside the closed unit ball D in $\mathbf{R}^3$. These fluids are separated by a surface H consisting of C^1 surfaces meeting only along finitely many curves and points. Such curves and points are called *nodes*, and it will always be clear from context which type of node we are considering. The surface H divides D into disjoint open regions $R_1, \ldots, R_n$, where fluid F_i occupies region R_i.

Definition: The *region* R_i assigned to each fluid F_i is defined as the union of one or several connected components of $D \setminus H$. Thus each region is an open, but not necessarily connected, set. We assume the regions R_i are disjoint.

Definition: To each region R_i we fix a *reference surface* $S_i = \partial R_i \cap \partial D$.

The curve, ∂S_i is a closed curve on the sphere, and $\cup_i \partial S_i$ defines a *frame* consisting of piecewise C^1 curves.

Definition: For two regions R_i and R_j we define the *interface* $H_{ij} = \partial R_i \cap \partial R_j$.

Definition: We define the *dividing surface* separating fluids $F_1, \ldots, F_n$ as the collection of the interfaces H_{ij}, denoted $H = \{H_{ij}\}$.

We often say a dividing surface H *separates the* S_i, meaning that $\partial D \cap H = \cup \partial S_i$. A *comparison surface* is a dividing surface separating the S_i.

We assume no volume constraints on the fluids, so we may think of the regions R_i as connected to external tanks of the various fluids at equal pressures, as suggested by [1]. We think of fluid F_i entering region R_i through the fixed boundary region S_i. Later we will allow additional fluid not present among those represented in the S_i. That is, we may allow one or more fluids with no reference surfaces.

To each pair of fluids F_i, F_j we assign a *cost-constant* $a_{ij} = a_{ji} > 0$. We assume a *triangle inequality* on these constants as follows: For any fluids F_i, F_j, F_k,

$$a_{ik} \leq a_{ij} + a_{jk}. \tag{1}$$

The triangle inequality assures us that no infinitesimal layer of one fluid could be placed between two others to reduce cost (figure 4).

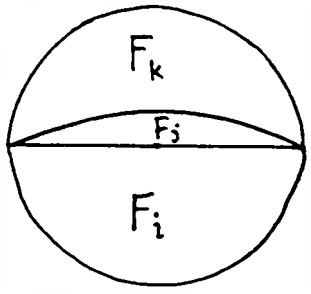


Figure 4: The triangle inequality prevents a network with a small layer of fluid separating two others from reducing cost.

The interface energy, or *cost* of H is computed by taking the weighted areas of the interfaces:

$$\text{cost}(H) = \sum_{i<j} a_{ij} \text{ area}(H_{ij}). \tag{2}$$

2.2 Networks in the Plane

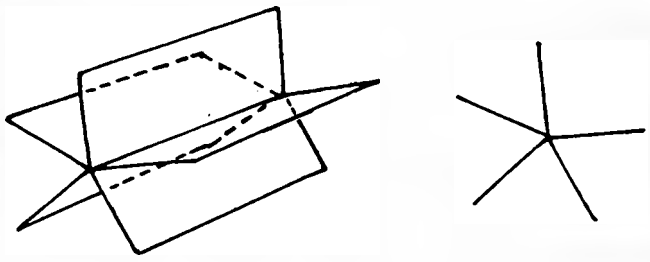


Figure 5: Looked at from a certain angle, a cone of planes meeting along a line in space looks like a network in the plane.

It will be useful to refer to the two-dimensional version of our problem, as we'll reduce one case of the three dimensional problem to two dimensions (see figure 5). Much of the notation and terminology is identical, but this should cause no confusion in practice.

In the planar case we consider finitely many immiscible fluids $F_1, \ldots, F_n$ meeting in the unit disk D. On the perimeter of D, we fix several points dividing the boundary of D into arcs, each of which is associated with a fluid. The fluids are separated by a *network* N of C^1 curves meeting only at their endpoints at finitely many *nodes*. Unlike the three dimensional case, nodes can only be points.

As before, we have the following definitions:

Definition: The region R_i assigned to each fluid F_i is the union of one or several connected components of $D \setminus N$. Each region is an open, not necessarily connected set, and we assume the regions R_i are disjoint.

Definition: To each region R_i is assigned a *reference arc* $S_i = \partial R_i \cap \partial D$.

Definition: For two regions R_i and R_j we define the *interface* $L_{ij} = \partial R_i \cap \partial R_j$.

A network N thus consists of the collection of interfaces; that is, $N = \{L_{ij}\}$. We say that N separates the S_i, and we say another network $M = \{M_{ij}\}$ is a *comparison network* if M separates the arcs S_i the same way N does.

The costs in the planar case are proportional to length instead of area, and we compute the cost of a network $N = \{L_{ij}\}$ by summing the weighted lengths:

$$\text{cost}(N) = \sum_{i<j} a_{ij} \text{ length}(L_{ij}). \tag{3}$$

We also assume the triangle inequality for the cost constants,

$$a_{ik} \leq a_{ij} + a_{jk}. \tag{4}$$

2.3 Existence and Regularity

Given a boundary condition, we would like to know that a minimizing configuration exists, and we would also hope that it consists of sufficiently nice surfaces.

Brian White [9], [10] has established that the boundaries of cost-minimizing clusters of prescribed volumes in $\mathbf{R}^n$ are smooth constant-mean-curvature surfaces which meet at singularities of Hausdorff dimension at most $n - 2$ [7]. In $\mathbf{R}^3$ we thus have surfaces meeting along curves or at points in minimizing configurations.

The following theorem establishes the existence and regularity of minimizing networks in the plane, discussed in greater detail in section 2.2.

Theorem 2.1 *[7, Theorem 3.3 and Corollary 4.5] Let $F_1, \ldots, F_n$ be a set of fluids with cost-constants a_{ij} satisfying strict triangle inequalities $a_{ik} < a_{ij} + a_{jk}$. Let $S_1, \ldots, S_n$ be perimeter segments of D, with each S_i assigned to fluid F_i. Then there exists a network N that is a minimizer for $F_1, \ldots, F_n$. Moreover, N consists of finitely many line segments.*

Frank Morgan [7] considers a more general problem with area constraints on the fluids in the context of flat chains with coefficients in a group representing the fluids. He uses geometric measure theory to prove that the minimizer consists of constant-curvature arcs. Since our problem places no area constraints on the fluids, the constant-curvature arcs are of course straight lines.

3 Flux

One particularly useful and general way of comparing the costs of various dividing surfaces uses the idea of a calibration. A calibration allows us to compare the cost of two dividing surfaces using the divergence theorem, so it will be useful to recall some basic properties of flux. Our discussion is borrowed largely from [2].

Given a piecewise smooth orientable surface Φ oriented by its unit normal vector field n, we define the flux of a vector field p through Φ to be

$$\text{flux}_\Phi(p) = \int_\Phi p \cdot n.$$

3.1 Properties of Flux

Elementary properties of the integral imply: For constants $a, b \in \mathbf{R}$ and vector fields p, q,

- $\text{flux}_\Phi(ap + bq) = a\text{flux}_\Phi(p) + b\text{flux}_\Phi(q)$.
- If $-\Phi$ denotes the surface oriented by the vector field $-n$, then
$$\text{flux}_{-\Phi}(p) = -\text{flux}_\Phi(p) = \text{flux}_\Phi(-p).$$
- If Ω is another surface and $\Omega \cap \Phi$ has measure zero, then $\text{flux}_\Phi(p) + \text{flux}_\Omega(p) = \text{flux}_{\Phi\cup\Omega}(p)$.
- If $\Phi = \partial R$ for some region R and p is divergence-free, then $\text{flux}_\Phi(p) = 0$.

3.2 Calculations with Flux

Consider a dividing surface $H = \{H_{ij}\}$ separating regions $\{R_i\}$ in the closed unit ball D with reference surfaces $S_i = \partial R_i \cap \partial D$. Define n_{ij} to be the unit normal vector field to H_{ij} pointing from R_i into R_j. Using this notation, $H_{ij} = -H_{ji}$ as oriented surfaces. Given the unit normal vector field orienting S_i pointing out from D, the vector fields $\{n_{ij} : j \neq i\}$ orient $\partial R_i = S_i \cup \bigcup_{j\neq i} H_{ij}$, and ∂R_i is an oriented manifold.

Suppose that p_i are divergence-free vector fields such that

$$p_j - p_i = a_{ij} n_{ij} \tag{5}$$

on H_{ij} whenever $H_{ij} \neq \emptyset$. Then

$$\text{flux}_{H_{ij}}(p_j - p_i) = \int_{H_{ij}} (p_j - p_i) \cdot n_{ij} = \int_{H_{ij}} a_{ij} = a_{ij}\ \text{area}(H_{ij}).$$

These types of vector fields allow us to compare the costs of two dividing surfaces, as partially demonstrated by the following Lemma:

Lemma 3.1 *With the p_i as in (5),*

$$\sum_{i<j} a_{ij}\ \text{area}(H_{ij}) = \sum_i \text{flux}_{S_i}(p_i).$$

Proof: Since each p_i is divergence-free, and by definition of ∂R_i,

$$\text{flux}_{S_i}(p_i) = \text{flux}_{\partial R_i}(p_i) - \text{flux}_{\cup_{j\neq i} H_{ij}}(p_i) = -\text{flux}_{\cup_{j\neq i} H_{ij}}(p_i).$$

Calculating gives

$$\begin{aligned}
\sum_{i<j} a_{ij}\ \text{area}(H_{ij}) &= \sum_{i<j} \text{flux}_{H_{ij}}(p_j - p_i) \\
&= \sum_{i<j} (\text{flux}_{H_{ij}}(p_j) - \text{flux}_{H_{ij}}(p_i)) \\
&= \sum_{i<j} (-\text{flux}_{H_{ji}}(p_j) - \text{flux}_{H_{ij}}(p_i)) \\
&= \sum_i (\sum_{j\neq i} -\text{flux}_{H_{ij}}(p_i)) \\
&= \sum_i -\text{flux}_{\cup_{j\neq i} H_{ij}}(p_i) \\
&= \sum_i \text{flux}_{S_i}(p_i).
\end{aligned}$$

□

Thus if we have two different dividing surfaces that separate the S_i and can find p_i as in (5) for both surfaces, we can use Lemma 3.1 to compare their costs. We will use this Lemma in section 2.2 for planar networks, and although it is not stated for networks, the same proof is valid.

4 The Local Behavior of Cost-Minimizing Networks in the Plane

Given a network, we wish to recognize when it is minimizing. For any singularity in this network, we can look in a neighborhood of this singularity, chosen to be a unit disk with line segments meeting at the center. If the network is minimizing, then the subnetwork in this disk will also be minimizing. We will from now on focus on networks consisting of line segments meeting at a node in the center of D.

The line segments meeting in the center of D all have unit length to make the calculations in Lemmas 4.2 and 4.4 work. Later it will be useful to change from working in D to working in a polygon P (see section 5.1). Our proofs for D also work for P. The only issue is Lemma 3.1, which applies to P since ∂P is piecewise-smooth.

4.1 A Sufficient Condition for Cost-Minimization

Gary Lawlor and Frank Morgan [5] prove that N minimizes cost among all comparison networks containing fluids $F_1, \ldots, F_n$ if and only if certain point-placing conditions are satisfied (figure 6).

