Cobordism Obstructions to Complex Sections

by

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DISSERTATION ABSTRACT

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There is a classical problem to determine whether a manifold admits r linearly independent tangent vector fields. The first results, due to Poincare and Hopf, show that an oriented manifold admits an everywhere non-zero vector field if and only if its Euler characteristic is zero. Thomas, Mayer, Atiyah and Dupont did further work showing the existence of obstructions whose vanishing was a necessary condition for a manifold to admit r vector fields. To solve this problem up to cobordism, Bökstedt and Svane defined and studied a notion of vector field cobordism for oriented manifolds and a corresponding cobordism category.

The main goal of this thesis is to study the complex version of this problem, namely finding linearly independent complex tangent sections of almost complex manifolds. We define the complex section cobordism groups and the related cobordism categories. We identify an obstruction to finding a manifold in the same complex cobordism class as a given manifold with r complex sections. This obstruction is an element of a relevant bordism group. The vanishing of this obstruction is both necessary and sufficient to show a cobordism class contains a manifold which can be equipped with r linearly independent complex sections. Up to torsion, we completely describe this obstruction in terms of the Chern characteristic numbers. Further, calculations with the Adams-Novikov spectral sequence for particular Thom spectra allow us to show the torsion in the obstruction vanishes for low values of r. For prime $p \geq 3$, we show that torsion obstructions of order p for finding r complex sections vanish for $r < p^2 - p$ and that all torsion obstructions for finding 2 or 3 linearly independent complex sections vanish. Finally, we show that this obstruction vanishes for certain multiplicative generators in the complex cobordism ring.

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For my Parents

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CHAPTER 1

INTRODUCTION

1.1 Motivation

Smooth manifolds are the principal object of study in geometric topology. There is a classical problem to determine whether a manifold admits r everywhere linearly independent tangent vector fields, that is, when the tangent bundle splits off a trivial bundle of rank r. Poincare [24] proved that the sphere S^2 does not admit a non-zero tangent vector field. For higher dimensional spheres, Adams [2] found the maximal value of r and in the process developed much of K-theory. This determination solved important mathematical questions regarding the classification of Clifford algebras and the Hopf invariant one problem. However, for an arbitrary manifold there is no straightforward way to determine how many linearly independent vector fields can be constructed. The ideal result is to determine an obstruction whose vanishing is both a necessary and sufficient condition for a manifold to have r linearly independent tangent vector fields. In the case of one everywhere non-zero vector field, $r = 1$, this problem was solved by Hopf.

Theorem 1.1 ([14, Poincare-Hopf Theorem]**.** *A compact manifold M admits an everywhere non-zero tangent field if and only if the Euler characteristic of M is zero.*

Further work by Thomas [34], Mayer [19], and Atiyah and Dupont [3] found necessary conditions for finding r linearly independent tangent vector fields. However, sufficient conditions have remained elusive.

A classical technique in geometric topology is to perform surgery on a manifold in order to give it a particular property. This is equivalent to studying the cobordism class of a manifold. Bökstedt, Dupont and Svane [5] used this approach for the problem of finding r linearly independent tangent vector fields. That is, they discovered a complete obstruction for determining when a cobordism class contains a manifold with r linearly independent vector fields. They were able to fully determine the obstruction when $r = 2, 3$ and determine the obstruction in some cases when $r = 4, 5, 6$. They proved these results using the Thom spectra which classify the homotopy type of the cobordism category, constructed by Galatius, Tillmann, Madsen and Weiss [12]. We generalize their work to the complex case by attempting to determine a necessary and sufficient condition for finding a manifold in a complex cobordism class with r linearly independent complex tangent sections. We completely determined the rational obstruction in terms of the Chern characteristic classes. We then used the Adams-Novikov spectral sequence to determine the torsion obstruction for small values of r.

1.2 Background

Vector Bundles

We begin by overviewing vector bundles and structures on vector bundles.

Definition 1.2. A vector bundle of dimension n over a space B consists of a total space E and a projection $\pi : E \to B$ such that the fiber, $\pi^{-1}(b)$, is isomorphic to the vector space $\mathbb{R}^n$ for any $b \in B$. The projection must satisfy the property that for any $b \in B$, there is an open neighborhood U of b and a homeomorphism $h : U \times \mathbb{R}^n \to \pi^{-1}(U)$ which reduces to an isomorphism at each point in U.

Definition 1.3. A bundle map from $\pi : E \to B$ to $\pi' : E' \to B'$ consists of maps $F : E \to E'$ and $f : B \to B'$ making the following square commute:

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & B' \end{array}$$

and such that F reduces to an isomorphism on each fiber.

There is a natural way to define direct sums, tensor products and other constructions on bundles, see [22, Chapter 3].

Definition 1.4. The Grassmannian $G(d, n)$ is the space of d dimensional linear subspaces of $\mathbb{R}^{d+n}$. There is a tautological bundle over the Grassmannian, where the fiber over any d-plane (which is a point in the Grassmannian) is the d dimensional set of vectors in that plane. We denote the total space of this bundle by $U_{d,n}$.

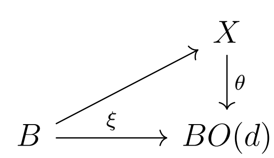
We note that there are many competing choices of notation and indexing for the Grassmannian; the ones chosen in the previous theorem will be used throughout this work. There is a universal or tautological bundle over the Grassmannian such that all bundles are pullback of this universal bundle. [22, Chapter 5] There is a natural map

$G(d, n) \to G(d, n+1)$ which takes a plane to the same plane embedded in the next higher dimensional space. The limit $\lim_{n\to\infty} G(d, n)$ is, up to homotopy, the classifying space $BO(d)$. We call the limit of the tautological bundle U_d. The bundle U_d is the universal d-dimensional bundle by the following Theorem [22, Theorems 5.6 , 5.7]:

Theorem 1.5. *There is a bijection between the set of homotopy classes $[B, BO(d)]$ and the set of isomorphism classes of d-dimensional vector bundles over a paracompact space B. This bijection is defined by pulling back the bundle U_d.*

In particular, if the base space is a manifold M^d and the bundle is the tangent bundle TM, then there is a canonical map, up to homotopy, $M \to BO(d)$ which is called the Gauss map. This map can be described as follows. Embed the manifold M^d into a Euclidean space $\mathbb{R}^{d+n}$; each tangent space is a plane, that is, an element in $G(d, n)$. The Gauss map takes each point to its tangent space.

It is frequently useful to examine when a vector bundle has additional structure. A structure is a fibration $\theta : X \to BO(d)$; a θ structure on a bundle over a paracompact space B is a homotopy lift $B \to X$ such that the following diagram commutes:



where ξ is the map defined by Theorem 1.5. The most common structures on a vector bundle are G-structures where G is a subgroup of $O(d)$. They come from the induced fibration $BG \to BO(d)$. There is a tautological bundle over BG which is defined as the pullback of U_d. There is a generalization of Theorem 1.5 for bundles with a G structure.

Theorem 1.6 ([20, Theorem 3.1]**.** *There is a bijection between the set of homotopy classes $[B, BG]$ and the set of isomorphism classes of d-dimensional vector bundles over a paracompact space B with G structure. This bijection is defined by pulling back the tautological bundle over BG.*

There is a canonical map $BO(d-1) \to BO(d)$ induced by the group homomorphism $O(d-1) \hookrightarrow O(d)$. This map induces an operation on vector bundles which is equivalent to adding a single trivial bundle. We can ask whether a bundle, that is, a map $B \to BO(d)$, can be lifted over the map $BO(d-1) \to BO(d)$. In other words, we can ask when a bundle splits into a trivial bundle and a complement. This is equivalent to the geometric question of finding a section $s : B \to E$ of the bundle $\pi : E \to B$ such that $\pi \circ s$ is the identity and such that $s(b)$ is not the zero element in $\pi^{-1}(b)$ for any $b \in B$. Similarly, lifting over the map $BO(d-r) \to BO(d)$ is equivalent to finding r sections $s_i : B \to E$ such that the set of vectors $\{s_i(b) \mid 1 \leq i \leq r\}$ is linearly independent. A section of the tangent bundle of a manifold is often called a vector field.

A final way to rephrase the question uses the Stiefel or frame manifold $V_{d,r}$. The manifold $V_{d,r}$ is the space of orthogonal frames of r vectors in $\mathbb{R}^d$. It can be defined equivalently as $V_{d,r} = O(d)/O(d-r)$. For any d dimensional bundle $\pi : E \to B$, there is a fibration $V_r(E) \to B$ where the fiber over b is $V_{d,r}$ the space of orthogonal frames of r vectors in $\pi^{-1}(b)$. Using the Gram-Schmidt algorithm, the structure of r linearly independent sections of a bundle is equivalent to a section of the fibration $V_r(E) \to B$ or a lift of the classifying map over $V_r(U_d) \to BO(d)$.

The main focus of this work will be the complex versions of these constructions. In particular, the subgroup $U(d) \subseteq O(2d)$ defines a complex structure on a bundle. The classifying space $BU(d)$ is constructed similarly to $BO(d)$ except with complex instead of real vector spaces. Denote the complex Grassmannian of d dimensional complex subspaces of $\mathbb{C}^{d+n}$ by $G_{\mathbb{C}}(d,n)$. Let $U_{\mathbb{C},d,n} \to G_{\mathbb{C}}(d,n)$ be the tautological d dimensional complex vector bundle. Then $\lim_{n\to\infty} G_{\mathbb{C}}(d,n) \simeq BU(d)$. Analogous to the real case, there is a complex frame bundle of $V_{\mathbb{C},r}(E) \to B$ whose fiber $V_{\mathbb{C},d,r}$ is the space of r-frames in $\mathbb{C}^d$. An equivalent geometric defintion of a complex bundle is:

Definition 1.7 ([22, Section 13]**.** A d-dimensional complex vector bundle is a $2d$-dimensional real vector bundle $\pi : E \to B$ along with a fiberwise operator $J : E \to E$ such that $J^2 = -I$ where I is the identity. This definition is equivalent to a lift of the map $B \to BO(2d)$ to $BU(d)$.

If a manifold M^{2d} has a tangent bundle with $U(d)$ structure, we call the manifold almost-complex or weakly complex. If a manifold M^n has a tangent bundle with $U(d)$ structure after stabiliztion with trivial bundles, we call the manifold stably complex.

Remark 1.8. The structure of a complex manifold, which additionally requires that the structure be integrable, is strictly stronger than the structure of an almost complex manifold. We will not deal with this stricter notion at all in this paper.

There are a few equivalent definitions of complex sections, which we use interchangeably throughout our work.

Definition 1.9. A d dimensional complex vector bundle $\pi : E \to B$ has r linearly independent complex sections if one of the following equivalent conditions hold:

- There are r sections $s_i : B \to E$ with $\pi \circ s_i = \mathrm{Id}$ such that the set of vectors $\{s_i(b) \mid 1 \leq i \leq r\}$ is (complex) linearly independent.
- There is a section $s : B \to V_{\mathbb{C},r}(E)$ of $p_{V_{\mathbb{C},r}} : V_{\mathbb{C},d,r}(E) \to B$ where $p_{V_{\mathbb{C},r}} \circ s = \mathrm{Id}$.
- The vector bundle splits as $E \cong E' \oplus \mathbb{C}^r$ for some $d-r$ dimensional complement bundle E'.
- The classifying map of the bundle $B \to BU(d)$ factors as the composition of $B \to BU(d-r)$ and $BU(d-r) \to BU(d)$ for some map $B \to BU(d-r)$ where $BU(d-r) \to BU(d)$ is the canonical map.
- There is a homotopy lift of the classifying map of the bundle $B \to BU(d)$ to a map $B \to V_{\mathbb{C},r}(U_d)$ making the following diagram commute:

$$\begin{array}{ccc} & & V_{\mathbb{C},r}(U_d) \\ & \nearrow & \downarrow \\ B & \xrightarrow{\xi} & BU(d) \end{array}$$

The Euler Characteristic

The original form of the problem addressed in this paper (which predates the definition of vector bundles) asked whether the sphere S^2 admits an everywhere non-zero vector field. In 1885, Poincare [24] proved that it did not; this result is often referred to as the hairy ball theorem:

Theorem 1.10 ([24, Poincare Theorem]**.** *There is no everywhere non-zero vector field on the sphere S^2.*

However, some manifolds like T^2 and S^3 do admit everywhere non-zero vector fields. (In fact, both those manifolds have trivial tangent bundles.) The question of when a manifold has an everywhere non-zero vector field was answered by Hopf [14] using the Euler characteristic.

Definition 1.11. The Euler characteristic of a topological space, denoted $\chi(M)$ is:

$$\chi(M) = \sum_{i=0}^{\infty}(-1)^i \dim(H_i(X;\mathbb{Q})),$$

assuming the sum converges.

Hopf, in Theorem 1.1, generalized the Poincare theorem and proved that the Euler characteristic was a complete obstruction to the existence of a non-zero vector field.

Further progress in this field depends on the study of the characteristic classes of a bundle. This material is treated in detail in [22]. A characteristic class c of a bundle is a natural assignment for every bundle $\pi : E \to B$, an element $c(E) \in H^*(B)$. (More generally, characteristic classes can sit inside any generalized cohomology theory of B, which is why we are suppressing coefficients. See [9] for details on characteristic classes in other cohomology theories.) That, is if $\pi' : E' \to B'$ is another bundle and $f : (E, B) \to (E', B')$ induces a bundle map, then $f^*(c(E')) = c(E)$. Since every d dimensional bundle over a paracompact space is the pull back of the universal bundle over $BO(d)$, every characteristic class is represented by an element of $H^*(BO(d))$. In particular, for any oriented vector bundle $\pi : E \to B$, there is a class called the *Euler class* $\chi(E) \in H^d(B;\mathbb{Z})$ [22, Section 9]. This class is closely related to the Euler charactersitic by the following proposition [22, Corollary 11.12]:

Proposition 1.12. *Let M^d be a compact oriented manifold and $[M] \in H_*(M^d;\mathbb{Z})$ be the fundamental class. Then, the Euler class of the tangent bundle TM is related to the Euler characteristic by:*

$$\chi(M^d) = \pm\langle \chi(TM), [M]\rangle.$$

The sign is determined by the choice of orientation.

The previous proposition gives a sign to the Euler characteristic depending on the orientation of the manifold. For the rest of this paper we will use the signed (orientation dependent) Euler characteristic, $\langle \chi(TM), [M]\rangle$ and refer to χ as a map to $\mathbb{Z}$. The Euler class can be used to generalize the Hopf theorem:

Theorem 1.13 ([22, Theorem 12.5]**.** *Let $\pi : E \to B$ be an oriented vector bundle of rank d. Then E has an everywhere non-zero section only if $\chi(E) = 0$. The converse holds if B is a CW complex with cells of dimension less than or equal to d.*

This is one of the few known necessary and sufficient conditions for finding linearly independent sections. In general, for larger numbers of sections, the vanishing of an obstruction is only a necessary condition. One significant result in this direction was proven by Thomas:

Theorem 1.14 ([34, Table 1]**.** *Let M^d be an oriented manifold of dimension d. M^d admits 2 vector fields only if $\mathrm{Ind}(M) = 0$ where:*

- *If $d \equiv 0 \bmod 4$, $\mathrm{Ind}(M) = \chi(M) \oplus \frac{1}{2}(\sigma(M) - (-1)^{d/4}\chi(M)) \in \mathbb{Z} \oplus \mathbb{Z}/2$,*
- *If $d \equiv 1 \bmod 4$, $\mathrm{Ind}(M) = k(M) \in \mathbb{Z}/2$,*
- *If $d \equiv 2 \bmod 4$, $\mathrm{Ind}(M) = \chi(M) \in \mathbb{Z}$,*
- *If $d \equiv 3 \bmod 4$, $\mathrm{Ind}(M) = 0$,*

where $\sigma(M)$ is the signature of M [13] and $k(M)$ is the Kervaire semi-characteristic [16] defined as $k(M) = \sum_i \dim(H^{2i}(M;\mathbb{R})) \bmod 2$.

The index invariant described in the previous theorem has geometric meaning, which we will not delve into here, but is described in [5], [3] and [29]. The signature of M will be described in more detail in the next section. This theorem was further generalized by Atiyah and Dupont or by Mayer [19]:

Theorem 1.15 ([3, Corollary 6.6]**.** *Let M^d be an oriented manifold of dimension $4k$. M^d admits r vector fields only if the signature of M, $\sigma(M)$, is divisible by b_r where b_r is given by:*

r	1	2	3	4	5	6	7	8
b_r	2	4	8	16	16	16	16	32

and where $b_{r+8} = 16b_r$.

Chern classes

We will now turn to the complex case. The Chern characteristic classes of complex bundles are the elements of $H^*(BU(d))$.

Proposition 1.16 ([22, Theorem 14.5]. *The cohomology ring $H^*(BU(d);\mathbb{Z})$ is a polynomial ring with generators in degree $2i$, $1 \leq i \leq d$. That is,*

$$H^*(BU(d);\mathbb{Z}) \cong \mathbb{Z}[c_1, ..., c_d],$$

where c_i has degree $2i$. By convention, we set $c_0 = 1$.

The classes c_i are called the Chern classes and were originally discovered by Chern [8]. These classes have the following properties, among others:

Proposition 1.17 ([22, Section 14]. *Let $\pi : E \to B$ be a d dimensional complex bundle.*

- *The classes $c_0(E) = 1 \in H^0(B)$, $c_d(E) = \chi(E)$, and $c_i(E) = 0$ for $i > d$.*
- *If $D \to B$ is another bundle, $c_i(E \oplus D) = \sum_{j=0}^{i} c_j(E)c_{i-j}(D)$.*
- *If $f : B' \to B$ is a map, then $f^*(c_i(E)) = c_i(f^*(E))$.*
- *For any trivial bundle $c_i(\mathbb{C}^r) = 0$ for $i > 0$ and $c_i(E \oplus \mathbb{C}^r) = c_i(E)$.*

As an immediate corollary, there is the following necessary condition:

Corollary 1.18. *A d dimensional complex bundle $\pi : E \to B$ has r linearly independent complex sections only if $c_i(E) = 0$ for all $i > d - r$.*

If the bundle is the tangent bundle of a manifold, these classes define a collection of integers which are called the characteristic numbers of the manifold. (See [33] and [22, p. 188].). Let $\omega = (i_1, i_2, ..., i_k)$ where $i_1 \geq i_2 \geq ... \geq i_k \geq 1$, be a partition of d. That is, $i_1 + ... + i_k = d$. Define the length of ω to be $l(\omega) = k$. Let $f_\omega(t_1, ..., t_d)$ be the symmetric polynomial with the fewest monomial terms and lowest degree in variables $t_1, ..., t_d$ with $t_1^{i_1}...t_k^{i_k}$ as a summand. The ring of symmetric polynomials is a polynomial ring:

Proposition 1.19 ([22, p. 84]. *Define the elementary symmetric polynomials by:*

$$\sigma_i(t_1, ..., t_d) = \sum_{1 \leq j_1 < j_2 < ... < j_i \leq d} t_{j_1} t_{j_2} ... t_{j_i}.$$

Every symmetric polynomial $f(t_1, ..., t_d)$ can be written uniquely as a polynomial in the variables $\sigma_1, ..., \sigma_d$.

