Griffiths Residues and Smooth Hypersurfaces in Grassmannians

by

Andrew Huchala

A dissertation accepted and approved in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
in Mathematics

Dissertation Committee:
Nicolas Addington, Chair
Alexander Polishchuk, Core Member
Benjamin Young, Core Member
Andrew Hanlon, Core Member
Jake Searcy, Institutional Representative

University of Oregon
Spring 2026

© 2026 Andrew Huchala
This work is openly licensed via CC BY-SA 4.0.

DISSERTATION ABSTRACT

Andrew Huchala

Doctor of Philosophy in Mathematics

Title: Griffiths Residues and Smooth Hypersurfaces in Grassmannians

We extend Griffiths–Dwork reduction for hypersurfaces in projective space to the case of Grassmannians by giving an explicit construction of the generalized Griffiths ring. As an application, we construct a new Debarre–Voisin fourfold with A_7 symmetry and maximal Picard rank.

Much of this work relies on cohomological vanishings, which are established using the Borel–Weil–Bott theorem. Specifically, we define a new invariant of bounded partitions—the t-hook ratio—and use it to produce a closed-form formula for the twisted Hodge numbers $h^i(\mathrm{Gr}(k,n),\Omega^j(t))$. This allows us to prove new identities involving sums over bounded partitions. We also conjecture a q-analog of one such identity, in terms of the q-binomial coefficient.

This dissertation includes coauthored material.

ACKNOWLEDGMENTS

This thesis is the product of a long journey through graduate school. I would like to thank everyone who helped me along this path, academically and otherwise.

I'd like to thank Fern Gossow for their collaboration and insights into the combinatorics of core partitions. I thank Arya Yae for her many contributions and perspectives on residues and GIT constructions. I thank Cruz Godar for endless coding help and support. I thank my advisor Nicolas Addington for the original idea for this project, invaluable guidance through proofs, and constantly helping me stay on track. Thanks to Amy Massoud for being such a source of positivity in the department. I thank my family for their unconditional support throughout my life. And finally, I thank Nyda Ouch for being by my side, through quarantine, surgeries, and everything else along the way.

I was partially supported by RTG 2039316.

For Hal and Joanne.

TABLE OF CONTENTS

Chapter Page

1. Introduction and Notation . . . 8
   1.1. Motivation . . . 8
   1.2. Main Results . . . 9
   1.3. Past Work . . . 11
      Griffiths Rings for Partial Flag Varieties . . . 11
      Borel–Weil–Bott and Snow's Bounds . . . 12
   1.4. Outline . . . 12
   1.5. Notation . . . 13

2. Hypersurfaces and Residues . . . 14
   2.1. Classical Residues . . . 14
      Residues in One Variable . . . 15
   2.2. Smooth Hypersurfaces in a Smooth Projective Variety . . . 16
      Vanishing vs. Primitive Cohomology . . . 17
      The Logarithmic de Rham Complex . . . 19
      Hypercohomology of the Logarithmic de Rham Complex . . . 21
      Hypercohomology Spectral Sequence Calculation . . . 22
      Smooth Hypersurfaces in $\mathrm{Gr}(k, n)$ . . . 26
   2.3. A Global Section of $K_X(n)$ . . . 28
      Overview . . . 28
      GIT Construction . . . 28
   2.4. Griffiths–Dwork Reduction for Grassmannians . . . 32

3. Combinatorial Results . . . 36
   3.1. Introduction . . . 36
   3.2. Background . . . 38
      Sequences . . . 38
      Permutations . . . 39
      Bounded partitions . . . 39
      Tableaux . . . 41
      Representation theory . . . 42
      Schur Functors . . . 44
   3.3. Computation of the Hodge numbers . . . 45
      BWB for the Grassmannian . . . 45
      The BWB weight . . . 47
      A hook-product formula . . . 50

Examples . . . 53
3.4. Restrictions on bounded core partitions . . . 55
Skew-linking diagrams . . . 55
Nakano vanishing . . . 57
Snow's bounds . . . 60
3.5. Hook Lengths and Euler Characteristic . . . 62
Algebraic proof . . . 62
Combinatorial interpretation . . . 64
3.6. A q-analog . . . 64
Setup . . . 65
Calculation of the q-dimension . . . 65
Examples . . . 68
3.7. Deferred Combinatorial Proofs About Cohomology . . . 70
Vector Bundle Cohomology . . . 70
Twisted Hodge Numbers when $i + j \geq N - 1$ . . . 73
3.8. Further Combinatorial Results . . . 74
Holomorphic Euler Characteristic of a Smooth Complete Intersection 74
Primitive Cohomology of the Grassmannian . . . 79
Remark About Grassmannian Cohomology Vanishings . . . 79
Conjectured Strengthening of Known Bounds for Snow Partitions . . . 80

4. Examples . . . 82
4.1. Example: Griffiths ring with nontrivial primitive cohomology . . . 82
4.2. Example: a quadric in $\mathrm{Gr}(2, 5)$ . . . 84
4.3. Non-example: a cubic in $\mathrm{Gr}(2, 5)$ . . . 86
4.4. Non-examples: hypersurfaces of degree d in $\mathrm{Gr}(k, n)$ with $d|n|kd$, etc. 87
4.5. Smooth Hyperplane Sections of $\mathrm{Gr}(3, 10)$ . . . 88
The Debarre–Voisin Construction . . . 88
Constructing Debarre–Voisin Fourfolds with Maximal Picard Rank . . . 90
Song's Construction for $\mathrm{PSL}(2, \mathbb{F}_{11})$ . . . 91
Our Construction for $\mathrm{PSL}(2, \mathbb{F}_{11})$ . . . 92
A Hyperplane Section with A_7 *Symmetry* . . . 95

REFERENCES CITED . . . 99

CHAPTER 1

INTRODUCTION AND NOTATION

1.1 Motivation

In [Hir56], Hirzebruch computed the Hodge numbers $h^{p,q}(Z)$ of a smooth hypersurface Z in $\mathbb{P}^n$, and in [Gri69], Griffiths produced the residue map from the space of polynomials to differential forms. If a group G acts linearly on a smooth hypersurface Z, the Griffiths residue map is G-equivariant. This map allows us to describe the primitive cohomology of Z in terms of graded pieces of the Griffiths ring

$$H^{n-p,p-1}_{\text{prim}}(Z) \cong (R/J_f)_{pd-n-1},$$

where $R = \mathbb{C}[x_0, \ldots, x_n]$ and J_f is the Jacobian ideal

$$J_f = \left(\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right). \tag{1.1.1}$$

The graded ring R/J_f is called the Griffiths ring, or sometimes the Jacobian ring. A pole-order reduction algorithm for the primitive cohomology was introduced by Griffiths in [Gri69] and attributed to Dwork, and is known as Griffiths–Dwork reduction. This is explained in detail in [Voi03, §6], and we only sketch the idea here. First, we may recover the cohomology of a projective hypersurface Z by studying the cohomology of $U := X \setminus Z$ where $X = \mathbb{P}^n$. Consider the short exact sequence

$$0 \to H^n_{\text{prim}}(X, \mathbb{C}) \to H^n(U, \mathbb{C}) \xrightarrow{\text{Res}} H^{n-1}_{\text{van}}(Z, \mathbb{C}) \to 0.$$

Unlike a Grassmannian $\text{Gr}(k, n)$ with $2 \leq k \leq n-2$, $\mathbb{P}^n$ has trivial primitive cohomology, so the residue map Res is an isomorphism. In fact, the pole order filtration on $H^n(U, \mathbb{C})$ induces the Hodge filtration on $H^{n-1}_{\text{van}}(Z, \mathbb{C})$: given a nowhere-vanishing section $\Omega \in H^0(K_X(n+1))$ of the twisted canonical bundle of X, for any $p \geq 1$ we may present $H^{n-p,p-1}_{\text{van}}(Z)$ as the vector space

$$\left\langle \frac{g\Omega}{f^p} : \, g \in H^0(\mathcal{O}_X(pd-n-1)) \right\rangle,$$

modulo the sum of the subspaces

$$\left\langle \frac{gf}{f^{p+1}}\Omega : \, g \in H^0(X, \mathcal{O}_X((p-1)d-n-1)) \right\rangle$$

and

$$\left\langle \left(f\frac{\partial g}{\partial x_i} - (p-1)g\frac{\partial f}{\partial x_i} \right) \frac{\Omega}{f^p} : \, g \in H^0(\mathcal{O}_X((p-1)d-n)) \right\rangle.$$

Explicitly, this means that a form $\frac{g}{f^p}\Omega$ is cohomologous to a form of pole order $p-1$ as forms on U if and only if $g \in J_f$. An analog in rigid cohomology has been used to efficiently compute the Hasse–Weil zeta function of smooth hypersurfaces in positive characteristic, as in [CHK19].

1.2 Main Results

We extend the Griffiths construction from the previous section to smooth hypersurfaces in Grassmannians. Recall that $\mathfrak{sl}(n, \mathbb{C})$ is generated by $\{E_{ij}\}$, where E_{ij} is given by

$$E_{ij} := \begin{cases} e_{i,j} & i \neq j, \\ e_{i,i} - e_{i+1,i+1} & i = j < n. \end{cases} \tag{1.2.1}$$

By convention we write $E_{n,n} = 0$. When $X = \mathrm{Gr}(k, n)$, then $H^0(X, T_X) \cong \mathfrak{sl}(n, \mathbb{C})$ has a natural action on $H^0(X, \mathcal{O}_X(1)) \cong \bigwedge^k(\mathbb{C}^n)^\vee$, which extends to an action by derivations on $\mathrm{Sym}^d \bigwedge^k(\mathbb{C}^n)^\vee$.

Theorem 1.1 (Main Result)**.** *Let $X = \mathrm{Gr}(k, n)$ with $2 \leq k \leq n-2$ and let $Z \subset X$ be a smooth hypersurface of degree d with defining polynomial f. Let R denote the homogeneous coordinate ring of X. Assume the following constraints on k, n, d:*

1. *$d \nmid n$ or $\gcd(k, n/d) = 1$, and*
2. *$k \notin \{2, n-2\}$ or $2d \nmid (n+1)$.*

Define the Jacobian ideal J_f to be the ideal of R generated by f and all $\{E_{ij}(f)\}$, and define the Griffiths ring as the quotient R/J_f. Then for each $p \geq 1$,

$$(R/J_f)_{pd-n} \cong H^{N-p,p-1}_{\mathrm{van}}(Z) \oplus H^{N-p+1,p-1}_{\mathrm{prim}}(\mathrm{Gr}(k, n)).$$

Theorem 1.2 (Griffiths–Dwork Reduction)**.** *With the same assumptions and notation as Theorem 1.1, let $\Omega \in H^0(K_X(n))$ be a nonzero section of a twist of the canonical bundle $K_X \cong \mathcal{O}_{\mathrm{Gr}(k,n)}(-n)$. Let $\gamma = \frac{g}{f^p}\Omega$ be an arbitrary section of $K_X(pZ)$ for $p > 1$. Suppose there exist functions h^{ij} and c such that*

$$\gamma = \frac{\sum_{ij} h^{ij} E_{ij}(f) + cf}{f^p}\Omega.$$

Then

$$\gamma \equiv \frac{\sum_{ij} E_{ij}(h^{ij}) + (p-1)c}{(p-1)f^{p-1}}\Omega$$

as meromorphic forms on $U := X \setminus Z$ (modulo exact forms).

Explicitly, we may present $H^{N-p,p-1}_{\text{van}}(Z) \oplus H^{N-p+1,p-1}_{\text{prim}}(X)$ as the quotient of the vector space generated by

$$\left\langle \frac{g}{f^p}\Omega : g \in H^0(X, \mathcal{O}_X(pd-n)) \right\rangle,$$

modulo the sum of the subspaces

$$\left\langle \frac{gf}{f^{p+1}}\Omega : g \in H^0(X, \mathcal{O}_X((p-1)d-n)) \right\rangle$$

and

$$\left\langle \frac{(p-1)gE_{ij}(f) - fE_{ij}(g)}{f^p}\Omega : g \in H^0(X, \mathcal{O}_X((p-1)d-n)) \right\rangle.$$

All code for calculations performed in this thesis is available on a GitHub repository [Huc26]. For example, we provide an implementation of the calculation of the Griffiths ring of an arbitrary smooth hypersurface in $\mathrm{Gr}(k,n)$ (which satisfies the divisibility hypothesis of Theorem 1.1) in *GriffithsRing.M2*.

Example 1.3. As an application of Theorem 1.1, we produce a new Debarre–Voisin fourfold of maximal Picard rank with A_7 symmetry (see Section 4.5). The only other example of a Debarre–Voisin fourfold of maximal Picard rank that the author is aware of was constructed by Song in [Son23], which had symmetry group $\mathrm{PSL}_2(\mathbb{F}_{11})$. Our proof differs from that of Song, although we also confirm his calculation.

The divisibility constraints on k, n, d of Theorem 1.1 come from conditions ensuring the vanishing of various twisted Hodge numbers on the Grassmannian. We obtain a closed-form formula for these twisted Hodge numbers, of interest in its own right:

Theorem 1.4. *Let λ be a partition bounded by a $k \times (n-k)$ box, so $\lambda = (\lambda_1, \ldots, \lambda_k)$ is an integer sequence with $n-k \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0$. We let $\lambda^{\mathsf{C}} := (n-k-\lambda_k, \ldots, n-k-\lambda_1)$ denote the complement partition. For $1 \leq a \leq k$ and $1 \leq b \leq \lambda_a$, let $h_\lambda(a,b)$ denote the hook length of the box $(a,b) \in \lambda$.*

For all integers $i, j, t \geq 0$:

$$h^i(\mathrm{Gr}(k,n), \Omega^j(t)) = (-1)^{i+j} \sum_{\lambda} \prod_{(a,b)\in\lambda} \frac{h_\lambda(a,b) - t}{h_\lambda(a,b)} \prod_{(a,b)\in\lambda^{\mathsf{C}}} \frac{h_{\lambda^{\mathsf{C}}}(a,b) + t}{h_{\lambda^{\mathsf{C}}}(a,b)},$$

where the sum is over all partitions λ such that:

- *λ is bounded in a $k \times (n-k)$ box,*

- *λ has no hook length equal to t,*
- *λ has j boxes in total,*
- *λ has exactly i boxes with hook length greater than t.*

We point out that the sign $(-1)^{i+j}$ forces the sum to be nonnegative (see Remark 3.30).

1.3 Past Work

General Theory of Griffiths Rings of Hypersurfaces in Partial Flag Varieties

In [FM21], Fatighenti and Mongardi constructed a Griffiths-type ring for the cohomology of a smooth complex hypersurface $Z \subset X = \mathrm{Gr}(k, n)$ of degree d, subject to some cohomological vanishings. For a general statement ([FM21, Theorem 3.8]), they assume $d \geq n - 1$, which guarantees all desired vanishings (see [Sno86, Theorem 3.2]). Observe that the assumption $d \geq n - 1$ automatically satisfies the hypotheses of Theorem 1.1. For $d < n - 1$ ([FM21, Corollary 3.9]), Fatighenti and Mongardi assert that a Griffiths ring calculation requires an understanding of the nonvanishing of the groups $H^i(X, \Omega^j_X(t))$.

In Chapter 3, we classify the values of k, n, d which satisfy the vanishings of Fatighenti and Mongardi, and make explicit which of these are sufficient in Chapter 2.

In [Hua+18], Huang, Lian, Yau, and Yu generalize Griffiths' residue theorem to the zero loci of vector bundles on partial flag varieties assuming some cohomological vanishings. Some vanishings and assumptions are implicit, and the construction appears to require ampleness of the vector bundle in question. We emphasize that their construction is done at a very abstract level. In the case of $X = G/P$ a generalized flag variety and L an ample line bundle, they obtain an analog of Griffiths residues and Griffiths–Dwork reduction [Hua+18, Proposition 2.3] assuming the following cohomological vanishings [Hua+18, Theorem 2.4]:

Assumption 1.5. *Let $0 \leq k \leq n - 1$ and assume L satisfies the following vanishing conditions for (i, j, l) in the following set: $\{1 \leq i \leq k, j = N - i, l = k - i + 1\} \cup \{1 \leq i \leq k - 1, j = N - i - 1, l = k - i\} \cup \{1 \leq i \leq k - 1, j = N - i, l = k - i\}$:*

- $H^i(X, \Omega^j_X \otimes L^l) = 0$
- $H^1(X, (G \times_{ad_P} \mathfrak{p}) \otimes L^k \otimes K_X) = 0$

However, even in the case of an ample line bundle on a Grassmannian, determining whether these vanishings hold appears difficult. We make this explicit when $X = \mathrm{Gr}(k, n)$ and $L = \mathcal{O}_X(d)$, and use this to give an explicit pole-order reduction algorithm for smooth hypersurfaces in X, as well as concise combinatorial criteria for when the algorithm applies.

Borel–Weil–Bott and Snow's Bounds

The Borel–Weil–Bott theorem is an algorithm to compute the sheaf cohomology of a line bundle on the full flag variety, and this in turn can be used to compute the cohomology of the sheaf of twisted holomorphic forms on a Grassmannian $\mathrm{Gr}(k, n)$. In [Sno86], Snow gave five inequalities on parameters k, n, j, t, i such that $h^i(\mathrm{Gr}(k, n), \Omega^j(t)) = 0$. Improving upon his bounds, we develop this theory further and produce a closed formula for $h^i(\mathrm{Gr}(k, n), \Omega^j(t))$ (see Theorem 1.4).

1.4 Outline

In Chapter 2, we construct the Griffiths map assuming cohomological vanishings and present the associated spectral sequence argument which shows why these are sufficient. We then produce the GIT construction of the Grassmannian $X = \mathrm{Gr}(k, n)$ and give an explicit formula to construct a nowhere-vanishing global section of its twisted canonical bundle $K_X(n)$. This allows us to define the Griffiths ring of a hypersurface in X, and as a byproduct of this approach we get a generalized version of Griffiths–Dwork reduction.

In Chapter 3, we completely classify which hypersurfaces in $\mathrm{Gr}(k, n)$ satisfy the vanishing assumptions of Chapter 2. In doing this, we use the Borel–Weil–Bott theorem to define an integer-valued polynomial we call the t-hook ratio of a partition λ. Interpreting this polynomial algebraically, we obtain some surprising combinatorial identities. We also obtain a recurrence for computing the Euler–Hodge generating function of a smooth complete intersection in $\mathrm{Gr}(k, n)$.

In Chapter 4, we combine the results of the previous chapters to work out the Griffiths ring of a few interesting hypersurfaces. Each smooth hyperplane section $Z_\sigma \subset \mathrm{Gr}(3, 10)$ defined by a linear form σ has an associated fourfold Y_σ; for generic σ, Y_σ is a hyper-Kähler fourfold of $\mathrm{K3}^{[2]}$-type, called a Debarre–Voisin fourfold. Debarre–Voisin fourfolds are deformation equivalent to the Hilbert scheme of two points on a K3 surface. We construct such a hyperplane section with a large automorphism group A_7, and show that the associated Debarre–Voisin fourfold has maximal Picard rank.

The results in Chapter 3 are based on joint work with Fern Gossow. All remaining chapters represent the author's independent work.

1.5 Notation

Fix $V = \mathbb{C}^n$. The Grassmannian $\mathrm{Gr}(k, V)$ of k-dimensional subspaces of V will be denoted by $X = \mathrm{Gr}(k, n)$. Consider the tautological sequence

$$0 \to S \to \mathcal{O}_X \otimes V \to Q \to 0$$

for S and Q the tautological subbundle and quotient bundle, respectively. The dual of a bundle E is denoted $E^\vee$. Recall that the cotangent bundle of X is given by $\Omega_X \cong S \otimes Q^\vee$. We define $\mathcal{O}_X(1) := \bigwedge^{n-k}(Q) = \det(Q)$ and observe that $\det Q \cong \det S^\vee$. We define $\mathcal{O}_X(d) := \mathcal{O}_X(1)^{\otimes d}$, and $\mathcal{O}_X(-d) := (\mathcal{O}_X(d))^\vee$. Let $\Omega^j_X := \bigwedge^j(\Omega_X)$ and $\Omega^j_X(t) := \Omega^j_X \otimes \mathcal{O}_X(t)$. The tangent bundle T_X is naturally identified with $S^\vee \otimes Q$ and its exterior powers similarly. By a hypersurface of degree d we mean the vanishing locus of a section of $\mathcal{O}_X(d)$. All varieties are smooth unless otherwise noted. If $\dim X = N$ we define the canonical bundle $K_X := \bigwedge^N(\Omega_X)$, and in the case of $X = \mathrm{Gr}(k, n)$ we note that $K_X \cong \mathcal{O}_X(-n)$.

We use Einstein summation notation; repeated indices indicate an implicit sum.

CHAPTER 2

HYPERSURFACES AND RESIDUES

This chapter has two goals. First, we prove a general residue theorem (Theorem 2.1) for smooth ample hypersurfaces in any smooth projective variety, assuming explicit cohomological vanishings weaker than those of [Voi03, §6.1.2]. Second, we specialize to $X = \mathrm{Gr}(k, n)$ and prove the main result of this thesis (Theorem 1.1), restated from the introduction.

Theorem 2.1. *Suppose $Z \hookrightarrow X$ is a smooth ample hypersurface in a smooth projective variety X of dimension N, and moreover that we have the following vanishings:*

1. $H^i(X, \Omega_X^{N-i}((p-i)Z)) = 0$ *for every* $0 < i < p \leq N$ *(2.9), and*
2. $H^i(X, \Omega_X^{N-i-1}((p-i-1)Z)) = 0$ *for all* $0 < i+1 < p \leq N$ *(2.10).*

Then for all $p \geq 1$,

$$H^{N-p,p-1}_{\mathrm{van}}(Z) \oplus H^{N-p+1,p-1}_{\mathrm{prim}}(X) \cong H^0(X, K_X(pZ))/\left(f \cdot H^0(X, K_X((p-1)Z)) + \mathrm{d}H^0(\Omega_X^{N-1}((p-1)Z))\right).$$

In [Voi03, §6.1.2], Voisin proves a version of this theorem assuming instead much stronger vanishing conditions:

3. $H^i(\Omega_X^j(pZ)) = 0$ *for all* $p > 0$, $i > 0$, $j \geq 0$

Observe that while condition 3 is satisfied for smooth hypersurfaces in projective space due to Serre's vanishing theorem (see [Voi02, Exercise 7.2]), condition 3 does not generally hold in the Grassmannian case when Z is Fano. Indeed, when $X = \mathrm{Gr}(k, n)$ with $k \neq 1, n-1$, condition 3 only holds when $\deg Z \geq n-1$, see Section 3.8. In this section, we take care to demonstrate that vanishings 1 and 2 are sufficient for Theorem 2.1. In Chapter 3 we give explicit conditions in the Grassmannian case for values of k, n, and $\deg Z$ for which vanishings 1 and 2 hold.

2.1 Classical Residues

In [Gri69], Griffiths is interested in computing integrals of the form

$$\int_\Gamma \frac{P(x)}{Q(x)}\, \mathrm{d}x_1 \ldots \mathrm{d}x_n,$$

where $P(x), Q(x) \in \mathbb{C}[x_1, \ldots, x_n]$ are polynomials and $\Gamma \subset \mathbb{C}^n$ is an n-cycle on $\mathbb{C}^n$ which avoids the zero locus V of $Q(x)$. Such integrals will be called periods, and we call V the corresponding polar locus.

De Rham's theorem states that the ring homomorphism from de Rham cohomology to singular cohomology of a manifold M given by integration is an isomorphism.

In particular, if we replace Γ with a homologous cycle Γ' we obtain the same period:

$$\int_{\Gamma'} \frac{P(x)}{Q(x)}\,\mathrm{d}x_1 \ldots \mathrm{d}x_n = \int_{\Gamma} \frac{P(x)}{Q(x)}\,\mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

Similarly, if we replace $\dfrac{P(x)}{Q(x)}\,\mathrm{d}x_1 \ldots \mathrm{d}x_n$ with a cohomologous form (e.g. add an exact form) we again obtain the same period:

$$\int_{\Gamma} \left(\frac{P(x)}{Q(x)}\,\mathrm{d}x_1 \ldots \mathrm{d}x_n + \mathrm{d}g(x) \right) = \int_{\Gamma} \frac{P(x)}{Q(x)}\,\mathrm{d}x_1 \ldots \mathrm{d}x_n,$$

where d is the exterior derivative and g is any rational $n-1$ form with polar locus V.

Below we recall how to compute periods in one variable, where every period is cohomologous to one with simple poles. In higher dimensions a residual obstruction appears—the Griffiths ring—and the rest of this chapter is dedicated to making it explicit.

Residues in One Variable

Cauchy's residue formula computes periods in the one-dimensional case:

Theorem 2.2 (Cauchy Residue Formula)**.** *Assume $f := \dfrac{P(x)}{Q(x)}$ is a holomorphic function on $X := \mathbb{C} \setminus \{\lambda_1, \ldots, \lambda_k\}$ where $\{\lambda_1, \ldots, \lambda_k\}$ are the roots of $Q(x)$. For an oriented closed curve Γ on X, let $\mu(\Gamma, \lambda_j)$ be the winding number of Γ around λ_j. Then*

$$\int_{\Gamma} f(x)\,\mathrm{d}x = 2\pi i \sum_j \mu\left(\Gamma, \lambda_j\right) \mathrm{Res}\left(f(x), \lambda_j\right),$$

where $\mathrm{Res}\left(f(x), \lambda_j\right)$ denotes the residue of $f(x)$ at λ_j, given by the coefficient a_{-1} in the Laurent expansion of $f(x)$ at λ_j:

$$f(x) = \sum_{n=-\infty}^{\infty} a_n (x - \lambda_j)^n.$$

Rather than compute the Laurent coefficient directly, we may reduce to the simple-pole case where the residue is a single limit. If $p > 1$, in a neighborhood of

λ_j write $f(x) = g(x)/(x-\lambda_j)^p$ with $g(\lambda_j) \neq 0$. Then, differentiating $g(x)/(x-\lambda_j)^{p-1}$ and rearranging, we have

$$\frac{g(x)}{(x-\lambda_j)^p} = \frac{1}{p-1}\frac{g'(x)}{(x-\lambda_j)^{p-1}} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g(x)}{(p-1)(x-\lambda_j)^{p-1}}\right).$$

Iterating this process, we may assume that $g(x)/(x-\lambda_j)^p\,\mathrm{d}x$ is cohomologous to a meromorphic form with a simple pole at λ_j. This is the one-dimensional version of Griffiths–Dwork reduction (see Section 2.4).

2.2 Smooth Hypersurfaces in a Smooth Projective Variety

It is natural to consider the construction in the previous section in higher dimension. In this section, we define the residue map in higher dimensions, and consequently introduce the Hodge filtration on the cohomology of the complement of a hypersurface, following [Voi03, §6.3]. Let $Z \overset{\ell}{\hookrightarrow} X$ be a smooth ample hypersurface in a smooth projective variety of dimension N. Let $U = X \setminus Z \overset{j}{\hookrightarrow} X$ be the complement of Z in X. Then we have a long exact sequence of relative cohomology of the pair (X, U):

$$\ldots \to H^k(X,U,\mathbb{Z}) \to H^k(X,\mathbb{Z}) \to H^k(U,\mathbb{Z}) \to H^{k+1}(X,U,\mathbb{Z}) \to H^{k+1}(X,\mathbb{Z}) \to \ldots.$$

By the Thom isomorphism, we have

$$H^{k+1}(X,U,\mathbb{Z}) \cong H^{k-1}(Z,\mathbb{Z}).$$

The map

$$H^{k-1}(Z,\mathbb{Z}) \to H^{k+1}(X,\mathbb{Z}) \tag{2.2.1}$$

is called the Gysin morphism ℓ_* associated to the inclusion $Z \overset{\ell}{\hookrightarrow} X$. Finally, the connecting homomorphism

$$H^k(U,\mathbb{Z}) \to H^{k-1}(Z,\mathbb{Z})$$

is called the *residue map* Res (defined explicitly below). Passing to rational cohomology, we have the long exact sequence

$$\cdots \longrightarrow H^{N-2}(Z,\mathbb{Q}) \xrightarrow{\ell_*} H^N(X,\mathbb{Q}) \xrightarrow{j^*} H^N(U,\mathbb{Q}) \xrightarrow{\text{Res}} H^{N-1}(Z,\mathbb{Q}) \xrightarrow{\ell_*} H^{N+1}(X,\mathbb{Q}) \xrightarrow{j^*} \cdots$$

By the Lefschetz decomposition associated to the polarization given by $\mathcal{O}_X(Z)$ we have the *primitive cohomology* of X

$$H^N_{\text{prim}}(X,\mathbb{Q}) = H^N(X,\mathbb{Q})/\ell_* H^{N-2}(Z,\mathbb{Q}),$$

and define the *vanishing cohomology* of Z

$$H^{N-1}_{\text{van}}(Z,\mathbb{Q}) := \ker(\ell_* : H^{N-1}(Z,\mathbb{Q}) \to H^{N+1}(X,\mathbb{Q})).$$

We then have a short exact sequence

$$0 \to H^N_{\text{prim}}(X,\mathbb{Q}) \xrightarrow{j^*} H^N(U,\mathbb{Q}) \xrightarrow{\text{Res}} H^{N-1}_{\text{van}}(Z,\mathbb{Q}) \to 0. \tag{2.2.2}$$

For an example of a calculation of the primitive cohomology of a hypersurface where $H^N_{\text{prim}}(\text{Gr}(k,n),\mathbb{C}) \neq 0$ see Section 4.1.

Vanishing vs. Primitive Cohomology

Although we will not need this until Chapter 4, it is convenient to prove here that the vanishing and primitive cohomology of Z are closely related; we describe this relationship now.

We have the following identity:

$$\ell_*\ell^*\alpha = \alpha h \quad \text{for } \alpha \in H^\bullet(X,\mathbb{Q}),$$

where h is the hyperplane class of X. By the self-intersection formula,

$$\ell^*\ell_*\beta = \beta\ell^* h \quad \text{for } \beta \in H^\bullet(Z,\mathbb{Q}).$$

We define the *primitive cohomology* of Z by

$$H^{N-1}_{\text{prim}}(Z,\mathbb{Q}) := H^{N-1}(Z,\mathbb{Q})/\left(\ell^* h \cdot H^{N-3}(Z,\mathbb{Q})\right).$$

The following argument is illustrated in Figure 2.1.

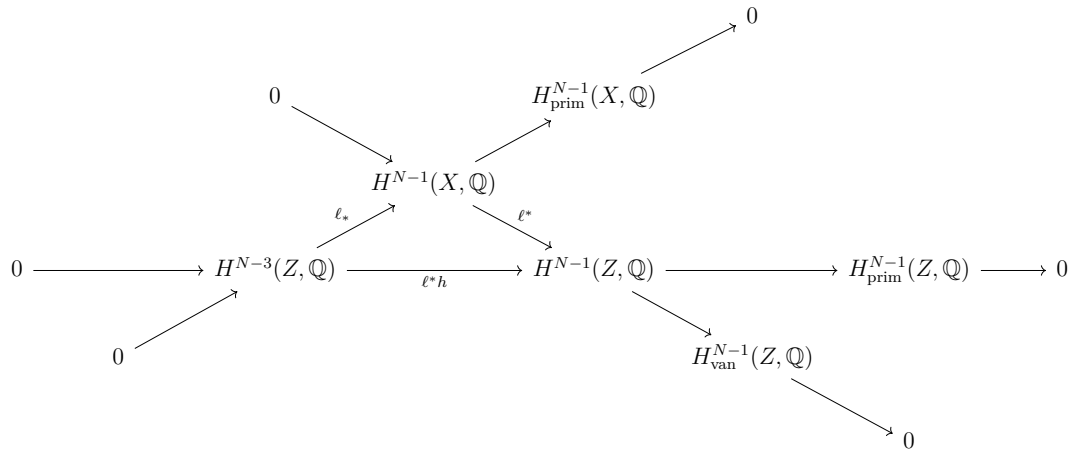


Figure 2.1. Relationship between $H^{N-1}_{\text{van}}(Z)$ and $H^{N-1}_{\text{prim}}(Z)$.