Definition: Suppose that a network N consisting of line segments L_{ij} separates regions $R_1, \ldots, R_n$. For each $L_{ij} \neq \emptyset$, let n_{ij} be its unit normal, pointing from region R_i into R_j. We say $\{p_1, \ldots, p_n\}$ *calibrate* N with respect to $F_1, \ldots, F_n$ if

(P1) $p_j - p_i = a_{ij} n_{ij}$ when $L_{ij} \neq \emptyset$, and

(P2) $|p_j - p_i| \leq a_{ij}$ for all pairs (i, j).

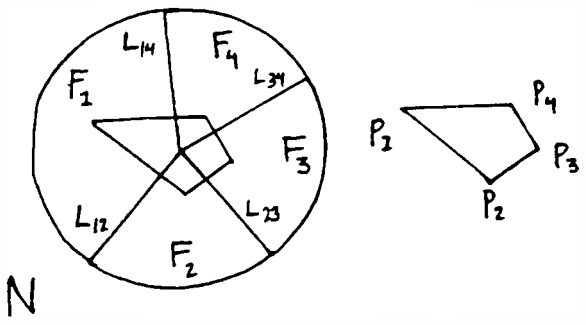


Figure 6: Calibrating points for a network $N = \{L_{ij}\}$.

Proposition 4.1 *[5, Theorem 2.1] Consider a network N consisting of line segments L_{ij} with unit normals n_{ij}. If N is calibrated with respect to $F_1, \ldots, F_n$, then for any comparison network $M = \{M_{ij}\}$,*

$$\text{cost}(N) \leq \text{cost}(M).$$

It may be that we have p_i even though $R_i = \emptyset$, in which case such a network is minimizing even among fluids which might not be present.

Proof: If there are points which calibrate N with respect to $F_1, \ldots, F_n$, then for any comparison network $M = \{M_{ij}\}$ we have

$$\begin{aligned}
\text{cost}(N) &= \sum_{i<j} a_{ij} \,\text{length}(L_{ij}) \\
&= \sum_{i<j} \text{flux}_{L_{ij}}(p_j - p_i) \\
&= \sum_{i} \text{flux}_{S_i}(p_i) \qquad \text{(by Lemma 3.1)} \\
&= \sum_{i<j} \text{flux}_{M_{ij}}(p_j - p_i) \\
&\leq \sum_{i<j} a_{ij} \cdot \text{length}(M_{ij}) \\
&= \text{cost}(M).
\end{aligned}$$

Hence if N is calibrated, it is a minimizer. □

We now derive some necessary conditions for minimization.

4.2 Necessary Conditions for Cost-Minimization

We will now derive some necessary conditions for minimization at a node, which will show that if a node is minimizing, then it can be calibrated, proving that locally minimization and the existence of a calibration are equivalent. First we need an easily proved identity from vector calculus. If we move the head of a unit vector L_{ij} in the direction of any vector u, then the initial rate of change of length is

$$\frac{d}{dt}\text{length}(L_{ij}) = u \cdot L_{ij}.$$

Since cost is proportional to length, this also produces the formula

$$\frac{d}{dt}\text{cost}(L_{ij}) = u \cdot a_{ij} L_{ij}.$$

Lemma 4.2 *Suppose a network N, consisting of n line segments $L_{i(i+1)}$ that meet at a node, is a minimizer for fluids $F_1, \ldots, F_n$. Then (indexing modulo n),*

$$\sum_{i=1}^{n} a_{i(i+1)} n_{i(i+1)} = 0.$$

Proof: If we move the node in the direction of any vector u, then the initial rate of change of cost will be

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \text{cost}(N) &= \sum_{i=1}^{n} u \cdot a_{i(i+1)} L_{i(i+1)} \\
&= u \cdot \sum_{i=1}^{n} a_{i(i+1)} L_{i(i+1)}.
\end{aligned}$$

Since N is a minimizer, we know this initial rate of change must be zero. Since u was arbitrary, it follows that $\sum_{i=1}^{n} a_{i(i+1)} L_{i(i+1)} = 0$. By rotation, $\sum_{i=1}^{n} a_{i(i+1)} n_{i(i+1)} = 0$. □

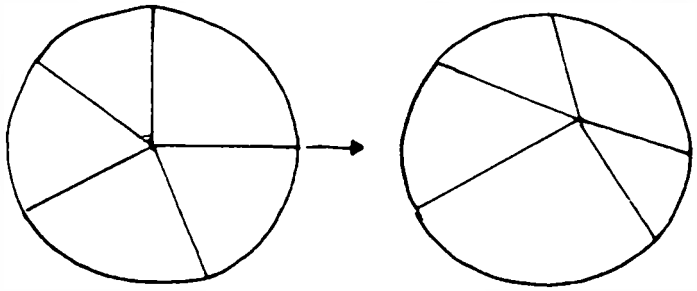


Figure 7: Moving the node of a network will not reduce cost if the network is minimizing.

Now we prove a different version of Lemma 4.2 to show that if the sum of the weighted unit normals is not zero, then we can move the node and achieve a first order savings in cost.

Lemma 4.3 *Consider a network N of n line segments $L_{i(i+1)}$ of unit length with unit normals $n_{i(i+1)}$ that meet at a node. If*

$$\sum_{i=1}^{n} a_{i(i+1)} n_{i(i+1)} \neq 0$$

(indexing modulo n), then there is a continuous family of networks $N(t) = \{L_{i(i+1)}\}_{i=1}^{n}$ such that

$$\left.\frac{d}{dt}\right|_{t=0} \text{cost}(N(t)) \quad < \quad 0.$$

Proof: If $\sum_{i=1}^{n} a_{i(i+1)} n_{i(i+1)} \neq 0$, then by rotation, $\sum_{i=1}^{n} a_{i(i+1)} L_{i(i+1)} \neq 0$. Let u be a unit vector in the direction $-\sum_{i=1}^{n} a_{i(i+1)} L_{i(i+1)}$. Consider the initial rate change of cost of a continuous family of networks $N(t)$ (where $N(0) = N$) as we displace the node in direction u.

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \text{cost}(N(t)) &= \sum_{i=1}^{n} u \cdot a_{i(i+1)} L_{i(i+1)} \\
&= u \cdot \sum_{i=1}^{n} a_{i(i+1)} L_{i(i+1)} \\
&= -\left| \sum_{i=1}^{n} a_{i(i+1)} L_{i(i+1)} \right| \\
&< 0.
\end{aligned}$$

Thus N is not minimizing and this parametrization gives a first order reduction in cost. □

The geometric information used in the following Lemma and indicated in figure 8 is also contained in the idea of supercalibration (section 4.3) for present fluids.

Lemma 4.4 *Suppose a network N, consisting of n line segments $L_{i(i+1)}$ that meet at a node, is a minimizer for fluids $F_1, \ldots, F_n$. Then for any $s < n$,*

$$|a_{12} n_{12} + \cdots + a_{(s-1)s} n_{(s-1)s}| \leq a_{1s}.$$

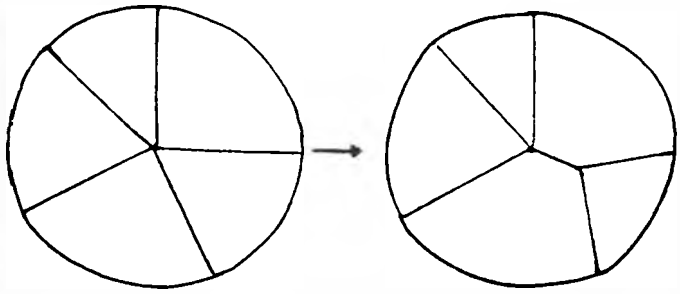


Figure 8: Creating a new segment in a minimizing network increases cost.

Proof: Let u be a unit vector in the direction of $-(a_{12}L_{12} + \cdots + a_{(s-1)s}L_{(s-1)s})$. Consider the initial rate of change as we move the associated line segments $L_{12}, \ldots, L_{(s-1)s}$ away from the original node, creating a new node. Parametrizing this change by the length of the new line segment L_{1s}, the initial change in cost is

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \text{cost}(N(t)) &= \sum_{i=1}^{s-1} u \cdot a_{i(i+1)}L_{i(i+1)} + a_{1s} \\
&= u \cdot \sum_{i=1}^{s-1} a_{i(i+1)}L_{i(i+1)} + a_{1s} \\
&= -\left|\sum_{i=1}^{s-1} a_{i(i+1)}L_{i(i+1)}\right| + a_{1s} \\
&\geq 0.
\end{aligned}$$

since N is a minimizer. Thus $|a_{12}L_{12} + \cdots + a_{(s-1)s}L_{(s-1)s}| \leq a_{1s}$, and by rotation $|a_{12}n_{12} + \cdots + a_{(s-1)s}n_{(s-1)s}| \leq a_{1s}$. □

Proposition 4.5 *Consider a network N consisting of n line segments $L_{i(i+1)}$ that meet at a node. If N is minimizing, then there are points $\{p_1, \ldots, p_n\}$ which calibrate N.*

Proof: We will describe an algorithm for finding the calibrating points. Let $p_1 = 0$, and define p_j inductively by $p_j = p_{j-1} + a_{(j-1)j}n_{(j-1)j}$ for all $1 < j \leq n$. Then of course $p_{j+1} - p_j = a_{(j+1)j}n_{(j+1)j}$, and Lemma 4.2 gives $p_1 - p_n = a_{1n}n_{1n}$, so the points $\{p_1, \ldots, p_n\}$ satisfy (P1). By Lemma 4.4,

$$|p_j - p_i| = |a_{i(i+1)}n_{i(i+1)} + \cdots + a_{(j-1)j}n_{(j-1)j}| \leq a_{ij}.$$

Hence (P2) is satisfied as well, so that the points $\{p_1, \ldots, p_n\}$ calibrate N, and thus N is minimizing for fluids $F_1, \ldots, F_n$. □

Remark: Note that we gain no information from fluids that are allowed but do not exist in the minimizing configuration from Lemmas 4.2 and 4.4. Thus the necessity of point-placing only follows for present fluids. We also give an alternative algorithm for placing points in Theorem 6.1 which generalizes to higher dimensions quite easily.

Remark: For a minimizing network N of line segments $L_{i(i+1)}$ meeting at a node, we can place points $\{p_1,\ldots,p_n\}$ so that $p_i \in R_i$ for each region R_i if we allow ourselves to normalize the cost constants, keeping their proportions the same. First normalize the cost constants so that all points will lie in D. Second, note that no two adjacent line segments meet at an angle of more than 180° in a minimizing configuration, else we could reduce cost by moving the node. It follows that the points $\{p_1,\ldots,p_n\}$ which calibrate N form a convex polygon with edges $\overline{p_ip_{i+1}}$ (indexing modulo n). Choose a point c in the interior of this polygon. Translate and renormalize the cost constants if necessary so that c lies at the node of N and all points still lie in D. For each pair of adjacent vertices p_i, p_{i+1}, the interior angles of the triangle formed by c, p_i, p_{i+1} are all acute. Hence the perpendicular bisector to $\overline{p_ip_{i+1}}$ through c lies inside this triangle. These perpendicular bisectors must match the line segments in the network exactly, so that p_i lies between the line segments $L_{(i-1)i}$ and $L_{i(i+1)}$, and hence in the region R_i.