Thus, we can write $f_\omega(t_1, ..., t_k)$ as a polynomial s_ω of the elementary symmetric polynomials so that $f_\omega(t_1, ..., t_k) = s_\omega(\sigma_1, ...\sigma_d)$. Let M^{2d} be an almost complex manifold. Then, the characteristic numbers $s_\omega(M^{2d})$ are defined by

$$s_\omega(M^{2d}) = \langle s_\omega(c_1(TM), ..., c_d(TM)), [M^{2d}] \rangle.$$

A special example is $s_{1,1,..,1}(M^{2d})$; the least symmetric polynomial in variables $t_1, ..., t_d$ with $t_1...t_d$ as a summand is the elementary symmetric polynomial $t_1t_2...t_d$ itself. Hence, $s_{1,1,...,1}(c_1, ..., c_d) = c_d$ and $s_{1,1,..,1}(M^{2d}) = \chi(M^{2d})$. The following algebraic property of these classes will be useful later:

Lemma 1.20. *Let ω be such that $l(\omega) \geq N$, then the polynomial $s_\omega(t_1, ..., t_d)$ vanishes if $c_k = \sigma_k(t_1, ..., t_d) = 0$ for all $k \geq N$.*

Proof. Suppose $c_k = 0$ for all $k \geq N$. Let ω have length N. The polynomial $f_\omega(t_1, ..., t_d)$ is a sum of monomials, up to permutation of the variables, of the form $t_1^{i_1}...t_N^{i_N}$ where $\omega = \{i_1 \geq ... \geq i_N \neq 0\}$. Since, $c_N = \sigma_N(t_1, ..., t_d) = 0$, we can write $t_1...t_N = -\sum t_{j_1}...t_{j_N}$ where the summation is over $\{j_1, ..., j_N\} \neq \{1, ..., N\}$. In particular:

$$t_1^{i_1}...t_N^{i_N} = -t_1^{i_1-1}...t_N^{i_N-1} \sum t_{j_1}...t_{j_N}.$$

Notice that each monomial in this sum has more than N distinct t_j. Thus we may repeat the process (along with symmetry), using $c_M = 0$ for $M > N$, to increase the number of distinct t_j in the polynomial up to d. Thus $f_\omega(t_1, ..., t_d) = at_1...t_d = ac_d = 0$ for some integer a, if $c_k = 0$ for all $k \geq N$. We conclude that $s_\omega(c_1, ..., c_d) = 0$ for all ω such that $l(\omega) = N$. Moreover, the same argument (with weaker assumptions) for larger values of N shows $s_\omega(c_1, ..., c_d) = 0$ for all ω such that $l(\omega) \geq N$. $\square$

We conclude this section by describing the signature $\sigma(M)$ [22, Chapter 19]. For a connected, compact, orientable manifold M^{4k}, there is a real symmetric bilinear form $H^{2k}(M; \mathbb{R}) \otimes H^{2k}(M; \mathbb{R}) \rightarrow H^{4k}(M; \mathbb{R}) \cong \mathbb{R}$. Using a change of basis, this form can be diagonalized. Moreover, by Poincare duality, this form is non-degenerate and so has no zero terms on the diagonal. The signature is the difference between the number of positive diagonal entries and the number of negative diagonal entries. Hirzebruch discovered $\sigma(M)$ had a formula in terms of the Pontryagin classes [13]; and for an almost complex manifold, the Pontryagin classes can be written in terms of the Chern classes.

Cobordism Theory

We will give a brief overview of (complex) cobordism theory and related concepts that we will use throughout the paper. Complex cobordism theory is an extraordinary cohomology theory, which is defined by the spectrum MU. Spectra are the key object of study in stable homotopy theory.

Definition 1.21. A spectrum X is a sequence of pointed topological spaces X_n together with a structure map $\Sigma X_n \to X_{n+1}$. A map of spectra $X \to Y$ is a sequence of maps $X_n \to Y_n$ commuting with the structure maps.

There are a few important examples of spectra. Any topological space X has a suspension spectrum $\Sigma^\infty X$; this spectrum has spaces $(\Sigma^\infty X)_n = \Sigma^n X$ and the spectrum structure maps are the identities. There is an adjoint construction, the infinite loop space of a spectrum Y, which is $\Omega^\infty Y = \mathrm{colim}_{n\to\infty} \Omega^n Y_n$. The homotopy groups of a spectrum Y are defined as the homotopy classes of maps $\Sigma^\infty S^0 \to Y$. Another important example comes from Thom spaces.

Let $\pi : E \to B$ be a bundle over a paracompact base space. We can equip E with a Euclidean metric. There is a disk bundle $D(E)$ consisting of all vectors of norm less than or equal to 1 and a sphere bundle $S(E)$ consisting of all vectors of norm equal to 1.

Definition 1.22. The Thom space of a bundle is defined as: $\mathrm{Th}(E) = D(E)/S(E)$. This construction is natural with respect to bundle maps.

One of the key properties of the Thom space is the Thom isomorphism theorem:

Theorem 1.23 ([33, Thom Isomorphism Theorem]**.** *For an oriented d dimensional bundle $\pi : E \to B$ and ring R, there is an isomorphism:*

$$H^*(B; R) \to \tilde{H}^{*+n}(\mathrm{Th}(E); R).$$

This isomorphism is the cup product with the Thom class $u \in H^n(\mathrm{Th}(E); R)$.

For the bundle $U_{\mathbb{C},d} \to BU(d)$, the Thom space is called $\mathbf{MU}(d)$. The pullback of $U_{\mathbb{C},d} \to BU(d)$ over $BU(d-1) \to BU(d)$ is $U_{\mathbb{C},d-1} \oplus \mathbb{C} \to BU(d-1)$. This map defines a map of Thom spaces $\mathrm{Th}(U_{\mathbb{C},d-1} \oplus \mathbb{C}) \to \mathrm{Th}(U_{\mathbb{C},d})$.

Definition 1.24. $\mathbf{MU}$ is defined as the spectrum with the spaces $\mathbf{MU}_{2n} = \mathbf{MU}(n)$ and $\mathbf{MU}_{2n+1} = \Sigma\mathbf{MU}(n)$ with structure maps:

$$\Sigma\mathbf{MU}(n) \to \Sigma\mathbf{MU}(n),\ \Sigma^2\mathbf{MU}(n) \to \mathbf{MU}(n+1),$$

defined respectively as the identity and the map of Thom spaces:

$$\Sigma^2\mathrm{Th}(U_{\mathbb{C},d-1}) \simeq \mathrm{Th}(U_{\mathbb{C},d-1} \oplus \mathbb{C}) \to \mathrm{Th}(U_{\mathbb{C},d}).$$

Any spectrum Y defines an extraordinary homology and cohomology theory [7, Theorem 1].

Definition 1.25. For a spectrum Y and space X, the extraordinary homology and cohomology theory of X are:

- $Y_n(X) = \pi_n(Y \wedge X)$.
- $Y^n(X) = [\Sigma^{-n}X, Y]$

If the groups $\pi_*(Y)$ are known, the extraordinary homology of any space X can be computed using the Atiyah-Hirzebruch spectral sequence [4, p. 435]:

Theorem 1.26 ([4, Atiyah-Hirzebruch Spectral Sequence]**.** *For a spectrum Y and space X, there exists a spectal sequence with $E^2_{p,q} = H_p(X; \pi_q(Y))$ and differential $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ such that the spaces $\{E^\infty_{p,n-p}\}$ are a filtration of $Y_n(X)$.*

In particular, $\mathbf{MU}$ defines the theory of complex (co)bordism. The homotopy groups $\pi_*(\mathbf{MU})$ which we also denote Ω^U_* were computed by Milnor [21]:

Theorem 1.27 ([21, Theorem 3]**.** *The homotopy groups $\pi_*(\mathbf{MU})$ are a polynomial ring with generators b_i in degree $2i$:*

$$\pi_*(\mathbf{MU}) \cong \mathbb{Z}[b_1, b_2, ...].$$

Thom gave a geometric interpretation of these spectra in [32].

Theorem 1.28. *The ring $\pi_*(\mathbf{MU})$ is the set of complex cobordism classes of stably complex manifolds. Two manifolds M and N with stably complex structure are defined to be complex cobordant, if there is a cobordism W with boundary $M \cup \overline{N}$ such that $TW \oplus \mathbb{R}^n$ has complex structure compatible with the complex structures on $TM \oplus \mathbb{R}^{n-1}$ and $T\overline{N} \oplus \mathbb{R}^{n-1}$ for some large n. ($\overline{N}$ is N with the opposite orientation.) The operation of addition is given by disjoint union and the operation of multiplication is Cartesian product.*

There is an analogous (and earlier) construction of the oriented cobordism groups Ω_*^{SO}. The Chern numbers are a complete set of invariants up to complex cobordism. Moreover, Stong proved that:

Theorem 1.29 ([30, Theorem 1]**.** *The map* $s_\omega : \Omega_{2d}^U \to \mathbb{Z}$ *for* ω *a partition of* d *is a well defined homomorphism and the set* $\{s_\omega\}$ *of all characteristic numbers is a (integral) basis for* $\mathbf{Hom}_\mathbb{Z}(\Omega_{2d}^U, \mathbb{Z})$*.*

Bökstedt, Dupont and Svane [5] extended the question of when a manifold can be equipped with r vector fields to instead ask when a cobordism class contains a manifold with r linearly independent vector fields. To answer this question, they defined a notion of vector field cobordism [6, Theorem 1.1]:

Definition 1.30. Let $r < d/2$. Two oriented manifolds M and N with r linearly independent vector fields are vector field cobordant if there is a cobordism with $r+1$ linearly independent vector fields, where the first r vector fields are compatible on the boundary with the vector fields of M and N. The last vector field corresponds to the inward and outward normal on M and N respectively. The vector field cobordism groups $\Omega_{d,r}^{SO}$ are the equivalence classes of manifolds under vector field cobordism

Even with $r = 0$, this definition turns out to be subtly different from oriented cobordism; it is equivalent to Reinhart cobordism. The Reinhart cobordism groups are the groups $\Omega_{d,0}^{SO}$; that is, they are the cobordism groups of oriented manifolds, where two manifolds are Reinhart cobordant if there is a cobordism between them equipped with an everywhere non-zero vector field which is the inward and outward normal on the incoming and outgoing manifold respectively.

Theorem 1.31 ([26, Proposition 2]**.** *The Reinhart cobordism groups, that is* $\Omega_{d,0}^{SO}$*, are:*

- *If* $d \equiv 0 \bmod 2$, $\Omega_{d,0}^{SO} = \Omega_d^{SO} \oplus \mathbb{Z}$.
- *If* $d \equiv 1 \bmod 4$, $\Omega_{d,0}^{SO} = \Omega_d^{SO} \oplus \mathbb{Z}/2$.
- *If* $d \equiv 3 \bmod 4$, $\Omega_{d,0}^{SO} = \Omega_d^{SO}$.

In [5, Section 4], the groups $\Omega_{d,1}^{SO}$ were computed and filtrations of $\Omega_{d,2}^{SO}$ and $\Omega_{d,2}^{SO}$ were found using the exact sequence in homotopy for fibrations and the Adams spectral sequence.

Theorem 1.32 ([5, Propositions 4.1, 4.2]**.** *The groups* $\Omega^{SO}_{d,1}$*, are:*

- *If* $d \equiv 0 \bmod 4$*, there is a short exact sequence* $0 \to \Omega^{SO}_{d,1} \to \Omega^{SO}_d \xrightarrow{\chi} \mathbb{Z}/2 \to 0$.
- *If* $d \equiv 1 \bmod 4$*,* $\Omega^{SO}_{d,1} = \Omega^{SO}_d \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$.
- *If* $d \equiv 2 \bmod 4$*,* $\Omega^{SO}_{d,1} = \Omega^{SO}_d \oplus \mathbb{Z}/2$.
- *If* $d \equiv 3 \bmod 4$*,* $\Omega^{SO}_{d,1} = \Omega^{SO}_d$.

Theorem 1.33 ([5, Theorems 4.3, 4.4, 4.5, 4.6]**.** *There exist short exact sequences for the groups* $\Omega^{SO}_{d,2}$ *with* $d \geq 4$*:*

- *If* $d \equiv 0 \bmod 4$*, there is a short exact sequence*
$$0 \to \Omega^{SO}_{d,2} \to \Omega^{SO}_d \xrightarrow{\sigma} \mathbb{Z}/4 \to 0.$$
- *If* $d \equiv 1 \bmod 4$*, there is a short exact sequence*
$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \Omega^{SO}_{d,2} \to \Omega^{SO}_d \to 0.$$
- *If* $d \equiv 2 \bmod 4$*, there is a short exact sequence*
$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \Omega^{SO}_{d,2} \to \Omega^{SO}_d \to 0.$$
- *If* $d \equiv 3 \bmod 4$*, there is a short exact sequence*
$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \Omega^{SO}_{d,2} \to \Omega^{SO}_d \to 0.$$

Theorem 1.34 ([5, Theorems 4.7, 4.8, 4.9, 4.10]**.** *There exist short exact sequences for the groups* $\Omega^{SO}_{d,3}$ *with* $d \geq 5$*:*

- *If* $d \equiv 0 \bmod 4$*, there is a short exact sequence*
$$0 \to \Omega^{SO}_{d,3} \to \Omega^{SO}_d \xrightarrow{\sigma} \mathbb{Z}/8 \to 0.$$
- *If* $d \equiv 1 \bmod 4$*, there is a short exact sequence*
$$0 \to \mathbb{Z}/24 \to \Omega^{SO}_{d,3} \to \Omega^{SO}_d \to \mathbb{Z}/2 \to 0.$$
- *If* $d \equiv 2 \bmod 4$*, there is a short exact sequence*
$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \Omega^{SO}_{d,3} \to \Omega^{SO}_d \to 0.$$
- *If* $d \equiv 3 \bmod 4$*, there is a short exact sequence*
$$0 \to \mathbb{Z}/24 \to \Omega^{SO}_{d,3} \to \Omega^{SO}_d \to 0.$$

They reproduced the earlier results of Theorems 1.14 and 1.15 with this theory.

1.3 Main Results

We apply the techniques of Bökstedt, Dupont and Svane to almost complex manifolds in order to determine the obstruction to finding linearly independent complex sections of the tangent bundle of almost complex manifolds. We define the complex section cobordism groups as:

Definition 1.35 (Definition 2.27)**.** Two $2d$ dimensional manifolds M and N with almost complex structure and r complex sections on TM and TN are defined to be complex section cobordant, if there is a cobordism W with boundary $M \cup \overline{N}$ such that $TW \oplus \mathbb{R}$ has complex structure and r linearly independent complex sections compatible with the structures on $TM \oplus \mathbb{C}$ and $TN \oplus \mathbb{C}$. ($\overline{N}$ is N with the opposite orientation.) The even dimensional complex section cobordism groups $\Omega^U_{2d,r}$ are the equivalence classes under this relation.

Definition 1.36 (Definition 2.17)**.** Two $2d-1$ dimensional manifolds M and N such that $TM \oplus \mathbb{R}$ and $TN \oplus \mathbb{R}$ are equipped with a complex structure and r linearly independent complex sections are defined to be complex section cobordant, if there is a cobordism W with boundary $M \cup \overline{N}$ such that TW has complex structure and r linearly independent complex sections compatible with the structures on $TM \oplus \mathbb{R}$ and $TN \oplus \mathbb{R}$. The odd dimensional complex section cobordism groups, $\Omega^U_{2d-1,r}$, are the equivalence classes under this relation.

We will use $\oplus\mathbb{R}$ and $\oplus\mathbb{C}$ to indicate the sum with a trivial real and complex bundle respectively. Note that a priori, even for $r = 0$, these definitions are not the same as the definition of the classical complex cobordism groups, since the classical complex cobordism groups allow stabilization of arbitrarily high dimension. A cobordism obstruction to finding r-linearly independent complex sections is an element of the cokernel of the forgetful map $\Omega^U_{2d,r} \to \Omega^U_{2d}$, which we call $Ob_{2d,r}$. The group $Ob_{2d,r}$ is a finitely generated abelian group and splits into a free and torsion part. We call the free part the rational obstruction, since it persists after tensoring with $\mathbb{Q}$. We were able to describe the rational obstruction to r linearly independent complex sections of the tangent bundle of an almost complex manifold up to cobordism. This obstruction is given in terms of Chern characteristic numbers as follows:

Theorem 1.37 (Theorem 2.38)**.** *Let M^{2d} be a d-dimensional almost-complex manifold. Then there exists non-zero integer c such that the cobordism class $c[M^{2d}]$ con-*

tains a manifold N^{2d} with r complex sections on the tangent bundle TN if and only if $s_\omega(M^{2d}) = 0$ for all ω of length greater than $d - r$.

We can describe how many distinct ways we can equip a cobordant manifold with the complex sections. This is equivalent to describing the kernel of the forgetful map from the complex section cobordism group to the complex cobordism group. We proved that this kernel is finite:

Theorem 1.38 (Corollary 2.35)**.** *Let $[M] \in \Omega^U_{2d}$, be such that M can be equipped with r linearly independent complex sections on TM. Then, there are only finitely many ways to equip a manifold $N \in [M]$ with r linearly independent complex sections up to complex section cobordism.*

We note that this kernel is not necessarily finite in the oriented case as observed in Theorems 1.31 and 1.32. The ways of equipping a manifold with r complex sections are indexed by a group which is computed in Theorem 2.56. In odd dimensions, it is well known that the complex cobordism group is zero (cf. Theorem 1.27), thus the odd dimensional complex section cobordism group parameterizes all the ways of equipping a manifold in the unique cobordism class with r complex sections. The odd dimensional complex section cobordism group is in general more difficult to compute than the even dimensional complex section cobordism group, however we note that:

Theorem 1.39 (Proposition 2.36)**.** *The odd dimensional complex section cobordism group is finite.*

Next, we determined the torsion obstruction for small values of r; we simplified the problem by splitting the torsion obstruction by prime p.

Definition 1.40. For prime p, the p-torsion obstruction to finding r linearly independent complex sections is the p-primary torsion in the abelian group $Ob_{2d,r}$.

We showed in Proposition 2.48 that p torsion obstructions correspond to non-vanishing differentials in the p-primary Adams-Novikov spectral sequence for a particular spectrum. We showed that for small r, these obstructions vanish.

Theorem 1.41 (Theorem 2.54)**.** *The p-torsion obstruction to the existence of r linearly independent complex sections of the tangent bundle of an almost complex manifold vanishes when $r < p^2 - p$.*

This theorem is vacuous when $p = 2$, so that case was treated separately. We showed that the 2-torsion obstruction vanishes for $r = 2$ or $r = 3$ linearly independent complex sections. Thus, we can give a necessary and sufficient condition for when a cobordism class admits 2 or 3 complex sections.

Theorem 1.42 (Theorem 2.58)**.** *Let M^{2d} be a d-dimensional almost-complex manifold with $d \geq 6$. There is a manifold $N^{2d} \in [M^{2d}]$ with 2 linearly independent complex sections of TN if and only if $\chi([M^{2d}]) = 0$ and $s_{2,1,\ldots,1}([M^{2d}]) = 0$.*

There is a manifold $N^{2d} \in [M^{2d}]$ with 3 linearly independent complex sections of TN if and only if $s_{2,2,1\ldots,1}([M^{2d}]) = 0$, $s_{3,1,\ldots,1}([M^{2d}]) = 0$, $s_{2,1,\ldots,1}([M^{2d}]) = 0$, and $\chi([M^{2d}]) = 0$.

In the case d is odd, we extend this result to 4 linearly independent complex sections.

Theorem 1.43 (Theorem 2.59)**.** *If $d > 6$ is odd, there is a manifold $N^{2d} \in [M^{2d}]$ with 4 linearly independent complex sections of TN if and only if $s_{4,1\ldots,1}([M^{2d}]) = 0$, $s_{3,2,1\ldots,1}([M^{2d}]) = 0$, $s_{2,2,2,\ldots,1}([M^{2d}]) = 0$, $s_{2,2,1\ldots,1}([M^{2d}]) = 0$, $s_{3,1,\ldots,1}([M^{2d}]) = 0$, $s_{2,1,\ldots,1}([M^{2d}]) = 0$, and $\chi([M^{2d}]) = 0$.*

Our final question is finding multiplicative generators of the complex cobordism ring which can be equipped with r complex sections. While we are unable to give an integral result stating which dimensions will have a generator with the obstruction vanishing, we show that they could be found in rational cobordism.

Theorem 1.44 (Theorem 2.42)**.** *Let $r < d$. There exists a manifold M^{2d} in Ω^U_{2d} which can be equipped with r linearly independent complex sections on TM and whose image in $\Omega^U_{2d} \otimes \mathbb{Q}$ is a multiplicative generator.*

Moreover, we identify integral representatives of these rational generators in Theorem 2.66.

This work in the body (but not the appendix) of this dissertation is awaiting publication in the journal *Homotopy, Homology, and Applications*, split into two papers. The results up to Theorem 1.39 will appear in the first paper, and the remaining results in the second.