By weak Lefschetz,

$$\ell^* : H^{N-3}(X,\mathbb{Q}) \to H^{N-3}(Z,\mathbb{Q})$$

is an isomorphism and

$$\ell^* : H^{N-1}(X,\mathbb{Q}) \to H^{N-1}(Z,\mathbb{Q})$$

is an injection. Therefore we have an equality

$$\ell_* H^{N-3}(Z,\mathbb{Q}) = h \cdot H^{N-3}(X,\mathbb{Q}).$$

Now,

$$\ell^*\ell_* H^{N-3}(Z,\mathbb{Q}) = \ell^* h \cdot H^{N-3}(Z,\mathbb{Q})$$

by the self-intersection formula. Therefore

$$\ell^* H^{N-1}_{\text{prim}}(X,\mathbb{Q}) \cong \ell^* H^{N-1}(X,\mathbb{Q})/(\ell^* h \cdot H^{N-3}(Z,\mathbb{Q})).$$

By Hard Lefschetz,

$$\ell_*\ell^* = L : H^{N-1}(X,\mathbb{Q}) \to H^{N+1}(X,\mathbb{Q})$$

is an isomorphism, so ℓ_* restricts to an injection on $\ell^* H^{N-1}(X,\mathbb{Q})$, and its image is all of $H^{N+1}(X,\mathbb{Q})$. Hence

$$H^{N-1}(Z,\mathbb{Q}) \cong H^{N-1}_{\text{van}}(Z,\mathbb{Q}) \oplus \ell^* H^{N-1}(X,\mathbb{Q}).$$

Then by the third isomorphism theorem,

$$H^{N-1}_{\text{prim}}(Z,\mathbb{Q})/\ell^* H^{N-1}_{\text{prim}}(X,\mathbb{Q}) \cong H^{N-1}_{\text{van}}(Z,\mathbb{Q}). \tag{2.2.3}$$

The Logarithmic de Rham Complex

We begin this section with a review of some details about the logarithmic de Rham complex of a complex manifold X, which was first introduced by Deligne in [Del70]. We will follow the exposition from [Voi02, §8.2] and demonstrate how to explicitly compute $H^N(U, \mathbb{Q})$ using the logarithmic de Rham complex.

Let Z be a smooth divisor on X. Define $\Omega^k_X(\log Z)$ as the subsheaf of $\Omega^k_X(Z)$ of meromorphic forms on X, holomorphic on $X \setminus Z$, defined by the following condition:

- If α is a meromorphic form on some open set $U \subset X$, holomorphic on $U \setminus (Z \cap U)$, then $\alpha \in \Omega^k_X(\log Z)|_U$ if α admits a pole of order at most 1 along Z, and the same holds for $\mathrm{d}\alpha$.

Proposition 2.3. *With the notation as above, if $z_1, \ldots, z_n$ are local coordinates for U for which Z is defined by the equation $z_1 = 0$, then $\Omega^k_X(\log Z)|_U$ is a sheaf of free $\mathcal{O}_U$ modules with basis given by*

$$\mathrm{d}z_{j_1} \wedge \cdots \wedge \mathrm{d}z_{j_k} \quad \textit{and} \quad \frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{j_2} \wedge \cdots \wedge \mathrm{d}z_{j_k},$$

where $\{j_r\}$ runs over all subsets of $(1, 2, \ldots, n)$ of size k.

Proof. See [Voi02, Lemma 8.16]. □

Corollary 2.4. *The sheaves $\Omega^k_X(\log Z)$ are sheaves of locally free $\mathcal{O}_X$-modules. Moreover, for $k = 1$, $\Omega_X(\log Z)$ is locally generated by Ω_X and $\dfrac{\mathrm{d}f}{f}$ where f is a local holomorphic equation for Z.*

Proposition 2.5. *An equivalent construction of $\Omega^k_X(\log Z)$ is*

$$\Omega^k(\log Z) := \bigwedge^k \Omega_X(\log Z).$$

Proposition 2.6. *If α is a section of $\Omega^k_X(\log Z)$ on some open set $U \subset X$ then $\mathrm{d}\alpha = \partial\alpha$ is in $\Omega^{k+1}_X(\log Z)$.*

Proof. $\mathrm{d}\alpha$ is meromorphic, with a pole of order at most 1 along Z, and $\mathrm{d}\alpha$ is closed. □

Thus $(\Omega^\bullet_X(\log Z), \partial)$ is a complex of sheaves over X, called the *logarithmic de Rham complex*.

Proposition 2.7. *Let*

$$\Omega_X^{k,c}(Z) := \ker\left(\mathrm{d} : \Omega_X^k(Z) \to \Omega_X^{k+1}(2Z)\right)$$

denote the subsheaf of $\Omega_X^k(Z)$ *consisting of closed* k*-forms, and similarly define*

$$\Omega_X^{k,c}(\log Z) := \ker\left(\mathrm{d} : \Omega_X^k(\log Z) \to \Omega_X^{k+1}(\log Z)\right)$$

as the closed subsheaf of $\Omega_X^k(\log Z)$ *of closed logarithmic* k*-forms. Then*

$$\Omega_X^{k,c}(\log Z) = \Omega_X^{k,c}(Z).$$

Proof. If α is a section of $\Omega_X^{k,c}(\log Z)$ on some open set $U \subset X$, then α is a meromorphic form on $U \setminus (Z \cap U)$ with a pole of order at most 1 along Z and $\mathrm{d}\alpha = 0$.

Conversely, if $\alpha \in \Omega_X^{k,c}(Z)$, then α has pole order at most one along Z, and $\mathrm{d}\alpha = 0$ has pole order at most one along Z. Hence $\alpha \in \Omega_X^k(\log Z)$ and is closed, so $\alpha \in \Omega_X^{k,c}(\log Z)$.

□

We may organize this information in the following diagram.

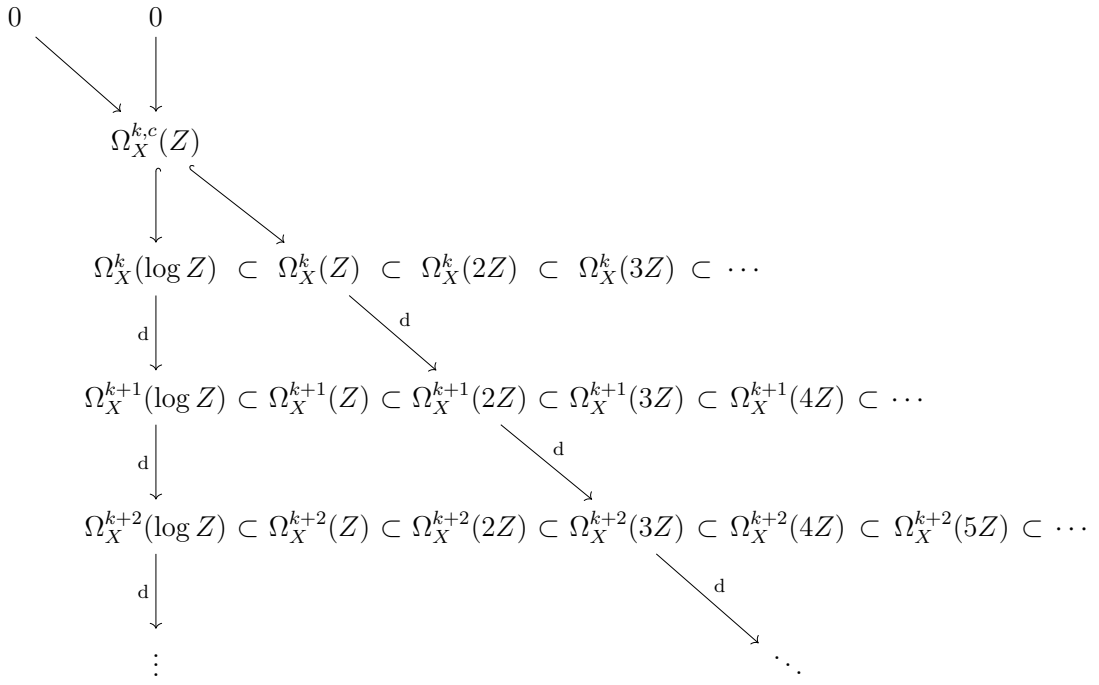


Figure 2.2. Chain of containments for the logarithmic and holomorphic de Rham complexes and the associated differentials.

Finally, we observe that the logarithmic de Rham complex is exact in degree ≥ 2 (see [Voi03, Corollary 6.7]).

With $U = X \setminus Z$ as before, let $j : U \hookrightarrow X$ denote the inclusion. Recall that $\mathcal{A}_U^\bullet$ denotes the complex of smooth forms on U. Then we have an inclusion of sheaves

$$\Omega_X^k(\log Z) \subset j_*(\Omega_U^k),$$

which is compatible with differentials, and moreover from the inclusion of complexes $\Omega_U^\bullet \subset \mathcal{A}_U^\bullet$ we obtain a morphism of complexes

$$\Omega_X^\bullet(\log Z) \to j_*\mathcal{A}_U^\bullet. \tag{2.2.4}$$

The main reason for introducing the logarithmic de Rham complex comes from the following theorem, due to Griffiths [Gri69] and Deligne [Del71]:

Theorem 2.8. *The morphism from Equation (2.2.4) is a quasi-isomorphism.*

The de Rham complex $(\mathcal{A}_U^\bullet, \mathrm{d})$ is a resolution of the constant sheaf $\underline{\mathbb{C}}$ on U by fine sheaves which are thus acyclic for the functor j_*. Thus, we obtain

$$H^k(U, \mathbb{C}) \cong \mathbb{H}^k(X, j_*\mathcal{A}_U^\bullet) \cong \mathbb{H}^k(X, \Omega_X^\bullet(\log Z)).$$

We define the Hodge filtration $F^\bullet$ on each $H^k(U, \mathbb{C})$ as the one induced by the naïve filtration on the logarithmic de Rham complex $\Omega_X^\bullet(\log Z)$ given by

$$F^k\Omega_X^\bullet(\log Z) = 0 \to \Omega^k(\log Z) \to \Omega^{k+1}(\log Z) \to \dots.$$

We define the residue map on complexes

$$\mathrm{Res} : \Omega_X^\bullet(\log Z) \to \Omega_Z^{\bullet-1},$$

which induces the map

$$H^k(U, \mathbb{C}) \to H^{k-1}(Z, \mathbb{C}),$$

via the residue map on forms

$$\alpha \wedge \frac{\mathrm{d}f}{f} \mapsto 2\pi i \alpha|_Z.$$

Now, j^* is strictly compatible with the Hodge filtration, and so is the residue map Res after shifting by bidegree $(1, 1)$. We may then recover the vanishing cohomology of Z from the cohomology of U by examining the weight filtration on $H^N(U, \mathbb{Q})$:

$$W_0(H^N(U, \mathbb{Q})) = j^*H^N_{\mathrm{prim}}(X, \mathbb{Q}), \quad W_1(H^N(U, \mathbb{Q})) = H^N(U, \mathbb{Q}),$$

where we use the weight index notation of Voisin ([Voi02, §8.4]).

We construct a resolution $\mathcal{C}_k^\bullet$ of $\Omega_X^{k,c}(Z)$:

$$\mathcal{C}_k^\bullet := 0 \to \Omega_X^k(Z) \xrightarrow{\mathrm{d}} \Omega_X^{k+1}(2Z) \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Omega_X^N((N-k+1)Z) \to 0.$$

Here we note that $\Omega_X^{k,c}(\log Z) \cong \Omega_X^{k,c}(Z)$ so $\mathcal{C}_k^\bullet$ is indeed a resolution of $\Omega_X^{k,c}(\log Z)$. We consider both complexes $F^k(\Omega_X^\bullet(\log Z))$ and $\mathcal{C}_k^\bullet$ to be supported in degrees k through N. Because both complexes are resolutions of $\Omega_X^{k,c}(\log Z)$ and the logarithmic de Rham complex is exact in degree ≥ 2, there is a quasi-isomorphism between them, which induces an isomorphism on hypercohomology

$$\mathbb{H}^N(X, F^k(\Omega_X^\bullet(\log Z))) \cong \mathbb{H}^N(X, \mathcal{C}_k^\bullet),$$

for all $k \geq 1$. See [Huy06, §3.3] for an introduction to the hypercohomology of a complex or [Voi02, §8.1] for an introduction to the hypercohomology spectral sequence. By the definition of the hypercohomology of a filtered complex (see [Voi02, §8.4.3, especially Theorem 8.35b]) we have

$$F^{N-p+1}H^N(U, \mathbb{C}) \cong \mathrm{Im}(\mathbb{H}^N(F^{N-p+1}\Omega_X^\bullet(\log Z)) \to \mathbb{H}^N(\Omega_X^\bullet(\log Z))).$$

Now, the morphism on the right is injective due to the degeneration of the Frölicher spectral sequence (see [Voi02, Theorem 8.3]). Thus we have the identification

$$F^{N-p+1}H^N(U, \mathbb{C}) \cong \mathbb{H}^N(X, F^{N-p+1}(\Omega_X^\bullet(\log Z))) \cong \mathbb{H}^N(X, \mathcal{C}_{N-p+1}^\bullet).$$

We may thus compute the Hodge filtration on $H^N(U, \mathbb{C})$ using the hypercohomology spectral sequence associated to the naïve filtration on $\mathcal{C}^\bullet$.

Hypercohomology Spectral Sequence Calculation

To ease the calculation of the hypercohomology spectral sequence of the complex $\mathcal{C}_{N-p+1}^\bullet$ we assume some vanishings of certain cohomology groups throughout the rest of this section:

Assumption 2.9. *$H^i(X, \Omega_X^{N-i}((p-i)Z)) = 0$ for all $0 < i < p \leq N$.*

Assumption 2.10. *$H^i(X, \Omega_X^{N-i-1}((p-i-1)Z)) = 0$ for all $0 < i+1 < p \leq N$.*

These vanishings, along with Nakano vanishing, are illustrated in Figure 2.4. Now fix p. Taking the hypercohomology spectral sequence of $\mathcal{C}_{N-p+1}^\bullet$ we have an E_1 page

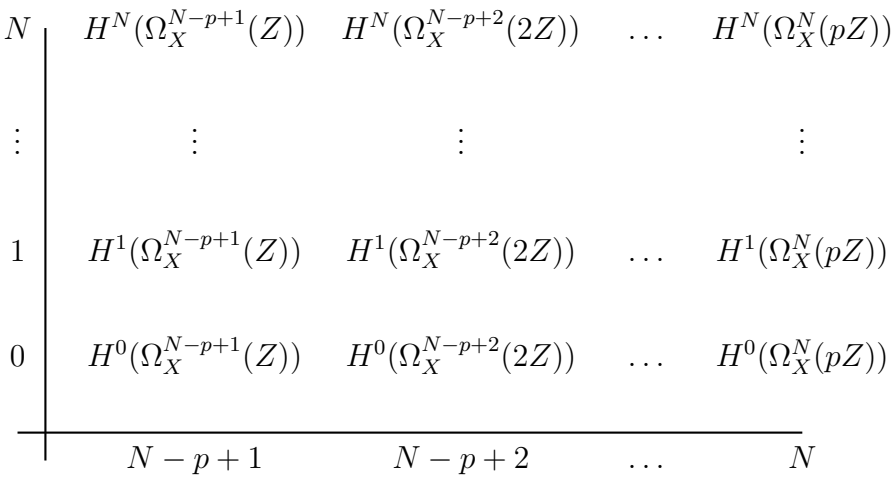


Figure 2.3. E_1 page for the hypercohomology spectral sequence of $\mathcal{C}^\bullet_{N-p+1}$.

Let us look at all differentials d_r whose target is $E_r^{N,0}$ for some r. When $r = 1$, we have the map

$$\mathrm{d}_1 : H^0(\Omega_X^{N-1}((p-1)Z)) \to H^0(\Omega_X^N(pZ)).$$

For $r \geq 2$, the source of a differential landing in $E_r^{N,0}$ is a subquotient of

$$H^{r-1}(\Omega_X^{N-r}((p-r)Z)).$$

If $r < p$, this vanishes by Assumption 2.10 with $i = r-1$; if $r \geq p$, the source lies outside the complex. Thus no differential d_r with $r \geq 2$ contributes to $E_r^{N,0}$. Therefore, for all $r > 1$,

$$E_r^{N,0} = E_\infty^{N,0} = H^0(\Omega_X^N(pZ))/\mathrm{d}_1(H^0(\Omega_X^{N-1}((p-1)Z))).$$

Taking the Nth graded piece of the total complex $E_\infty^{p,q}$, we have

$$\mathbb{H}^N(\mathcal{C}^\bullet_{N-p+1}) \cong E_\infty^{N,0} \oplus E_\infty^{N-1,1} \oplus \cdots \oplus E_\infty^{0,N}.$$

But for $i > 0$, $E_r^{N-i,i}$ is a quotient of $H^i(\Omega^{N-i}(p-i)Z)$, which is zero by Assumption 2.9. Thus

$$\mathbb{H}^N(\mathcal{C}^\bullet_{N-p+1}) \cong H^0(\Omega_X^N(pZ))/\mathrm{d}_1(H^0(\Omega_X^{N-1}((p-1)Z))). \tag{2.2.5}$$

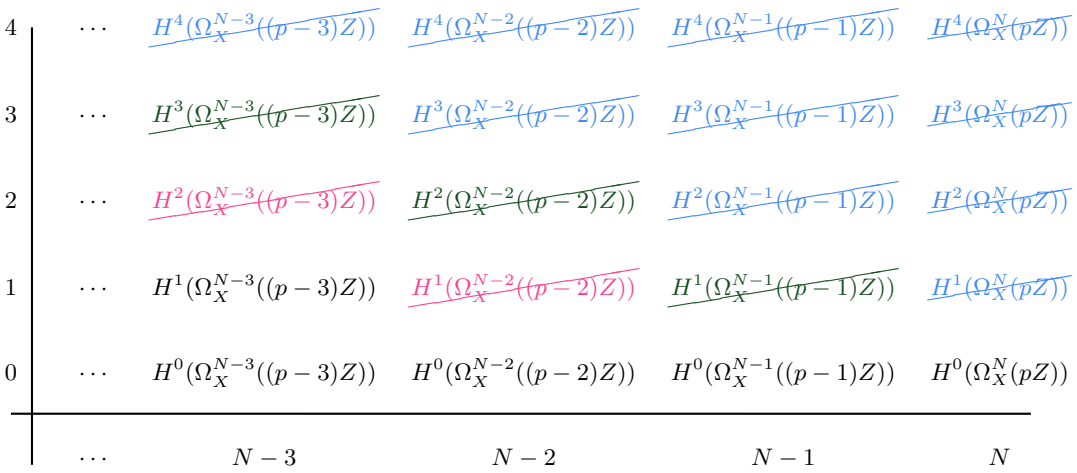


Figure 2.4. Terms above the antidiagonal vanish by Nakano (see Section 3.4). Terms on the antidiagonal with $q > 0$ vanish by Assumption 2.9. Terms immediately to the left of the antidiagonal with $q > 0$ vanish by Assumption 2.10.

Generalizing Griffiths' construction for $X = \mathbb{P}^N$ in [Gri69], we have a natural map

$$\alpha_p : H^0(X, K_X(pZ)) \to H^N(U, \mathbb{C}), \tag{2.2.6}$$

which sends a section (regarded as a meromorphic form on X with a pole of order at most p along Z) to its de Rham cohomology class in $F^{N-p+1}H^N(U)$.

By strictness of j^* and Res with respect to the Hodge filtration, equation (2.2.2) induces short exact sequences on each step of the filtration. Passing to the associated graded for the Hodge filtration gives

$$\mathrm{Gr}_F^{N-p+1} H^N(U, \mathbb{C}) \cong \mathrm{Gr}_F^{N-p+1} H^N_{\mathrm{prim}}(X, \mathbb{C}) \oplus \mathrm{Gr}_F^{N-p} H^{N-1}_{\mathrm{van}}(Z, \mathbb{C}),$$

where the shift in the second term comes from the residue map having bidegree $(-1, -1)$. Equivalently,

$$\frac{F^{N-p+1}H^N(U, \mathbb{C})}{F^{N-p+2}H^N(U, \mathbb{C})} \cong H^{N-p+1,p-1}_{\mathrm{prim}}(X) \oplus H^{N-p,p-1}_{\mathrm{van}}(Z).$$

We seek to prove the main result of the section:

Theorem 2.1. *Suppose $Z \hookrightarrow X$ is a smooth ample hypersurface in a smooth projective variety X of dimension N, and moreover that we have the following vanishings:*

1. *$H^i(X, \Omega_X^{N-i}((p-i)Z)) = 0$ for every $0 < i < p \leq N$ (2.9), and*

2. $H^i(X, \Omega_X^{N-i-1}((p-i-1)Z)) = 0$ *for all* $0 < i+1 < p \leq N$ *(2.10).*

Then for all $p \geq 1$,

$$H^{N-p,p-1}_{\text{van}}(Z) \oplus H^{N-p+1,p-1}_{\text{prim}}(X) \cong H^0(X, K_X(pZ))/\left(f \cdot H^0(X, K_X((p-1)Z)) + \mathrm{d}H^0(\Omega_X^{N-1}((p-1)Z))\right).$$

Proof. This proof has essentially the same flavor as [Voi03, Theorem 6.10]. Observe that by Equation (2.2.5), the image of α_p is $F^{N-p+1}H^N(U)$:

$$F^{N-p+1}H^N(U, \mathbb{C}) = \operatorname{im}\left(\alpha_p : H^0(K_X(pZ)) \to H^N(U, \mathbb{C})\right).$$

Now, for any $Q \in H^0(K_X((p-1)Z))$ we have

$$\alpha_{p-1}(Q) = \alpha_p(fQ),$$

because

$$\frac{Q}{f^{p-1}}\Omega = \frac{Qf}{f^p}\Omega$$

as meromorphic forms on U. Thus

$$\operatorname{im}(\alpha_{p-1}) = \alpha_p\left(f \cdot H^0(K_X((p-1)Z))\right).$$

Let $\overline{\alpha_p}$ denote the composition of α_p with the projection to the associated graded.

$$\begin{aligned} \overline{\alpha_p} \colon H^0(X, K_X(pZ)) &\twoheadrightarrow \frac{F^{N-p+1}H^N(U, \mathbb{C})}{F^{N-p+2}H^N(U, \mathbb{C})} \\ &\cong H^{N-p,p-1}_{\text{van}}(Z) \oplus H^{N-p+1,p-1}_{\text{prim}}(X). \end{aligned} \tag{2.2.7}$$

Now suppose $P \in \ker \overline{\alpha}_p$. Then

$$\alpha_p(P) \in F^{N-p+2}H^N(U),$$

so

$$\alpha_p(P) = \alpha_p(fQ) \text{ for some } Q \in H^0(K_X((p-1)Z)),$$

hence

$$\alpha_p(P - fQ) = 0,$$

so

$$P - fQ \in \ker \alpha_p.$$

Now by the spectral sequence Equation (2.2.5),

$$F^{N-p+1}H^N(U) \cong H^0(\Omega_X^N(pZ))/dH^0(\Omega_X^{N-1}((p-1)Z)),$$

so

$$\ker \alpha_p = dH^0(\Omega_X^{N-1}((p-1)Z)).$$

Therefore

$$\ker \overline{\alpha_p} = f \cdot H^0(X, K_X((p-1)Z)) + dH^0(\Omega_X^{N-1}((p-1)Z)). \qquad \square$$

In order to produce an explicit example calculating $\mathrm{d}H^0(\Omega_X^{N-1}((p-1)Z))$, we specialize to the case of a smooth hypersurface Z in X of degree d, where $X = \mathrm{Gr}(k,n)$ is the Grassmannian of k-planes in $V = \mathbb{C}^n$. In Chapter 3 we characterize the values k, n, d for which Assumptions 2.9 and 2.10 hold in the Grassmannian case.

Proposition 2.11. *For $2 \leq k \leq n-2$, suppose the following:*

- *$d \nmid n$ or $\gcd(k, n/d) = 1$ (2.9), and*
- *$k \notin \{2, n-2\}$ or $2d \nmid (n+1)$ (2.10)*

Then Assumption 2.9 and Assumption 2.10 hold, i.e.

$$H^i(X, \Omega_X^{N-i}((p-i)Z)) = 0 \text{ for every } 0 < i < p \leq N$$

$$H^i(X, \Omega_X^{N-i-1}((p-i-1)Z)) = 0 \text{ for every } 0 < i+1 < p \leq N.$$

Proof. Deferred to Theorem 3.71 and Theorem 3.72. □

We recall that X has canonical bundle $K_X \cong \mathcal{O}(-n)$, and tautological subbundle S and quotient bundle Q given by the tautological sequence

$$0 \to S \to \mathcal{O}_X \otimes V \to Q \to 0. \tag{2.2.8}$$

Recall that $S^\vee \otimes Q \cong T_X$. For any $p > 0$ we produce two exact sequences: first by dualizing the tautological sequence Equation (2.2.8) and tensoring with $K_X(pZ) \otimes V$:

$$0 \to Q^\vee \otimes K_X(pZ) \otimes V \to K_X(pZ) \otimes \mathrm{End}(V) \xrightarrow{\varphi_1} S^\vee \otimes K_X(pZ) \otimes V \to 0, \tag{2.2.9}$$

and second by tensoring the tautological sequence with $S^\vee \otimes K_X(pZ)$:

$$0 \to \mathcal{E}nd(S) \otimes K_X(pZ) \to S^\vee \otimes K_X(pZ) \otimes V \xrightarrow{\varphi_2} T_X \otimes K_X(pZ) \to 0. \tag{2.2.10}$$

Observe that if we have the following vanishings:

- $H^1(X, Q^\vee \otimes K_X(pZ) \otimes V) = 0$,
- $H^1(X, \mathcal{E}nd(S) \otimes K_X(pZ)) = 0$,

then $\varphi_2 \circ \varphi_1$ induces the surjective map

$$\varphi : H^0(X, \mathcal{O}_X(pd-n)) \otimes \mathrm{End}(V) \twoheadrightarrow H^0(X, T_X \otimes K_X(pZ)). \tag{2.2.11}$$

Here we have identified holomorphic sections of $\mathcal{O}_X(pd-n)$ with $H^0(K_X(pZ))$, meromorphic N forms on X, holomorphic on $X \setminus Z$ with a pole of order at most p along Z. We defer the combinatorics to Chapter 3 (see Theorems 3.68 and 3.69), and only state the corollaries that follow from these calculations:

Corollary 2.12. *For any line bundle L on $X = \mathrm{Gr}(k,n)$ with $2 \le k \le n-1$,*

$$H^1(X, Q^\vee \otimes V \otimes K_X \otimes L) = 0.$$

Corollary 2.13. *For any line bundle L on $X = \mathrm{Gr}(k,n)$ with $1 \le k \le n-2$,*

$$H^1(X, \mathcal{E}nd(S) \otimes K_X \otimes L) = 0.$$

Let us assume $2 \le k \le n-2$. Then φ as constructed in Equation (2.2.11) is indeed surjective.

Let e_{ij} be the standard basis for $\mathrm{End}(V) \cong \mathbb{C}^{n^2}$, and recall that $\mathfrak{sl}(n,\mathbb{C})$ is generated by $\{E_{ij}\}$, where as in Equation (1.2.1), E_{ij} are given by

$$E_{ij} := \begin{cases} e_{i,j} & i \neq j, \\ e_{i,i} - e_{i+1,i+1} & i = j < n, \end{cases}$$

with the convention $E_{n,n} = 0$. Then $e_{ij} \mapsto E_{ij}$ is a surjective map. Writing Z as the zero set of a polynomial f, we may write φ as a map

$$\varphi : P^{ij} \otimes e_{ij} \mapsto \frac{P^{ij} E_{ij}}{f^p},$$

where P^{ij} is a polynomial of degree $pd-n$.

Because $K_X \cong \mathcal{O}_X(-n)$, there exists a global nowhere vanishing section $\Omega \in H^0(K_X(n))$, and via the interior product, such an Ω gives rise to an isomorphism

$$T_X \xrightarrow{\sim} \Omega_X^{N-1}(n),$$

$$v \mapsto \mathrm{int}(v)(\Omega).$$

We write $\mathrm{int}(v)(w)$ as $v \lrcorner w$, or sometimes $\iota_v(w)$. Post-composing φ with the interior product with Ω we obtain a surjection

$$\varphi' : H^0(X, \mathcal{O}_X(pd-n)) \otimes \mathrm{End}(V) \twoheadrightarrow H^0(X, \Omega_X^{N-1}(pd)), \tag{2.2.12}$$

$$P^{ij} \otimes e_{ij} \mapsto \frac{P^{ij} E_{ij} \lrcorner \Omega}{f^p}.$$

2.3 A Global Section of $K_X(n)$

Overview

The prototypical example of Ω is

$$\Omega = x\,\mathrm{d}y \wedge \mathrm{d}z - y\,\mathrm{d}x \wedge \mathrm{d}z + z\,\mathrm{d}x \wedge \mathrm{d}y$$

on $\mathbb{P}^2$. This can be constructed by taking the standard volume form $\widetilde{\Omega} := \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ on $\mathbb{C}^3$ and contracting it with the Euler vector field

$$\xi = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

The case on Grassmannians is similar: let $\widetilde{\Omega}$ denote the standard algebraic volume form on $\mathbb{C}^{nk}$. In this case, however, we have

$$\Omega := \left(\bigwedge_{F_{ij} \in B} F_{ij} \right) \lrcorner\, \widetilde{\Omega},$$

where B is the standard basis for $\mathfrak{gl}(k, \mathbb{C})$.