Using Propositions 4.1 and 4.5 we have

Theorem 4.6 *[5, Theorems 2.1 and 2.5] A network N consisting of n line segments $L_{i(i+1)}$ meeting at a node is minimizing with respect to $F_1,\ldots,F_n$ if and only if it is calibrated with respect to those fluids.*

4.3 Supercalibration

Definition: Consider a network $N = \{L_{ij}\}$ calibrated by $\{p_1,\ldots,p_n\}$, and a comparison network $M = \{K_{ij}\}$. Let n_{ij} be a unit normal to K_{ij}, θ_{ij} the angle between K_{ij} and L_{ij}. We say that $\{p_1,\ldots,p_{n+m}\}$ *supercalibrate* M with respect to N and $\{p_1,\ldots,p_n\}$ if

(SC1) If $i,j \le n$ and $K_{ij} \neq \emptyset$, $(p_j - p_i)\cdot n_{ij} = a_{ij}\cos\theta_{ij}$ (θ_{ij} is typically some small angle).

(SC2) If $i > n$ or $j > n$ and $K_{ij} \neq \emptyset$, then $p_j - p_i = \alpha_{ij}n_{ij}$ with $\alpha_{ij} \ge a_{ij}$ and strict inequality for at least one pair (i,j).

Condition (SC1) follows from basic geometry, but we include it to stress the fact that, to first order, L_{ij} and K_{ij} have essentially the same length when θ_{ij} is small. Supercalibrated networks arise when we consider allowing additional fluids (also see the remark following Theorem 4.7). The new edges inserted produce a first order reduction in cost, whereas tilting the original edges to accommodate them produces a second order change in cost (figure 9).

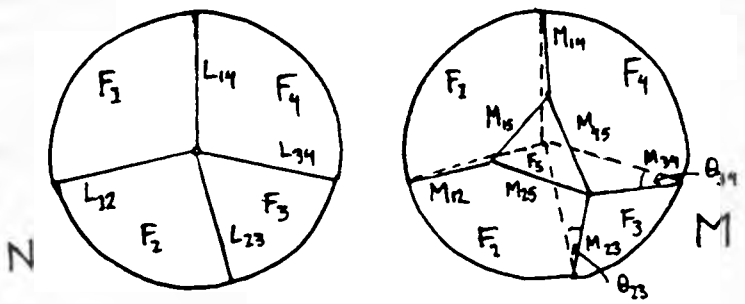


Figure 9: A supercalibrated network K with respect to a calibrated network N.

Theorem 4.7 *[2, Theorem 3.9] Suppose that $\{p_1,\ldots,p_n\}$ calibrate network N that consists of line segments L_{ij} $(i,j<n)$ meeting at a node. Let $\{N(t)\}$ be a continuous family of networks consisting of line segments $K_{ij}(t)$. Suppose that $N(0)=N$ and that $\{N(t)\}$ is parametrized by the length of some non-empty segment K_{bc} where $c>n$. Suppose that each $N(t)$ is supercalibrated by points $\{p_1,\ldots,p_{n+m}\}$, where the associated functions $\theta_{ij}(t)$ are smooth and $\theta_{ij}(0)=0$.*

Then there exists $t_0>0$ such that $\mathrm{cost}(N(t))<\mathrm{cost}(N)$ for all $0<t<t_0$.

Proof: Define functions $B_{ij}(t)=\mathrm{length}(K_{ij}(t))$ and consider the costs of N and $N(t)$.

$$\begin{aligned}
\mathrm{cost}(N) &= \sum_{i<j\le n} a_{ij}\,\mathrm{length}(L_{ij}) \\
&= \sum_{i<j\le n} \mathrm{flux}_{L_{ij}}(p_j-p_i) \\
&= \sum_{i=1}^{n} \mathrm{flux}_{S_i}(p_i) \qquad \text{(by Lemma 3.1)} \\
&= \sum_{i=1}^{n+m} \mathrm{flux}_{S_i}(p_i) \qquad \text{(because } S_i=\emptyset \text{ for } i>n\text{)} \\
&= \sum_{i<j\le n+m} \mathrm{flux}_{K_{ij}}(p_j-p_i). \\
\mathrm{cost}(N(t)) &= \sum_{i<j\le n+m} a_{ij}B_{ij}(t) \\
&= \sum_{i<j\le n} a_{ij}B_{ij}(t) + \sum_{\substack{i<j\\ n<j}} a_{ij}B_{ij}(t).
\end{aligned}$$

Combining the above we obtain

$$\begin{aligned}
\mathrm{cost}(N(t))-\mathrm{cost}(N) &= \sum_{i<j\le n} (a_{ij}B_{ij}-\mathrm{flux}_{K_{ij}}(p_j-p_i)) \\
&\quad + \sum_{\substack{i<j\\ n<j}} (a_{ij}B_{ij}-\mathrm{flux}_{K_{ij}}(p_j-p_i)) \\
&= \sum_{i<j\le n} a_{ij}(1-\cos\theta_{ij})B_{ij} + \sum_{\substack{i<j\\ n<j}} (a_{ij}-\alpha_{ij})B_{ij}.
\end{aligned}$$

We now calculate the derivative of $\mathrm{cost}(N(t))$ at zero. Recall that $\theta_{ij}(0)=0$, and that $N(0)=N$, and thus $K_{ij}(0)=\emptyset$ if $j>n$. Thus $B'_{ij}(0)\ge 0$ for all $j>n$. Furthermore, since $\{N(t)\}$ is parametrized by the length of some non-empty segment K_{bc} where $c>n$, we know that $B'_{bc}(0)>0$.

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \mathrm{cost}(N(t)) &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{cost}(N(t))-\mathrm{cost}(N) \\
&= \sum_{i<j\le n} a_{ij}\left(\sin\theta_{ij}(0)\,\theta'_{ij}(0)\,B_{ij}(0)+(1-\cos\theta_{ij}(0))\,B'_{ij}(0)\right) \\
&\quad + \sum_{\substack{i<j\\ n<j}} (a_{ij}-\alpha_{ij})B'_{ij}(0) \\
&= \sum_{\substack{i<j\\ n<j}} (a_{ij}-\alpha_{ij})B'_{ij}(0)
\end{aligned}$$

$$< \quad 0.$$

Thus for sufficiently small $t_0 > 0$, $\text{cost}(N(t)) < \text{cost}(N)$ for $0 < t < t_0$.

□

Remark: We can prove a similar theorem with this method providing a new proof of the equivalence of point-placing and minimization in Theorem 4.6 when no new fluids are allowed. If we only assume (P1) is true for a given network $N = \{L_{ij}\}$, we know that there are points $\{p_1, \ldots, p_n\}$ such that $p_i - p_j = a_{ij}n_{ij}$ whenever $L_{ij} \neq \emptyset$, where n_{ij} the unit normal to L_{ij}. If such a network is minimizing, we can then prove that for any pair (i,j), $|p_i - p_j| \leq a_{ij}$, as stated below.

Theorem 4.8 *Suppose that a network $N = \{L_{ij}\}$ with unit normals n_{ij} consisting of n line segments meeting at a node is minimizing. Then there are points $\{p_1, \ldots, p_n\}$ such that $p_j - p_i = a_{ij}n_{ij}$ whenever $L_{ij} \neq \emptyset$, and for any pair (i,j), $|p_i - p_j| \leq a_{ij}$.*

Proof: In the proof of Proposition 4.5, we showed that Lemma 4.2 implied the existence of points $\{p_1, \ldots, p_n\}$ such that $p_i - p_j = a_{ij}n_{ij}$ whenever $L_{ij} \neq \emptyset$. If there is a pair (b,c) such that $|p_b - p_c| > a_{bc}$, we parametrize a continuous family of networks $\{N(t)\}$ consisting of line segments $K_{ij}(t)$ parametrized by $t = \text{length}(K_{bc})$, where $p_b - p_c = \alpha_{bc}n_{bc}$, and $\alpha_{bc} > a_{bc}$ (see figure 10).

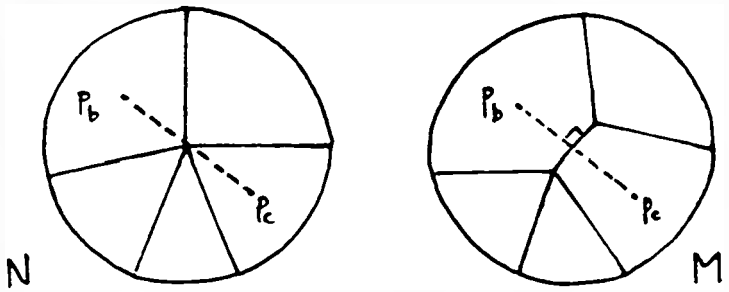


Figure 10: When $|p_b - p_c| > a_{bc}$, we can parametrize a continuous family of supercalibrated comparison networks.

Now $N(0) = N$, and the associated functions $B_{ij}(t) = \text{length}(K_{ij}(t))$ satisfy

$$a_{ij}B_{ij} - \text{flux}_{K_{ij}}(p_j - p_i) = a_{ij}(1 - \cos\theta_{ij})B_{ij}.$$

We differentiate

$$\text{cost}(N(t)) - \text{cost}(N) = \sum_{\substack{i<j \\ (i,j)\neq(b,c)}} a_{ij}(1 - \cos\theta_{ij})B_{ij} + (a_{bc} - \alpha_{bc})t$$

as in Theorem 4.7, and obtain

$$\left.\frac{d}{dt}\right|_{t=0} \text{cost}(N(t)) < 0.$$

Thus there is t_0 such that $\text{cost}(N(t)) < \text{cost}(N)$ for $0 < t < t_0$, which contradicts the assumption that N is minimizing. Hence $|p_i - p_j| \leq a_{ij}$ for all pairs (i,j). □

4.4 Adding One Fluid to n in the Plane

The proof of sufficiency of (P1) and (P2) in Theorem 4.6 tells us that if our configuration is calibrated for $F_1, \ldots, F_{n+m}$ it is a minimizer for $F_1, \ldots, F_{n+m}$. However, Lemmas 4.2 and 4.4 used to prove necessity do not recover any information about fluids not present in our configuration, so we don't know if a minimizer can be calibrated for fluids not present. Theorem 4.10 considers the case of allowing one additional fluid, so that we assume a network N is calibrated for fluids $F_1, \ldots, F_n$, but assume no choice of p_{n+1} can be made to calibrate N for $F_1, \ldots, F_{n+1}$. In such a situation, it suffices to prove that we can find a cheaper comparison network. Such networks will consist of central triangles, indicated in figure 12.

We can restate the hypothesis of allowing an additional fluid as follows. Construct disks $D_i = D(p_i, a_{i(n+1)})$ for each $i \in \{1, \ldots, n\}$. If $\cap_i D_i \neq \emptyset$, then we can place p_{n+1} to satisfy (P2). Corresponding to the case we are interested in is the case when $\cap_i D_i = \emptyset$. In such a case, it will be useful to employ Helly's Theorem:

Theorem 4.9 *[6, Theorem 6.2] Let $F = \{B_1, \ldots, B_n\}$ be a family of n convex sets in $\mathbf{R}^2$ with $n \geq 3$. If every subfamily of three sets in F has nonempty intersection, then $\cap_{i=1}^n B_i \neq \emptyset$.*

We also need the fact that the triangle inequality implies the disks intersect pairwise. Since $|p_j - p_i| = \alpha \leq a_{ij}$, the triangle inequality implies that $\alpha \leq a_{ij} \leq a_{i(n+1)} + a_{j(n+1)}$, so that $D_i \cap D_j \neq \emptyset$ (see figure 11).