CHAPTER 2

COBORDISM OBSTRUCTIONS

This work is awaiting publication in the journal *Homotopy, Homology, and Applications.*

2.1 Construction of Spectra

The spectra MTU(d) and MTU(d,r)

Here we introduce the spectra $\mathbf{MTU}(d)$ and $\mathbf{MTU}(d,r)$. The homotopy groups of these spectra are the natural objects of study. The real case is described in [5, Section 2] and [12, Definition 5.3]. There is some disagreement about the proper indexing of these spectra between [5] and [12]. We follow the convention given in Bökstedt, Dupont and Svane in [5, Definition 2.1]. Let $U^{\perp}_{\mathbb{C},d,n} \to G_{\mathbb{C}}(d,n)$ be the n dimensional orthogonal complement bundle over $G_{\mathbb{C}}(d,n)$ of $U_{\mathbb{C},d,n}$. There exists a canonical map $G_{\mathbb{C}}(d,n) \to G_{\mathbb{C}}(d,n+1)$ defined using the composition:

$$\mathbb{C}^d \to \mathbb{C}^{d+n} \hookrightarrow \mathbb{C} \oplus \mathbb{C}^{d+n} \cong \mathbb{C}^{d+n+1}.$$

The restriction of the bundle $U^{\perp}_{\mathbb{C},d,n+1}$ to $G_{\mathbb{C}}(d,n)$ under this map is $\mathbb{C} \oplus U^{\perp}_{\mathbb{C},d,n}$. In other words, there is a bundle map $U^{\perp}_{\mathbb{C},d,n} \oplus \mathbb{C} \to U^{\perp}_{\mathbb{C},d,n+1}$ covering the canonical map $G(d,n) \to G(d,n+1)$. This map induces a map of Thom spaces.

Definition 2.1. Define $\mathbf{MTU}(d)$ to be the spectrum whose $2n$-th space is:

$$\mathbf{MTU}(d)_{2n} = \mathrm{Th}(U^{\perp}_{\mathbb{C},d,n})$$

and whose structure map $\Sigma^2\mathbf{MTU}(d)_{2n} \to \mathbf{MTU}(d)_{2n+2}$ is defined by:

$$\Sigma^2(\mathrm{Th}(U^{\perp}_{\mathbb{C},d,n})) \simeq S^2 \wedge \mathrm{Th}(U^{\perp}_{\mathbb{C},d,n}) \simeq \mathrm{Th}(\mathbb{C} \oplus U^{\perp}_{\mathbb{C},d,n}) \to \mathrm{Th}(U^{\perp}_{\mathbb{C},d,n+1}).$$

We take the odd indexed spaces of $\mathbf{MTU}(d)$ to be $\mathbf{MTU}(d)_{2n+1} = \Sigma\mathbf{MTU}(d)_{2n}$ and the structure map to be the suspension map. There is also a canonical map $G_{\mathbb{C}}(d-r,n) \to G_{\mathbb{C}}(d,n)$ which takes a $(d-r)$-complex plane $P \subset \mathbb{C}^{d-r+n}$ to the direct sum, the d-plane $P \oplus \mathbb{C}^r \subset \mathbb{C}^{d-r+n} \oplus \mathbb{C}^r$. Under this map, the pullback of $U^{\perp}_{\mathbb{C},d,n}$ is $U^{\perp}_{\mathbb{C},d-r,n}$. Thus, there are maps of Thom spaces $\mathbf{MTU}(d-r)_{2n} \to \mathbf{MTU}(d)_{2n}$. Since this map commutes with the structure map, it defines a map of spectra:

$$\mathbf{MTU}(d-r) \to \mathbf{MTU}(d). \tag{2.1}$$

Definition 2.2. Let $\mathbf{MTU}(d,r)$ be the cofiber of the map (2.1).

We note that, having defined $\mathbf{MTU}(d,r)$, we immediately get a cofibration for $k \leq d-r$.

$$\mathbf{MTU}(d-r,k) \to \mathbf{MTU}(d,r+k) \to \mathbf{MTU}(d,r) \tag{2.2}$$

This cofibration reduces to the definition when $k = d-r$.

There is a second construction of the spectrum $\mathbf{MTU}(d)$ due to [5, Theorem 2.4] which will be used later to study the complex section cobordism category.

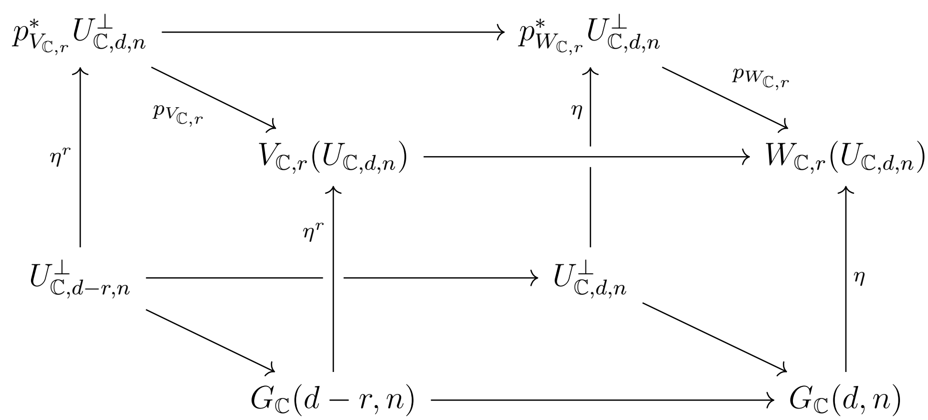
Recall that, for any d-dimensional complex fiber bundle $E \to B$ equipped with a Hermitian inner product, there is a complex frame bundle $V_{\mathbb{C},r}(E) \to B$ with fiber $V_{\mathbb{C},d,r}$. There is a related bundle $W_{\mathbb{C},r}(E) \to B$, whose fiber is the space of ordered r-tuples in $\mathbb{C}^d$ which are (hermitian) orthogonal and of the same length in the interval $[0,1]$. The fiber $W_{\mathbb{C},d,r}$ is the cone over $V_{\mathbb{C},d,r}$, by construction. The subspace where the norm is 1 is the base of the cone and isomorphic to $V_{\mathbb{C},d,r}$, and the cone point is the single tuple where all vectors are 0.

Now consider the specific case where the bundle is $U_{\mathbb{C},d,n} \to G_{\mathbb{C}}(d,n)$. Elements of $V_{\mathbb{C},r}(U_{\mathbb{C},d,n})$ consist of a complex d dimensional plane $P \subset \mathbb{C}^{d+n}$ along with an r complex frame in that plane. There is a map

$$\eta^r : G_{\mathbb{C}}(d-r,n) \to V_{\mathbb{C},r}(U_{\mathbb{C},d,n})$$

which takes a $(d-r)$-plane $P \subset \mathbb{C}^{d-r+n}$ to $P \oplus \mathbb{C}^r \subset \mathbb{C}^{d-r+n} \oplus \mathbb{C}^r$ with r frame consisting of the standard basis vectors of $\mathbb{C}^r$.

We extend η^r to a section $\eta : G_{\mathbb{C}}(d,n) \to W_{\mathbb{C},2r}(U_{\mathbb{C},d,n})$ as follows. Choose a metric on $G_{\mathbb{C}}(d,n)$, and construct an ε tubular neighborhood N of $G_{\mathbb{C}}(d,n)$ in $G_{\mathbb{C}}(d-r,n)$. Let $R : N \to [0,\infty)$ be the distance from a point to $G_{\mathbb{C}}(d-r,n)$, which is continuous by [18, Theorem 6.31]. For each point $x \in N$, define the unit speed radial geodesic $\gamma_x(t)$ to be the radial geodesic starting on $G_{\mathbb{C}}(d-r,n)$ passing through x and let $p : N \to G_{\mathbb{C}}(d,n)$ be defined by $p(x) = \gamma_x(0)$; these exist and are unique by [18, Theorem 6.40]. Choose a connection on the bundle $V_{\mathbb{C},r}(U_{\mathbb{C},d,n})$, and define $\tilde{\eta}(x)$ for $x \in N$ to be the parallel transport of $\eta(p(x))$ from $p(x)$ to x along $\gamma_x(t)$. Since the paths γ_x are a continuous family, this map is continuous. Then define η by $\eta(x) = \frac{1}{\varepsilon}(\varepsilon - R(x))\tilde{\eta}(x)$ for $x \in N$. On the boundary of N, all points (which represent planes) are sent to the zero frame. (The zero frame is the frame of all zero vectors, that is, the cone point of the fiber.) Thus we can extend this map to all of $G_{\mathbb{C}}(d,n)$ by mapping planes outside of N to the plane equipped with the zero frame. This section $\eta : G_{\mathbb{C}}(d,n) \to W_{\mathbb{C},r}(U_{\mathbb{C},d,n})$ is the right vertical map in the left diagram below.



The maps $p_{V_{\mathbb{C},r}} : V_{\mathbb{C},r}(U_{\mathbb{C},d,n}) \to G_{\mathbb{C}}(d,n)$ and $p_{W_{\mathbb{C},r}} : W_{\mathbb{C},r}(U_{\mathbb{C},d,n}) \to G_{\mathbb{C}}(d,n)$ are the corresponding projections. Then we have the Thom spaces $\mathrm{Th}(p^*_{V_{\mathbb{C},r}} U^{\perp}_{\mathbb{C},d,n})$ and $\mathrm{Th}(p^*_{W_{\mathbb{C},r}} U^{\perp}_{\mathbb{C},d,n})$ which we form into spectra $\mathbf{MTU}(d)_{V_r}$ and $\mathbf{MTU}(d)_{W_r}$ respectively. The maps η^r and η determine maps of spectra: $\mathbf{MTU}(d)_{V_r} \to \mathbf{MTU}(d-r)$ and $\mathbf{MTU}(d)_{W_r} \to \mathbf{MTU}(d)$. The top horizontal maps in the above diagram induce a map of spaces $\mathrm{Th}(p^*_{V_{\mathbb{C},r}} U^{\perp}_{\mathbb{C},d,n}) \to \mathrm{Th}(p^*_{W_{\mathbb{C},r}} U^{\perp}_{\mathbb{C},d,n})$. The commutative square of bundles shown below:

$$\begin{array}{ccc} p^*_{V_{\mathbb{C},r}} U^{\perp}_{\mathbb{C},d,n} & \longrightarrow & p^*_{W_{\mathbb{C},r}} U^{\perp}_{\mathbb{C},d,n} \\ \big\downarrow & & \big\downarrow \\ p^*_{V_{\mathbb{C},r}} U^{\perp}_{\mathbb{C},d,n+1} & \longrightarrow & p^*_{W_{\mathbb{C},r}} U^{\perp}_{\mathbb{C},d,n+1} \end{array}$$

defines a square of Thom spaces which define the map $\mathbf{MTU}(d)_{V_r} \to \mathbf{MTU}(d)_{W_r}$. From this map we form a cofiber which we call $\mathbf{MTU}'(d,r)$.

Proposition 2.3. *In the below commutative diagram, all vertical maps are homotopy equivalences.*

$$\begin{array}{ccccc} \mathbf{MTU}(d-r) & \longrightarrow & \mathbf{MTU}(d) & \longrightarrow & \mathbf{MTU}(d,r) \\ \big\uparrow{\scriptstyle \eta^r} & & \big\uparrow{\scriptstyle \eta} & & \big\uparrow{\scriptstyle \overline{\eta}} \\ \mathbf{MTU}(d)_{V_r} & \longrightarrow & \mathbf{MTU}(d)_{W_r} & \longrightarrow & \mathbf{MTU}'(d,r) \end{array}$$

Proof. We know the section $\eta : G_{\mathbb{C}}(d,n) \to W_{\mathbb{C},r}(U_{\mathbb{C},d,n})$ is a homotopy inverse of $p_{W_{\mathbb{C},r}}$ because $W_{\mathbb{C},r}(U_{\mathbb{C},d,n})$ has contractible fibers. There is a fiber bundle:

$$V_{\mathbb{C},r}(U_{\mathbb{C},d,n}) \to V_{\mathbb{C},d+n,r}.$$

The total space $V_{\mathbb{C},r}(U_{\mathbb{C},d,n})$ consists of a d dimensional plane in $\mathbb{C}^{d+n}$ and an r-frame in that plane. The projection forgets the plane leaving only the frame. For a given frame in $V_{\mathbb{C},d+n,r}$, the fiber consists of any d dimensional complex plane containing the frame. This is equivalent to choosing a $(d-r)$-plane in the orthogonal complement of the frame. The orthogonal complement is $\mathbb{C}^{d+n-r}$, so the fiber is $G_{\mathbb{C}}(d-r,n)$. The fiber inclusion takes a complex plane $P^{d-r} \subseteq \mathbb{C}^{n+d-r}$ to the plane $P^{d-r} \oplus \mathbb{C}^r \subseteq \mathbb{C}^{n+d}$ equipped with the frame of the r standard basis vectors in $\mathbb{C}^r$. This is the map η^r.

We know the base of the fibration, $V_{\mathbb{C},d+n,r}$, is $(2n+2d-2r-1)$ connected and thus the pair of spaces $(V_{\mathbb{C},r}(U_{\mathbb{C},d,n}), G_{\mathbb{C}}(d-r,n))$ has the same connectivity. By the Thom isomorphism theorem, we find the pair $(\mathrm{Th}(p^*_{V_{\mathbb{C},r}} U^{\perp}_{\mathbb{C},d,n}), \mathrm{Th}(U^{\perp}_{\mathbb{C},d-r,n}))$ is $(4n+2d-2r-1)$ connected.

Letting $n \to \infty$, we see that $\eta^r : \mathbf{MTU}(d)_{V_r} \to \mathbf{MTU}(d-r)$ is a homotopy equivalence. By the five lemma, the right vertical map is also a homotopy equivalence. ☐

Using these isomorphisms, the space $\mathbf{MTU}(d,1)$ can be given a simpler description which is useful for computing the homotopy groups: (cf. [5, Proposition 2.6])

Theorem 2.4. *There is a homotopy equivalence:*

$$\mathbf{MTU}(d,1) \simeq \Sigma^{\infty} S^{2d} \vee \Sigma^{\infty+2d} BU(d).$$

In particular,

$$\pi_{q+2d}(\mathbf{MTU}(d,1)) \simeq \pi^s_q(S^0) \oplus \pi^s_q(BU(d)).$$

Proof. We observe that $\mathbf{MTU}(d,1)_{2n} = \mathrm{Th}(p^*_{W_{\mathbb{C},1}} U^{\perp}_{\mathbb{C},d,n}) / \mathrm{Th}(p^*_{V_{\mathbb{C},1}} U^{\perp}_{\mathbb{C},d,n})$ by construction where $p^*_{V_{\mathbb{C},1}}$ is the projection $p^*_{V_{\mathbb{C},1}} : V_1(U_{\mathbb{C},d,n}) \simeq S(U_{\mathbb{C},d,n}) \to G_{\mathbb{C}}(d,n)$ and $p^*_{W_{\mathbb{C},1}}$ is the projection $p^*_{W_{\mathbb{C},1}} : W_1(U_{\mathbb{C},d,n}) \simeq D(U_{\mathbb{C},d,n}) \to G_{\mathbb{C}}(d,n)$. For two vector bundles V_1 and V_2 over the same base, we can pull back V_2 over the disk and sphere bundles of V_1. The Thom space of the sum of the bundles is homotopic as follows:

$$\mathrm{Th}(V_1 \oplus V_2) \simeq \mathrm{Th}(V_2|_{D(V_1)}) / \mathrm{Th}(V_2|_{S(V_1)}).$$

Thus,

$$\mathbf{MTU}(d,1)_{2n} \simeq \mathrm{Th}(U_{\mathbb{C},d,n} \oplus U_{\mathbb{C},d,n}^{\perp}).$$

Since, $U_{\mathbb{C},d,n} \oplus U_{\mathbb{C},d,n}^{\perp}$ is a trivial $d+n$ dimensional complex bundle,

$$\mathbf{MTU}(d,1)_{2n} \simeq S^{2d+2n} \vee \Sigma^{2d+2n} G_{\mathbb{C}}(d,n).$$

Moreover these isomorphisms commute with the structure maps:

$$\begin{array}{ccccc}
\mathbf{MTU}(d,1)_{2n} & \longrightarrow & \mathrm{Th}(U_{\mathbb{C},d,n} \oplus U_{\mathbb{C},d,n}^{\perp}) & \longrightarrow & S^{2d+2n} \vee \Sigma^{2d+2n} G_{\mathbb{C}}(d,n) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{MTU}(d,1)_{2n+2} & \longrightarrow & \mathrm{Th}(U_{\mathbb{C},d,n+1} \oplus U_{\mathbb{C},d,n+1}^{\perp}) & \longrightarrow & S^{2d+2n+2} \vee \Sigma^{2d+2n+2} G_{\mathbb{C}}(d,n+1)
\end{array}$$

The central vertical map is defined by the canonical bundle maps $U_{\mathbb{C},d,n} \to U_{\mathbb{C},d,n+1}$, and $U_{\mathbb{C},d,n}^{\perp} \to U_{\mathbb{C},d,n+1}^{\perp}$ over $G_{\mathbb{C}}(d,n) \to G_{\mathbb{C}}(d,n+1)$. Thus, the right vertical map is equivalent to the structure map for $\Sigma^{\infty} S^{2d} \vee \Sigma^{\infty+2d} BU(d)$, proving the result. □

Cohomology of the spectra

We next compute the cohomology of the spectra $\mathbf{MTU}(d)$ and $\mathbf{MTU}(d,r)$. For the rest of this section, all cohomology will be assumed to have $\mathbb{Z}$ coefficients.

Proposition 2.5. *There is a Thom isomorphism:*

$$\phi : \mathbb{Z}[c_1, c_2, ..., c_d] \cong H^*(BU(d);\mathbb{Z}) \to H^*(\mathbf{MTU}(d);\mathbb{Z}),$$

which endows $H^*(\mathbf{MTU}(d);\mathbb{Z})$ with a $H^*(BU(d);\mathbb{Z})$ module structure.

Proof. There is a Thom class $\overline{u}_{d,n} \in H^{2n}(\mathrm{Th}(U_{\mathbb{C},d,n}^{\perp}))$ associated to the bundle $U_{\mathbb{C},d,n}^{\perp}$. In the $2n$-th space $\mathbf{MTU}(d)_{2n} = \mathrm{Th}(U_{\mathbb{C},d,n}^{\perp})$, this class defines a Thom isomorphism $H^*(G_{\mathbb{C},d,n}) \to \tilde{H}^{*+2n}(\mathrm{Th}(U_{\mathbb{C},d,n}^{\perp}))$. In the limit, there is a stable Thom class in degree 0, $\overline{u}_d \in H^0(\mathbf{MTU}(d))$. The Thom isomorphism theorem states that $H^*(\mathbf{MTU}(d))$ is the dimension 1 free module over $H^*(BU(d))$ generated by the Thom class. □

Theorem 2.6. *The map $H^*(\mathbf{MTU}(d,r)) \to H^*(\mathbf{MTU}(d))$ is injective with image the $H^*(BU(d))$ module generated by $\phi(c_{d-r+1}), ..., \phi(c_d)$.*

Proof. Observe the following commutative diagram:

$$\begin{array}{ccccc} H^{k+2n}(\mathrm{Th}(U^{\perp}_{\mathbb{C},d,n}), \mathrm{Th}(U^{\perp}_{\mathbb{C},d-r,n})) & \longrightarrow & \tilde{H}^{k+2n}(\mathrm{Th}(U^{\perp}_{\mathbb{C},d,n})) & \longrightarrow & \tilde{H}^{k+2n}(\mathrm{Th}(U^{\perp}_{\mathbb{C},d-r,n})) \\ \uparrow & & \uparrow & & \uparrow \\ H^k(G_{\mathbb{C}}(d,n), G_{\mathbb{C}}(d-r,n)) & \longrightarrow & H^k(G_{\mathbb{C}}(d,n)) & \longrightarrow & H^k(G_{\mathbb{C}}(d-r,n)) \end{array}$$

The horizontal maps come from the exact sequence in cohomology. As $n \to \infty$ the bottom right map is $\mathbb{Z}[c_1, ..c_d] \cong H^k(BU(d)) \to H^k(BU(d-r)) \cong \mathbb{Z}[c_1, ..., c_{d-r}]$ by [22, Theorem 14.5]. This map is a surjection mapping $c_i \to c_i$ for $i \leq d-r$. So $H^*(BU(d), BU(d-r))$ is the kernel, namely the $H^*(BU(d))$ submodule generated by $c_{d-r+1}, ..., c_d$.

