Double Boxes and Double Dimers

by

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A dissertation accepted and approved in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
in Mathematics

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University of Oregon
Summer 2025

ABSTRACT

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Doctor of Philosophy in Mathematics

Title: Double Boxes and Double Dimers

We give a combinatorial proof of a result in rank 2 Donaldson-Thomas theory, which states that the generating function for certain plane-partition-like objects, called double-box configurations, is equal to a product of MacMahon's generating function for (boxed) plane partitions. We first give the correspondence between double-box configurations and double-dimer configurations on the hexagon lattice with a particular tripartite node pairing. Using this correspondence, we apply graphical condensation and double-dimer condensation in our proof.

This dissertation includes previously published co-authored material [BY25].

ACKNOWLEDGEMENTS

I would first like to express my gratitude to my advisor, Ben Young, for the opportunity to work on this project and for your continual support and kindness. I appreciate all your help and encouragement throughout this process.

I will always be thankful to Pam Maslyk and Iva Stavrov for inspiring me and helping me become the mathematician I am today. I'm especially grateful to Iva for guiding and challenging me to develop as a mathematician.

I'm grateful to my mentors at Pacific Northwest National Laboratory, Sinan Aksoy and Rebecca Jones, for their mentorship and continued guidance. I also want to thank Helen Jenne for her mentorship, valuable advice, and friendship throughout grad school and during my time at PNNL.

To Leigh and Agus, I'll always be grateful for your friendship and your support during the most challenging years. You made even the most difficult parts of grad school enjoyable.

I'm also thankful to my roommates, Megan, Caela, Magda, Jake, and Emily, for bringing balance to my life outside of math. I'm especially grateful to Magda for the many hikes and climbs, and all those runs in the dark, cold, and rain.

A big thanks to everyone at Lane 5 CrossFit, especially the 5:30 crew, for all the fun and difficult workouts over the past year.

To Dylan, thank you for your understanding and acceptance.

I'm deeply grateful to my grandma Dottie and grandpa Ed for their support and confidence in me.

Lastly, I can never fully express my gratitude to my mom and dad, Lee Ann and Barry, and my brother, Nicholai. Thank you for believing in me and always being there for me.

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Chapter 1

Introduction

The work in this chapter was published in Séminaire Lotharingien de Combinatoire **93B** (2025), Article # 39, with coauthor Benjamin Young [BY25].

In this dissertation, we enumerate certain plane-partition-like objects called *double-box configurations*. These objects were introduced by Gholampour, Kool and Young [GKY18] for the purpose of computing the rank 2 Donaldson-Thomas (DT) invariants of a Calabi-Yau threefold. We define double-box configurations in Definition 3, as well as their generating function, $Z^{DB}_{a,b,c}(q)$, in Equation (2.2) (where DB stands for double-box), and we give a combinatorial proof of the following geometrically motivated theorem:

Theorem 1. *Let* $a, b, c \in \mathbb{N}$*, then*

$$Z^{DB}_{a,b,c}(q) = M(q)^2 M_{a,b,c}(q) \tag{1.1}$$

where $Z^{DB}_{a,b,c}(q)$ *denotes the generating function for double-box configurations, and*

$$M(q) = \prod_{i=1}^{\infty} \frac{1}{(1-q^i)^i}$$

is MacMahon's generating function for plane partitions, and

$$M_{a,b,c}(q) = \prod_{s=1}^{a} \prod_{t=1}^{b} \prod_{r=1}^{c} \frac{1-q^{s+t+r-1}}{1-q^{s+t+r-2}}$$

is MacMahon's generating function for boxed $a \times b \times c$ *plane partitions.*

Note that this formula already has a geometric proof [GK17]. In this dissertation, we outline our combinatorial proof of this formula, which surprisingly uses the tripartite double-dimer model of Kenyon and Wilson [KW09]. We follow the general strategy of [JWY20], where the tripartite double-dimer model also appears (though for apparently completely different reasons). Our proof consists of two main components. First, we give a correspondence between double-box configurations and tripartite double-dimer configurations on the hexagon lattice; this correspondence is many-to-one, yet still weight preserving. Next, we use a quadratic recurrence relation called condensation to prove our main result. We show that to the left hand side of Equation (1.1), $Z^{DB}_{a,b,c}(q)$, we may apply a result by Jenne [Jen21], which states that under certain conditions the generating function for tripartite double-dimer configurations satisfies a recurrence relation related to the Desnanot-Jacobi identity from linear algebra. Then, using Kuo condensation [Kuo04] (also related to the Desnanot-Jacobi identity), we show that $M(q)^2 M_{a,b,c}(q)$ satisfies the same recurrence relation. Finally, we show that both sides of Equation (1.1) satisfy the same initial conditions.

This dissertation is organized as follows. In Chapter 1.1, we give some background on plane partitions, dimer configurations, and the relation between them. In Chapter 2, we define the double-box configurations and the corresponding tripartite double-dimer configurations. In Chapter 3 we prove the equivalence of the generating functions for the double-box configurations and the tripartite double-dimer configurations. Finally, in Chapter 4 we prove Theorem 1 using the recurrences of Jenne and Kuo.

1.1 Background

A *plane partition* is a two-dimensional array of nonnegative integers $\pi_{i,j}$ for $i, j \geq 0$, with $\pi_{i,j+1} \geq \pi_{i,j}$ and $\pi_{i+1,j} \geq \pi_{i,j}$ for all i, j, with finitely many $\pi_{i,j}$ being nonzero. We can visualize a plane partition π as a stack of boxes in the corner of a room, with the number of boxes in each stack given by the entries of π. A *dimer configuration* (also called a *perfect matching*) on a graph $G = (V, E)$ is a collection of edges $E' \subseteq E$ such that every vertex in V is covered exactly once. There is a bijection between plane partitions and dimer configurations on the hexagon graph, sometimes referred to as the *folklore bijection* (see

Figure 1.1). The stacks of boxes representing a plane partition π can be viewed as a lozenge tiling of a hexagonal region of triangles, corresponding to the visible faces of the boxes and the tiles on the walls and floor of the room. Note that the triangular lattice is dual to the hexagon lattice. Moreover, each lozenge is made of two equilateral triangles that share an edge. If we join the centers of these two triangles with the corresponding dual edge, and do this for all tiles in the tiling, we get a perfect matching on the hexagon graph.

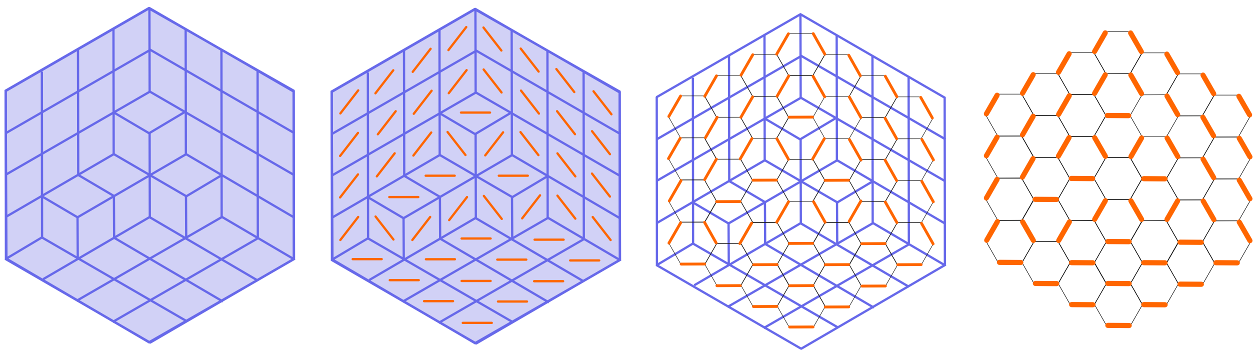

Figure 1.1: Folklore bijection between plane partitions (left-most) and dimer configurations on the hexagon graph (right-most)

Overlaying two perfect matchings of a graph $G = (V, E)$ gives a *double-dimer configuration*, which consists of doubled edges and loops. If in addition we have defined a set of nodes $\mathbf{N} \subset V$, that is, a special set of vertices, then the double-dimer configuration on $G = (V, E)$ with node set $\mathbf{N}$ is a multiset of edges of E such that each vertex in $V \setminus \mathbf{N}$ is covered exactly twice, and each node in $\mathbf{N}$ is covered exactly once. In this case, the double-dimer configuration consists of doubled edges, loops, and paths between the nodes in $\mathbf{N}$ (see Figure 1.2).

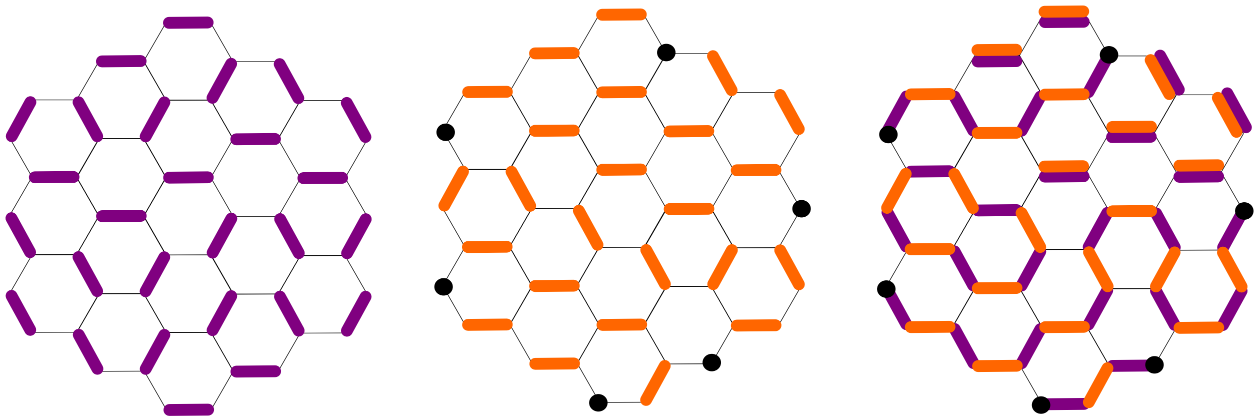

Figure 1.2: Double-dimer configuration with nodes on the hexagon graph (right-most) from two single-dimer configurations, one on the hexagon graph (left-most) and one on the hexagon graph minus the defined nodes (middle). We will often omit the hexagon graph in the background for clarity.

Chapter 2

Definitions

2.1 Double-Box Configurations

Most of the work in this chapter was published in Séminaire Lotharingien de Combinatoire **93B** (2025), Article # 39, with coauthor Benjamin Young [BY25].

In this section we define the double-box configurations [GKY18], and provide some examples. Throughout this section we let $a, b, c \in \mathbb{N}$ be fixed. We identify the point $(i, j, k) \in \mathbb{Z}^3$ with the unit cube $[i, i+1] \times [j, j+1] \times [k, k+1] \in \mathbb{R}^3$. We refer to this unit cube as the box (i, j, k). We consider plane partitions as stacks of unit cubes (i.e. boxes) placed in $\mathbb{R}^3$. A plane partition π is said to be based at (l, m, n) in $\mathbb{R}^3$ if the bottommost box in the stack of boxes corresponding to the entry in the first row and column of π is the box (l, m, n). In other words, the back corner of the room where the boxes are stacked is placed at (l, m, n).

Consider triples of plane partitions $\eta = (\eta_1, \eta_2, \eta_3)$ such that η_1 is based at $(0, b, c)$, η_2 is based at $(a, 0, c)$, and η_3 is based at $(a, b, 0)$ in $\mathbb{R}^3$ (see Figure 2.1). We say that a box (i, j, k) is in the *intersection space* if $i \geq a$, $j \geq b$, and $k \geq c$. We denote all the boxes in the intersection space by η_{int}.

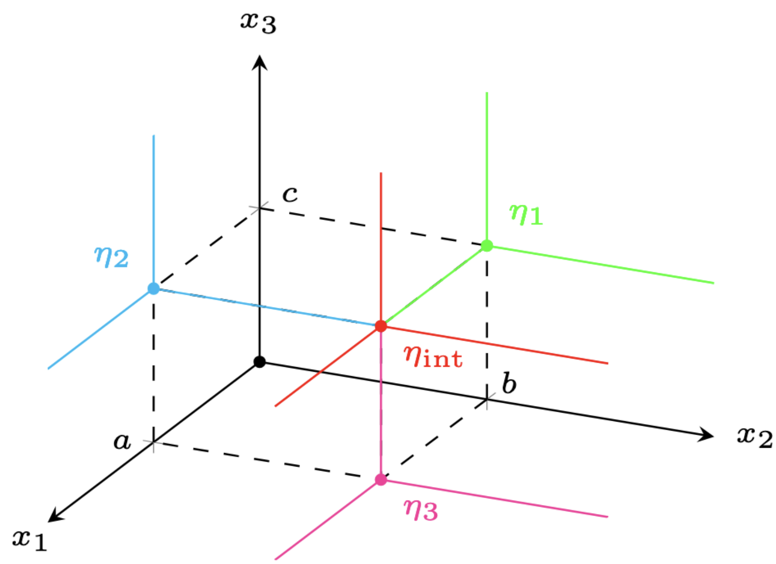

Figure 2.1: Basepoints of plane partitions η_1, η_2, η_3 in $\mathbb{R}^3$

We define different types of boxes based on the number of plane partitions they are contained in as:

Definition 2. We say that a box $(i, j, k) \in \eta = (\eta_1, \eta_2, \eta_3)$ is:

- a type III box if $(i, j, k) \in \eta_1, \eta_2, \eta_3$ (triple intersection boxes)
- a type II box if $(i, j, k) \in \eta_l, \eta_m$ and $(i, j, k) \notin \eta_n$ for $\{l, m, n\} = \{1, 2, 3\}$ (double intersection boxes)
- a type I box if $(i, j, k) \in \eta_l$ and $(i, j, k) \notin \eta_m, \eta_n$ for $\{l, m, n\} = \{1, 2, 3\}$ (boxes in only one of the plane partitions)

Let η_{in} denote the set of type III boxes, and let η_{out} denote the set of type II boxes. Note that $\eta_{\text{in}} \cup \eta_{\text{out}}$ is a plane partition based at (a, b, c), with $\eta_{\text{in}} \cup \eta_{\text{out}} \subseteq \eta_{\text{int}}$. We want to consider triples of plane partitions $\eta = (\eta_1, \eta_2, \eta_3)$ such that the following Criterion is satisfied:

Criterion 1. $\eta_{\text{int}} = \eta_{\text{in}} \cup \eta_{\text{out}}$,

that is,

$$\eta_{\text{int}} = (\eta_1 \cap \eta_2) \cup (\eta_1 \cap \eta_3) \cup (\eta_2 \cap \eta_3). \tag{2.1}$$

We define an equivalence relation on triples of plane partitions satisfying Criterion 1 as follows. If $\eta = (\eta_1, \eta_2, \eta_3)$ and $\tilde{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3)$, we say that $\eta \sim \tilde{\eta}$ if they have the same multiset of boxes. That is, $\eta \sim \tilde{\eta}$ if:

- $\eta_{\text{in}} = \tilde{\eta}_{\text{in}}$ (type III boxes the same)
- $\eta_{\text{out}} = \tilde{\eta}_{\text{out}}$ (type II boxes the same, ignoring which two partitions they came from)
- η_1 agrees with $\tilde{\eta}_1$ on $[0, a) \times [b, \infty) \times [c, \infty)$
- η_2 agrees with $\tilde{\eta}_2$ on $[a, \infty) \times [0, b) \times [c, \infty)$
- η_3 agrees with $\tilde{\eta}_3$ on $[a, \infty) \times [b, \infty) \times [0, c)$

The last three conditions ensure that all type I boxes, that is, those not in the intersection space (by Criterion 1), are the same. We are now ready to define double-box configurations as:

Definition 3. Given $(a, b, c) \in \mathbb{N}^3$, an equivalence class of triples of plane partitions $\eta = (\eta_1, \eta_2, \eta_3)$ satisfying Criterion 1 under the equivalence relation $\sim$ is called a *double-box configuration*. Note that we often denote such an equivalence class by η, rather than $[\eta]$.

We denote the set of all double-box configurations by $DB_{a,b,c}$. For each $\eta \in DB_{a,b,c}$, we define the following:

Definition 4. The *weight* of a double-box configuration $\eta = (\eta_1, \eta_2, \eta_3)$ is defined as

$$|\eta| = |\eta_1| + |\eta_2| + |\eta_3| - |\eta_{\text{int}}|$$

$$= \#\{\text{type I boxes}\} + \#\{\text{type II boxes}\} + 2 \cdot \#\{\text{type III boxes}\}.$$

Note that this quantity is well-defined on equivalence classes.

Next, we consider elements within the equivalence class of a double-box configuration $[\eta]$. To do this, we first make the following definition:

Definition 5. A box $(i,j,k) \in \eta_{\text{out}}$ is said to be *moveable* if there exists $\hat{\eta} \neq \tilde{\eta} \in [\eta]$ and two indices $m \neq n \in \{1,2,3\}$ such that $(i,j,k) \notin \hat{\eta}_m$ and $(i,j,k) \notin \tilde{\eta}_n$.

Triples of plane partitions within an equivalence class (i.e. a double-box configuration), may differ by which two plane partitions a moveable box is contained in. We consider several examples to illustrate this.

Example 6. Let $(a,b,c) = (1,1,1)$. Consider $\eta = (\eta_1, \eta_2, \eta_3)$ and $\tilde{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3)$ as defined in Figure 2.2.

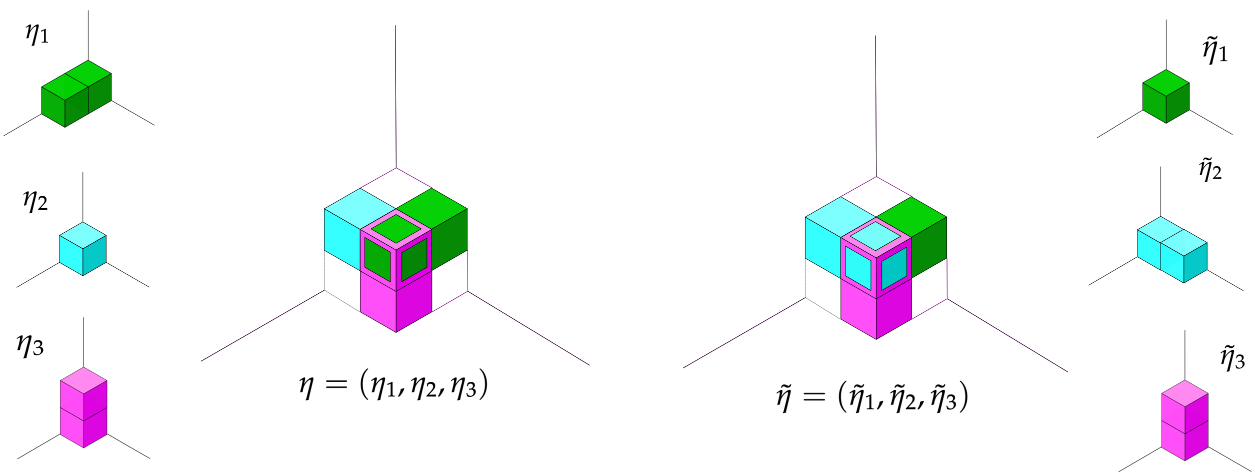

Figure 2.2: Example of $\eta \neq \tilde{\eta}$ with $[\eta] = [\tilde{\eta}]$

There is one type II box in this double-box configuration at $(1,1,1)$. In η, this box is contained in η_1 and η_3, and in $\tilde{\eta}$ this box is contained in $\tilde{\eta}_2$ and $\tilde{\eta}_3$. Since η and $\tilde{\eta}$ contain the same multiset of boxes, we have that $[\eta] = [\tilde{\eta}]$, and so the type II box at $(1,1,1)$ is a moveable box.

Example 7. Let $(a,b,c) = (1,1,1)$, and consider $\eta = (\eta_1, \eta_2, \eta_3)$ as defined in Figure 2.3.

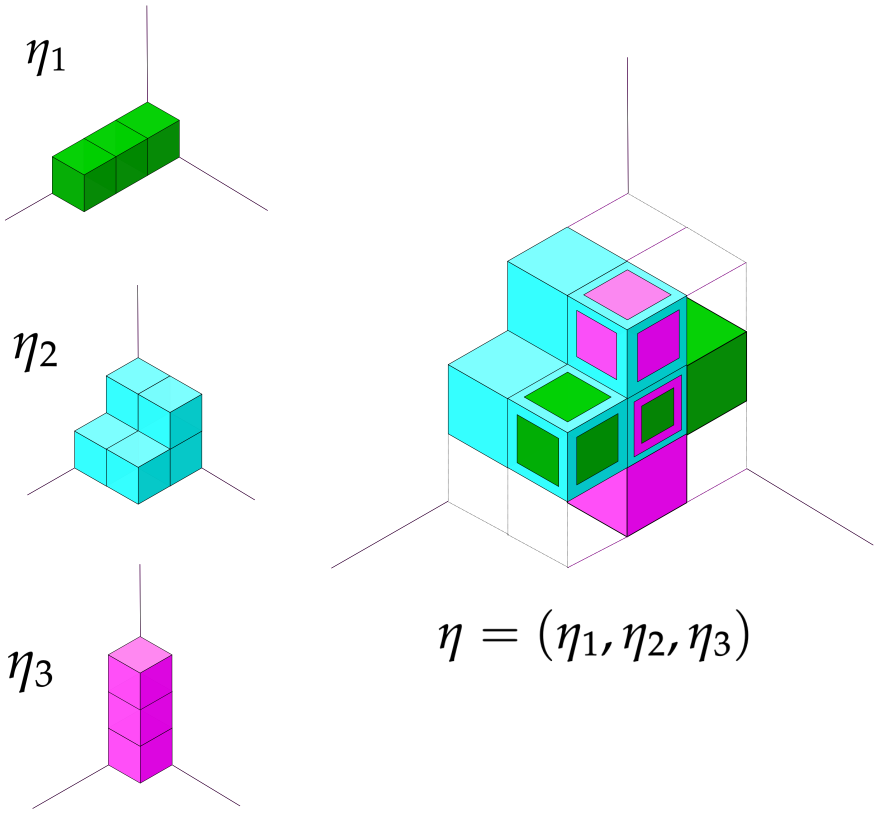

Figure 2.3: Example of a double-box configuration

In this example, there is one type III box at $(1,1,1)$, and two type II boxes, one at $(2,1,1)$ contained in η_1 and η_2, and one at $(1,1,2)$ contained in η_2 and η_3. Both of these type II boxes are not moveable. The box at $(1,1,2)$ cannot be contained in η_1 because $(0,1,2) \notin \eta_1$, and the box at $(2,1,1)$ cannot be contained in η_3 because $(2,1,0) \notin \eta_3$.

To define the generating function for double-box configurations, we first make the following definition:

Definition 8. The *contribution* of a double-box configuration η is defined as

$$\chi(\eta) = 2^m$$

where m is the number of connected components of moveable boxes in η. Two boxes are connected if they share a face.

Finally, we define the generating function for double-box configurations as

$$Z^{DB}_{a,b,c}(q) = \sum_{\eta \in DB_{a,b,c}} \chi(\eta) q^{|\eta|}, \tag{2.2}$$

where $|\eta|$ is defined in Definition 4, and $\chi(\eta)$ is defined in Definition 8.

2.2 Tripartite Double-Dimer Configurations

In this section we define the tripartite node pairing of the double-dimer configurations on the hexagon graph that are in correspondence with the double-box configurations. Let $H(n)$ be the hexagon graph of size $n \times n \times n$, for $n \in \mathbb{N}$. That is, project the points $\{0, \ldots, n\}^3 \subset \mathbb{N}^3$ onto the plane $P : \{x + y + z = 0\}$ to obtain the vertices of a hexagon-shaped piece of the triangular lattice; $H(n)$ is the planar dual of this graph, without an external vertex. Choose coordinates (x, y) for P such that a third of the edges are parallel to the x axis - "horizontal" - and the others have slope $\pm\sqrt{3}/2$. For convenience, we will use standard "compass coordinates" to describe directions on this picture - so "North" is the positive y direction, "West" is negative x, and so on (see Figure 2.6).

Let q be an indeterminate, and define a weight function $w : E \to \mathbb{Q}[q]$ on the edge set E of $H(n)$. If $e \in E$ is not a horizontal edge, then we define $w(e)$ to be 1. Otherwise, define $w(e) = q^k$, where k is the number of horizontal edges between e and the SW-NE line bounding the southeast edge of $H(n)$ (see Figure 2.5). This edge weighting ensures that the generating function for plane partitions is the same as the one for single-dimer configurations on $H(n)$ up to a constant. We can see this by noting that if a perfect matching consists of the bottom edge (weighted q^i) and two other edges of a 6-cycle, then replacing those edges with the other three edges corresponds to dropping the q^i-weighted edge in favor of the q^{i+1}-weighted edge. The matching would then gain a factor of q, resembling the action of adding a new box (weighted q) to a plane partition (see Figure 2.4).

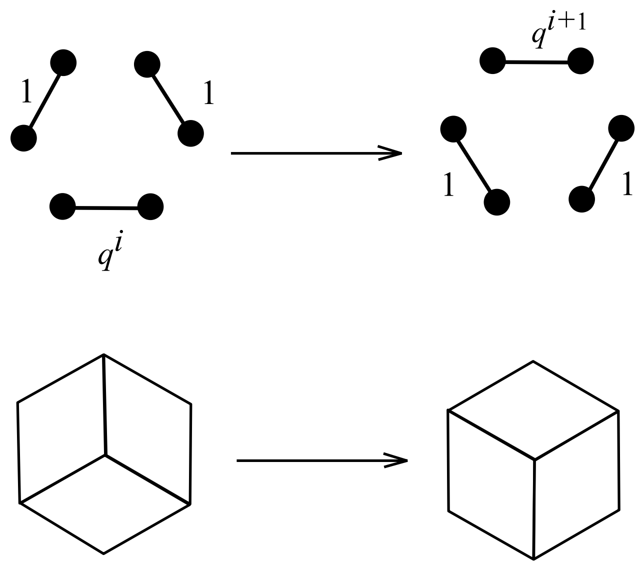

Figure 2.4: Adding/removing a box. Image source: [Kuo04].

Define sectors of $H(n)$ as in Figure 2.5 below. Note that this division makes sense as $n \to \infty$.

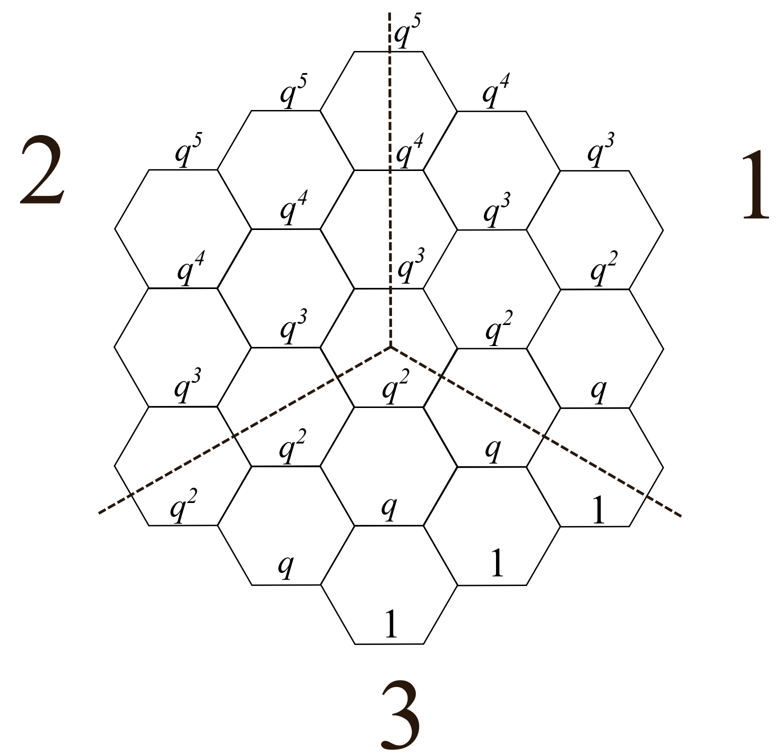

Figure 2.5: Sectors and horizontal edge weights on $H(3)$

Next, let A be the southwest corner of $H(n)$ (that is, the intersection of the lines L_1 and

L_2 in Figure 2.6), let B be the southeast corner (intersection of L_3 and L_4), and let C be the north corner of $H(n)$ (intersection of L_5 and L_6). We define the following sets of nodes, i.e. special vertices, on the boundary of $H(n)$

$$R = \{b \text{ nodes on } L_1 \text{ closest to } A\} \cup \{c \text{ nodes on } L_2 \text{ closest to } A\}$$

$$G = \{c \text{ nodes on } L_3 \text{ closest to } B\} \cup \{a \text{ nodes on } L_4 \text{ closest to } B\}$$

$$B = \{a \text{ nodes on } L_5 \text{ closest to } C\} \cup \{b \text{ nodes on } L_6 \text{ closest to } C\}$$

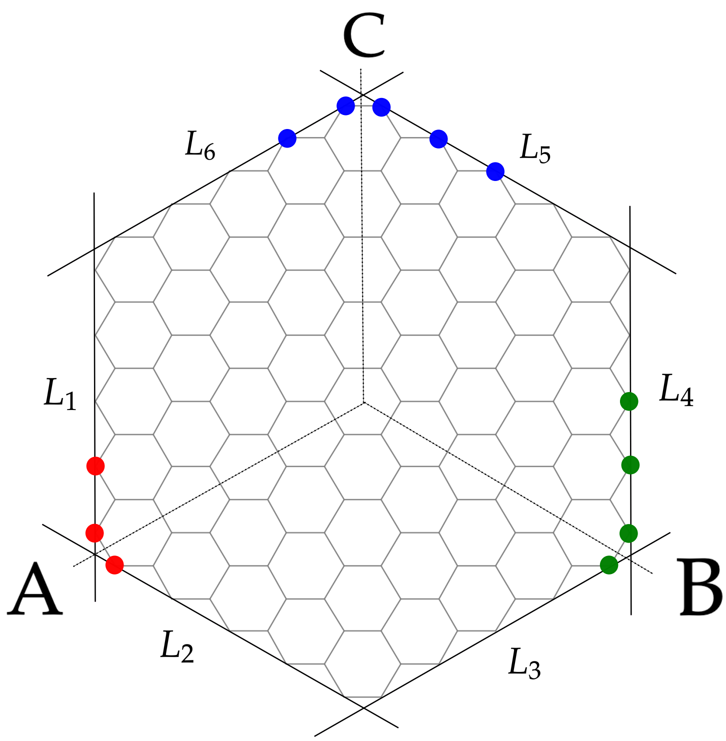

Figure 2.6: Red, green, and blue nodes on $H(5)$, with $a = 3$, $b = 2$, and $c = 1$

Note that $|R| = b + c$, $|G| = a + c$, and $|B| = a + b$, satisfy the triangle inequality, and so there is a unique planar *tripartite pairing* of the nodes, we call this pairing $\sigma_{a,b,c}$. The planar tripartite pairing $\sigma_{a,b,c}$ matches the b red nodes on L_1 with the b blue nodes on L_6, the a blue nodes on L_5 with the a green nodes on L_4, and the c red nodes on L_2 with the c green nodes on L_3, so that each node is paired with another node of a different color.

Let $DD_n(\sigma_{a,b,c})$ be the set of all double-dimer configurations on $H(n)$ with node set $\mathbf{N} = R \cup G \cup B$ and tripartite node pairing $\sigma_{a,b,c}$. We define the generating function for elements in $DD_n(\sigma_{a,b,c})$ as

$$Z^{DD}_{n;a,b,c}(q) = \frac{1}{w(\pi_0)} \sum_{\pi \in DD_n(\sigma_{a,b,c})} 2^{\ell(\pi)} w(\pi) \tag{2.3}$$

where $\ell(\pi)$ is the number of closed loops of π, and the configuration $\pi_0 \in DD_n(\sigma_{a,b,c})$ has minimal weight (see Figure 2.7). The weight of a double-dimer configuration π is given by

$$w(\pi) = \prod_{e \in \pi} w(e)$$

for the edge weighting function $w : E \to \mathbb{Q}[q]$ (see Figures 2.4 and 2.5).

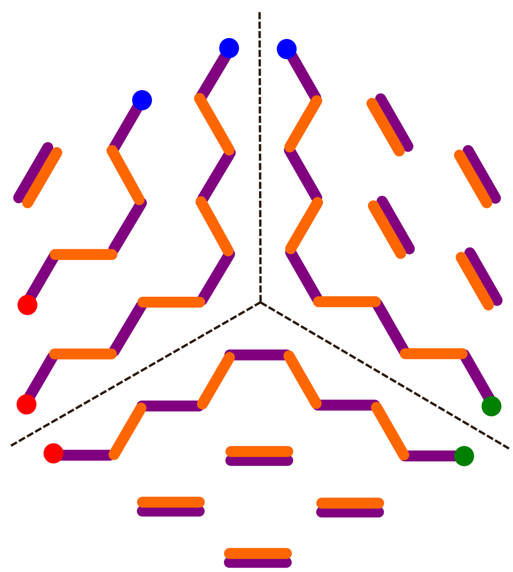

Figure 2.7: Minimal configuration $\pi_0 \in DD_3(\sigma_{1,2,1})$. Note we have removed the hexagon graph for clarity.

We now make the following definition:

Definition 9. Let $DD(\sigma_{a,b,c})$ be the set of all double-dimer configurations on the infinite hexagon lattice $H = \lim_{n \to \infty} H(n)$ such that for each $\pi \in DD(\sigma_{a,b,c})$, there exists $N \in \mathbb{N}$ such that π truncated to $H(M)$ for all $M \geq N$ has the node set $\mathbf{N}$ and the tripartite node pairing $\sigma_{a,b,c}$ (see Figure 2.6).