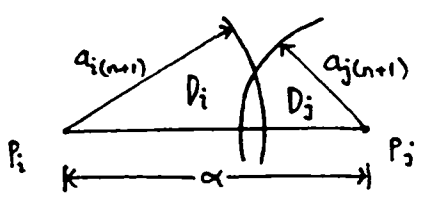


Figure 11: The triangle inequality implies disks D_j and D_i intersect pairwise.

A more involved discussion of allowing additional fluid appears in [2]. They conjecture a counterexample to the extending Theorem 4.6 to allow two additional fluids. We now prove that even in the presence of fluid with no reference surface, the existence of a calibration is still necessary and sufficient for minimization.

Theorem 4.10 *[2, Theorem 5.1] Consider a network N consisting of n line segments $L_{i(i+1)}$ that meet at a node. Then N is a minimizer for fluids $F_1, \ldots, F_{n+1}$ if and only if there are points $\{p_1, \ldots, p_n, p_{n+1}\}$ that calibrate it with respect to these fluids.*

Proof: Theorem 4.6 proves that if the calibrating points $\{p_1, \ldots, p_n, p_{n+1}\}$ exist, then N is minimizing. Note that F_{n+1} does not appear in the original configuration. Conversely, we assume that N cannot be calibrated with respect to $F_1, \ldots, F_{n+1}$. Again, Theorem 4.6 assures us that a minimizing network N can be calibrated for $F_1, \ldots, F_n$. Thus we only need to consider when no choice of p_{n+1} can be made to satisfy (P1) and (P2) given points $\{p_1, \ldots, p_n\}$ which calibrate N with respect to $F_1, \ldots, F_n$. We will describe a general procedure for producing a cheaper network in such a case.

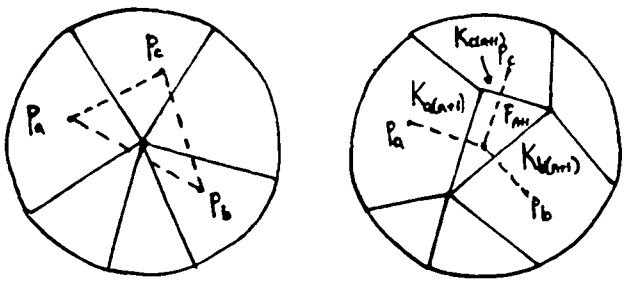


Figure 12: The construction of a cheaper network with additional fluid.

Construct disks $D_i = D(p_i, a_{i(n+1)})$ about points $p_1, \ldots, p_n$. Since no point can be chosen for fluid F_{n+1}, we know that $\cap_{i=1}^{n} D_i = \emptyset$. By Helly's Theorem 4.9, there exist $1 \leq a < b < c \leq n$ such that $D_a \cap D_b \cap D_c = \emptyset$. These disks bound a mutually omitted region, and we may choose coordinates so that this region contains the origin.

Place p_{n+1} at the origin. Construct triangle $T = \{K_{a(n+1)}, K_{b(n+1)}, K_{c(n+1)}\}$ with $K_{j(n+1)} \perp \overline{p_j p_{n+1}}$ for $j \in \{a, b, c\}$. Let fluid F_{n+1} occupy the interior of this triangle. Parametrize a continuous family of networks $\{N(t)\} = \{K_{ij}(t)\}$ such that $N(0) = N$ and $t = \text{length}(K_{a(n+1)})$. $K_{i(i+1)}$ ends at $K_{a(n+1)} \cap K_{b(n+1)}$ for $a \leq i < b$, at $K_{b(n+1)} \cap K_{c(n+1)}$ for $b \leq i < c$, and at $K_{c(n+1)} \cap K_{a(n+1)}$ for $c \leq i < a$ (see figure 12).

If θ_{ij} is the angle between $L_{ij} = L_{ij}(0)$ and $L_{ij}(t)$, then $\theta_{ij}(0) = 0$ and the θij are smooth. Finally let $\alpha_{i(n+1)} = |p_{n+1} - p_i|$ for $i \in \{a, b, c\}$. Then the networks $\{N(t)\}$ are supercalibrated for small t and by Theorem 4.7 there is a $t_0 > 0$ such that $\text{cost}(N(t)) < \text{cost}(N)$ for all $0 < t < t_0$. □

4.4.1 Minimizers Need Not Be Unique

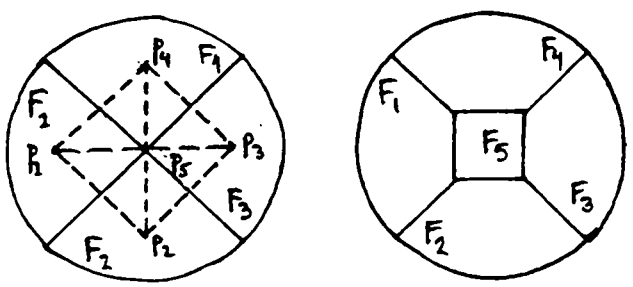


Figure 13: Points $p_1, \ldots, p_5$ calibrate a continuous family of networks with a possibly empty central square containing fluid F_5.

Consider five fluids $F_1, \ldots, F_5$, with costs

- $a_{12} = a_{23} = a_{34} = a_{14} = 1$,

- $a_{13} = a_{24} = \sqrt{2}$, and
- $a_{i5} = 1/\sqrt{2}$ for $i = 1, \ldots, 4$.

Let the boundary condition on D be given by letting S_i be one fourth an arc of the circle for fluids F_1, F_2, F_3, F_4. Then placing points p_1, p_2, p_3, p_4 in a square of unit length and point p_5 in the center of this square calibrates a continuous family of networks, seen in figure 13. Note that the disks D_i for fluid five as used in Theorem 4.10 intersect in a single point. All networks in this family consist of part of an X with a (possibly empty) central square whose vertices lie on the $L_{i(i+1)}$ with fluid F_5 occupying the interior. All of these networks must have the same cost, since they are all calibrated.

4.4.2 The Steiner Problem

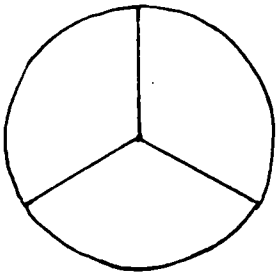


Figure 14: In length minimizing networks, segments meet in threes at 120° angles.

When costs to separate all fluids are 1, then given any number of fluids and any boundary condition on the boundary of the unit disk, it follows that any node must consist of three segments meeting at 120°, as in figure 14. This is because the only points which can calibrate such a node must satisfy $p_j - p_i = n_{ij}$ when $L_{ij} \neq \emptyset$ and $|p_j - p_i| \leq 1$ for all pairs (i, j). Whenever $L_{ij} \neq \emptyset$, we may consider the associated line segment $\overline{p_i p_j}$ and thus the unit polygon P formed by such points. It is easy to prove that the calibrating points form a convex polygon. The only convex unit polygon P whose vertices p_i satisfy $|p_j - p_i| \leq 1$ is an equilateral triangle by an easy angle argument. Thus the calibrating points for an auxiliary node in a Steiner network form an equilateral triangle, and it follows that such a node consists of three segments meeting at 120° angles. Our problem is just one of many generalizations of the Steiner problem. Two good resources containing many more generalizations are [3] and [4].

5 Cones of Planes in Space

As in the case of a network in the plane, given a configuration of planes and a node, we place a ball or polyhedron around that node and reason that if the original configuration is minimizing, then so is the cone inside the magnified region. In three dimensions, there are two types of singularities: planes can meet along a line, or several planes can meet along lines which meet at a point. We thus suppose that our surface C, called a *polyhedral cone*, or just a *cone*, consists of planes meeting along a line through the center of the unit ball D, or that C consists of several planes meeting along lines meeting at the center of D.

5.1 A Sufficient Condition for Minimization

Definition: Consider a surface H of planar interfaces H_{ij} that separates regions $R_1, \ldots, R_n$. For each $H_{ij} \neq \emptyset$, let n_{ij} be its unit normal, pointing from R_i into R_j. We say that points $\{p_1, \ldots, p_n\}$ *calibrate* C if

(P1) $p_j - p_i = a_{ij} n_{ij}$ when $H_{ij} \neq \emptyset$, and

(P2) $|p_j - p_i| \leq a_{ij}$ when $H_{ij} = \emptyset$.

Of course, if H_{ij} is not connected, then each connected piece must be parallel and oriented the same way. If we think of the p_i as constant vector fields, they satisfy the hypotheses of Lemma 3.1.

Proposition 5.1 *[5, Theorem 2.1] Consider a surface H consisting of planes H_{ij} with unit normals n_{ij}. If H is calibrated with respect to $F_1, \ldots, F_n$, then for any comparison surface $K = \{K_{ij}\}$,*

$$\text{cost}(H) \leq \text{cost}(K).$$

Proof: Let $\{p_1, \ldots, p_n\}$ calibrate H. Then for any comparison surface $K = \{K_{ij}\}$ we can compute

$$\begin{aligned}
\text{cost}(H) &= \sum_{i<j} a_{ij} \,\text{area}(H_{ij}) \\
&= \sum_{i<j} \text{flux}_{H_{ij}}(p_j - p_i) \\
&= \sum_{i} \text{flux}_{S_i}(p_i) \qquad \text{(by Lemma 3.1)} \\
&= \sum_{i<j} \text{flux}_{K_{ij}}(p_j - p_i) \\
&\leq \sum_{i<j} a_{ij} \,\text{area}(K_{ij}) \\
&= \text{cost}(K).
\end{aligned}$$

Thus H is minimizing among surfaces separating the S_i the same way H does. □

Again it may be that we have a point p_i for a region $R_i = \emptyset$, so that H is minimizing even in the presence of fluid F_i which is not present in the configuration.

Though technically we work in the closed unit ball D with a boundary condition, it will be convenient to change to working in a prism. Since much of our work consists of finding comparison surfaces, if we place a prism Ψ inside the ball D and let it inherit the boundary condition given by its intersection with the polyhedral cone C, then if we can find appropriate comparison surfaces in Ψ, they are also comparison surfaces in D. The only issue is applying Lemma 3.1, but we can use it since the boundary of Ψ is piecewise smooth. We will also find it convenient to rescale Ψ to a size suitable for carrying out our computations.

5.2 Cones of Planes Meeting Along a Line

Now we concentrate on cones of planes meeting along a line. When looked at on end, such cones look like networks in the plane (figure 16). We will reduce this case to the case of networks in the plane. We use Theorems 4.6, 4.7, and 4.10 to prove the equivalence of minimization and point-placing. In order to facilitate the statements of theorems, we will introduce some terminology first.

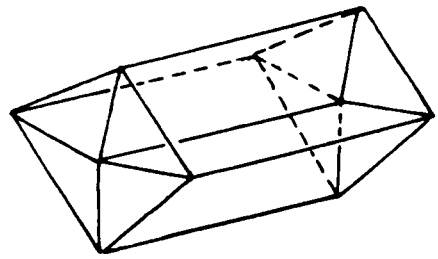


Figure 15: A cone of planes meeting along a line.