The vertical maps are the Thom isomorphisms. Thus we conclude that the map $H^*(\mathbf{MTU}(d,r)) \to H^*(\mathbf{MTU}(d))$ is injective with image the $H^*(BU(d))$ submodule generated by $\phi(c_{d-r+1}), ..., \phi(c_d)$. □

Corollary 2.7. *The spectrum* $\mathbf{MTU}(d,r)$ *is* $(2(d-r)+1)$ *connected.*

As a corollary, there is a stabilization isomorphism for $\mathbf{MTU}(d,r)$:

Theorem 2.8. $\pi_q(\mathbf{MTU}(d,r)) \cong \pi_q(\mathbf{MTU}(d+k, r+k))$ *for* $q \leq 2d$.

Proof. There is a homotopy exact sequence induced by the cofibration 2.2:

$$\begin{aligned} ... \to &\pi_{q+1}(\mathbf{MTU}(d+k,k)) \to \pi_q(\mathbf{MTU}(d,r)) \\ &\to \pi_q(\mathbf{MTU}(d+k, r+k)) \to \pi_q(\mathbf{MTU}(d+k,k)) \to ... \end{aligned}$$

By the Hurewicz theorem, $\mathbf{MTU}(d+k,k)$ is $2(d+k-k)+1$ connected. So there is an isomorphism, $\pi_q(\mathbf{MTU}(d,r)) \cong \pi_q(\mathbf{MTU}(d+k,r+k))$ for $q \leq 2d$. □

We will connect the homotopy groups of $\mathbf{MTU}(d)$ to the homotopy groups of $\mathbf{MU}$, which are the complex cobordism groups. There is a sequence of spectra:

$$\mathbf{MTU}(d) \to \mathbf{MTU}(d+1) \to \mathbf{MTU}(d+2) \to ...$$

Let us call the colimit of this spectrum $\mathbf{MTU}$; this spectrum is the Thom spectrum of the bundle $U^{\perp}_{\mathbb{C}}$ over BU. There is an inversion map $BU \to BU$ which takes a vector space to its complement. This map is covered by a bundle map $U^{\perp}_{\mathbb{C}} \to U_{\mathbb{C}}$ which defines a map $\mathbf{MTU} \to \mathbf{MU}$. Since there is also an inverse bundle map $U_{\mathbb{C}} \to U^{\perp}_{\mathbb{C}}$, the spectra $\mathbf{MU} \to \mathbf{MTU}$ is a weak homotopy equivalence. (The real version of this construction is described in [10, p. 26] among others.) In particular, there is a well defined map $\mathbf{MTU}(d) \to \mathbf{MU}$. The computation in Theorem 2.6 implies that the map $\mathbf{MTU}(d) \to \mathbf{MU}$ is $2d+1$ connected. (The map on cohomology is the quotient $\mathbb{Z}[c_1, .., c_d, ...] \to \mathbb{Z}[c_1, ..., c_d]$.) We extend the results on $\mathbf{MTU}(d, r)$ to a new spectrum, which will be more convenient for the calculations in later sections.

Definition 2.9. Let $\overline{\mathbf{MTU}}(d)$ be the cofiber of the map $\mathbf{MTU}(d) \to \mathbf{MU}$.

Proposition 2.10. *The map* $H^*(\overline{\mathbf{MTU}}(d))) \to H^*(\mathbf{MU})$ *is injective. The ring* $H^*(\overline{\mathbf{MTU}}(d)))$ *is torsion-free and has non-zero cohomology only in even degrees.*

Proof. There is an exact sequence (with integer coefficients):

$$H^{q-1}(\mathbf{MU}) \to H^{q-1}(\mathbf{MTU}(d)) \to H^q(\overline{\mathbf{MTU}}(d))) \to H^q(\mathbf{MU}) \to H^q(\mathbf{MTU}(d)).$$

If q is even, then $H^{q-1}(\mathbf{MTU}(d))$ is 0 so the map is injective. If q is odd, then the map $H^{q-1}(\mathbf{MU}, \mathbb{Z}) \to H^{q-1}(\mathbf{MTU}(d))$ coincides with $\mathbb{Z}[c_1, c_2, ...] \to \mathbb{Z}[c_1, ..., c_d]$ on degree $q-1$ parts and is surjective. Thus, $H^q(\overline{\mathbf{MTU}}(d)), \mathbb{Z}) \to H^q(\mathbf{MU}, \mathbb{Z})$ is injective. □

We will also need to describe the homology of the spectra $\mathbf{MTU}(d)$.

Proposition 2.11. *There is a Thom isomorphism:*

$$H_*(BU(d); \mathbb{Z}) \cong H_*(\mathbf{MTU}(d); \mathbb{Z})$$

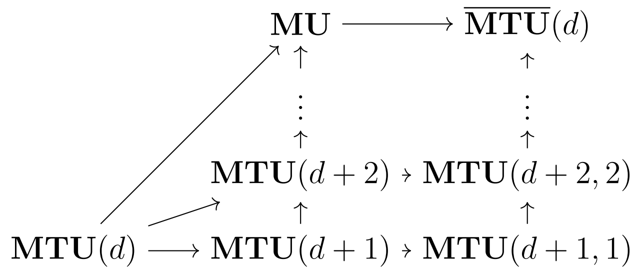
and the map $H_*(\mathbf{MTU}(d); \mathbb{Z}) \to H_*(\mathbf{MU}; \mathbb{Z})$ *is an injection. Moreover, the image of* $H_*(\mathbf{MTU}(d))$ *in* $H_*(\mathbf{MU}; \mathbb{Z}) \cong \mathbb{Z}[b_1, b_2, ...]$ *is generated as a group by monomials of degree less than* d.

Proof. This result follows by dualizing the corresponding result, Proposition 2.5, for the cohomology of $\mathbf{MTU}(d)$. □

There is an alternative interpretation of $\overline{\mathbf{MTU}}(d)$ as the colimit of the sequence of spectra:

$$\mathbf{MTU}(d+1, 1) \to \mathbf{MTU}(d+2, 2) \to ...$$

The following commutative diagram makes it clear that these definitions are the same.



As a corollary of Theorem 2.8,

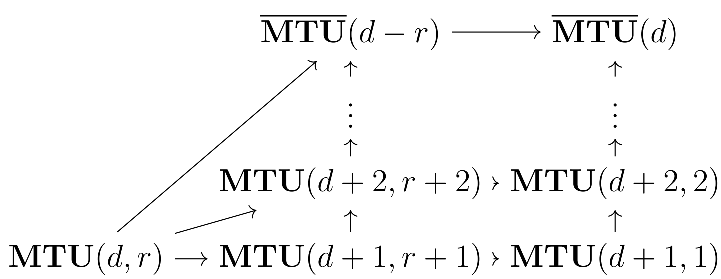
Corollary 2.12. *There is an isomorphism for all d and r:*

$$\pi_{2d+1}(\mathbf{MTU}(d+1, r+1)) \cong \pi_{2d+1}(\overline{\mathbf{MTU}}(d-r)).$$

This corollary will be helpful when we want to compute the homotopy groups $\pi_{2d+1}(\overline{\mathbf{MTU}}(d-r))$. The following proposition will allow us to study the homotopy of $\overline{\mathbf{MTU}}(d-r)$ one r at a time. (cf. Theorem 2.59.)

Proposition 2.13. *There is a cofibration* $\mathbf{MTU}(d, r) \to \overline{\mathbf{MTU}}(d-r) \to \overline{\mathbf{MTU}}(d)$.

Proof. There is a commutative diagram as below. The columns are the sequences of spectra used to define $\overline{\mathbf{MTU}}(d-r)$ and $\overline{\mathbf{MTU}}(d)$. The rows are each cofibrations.



The colimit of the cofibrations defined by each row gives the desired cofibration. □

As a further consequence of Corollary 2.7, we conclude the following lemma.

Lemma 2.14. *There is an isomorphism* $\pi_q(\mathbf{MTU}(d)) \cong \pi_q(\mathbf{MU})$ *for* $q \leq 2d$.

This lemma shows that there is an isomorphism $\Omega^U_{2d,0} \cong \Omega^U_{2d}$ in even dimensions. That is, for any two even dimensional complex cobordant manifolds, there is a cobordism between them which can be equipped with a complex section which matches the inward and outward normals on the boundary. This is in contrast to the oriented case, Reinhart cobordism, Theorem 1.31.

2.2 The cobordism category

General cobordism categories

In order to provide a geometric interpretation for the spectra constructed in the previous section, we discuss cobordism categories. These categories were originally studied by [12] and [6]. Here we specify what happens in the complex case. We will reference the spectra $\mathbf{MTO}(d)$ and $\mathbf{MTSO}(d)$, which are the real analogues for $\mathbf{MTU}(d)$. For more details on their specific construction, see [12, Section 5] and [6, Section 2]. We start with the general cobordism category with tangential structure.

Definition 2.15. Let $\theta : X \to BO(d)$ be a fibration. The cobordism category $\mathcal{C}^{\theta}_{d,n+d}$ has as objects $(d-1)$ dimensional manifolds $M \subset (-1,1)^{n+d-1} \subset \mathbb{R}^{n+d-1}$ with M closed in $\mathbb{R}^{n+d-1}$ without boundary along with the data of a chosen lift of the classifying map $\xi : M \to G(d-1,n) \to G(d,n)$ to $\theta^*(G(d,n))$. The space of morphisms from $M_0 \to M_1$ is the disjoint union of the identity morphism along with pairs (W,a) with the following properties. $W \subseteq (-1,1)^{n+d-1} \times \mathbb{R} \subset \mathbb{R}^{n+d-1} \times \mathbb{R}$ is a manifold of dimension d, which is closed in $\mathbb{R}^{n+d}$, $a \in (0,\infty)$ and for some $\varepsilon > 0$:

$$W \cap (\mathbb{R}^{n+d-1} \times (-\infty,\varepsilon)) = M_0 \times (-\infty,\varepsilon),$$

$$W \cap (\mathbb{R}^{n+d-1} \times (a-\varepsilon,\infty)) = M_1 \times (a-\varepsilon,\infty).$$

Additionally W is equipped with a chosen lift of the classifying map $W \to G(d,n)$ to $\theta^*(G(d,n))$ which is compatible with the structures on the cobordant manifolds. Composition is defined by concatenation of cobordisms.

Since no cobordism will compose as the identity, we need to formally add an identity morphism to the set of morphism, hence the disjoint union.

There is a canonical map $\mathcal{C}^{\theta}_{d,n+d} \to \mathcal{C}^{\theta}_{d,n+d+1}$ which embeds manifolds and morphisms into one higher dimensional space. In this paper, we will use the limit category as $n \to \infty$ of the categories $\mathcal{C}^{\theta}_{d,n+d}$ and write it as $\mathcal{C}^{\theta}_{d}$.

Every category has a classifying space, and nerve. The nerve of a category is the simplicial set constructed by: 0-simplicies are the objects, 1-simplices are added for each morphism, 2 simplices are added for the composition of two morphisms, 3 simplices are added for the composition of three morphisms etc. The classifying space is the geometric realization of the nerve. We will denote the classifying space of category $\mathcal{C}$ by $B\mathcal{C}$. Galatius, Tillmann, Madsen, and Weiss found the homotopy type of the classifying space of the cobordism category:

Theorem 2.16 ([12, Main Theorem]**.** *There is a (weak) homotopy equivalence:*

$$B\mathcal{C}_d^\theta \simeq \Omega^{\infty-d-1}\theta^*(\mathbf{MTO}(d)).$$

The spectrum $\theta^*(\mathbf{MTO}(d))$ is the pullback of the construction of $\mathbf{MTO}(d)$ over θ i.e. the spectrum whose n-th space is given by $\mathrm{Th}(\theta^* U_{d,n}^\perp)$. In the complex case, the structure is $\theta : BU(d) \to BO(2d)$ and the spectrum, by construction, is $\theta^*(\mathbf{MTO}(d)) = \mathbf{MTU}(d)$. So the $U(d)$ cobordism category has as objects $2d-1$ dimensional manifolds M with $U(d)$ structure on $TM \oplus \mathbb{R}$ and has as morphisms $2d$ dimensional cobordisms with $U(d)$ structure.

The even dimensional complex section cobordism category

In order to describe the complex section cobordism category, that is the category of almost complex manifolds equipped with linearly independent complex sections, we consider the even and odd dimensional cases separately. We need to split into even and odd cases because even dimensional manifolds may be equipped with an almost complex structure but odd dimensional manifolds cannot be. Define the even dimensional complex cobordism category as the category from Definition 2.15, with tangential structure given by $\theta_{\mathbb{C},r} : V_{\mathbb{C},r}(U_{\mathbb{C},d}) \to BO(2d)$ which is defined by the following composition:

$$\begin{array}{ccc} V_{\mathbb{C},r}(U_{\mathbb{C},d}) & \longrightarrow & V_r(U_{2d}) \\ \downarrow{\scriptstyle i_r} & & \downarrow{\scriptstyle i_r} \\ BU(d) & \longrightarrow & BO(2d) \end{array}$$

The map $V_{\mathbb{C},r}(U_{\mathbb{C},d}) \to V_r(U_{2d})$ is the natural map which forgets the complex structure. (Note that this diagram is not a pull back square.) The category $\mathcal{C}_{2d}^{\theta_{\mathbb{C},r}}$ with this tangential structure has as objects $2d-1$ dimensional manifolds M with a complex structure on the bundle $TM \oplus \mathbb{R}$ and r linearly independent complex sections. A morphism $W : M \to N$ is a $2d$ dimensional almost complex cobordism equipped with r linearly independent complex sections such that the structures are compatible as given in the definition below. We define two manifolds to be complex section cobordant if there is a morphism between them in this category.

Definition 2.17. Two $2d-1$ dimensional manifolds M and N such that $TM\oplus\mathbb{R}$ and $TN\oplus\mathbb{R}$ are equipped with a complex structure and r linearly independent complex sections are defined to be complex section cobordant, if there is a cobordism W with boundary $M\cup\overline{N}$ such that TW has complex structure and r linearly independent complex sections compatible with the structures on $TM\oplus\mathbb{R}$ and $TN\oplus\mathbb{R}$.

Proposition 2.18. *The relation defined in Definition 2.17 is an equivalence relation.*

Proof. This relation is obviously reflexive and transitive. It remains to show that it is symmetric. Suppose M and N are $2d-1$ dimensional manifolds with complex structure and r linearly independent complex sections on $TM\oplus\mathbb{R}$ and $TN\oplus\mathbb{R}$ and suppose that there is a cobordism W with boundary $M\cup\overline{N}$ such that TW has complex structure and r linearly independent complex sections compatible with the structures on $TM\oplus\mathbb{R}$ and $TN\oplus\mathbb{R}$.

We may reverse W and get a complex cobordism between N and M, which we call $\tilde{W}$. However the complex sections will be reversed on the boundary. We will correct this reversal by showing there exists a cobordism reversing the sections. Let $\nu: M\to V_{\mathbb{C},r}(TM\oplus\mathbb{R})$ be a map representing complex sections and $-\nu$ be their reverse. Define a section $\tilde{\nu}: M\times[0,1]\to V_{\mathbb{C},r}(TM\times\mathbb{R})\simeq V_{\mathbb{C},r}(TM\oplus\mathbb{R})\times[0,1]$ by the formula $\tilde{\nu}(x,t)=(e^{\pi it}\nu(x),t)$. This section restricts to ν on the incoming boundary and $-\nu$ on the outgoing boundary. We may perform a similar construction on N. Composing these cobordisms with $\tilde{W}$, we get a cobordism from N to M with the correct complex section structure. □

Definition 2.19. The odd dimensional complex section cobordism groups, $\Omega^U_{2d-1,r}$, are the equivalence classes under this relation.

We can interpret the complex section cobordism groups as the connected components of the classifying space $B\mathcal{C}_{2d}^{\theta_{\mathbb{C},r}}$.

Proposition 2.20. *The connected components of $B\mathcal{C}_{2d}^{\theta_{\mathbb{C},r}}$ are the equivalence classes under complex section cobordism.*

Proof. If two manifolds are equivalent, then there is a morphism in $\mathcal{C}_{2d}^{\theta_{\mathbb{C},r}}$ connecting them and so they are in the same connected component of $B\mathcal{C}_{2d}^{\theta_{\mathbb{C},r}}$. If two manifolds are in the same connected component of $B\mathcal{C}_{2d}^{\theta_{\mathbb{C},r}}$, then there is a zig zag of morphisms connecting them by [6, Theorem 3.4]. By Proposition 2.18, this means that the two manifolds are complex section cobordant. □

Remark 2.21. Note that we index the cobordism category by the dimension of the cobordisms but we index the cobordism groups by the dimension of the object manifolds.

Next we modify the results from [6], specializing to the complex case in order to connect these groups to the Thom spectra constructed in the previous section. This relies on the calculation of the homotopy type of the cobordism category in Theorem 2.16. Consider the structure $p_{V_{\mathbb{C},r}} : V_{\mathbb{C},r}(U_{\mathbb{C},2d}) \to BU(d)$. We observe that $p^*_{V_{\mathbb{C},r}}\mathbf{MTU}(d)$ is, by construction, $\mathbf{MTU}(d)_{V_r}$, and we proved in Theorem 2.3 that $\mathbf{MTU}(d)_{V_r} \simeq \mathbf{MTU}(d-r)$. Note that $\theta_{\mathbb{C},r} = \theta \circ p_{V_{\mathbb{C},r}}$ where the maps $\theta_{\mathbb{C},r}$ and $p_{V_{\mathbb{C},r}}$ are as above. The next theorem and its corollary will allow us to study and compute the complex section cobordism groups by studying the homotopy groups of the Thom spectrum $\mathbf{MTU}(d)$.

Theorem 2.22. *The spaces $B\mathcal{C}_{2d}^{\theta_{\mathbb{C},r}}$ and $\Omega^{\infty+2d-1}\mathbf{MTU}(d-r)$ are weakly homotopy equivalent.*

Proof. Theorem 2.16 shows that there is a weak homotopy equivalence

$$B\mathcal{C}_{2d}^{\theta_{\mathbb{C},r}} \to \Omega^{\infty+2d-1}\theta^*_{\mathbb{C},r}\mathbf{MTO}(2d).$$

By construction, $\theta^*_{\mathbb{C},r}\mathbf{MTO}(2d) \simeq \mathbf{MTU}(d)_{V_r}$. Theorem 2.3 shows that $\mathbf{MTU}(d)_{V_r}$ is homotopy equivalent to $\mathbf{MTU}(d-r)$. It follows that:

$$B\mathcal{C}_{2d}^{\theta_{\mathbb{C},r}} \simeq \Omega^{\infty+2d-1}\mathbf{MTU}(d-r).$$

□

Remark 2.23. The indexing of our spectra differs from the indexing in [12], leading to the $2d$ in the shift of the loop space $\Omega^{\infty+2d-1}\mathbf{MTU}(d-r)$.

Corollary 2.24. *There is an isomorphism: $\pi_{2d+2r-1}(\mathbf{MTU}(d)) \cong \pi_0 B\mathcal{C}_{2(d+r)}^{\theta_{\mathbb{C},r}}$.*

The above corollary classifies all odd homotopy groups of $\mathbf{MTU}(d)$. (The lower homotopy groups of $\mathbf{MTU}(d)$ are classified by Lemma 2.14).

Remark 2.25. We note that because the dimension of our manifolds is odd, we are not finding sections on M^{2d-1} but on $M^{2d-1} \times I$ which has a genuine complex structure. Cobordisms in this category are equivalences of these cylinders not of the lower dimensional manifold.