Informally, elements of $DD(\sigma_{a,b,c})$ are double-dimer configurations on the hexagon lattice that may be truncated so that they have the node set $\mathbf{N}$ and the tripartite node pairing $\sigma_{a,b,c}$. For elements of $DD(\sigma_{a,b,c})$, we define the generating function as

$$Z^{DD}_{a,b,c}(q) := \lim_{n\to\infty} Z^{DD}_{n;a,b,c}(q). \tag{2.4}$$

Theorem 10. *The generating function $Z^{DD}_{a,b,c}(q)$ is well-defined.*

Proof. Let $n, m \in \mathbb{N}$ with $m > n$. Consider $Z^{DD}_{n;a,b,c}(q)$ and $Z^{DD}_{m;a,b,c}(q)$ as the formal power series

$$Z^{DD}_{n;a,b,c}(q) = \sum_{i\in\mathbb{N}} a_i q^i,$$

$$Z^{DD}_{m;a,b,c}(q) = \sum_{j\in\mathbb{N}} b_j q^j.$$

We define an injective map $DD_n(\sigma_{a,b,c}) \to DD_m(\sigma_{a,b,c})$ as follows. First, consider $\pi \in DD_n(\sigma_{a,b,c})$ on $H(m)$ with the center hexagons aligned. Then extend the matching as in Figure 2.8, giving a configuration in $DD_m(\sigma_{a,b,c})$. Thus we have $a_k \leq b_k$ for every $k \geq 0$.

Let $s = \min\{n-a, n-b, n-c\}$, and consider the terms $a_{s+1}q^{s+1}$ in $Z^{DD}_{n;a,b,c}(q)$, and $b_{s+1}q^{s+1}$ in $Z^{DD}_{m;a,b,c}(q)$. We will show that this is the first term where $Z^{DD}_{m;a,b,c}(q)$ differs from $Z^{DD}_{n;a,b,c}(q)$ (where we order terms by increasing exponent of q).

First, we show that $b_k \leq a_k$ for $k \leq s$, and thus $a_k = b_k$ for all $k \leq s$. Let $\pi \in DD_m(\sigma_{a,b,c})$ such that $w(\pi)/w(\pi_0) = q^t$ where $t \leq s$. Then the configuration π is obtained by performing t local moves from the minimally weighted configuration π_0 (see Figure 2.7 for the minimal configuration, see Figure 2.9 for the local moves). Performing t local moves stays within the boundary of $H(n)$ since $t \leq s$, and so we can truncate π to obtain a configuration in $DD_n(\sigma_{a,b,c})$ (see Example 11). Thus $b_k \leq a_k$ for $k \leq s$, and so $a_k = b_k$ for $k \leq s$.

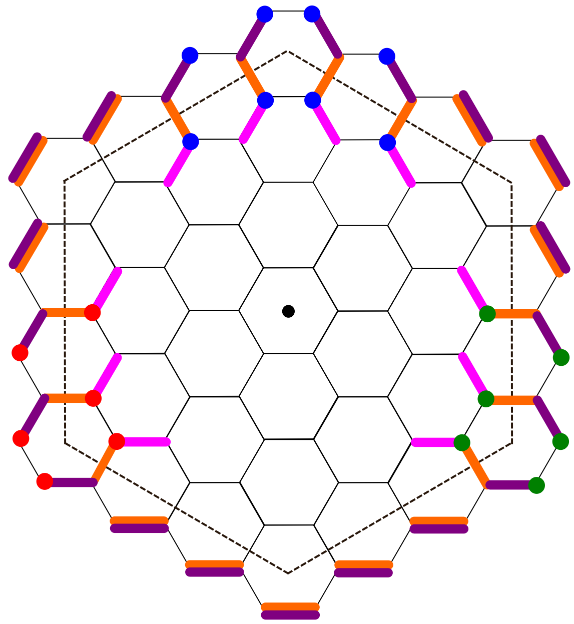

Figure 2.8: For any $\pi \in DD_3(\sigma_{2,2,1})$ (on the hexagon graph inside the dashed lines), the nodes will be matched as in this example (the pink edges). To extend π to a configuration in $DD_4(\sigma_{2,2,1})$, we can extend the matching by adding the matched edges that are crossing and outside of the dashed lines (orange and purple edges). We may repeat this move as many times as needed to produce a configuration in $DD_m(\sigma_{2,2,1})$ for any $m > 4$.

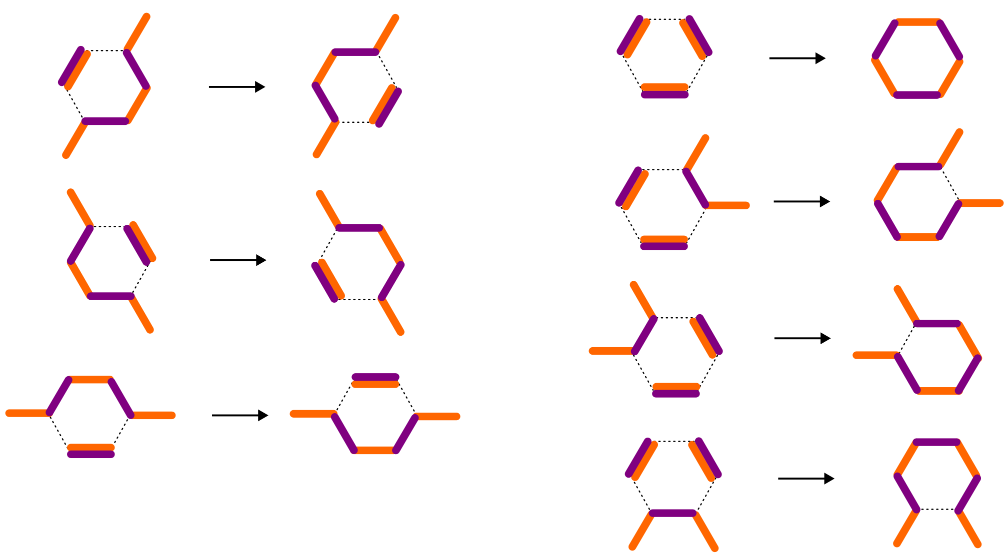

Figure 2.9: Local moves on the double-dimer configuration which increase the weight by a power of q

Example 11. Let $m = 4, n = 3$, and $(a, b, c) = (2, 2, 1)$, then $s = 1$. We consider a configuration $\pi \in DD_4(\sigma_{2,2,1})$ with $w(\pi)/w(\pi_0) = q$ (that is, one local move has been made on the minimally weighted configuration π_0 to obtain π). In Figure 2.10, in the leftmost image, we have the minimal configuration on $H(4)$. In the middle image, we have the configuration obtained by applying a local move to the minimal configuration at the location indicated by the pink dot. In the rightmost image, we have truncated the configuration that was obtained by this local move to $H(3)$. Note that this process gives a valid configuration in $DD_3(\sigma_{2,2,1})$.

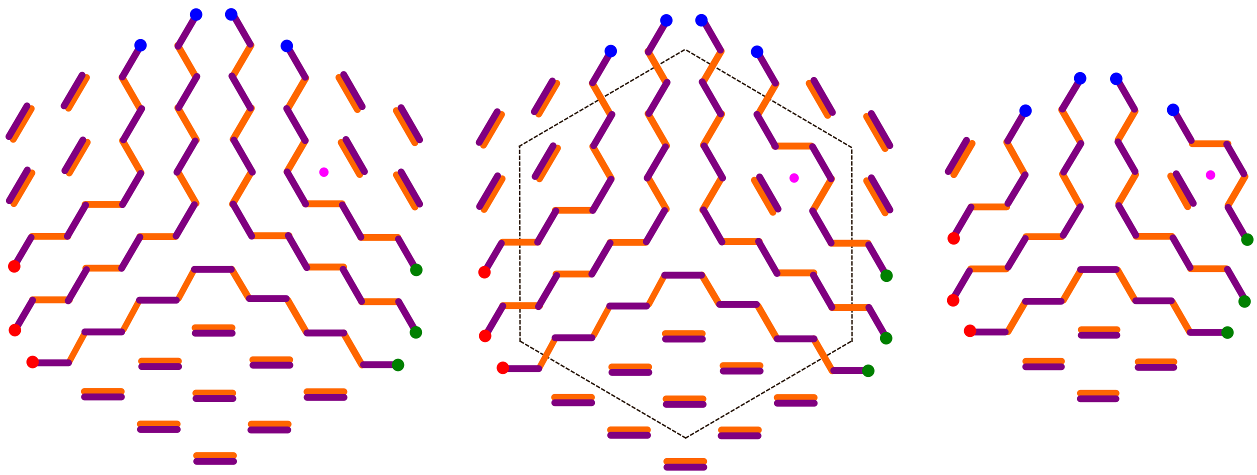

Figure 2.10: Minimal configuration on $H(4)$ (left), pink dot indicates where the local move is performed (middle). The configuration is then truncated to $H(3)$ (right).

Next, we show that $a_k < b_k$ for $k > s$. There are several cases to consider.

Case 1. Suppose $s = n-b$. Consider the configuration in $DD_m(\sigma_{a,b,c})$ obtained by performing $s+1$ local moves as in Example 12. This produces a configuration which cannot be truncated to a valid configuration in $DD_n(\sigma_{a,b,c})$, and so $a_k < b_k$.

Example 12. Let $m = 5, n = 4$ and $(a, b, c) = (1, 2, 1)$. Then $s = 2$. In Figure 2.11 (bottom row left), we show a configuration in $\pi \in DD_5(\sigma_{1,2,1})$ such that $w(\pi)/w(\pi_0) = s + 1$. In the bottom row middle image we see that truncating π to $H(4)$ does not produce a configuration in $DD_4(\sigma_{1,2,1})$.

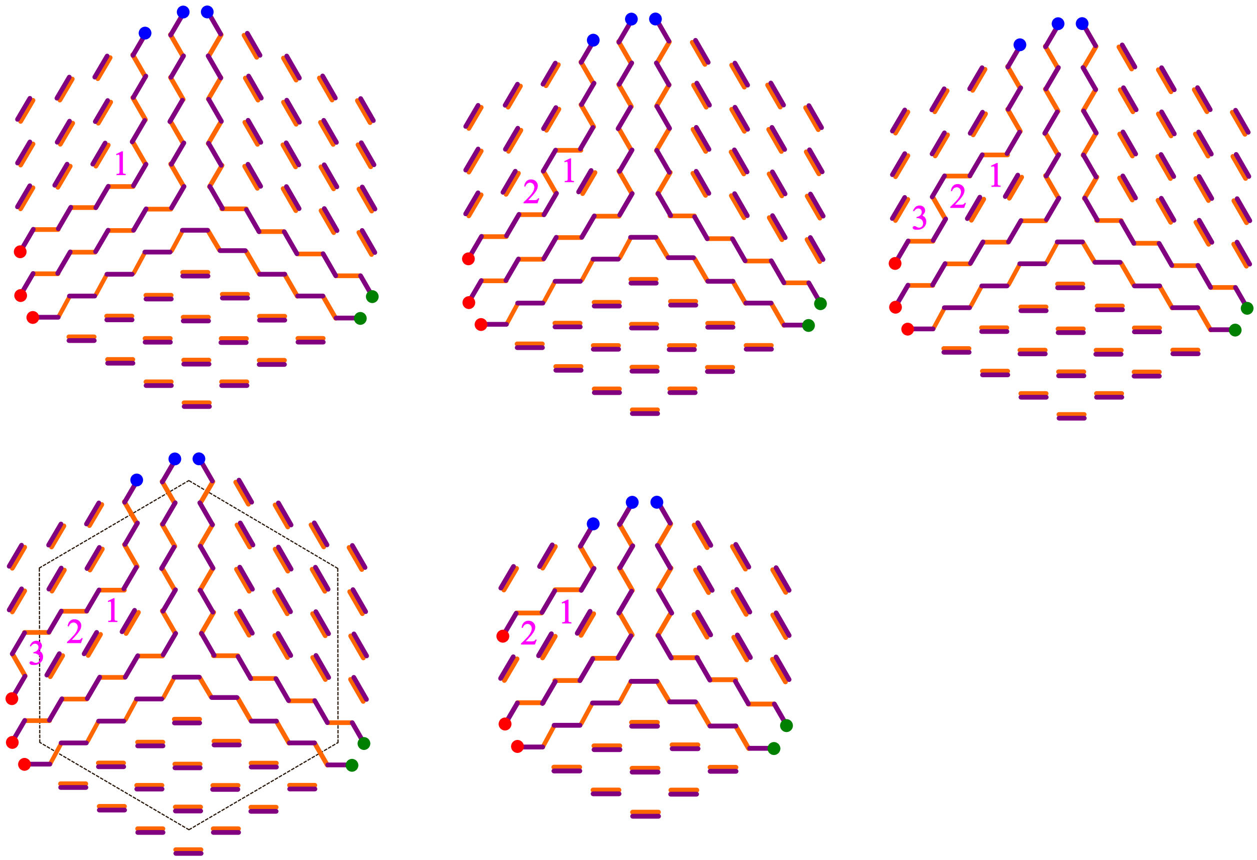

Figure 2.11: We obtain a configuration $\pi \in DD_5(\sigma_{1,2,1})$ (bottom left) from the minimal configuration on $H(5)$ (top left) by applying local moves at the locations indicated by the pink numbers, which we then truncate to $H(4)$ (bottom middle). This configuration is not in $DD_4(\sigma_{1,2,1})$ because it has the wrong red node set.

We omit Case 2 ($s = n - a$) and Case 3 ($s = n - c$) as they follow the same idea as Case 1 (for Case 2, perform local moves in the SE direction, and for Case 3, perform local moves in the S direction).

We have shown that for any $k > s$, we have a configuration that exists in $DD_m(\sigma_{a,b,c})$ which does not exist in $DD_m(\sigma_{a,b,c})$, and so $a_k < b_k$ for $k > s$.

Since $a_k = b_k$ for $k \leq s$, and $a_k < b_k$ for $k > s$, the limit is well-defined.

$\square$

Chapter 3

An Equivalence of Generating Functions

3.1 Part 1

Some of the work in this chapter was published in Séminaire Lotharingien de Combinatoire **93B** (2025), Article # 39, with coauthor Benjamin Young [BY25].

Fix $(a, b, c) \in \mathbb{N}^3$ throughout this section.

Definition 13. Let $\mathbf{M}$ be the infinite collection of boxes

$$\mathbf{M} = ([0, a) \times [0, b) \times \mathbb{Z}_{\geq 0}) \cup (\mathbb{Z}_{\geq 0} \times [0, b) \times [0, c)) \cup ([0, a) \times \mathbb{Z}_{\geq 0} \times [0, c)) \,.$$

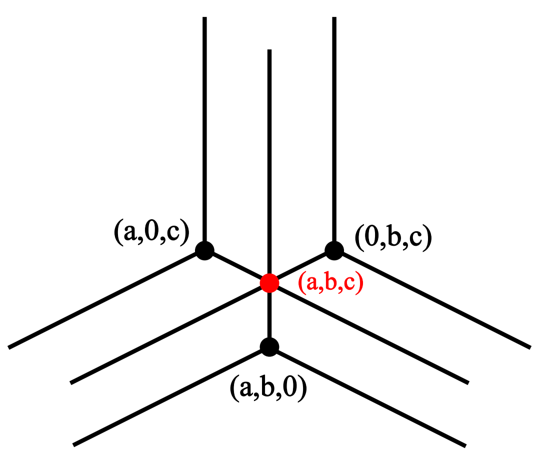

Figure 3.1: $\mathbf{M}$

Definition 14. Let π_1 be an order ideal under the product order on $\mathbb{Z}^3_{\geq 0}$, and let π_2 be an order ideal under the product order on $\mathbb{Z}^3_{\geq 0}$ which contains $\mathbf{M}$. Let $\pi_1 \cup \pi_2$ be the configuration obtained by superimposing π_1 and π_2, such that the coordinate $(0, 0, 0)$ in π_1 (the back corner of the room) and (a, b, c) in π_2 are aligned (see Figure 3.2). In the configuration $\pi_1 \cup \pi_2$, we let this aligned point be (a, b, c) (that is, we keep the coordinates of π_2). Note that there may exist boxes that overlap between π_1 and π_2 in this configuration. Let the *intersection space* of $\pi_1 \cup \pi_2$, denoted by *Int*, be the boxes $(i, j, k) \in \pi_1 \cup \pi_2$ such that $i \geq a, j \geq b$, and $k \geq c$.

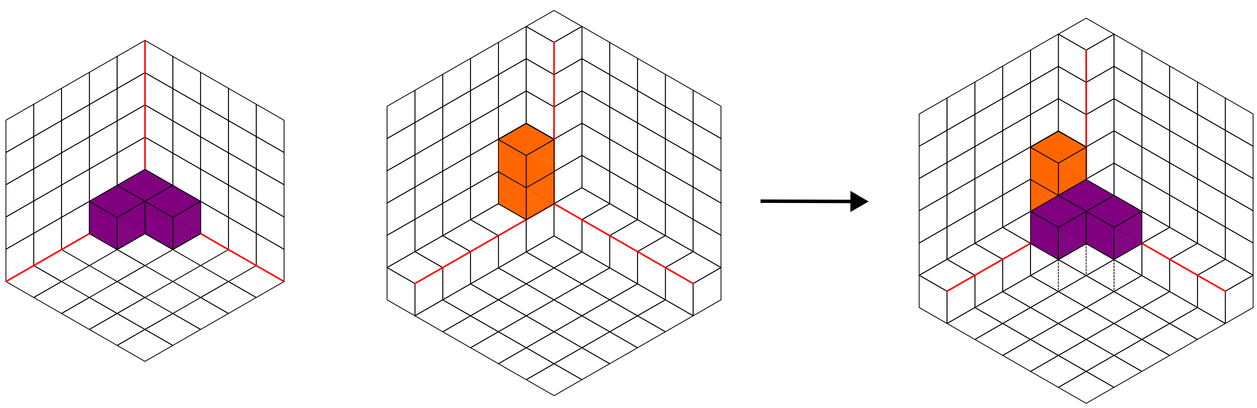

Figure 3.2: Let $a, b, c = 1$. In the leftmost image is a configuration π_1, in the middle is a configuration π_2, and on the right is $\pi_1 \cup \pi_2$.

Definition 15. Let (π_1, π_2) be defined as in Definition 14, and consider $\pi_1 \cup \pi_2$. Let

- $I_1^- = \{(i, j, k) \in \mathbb{Z}^3 \mid 0 \leq i < a,\ j \geq b,\ k \geq c\}$
- $I_2^- = \{(i, j, k) \in \mathbb{Z}^3 \mid i \geq a,\ 0 \leq j < b,\ k \geq c\}$
- $I_3^- = \{(i, j, k) \in \mathbb{Z}^3 \mid i \geq a,\ j \geq b,\ 0 \leq k < c\}$

We refer to I_1^-, I_2^-, I_3^- as *negative spaces*, and we refer to boxes in $I^- := (I_1^- \cup I_2^- \cup I_3^-) \cap (\pi_1 \cup \pi_2)$ as *negative space boxes.*

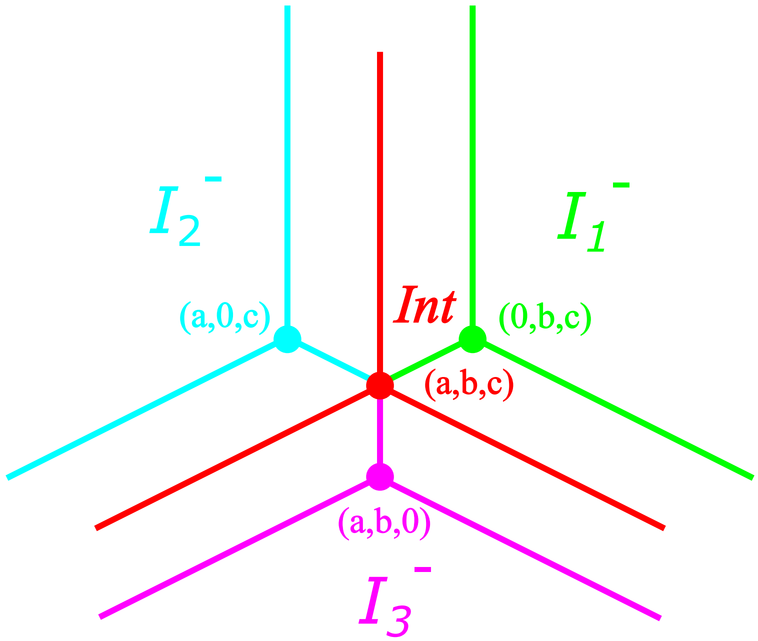

Figure 3.3: Negative spaces I_1^-, I_2^-, and I_3^-

Definition 16. Let (π_1, π_2) be defined as in Definition 14. Consider the configuration $\pi_1 \cup \pi_2$. Let $w = (w_1, w_2, w_3) \in Int$, and let

$$T_-(w) = \{(t, w_2, w_3) \in \pi_1 \cup \pi_2 \ \mid 0 \leq t < a\} \subset I_1^-$$

$$S_-(w) = \{(w_1, s, w_3) \in \pi_1 \cup \pi_2 \ \mid 0 \leq s < b\} \subset I_2^-$$

$$R_-(w) = \{(w_1, w_2, r) \in \pi_1 \cup \pi_2 \ \mid 0 \leq r < c\} \subset I_3^-$$

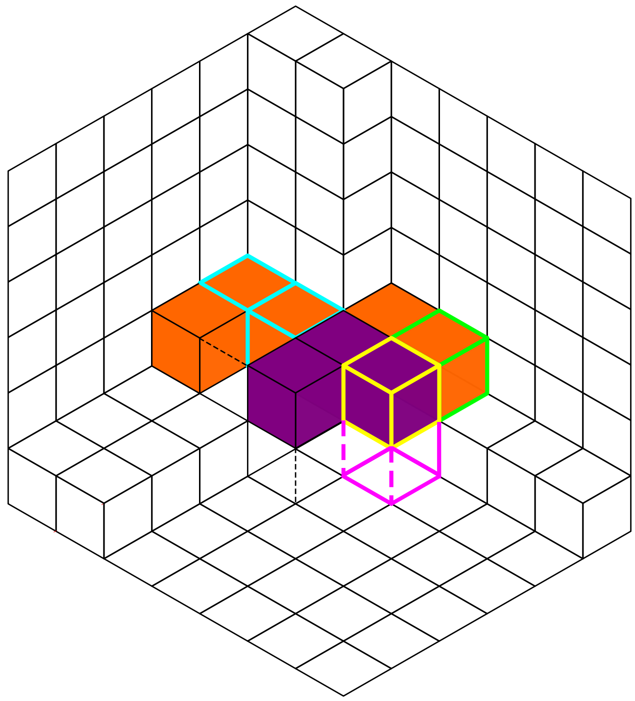

Figure 3.4: Example with $a = 1, b = 2, c = 1$ of a configuration $\pi_1 \cup \pi_2$, with boxes from π_1 in purple and boxes from π_2 in orange. Consider the box $w = (1, 3, 1)$ (purple box outlined in yellow). Then $T_-(w) = \{(0, 3, 1))\}$ (orange box with green outline), $S_-(w) = \{(1, 0, 1), (1, 1, 1)\}$ (orange boxes with blue outline) and $R_-(w) = \emptyset$ (empty space with pink outline).

Definition 17. We define what it means for a box $w = (w_1, w_2, w_3) \in Int$ to be *supported*, *weakly supported*, or *completely supported* in a negative space I_i^- for $i \in \{1, 2, 3\}$ by the following table:

	in I_1^-	in I_2^-	in I_3^-
supported (s.)	$(a-1, w_2, w_3) \in \pi_2$	$(w_1, b-1, w_3) \in \pi_2$	$(w_1, w_2, c-1) \in \pi_2$
weakly supported (w.s.)	w not s. in I_1^- and $\lvert T_-(w)\rvert \geq 1$	w not s. in I_2^- and $\lvert S_-(w)\rvert \geq 1$	w not s. in I_3^- and $\lvert R_-(w)\rvert \geq 1$
completely not supported (c.n.s)	$\lvert T_-(w)\rvert = 0$	$\lvert S_-(w)\rvert = 0$	$\lvert R_-(w)\rvert = 0$

Table 3.1: Supported (s.), weakly supported (w.s.), completely not supported (c.n.s) boxes

We say a box w is *not supported (n.s.)* in a negative space I_i^- if w is either weakly supported in or completely not supported in I_i^-.

Example 18. Let $a = 1, b = 2$, and $c = 1$. Consider the configuration $\pi_1 \cup \pi_2$ given in Figure 3.5 (boxes of π_1 are in purple, boxes of π_2 are in orange). Let $w = (1, 2, 1)$ (the purple box outlined in yellow), $u = (2, 2, 1)$ (the purple box outlined in blue), and $v = (1, 3, 1)$ (the purple box outlined in green). Then w is supported in I_1^-, I_2^- and I_3^-, u is supported in I_1^- and I_3^-, and weakly supported in I_2^-, and v is supported in I_1^- and I_2^- and completely not supported in I_3^-.

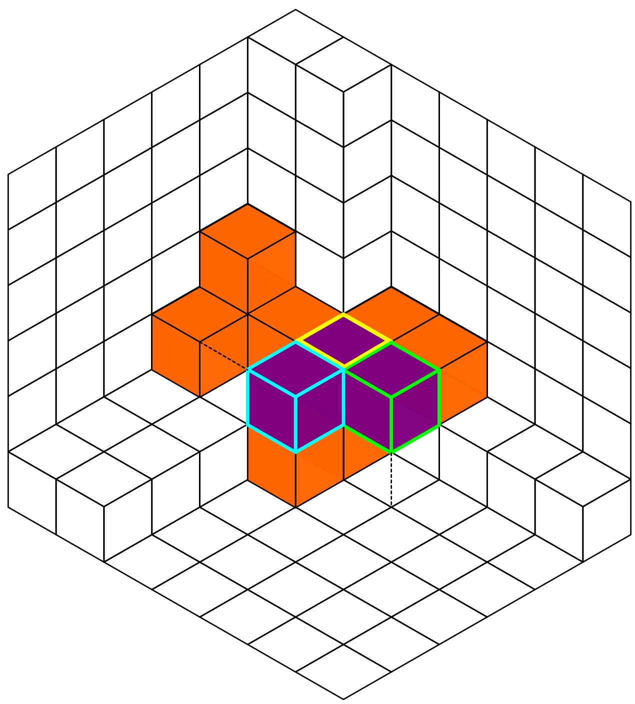

Figure 3.5: A configuration $\pi_1 \cup \pi_2$ with $(a, b, c) = (1, 2, 1)$.

Definition 19. Let $DP_{a,b,c}$ (for double plane partition) denote the set of all possible pairs (π_1, π_2), where π_1 and π_2 are defined as in Definition 14, such that every box $w \in Int$ is at least weakly supported in at least two negative spaces.

Example 20. Let $a = 1, b = 2$, and $c = 1$. Consider the configurations $\pi_1 \cup \pi_2$ and $\tilde{\pi}_1 \cup \tilde{\pi}_2$ given in Figure 3.6, where boxes of π_1 and $\tilde{\pi}_1$ are in purple, boxes of π_2 and $\tilde{\pi}_2$ are in orange. Let $w = (1, 2, 1)$ (the purple box outlined in yellow), $u = (2, 2, 1)$ (the purple box outlined in blue), and $v = (1, 3, 1)$ (the purple box outlined in green), which all exist in both $\pi_1 \cup \pi_2$

and $\tilde{\pi}_1 \cup \tilde{\pi}_2$. In $\pi_1 \cup \pi_2$, all the boxes in the intersection space (the purple boxes) are at least weakly supported in at least two negative spaces, and so the configuration is an element of $DP_{1,2,1}$. In $\tilde{\pi}_1 \cup \tilde{\pi}_2$, v is completely not supported in two negative spaces, I_1^- and I_3^-, and so this configuration is not an element of $DP_{1,2,1}$.

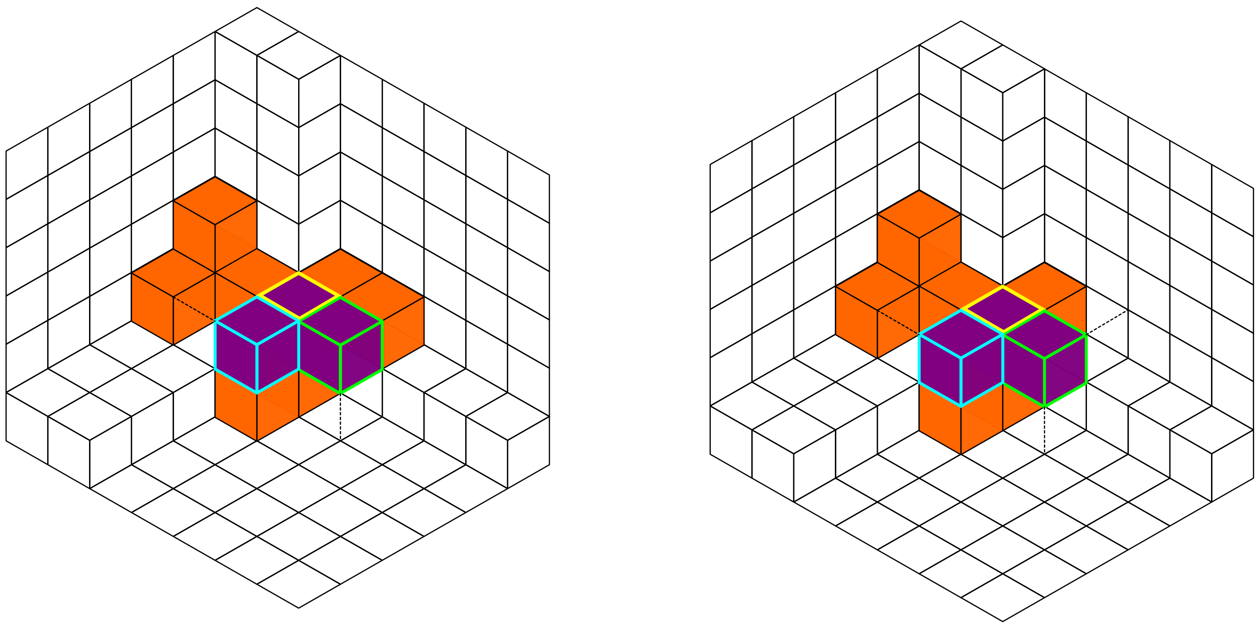

Figure 3.6: Left: $\pi_1 \cup \pi_2 \in DP_{1,2,1}$. Right: $\tilde{\pi}_1 \cup \tilde{\pi}_2 \notin DP_{1,2,1}$ (because v, outlined in green, is completely not supported in two negative spaces).

Definition 21. We say that $(\pi_1, \pi_2) \in DP_{a,b,c}$ *is associated to* $\eta \in DB_{a,b,c}$ if it satisfies the following conditions:

(1) The set of coordinates supporting boxes in $\pi_1 \cup \pi_2$ equals the set of coordinates supporting boxes in η, where we associate the point (a, b, c) in $\pi_1 \cup \pi_2$ and in η, and

(2) $\pi_1 \cap \pi_2 =$ type III boxes of η.

Definition 22. Let $\mathcal{DP}_{a,b,c}(\eta)$ denote the set of all $(\pi_1, \pi_2) \in DP_{a,b,c}$ such that (π_1, π_2) is associated to $\eta \in DB_{a,b,c}$.

Lemma 23. *$\mathcal{DP}_{a,b,c}(\eta)$ and $\mathcal{DP}_{a,b,c}(\tilde{\eta})$ are disjoint for $\eta, \tilde{\eta} \in DB_{a,b,c}$, $\eta \neq \tilde{\eta}$.*

Proof. Suppose there exists an element $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}(\eta) \cap \mathcal{DP}_{a,b,c}(\tilde{\eta})$ for $\eta \neq \tilde{\eta} \in DB_{a,b,c}$. Then by Definition 22, η and $\tilde{\eta}$ have the same set of coordinates which support boxes, and the same type III boxes. Since $\eta, \tilde{\eta} \in DB_{a,b,c}$, we have that for both η and $\tilde{\eta}$, the type I boxes only exist in the negative spaces, and type II boxes only exist in the intersection spaces. Thus η and $\tilde{\eta}$ have the same type II and type I boxes, and so $\eta = \tilde{\eta}$. □

Definition 24. Let

$$\mathcal{DP}_{a,b,c} = \bigcup_{\eta \in DB_{a,b,c}} \mathcal{DP}_{a,b,c}(\eta)$$

Note that for any $\eta \in DB_{a,b,c}$, we have that $\mathcal{DP}_{a,b,c}(\eta) \subset \mathcal{DP}_{a,b,c} \subset DP_{a,b,c}$.

Definition 25. Let $(\pi_1, \pi_2) \in DP_{a,b,c}$. We call a box $w = (w_1, w_2, w_3) \in \pi_1 \cup \pi_2$:

- a type III_{DP} box if $w \in \pi_1 \cap \pi_2$ (in this case w must be supported in all three negative spaces since it is present in π_2),
- a type I^-_{DP} box if $w \notin \mathit{Int}$ (in this case, we must have that $w \notin \pi_1$ and $w \in \pi_2$),
- a type II_{DP} box if $w \in \mathit{Int} \setminus \mathrm{III}_{DP}$ (i.e. all the other boxes left).