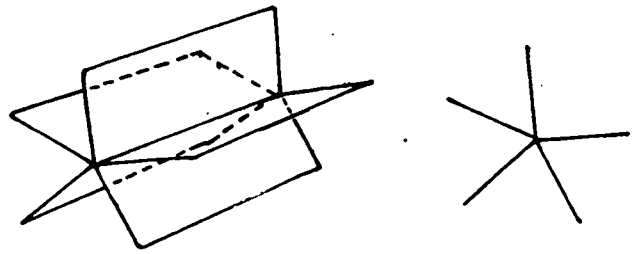


Figure 16: When looked at on end, cones of planes look like networks in the plane.

Definition: We say a convex polygon P with vertices $\{v_1, \ldots, v_n\}$ is a *unigon* if there is a point p in the interior of P such that $d(p, v_i) = 1$ for all i. We refer to p as the *center* of P. The various segments $\overline{v_i v_j}$ are the *faces* of P.

Definition: The *n-agonal prism* Ψ is the surface of the solid $P \times [0,1]$ where P is a unigon.

We call $P_0 = P \times \{0\}$, $P_1 = P \times \{1\}$ the *caps* of an n-agonal prism Ψ.

L will always be the line joining the centers of caps P_0, P_1 of Ψ, and we call a *long edge* of P is one joining a vertex of P_0 to the matching vertex of P_1.

One final definition:

Definition: The *line-cone* over the n-agonal prism Ψ is the set of n half planes H_{ij} meeting along L, each passing through exactly one long edge of Ψ.

The boundary condition we impose is that given by the intersection of the various half planes with the n-agonal prism Ψ. The intersection $\partial S_i = (\bigcup_{j \neq i} H_{ij}) \cap \Psi$ are curves which define disjoint, open connected regions of Ψ which we again call the reference surfaces S_i. Note that the S_i are piecewise smooth. For a line cone C there is an associated network $N = C \cap P_0$. N consists of line segments L_{ij} of unit length separating the faces of P_0. We shall label the vertices of P_0 v_{ij} where L_{ij} is the line segment beginning at v_{ij}. Notice that

$$\text{cost}(C) = \text{cost}(N) \tag{6}$$

since the prism Ψ has unit length.

As in the two dimensional case, we will prove that the existence of a calibration is necessary and sufficient for minimization. Equation (6) suggests we reduce the case of planes meeting along a line to the two dimensional version of lines meeting at a point. However, the two dimensional comparison surfaces would change the boundary condition on the caps of Ψ, a problem we fix by working in a smaller version of the prism on which we have no such boundary condition. However, we must find a surface connecting the new network on the caps of the smaller prism to the original network on the caps of Ψ to satisfy the boundary condition. For this reason we will need to discuss a special class of surfaces called ruled surfaces.

5.3 Ruled Surfaces

Consider two line segments $L_1(s)$, $L_2(s)$ contained in parallel planes in space, parametrized by the same variable $s \in [0,1]$. The *ruled surface joining* L_1 *and* L_2 is the collection of line segments $\{\ell(s,u)\}$ for $(s,u) \in [0,1]^2$ such that

$$\ell(s,0) = L_1(s) \text{ and } \ell(s,1) = L_2(s). \tag{7}$$

where

$$\ell(s,u) = u\ell(s,0) + (1-u)\ell(s,1). \tag{8}$$

Consider a network N consisting of line segments $L_{i(i+1)}$ that meet at a node $\bar{o}$ in a unigon $P \times \{0\}$. For $L_{ij} = \emptyset$, define $L_{ij} = \bar{o}$. Let $N(t)$ be a continuous family of comparison networks in $P \times \{\delta\}$ consisting of line segments $L_{ij}(t)$ such that $N(0) = N$ and $L_{ij}(t) \neq \emptyset$ whenever $L_{ij} \neq \emptyset$. When $L_{ij}(t) \neq \emptyset$, we can define a ruled surface $K^t_{ij}(t)$ joining L_{ij} to $L_{ij}(t)$.

Since N and $N(t)$ separate the boundary of P the same way, the projection of $L_{ij}(t)$ into $P \times \{0\}$ is a line segment emanating from the same point v_{ij} on ∂P as L_{ij}. Note that $L_{ij}(0) = L_{ij}$, and parametrize L_{ij} and $L_{ij}(t)$ by

$$\begin{aligned} L_{ij}(s,0) &= (1-s)v_{ij} + s\bar{o} \\ L_{ij}(s,t) &= (1-s)v_{ij} + s\bar{o}(t). \end{aligned}$$

We'll consider the above as the parametrization of $L_{ij}(t)$ both in $P \times \{\delta\}$ and in its projection into $P \times \{0\}$. Here $\bar{o}(t)$ is the (possibly new) node at which $L_{ij}(t)$ ends. If $L_{ij} = \emptyset$, then $v_{ij} = \bar{o}$. Of course, $N(t)$ may have several new nodes, so we will number them in the network to differentiate between them.

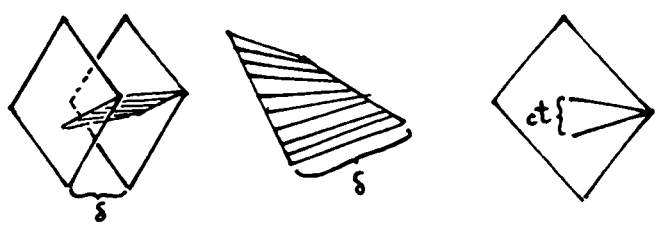


Figure 17: A ruled surface joining two parallel line segments in space for which we have an upper bound on the area.

As we noted before, the projection of $L_{ij}(t)$ into $P \times \{0\}$ is a line segment emanating from the same boundary point v_{ij} as L_{ij}. Since $d(P \times \{0\}, P \times \{\delta\}) = \delta$, it follows that the longest line

segment in the ruled surface is $\ell(1,u)$ with the $\ell(s,u)$ as in equation (7). If the parameter t is chosen so that $d(\overline{o},\overline{o}(t)) = ct$ for some constant $c > 0$ (here we consider $\overline{o}$, $\overline{o}(t) \in P \times \{0\}$), then we can calculate length$(\ell(1,u)) = \sqrt{\delta^2 + (ct)^2}$. Let $M_{ij}(t) = \max\{\text{length}(L_{ij}), \text{length}(L_{ij}(t))\}$. Then we get an upper bound on the area of the ruled surface (see figure 17):

$$\text{area}(K'_{ij}(t)) \leq M_{ij}\sqrt{\delta^2 + (ct)^2}. \tag{9}$$

Note that equality holds in equation (9) when $t = 0$ since $N(0) = N$ at $t = 0$. Let $K'(t) = \{K'_{ij}(t)\}$. Since area$(K'_{ij}(t))$ is an increasing function of t, we know that the function

$$\text{cost}(M(t)) = \sum_{i<j} a_{ij} M_{ij}(t)\sqrt{\delta^2 + (ct)^2}, \tag{10}$$

where $M(t) = \sum_{i<j} M_{ij}(t)$ has the following properties:

- cost$(K(t)) \leq$ cost$(M(t))$ for all t.
- cost$(N)\delta =$ cost$(K(0)) =$ cost$(M(0))$.

Since the functions cost$(K(t))$ and cost$(M(t))$ are differentiable, we find that

$$\left.\frac{d}{dt}\right|_{t=0} \text{cost}(K(t)) \leq \left.\frac{d}{dt}\right|_{t=0} \text{cost}(M(t)). \tag{11}$$

by an easy limit argument.

We can easily calculate

$$\left.\frac{d}{dt}\right|_{t=0} \text{cost}(M(t)) = M'(0)\delta.$$

Thus we obtain

$$\left.\frac{d}{dt}\right|_{t=0} \text{cost}(K(t)) \leq M'(0)\delta. \tag{12}$$

This discussion allows us to prove a very strong theorem with some easy calculations.

5.4 A Reduction to the Planar Case

Theorem 5.2 *Consider a line-cone C in an n-agonal prism Ψ consisting of n half-planes $H_{i(i+1)}$ meeting along a line L. Let $n_{i(i+1)}$ be the unit normal to $H_{i(i+1)}$. Then C is minimizing for fluids $F_1, \ldots, F_{n+1}$ if and only if it is calibrated with respect to those fluids. Furthermore, if C is calibrated with respect to fluids $F_1, \ldots, F_{n+m}$, then it is minimizing.*

Proof: If C is calibrated for fluids $F_1, \ldots, F_{n+m}$, then there are points $\{p_1, \ldots, p_{n+m}\}$ satisfying conditions (P1) and (P2). Proposition 5.1 shows that C is cost-minimizing.

For the converse, let $N = C \cap P_0$ be the associated network on the cap P_0 of $\Psi = P \times [0,1]$. We note first that

$$\text{cost}(C) = \text{cost}(N)$$

since Ψ has unit length. Also note that each line segment L_{ij} in the associated network has unit length since P_0 is a unigon. Thus we may use Lemmas 4.3, 4.4, and Theorems 4.7 and 4.10 for the network N. Notice that F_{n+1} does not exist in the original configuration. If no points can be found to calibrate C (and thus N) with respect to $F_1, \ldots, F_{n+1}$, then it can be for one of three reasons:

1. No choice of points can be made to satisfy (P1).
2. No choice of points can be made to satisfy (P2).

3. No choice of point p_{n+1} exists for the calibration $\{p_1, \ldots, p_n\}$ calibrating C for fluids $F_1, \ldots, F_n$.

In each case it suffices to find a comparison surface which has less cost than C.

We will break up the cone into three sections of the prism: $P \times [0, \delta]$, $P \times [\delta, 1-\delta]$, and $P \times [1-\delta, 1]$. We work under the assumption that no calibrating points can be found. We will employ Lemmas 4.4, 4.3, and Theorem 4.10 to find cheaper comparisons in $P \times [\delta, 1-\delta]$, and employ ruled surfaces inside $P \times [0, \delta]$ and $P \times [1-\delta, 1]$ to ensure that our comparisons satisfy the boundary condition given by the original cone C. We will parametrize a continuous family of comparison surfaces $C(t)$ such that

- C(0)=C
- The associated network $N(t) = C(t) \cap (P \times \{\delta\})$ determines the cost of $C(t)$ in $P \times [\delta, 1-\delta]$ via the formula
$$\text{cost}(C(t))|_{P \times [\delta, 1-d]} = \text{cost}(N(t))(1-2\delta).$$
- $C(t)$ consists of a ruled surface $K'(t)$ connecting N in $P \times \{0\}$ and $N(t)$ in $P \times \{\delta\}$ and a reflected version of $K'(t)$ connecting $N(t)$ in $P \times \{1-\delta\}$ to N in $P \times \{1\}$.

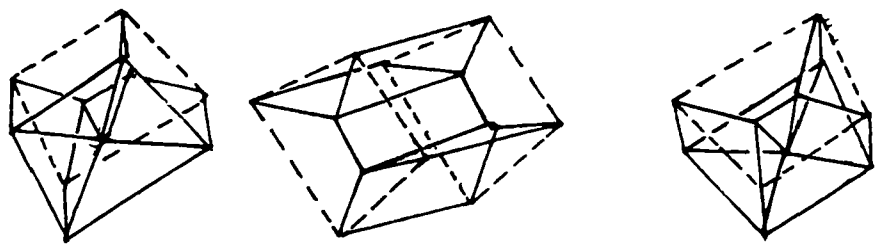


Figure 18: A comparison surface consisting of pieces of planes and ruled surfaces used in Theorem 5.2

.