We define a complex cobordism theory with cobordisms of dimension $2d+1$ and objects of dimension $2d$. While this is a specific case of Definition 2.15 with structure $V_{\mathbb{C},r}(U_{\mathbb{C},d+1}) \to BO(2d+2)$, we include it in detail for clarity and to point out the necessity of Proposition 2.33.

Definition 2.26. The cobordism category $\mathcal{C}^{U,r}_{2d+1,2n+2d+1}$ has as objects, $2d$ dimensional manifolds without boundary $M \subset (-1,1)^{2n+2d} \subset \mathbb{R}^{2n+2d}$ with M closed in $\mathbb{R}^{2n+2d}$ along with a chosen lift of the map $\xi : M \to G(2d,2n) \to G(2d+2,2n)$ to $V_{\mathbb{C},r}(U_{\mathbb{C},d+1,n})$. The morphisms from $M_0 \to M_1$ are the disjoint union of the identity morphism along with pairs (W,a) with the following properties. The real number $a \in (0,\infty)$ and $W \subseteq (-1,1)^{2n+2d} \times \mathbb{R} \subset \mathbb{R}^{2n+2d} \times \mathbb{R}$ is a manifold of dimension $2d+1$, which is closed in $\mathbb{R}^{2n+2d+1}$ and such that for some $\varepsilon > 0$:

$$W \cap (\mathbb{R}^{2n+2d} \times (-\infty,\varepsilon)) = M_0 \times (-\infty,\varepsilon),$$

$$W \cap (\mathbb{R}^{2n+2d} \times (a-\varepsilon,\infty)) = M_1 \times (a-\varepsilon,\infty).$$

Additionally W is equipped with the data of a chosen lift of the classifying map $W \to G(2d+1,2n) \to G(2d+2,2n)$ to $V_{\mathbb{C},r}(U_{\mathbb{C},d+1,n})$ which is compatible with the structures on the cobordant manifolds.

For more details on the definition see [11, Definition 3.7] and [6, Definition 2.2]. We typically consider the limit as $n \to \infty$ with and abbreviate the notation for this category as $\mathcal{C}^{U,r}_{2d+1}$. Unraveling the definition, objects are manifolds M^{2d} with complex structure and r linearly independent complex sections on $TM \oplus \mathbb{R}^2$. Cobordisms W are $2d+1$ dimensional manifolds such that $TW \oplus \mathbb{R}$ is equipped with a complex structure and r linearly independent complex sections, compatible with the structure on the cobordant manifolds. We can define even dimensional complex section cobordism in a similar way to the odd case.

Definition 2.27. Two $2d$ dimensional manifolds M and N with almost complex structure and r linearly independent complex sections on $TM \oplus \mathbb{R}^2$ and $TN \oplus \mathbb{R}^2$ are defined to be complex section cobordant, if there is a cobordism W with boundary $M \cup \overline{N}$ such that $TW \oplus \mathbb{R}$ has complex structure and r linearly independent complex sections compatible with the structures on $TM \oplus \mathbb{R}^2$ and $TN \oplus \mathbb{R}^2$.

Once again, this relation is equivalent to the existence of a morphism in $\mathcal{C}^{U,r}_{2d+1}$.

Proposition 2.28. *The relation defined in Definition 2.27 is an equivalence relation.*

Definition 2.29. The even dimensional complex section cobordism groups, $\Omega^U_{2d,r}$, are the equivalence classes under this relation.

Theorem 2.30. *The connected components of $B\mathcal{C}^{U,r}_{2d+1}$ are the equivalence classes of $2d$ dimensional manifolds under complex section cobordism.*

The proofs of the proposition and theorem above are identical to the odd case.

Theorem 2.31. *There is a homotopy equivalence*

$$B\mathcal{C}^{U,r}_{2d+1} \to \Omega^{\infty+2d}\mathbf{MTU}(d+1-r).$$

Proof. By Theorem 2.16, there is a weak homotopy equivalence:

$$B\mathcal{C}^{U,r}_{2d+1} \to \Omega^{\infty+2d}\theta^*_{\mathbb{C},r}\mathbf{MTO}(2d+2).$$

Since $\theta^*_{\mathbb{C},r}\mathbf{MTO}(2d+2) \simeq \mathbf{MTU}(d+1)_{V_r} \simeq \mathbf{MTU}(d-r+1)$ as in the even case, the homotopy equivalence follows. □

Corollary 2.32. *There is an isomorphism*

$$\pi_0\left(B\mathcal{C}^{U,r}_{2d+1}\right) \cong \pi_{2d}(\mathbf{MTU}(d+1-r)).$$

This corollary implies that $\pi_{2d}(\mathbf{MTU}(d+1-r))$ is the even complex section cobordism group. Note that objects of the $2d$ dimensional complex section cobordism group are manifolds M^{2d} with complex structures and r linearly independent complex sections on $TM \oplus \mathbb{R}^2$. Proposition 2.18 shows that the conditions in [6, Theorem 3.5] are satisfied, namely that morphisms are reversible. Thus, looking instead at the morphisms of the even complex section cobordism group, the elements of the group $\pi_1(B\mathcal{C}^{\theta_{\mathbb{C},r}}_{2d})$ can be represented by $2d$ dimensional almost complex manifolds with r linearly independent complex tangent sections. By using the established homotopy equivalences in Corollary 2.24, we arrive at the following isomorphism:

Proposition 2.33. *There is an isomorphism*

$$\pi_1(B\mathcal{C}^{\theta_{\mathbb{C},r}}_{2d}) \cong \pi_{2d}\mathbf{MTU}(d-r).$$

Moreover every class in $\Omega^U_{2d,r}$, the even dimensional complex cobordism group, has a representative which is a $2d$ dimensional manifold with r linearly independent complex sections on the tangent bundle itself.

This group is the main object of study in the next section. There is a geometric description of this isomorphism. Suppose we have an (almost complex) manifold M^{2d} embedded in $\mathbb{R}^{2d+2n}$ equipped with r linearly independent complex sections. Then we get a Gauss map $M^{2d} \to G(2d, 2n)$ which lifts as below.

$$\begin{array}{ccc} & & V_{\mathbb{C},r}(U_{\mathbb{C},d,n}) \\ & \nearrow & \downarrow \\ & \nearrow & G_{\mathbb{C}}(d,n) \\ & & \downarrow \\ M^{2d} & \longrightarrow & G(2d,2n) \end{array}$$

If ν is the normal bundle of the embedding $M^{2d} \to \mathbb{R}^{2d+2n}$, then we can construct the following commutative square.

$$\begin{array}{ccccc} \nu & \longrightarrow & p^*_{V_{\mathbb{C},r}} U^{\perp}_{\mathbb{C},d,n} & \longrightarrow & U^{\perp}_{\mathbb{C},d,n} \\ \downarrow & & \downarrow & & \downarrow \\ M^{2d} & \longrightarrow & V_{\mathbb{C},r}(U_{\mathbb{C},d,n}) & \longrightarrow & G_{\mathbb{C}}(d,n) \end{array}$$

If we add one point to $\mathbb{R}^{2d+2n}$, we can consider an embedding $M^{2d} \to S^{2d+2n}$. A small tubular neighborhood of M is diffeomorphic to the total space of ν. If we collapse outside this tubular neighborhood we get the Pontryagin-Thom map $S^{2d+2n} \to \mathrm{Th}(\nu)$. Then, the above bundle maps give the composition

$$S^{2d+2n} \to \mathrm{Th}(\nu) \to \mathrm{Th}(p^*_{V_{\mathbb{C},r}}(U^{\perp}_{\mathbb{C},d,n})).$$

This composition represents $[M^{2d}] \in \pi_{2d}(\mathbf{MTU}_{V_r}(d)) \cong \pi_{2d}(\mathbf{MTU}(d-r)) \cong \Omega^U_{2d,r}$.

2.3 The cobordism obstruction

In the previous section, we reduced the study of the complex section cobordism groups to the study of the homotopy groups of $\mathbf{MTU}(d)$ using Corollaries 2.24 and 2.32. We had previously shown that there is a homomorphism:

$$\pi_q(\mathbf{MTU}(d)) \to \pi_q(\mathbf{MU}) \cong \Omega^U_q.$$

When $q \leq 2d$, we showed that this map is an isomorphism. When $q > 2d$, it is the forgetful map, which forgets the structure of the complex sections in $\pi_q(\mathbf{MTU}(d))$.

To understand complex section cobordism, it is sufficient to understand the kernel and image of this map. The kernel represents the possible distinct structures of linearly independent complex sections on a manifold. The image will be the set of almost complex manifolds admitting a structure of r complex sections. The rest of this section will focus on computing these groups using the long exact sequence in homotopy from cofibration of spectra in Definition 2.9. We consider the following segment:

$$\begin{gathered} \dots \to \pi_{2d+1}(\mathbf{MU}) \to \pi_{2d+1}(\overline{\mathbf{MTU}}(d-r)) \to \pi_{2d}(\mathbf{MTU}(d-r)) \to \\ \pi_{2d}(\mathbf{MU}) \xrightarrow{\gamma^r} \pi_{2d}(\overline{\mathbf{MTU}}(d-r)) \to \pi_{2d-1}(\mathbf{MTU}(d-r)) \to \pi_{2d-1}(\mathbf{MU}) \to \dots \end{gathered}$$

Theorem 1.27 tells us that $\pi_{2d+1}(\mathbf{MU}) \cong \Omega^U_{2d+1} = 0$ and $\pi_{2d-1}(\mathbf{MU}) \cong \Omega^U_{2d-1} = 0$, leaving a five term exact sequence. Moreover, we know that $\pi_{2d}(\mathbf{MU}) \cong \Omega^U_{2d}$ and Corollary 2.32 shows that $\pi_{2d}(\mathbf{MTU}(d-r))$ is $\Omega^U_{2d,r}$. So, the above exact sequence reduces to:

$$0 \to \pi_{2d+1}(\overline{\mathbf{MTU}}(d-r)) \to \Omega^U_{2d,r} \to \Omega^U_{2d} \xrightarrow{\gamma^r} \pi_{2d}(\overline{\mathbf{MTU}}(d-r)) \to \Omega^U_{2d-1,r} \to 0\ .$$

For the rest of this section let $i_{d,r}$ be the dimension of $H^{2d}(\mathbf{MTU}(d-r);\mathbb{Q})$ and $j_{d,r}$ be the dimension of $H^{2d}(\overline{\mathbf{MTU}}(d-r);\mathbb{Q})$. A classical result of homotopy theory states that $H^*(X;\mathbb{Q}) \cong \pi_*(X) \otimes \mathbb{Q}$ for a spectrum X of finite type, i.e. a spectrum whose homotopy groups are finitely generated in each degree. Thus, we determine the dimensions of the homotopy groups of $\mathbf{MTU}(d)$ and $\overline{\mathbf{MTU}}(d-r)$ by using Propositions 2.5 and 2.10. Specifically, $i_{d,r} = \dim(\pi_{2d}(\mathbf{MTU}(d-r))) = \dim(\mathrm{Ker}(\gamma^r))$. Since the dimension of $\pi_q(\overline{\mathbf{MTU}}(d-r)) \otimes \mathbb{Q}$ is 0 for odd q, then $\pi_q(\overline{\mathbf{MTU}}(d-r))$ is a finite group for q odd.

It follows from the above sequence that the map $\pi_{2d+1}(\overline{\mathbf{MTU}}(d-r)) \to \Omega^U_{2d,r}$ is an injection. So, the forgetful map $\Omega^U_{2d,r} \to \Omega^U_{2d}$ must have kernel $\pi_{2d+1}(\overline{\mathbf{MTU}}(d-r))$. Moreover, since Ω^U_{2d} is free abelian by Theorem 1.27, the kernel must be the entire torsion of $\Omega^U_{2d,r}$. Thus $\Omega^U_{2d,r}$ splits as $\pi_{2d+1}(\overline{\mathbf{MTU}}(d-r)) \oplus \mathbb{Z}^{\oplus i_{d,r}}$. This proves the following theorem.

Theorem 2.34. *There is an isomorphism*

$$\Omega^U_{2d,r} \cong \pi_{2d+1}(\overline{\mathbf{MTU}}(d-r)) \oplus \mathbb{Z}^{\oplus i_{d,r}}$$

where $i_{d,r}$ is the rank of $\mathbb{Z}[c_1, ..., c_{d-r}]$ in degree $2d$.

We conclude the following result:

Corollary 2.35. *Let* $[M^{2d}] \in \Omega^U_{2d}$*, be such that* M^{2d} *is equipped with* r *linearly independent complex sections on* TM*. Then, there are only finitely many pairs, up to complex section cobordism,* (N^{2d}, s) *where the manifold* $N^{2d} \in [M^{2d}]$ *and the map* $s : N \to V_{\mathbb{C},r}(TN)$ *is the structure of* r *linearly independent complex sections on* N^{2d}*. Moreover, such pairs are indexed by the group* $\pi_{2d+1}(\overline{\mathbf{MTU}}(d-r))$*.*

A similar result is also true for odd dimensional manifolds. The above corollary and following proposition provide the proofs of Theorems 1.38 and 1.39.

Proposition 2.36. *The odd dimensional complex section cobordism group is finite.*

Proof. Since $H^{2d+2r-1}(\mathbf{MTU}(d);\mathbb{Q}) = 0$, the vector space $\pi_{2d+2r-1}(\mathbf{MTU}(d)) \otimes \mathbb{Q}$ is zero. By definition, the homotopy group $\pi_{2d+2r-1}(\mathbf{MTU}(d))$ is $\Omega^U_{2d+2r-1,r}$. Thus $\Omega^U_{2d+2r-1,r}$ is finite. □

Let $\gamma^r : \Omega^U_{2d} \to \pi_{2d}(\overline{\mathbf{MTU}}(d-r))$ be the map induced by $\mathbf{MU} \to \overline{\mathbf{MTU}}(d-r)$. By the homotopy exact sequence, a manifold $M \in \Omega^U_{2d}$ lifts to $\Omega^U_{2d,r}$ if and only if $\gamma^r([M]) = 0$. We conclude that:

Theorem 2.37. *A manifold* M *is complex cobordant to a manifold equipped with* r *linearly independent complex sections on* TM *if and only if* $\gamma^r([M]) = 0$*.*

Importantly, the vanishing of $\gamma^r([M])$ is a sufficient, not just a necessary condition for finding r linearly independent complex sections. To describe the obstruction map γ^r, we use the characteristic classes s_ω. We already know by Theorem 1.13 that $s_{1,1,\dots,1}(M) = \chi(M)$ is the only obstruction to 1 complex section. We can now establish the obstruction, Theorem 1.37, in terms of the characteristic numbers for $r > 1$.

Theorem 2.38. *Let* M^{2d} *be a* d*-dimensional almost-complex manifold with* $r < d$*. Let*

$$\gamma^r_{\mathbb{Q}} := \gamma^r \otimes \mathbb{Q} : \Omega^U_{2d} \otimes \mathbb{Q} \to \pi_{2d}(\overline{\mathbf{MTU}}(d-r)) \otimes \mathbb{Q}.$$

Then $\gamma^r_{\mathbb{Q}}(M^{2d}) = 0$ *if and only if* $s_\omega(M^{2d}) = 0$ *for all* ω *of length greater than* $d - r$*.*

Corollary 2.39. *Let* M^{2d} *be a* $2d$*-dimensional almost-complex manifold. Then there exists non-zero* c *such that the cobordism class* $c[M^{2d}]$ *contains a manifold* N^{2d} *with* r *complex sections on* TN *if and only if* $s_\omega(M^{2d}) = 0$ *for all* ω *with* $l(\omega) \geq N$*.*

Proof of Theorem 2.38. We start by proving the forward direction, which is a consequence of known results. Suppose $\gamma_r(M^{2d}) = 0$. By Theorem 2.37, there is a cobordant manifold N^{2d} such that TN^{2d} has r linearly independent complex sections. Thus, TN^{2d} splits into $E \oplus \mathbb{C}^r$ where E has complex dimension $d - r$. Thus $c_k(TN^{2d}) = 0$ for $k \geq d - r + 1$. By Lemma 1.20, $s_\omega(N^{2d}) = 0$ for ω of length greater than $d - r$. Since cobordant manifolds have the same Chern numbers, $s_\omega(M^{2d}) = s_\omega(N^{2d}) = 0$.

For the other direction, we know $\gamma^r_{\mathbb{Q}} : \Omega^U_{2d} \otimes \mathbb{Q} \to \pi_{2d}(\overline{\mathbf{MTU}}(d - r)) \otimes \mathbb{Q} \cong \mathbb{Q}^{j_{d,r}}$ has rank $j_{d,r}$ by definition of $j_{d,r}$ and so is surjective. Each summand of this map can be written as a linear combination of s_ω by Theorem 1.29. Let $S' \subseteq \mathbf{Hom}(\Omega^U_{2d} \otimes \mathbb{Q}, \mathbb{Q})$ be the span of all s_ω such that $l(\omega) \geq d - r + 1$. Let $S \subseteq \Omega^U_{2d} \otimes \mathbb{Q}$ be the vector space of all elements x such that $r(x) = 0$ for every element $r \in S'$. Let $[M^{2d}] \notin S$. Then, for some ω with $l(\omega) \geq d - r + 1$, $s_\omega([M^{2d}]) \neq 0$. Moreover for any integer n, $s_\omega(n[M^{2d}]) \neq 0$. By Lemma 1.20, $c_k(n[M^{2d}]) \neq 0$ for some $k \geq d - r + 1$. Thus, no manifold in the class $n[M^{2d}]$ may have r sections for any integer n, because it has a non-zero characteristic class in dimension greater than $d-r$. By Theorem 2.37, every element $[M^{2d}]$ not in S must have $\gamma^r_{\mathbb{Q}}([M^{2d}]) \neq 0$. We conclude that $\mathrm{Ker}(\gamma^r_{\mathbb{Q}}) \subseteq S$.

It remains to show that the space S has dimension $i_{d,r}$. We recall that $i_{d,r}$ is the dimension of the vector space $H^*(\mathbf{MTU}(d - r); \mathbb{Q}) \cong \mathbb{Q}[c_1, ..., c_{d-r}]$ in degree $2d$. This dimension is equivalent to partitions of d by integers less than or equal to $d-r$. The dimension of S is the dimension of $\Omega^U_{2d} \otimes \mathbb{Q}$ minus the dimension of S'. Since the dimension of $\Omega^U_{2d} \otimes \mathbb{Q}$ is the number of partitions of d and the dimension of S' is the number of partitions of d with length greater than $d - r$, we conclude that the dimension of S is the number of partitions of length less than or equal to $d-r$. Thus, both S and $\mathrm{Ker}(\gamma^r_{\mathbb{Q}})$ have dimension $i_{d,r}$. Since $\mathrm{Ker}(\gamma^r_{\mathbb{Q}})$ is a finite dimensional vector space, $\mathrm{Ker}(\gamma^r_{\mathbb{Q}}) = S$. □

We now discuss the multiplicative structure of the complex cobordism ring. The following result is used to verify whether certain manifolds are multiplicative generators. [31, p. 128]

Theorem 2.40. *The complex cobordism ring Ω^U_* is isomorphic to the polynomial ring $\mathbb{Z}[b_1, b_2, ...]$ with generators b_i in dimension $2i$. If $i \neq p^q - 1$ for any prime p, then a manifold M^{2i} can be taken to be the generator if and only if $s_i([M^{2i}]) = \pm 1$ where s_i is the characteristic class corresponding to the trivial partition. If $i = p^q - 1$ for some prime p, then M^{2i} can be taken to be the generator if and only if $s_i([M^{2i}]) = \pm p$.*

As a corollary, when we look at the rational complex cobordism ring, the generators are identified by the following theorem. This corollary is well-known, however we were unable to find a satisfactory reference so it is reproved here.