Note that for $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}(\eta)$ for $\eta \in DB_{a,b,c}$, we have that

$$\{\text{type } \mathrm{I}^-_{DP} \text{ boxes in } \pi_1 \cup \pi_2\} = \{\text{type I boxes in } \eta\},$$

$$\{\text{type } \mathrm{II}_{DP} \text{ boxes in } \pi_1 \cup \pi_2\} = \{\text{type II boxes in } \eta\},$$

$$\{\text{type } \mathrm{III}_{DP} \text{ boxes in } \pi_1 \cup \pi_2\} = \{\text{type III boxes in } \eta\}.$$

Algorithm 1. (DP Labeling Algorithm) Let $(\pi_1, \pi_2) \in DP_{a,b,c}$.

1. Label boxes in $\mathit{Int} \setminus \mathrm{III}_{DP}$ by the subscript of the negative space(s) I_i^- that they are weakly supported in or completely not supported in (i.e. not supported in).
   (a) If any boxes are assigned more than one label in this step, terminate with failure.

(b) Label unlabeled boxes as follows. If at any point an unlabeled box receives more than one label, terminate with failure.

i. Label unlabeled boxes in $Int \setminus Ш_{DP}$ by the labels of any boxes they are connected to which were labeled in Step 1a.

ii. Repeat this process, labeling any unlabeled boxes by the labels of any boxes they are connected to, until there are no unlabeled boxes to label.

2. If any connected component of $Int \setminus Ш_{DP}$ has conflicting labels, terminate with failure.

Note that when labeling $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}(\eta)$ for $\eta \in DB_{a,b,c}$, any remaining unlabeled boxes after the algorithm has completed are moveable type Ⅱ boxes in η.

Example 26. Let $(a, b, c) = (1, 2, 1)$. Consider the configuration $\pi_1 \cup \pi_2$ for $(\pi_1, \pi_2) \in DP_{1,2,1}$ given by Figure 3.7, which fails the DP Labeling Algorithm in Step 1a. We have $Int \setminus Ш_{DP} = \{(1, 2, 1), (2, 2, 1), (1, 3, 1)\}$ (the purple boxes in the rightmost image, coming from π_1 in the leftmost image). The box $(1, 2, 1)$ (outlined in yellow) is supported in I_1^-, I_2^-, and I_3^-, so it is unlabeled. The box $(1, 3, 1)$ (outlined in green) is supported in I_2^- and I_3^- and completely not supported in I_1^-, so it gets label 1. The box $(2, 2, 1)$ (outlined in blue) is supported in I_1^-, weakly supported in I_2^-, and completely not supported I_3^-, and so it gets labels 2 and 3. Thus it has failed the labeling algorithm in Step 1a.

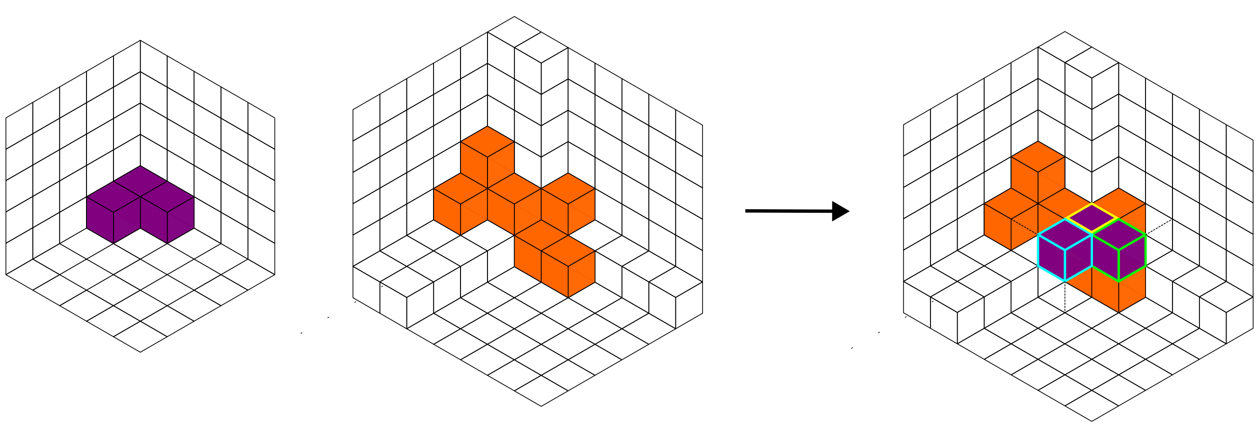

Figure 3.7: Left: π_1, Middle: π_2, Right: $\pi_1 \cup \pi_2$

Example 27. Let $(a, b, c) = (1, 2, 1)$. Consider the configuration $\pi_1 \cup \pi_2$ for $(\pi_1, \pi_2) \in DP_{1,2,1}$ given by Figure 3.8, which fails the DP Labeling Algorithm in Step 1b. In Step 1a, the box $(1, 2, 1)$ (outlined in yellow) is unlabeled, the box $(2, 2, 1)$ (outlined in blue) gets label 3, and the box $(1, 3, 1)$ (outlined in green) gets label 1. Then in Step 1b, the box $(1, 2, 1)$ (outlined in yellow) gets labels 1 and 3, thus it fails.

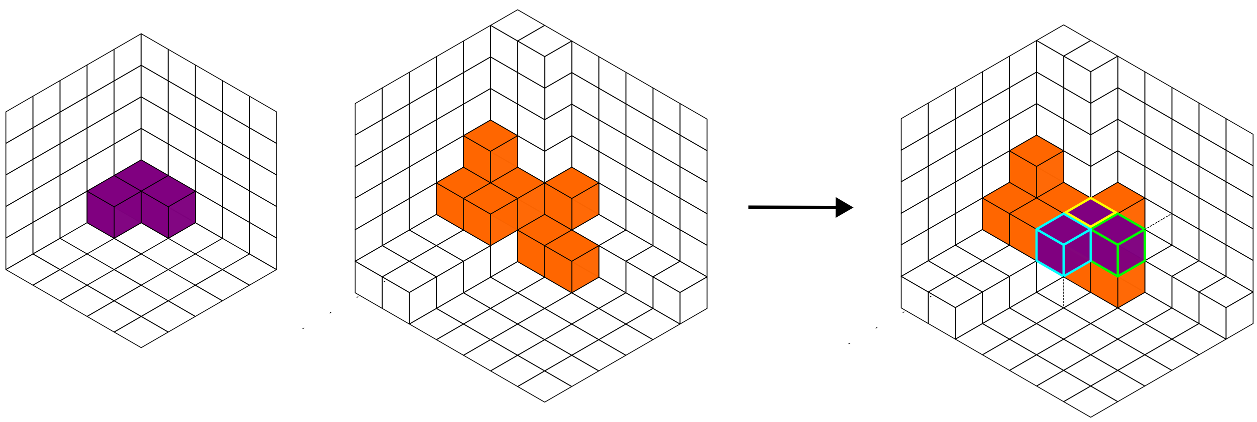

Figure 3.8: Left: π_1, Middle: π_2, Right: $\pi_1 \cup \pi_2$

Example 28. Let $(a, b, c) = (1, 2, 1)$. Consider the configuration $\pi_1 \cup \pi_2$ for $(\pi_1, \pi_2) \in DP_{1,2,1}$ given by Figure 3.9, which passes the DP Labeling Algorithm. The box $(1, 2, 1)$ (outlined in yellow) remains unlabeled since it is type III_{DP} (that is, $(1, 2, 1) \in \pi_1 \cap \pi_2$). The box $(2, 2, 1)$ (outlined in blue) is unlabeled since is is supported in all three negative spaces, and the box $(1, 3, 1)$ (outlined in green) gets label 1. In Step 1b, the box $(2, 2, 1)$ remains unlabeled since it is not connected to any boxes with labels.

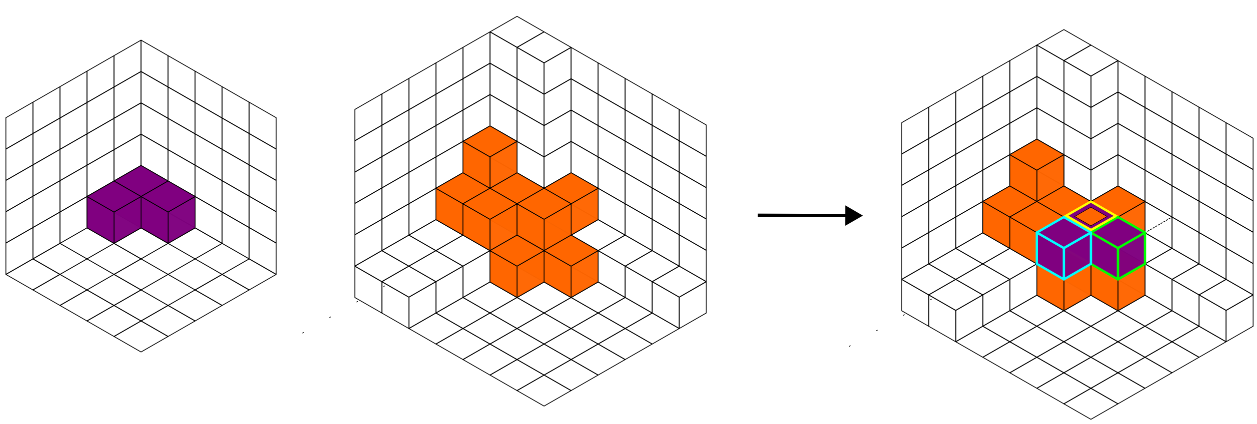

Figure 3.9: Left: π_1, Middle: π_2, Right: $\pi_1 \cup \pi_2$

Definition 29. Let $w = (w_1, w_2, w_3)$ be a box in $\mathbb{Z}^3$. Define the *back neighbors of* w to be

$$BN(w) = \{(w_1 - 1, w_2, w_3), (w_1, w_2 - 1, w_3), (w_1, w_2, w_3 - 1)\},$$

and define the *front neighbors* of w to be

$$FN(w) = \{(w_1 + 1, w_2, w_3), (w_1, w_2 + 1, w_3), (w_1, w_2, w_3 + 1)\}.$$

Definition 30. Let $(\pi_1, \pi_2) \in DP_{a,b,c}$, and let $w = (w_1, w_2, w_3) \in Int$. Define the following:

- $w^1 = (a - 1, w_2, w_3) \in I_1^-$,
- $w^2 = (w_1, b - 1, w_3) \in I_2^-$, and
- $w^3 = (w_1, w_2, c - 1) \in I_3^-$.

Example 31. Let $(a, b, c) = (1, 2, 1)$, and consider the configuration $\pi_1 \cup \pi_2$ for $(\pi_1, \pi_2) \in DP_{a,b,c}$ given by Figure 3.10. Consider $w = (1, 3, 1)$ (outlined in yellow). Then $w^1 = (0, 3, 1) \in I_1^-$ (outlined in green), $w^2 = (1, 1, 1) \in I_2^-$ (outlined in blue), and $w^3 = (1, 3, 0) \in I_3^-$ (outlined in pink).

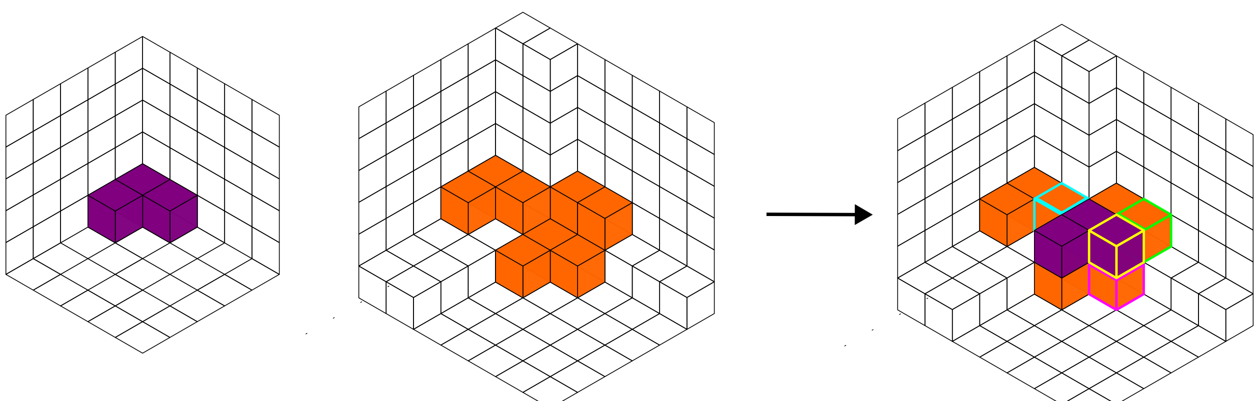

Figure 3.10: Left: π_1, Middle: π_2, Right: $\pi_1 \cup \pi_2$

Lemma 32. *Let $(\pi_1, \pi_2) \in DP_{a,b,c}$, and let $w = (w_1, w_2, w_3) \in Int \setminus III_{DP}$ such that there exists $u \in BN(w) \cap (\pi_1 \cup \pi_2)$.*

We have the following cases:

1. *If* $u = (w_1, w_2 - 1, w_3)$*, then* $u^2 = w^2$*,* $u^1 \in BN(w^1)$*, and* $u^3 \in BN(w^3)$*.*

2. *If* $u = (w_1, w_2, w_3 - 1)$*, then* $u^3 = w^3$*,* $u^1 \in BN(w^1)$*, and* $u^2 \in BN(w^2)$*.*

3. *If* $u = (w_1 - 1, w_2, w_3)$*, then* $u^1 = w^1$*,* $u^2 \in BN(w^2)$*, and* $u^3 \in BN(w^3)$*.*

Proof. By Definitions 30 and 29. □

Theorem 33. *Let* $(\pi_1, \pi_2) \in DP_{a,b,c}$*, then* $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}$ *iff* (π_1, π_2) *passes the DP Labeling Algorithm 1.*

Proof. Suppose that $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}$. We want to show that (π_1, π_2) passes Algorithm 1. Since $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}$, we know it is associated to some $\eta \in DB_{a,b,c}$. Then, since every box in the intersection space of a double-box configuration must be contained in at least two plane partitions, we have that every box in the intersection space of (π_1, π_2) is supported in at least two negative spaces, and so it passes Step 1a of Algorithm 1.

Lemma 34. *Let* $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}$*, and let* $u, v \in Int \setminus III_{DP}$ *be connected boxes such that* u *was labeled* i *in Step 1a of Algorithm 1 and* v *was unlabeled in Step 1a of Algorithm 1. Then* $v \in BN(u)$*.*

Proof. Suppose $v \in FN(u)$. Since u was labeled i in Step 1a, u is not supported in I_i^-, and so $u^i \notin \pi_1 \cup \pi_2$. Since $u^i \notin \pi_1 \cup \pi_2$ and $v^i \in \pi_1 \cup \pi_2$ (since v was unlabeled in Step 1a, that means it is supported in all three negative spaces), we have that $u^i \neq v^i$, and so by Lemma 32 we have that $u^i \in BN(v^i)$. Then we have a contradiction with π_2 being an order ideal under the product order on $\mathbb{Z}^3_{\geq 0}$ containing M. Thus $v \in BN(u)$.

□

Suppose the Labeling Algorithm fails in Step 1b. Then there exists a box $w \in Int \setminus \mathrm{III}_{DP}$ which is supported in all three negative spaces, and is connected to at least two other boxes in $Int \setminus \mathrm{III}_{DP}$ that give conflicting labels. Let these boxes be u and v.

Case 1. Suppose both u, v were labeled in Step 1a. Suppose u was labeled i, that is, u is not supported in I_i^-, and suppose v was labeled j, that is, v is not supported in I_j^-, with

$i \neq j \in \{1,2,3\}$. By Lemma 34, we have that both $u, v \in FN(w)$. Then since u is not supported in I_i^-, and since (π_1, π_2) is associated to the double-box configuration $\eta = (\eta_1, \eta_2, \eta_3)$, u must be contained in the plane partitions η_j and η_k, for $\{i, j, k\} = \{1, 2, 3\}$. Since v is not supported in I_j^-, it must be contained in the plane partitions η_i and η_k. Since $w \in BN(u)$, we must have w in η_j and η_k as well. Since $w \in BN(v)$, we must have that $w \in \eta_i$ and $w \in \eta_k$. Thus w is a type III_{DP} box, however this is a contradiction since $w \in Int \setminus \text{III}_{DP}$.

Case 2. Suppose u was labeled i in Step 1a, and v was labeled j in Step 1b(i), that is, v received its label from a connected box x which was labeled j in Step 1a. Since u was labeled i in Step 1a, $u \in \eta_j, \eta_k$, and since x was labeled j in Step 1a, $x \in \eta_i, \eta_k$. By Lemma 34, $u \in FN(w)$, and so $w \in \eta_j, \eta_k$. Also by Lemma 34, we have $x \in FN(v)$, and so $v \in \eta_i, \eta_k$. Suppose $v \in FN(w)$. Then $w \in \eta_i, \eta_k$, and so $w \in \eta_i, \eta_j, \eta_k$, a contradiction since this implies w is a type III_{DP} box. Thus $v \in BN(w)$. But then we have $v \in \eta_j, \eta_k$, and so $v \in \eta_i, \eta_j, \eta_k$, also a contradiction.

Lemma 35. *Let $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}(\eta)$, and let $C \subset Int \setminus \text{III}_{DP}$ be a connected component of boxes that were unlabeled after Step 1a of Algorithm 1. After Step 1b of Algorithm 1, if any box in C is labeled i, then every box of C has the same label, and are all contained in η_j, η_k for $\{i, j, k\} = \{1, 2, 3\}$ where (η_1, η_2, η_3) is any representative for η.*

Proof. Suppose there exist two boxes $u, v \in C$, such that u is labeled i and v is labeled j in Step 1b of Algorithm 1, for $i \neq j$. For every box $x \in C$, since $x \in Int \setminus \text{III}_{DP}$ and since (π_1, π_2) is associated to $\eta \in DB_{a,b,c}$, we know that x is in exactly two of η_1, η_2, η_3. Since u was labeled i in Step 1b, there exists a box $x_1 \in Int \setminus \text{III}_{DP}$ such that x_1 was labeled i in Step 1a, and x_1 is connected to a box $y_1 \in C$. Since v was labeled j in Step 1b, there exists a box $x_2 \in Int \setminus \text{III}_{DP}$ such that x_2 was labeled j in Step 1a, and x_2 is connected to a box $y_2 \in C$. By Lemma 34, we have $y_1 \in BN(x_1)$ and $y_2 \in BN(x_2)$. Then since x_1 was labeled i in Step 1a, and since $x_1 \in Int$, and since η is a double-box configuration, we must have that $x_1 \in \eta_j, \eta_k$. Then since $y_1 \in BN(x_1)$, we must have that $y_1 \in \eta_j, \eta_k$. Similarly, we have that $y_2 \in \eta_i, \eta_k$. Since every box in C must be contained in exactly two of η_1, η_2, η_3, at some point in C there exist boxes z_1 and z_2, such that $z_1 \in \eta_s, \eta_t$ and $z_2 \in \eta_s, \eta_r$ for $\{s, t, r\} = \{1, 2, 3\}$. This is a contradiction, since whichever box is behind the other would be a type III_{DP} box. Thus there must exist only one of x_1, x_2, that is, a box x labeled s in Step 1a that is connected to a box in C, with every box in C labeled s, and contained in η_t, η_r for $\{s, t, r\} = \{1, 2, 3\}$. □

Case 3. Suppose u was labeled i in Step 1a, and v was labeled j in Step 1b(ii), that is, v is contained in a connected component C of unlabeled boxes that received label j from a box x labeled j in Step 1a. By Lemma 34, $u \in FN(w)$, and so $w \in \eta_j, \eta_k$. By Lemma 35, we have that $v \in \eta_i, \eta_k$. Then we have $v, w \in C$ contained in two different pairs of η_1, η_2, η_3, a contradiction.

In the cases where u and v are both labeled in Step 1b, we have a contradiction by Lemma 35.

Suppose the DP Labeling Algorithm fails in Step 2. Since (π_1, π_2) passed Step 1b, we know that any connected components containing boxes that were unlabeled in Step 1a are consistently labeled. Thus we consider the case where we have a connected component C, which contains boxes u and v with different labels. Similarly to the argument in the proof of Lemma 35, at some point there are connected boxes which are in two different pairs of η_1, η_2, η_3, a contradiction.

Next, suppose that $(\pi_1, \pi_2) \in DP_{a,b,c}$ passes the DP Labeling Algorithm. We want to show that $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}$, that is, we want to show that (π_1, π_2) is associated to some $\eta \in DB_{a,b,c}$. We will show that the boxes in $\pi_1 \cup \pi_2$ can be partitioned into three plane partitions η_1, η_2 and η_3 such that (η_1, η_2, η_3) create a double-box configuration.

Let

$$\eta_i = (\pi_2 \cap I_i^-) \cup (\pi_1 \cap \pi_2)$$

for each $i \in \{1,2,3\}$. Note that each η_i is an order ideal under the product order on $\mathbb{Z}^3_{\geq 0}$, where we associate the point $(0,b,c)$ for η_1, $(a,0,c)$ for η_2, and $(a,b,0)$ for η_3, with the point $(0,0,0)$.

If $Int \setminus Ш_{DP}$ is empty, then we are done. If not, then we consider boxes $w \in Int \setminus Ш_{DP}$ one at a time, in the order given by the Traversing Algorithm 2, and add them to our plane partitions η_1, η_2, and η_3 according to the steps below.

Algorithm 2. (Traversing Algorithm) Let $(\pi_1, \pi_2) \in DP_{a,b,c}$.

$$z_{max} = \underset{r}{max}\{r \mid (s,t,r) \in Int \setminus Ш_{DP}\}$$

for k in range$(c, z_{max}+1)$:

$$y_{max} = \underset{t}{max}\{t \mid (s,t,k) \in Int \setminus Ш_{DP}\}$$

for j in range$(b, y_{max}+1)$:

$$x_{min} = \underset{s}{min}\{s \mid (s,j,k) \in Int \setminus Ш_{DP}\}$$
$$x_{max} = \underset{s}{max}\{s \mid (s,j,k) \in Int \setminus Ш_{DP}\}$$

for i in range(x_{min}, x_{max}):

$$w = (i,j,k)$$

For each box w visited in the Traversing Algorithm, complete the steps below. If the process terminates, move to the next box.

1. If w is labeled i by Algorithm 1, place w into η_j and η_k for $\{i,j,k\} = \{1,2,3\}$.

2. If w is unlabeled and if $w \in \eta_1 \cup \eta_2 \cup \eta_3$ (this means we have already put w into plane partitions because it was connected to an unlabeled box that was already processed),

terminate. Else, choose any two of η_1, η_2, η_3 to place w in. Place all boxes in the connected component containing unlabeled boxes of $\mathit{Int} \setminus \text{III}_{DP}$ and w into the same two plane partitions.

By Algorithm 1, in each connected component of $\mathit{Int} \setminus \text{III}_{DP}$, every box has the same label, and thus gets assigned to the same two plane partitions by the process above. Thus we have three plane partitions η_1, η_2, η_3. Also note that we have assigned every box in the intersection space to two or three plane partitions, thus the condition on double-box configurations is satisfied.

□

Definition 36. For $(\pi_1, \pi_2) \in DP_{a,b,c}$, let $|\pi_1|$ be the number of boxes in π_1, and let $|\pi_2|$ be the number of boxes in π_2 minus those in $\mathbf{M}$ (see Definition 13).

Definition 37. Given $\eta \in DB_{a,b,c}$, define $(\pi_1, \pi_2)_\eta = (\pi_1^\eta, \pi_2^\eta) \in \mathcal{DP}_{a,b,c}(\eta)$ by:

- Let π_1^η contain the type II and type III boxes of η, i.e. the boxes in the intersection space.
- Let π_2^η contain the type I and type III boxes of η plus those in $\mathbf{M}$.

Note that for $\eta \in DB_{a,b,c}$, $(\pi_1^\eta, \pi_2^\eta) \in \mathcal{DP}_{a,b,c}(\eta)$, and $|\eta| = |\pi_1^\eta| + |\pi_2^\eta|$. Also note that (π_1^η, π_2^η) is well-defined (that is, there is only one for each double-box configuration η). Finally, note that by Lemma 23, for $\eta \neq \tilde{\eta} \in DB_{a,b,c}$, $(\pi_1^\eta, \pi_2^\eta) \neq (\pi_1^{\tilde{\eta}}, \pi_2^{\tilde{\eta}})$.

For $(\pi_1, \pi_2) \neq (\tilde{\pi}_1, \tilde{\pi}_2) \in \mathcal{DP}_{a,b,c}(\eta)$, since they have the same multiset of boxes, we have that $|\pi_1| + |\pi_2| = |\tilde{\pi}_1| + |\tilde{\pi}_2|$.

Next, we define the generating function for objects of $\mathcal{DP}_{a,b,c}$ as:

Definition 38.

$$Z^{\mathcal{DP}}_{a,b,c}(q) = \sum_{(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}} q^{|\pi_1| + |\pi_2|}$$

Lemma 39. *For $\eta \in DB_{a,b,c}$, $|\mathcal{DP}_{a,b,c}(\eta)| = 2^m$ where m is the number of connected components of moveable type II boxes in η.*

Proof. Let $(\pi_1, \pi_2), (\tilde{\pi}_1, \tilde{\pi}_2) \in \mathcal{DP}_{a,b,c}(\eta)$ for $\eta \in DB_{a,b,c}$, with $(\pi_1, \pi_2) \neq (\tilde{\pi}_1, \tilde{\pi}_2)$. By Theorem 33, we have that both (π_1, π_2) and $(\tilde{\pi}_1, \tilde{\pi}_2)$ pass the DP Labeling Algorithm. Consider (π_1, π_2). We will show that any labeled box must be in π_1. Suppose $w \in Int \setminus \text{III}_{DP}$ was labeled i in Step 1a of the DP Labeling Algorithm. Then w is not supported in I_i^-, and thus $w \notin \pi_2$. Suppose $w \in Int \setminus \text{III}_{DP}$ was labeled i in Step 1b of the Labeling Algorithm. Then there exists a box $x \in Int \setminus \text{III}_{DP}$ that was labeled i in Step 1a, which is connected to a unlabeled box y in the same connected component C of unlabeled boxes as w. Then since x must be in π_1, and since $x \in FN(y)$ by Lemma 34, all the boxes in C must also be in π_1. Similarly for $(\tilde{\pi}_1, \tilde{\pi}_2)$.

Thus, the only thing that can differ between (π_1, π_2) and $(\tilde{\pi}_1, \tilde{\pi}_2)$ corresponds to the location of the moveable type II boxes, i.e. the unlabeled boxes. For example, a moveable type II box may be in π_1 in (π_1, π_2), and in $\tilde{\pi}_2$ in $(\tilde{\pi}_1, \tilde{\pi}_2)$.

Suppose w is a moveable type II box (i.e. w is unlabeled in $\pi_1 \cup \pi_2$ and $\tilde{\pi}_1 \cup \tilde{\pi}_2$), and suppose $w \in \pi_i$ for $i \in \{1, 2\}$. Let C be the connected component of moveable type II boxes containing w. Consider $v \in C$ such that v is connected to w, and suppose $v \in \pi_j$, for $\{i, j\} = \{1, 2\}$. Then if $w \in BN(v)$, then π_j is not an order ideal under the product order on $\mathbb{Z}^3_{\geq 0}$, and if $w \in FN(v)$, then π_i is not an order ideal under the product order on $\mathbb{Z}^3_{\geq 0}$ containing $\mathbf{M}$.

If there exist two boxes in the same connected component C of moveable type II boxes such that one is in π_1 and the other is in π_2, then there exists two connected boxes $u, v \in C$ where u is in one of π_1, π_2 and v is in the other. Then by the argument above we have a contradiction, and so every element of C must be in the same plane partition π_1 or π_2. Thus $|\mathcal{DP}_{a,b,c}(\eta)| = 2^m$ where m is the number of connected components of moveable type II boxes in η.

$\square$

Theorem 40.

$$Z^{DB}_{a,b,c}(q) = Z^{\mathcal{DP}}_{a,b,c}(q)$$

Proof. Recall Definition 38, which states

$$Z^{\mathcal{DP}}_{a,b,c}(q) = \sum_{(\pi_1,\pi_2)\in\mathcal{DP}_{a,b,c}} q^{|\pi_1|+|\pi_2|}.$$

Next, since $\mathcal{DP}_{a,b,c} = \bigsqcup_{\eta\in DB_{a,b,c}} \mathcal{DP}_{a,b,c}(\eta)$ (by Lemma 23), and since the weight for every element in $\mathcal{DP}_{a,b,c}(\eta)$ is the same, we have

$$\sum_{(\pi_1,\pi_2)\in\mathcal{DP}_{a,b,c}} q^{|\pi_1|+|\pi_2|} = \sum_{\eta\in DB_{a,b,c}} |\mathcal{DP}_{a,b,c}(\eta)| q^{|\pi_1^\eta|+|\pi_2^\eta|},$$

then by Lemma 39,

$$= \sum_{\eta\in DB_{a,b,c}} 2^m q^{|\eta|} = Z^{DB}_{a,b,c}(q),$$

where m is the number of connected components of moveable type II boxes in η. □

3.2 Part 2

In this section, we show that the generating functions $Z^{\mathcal{DP}}_{a,b,c}(q)$ and $Z^{DD}_{a,b,c}(q)$ are equal.

Definition 41. Let π be a plane partition. We denote the corresponding dimer configuration via the folklore bijection (see Figure 1.1) by D_π. For $(\pi_1,\pi_2) \in DP_{a,b,c}$, we denote the double-dimer configuration corresponding to superimposing D_{π_1} and D_{π_2} by $D_{(\pi_1,\pi_2)}$ (with the hexagons corresponding to (a,b,c) in both aligned, we call this the "center" hexagon).

We define the following map $P : DP_{a,b,c} \to DP_{a,b,c}$ such that P takes pairs (π_1,π_2) that are associated to some $\eta \in DB_{a,b,c}$, to the element $(\pi_1,\pi_2)_\eta \in \mathcal{DP}_{a,b,c}(\eta)$ (see Definition 37), that is, $P(\mathcal{DP}_{a,b,c}(\eta)) = (\pi_1,\pi_2)_\eta$.

Definition 42. We define the map $P : DP_{a,b,c} \to DP_{a,b,c}$ as follows. Informally, we construct $P(\pi_1,\pi_2)$ by moving every box from π_2 that we can into π_1 such that we still have an element in $DP_{a,b,c}$.

Consider the set of boxes $W = \{(w_1,w_2,w_3) \in \pi_2 \mid w_1 \geq a, w_2 \geq b, w_3 \geq c\} \setminus \pi_1$ (these are boxes in the intersection space of $\pi_1 \cup \pi_2$ that are in only π_2). We choose boxes in this set one at at time in the order given by the Traversing Algorithm 2, replacing $Int \setminus \text{Ш}_{DP}$ with W. For each box $w \in W$ reached in the algorithm, we apply the following process (if the process is terminated, we move to the next box):

1. If $w \in \pi_1$ (if we have already moved w previously by moving a connected component of a box), terminate.
2. If $\pi_1 + w$ is not an order ideal under the product order on $\mathbb{Z}^3_{\geq 0}$, terminate.
3. If $\pi_1 + w$ is an order ideal under the product order on $\mathbb{Z}^3_{\geq 0}$, then complete the following step.
   (a) Let G be the connected component of W containing w. Let $\pi_1 = \pi_1 + G$, and let $\pi_2 = \pi_2 - G$.

Note that moving connected components $G \subset W$ ensures that the constructed (π_1, π_2) is contained in $DP_{a,b,c}$ at each step.

Next, we are going to map elements $(\pi_1, \pi_2) \in DP_{a,b,c}$ to an AB configuration – a plane-partition-like object defined in [JWY20]. In [JWY20], Jenne, Webb, and Young, show that AB configurations which pass their labeling algorithm (see Algorithm 3 below) are a discrete version of the labelled box configurations defined by Pandharipande and Thomas in [PT09]. Jenne, Webb, and Young map the AB configurations to tripartite double-dimer configurations in order to prove a relation of two generating functions (the DT topological vertex and the PT topological vertex). Our proof of 1 follows a similar strategy as the one in [JWY20].

We define labelled AB configurations here as given in [JWY20].

Definition 43. (Jenne, Webb, Young, [JWY20])
Fix partitions μ_1, μ_2, μ_3, and identify μ_i with the coordinates of the boxes of its Young diagram, with the corner of the diagram located at $(0,0)$. Define the following subsets of $\mathbb{Z}^3$, thought of as sets of boxes: $\text{Cyl}_1 = \{(x,u,v) \in \mathbb{Z}^3 \mid (u,v) \in \mu_1\}, \text{Cyl}_2 = \{(v,y,u) \in \mathbb{Z}^3 \mid (u,v) \in \mu_2\}$, and $\text{Cyl}_3 = \{(u,v,z) \in \mathbb{Z}^3 \mid (u,v) \in \mu_3\}$.

Moreover, let $\mathbb{Z}^3_{\geq 0}$ denote the integer points in the first octant (including the coordinate planes and axes). Let $\text{Cyl}_i^+ = \text{Cyl}_i \cap \mathbb{Z}^3_{\geq 0}$ and $\text{Cyl}_i^- = \text{Cyl}_i \setminus \mathbb{Z}^3_{\geq 0}$. Finally, let

$$\begin{aligned} & & \mathrm{II}_{\bar{1}} &= \text{Cyl}_2 \cap \text{Cyl}_3 \setminus \text{Cyl}_1, & & \\ \mathrm{I}^- &= \text{Cyl}_1^- \cup \text{Cyl}_2^- \cup \text{Cyl}_3^-, & \mathrm{II}_{\bar{2}} &= \text{Cyl}_3 \cap \text{Cyl}_1 \setminus \text{Cyl}_2, & \mathrm{II} &= \mathrm{II}_{\bar{1}} \cup \mathrm{II}_{\bar{2}} \cup \mathrm{II}_{\bar{3}}, \\ \mathrm{I}^+ &= \text{Cyl}_1^+ \cup \text{Cyl}_2^+ \cup \text{Cyl}_3^+, & \mathrm{II}_{\bar{3}} &= \text{Cyl}_1 \cap \text{Cyl}_2 \setminus \text{Cyl}_3, & \mathrm{III} &= \text{Cyl}_1 \cap \text{Cyl}_2 \cap \text{Cyl}_3. \end{aligned}$$

Definition 44. (Jenne, Webb, Young, [JWY20]) If $A \subseteq \mathrm{I}^- \cup \mathrm{III}$ and $B \subseteq \mathrm{II} \cup \mathrm{III}$ are finite sets of boxes, then (A, B) is an AB *configuration* if the following condition is satisfied:
If $w = (w_1, w_2, w_3)$ is a cell (i.e. box) in $\mathrm{I}^- \cup \mathrm{III}$ (resp. $w \in \mathrm{II} \cup \mathrm{III}$) and any cell in $\{(w_1 - 1, w_2, w_3), (w_1, w_2 - 1, w_3), (w_1, w_2, w_3 - 1)\}$ supports a box in A (resp. B), then w must support a box in A (resp. B).