GIT Construction

We shall obtain the section $\Omega \in H^0(X, K_X(n))$ by regarding $X = \mathrm{Gr}(k, n)$ as the GIT quotient of $M := \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ by the right action of $\mathrm{GL}(k, \mathbb{C})$ on M, and show that the standard algebraic volume form $\widetilde{\Omega}$ on M descends via the quotient map $M^{\mathrm{st}} \xrightarrow{q} X$. Here M^{st} denotes the stable locus consisting of full rank matrices, and we take the GIT quotient $X = M/\!/_{\chi}\mathrm{GL}(k, \mathbb{C})$ with the character $\chi(g) = \det(g)$. Let $(x_{11}, x_{12}, \ldots, x_{nk})$ be the standard coordinates on M. Let $\widetilde{\Omega} \in H^0(M, \Omega^{nk}(M))$ be the standard algebraic volume form on M:

$$\widetilde{\Omega} = \mathrm{d}x_{11} \wedge \mathrm{d}x_{12} \wedge \ldots \wedge \mathrm{d}x_{nk}.$$

The infinitesimal action of the elementary matrices $f_{ij} \in \mathfrak{gl}(k, \mathbb{C})$ generates vector fields

$$F_{ij} = \sum_{\ell=1}^{n} x_{\ell i}\, \frac{\partial}{\partial x_{\ell j}}.$$

First we prove the following lemma:

Lemma 2.14.

$$\mathcal{L}_{F_{ij}} \widetilde{\Omega} = n\delta_{ij}\widetilde{\Omega}.$$

Proof. By Cartan's magic formula (see, e.g., [Lee13, Theorem 14.3.5]),

$$\mathcal{L}_{F_{ij}}\widetilde{\Omega} = F_{ij} \lrcorner \mathrm{d}\widetilde{\Omega} + \mathrm{d}(F_{ij} \lrcorner \widetilde{\Omega}).$$

(We read contraction right-associatively.) Now, $\mathrm{d}\widetilde{\Omega} = 0$ as $\widetilde{\Omega}$ is a top form. We proceed computing directly:

$$\mathrm{d}(F_{ij} \lrcorner \widetilde{\Omega}) = \mathrm{d}\left(\sum_{\ell=1}^{n} x_{\ell i} \frac{\partial}{\partial x_{\ell j}} \lrcorner \widetilde{\Omega}\right)$$

$$\mathrm{d}\left(\sum_{\ell=1}^{n} (-1)^{\ell+j} x_{\ell i}\, \mathrm{d}x_{11} \wedge \mathrm{d}x_{12} \cdots \wedge \widehat{\mathrm{d}x_{\ell j}} \wedge \cdots \wedge \mathrm{d}x_{nk}\right),$$

where $\widehat{\mathrm{d}x_{\ell j}}$ denotes omitting the term $\mathrm{d}x_{\ell j}$ from the wedge product. But by the definition of the exterior derivative we have

$$\begin{aligned}
&\mathrm{d}\left(\sum_{\ell=1}^{n} (-1)^{\ell+j} x_{\ell i}\, \mathrm{d}x_{11} \wedge \mathrm{d}x_{12} \cdots \wedge \widehat{\mathrm{d}x_{\ell j}} \wedge \cdots \wedge \mathrm{d}x_{nk}\right) \\
&\quad = \delta_{ij} \sum_{\ell=1}^{n} \mathrm{d}x_{11} \wedge \mathrm{d}x_{12} \cdots \wedge \mathrm{d}x_{\ell j} \wedge \cdots \wedge \mathrm{d}x_{nk} \\
&\quad = n\delta_{ij}\widetilde{\Omega}.
\end{aligned}$$

Observe that the sign $(-1)^{\ell+j}$ in the first expression was produced by the interior product; reinserting $\mathrm{d}x_{\ell j}$ into its standard position in the wedge product produces a second factor of $(-1)^{\ell+j}$, and the signs cancel. □

On the other hand, $\mathrm{GL}(n, \mathbb{C})$ acts on M on the left by post-composition. The center acts trivially on the quotient X, so instead we consider the action of $\mathrm{SL}(n, \mathbb{C})$. The infinitesimal action of $\mathfrak{sl}(n, \mathbb{C})$ generates vector fields

$$E_{ij} = \begin{cases} \displaystyle\sum_{\ell=1}^{k} x_{i\ell} \frac{\partial}{\partial x_{j\ell}} & i \neq j, \\ \displaystyle\sum_{\ell=1}^{k} x_{i\ell} \frac{\partial}{\partial x_{j\ell}} - \sum_{\ell=1}^{k} x_{i+1,\ell} \frac{\partial}{\partial x_{j+1,\ell}} & i = j < n. \end{cases}$$

By convention we write $E_{n,n} = 0$. Observe that the left and right actions commute, hence so do the actions of the Lie algebras. Consider the "twisted volume form" Ω defined by

$$\Omega := F_{11} \lrcorner F_{12} \lrcorner \ldots \lrcorner F_{kk} \lrcorner \widetilde{\Omega}.$$

We prove the following lemma:

Lemma 2.15. *Ω is a semi-invariant form on X with weight n with respect to the $\mathrm{GL}(k, \mathbb{C})$ action.*

Proof. A function f on M^{st} descends to a section of $\mathcal{O}_{\mathrm{Gr}}(m)$ if and only if $A \cdot f = (\det A)^m f$ for all A in $\mathrm{GL}(k, \mathbb{C})$. A standard reference is [Lee13, §14]. Then we say f is a semi-invariant function with weight m. Taking the derivative, in order to show that Ω is a semi-invariant form with weight n we must show that

$$\mathcal{L}_{F_{ij}} \Omega = n \operatorname{Tr}(F_{ij})\, \Omega,$$

where $\operatorname{Tr}(F_{ij}) = \delta_{ij}$. Expanding $\mathcal{L}_{F_{ij}} \Omega$, we have

$$\mathcal{L}_{F_{ij}} \Omega = \mathcal{L}_{F_{ij}} \left(F_{11} \lrcorner F_{12} \lrcorner \ldots \lrcorner F_{kk} \lrcorner \widetilde{\Omega} \right).$$

Now, we recall the following standard result (see e.g. [KN63, §1.3]):

$$[\mathcal{L}_X, \iota_Y] \alpha = \iota_{[X,Y]} \alpha, \tag{2.3.1}$$

and a standard result from linear algebra:

$$[F_{ij}, F_{k\ell}] = \delta_{jk} F_{i\ell} - \delta_{i\ell} F_{kj}.$$

Thus

$$\begin{aligned} &\mathcal{L}_{F_{ij}} \left(F_{11} \lrcorner F_{12} \lrcorner \cdots \lrcorner F_{kk} \lrcorner \widetilde{\Omega} \right) \\ &\quad = F_{11} \lrcorner \mathcal{L}_{F_{ij}} \left(F_{12} \lrcorner \cdots \lrcorner F_{kk} \lrcorner \widetilde{\Omega} \right) + [F_{ij}, F_{11}] \lrcorner F_{12} \lrcorner \cdots \lrcorner F_{kk} \lrcorner \widetilde{\Omega}. \end{aligned}$$

We unpack the right-hand side. If $X = F_{ab}$ then $X \lrcorner F_{12} \lrcorner \ldots \lrcorner F_{kk} \lrcorner \widetilde{\Omega}$ is zero unless $a = b = 1$. By the commutator identity, $[F_{ij}, F_{11}] = \delta_{j1} F_{i1} - \delta_{i1} F_{1j}$. If $(i, j) \neq (1, 1)$, the surviving summand is some $(c, d) \in \{(1, 2), (1, 3), \cdots, (k, k)\}$, so contraction with $F_{12} \lrcorner F_{13} \lrcorner \cdots \lrcorner F_{kk} \lrcorner \widetilde{\Omega}$ has a repeated index, thus resulting in zero. If $(i, j) = (1, 1)$, the commutator itself vanishes. In either case,

$$[F_{ij}, F_{11}] \lrcorner F_{12} \lrcorner \ldots \lrcorner F_{kk} \lrcorner \widetilde{\Omega} = 0,$$

for all i, j.

Proceeding with another application of Equation (2.3.1), we have

$$\begin{aligned} &F_{11} \lrcorner \mathcal{L}_{F_{ij}} \left(F_{12} \lrcorner \ldots \lrcorner F_{kk} \lrcorner \widetilde{\Omega} \right) \\ &\qquad = F_{11} \lrcorner F_{12} \lrcorner \mathcal{L}_{F_{ij}} \left(F_{13} \lrcorner \ldots \lrcorner F_{kk} \lrcorner \widetilde{\Omega} \right) \\ &\qquad\quad + F_{11} \lrcorner [F_{ij}, F_{12}] \lrcorner F_{13} \lrcorner \ldots \lrcorner F_{kk} \lrcorner \widetilde{\Omega}. \end{aligned}$$

Now,

$$[F_{ij}, F_{12}] = \delta_{j1} F_{i2} - \delta_{i2} F_{1j}.$$

By a similar argument as before, $F_{11} \lrcorner [F_{ij}, F_{12}] \lrcorner F_{13} \lrcorner \ldots \lrcorner F_{kk} \lrcorner \widetilde{\Omega}$ will only be nonzero when $i = j = 1$ or $i = j = 2$. Suppose we take $i = j = 1$ (or $i = j = 2$, either will have the same result). Observe that we may proceed with the above process until we compute

$$F_{11} \lrcorner F_{12} \lrcorner \ldots \lrcorner F_{1k} \lrcorner \mathcal{L}_{F_{ij}} \left(F_{21} \lrcorner \ldots \lrcorner F_{kk} \lrcorner \widetilde{\Omega} \right).$$

This will equal

$$\begin{aligned} &F_{11} \lrcorner F_{12} \lrcorner \ldots \lrcorner F_{1k} \lrcorner F_{21} \lrcorner \mathcal{L}_{F_{ij}} \left(F_{22} \lrcorner \ldots \lrcorner F_{kk} \lrcorner \widetilde{\Omega} \right) \\ &+ F_{11} \lrcorner F_{12} \lrcorner \ldots \lrcorner F_{1k} \lrcorner [F_{ij}, F_{21}] \lrcorner F_{22} \lrcorner \ldots \lrcorner F_{kk} \lrcorner \widetilde{\Omega}. \end{aligned}$$

Regardless of whether we chose $i = j = 1$ or $i = j = 2$, this will be equal to $F_{11} \lrcorner [F_{ij}, F_{12}] \lrcorner F_{13} \lrcorner \ldots \lrcorner F_{kk} \lrcorner \widetilde{\Omega}$ but with opposite sign, hence these contributions will cancel out.

We may repeat the above argument to show that if $i \neq j$ then $\mathcal{L}_{F_{ij}} \Omega = 0$, and even when $i = j$ all contributions arising from the commutators cancel out, and we are left with

$$\mathcal{L}_{F_{ij}} \left(F_{11} \lrcorner F_{12} \lrcorner \ldots \lrcorner F_{kk} \lrcorner \widetilde{\Omega} \right) = \delta_{ij} F_{11} \lrcorner F_{12} \lrcorner \ldots \lrcorner F_{kk} \lrcorner \mathcal{L}_{F_{ij}} \widetilde{\Omega}.$$

We then conclude the proof by citing Lemma 2.14. □

Therefore Ω is a semi-invariant form for $\mathrm{GL}(k, \mathbb{C})$ of weight n, and thus descends to X as an element of $H^0(X, K_X(n))$.

We show an analogous result for E_{ij}, although it results in zero as E_{ij} is traceless.

Theorem 2.16.

$$\mathcal{L}_{E_{ij}}(\Omega) = 0.$$

Proof. By Equation (2.3.1), because $[E_{ij}, F_{k\ell}] = 0$,

$$\begin{aligned} \mathcal{L}_{E_{ij}}(\Omega) &= \mathcal{L}_{E_{ij}} \left(F_{11} \lrcorner F_{12} \lrcorner \ldots \lrcorner F_{kk} \lrcorner \widetilde{\Omega} \right) \\ &= F_{11} \lrcorner F_{12} \lrcorner \ldots \lrcorner F_{kk} \lrcorner \mathcal{L}_{E_{ij}} \widetilde{\Omega}. \end{aligned}$$

Applying Cartan's magic formula again, we have

$$\mathcal{L}_{E_{ij}} \widetilde{\Omega} = \mathrm{d}(E_{ij} \lrcorner \widetilde{\Omega}) + E_{ij} \lrcorner \mathrm{d}\widetilde{\Omega},$$

where again $\mathrm{d}\widetilde{\Omega} = 0$ for dimension reasons. Now, we consider two cases. If $i \neq j$,

$$E_{ij} \lrcorner \widetilde{\Omega} = \sum_{\ell=1}^{k} x_{i\ell} \frac{\partial}{\partial x_{j\ell}} \lrcorner \mathrm{d}x_{11} \wedge \mathrm{d}x_{12} \wedge \ldots \wedge \mathrm{d}x_{nk}$$

$$= \sum_{\ell=1}^{k} (-1)^{j+\ell} x_{i\ell}\, \mathrm{d}x_{11} \wedge \mathrm{d}x_{12} \wedge \ldots \wedge \widehat{\mathrm{d}x_{j\ell}} \wedge \ldots \wedge \mathrm{d}x_{nk},$$

where again $\widehat{\mathrm{d}x_{j\ell}}$ denotes omitting the term $\mathrm{d}x_{j\ell}$ from the wedge product. Thus

$$\begin{aligned}
\mathrm{d}(E_{ij} \lrcorner \widetilde{\Omega}) &= \mathrm{d}\left(\sum_{\ell=1}^{k} (-1)^{j+\ell} x_{i\ell} \mathrm{d}x_{11} \wedge \mathrm{d}x_{12} \wedge \ldots \wedge \widehat{\mathrm{d}x_{j\ell}} \wedge \ldots \wedge \mathrm{d}x_{nk} \right) \\
&= \delta_{ij} \left(\sum_{\ell=1}^{k} \mathrm{d}x_{11} \wedge \mathrm{d}x_{12} \wedge \ldots \wedge \mathrm{d}x_{j\ell} \wedge \ldots \wedge \mathrm{d}x_{nk} \right) \\
&= k\delta_{ij} \widetilde{\Omega} = 0,
\end{aligned}$$

since by assumption $i \neq j$.

If $i = j$ we repeat the same argument and observe that

$$\mathrm{d}(E_{ii} \lrcorner \widetilde{\Omega}) = k\delta_{ii}\widetilde{\Omega} - k\delta_{i+1,i+1}\widetilde{\Omega} = 0. \qquad \square$$

2.4 Griffiths–Dwork Reduction for Grassmannians

Suppose we have a form $\gamma \in H^0(\mathrm{Gr}(k,n), \Omega^{N-1}((p-1)d))$, considered as a form with a pole of order at most $p-1$ along a hypersurface Z of degree d. We have shown that γ may be written as

$$\frac{P^{ij} E_{ij} \lrcorner \Omega}{f^{p-1}},$$

where f is a defining equation for Z.

Theorem 2.17.

$$\mathrm{d}\gamma = -(p-1)\frac{(P^{ij} E_{ij} \lrcorner \mathrm{d}f)\Omega}{f^p} + \frac{(E_{ij} \lrcorner \mathrm{d}P^{ij})\Omega}{f^{p-1}}.$$

Proof. Let $g = \dfrac{P^{ij}}{f^{p-1}}$. Then $\gamma = gE_{ij} \lrcorner \Omega = E_{ij} \lrcorner g\Omega$. We shall show that

$$\mathrm{d}\gamma = E_{ij}(g)\, \Omega,$$

and the desired result follows from the quotient rule. We compute directly:

$$\mathrm{d}\gamma = \mathrm{d}(E_{ij} \lrcorner g\Omega) = \mathcal{L}_{E_{ij}}(g\Omega) - E_{ij} \lrcorner \mathrm{d}(g\Omega),$$

by Cartan's magic formula. Now, $g\Omega$ is a meromorphic N-form on the Grassmannian, hence $\mathrm{d}(g\Omega) = 0$. The Lie derivative is a derivation, hence

$$\mathcal{L}_{E_{ij}}(g\Omega) = \mathcal{L}_{E_{ij}}(g)\Omega + g\mathcal{L}_{E_{ij}}(\Omega).$$

All that remains is to show that $\mathcal{L}_{E_{ij}}(\Omega) = 0$. But this follows by Theorem 2.16.

$\square$

Lemma 2.18. *For all $1 \leq i, j \leq n$, E_{ij} has a natural action on R, the homogeneous coordinate ring of the Grassmannian.*

Proof. A function h on M^{st} descends to a section of $\mathcal{O}_X(1)$ if $F_{ij}(h) = \delta_{ij}h$ for all $1 \leq i, j \leq k$. Let h be such a section, and consider the function $E_{k\ell}(h)$. We have

$$\begin{aligned} F_{ij}(E_{k\ell}(h)) &= E_{k\ell}(F_{ij}(h)) \\ &= \delta_{ij}E_{k\ell}(h). \end{aligned}$$

□

We can now prove the main result of this thesis:

Theorem 1.1 (Main Result)**.** *Let $X = \mathrm{Gr}(k, n)$ with $2 \leq k \leq n-2$ and let $Z \subset X$ be a smooth hypersurface of degree d with defining polynomial f. Let R denote the homogeneous coordinate ring of X. Assume the following constraints on k, n, d:*

1. *$d \nmid n$ or $\gcd(k, n/d) = 1$, and*
2. *$k \notin \{2, n-2\}$ or $2d \nmid (n+1)$.*

Define the Jacobian ideal J_f to be the ideal of R generated by f and all $\{E_{ij}(f)\}$, and define the Griffiths ring as the quotient R/J_f. Then for each $p \geq 1$,

$$(R/J_f)_{pd-n} \cong H^{N-p,p-1}_{\mathrm{van}}(Z) \oplus H^{N-p+1,p-1}_{\mathrm{prim}}(\mathrm{Gr}(k, n)).$$

Proof. By Theorem 2.1, we have a surjective map

$$\overline{\alpha_p} : H^0(X, K_X(pZ)) \twoheadrightarrow H^{N-p,p-1}_{\mathrm{van}}(Z) \oplus H^{N-p+1,p-1}_{\mathrm{prim}}(\mathrm{Gr}(k, n)),$$

whose kernel equals

$$f \cdot H^0(X, K_X((p-1)Z)) + \mathrm{d}H^0(\Omega^{N-1}_X((p-1)Z)).$$

If $p = 1$ then the spectral sequence argument is trivial: all differentials on Figure 2.4 vanish, so

$$(R/J_f)_{d-n} \cong R_{d-n} \cong H^0(\Omega^N_X(Z)) \cong H^{N-1,0}_{\mathrm{van}}(Z) \oplus H^{N,0}_{\mathrm{prim}}(X).$$

For $p > 1$, take an arbitrary element $\gamma \in H^0(X, \Omega^{N-1}((p-1)Z))$. By equation Equation (2.2.12), we may write γ as

$$\gamma = \frac{P^{ij}E_{ij} \lrcorner \Omega}{f^{p-1}}$$

for $P^{ij} \in H^0(X, \mathcal{O}_X((p-1)d-n))$. By Theorem 2.17,

$$\mathrm{d}\gamma = -(p-1)\frac{(P^{ij}E_{ij} \lrcorner \mathrm{d}f)\Omega}{f^p} + \frac{(fE_{ij} \lrcorner \mathrm{d}P^{ij})\Omega}{f^p}.$$

On the other hand, relabeling P^{ij} by g, elements of $H^0(X, K_X((p-1)Z))$ are of the form $\frac{g}{f^{p-1}}\Omega = \frac{gf}{f^p}\Omega$. Let R_{pd-n} denote the $(pd-n)$th graded piece of the coordinate ring R of X. Then

$$H^0(X, K_X(pZ)) \cong \left\{ \frac{g}{f^p}\Omega \,:\, g \in R_{pd-n} \right\},$$
$$H^0(X, K_X((p-1)Z)) \cong \left\{ \frac{gf}{f^p}\Omega \,:\, g \in R_{(p-1)d-n} \right\}.$$

$$\mathrm{d}H^0(\Omega_X^{N-1}((p-1)Z)) \cong \left\{ -(p-1)\frac{(gE_{ij} \lrcorner \mathrm{d}f)\Omega}{f^p} + \frac{(fE_{ij} \lrcorner \mathrm{d}g)\Omega}{f^p} : g \in R_{(p-1)d-n} \right\}.$$

By considering the associated holomorphic sections, we have the following identifications:

$$H^0(X, K_X(pZ)) \cong \{g \in R_{pd-n}\},$$
$$H^0(X, K_X((p-1)Z)) \cong \{fg : g \in R_{(p-1)d-n}\},$$
$$\mathrm{d}H^0(\Omega_X^{N-1}((p-1)Z)) \cong \left\{ -(p-1)(gE_{ij}(f)) + (fE_{ij}(g)) : g \in R_{(p-1)d-n} \right\}.$$

Therefore we may identify

$$H^0(X, K_X(pZ))/\left(f \cdot H^0(X, K_X((p-1)Z)) + \mathrm{d}H^0(\Omega_X^{N-1}((p-1)Z)) \right)$$

with the $(pd-n)$th graded piece of the quotient ring (R/I), where I is the ideal generated by $f \cdot R_{(p-1)d-n}$ and

$$(-(p-1)(gE_{ij}(f)) + (fE_{ij}(g)) : g \in R_{(p-1)d-n}).$$

Now, $fE_{ij}(g) \in I$ so we may write I as the ideal generated by f and $E_{ij}(f)$. This is the definition of the Jacobian ideal, so $I = J_f$, and moreover we have an identification

$$(R/J_f)_{pd-n} \cong H^{N-p,p-1}_{\mathrm{van}}(Z) \oplus H^{N-p+1,p-1}_{\mathrm{prim}}(\mathrm{Gr}(k,n)),$$

as desired. □

As mentioned in Chapter 1, this is an extension of the work of Fatighenti and Mongardi which includes the Fano case. Moreover, we have given an explicit construction of the Griffiths–Dwork pole order reduction map in this setting.

Theorem 1.2 (Griffiths–Dwork Reduction)**.** *With the same assumptions and notation as Theorem 1.1, let* $\Omega \in H^0(K_X(n))$ *be a nonzero section of a twist of the canonical bundle* $K_X \cong \mathcal{O}_{\mathrm{Gr}(k,n)}(-n)$*. Let* $\gamma = \dfrac{g}{f^p}\Omega$ *be an arbitrary section of* $K_X(pZ)$ *for* $p > 1$*. Suppose there exist functions* h^{ij} *and* c *such that*

$$\gamma = \frac{\sum_{ij} h^{ij} E_{ij}(f) + cf}{f^p}\Omega.$$

Then

$$\gamma \equiv \frac{\sum_{ij} E_{ij}(h^{ij}) + (p-1)c}{(p-1)f^{p-1}}\Omega$$

as meromorphic forms on $U := X \setminus Z$ *(modulo exact forms).*

Remark 2.19. If Assumption 2.10 fails, it appears to only affect a single Hodge number (see Theorem 3.44). Moreover, Assumption 2.10 only fails when $k = 2$ or $k = n - 2$, so $k(n-k)$ is even, so in such cases we may always use Hodge symmetry to recover the full Hodge diamond. For an example of this, see Section 4.3.

CHAPTER 3

COMBINATORIAL RESULTS

The results of this chapter are based on joint work with Fern Gossow [GH26].

3.1 Introduction

The Borel–Weil–Bott (BWB) theorem [Ser95] is a tool for computing the higher sheaf cohomology associated to certain vector bundles, by a decomposition of each cohomology group into irreducible representations of a Lie group. In Type A, this Lie group is the group $\mathrm{GL}(n,\mathbb{C})$ of complex invertible matrices, and its irreducible representations have an underlying combinatorial structure which can be neatly described in terms of partitions and Young tableaux.

Fix integers $1 \leq k < n$ and let $X := \mathrm{Gr}(k,n)$ denote the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$. Let $\Omega^j_X(t) := \Omega^j_X \otimes \mathcal{O}_X(t)$ be the sheaf of twisted holomorphic j-forms on X. We are interested in computing the Hodge number

$$h^{j,i}(t) := \dim H^i(\Omega^j_X(t))$$

for integers $j, i \geq 0$ and t. The cohomology has a natural action of $\mathrm{GL}(n,\mathbb{C})$ induced from the action on $\mathbb{C}^n$. For a dominant integral weight β, let L_β denote the irreducible representation of highest weight β and $L^\vee_\beta$ its dual. In Section 3.3 we use the formulation of the BWB theorem from [Wey03, §4.1] to observe that

$$H^i(\Omega^j_X(t)) \cong \bigoplus_\lambda L^\vee_{\beta_\lambda(t)}$$

where $\beta_\lambda(t)$ is a dominant integral weight associated to λ (see Definition 3.15), and the direct sum is over all partitions λ such that:

- λ is $(k, n-k)$-bounded (meaning λ has at most k rows and $n-k$ columns),
- λ has j boxes,
- no boxes in λ have hook length t, and
- exactly i boxes have hook length exceeding t.

The positivity of $h^{j,i}(t)$ is then equivalent to the existence of such a partition. This fact was first observed by Snow [Sno86], and we call λ a *Snow partition* with parameters (k,n,j,t,i).

A partition with no boxes of hook length t is known as a t-core partition, and the properties and applications of these objects are well-studied in the literature

(see [GKS90] for associated generating functions, [GO96] for an approach based on representation theory, or [Cho+21] for a survey). Partitions which are both t-core and bounded in a rectangle were studied in [AS24], but that analysis does not include the size of the partition or its interior.

A set of five inequalities on the parameters that each imply the nonexistence of Snow partitions was given by Snow, and in Section 3.4 we generalize and improve these bounds by mapping each Snow partition to a plane partition using a construction of Chen [Che10]. In particular, we obtain a combinatorial proof of the Nakano vanishing theorem [AN54] for $\Omega_X^j(t)$, which is equivalent to every Snow partition with $t > 0$ satisfying $i + j \leq N$.

Set $d_\lambda(t) = \dim L^\vee_{\beta_\lambda(t)}$, so $h^{j,i}(t) = \sum_\lambda d_\lambda(t)$ with the sum over all Snow partitions with parameters (k, n, j, t, i). Our main results are two formulae for computing $d_\lambda(t)$ for every Snow partition (previously only special cases were known, such as $t = 2$ in [Sno86]). In the following we state the results for $t > 0$, but our proofs allow for all integers t by extending some combinatorial definitions (in any case, Serre duality implies that $h^{j,i}(t) = h^{N-j,N-i}(-t)$).

The first formula for $d_\lambda(t)$ is based on a novel integer-valued hook product statistic on bounded partitions (Definition 3.28), which is nonzero if and only if λ is t-core. Let $\lambda^\mathsf{c} := (n - k - \lambda_k, \ldots, n - k - \lambda_1)$ denote the complement partition to λ in its $k \times (n - k)$ bounding rectangle, and $h_\lambda(a, b)$ the hook length of the box in row a and column b of λ.

Theorem 3.1. *For a Snow partition λ with $t > 0$,*

$$d_\lambda(t) = (-1)^{i+j} \prod_{(a,b)\in\lambda} \frac{h_\lambda(a, b) - t}{h_\lambda(a, b)} \prod_{(a,b)\in\lambda^\mathsf{c}} \frac{h_{\lambda^\mathsf{c}}(a, b) + t}{h_{\lambda^\mathsf{c}}(a, b)}.$$

The second formula for $d_\lambda(t)$ follows from explicitly calculating $\beta_\lambda(t)$, and observing that $L^\vee_{\beta_\lambda(t)}$ is a polynomial representation of $\mathrm{GL}(n, \mathbb{C})$ when $t > 0$ (concretely, all entries of $\beta_\lambda(t)$ are nonpositive), so $d_\lambda(t)$ is enumerated by semistandard Young tableaux.

Given a Snow partition λ with $t > 0$, let $\partial_\lambda(t)_a$ for $1 \leq a \leq k$ be the number of boxes in row a of λ with hook length less than t.

Theorem 3.2. *For a Snow partition λ with $t > 0$, let γ be the weakly decreasing sequence of integers with entries given by the multiset union*

$$\{\partial_\lambda(t)_a \mid 1 \leq a \leq k\} \cup \{t - \partial_{\lambda^\mathsf{T}}(t)_b \mid 1 \leq b \leq n - k\}.$$

Then γ is an (n, t)-bounded partition of size $(n-k)t$, and $d_\lambda(t)$ is equal to the number of semistandard Young tableaux of shape γ with entries $1, \ldots, n$.

We apply these formulae to compute Hodge numbers in specific cases. This involves first finding all Snow partitions λ for a given set of parameters, and then calculating $d_\lambda(t)$ for each of these. For example, when $j = 0$ the only Snow partition is $\lambda = (0, \ldots, 0)$ with $i = 0$, and comparing the calculation from Theorem 3.1 with a formula of MacMahon [Mac86] gives that $h^{0,0}(t)$ is equal to the number of plane partitions bounded in a $k \times (n-k) \times t$ box.

We also examine the cases $i + j = N$ and $i + j = N - 1$, which are extremal cases for Nakano vanishing. These are related to Assumptions 2.9 and 2.10, and in Theorems 3.42 and 3.44 we provide a complete classification of values k, n, t for which these vanishings occur.

Applying Stembridge's notion of q-dimension [Ste94] to representations of $\mathrm{GL}(n, \mathbb{C})$, we prove the following q-analog of Theorem 3.1 in Section 3.6. If

$$[m]_q := \frac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}}$$

is the (symmetrized) q-analog of an integer m, then for every Snow partition λ with $t > 0$:

$$\dim_q L^\vee_{\beta_\lambda(t)} = (-1)^{i+j} \prod_{(a,b)\in\lambda} \frac{[h_\lambda(a,b) - t]_q}{[h_\lambda(a,b)]_q} \prod_{(a,b)\in\lambda^\mathsf{c}} \frac{[h_{\lambda^\mathsf{c}}(a,b) + t]_q}{[h_{\lambda^\mathsf{c}}(a,b)]_q}.$$

In Section 3.5 we give an algebraic approach for the computation of the Euler characteristic $\chi(\Omega^j_X(t))$, and discuss the combinatorial implications for bounded partitions. Conjecture 1 suggests a q-analog for this formula based on the q-binomial coefficient.

3.2 Background

Sequences

For sequences $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ and $b = (b_1, \ldots, b_m) \in \mathbb{Z}^m$, we let $a \smile b := (a_1, \ldots, a_n, b_1, \ldots, b_m) \in \mathbb{Z}^{n+m}$ denote concatenation. For $t \in \mathbb{Z}$, let $a + t := (a_1 + t, \ldots, a_n + t)$. If $m = n$, define $a + b := (a_1 + b_1, \ldots, a_n + b_n)$. We also define $-a := (-a_1, \ldots, -a_n)$ and $\mathrm{rev}(a) := (a_n, \ldots, a_1)$.

We sometimes represent sequences in exponential notation, with the exponent denoting a repeated entry. For example:

$$(1, 1, 1, 1, 0, 0, 5, 5, 5, 0) = (1^4, 0^2, 5^3, 0^1).$$

Permutations

For $n \geq 1$, let $\mathfrak{S}_n$ denote the symmetric group on n letters, whose elements (known as *permutations*) are bijections on $\{1, \ldots, n\}$. We express the permutation $\sigma \in \mathfrak{S}_n$ by its *one-line notation* $\sigma(1)\sigma(2)\cdots\sigma(n)$. An *inversion* is a pair (a, b) with $1 \leq a < b \leq n$ and $\sigma(a) > \sigma(b)$. This inversion pair is a *descent* (at ℓ) if $(a, b) = (\ell, \ell + 1)$. A permutation is *Grassmannian* if it has at most one descent. The *Coxeter length* of a permutation is equal to its total number of inversions. For more on the Coxeter theory of permutations and its connections to representation theory, see [BB05].

We let $\mathfrak{S}_n$ act on the sequence $a \in \mathbb{Z}^n$ by $\sigma(a) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$.

Bounded partitions

The primary objects of our combinatorial study are *bounded partitions*, which include their bounding rectangle as part of their combinatorial data. This additional information is essential for some of our definitions.

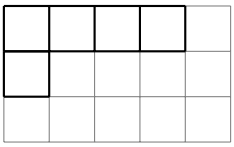
Fix integers $1 \leq k < n$. A $(k, n-k)$-*bounded partition* (or simply, a bounded partition when k and n are understood) is an integer sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that

$$n - k \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0.$$

In some cases we extend our definition to allow $n = \infty$, and simply refer to k-bounded partitions.

Consider the rectangular grid of boxes (a, b) for $1 \leq a \leq k$ and $1 \leq b \leq n - k$ specified in matrix coordinates. Let $N := k(n-k)$ denote the total number of boxes. We say $(a, b) \in \lambda$ if $b \leq \lambda_a$, and draw the *diagram* for λ by outlining all such boxes.

Example 3.3. The $(3, 8)$-bounded partition $\lambda = (4, 1, 0)$ has diagram:



The *transpose* of λ is the $(n-k, k)$-bounded partition λ^{T} whose diagram is obtained by reflecting the diagram for λ along $(a, b) \mapsto (b, a)$. The *complement* λ^{C} is the $(k, n-k)$-bounded partition obtained by rotating the bounding box of λ by $180°$ and taking the set-theoretic complement of λ. That is,

$$\lambda^{\mathsf{C}} = (n - k - \lambda_k, \ldots, n - k - \lambda_1).$$

For $1 \leq a \leq k$ and $1 \leq b \leq n-k$, define the *hook length* of the box (a,b) by

$$h_\lambda(a,b) := \lambda_a + \lambda_b^\mathsf{T} - a - b + 1.$$

We make special note that this definition applies to boxes both inside and outside of λ. When $(a,b) \in \lambda$, $h_\lambda(a,b)$ is the length of the south-east hook originating from the box (a,b) to the boundary of λ. Otherwise, the hook length is negative, and its magnitude is the associated hook length in λ^C.

Example 3.4. For the $(3,8)$-bounded partition $\lambda = (4,1,0)$ we notate every box by its hook length.

5	3	2	1	−1
1	−1	−2	−3	−5
−1	−3	−4	−5	−7

For *any* integer t, a bounded partition λ is called *t-core* if it has no boxes of hook length t. Note that λ is $(-t)$-core if and only if λ^C is t-core. By a standard argument using the abacus construction (see [JK81, Theorem 2.7.40]), if a partition is t-core then it is also (mt)-core for every $m \geq 1$.

Define the *t-interior* $\iota_\lambda(t)$ of λ to be the $(k, n-k)$-bounded partition made up of all boxes (a,b) such that $h_\lambda(a,b) > t$. Also let $|\lambda| := \lambda_1 + \cdots + \lambda_k$ denote the *size* of a partition, which counts its boxes. We now define the objects which index the summands of the decomposition of $H^i(\Omega_X^j(t))$ into irreducible representations of $\mathrm{GL}(n,\mathbb{C})$, as mentioned in the introduction.

Definition 3.5. Fix a tuple of integers (k,n,j,t,i) such that $1 \leq k < n$ and $j,i \geq 0$. A *Snow partition* with *parameters* (k,n,j,t,i) is a $(k,n-k)$-bounded partition with $|\lambda| = j$, which is t-core with $|\iota_\lambda(t)| = i$.

We are interested in a classification of the parameters (k,n,j,t,i) for which such partitions exist, as these are the parameters for which $h^{j,i}(t) > 0$ (see Theorem 3.16). It is clear that $0 \leq i,j \leq N$, so $h^{j,i}(t) = 0$ if $j > N$ or $i > N$. If $t \geq 0$, then $\iota_\lambda(t) \subseteq \lambda$, so $i \leq j$. If $t \leq 0$, then $\iota_\lambda(t) \supseteq \lambda$ so $i \geq j$. Since the only 1-core partition is empty, if $t = 1$ we must have $i = j = 0$.

Definition 3.6. For a Snow partition λ with $t \geq 0$, set $\partial_\lambda(t) := \lambda - \iota_\lambda(t)$.