We will utilize the equation

$$\text{cost}(C(t)) - \text{cost}(C) = [\text{cost}(N(t)) - \text{cost}(N)](1-2\delta) + [\text{cost}(K'(t)) - \text{cost}(N)]2\delta. \quad (13)$$

1. If no calibrating points can be found for C so that $p_{i+1} - p_i = a_{i(i+1)} n_{i(i+1)}$, then we must have
$$\sum_{i=1}^{n} a_{ij} n_{ij} \neq 0.$$
Lemma 4.3 provides a continuous family of comparison networks $N(t)$ such that
$$\left.\frac{d}{dt}\right|_{t=0} \text{cost}(N(t)) - \text{cost}(N) = -\alpha < 0.$$
Using equation (12) and differentiating equation (13) we obtain
$$\left.\frac{d}{dt}\right|_{t=0} \text{cost}(C(t)) - \text{cost}(C) \leq -\alpha(1-2\delta) + M'(0)2\delta. \quad (14)$$
Since $M'(0)$ is a constant, we may choose $\delta > 0$ so small that $-\alpha(1-2\delta) + M'(0)2\delta < 0$. Thus $\text{cost}(C(t)) < \text{cost}(C)$ for some $t > 0$.

2. If the cone has the property that $p_{i+1} - p_i = a_{i(i+1)}n_{i(i+1)}$, and yet for some pair (b,c) $|p_b - p_c| = \alpha_{bc} > a_{bc}$, then Lemma 4.4 or Theorem 4.8 give a continuous family of comparison networks $N(t)$ such that

$$\left.\frac{d}{dt}\right|_{t=0} \text{cost}(N(t)) - \text{cost}(N) = -\alpha < 0.$$

Using equation (12) and differentiating equation (13) we obtain

$$\left.\frac{d}{dt}\right|_{t=0} \text{cost}(C(t)) - \text{cost}(C) \leq -\alpha(1-2\delta) + M'(0)2\delta. \tag{15}$$

Since $M'(0)$ is a constant, we may choose $\delta > 0$ so small that $-\alpha(1-2\delta) + M'(0)2\delta < 0$. Thus $\text{cost}(C(t)) < \text{cost}(C)$ for some $t > 0$.

3. If the cone C is calibrated for $F_1, \ldots, F_n$ but no choice of p_{n+1} can be made to calibrate it for $F_1, \ldots, F_{n+1}$, then by Theorem 4.10 there is a continuous family comparison networks $N(t)$ such that

$$\left.\frac{d}{dt}\right|_{t=0} \text{cost}(N(t)) - \text{cost}(N) = -\alpha < 0.$$

Using equation (12) and differentiating equation (13) we obtain

$$\left.\frac{d}{dt}\right|_{t=0} \text{cost}(C(t)) - \text{cost}(C) \leq -\alpha(1-2\delta) + M'(0)2\delta. \tag{16}$$

Since $M'(0)$ is a constant, we may choose $\delta > 0$ so small that $-\alpha(1-2\delta) + M'(0)2\delta < 0$. Thus $\text{cost}(C(t)) < \text{cost}(C)$ for some $t > 0$.

□

6 Minimal Cones of Planes Meeting at a Point

We will now establish the sufficiency of (P1) and (P2) in cones consisting of several planes meeting along lines which meet at a point in the center of the unit ball D.

Theorem 6.1 *[5, Theorem 2.5] Consider a cone C consisting of m planes H_{ij} with unit normals n_{ij} meeting along lines which meet at a point. Suppose that*

- *Whenever k planes $H_{i_1 i_2}, \ldots, H_{i_k i_1}$ meet along a line,*

$$\sum_{j=1}^{k} a_{i_j i_{j+1}} n_{i_j i_{j+1}} = 0 \tag{17}$$

(indexing modulo k), and

- *For any integers $1 \leq i_1, \ldots, i_s \leq m$,*

$$|a_{i_1 i_2} n_{i_1 i_2} + \cdots + a_{i_{s-1} i_s} n_{i_{s-1} i_s}| \leq a_{i_1 i_s} \tag{18}$$

when the normals $n_{i_j i_{j+1}}$ exist because $H_{i_j i_{j+1}} \neq \emptyset$.

Then for any comparison surface $M = \{M_{ij}\}$,

$$\text{cost}(C) \leq \text{cost}(M).$$

Proof: We will describe an algorithm for constructing points which calibrate C. Let $p_1 = 0$. Consider a generic path γ_0 from R_1 to R_j passing through regions $R_1 = R_{i_1}, R_{i_2}, \ldots, R_{i_s} = R_j$, such that γ_0 never passes through the intersection of any of the H_{ij}. Define

$$p_j = a_{i_1 i_2} n_{i_1 i_2} + \cdots + a_{i_{s-1} i_s} n_{i_{s-1} i_s}. \tag{19}$$

To see that the definition of p_j is independent of the choice of the path, consider any other path β passing through regions $R_1 = R_{l_1}, R_{l_2}, \ldots, R_{l_r} = R_j$. Here p_j is defined as

$$p_j = a_{l_1 l_2} n_{l_1 l_2} + \cdots + a_{l_{r-1} l_r} n_{l_{r-1} l_r}. \tag{20}$$

To show the difference of the two equations (19) and (20) is zero, it suffices to prove that the loop $\Gamma_0 = \gamma_0 * \overline{\beta}$ is equivalent to the stationary loop at R_1 by a sequence of deformations which will not change the value Γ_0 assigns to p_1. Here $\overline{\beta}$ is defined as $\beta(1-t)$ given a suitable parametrization for β on the unit interval, and $*$ denotes path multiplication. See figure 19 for the planar interpretation.

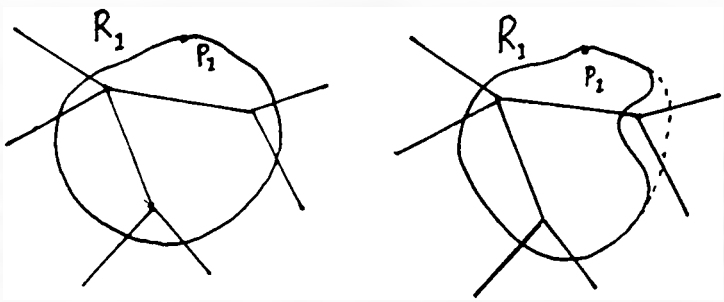


Figure 19: We may deform a loop to a point without changing the value it assigns to that point because the sum of the weighted unit normals around a singularity is 0.

Γ_0 encircles several lines along which planes meet. Moving a piece of Γ_0 through a piece of C consisting of planes meeting along a line does not change the value it gives to p_1 by equation (17). If two successive piercings of planes of Γ_0 occur through a single plane H_{ij}, then it will pass through it once with positive orientation and once with negative orientation, so that eliminating such a portion of Γ_0 does not change the value it gives to p_1 either. Repeating this process allows us to deform Γ_0 to a loop completely contained in R_1 and thus a trivial loop at $p_1 = 0$, so that the definition of p_j is independent of the choice of path γ_0.

If γ is a path from region R_i into R_j passing through regions $R_{k_1}, \ldots, R_{k_s}$, then

$$p_j - p_i = a_{k_1 k_2} n_{k_1 k_2} + \cdots + a_{k_{s-1} k_s} n_{k_{s-1} k_s}.$$

Inequality (18) gives us that $|p_j - p_i| \leq a_{ij}$. Moreover, if $H_{ij} \neq \emptyset$, then there is a direct path from R_i into R_j and

$$p_j - p_i = a_{ij} n_{ij}.$$

Thus the points $\{p_1, \ldots p_m\}$ calibrate C. Thus for any comparison surface $M = \{M_{ij}\}$, we can compute

$$\text{cost}(C) \quad = \quad \sum_{i<j} a_{ij} \, \text{area}(H_{ij})$$

$$
\begin{aligned}
&= \sum_{i<j} \text{flux}_{H_{ij}}(p_j - p_i) \\
&= \sum_{i} \text{flux}_{S_i}(p_i) \qquad \text{(by Lemma 3.1)} \\
&= \sum_{i<j} \text{flux}_{M_{ij}}(p_j - p_i) \\
&\leq \sum_{i<j} a_{ij}\, \text{area}(M_{ij}) \\
&= \text{cost}(M).
\end{aligned}
$$

Thus C is minimizing among surfaces separating the S_i the same way C does. □

If a cone C of planes meeting along lines meeting at a point is minimizing, then we may assume that there are points which calibrate each collection of planes meeting along a line. If not, then we could place a small prism on such a line-cone and proceed as in Theorem 5.2 to reduce its cost. Hence there are points $\{p_1, \ldots, p_m\}$ such that $p_j - p_i = a_{ij} n_{ij}$ when $H_{ij} \neq \emptyset$. If two fluids F_a, F_b face each other across a line, then again by Theorem 5.2, $|p_a - p_b| \leq a_{ab}$. Thus we need to verify that $|p_j - p_i| \leq a_{ij}$ for fluids F_i, F_j for which $H_{ij} = \emptyset$ and which do not face each other across a line. We tackle a special case of this in section 6.2.

6.1 Soap Films Meet in Two Ways

Minimizing configurations of immiscible fluids can meet either along lines or at a point, as we discussed in section 2.3. When all the cost-constants are 1, we have so-called area-minimizing surfaces, modeled by soap films. Since the calibrating points for a line-cone lie in a plane, we may use the fact that the only convex unit polygon whose vertices are all less than or equal to a unit distance apart is the equilateral triangle, derived in section 4.4.2, to conclude that only the only way planes can meet along a line is in threes at 120° angles.

We now establish that the only possible configuration of points calibrating a area-minimizing surface locally must be a tetrahedron. Call a convex polyhedron P a *unit polyhedron* if for any two vertices p_i, p_j, $|p_j - p_i| \leq 1$ with equality if the segment between them is an edge of P. We claim that the only such polyhedron is a tetrahedron. It is easy to see that one can place four points $\{p_1, \ldots, p_4\}$ a unit distance from one another. For a new point p_5 to be a vertex of a unit polyhedron, we must have $|p_5 - p_i| = 1$ for at least three different $i \in \{1, 2, 3, 4\}$, say $i = 1, 2, 3$. The only such point which satisfies $|p_5 - p_i| \leq 1$ for all $i \in \{1, 2, 3, 4\}$ is the remaining point p_4, so the only convex unit polyhedron is the tetrahedron. We also claim, without proof, that the calibrating points for a cone of planes meeting at a point always form a convex polyhedron. If it were known that minimization for cones of planes meeting at a point implied point-placing, it would follow that there is only one way for soap-films to meet in space. The only calibration could be be a tetrahedron, which calibrates a cone consisting of six planes meeting along four lines meeting at a point at angles of about 109°. Jean Taylor [8] proves this by ruling out a large class of possible cones and finding cheaper comparisons for the other possibilities to deduce that the tetrahedron is the only cone of planes meeting at a point which locally minimizes area.