Let AB denote the set of all AB configurations.

The labeling algorithm for AB configurations is given by:

Algorithm 3. (Jenne, Webb, Young, [JWY20]) (AB Labeling Algorithm)

1. If a connected component of $(\mathrm{I}^- \cap A) \cup (\mathrm{II} \setminus B) \cup (\mathrm{III} \cap (A \Delta B))$ contains a box in $\text{Cyl}_i^- \cup \mathrm{II}_{\bar{i}}$ and a box in $\text{Cyl}_j \cup \mathrm{II}_{\bar{j}}$, where $i \neq j$, terminate with failure.

2. For each connected component C of $(\mathrm{I}^- \cap A) \cup (\mathrm{II} \setminus B) \cup (\mathrm{III} \cap (A \Delta B))$ that contains a box in $\text{Cyl}_i^- \cup \mathrm{II}_{\bar{i}}$, label each element of C by i.

3. For each remaining connected component C of $(\mathrm{I}^- \cap A) \cup (\mathrm{II} \setminus B) \cup (\mathrm{III} \cap (A \Delta B))$, label each element of C by the same freely chosen element of $\mathbb{P}^1$.

Definition 45. Denote all AB configurations which pass Algorithm 3, that is, labelled AB configurations, by $\mathcal{AB}$. Denote the set $(\mathrm{I}^- \cap A) \cup (\mathrm{II} \setminus B) \cup (\mathrm{III} \cap (A \Delta B))$ of boxes potentially labeled by Algorithm 3 by $\mathcal{L}(A, B)$.

In this paper, we will denote $\mathrm{I}^-, \mathrm{I}^+, \mathrm{II}$ and III as defined above with the subscripts $\mathrm{I}^-_{AB}, \mathrm{I}^+_{AB}, \mathrm{II}_{AB}$ and III_{AB} to differentiate them from our other notions of types of boxes (as in double-box configurations, where we use no subscripts, and configurations $\pi_1 \cup \pi_2$ for

$(\pi_1, \pi_2) \in DP_{a,b,c}$, where we use the subscript DP).

Next we define a map γ which takes an element $(\pi_1, \pi_2) \in DP_{a,b,c}$, and maps it to an AB configuration, which we will show is a labelled AB configuration if and only if $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}$. We will later use this to show that $D_{(\pi_1,\pi_2)} \in DD(\sigma_{a,b,c})$ for $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}$.

Definition 46. Let $(\pi_1, \pi_2) \in DP_{a,b,c}$. Define a map $\gamma : DP_{a,b,c} \to AB$ as follows:

1. First, map (π_1, π_2) to $P(\pi_1, \pi_2)$, which we will denote by (π_1^p, π_2^p). In what follows, we often refer to π_1^p and π_2^p by π_1 and π_2.

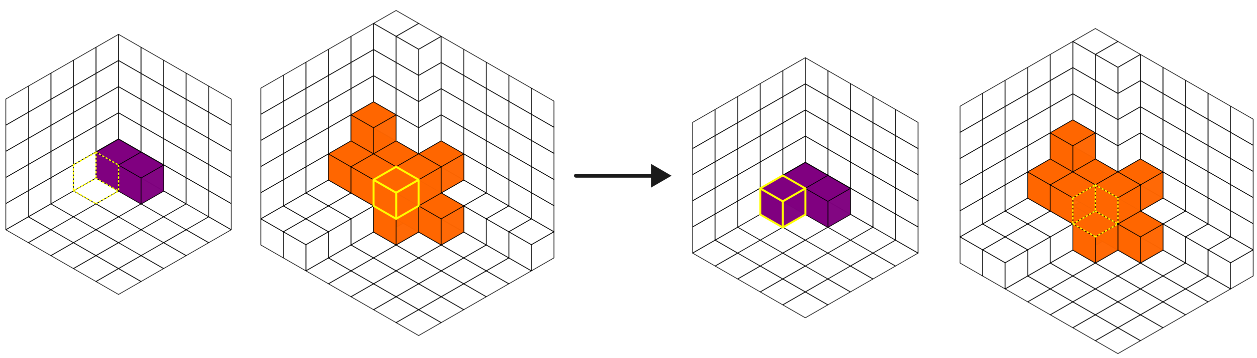

Figure 3.11: Move all possible boxes to π_1. Left: $(\pi_1, \pi_2) \in DP_{a,b,c}$. Right: (π_1^p, π_2^p).

2. Define π_1', π_2', π_3' to be the plane partitions in the negative spaces of π_2, that is,

   - $\pi_1' = I_1^- \cap \pi_2$
   - $\pi_2' = I_2^- \cap \pi_2$
   - $\pi_3' = I_3^- \cap \pi_2$

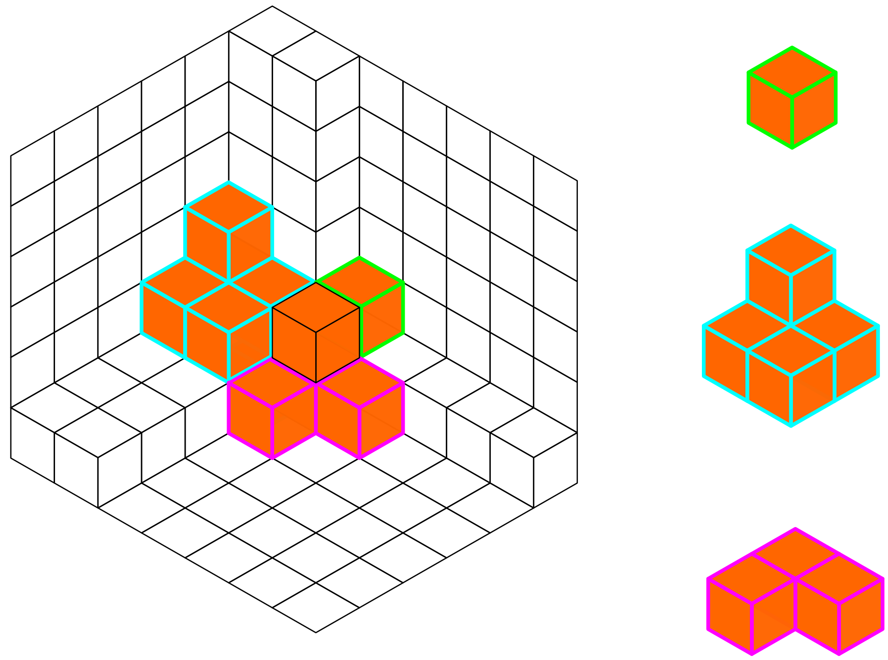

Figure 3.12: Plane partitions π'_1 (outlined in green), π'_2 (outlined in blue), π'_3 (outlined in pink) defined from π_2.

3. Define partitions μ_1, μ_2, μ_3 as follows

   - $\mu_1 =$ (max(col 1 of π'_1), max(col 2 of π'_1), ...)
   - $\mu_2 =$ (max(row 1 of π'_2), max(row 2 of (π'_2), ...)
   - $\mu_3 =$ (length(row 1 of π'_3), length(row 2 of π'_3), ...)

   Note that μ_1, μ_2, and μ_3 do not depend on boxes in the intersection space, so every element in $\mathcal{DP}_{a,b,c}(\eta)$ gets the same partitions. Also note that since π'_1, π'_2, and π'_3 are plane partitions, we have that μ_1, μ_2, μ_3 are well-defined.

4. Consider $\mathrm{Cyl}_1, \mathrm{Cyl}_2$, and Cyl_3 as defined in Definition 43, using the partitions μ_1, μ_2, μ_3 defined in the previous step.

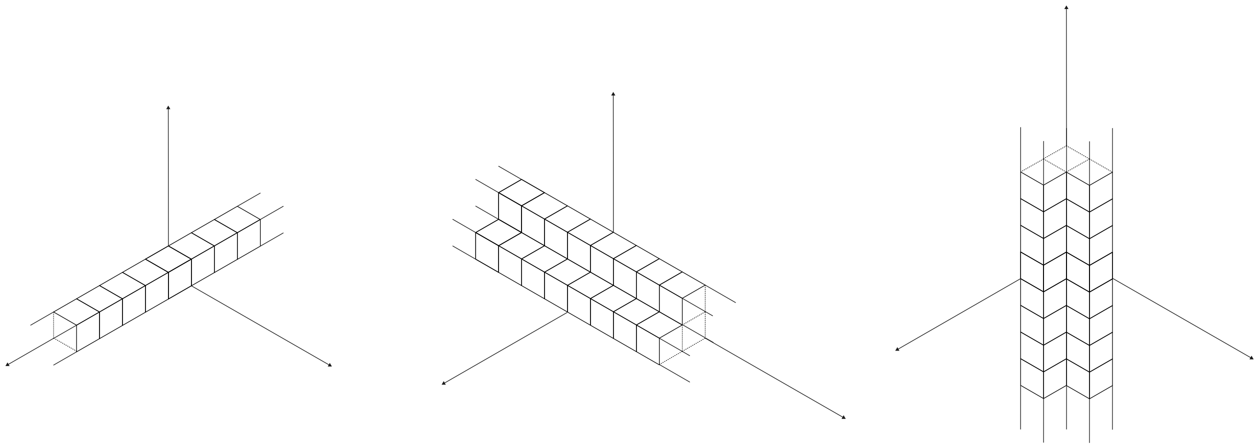

Figure 3.13: Cylinders Cyl_1 (left), Cyl_2 (middle), Cyl_3 (right) created from partitions $\mu_1 = (1), \mu_2 = (2,1), \mu_3 = (2,1)$.

5. Base π_1' at $(-a, 0, 0)$, π_2' at $(0, -b, 0)$, and π_3' at $(0, 0, -c)$ in $\mathbb{R}^3$. Define the following cylinder extensions (informally, take the plane partitions π_i', and extend in the negative axes directions with boxes in the μ_i extended cylinder). Figure 3.14 shows the cylinder extensions on the subset of $\mathbb{Z}^3$ consisting of the tiles that have at least two negative coordinates (the floors and walls).

   - $\mathrm{Cyl}_1^e = \pi_1' \cup \{(i,j,k) \in \mathbb{Z}^3 \mid (i,j,k) \in \mathrm{Cyl}_1^- \text{ and } i < -a\}$
   - $\mathrm{Cyl}_2^e = \pi_2' \cup \{(i,j,k) \in \mathbb{Z}^3 \mid (i,j,k) \in \mathrm{Cyl}_2^- \text{ and } j < -b\}$
   - $\mathrm{Cyl}_e^3 = \pi_3' \cup \{(i,j,k) \in \mathbb{Z}^3 \mid (i,j,k) \in \mathrm{Cyl}_3^- \text{ and } k < -c\}$

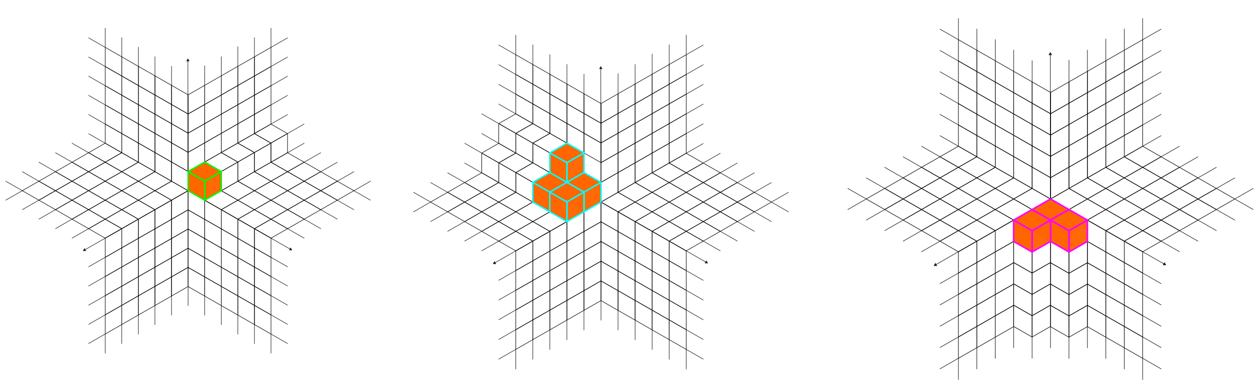

Figure 3.14: The cylinder extensions Cyl_1^e (left), Cyl_2^e (middle), Cyl_3^e (right).

6. Consider the configuration $\mathrm{Cyl}_{AB} = \mathrm{Cyl}_1^- \cup \mathrm{Cyl}_2^- \cup \mathrm{Cyl}_3^- \cup (\mathrm{Cyl}_1 \cap \mathrm{Cyl}_2) \cup (\mathrm{Cyl}_1 \cap \mathrm{Cyl}_3) \cup (\mathrm{Cyl}_2 \cup \mathrm{Cyl}_3)$.

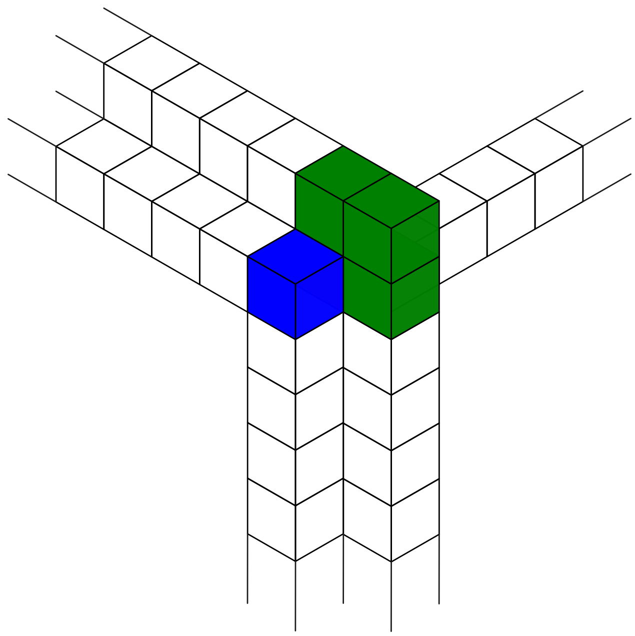

Figure 3.15: Type II_{AB} (green) and III_{AB} (blue) boxes in Cyl_{AB}.

Let

$$A = \left(\bigcup_{i \in \{1,2,3\}} Cyl_i^- \setminus Cyl_i^e \right) \cup (\mathrm{III}_{AB} \setminus \pi_2^p)$$

$$B = (\mathrm{II}_{AB} \cup \mathrm{III}_{AB}) \setminus \pi_1^p$$

where we identify the point (a, b, c) in π_2^p, and the point $(0, 0, 0)$ in π_1^p, with $(0, 0, 0)$ in Cyl_{AB}.

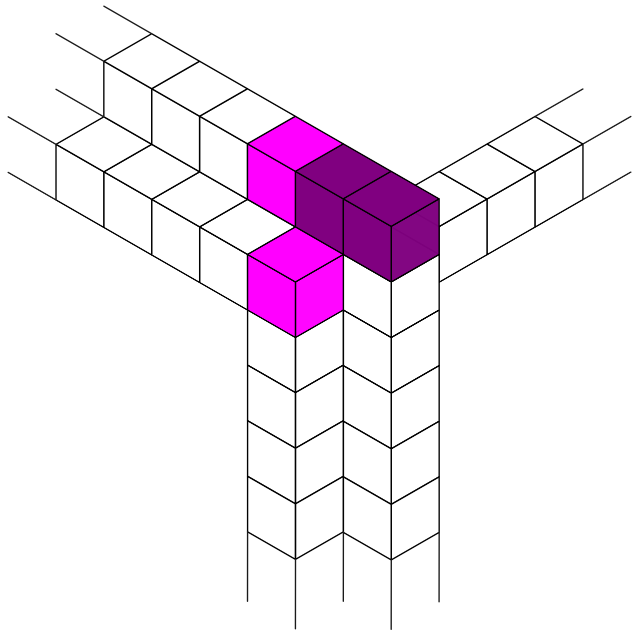

Figure 3.16: Boxes in A (pink) and in B (purple).

Finally, let $\gamma(\pi_1, \pi_2) = (A, B)$.

Note that γ is well-defined since we first map by P (and also by Theorem 48).

Lemma 47. *Let $(\pi_1, \pi_2) \in DP_{a,b,c}$, and consider the AB configuration given by $\gamma(\pi_1, \pi_2)$. Let $w = (w_1, w_2, w_3) \in I^-_{AB} \cup II_{AB} \cup III_{AB}$.*

1. *If $w \in A$, then $w \notin \pi_2^p$.*

2. *If $w \in I^-_{AB} \cup III_{AB}$, $w \notin A$, $i \geq -a, j \geq -b$, and $k \geq -c$, then $w \in \pi_2^p$.*

3. *If $w \in B$, then $w \notin \pi_1^p$.*

4. *If $w \in II_{AB} \cup III_{AB}$ and $w \notin B$ then $w \in \pi_1^p$.*

Proof. 1. By definition of A given by γ.

2. Suppose $w \in \mathrm{I}^-_{AB}$. Then since $w \notin A$, by definition of γ we have that $w \in \mathrm{Cyl}_i^e$ for some $i \in \{1, 2, 3\}$. Since $w_1 \geq -a, w_2 \geq -b, w_3 \geq -c$, we must have that $w \in \pi_i'$,

thus $w \in \pi_2^p$. Suppose $w \in \mathrm{III}_{AB}$. Then since $w \notin A$, by definition of γ, we have that $w \in \pi_2^p$.

3. By definition of B given by γ.

4. By definition of B given by γ.

$\square$

Next we show that the image of γ is in fact an AB configuration.

Theorem 48. *For $(\pi_1, \pi_2) \in DP_{a,b,c}$, we have that $\gamma(\pi_1, \pi_2) \in AB$.*

Proof. To show that $(A, B) = \gamma(\pi_1, \pi_2)$ for $(\pi_1, \pi_2) \in DP_{a,b,c}$ is an AB configuration, we need to show that it satisfies the following conditions from Definition 44:

(1) If $w \in \mathrm{I}_{AB}^{-} \cup \mathrm{III}_{AB}$ and $BN(w) \cap A \neq \emptyset$, then $w \in A$.
(2) If $w \in \mathrm{II}_{AB} \cup \mathrm{III}_{AB}$ and $BN(w) \cap B \neq \emptyset$, then $w \in B$.

Suppose that Condition (1) is not satisfied. Then there exists a box $w = (w_1, w_2, w_3) \in \mathrm{I}_{AB}^{-} \cup \mathrm{III}_{AB}$ with $BN(w) \cap A \neq \emptyset$ and $w \notin A$. Let $u \in BN(w) \cap A$. If $u = (u_1, u_2, u_3) \in \mathrm{III}_{AB}$, then $u_1 \geq -a, u_2 \geq -b$ and $u_3 \geq -c$. Suppose $u \in \mathrm{I}_{AB}^{-}$. Then since $u \in A$, by definition of γ, we have that $u \notin \mathrm{Cyl}_i^e$ for each $i \in \{1, 2, 3\}$. Thus $u_1 \geq -a, u_2 \geq -b$ and $u_3 \geq -c$. Then since $u \in BN(w)$, we have that $w_1 \geq -a, w_2 \geq -b$ and $w_3 \geq -c$. Then since $w \notin A$ and $w \in \mathrm{I}_{AB}^{-} \cup \mathrm{III}_{AB}$, by Lemma 47, we have that $w \in \pi_2^p$. And since $u \in A$, by Lemma 47, we have that $u \notin \pi_2^p$. However, since $u \in BN(w)$, this is a contradiction with π_2^p being an order ideal under the product order on $\mathbb{Z}_{\geq 0}^3$ containing $\mathbf{M}$.

Suppose that Condition (2) is not satisfied. Then there exists a box $w \in \mathrm{II}_{AB} \cup \mathrm{III}_{AB}$ with $BN(w) \cap B \neq \emptyset$, and $w \notin B$. Let $v \in BN(w) \cap B$. Since $B \subseteq \mathrm{II}_{AB} \cup \mathrm{III}_{AB}$, we know that $v \in \mathrm{II}_{AB} \cup \mathrm{III}_{AB}$. Then $w \in (\mathrm{II}_{AB} \cup \mathrm{III}_{AB}) \setminus B$ and $v \notin (\mathrm{II}_{AB} \cup \mathrm{III}_{AB}) \setminus B$, and so by Lemma 47, $w \in \pi_1^p$ and $v \notin \pi_1^p$, a contradiction with π_1^p being an order ideal under the product order on $\mathbb{Z}_{\geq 0}^3$.

□

Next we have a lemma and corollary that are used in the proof of our next theorem.

Lemma 49. *Let* $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}$*. Let* $w \in (II_{AB} \setminus B) \cup (III_{AB} \cap (A\Delta B))$ *for* $(A, B) = \gamma(\pi_1, \pi_2)$*. Then* $w \in Int \setminus III_{DP}$*.*

Proof. First, note that if $w \in \text{II}_{AB} \cup \text{III}_{AB}$ in (A, B) and $w \in \pi_1 \cup \pi_2$, then $w \in Int$ of $\pi_1 \cup \pi_2$.

Suppose $w \in \text{II}_{AB} \setminus B$, and suppose w is not contained in Cyl_j for $j \in \{1, 2, 3\}$. Since $w \notin B$, by Lemma 47, we have that $w \in \pi_1^p$, and thus $w \in \pi_1 \cup \pi_2$, and so $w \in Int$. Next, since $w \in \text{II}_{AB}$, by construction of γ, w cannot be a type III_{DP} box in (π_1, π_2) (because it would not be supported in I_j^-). Thus $w \in Int \setminus \text{III}_{DP}$. Note that since w is not in Cyl_j, it is not supported in I_j^-, and thus would get label j by Algorithm 1.

Suppose $w \in \text{III}_{AB} \cap (A\Delta B)$. First, we know that w is a box in (π_1, π_2) because w is *not* in one of A, B, thus it *is* in one of π_1, π_2 by Lemma 47. Next, since w is *not* in one of π_1, π_2, it is not a type III_{DP} box in (π_1, π_2). Note that w could be a moveable or unmovable type II box in (π_1, π_2) (and so it could be labeled or unlabeled).

□

Corollary 50. *Let* $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}$ *and* $(A, B) = \gamma(\pi_1, \pi_2)$*. Any connected component of* $(II_{AB} \setminus B) \cup (III_{AB} \cap (A\Delta B))$ *is also connected in* $Int \setminus III_{DP}$*.*

Next we show that if $(\pi_1, \pi_2) \in DP_{a,b,c}$ is associated to some $\eta \in DB_{a,b,c}$, then γ maps (π_1, π_2) to a labelled AB configuration.

Theorem 51. *If* $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}$*, then* $\gamma(\pi_1, \pi_2) \in \mathcal{AB}$*.*

Proof. We need to show that $(A, B) = \gamma(\pi_1, \pi_2)$ satisfies the AB Labeling Algorithm 3 of Jenne, Webb, and Young from [JWY20]. The AB Labeling Algorithm will be satisfied if it does not fail in the first step.

Let $\mathcal{L}(A, B) = (\mathrm{I}^-_{AB} \cap A) \cup (\mathrm{II}_{AB} \setminus B) \cup (\mathrm{III}_{AB} \cap (A \Delta B))$ as in [JWY20].

Case 1. The algorithm fails because there is a box $w \in \mathrm{I}^-_{AB} \cap A$ giving label i, and a box $w' \in \mathrm{II}_{AB} \setminus B$ giving label j, with $i \neq j$, with w and w' in the same connected component of $\mathcal{L}(A, B)$. Note that w' is in Cyl_i and in Cyl_k, not in Cyl_j, for $\{i, j, k\} = \{1, 2, 3\}$, since it was assigned label j. By Lemma 49, since $w' \in (\mathrm{II}_{AB} \setminus B) \cup (\mathrm{III}_{AB} \cap (A \Delta B))$, we have that $w' \in Int \setminus \mathrm{III}_{DP}$ in (π_1, π_2). Note that since $w' \notin \mathrm{Cyl}_j$, w' is not supported in I^-_j, and so it gets assigned label j by the DP Labeling Algorithm 1.

Let C be the connected component of $\mathcal{L}(A, B)$ containing w and w'. Let $u \in C \cap Cyl^-_i$ such that $FN(u) \cap (\mathrm{II}_{AB} \cup \mathrm{III}_{AB}) \cap C \neq \emptyset$, let $v \in FN(u) \cap (\mathrm{II}_{AB} \cup \mathrm{III}_{AB}) \cap C$ (that is, u is the box in the negative part of Cyl_i that is connected to a box v in the intersection space). We know that u and v exist in C because w and w' are in the same connected component C of $\mathcal{L}(A, B)$, and $w \in \mathrm{Cyl}^-_i$ and $w' \in \mathrm{II}_{AB}$. By Lemma 47, since $u \in A$, we have that $u \notin \pi^p_2$, thus v is not supported in I^-_i (since $v^i = u$). Note that since v is in the intersection space and in C, v is either in $\mathrm{II}_{AB} \setminus B$ or $\mathrm{III}_{AB} \cap (A \Delta B)$. Thus by Lemma 49, we have that $v \in Int \setminus III_{DP}$ in (π_1, π_2). Since v is not supported in I^-_i, it is assigned label i by the DP Labeling Algorithm 1.

We define a path of boxes contained in C from v to w'. Let $p_1 = v$, and let $p_2 \in C \cap (FN(p_1) \cup BN(p_1))$, $p_3 \in C \cap (FN(p_2) \cup BN(p_2))$, and so on. A path $p_1, ..., p_n$ with $p_1 = v$ and $p_n = w'$ exists since v and w' are in the same connected component C. Since $C \subset \mathcal{L}(A, B)$, and since $v, w' \in (\mathrm{II}_{AB} \setminus B) \cup (\mathrm{III}_{AB} \cap (A \Delta B))$, we may choose a path $p_1, ..., p_n$ such that $p_1, ..., p_n \in (\mathrm{II}_{AB} \setminus B) \cup (\mathrm{III}_{AB} \cap (A \Delta B))$. Then by Lemma 49, we have that $p_1, ..., p_n \in Int \setminus \mathrm{III}_{DP}$ in $\pi_1 \cup \pi_2$. Thus we have a box $v \in Int \setminus \mathrm{III}_{DP}$ with label i in the same connected component as a box $w' \in Int \setminus \mathrm{III}_{DP}$ with label j. And so we have a contradiction with Step 2 of the DP Labeling Algorithm 1.

Case 2. The algorithm fails because there are boxes $w, w' \in \mathrm{I}^-_{AB} \cap A$ in the same connected component C of $\mathcal{L}(A, B)$ giving conflicting labels. Let $w \in \mathrm{Cyl}^-_i$ and $w' \in \mathrm{Cyl}^-_j$ with $i \neq j$.

We follow a similar argument as in Case 1. Let $u \in \mathrm{Cyl}_i^- \cap C$ such that $FN(u) \cap (\mathrm{II}_{AB} \cup \mathrm{III}_{AB}) \cap C \neq \emptyset$, and let $v \in FN(u) \cap (\mathrm{II}_{AB} \cup \mathrm{III}_{AB}) \cap C$. Let $u' \in \mathrm{Cyl}_j^- \cap C$ be such that $FN(u') \cap (\mathrm{II}_{AB} \cup \mathrm{III}_{AB}) \cap C \neq \emptyset$, and let $v' \in FN(u') \cap (\mathrm{II}_{AB} \cup \mathrm{III}_{AB}) \cap C$. Since w and w' are in the same connected component C of $\mathcal{L}(A, B)$, and since $w \in \mathrm{Cyl}_i^-$ and $w' \in \mathrm{Cyl}_j^-$ with $i \neq j$, we know that u, v, u', v' exist.

Since v and v' are in the intersection space and in C, v and v' are either in $\mathrm{II}_{AB} \setminus B$ or $\mathrm{III}_{AB} \cap (A \Delta B)$. Thus by Lemma 49, we have that $v, v' \in \mathit{Int} \setminus III_{DP}$ in (π_1, π_2). We may define a path of boxes $p_1, ..., p_n$ contained in $C \cap (\mathrm{II}_{AB} \setminus B) \cup (\mathrm{III}_{AB} \cap (A \Delta B))$ such that $p_1 = v$ and $p_n = v'$ as in Case 1. By Lemma 49, $p_1, ..., p_n \in \mathit{Int} \setminus \mathrm{III}_{DP}$ in $\pi_1 \cup \pi_2$. Then, since $v_i = u \in A$ and $v'_j = u' \in A$, by Lemma 47, we have that $u, u' \notin \pi_2^p$, and so v gets label i by the DP Labeling Algorithm 1, and v' gets label j by the DP Labeling Algorithm 1. Thus we have two boxes v and v' in the same connected component of $\mathit{Int} \setminus \mathrm{III}_{DP}$ of $\pi_1 \cup \pi_2$ with conflicting labels, a contradiction with Step 2 of the DP Labeling Algorithm.

Case 3. The AB Labeling Algorithm fails because there are boxes $w, w' \in \mathrm{II}_{AB} \setminus B$ giving conflicting labels. Let $w \notin \mathrm{Cyl}_i$, and $w' \notin \mathrm{Cyl}_j$, with $i \neq j$.

By Lemma 49, since $w, w' \in C \cap \mathrm{II}_{AB}$, we have that $w, w' \in \mathit{Int} \setminus \mathrm{III}_{DP}$. Since $w \notin \mathrm{Cyl}_i$, we have that w is not supported in I_i^-, thus gets label i by the DP Labeling Algorithm 1. Since $w' \notin \mathrm{Cyl}_j$, we have that w' is not supported in I_j^-, thus gets label j by the DP Labeling Algorithm 1. By creating a path of boxes between w and w' contained in $\mathit{Int} \setminus \mathrm{III}_{DP}$ as in Case 1, we have two boxes, w and w' in the same connected component of $\mathit{Int} \setminus \mathrm{III}_{DP}$ with conflicting labels, a contradiction with Step 2 of the DP Labeling Algorithm.

□

Definition 52. (Jenne, Webb, Young [JWY20]) Let (A, B) be an AB configuration. We consider A and B separately. For A, we view the surface $(\mathrm{I}_{AB}^- \cup \mathrm{III}_{AB}) \setminus A$ as a lozenge tiling (see Figure 3.17 top row, middle image). In other words, we take the set of boxes

$A \subseteq \mathrm{I}^{-}_{AB} \cup \mathrm{III}_{AB}$ and draw the tiles corresponding to cells (i.e. boxes) that are <u>not</u> in A. Similarly for B, we view the surface $(\mathrm{II}_{AB} \cup \mathrm{III}_{AB}) \setminus B$ as a lozenge tiling (see Figure 3.17 bottom row, middle image). We then extend each of these tilings to tilings of the entire plane (see Figure 3.17 rightmost column). Then, these lozenge tilings are equivalent to dimer configurations of the infinite hexagon graph H. Let M_A (resp. M_B) denote the dimer configuration of H corresponding to the infinite tiling obtained from A (resp. B). Superimposing M_A and M_B so that boxes in III_{AB} are in the same place in the two pictures produces a double-dimer configuration $D_{(A,B)}$ on H (see Figure 3.18).

