$RO(C_2)$-graded Stable Stems and Equivariant Framed Bordism

by

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DISSERTATION ABSTRACT

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Doctor of Philosophy in Mathematics

Title: $RO(C_2)$-graded Stable Stems and Equivariant Framed Bordism

The purpose of this dissertation is to prove fundamental relations in the $RO(C_2)$-graded stable equivariant homotopy groups of spheres $\pi_{*,*}$ using geometric methods. The main tool we use is a singular version of the Pontryagin-Thom isomorphism which holds in the equivariant setting. Our work then consists of writing down explicit bordisms between manifold representatives of homotopy classes. Selected relations include $\epsilon\eta = \eta$, $\rho\eta = 1 + \epsilon$, and $24\nu = 0$ where η and ν are equivariant Hopf maps, ϵ is a unit in $\pi_{0,0}$, and ρ is the generator of $\pi_{-1,-1}$. We also completely characterize the periodic portion of the topological zero-stem $\pi_{0,*}$ using singular manifold representatives which are the products $C_2 \times D^k$ equipped with various C_2-actions. While we focus on C_2, most of the theory we develop applies to $RO(G)$-graded homotopy groups for arbitrary finite groups G.

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The list of other teachers I've had throughout my life who have encouraged me on my academic journey is too long to give here, but I would at least thank my committee for the various roles they have played in my time at UO and especially Boris for his friendship.

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For my Mom

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CHAPTER 1

INTRODUCTION

Understanding the stable homotopy groups of spheres π_k^s is a foundational problem in algebraic topology. One way to approach these groups is with the famous Pontryagin-Thom isomorphism [21, 25, 30] which reinterprets the homotopy ring π_*^s as the framed bordism ring Ω_*^{fr}. Our goal is to explore the $RO(G)$-graded stable homotopy groups π_V^G, an equivariant analog of π_k^s, using a version of this isomorphism [19]. In the equivariant world, G acts on everything in sight, homotopy groups are defined using **representation spheres** $S^V = V \cup \{\infty\}$ for finite-dimensional real orthogonal G-reps V, and we frame manifolds using equivariant bundle equivalences $\Phi : \underline{V} = M \times V \xrightarrow{\sim} TM$ called **V-frames**. Much of the theory we develop around V-framed manifolds works for arbitrary finite groups G, but we focus on the order 2 cyclic group C_2. Our main objective is to give geometric explanations for some of the known features of $\pi_*^{C_2}$ using hands-on constructions in equivariant framed bordism.

1.1 Review of $RO(C_2)$-graded Homotopy Groups

In the case $G = C_2$ we'll use some special notation. As an abelian group the real representation ring $RO(C_2)$ is generated by the one-dimensional trivial and sign representations $\mathbb{R}^{1,0}$ and $\mathbb{R}^{1,1}$. Any other finite-dimensional C_2-rep is isomorphic to some $\mathbb{R}^{p,q} = (\mathbb{R}^{1,0})^{\oplus p-q} \oplus (\mathbb{R}^{1,1})^{\oplus q}$, $0 \leq q \leq p$. The corresponding representation spheres are denoted $S^{p,q}$. So we can write the $RO(C_2)$-grading $\pi_{\mathbb{R}^{p,q}}^{C_2}$ as a bi-grading $\pi_{p,q}$ using **topological stem** p and **weight** q.[1]

The C_2-equivariant groups, and their close relatives the $\mathbb{R}$- and $\mathbb{C}$-motivic homotopy groups, have been extensively studied using various algebraic methods [1–3, 11, 12, 14–16]. These computations typically involve many spectral and exact sequences. While these methods are fruitful, the machinery involved can be pretty heavy. Our objective is to explain some features of $\pi_{p,q}$ from a geometric perspective.