Corollary 2.41. *The rational complex cobordism ring is* $\Omega^U \otimes \mathbb{Q} \cong \mathbb{Q}[b_1, b_2, ...]$ *with generators* b_i *in dimension* $2i$*. A manifold* $M^{2i} \in \Omega_{2i} \otimes \mathbb{Q}$ *can be taken to be the multiplicative generator if and only if* $s_i([M^{2i}]) \neq 0$*.*

Proof. The statement $\Omega^U \otimes \mathbb{Q} \cong \mathbb{Q}[b_1, b_2, ...]$ is an immediate consequence of Theorem 2.40. Let $k_i = 1$ if $i \neq p^q - 1$ for all primes p and $k_i = p$ if $i = p^q - 1$ for some prime p. Suppose M^{2i} is such that $s_i([M^{2i}]) = k \in \mathbb{Q}^\times$. Choose sufficiently large integer n such that k_i divides nk. Choose any $[\tilde{M}^{2i}]$ which is a representative of a generator in Ω^U. Since $s_i(\tilde{M}^{2i}) = k_i$, there exists m such that $s_i(m[\tilde{M}^{2i}] - n[M^{2i}]) = 0$. This manifold can be written as a polynomial in the generators $b_1, ..., b_i$. The only possible linear term is b_i for degree reasons, but this term must have coefficient 0, since s_i vanishes on all product terms by [22, Corollary 16.7]. So $m[\tilde{M}^{2i}] - n[M^{2i}]$ is decomposable into a sum of products. Thus $n[M^{2i}]$ is a generator, since it differs from a generator by decomposables, and $[M^{2i}]$ is also a rational generator. □

We finish by proving Theorem 1.44

Theorem 2.42. *Let* $r < d$*. There exists a manifold* M^{2d} *in* Ω^U_{2d} *which can be equipped with* r *linearly independent complex sections on* TM *and whose image in* $\Omega^U_{2d} \otimes \mathbb{Q}$ *is a multiplicative generator.*

Proof. Let $p(d)$ be the number of partitions of d. By [22, Theorem 16.7], the maps s_ω span the $p(d)$-dimensional vector space $\mathbf{Hom}_{\mathbb{Q}}(\Omega^U_{2d} \otimes \mathbb{Q}, \mathbb{Q})$. Since there are $p(d)$ characteristic numbers, they must also be linearly independent. Moreover, for every partition ω, we can find a dual object $M^{2d}_\omega \in \Omega^U_{2d} \otimes \mathbb{Q}$ such that $s_\omega(M^{2d}_\omega) = 1$ and all other characteristic numbers are 0. If we choose ω to be the partition d, then we get a rational generator M^{2d}_d by Corollary 2.41. We can choose integer c such that $M^{2d} := cM^{2d}_d \in \Omega^U_{2d}$. Then $s_\omega(M^{2d}) = 0$ for $\omega \neq d$ and Theorem 2.38 shows that some manifold $\tilde{N}^{2d} \in [M^{2d}]$ has r linearly independent complex sections. □

2.4 The Adams-Novikov Spectral Sequence

Overview of **MU** *and* **BP** *theory*

This section will be devoted to identifying which elements of $\pi_*(\mathbf{MU})$ can be lifted to $\pi_*(\mathbf{MTU}(d))$. We need to overview classical results of **MU** theory first. Recall that there is the Lazard ring [17] over which formal group laws are defined. This ring $L \cong \mathbb{Z}[x_1, x_2, ...]$ has a map to a ring $M \cong \mathbb{Z}[m_1, m_2, ...]$. (See [25, Theorem A2.1.10] for details.) There is a commutative square [25, Theorem 4.1.6] as follows:

$$\begin{array}{ccc} L & \longrightarrow & M \\ \downarrow & & \downarrow \\ \pi_*(\mathbf{MU}) & \longrightarrow & H_*(\mathbf{MU}) \end{array}$$

The vertical maps are isomorphisms and the bottom horizontal map is the Hurewicz homomorphism and is an injection. The Atiyah-Hirzebruch spectral sequence, Theorem 1.26, for $MU_*(\mathbf{MU})$ collapses and so $MU_*(\mathbf{MU}) \cong \pi_*(\mathbf{MU}) \otimes H_*(\mathbf{MU})$. We will use the following notations. Let $B_i \in MU_*(\mathbf{MU}) \cong \pi_*(\mathbf{MU}) \otimes H_*(\mathbf{MU})$, be the generators (as a $\pi_*(\mathbf{MU})$ algebra). These B_i correspond to the generator elements $b_i \in H_*(\mathbf{MU}) \cong \mathbb{Z}[b_1, b_2, ...]$. An explicit construction of the generators b_i and B_i is given in [27, p. 316]. By slight abuse of notation, we call the elements of $\pi_*(\mathbf{MU})$ by their image in $H_*(\mathbf{MU})$. We begin by stating the following results about **MU** theory originally developed by Novikov in [23] and described in detail in [25, Appendix 2.1].

Theorem 2.43. *The pair* $(\pi_*(\mathbf{MU}), MU_*(\mathbf{MU}))$ *is a Hopf algebroid with the following structures:*

- *An augmentation map* $\varepsilon : MU_*(\mathbf{MU}) \to \pi_*(\mathbf{MU})$*, defined by* $\varepsilon(B_i) = 0$.
- *A left unit* $\eta_L : \pi_*(\mathbf{MU}) \to MU_*(\mathbf{MU})$*, induced by the map from* $\Sigma^\infty S^0 \to \mathbf{MU}$ *and corresponds to the map* $\pi_*(\mathbf{MU}) \to \pi_*(\mathbf{MU}) \otimes H_*(\mathbf{MU})$.
- *A right unit,* $\eta_R : \pi_*(\mathbf{MU}) \to MU_*(\mathbf{MU})$*, corresponding to the Hurewicz homomorphism, defined by the formula from [27, p. 316]:*

$$\sum_i \eta_R(b_i) x^{i+1} = \sum_j B_j \left(\sum_k b_k^{i+1} \right)^{j+1}.$$

- *A product structure defined by the tensor product of rings* $\pi_*(\mathbf{MU}) \otimes H_*(\mathbf{MU})$.
- *A coproduct defined by:*

$$\sum_{i\geq 0} \Delta(B_i) = \sum_{j\geq 0} \left(\sum_{i\geq 0} B_i \right)^{j+1} \otimes B_j.$$

It is typically convenient to localize at a prime p. The structure of $\mathbf{MU}$ localized at p is described as follows. See [25, Appendix 2.1] for more details.

Theorem 2.44. *For each prime p, there is an associative ring spectrum* $\mathbf{BP}$ *which is a retract of* $\mathbf{MU}_{(p)}$*. Moreover,* $\mathbf{MU}_{(p)}$ *splits into copies of suspensions of the spectrum* $\mathbf{BP}$*. The homotopy ring* $\pi_*(\mathbf{BP})$ *is* $\mathbb{Z}_{(p)}[v_1, v_2, ...]$ *where* $v_i \in \pi_{2(p^i-1)}(\mathbf{BP})$*. The pair* $(\pi_*(\mathbf{BP}), BP_*(\mathbf{BP}))$ *is a Hopf algebroid. We will write* $BP_*(\mathbf{BP}) = \pi_*(\mathbf{BP})[t_1, t_2, ...]$ *where element* t_i *is in degree* $2(p^i - 1)$*. The algebroid has the structures:*

- *The generators* $l_i \in \pi_*(\mathbf{BP}) \otimes \mathbb{Q}$ *are the images of* $m_{p^i-1} \in \pi_*(\mathbf{MU}) \otimes \mathbb{Q}$.
- *The generators* v_i *are defined in [25, Appendix 2.2.1] by*

$$pl_n = \sum_{i=0}^{n-1} l_i v_{n-i}^{p^i}.$$

- *The augmentation map* $\varepsilon : BP_*(\mathbf{BP}) \to \pi_*(\mathbf{BP})$ *is defined on generators by* $\varepsilon(t_i) = 0$ *and* $\varepsilon(v_i) = v_i$.
- *The left unit* $\eta_L : \pi_*(\mathbf{BP}) \to BP_*(\mathbf{BP})$ *is induced by the map from* $\Sigma^\infty S^0 \to \mathbf{BP}$ *and corresponding to the map* $\pi_*(\mathbf{BP}) \to \pi_*(\mathbf{BP}) \otimes H_*(\mathbf{BP})$.
- *The right unit* $\eta_R : \pi_*(\mathbf{BP}) \to BP_*(\mathbf{BP})$*, corresponding to the Hurewicz homomorphism, is defined by the formula:*

$$\eta_R(l_i) = \sum_{j=0}^{n} l_j t_{i-j}^{p^j}.$$

- *The product structure is defined by the tensor product of rings* $\pi_*(\mathbf{BP}) \otimes H_*(\mathbf{BP})$.
- *The coproduct is determined by:*

$$\sum_{i,j\geq 0} l_i \Delta(t_j)^{p^i} = \sum_{i,j,k\geq 0} l_i t_j^{p^i} \otimes t_k^{p^{i+j}}.$$

We specifically note that $v_1 = pl_1$, $\eta_R(v_1) = v_1 + pt_1$, and $\Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1$.

Proposition 2.45. *Let X be a spectrum with torsion-free homology concentrated in even degrees. Let E be either* $\mathbf{MU}$ *or* $\mathbf{BP}$ *for some prime p. Then, there is an isomorphism $E_*(X) \cong \pi_*(E) \otimes_{\mathbb{Z}} H_*(X; \mathbb{Z})$, and $E_*(X)$ is torsion-free, and concentrated in even degrees.*

Proof. The Atiyah-Hirzebruch spectral sequence, Theorem 1.26, collapses for degree reasons, giving the stated result. □

Using Proposition 2.11 and 2.45, we immediately conclude:

Corollary 2.46. *Let D be either* $\mathbf{MU}$ *or* $\mathbf{BP}$ *for some prime p. The map*

$$D_*(\mathbf{MTU}(d)) \to D_*(\mathbf{MU}) = \pi_*(D)[B_1, B_2, ...]$$

is an injection, which is an isomorphism in degrees less than $2d + 2$. Moreover, both groups are concentrated in even degrees. $D_*(\mathbf{MTU}(d))$ is a $D_*(D)$ subcomodule of $D_*(\mathbf{MU})$. The image of $D_*(\mathbf{MTU}(d))$ maps onto polynomials of degree d or less.

$$D_*(\mathbf{MU}) \to D_*(\overline{\mathbf{MTU}}(d))$$

is the quotient map killing all polynomials of degree d or less. $D_*(\overline{\mathbf{MTU}}(d))$ is a quotient $D_*(D)$ comodule of $D_*(\mathbf{MU})$.

We will use the Adams-Novikov spectral sequence as our main tool to compute the cokernel of the map $\pi_*(\mathbf{MTU}(d)) \to \pi_*(\mathbf{MU})$. The p-local Adams-Novikov spectral sequence computes the homotopy groups of a finite type spectrum X as follows, see [23, Theorem 1.1] or [25, Theorem 4.4.1].

Theorem 2.47. *For a spectrum of finite type X, there is a spectral sequence $\{E_k^{s,t}(X), d_k\}$ with differentials $d_k : E_k^{s,t}(X) \to E_k^{s+k,t+k-1}(X)$ such that:*

1. *$E_2^{s,t}(X) \cong \mathbf{Ext}^{s,t}_{BP_*(\mathbf{BP})}(\pi_*(\mathbf{BP}), BP_*(X))$.*
2. *If X is connective and p local, there is a filtration*

$$\pi_{t-s}(X) = F^{0,t-s} \supseteq ... \supseteq F^{s,t} \supseteq F^{s+1,t+1} \supseteq ...$$

such that $E_\infty^{s,t}(X) = F^{s,t}/F^{s+1,t+1}$ and $\bigcap_{i=-s}^{\infty} F^{s+i,t+i} = 0$.

Moreover, this construction is natural with respect to maps of spectra. For an element in $E_q^{s,t}$, we will refer to s as the Adams-Novikov line of the element and t as the degree. Note that, any element in $\pi_{t-s}(X)$ defines an element of $E_\infty^{0,t-s}$. Moreover, the image of d_k in $E_*^{0,t-s}$ must be zero for degree reasons, so we may view the 0-line as subsets: $E_2^{0,t-s} \supseteq E_3^{0,t-s} \supseteq ... \supseteq E_\infty^{0,t-s}$.

Computations for $\mathbf{MTU}(d)$

Consider the Adams-Novikov spectral sequence for $\mathbf{MU}_{(p)}$. Since $\mathbf{MU}_{(p)}$ splits as a sum of suspensions of $\mathbf{BP}$, $E_2^{s,*}(\mathbf{MU}_{(p)}) = 0$ for $s > 0$. So the spectral sequence collapses, and tells us $E_2^{0,*}(\mathbf{MU}_{(p)}) \cong \pi_*(\mathbf{MU}_{(p)})$. In particular, it is a free $\mathbb{Z}_{(p)}$ module. Moreover, by basic properties of **Ext** groups,

$$E_2^{0,*}(\mathbf{MTU}(d)_{(p)}) \cong \mathbf{Hom}_{BP_*(\mathbf{BP})}(\pi_*(\mathbf{BP}), BP_*(\mathbf{MTU}(d)_{(p)}))$$

The image of $\pi_*(\mathbf{MTU}(d)_{(p)}) \to \pi_*(\mathbf{MU}_{(p)})$ is generated by elements from the 0 line. Thus, we first need to know the image of $E_2^{0,*}(\mathbf{MTU}(d)_{(p)}) \to E_2^{0,*}(\mathbf{MU}_{(p)})$. This is the map, induced by $\mathbf{MTU}(d)_{(p)} \to \mathbf{MU}_{(p)}$:

$$\mathbf{Hom}_{BP_*(\mathbf{BP})}(\pi_*(\mathbf{BP}), BP_*(\mathbf{MTU}(d)_{(p)})) \to \mathbf{Hom}_{BP_*(\mathbf{BP})}(\pi_*(\mathbf{BP}), BP_*(\mathbf{MU}_{(p)})).$$

Its cokernel is a free $\mathbb{Z}_{(p)}$ module because $BP_*(\mathbf{MTU}(d)_{(p)}) \to BP_*(\mathbf{MU}_{(p)})$ has free cokernel as a $\mathbb{Z}_{(p)}$ module. Since all differentials are zero in the Adams-Novikov spectral sequence for $\mathbf{MU}_{(p)}$, there is an isomorphism:

$$E_2^{0,*} \cong \mathbf{Hom}_{BP_*(\mathbf{BP})}(\pi_*(\mathbf{BP}), BP_*(\mathbf{MU}_{(p)})) \to \pi_*(\mathbf{MU}_{(p)}).$$

From now on localization at p will be assumed.

Proposition 2.48. *Fix prime p. Suppose $p[M] \in \pi_*(\mathbf{MU})$ lifts to $\pi_*(\mathbf{MTU}(d))$. Then $[M]$ lifts to an element $x \in E_2^{0,*}(\mathbf{MTU}(d))$ via the map:*

$$\mathbf{Hom}_{BP_*(\mathbf{BP})}(\pi_*(\mathbf{BP}), BP_*(\mathbf{MTU}(d)_{(p)})) \to \pi_*(\mathbf{MU}_{(p)}).$$

Moreover, $[M]$ lifts to $\pi_*(\mathbf{MTU}(d))$ if and only if there is no Adams-Novikov differential originating from x.

Proof. Let $[M] \in \pi_*(\mathbf{MU})$, where $p[M] \in \pi_*(\mathbf{MU})$ lifts to $\pi_*(\mathbf{MTU}(d))$. There is a unique lift of $[M]$ to an element $z \in E_2^{0,*}(\mathbf{MU})$. We know $p[M]$ lifts to an element in $[N] \in \pi_*(\mathbf{MTU}(d))$ which defines an element in $E_\infty^{0,*}(\mathbf{MU})$ which lifts to $y \in E_2^{0,*}(\mathbf{MTU}(d))$. Suppose there is another lift $[N'] \in \pi_*(\mathbf{MTU}(d))$ of $p[M]$. By Theorem 2.35, this element differs from $p[\mathbf{MU}]$ by torsion. However $E_\infty^{0,*}(\mathbf{MTU}(d))$ is free, so the difference between the elements comes from a higher filtration. Thus $[N]$ and $[N']$ define the same element of $y \in E_2^{0,*}(\mathbf{MTU}(d))$. By naturality of the Adams Novikov spectral sequence $y \mapsto pz$. The map $E_2^{0,*}(\mathbf{MTU}(d)) \to E_2^{0,*}(\mathbf{MU})$ has free cokernel so there must be an element $x \mapsto z$. (Otherwise z would generate p-torsion in the cokernel.) The element x is unique since $E_2^{0,*}(\mathbf{MTU}(d)) \to E_2^{0,*}(\mathbf{MU})$ is injective. The element x survives to the E_∞ page if and only if all Adams-Novikov differentials vanish on x. □

Thus, the differentials in the Adams-Novikov spectral sequence for $\mathbf{MTU}(d)$ correspond with torsion obstructions. We will use the cobar complex in order to compute the E_2 page. This construction is described in more detail in [25, Appendix 1]. If we wished to extend this computation further, it would be beneficial to use a more sophisticated method. Recall that $BP_*(\mathbf{BP})$ has an action of $\pi_*(\mathbf{BP})$ given by multiplication by the image of η_R. Let $\overline{\Gamma}$ be the cokernel of $\eta_L : \pi_*(\mathbf{BP}) \to BP_*(\mathbf{BP})$.

Definition 2.49 ([25, Definition A1.2.11]**.** Let (M, ψ) be a left $BP_*(\mathbf{BP})$ comodule. Define the cobar complex as the tensor product $C_{BP}^s(M) = \overline{\Gamma}^{\otimes s} \otimes_{\pi_*(\mathbf{BP})} M$. The differential in the complex is $d_1 : C_{BP}^s(M) \to C_{BP}^{s+1}(M)$, given by:

$$d_1(\gamma_1 \otimes ... \otimes \gamma_s \otimes m) = \tilde{d}_1(\gamma_1 \otimes ... \otimes \gamma_s) \otimes m - (-1)^s \gamma_1 \otimes ... \otimes \gamma_s \otimes \psi(m).$$

The differential $\tilde{d}_1 : \overline{\Gamma}^{\otimes s} \to \overline{\Gamma}^{\otimes s+1}$ is defined as:

$$\tilde{d}_1(\gamma_1 \otimes ... \otimes \gamma_s) = \sum_{i=1}^{s} (-1)^i \gamma_1 \otimes ... \otimes \Delta(\gamma_i) \otimes ... \otimes m.$$

Proposition 2.50 ([25, Corollary A1.2.12]**.** *The homology of the cobar complex is the Ext groups of M, specifically:*

$$H(C_{BP}^*(M)) = \mathbf{Ext}_{BP_*(\mathbf{BP})}(\pi_*(\mathbf{BP}), M).$$

It will typically be convenient to view the cobar complex as the E_1 page of the Adams-Novikov spectral sequence where the differential corresponds to d_1. We observe that the 0 line of the E_2 page of the Adams-Novikov spectral sequence is generated by primitives, i.e. elements of the form $\psi(x) = 1 \otimes x$ where ψ is the coaction. The following proposition will be used to compute $E_2(\mathbf{MTU}(d))$.

Proposition 2.51. *There is an isomorphism for $s \geq 1$:*

$$E_*^{s,t}(\overline{\mathbf{MTU}}(d)) \cong E_*^{s+1,t}(\mathbf{MTU}(d)).$$

Proof. By [25, Theorem 2.3.4], there is an exact sequence of spectral sequences:

$$\ldots \to E_*^{s,t}(\mathbf{MTU}(d)) \to E_*^{s,t}(\mathbf{MU}) \to E_*^{s,t}(\overline{\mathbf{MTU}}(d)) \to E_*^{s+1,t}(\mathbf{MTU}(d)) \to \ldots$$

Since $E_*^{s,t}(\mathbf{MU})$ for $s > 0$, the result follows. □

We summarize the different conditions that we use to prove there are no p-torsion obstructions.