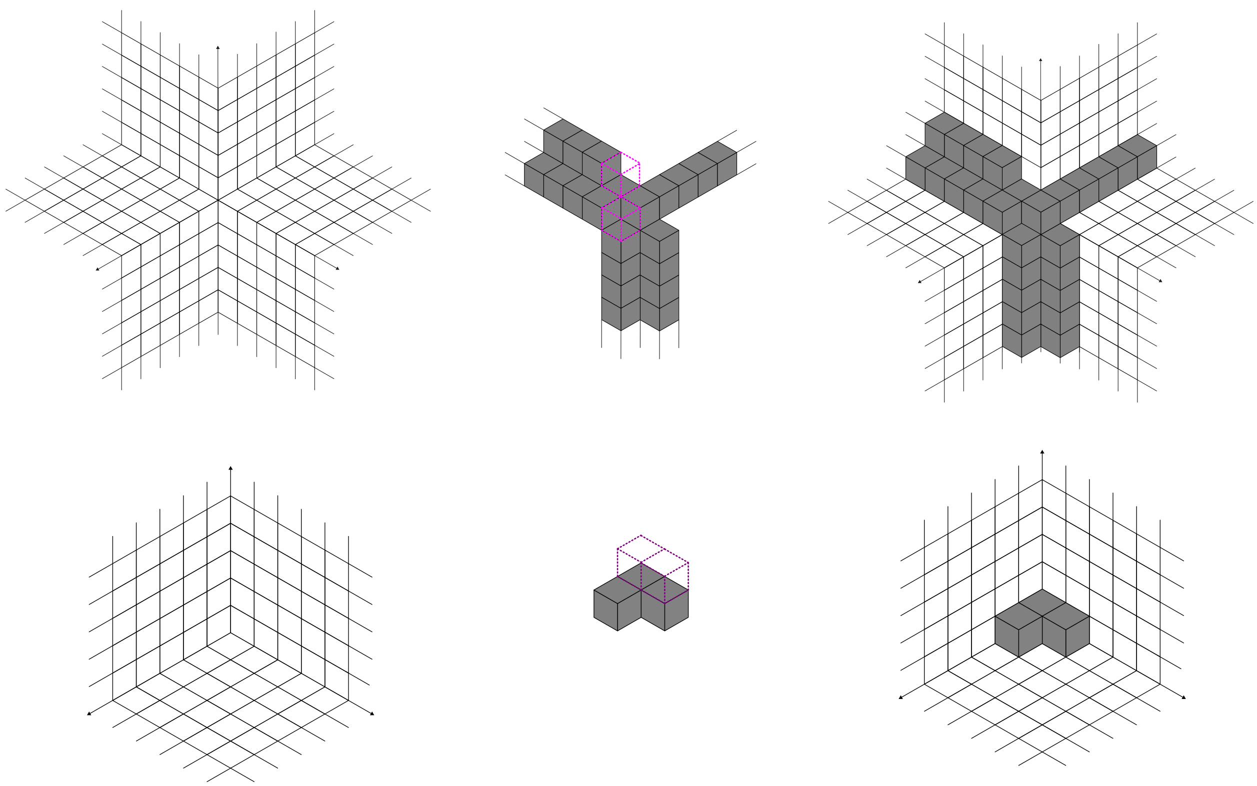

Figure 3.17: Top row: $(\mathrm{I}^{-}_{AB} \cup \mathrm{III}_{AB}) \setminus A$ (middle), with boxes in A that were removed outlined in dashed pink lines, extended to a tiling of the entire plane (right) using the tiles in the leftmost image. Bottom row: $(\mathrm{II}_{AB} \cup \mathrm{III}_{AB}) \setminus B$ (middle), boxes in B that were removed outlined in dashed purple lines, extended to a tiling of the entire plane (right) using the tiles of the leftmost image.

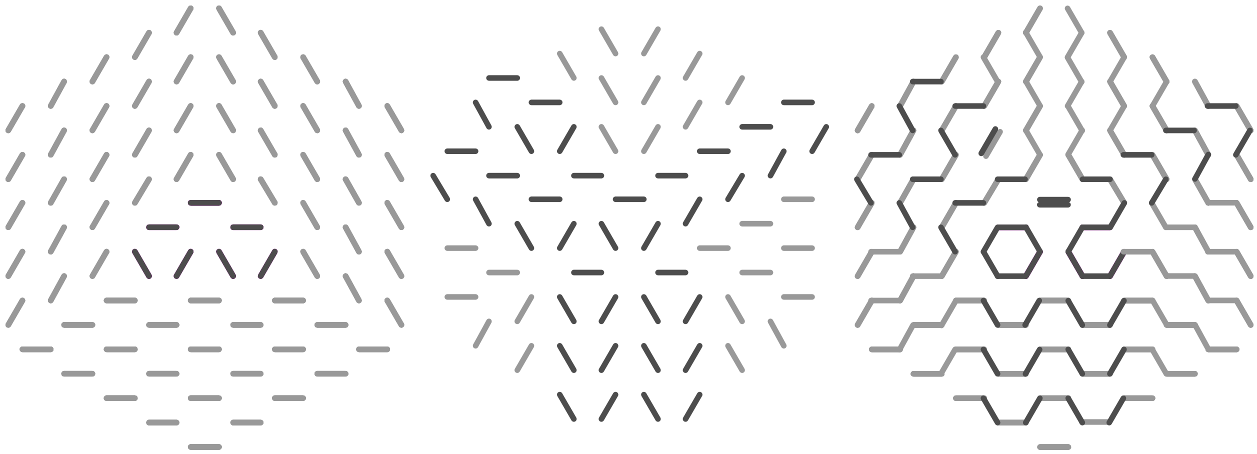

Figure 3.18: M_B (left), M_A (middle), $D_{(A,B)}$ (right)

Definition 53. Given $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}$, we define $D_{(\pi_1,\pi_2^e)}$ from $D_{(\pi_1,\pi_2)}$ by:

- Let the *extended* π_2 be defined by $\pi_2^e = \mathrm{Cyl}_1^e \cup \mathrm{Cyl}_2^e \cup \mathrm{Cyl}_3^e \cup \mathrm{III}_{DP} = (\mathrm{I}_{AB}^- \cup \mathrm{III}_{AB}) \setminus A$ (see Figure 3.19). Then $M_A = D_{\pi_2^e}$. Note then that $D_{(\pi_1,\pi_2^e)} = D_{(A,B)}$ (compare Figure 3.18 and 3.22).

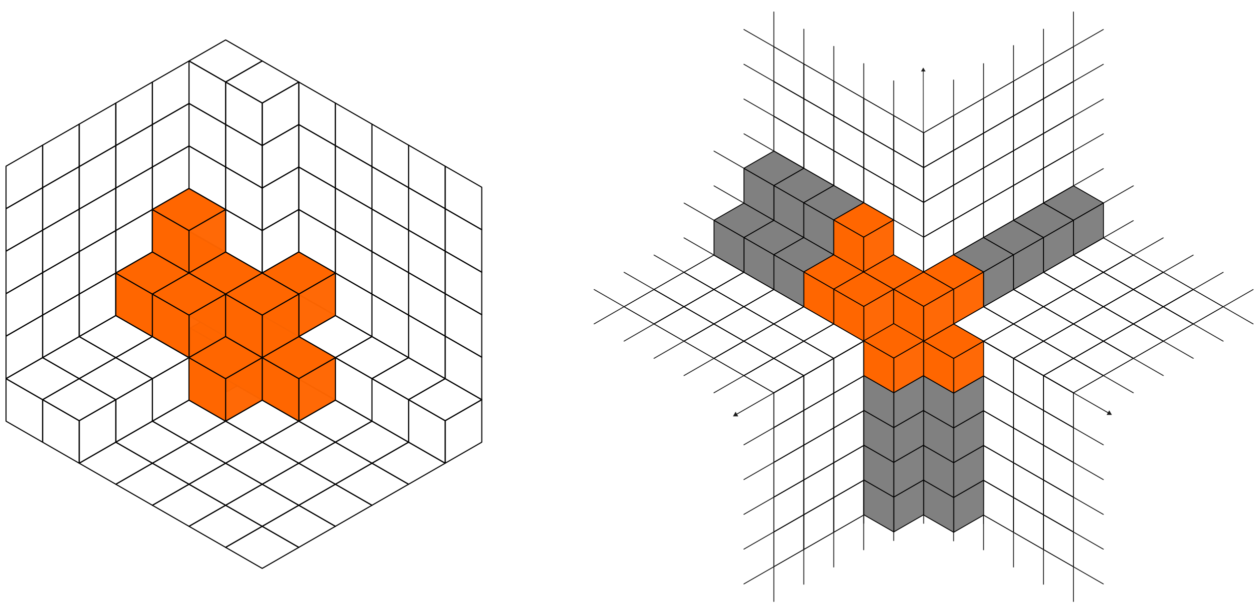

Figure 3.19: π_2 (left) and π_2^e (right)

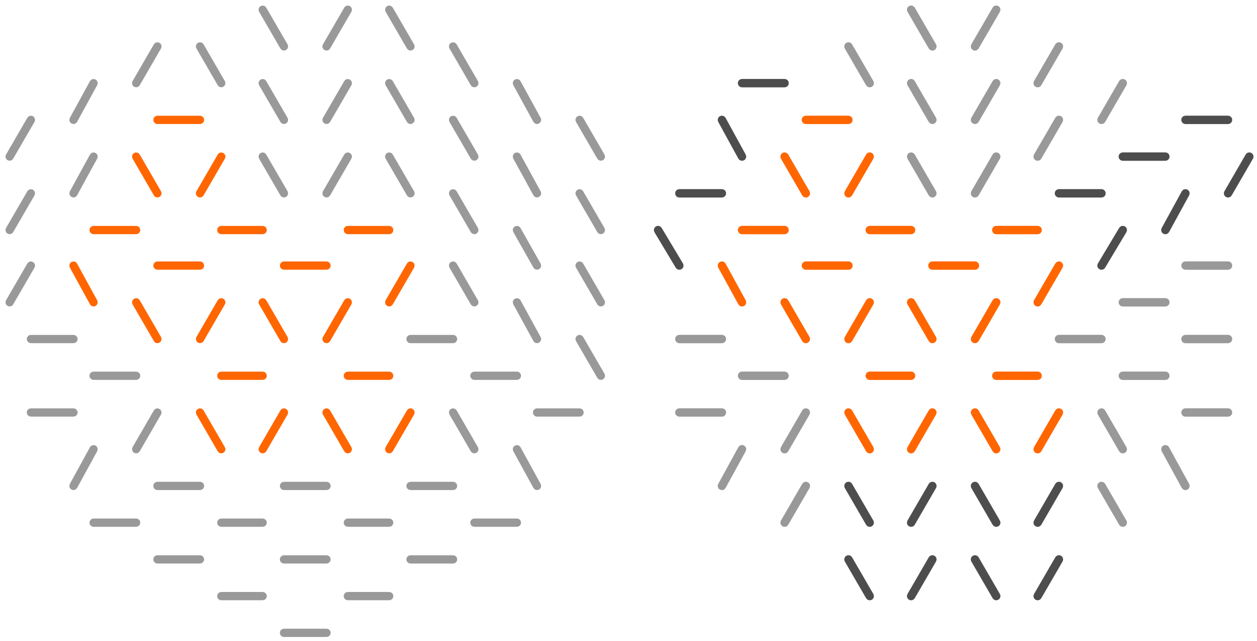

Figure 3.20: D_{π_2} on $H(5)$ (left) and $D_{\pi_2^e}$ on $H(5)$ (right) with orange dimers from the orange boxes, dark gray dimers from the gray boxes, and light gray dimers from the wall and floor tiles.

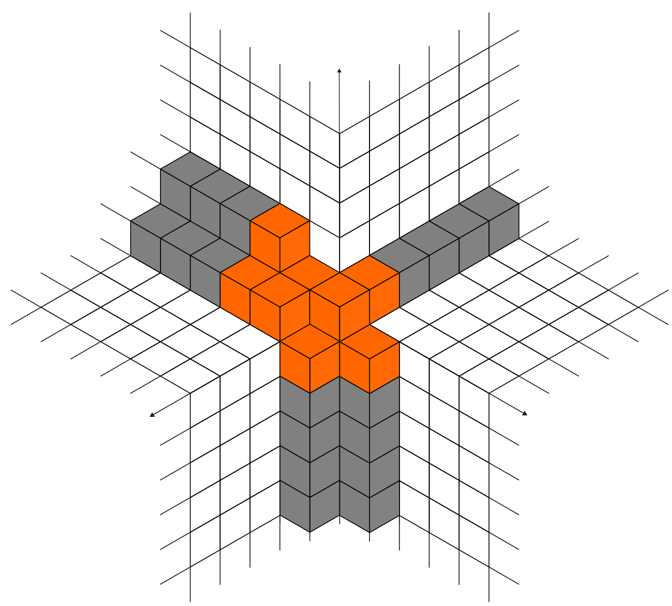

Figure 3.21: π_1 (left) and π_2^e (right).

Figure 3.22: D_{π_1} (left), $D_{\pi_2^e}$ (middle), and $D_{(\pi_1,\pi_2^e)}$ (right)

In [JWY20], Jenne, Webb, and Young show that for $N \geq M$, where M is defined to be

$$M = \max\{(\mu_1)_1, \ell(\mu_1), (\mu_2)_1, \ell(\mu_2), (\mu_3)_1, \ell(\mu_3)\}, \tag{3.1}$$

and for $(A, B) \in \mathcal{AB}$ (that is, (A, B) passes the AB Labeling Algorithm), each path in $D_{(A,B)}(N)$ (that is, $D_{(A,B)}$ restricted to $H(N)$) starts and ends in the same sector (Theorem 4.4.17 of [JWY20]), and that the nodes of $D_{(A,B)}(N)$ are paired according to the rainbow pairing (Theorem 4.4.19 of [JWY20]). Thus if $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}$, by Theorem 51 $(A, B) = \gamma(\pi_1, \pi_2) \in \mathcal{AB}$, and so $D_{(A,B)}(N) = D_{(\pi_1,\pi_2^e)}(N)$ for $N \geq M$ has all paths starting and ending in the same sector.

Let M be defined as in Equation (3.1), and let

$$M' = \max\{M + a, M + b, M + c\}.$$

Informally, M is defined to include all the vertices on the exterior face of $H(N)$ that are *not* nodes of $D_{(A,B)}$, and so M is the farthest non-included node that there is, and we need a, or b, or c more than that to be included so that we have all the endpoints of the paths in $D_{(\pi_1,\pi_2^e)}$.

For $N \geq M'$, we may transform $D_{(\pi_1,\pi_2^e)}$ into $D_{(\pi_1,\pi_2)}$ by changing all the paths that are in $D_{(\pi_1,\pi_2^e)}$ and not in $D_{(\pi_1,\pi_2)}$ into doubled edges in such a way that we remove all nodes that are in $D_{(\pi_1,\pi_2^e)}$ but not in $D_{(\pi_1,\pi_2)}$ (see Figure 3.24).

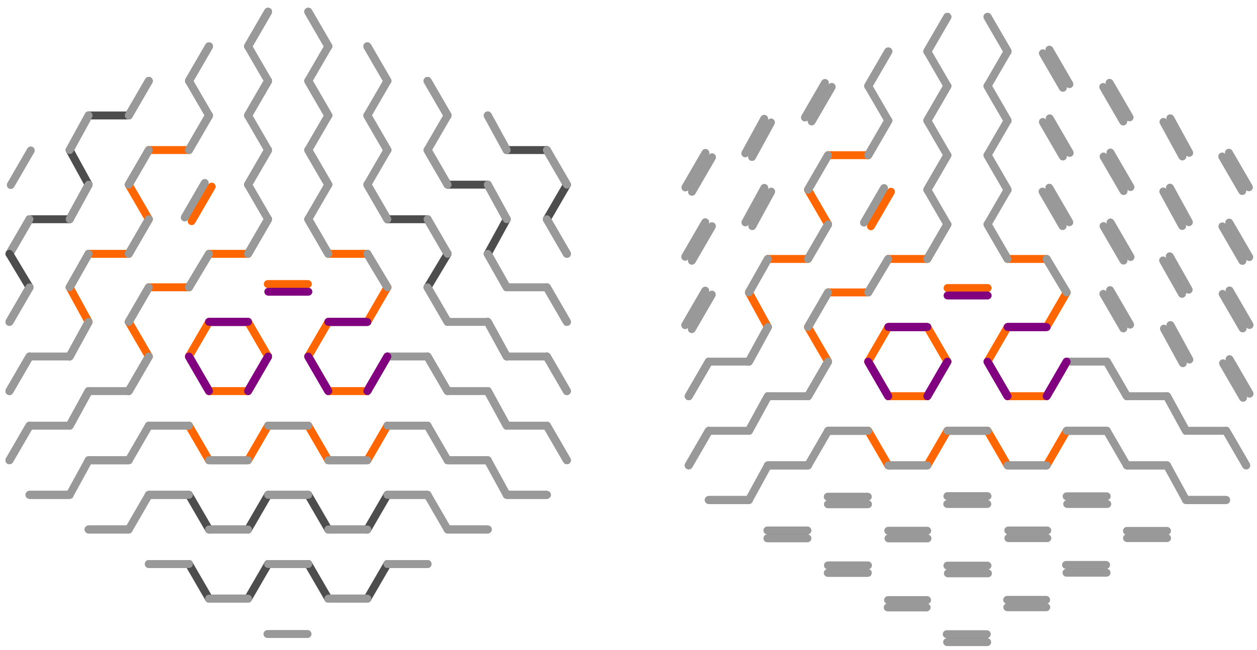

Figure 3.23: Transforming $D_{(\pi_1,\pi_2^e)}$ (left) to $D_{(\pi_1,\pi_2)}$ (right). All paths that include dark gray edges become doubled edges as in the minimal configuration.

Theorem 54. *Let $N \geq M'$, then $D_{(\pi_1,\pi_2^e)}(N)$ has all paths starting and ending in the same sector if and only if $D_{(\pi_1,\pi_2)}(N)$ does.*

Proof. Suppose $D_{(\pi_1,\pi_2^e)}(N)$ has all paths starting and ending in the same sectors. Since the paths of $D_{(\pi_1,\pi_2)}(N)$ are the same as the interior paths of $D_{(\pi_1,\pi_2^e)}(N)$, all paths in $D_{(\pi_1,\pi_2)}(N)$ also start and end in the same sectors.

Suppose $D_{(\pi_1,\pi_2)}(N)$ has all paths starting and ending in the same sector. Since the nodes in $D_{(\pi_1,\pi_2)}(N)$ are the ones closest to the southwest, southeast, and north corners of $H(N)$ (see A, B, C in Figure 2.6), if any of the paths in $D_{(\pi_1,\pi_2^e)}(N)$ started and ended in different sectors, they would cross the paths in $D_{(\pi_1,\pi_2)}(N)$ and there would be a node of degree 4. □

We now have that if $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}$, then $D_{(\pi_1,\pi_2)}(N)$ for $N \geq M'$ has all paths starting and ending in the same sectors, i.e. it has the tripartite node pairing $\sigma_{a,b,c}$, thus

$D_{(\pi_1,\pi_2)} \in DD(\sigma_{a,b,c})$.

Next we want to show that for every $\pi \in DD(\sigma_{a,b,c})$, there exists $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}$ such that $D_{(\pi_1,\pi_2)} = \pi$.

Theorem 55. *Let $(\pi_1, \pi_2) \in DP_{a,b,c}$. Let w be a box in the intersection space of $\pi_1 \cup \pi_2$. Consider the AB configuration $(A, B) = \gamma(\pi_1, \pi_2)$.*

1. *If w is type III_{DP} in (π_1, π_2), then w is type III_{AB} in (A, B), $w \notin B$, $w \notin A$, and $w \notin \mathcal{L}(A, B)$.*
2. *If w is type I^-_{DP} in (π_1, π_2), then w is type I^-_{AB} in (A, B), $w \notin A$, $w \notin B$, and $w \notin \mathcal{L}(A, B)$.*
3. *If w is type II_{DP} in (π_1, π_2), there are several cases.*
   (a) *Suppose $w \in \pi_1$ and $w \notin \pi_2$.*
      i. *If w is type II_{AB} in (A, B), then $w \notin B, w \notin A$, and $w \in \mathcal{L}(A, B)$.*
      ii. *If w is type III_{AB} in (A, B), then $w \notin B, w \in A$, and $w \in \mathcal{L}(A, B)$.*
   (b) *Suppose $w \notin \pi_1$ and $w \in \pi_2$.*
      i. *If w is type II_{AB} in (A, B), then $w \in B, w \notin A$, and $w \notin \mathcal{L}(A, B)$.*
      ii. *If w is type III_{AB} in (A, B), then $w \in B, w \notin A$, and $w \in \mathcal{L}(A, B)$.*

Proof. 1. Suppose w is a type III_{DP} box in (π_1, π_2), then $w \in \pi_1$ and $w \in \pi_2$. Since $w \in \pi_2$, it must be fully supported in all three negative spaces, thus w is type III_{AB} in (A, B). Then since $w \in \pi_1$ and $w \in \mathrm{II}_{AB} \cup \mathrm{III}_{AB}$, we have that $w \notin B$. And since $w \in \pi_2$ and $w \in \mathrm{I}^-_{AB} \cup \mathrm{III}_{AB}$, we have that $w \notin A$. Then since w is type III_{AB} in (A, B) and it is in neither of A, B, we have that $w \notin \mathcal{L}(A, B)$.

2. Suppose w is a type I^-_{DP} box in (π_1, π_2). Then $w \in \pi_2$ and $w \notin \pi_1$, and w is a type I^-_{AB} box in (A, B). Since $w \in \pi_2$ and $w \in \mathrm{I}^-_{AB} \cup \mathrm{III}_{AB}$, we have that $w \notin A$. And since $w \notin \mathrm{II}_{AB} \cup \mathrm{III}_{AB}$, we have that $w \notin B$. Since $w \in \mathrm{I}^-_{AB}$ and $w \notin A$, we have that $w \notin \mathcal{L}(A, B)$.

type	$w \in \pi_1, w \notin \pi_2$	$w \notin \pi_1, w \in \pi_2$
II_{AB}	$w \in \pi_1$ and $w \in \mathrm{II}_{AB} \cup \mathrm{III}_{AB} \implies w \notin B$ $w \notin \pi_2$ and $w \notin \mathrm{I}^-_{AB} \cup \mathrm{III}_{AB} \implies w \notin A$ $w \in \mathrm{II}_{AB}$ and $w \notin B \implies w \in \mathcal{L}(A,B)$	$w \notin \pi_1$ and $w \in \mathrm{II}_{AB} \cup \mathrm{III}_{AB} \implies w \in B$ $w \in \pi_2$ and $w \notin \mathrm{I}^- \cup \mathrm{III}_{AB} \implies w \notin A$ $w \in \mathrm{II}_{AB}$ and $w \in B \implies w \notin \mathcal{L}(A,B)$
III_{AB}	$w \in \pi_1$ and $w \in \mathrm{II}_{AB} \cup \mathrm{III}_{AB} \implies w \notin B$ $w \notin \pi_2$ and $w \in \mathrm{I}^-_{AB} \cup \mathrm{III}_{AB} \implies w \in A$ $w \in \mathrm{III}_{AB}$ and $w \in \Delta(A,B) \implies w \in \mathcal{L}(A,B)$	$w \notin \pi_1$ and $w \in \mathrm{II}_{AB} \cup \mathrm{III}_{AB} \implies w \in B$ $w \in \pi_2$ and $w \in \mathrm{I}^- \cup \mathrm{III}_{AB} \implies w \notin A$ $w \in \mathrm{III}_{AB}$ and $w \in \Delta(A,B) \implies w \in \mathcal{L}(A,B)$

Table 3.2: Type II_{AB} and III_{AB} boxes in π_1 and π_2

3. Suppose w is type II_{DP} in (π_1, π_2). Then $w \in \mathit{Int} \setminus \mathrm{III}_{DP}$. There are two options, either $w \in \pi_1, w \notin \pi_2$ or $w \notin \pi_1, w \in \pi_2$. Also w may be type II_{AB} or type III_{AB} in (A, B).

In each cell of the table above, the first statement is by definition of B given by γ, the second statement is by definition of A given by γ, and the third statement is by definition of $\mathcal{L}(A, B)$ as given by [JWY20]. □

Definition 56. Let $(\pi_1, \pi_2) \in DP_{a,b,c}$, and let $w = (w_1, w_2, w_3) \in \mathit{Int}$. Define the following

$$T_+(w) = \{(t, w_2, w_3) \in \pi_1 \cup \pi_2 \mid a \leq t < w_1\}$$

$$S_+(w) = \{(w_1, s, w_3) \in \pi_1 \cup \pi_2 \mid b \leq s < w_2\}$$

$$R_+(w) = \{(w_1, w_2, r) \in \pi_1 \cup \pi_2 \mid c \leq r < w_3\}$$

that is, $T_+(w)$ is the set of boxes in between (not including) w and w_1, $S_+(w)$ is the set of boxes in between (not including) w and w_2, and $R_+(w)$ is the set of boxes in between (not including) w and w_3.

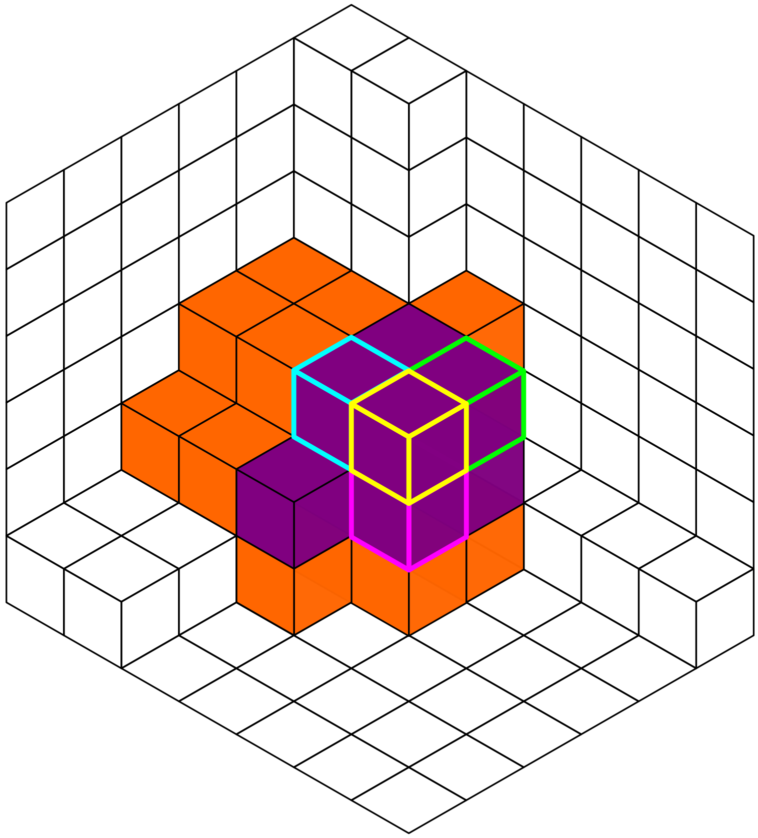

Figure 3.24: Let $w = (2, 3, 2)$ be the box outlined in yellow. Then $T_+(w) = \{(1, 3, 2)\}$ (box outlined in green), $S_+(w) = \{(2, 2, 2)\}$ (box outlined in blue), and $R_+(w) = \{(2, 3, 1)\}$ (box outlined in pink).

Next, we state several lemmas that we use in the proof of our next theorem.

Lemma 57. *Let* $(\pi_1, \pi_2) \in DP_{a,b,c}$. *Let* $w = (w_1, w_2, w_3) \in II_{AB} \cup III_{AB}$ *of* $(A, B) = \gamma(\pi_1, \pi_2)$. *If* $w \notin B$, *then* $(T_+(w) \cup S_+(w) \cup R_+(w)) \cap B = \emptyset$.

Proof. First, note that $T_+(w) \cup S_+(w) \cup R_+(w) \subset \mathrm{II}_{AB} \cup \mathrm{III}_{AB}$ of (A, B). Suppose there exists $u = (t, w_2, w_3) \in T_+(w) \cap B$. Then by the AB configuration condition 3.2, the box $u_1 := (t + 1, w_2, w_3) \in FN(u) \cap T_+(w)$ is in B. Then by the same condition, the box $u_2 := (t + 2, w_2, w_3) \in B$, and so on until $u_{w_1 - t} = (w_1, w_2, w_3) \in B$, a contradiction. The same process shows that boxes in $S_+(w)$ and $R_+(w)$ are also not contained in B.

□

Lemma 58. *Let* $(\pi_1, \pi_2) \in DP_{a,b,c}$. *Let* $w = (w_1, w_2, w_3) \in II_{AB} \cup III_{AB}$ *of* $(A, B) = \gamma(\pi_1, \pi_2)$. *If* $w^1 \in A$ *and if every box in* $T_+(w)$ *is type* III_{AB} *in* (A, B), *then* $T_+(w) \subset A$. *If* $w^2 \in A$ *and* *if every box in* $S_+(w)$ *is type* III_{AB} *in* (A,B)*, then* $S_+(w) \subset A$*. If* $w^3 \in A$ *and if every box in* $R_+(w)$ *is type* III_{AB} *in* (A,B)*, then* $R_+(w) \subset A$*.*

Proof. Consider the case $w^1 \in A$, and every box in $T_+(w)$ is type III_{AB} in (A,B). Then since $w^1 = (a-1, w_2, w_3) \in A$, by the AB configuration condition 3.2 we have that $u := (a, w_2, w_3) \in FN(u) \cap T_+(w)$ is in A. By the same condition, every element of $T_+(w)$ is in A. Similarly for $S_+(w)$ and $R_+(w)$.

□

Lemma 59. *Let* $(\pi_1, \pi_2) \in DP_{a,b,c}$*, and let* $(A,B) = \gamma(\pi_1, \pi_2)$*. If* $w \in Int \setminus III_{DP}$ *gets assigned more than two labels from the Step 1a of the labeling algorithm, then there are four options for* w*:*

- *1 w.s., 1 s., 1 c.n.s.* $\implies$ *w is type* II_{AB}
- *2 w.s., 1 s.* $\implies$ *w is type* III_{AB}
- *2 w.s., 1 c.n.s.* $\implies$ *w is type* II_{AB}
- *3 w.s.* $\implies$ *w is type* III_{AB}

If w gets assigned more than two labels from Step 1b of the labeling algorithm, then

- *3 s.* $\implies$ *w is type* III_{AB}

where w.s. means weakly supported, s. means supported, and c.n.s. means completely not supported (see Definition 17).

Proof. In the first case, we get these options from: the condition on elements of $DP_{a,b,c}$ says that each box in the intersection space is at least weakly supported in at least two negative spaces (i.e. we cannot have two negative spaces where w is completely not supported). Since w was assigned two labels in the DP Labeling Algorithm, these come from spaces where w is weakly supported or completely not supported, so we cannot have two negative spaces where w is supported.

In the second case, if w did not get assigned any labels in Step 1a, then it must be supported in all three negative spaces. If w gets assigned at least two labels in Step 1b, then it is connected to two boxes with different labels. $\square$

Theorem 60. *For* $(\pi_1, \pi_2) \in DP_{a,b,c}$*, if* $\gamma(\pi_1, \pi_2) \in \mathcal{AB}$*, then* $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}$*.*

Proof. If (π_1, π_2) passes the DP Labeling Algorithem 1, then it is in $\mathcal{DP}_{a,b,c}$. Suppose it does not pass the DP Labeling Algorithm. In this proof, when we refer to π_1 and π_2, we mean π_1^p and π_2^p.

Case 1. Suppose the DP Labeling Algorithm fails in Step 1a. Then there is a box $w = (w_1, w_2, w_3) \in Int \setminus \mathrm{III}_{DP}$ which is not supported in two or more negative spaces. Since $w \notin \mathrm{III}_{DP}$ in (π_1, π_2), we know that it is not the case that $w \in \pi_1$ and $w \in \pi_2$. Thus we have either $w \in \pi_1$ and $w \notin \pi_2$, or $w \in \pi_2$ and $w \notin \pi_1$. Since w is not supported in two or more negative spaces, we know that w can't be in π_2 (due to π_2 being an order ideal under the product order on $\mathbb{Z}^3$ containing $\mathbf{M}$), thus we must have that $w \in \pi_1, w \notin \pi_2$. Since $w \in \pi_1$ and $w \in \mathrm{II}_{DP}$, by Theorem 55 3(a) (in either case (i) or (ii)), we know that $w \notin B$. Next, with $w \in Int$ and $w \notin \pi_2$, we have two cases to consider: w a type II_{AB} box in (A, B), or w a type III_{AB} box in (A, B).

Subcase 1. Suppose w is a type II_{AB} box in (A, B). Since w was assigned more than one label in Step 1a, by Lemma 59 part 1, we have that w is completely not supported in one negative space, and is weakly supported in at least one negative space. For the following, we will show the case when w is completely not supported in I_3^- (corresponding to Cyl_3), and weakly supported in I_1^- (corresponding to Cyl_1). We omit the other possible cases since they follow the same argument.

Note that since w is weakly supported in I_1^-, we have that $w^1 \notin \pi_2$ by definition. Thus by the definition of A given by γ, we have that $w^1 \in A$. Consider applying the AB Labeling Algorithm 3 to $\gamma(\pi_1, \pi_2)$. Since $w^1 \in \mathrm{I}_{AB}^- \cap A$, we have that w^1 receives a label of 1. Also, since $w \in \mathrm{II}_{AB} \setminus B$ and $w \notin \mathrm{Cyl}_3$, we have that w receives label 3. In the following cases, we will show that $T_+(w) \subset \mathcal{L}(A, B)$, thus showing that w and w^1 are part of the same connected component of $\mathcal{L}(A, B)$, thus giving a contradiction with our assumption that $\gamma(\pi_1, \pi_2)$ passes the AB Labeling Algorithm 3. Note that since $w \notin B$, by Lemma 57 we have that $T_+(w) \cap B = \emptyset$.

Subcase 1a. Suppose $T_+(w) \subset \text{III}_{AB}$ in (A, B). By Lemma 58, we have that $T_+(w) \subset A$. Then since $T_+(w) \cap B = \emptyset$, we have that $T_+(w) \subset \text{III}_{AB} \cap (A \Delta B)$, and so $T_+(w) \subset \mathcal{L}(A, B)$.

Subcase 1b. Suppose $T_+(w) \subset \text{II}_{AB}$. Since $T_+(w) \cap B = \emptyset$, we have that $T_+(w) \subset \text{II}_{AB} \setminus B$, and so $T_+(w) \subset \mathcal{L}(A, B)$.

Subcase 1c. Suppose $T_+(w) \cap \text{II}_{AB} \neq \emptyset$ and $T_+(w) \cap \text{III}_{AB} \neq \emptyset$. Let $u \in T_+(w) \cap \text{III}_{AB}$ such that $FN(u) \cap T_+(w) \cap \text{II}_{AB} \neq \emptyset$. Let $v \in FN(u) \cap T_+(w) \cap \text{II}_{AB}$. By Lemma 58 applied to v, we have that $T_+(v) \subset A$, and so $T_+(v) \subset \text{III}_{AB} \cap (A \Delta B)$. Then since $T_+(w) \setminus T_+(v) \subset \text{II}_{AB} \setminus B$, we have that $T_+(w) \subset \mathcal{L}(A, B)$.