It turns out that $\partial_\lambda(t)$ is always a partition. It is obtained by taking the boxes in the *t-boundary* skew partition $\lambda/\iota_\lambda(t)$ and sliding them to the left. Moreover, the map $\lambda \mapsto \partial_\lambda(t)$ can be inverted.

Lemma 3.7 ([LM05, Theorem 7]). *For $t > 0$, the map $\lambda \mapsto \partial_\lambda(t)$ is a bijection between k-bounded t-core partitions and $(k, t-1)$-bounded partitions.*

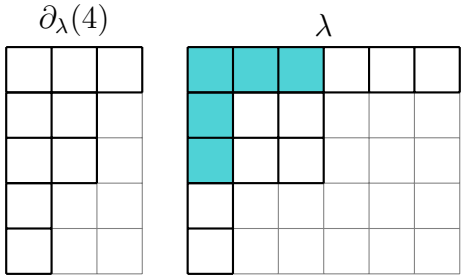


Figure 3.1. A Snow partition $\lambda = (6, 3, 3, 1, 1)$ for the parameters $k = 5$, $n = 11$, $j = 14$, $t = 4$ and $i = 5$, with its 4-interior $\iota_\lambda(4) = (3, 1, 1, 0, 0)$ shaded. The $(5, 3)$-bounded partition $\partial_\lambda(4)$ is on the left.

Tableaux

For a partition λ, a *tableau* of shape λ is a labeling of the boxes of λ by positive integers. The size $|T|$ of a tableau T is the sum of its labels. We give some basic enumerations of classes of tableaux below, and refer to [Sta99] for the general theory.

Fix a partition λ of size j. For an integer $m \geq 1$, a *semistandard Young tableau* is a labeling of the boxes of λ by the integers $1, \ldots, m$ such that the labels increase weakly along rows and strictly down columns. The number of such semistandard Young tableaux can be counted as

$$\#\mathrm{SSYT}(\lambda; m) = \prod_{1 \leq p < q \leq m} \frac{\lambda_p - \lambda_q + q - p}{q - p} = \prod_{(a,b) \in \lambda} \frac{m + b - a}{h_\lambda(a, b)}. \tag{3.2.1}$$

The *content* of a semistandard tableau is the sequence $\mu = (\mu_1, \ldots, \mu_m)$ where μ_ℓ is the number of boxes with label ℓ. The generating function for semistandard Young tableaux weighted by content is the *Schur polynomial*

$$s_\lambda(x_1, \ldots, x_m) = \sum_{\mu \in \mathbb{Z}^m} \#\mathrm{SSYT}(\lambda; \mu)\, x_1^{\mu_1} \cdots x_m^{\mu_m}$$

where $\mathrm{SSYT}(\lambda; \mu)$ is the set of semistandard Young tableaux of shape λ and content μ. Schur polynomials are symmetric in the variables $x_1, \ldots, x_m$.

Example 3.8. If $\lambda = (2,2)$, the elements of $\mathrm{SSYT}(\lambda; 3)$ are

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array}$$

so $s_\lambda(x_1, x_2, x_3) = x_1^2x_2^2 + x_1^2x_2x_3 + x_1^2x_3^2 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_2^2x_3^2$.

A *plane partition* is a tableau where the labels weakly decrease both along rows and down columns. We call a plane partition (k_1, k_2, k_3)-bounded if its shape λ is (k_1, k_2)-bounded and every label is at most k_3. MacMahon in [Mac86] proved that the set of such plane partitions is enumerated by

$$\#\mathrm{PP}(k_1, k_2, k_3) = \prod_{a=1}^{k_1}\prod_{b=1}^{k_2}\frac{a+b+k_3-1}{a+b-1}.$$

Representation theory

We recall some of the standard Lie theory in Type A, including a classification of (finite dimensional) rational representations for the matrix Lie group $\mathrm{GL}(n, \mathbb{C})$ of invertible matrices over $\mathbb{C}$. The general theory can be found in [Hum72] and [FH91].

Given a finite dimensional complex vector space V of dimension m and an action of $\mathrm{GL}(n, \mathbb{C})$ on V, we can construct an explicit homomorphism $\mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(m, \mathbb{C})$ by choosing a basis for V. We call the representation V *rational* (resp. *polynomial*) if this map is a rational function (resp. polynomial) in terms of the entries of the input matrix. This classification is independent of the choice of basis on V.

The *weight lattice* of $\mathrm{GL}(n, \mathbb{C})$ is identified with $\mathbb{Z}^n$. We let e_p be the p^{th} standard basis vector and $\langle -, - \rangle$ the usual inner product. Every rational representation V can be decomposed into *weight spaces* as $V = \bigoplus_{\mu\in\mathbb{Z}^n} V[\mu]$, where $V[\mu]$ is the subspace of elements $v \in V$ such that every diagonal matrix acts by

$$\mathrm{diag}(x_1, \ldots, x_n) \cdot v = x_1^{\mu_1} \cdots x_n^{\mu_n} v.$$

The *Weyl group* of $\mathrm{GL}(n, \mathbb{C})$ is $\mathfrak{S}_n$. It acts by linear isomorphism on the weight spaces, so $\dim V[\sigma(\mu)] = \dim V[\mu]$ for every $\mu \in \mathbb{Z}^n$ and $\sigma \in \mathfrak{S}_n$. The dual representation $V^\vee$ satisfies $\dim V^\vee[\mu] = \dim V[-\mu]$ for every $\mu \in \mathbb{Z}^n$ (see [Kap84, §0.2] or [Wey39, Theorem IV.5.A]).

The *positive roots* of $\mathrm{GL}(n, \mathbb{C})$ are $\Phi^+ := \{e_r - e_s \mid 1 \le r < s \le n\}$ and we define a partial order on $\mathbb{Z}^n$ by $\mu \preceq \lambda$ if $\lambda - \mu$ is a sum of positive roots. The *Weyl vector* is $\varrho = \frac{1}{2}\sum_{\phi\in\Phi^+} \phi$, or explicitly,

$$\varrho = \left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{-n+1}{2}\right).$$

Every finite dimensional rational representation of $\mathrm{GL}(n,\mathbb{C})$ is a direct sum of irreducible representations. The isomorphism classes of irreducible representations of $\mathrm{GL}(n,\mathbb{C})$ are indexed by the set of *dominant integral weights*

$$\Lambda_n := \{\beta \in \mathbb{Z}^n \mid \beta_1 \geq \beta_2 \geq \cdots \geq \beta_n\}.$$

The irreducible representation L_β associated to β is a *highest-weight representation* for β, meaning $\dim L_\beta[\beta] = 1$ and $\dim L_\beta[\mu] = 0$ unless $\mu \preceq \beta$. We provide an explicit construction of these representations in Section 3.2. Below are some common examples:

β	L_β
(0^n)	trivial representation
(1^n)	determinant representation
$(1^m, 0^{n-m})$	$\bigwedge^m \mathbb{C}^n$
$(m, 0^{n-1})$	$\mathrm{Sym}^m \mathbb{C}^n$

The dimension of L_β is given by the Weyl dimension formula:

$$\dim L_\beta = \prod_{\phi \in \Phi^+} \frac{\langle \phi, \beta + \varrho \rangle}{\langle \phi, \varrho \rangle} = \prod_{1 \leq r < s \leq n} \frac{\beta_r - \beta_s + s - r}{s - r}.$$

Remark 3.9. For most of this chapter we will replace ϱ by the *shifted Weyl vector* $\varrho' = (n, \ldots, 2, 1)$ for ease in calculations. It is often the case that the choice between ϱ and ϱ' is inconsequential, such as when using the Weyl dimension formula.

The representation L_β is polynomial if and only if β belongs to the set

$$\Lambda_n^+ := \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0\},$$

which we identify with the set of n-bounded partitions. For a weight $\lambda \in \Lambda_n^+$ we have $\dim L_\lambda = \#\mathrm{SSYT}(\lambda; n)$, and more precisely, $\dim L_\lambda[\mu] = \#\mathrm{SSYT}(\lambda; \mu)$ for every weight $\mu \in \mathbb{Z}^n$.

Remark 3.10. Because λ is a *bounded partition*, there is no ambiguity about which group is acting on L_λ (or L_β). For example, $L_{(2,1)}$ is a representation of $\mathrm{GL}(2,\mathbb{C})$, and $L_{(2,1,0,0)}$ is a representation of $\mathrm{GL}(4,\mathbb{C})$.

We state some relationships amongst the irreducible representations L_β which are relevant in this work. The dual of L_β is isomorphic to $L_{-\mathrm{rev}(\beta)}$, and

$$L_\beta \otimes L_{(t^n)} \cong L_{\beta + t} \tag{3.2.2}$$

for any $\beta \in \Lambda_n$ and $t \in \mathbb{Z}$ as a special case of the Littlewood–Richardson rule. Finally, we have the Cauchy decomposition [FH91, Exercise 6.11] which gives the isomorphism of representations of $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(n-k,\mathbb{C})$:

$$\bigwedge^j \left(\mathbb{C}^k \otimes \mathbb{C}^{n-k}\right) \cong \bigoplus_{|\lambda|=j} L_\lambda \otimes L_{\lambda^\intercal} \tag{3.2.3}$$

where λ runs over all $(k, n-k)$-bounded partitions of size j.

Schur Functors

We give an explicit construction of the irreducible $\mathrm{GL}(n,\mathbb{C})$ representation L_β for any $\beta \in \Lambda_n$. Importantly, this construction is *functorial*, and will be applied to vector bundles in the setup of the Borel–Weil–Bott theorem.

For every $\beta \in \Lambda_n$, we construct a functor Σ^β from the category of rational representations of $\mathrm{GL}(n,\mathbb{C})$ to itself, such that $\Sigma^\beta(\mathbb{C}^n) \cong L_\beta$ for every $\beta \in \Lambda_n$, where $\mathbb{C}^n$ is equipped with the standard action. For a more detailed version of this construction, see [FH91, §6].

We start with $\lambda \in \Lambda_n^+$, so λ is an n-bounded partition of some size j. Consider the standard tableau T_λ obtained by labeling the boxes of λ by $1, \ldots, j$ row-by-row from left to right, then top to bottom. Define subgroups of $\mathfrak{S}_j$ by:

$$P_\lambda := \{\sigma \in \mathfrak{S}_j \mid \sigma \text{ preserves the rows of } T_\lambda\}$$
$$Q_\lambda := \{\sigma \in \mathfrak{S}_j \mid \sigma \text{ preserves the columns of } T_\lambda\}$$

The group algebra $\mathbb{C}[\mathfrak{S}_j]$ consists of formal $\mathbb{C}$-linear combinations of the elements of $\mathfrak{S}_j$. Let $\ell(\sigma)$ be the Coxeter length of each permutation $\sigma \in \mathfrak{S}_j$, and define the *Young symmetrizer*

$$c_\lambda = \sum_{\sigma \in P_\lambda} \sigma \sum_{\tau \in Q_\lambda} (-1)^{\ell(\tau)} \tau \in \mathbb{C}[\mathfrak{S}_j].$$

Let V be a rational representation of $\mathrm{GL}(n,\mathbb{C})$. The space $V^{\otimes j} := V \otimes \cdots \otimes V$ has a left action of $\mathrm{GL}(n,\mathbb{C})$ (acting on each factor simultaneously) and a right action of $\mathfrak{S}_j$ (permuting the tensor factors). These actions commute, and we define $\Sigma^\lambda(V)$ as the image of the action of c_λ on $V^{\otimes j}$. To obtain all dominant integral weights $\beta \in \Lambda_n$ we use the definition

$$\Sigma^\beta(V) := \Sigma^{\beta+r}(V) \otimes \Sigma^{(r^n)}(V^\vee)$$

for $r \geq 0$. This is well-defined, and we can use this to "translate" any dominant integral weight to one with nonnegative values by making r sufficiently large.

We state a number of facts about this construction, which mirror isomorphisms between the irreducible representations L_β. Applying Σ^β to the dual of V gives

$\Sigma^\beta(V^\vee) \cong \Sigma^\beta(V)^\vee \cong \Sigma^{-\mathrm{rev}(\beta)}(V)$, and $\Sigma^{\beta+t}(V) \cong \Sigma^\beta(V) \otimes \Sigma^{(t^n)}(V)$. We also have the Cauchy decomposition (see (2.3.3) in [Wey03]): for V and W representations of $\mathrm{GL}(k,\mathbb{C})$ and $\mathrm{GL}(n-k,\mathbb{C})$, respectively,

$$\bigwedge^j (V \otimes W) \cong \bigoplus_{|\lambda|=j} \Sigma^\lambda(V) \otimes \Sigma^{\lambda^\mathsf{T}}(W) \tag{3.2.4}$$

as representations of $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(n-k,\mathbb{C})$, where the sum is over all $(k, n-k)$-bounded partitions λ of size j. Finally, if $V = \mathbb{C}^n$ is the standard representation of $\mathrm{GL}(n,\mathbb{C})$, then $\Sigma^\beta(V) \cong L_\beta$ [Kap84].

This construction can be extended to vector bundles (see [Wey03], and note that their notation $L_\lambda E$ corresponds to our $\Sigma^{\lambda^\mathsf{T}}(E)$).

3.3 Computation of the Hodge numbers

BWB for the Grassmannian

The Borel–Weil–Bott (BWB) theorem allows us to decompose the cohomology of twisted holomorphic j-forms on the Grassmannian into irreducible representations of $\mathrm{GL}(n,\mathbb{C})$. Our construction follows [Wey03, §4.1]; an approachable account of this is available in [Nik25, Theorem A.9].

First, a brief reminder on notation. Fix $V = \mathbb{C}^n$, equipped with the standard action of $\mathrm{GL}(n,\mathbb{C})$. The Grassmannian $\mathrm{Gr}(k,V)$ of k-dimensional subspaces of V is denoted by $X = \mathrm{Gr}(k,n)$. The structure sheaf of X is denoted $\mathcal{O}_X$, and the sheaf of holomorphic j-forms on X is denoted Ω^j_X. For an overview of complex geometry in this setting, see [Voi02, §10.1.1]. Consider the tautological sequence

$$0 \to S \to \mathcal{O}_X \otimes V \to Q \to 0$$

for S and Q the tautological subbundle and quotient bundle, respectively. The dual of a bundle E is denoted $E^\vee$. Recall that $\Omega_X \cong S \otimes Q^\vee$. There is an isomorphism $\mathcal{O}_X(1) \cong \bigwedge^{n-k} Q = \det Q$, and we define $\mathcal{O}_X(d) := \mathcal{O}_X(1)^{\otimes d}$. Let $\Omega^j_X := \bigwedge^j \Omega_X$ and $\Omega^j_X(t) := \Omega^j_X \otimes \mathcal{O}_X(t)$. Applying Equation (3.2.4) to the twisted cotangent bundle $\Omega^j_X(t)$ gives

$$\begin{aligned}
\Omega^j_X(t) &\cong \bigoplus_\lambda \Sigma^\lambda S \otimes \Sigma^{\lambda^\mathsf{T}}(Q^\vee) \otimes \Sigma^{(t^{n-k})}(Q) \\
&\cong \bigoplus_\lambda \Sigma^\lambda S \otimes \Sigma^{\lambda^\mathsf{T} - t}(Q^\vee) \\
&\cong \bigoplus_\lambda \Sigma^{-\mathrm{rev}(\lambda)}(S^\vee) \otimes \Sigma^{\lambda^\mathsf{T} - t}(Q^\vee)
\end{aligned} \tag{3.3.1}$$

with the sum over all $(k, n-k)$-bounded partitions λ of size j. Under this isomorphism $\Omega^j_X(t)$ is a representation of $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(n-k,\mathbb{C})$. The weight of such a representation can be described by a *Levi dominant weight* $\alpha = \delta \smile \epsilon \in \mathbb{Z}^n$ where $\delta \in \Lambda_k$ and $\epsilon \in \Lambda_{n-k}$. In particular, α satisfies the property that $\alpha_i \geq \alpha_{i+1}$ for all $i \neq k$.

We now state the Borel–Weil–Bott theorem for the Grassmannian. Recall that $V = \mathbb{C}^n$ has the standard $\mathrm{GL}(n,\mathbb{C})$ action and $\varrho' = (n, \ldots, 1)$.

Theorem 3.11 (Borel–Weil–Bott, adapted from [Wey03, Corollary 4.1.9])**.** *Set $X = \mathrm{Gr}(k,n)$ and suppose $\alpha = \delta \smile \epsilon \in \mathbb{Z}^n$ is a Levi dominant weight, where $\delta \in \Lambda_k$ and $\epsilon \in \Lambda_{n-k}$. Let $\mathcal{V}(\alpha)$ denote the vector bundle*

$$\mathcal{V}(\alpha) = \Sigma^\delta(S^\vee) \otimes \Sigma^\epsilon(Q^\vee).$$

Then one of two mutually exclusive possibilities occurs:

1. *$\alpha + \varrho'$ has repeated entries, and the cohomology of $\mathcal{V}(\alpha)$ vanishes; or*
2. *let $\sigma \in \mathfrak{S}_n$ be the (unique) permutation which orders $\alpha + \varrho'$ into strictly decreasing order, and ℓ its Coxeter length. Set $\beta := \sigma(\alpha + \varrho') - \varrho' \in \Lambda_n$. Then the cohomology of $\mathcal{V}(\alpha)$ is zero in all degrees except ℓ, and*

$$H^\ell(X, \mathcal{V}(\alpha)) \cong \Sigma^\beta(V^\vee) \cong L^\vee_\beta.$$

Remark 3.12. The notation used in [Wey03] describes $H^\ell(X, \mathcal{V}(\alpha))$ in terms of the (skew) Weyl functor K_β rather than the (skew) Schur functor; these two functors are isomorphic over characteristic zero [Wey03, Proposition 2.1.18].

We will adapt this theorem for $\Omega^j_X(t)$ through Equation (3.3.1). We first define the Levi dominant weight α which corresponds to each partition λ indexing the direct sum.

Definition 3.13. For a $(k, n-k)$-bounded partition λ and an integer t, set

$$\alpha_\lambda(t) := -\mathrm{rev}(\lambda) \smile (\lambda^\mathsf{T} - t) \in \mathbb{Z}^n.$$

Lemma 3.14. *The sequence $\alpha_\lambda(t) + \varrho'$ has unique entries if and only if λ is t-core. In this case, the permutation σ which reorders $\alpha_\lambda(t) + \varrho'$ in descending order has Coxeter length $\ell = |\iota_\lambda(t)|$.*

Proof. Since $-\mathrm{rev}(\lambda)$ and $\lambda^\mathsf{T} - t$ are both weakly decreasing and ϱ' is strictly decreasing, we only need to compare between the first k and last $n-k$ terms of $\alpha + \varrho'$. For $1 \leq a \leq k$ and $1 \leq b \leq n-k$, we have

$$(\alpha + \varrho')_{k-a+1} = -\lambda_a + n - k + a \text{ and } (\alpha + \varrho')_{k+b} = \lambda^\mathsf{T}_b - t + n - k - b + 1.$$

By rearranging, these entries are equal if and only if

$$\lambda_a + \lambda_b^{\mathsf{T}} - a - b + 1 = t,$$

so λ has a hook length of t. The Coxeter length of σ is the number of pairs of entries in $\alpha_\lambda(t) + \varrho'$ which are already in ascending order. This happens precisely when $(\alpha + \varrho')_{k-a+1} < (\alpha + \varrho')_{k+b}$, which is equivalent to $h_\lambda(a, b) > t$ by the above calculation. □

Definition 3.15. For a bounded t-core partition λ, define the *BWB weight*

$$\beta_\lambda(t) := \sigma(\alpha_\lambda(t) + \varrho') - \varrho' \in \Lambda_n$$

and set $d_\lambda(t) := \dim L^\vee_{\beta_\lambda(t)}$.

We now simplify the statement of Theorem 3.11. This directly implies Snow's combinatorial condition on the positivity of Hodge numbers [Sno86, §3.1].

Theorem 3.16. *Fix integers (k, n, j, t, i) such that $1 \le k < n$ and $j, i \ge 0$. Then*

$$H^i(\mathrm{Gr}(k, n), \Omega^j(t)) \cong \bigoplus_\lambda L^\vee_{\beta_\lambda(t)}$$

as representations of $\mathrm{GL}(n, \mathbb{C})$, with the direct sum over all Snow partitions λ with parameters (k, n, j, t, i). In particular, $h^{j,i}(t) = \sum_\lambda d_\lambda(t)$.

The BWB weight

Fix a Snow partition λ with parameters (k, n, j, t, i). In this section we explicitly determine the permutation σ which reorders $\alpha_\lambda(t) + \varrho'$, and hence the BWB weight $\beta_\lambda(t) \in \Lambda_n$. This gives a formula for $d_\lambda(t)$ based on semistandard Young tableaux. In the next section we use a different approach to obtain the explicit formula for $d_\lambda(t)$ from Theorem 3.1.

Lemma 3.17. *For a bounded t-core partition λ, the permutation σ which rearranges $\alpha_\lambda(t) + \varrho'$ into descending order is given by*

$$\sigma(k - a + 1) = k - a + 1 + \iota_\lambda(t)_a \text{ and } \sigma(k + b) = k + b - \iota_\lambda(t)_b^{\mathsf{T}}$$

for every $1 \le a \le k$ and $1 \le b \le n - k$.

Proof. We consider σ as a sequence of adjacent transpositions which swap ascending pairs, until the sequence is weakly decreasing.

The entry $(\alpha+\varrho')_k$ swaps with every entry $(\alpha+\varrho')_{k+b}$ such that $(\alpha+\varrho')_k < (\alpha+\varrho')_{k+b}$, or equivalently, $h_\lambda(1,b) > t$ by the proof of Lemma 3.14, so it moves to the right $\iota_\lambda(t)_1$ times. Next, $(\alpha+\varrho')_{k-1}$ swaps with every entry $(\alpha+\varrho')_{k+b}$ such that $h_\lambda(2,b) > t$, so it moves to the right $\iota_\lambda(t)_2$ times, and so on.

Alternatively, we can first swap $(\alpha+\varrho')_{k+1}$ with every $(\alpha+\varrho')_{k-a+1}$ such that $(\alpha+\varrho')_{k+1} > (\alpha+\varrho')_{k-a+1}$, or equivalently $h_\lambda(a,1) > t$, which moves it to the left $\iota_\lambda(t)_1^{\mathsf{T}}$ times. Then $(\alpha+\varrho')_{k+2}$ moves to the left $\iota_\lambda(t)_2^{\mathsf{T}}$ times and so on. □

Example 3.18. Take the $(4,5)$-bounded partition $\lambda = (5,2,2,1)$, which we consider as both a 0-core and 5-core partition. We have $\alpha_\lambda(0)+\varrho' = (8,6,5,1,9,7,4,3,2)$ with $\sigma = 245913678$, which records that 8 is the second largest entry of $\alpha_\lambda(0)+\varrho'$, 6 is the fourth largest, and so on. Similarly, $\alpha_\lambda(5)+\varrho' = (8,6,5,1,4,2,-1,-2,-3)$ with $\sigma = 123645789$.

In Figure 3.2 we draw the crossing diagram for both permutations. Drawing a rotated box around each crossing recovers the shapes $\iota_\lambda(0) = \lambda$ and $\iota_\lambda(5)$.

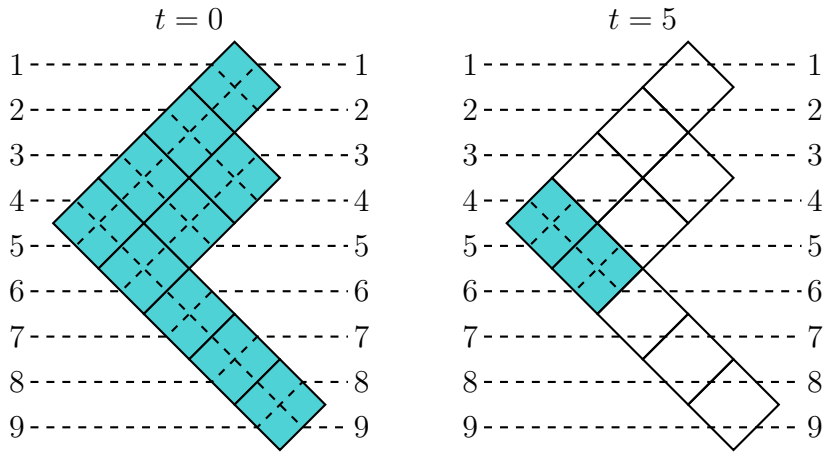


Figure 3.2. The left-to-right line diagrams for the permutations rearranging $\alpha_{(5,2,2,1)}(t)+\varrho'$ into descending order when $t=0$ and $t=5$, where all crossings occur as early as possible. The diagram for λ (vertically flipped then rotated by 45° clockwise) is drawn on each side, with the t-interior shaded.

Remark 3.19. For $t=0$, the map $\lambda \mapsto \sigma$ recovers the well-known bijection between $(k,n-k)$-bounded partitions and Grassmannian permutations in $\mathfrak{S}_n$ whose only descent (if it has one) is at k (see §19 of [Pos06]).

Theorem 3.20. *For a bounded t-core partition λ, set $\iota := \iota_\lambda(t)$. The BWB weight*

$\beta_\lambda(t)$ is given by rearranging the sequence

$$(-\lambda + \iota) \smile (\lambda^\mathsf{T} - \iota^\mathsf{T} - t)$$

into weakly decreasing order.

Proof. Set $\alpha := \alpha_\lambda(t)$, $\beta := \beta_\lambda(t)$ and recall that $\beta = \sigma(\alpha + \varrho') - \varrho'$. If $\sigma(i) = j$, then

$$\beta_j = \alpha_i + \varrho'_i - \varrho'_j = \alpha_i + (j - i).$$

By the calculation of σ in Lemma 3.17, we have

$$\beta_{k-a+1+\iota_a} = \alpha_{k-a+1} + \iota_a = -\lambda_a + \iota_a$$

and

$$\beta_{k+b-\iota_b^\mathsf{T}} = \alpha_{k+b} - \iota_b^\mathsf{T} = \lambda_b^\mathsf{T} - t - \iota_b^\mathsf{T}$$

for every $1 \le a \le k$ and $1 \le b \le n-k$. Since σ was a bijection, this gives every entry of β. These entries appear in weakly decreasing order by the definition of σ. □

Corollary 3.21. *For a bounded t-core partition λ, we have $-\mathrm{rev}(\beta_\lambda(t)) = \beta_{\lambda^\mathsf{C}}(-t)$. If $t \ge 0$, then $-\mathrm{rev}(\beta_\lambda(t)) \in \Lambda_n^+$.*

Proof. We have $\iota_{\lambda^\mathsf{C}}(-t) = \iota_\lambda(t)^\mathsf{C}$. Therefore, the entries of $\beta_{\lambda^\mathsf{C}}(-t)$ are the union of the entries in

$$-\lambda^\mathsf{C} + \iota^\mathsf{C} = -(n - k - \mathrm{rev}(\lambda)) + (n - k - \mathrm{rev}(\iota)) = \mathrm{rev}(\lambda - \iota)$$

and

$$\lambda^{\mathsf{C}\mathsf{T}} - \iota^{\mathsf{C}\mathsf{T}} + t = (k - \mathrm{rev}(\lambda^\mathsf{T})) - (k - \mathrm{rev}(\iota^\mathsf{T})) + t = t - \mathrm{rev}(\lambda^\mathsf{T} - \iota^\mathsf{T})$$

which are the entries of $-\mathrm{rev}(\beta_\lambda(t))$. Since both sequences are weakly decreasing, they must be equal. If $t > 0$, the sequences $\lambda - \iota$ and $\lambda^\mathsf{T} - \iota^\mathsf{T}$ have entries in $\{0, 1, \ldots, t-1\}$ by Lemma 3.7, so $-\mathrm{rev}(\beta_\lambda(t)) \in \Lambda_n^+$ since it has only nonnegative entries. If $t = 0$, then $\iota = \lambda$ so $-\mathrm{rev}(\beta_\lambda(0)) = (0^n) \in \Lambda_n^+$. □

As a consequence, $L^\vee_{\beta_\lambda(t)}$ is a polynomial representation for $t \ge 0$, and its dual is polynomial when $t \le 0$ (when $t = 0$ it is the trivial representation). In particular, the dimension $d_\lambda(t)$ in each case is the enumeration of semistandard tableaux of a given shape.

Definition 3.22. For a bounded t-core partition λ, define $\gamma_\lambda(t) \in \Lambda_n^+$ by

$$\gamma_\lambda(t) = \begin{cases} -\mathrm{rev}(\beta_\lambda(t)) & \text{if } t \ge 0, \\ \beta_\lambda(t) & \text{if } t < 0. \end{cases}$$

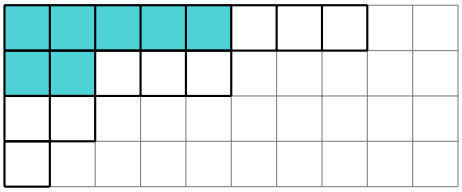
By applying the above results, we can give an explicit enumeration for $d_\lambda(t)$ in terms of semistandard tableaux since for $t \geq 0$, $L^\vee_{\beta_\lambda(t)} \cong L_{-\mathrm{rev}(\beta_\lambda(t))} = L_{\gamma_\lambda(t)}$ is a polynomial representation. Recall the definition of the partition $\partial_\lambda(t) := \lambda - \iota_\lambda(t)$ for $t > 0$ from Definition 3.6. This recovers Theorem 3.2.

Theorem 3.23. *For a bounded t-core partition λ, $d_\lambda(t) = \#\mathrm{SSYT}(\gamma_\lambda(t); n)$. If $t > 0$, then $\gamma_\lambda(t)$ is obtained by rearranging the sequence*

$$\partial_\lambda(t) \smile (t - \partial_{\lambda^\intercal}(t))$$

into weakly descending order, and $\gamma_\lambda(-t) = \gamma_{\lambda^\mathrm{c}}(t)$. For all integers t, $\gamma_\lambda(t)$ is an $(n, |t|)$-bounded partition of size $(n-k)|t|$.

Example 3.24. Let $\lambda = (8, 5, 2, 1)$ be the Snow partition with parameters $k = 4$, $n = 14$, $j = 16$, $t = 4$ and $i = 7$.



We have $\partial_\lambda(4) = (3^2, 2, 1)$, $\partial_{\lambda^\intercal}(4) = (2, 1^7, 0^2)$ and $4 - \partial_{\lambda^\intercal}(4) = (2, 3^7, 4^2)$, hence $\gamma_\lambda(4) = (4^2, 3^9, 2^2, 1)$. Then $d_\lambda(4) = \#\mathrm{SSYT}(\gamma_\lambda(4); 14) = 354900$.

A hook-product formula

In this section we prove Theorem 3.1, naturally extending the result to all integers t. Recall the definitions of $\alpha_\lambda(t)$, $\beta_\lambda(t)$ and $d_\lambda(t)$ from Section 3.3.

In this section, a Snow partition λ will always have fixed parameters (k, n, j, t, i).

Lemma 3.25. *For a Snow partition λ,*

$$d_\lambda(t) = (-1)^i \prod_{1 \leq r < s \leq n} \frac{\alpha_\lambda(t)_r - \alpha_\lambda(t)_s + s - r}{s - r}.$$

Proof. Set $\alpha := \alpha_\lambda(t)$ and $\beta := \beta_\lambda(t)$. By the Weyl dimension formula,

$$d_\lambda(t) = \dim L^\vee_\beta = \dim L_\beta = \prod_{\phi \in \Phi^+} \frac{\langle \phi, \beta + \varrho' \rangle}{\langle \phi, \varrho' \rangle}.$$

Since the inner product is invariant under reordering of the elements,

$$\langle \phi, \beta + \varrho' \rangle = \langle \phi, \sigma(\alpha + \varrho') \rangle = \langle \sigma^{-1}(\phi), \alpha + \varrho' \rangle.$$

The permutation σ^{-1} acts bijectively on the set $\{\{\phi, -\phi\} \mid \phi \in \Phi^+\}$ and swaps a positive root for a negative root for each inversion of σ. Therefore, since σ (and hence σ^{-1}) has i inversions by Lemma 3.14 we obtain

$$\dim L_\beta = (-1)^i \prod_{\phi \in \Phi^+} \frac{\langle \phi, \alpha + \varrho' \rangle}{\langle \phi, \varrho' \rangle} = (-1)^i \prod_{1 \le r < s \le n} \frac{\alpha_r - \alpha_s + s - r}{s - r}. \qquad \square$$

We evaluate the product in the above lemma by splitting it into three parts. Two of these parts will then be consolidated using the result below.