6.2 Comparisons Surfaces for a Regular Symmetric Prism

Consider a regular n-gon P. The *regular symmetric prism* obtained from P is $P \times [-1, 1]$. We think of the interval $[-1, 1]$ as lying along the z-axis, so that P lies parallel to the xy-plane. The ends of the prism are thus bounded above and below by the $z = \pm 1$ planes. Each of the remaining pieces of the frame lies parallel to the z-axis and the distance from the origin to the center of the faces of the

n-gon in the xy-plane is 1. Thus this is a prism over the regular unigon. It follows that the width of a face of the n-gon is $2\tan(\pi/n)$. The node of the polyhedral cone over this prism lies at the origin.

There are natural costs to separate the fluids associated with the various faces of the regular symmetric prism, which we list below:

- The cost of separating fluids associated with the $z = 1$ and $z = -1$ faces is $1/\sin(\pi/n)$.
- The cost of separating fluids associated with adjacent faces of the n-gon is 1.
- The cost of separating fluids associated with nonadjacent faces of the n-gon, although not calculated, is assumed to be a value such that equality holds in condition (P2).
- The cost of separating either of the $z = \pm 1$ faces with a face of the n-gon is $1/\sqrt{2}\sin(\pi/n)$.

We will often refer to the cost of separating two faces, by which we mean the same thing as the cost of separating the fluids associated with these faces. These costs are natural in the sense that we can find a calibration and the calibrating points for the cone with node at the origin satisfy $|p_j - p_i| = a_{ij}$ for all pairs (i, j). When $n = 4$, we have points calibrating the cone over the cube, shown in figure 20

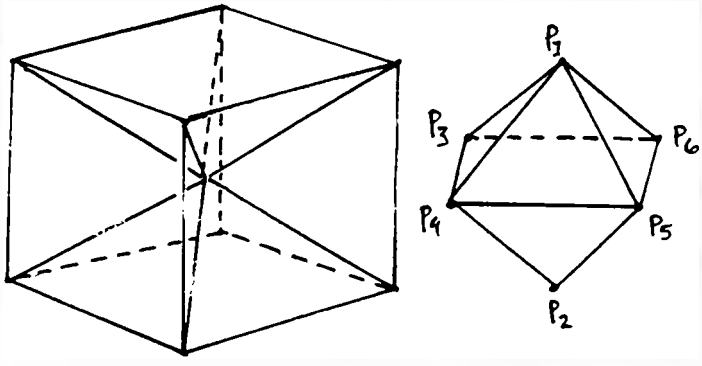


Figure 20: When $n = 4$, the points calibrating the cone over the cube form a unit octahedron, where points not directly joined by an edge are a distance $\sqrt{2}$ apart. (picture taken from [5]).

The cone C over this prism has cost

$$\text{cost}(C) = \frac{3n}{\cos(\pi/n)}.$$

If we let $0 < T < 1/\sin(\pi/n)$ be the cost of separating the $z = \pm 1$ faces, then the next theorem shows that the cone is not minimal. This partially resolves the question of whether point-placing and minimization are equivalent in this case (and actually solves it when $n = 3, 4$, since there are no other new possible interfaces). Our comparison is a generalization of that used by Ken Brakke [1] for the cone over the cube.

Theorem 6.2 *For all $T < 1/\sin(\pi/n)$ there exists a surface H separating the faces of the regular symmetric prism such that* $\text{cost}(H) < \text{cost}(C)$.

Proof: The general comparison H has a small n-gon parallel to and separating the $z = \pm 1$ faces. It is centered at the origin and the distance from the origin to the center of an edge is $y_0 > 0$. The surfaces joining the n-gon to the frame are cylinders with generator $z = y + c\log y$, where $c > 0$ chosen so that $0 = y_0 + c\log y_0$. At $y = 1$, $z = 1$. so the comparison fits the frame. The first integral represents the area of the $2n$ curved pieces separating a $z = \pm 1$ face from an adjacent face, with a

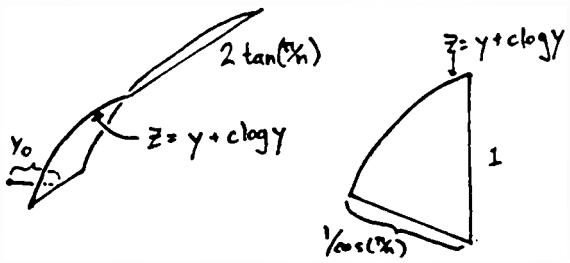


Figure 21: Pieces of the comparison surface used in Theorem 6.2.

scaling factor of $2\tan(\pi/n)$ for the size of the frame, as well as the cost. We multiply the arc length by the width to obtain the area of such a piece. The second integral represents the cost of one of the n pieces separating the other adjacent faces of the prism, and we have a scaling factor of $2/\cos(\pi/n)$ introduced for the size of the frame. The final term is the cost of the central n-gon. See figure 21 for a partial picture. Now we compute the cost of H.

$$\text{cost}(H) = \frac{4n\tan(\pi/n)}{\sqrt{2}\sin(\pi/n)}\int_{y_0}^{1}\sqrt{1+z'^2}y\,dy + \frac{2n}{\cos(\pi/n)}\int_{y_0}^{1} z\,dy + n\tan(\pi/n)Ty_0^2.$$

Now we show that the cost of the comparison is less than the cost of the cone for $T < 1/\sin(\pi/n)$.

$$\begin{aligned}
\text{cost}(H) &= \frac{4n}{\sqrt{2}\cos(\pi/n)}\int_{y_0}^{1}\sqrt{1+z'^2}y\,dy + \frac{2n}{\cos(\pi/n)}\int_{y_0}^{1} z\,dy \\
&\quad + n\tan(\pi/n)Ty_0^2 \\
&= \frac{4n}{\sqrt{2}\cos(\pi/n)}\sqrt{2}\int_{y_0}^{1}\sqrt{y^2+cy+\frac{c^2}{2}}\,dy \\
&\quad + \frac{2n}{\cos(\pi/n)}\int_{y_0}^{1} y + c\log y\,dy + n\tan(\pi/n)Ty_0^2 \\
&\le \frac{4n}{\cos(\pi/n)}\int_{y_0}^{1} y + \frac{c}{2} + \frac{c^2}{8y}\,dy \\
&\quad + \frac{2n}{\cos(\pi/n)}\int_{y_0}^{1} y + c\log y\,dy + n\tan(\pi/n)Ty_0^2 \\
&= \frac{4n}{\cos(\pi/n)}\left[\frac{1-y_0^2}{2} + \frac{c(1-y_0)}{2} - \frac{c^2}{8}\log y_0\right] \\
&\quad + \frac{2n}{\cos(\pi/n)}\left[\frac{1-y_0^2}{2} - c - cy_0\log y_0 + cy_0\right] + n\tan(\pi/n)Ty_0^2 \\
&= \frac{3n}{\cos(\pi/n)}(1-y_0^2) - \frac{n(c^2/2)\log y_0 + 2ncy_0\log y_0}{\cos(\pi/n)} \\
&\quad + n\tan(\pi/n)Ty_0^2 \\
&= \frac{3n}{\cos(\pi/n)} - \frac{ny_0^2}{\cos(\pi/n)} - \frac{ny_0^2}{2\log y_0\cos(\pi/n)} + n\tan(\pi/n)Ty_0^2.
\end{aligned}$$

Thus we get cost less than the cone for

$$-\frac{ny_0^2}{\cos(\pi/n)} - \frac{ny_0^2}{2\log y_0 \cos(\pi/n)} + n\tan(\pi/n)Ty_0^2 < 0$$

and thus for

$$T < \frac{1}{\sin(\pi/n)} + \frac{1}{2\sin(\pi/n)\log y_0}.$$

Any choice of $y_0 > 0$ gives $c > 0$, so that we may choose y_0 sufficiently small to get comparison surfaces for all $T < 1/\sin(\pi/n)$. □

Clearly such calculations would be even more difficult to carry out in general, making this not the preferred method. Since differentiation is our favorite tool, we should note that the cost function described in Theorem 6.2 fails to be twice differentiable at $y_0 = 0$. We now move back to using planar comparison surfaces.

6.3 Planar Comparisons for the Cone over the Cube: a First Look

Consider the cone over the cube, which can be calibrated when the cost of separating opposite regions is $\sqrt{2}$ and the cost of separating adjacent regions is 1. The calibrating points for the cone form a octahedron of unit length, where points not connected by an edge are exactly $\sqrt{2}$ apart. These costs are minimal in the sense that $|p_j - p_i| = a_{ij}$ for all pairs (i,j). We would like to use comparison surfaces consisting of planar pieces in an attempt to establish the necessity of (P1) and (P2) because they are easy to do computations with. We suspect the actual minimizer to consist of curved surfaces. Since we can approximate a curved surface with pieces of planes, we will restrict to using a single piece of a plane for each interface. We will find, however, that these comparisons are not good enough, as suggested by the subtle calculation in section 6.2.

Let $T < \sqrt{2}$ be the cost to separate the opposite regions. We ask whether cost can be reduced by allowing a small central square separating two of the opposite regions connected to the frame by pieces of planes. Interestingly enough, these planar comparisons only have cost cheaper than the cone for $T < 3/4\sqrt{2}$, although using curved surfaces works for all $T < \sqrt{2}$, as we show in section 6.2. Thus it shows the cone is not area minimizing, but it fails to show the cone is not cost-minimizing for some larger $T < \sqrt{2}$. This suggests that general comparisons will have to be made up of curved interfaces rather than the easily computable planar comparisons. We give a more complete analysis of the cone over the cube in section 6.4.

For convenience, the cubical frame will be centered at the origin in $\mathbf{R}^3$ with side length 1. The cost of separating adjacent faces is 1, and the cost of separating opposite faces is $\sqrt{2}$. The cost of the cone over this cube is $3\sqrt{2}$. We parametrize a continuous family of planar comparisons $K(t)$ with parameter t being the length of the side of the central square. Of course, $0 \le t < 1$. Note that when $t = 0$, we have the cost of the original cone C, since $K(0) = C$. The cost of the comparison is thus

$$\text{cost}(K(t)) = \sqrt{2}(1-t) + 2(t+1)\sqrt{t^2-2t+2} + Tt^2,$$

as explained by figure 22.

Computing the first derivative at $t = 0$ we obtain

$$\left.\frac{d}{dt}\right|_{t=0} \text{cost}(K(t)) \;=\; 0$$

which is not surprising considering the central square is a second order effect.

Taking the second derivative, however, gives

$$\frac{d^2}{dt^2}\text{cost}(K(t)) \;=\; \frac{2(t-1)}{\sqrt{t^2-2t+2}} + \frac{4t(t^2-2t+2) - 2(t-1)^2(t+1)}{(t^2-2t+2)^{(3/2)}} + 2T.$$

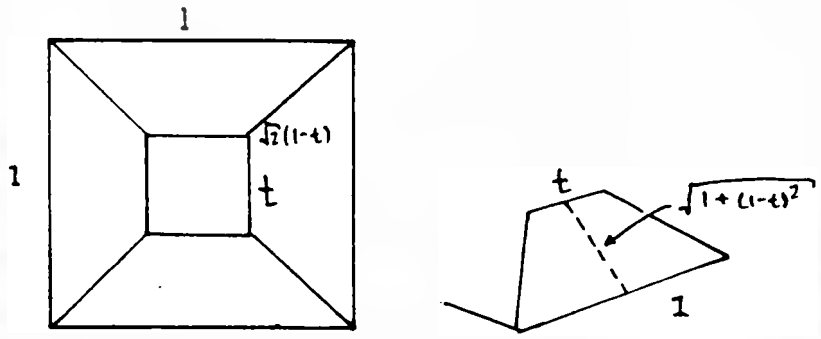


Figure 22: Parametrizing comparisons with a central square equidistant from the faces it separates.

and thus

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \text{cost}(K(t)) \;=\; -\frac{3}{2}\sqrt{2}+2T.$$

Thus

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \text{cost}(K(t)) \;<\; 0$$

if and only if $T < (3/4)\sqrt{2}$. Hence there is some $t > 0$ such that $\text{cost}(K(t)) < \text{cost}(C)$.