Subcase 1d. Suppose $T_+(w) = \emptyset$, then $w^1 \in BN(w)$, and so w and w^1 are in the same connected component of $\mathcal{L}(A, B)$.

Subcase 2. Suppose w is a type III_{AB} box in (A, B). By Theorem 55 part 3(a)(ii), we have that $w \in A$, and since $w \notin B$, we have that $w \in \mathcal{L}(A, B)$. Since $w \in \text{III}_{AB}$, we know that w is not c.n.s. in any negative spaces. Then, by Lemma 59, we have that w has at least two spaces where it is weakly supported. For the following, we will show the case where w is weakly supported in I_1^- and I_2^-. We omit the other possible cases since they follow the same argument.

Since w is weakly supported in I_1^- and I_2^-, we have that $w^1, w^2 \notin \pi_2$. Then since w^1 and w^2 are contained in $\text{Cyl}_1 \cup \text{Cyl}_2 \cup \text{Cyl}_3$ (because w is weakly supported in both these spaces), we have that $w^1, w^2 \in A$ (by definition of γ).

Since w is type III_{AB} in (A, B), we must have that every box in $T_+(w) \cup S_+(w) \cup R_+(w)$ is also type III_{AB} in (A, B) (this is by construction of the cylinders from plane partitions). Then by Lemma 58, we have that $T_+(w) \cup S_+(w) \subset A$, and by Lemma 57, $(T_+(w) \cup S_+(w)) \cap B = \emptyset$.

Then, since the boxes in $T_+(w) \cup S_+(w)$ are type III_{AB} boxes in A and not in B, we have that $T_+(w) \cup S_+(w) \subset \mathcal{L}(A, B)$. Apply the AB Labeling Algorithm 3 to $\gamma(\pi_1, \pi_2)$. Since $w^1, w^2 \in \text{I}^-_{AB} \cap A$ we assign label 1 to w^1, and label 2 to w^2. Since $T_+(w) \cup S_+(w) \subset \mathcal{L}(A, B)$, we have that $T_+(w) \cup S_+(w) \cup \{w^1, w^2\} \cup w$ are all contained in the same connected component of $\mathcal{L}(A, B)$. However, w^1 and w^2 induce conflicting labels, a contradiction.

Case 2. Suppose there exists a connected component C of $Int \setminus \text{III}_{DP}$ with conflicting labels (note that this includes the case of failing in Step 1b and Step 2. Let $u, v \in C$, where u has label e, and v has label f for $e \neq f \in \{1, 2, 3\}$, such that u and v were labeled in Step 1a.

Since u and v were labeled in Step 1a of the DP Labeling Algorithm 1, we know that both u and v are either weakly supported or completely not supported in exactly one negative space. Let u be weakly supported or completely not supported in I^-_e, and let v be weakly supported or completely not supported in I^-_f for $e \neq f \in \{1, 2, 3\}$ (we know that $e \neq f$ because u and v induce different labels on w by our assumption).

Let $z \in C$. We will show that $z \in \mathcal{L}(A, B)$. Since $z \in Int \setminus \text{III}_{DP}$, we have that $z \in \text{II}_{AB} \cup \text{III}_{AB}$. Consider $z \in \text{II}_{AB}$, and suppose $z \in B$. Then $z \notin \pi_1$, and since $z \notin I^-_{AB} \cup \text{III}_{AB}$, we also have $z \notin \pi_2$. Thus $z \notin \pi_1 \cup \pi_2$, a contradiction with our assumption. Thus if $z \in \text{II}_{AB}$, then $z \notin B$, and so $z \in \text{II}_{AB} \setminus B$. Next, suppose $z \in \text{III}_{AB}$. Since $z \notin \pi_1 \cap \pi_2$, we have that $z \in A \Delta B$. Thus if $z \in \text{III}_{AB}$, $z \in (\text{III}_{AB} \cap (A \Delta B)$.

Lemma 61. *Let $u \in Int \setminus III_{DP}$ such that u was labeled e by the DP Labeling Algorithm in Step 1a. If $u \in III_{AB}$, then $\{u_e\} \cup E_+(u) \subset \mathcal{L}(A, B)$.*

Proof. Since $u \in \text{III}_{AB}$, and since u was labeled e in Step 1a of the DP Labeling Algorithm, we know that it is weakly supported in I_e^-, and so $u^e \in A$. Since $u \in \text{Ш}_{AB}$, we must have that $E_+(u) \subset \text{Ш}_{AB}$ (by construction of the cylinders from plane partitions), where $E_+ \in \{T_+, S_+, R_+\}$ corresponding to e). Next, by Lemma 58, since $u \in \text{Ш}_{AB}$, $u^e \in A$, and $E_+(u) \subset \text{Ш}_{AB}$, we have that $E_+(u) \subset A$. By Lemma 57, since $u \in \text{Ш}_{AB}$, and $u \notin B$, we have that $E_+(u) \cap B = \emptyset$. Thus $\{u^e\} \cup E_+(u) \subset \mathcal{L}(A, B)$. Note that we may also apply this proof to v to show that $\{v^f\} \cup F_+(v) \subset \mathcal{L}(A, B)$. □

Subcase 1. Suppose $u, v \in \text{Ш}_{AB}$. Then u is weakly supported in I_e^- and v is weakly supported in I_f^- for $e \neq f \in \{1, 2, 3\}$. Then $u^e \in A$ and $v^f \in A$. By Lemma 61, we have that $\{u^e, v^f\} \cup E_+(u) \cup F_+(v) \subset \mathcal{L}(A, B)$. Thus we have a connected component of $\mathcal{L}(A, B)$ containing u^e and v^f, which fails Algorithm 3, a contradiction.

Subcase 2. Suppose $u, v \in \text{II}_{AB}$. Then u is completely not supported in I_e^- and v is completely not supported in I_f^- for $e \neq f \in \{1, 2, 3\}$, thus $u \notin \text{Cyl}_e$ and $v \notin \text{Cyl}_f$. Then $u, v \in \text{II}_{AB} \setminus B$ with conflicting labels, a contradiction.

Subcase 3. Suppose one of u, v is type II_{AB} in (A, B) and the other is type Ш_{AB} in (A, B). Then by Subcase 1 and Subcase 2, we have a contradiction.

□

Definition 62. Define the map $\varphi : \mathcal{DP}_{a,b,c} \to DD(\sigma_{a,b,c})$ by $\varphi(\pi_1, \pi_2) = D_{(\pi_1, \pi_2)}$.

Lemma 63. *φ is well-defined.*

Proof. Let $N \geq M'$. Let $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}$, then $D_{(\pi_1, \pi_2^e)}(N)$ has all paths starting and ending in the same sectors. By design of M', we have that the nodes of $D_{(\pi_1, \pi_2)}(N)$ are also nodes in $D_{(\pi_1, \pi_2^e)}(N)$. Thus when we perform the conversion from $D_{(\pi_1, \pi_2^e)}(N)$ to $D_{(\pi_1, \pi_2)}(N)$ (as in Figure 3.24), which removes nodes in $D_{(\pi_1, \pi_2^e)}(N)$ which are not nodes in $D_{(\pi_1, \pi_2)}(N)$ by converting paths into doubled edges, we have that $D_{(\pi_1, \pi_2)}$ has the node set $\mathbf{N}$, and the nodes are connected as in $\sigma_{a,b,c}$, thus $D_{(\pi_1, \pi_2)} \in DDC(\sigma_{a,b,c})$. □

Lemma 64. *φ is surjective.*

Proof. Let $\pi \in DD(\sigma_{a,b,c})$. We will show that there exists $(\pi_1, \pi_2) \in \mathcal{DP}_{a,b,c}$ such that $\varphi(\pi_1, \pi_2) = \pi$. Truncate π to $H(N)$ for $N \in \mathbb{N}$ such that π on $H(M)$ for $M > N$ differs from π on $H(N)$ by doubled edges as in the minimal configuration.

First, we split π into two single-dimer configurations, π_1 on $H(N)$ and π_2 on $H(N) - \mathbf{N}$, assigning the top horizontal edges of any loop to π_1. Then we have a pair $(\pi_1, \pi_2) \in DP_{a,b,c}$ such that $D_{(\pi_1,\pi_2)} = \pi$.

Consider $(A, B) = \gamma(\pi_1, \pi_2)$. Since $\pi \in DD(\sigma_{a,b,c})$, all paths in $D_{(\pi_1,\pi_2)}$ start and end in the same sector. Then by Theorem 54, all paths of $D_{(\pi_1,\pi_2^e)} = D_{(A,B)}$ start and end in the same sector. Then by Theorem 4.4.19 of [JWY20], we have that $(A, B) \in \mathcal{AB}$, and then by Theorem 60, we have that $(\pi_1, \pi_2) \in \mathcal{DD}$. Note that this holds for any choice of π_1, π_2 constructed by assigning dimers from the loops of π.

□

Note that by our edge weighting on the hexagon graph, φ is weight-preserving.

Lemma 65. *Let $\pi \in DD(\sigma_{a,b,c})$. Then $|\varphi^{-1}(\pi)| = 2^{\ell(\pi)}$, where $\ell(\pi)$ is the number of closed loops of π.*

Proof. Let $\pi \in DD(\sigma_{a,b,c})$. Split π into two single-dimer configurations, π_1 on $H(N)$ and π_2 on $H(N) - \mathbf{N}$. The number of possible elements $(\pi_1, \pi_2) \in DP_{a,b,c}$ constructed in this way is $2^{\ell(\pi)}$, and for each one $D_{(\pi_1,\pi_2)} = \pi$. Thus $|\varphi^{-1}(\pi)| = 2^{\ell(\pi)}$. □

Theorem 66.

$$Z^{\mathcal{DP}}_{a,b,c}(q) = Z^{DD}_{a,b,c}(q)$$

Proof. As defined in Definition 38, we have

$$Z^{\mathcal{DP}}_{a,b,c}(q) = \sum_{(\pi_1,\pi_2)\in\mathcal{DP}_{a,b,c}} q^{|\pi_1|+|\pi_2|},$$

by Lemma 64 and since φ is weight-preserving, we have

$$= \sum_{\pi \in DD(\sigma_{a,b,c})} |\varphi^{-1}(\pi)| w(\pi),$$

then by Lemma 65, we have

$$= \sum_{\pi \in DD(\sigma_{a,b,c})} 2^{\ell(\pi)} w(\pi),$$

by Equation 9, we have

$$= Z_{DD}^{a,b,c}(q)$$

□

By Theorem 40 and Theorem 66, we have one of our main results:

Theorem 67.

$$Z_{a,b,c}^{DB}(q) = Z_{a,b,c}^{DD}(q).$$

Chapter 4

Induction by Condensation

Some of the work in this chapter was published in Séminaire Lotharingien de Combinatoire **93B** (2025), Article # 39, with coauthor Benjamin Young [BY25].

In this section we will prove Theorem 1 by showing that $Z^{DD}_{a,b,c}(q)$ and $M(q)^2 M_{a,b,c}(q)$ satisfy the same recurrence relation in a, b and c, given by

$$\begin{aligned} X(a,b,c)X(a+1,b+1,c) = {} & X(a+1,b,c)X(a,b+1,c) \\ & + q^K X(a+1,b+1,c-1)X(a,b,c+1) \end{aligned} \tag{4.1}$$

where $K = a+b+1$. We also show that $Z^{DD}_{a,b,c}(q)$ and $M(q)^2 M_{a,b,c}(q)$ satisfy the same initial conditions.

Our first job is to explain why $M(q)^2 M_{a,b,c}(q)$ satisfies recurrence (4.1). Essentially this is because $M_{a,b,c}(q)$ is q-counting boxed $a \times b \times c$ plane partitions, so we can modify an argument of Kuo [Kuo04]. Our second job is to convert the double-dimer recurrence in Jenne [Jen21] (which comes from the Desnanot-Jacobi relation applied to Jenne's matrix) into a recurrence for the associated double-box configurations, and observe that the result is, again, recurrence (4.1). This involves renormalizing all six double-dimer generating functions so that the weight of the minimal configuration is 1 (i.e. dividing the entire generating function by the power of q in its leading term), which allows us to take $n \to \infty$. Computing the leading terms of the double-dimer partition functions — which we call *normalization* *constants* due to the way they are used in the computation— is in principle elementary but in practice very detail-heavy, and we defer the actual computations to Appendix A.2.

4.1 Graphical Condensation

We first consider $M(q)^2 M_{a,b,c}(q)$ where $M(q) = \prod_{k>0} 1/(1-q^k)^k$, and

$$M_{a,b,c}(q) = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

We will show that $M(q)^2 M_{a,b,c}(q)$ satisfies the recurrence relation given by Equation (4.1), using Kuo's graphical condensation (Theorem 6.2 in [Kuo04]), with a slight modification which affects the factor of q^K. We will first explain Kuo's graphical condensation and his proof, in order to then discuss our slight modification.

Theorem 68. *(Kuo 2004, [Kuo04]) Define* $P(r,s,t)$ *as the generating function for plane partitions that fit in* $\mathcal{B}(r,s,t)$ *(the box of size* $r \times s \times t$*). Then*

$$P(r,s,t) = \prod_{i=1}^{r}\prod_{j=1}^{s} \frac{1-q^{i+j+t-1}}{1-q^{i+j-1}}$$

Comparing Kuo's notation to ours, note that $P(r,s,t) = M_{a,b,c}(q)$.

Theorem 69. *(Kuo 2004, [Kuo04])*

$$P(r+1,s+1,t)P(r,s,t) = q^t P(r,s+1,t)P(r+1,s,t) + P(r+1,s+1,t-1)P(r,s,t+1)$$

In the proof of Theorem 69, Kuo creates four outer strips on the $(r+1)\times(s+1)\times t$ hexagon graph as in Figure 4.1.

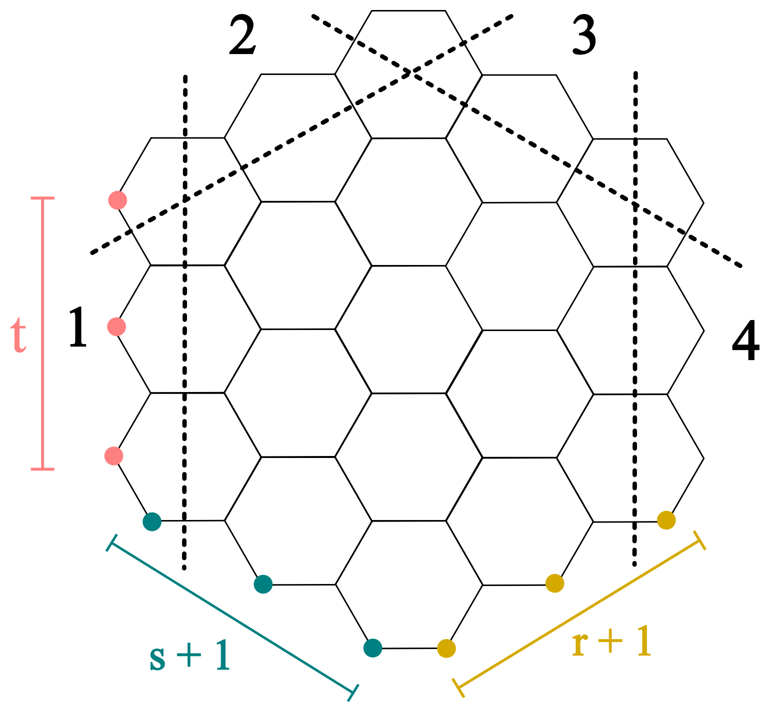

Figure 4.1: Kuo's hexagon strips 1-4 on $H(r+1, s+1, t)$. Image source: [Kuo04].

Kuo superimposes a single-dimer configuration on $H(r, s, t)$ with one on $H(r+1, s+1, t)$ such that the bottom hexagon is aligned, resulting in a configuration called H on $H(r+1, s+1, t)$ where the nodes in the four outer strips have degree one, and the nodes in the inner $H(r, s, t)$ graph have degree two. Each strip must have a matched edge which connects to the nodes of degree two in the middle $H(r, s, t)$ hexagon graph, thus creating paths between the strips. These paths must be connected by: strip 1 and strip 2, strip 3 and strip 4, or by strip 1 and strip 4, strip 2 and strip 3 (otherwise you get a node of degree four where the paths intersect).

Kuo then shows that if the paths are between strip 1 and strip 2, strip 3 and strip 4, then you can split the configuration H into a single-dimer configuration on $H(r+1, s, t)$ (by deleting strips 1 and 2) and a single-dimer configuration on $H(r, s+1, t)$ (by deleting strips 3 and 4), see Figure 4.2.

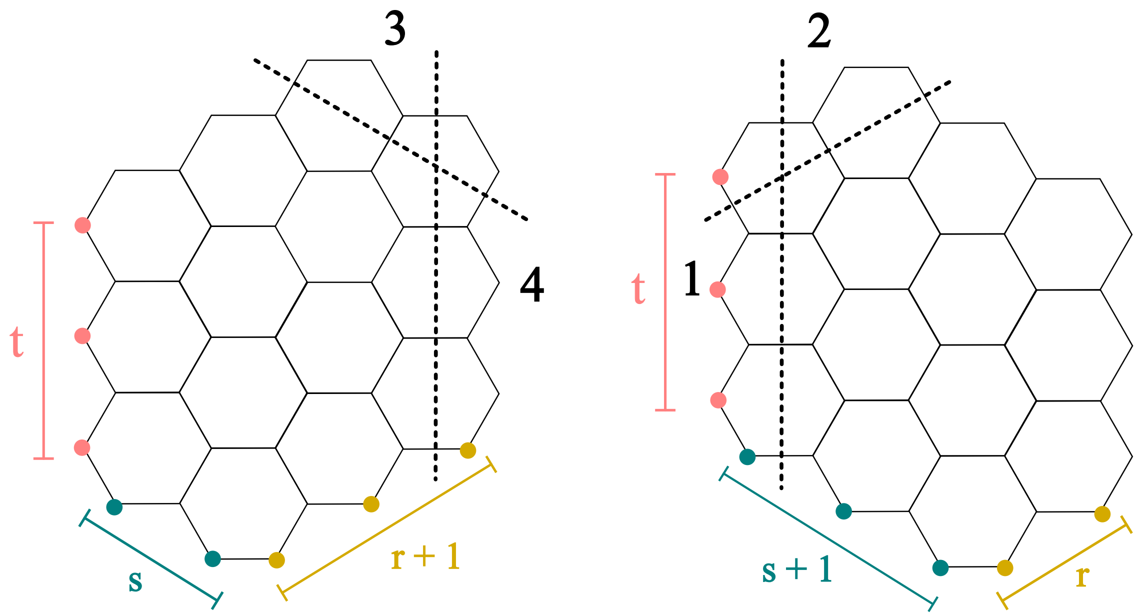

Figure 4.2: Paths connect strips 1 and 2, 3 and 4. Left: Strips 1 and 2 deleted giving $H(r+1, s, t)$. Right: Strips 3 and 4 deleted giving $H(r, s+1, t)$. Image source: [Kuo04].

If the paths are between strip 1 and strip 4, strip 2 and strip 3, Kuo splits the configuration into a single-dimer configuration on $H(r, s, t+1)$ (by deleting strips 1 and 4), and a single-dimer configuration on $H(r+1, s+1, t-1)$ (by deleting strips 2 and 3), see Figure 4.3.

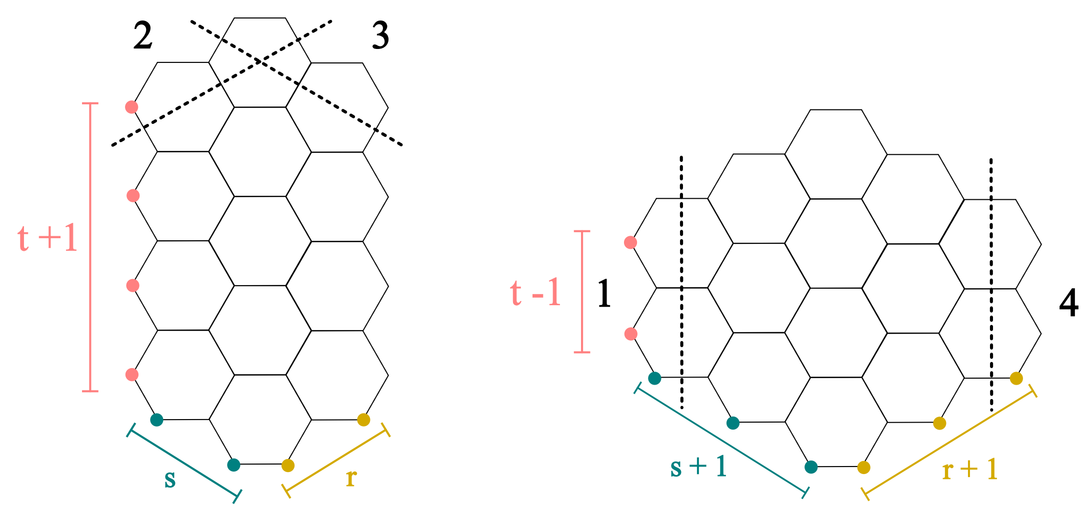

Figure 4.3: Paths connect strips 1 and 4, 2 and 3. Left: Strips 1 and 4 deleted giving $H(r, s, t+1)$. Right: Strips 2 and 3 deleted giving $H(r+1, s+1, t-1)$. Image source: [Kuo04].

Since the paths can only connect the strips in these two possible ways, we may split H in two possible ways, giving the following relation

$$\begin{aligned}&q^{(r+1)(s+1)s/2}P(r+1,s+1,t)\cdot q^{rs(s-1)/2}P(r,s,t)\\&=q^{s+t}\cdot q^{r(s+1)s/2}P(r,s+1,t)\cdot q^{(r+1)s(s-1)/s}P(r+1,s,t)\\&+q^{(r+1)(s+1)s/2}P(r+1,s+1,t-1)\cdot q^{rs(s-1)/2}P(r,s,t+1).\end{aligned}$$

Each generating function P for boxed plane partitions is multiplied by the weight of the minimal configuration on the corresponding hexagon graph (thus giving the generating function for single-dimer configurations on the hexagon graph). The factor of q^{s+t} comes from the edge of weight q^{s+t} that was not covered by either subgraph $H(r,s+1,t)$ or $H(r+1,s,t)$ (Figure 4.2).

Finally, Kuo simplifies this relation by dividing through by $q^{(r+1)(s+1)s/2+rs(s-1)/2}$, giving his result stated in Theorem 6.2.

In order to get the recurrence relation given in Equation (4.1), we apply Kuo's proof with the hexagon strips created as in Figure 4.4.

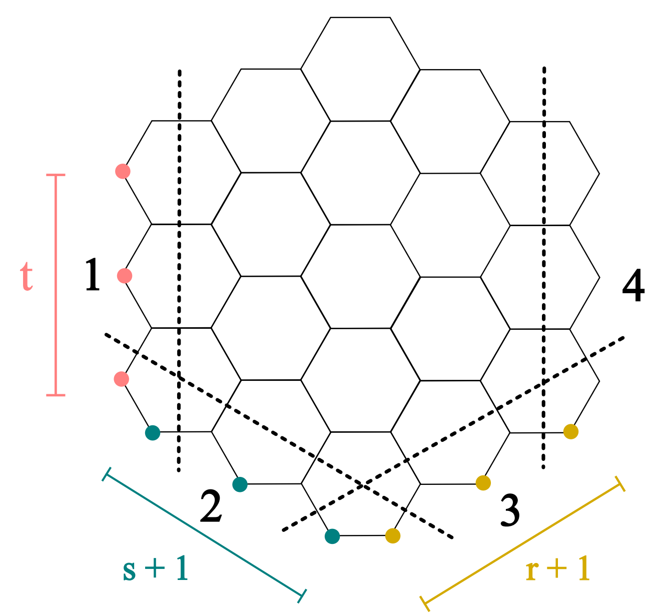

Figure 4.4: Re-define the strips on the $H(r+1,s+1,t)$ hexagon graph.

When we multiply each term in the recurrence by the weight of the minimal configuration in this case, we get

$$
\begin{aligned}
&q^{(r+1)(s+1)s/2}P(r+1,s+1,t)\cdot q^{rs}\cdot q^{rs(s-1)/2}P(r,s,t)\\
&= q^{r(s+1)s/2}P(r,s+1,t)\cdot q^{s(r+1)}\cdot q^{(r+1)s(s-1)/2}P(r+1,s,t)\\
&+ q^{rs(s-1)/2}P(r,s,t+1)\cdot q^{(r+1)(s+1)}\cdot q^{(r+1)(s+1)s/2}P(r+1,s+1,t-1)
\end{aligned}
$$

where the factors q^{rs}, $q^{s(r+1)}$, and $q^{(r+1)(s+1)}$ come from adding the additional edge weights resulting in the shift up by one hexagon when deleting strips 2 and 3, or just strip 3 (see Figures 4.5 and 4.6).

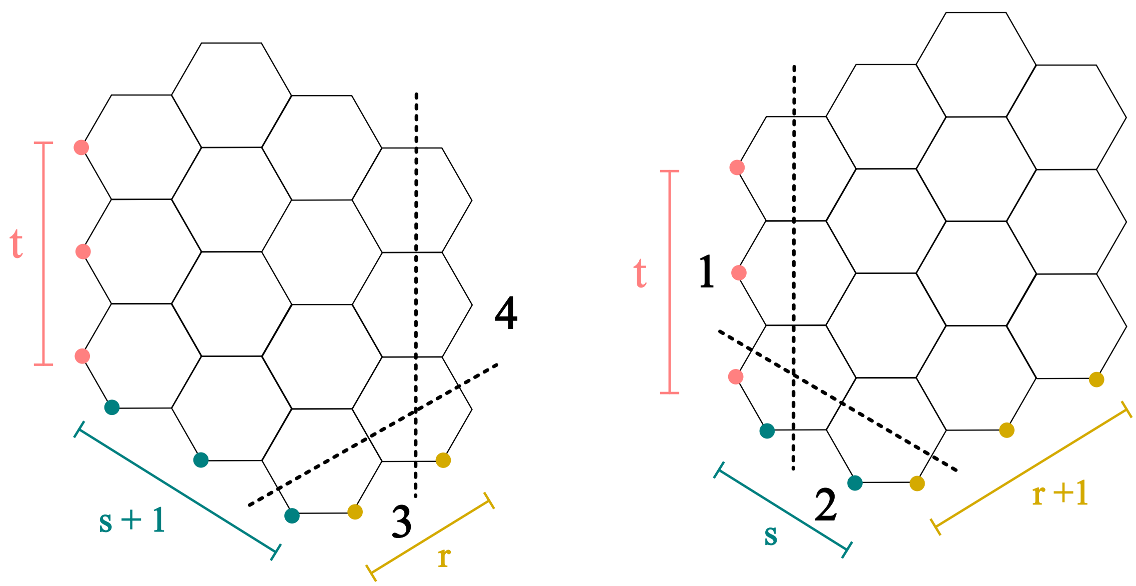

Figure 4.5: Path connect strips 1 and 2, 3 and 4. Left: Strips 1 and 2 deleted giving $H(r,s+1,t)$. Right: Strips 3 and 4 deleted giving $H(r+1,s,t)$.

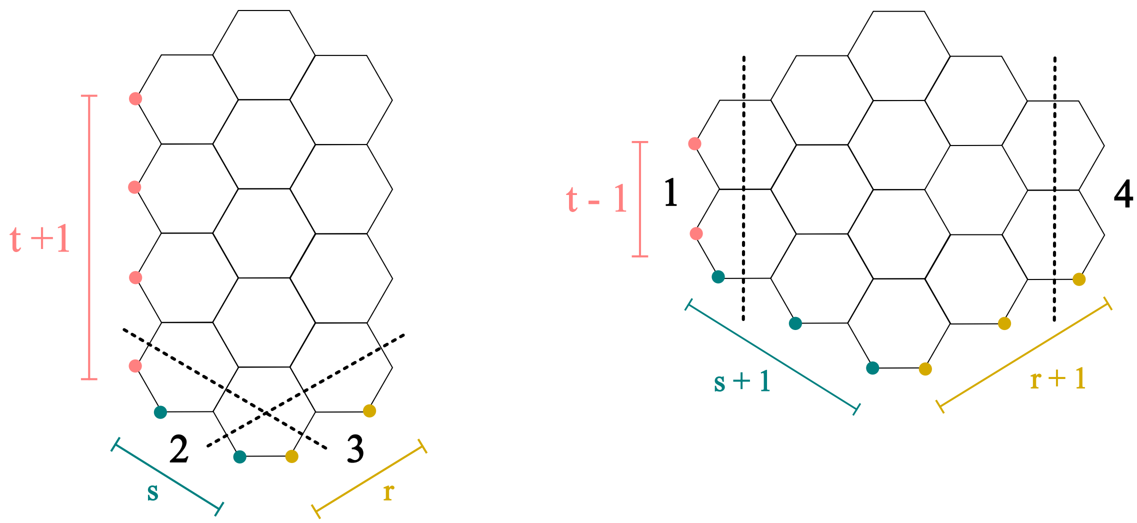

Figure 4.6: Paths connect strips 1 and 4, 2 and 3. Left: Strips 1 and 4 deleted giving $H(r, s, t+1)$. Right: Strips 2 and 3 deleted giving $H(r+1, s+1, t-1)$.

We simplify by dividing both sides by $q^{(r+1)(s+1)s/2+rs+rs(s-1)/2}$, which gives

$$P(r+1, s+1, t)P(r, s, t) = P(r, s+1, t)P(r+1, s, t) + q^{r+s+1}P(r, s, t+1)P(r+1, s+1, t-1)$$

Details of these calculations are in Appendix A.1.

We have shown that $M_{a,b,c}(q)$ satisfies Equation (4.1). Note that $M(q)^2M_{a,b,c}(q)$ does as well, since $M(q)^2$ factors out and cancels.

4.2 Double-Dimer Condensation

We will use a result of Jenne [Jen21] to show that $Z^{DD}_{n;a,b,c}(q)$ satisfies the same recurrence relation (4.1).

Definition 70. (Jenne, 2021. [Jen21]) Let $Z^{DD}_{\sigma}(G, \mathbf{N})$ be the weighted sum of all double-dimer configurations on graph G with node set $\mathbf{N}$ with pairing σ.

Theorem 71. *(Jenne, 2021. [Jen21]) Let $G = (V_1, V_2, E)$ be a finite edge-weighted planar bipartite graph with a set of nodes N. Divide the nodes into three circularly contiguous sets*

R, G, and B such that $|R|$, $|G|$, and $|B|$ satisfy the triangle inequality and let σ be the corresponding tripartite pairing. Let x, y, w, v be nodes appearing in a cyclic order such that the set $\{x, y, w, v\}$ contains at least one node of each RGB color. If $x, w \in V_1$ and $y, v \in V_2$ then

$$\begin{aligned} Z^{DD}_{\sigma_{\{x,y,w,v\}}}(G, \mathbf{N} - \{x, y, w, v\}) Z^{DD}_{\sigma}(G, \mathbf{N}) = \quad & Z^{DD}_{\sigma_{x,y}}(G, \mathbf{N} - \{x, y\}) Z^{DD}_{\sigma_{w,v}}(G, \mathbf{N} - \{w, v\}) \\ + \quad & Z^{DD}_{\sigma_{y,w}}(G, \mathbf{N} - \{x, v\}) Z^{n}_{DD}(G, \mathbf{N} - \{y, w\}) \end{aligned} \tag{4.2}$$

where for $i, j \in \{x, y, w, v\}$, σ_{ij} is the unique planar pairing on $\mathbf{N} - \{i, j\}$ in which like colors are not paired together.

We will show that $Z^{DD}_{n;a,b,c}(q)$, for $n \in \mathbb{N}$, which we will write as $Z^{DD}_n(a, b, c)$ here to emphasize the change in a, b, c, satisfies the following recurrence relation

$$\begin{aligned} Z^{DD}_n(a, b, c) Z^{DD}_n(a + 1, b + 1, c) = & \, Z^{DD}_n(a, b + 1, c) Z^{DD}_n(a + 1, b, c) \\ & + q^K Z^{DD}_{n,down}(a + 1, b + 1, c - 1) Z^{DD}_{n,up}(a, b, c + 1) \end{aligned} \tag{4.3}$$

for $K = a + b + 1$, which we note does not depend on n.

First we consider the unnormalized and normalized generating functions

$$\tilde{Z}^{DD}_n(a, b, c) = \sum_{\pi \in DD(\sigma_{a,b,c})} 2^{\ell(\pi)} w(\pi)$$

$$Z^{DD}_n(a, b, c) = \frac{1}{w(\pi_0)} \tilde{Z}^{DD}_n(a, b, c)$$

Let $G = H(N)$ for $N \geq \max\{a + 1, b + 1, c\}$. Define the following special nodes x, y, w, v by: (see Figure 4.7)

- $x = (b + 1)$ vertex on L_1
- $y = (b + 1)$ vertex on L_6

- $w = (a+1)$ vertex on L_5
- $v = (a+1)$ vertex on L_4

Define the node set $\mathbf{N}^*$ of $H(N)$ to be $\mathbf{N}^* = R^* \cup G^* \cup B^*$ where $R^* = R \cup \{x\}, B^* = B \cup \{y, w\}, G^* = G \cup \{v\}$, for R, G, B defined as in Section 2.2. Note that the set $\{x, y, w, v\}$ contains at least one node of each RGB color, and the pairs of nodes (x, w) and (y, v) have the same coloring in the bipartite coloring on $H(N)$.