Proposition 2.52. *There are no p-torsion obstructions to the existence of r linearly independent complex sections on a $2d$ dimensional manifold if any of the following are true. Here $E_*^{s,t}(X)$ is the p-primary Adams-Novikov spectral sequence for spectrum X.*

1. *The differential $d_s : E_s^{0,2d-2r}(\mathbf{MTU}(d-r)) \to E_{s+1}^{s,2d-2r+q-1}(\mathbf{MTU}(d-r))$ is zero for all $s \geq 3$.*
2. *For some q, the group $E_q^{s,2d-2r+s-1}(\mathbf{MTU}(d-r))$ is 0 for all $s \geq 3$.*
3. *For some q, the group $E_q^{s,2d-2r+s-1}(\overline{\mathbf{MTU}}(d-r))$ is 0 for all $s \geq 2$.*
4. *The group $\pi_{2d}(\overline{\mathbf{MTU}}(d-r))$ has no p-torsion.*

Proof. The first condition is an immediate consequence of Proposition 2.48. The second condition immediately implies the first. The second and third conditions are equivalent by Proposition 2.51. The final condition follows because, by constuction, $\mathrm{Ob}_{2d,r} \subseteq \pi_{2d}(\overline{\mathbf{MTU}}(d-r))$ from the discussion preceding Theorem 2.37. □

Remark 2.53. Note that only condition 1 of the previous proposition is a necessary condition for the vanishing of p-torsion obstructions. The others are merely sufficient conditions.

We now prove Theorem 1.41.

Theorem 2.54. *The p-torsion obstruction for $p > 2$ to the existence of r linearly independent complex sections vanishes when $r < p^2 - p$.*

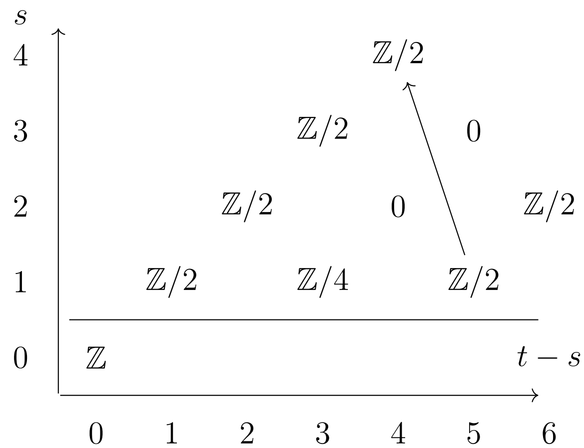
Proof. Fix prime $p > 2$. Consider the cobar complex for $\overline{\mathbf{MTU}}(d)$. Choose a $\pi_*(\mathbf{BP})$ basis $\{x_i\}$ for $BP_*(\overline{\mathbf{MTU}}(d))$ where the degrees of $\{x_i\}$ are bounded below. (Corollary 2.46 shows there is such a basis.) Suppose we have some element $\beta \in E_2^{2,t}(\overline{\mathbf{MTU}}(d))$ for $t < 2(p^2 - p + d + 1)$. This element is represented by the tensor sum $\sum_k \mu_k \otimes x_k \in \overline{\Gamma}^{\otimes 2} \otimes BP_*(\overline{\mathbf{MTU}}(d))$ such that the element sum $\sum_k \mu_k \otimes x_k$ is in $\mathrm{Ker}(d_1)$. Since $\{x_i\}$ is a basis, we may assume that each term has a unique x_k. Let $\{x_i \mid i \in J\}$ be the basis factors from $\sum_k \mu_k \otimes x_k$ where x_i has maximal degree n. We will induct down on n to show this element is zero.

The sum $d_1(\sum_k \mu_k \otimes x_k)$, has maximal degree elements $\sum_{i\in J} \tilde{d}_1(\mu_i) \otimes x_i$. Note that $\psi(x_i)$ is equivalent to $1 \otimes x_i$ modulo terms of lower degree in the second factor, $BP_*(\overline{\mathbf{MTU}}(d))$. So, the ψ term only contributes lower degree terms (in the second factor) because $\mu_i \otimes 1 \otimes x_i = 0 \in \overline{\Gamma}^{\otimes 3} \otimes BP_*(\overline{\mathbf{MTU}}(d))$. Since the term of maximal degree $\sum_{i\in J} \tilde{d}_1(\mu_i) \otimes x_i$ must vanish, $\tilde{d}_1(\mu_i)$ must be 0 for all $i \in J$, because we chose $\{x_k\}$ to be a basis. We know μ_i has degree less than $2(p^2 - p)$ because x_i has degree greater than or equal to $2d + 2$. The element $\mu_i \otimes 1$ can be considered as an element in the cobar complex for the sphere spectrum. Call the differential in this complex d_1'. Then, by Definition 2.49 the differential $d_1'(\mu_i \otimes 1)$ evaluates to $\tilde{d}_1(\mu_i) \otimes 1 = 0$. So, $\mu_i \otimes 1$ is in $\mathrm{Ker}(d_1')$ and it represents an element in $E_2(\Sigma^\infty S^0)$. By [35, Theorem 7.1], this element must be zero. Thus $\mu_i \otimes 1$ is in the image of $\sum_j \tilde{\mu}_j \otimes \nu_j$ where $\tilde{\mu}_j \in \overline{\Gamma}^{\otimes 1}$ and $\nu_j \in \pi_*(BP)$.

So, $d_1(\sum_j \tilde{\mu}_j \otimes \nu_j x_i)$ is equivalent to $\mu_i \otimes x_i$ up to elements of strictly lower degree in the second factor. (These elements come from the ψ part of the differential.) Thus, for every $i \in J$, the element $\mu_i \otimes x_i$ can be replaced by a sum of elements with degree less than n. So β can be represented by a new sum $\sum_{k'} \mu_{k'} \otimes x_{k'}$ where the maximal degree of $x_{k'}$ is strictly less than n. By induction on n, β can be represented by 0, and thus $\beta = 0$. (We choose the letter β suggestively, as the first potentially non zero β corresponds to β_1 constructed in [25, Definition 1.3.14]). Thus, $E_2^{2,t}(\overline{\mathbf{MTU}}) = 0$ for $t < 2(p^2 - p + d + 1)$. A similar argument shows that $E_2^{s,t}(\overline{\mathbf{MTU}}) = 0$ for $t < 2(p^2 - p + d + s - 1)$ and $s \geq 2$. By Proposition 2.52, the p-torsion obstruction to $r < p^2 - p$ linearly independent complex sections vanishes. □

Remark 2.55. This result is not sharp, and we can improve it significantly by closer analysis. However, this bound is sufficient for this paper, since it will show the vanishing of all $p > 2$ torsion obstructions for $r < 6$.

We now focus on the case $p = 2$. For the rest of this subsection all homotopy groups are assumed to be 2-primary. We will need to compute the homotopy groups of $\overline{\mathbf{MTU}}(d)$. To do so we remind ourselves of the Adams-Novikov spectral sequence for the sphere spectrum: [35, Figure 1]



In particular, we will call the generators of $\pi_1^s(S^0)$ and $\pi_3^s(S^0)$, h_1 and h_2 respectively. These can be represented by the Hopf maps.

Theorem 2.56. *The 2-primary homotopy groups of* $\overline{\mathbf{MTU}}(d)$ *for* $d \geq 3$ *are:*

q	$2d+2$	$2d+3$	$2d+4$	$2d+5$	$2d+6$
$d \equiv 0 \pmod 2$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}^2$	?	$\mathbb{Z}^4$
$d \equiv 1 \pmod 2$	$\mathbb{Z}$	0	$\mathbb{Z}^2$	0	$\mathbb{Z}^4$

Proof. The coaction $\psi : BP_*(\overline{\mathbf{MTU}}(d)) \to BP_*(\mathbf{MU}) \otimes_{\pi_*(\mathbf{BP})} BP_*(\overline{\mathbf{MTU}}(d))$ is the natural quotient of the coaction in $BP_*(\mathbf{MU})$ induced by the map $\mathbf{MU} \to \overline{\mathbf{MTU}}(d)$. Note that the right action of $\pi_*(\mathbf{BP})$ on $BP_*(\mathbf{MU})$ is given by multiplication by the image of the Hurewicz homomorphism η_R. We compute that $\eta_R(v_1) = v_1 + 2t_1$. (See [25, Appendix 2.2].)

We start by computing the following coactions in $BP_*(\mathbf{MU})$.

$$\psi(B_1) = 1 \otimes B_1 + t_1 \otimes 1$$
$$\psi(B_2) = 1 \otimes B_2 + (2t_1) \otimes B_1 + t_1^2 \otimes 1$$

We can then compute in $BP_*(\overline{\mathbf{MTU}}(d))$ using Corollary 2.46:

$$\psi(B_1^{d+1}) = 1 \otimes B_1^{d+1},$$
$$\psi(B_2 B_1^d) = 2t_1 \otimes B_1^{d+1} + 1 \otimes B_2 B_1^d,$$

$$\psi(B_1^{d+2}) = 1 \otimes B_1^{d+2} + (d+2)t_1 \otimes B_1^{d+1},$$
$$\psi(v_1 B_1^{d+1}) = v_1 \otimes B_1^{d+1} = 1 \otimes v_1 B_1^{d+1} - 2t_1 \otimes B_1^{d+1}.$$

We have two cases, if d is even, then the cokernel of $d_1 : E_1^{0,2d+4} \to E_1^{1,2d+4}$ is $\mathbb{Z}/2$ generated by $t_1 \otimes B_1^{d+1}$ and if d is odd, the cokernel is zero. Next we compute the cokernel of $d_1 : E_1^{1,2d+6} \to E_1^{2,2d+6}$.

$$d_1(t_1 \otimes B_2 B_1^d) = 2t_1 \otimes t_1 \otimes B_1^{d+1} \qquad d_1(t_1 \otimes B_1^{d+2}) = (d+2)t_1 \otimes t_1 \otimes B_1^{d+1}$$
$$d_1(t_1 \otimes v_1 B_1^{d+1}) = -2t_1 \otimes t_1 \otimes B_1^{d+1} \qquad d_1(t_1^2 \otimes B_1^{d+1}) = -2t_1 \otimes t_1 \otimes B_1^{d+1}$$

Once again, if d is even, then the cokernel of $d_1 : E_1^{1,2d+6} \to E_1^{2,2d+6}$ is $\mathbb{Z}/2$, and if d is odd, the cokernel is zero. For similar reasons, $d_1 : E_1^{2,2d+8} \to E_1^{3,2d+8}$ has cokernels $\mathbb{Z}/2$ or 0 when d is even or odd respectively. We now compute the image of the map $d_1 : E_1^{0,2d+6} \to E_1^{1,2d+6}$. We will write this as a table giving of coefficients in the basis $t_1 \otimes B_2 B_1^d, t_1 \otimes B_1^{d+2}, t_1 \otimes v_1 B_1^{d+1}, t_1^2 \otimes B_1^{d+1}$.

$B_3 B_1^d$	3	0	-2	5
$B_2^2 B_1^{d-1}$	4	0	0	4
$B_2 B_1^{d+1}$	$(d+1)$	2	0	$2d+3$
B_1^{d+3}	0	$(d+3)$	0	$\frac{1}{2}(d+3)(d+2)$
$v_1 B_2 B_1^d$	-2	0	2	-4
$v_1 B_1^{d+2}$	0	-2	$(d+2)$	$2(d+2)$
$v_1^2 B_1^{d+1}$	0	0	-4	4

The kernel of $d_1 : E_1^{0,2d+6} \to E_1^{1,2d+6}$ is generated by elements:

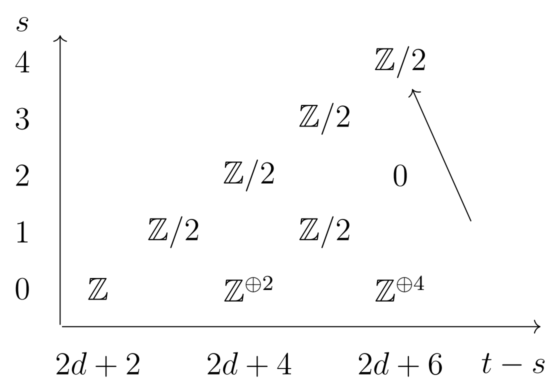
$$a_1 t_1 \otimes B_2 B_1^d + a_2 t_1 \otimes B_1^{d+2} + a_3 t_1 \otimes v_1 B_1^{d+1} + a_4 t_1^2 \otimes B_1^{d+1}$$

where $2a_1 + (d+2)a_2 - 2(a_3 + a_4) = 0$. Thus, we have two cases for $E_2^{1,2d+4}$: if d is even, $E_2^{1,2d+4} = \mathbb{Z}/2$ and if d is odd, $E_2^{1,2d+4} = 0$. Similarly we can compute the image of $d_1 : E_1^{1,2d+8} \to E_1^{2,2d+8}$ in the basis:

$$t_1 \otimes t_1 \otimes B_2 B_1^d, t_1 \otimes t_1 \otimes B_1^{d+2}, t_1 \otimes t_1 \otimes v_1 B_1^{d+1}, t_1 \otimes t_1^2 \otimes B_1^{d+1}, t_1^2 \otimes t_1 \otimes B_1^{d+1}.$$

$t_1 \otimes B_3 B_1^d$	3	0	-2	5	0
$t_1 \otimes B_2^2 B_1^{d-1}$	4	0	0	4	0
$t_1 \otimes B_2 B_1^{d+1}$	$(d+1)$	2	0	$2d+3$	0
$t_1 \otimes B_1^{d+3}$	0	$(d+3)$	0	$\frac{1}{2}(d+3)(d+2)$	0
$t_1 \otimes v_1 B_2 B_1^d$	-2	0	2	-4	0
$t_1 \otimes v_1 B_1^{d+2}$	0	-2	$(d+2)$	$2(d+2)$	0
$t_1 \otimes v_1^2 B_1^{d+1}$	0	0	-4	4	0
$t_1^2 \otimes B_2 B_1^d$	-2	0	0	0	2
$t_1^2 \otimes B_1^{d+2}$	0	-2	0	0	$(d+2)$
$t_1^2 \otimes v_1 B_1^{d+1}$	0	0	-2	0	-2
$t_2 \otimes B_1^{d+1}$	0	0	-1	3	2
$t_1^3 \otimes B_1^{d+1}$	0	0	0	3	3

We find that $E_2^{2,2d+6}$ is always zero, and the Adams-Novikov chart for d even is:



The non-trivial d_3 differential can be observed by looking at the map of spectra $\Sigma^\infty S^{2d} \to \overline{\mathbf{MTU}}(d)$ given by the generator of $\pi_{2d}(\overline{\mathbf{MTU}}(d))$. There may be an additional differential killing $E_2^{2,2d+8}$ but since this element is in odd degree, we will not need to know it. The Adams-Novikov spectral sequence for d odd is trivial in this range. □

Remark 2.57. The vanishing of the torsion obstruction to r complex sections does not imply the vanishing of the torsion obstruction to $r-1$ complex sections, thus each result must be proved separately.

We use these Adams-Novikov charts and homotopy groups to prove:

Theorem 2.58. *There is no torsion obstruction to the existence of 2 or 3 linearly independent complex sections on M^{2d} for $d \geq 6$.*

Proof. Since $\pi_{2d}(\overline{\mathbf{MTU}}(d-3))$ is free abelian, there is no possible target for a torsion obstruction to 3 complex sections by Proposition 2.52. Similarly, there is no possible target for a torsion obstruction to 2 complex sections when d is odd, because $\pi_{2d}(\overline{\mathbf{MTU}}(d-2))$ is free abelian. We recall from [15, Theorem 2.2] that $\Sigma^\infty S^{2(d-2)}$ splits off $\mathbf{MTU}(d-2)$ for d even. We conclude that the non-zero element in $E_2^{3,2d+2}(\mathbf{MTU}(d-2)) \cong E_2^{2,2d+2}\overline{\mathbf{MTU}}(d-2))$ comes from this split $\Sigma^\infty S^{2(d-2)}$ (specifically, it can be identified with h_1^3), and so cannot be the target of a differential. □

Theorem 2.59. *There is no torsion obstruction to* 4 *linearly independent complex sections when* $d > 6$ *is odd.*

Proof. Let $M \in \pi_{2d}(\mathbf{MU})$ with vanishing rational obstruction. Recall that the image of M in the group $\pi_{2d}(\overline{\mathbf{MTU}}(d-4))$ is the obstruction to 4 complex sections. Thus the image is torsion. By Theorem 2.58, there is no torsion obstruction to 3 complex sections so the image of M in $\pi_{2d}(\overline{\mathbf{MTU}}(d-3))$ is zero. Thus the obstruction to 4 complex sections lifts to $\pi_{2d}(\mathbf{MTU}(d-3,1)) \cong \pi_{2d}^s(S^{2d-6}) \oplus \pi_6^s(BU(d-3))$.

$$
\begin{array}{ccccc}
 & & & & \pi_{2d}(\mathbf{MTU}(d-3,1)) \\
 & & & & \downarrow \\
\pi_{2d}(\mathbf{MTU}(d-4)) & \longrightarrow & \pi_{2d}(\mathbf{MU}) & \longrightarrow & \pi_{2d}(\overline{\mathbf{MTU}}(d-4)) \\
\downarrow & & | & & \downarrow \\
\pi_{2d}(\mathbf{MTU}(d-3)) & \longrightarrow & \pi_{2d}(\mathbf{MU}) & \longrightarrow & \pi_{2d}(\overline{\mathbf{MTU}}(d-3))
\end{array}
$$

For $d > 6$, there is an isomorphism $\pi_6^s(BU(d-3)) \to \pi_6^s(BU)$ and the group $\pi_6^s(BU)$ is torsion free by [28, Theorem 4.2]. The torsion element in $\pi_6^s(S^0)$ is $h_2 \cdot h_2$. The map $\pi_{2d}^s(S^{2d-6}) \to \pi_{2d}(\mathbf{MTU}(d-3))$ is a $\pi_*^s(S^0)$ module map. In particular, it must take h_2 to 0 if d is odd and we conclude that $h_2 \cdot h_2 \mapsto h_2 \cdot 0 = 0$. Thus the torsion obstruction to M having 4 linearly independent complex sections vanishes. □

Remark 2.60. For d even, there is a potential obstruction of order 2. We expect that this obstruction is non-vanishing and, based on similar results in [3, Corollary 6.6] and [5, Theorem 1.2], we expect that the obstruction is a divisibility condition on the signature of the manifold.

The calculations of Theorem 2.56 also allow us to enumerate the number of distinct ways to equip a manifold with 1 complex section.

Corollary 2.61. *Suppose that almost complex manifold M^{2d} has Euler characteristic zero. If d is odd, then there are two manifolds $N^{2d}, \tilde{N}^{2d} \in [M^{2d}]$ with 1 complex tangent section which are not complex section cobordant. If d is even, then there is a unique way to equip a cobordant manifold N^{2d} with 1 complex tangent section.*

Proof. By Theorem 1.1, the obstruction to finding a complex section on a manifold in $[M^{2d}]$ vanishes. The result follows immediately from Theorems 2.56 and 2.34. □

We finish this section by briefly discussing the odd complex section cobordism group. Consider an odd dimensional manifold M, with complex structure J on $TM \oplus \mathbb{R}$. The bundle splits off a trivial complex bundle $\mathbb{R} \oplus J\mathbb{R}$, thus it must always have one everywhere non-zero complex section. All stably complex odd dimensional manifolds are cobordant, however we can show that they are not necessarily complex section cobordant.

Theorem 2.62. *The odd complex section cobordism groups can be computed as:*

- $\Omega^U_{2d+1,1} \cong 0$ *if d is odd.*
- $\Omega^U_{2d+1,1} \cong \mathbb{Z}/2$ *if d is even.*

Proof. Recall the homotopy exact sequence:

$$\pi_{2d+2}(\mathbf{MU}) \to \pi_{2d+2}(\overline{\mathbf{MTU}}(d)) \to \pi_{2d+1}(\mathbf{MTU}(d)) \to \pi_{2d+1}(\mathbf{MU}).$$

Substituting the known groups from Theorem 2.56 and Lemma 2.14, we get a short exact sequence:

$$\Omega^U_{2d+2} \xrightarrow{\chi} \mathbb{Z} \to \Omega^U_{2d+1,1} \to 0.$$

The left most map is the Euler characteristic (see [5, Theorem 2.17, Remark 2.18]). If d is odd, $(S^2)^{d+1}$ has an almost complex structure with Euler characteristic 2^{d+1} and $\mathbb{C}P^{d+1}$ has Euler characteristic $d+2$ which is odd. The $\gcd(2^{d+1}, d+2) = 1$ and thus χ is surjective.