Lemma 3.26. *For a $(k, n-k)$-bounded partition λ,*

$$\dim L_\lambda \cdot \dim L_{\lambda^\intercal} = (-1)^{j-N} \prod_{a=1}^{k} \prod_{b=1}^{n-k} \frac{a+b-1}{h_\lambda(a,b)}.$$

Proof. By [PŚ18, §2.1],

$$\dim L_\lambda \cdot \dim L_{\lambda^\intercal} = \frac{\prod_{(a,b) \in (n-k)^k} h_{k^{(n-k)}}(a,b)}{\prod_{(a,b) \in \lambda} h_\lambda(a,b) \prod_{(a,b) \in \lambda^\complement} h_{\lambda^\complement}(a,b)}.$$

The hook lengths in $\lambda^\complement$ are precisely those that appear as negative hook lengths in λ when ranging over all $1 \le a \le k$ and $1 \le b \le n-k$, of which there are $N - j$. This gives the denominator and the sign of our formula. For the numerator, we have

$$\prod_{a=1}^{k} \prod_{b=1}^{n-k} \Big((n-k) + k - a - b + 1 \Big) = \prod_{a=1}^{k} \prod_{b=1}^{n-k} (a + b - 1)$$

after reindexing a for $k - a + 1$ and b for $n - k - b + 1$. $\square$

Lemma 3.27. *For a Snow partition λ,*

$$\prod_{1 \le r < s \le n} \frac{\alpha_\lambda(t)_r - \alpha_\lambda(t)_s + s - r}{s - r} = (-1)^j \prod_{a=1}^{k} \prod_{b=1}^{n-k} \frac{h_\lambda(a,b) - t}{h_\lambda(a,b)}.$$

Proof. We break the product on the right into three parts. If both r and s are at most k, we obtain

$$\prod_{1 \le r < s \le k} \frac{(-\lambda_{k-r+1}) - (-\lambda_{k-s+1}) + s - r}{s - r}$$

which after reindexing r for $k - s + 1$ and s for $k - r + 1$ becomes

$$\prod_{1 \le r < s \le k} \frac{\lambda_r - \lambda_s + s - r}{s - r} = \dim L_\lambda.$$

Similarly, if both r and s are greater than k we get

$$\prod_{k<r<s\leq n} \frac{(\lambda^{\mathsf{T}}_{r-k} - t) - (\lambda^{\mathsf{T}}_{s-k} - t) + s - r}{s - r}$$

and reindexing r for $r - k$ and s for $s - k$ gives

$$\prod_{1\leq r<s\leq n-k} \frac{\lambda^{\mathsf{T}}_r - \lambda^{\mathsf{T}}_s + s - r}{s - r} = \dim L_{\lambda^{\mathsf{T}}}.$$

Finally, we consider the mixed terms where $r \leq k$ and $s > k$. Reindexing a for $k - r + 1$ and b for $s - k$ gives $s - r = a + b - 1$ and hence

$$\prod_{a=1}^{k}\prod_{b=1}^{n-k} \frac{-\lambda_a - (\lambda^{\mathsf{T}}_b - t) + a + b - 1}{a + b - 1} = (-1)^N \prod_{a=1}^{k}\prod_{b=1}^{n-k} \frac{h_\lambda(a, b) - t}{a + b - 1}.$$

Multiplying the three terms together and directly applying Lemma 3.26 gives the required result. □

We can now state and immediately prove the main theorem for this section, which extends Theorem 3.1 to all integers t.

Definition 3.28. Suppose λ is a $(k, n - k)$-bounded partition and t is an integer. We define the *t-hook ratio* of λ to be

$$f_\lambda(t) := \prod_{a=1}^{k}\prod_{b=1}^{n-k} \frac{h_\lambda(a, b) - t}{h_\lambda(a, b)},$$

where $h_\lambda(a, b) := \lambda_a + \lambda^{\mathsf{T}}_b - a - b + 1$.

Theorem 3.29. *For a Snow partition λ, $d_\lambda(t) = (-1)^{i+j} f_\lambda(t)$. Hence,*

$$h^{j,i}(t) = (-1)^{i+j} \sum_\lambda f_\lambda(t)$$

where the sum is over all Snow partitions with parameters (k, n, j, t, i).

Proof. This follows from Theorem 3.16 and Lemmas 3.25 and 3.27. □

We conclude this section with some comments on the t-hook ratio. For a bounded partition λ, $f_\lambda(t)$ is a rational polynomial whose constant term is 1, and whose roots are the (positive and negative) hook lengths of λ, counted with multiplicity. When $\lambda = \lambda^{\mathsf{C}}$, then $f_\lambda(t)$ can be written

$$\prod_{(a,b)\in\lambda} \left(1 - \frac{t^2}{h_\lambda(a, b)^2}\right)$$

which appears in [Han10, Theorem 1.1] and is closely related to the Nekrasov–Okounkov hook length formula [NO06, Theorem 6.12].

Remark 3.30. Extend the definition of $d_\lambda(t)$ to be zero when λ is not t-core, in which case $f_\lambda(t)$ is also zero. Then $d_\lambda(t) = |f_\lambda(t)|$ for every bounded partition λ and integer t. In particular, the t-hook ratio is an integer-valued polynomial in t. However, $d_\lambda(t)$ and $h^{j,i}(t)$ are not polynomials in general, since the sign $(-1)^{i+j}$ (and the i term in particular) depends on both λ and t.

Examples

To compute the Hodge numbers $h^{j,i}(t)$ by the above methods, we first have to classify all Snow partitions λ with parameters (k, n, j, t, i). We then have three options for computing $d_\lambda(t)$: using the Weyl dimension formula on $\beta_\lambda(t)$ directly requires a product of $n(n-1)/2$ ratio terms, calculating $\#\mathrm{SSYT}(\gamma_\lambda(t); n)$ by Equation (3.2.1) requires $(n-k)|t|$ terms, and evaluating $f_\lambda(t)$ requires $N = k(n-k)$ terms. The latter calculation can sometimes be dramatically simplified via a simple observation.

Lemma 3.31. *Suppose h and h' satisfy $h + h' = t$. Then*

$$\frac{h-t}{h} = \left(\frac{h'-t}{h'}\right)^{-1},$$

so the ratios corresponding to boxes in λ with hook lengths h and h' cancel in the evaluation of $f_\lambda(t)$.

Example 3.32 (Duality)**.** Since λ and λ^T have the same hook lengths, the t-hook ratios $f_\lambda(t)$ and $f_{\lambda^\mathsf{T}}(t)$ are equal. This arises from the duality between $\mathrm{Gr}(k, n)$ and $\mathrm{Gr}(n-k, n)$. The hook lengths of λ^C are the negatives of the hook lengths of λ, which gives the equality $f_\lambda(-t) = f_{\lambda^\mathsf{C}}(t)$. This gives $h^{j,i}(t) = h^{N-j,N-i}(-t)$, which is implied by Serre duality.

From these equalities, when calculating $h^{j,i}(t)$ we can assume that $t \geq 0$ and $k \leq n/2$ without loss of generality.

Example 3.33 ($j = 0$)**.** The only Snow partition is $\lambda = (0^k)$. For $t \geq 0$, we have $i = 0$ and $h_\lambda(a, b) = -a - b + 1$ for every box (a, b), so

$$h^{0,0}(t) = d_\lambda(t) = \prod_{a=1}^{k} \prod_{b=1}^{n-k} \frac{-a-b+1-t}{-a-b+1} = \prod_{a=1}^{k} \prod_{b=1}^{n-k} \frac{a+b+t-1}{a+b-1}$$

which is the number of $(k, n-k, t)$-bounded plane partitions [Mac86].

Alternatively, for $t \geq 0$ we have $\gamma_\lambda(t) = (t^{n-k}, 0^k)$, so $f_\lambda(t) = \#\mathrm{SSYT}((t^{n-k}); n)$. Subtracting κ from every label in row κ of the tableau for every $1 \leq \kappa \leq n-k$ gives a bijection to plane partitions which fit inside a $(n-k) \times t \times k$ box, since entries now weakly decrease along rows and down columns.

Example 3.34 ($t = 1$)**.** The only 1-core partition is again $\lambda = (0^k)$, so

$$\begin{aligned} h^{0,0}(1) &= \#\{\text{plane partitions in a } k \times (n-k) \times 1 \text{ box}\} \\ &= \#\{\text{bounded partitions}\} = \binom{n}{k}, \end{aligned}$$

and $h^{j,i}(1) = 0$ otherwise.

Example 3.35 ($t = 0$)**.** Every bounded partition λ is 0-core with $i = j$. The BWB weight is $\beta_\lambda(0) = (0^n)$ and hence $d_\lambda(0) = 1$. Equivalently, $f_\lambda(0) = 1$ since every term in the product is 1. Therefore,

$$h^{j,j}(0) = \#\{\text{bounded partitions of size } j\},$$

which is the coefficient of q^j in the Gaussian binomial coefficient

$$\frac{(1-q^n)\cdots(1-q)}{(1-q^k)\cdots(1-q)(1-q^{n-k})\cdots(1-q)} \in \mathbb{Z}[q].$$

Otherwise, $h^{j,i}(0) = 0$ for $i \neq j$.

Example 3.36 ($t = 2$)**.** Assume without loss of generality that $k \leq n/2$. The bounded 2-core partitions are the staircase partitions $\Delta_m := (m, \ldots, 2, 1)$ for every $0 \leq m \leq k$. For a fixed such m set $\beta := \beta_{\Delta_m}(2)$. Since $\iota_{\Delta_m}(2) = \Delta_{m-1}$, we have

$$\gamma_{\Delta_m}(2) = (2^{n-k-m}, 1^{2m}, 0^{k-m}).$$

Applying Equation (3.2.1) to this shape gives

$$d_{\Delta_m}(2) = \frac{n\cdots(n-k-m+1)(n+1)\cdots(n-k+m+2)}{(2m)!(k-m)!(k+m+1)\cdots(2m+2)}$$

which can be simplified to

$$\frac{2m+1}{n+1}\binom{n+1}{k+m+1}\binom{n+1}{n-k+m+1},$$

recovering [Sno86, Theorem 3.3]. In total, we obtain

$$h^{m(m+1)/2,m(m-1)/2}(2) = \frac{2m+1}{n+1}\binom{n+1}{k+m+1}\binom{n+1}{n-k+m+1}$$

for every $0 \leq m \leq \min(k, n-k)$, and $h^{j,i}(2) = 0$ otherwise.

3.4 Restrictions on bounded core partitions

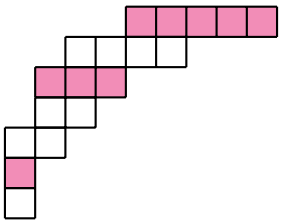
We have seen that positivity of the twisted Hodge number $h^{j,i}(t)$ is equivalent to the existence of a Snow partition λ. In this section we prove necessary conditions for the existence of Snow partitions with given parameters. When these restrictions are not satisfied, the corresponding cohomology will vanish.

Skew-linking diagrams

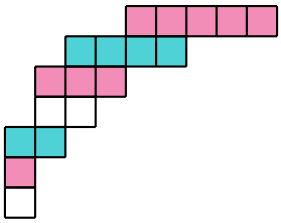
We use a construction of Chen [Che10] to define a map $\lambda \mapsto \mathcal{P}_\lambda(t)$ from Snow partitions to plane partitions (which were defined in Section 3.2). Many of our restrictions on Snow partitions then derive from simple arguments about plane partitions.

Recall that for a Snow partition λ with $t > 0$, the skew partition $\lambda/\iota_\lambda(t)$ has weakly decreasing row and column lengths by Lemma 3.7. Skew partitions with this property are called *skew-linking diagrams*. Note that there are skew-linking diagrams which are not of the form $\lambda/\iota_\lambda(t)$.

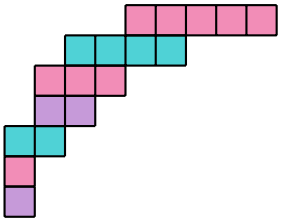
We build a plane partition P out of an arbitrary skew-linking diagram λ/μ with an iterative procedure. In the first step, read down the rows of λ/μ and select any row which does not overlap in the columns of an already-selected row. We highlight these rows in the below skew-linking diagram, which were selected top-to-bottom:



The labels of the first row of P are determined by 'stacking' these rows. More precisely, $P_{1,s}$ for $s \geq 1$ is the number of rows selected in this step with length at least s. We then label the second row of P by the same procedure, but ignore any rows in λ/μ which have already been selected. Below we shade the selected rows in the second step.



We continue until all rows have been exhausted, at which point we obtain P.



3	2	2	1	1	
2	2	1	1		
2	1				

We could create another plane partition P' by swapping the roles of rows and columns in the construction, and labelling P' one column at a time by selecting columns in λ/μ which do not overlap in their rows. Surprisingly, it turns out that $P = P'$ [Che10, Corollary 3.4.1]. Moreover, we obtain the following properties:

Theorem 3.37. *Let λ/μ be a skew-linking diagram, and let P be the plane partition obtained from it by the above construction. Then:*

- *The length of the first row/column of P is the length of the first row/column of λ/μ.*
- *The sum of the first row/column of P is the number of columns/rows in λ/μ.*
- *The sum $|P|$ of labels in P is equal to the number of boxes in λ/μ.*
- *Let ΔP be the plane partition obtained by replacing every label ℓ in P by $1 + 2 + \cdots + (\ell - 1)$. Then $|\Delta P| = |\mu|$.*

Proof. The first (and longest) row of λ/μ is always the first row selected in the construction, which adds this many 1s to the first row of P. The statement for columns follows from $P = P'$. The first box of every row in λ/μ increments a label in the first column of P. Similarly for columns and rows by $P = P'$.

Every box of λ/μ increments a label in P, which gives the third statement. For the fourth, see [Che10, Lemma 2.3.6]. □

Definition 3.38. For a Snow partition λ with $t > 0$, let $\mathcal{P}_\lambda(t)$ denote the plane partition obtained from $\lambda/\iota_\lambda(t)$ using the above construction.

Corollary 3.39. *For a Snow partition λ with parameters (k, n, j, t, i) and $t > 0$, let $P := \mathcal{P}_\lambda(t)$. Then:*

- *P is $(t-1, t-1, k)$-bounded.*
- *The sum of the first column of P is at most k.*
- *The sum of the first row in P is at most $n - k$.*
- *If ΔP is defined as above, then $i = |\Delta P|$ and $j = |P| + |\Delta P|$.*

Proof. Recall that $\lambda/\iota(t)$ is a skew-linking diagram whose first row and first column have length at most $t-1$ by Lemma 3.7. Also, the number of rows and columns in λ are at most k and $n-k$. Finally, we have $j = i + (j-i) = |\Delta P| + |P|$. □

Nakano vanishing

The Nakano vanishing theorem (see [AN54]), sometimes called the Kodaira–Nakano or Akizuki–Nakano vanishing theorem, asserts that for a compact Kähler manifold X and an ample line bundle L,

$$H^i(X, \Omega_X^j \otimes L) = 0 \quad \text{whenever} \quad i + j > \dim X.$$

In the case $X = \mathrm{Gr}(k, n)$, this gives $H^i(X, \Omega^j(t)) = 0$ whenever $t > 0$ and $i + j > N$, where $N := k(n-k)$. We prove this inequality for all plane partitions, where the values k, n, j, i are associated to a plane partition based on Corollary 3.39. We then obtain the same inequality for Snow partitions using the map $\lambda \mapsto \mathcal{P}_\lambda(t)$ from the previous section.

Lemma 3.40. *For a plane partition P, choose k, n, j, i such that*

- *k is the sum of the first column of P,*
- *$n - k$ is the sum of the first row of P,*
- *$i = |\Delta P|$ and $j = |\Delta P| + |P|$.*

Then $i + j \le k(n-k)$.

Proof. We have

$$k(n-k) = \sum_r P_{r,1} \sum_s P_{1,s}$$

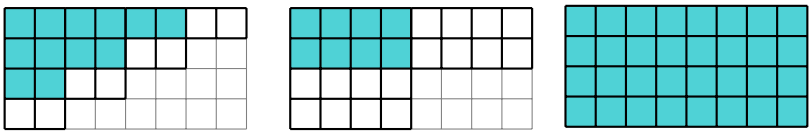
by definition. Since the entries in a plane partition are weakly decreasing along rows and down columns, we get

$$\sum_r P_{r,1} \sum_s P_{1,s} \geq \sum_{r,s} P_{r,s}^2$$

and by the identity $\ell^2 = 2(1 + 2 + \cdots + (\ell - 1)) + \ell$, the right-hand side of this inequality is equal to $2|\Delta P| + |P| = i + j$. □

Remark 3.41. In general, the values of k and $n - k$ for the Snow partition λ might be greater than the associated values for $\mathcal{P}_\lambda(t)$, with equality when λ is *tightly* bounded in its $k \times (n - k)$ rectangle, or equivalently, when λ and λ^T have no zero entries. The proof using plane partitions is sufficient because increasing k and $n - k$ preserves the inequality $i + j \leq k(n - k)$.

In the next theorem, we calculate the Hodge numbers when $i + j = N$ and observe that the Snow partitions which satisfy this condition are "generalized staircases" (see Figure 3.3).



$t = 3$ $t = 6$ $t \geq 12$

Figure 3.3. All Snow partitions for $t > 0$ with $k = 8$, $n = 12$ and $i + j = N$. The t-interior of each is shaded with the possible values of t given.

Theorem 3.42. *Suppose $i, t > 0$ and $i + j = N$. Then $h^{j,i}(t) \neq 0$ if and only if $t \mid n \mid kt$ and $i = N(n - t)/(2n)$. In this case, $h^{j,i}(t) = 1$.*

Proof. To get equality in the proof of Lemma 3.40, we need λ to be tightly bounded (Remark 3.41), and for $P_{r,s}^2 = P_{r,1}P_{1,s}$ for all r and s, which means P takes a constant value ℓ on some $a \times b$ rectangle with no other labels. The skew-linking diagram which maps to this partition consists of ℓ diagonally arranged blocks of size $a \times b$. We have $k = \ell a$ and $n - k = \ell b$, so $n = \ell(a + b)$.

The largest hook length on each diagonal block is $a + b - 1$, and the smallest hook lengths above the diagonal blocks are $a + b + 1$, so $t = a + b$ and $n = \ell t$. Together we get $t \mid n \mid kt$. The size of the interior is

$$i = \frac{ab\ell(\ell - 1)}{2} = \frac{N(\ell - 1)}{2\ell} = \frac{N(n - t)}{2n}.$$

Conversely, if $t \mid n \mid kt$ we set $\ell = n/t$, $a = k/\ell$, $b = (n-k)/\ell$, which are integers. We let λ be the generalized staircase with ℓ diagonal blocks of size $a \times b$.

We can prove that $d_\lambda(t) = 1$ either using the t-hook ratio and the cancelling of boxes in $\iota_\lambda(t)$ and λ^{C} using Lemma 3.31, or by observing that $\gamma_\lambda(t) = (b^n)$ and $\#\mathrm{SSYT}((b^n); n) = 1$ since every column must contain all values $1, \ldots, n$. □

Example 3.43. For $k = 2$ and $n = 4$ (so $N = 4$), the only Snow partition with $i + j = 4$ is $\lambda = (2, 1)$ with $j = 3$ and $i = 1$. Below is the Hodge diamond for $\mathrm{Gr}(2, 4)$ twisted by $\mathcal{O}(2)$, showcasing $h^{3,1}(2) = 1$.

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
\boxed{0} & & \boxed{1} & & \boxed{0} & & \boxed{0} & & \boxed{0} \quad i+j=4 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & \\
 & & & 15 & & 0 & & & \\
 & & & & 20 & & & &
\end{array}$$

The case $i + j = N - 1$ turns out to be far more restrictive. In this case we get at most two Snow partitions satisfying $i, t > 0$ and $i + j = N - 1$ for any fixed value of n (see Figure 3.4).



Figure 3.4. A Snow partition λ satisfying $i + j = N - 1$ for $n = 11$ and $i > 0$. The only other such Snow partition is λ^{T}. Both require that $i = 4$ and $t = 6$.

Theorem 3.44. *Suppose $i, t > 0$ and $i + j = N - 1$. Then $h^{j,i}(t) = n$ if $n \geq 5$, $k \in \{2, n-2\}$, $i = (n-3)/2$ (hence n is odd) and $t = (n+1)/2$. Otherwise, $h^{j,i}(t) = 0$.*

Proof. Fix $1 \leq k < n$ and $t > 0$. We first prove that there are at most two Snow partitions when $i + j = N - 1$ and $i > 0$.

Consider the chain of inequalities in Lemma 3.40. If $i > 0$, we have both $k > 1$ and $n - k > 1$. Hence, if λ or λ^{T} had zero entries, we would have $k > \sum_r P_{r,1}$ or $n - k > \sum_s P_{1,s}$, and $k(n-k) \geq i + j + 2$ in either case.

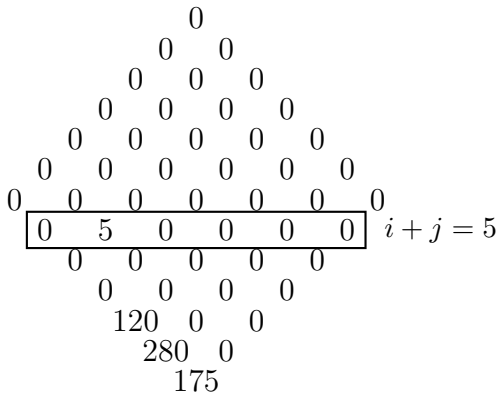
Hence, we have $\sum_{r,s} P_{r,1} P_{1,s} = \sum_{r,s} P_{r,s}^2 + 1$, so

$$P = \begin{array}{|c|c|c|c|c|c|}\hline 2 & 2 & 2 & \cdots & 2 & 1 \\ \hline \end{array}$$

or its transpose. If there are r copies of 2 in this plane partition, then $\lambda = (2r+1, r)$ (or its transpose) which is a Snow partition for $n = 2r+3$, $i = r$ and $t = r+2$, satisfying the conditions in the theorem. Conversely, given parameters matching the conditions in the theorem we can set $r = t-2$ and $\lambda = (2r+1, r)$ when $k = 2$, or its transpose when $k = n-2$, which both satisfy $i + j = N - 1$.

We can calculate that $d_\lambda(t) = n$ for $\lambda = (2r+1, r)$ either using the t-hook ratio, or by observing that $\gamma_\lambda(t) = ((r+1)^{n-1}, r)$, and $\#\mathrm{SSYT}(\gamma_\lambda(t); n) = n$ since each tableau is determined by the entry missing from the final column. □

Example 3.45. For $k = 2$ and $n = 5$ (so $N = 6$), the only Snow partition satisfying $i, t > 0$ and $i + j = 6 - 1$ is $\lambda = (3, 1)$ with $j = 4$ and $i = 1$. Below is the Hodge diamond for $\mathrm{Gr}(2, 5)$ twisted by $\mathcal{O}(3)$, showcasing $h^{4,1}(3) = 5$.



Snow's bounds

In [Sno86, §3.4], Snow lists five inequalities in terms of the parameters (k, n, j, t, i) which guarantee the nonexistence of Snow partitions. In this section we strictly improve four of these bounds by turning them into necessary inequalities for the existence of Snow partitions. The bounds we provide are sharp in the sense that there are infinitely many Snow partitions satisfying the associated equality. Our proofs are given on the level of plane partitions where possible, using the construction in Section 3.4.

Recall that by applying the symmetries $\lambda \mapsto \lambda^\mathsf{T}$ and $\lambda \mapsto \lambda^\mathsf{C}$ we can effectively assume that $t > 0$ and $k \leq n/2$. Under these assumptions, Snow proves that no Snow partitions exist if any of the following are satisfied:

(S1) $ki \geq (k-1)j > 0$,

(S2) $i > N - j$,

(S3) $j > k(n-k-1)$ if $(k,n) \neq (2,4)$,

(S4) $j \leq t$ and $i > 0$,

(S5) $(2k-1)i \geq (k-1)N$.

We now state improved versions of these bounds. These hold under the assumption that $t > 0$. Note that the inequalities are reversed, since these are necessary conditions for the existence of Snow partitions.

(P1) $i \leq \dfrac{k-1}{k+1} j$,

(P2) $i + j \leq N$,

(P3) $j \leq \dfrac{1}{2} k(n-k+t-1)$,

(P4) $j - i \geq t$ if $i > 0$,

(P5) $i \leq \dfrac{k-1}{2k} N$.

Example 3.46. The table below provides examples of parameters (k, n, j, t, i) for which no Snow partitions exist. Each of these parameter sets is ruled out by our constraints (P1)–(P5), but not by Snow's (S1)–(S5).

k	n	j	t	i	(P1)	(P2)	(P3)	(P4)	(P5)
2	7	5	2	2	False	True	True	True	True
2	4	3	2	1	True	True	False	True	True
3	6	4	3	2	True	True	True	False	True
2	7	7	2	3	False	True	False	True	False

Proposition 3.47. *All Snow partitions with $i, t > 0$ satisfy (P4).*

Proof. Choose a box $(a, b) \in \iota_\lambda(t)$ whose south and east neighbors are not in the t-interior. Then all boxes in the hook with vertex (a, b) except for the vertex itself lie outside the t-interior, so

$$j - i \geq h_\lambda(a, b) - 1 \geq t + 1 - 1 \geq t. \qquad \square$$

We prove the remaining inequalities on the level of plane partitions by the same approach as Lemma 3.40. These inequalities then hold for all Snow partitions by the same reasoning (also see Remark 3.41, which applies to these inequalities in the same way).

Proposition 3.48. *Every $(t-1, t-1)$-bounded plane partition P satisfies (P1), (P2), (P3) and (P5), with k, n, j, i as in Lemma 3.40. Moreover, we have*

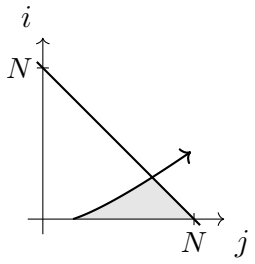
$$i \leq \frac{\sqrt{N}(\sqrt{N}-1)}{2}.$$

Proof. Lemma 3.40 gives $i + j \leq N$, which is (P2).

We have the inequality $j - i \leq k(t-1)$ by replacing each column of P by the first column. Combining this with $i + j \leq N$ gives (P3).

We have $j - i \geq \max(k, n-k)$ (the total sum of entries compared to the first row/column) so with $i + j \leq N$ we get $i + j \leq k(j-i)$, which after rearranging gives (P1). Applying $i + j \leq N$ to replace j with $N - i$ in (P1) and rearranging gives (P5).

Replacing both k and $n-k$ with $j-i$ in $i + j \leq N$ gives $i + j \leq (j-i)^2$. By finding the intersection of this region with $i + j \leq N$ (see the diagram below) we obtain $i \leq \sqrt{N}(\sqrt{N}-1)/2$.



□

All bounds in the above proposition are sharp for at least the staircase partitions $(k, \ldots, 2, 1)$ where $n = 2k$ and $t = 2$.

3.5 Hook Lengths and Euler Characteristic

Algebraic proof

We give an algebraic proof that the sum of t-hook ratios over all bounded partitions is independent of t. Recall $f_\lambda(t)$ from Definition 3.28.

Theorem 3.49. *For every $t \in \mathbb{Z}$,*

$$\sum_\lambda f_\lambda(t) = \binom{n}{k}$$

with the sum over all $(k, n-k)$-bounded partitions.

Proof. We define the twisted holomorphic Euler characteristic of $\mathrm{Gr}(k,n)$ (see [Hir95, §2.10]) by

$$\chi(\mathrm{Gr}(k,n),\Omega^j(t)) := \sum_{i\geq 0}(-1)^i h^i(\mathrm{Gr}(k,n),\Omega^j(t)).$$

When $t=0$, the holomorphic Euler characteristic is related to the standard (topological) Euler characteristic $\chi(\mathrm{Gr}(k,n))$ via

$$\chi(\mathrm{Gr}(k,n)) = \sum_{j\geq 0}(-1)^j \chi(\mathrm{Gr}(k,n),\Omega^j).$$

One readily shows that $\chi(\mathrm{Gr}(k,n)) = \binom{n}{k}$. Now, let $X = \mathrm{Gr}(k,n)$ and consider a generic section $\xi \in H^0(T_X)$. This vanishes at $\binom{n}{k}$ points, corresponding to the zero locus D (see e.g. [Ful84, §14.1]). We consider the Koszul resolution (see e.g. [Eis95, §17.2]) of the sheaf $\mathcal{O}_D$:

$$0 \to \Omega_X^{k(n-k)} \xrightarrow{\lrcorner\xi} \cdots \xrightarrow{\lrcorner\xi} \Omega_X^2 \xrightarrow{\lrcorner\xi} \Omega_X^1 \xrightarrow{\lrcorner\xi} \mathcal{O}_X \to \mathcal{O}_D \to 0$$

which is exact because ξ is a regular section. By the additivity of the Euler characteristic,

$$\sum(-1)^j \chi(\Omega_X^j) = \chi(\mathcal{O}_D) = \binom{n}{k}.$$

We note that $\chi(\mathcal{O}_D) = \chi(\mathcal{O}_X(t)\otimes\mathcal{O}_D)$ since $\mathcal{O}_D$ is a sum of skyscraper sheaves and $\mathcal{O}_X(t)$ is a line bundle, hence tensoring the Koszul resolution by $\mathcal{O}_X(t)$ we obtain

$$\sum_{j\geq 0}(-1)^j \chi(\Omega_X^j(t)) = \binom{n}{k}.$$

By Theorem 3.29,

$$\sum_{|\lambda|=j} f_\lambda(t) = (-1)^j \chi(\Omega^j(t))$$

with the sum over all bounded partitions of size j, and hence

$$\sum_{\lambda} f_\lambda(t) = \sum_{j\geq 0}(-1)^j \chi(\Omega^j(t)) = \binom{n}{k}$$

with the first sum now over all bounded partitions. □

Example 3.50 ($t=0$)**.** All partitions are 0-core, and $f_\lambda(0)=1$ for every λ. The sum $\sum_\lambda f_\lambda(0)$ is the number of bounded partitions $\binom{n}{k}$, and the holomorphic Euler characteristic is $\sum(-1)^j\chi(\Omega^j(0)) = \chi(\mathrm{Gr}(k,n))$.

Example 3.51 ($t=1$)**.** The only 1-core partition is the empty partition (0^k), and $f_{(0^k)}(1) = \binom{n}{k}$ by Example 3.33. Algebraically this follows from

$$h^0(\mathrm{Gr}(k,n),\mathcal{O}(1)) = \binom{n}{k}.$$

Combinatorial interpretation

Because the t-hook ratio $f_\lambda(t)$ is an integer-valued polynomial in t (Remark 3.30), so is $\sum_\lambda f_\lambda(t)$ with the sum over all bounded partitions λ. By Theorem 3.49, this polynomial is constant.

Suppose $(a_1, b_1), \ldots, (a_N, b_N)$ is an arbitrary linear ordering on the boxes of the $k \times (n-k)$ bounding rectangle. By expanding $f_\lambda(t)$, we see that the coefficient of t^d is given by

$$[t^d] f_\lambda(t) = \sum_{1 \le \ell_1 < \cdots < \ell_d \le N} \frac{(-1)^d}{h_\lambda(a_{\ell_1}, b_{\ell_1}) \cdots h_\lambda(a_{\ell_d}, b_{\ell_d})},$$

where $h_\lambda(a, b) = \lambda_a + \lambda_b^\mathsf{T} - a - b + 1$ allows for both positive and negative hook lengths as in Section 3.2. The following statement (plus the calculation for $t = 0$) is equivalent to Theorem 3.49.

Corollary 3.52. *For every $d \ge 1$,*

$$\sum_\lambda \sum_{1 \le \ell_1 < \cdots < \ell_d \le N} \frac{1}{h_\lambda(a_{\ell_1}, b_{\ell_1}) \cdots h_\lambda(a_{\ell_d}, b_{\ell_d})} = 0,$$

where the first sum is over all $(k, n-k)$-bounded partitions.

We are unaware of a direct combinatorial proof of this fact. The case where d is odd follows from the duality $\lambda \mapsto \lambda^\mathsf{C}$, since any d-fold product of hook lengths in λ can be exchanged for their negatives in λ^C, which introduces a negative overall.