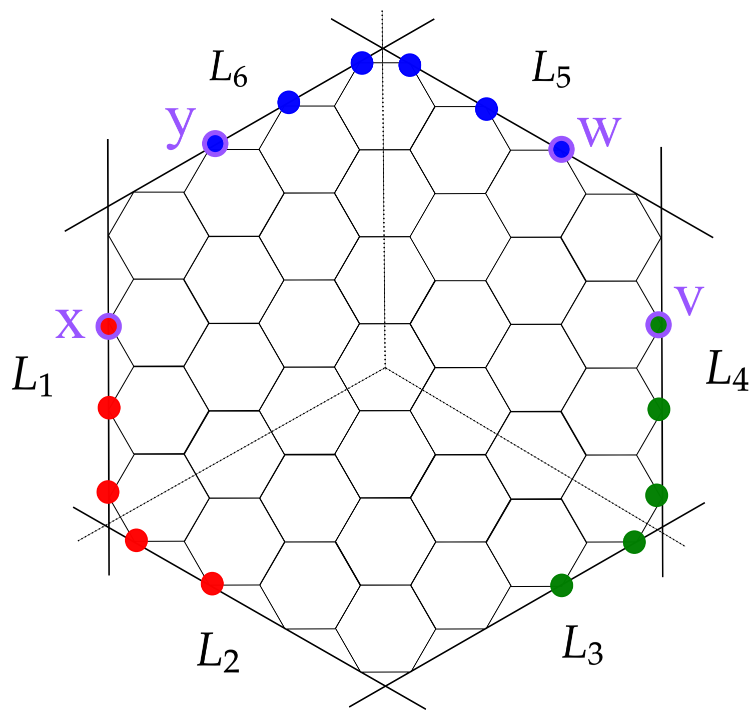

Figure 4.7: Let $a, b, c = 2$. Define nodes $x, y, w, v \in \mathbf{N}^*$.

We compare our notation to the notation used by Jenne on the left-hand side of Equation 4.2 in Figure 4.8.

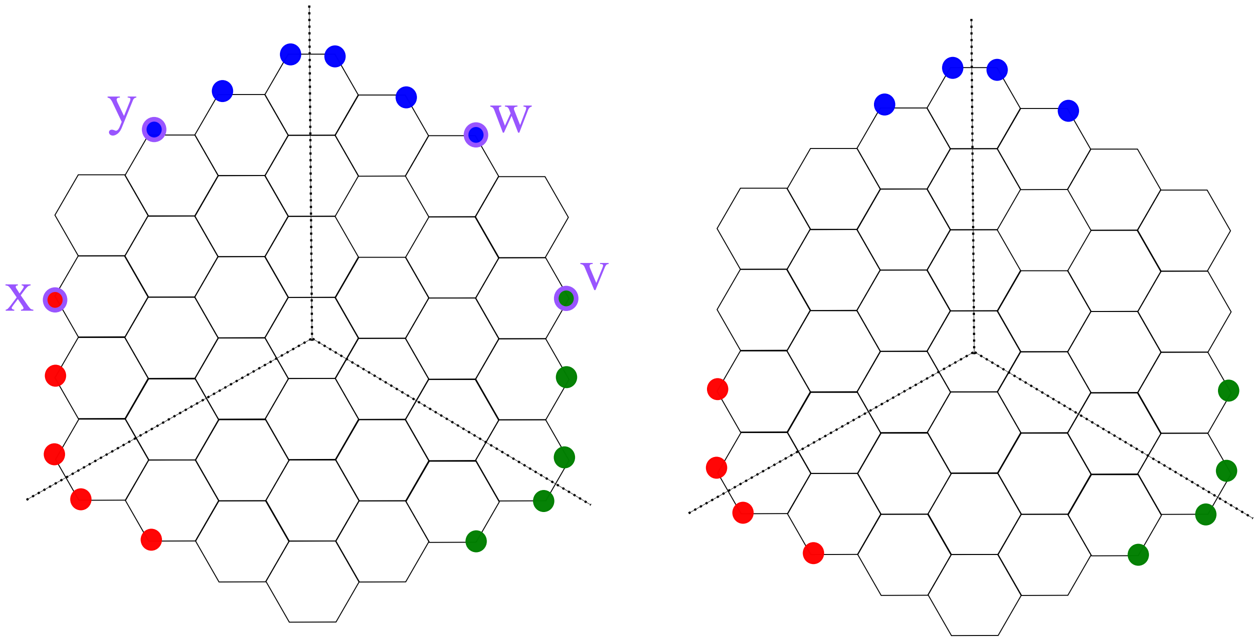

Figure 4.8: Left: $Z^{DD}_{\sigma}(G, \mathbf{N}) = \tilde{Z}^{DD}_{n}(a+1, b+1, c)$. Right: $Z^{DD}_{\sigma_{x,y,w,v}}(G, \mathbf{N} - \{x, y, w, v\}) = \tilde{Z}^{DD}_{n}(a, b, c)$.

Next, we compare the first term on the right-hand side of Equation 4.2 in Figure 4.9.

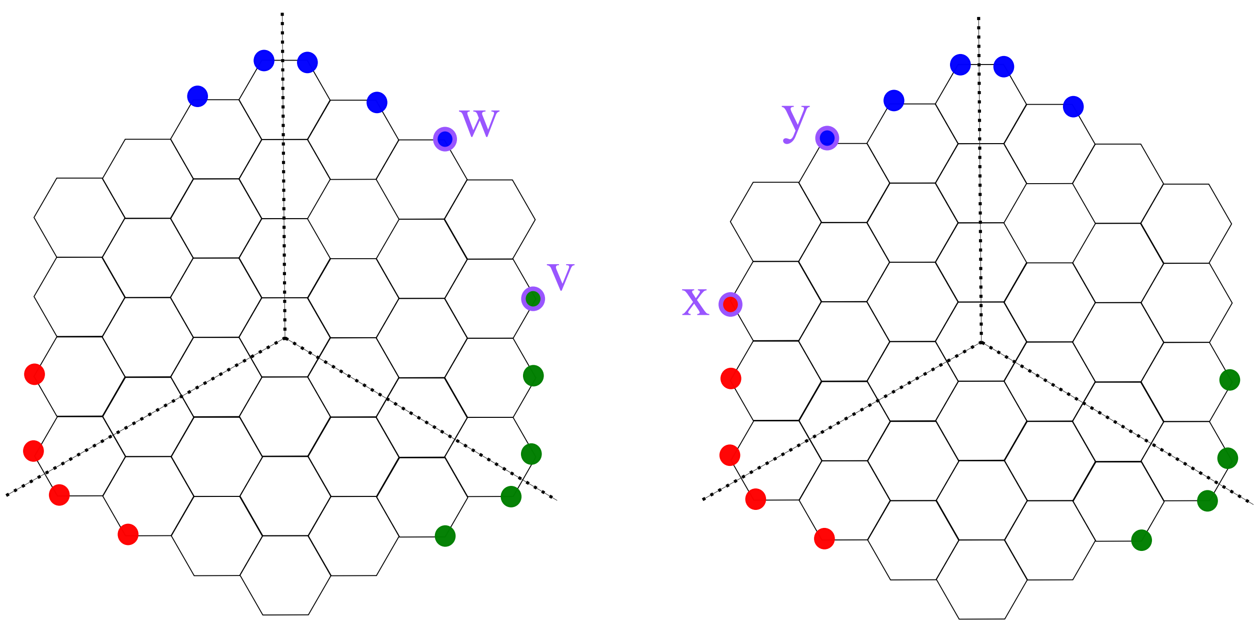

Figure 4.9: Left: $Z^{DD}_{\sigma_{x,y}}(G, \mathbf{N} - \{x, y\}) = \tilde{Z}^{DD}_{n}(a+1, b, c)$. Right: $Z^{DD}_{\sigma_{w,v}}(G, \mathbf{N} - \{w, v\}) = \tilde{Z}^{DD}_{n}(a, b+1, c)$.

Note that in the two cases above (Figure 4.9) we delete both endpoints of a path, since nodes x and y are endpoints of the $(b+1)$-st path in sector 2, and nodes w and v are endpoints of the $(a+1)$-st path in sector 1.

Finally, we compare the second term on the right-hand side of Equation 4.2 in Figure 4.10.

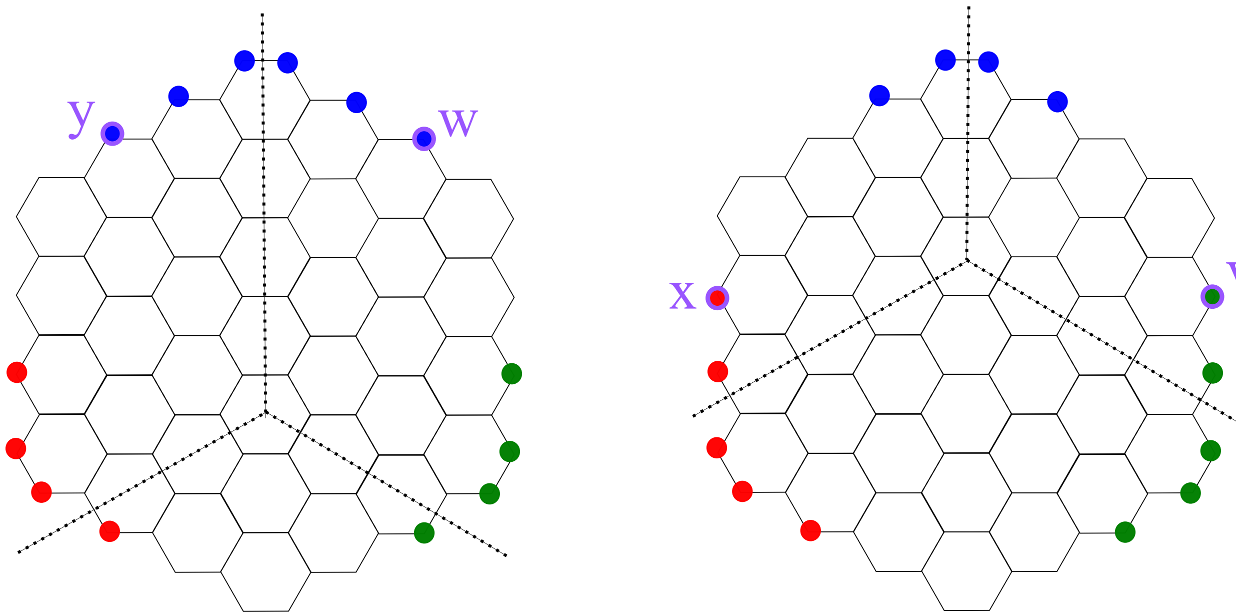

Figure 4.10: Left: $Z^{DD}_{\sigma_{x,v}}(G, \mathbf{N}-\{x,v\}) = \tilde{Z}^{DD}_{n,down}(a+1, b+1, c)$. Right: $Z^{DD}_{\sigma_{y,w}}(G, \mathbf{N}-\{y,w\}) = \tilde{Z}^{DD}_{n,up}(a, b, c+1)$.

In these cases (Figure 4.10), we have to move the center of the hexagon up or down. Note that this is well-defined when we take the limit as n goes to infinity, that is

$$\lim_{n\to\infty} \left(Z^{DD}_{n,down}(a+1, b+1, c)\right) = Z^{DD}(a+1, b+1, c),$$

$$\lim_{n\to\infty} \left(Z^{DD}_{n,up}(a, b, c+1)\right) = Z^{DD}(a, b, c+1).$$

We have shown how the unnormalized generating function for double-dimer configurations $\tilde{Z}^n_{DD}(a, b, c)$ satisfies Jenne's recurrence, that is, by Theorem 71, we have

$$\tilde{Z}_n^{DD}(a,b,c)\tilde{Z}_n^{DD}(a+1,b+1,c) = \tilde{Z}_n^{DD}(a,b+1,c)\tilde{Z}_n^{DD}(a+1,b,c) + \tilde{Z}_{n,down}^{DD}(a+1,b+1,c-1)\tilde{Z}_{n,up}^{DD}(a,b,c+1).$$

Next, we compute the normalization constants for each of the six generating functions (that is, the weight of the minimal configuration for each hexagon graph), which will give us the value of the constant K in Equation (4.3).

Computing Normalization Constants

In this section we compute the weight of the unique minimally weighted configuration $\pi_0 \in DD_n(\sigma_{a,b,c})$(see Figure 2.7). For notational convenience, we define $\text{qexp}(X) := q^X$.

We compute $w(\pi_0) \in DD_{r,s,t}(\sigma_{a,b,c})$ as follows: (that is, for the $r \times s \times t$ hexagon graph $H(r,s,t)$)

Horizontal edges from the floor: (all included, second term comes from the doubled edges that are not part of a path)

$$\text{qexp}\left(\frac{rs(s-1)}{2} + \left(\frac{(r-c)(s-c)(s-c-1)}{2}\right)\right)$$

Horizontal edges from left wall: (coming from the b paths)

$$\text{qexp}\left(\sum_{i=0}^{b-1}(s+i)(r-1-i)\right)$$

Horizontal edges from right wall: (coming from the a paths)

$$\text{qexp}\left(\sum_{j=1}^{a}\left(\frac{(s-1)s}{2}-\frac{(j-1)j}{2}\right)\right)$$

(upper value on summations included).

Thus

$$\begin{aligned} w(\pi_0) =& \text{qexp}\left(\frac{rs(s-1)}{2}+\left(\frac{(r-c)(s-c)(s-c-1)}{2}\right)+\sum_{i=0}^{b-1}(s+i)(r-1-i)\right. \\ & \left.+\sum_{j=1}^{a}\left(\frac{(s-1)s}{2}-\frac{(j-1)j}{2}\right)\right) \end{aligned}$$

We compute the weight of the minimal configuration for each of the six terms in Equation (4.3) (letting $r = s = t = n$ for $n \geq \max\{a+1, b+1, c\}$), and we get

$$\begin{aligned} \tilde{Z}_n^{DD}(a,b,c) = \text{qexp}&\left[\frac{n^2(n-1)}{2}+\frac{(n-c)^2(n-c-1)}{2}+\sum_{i=0}^{b-1}(n+i)(n-1-i)\right. \\ & \left.+\sum_{j=1}^{a}\left(\frac{(n-1)n}{2}-\frac{(j-1)j}{2}\right)\right] Z_n^{DD}(a,b,c) \end{aligned} \tag{4.4}$$

$$\begin{aligned} \tilde{Z}_n^{DD}(a+1,b+1,c) = \text{qexp}&\left[\frac{n^2(n-1)}{2}+\frac{(n-c)^2(n-c-1)}{2}+\sum_{i=0}^{b}(n+i)(n-1-i)\right. \\ & \left.+\sum_{j=1}^{a+1}\left(\frac{(n-1)n}{2}-\frac{(j-1)j}{2}\right)\right] Z_n^{DD}(a+1,b+1,c) \end{aligned} \tag{4.5}$$

$$\tilde{Z}_n^{DD}(a+1,b,c) = \text{qexp}\left[\frac{n^2(n-1)}{2} + \frac{(n-c)^2(n-c-1)}{2} + \sum_{i=0}^{b-1}(n+i)(n-1-i) + \sum_{j=1}^{a+1}\left(\frac{(n-1)n}{2} - \frac{(j-1)j}{2}\right)\right] Z_n^{DDC}(a,b+1,c) \tag{4.6}$$

$$\tilde{Z}_n^{DD}(a,b+1,c) = \text{qexp}\left[\frac{n^2(n-1)}{2} + \frac{(n-c)^2(n-c-1)}{2} + \sum_{i=0}^{b}(n+i)(n-1-i) + \sum_{j=1}^{a}\left(\frac{(n-1)n}{2} - \frac{(j-1)j}{2}\right)\right] Z_n^{DD}(a+1,b,c) \tag{4.7}$$

$$\tilde{Z}_{n,down}^{DD}(a+1,b+1,c-1) = \text{qexp}\left[\frac{(n-1)^2(n-2)}{2} + \frac{(n-c)^2(n-c)}{2} + \sum_{i=0}^{b}(n-1-i)(n-1+i) + \sum_{j=1}^{a}\left(\frac{(n-2)(n-1)}{2} - \frac{j(j-1)}{2}\right) + \frac{(n-2)(n-1)}{2}\right] Z_{n,down}^{DD}(a+1,b+1,c) \tag{4.8}$$

$$\tilde{Z}_{n,up}^{DD}(a,b,c+1) = \text{qexp}\left[\frac{n^2(n-1)}{2} + \frac{(n-c)^2(n-c-1)}{2} + n^2 + \frac{n(n-1)}{2} + \sum_{i=0}^{b-1}(n-1-i)(n+1+i) + \sum_{j=1}^{a}\left(\frac{n(n+1)}{2} - \frac{j(j+1)}{2}\right)\right] Z_{n,up}^{DD}(a,b,c+1) \tag{4.9}$$

Then we have

$$\begin{aligned} q^A Z_n^{DD}(a,b,c) Z_n^{DD}(a+1,b+1,c) &= q^B Z_n^{DD}(a,b+1,c) Z_n^{DD}(a+1,b,c) \\ &+ q^C Z_n^{DD}(a+1,b+1,c) Z_n^{DD}(a,b,c+1), \end{aligned} \tag{4.10}$$

for some explicit integers A,B,C, depending on a, b, c and in principle n, which are computed in Appendix A.2. The computation there is a careful accounting of the normalization terms in Equations (4.4)–(4.9). Indeed, in that computation, we find that A, B, C do *not* depend on n, that $A = B$, and when we multiply both sides by q^{-A} we get Equation (4.3) with

$$K = C - A = a + b + 1.$$

4.3 Initial Conditions

Let $D(G)$ be the set of all single-dimer configurations on a finite graph G. Let

$$\tilde{Z}^D_{r,s,t}(q) = \sum_{\pi \in D(H(r,s,t))} w(\pi)$$

$$Z^D_{r,s,t}(q) = \frac{1}{w(\pi_0)} \tilde{Z}^D_{r,s,t}(q)$$

be the unnormalized and normalized (respectively) generating functions for single-dimer configurations on the $r \times s \times t$ hexagon graph $H(r,s,t)$ with the same edge weighting as in Figure 2.5, where π_0 is the single-dimer configuration with minimal weight, which is given by

$$w(\pi_0) = q^{rs(s-1)/2}. \tag{4.11}$$

Then let

$$Z^D(q) = \lim_{n \to \infty} Z^D_{n,n,n}(q)$$

be the generating function for single-dimer configurations on the infinite hexagon lattice $H = \lim_{n \to \infty} H(n)$. Also let $D(H) = \lim_{n \to \infty} D(H(n))$ (this makes sure that when we say $D(H)$, we mean single-dimer configurations that correspond to plane partitions via the folklore bijection).

Note that $M_{a,b,c}(q) = Z^D_{a,b,c}(q)$ and $M(q) = Z^D(q)$.

For each of the cases below, we want to show that

$$Z^{DD}_{a,b,c}(q) = Z^D(q)^2 Z^D_{a,b,c}(q).$$

Let $\pi \in DD(\sigma_{a,b,c})$, and let $\pi_1, \pi_2 \in D(H)$.

1. Let $a = b = c = 0$. We want to show that $Z^{DD}_{0,0,0}(q) = Z^D(q)^2 Z^{DC}_{0,0,0}(q)$, where $Z^D_{0,0,0}(q) = 1$.

   (a) The double-dimer configuration $\pi \in DD(\sigma_{0,0,0})$ consists of only doubled edges and loops, no paths, and so π can be split into two single-dimer configurations on H in $2^{\ell(\pi)}$ ways.

   (b) Superimposing π_1 and π_2 such that their centers overlap gives a double-dimer configuration π' with no nodes, thus $\pi' \in DD(\sigma_{0,0,0})$.

2. Let one of $a, b, c = 0$.

   (a) Let $b \neq 0$, $a, c = 0$.

      i. We can split π into two dimer configurations π'_1, π'_2 on $H(n)$, for $n \geq a$, as follows. Doubled edges are split in one possible way, loops can be split in $2^{\ell(\pi)}$ ways, and the edges in the b paths are split as follows.

      In π, we have b paths, each connecting a blue node to a red node. In each path, place the dimer connected to the blue node into π'_1. Then, alternating edges go in π'_2, π'_1 (see Figure 4.11). Then $\pi'_1, \pi'_2 \in D(H(n))$. Note that assigning the dimers in the opposite way just swaps π'_1 and π'_2, it does not give additional configurations.

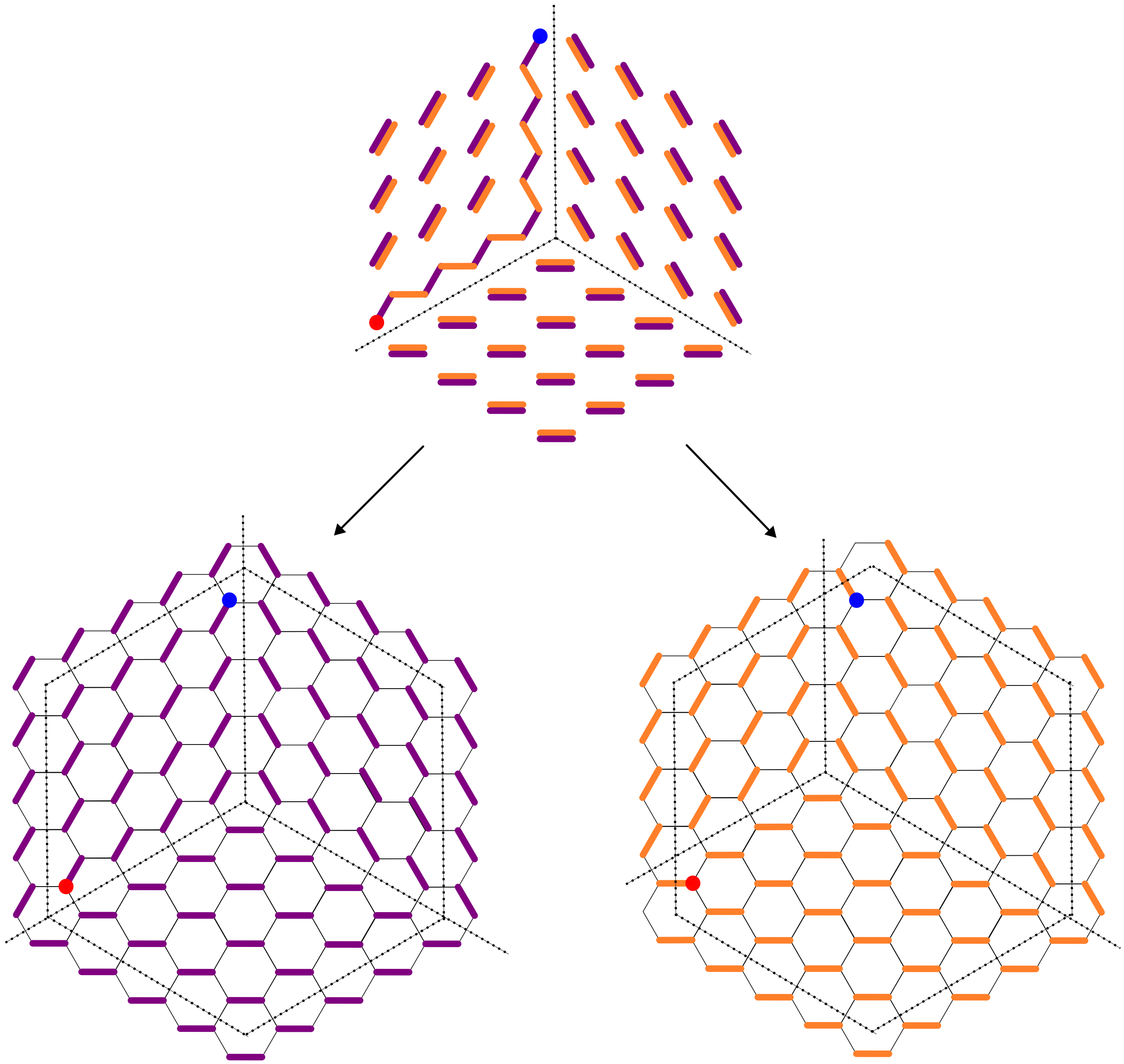

Figure 4.11: Let $b = 1$. Split π (top) into π_1' (bottom left) and π_2' (bottom right).

(ii) Superimpose π_1 and π_2 such that the center of π_2 is b hexagons NW of the center of π_1. Then far enough away from the centers (consider $H(n)$ such that π_1 and π_2 from $H(n)$ to $H(m)$ for $m > n$ are extended in the minimally weighted way), the walls and floors of π_1 and π_2 overlap, thus giving a configuration in $DD(\sigma_{0,b,0})$ (see Figure 4.12).

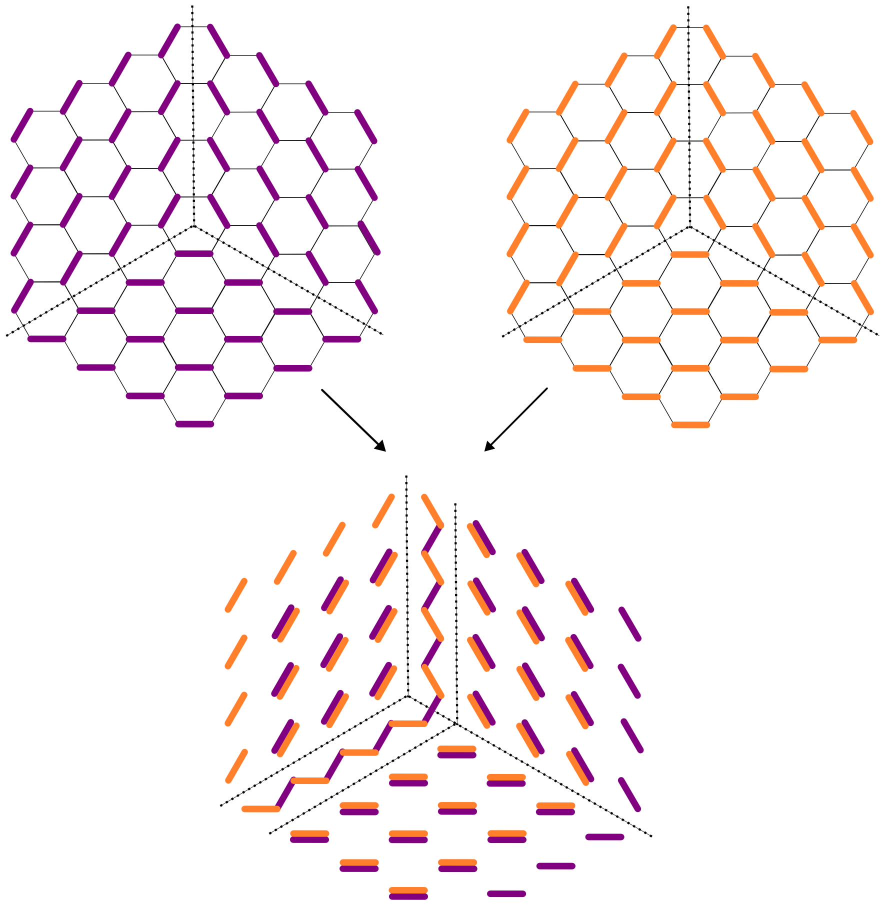

Figure 4.12: Let $b = 1$. Superimpose π_1 (top left) and π_2 (top right).

(b) Let $a \neq 0$, $b, c = 0$.

i. Split π into two dimer configurations π'_1, π'_2 on H in $2^{\ell(\pi)}$ possible ways as follows. Doubled edges are split in one possible way, loops can be split in $2^{\ell(\pi)}$ ways, and the edges in the a paths are split as follows.

Consider truncating H at a large n, such that extending π from $H(n)$ to $H(m)$ for $m > n$ extends in the minimally weighted way. We get a paths, each connecting a blue node to a green node. In each path, place the dimer connected to the blue node into π'_1. Then alternating edges go in π'_2, π'_1. Then $\pi'_1, \pi'_2 \in D(H)$. Note that assigning the dimers in the opposite way just swaps π'_1 and π'_2, it does not give additional configurations.

ii. Superimpose π_1 and π_2 such that the center of π_2 is a hexagons NE of the center of π_1. Then far enough away from the centers, the walls and floors of π_1 and π_2 overlap, thus giving a configuration in $DD(\sigma_{a,0,0})$.

(c) Let $c \neq 0$, $a, b = 0$.

i. Split π into two dimer configurations π'_1, π'_2 on H in $2^{\ell(\pi)}$ possible ways as follows. Doubled edges are split in one possible way, loops can be split in $2^{\ell(\pi)}$ ways, and the edges in the c paths are split as follows.

Consider truncating H at a large n, such that extending π from $H(n)$ to $H(m)$ for $m > n$ extends in the minimally weighted way. We get c paths, each connecting a red node to a green node. In each path, place the dimer connected to the red node into π'_1. Then alternating edges go in π'_2, π'_1. Then $\pi'_1, \pi'_2 \in D(H)$. Note that assigning the dimers in the opposite way just swaps π'_1 and π'_2, it does not give additional configurations.

ii. Superimpose π_1 and π_2 such that the center of π_2 is c hexagons south of the center of π_1. Then far enough away from the centers, the walls and floors of π_1 and π_2 overlap, thus giving a configuration in $DD(\sigma_{0,0,c})$.

3. Two of $a, b, c \neq 0$.

(a) Let $a, b \neq 0$, $c = 0$.

i. Split π into two single dimer configurations π'_1, π'_2 on H in $2^{\ell(\pi)}$ possible ways as follows. Doubled edges are split in one possible way, loops can be split in $2^{\ell(\pi)}$ possible ways, and the edges in the paths are split as follows.

Consider truncating H at a large n, such that extending π from $H(n)$ to $H(m)$ for $m > n$ extends in the minimally weighted way. We get b paths connecting the nodes in sector 2, and a paths connecting the nodes in sector

1. In sector 2, place the dimers connected to the blue nodes into π'_1, then alternate placing dimers in the path into π'_2, π'_1. In sector 1, place the dimers connected to the blue nodes into π'_2, then alternate placing dimers into π'_1, π'_2 (see Figure 4.13). Then $\pi'_1, \pi'_2 \in D(H)$. Note that assigning the dimers in the paths in any other way does not produce two configurations π'_1, π'_2 in $D(H)$.

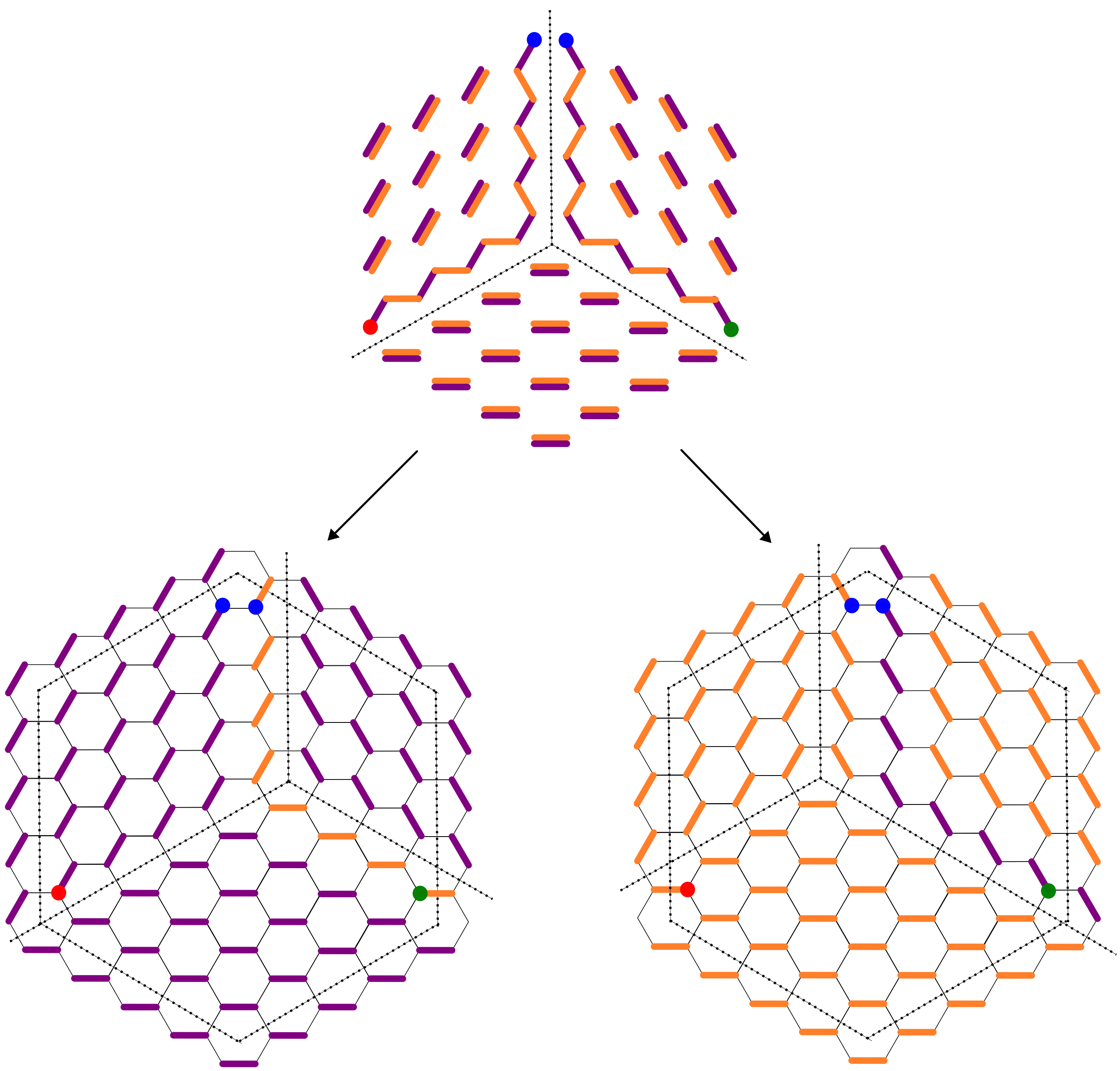

Figure 4.13: Let $a = b = 1$. Split π (top) into π'_1 (bottom left) and π'_2 (bottom right).

ii. Superimpose π_1 and π_2 such that π_1 is b hexagons NE of the center, and π_2 is a hexagons NW of the center (see Figure 4.14). Then far enough away from the centers, the walls and floors of π_1 and π_2 overlap, thus giving a configuration in $DD(\sigma_{a,b,0})$.