Table 1.1 below summarizes these computations in a range. Each entry contains a shorthand expression for a direct sum of abelian groups where b^n is $(\mathbb{Z}/b\mathbb{Z})^n$, ∞ is $\mathbb{Z}$, π_k is the kth classical stable stem π_k^s, and 0 is the trivial group. Empty entries are also 0. For $q \leq \frac{1}{2}p$, the group $\pi_{p,q}$ contains a split copy of π_{p-q}^s. Ignoring those copies,

[1] This is not the only convention. *E.g.* Araki and Iriye [1] use p as weight and q as coweight when they write "$\pi_{p,q}$".

when $p \geq 0$ each column is periodic in q excluding an exceptional region (marked in red) between $\frac{1}{2}p < q \leq p+1$. We refer to these interrupted periodicities as **pseudo-periodicities**. In the first column $p = 0$ the pseudo-periodicity has period 2. It is accentuated with yellow highlighting. As p increases the periods become increasingly large powers of 2 which cannot be clearly indicated in the range of Table 1.1.

	-2	-1	$p=0$	1	2	3	4	5
4			∞	2^2	2^2	12	∞	0
3			2	2	0	∞	0	2
2			∞	0	∞	24	$0 \cdot \pi_2$	$8 \cdot \pi_3$
1			0	∞	$2 \cdot \pi_1$	$2 \cdot \pi_2$	$8 \cdot \pi_3$	$2 \cdot \pi_4$
$q=0$			$\infty \cdot \pi_0$	$2^2 \cdot \pi_1$	$2^2 \cdot \pi_2$	$24 \cdot 8 \cdot \pi_3$	$2 \cdot \pi_4$	$0 \cdot \pi_5$
-1		π_0	$2 \cdot \pi_1$	$2 \cdot \pi_2$	$8 \cdot \pi_3$	$2 \cdot \pi_4$	$0 \cdot \pi_5$	$2 \cdot \pi_6$
-2	π_0	π_1	$\infty \cdot \pi_2$	$4 \cdot \pi_3$	$0 \cdot \pi_4$	$12 \cdot \pi_5$	$2 \cdot \pi_6$	$16 \cdot \pi_7$
-3	π_1	π_2	$2 \cdot \pi_3$	$0 \cdot \pi_4$	$2 \cdot \pi_5$	$2^2 \cdot \pi_6$	$16 \cdot \pi_7$	$2 \cdot \pi_8$

Table 1.1. The C_2-equivariant homotopy groups in a range.

An example of an element in $\pi_{p,q}$ is the equivariant Hopf map $S(\mathbb{C}^2) \to \mathbb{C}P^1$ where C_2 acts by complex conjugation. Here $S(\mathbb{C}^2)$ is the unit sphere in $\mathbb{C}^2$. After carefully choosing equivalences $S(\mathbb{C}^2) \cong S^{3,2}$ and $\mathbb{C}P^1 \cong S^{2,1}$ we get an element $\eta \in \pi_{1,1}$. The class η can be represented in terms of framed bordism by the C_2-manifold $U(1)$ equipped with the frame ∂_θ. The action on $U(1)$ is also complex conjugation. The frame is not invariant under the action, but rather gets sent to $-\partial_\theta$. This means ∂_θ is an $\mathbb{R}^{1,1}$-frame. There are also variants of the Hopf map $\eta_{\text{top}}, \eta_{\text{free}} \in \pi_{1,0}$ which use the trivial and antipodal actions on $U(1)$. We can also replace $\mathbb{C}$ with $\mathbb{H}$ or $\mathbb{O}$ to obtain similar elements $\nu \in \pi_{3,2}$, $\sigma \in \pi_{7,4}$, $\nu_{\text{top}} \in \pi_{3,0}$, *etc.*, which are represented by framed spheres S^3 and S^7 with various actions.

Other important elements include the units $1, \epsilon \in \pi_{0,0}$, the fixed-point inclusion $\rho : S^{0,0} \hookrightarrow S^{1,1}$ in $\pi_{-1,-1}$, and the equivariant squares $Sq(\alpha) \in \pi_{2k,k}$ of classical elements $\alpha \in \pi_k^s$. Manifold representatives for 1 and $Sq(\alpha)$ might be guessed by the reader,[2] but ϵ and ρ turn out to be more subtle.

[2]Answers: 1) a point, 2) $M \times M$ with twist action, where M is a framed manifold representative of α.

1.2 Preview of Results

The elements described above satisfy a variety of relations. For example:

- $\epsilon\eta = \eta$
- $\rho\eta = 1 + \epsilon$
- $\rho\, Sq(\eta_{\text{top}}) = \eta_{\text{top}} + \epsilon\eta_{\text{free}}$
- $\eta\eta_{\text{free}} \in \langle 1-\epsilon, \eta, 1-\epsilon\rangle$
- $\epsilon\nu = -\nu$
- $24\nu = 0$

Our goal is to witness such relations by explicitly writing down bordisms between the left- and right-hand sides of each equation.