Example 3.53. For $k = 2$ and $n = 4$ we have the partitions:

λ	$\begin{array}{\|c\|c\|}\hline 3 & 2 \\ \hline 2 & 1 \\ \hline \end{array}$	$\begin{array}{\|c\|c\|}\hline 3 & 1 \\ \hline 1 & -1 \\ \hline \end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & -1 \\ \hline 1 & -2 \\ \hline \end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 1 \\ \hline -1 & -2 \\ \hline \end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & -1 \\ \hline -1 & -3 \\ \hline \end{array}$	$\begin{array}{\|c\|c\|}\hline -1 & -2 \\ \hline -2 & -3 \\ \hline \end{array}$
$[t^2] f_\lambda(t)$	$\frac{23}{12}$	$-\frac{2}{3}$	$-\frac{5}{4}$	$-\frac{5}{4}$	$-\frac{2}{3}$	$\frac{23}{12}$

and the sum of the values $[t^2] f_\lambda(t)$ is zero.

3.6 A q-analog

In this section, we compute the q-dimension (as a $\mathrm{GL}(n, \mathbb{C})$ representation) of the cohomology $H^i(\Omega_X^j(t))$, discuss some examples, and conjecture a q-analog of Theorem 3.49. We recall the notation and definitions from Section 3.2.

Setup

Following [Ste94], the q-dimension of a (finite dimensional) rational representation V of $\mathrm{GL}(n,\mathbb{C})$ is defined as the Laurent polynomial

$$\dim_q(V) := \sum_{\mu\in\mathbb{Z}^n} \dim(V[\mu])q^{\langle\varrho,\mu\rangle} \in \mathbb{Z}[q^{\pm 1/2}].$$

We make special note of the use of ϱ as the Weyl vector here, rather than the shifted version ϱ' that has been employed so far. The benefit of this choice is the identity

$$\dim_q(V) = \dim_{q^{-1}}(V) = \dim_q(V^\vee).$$

This holds by recalling that $V[\mu]$, $V[\mathrm{rev}(\mu)]$ and $V^\vee[-\mu]$ have the same dimension for every $\mu \in \mathbb{Z}^n$, and computing that $\langle \mathrm{rev}(\mu), \varrho\rangle = -\langle \mu, \varrho\rangle = \langle -\mu, \varrho\rangle$.

Definition 3.54. For a bounded t-core partition λ, set $[d_\lambda(t)]_q := \dim_q L^\vee_{\beta_\lambda(t)}$.

The following is immediate by taking the q-dimension of both sides of the equality in Theorem 3.11.

Corollary 3.55. *For integers (k,n,j,t,i) with $1 \le k < n$ and $j,i \ge 0$:*

$$[h^{j,i}(t)]_q := \dim_q(H^i(\mathrm{Gr}(k,n),\Omega^j(t))) = \sum_\lambda [d_\lambda(t)]_q$$

with the sum over all Snow partitions λ with parameters (k,n,j,t,i).

We observe that $[d_\lambda(t)]_q$ and $[h^{j,i}(t)]_q$ are always polynomials in $\mathbb{Z}[q^{\pm 1/2}]$ which are invariant under $q \mapsto q^{-1}$.

Calculation of the q-dimension

We first calculate $[d_\lambda(t)]_q$ in terms of the BWB weight $\beta_\lambda(t) \in \Lambda_n$ (see Definition 3.15) using the q-analog of the Weyl dimension formula. For every integer m, define the (symmetrized) q-analog

$$[m]_q := \frac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}} = q^{-(m-1)/2}\frac{1-q^m}{1-q} \in \mathbb{Z}[q^{\pm 1/2}]$$

which satisfies $[m]_q = [m]_{q^{-1}}$ and $[-m]_q = -[m]_q$.

Theorem 3.56. *For a Snow partition λ,*

$$[d_\lambda(t)]_q = \prod_{1\le r<s\le n} \frac{[\beta_\lambda(t)_r - \beta_\lambda(t)_s + s - r]_q}{[s-r]_q}.$$

Proof. We have $[d_\lambda(t)]_q = \dim_q L^\vee_{\beta_\lambda(t)} = \dim_q L_{\beta_\lambda(t)}$. By [Ste94, p. 2.7], for any $\beta \in \Lambda_n$ we have

$$\dim_q L_\beta = q^{-\langle \beta, \varrho \rangle} \prod_{\phi \in \Phi^+} \frac{1 - q^{\langle \beta + \varrho, \phi \rangle}}{1 - q^{\langle \varrho, \phi \rangle}} = q^{-\langle \beta, \varrho \rangle} \prod_{1 \le r < s \le n} \frac{1 - q^{\beta_r - \beta_s + s - r}}{1 - q^{s-r}}.$$

Converting each term in the product to the symmetrized q-analog gives

$$\dim_q L_\beta = q^z \prod_{1 \le r < s \le n} \frac{[\beta_r - \beta_s + s - r]_q}{[s-r]_q}$$

for some shift z. Because the left-hand side is invariant under $q \mapsto q^{-1}$, as is every term of the form $[m]_q$, we must have $z = 0$. □

Remark 3.57. In the remaining proofs in this section, we will repeat this trick of denoting by q^z some undetermined q-shift, which we reason to be q^0 by showing that $[d_\lambda(t)]_q$ is equal to the product of q^z and terms of the form $[m]_q$.

In the case of a polynomial representation L_γ for $\gamma \in \Lambda^+_n$, we can count tableaux of shape γ by their size. Recall that $L^\vee_{\beta_\lambda(t)}$ was either isomorphic or dual to $L_{\gamma_\lambda(t)}$ depending on the sign of t (see Definition 3.22 and its preceding discussion).

Theorem 3.58. *For a Snow partition λ, set $\gamma := \gamma_\lambda(t)$. Then*

$$[d_\lambda(t)]_q = q^{-(n+1)(n-k)|t|/2} \sum_{T \in \mathrm{SSYT}(\gamma; n)} q^{|T|} = \prod_{(a,b) \in \gamma} \frac{[n + b - a]_q}{[h_\gamma(a,b)]_q}.$$

Proof. If $t \ge 0$, then $L^\vee_{\beta_\lambda(t)} \cong L_\gamma$, so $[d_\lambda(t)]_q = \dim_q L_\gamma$. If $t < 0$, then $L^\vee_{\beta_\lambda(t)}$ is dual to L_γ so $[d_\lambda(t)]_q = \dim_{q^{-1}} L_\gamma = \dim_q L_\gamma$. In either case it remains to calculate the q-dimension of L_γ.

For any $\gamma \in \Lambda^+_n$, we have $\dim L_\gamma[\mu] = \#\mathrm{SSYT}(\gamma; \mu)$ for every $\mu \in \mathbb{Z}^n$ and so

$$\begin{aligned} \dim_q(L_\gamma) &= \sum_{\mu \in \mathbb{Z}^n} \#\mathrm{SSYT}(\gamma; \mu) q^{\langle \mu, \varrho \rangle} \\ &= s_\gamma(q^{(n-1)/2}, q^{(n-3)/2}, \ldots, q^{-(n-1)/2}) \\ &= q^{-(n-1)|\gamma|/2} s_\gamma(1, q, \ldots, q^{n-1}), \end{aligned}$$

where we use the fact that $s_\gamma(x_1, \ldots, x_n)$ is a symmetric polynomial of homogeneous degree $|\gamma|$. Reintroducing a factor of q in each variable gives

$$q^{-(n+1)|\gamma|/2} s_\gamma(q, q^2, \ldots, q^n) = q^{-(n+1)|\gamma|/2} \sum_{T \in \mathrm{SSYT}(\gamma; n)} q^{|T|}.$$

Recall that $|\gamma_\lambda(t)| = (n-k)|t|$ by Theorem 3.23, which gives the first equality.

For the second, [Mac86, §3 Example 1] gives

$$s_\gamma(1, q, \ldots, q^{n-1}) = q^{z_1} \prod_{(a,b)\in\gamma} \frac{1-q^{n+b-a}}{1-q^{h_\gamma(a,b)}}$$

for some z_1, and so after applying the relevant q-shift we get

$$[d_\lambda(t)]_q = q^{z_2} \prod_{(a,b)\in\gamma} \frac{[n+b-a]_q}{[h_\gamma(a,b)]_q}$$

for some z_2. But by $q \mapsto q^{-1}$ symmetry, we must have $z_2 = 0$. □

Finally, we prove $[d_\lambda(t)]_q$ can also be calculated by the expected q-analog of the t-hook ratio.

Definition 3.59. For a bounded partition λ and an integer t, set

$$[f_\lambda(t)]_q = \prod_{a=1}^{k} \prod_{b=1}^{n-k} \frac{[h_\lambda(a,b)-t]_q}{[h_\lambda(a,b)]_q}.$$

Remark 3.60. Using $[m]_q = q^{-(m-1)/2}(1-q^m)/(1-q)$ we can rewrite the q-analog of the t-hook ratio as

$$[f_\lambda(t)]_q = q^{-Nt/2} \prod_{a=1}^{k} \prod_{b=1}^{n-k} \frac{1-q^{h_\lambda(a,b)-t}}{1-q^{h_\lambda(a,b)}}.$$

We start with a q-analog of Lemma 3.26.

Lemma 3.61. *For a bounded partition λ:*

$$\dim_q L_\lambda \cdot \dim_q L_{\lambda^\intercal} = (-1)^{j-N} \prod_{a=1}^{k} \prod_{b=1}^{n-k} \frac{[a+b-1]_q}{[h_\lambda(a,b)]_q}$$

Proof. We follow the proof of [PŚ18, Theorem 1.1], but apply the q-analog. There is a bijection $c : \mathrm{SSYT}(\lambda; k) \to \mathrm{SSYT}(\lambda^\complement; k)$ since every column must contain every entry between 1 and k. Moreover, $|c(T)| = (n-k)(1+\cdots+k) - |T|$. In particular, we obtain that

$$\begin{aligned}
\dim_q L_\lambda &= q^{z_1} s_\lambda(1, q, \ldots, q^{k-1}) \\
&= q^{z_2} s_{\lambda^\complement}(1, q^{-1}, \ldots, q^{-(k-1)}) \\
&= q^{z_3} s_{\lambda^\complement}(1, q, \ldots, q^{k-1}) \\
&= q^{z_4} \dim_q L_{\lambda^\complement}
\end{aligned}$$

and by $q \mapsto q^{-1}$ symmetry we have $z_4 = 0$. By applying the calculation in the proof of Theorem 3.58 to λ^{TC} we have

$$\dim_q L_{\lambda^{\mathsf{T}}} = \dim_q L_{\lambda^{\mathsf{TC}}} = \prod_{(a,b)\in\lambda^{\mathsf{TC}}} \frac{[n-k+b-a]_q}{[h_{\lambda^{\mathsf{TC}}}(a,b)]_q}.$$

Reindexing the numerator by associating the boxes of λ^{TC} with the boxes of the skew partition $(n-k)^k/\lambda$ under the map $(a,b) \mapsto (k-b+1, n-k-a+1)$ turns $n-k+b-a$ into $k+b-a$ and gives

$$\dim_q L_{\lambda^{\mathsf{T}}} = \frac{\prod_{(a,b)\in(n-k)^k/\lambda}[k+b-a]_q}{\prod_{(a,b)\in\lambda^{\mathsf{C}}}[h_{\lambda^{\mathsf{C}}}(a,b)]_q}.$$

Hence,

$$\dim_q L_\lambda \cdot \dim_q L_{\lambda^{\mathsf{T}}} = \frac{\prod_{a=1}^{k}\prod_{b=1}^{n-k}[k+b-a]_q}{\prod_{(a,b)\in\lambda}[h_\lambda(a,b)]_q \prod_{(a,b)\in\lambda^{\mathsf{C}}}[h_{\lambda^{\mathsf{C}}}(a,b)]_q}.$$

The terms in the numerator become $[a+b-1]_q$ after reindexing a for $k-a+1$. In the denominator, the hook lengths in λ^{C} correspond to the negative hook lengths of λ, so the denominator is $(-1)^{N-j}\prod_{a=1}^{k}\prod_{b=1}^{n-k}[h_\lambda(a,b)]_q$ as required. □

Theorem 3.62. *For a Snow partition λ,*

$$[d_\lambda(t)]_q = (-1)^{i+j}[f_\lambda(t)]_q.$$

Proof. We follow the proof of Theorem 3.29. Since $\beta_\lambda(t) = \sigma(\alpha_\lambda(t)+\varrho)-\varrho$ where σ is a permutation of length i, we have

$$[d_\lambda(t)]_q = (-1)^i \prod_{1\le r<s\le n} \frac{[\alpha_\lambda(t)_r - \alpha_\lambda(t)_s + s - r]_q}{[s-r]_q}.$$

We split this product into three terms. When $r, s \le k$ we get $\dim_q L_\lambda$ and when $r, s > k$ we get $\dim_q L_{\lambda^{\mathsf{T}}}$. The final term is

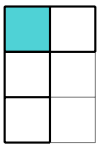
$$(-1)^N \prod_{a=1}^{k}\prod_{b=1}^{n-k} \frac{[h_\lambda(a,b)-t]_q}{[a+b-1]_q}$$

so multiplying these terms together and applying the above lemma gives the result. □

Examples

We begin with a generic example to showcase various methods for calculating $[d_\lambda(t)]_q$, before taking the q-analog of some examples in Section 3.3.

Example 3.63. Let $\lambda = (2,1,1)$, which is the only Snow partition with parameters $k = 3$, $n = 5$, $j = 4$, $t = 3$ and $i = 1$.



Then $\dim_q H^1(\mathrm{Gr}(3,5), \Omega^4(3)) = [d_\lambda(3)]_q$. We first calculate via $\gamma_\lambda(t)$. Because $\partial_\lambda(3) = (1,1,1)$ and $3 - \partial_{\lambda^\intercal}(3) = (1,2)$ we have $\gamma_\lambda(t) = (2,1^4)$. Hence, we want the weights of all tableaux in $\mathrm{SSYT}((2,1^4);5)$.

T	$\begin{array}{\|c\|c\|}\hline 1&1\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1} 5\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1} 5\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1} 5\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1} 5\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&5\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1} 5\\\cline{1-1}\end{array}$
$\lvert T\rvert$	16	17	18	19	20

Since $-(n+1)(n-k)|t|/2 = -18$ we have

$$[d_\lambda(t)]_q = q^{-18}(q^{16} + q^{17} + q^{18} + q^{19} + q^{20}) = q^{-2} + q^{-1} + 1 + q + q^2.$$

Notice that the shift symmetrizes $\sum_T q^{|T|}$ about $q \mapsto q^{-1}$. Alternatively, we could have calculated $[d_\lambda(t)]_q$ using the second formula in Theorem 3.58. Instead, we use the t-hook ratio for λ and Theorem 3.62 to get

$$[d_\lambda(t)]_q = (-1)^{1+4}[f_\lambda(3)]_q = -\frac{[1]_q}{[4]_q} \cdot \frac{[-1]_q}{[2]_q} \cdot \frac{[-2]_q}{[1]_q} \cdot \frac{[-2]_q}{[1]_q} \cdot \frac{[-4]_q}{[-1]_q} \cdot \frac{[-5]_q}{[-2]_q}$$

which after cancelling terms is $-[-5]_q = [5]_q = q^{-2} + q^{-1} + 1 + q + q^2$.

Example 3.64 ($j = 0$)**.** Following Example 3.33, the only Snow partition with $j = 0$ is $\lambda = (0^k)$, and for every $t \geq 0$ we obtain

$$\begin{aligned}
[d_\lambda(t)]_q &= \prod_{a=1}^{k} \prod_{b=1}^{n-k} \frac{[a+b+t-1]_q}{[a+b-1]_q} \\
&= q^{-Nt/2} \prod_{a=1}^{k} \prod_{b=1}^{n-k} \frac{1-q^{a+b+t-1}}{1-q^{a+b-1}} \\
&= q^{-Nt/2} \sum_{P} q^{|P|}
\end{aligned}$$

where the final sum is over all $(k, n-k, t)$-bounded plane partitions (see [Mac86]).

Definition 3.65. For integers $1 \leq k < n$, define the (symmetrized) q-analog of the binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q \cdots [1]_q}{[k]_q \cdots [1]_q [n-k]_q \cdots [1]_q}$$

Example 3.66 ($t = 2$)**.** Following Example 3.36, assume that $k \leq n/2$, and set $\Delta_m := (m, \ldots, 2, 1)$ for $0 \leq m \leq k$. We have

$$\gamma_{\Delta_m}(2) = (2^{n-k-m}, 1^{2m}, 0^{k-m}).$$

Applying Theorem 3.58 to this shape we get

$$[h^{m(m+1)/2, m(m-1)/2}(2)]_q = \frac{[2m+1]_q}{[n+1]_q} \begin{bmatrix} n+1 \\ k+m+1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ n-k+m+1 \end{bmatrix}_q$$

for every $0 \leq m \leq \min(k, n-k)$, and $[h^{j,i}(2)]_q = 0$ otherwise.

Finally, we conjecture a q-analog for Theorem 3.49, which we hope can lead to a combinatorial proof of that result.

Conjecture 1. Fix $1 \leq k < n$. Then for every integer t,

$$\sum_{\lambda} [f_\lambda(t)]_q = \begin{bmatrix} n \\ k \end{bmatrix}_{q^t}$$

where the sum is over all $(k, n-k)$-bounded partitions λ.

Remark 3.67. This conjecture has been computationally verified for all $n \leq 20$; code in *SnowBoundConjecture.sage*.

3.7 Deferred Combinatorial Proofs About Cohomology

Vector Bundle Cohomology

We deferred the proof of several lemmas in Chapter 2. We are now ready to prove them.

For the vanishings related to Equation (2.2.10) we compute the cohomology of $\mathrm{End}(S) \otimes \mathcal{O}(t)$:

Theorem 3.68. *Let $X = \mathrm{Gr}(k, n)$ with tautological subbundle S of rank k for $n \geq 3$, and let $t \in \mathbb{Z}$. Then*

$$H^1(X, \mathcal{E}nd(S) \otimes \mathcal{O}_X(t)) \neq 0 \text{ if and only if } t = -1 \text{ and } k = n-1.$$

Proof. If $k = 1$ then S is a line bundle hence $\mathcal{E}nd(S) = \mathcal{O}_X$, so $\mathcal{E}nd(S) \otimes \mathcal{O}(t)$ is supported in degrees 0 or $k(n-k)$. Otherwise, we assume $k \geq 2$, and

$$\mathcal{E}nd(S) \cong \Sigma^{(1,0,\ldots,0,0)} S \otimes \Sigma^{(0,0,\ldots,0,-1)} S \cong \mathcal{O}_X \oplus \Sigma^{(1,0,\ldots,0,-1)} S^\vee,$$

by the Pieri rule (see, e.g., [Wey03, Corollary 2.3.5] or [Nik25, Theorem A.5]).

The cohomology of $\mathcal{E}nd(S) \otimes \mathcal{O}(t)$ can then be computed using the Borel–Weil–Bott theorem (Theorem 3.11). $H^i(\mathcal{O}_X(t))$ is straightforward and concentrated in degrees 0 or $k(n-k)$, and we omit its calculation. Now take $\delta = (1, 0, \ldots, 0, -1)$ and $\epsilon = (-t, -t, \ldots, -t)$. Then we compute the cohomology of

$$\Sigma^\delta S^\vee \otimes \Sigma^\epsilon Q^\vee,$$

by considering

$$\alpha := \delta \smile \epsilon = (1, 0, \ldots, 0, -1, -t, -t, \ldots, -t, -t),$$

hence

$$\alpha + \varrho' = (n+1, n-1, \ldots, n-k+2, n-k, n-k-t, n-k-t-1, \ldots, 2-t, 1-t).$$

If $\alpha + \varrho'$ has exactly one non-inversion, it must occur at entry k, so then

$$\beta + \varrho' = (n+1, \ldots, n-k-t, n-k, n-k-t-1, \ldots, 2-t, 1-t),$$

(since we assume $k > 1$, hence both $n+1$ and $n-k$ appear in $\alpha + \varrho'$). But uniqueness of terms in $\alpha + \varrho'$ forces $n - k = 1$, and consequently β is of the form

$$\beta = (n+1, \ldots, n-k+2, n-k-t-1, n-k),$$

hence $t = -1$.

Conversely, if $k = n-1$ and $t = -1$, then

$$\alpha + \varrho' = (n+1, n-1, n-2, \ldots, 3, 1, 2).$$

This sequence has no repeated entries, and it has exactly one non-inversion. Thus Theorem 3.11 gives nonzero cohomology in degree 1 for the summand

$$\Sigma^{(1,0,\ldots,0,-1)} S^\vee \otimes \mathcal{O}_X(-1),$$

hence

$$H^1(X, \mathcal{E}nd(S) \otimes \mathcal{O}_X(-1)) \neq 0.$$

□

Corollary 2.13. *For any line bundle L on $X = \mathrm{Gr}(k,n)$ with $1 \leq k \leq n-2$,*

$$H^1(X, \mathcal{E}nd(S) \otimes K_X \otimes L) = 0.$$

For the vanishings related to Equation (2.2.9) we compute the cohomology of $Q^\vee \otimes \mathcal{O}(t)$:

Theorem 3.69. *Let $X = \mathrm{Gr}(k,n)$ with tautological quotient bundle Q. Then*

$$H^1(X, Q^\vee \otimes \mathcal{O}_X(t)) \neq 0 \text{ if and only if } k = 1 \text{ and } t = -1.$$

Proof. For computational convenience, we take $\mathcal{O}_X(t) = \Sigma^{(t,t,\ldots,t)} S^\vee$. By Borel–Weil–Bott, we consider

$$\Sigma^{(t,\ldots,t)} S^\vee \otimes \Sigma^{(1,0,\ldots,0)} Q^\vee.$$

Then

$$\alpha + \varrho' = (n+t, \ldots, n-k+t+1, n-k+1, n-k-1, \ldots, 1).$$

If $\alpha + \varrho'$ has a unique non-inversion (i.e. the associated BWB permutation has Coxeter length one) then since $\alpha + \varrho'$ only can have descent at entry k it must be that $k = 1$, i.e.

$$\alpha + \varrho' = (n+t, \ldots, n, n-2, \ldots, 1)$$

so

$$\beta + \varrho' = (n, n+t, n-2, \ldots, 2, 1).$$

Then uniqueness of entries of $\alpha + \varrho'$ would force $t = -1$.

Conversely, if $k = 1$ and $t = -1$, $Q^\vee \otimes \mathcal{O}_X(-1) \cong Q^\vee \otimes S \cong \Omega_X$ is the cotangent bundle on X, so the group in question is $H^i(X, \Omega^1_X)$, which by Hodge theory (or Theorem 3.29) must be $\mathbb{C}$ if $i = 1$ else 0.

□

Corollary 2.12. *For any line bundle L on $X = \mathrm{Gr}(k,n)$ with $2 \leq k \leq n-1$,*

$$H^1(X, Q^\vee \otimes V \otimes K_X \otimes L) = 0.$$

Remark 3.70. The reader may be interested that Theorems 3.68 and 3.69 fail when $k = 1, n-1$ and $t = -1$; in the context of Equation (2.2.11) this amounts to a failure of surjectivity for φ when $pd - n = -1$. In such cases, the correct Jacobian ideal is the original one constructed by Griffiths himself (Equation (1.1.1)).

Twisted Hodge Numbers when $i + j \geq N - 1$

In this section, let $X = \mathrm{Gr}(k, n)$ with $2 \leq k \leq n - 2$ and let $Z \subset X$ be a smooth hypersurface of degree d. Let $N = k(n - k) = \dim(X)$.

Theorem 3.71. *$H^i(X, \Omega_X^{N-i}((p-i)Z)) \neq 0$ for some p in the range $0 < i < p \leq N$ if and only if $d|n$ and $\gcd(k, n/d) > 1$.*

Proof. Set $q := p - i$, so $q > 0$ and

$$H^i(X, \Omega_X^{N-i}((p-i)Z)) = H^i(X, \Omega_X^{N-i}(qd)).$$

Thus we are considering twisted Hodge numbers $h^{j,i}(t)$ with $i + j = N$ and $t = qd$. By Theorem 3.42, such a group is nonzero if and only if

$$qd \mid n, \qquad n \mid kqd, \qquad i = \frac{N(n - qd)}{2n}.$$

Since the theorem assumes $i > 0$, we also have $qd < n$.

It remains to translate the existence of such a positive integer q into a condition on d, k, n. If such a q exists, then $qd \mid n$, so $d \mid n$. Write $M = n/d$. The conditions become

$$q \mid M, \qquad M \mid kq, \qquad q < M.$$

Writing $M = qr$, the condition $q < M$ says $r > 1$, and $M \mid kq$ says $r \mid k$. Hence r is a common divisor of k and $M = n/d$ greater than 1, so $\gcd(k, n/d) > 1$.

Conversely, suppose $d \mid n$ and let $r > 1$ be a common divisor of k and $M = n/d$. Set $q = M/r$. Then $q > 0$, $qd = n/r < n$, and

$$qd \mid n, \qquad n \mid kqd$$

because $r \mid k$. Therefore Theorem 3.42 gives a nonzero group for

$$i = \frac{N(n - qd)}{2n}.$$

The corresponding $p = i + q$ satisfies $0 < i < p \leq N$, so this produces a nonvanishing in the required range.

□

Theorem 3.72. *$H^i(X, \Omega_X^{N-i-1}((p-i-1)Z)) \neq 0$ for some p in the range $0 < i+1 < p \leq N$ if and only if $k \in \{2, n-2\}$ and $2d|(n+1)$.*

Proof. Set $q := p - i - 1$, so $q > 0$ and $t = qd$. Then we are interested in twisted Hodge numbers $h^{j,i}(t)$ with $i + j = N - 1$:

$$H^i(X, \Omega_X^{N-i-1}((p-i-1)Z)) = H^i(X, \Omega_X^{N-i-1}(qd)).$$

Suppose first that such a group is nonzero. By Theorem 3.44, we must have

$$k \in \{2, n-2\}, \qquad i = \frac{n-3}{2}, \qquad qd = \frac{n+1}{2}.$$

In particular, $2d \mid (n+1)$.

Conversely, suppose $k \in \{2, n-2\}$ and $2d \mid (n+1)$. Set

$$q = \frac{n+1}{2d}, \qquad i = \frac{n-3}{2}, \qquad p = i + 1 + q.$$

Then q is a positive integer and $qd = (n+1)/2$. Moreover, n is odd, and since $2 \le k \le n-2$ we have $n \ge 5$, so $i > 0$. By Theorem 3.44,

$$H^i(X, \Omega_X^{N-i-1}(qd)) \neq 0.$$

Finally, $p > i + 1$ by construction, and

$$p = \frac{n-1}{2} + \frac{n+1}{2d} \le n \le 2(n-2) = N,$$

where the last equality uses $k \in \{2, n-2\}$. Thus the nonzero group occurs for some p in the required range. □

3.8 Further Combinatorial Results

In this section we collect some combinatorial results that did not naturally fit into the previous discussion.

Holomorphic Euler Characteristic of a Smooth Complete Intersection

With $X = \mathrm{Gr}(k, n)$ as above, let $Z \hookrightarrow X$ be a smooth complete intersection of multidegree $(d_1, \dots, d_r)$. We give a formula to compute $\chi(\Omega_Z^j(t))$. Note that due to the Lefschetz hyperplane theorem (see e.g. [Huy05, §3]), when $t = 0$ the holomorphic Euler characteristics of Z and X completely determine the Hodge numbers of Z.

For a smooth closed complex variety Y define the Euler–Hodge generating function

$$\Phi_Y(x, t) := \sum_j (-1)^j x^j \chi(\Omega_Y^j(t)) \in \mathbb{Q}[x, t].$$

Note that this is related to Hirzebruch's χ_y-genus ([Hir95, §15.5]):

$$\chi_y(Y,t) := \sum_j y^j \chi(\Omega^j_Y(t)) = \Phi_Y(-y,t).$$

Also note that $\Phi_Y(x,t)$ enjoys a functional equation

$$\Phi_Y(x,t) = x^{\dim Y}\Phi_Y(1/x,-t),$$

due to Poincaré duality.

$\Phi_Y(x,t)$ is related to the holomorphic Euler characteristic $\chi(Y,t)$ by

$$\chi(Y,\Omega^j(t)) = (-1)^j[x^j]\Phi_Y(x,t).$$

We have shown (Theorem 3.29) that when $X = \mathrm{Gr}(k,n)$, we have

$$\Phi_X(x,t) = \sum_{\lambda\subset(n-k)^k} x^{|\lambda|} f_\lambda(t).$$

Theorem 3.73. *Let $Z = V(f_1,\dots,f_r) \hookrightarrow X = \mathrm{Gr}(k,n)$ denote a smooth complete intersection in X of multidegree $(d_1,\dots,d_r)$. Define the shift-operator T_d by*

$$T_d\Phi_Y(x,t) := \Phi_Y(x,t-d).$$

Then

$$\left(\prod_{i=1}^r (1-xT_{d_i})\right)\Phi_Z(x,t) = \left(\prod_{i=1}^r (1-T_{d_i})\right)\Phi_X(x,t).$$

$\Phi_Z(x,t)$ has degree $\dim Z = \dim X - r$ in t and is uniquely determined by this recurrence.

Proof. Let us start with the case that Z is a smooth hypersurface of degree d in $X = \mathrm{Gr}(k,n)$. Consider the ideal sheaf sequence

$$0 \to \Omega^j_X(t-d) \to \Omega^j_X(t) \to \Omega^j_X|_Z(t) \to 0,$$

and the conormal sequence

$$0 \to \Omega^{j-1}_Z(t-d) \to \Omega^j_X|_Z(t) \to \Omega^j_Z(t) \to 0.$$

Taking Euler characteristics and combining we obtain

$$\chi(\Omega^j_Z(t)) = \chi(\Omega^j_X(t)) - \chi(\Omega^j_X(t-d)) - \chi(\Omega^{j-1}_Z(t-d)), \tag{3.8.1}$$

where we use the convention $\chi(\Omega^{-1}_Z) = 0$. Multiplying by $(-x)^j$ and summing over j we have

$$\Phi_Z(x,t) = \Phi_X(x,t) - \Phi_X(x,t-d) + x\Phi_Z(x,t-d),$$

hence

$$(1 - xT_d)\Phi_Z(x,t) = (1 - T_d)\Phi_X(x,t).$$

Now we repeat this process for $i = 1 \dots r$ taking $Z_0 := X$ and $Z_i := V(f_1, \dots, f_i)$. Each $Z_i \hookrightarrow Z_{i-1}$ is a smooth hypersurface cut out by a section of $\mathcal{O}_X(d_i)|_{Z_{i-1}}$, so the argument gives

$$\left(\prod_{i=1}^{r}(1 - xT_{d_i})\right)\Phi_Z(x,t) = \left(\prod_{i=1}^{r}(1 - T_{d_i})\right)\Phi_X(x,t).$$

Note that the T_{d_i} commute so we may arrange them as a single product on the left. □

Example 3.74. Let $X = \mathrm{Gr}(2,7)$ and let $Z = V(1,1,1,1,1,1,1)$ be an intersection of 7 generic hyperplanes, which is a Calabi–Yau threefold. The Hodge numbers of Z were computed in [Rød00], following an involved calculation. We use a generating function calculation to recover its Hodge numbers. Then by Theorem 3.73 we have

$$\left(\prod_{i=1}^{7}(1 - xT_1)\right)\Phi_Z(x,t) = \left(\prod_{i=1}^{7}(1 - T_1)\right)\Phi_X(x,t),$$

so

$$(1 - xT_1)^7\,\Phi_Z(x,t) = (1 - T_1)^7\,\Phi_X(x,t),$$

hence

$$\sum_{j=0}^{7}\binom{7}{j}(-x)^j\Phi_Z(x,t-j) = \sum_{j=0}^{7}\binom{7}{j}(-1)^j\Phi_X(x,t-j).$$

When $t = 0$, the Hodge numbers of X are diagonal ($h^{p,q}(X) = 0$ if $p \neq q$) and $f_\lambda(0) = 1$ for all λ, so

$$\Phi_X(x,0) = \sum_{\lambda \subset (5)^2} x^{|\lambda|}$$

$$= 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 3x^6 + 2x^7 + 2x^8 + x^9 + x^{10}.$$

(i.e. the Gaussian binomial coefficients $\binom{7}{2}_q$ with $q = x$). For general t we compute in Sage and find

$$\Phi_X(x,t) = 1/86400x^{10}t^{10} - 7/17280x^{10}t^9 - 1/8640x^9t^{10} + \dots.$$

Summing greatly simplifies this:

$$\sum_{j=0}^{7}\binom{7}{j}(-1)^j\Phi_X(x,t-j)$$
$$= 7(x-1)^7(xt - 7x - t)(x^2t^2 - 14x^2t - 2xt^2 + 50x^2 + 14xt + t^2 - 11x + 1).$$

When $t = 0$ one extracts

$$\Phi_Z(x, 0) = -49x - 49x^2,$$

thus Z has primitive Hodge numbers $(1, 50, 50, 1)$ by the Lefschetz hyperplane theorem.