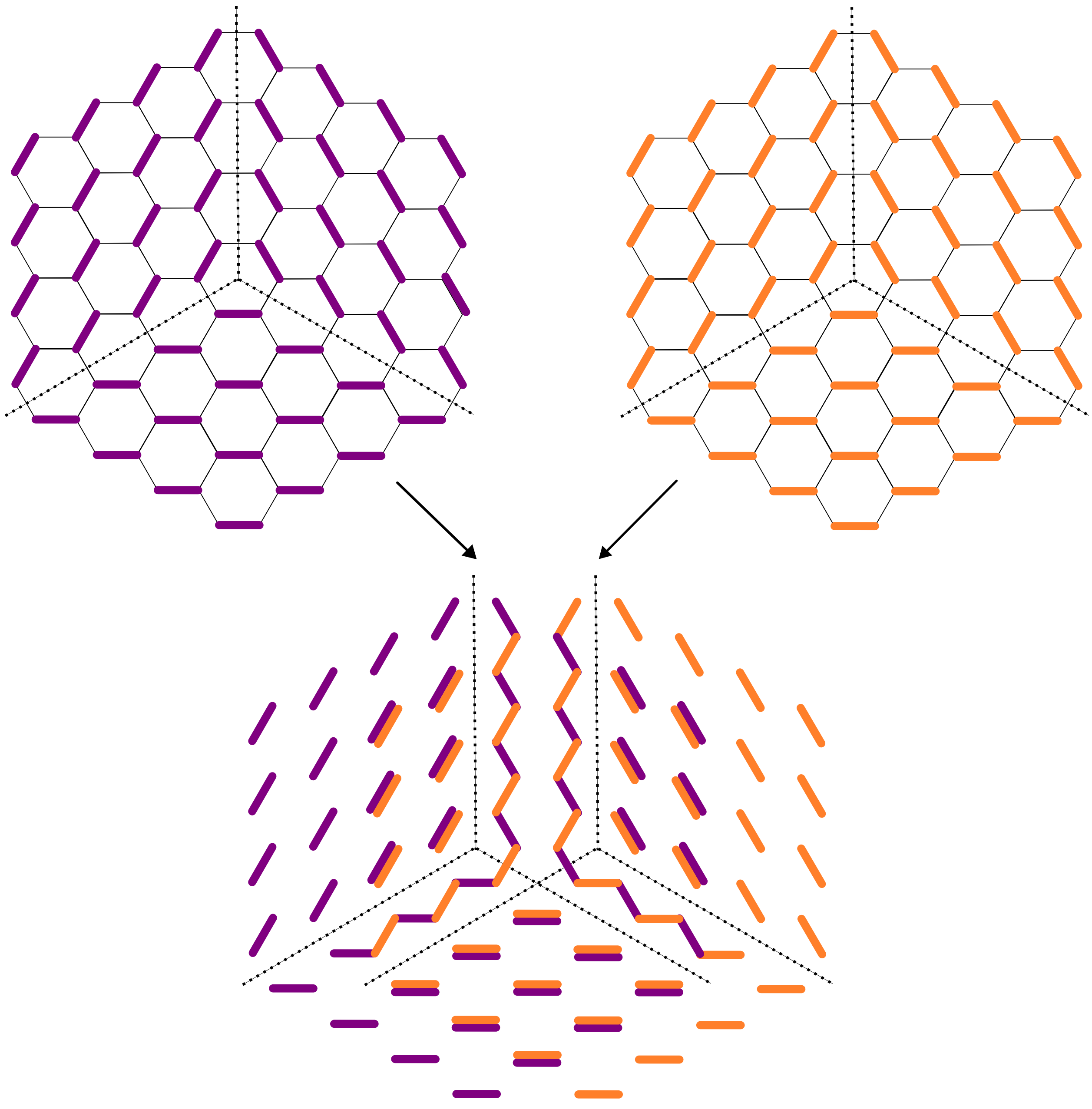

Figure 4.14: Let $a = b = 1$. Superimpose π_1 (top left) and π_2 (top right).

(b) Let $b, c \neq 0$, $a = 0$.

i. Split π into two single dimer configurations in the same way π_1', π_2' on H in $2^{\ell(\pi)}$ possible ways as follows. Doubled edges are split in one possible way, loops can be split in $2^{\ell(\pi)}$ possible ways, and the edges in the paths are split as follows.

Consider truncating H at a large n, such that extending π from $H(n)$ to $H(m)$ for $m > n$ extends in the minimally weighted way. We get b paths connecting the nodes in sector 2, and c paths connecting the nodes in sector 3. In sector 2, place the dimers connected to the blue nodes into π_1', then alternate placing dimers in the path into π_2', π_1'. In sector 3, place the dimers connected to the red nodes into π_2', then alternate placing dimers into π_1', π_2'. Then $\pi_1', \pi_2' \in D(H)$. Note that assigning the dimers in the paths in any other way does not produce two configurations π_1', π_2' in $D(H)$.

ii. Superimpose π_1 and π_2 such that π_1 is b hexagons NW of the center, and π_2 is c hexagons S of the center. Then far enough away from the centers, the walls and floors of π_1 and π_2 overlap, thus giving a configuration in $DD(\sigma_{0,b,c})$.

(c) Let $a, c \neq 0$, $b = 0$.

i. Split π into two single dimer configurations in the same way π_1', π_2' on H in $2^{\ell(\pi)}$ possible ways as follows. Doubled edges are split in one possible way, loops can be split in $2^{\ell(\pi)}$ possible ways, and the edges in the paths are split as follows.

Consider truncating H at a large n, such that extending π from $H(n)$ to $H(m)$ for $m > n$ extends in the minimally weighted way. We get a paths connecting the nodes in sector 1, and c paths connecting the nodes in sector 3. In sector 1, place the dimers connected to the blue nodes into π_1', then alternate placing dimers in the path into π_2', π_1'. In sector 3, place the dimers connected to the red nodes into π_2', then alternate placing dimers into π_1', π_2'. Then $\pi_1', \pi_2' \in DC(H)$. Note that assigning the dimers in the paths in any other way does not produce two configurations π_1', π_2' in $D(H)$.

ii. Superimpose π_1 and π_2 such that π_1 is c hexagons S of the center, and π_2 is a hexagons NE of the center. Then far enough away from the centers, the walls and floors of π_1 and π_2 overlap, thus giving a configuration in $DD(\sigma_{a,0,c})$.

Appendix A

Condensation Details

A.1 Kuo's Graphical Condensation Modification

When applying Kuo's proof of Theorem 6.2 to our hexagon graph with modified strips, we found

$$\begin{aligned}
&q^{(r+1)(s+1)s/2}P(r+1,s+1,t)\cdot q^{rs}\cdot q^{rs(s-1)/2}P(r,s,t) \\
&= q^{r(s+1)s/2}P(r,s+1,t)\cdot q^{s(r+1)}\cdot q^{(r+1)s(s-1)/2}P(r+1,s,t) \\
&+ q^{rs(s-1)/2}P(r,s,t+1)\cdot q^{(r+1)(s+1)}\cdot q^{(r+1)(s+1)s/2}P(r+1,s+1,t-1).
\end{aligned} \tag{A.1}$$

Here we show the simplification when dividing through by $q^{(r+1)(s+1)s/2+rs+rs(s-1)/2}$.

First we re-write Equation (A.1), combining the factors in each term:

$$\begin{aligned}
&q^{(r+1)(s+1)s/2+rs+rs(s-1)/2}P(r+1,s+1,t)P(r,s,t) \\
&= q^{r(s+1)s/2+s(r+1)+(r+1)s(s-1)/2}P(r,s+1,t)P(r+1,s,t) \\
&+ q^{rs(s-1)/2+(r+1)(s+1)+(r+1)(s+1)s/2}P(r,s,t+1)P(r+1,s+1,t-1).
\end{aligned} \tag{A.2}$$

Next we divide through by $q^{(r+1)(s+1)s/2+rs+rs(s-1)/2}$. Note that on the left hand side this clears the entire factor of q. For the first term on the right hand side we have

$$\begin{aligned}
&\mathrm{qexp}\left((s+1)s/2+s(r+1)+(r+1)s(s-1)/2\right)\cdot\mathrm{qexp}\left(-(r+1)(s+1)s/2-rs-rs(s-1)/2\right)\\
&=\mathrm{qexp}([rs(s+1)+2s(r+1)+(r+1)s(s-1)-(r+1)(s+1)s-2rs-rs(s-1)]/2)\\
&=\mathrm{qexp}([rs^2+rs+2rs+2s+s(rs-r+s-1)-s(rs+r+s+1)-2rs-rs^2+rs]/2)\\
&=\mathrm{qexp}([rs^2+rs+2rs+2s+rs^2-rs+2s-s-rs^2-rs-s^2-s-2rs-rs^2+rs]/2)\\
&=\mathrm{qexp}(0/2)=1.
\end{aligned}$$

For the second term on the right hand side we have

$$\begin{aligned}
&\mathrm{qexp}(rs(s-1)/2+(r+1)(s+1)+(r+1)(s+1)s/2)\cdot\mathrm{qexp}(-(r+1)(s+1)s/2-rs-rs(s-1)/2)\\
&=\mathrm{qexp}([rs(s-1)+2(r+1)(s+1)+(r+1)(s+1)s-(r+1)(s+1)s-2rs-rs(s-1)]/2)\\
&=\mathrm{qexp}([rs^2-rs+2(rs+r+s+1)+s(rs+r+s+1)-s(rs+r+s+1)-2rs-rs^2+rs]/2)\\
&=\mathrm{qexp}([rs^2-rs+2rs+2r+2s+2+rs^2+rs+s^2+s-rs^2-rs-s^2-s-2rs-rs^2+rs]/2)\\
&=\mathrm{qexp}([2r+2s+2]/2)\\
&=\mathrm{qexp}(r+s+1).
\end{aligned}$$

A.2 Computing Double-Dimer Normalization Constants

Here we compute A, B, and C from Equation (A.3), given by

$$\begin{aligned}
q^A Z_n^{DD}(a,b,c)Z_n^{DD}(a+1,b+1,c) &= q^B Z_n^{DD}(a,b+1,c)Z_n^{DD}(a+1,b,c) \\
&+ q^C Z_n^{DD}(a+1,b+1,c)Z_n^{DD}(a,b,c+1).
\end{aligned} \tag{A.3}$$

To compute A, we have the following normalization factors (from the weights of the minimally weighted configurations) from Equations (4.4) and (4.5):

$\text{qexp}(A) =$

$$\text{qexp}\left(\frac{n^2(n-1)}{2}+\frac{(n-c)^2(n-c-1)}{2}+\sum_{i=0}^{b-1}(n+i)(n-1-i)+\sum_{j=1}^{a}\left(\frac{(n-1)n}{2}-\frac{(j-1)j}{2}\right)\right)$$

$$\cdot\,\text{qexp}\left(\frac{n^2(n-1)}{2}+\frac{(n-c)^2(n-c-1)}{2}+\sum_{i=0}^{b}(n+i)(n-1-i)+\sum_{j=1}^{a+1}\left(\frac{(n-1)n}{2}-\frac{(j-1)j}{2}\right)\right)$$

$$= \text{qexp}\left(\frac{n^2(n-1)}{2}+\frac{(n-c)^2(n-c-1)}{2}+\sum_{i=0}^{b-1}(n+i)(n-1-i)+\sum_{j=1}^{a}\left(\frac{(n-1)n}{2}-\frac{(j-1)j}{2}\right)\right.$$

$$\left.+\frac{n^2(n-1)}{2}+\frac{(n-c)^2(n-c-1)}{2}+\sum_{i=0}^{b}(n+i)(n-1-i)+\sum_{j=1}^{a+1}\left(\frac{(n-1)n}{2}-\frac{(j-1)j}{2}\right)\right).$$

We combine the weights from the floor horizontal edges (we keep these in the first row in the following), those from the b paths on the left (we keep these in the second row), and those from the a paths on the right (we keep these in the third row), and we get:

$$\begin{aligned} A = {} & n^2(n-1)+(n-c)^2(n-c-1) \\ & + 2\sum_{i=0}^{b-1}\left((n+i)(n-1-i)\right)+(n+b)(n-1-b) \\ & + \sum_{j=1}^{a}\left(n(n-1)-j(j-1)\right)+\left(\frac{n(n-1)}{2}-\frac{a(a+1)}{2}\right). \end{aligned}$$

Next, to compute B, we have the following normalization factors (from the weights of the minimally weighted configurations) from Equations (4.7) and (4.6)

$\text{qexp}(B) =$

$$\text{qexp}\left(\frac{n^2(n-1)}{2} + \frac{(n-c)^2(n-c-1)}{2} + \sum_{i=0}^{b}(n+i)(n-1-i) + \sum_{j=1}^{a}\left(\frac{(n-1)n}{2} - \frac{(j-1)j}{2}\right)\right)$$

$$\cdot\,\text{qexp}\left(\frac{n^2(n-1)}{2} + \frac{(n-c)^2(n-c-1)}{2} + \sum_{i=0}^{b-1}(n+i)(n-1-i) + \sum_{j=1}^{a+1}\left(\frac{(n-1)n}{2} - \frac{(j-1)j}{2}\right)\right)$$

$$= \text{qexp}\left(\frac{n^2(n-1)}{2} + \frac{(n-c)^2(n-c-1)}{2} + \sum_{i=0}^{b}(n+i)(n-1-i) + \sum_{j=1}^{a}\left(\frac{(n-1)n}{2} - \frac{(j-1)j}{2}\right)\right.$$

$$\left. + \frac{n^2(n-1)}{2} + \frac{(n-c)^2(n-c-1)}{2} + \sum_{i=0}^{b-1}(n+i)(n-1-i) + \sum_{j=1}^{a+1}\left(\frac{(n-1)n}{2} - \frac{(j-1)j}{2}\right)\right).$$

Again, we combine the weights from the floor horizontal edges (we keep these in the first row in the following), those from the b paths on the left (we keep these in the second row), and those from the a paths on the right (we keep these in the third row), and we get:

$$\begin{aligned} B = {} & n^2(n-1) + (n-c)^2(n-c-1) \\ & + 2\sum_{i=0}^{b-1}\left((n+i)(n-1-i)\right) + (n+b)(n-1-b) \\ & + \sum_{j=1}^{a}\left(n(n-1) - j(j-1)\right) + \left(\frac{n(n-1)}{2} - \frac{a(a+1)}{2}\right). \end{aligned}$$

Note that $A = B$. We further simplify A as

$$
\begin{aligned}
A &= n^3 - n^2 + (n^2 - 2nc + c^2)(n - c - 1) \\
&\quad + 2\sum_{i=0}^{b-1}(n^2 - n - i - i^2) + n^2 - n - b - b^2 \\
&\quad + \sum_{j=1}^{a}(n^2 - n - j^2 + j) + \frac{n^2 - n}{2} - \frac{a^2 + a}{2}
\end{aligned}
$$

$$
\begin{aligned}
&= 2n^3 - 2n^2 - 3n^2c + 3nc^2 + 2nc - c^3 - c^2 \\
&\quad + 2b(n^2 - n) - 2\left(\sum_{i=1}^{b-1} i^2 + \sum_{i=1}^{b-1} i\right) + n^2 - n - b - b^2 \\
&\quad + a(n^2 - n) - \sum_{j=1}^{a} j^2 + \sum_{j=1}^{a} j + \frac{1}{2}n^2 - \frac{1}{2}n - \frac{1}{2}a^2 - \frac{1}{2}a.
\end{aligned}
$$

Using the following,

$$
\begin{aligned}
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6}, \\
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2},
\end{aligned}
$$

we have:

$$
\begin{aligned}
&= 2n^3 - 2n^2 - 3n^2c + 3nc^2 + 2nc - c^3 - c^2 \\
&\quad + 2bn^2 - 2bn - 2\left(\frac{(b-1)(b)(2(b-1)+1)}{6} + \frac{(b-1)b}{2}\right) + n^2 - n - b - b^2 \\
&\quad + an^2 - an - \frac{a(a+1)(2a+1)}{6} + \frac{a(a+1)}{2} + \frac{1}{2}n^2 - \frac{1}{2}n - \frac{1}{2}a^2 - \frac{1}{2}a
\end{aligned}
$$

$$
\begin{aligned}
&= 2n^3 - 2n^2 - 3n^2c + 3nc^2 + 2nc - c^3 - c^2 \\
&\quad + 2bn^2 - 2bn - 2\left(\frac{(b^2-b)(2b-1)}{6} + \frac{b^2-b}{2}\right) + n^2 - n - b - b^2 \\
&\quad + an^2 - an - \frac{(a^2+a)(2a+1)}{6} + \frac{a^2+a}{2} + \frac{1}{2}n^2 - \frac{1}{2}n - \frac{1}{2}a^2 - \frac{1}{2}a
\end{aligned}
$$

$$
\begin{aligned}
&= 2n^3 - 2n^2 - 3n^2c + 3nc^2 + 2nc - c^3 - c^2 \\
&\quad + 2bn^2 - 2bn - 2\left(\frac{2b^3 - 3b^2 + b}{6} + \frac{b^2-b}{2}\right) + n^2 - n - b - b^2 \\
&\quad + an^2 - an - \frac{1}{6}(2a^3 + 3a^2 + a) + \frac{1}{2}a^2 + \frac{1}{2}a + \frac{1}{2}n^2 - \frac{1}{2}n - \frac{1}{2}a^2 - \frac{1}{2}a
\end{aligned}
$$

$$
\begin{aligned}
&= 2n^3 - 2n^2 - 3n^2c + 3nc^2 + 2nc - c^3 - c^2 \\
&\quad + 2bn^2 - 2bn - \frac{2}{3}b^3 - b^2 - \frac{1}{3}b + n^2 - n \\
&\quad + an^2 - an - \frac{1}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a + \frac{1}{2}n^2 - \frac{1}{2}n \\
&\quad = A.
\end{aligned}
$$

Finally, to compute C, we have the following normalization factors (from the weights of the minimally weighted configurations) from Equations (4.8) and (4.9):

$$\begin{aligned}
&\mathrm{qexp}(C) = \qquad \text{(A.4)}\\
&\mathrm{qexp}\left(\frac{(n-1)^2(n-2)}{2} + \frac{(n-c)^2(n-c-1)}{2} + \sum_{i=0}^{b}(n-1-i)(n-1+i)\right.\\
&\left. + \sum_{j=1}^{a}\left(\frac{(n-2)(n-1)}{2} - \frac{j(j-1)}{2}\right) + \frac{(n-2)(n-1)}{2}\right)\\
&\cdot \mathrm{qexp}\left(\frac{n^2(n-1)}{2} + \frac{(n-c)^2(n-c-1)}{2} + n^2 + \frac{n(n-1)}{2} + \sum_{i=0}^{b-1}(n-1-i)(n+1+i)\right.\\
&\left. + \sum_{j=1}^{a}\left(\frac{n(n+1)}{2} - \frac{j(j+1)}{2}\right)\right).
\end{aligned}$$

We consider just the exponent of q from Equation (A.4) to compute C:

$$\begin{aligned}
C = &\frac{(n-1)^2(n-2)}{2} + \frac{n^2(n-1)}{2} + (n-c)^2(n-c-1) + n^2 + \frac{n(n-1)}{2}\\
&+ \sum_{i=0}^{b-1}((n-1-i)(n-1+i) + (n-1-i)(n+1+i)) + (n-1-b)(n-1+b)\\
&+ \frac{1}{2}\sum_{j=1}^{a}((n-2)(n-1) - j(j-1) + n(n+1) - j(j+1)) + \frac{(n-2)(n-1)}{2}
\end{aligned}$$

$$\begin{aligned}
=&\frac{n^3-4n^2+5n-2}{2} + \frac{n^3-n^2}{2} + \frac{2(n^3-3n^2c+3nc^2+2nc-n^2-c^3-c^2)}{2} + \frac{2n^2}{2} + \frac{n^2-n}{2}\\
&+ \sum_{i=0}^{b-1}(2n^2-2n-2i-2i^2) + (n-1-b)(n-1+b)\\
&+ \frac{1}{2}\sum_{j=1}^{a}(2n^2-2n+2-2j^2) + \frac{1}{2}(n^2-3n+2)
\end{aligned}$$

$$\begin{aligned}
=&2n^3-2n^2+2n-1-3n^2c+3nc^2+2nc-c^3-c^2\\
&+ b(2n^2-2n) - 2\left(\sum_{i=1}^{b-1} i^2 + \sum_{i=1}^{b-1} i\right) + n^2 - 2n - b^2 + 1\\
&= \frac{1}{2}\cdot a(2n^2-2n+2) - \sum_{j=1}^{a} j^2 + \frac{1}{2}n^2 - \frac{3}{2}n + 1.
\end{aligned}$$

Using the following,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$

we have:

$$\begin{aligned}
=&2n^3 - 2n^2 + 2n - 1 - 3n^2c + 3nc^2 + 2nc - c^3 - c^2 \\
&+ 2bn^2 - 2bn - 2\left(\frac{(b-1)(b)(2(b-1)+1)}{6} + \frac{(b-1)b}{2}\right) + n^2 - 2n - b^2 + 1 \\
&+ an^2 - na + a - \left(\frac{a(a+1)(2a+1)}{6}\right) + \frac{1}{2}n^2 - \frac{3}{2}n + 1
\end{aligned}$$

$$\begin{aligned}
=&2n^3 - 2n^2 + 2n - 1 - 3n^2c + 3nc^2 + 2nc - c^3 - c^2 \\
&2bn^2 - 2bn - \frac{1}{3}((b^2 - b)(2b-1)) - b^2 + b + n^2 - 2n - b^2 + 1 \\
&an^2 - na + a - \frac{1}{6}(2a^3 + 3a^2 + a) + \frac{1}{2}n^2 - \frac{3}{2}n + 1
\end{aligned}$$

$$\begin{aligned}
=&2n^3 - 2n^2 + 2n - 1 - 3n^2c + 3nc^2 + 2nc - c^3 - c^2 \\
&+ 2bn^2 - 2bn - \frac{2}{3}b^3 - b^2 + n^2 + \frac{2}{3}b - 2n + 1 \\
&+ an^2 - na + a - \frac{1}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a + \frac{1}{2}n^2 - \frac{3}{2}n + 1 \\
&= C
\end{aligned}$$

Since $A = B$, when we multiply Equation (A.3) by q^{-A} we get Equation (4.3) with

$$
\begin{aligned}
K =& C - A \\
=& 2n^3 - 2n^2 + 2n - 1 - 3n^2c + 3nc^2 + 2nc - c^3 - c^2 \\
&+ 2bn^2 - 2bn - \frac{2}{3}b^3 - b^2 + n^2 + \frac{2}{3}b - 2n + 1 \\
&+ an^2 - na + a - \frac{1}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a + \frac{1}{2}n^2 - \frac{3}{2}n + 1 \\
&- (2n^3 - 2n^2 - 3n^2c + 3nc^2 + 2nc - c^3 - c^2) \\
&- (2bn^2 - 2bn - \frac{2}{3}b^3 - b^2 - \frac{1}{3}b + n^2 - n) \\
&- (an^2 - an - \frac{1}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a + \frac{1}{2}n^2 - \frac{1}{2}n) \\
\\
=& -1 + 2n \\
&+ b - n + 1 \\
&+ a - n + 1 \\
&= a + b + 1.
\end{aligned}
$$

Thus $K = a + b + 1$.

Appendix B

Notation and Definitions Index

B.1 Double-Box Configurations (DB)

$DB_{a,b,c}$	the set of all double-box configurations given $a, b, c \in \mathbb{N}$ (Definition 3)
type I, II, III boxes	different types of boxes in a double-box configuration depending on how many plane partitions they are contained in (Definition 2)
moveable type II box	a type II box in a double-box configuration that can be from different pairs of plane partitions (Definition 5)
$Z^{DB}_{a,b,c}(q)$	the generating function for double-box configurations (Definition (2.2))

B.2 Double-Dimer Configurations (DD)

$H(n)$	the $n \times n \times n$ hexagon graph (first paragraph of Section 2.2)
$w(e)$	edge weighting on the hexagon graph (Figures 2.4 and 2.5)
$\mathbf{N}$	node set of red, green, and blue nodes defined on $H(n)$, depends on $a, b, c \in \mathbb{N}$ (Figure 2.6)
$\sigma_{a,b,c}$	the tripartite node pairing of the nodes $\mathbf{N}$ (paragraph below Figure 2.6)

$DD_n(\sigma_{a,b,c})$ — the set of all double-dimer configurations on $H(n)$ with node set $\mathbf{N}$ and tripartite node pairing $\sigma_{a,b,c}$

$Z^{DD}_{n;a,b,c}(q)$ — the generating function for elements of $DD_n(\sigma_{a,b,c})$ (Equation (2.3))

$DD(\sigma_{a,b,c})$ — the set of all double-dimer configurations π on the infinite hexagon graph, such that there exists some $M \in \mathbb{N}$, such that $H(N)$ for $N \geq M$, π restricted go $H(N)$ has node set $\mathbf{N}$ and tripartite node pairing $\sigma_{a,b,c}$ (Definition 9)

$Z^{DD}_{a,b,c}(q)$ — the generating function for elements of $DD(\sigma_{a,b,c})$ (Equation (2.4))

π_0 — minimal configuration in $DD_n(\sigma_{a,b,c})$ (Figure 2.7)

B.3 Double Plane Partitions (DP)

$\mathbf{M}$ — $([0,a) \times [0,b) \times \mathbb{Z}_{\geq 0}) \cup (\mathbb{Z}_{\geq 0} \times [0,b) \times [0,c)) \cup ([0,a) \times \mathbb{Z}_{\geq 0} \times [0,c))$ (Definition 13)

(π_1, π_2) — π_1 is an order ideal under the product order on $\mathbb{Z}^3_{\geq 0}$, and π_2 is an order ideal under the product order on $\mathbb{Z}^3_{\geq 0}$ which contains $\mathbf{M}$ (Definition 14)

$\pi_1 \cup \pi_2$ — π_1 and π_2 superimposed such that $(0,0,0)$ in π_1 and (a,b,c) in π_2 are identified (see Figure 3.2)

Int — the intersection space of $\pi_1 \cup \pi_2$, $\{(w_1,w_2,w_3) \in \pi_1 \cup \pi_2 \mid w_1 \geq a, w_2 \geq b, w_3 \geq c\}$ (Definition 14)

I_1^- — $\{(i,j,k) \in \mathbb{Z}^3 \mid 0 \leq i < a,\ j \geq b,\ k \geq c\}$ (Definition 15)

I_2^- — $\{(i,j,k) \in \mathbb{Z}^3 \mid i \geq a,\ 0 \leq j < b,\ k \geq c\}$ (Definition 15)

I_3^- — $\{(i,j,k) \in \mathbb{Z}^3 \mid i \geq a,\ j \geq b,\ 0 \leq k < c\}$ (Definition 15)

Let $w = (w_1, w_2, w_3) \in Int$:

$T_-(w)$ — $\{(t, w_2, w_3) \in \pi_1 \cup \pi_2 \mid 0 \leq t < a\} \subset I_1^-$ (Definition 16)

$S_-(w)$	$\{(w_1, s, w_3) \in \pi_1 \cup \pi_2 \mid 0 \leq s < b\} \subset I_2^-$ (Definition 16)
$R_-(w)$	$\{(w_1, w_2, r) \in \pi_1 \cup \pi_2 \mid 0 \leq r < c\} \subset I_3^-$ (Definition 16)

s., w.s., c.n.s.

	in I_1^-	in I_2^-	in I_3^-
supported (s.)	$(a-1, w_2, w_3) \in \pi_2$	$(w_1, b-1, w_3) \in \pi_2$	$(w_1, w_2, c-1) \in \pi_2$
weakly supported (w.s.)	w not s. in I_1^- and $\lvert T_-(w)\rvert \geq 1$	w not s. in I_2^- and $\lvert S_-(w)\rvert \geq 1$	w not s. in I_3^- and $\lvert R_-(w)\rvert \geq 1$
completely not supported (c.n.s)	$\lvert T_-(w)\rvert = 0$	$\lvert S_-(w)\rvert = 0$	$\lvert R_-(w)\rvert = 0$

$DP_{a,b,c}$	for $a, b, c \in \mathbb{N}$, the set of all pairs (π_1, π_2) such that every box $w \in Int$ is at least weakly supported in at least two of the negative spaces I_1^-, I_2^-, I_3^- (Definition 19)
$\mathcal{DP}(\eta)_{a,b,c}$	the set of all elements $(\pi_1 \cup \pi_2) \in DP_{a,b,c}$ that are associated to $\eta \in DB_{a,b,c}$ (Definition 21)
$\mathcal{DP}_{a,b,c}$	the set of all elements of $(\pi_1 \cup \pi_2) \in DP$ that are associated to some $\eta \in DB_{a,b,c}$ (Definition 24)
III_{DP}	$\{w \in \pi_1 \cup \pi_2 \mid w \in \pi_1 \cap \pi_2\}$ (Definition 25)
II_{DP}	$\{w \in \pi_1 \cup \pi_2 \mid w \in Int \setminus \mathrm{III}_{DP}\}$ (Definition 25)
I_{DP}^-	$\{w \in \pi_1 \cup \pi_2 \mid w \notin Int\}$ (Definition 25)
$BN(w)$	the back neighbors of w, $\{(w_1-1, w_2, w_3), (w_1, w_2-1, w_3), (w_1, w_2, w_3-1)\}$ (Definition 29)
$FN(w)$	the front neighbors of w, $\{(w_1+1, w_2, w_3), (w_1, w_2+1, w_3), (w_1, w_2, w_3+1)\}$ (Definition 29)

Let $w = (w_1, w_2, w_3) \in Int$:

w^1	$(a-1, w_2, w_3)$ (Definition 30)
w^2	$(w_1, b-1, w_3)$ (Definition 30)

w^3	$(w_1, w_2, c-1)$ (Definition 30)
(π_1^η, π_2^η)	For $\eta \in DB_{a,b,c}$, π_1^η contains the type II and III boxes of η, π_2^η is an order ideal on $\mathbb{Z}^3_{\geq 0}$ containing $\mathbf{M}$ and the the type I^- and III boxes of η (Definition 37)

B.4 Labeled AB Configurations (AB)

Let $(\pi_1, \pi_2) \in DP_{a,b,c}$:

(π_1^p, π_2^p)	construct $(\pi_1^p, \pi_2^p) = P(\pi_1, \pi_2)$ by moving every box from π_2 that we can into π_1 such that we still have an element in $DP_{a,b,c}$ (Definition 42)
π_1'	$I_1^- \cap \pi_2$ (Definition 46 part 2)
π_2'	$I_2^- \cap \pi_2$ (Definition 46 part 2)
π_3'	$I_3^- \cap \pi_2$ (Definition 46 part 2)
Cyl_1^e	$\pi_1' \cup \{(i,j,k) \in \mathbb{Z}^3 \mid (i,j,k) \in \mathrm{Cyl}_1^- \text{ and } i < -a\}$ (Definition 46 part 5)
Cyl_2^e	$\pi_2' \cup \{(i,j,k) \in \mathbb{Z}^3 \mid (i,j,k) \in \mathrm{Cyl}_2^- \text{ and } j < -b\}$ (Definition 46 part 5)
Cyl_3^e	$\pi_3' \cup \{(i,j,k) \in \mathbb{Z}^3 \mid (i,j,k) \in \mathrm{Cyl}_3^- \text{ and } k < -c\}$ (Definition 46 part 5)
$(A,B) = \gamma(\pi_1, \pi_2)$	$A = \left(\bigcup_{i \in \{1,2,3\}} Cyl_i^- \setminus Cyl_i^e\right) \cup (\mathrm{III}_{AB} \setminus \pi_2^p)$ $B = (\mathrm{II}_{AB} \cup \mathrm{III}_{AB}) \setminus \pi_1^p$ (Definition 46 part 6)

Let $(\pi_1, \pi_2) \in DP_{a,b,c}$ and $w = (w_1, w_2, w_3) \in Int$:

$T_+(w)$	$\{(t, w_2, w_3) \in \pi_1 \cup \pi_2 \mid a \leq t < w_1\}$ (Definition 56)
$S_+(w)$	$\{(w_1, s, w_3) \in \pi_1 \cup \pi_2 \mid b \leq s < w_2\}$ (Definition 56)
$R_+(w)$	$\{(w_1, w_2, r) \in \pi_1 \cup \pi_2 \mid c \leq r < w_3\}$ (Definition 56)
π_2^e	$\mathrm{Cyl}_1^e \cup \mathrm{Cyl}_2^e \cup \mathrm{Cyl}_3^e \cup \mathrm{III}_{DP} = (\mathrm{I}_{AB}^- \cup \mathrm{III}_{AB}) \setminus A$ (Definition 53)

AB	the set of all AB configurations (Definition 44 [JWY20])
$\mathcal{AB}$	the set of all labeled AB configurations (Definition 45 [JWY20])
$\mathcal{L}(A, B)$	$(\mathrm{I}^-_{AB} \cap A) \cup (\mathrm{II}_{AB} \setminus B) \cup (\mathrm{III}_{AB} \cap (A \Delta B))$ (Definition 45 [JWY20])
M	$\max\{(\mu_1)_1, \ell(\mu_1), (\mu_2)_1, \ell(\mu_2), (\mu_3)_1, \ell(\mu_3)\}$ (Equation (3.1) [JWY20])
M'	$\max\{M + a, M + b, M + c\}$ for $a, b, c \in \mathbb{N}$ (Equation (3.2))

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