On the other hand, if d is even, the Euler characteristic is even. The almost complex manifolds $S^2 \times \mathbb{C}P^d$ and $\mathbb{C}P^{d+1}$ have Euler characteristics of $2d+2$ and $d+2$. Since $\gcd(2d+2, d+2) = 2$, the cokernel of χ is $\mathbb{Z}/2$. □

This theorem is the complex analogue to Theorem 1.31 on Reinhart cobordism groups.

We can give a bound on the possible torsion obstruction to the existence of r linearly independent complex sections. Recall the following theorem of Segal:

Theorem 2.63 ([27, Corollary 2.4]. *There are no non-zero differentials originating in the bottom row of the Adams-Novikov spectral sequence for* $\mathbf{MTU}(1) \simeq \Sigma^{\infty-2}\mathbb{C}P^{\infty}$.

Corollary 2.64. *There is no torsion obstruction to the existence of* $d-1$ *sections on* $[M^{2d}]$.

In particular, we can find an element of $N^{2d} \in \pi_{2d}(\mathbf{MTU}(1))$ with all characteristic classes zero except $s_d(N^{2d}) = (d+1)!/B_{2d}$ where B_{2d} is the Bernoulli number [30, Theorem 1]. These numbers are known to play a role in the theory of characteristic classes cf. [1, Theorem 1]. From Milnor's characterization of the generators of the complex cobordism ring, Theorem 2.40, the next proposition immediately follows.

Proposition 2.65. *For every* d*, define*

$$a_d = \begin{cases} (d+1)!/B_{2d} & d \neq p^i - 1 \\ (d+1)!/(pB_{2d}) & d = p^i - 1 \end{cases}$$

Then for any set of multiplicative generators $M^{2d} \in \Omega^U_{2d}$*, we can write the cobordism class as* $N^{2d} = \tilde{a}_d M^{2d} + \text{decomposables}$.

For any partition $I = \{i_1, ..., i_n\}$, define $a_I = \tilde{a}_{i_1}...\tilde{a}_{i_k}$. Let $M^{2i} \in \Omega^U_{2i}$ be any collection of multiplicative generators and write $M^I = M^{2i_1} \times ... \times M^{2i_k}$. Define a partial order on partitions of d by

$$\{i_1, ..., i_k\} > \{j_{1,1}, ..., j_{1,l_1}, j_{2,1}, ..., j_{2,l_1}, ..., j_{k,1}, ..., j_{k,l_k}\}$$

if $i_n = \sum_{m=1}^{l_m} j_{n,m}$.

Theorem 2.66. *Let* $K^{2d} = M^I + \sum_{J<I} b_J M^J$ *be a manifold with vanishing rational obstruction to* r *linearly independent complex sections. Then* $\text{lcm}(a_J \mid J < I)[K^{2d}]$ *contains a manifold which can be equipped with* r *linearly independent complex sections.*

Proof. Call $C = \text{lcm}(a_J \mid J < I)$. By Proposition 2.65, we can write

$$CK^{2d} = \frac{C}{a_J} N^I + C \sum_{J<I} b'_J M^J$$

for new constants b'_J. Choose maximal J with $b'_J \neq 0$. Then we can rewrite CM^J as $\frac{C}{a_J} \sum_{L<J} N^J$. By inducting down on the partial order, we can write

$$CK^{2d} = c_I N^I + \sum_{J<I} c_J N^J$$

for some integers c_J. Note that N^J has $s_J(N^J) \neq 0$ and all other characteristic classes equal to 0. So, in this sum, c_J must be zero for all J of length greater than or equal to $d - r + 1$ because we are assuming the rational obstruction vanishes. Note that by construction N^{2i} has $i - 1$ linearly independent complex sections, so N^J has $d - l(J) \geq r$ linearly independent complex sections. Thus $a_I K^{2d}$ also can be equipped with r linearly independent complex sections. □

Remark 2.67. We expect that the Bernoulli numbers are measuring unexpected obstructions to complex sections, by comparing the formula in Proposition 2.65 with the formula for the primitives in [27, p. 317]. Thus, we conjecture that they relate to the presence of non trivial differentials.

This result is not sharp, but it allows us to construct cobordism classes containing a manifold with r linearly independent complex sections for any r.

APPENDIX A

THE SPECTRUM $\widehat{\mathbf{MTU}}$

Unlike **MU**, the spectrum $\mathbf{MTU}(d)$ is not a ring spectrum. Observe that a product $\pi_{2(d+r)}(\mathbf{MTU}(d))\otimes\pi_{2(d+s)}(\mathbf{MTU}(d)) \to \pi_{2(2d+r+s)}(\mathbf{MTU}(d))$, if it existed, would be a map taking a $2(d+r)$ and a $2(d+s)$ dimensional manifold with r and s complex sections respectively, to a $2(2d+r+s)$ dimensional manifold with $d+r+s$ complex sections. There is no geometric reason why we should gain an additional d sections, and there is no natural way to define such a product. However, there is a natural product structure on almost complex manifolds equipped with complex sections. If M and N are two almost complex manifolds with r and s linearly independent complex sections respectively, then $M \times N$ is equipped with $r+s$ complex sections.

Proposition A.1. *For all d and d', there is a canonical map:*

$$\mathbf{MTU}(d) \wedge \mathbf{MTU}(d') \to \mathbf{MTU}(d+d').$$

This map induces the natural product structures:

$$\Omega_{2d,r} \otimes \Omega_{2d',s} \to \Omega_{2(d+d'),r+s},$$

$$\Omega_{2d+1,r} \otimes \Omega_{2d',s} \to \Omega_{2(d+d')+1,r+s},$$

$$\Omega_{2d,r} \otimes \Omega_{2d'+1,s} \to \Omega_{2(d+d')+1,r+s},$$

defined by the cartesian product of representatives of cobordism classes.

Proof. There is a canonical bundle map:

$$\begin{array}{ccc} U^{\perp}_{\mathbb{C},d,n} \times U^{\perp}_{\mathbb{C},d',n'} & \longrightarrow & U^{\perp}_{\mathbb{C},d+d',n+n'} \\ \downarrow & & \downarrow \\ G_{\mathbb{C}}(d,n) \times G_{\mathbb{C}}(d',n') & \longrightarrow & G_{\mathbb{C}}(d+d',n+n') \end{array}$$

which takes two subspaces of $\mathbb{C}^{d+n}$ and $\mathbb{C}^{d'+n'}$ to their span in $\mathbb{C}^{d+d'+n+n'}$ and takes two vectors in the orthogonal complement to their direct sum. This map commutes with the structure map and defines the map of spectra as $n, n' \to \infty$. Moreover, this map commutes with the map $G_{\mathbb{C}}(d,n) \to G_{\mathbb{C}}(d+1,n)$. So there is a commutative diagram of sequences. The bottom map is the product map in **MU** and the vertical maps are the forgetful maps in homotopy.

$$\begin{array}{ccc}
\mathbf{MTU}(d) \wedge \mathbf{MTU}(d') & \longrightarrow & \mathbf{MTU}(d+d') \\
\downarrow & & \downarrow \\
\mathbf{MTU}(d+1) \wedge \mathbf{MTU}(d'+1) & \longrightarrow & \mathbf{MTU}(d+d'+2) \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathbf{MU} \wedge \mathbf{MU} & \longrightarrow & \mathbf{MU}
\end{array}$$

Thus the product in complex section cobordism is induced by the cartesian product of cobordism classes. □

The product of two odd dimensional complex section cobordism groups is slightly different because of the indexing.

Proposition A.2. *There is a natural product:*

$$\Omega_{2d-1,r} \otimes \Omega_{2d'-1,s} \to \Omega_{2(d+d'-1),r+s-1}$$

Proof. From Proposition A.1, there is a bilinear map:

$$\pi_{2d-1}(\mathbf{MTU}(d-r)) \otimes \pi_{2d'-1}(\mathbf{MTU}(d'-s)) \to \pi_{2(d+d'-1)}(\mathbf{MTU}(d+d'-r-s)).$$

The product follows immediately from the definitions. □

Remark A.3. We lose one complex section, because, by definition, the odd complex section cobordism groups consist of manifolds M with sections on $TM \oplus \mathbb{R}$. Thus the product of two cobordism classes, is represented by a manifold $M \times N$ with sections on $T(M \times N) \oplus \mathbb{R}^2$. By Proposition 2.33, this product manifold can be represented by a manifold with 1 fewer section on the tangent bundle itself.

If all the complex section cobordism groups are put together they define a ring under this product. This ring is from the homotopy of a new ring spectrum $\widehat{\mathbf{MTU}}$.

Definition A.4. The spectrum $\widehat{\mathbf{MTU}}$ is defined as:

$$\widehat{\mathbf{MTU}} = \mathbf{MTU}(0) \vee \mathbf{MTU}(1) \vee \mathbf{MTU}(2) \vee ...$$

The product on $\widehat{\mathbf{MTU}}$ is defined by the maps in Proposition A.1.

Proposition A.5. *The split summand of $\pi_*(\widehat{\mathbf{MTU}})$ defined as $\bigoplus_{q \geq 2d} \pi_q(\mathbf{MTU}(d))$ is a bigraded subring and equivalent to $\Omega^U_{*,*}$.*

Proof. This proposition is an immediate consequence of the definitions. □

We observe that $\pi_*(\widehat{\mathbf{MTU}})$ contains many redundant groups. For any d, it contains infinite copies of Ω^U_{2d} from $\pi_{2d}(\mathbf{MTU}(d+n))$ for any $n \geq 0$. The spectrum $\mathbf{MTU}(0)$ is homotopy equivalent to $\Sigma^\infty S^0$. The stable homotopy groups of spheres correspond to the framed cobordism groups, [9, p. 92] and the multiplication corresponds to multiplication by a framed manifold. There is a map, defined as the wedge sum of the maps $\mathbf{MTU}(d) \to \mathbf{MU}$:

$$\widehat{\mathbf{MTU}} \to \mathbf{MU},$$

which induces a map of rings:

$$\pi_*(\widehat{\mathbf{MTU}}) \to \Omega^U_*.$$

Geometrically, this map is the forgetful map and it makes $\pi_*(\widehat{\mathbf{MTU}})$ into a Ω^U_* module. We can define a self map,

$$\Psi : \widehat{\mathbf{MTU}} \to \widehat{\mathbf{MTU}},$$

by taking the wedge sum of the maps $\mathbf{MTU}(d) \to \mathbf{MTU}(d+1)$ for all d. By composing Ψ with itself, there is a sequence:

$$\pi_*(\widehat{\mathbf{MTU}}) \to \Psi(\pi_*(\widehat{\mathbf{MTU}})) \to \Psi^2(\pi_*(\widehat{\mathbf{MTU}})) \to ...$$

Each composition forgets one complex section, and the limit of this sequence is $\pi_*(\mathbf{MU}) \cong \Omega^U_*$. By slight abuse of notation, we call the map $\widehat{\mathbf{MTU}} \to \mathbf{MU}$, Ψ^∞ and the following diagram commutes:

$$\begin{array}{ccc} \widehat{\mathbf{MTU}} & \xrightarrow{\Psi} & \widehat{\mathbf{MTU}} \\ & {\scriptstyle\Psi^\infty}\searrow \quad \swarrow{\scriptstyle\Psi^\infty} & \\ & \mathbf{MU} & \end{array}$$

The homology of $\widehat{\mathbf{MTU}}$ has a nice description. (Integer coefficients are assumed throughout this appendix.)

Theorem A.6. *The homology of $\widehat{\mathbf{MTU}}$ is $H_*(\widehat{\mathbf{MTU}}) \cong \mathbb{Z}[b_0, b_1, b_2...]$ where b_i is in degree $2i$. The map $\Psi^\infty : H_*(\widehat{\mathbf{MTU}}) \to H_*(\mathbf{MU})$ is the quotient by $b_0 = 1$. The homology is equipped with a coproduct defined by:*

$$\Delta(b_n) = \sum_{i+j=n} b_i \otimes b_j.$$

Proof. By Proposition 2.11, $H_*(\mathbf{MTU}(d))$ is the abelian group generated by monomials in the variables $b_1, ..., b_d$ of polynomial degree less than or equal to d. This is isomorphic to the group generated by monomials in the variables $b_0, b_1, ..., b_d$ of degree *equal* to d. We may choose b_0^d to be the inverse of 1 under the canonical map $\Psi^\infty : H_*(\mathbf{MTU}(d)) \to H_*(\mathbf{MU}) \cong \mathbb{Z}[b_1, ..., b_d]$. Therefore, as abelian groups $H_*(\widehat{\mathbf{MTU}}) \cong \mathbb{Z}[b_0, b_1, b_2...]$, using the known homology of the wedge sum of spectra. Then observe that $H_*(\mathbf{MTU}(1)) \cong \mathbb{Z}\langle b_0, b_1, ...\rangle$ where $b_i \in H_{2i}(\mathbf{MTU}(1))$. The product map $H_*(\mathbf{MTU}(1))^{\otimes d} \to H_*(\mathbf{MTU}(d))$ is surjective and is the natural product of d linear terms by the diagram:

$$\begin{array}{ccc} H_*(\mathbf{MTU}(1))^{\otimes d} & \longrightarrow & H_*(\mathbf{MTU}(d)) \\ \downarrow & & \downarrow \\ H_*(\mathbf{MU})^{\otimes d} & \longrightarrow & H_*(\mathbf{MU}). \end{array}$$

Thus, $\{b_0, b_1, ...\}$ generates $H_*(\widehat{\mathbf{MTU}})$ and therefore there is a ring isomorphism $H_*(\widehat{\mathbf{MTU}}) \cong \mathbb{Z}[b_0, b_1, b_2...]$.

Recall that $H^*(\mathbf{MTU}(1)) \cong \mathbb{Z}[c_1]$ and b_n is dual to c_1^n. By dualizing the product structure on $H^*(\mathbf{MTU}(1))$, the coproduct immediately follows. □

We note that $H_*(\widehat{\mathbf{MTU}})$ is not finite in each dimension because b_0 has cohomological degree 0. The cohomology, which is dual, is much larger than the homology.

Theorem A.7. *The cohomology of* $\widehat{\mathbf{MTU}}$ *is the ring:*

$$H^*(\widehat{\mathbf{MTU}}) \cong \mathbb{Z} \times \prod_{d=1}^{\infty} \mathbb{Z}[c_{1d}, ..., c_{dd}],$$

where c_{id} *is in degree* $2d$.

Proof. The cohomology of $\mathbf{MTU}(d)$ is $\mathbb{Z}[c_1, ..., c_d]$ for $d \geq 1$. In order to distinguish the classes, we will denote by c_{id}, the copy of c_i coming from $\mathbf{MTU}(d)$. The wedge sum of the cohomology is the direct *product* $\prod_d H^*(\mathbf{MTU}(d))$. Since $H^*(\mathbf{MTU}(0)) \cong \mathbb{Z}$, the result follows. □

Elements of the cohomology could be infinite sums (i.e. formal power series) of elements from the cohomologies of $\mathbf{MTU}(d)$. In particular, the map Ψ^∞ can only be described using such a sum. We first compute the map Ψ in homology:

Proposition A.8. *The map* $\Psi : H_*(\widehat{\mathbf{MTU}}) \to H_*(\widehat{\mathbf{MTU}})$ *is multiplication by* b_0.

Proof. Consider a monomial $\prod_{i\geq 0} b_i^{j_i}$ in $H_*(\mathbf{MTU}(d))$. The image of this element in $H_*(\mathbf{MU})$ is $\prod_{i>0} b_i^{j_i}$. In the following commutative diagram, the bottom map is the identity and the vertical maps are injective:

$$\begin{array}{ccc} H_*(\mathbf{MTU}(d)) & \longrightarrow & H_*(\mathbf{MTU}(d+1)) \\ \downarrow & & \downarrow \\ H_*(\mathbf{MU}) & \longrightarrow & H_*(\mathbf{MU}). \end{array}$$

For degree reasons, the only possible image of $\prod_{i\geq 0} b_i^{j_i}$ under the top horizontal map is $b_0 \prod_{i\geq 0} b_i^{j_i}$. By assembling all $\mathbf{MTU}(d)$ together, the result follows. □

The above map is obviously not a map of rings and thus Ψ is not a map of ring spectra.

Proposition A.9. *The map* $\Psi : H^*(\widehat{\mathbf{MTU}}) \to H^*(\widehat{\mathbf{MTU}})$ *is determined by the mappingss* $c_{i,d} \mapsto c_{i,d-1}$ *for* $i < d$ *and* $c_{d,d} \mapsto 0$.

The map $\Psi^\infty : H^*(\mathbf{MU}) \to H^*(\widehat{\mathbf{MTU}})$ *is determined by*

$$c_i \mapsto \sum_{d=i}^{\infty} c_{id}.$$

Proof. The map $H^*(\mathbf{MTU}(d)) \to H^*(\mathbf{MTU}(d-1))$ maps $c_{i,d}$ to $c_{i,d-1}$ for $i < d$ and $c_{d,d}$ to 0. By assembling the spectra together, the first result follows. Recall that the map Ψ^∞ is the wedge sum of the maps $\mathbf{MTU}(d) \to \mathbf{MU}$ for all d. By Proposition 2.5, on cohomology, $H^*(\mathbf{MU}) \to H^*(\mathbf{MTU}(d))$ is the quotient by c_i for $i > d$. Thus the map on cohomology $\Psi^\infty : \mathbb{Z}[c_1, c_2, ...] \to H^*(\widehat{\mathbf{MTU}})$ is defined by $c_i \mapsto \sum_{d=i}^{\infty} c_{id}$. □

Since $\widehat{\mathbf{MTU}}$ is a ring spectrum, its Adams-Novikov spectral sequence may be easier to study than $\mathbf{MTU}(d)$. Therefore, we computed for future reference the cooperations on $MU_*(\widehat{\mathbf{MTU}})$. By Theorem 2.46, $MU_*(\widehat{\mathbf{MTU}}) \cong \pi_*(\mathbf{MU})[B_0, B_1, B_2, ...]$. The cooperations $MU_*(\widehat{\mathbf{MTU}}) \to MU_*(\mathbf{MU}) \otimes MU_*(\widehat{\mathbf{MTU}})$ are defined by Theorem 2.46 except on B_0.

Proposition A.10. *The coaction* $\psi(B_0)$ *is* $1 \otimes B_0$.

Proof. Recall that $MU_*(\mathbf{MTU}(1)) \to MU_*(\mathbf{MU})$ takes B_0 to 1. Using the following commutative diagram:

$$\begin{array}{ccc} MU_*(\mathbf{MTU}(1)) & \longrightarrow & MU_*(\mathbf{MU}) \otimes MU_*(\mathbf{MTU}(1)) \\ \downarrow & & \downarrow \\ MU_*(\mathbf{MU}) & \longrightarrow & MU_*(\mathbf{MU}) \otimes MU_*(\mathbf{MU}), \end{array}$$

we can conclude that $\psi(B_0) = 1 \otimes B_0$, since the right vertical map is an injection. The result follows. $\square$

There are several interesting subrings of $\pi_*(\widehat{\mathbf{MTU}})$.

Proposition A.11. *The subgroup* $\bigoplus_{d\geq 0} \pi_{n\cdot d}(\mathbf{MTU}(m \cdot d))$ *is a subring of* $\pi_*(\widehat{\mathbf{MTU}})$ *for any* $m, n \in \mathbb{N}$.

Proof. This is an immediate consequence of the bigrading of $\pi_*(\widehat{\mathbf{MTU}})$. $\square$

In particular, if $m = 1$ and $n = 1$ this subring is isomorphic to Ω_*^U and gives a section of $\Psi^\infty : \pi_*(\widehat{\mathbf{MTU}}) \to \Omega_*^U$. If $m = 1$ and $n = 2$, the subring is isomorphic to the even dimensional complex cobordism groups. If $m = 1$ and $n = 4$, this subring is isomorphic to the cobordism ring of $4d$ dimensional manifolds whose tangent bundles split off a trivial bundle of half dimension, $2d$. A question for future research is whether these splittings are induced by a splitting of the spectrum $\widehat{\mathbf{MTU}}$.

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