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 50 & & 50 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Figure 3.5. Hodge diamond for $V(1^7) \subset \mathrm{Gr}(2, 7)$, a Calabi–Yau threefold.

Implementation of Grassmannian Smooth CI Hodge Number Calculator

In *Hodge.m2*, we produce a self-contained implementation of the construction described in the previous section in Macaulay2. Below are some sample calculations.

Example 3.75 (Quintic threefold in $\mathbb{P}^4$)**.**

```
hodgeDiamondPnCI(4, {5});
```

```
        1
      0   0
    0   1   0
  1  101 101  1
    0   1   0
      0   0
        1
```

Example 3.76 (Cubic fourfold in $\mathbb{P}^5$)**.**

```
hodgeDiamondPnCI(5, {3});
```

```
          1
        0   0
      0   1   0
    0   0   0   0
  0   1  21   1   0
    0   0   0   0
      0   1   0
        0   0
          1
```

Example 3.77 ($(2,2,2)$ CI in $\mathrm{Gr}(2,5)$)**.**

```
hodgeDiamondCI(2, 5, {2,2,2});
```

```
          1
        0    0
      0    1    0
   10  260  260   10
      0    1    0
        0    0
          1
```

Hodge.m2 also has a self-contained implementation of Theorem 3.29 to compute twisted Hodge diamonds of $\mathrm{Gr}(k,n)$:

Example 3.78 ($h^{p,q}(\mathrm{Gr}(2,5),\mathcal{O}(3))$)**.**

```
twistedHodgeNumbers(2, 5, 3);
```

```
              0
            0   0
          0   0   0
        0   0   0   0
      0   0   0   0   0
    0   0   0   0   0   0
  0   0   0   0   0   0   0
    0   5   0   0   0   0
      0   0   0   0   0
        0   0   0   0
       120  0   0
         280   0
           175
```

Primitive Cohomology of the Grassmannian

Much of this thesis concerns the primitive cohomology of hypersurfaces, but we are also interested in the primitive cohomology of the Grassmannian $X = \mathrm{Gr}(k, n)$ defined as

$$H^N_{\mathrm{prim}}(X, \mathbb{C}) := \ker(\ell_* : H^N(X, \mathbb{C}) \to H^{N+2}(X, \mathbb{C})),$$

where ℓ_* is the morphism given by wedging with the hyperplane class (cf. Equation (2.2.1)). By the Hard Lefschetz theorem, we have a Lefschetz decomposition

$$H^N(X, \mathbb{C}) = H^N_{\mathrm{prim}}(X, \mathbb{C}) \oplus \ell_* H^{N-2}(X, \mathbb{C}),$$

compatible with the Hodge structure. The primitive cohomology can be computed as

$$\dim H^N_{\mathrm{prim}}(X, \mathbb{C}) = \dim H^N(X, \mathbb{C}) - \dim H^{N-2}(X, \mathbb{C}).$$

In Example 3.35, we computed the Hodge numbers of $X = \mathrm{Gr}(k, n)$, which gives the generating function

$$\sum h^{p,p}(X) q^p = \binom{n}{k}_q = \prod_{j=1}^{k} \frac{1 - q^{n-k+j}}{1 - q^j},$$

(where $h^{p,q}(X) = 0$ for $p \neq q$). Thus, we have

$$\dim H^N_{\mathrm{prim}}(X, \mathbb{C}) = [q^{N/2}] \left(\binom{n}{k}_q - q \binom{n}{k}_q \right), \tag{3.8.2}$$

where $[q^{N/2}](f(q))$ denotes the coefficient of $q^{N/2}$ in the polynomial $f(q)$. We note that this implies that $\dim H^N_{\mathrm{prim}}(\mathrm{Gr}(k, n), \mathbb{C})$ also equals the dimension of the vector space of invariant polynomial functions of degree k on the space of binary forms of degree $n - k$ (i.e. $\dim \mathrm{Sym}^k \mathrm{Sym}^{n-k} \mathbb{C}^2$) via the Cayley–Sylvester formula [Dol03, §5.5]. The fact that this is also equal to $\dim H^N_{\mathrm{prim}}(\mathrm{Gr}(n - k, n), \mathbb{C})$ is an example of Hermite reciprocity.

Remark About Grassmannian Cohomology Vanishings

In this section, we show that the strong vanishing assumptions of Voisin in [Voi03, §6.1.2] rarely hold in the case of a Fano hypersurface in a Grassmannian, cf. condition 3.

Proposition 3.79. *Let $X = \mathrm{Gr}(k, n)$ with $2 \leq k \leq n - 2$ and let $Z \hookrightarrow X$ be a smooth ample hypersurface. Then*

$$H^i(\Omega^j_X(pZ)) = 0 \text{ for all } p > 0,\ i > 0,\ j \geq 0 \text{ if and only if } \deg Z \geq n - 1$$

Remark 3.80. If $k = 1$ or $k = n-1$ then this is a standard calculation of the cohomology of line bundles on $\mathbb{P}^{n-1}$: for $t > 0$, the only t-core partitions lying in a box of size $1 \times (n-1)$ have $i = 0$, hence the cohomology of $\Omega_{\mathbb{P}^{n-1}}(pZ)$ is concentrated in degree $i = 0$.

Proof. With the notation as in Proposition 3.79, assume $\deg Z = d > 0$. If $d \geq n-1$, then $H^i(\Omega^j(d)) = 0$ for all $i > 0$ by [Sno86, Theorem 3.2] (in short: the longest hook of a $(k, n-k)$-bounded partition is $n-1$).

If $d = 1$, let $p = 2$, otherwise take $p = 1$. Let $t = pd$. If $t \leq n-2$, let $\lambda \subset (n-k)^k$ be a hook partition with length $t+1$, i.e. λ is contained only in the first row and column of the bounding box, and $\lambda_1 > 1$, $\lambda_1^\mathsf{T} > 0$. Then $\iota_\lambda(t) = (1)$, hence $h^i(\Omega^j(t)) > 0$, so the vanishing assumption fails. □

Remark 3.81. See the erratum for [DV10] for an example of how this stronger vanishing assumption fails for hyperplane sections of $\mathrm{Gr}(3, 10)$.

Conjectured Strengthening of Known Bounds for Snow Partitions

We conclude this chapter with a collection of conjectured inequalities which sharpen those of Section 3.4. They have been verified for all $145,865,312,345$ Snow partitions with $k, n, j, t, i \leq 70$ by exhaustive computation, using the Sage script *SnowBoundConjectures.sage*.

In each conjecture below, let λ be a Snow partition with parameters k, n, j, t, i with $i, t > 0$. (The restriction $i > 0$ is needed for Conjecture 3, where the term $\sqrt{4i-3}$ would otherwise be undefined; the other conjectures hold trivially when $i = 0$). Then we have the following conjectured inequalities:

1. $$j - i \leq t + \lfloor (n-4)^2/4 \rfloor.$$

2. $$(j-i)^2 \geq (t-1)(j+i) - \lfloor (t-1)^2/4 \rfloor.$$

3. $$t \leq n - 1 - \lfloor \sqrt{4i-3} \rfloor.$$

4. $$i \leq \frac{k(n-k)(n-t)}{2n}.$$

5. $$j \leq \frac{k(n-k)(n+t)}{2n}.$$

6.
$$i \leq \frac{k-1}{k+1} j + \frac{k(k+1)(t-1)}{2} - j.$$

Remark 3.82. Each of the above conjectures is sharp: equality is achieved by many Snow partitions. For example, the staircase partition $\lambda = (2, 1)$ with parameters $(k, n, j, t, i) = (2, 4, 3, 2, 1)$ simultaneously saturates all six bounds.

CHAPTER 4

EXAMPLES

In this chapter we present a few calculations of Griffiths rings for hypersurfaces from Theorem 1.1. For the purpose of illustration, we also give a few calculations where the cohomological vanishings (Assumptions 2.9 and 2.10) fail.

In Section 4.1, we give an example of a Griffiths ring computation which contains a residual contribution from $H^N_{\text{prim}}(\text{Gr}(k,n),\mathbb{C})$, which slightly increases the dimension of a single graded piece of the Griffiths ring compared to the Hodge numbers of the underlying hypersurface. In Section 4.2, we give a straightforward calculation of the Griffiths ring where all vanishings occur, with no primitive contribution from the ambient Grassmannian. This contrasts Section 4.3, in which we give an example of a Griffiths ring calculation where Assumption 2.10 is violated, which also increases the dimension of a single graded piece of the Griffiths ring. Other non-examples are tabulated in Section 4.4, among which we see that the violation of Assumption 2.9 can alter multiple graded pieces of the Griffiths ring.

In Section 4.5, we conclude with a discussion of the Debarre–Voisin construction of hyper-Kähler fourfolds and associated hyperplane sections of $\text{Gr}(3,10)$, and compute the Picard rank of a few examples using the Griffiths ring. We note that in [DV10], Debarre and Voisin also use Griffiths theory for $\text{Gr}(3,10)$ in their work on the cohomology of a smooth hyperplane section.

All code from this section is available at the author's GitHub repository [Huc26].

4.1 An example of a quadric in $\text{Gr}(2,6)$ with nontrivial primitive Grassmannian cohomology

In this section, we take a smooth quadric f in $X = \text{Gr}(2,6)$ and compute the Griffiths ring of the associated hypersurface Z, so $k = 2$, $n = 6$, and $d = 2$. Because $\gcd(k, n/d) = 1$, Assumption 2.9 is satisfied, and because n is even, Assumption 2.10 is satisfied. The Hodge diamond of X is given in Figure 4.1. Noting that $h^7(X) = 0$, we conclude $H^{N-1}_{\text{prim}}(X) = 0$ where $N := \dim X = 8$. On the other hand, $h^4(X) = 3$ and $h^6(X) = 2$, so $H^N_{\text{prim}}(X) \cong \mathbb{C}$.

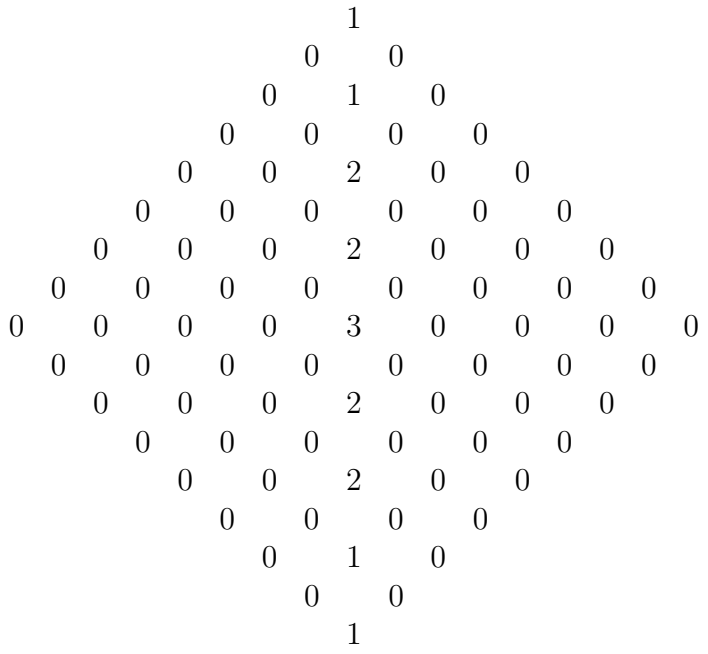


Figure 4.1. The Hodge diamond for $\mathrm{Gr}(2,6)$. Observe that $h^8_{\mathrm{prim}}(\mathrm{Gr}(2,6)) = 1$.

Therefore Theorem 1.1, in conjunction with Equation (2.2.3), gives the following identification:

$$(R/J_f)_{pd-n} \cong H^{N-p,p-1}_{\mathrm{prim}}(Z) \oplus \delta_{p,5}\mathbb{C}.$$

Explicitly, we may take

$$\begin{aligned} f = &-44p_{01}^2 - 43p_{02}^2 + 29p_{12}^2 + 27p_{03}^2 + 13p_{13}^2 + 5p_{23}^2 - 15p_{04}^2 - 41p_{14}^2 \\ &- 25p_{24}^2 - 43p_{34}^2 - 10p_{05}^2 + 6p_{15}^2 + 35p_{25}^2 - 8p_{35}^2 - 19p_{45}^2. \end{aligned}$$

We verify that f is smooth over $\mathbb{F}_{97}$ in *SmoothCheckQuadricGr26.M2*; by Proposition 4.2, f is also smooth over $\mathbb{C}$. We compute the Griffiths ring of f in *GriffithsRingQuadricGr26.M2*, obtaining the following basis for (R/J_f):

$$\begin{cases} 1, \\ p_{01}p_{24}, p_{01}p_{34}, p_{01}p_{25}, p_{01}p_{35}, p_{01}p_{45}, p_{02}p_{24}, \ldots, \\ p_{01}p_{24}p_{45}^2, p_{01}p_{34}p_{45}^2, p_{01}p_{25}p_{45}^2, p_{01}p_{35}p_{45}^2, \ldots, p_{35}^3p_{45}, \\ p_{45}^6. \end{cases}$$

We point out that every element in the basis for $(R/J_f)_4$ is p_{45}^2 times an element in the basis for $(R/J_f)_2$ *except for* $p_{35}^3p_{45}$, due to the contribution of $H^{4,4}_{\mathrm{prim}}(\mathrm{Gr}(2,6)) \cong$

$\mathbb{C}$. The Griffiths ring has dimensions

$$\dim(R/J_f)_{pd-n} = \begin{cases} 1 & p = 3, \\ 69 & p = 4, \\ 70 & p = 5, \\ 1 & p = 6, \\ 0 & \text{else}, \end{cases}$$

and we can see the contribution of $H^{4,4}_{\text{prim}}\text{Gr}(2,6) \cong \mathbb{C}$ to $\dim(R/J_f)_4$ in the middle row of Figure 4.2:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 0 & & 1 & & 0 & & & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 2 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
0 & & 0 & & 1 & & 69 & & 69 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 2 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 0 & & 2 & & 0 & & & \\
 & & & & 0 & & 0 & & 0 & & & & \\
 & & & & & 0 & & 1 & & & & & \\
 & & & & & & 0 & & & & & & \\
 & & & & & & 1 & & & & & &
\end{array}$$

Figure 4.2. The Hodge diamond for a smooth quadric $Z \subset \text{Gr}(2,6)$. Observe that $h^{3,4}(Z) = 69$, unlike $\dim(R/J_f)_4$.

4.2 Example: a quadric in $\text{Gr}(2,5)$

To contrast with the following section in which Assumption 2.10 fails, we consider the hypersurface Z defined by a generic quadric f in $X = \text{Gr}(2,5)$. $2 \nmid 5$ so Assumption 2.9 holds, and $2 \nmid ((5+1)/2)$ so Assumption 2.10 holds. The Hodge diamond of X is given in Figure 4.3.

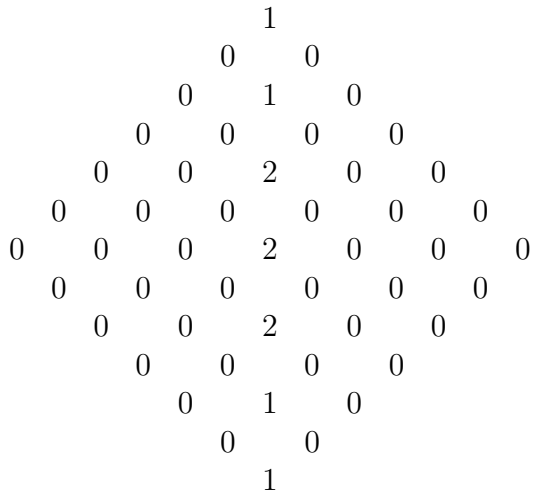


Figure 4.3. The Hodge diamond for $\mathrm{Gr}(2,5)$.

Let $N := \dim X = 6$. We observe that $h^N_{\mathrm{prim}}(X) = h^{N-1}_{\mathrm{prim}}(X) = 0$, so by Theorem 1.1 and Equation (2.2.3) we have the following identification:

$$(R/J_f)_{pd-n} \cong H^{N-p,p-1}_{\mathrm{prim}}(Z).$$

In *SmoothCheckQuadricGr25.M2* we verify that the following polynomial is smooth:

$$f = 12p_{01}^2 + 17p_{02}^2 + 44p_{12}^2 + 50p_{03}^2 + 56p_{13}^2 + 40p_{23}^2 + 71p_{04}^2 + 48p_{14}^2 + 79p_{24}^2 + 19p_{34}^2.$$

We compute a basis for the Griffiths ring of f in *GriffithsRingQuadricGr25.M2*, obtaining the following basis for (R/J_f):

$$\begin{cases} p_{01}, p_{02}, p_{12}, p_{03}, p_{13}, p_{23}, p_{04}, p_{14}, p_{24}, p_{34}, \\ p_{01}p_{34}^2, p_{02}p_{34}^2, p_{12}p_{34}^2, p_{03}p_{34}^2, p_{13}p_{34}^2, p_{23}p_{34}^2, p_{04}p_{34}^2, p_{14}p_{34}^2, p_{24}p_{34}^2, p_{34}^3. \end{cases}$$

We point out that every element in the basis for $(R/J_f)_3$ is p_{34}^2 times an element in the basis for $(R/J_f)_1$.

The Griffiths ring has dimensions

$$\dim(R/J_f)_{pd-n} = \begin{cases} 10 & p = 3, \\ 10 & p = 4, \\ 0 & \text{else}, \end{cases}$$

which agrees with the middle row of the Hodge diamond of Z (see Figure 4.4).

$$
\begin{array}{ccccccccccc}
 & & & & 1 & & & & & & \\
 & & & 0 & & 0 & & & & & \\
 & & 0 & & 1 & & 0 & & & & \\
 & 0 & & 0 & & 0 & & 0 & & & \\
0 & & 0 & & 2 & & 0 & & 0 & & \\
 & 0 & & 10 & & 10 & & 0 & & 0 & \\
0 & & 0 & & 2 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & & & \\
 & & & 0 & & 0 & & & & & \\
 & & & & 1 & & & & & &
\end{array}
$$

Figure 4.4. The Hodge diamond for a smooth quadric in $\mathrm{Gr}(2,5)$

4.3 Non-example: a cubic in $\mathrm{Gr}(2,5)$

Now we repeat the calculation in the previous section except with a cubic instead of a quadric, i.e. $k=2$, $n=5$, and $d=3$. Assumption 2.9 is still satisfied, but because $2d|(n+1)$, Assumption 2.10 does not hold.

In *SmoothCheckCubicGr25.M2* we verify that the following polynomial is smooth:

$$f = -9p_{0,1}^3 - 34p_{0,2}^3 - 47p_{1,2}^3 + 25p_{0,3}^3 - 37p_{1,3}^3 + 50p_{2,3}^3 - 9p_{0,4}^3 + 12p_{1,4}^3 + 16p_{2,4}^3 - 36p_{3,4}^3.$$

Computing the Griffiths ring of f in *GriffithsRingCubicGr25.M2* we find

$$\dim(R/J_f)_{pd-n} = \begin{cases} 10 & p=2, \\ 305 & p=3, \\ 300 & p=4, \\ 10 & p=5, \\ 0 & \text{else.} \end{cases}$$

Now, we may recover the correct Hodge numbers of Z using the fact that the values for $p \geq 4$ are correct and then applying Hodge symmetry. However, we may also look at Example 3.45 to see that $H^1(X, \Omega_X^4(Z)) \cong \mathbb{C}^5$. Comparing this to the middle row of the Hodge diamond of Z in Figure 4.5

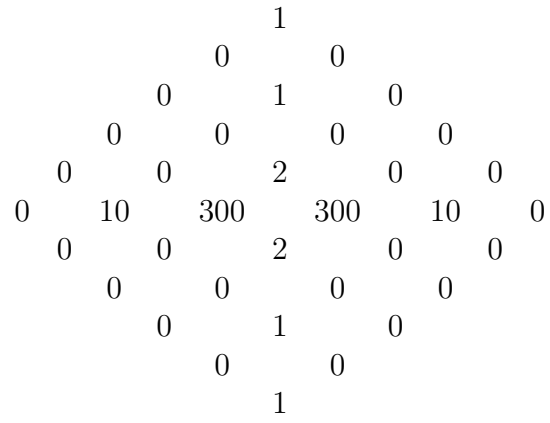


Figure 4.5. The Hodge diamond for a smooth cubic in $\mathrm{Gr}(2,5)$

we see the additional contribution to $\dim(R/J_f)_4$ coming from $H^1(X, \Omega^4_X(Z))$, the term responsible for the failure of Theorem 3.72.

4.4 Non-examples: hypersurfaces of degree d in $\mathrm{Gr}(k,n)$ with $d|n|kd$, etc.

We catalogue a few examples of when Assumptions 2.9 and 2.10 fail, and compute the dimensions of graded pieces of the Griffiths rings as well as actual Hodge numbers. The program to compute these tables is *NonExamples.m2*.

k	n	d	$h^{N-p,p-1}_{\text{prim}}(Z)$								
			$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$	$p=9$	$p=10$
2	4	2	2	2							
3	4	3	6								
2	5	1				(0)					
2	5	3	10	300	300	10					
2	6	3	1	455	5004	5004	455	1			
3	6	1				1					
3	6	2		1	140	601	140	1			
5	6	5	120	580	120						
2	7	2			21	441	441	21			
4	8	1							3	3	
3	9	1								2	2

Table 4.1. Dimensions of the primitive cohomology for a generic hypersurface of degree d in $\mathrm{Gr}(k,n)$.

k	n	d	$\dim(R/J_f)_{pd-n}$									issue
			$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$	$p=9$	$p=10$	
2	4	2	1	4*								$\gcd(k,n/d)=2$
3	4	3	10									$k=n-1$
2	5	1				1						$k=2,\ 2d\|(n+1)$
2	5	3	10	305	300	10						$k=2,\ 2d\|(n+1)$
2	6	3	1	454	5005	5005*	455	1				$\gcd(k,n/d)=2$
3	6	1					1					$\gcd(k,n/d)=3$
3	6	2		1	139	602	140	1				$\gcd(k,n/d)=3$
5	6	5	126	580	120							$k=n-1$
2	7	2			21	448	441	21				$k=2,\ 2d\|(n+1)$
4	8	1							1	6*		$\gcd(k,n/d)=4$
3	9	1								1	3	$\gcd(k,n/d)=3$

Table 4.2. Dimensions of graded pieces of the Griffiths ring for a generic hypersurface of degree d in $\mathrm{Gr}(k,n)$, demonstrating the effects of the failure of the assumptions of Theorem 1.1, cf. Table 4.1. Here a star indicates nontrivial contribution to $\dim(R/J_f)_{pd-n}$ from $H^{N-p+1,p-1}_{\mathrm{prim}}\mathrm{Gr}(k,n)$.

4.5 Smooth Hyperplane Sections of $\mathrm{Gr}(3,10)$

In [DV10] Debarre and Voisin produce a 20-dimensional family of hyper-Kähler fourfolds and prove that they are deformation equivalent to the Hilbert scheme of two points on a K3 surface by considering hyperplane sections of particular Grassmannians (construction below). In [Son23] Song constructs an associated hyper-Kähler fourfold with automorphism group $\mathrm{PSL}(2,\mathbb{F}_{11})$ and shows that it has maximal Picard rank using a detailed lattice-theoretic calculation. We give an alternative Griffiths ring computation, and construct more examples.

The Debarre–Voisin Construction

We follow [DV10, §2]. Let V_{10} be a complex vector space of dimension 10 and let σ be a generic element of $\bigwedge^3 V_{10}^\vee$. The 3-form σ determines a Plücker hyperplane section

$$Z_\sigma \subset \mathrm{Gr}(3,10) \subset \mathbb{P}(\bigwedge^3 V_{10}),$$

consisting of all 3-dimensional subspaces of V_{10} on which σ vanishes. Similarly, we may define another subvariety

$$Y_\sigma \subset \mathrm{Gr}(6,10),$$

defined as the set of 6-dimensional subspaces of V_{10} on which σ vanishes identically. One shows that Y_σ is a smooth fourfold with trivial canonical bundle. Moreover,

there is a natural correspondence between Z_σ and Y_σ: each point of Y_σ determines a 6-dimensional subspace W_6 on which σ vanishes identically, hence an inclusion $\mathrm{Gr}(3, W_6) \subset Z_\sigma$. Thus we may construct the variety

$$G_\sigma = \{([W_3], [W_6]) \in \mathrm{Gr}(3, V_{10}) \times \mathrm{Gr}(6, V_{10}) | W_3 \subset W_6,\, \sigma|_{W_6} = 0\}\,,$$

with two projections

$$Y_\sigma \xleftarrow{q} G_\sigma \xrightarrow{p} Z_\sigma.$$

The fibers of q are 9-dimensional Grassmannians $\mathrm{Gr}(3, W_6)$, and there is an induced cohomological correspondence

$$q_* p^* : H^{20}(Z_\sigma, \mathbb{Q}) \to H^2(Y_\sigma, \mathbb{Q}), \tag{4.5.1}$$

whose restriction on vanishing cohomology is denoted $(q_* p^*)_{\mathrm{van}}$. In particular, Debarre and Voisin show that $(q_* p^*)_{\mathrm{van}}$ is an injective morphism of Hodge structures [DV10, Theorem 2.2], and Y_σ is an irreducible hyper-Kähler fourfold with $b_2 = 23$. Moreover, Benedetti and Song show that $(q_* p^*)_{\mathrm{van}}$ is an isomorphism of polarized Hodge structures [BS21, Theorem 2.6] if one accounts for the Tate twist by (-1) on $H^2(Y_\sigma, \mathbb{Q})$.

Benedetti and Song also define the Peskine variety, given by

$$X_\sigma := \{[V_1] \in \mathrm{P}(V_{10}) : \mathrm{rank}([V_1] \lrcorner \sigma|_{V_{10}}) \leq 6\}.$$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Figure 4.6. Hodge diamond for a K3 surface.

$$
\begin{array}{ccccccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & & & \\
 & & & & & & & 0 & & 0 & & & & & & & & & \\
 & & & & & & 1 & & 21 & & 1 & & & & & & & & \\
 & & & & & 0 & & 0 & & 0 & & 0 & & & & & & & \\
 & & & & 1 & & 21 & & 232 & & 21 & & 1 & & & & & & \\
 & & & & & 0 & & 0 & & 0 & & 0 & & & & & & & \\
 & & & & & & 1 & & 21 & & 1 & & & & & & & & \\
 & & & & & & & 0 & & 0 & & & & & & & & & \\
 & & & & & & & & 1 & & & & & & & & & &
\end{array}
$$

Figure 4.7. Hodge diamond for a hyper-Kähler fourfold of $K3^{[2]}$ type. Note that $h^{1,1}_{\text{prim}} = 20$.

$$
\begin{array}{ccccccccccccccccccccc}
 & & & & & & & & h^{12,12}_{\text{prim}} & & & & & & & & & & \\
 & & & & & & & 0 & & 0 & & & & & & & & & \\
 & & & & & & 0 & & h^{11,11}_{\text{prim}} & & 0 & & & & & & & & \\
 & & & & & 0 & & 0 & & 0 & & 0 & & & & & & & \\
 & & & & 0 & & h^{11,9}_{\text{prim}} & & h^{10,10}_{\text{prim}} & & h^{9,11}_{\text{prim}} & & 0 & & & & & & \\
 & & & & & 0 & & 0 & & 0 & & 0 & & & & & & & \\
 & & & & & & 0 & & h^{9,9}_{\text{prim}} & & 0 & & & & & & & & \\
 & & & & & & & 0 & & 0 & & & & & & & & & \\
 & & & & & & & & h^{8,8}_{\text{prim}} & & & & & & & & & &
\end{array}
\quad = \quad
\begin{array}{ccccccccccccccccccccc}
 & & & & & & & & * & & & & & & & & & & \\
 & & & & & & & 0 & & 0 & & & & & & & & & \\
 & & & & & & 0 & & * & & 0 & & & & & & & & \\
 & & & & & 0 & & 0 & & 0 & & 0 & & & & & & & \\
 & & & & 0 & & 1 & & 20 & & 1 & & 0 & & & & & & \\
 & & & & & 0 & & 0 & & 0 & & 0 & & & & & & & \\
 & & & & & & 0 & & * & & 0 & & & & & & & & \\
 & & & & & & & 0 & & 0 & & & & & & & & & \\
 & & & & & & & & * & & & & & & & & & &
\end{array}
$$

Figure 4.8. We omit the full Hodge diamond of a smooth hyperplane section Z of $\mathrm{Gr}(3, 10)$ as it is very large, but it suffices to consider the primitive part of $H^{20}(Z, \mathbb{C})$. Observe that $H^{19}_{\text{prim}}(Z, \mathbb{C}) = 0$.

Constructing Debarre–Voisin Fourfolds with Maximal Picard Rank

It is natural to ask which hyper-Kähler fourfolds arising in the Debarre–Voisin construction (if any) have maximal Picard rank. Song produced one such example. Because the map $(q_* p^*)_{\text{van}}$ defined in Equation (4.5.1) is an isomorphism of Hodge structures, we say that Z_σ has *higher primitive Picard rank* r if the associated hyper-Kähler fourfold Y_σ has primitive Picard rank r; that is, if $H^{1,1}_{\text{prim}}(Y_\sigma) \cap H^2(Y_\sigma, \mathbb{Z}) \cong \mathbb{Z}^r$. Note that $H^1(Y_\sigma, \mathbb{Z}) = 0$, as shown in Figure 4.7, hence by hyper-Kähler simply connectedness there is no torsion part of $H^2(Y_\sigma, \mathbb{Z})$.

We will take an approach for Debarre–Voisin fourfolds similar to the one taken by Beauville [Bea14, §8] for cubic fourfolds: compute the Griffiths ring of a hyperplane section with automorphism group G, decompose the primitive cohomology

into irreducible representations, and deduce that it has maximal higher primitive Picard rank.

$$H^2(Y_\sigma, \mathbb{C}) \cong H^{2,0}(Y_\sigma) \oplus H^{1,1}(Y_\sigma) \oplus H^{0,2}(Y_\sigma),$$

and in rational cohomology we have the decomposition

$$H^2(Y_\sigma, \mathbb{Q}) \cong \mathbb{Q} \oplus H^2_{\text{prim}}(Y_\sigma, \mathbb{Q}).$$

The factor of $\mathbb{Q}$ is the Plücker class, which has type $(1,1)$ by the Lefschetz theorem on $(1,1)$ classes.

Suppose that Z_σ has an automorphism group containing G, which induces a representation of G on $H^{20}_{\text{van}}(Z_\sigma, \mathbb{Q})$. We would like to decompose $H^{20}_{\text{van}}(Z_\sigma, \mathbb{Q})$ into irreducible representations. In particular, if $H^{9,11}(Z_\sigma)$ and $H^{11,9}(Z_\sigma)$ are one-dimensional and $H^{10,10}_{\text{van}}(Z_\sigma)$ has no one-dimensional summands and is composed of representations realizable over $\mathbb{Q}$, then the splitting

$$H^{20}_{\text{van}}(Z_\sigma, \mathbb{C}) \cong H^{11,9}_{\text{van}}(Z_\sigma, \mathbb{C}) \oplus H^{10,10}_{\text{van}}(Z_\sigma, \mathbb{C}) \oplus H^{9,11}_{\text{van}}(Z_\sigma, \mathbb{C})$$

is actually defined over $\mathbb{Q}$, hence $H^{10,10}_{\text{van}}(Z_\sigma, \mathbb{Q}) \cong \mathbb{Q}^{20}$. That is, this suffices to show that Z_σ has maximal higher primitive Picard rank.

On the other hand, hyper-Kähler fourfolds of maximal Picard rank are isolated (see e.g. [Bea83, Theorem 5]), which suggests that we should consider 10-dimensional representations V of a group G such that $\bigwedge^3 V$ has exactly one trivial summand as candidates for automorphism groups of Z_σ.