In a similar fashion we can compute a planar comparison for the cone over the regular symmetric prism. Recall that we computed the cost of the cone over the regular symmetric prism in section 6.2 to be

$$\text{cost}(C) = \frac{3n}{\cos(\pi/n)}.$$

The general planar comparison has the same central n-gon, but has $2n$ trapezoids separating the $z = \pm 1$ faces from a face of the n-gon and has n triangular pieces separating the other adjacent faces of the prism. Using the same parameter y_0 as in the comparison with curved surfaces, we calculate the cost of this comparison $K(y_0)$,

$$\text{cost}(K(y_0)) = \frac{2n(1+y_0)\sqrt{1+(1-y_0)^2}}{\sqrt{2}\cos(\pi/n)} + \frac{n(1-y_0)}{\cos(\pi/n)} + Tn\tan(\pi/n)y_0^2.$$

Differentiating with respect to y_0 we obtain

$$\begin{aligned}\left.\frac{d}{dy_0}\right|_{t=0} \text{cost}(K(y_0)) \;&=\; \frac{2n(1-2y_0+2y_0^2)}{\sqrt{2}\cos(\pi/n)\sqrt{1+(1-y_0)^2}} - \frac{n}{\cos(\pi/n)} \\ &\quad +2nT\tan(\pi/n)y_0.\end{aligned}$$

Note that $\frac{d}{dy_0}\text{cost}(K(y_0)) = 0$, so the cone is a critical point of the cost function. Again the first derivative is not enough to tell us whether the comparison decreases cost because of the second order effect of the central n-gon. So we test the concavity at zero, in hopes that the cost function is indeed decreasing for small y_0. Differentiating again we find $\left.\frac{d^2}{dy_0^2}\right|_{t=0} \text{cost}(K(y_0)) < 0$ if and only if

$$\frac{-3n}{2\cos(\pi/n)} + 2nT\tan(\pi/n) < 0$$

or

$$T < \frac{3}{4\sin(\pi/n)}.$$

Thus we have proved

Proposition 6.3 *For all $T < 3/(4\sin(\pi/n))$ there exists a surface H consisting of single planar pieces and a regular symmetric n-gon separating the $z = \pm 1$ faces of the regular symmetric prism such that $\text{cost}(H) < \text{cost}(C)$, where C is the cone over the regular symmetric prism.*

We conjecture that the cone is minimizing for $T \geq 3/(4\sin(\pi/n))$ for comparisons consisting of single planar pieces separating each face. Recall that the cone is calibrated for $T = 1/\sin(\pi/n)$, so this type of comparison is not good enough to determine whether point-placing and minimization are equivalent in this case. We now proceed to discuss this in greater detail for the cone over the cube.

6.4 Planar Comparisons for the Cone over the Cube: a Second Look

Using a single piece of a plane for each interface is extremely restrictive. For example, comparisons for the cone over the cube consisting of an interface separating opposite regions must have that interface parallel to the sides of the prism, as we discuss below. We now work in the space of such comparisons for the cube in an attempt to prove that cheaper comparisons, consisting of a face separating opposite faces, must have the symmetries of a square.

We first show that any new piece separating these faces must be a rectangle. Note that the plane of every piece of the comparison which separates adjacent regions at the boundary of the cube has only one degree of freedom to move in, since it must always contain a side of the frame. This motivates the following terminology. A *hinge* is an edge of the frame on which the surface lies. We call the rectangular faces of the prism perpendicular to the xy-plane *side faces*. For a specified side face S_i, we call the hinge in the $z = 1$ plane the *upper hinge* of S_i, and the hinge in the $z = -1$ plane the *lower hinge* of S_i. The *side hinges* of S_i are those perpendicular to the xy-plane. The plane associated with the upper hinge of a side face will be called the *upper hinge plane*, and similarly the lower hinge has an associated *lower hinge plane*.

For any side face S_i, its upper hinge and lower hinge are parallel. Thus the upper hinge plane and the lower hinge plane of S_i can only intersect in a line parallel to S_i. Of course, this line can be reduced to a single point by moving the side hinge planes. This proves the following:

Proposition 6.4 *For a comparison surface to the cone over the cube consisting of single planar pieces, any new face separating opposite faces is a rectangle lying parallel to these planes with each side parallel to some side face S_i.*

Without loss of generality, we assume the comparison consists of a central rectangle parallel to the xy plane. Since the original cost function is continuous on $[0,1]$, for a fixed $T < (3/4)\sqrt{2}$, there is some s such that $\text{cost}(K(s)) < \text{cost}(K(y_0))$ for all $y_0 \in [0,1]$. So we assume s, T as above are given and parametrize the space of comparisons as follows. Note that we have changed from the center of the cube at $(0,0,0)$ to the center at $(1,1,1)$.

Let (x,y,z) be the point inside the cube $[0,2]^3$ at which the center of the rectangle lies. Let t, u be the distance from the center of the rectangle to the edges, t in the x direction and u in the y direction. We must have $0 \leq t \leq 1 - |x-1|$ and $0 \leq u \leq 1 - |y-1|$. Elementary geometry leads to the following parametrization of the space of comparisons:

$$\begin{aligned}\text{cost}(K(t,u,x,y,z)) \;=\; & (1+t)\sqrt{z^2+(y-u)^2} + (1+u)\sqrt{z^2+(x-t)^2} \\ & +(1+t)\sqrt{z^2+(2-y-u)^2} \\ & +(1+u)\sqrt{z^2+(2-x-t)^2}\end{aligned}$$

$$
\begin{aligned}
&+(1+t)\sqrt{(2-z)^2+(y-u)^2}\\
&+(1+u)\sqrt{(2-z)^2+(x-t)^2}\\
&+(1+t)\sqrt{(2-z)^2+(2-y-u)^2}\\
&+(1+u)\sqrt{(2-z)^2+(2-x-t)^2}\\
&+\sqrt{(x-t)^2+(y-u)^2}+\sqrt{(x-t)^2+(2-y-u)^2}\\
&+\sqrt{(2-x-t)^2+(y-u)^2}\\
&+\sqrt{(2-x-t)^2+(2-y-u)^2}\\
&+4Ttu.
\end{aligned}
$$

The configuration $(t,u,x,y,z) = (s,s,1,1,1)$, consisting of a central square in the center of the cube, is our proposed global minimizer. Figure 23 helps explain this parametrization, and we see the symmetries about the point $(x,y,z) = (1,1,1)$ reflected in the equations.

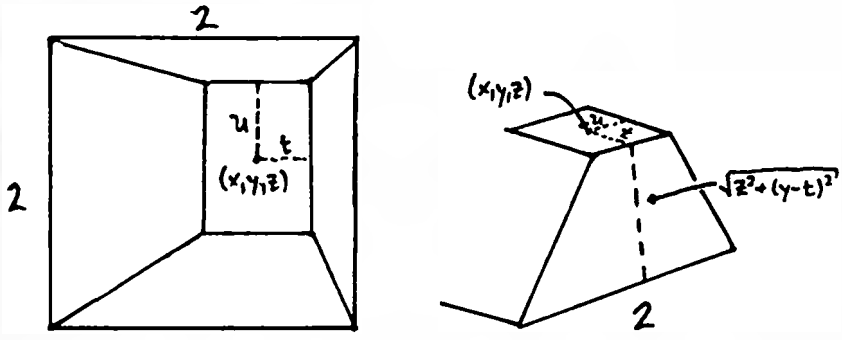


Figure 23: Parametrizing the space of planar comparison surfaces for the cone over the cube.

Since we are only interested in local minimization, we can eliminate two of the variables by a symmetry argument, starting with the central rectangle corresponding to $(t,u,1,1,1)$ and taking small variations. We'll refer to the plane parallel to the xz-plane passing thru the point $x=1$ as the $(x=1)z$-plane. Similarly we have the $(y=1)z$-plane. The configuration $(t,u,1+\delta,1+\epsilon,z)$ is not minimizing for the following reason. The $(x=1)z$-plane and the $(y=1)z$-plane cut the cube into four quadrants, each containing some small part of the displaced rectangle. One has a cheaper cost than the others, so replacing every other region with that of cheaper cost reduces total cost, while still separating the faces of the cube in the appropriate way and ensuring all interfaces are single pieces of planes. Note that this cost-reducing procedure does not depend on the height (z value) of the central rectangle. Therefore a local minimum with $\delta < t/2$ and $\epsilon < t/2$ has $\delta = \epsilon = 0$. Hence to prove local minimization is suffices to prove that $(t,u,z) = (s,s,1)$ is a local minimum of the following cost function:

$$
\begin{aligned}
\text{cost}(K(t,u,z)) \;=\; &(1+t)\sqrt{z^2+u^2}+(1+u)\sqrt{z^2+t^2}\\
&+(1+t)\sqrt{z^2+(2-u)^2}+(1+u)\sqrt{z^2+(2-t)^2}\\
&+(1+t)\sqrt{(2-z)^2+u^2}+(1+u)\sqrt{(2-z)^2+t^2}\\
&+(1+t)\sqrt{(2-z)^2+(2-u)^2}\\
&+(1+u)\sqrt{(2-z)^2+(2-t)^2}\\
&+\sqrt{t^2+u^2}+\sqrt{t^2+(2-u)^2}\\
&+\sqrt{(2-t)^2+u^2}+\sqrt{(2-t)^2+(2-u)^2}
\end{aligned}
$$

$$+4Ttu.$$

A necessary condition for $(s, s, 1)$ to be a critical point of the cost function is that $\frac{\partial}{\partial z}\text{cost}(K(u,t,z))$ vanish at $z = 1$. In fact, no matter what values we substitute for $t, u, \frac{\partial}{\partial z}\text{cost}(K(u,t,1)) = 0$. This can be seen by pairing terms which only differ in that one contains $(2-z)^2$ and the other contains z^2. To exploit the symmetries in t, u, we note that $\text{cost}(K(t,u,z)) = \text{cost}(K(u,t,z))$, so that

$$\frac{\partial}{\partial t}\text{cost}(K(u,t,z)) = \frac{\partial}{\partial t}\text{cost}(K(t,u,z)).$$

Hence (t_0, u_0, z_0) is a zero of

$$\frac{\partial}{\partial t}\text{cost}(K(t,u,z))$$

if and only if (u_0, t_0, z_0) is a zero of

$$\frac{\partial}{\partial u}\text{cost}(K(t,u,z)).$$

Thus if $(t_0, u_0, 1)$ is a zero of

$$\frac{\partial}{\partial t}\text{cost}(K(t,u,z)),$$

then so must $(u_0, t_0, 1)$ be a zero. Of course, this is all quite obvious from the symmetries of this problem. We conjecture that indeed we must have $t_0 = u_0$, which would prove that $(s, s, 1)$ is a local minimum of this cost function. Experiments on Mathematica seem to confirm our conjecture.

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