Mongardi ([Mon13] and [DM22]) constructs a special Eisenbud–Popescu–Walter sextic with a faithful $\mathrm{PSL}(2, \mathbb{F}_{11})$-action, from which one constructs a smooth hyper-Kähler fourfold with a large symmetry group. Along these lines, Song [Son23] considers the group $\mathrm{PSL}(2, \mathbb{F}_{11})$ in the context of Debarre–Voisin fourfolds, producing an explicit Debarre–Voisin fourfold with automorphism group $\mathrm{PSL}(2, \mathbb{F}_{11})$ and maximal Picard rank. We observe that A_7, along with $\mathrm{PSL}(2, \mathbb{F}_{11})$, has exactly one trivial summand in $\bigwedge^3 V_{10}$, for V_{10} a 10-dimensional representation. We explicitly construct a hyperplane section of $\mathrm{Gr}(3, 10)$ with automorphism group containing A_7, and show that the associated Debarre–Voisin fourfold has maximal Picard rank.

Song's Construction for $\mathrm{PSL}(2, \mathbb{F}_{11})$

In [Son23], Song considers the action of $G = \mathrm{PSL}(2, \mathbb{F}_{11})$ on $X = \mathrm{Gr}(3, 10)$, and shows that the space of G-invariant linear forms on X is one-dimensional and generated by the following linear form:

$$\sigma = p_{025} + p_{136} + p_{247} - p_{038} - p_{149} - p_{079} + p_{158} + p_{269} - p_{357} - p_{468}.$$

Conj. Class	(1)	$b^{-1}aba^{-1}b^{-1}a^{-2}$	$b^{-1}a^{-1}$	$aba^{-1}b^{-1}a^{-2}b$	b^{-1}	$a^{-2}b^{-1}a^{-2}$	a	a^2
Size	1	60	60	55	110	110	132	132
$\mathbb{1}$	1	1	1	1	1	1	1	1
V_5	5	$\frac{-1+\sqrt{-11}}{2}$	$\frac{-1-\sqrt{-11}}{2}$	1	-1	1	0	0
$V_5^\vee$	5	$\frac{-1-\sqrt{-11}}{2}$	$\frac{-1+\sqrt{-11}}{2}$	1	-1	1	0	0
V_{10}	10	-1	-1	-2	1	1	0	0
V'_{10}	10	-1	-1	2	1	-1	0	0
V_{11}	11	0	0	-1	-1	-1	1	1
V_{12}	12	1	1	0	0	0	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$
V'_{12}	12	1	1	0	0	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$

Table 4.3. Character table of $\mathrm{PSL}(2,\mathbb{F}_{11})$. Representatives of conjugacy classes are given in terms of a, b as in Section 4.5.

He does this by first considering the 10-dimensional irreducible representation V_{10} of G (see Table 4.3) and finding that the decomposition of $\bigwedge^3 V_{10}$ has precisely one trivial summand.

$$\bigwedge^3 V_{10} \cong \mathbb{1} \oplus V_5 \oplus V_5^\vee \oplus V_{10} \oplus V'_{10} \oplus V_{11}^4 \oplus V_{12}^2 \oplus V_{12}'^2.$$

Note that the third exterior power of the other 10-dimensional irreducible representation V'_{10} does not have a trivial summand:

$$\bigwedge^3 V'_{10} \cong V_{10}^4 \oplus V'_{10} \oplus V_{11}^2 \oplus V_{12}^2 \oplus V_{12}'^2.$$

Let Z_σ, Y_σ, and X_σ denote the Plücker hyperplane section, Debarre–Voisin fourfold, and Peskine variety as defined above.

Song shows that X_σ is not smooth and finds the 55 singular points, showing that G acts transitively on them. He produces a map to associate to each singular point of X_σ a divisor of the Debarre–Voisin fourfold Y_σ. Taking 21 such points, he computes the Beauville–Bogomolov pairing of these divisors, and concludes that it has rank 21 because the discriminant is square-free and the Picard lattice is a primitive sublattice of $\Lambda := H^2(Y_\sigma, \mathbb{Z})$. He concludes that the primitive cohomology decomposes as

$$H^2_{\mathrm{prim}}(Y_\sigma, \mathbb{Q}) \cong V_{10}'^{\oplus 2},$$

and therefore Y_σ has maximal primitive Picard rank 20, hence by the argument of Debarre and Voisin Z_σ has maximal higher primitive Picard rank.

Our Construction for $\mathrm{PSL}(2,\mathbb{F}_{11})$

This section reproves Song's results via Griffiths rings, requiring no lattice computations.

The group $\mathrm{PSL}(2,\mathbb{F}_{11})$ (in ATLAS, $\mathrm{L}_2(11)$) is given by the quotient of $\mathrm{SL}_2(\mathbb{F}_{11})$ by its center, where $\mathrm{SL}_2(\mathbb{F}_{11})$ is the space of 2×2 matrices over $\mathbb{F}_{11}$ with unit determinant. It is generated by two elements,

$$a = \begin{pmatrix} 0 & 6 \\ -2 & 8 \end{pmatrix}, \quad b = \begin{pmatrix} 6 & 4 \\ 3 & 4 \end{pmatrix}$$

modulo $\pm I$. Alternatively, we may describe $\mathrm{PSL}(2,\mathbb{F}_{11})$ by its permutation description:

$$\mathrm{PSL}(2,\mathbb{F}_{11}) = \langle a, b\rangle \text{ where } a = ((3,11,9,7,5)(4,12,10,8,6))$$
$$\text{and } b = ((1,2,8)(3,7,9)(4,10,5)(6,12,11)).$$

This explicitly shows that $a^5 = b^3 = 1$. A direct computation also gives

$$a^2ba^{-1}b^{-1}a^{-1}ba^2b^{-1} = (a(ab)^2)^2 = 1.$$

The character table of $\mathrm{PSL}(2,\mathbb{F}_{11})$ is in Table 4.3. There are two irreducible representations of $\mathrm{PSL}(2,\mathbb{F}_{11})$ of dimension 10; we consider the representation V_{10}, with explicit matrices given in Figure 4.9. Unlike V'_{10}, V_{10} has the desired property that $\bigwedge^3 V_{10}$ contains only a single one-dimensional irreducible summand.

$$a \mapsto A = \begin{pmatrix} -1 & 2 & 0 & 1 & 0 & -1 & 2 & -1 & 1 & -1 \\ -1 & 1 & 1 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & -1 & 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$b \mapsto B = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 & -2 & 2 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & -1 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 1 & -1 & 2 & -1 & 1 & 1 \\ -1 & 2 & 0 & 1 & 0 & -2 & 2 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Figure 4.9. Matrices for the representation V_{10} of $\mathrm{PSL}(2,\mathbb{F}_{11})$ on generators.

One readily verifies that $A^5 = B^3 = 1$. This induces an action on $\bigwedge^3 V_{10}$ in the usual way. Identifying $\bigwedge^3 V_{10}$ with $H^0(\mathcal{O}_X(1))$ where $X = \mathrm{Gr}(3,10)$, we obtain an action of $\mathrm{PSL}(2,\mathbb{F}_{11})$ on X and a 120-dimensional representation on $H^0(X,\mathcal{O}_X(1)) \cong \bigwedge^3 V_{10}$. In *PSL211Intersection.M2* we compute

$$\ker\left(\left(\wedge^3\rho_{V_{10}}(a)-1\right)\cap\left(\wedge^3\rho_{V_{10}}(b)-1\right)\right),$$

finding that there exists a one-dimensional space of $\mathrm{PSL}(2,\mathbb{F}_{11})$-invariant forms on X spanned by the following form:

$$\begin{aligned} f = {} & p_{0,1,2} - p_{0,1,4} + p_{0,1,5} + p_{0,1,8} - p_{0,2,3} - p_{0,2,4} + p_{0,2,6} \\ & - p_{0,3,9} + p_{0,4,5} - p_{0,4,6} - p_{0,4,7} + p_{0,5,6} + p_{0,5,7} + p_{0,5,8} \\ & + p_{0,6,7} - p_{0,7,9} - p_{0,8,9} - p_{1,2,5} + p_{1,2,7} + p_{1,2,8} + p_{1,3,4} \\ & - p_{1,3,5} + p_{1,3,7} - p_{1,3,9} + p_{1,5,7} + p_{1,5,8} + p_{1,8,9} + p_{2,3,4} \\ & - p_{2,3,8} + p_{2,4,5} - p_{2,4,7} + p_{2,5,6} + p_{2,6,7} + p_{2,6,8} + p_{3,4,5} \\ & - p_{3,5,7} + p_{5,7,9} + p_{5,8,9} - p_{6,8,9} - p_{7,8,9}. \end{aligned}$$

We check in *SmoothCheckPSL211.M2* that the zero locus of f is smooth. This shows that Z_f is smooth, and by [BS21, Lemma 2.1] this implies that Y_f is smooth. Song chose a basis to diagonalize the operator he calls ρ, so we end up with a different expression for f. Performing a suitable change of basis C (code in *songChangeOfBasis.sage*) we obtain

$$C(f) = p_{025} + p_{136} + p_{247} - p_{038} - p_{149} - p_{079} + p_{158} + p_{269} - p_{357} - p_{468}.$$

This is precisely the same as Song's generator. We verify that the divisibility constraints of Theorem 1.1 are satisfied for $k,n,d = 3,10,1$, as $n \nmid kd$ and n is even. Moreover, $\mathrm{Gr}(3,10)$ has no primitive cohomology in degrees N or $N-1$, so we may identify $H^{N-p,p-1}_{\mathrm{prim}}(Z)$ with $(R/J_f)_{pd-n}$ directly. The Griffiths ring (R/J_f) of $Z = V(f)$ is computed in *GriffithsRingPSL211.M2*, with basis in degrees 0,1, and 2 given by

$$\begin{aligned} (R/J_f)_0 &: \{1\} \\ (R/J_f)_1 &: \{p_{5,7,8}, p_{0,1,9}, p_{1,2,9}, p_{0,4,9}, p_{1,5,9}, p_{3,5,9}, p_{4,6,9}, p_{5,6,9}, p_{1,7,9}, p_{4,7,9}, \\ &\qquad p_{5,7,9}, p_{6,7,9}, p_{0,8,9}, p_{1,8,9}, p_{2,8,9}, p_{3,8,9}, p_{4,8,9}, p_{5,8,9}, p_{6,8,9}, p_{7,8,9}\} \\ (R/J_f)_2 &: \{p_{7,8,9}^2\} \end{aligned}$$

The image of the basis element $\{1\}$ under the isomorphism in Theorem 1.1 is the cohomology class of $\dfrac{1}{f^{10}}\Omega$, where Ω is the generator of $H^0(X,K_X(10))$.

Proposition 4.1. *Let G be a nonabelian simple group acting linearly on $X = \mathrm{Gr}(3, 10)$, and let Ω denote a nowhere-vanishing section of $K_X(10)$. Then G fixes Ω.*

Proof. G acts on the line $\mathbb{C} \cdot \Omega$, which induces a character χ from G to $\mathbb{C}^\times$. G is simple and nonabelian, so $[G, G] = G$, and any homomorphism from G to the abelian group $\mathbb{C}^\times$ factors through the abelianization $G/[G, G]$, hence the character is trivial. □

Therefore $G = \mathrm{PSL}(2, \mathbb{F}_{11})$ fixes both 1 and f, and it acts with trivial character on Ω by Proposition 4.1. Therefore G has the trivial action on $H^{11,9}_{\mathrm{prim}}(Z)$. By complex conjugation, it must also have trivial action on $H^{9,11}_{\mathrm{prim}}(Z)$. Now we investigate the action of G on the basis elements of $(R/J_f)_1$.

Direct computation in *PSL211IrrepDecomposition.M2* shows that this has a trace vector $(20, -2, -2, 4, 2, -2, 0, 0)$, which corresponds to two copies of the irreducible representation V'_{10} of G. V'_{10} is realizable over $\mathbb{Q}$ (see [Wil+b]), so Y_f has maximal Picard rank.

A Hyperplane Section with A_7 Symmetry

We tested finite simple groups with irreducible representations V of dimension 10 such that $\bigwedge^3 V$ has exactly one trivial summand, and found $\mathrm{PSL}(2, \mathbb{F}_{11})$ and the alternating group A_7. In fact, A_7 has two such representations, which we denote V_{10} and V'_{10} (see Table 4.4). The representations are complex conjugates of each other. A_7 is generated by the two permutations $a = (1, 2, 3)$ and $b = (3, 4, 5, 6, 7)$ (in cycle notation). A_7 has the character table given in Table 4.4.

	1	$ab^{-1}ab$	a	$a^{-1}bab$	$a^{-1}bab^2$	b	$ababab^2$	ab	$a^{-1}b$
$\mathbb{1}$	1	1	1	1	1	1	1	1	1
V_6	6	2	3	0	0	1	−1	−1	−1
V_{10}	10	−2	1	1	0	0	1	$\frac{-1+\sqrt{-7}}{2}$	$\frac{-1-\sqrt{-7}}{2}$
V'_{10}	10	−2	1	1	0	0	1	$\frac{-1-\sqrt{-7}}{2}$	$\frac{-1+\sqrt{-7}}{2}$
V_{14}	14	2	2	−1	0	−1	2	0	0
V'_{14}	14	2	−1	2	0	−1	−1	0	0
V_{15}	15	−1	3	0	−1	0	−1	1	1
V_{21}	21	1	−3	0	−1	1	1	0	0
V_{35}	35	−1	−1	−1	1	0	−1	0	0

Table 4.4. Character table for A_7. Representatives of each conjugacy class are given in terms of the generators $a = (1, 2, 3)$ and $b = (3, 4, 5, 6, 7)$ (in cycle notation).

We compute exterior products using the Newton identities

$$\chi_{\bigwedge^3 V}(g) = \frac{\chi_V(g)^3 - 3\chi_V(g)\chi_V(g^2) + 2\chi_V(g^3)}{6},$$

and find

$$\bigwedge^3 V_{10} \cong \mathbb{1} \oplus V_6 \oplus V_{14}^2 \oplus V_{14}' \oplus V_{15} \oplus V_{21} \oplus V_{35},$$

$$\bigwedge^3 V_{10}' \cong \mathbb{1} \oplus V_6 \oplus V_{14} \oplus V_{14}'^2 \oplus V_{15} \oplus V_{21} \oplus V_{35}.$$

Thus, there exist two one-dimensional spaces of invariant linear forms of A_7 on $\mathrm{Gr}(3,10)$, corresponding to each irreducible representation of dimension 10. We name the generators f_{10} and $f_{10'}$, respectively. We consider the case of f_{10}. To compute this, we need explicit matrices for V_{10}.

ATLAS [Wil+a] gives matrices for V_{10} as elements of $\mathbb{Q}(\zeta_7)$, where ζ_r is a primitive rth root of unity:

$$\rho_{V_{10}}(a) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & -\frac{w}{2} & -v-\frac{w}{2} & \frac{w}{2} & 0 & \frac{v}{2}-\frac{w}{4} & -\frac{w}{2} & -\frac{v}{2}-\frac{w}{4} & -\frac{w}{2} & -v-\frac{w}{2} \\
1 & -\frac{w}{2} & -\frac{w}{2} & -v-\frac{w}{2} & 1 & \frac{3v}{2}+\frac{3w}{4} & -\frac{w}{2} & \frac{v}{2}+\frac{3w}{4} & -\frac{w}{2} & -v-\frac{w}{2} \\
0 & w & 0 & v & -1 & 1 & -1 & -w & -v & v \\
-2v-w & v+\frac{3w}{2} & v+\frac{3w}{2} & v-\frac{w}{2} & -w & -\frac{v}{2}-\frac{3w}{4} & -v-\frac{w}{2} & -\frac{v}{2}+\frac{w}{4} & -v-\frac{w}{2} & \frac{w}{2} \\
-v+\frac{w}{2} & \frac{v}{2}-\frac{w}{4} & -\frac{v}{2}-\frac{3w}{4} & -\frac{3v}{2}-\frac{w}{4} & -2v-\frac{3w}{2} & \frac{3v}{4}-\frac{3w}{8} & -\frac{3v}{2}-\frac{5w}{4} & \frac{3v}{4}+\frac{13w}{8} & -\frac{v}{2}-\frac{3w}{4} & -\frac{v}{2}+\frac{5w}{4}
\end{pmatrix}$$

$$\rho_{V_{10}}(b) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-v-\frac{w}{2} & \frac{v}{2}+\frac{w}{4} & \frac{v}{2}+\frac{w}{4} & -\frac{v}{2}-\frac{3w}{4} & -\frac{w}{2} & \frac{v}{4}-\frac{w}{8} & -\frac{v}{2}-\frac{3w}{4} & -\frac{3v}{4}-\frac{w}{8} & \frac{v}{2}+\frac{w}{4} & -\frac{v}{2}-\frac{w}{4} \\
1 & -\frac{w}{2} & -v-\frac{w}{2} & \frac{w}{2} & 0 & \frac{v}{2}-\frac{w}{4} & -\frac{w}{2} & -\frac{v}{2}-\frac{w}{4} & -\frac{w}{2} & -v-\frac{w}{2} \\
v+\frac{w}{2} & \frac{v}{2}+\frac{3w}{4} & \frac{v}{2}+\frac{3w}{4} & \frac{v}{2}-\frac{w}{4} & v+\frac{w}{2} & -\frac{v}{4}+\frac{5w}{8} & \frac{v}{2}-\frac{w}{4} & -\frac{5v}{4}-\frac{11w}{8} & \frac{v}{2}+\frac{3w}{4} & -\frac{v}{2}-\frac{3w}{4} \\
-2 & -w & 0 & -v & 0 & 0 & 0 & 0 & 0 & 0 \\
w & 2 & w & v+2w & -v & -v-2w & -v & -2 & 1 & v+2w
\end{pmatrix}$$

where

$$v = \zeta_7 + \zeta_7^2 + \zeta_7^4, \quad w = \overline{v} = \zeta_7^3 + \zeta_7^5 + \zeta_7^6.$$

Matrices for $\rho_{V_{10}'}(a)$ and $\rho_{V_{10}'}(b)$ are given by complex conjugation. In *A7Intersection.M2* we compute

$$\ker\left(\left(\wedge^3\rho_{V_{10}}(a)-1\right)\cap\left(\wedge^3\rho_{V_{10}}(b)-1\right)\right),$$

finding a one-dimensional space spanned by the linear form

$$f_{10}=\sum_{i=1}^{120} r_i\ell_i,$$

where r_i runs over the generators of R in lexicographic order and ℓ is given by

$$\begin{aligned}\ell=[&-22v-11,\ 24v-30,\ 4v-5,\ 22v+11,\ -12v+15,\ 4v-5,\ -4v+5,\ -22v-11,\\ &-24v-47,\ 6v+31,\ 4v-5,\ 6v+31,\ -12v+15,\ 22v+11,\ -22v-11,\ -30v-1,\\ &-22v-11,\ 24v+47,\ 9v+8,\ -17v+79,\ 17v-2,\ -20v+25,\ -3v-54,\ -9v-8,\\ &17v-2,\ -22v-11,\ 12v-15,\ 37v+50,\ -22v-11,\ -4v+5,\ -28v+35,\ -21v-70,\\ &-77,\ 56v+7,\ 42v+63,\ -12v+15,\ -6v-31,\ 12v-15,\ 4v-5,\ -4v+5,\\ &-12v+15,\ 6v+31,\ -4v+5,\ 24v-30,\ 32v+37,\ -17v+2,\ -3v-54,\ 17v-2,\\ &37v+50,\ -17v+2,\ -5v-13,\ 17v-2,\ 4v-5,\ -4v+5,\ 22v+11,\ -35v-14,\\ &22v+11,\ -3v-54,\ 17v-2,\ 24v+47,\ -12v+15,\ -v-18,\ 12v-15,\ -4v+5,\\ &17v-2,\ -22v-11,\ -6v-31,\ 4v-5,\ 17v-2,\ -4v+5,\ 9v+8,\ 77,\\ &-35v-14,\ -21v-70,\ 21v-7,\ 56v+7,\ 14v+21,\ 4v-5,\ -5v-13,\ -17v+2,\\ &12v-15,\ 9v+8,\ -17v+2,\ 28v-35,\ 6v+31,\ -4v+5,\ 5v+13,\ -4v+5,\\ &6v+31,\ 4v-5,\ 12v-15,\ -12v+15,\ -12v+15,\ -32v-37,\ 12v-15,\ 32v+37,\\ &-4v+5,\ 11v+44,\ 12v-15,\ -22v-11,\ 17v-2,\ 4v-5,\ 6v+31,\ -4v+5,\\ &-21v+7,\ 30v+1,\ -4v+5,\ -22v-11,\ -11v-44,\ -24v-47,\ -37v-50,\ 6v+31,\\ &42v+63,\ -28v+35,\ -35v-14,\ -14v-21,\ 21v-7,\ 14v+21,\ 14v+21,\ 77]\end{aligned}$$

where as before $v=\zeta_7+\zeta_7^2+\zeta_7^4$. To verify the smoothness of f_{10}, we refer to the following proposition.

Proposition 4.2. *Let $X_{\mathbb{C}}=\mathrm{Gr}(k,n)$ and let R denote the coordinate ring of X. Let $g\in R$ be a polynomial defining a hypersurface $Z\subset X_{\mathbb{C}}$. Let $S\subset\mathbb{C}$ be any finitely-generated $\mathbb{Z}$-subalgebra containing all coefficients of g. Suppose there exists a maximal ideal $\mathfrak{m}\subset S$ such that the reduction $Z \mod \mathfrak{m}$ is smooth over the residue field $S/\mathfrak{m}$. Then Z is smooth over $\mathbb{C}$.*

Proof. This follows from spreading out; see EGA IV [Gro67, §8] or [Har77, §3.9]. □

In *SmoothCheckA7.M2* we verify that f_{10} is smooth over $\mathbb{F}_{113}$ (in which 49 is a seventh root of unity) and thus smooth over $\mathbb{C}$ by Proposition 4.2. Again, by [BS21, Lemma 2.1] this implies that $Y_{f_{10}}$ is smooth. By the same argument as in Section 4.5, we may identify $(R/J_f)_{pd-n}$ with $H^{N-p,p-1}_{\mathrm{van}}(Z)$. By the computation in

GriffithsRingA7.M2, the Griffiths ring of $f = f_{10}$ admits a basis for $(R/J_f)_1$ given by

$$p_{1,6,9}, p_{2,6,9}, p_{3,6,9}, p_{4,6,9}, p_{5,6,9}, p_{0,7,9}, p_{1,7,9}, p_{2,7,9}, p_{3,7,9}, p_{4,7,9},$$
$$p_{5,7,9}, p_{6,7,9}, p_{0,8,9}, p_{1,8,9}, p_{2,8,9}, p_{3,8,9}, p_{4,8,9}, p_{5,8,9}, p_{6,8,9}, p_{7,8,9}.$$

Computing traces over the conjugacy classes in *A7IrrepDecomposition.M2*, in the order of Table 4.4 we have

$$(20, 4, 2, 2, 0, 0, -2, -1, -1)$$

corresponding to $V_6 \oplus V'_{14}$. Both V_6 and V'_{14} are representations realizable over $\mathbb{Q}$ (see [Wil+a]), hence the hyperplane section cut out by f_{10} has maximal higher Picard rank.

It would be interesting to determine whether A_7 is the full automorphism group of the Debarre–Voisin fourfold associated to f_{10}; A_7 is one of the maximal finite symplectic automorphism groups of a hyper-Kähler fourfold of $\mathrm{K3}^{[2]}$ type identified by Höhn and Mason in [HM19, Theorem A].

[AN54] Y. Akizuki and S. Nakano, *Note on Kodaira-Spencer's proof of Lefschetz theorems*, Proc. Japan Acad. **30** (1954), 266–272.

[AS24] A. Ayyer and S. Sinha, *Cores of partitions in rectangles*, Electron. J. Combin. **31** (2024), no. 1, Paper No. 1.47, 14.

[BB05] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, *Vol. 231*, Graduate Texts in Mathematics, Springer, New York, 2005, xiv+363.

[Bea14] A. Beauville, *Some surfaces with maximal Picard number*, J. Éc. polytech. Math. **1** (2014), 101–116.

[Bea83] ———, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. **18** (1983), no. 4, 755–782.

[BS21] V. Benedetti and J. Song, *Divisors in the moduli space of Debarre-Voisin varieties*, arXiv: 2106.06859 [math.AG], 2021.

[Che10] L.-C. Chen, *Skew-Linked Partitions and a Representation-Theoretic Model for k-Schur Functions*, ProQuest, 2010, 71, Thesis (Ph.D.)–University of California, Berkeley.

[CHK19] E. Costa, D. Harvey, and K. S. Kedlaya, *Zeta functions of nondegenerate hypersurfaces in toric varieties via controlled reduction in p-adic cohomology*, Proceedings of the Thirteenth Algorithmic Number Theory Symposium, Vol. 2, Open Book Ser. Math. Sci. Publ., Berkeley, CA, 2019, 221–238.

[Cho+21] H. Cho et al., *A survey on t-core partitions*, Hardy-Ramanujan J. **44** (2021), 81–101.

[Del70] P. Deligne, *Équations différentielles à points singuliers réguliers*, *Vol. 163*, Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, 1970, iii+133.

[Del71] ———, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. (1971), no. 40, 5–57.

[DM22] O. Debarre and G. Mongardi, *Gushel-Mukai varieties with many symmetries and an explicit irrational Gushel-Mukai threefold*, Boll. Unione Mat. Ital. **15** (2022), no. 1-2, 133–161.

[Dol03] I. Dolgachev, *Lectures on invariant theory*, *Vol. 296*, London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2003, xvi+220.

[DV10] O. Debarre and C. Voisin, *Hyper-Kähler fourfolds and Grassmann geometry*, J. Reine Angew. Math. **649** (2010), 63–87.

[Eis95] D. Eisenbud, *Commutative algebra*, *Vol. 150*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1995, xvi+785, With a view toward algebraic geometry.

[FH91] W. Fulton and J. Harris, *Representation theory*, *Vol. 129*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1991, xvi+551, A first course, Readings in Mathematics.

[FM21] E. Fatighenti and G. Mongardi, *A note on a Griffiths-type ring for complete intersections in Grassmannians*, Math. Z. **299** (2021), no. 3-4, 1651–1672.

[Ful84] W. Fulton, *Intersection theory*, *Vol. 2*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], Springer-Verlag, Berlin, 1984, xi+470.

[GH26] F. Gossow and A. Huchala, *Bounded core partitions and Borel–Weil–Bott* (2026), arXiv: 2510.17239 [math.CO].

[GKS90] F. Garvan, D. Kim, and D. Stanton, *Cranks and t-cores*, Invent. Math. **101** (1990), no. 1, 1–17.

[GO96] A. Granville and K. Ono, *Defect zero p-blocks for finite simple groups*, Trans. Amer. Math. Soc. **348** (1996), no. 1, 331–347.

[Gri69] P. A. Griffiths, *On the periods of certain rational integrals. I, II*, Ann. of Math. (2) **90** (1969), 460–495, 90 (1969), 496–541.

[Gro67] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. (1967), no. 32, 361.

[Han10] G.-N. Han, *The Nekrasov-Okounkov hook length formula: refinement, elementary proof, extension and applications*, Ann. Inst. Fourier (Grenoble) **60** (2010), no. 1, 1–29.

[Har77] R. Hartshorne, *Algebraic geometry*, *Vol. No. 52*, Graduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1977, xvi+496.

[Hir56] F. Hirzebruch, *Der Satz von Riemann-Roch in Faisceau-theoretischer Formulierung: einige Anwendungen und offene Fragen*, Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, Erven P. Noordhoff N. V., Groningen, 1956, 457–473.

[Hir95] ———, *Topological methods in algebraic geometry*, English, Classics in Mathematics, Springer-Verlag, Berlin, 1995, xii+234, Translated from the German and Appendix One by R. L. E. Schwarzenberger, Appendix Two by A. Borel, Reprint of the 1978 edition.

[HM19] G. Höhn and G. Mason, *Finite groups of symplectic automorphisms of hyperkähler manifolds of type $K3^{[2]}$*, Bull. Inst. Math. Acad. Sin. (N.S.) **14** (2019), no. 2, 189–264.

[Hua+18] A. Huang et al., *Jacobian rings for homogenous vector bundles and applications*, arXiv: 1801.08261 [math.AG], 2018.

[Huc26] A. Huchala, *Griffiths-Dwork-Reduction*, https://github.com/Ahuchala/Griffiths-Dwork-Reduction, 2026.

[Hum72] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, *Vol. 9*, Graduate Texts in Mathematics, Springer-Verlag, New York-Berlin, 1972, xii+169.

[Huy05] D. Huybrechts, *Complex geometry*, Universitext, Springer-Verlag, Berlin, 2005, xii+309, An introduction.

[Huy06] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2006, viii+307.

[JK81] G. James and A. Kerber, *The representation theory of the symmetric group*, *Vol. 16*, Encyclopedia of Mathematics and its Applications, Addison-Wesley Publishing Co., Reading, MA, 1981, xxviii+510, With a foreword by P. M. Cohn, With an introduction by Gilbert de B. Robinson.

[Kap84] M. M. Kapranov, *Derived category of coherent sheaves on Grassmann manifolds*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 1, 192–202.

[KN63] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol I*, Interscience Publishers (a division of John Wiley & Sons, Inc.), New York-London, 1963, xi+329.

[Lee13] J. M. Lee, *Introduction to smooth manifolds*, Second, *Vol. 218*, Graduate Texts in Mathematics, Springer, New York, 2013, xvi+708.

[LM05] L. Lapointe and J. Morse, *Tableaux on $k+1$-cores, reduced words for affine permutations, and k-Schur expansions*, J. Combin. Theory Ser. A **112** (2005), no. 1, 44–81.

[Mac86] P. A. MacMahon, *Collected papers. Vol. II*, *Vol. 24*, Mathematicians of Our Time, MIT Press, Cambridge, MA, 1986, xxviii+952, Number theory, invariants and applications, Edited and with a preface by George E. Andrews, With an introduction by Gian-Carlo Rota.

[Mon13] G. Mongardi, *Automorphisms of hyperkähler manifolds*, arXiv: 1303.4670 [math.AG], 2013.

[Nik25] E. Nikolov, *The derived category of the Hilbert scheme of three points*, https://repo.uni-hannover.de/handle/123456789/19802, 2025.

[NO06] N. A. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, The unity of mathematics, Vol. 244, Progr. Math. Birkhäuser Boston, Boston, MA, 2006, 525–596.

[Pos06] A. Postnikov, *Total positivity, Grassmannians, and networks*, arXiv: math/0609764 [math.CO], 2006.

[PŚ18] G. Panova and P. Śniady, *Skew Howe duality and random rectangular Young tableaux*, Algebr. Comb. **1** (2018), no. 1, 81–94.

[Rød00] E. A. Rødland, *The Pfaffian Calabi-Yau, its mirror, and their link to the Grassmannian $G(2,7)$*, Compositio Math. **122** (2000), no. 2, 135–149.

[Ser95] J.-P. Serre, *Représentations linéaires et espaces homogènes kählériens des groupes de Lie compacts (d'après Armand Borel et André Weil)*, Séminaire Bourbaki, Vol. 2, Soc. Math. France, Paris, 1995, Exp. No. 100, 447–454.

[Sno86] D. M. Snow, *Cohomology of twisted holomorphic forms on compact Hermitian symmetric spaces*, *Vol. 24*, Journées SMF, Univ. Poitiers, Poitiers, 1986, 77–87.

[Son23] J. Song, *A special Debarre-Voisin fourfold*, Bull. Soc. Math. France **151** (2023), no. 2, 305–330.

[Sta99] R. P. Stanley, *Enumerative combinatorics. Vol. 2*, *Vol. 62*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1999, xii+581, With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[Ste94] J. R. Stembridge, *On minuscule representations, plane partitions and involutions in complex Lie groups*, Duke Math. J. **73** (1994), no. 2, 469–490.

[Voi02] C. Voisin, *Hodge theory and complex algebraic geometry. I*, *Vol. 76*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2002, x+322, Translated from the French original by Leila Schneps.

[Voi03] ———, *Hodge theory and complex algebraic geometry. II*, *Vol. 77*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2003, x+351, Translated from the French by Leila Schneps.

[Wey03] J. Weyman, *Cohomology of vector bundles and syzygies*, *Vol. 149*, Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2003, xiv+371.

[Wey39] H. Weyl, *The Classical Groups. Their Invariants and Representations*, Princeton University Press, Princeton, NJ, 1939, xii+302.

[Wil+a] R. Wilson et al., *ATLAS of Finite Group Representations, Version 3: Alternating group A_7*, URL: https://brauer.maths.qmul.ac.uk/Atlas/v3/alt/A7/ (visited on 05/11/2026).

[Wil+b] ———, *ATLAS of Finite Group Representations, Version 3: Linear group $L_2(11)$*, URL: https://brauer.maths.qmul.ac.uk/Atlas/v3/lin/L211/ (visited on 05/11/2026).