Table 1.1 shows that $\pi_{*,*}$ also has some large-scale structure, such as the alternating 2-∞ pseudo-periodicity in column $p = 0$. We will be able to characterize this piece entirely using framed bordism.

Theorem 1.1. *There are elements $\theta_n \in \pi_{0,n}$ which generate a subring of $\pi_{0,*}$ subject to the relations*

- $\theta_n\theta_m = 2\theta_{n+m}$,
- $2\theta_{2k+1} = 0$,
- $\theta_1 = 0$, *and*
- $\epsilon\theta_n = -\theta_n$.

Another obvious feature visible in Table 1.1 is the presence of the split copies of π^s_{p-q} for $q \leq \frac{1}{2}p$. Elements in these copies can be obtained by iterating ρ on the doubles $Sq(\alpha) \in \pi_{2k,k}$. Along $q = 0$, this is actually not the same as the naive embedding $\pi^s_k \hookrightarrow \pi_{k,0}$ which equips classes with the trivial action. That is, $\rho^k\, Sq(\alpha) \neq \alpha$ in general. We can "correct" this by literally cutting out the fixed-points of $\rho^k\, Sq(\alpha)$ to recover α using a framed bordism.

Theorem 1.2. *If M represents $\alpha \in \pi^s_k$, there is a canonical bordism*

$$W : \rho^k \cdot Sq(M) \rightsquigarrow M \sqcup F$$

where the terminal manifold is the disjoint union of M equipped with the trivial action and a C_2-free manifold F.

The relation $\rho\, Sq(\eta_{\text{top}}) = \eta_{\text{top}} + \epsilon\eta_{\text{free}}$ is an instance of this.

1.3 Singular Framed Bordism

Now, in summarizing the results we've been hiding an important detail. These geometric constructions actually take place in a theory called **singular framed bordism**. In addition to a V-frame Φ, we also equip a manifold M with a **singular map** $\sigma : (M, \partial M) \to (X, A)$ to some target pair of G-spaces (X, A). The map σ is singular in that it only has to be equivariant and send ∂M to A, there are no other conditions like smoothness. We will always have $(X, A) = (D(W), S(W))$ for some G-rep W. We define $\Omega^{fr}_V(D(W), S(W))$ to be the group of singular framed bordism classes of singular V-framed manifolds of $(D(W), S(W))$. The equivariant Pontryagin-Thom isomorphism for finite G then takes on the form [19],

$$\pi^G_{V-W} \cong \varinjlim_{U} \Omega^{fr}_{V\oplus U}(D(W \oplus U), S(W \oplus U)).$$

The limit on the right is called "stable framed bordism" [27]. The idea is that we are stabilizing the manifolds themselves; replacing M by $M \times D(U)$ for larger and larger representations U.

Why are singular maps necessary? In the classical Pontryagin-Thom story, one gets a framed manifold by first perturbing a map $f : S^{n+k} \to S^n$ to be smooth and transverse to $0 \in S^n$, and then setting $M = f^{-1}(0)$. This already breaks down for $G = C_2$. Consider $\rho : S^{0,0} \hookrightarrow S^{1,1}$. Since S^0 has smaller dimension than S^1 the only way to perturb ρ to be transverse to 0 is to miss it. This is impossible because equivariance forces fixed-points to be sent to fixed-points. The workaround is to include the singularity as part of the data. For ρ, doing this yields a geometric representative whose underlying manifold is a single fixed-point, but which is also equipped with the singular map $(\{*\}, \emptyset) \to (D(\mathbb{R}^{1,1}), S(\mathbb{R}^{1,1}))$. This may feel like a concession, but it turns out to be rather useful formalism even for elements like 1 and η which don't require singular representatives. The trick is to keep singular maps simple, *e.g.* piece-wise linear.

Singular bordisms contain a lot of data, so it's helpful to look at the simple example in Figure 1.1 which shows everything at once. This is also an opportunity to establish some of the conventions that we when illustrating these kinds of bordisms.

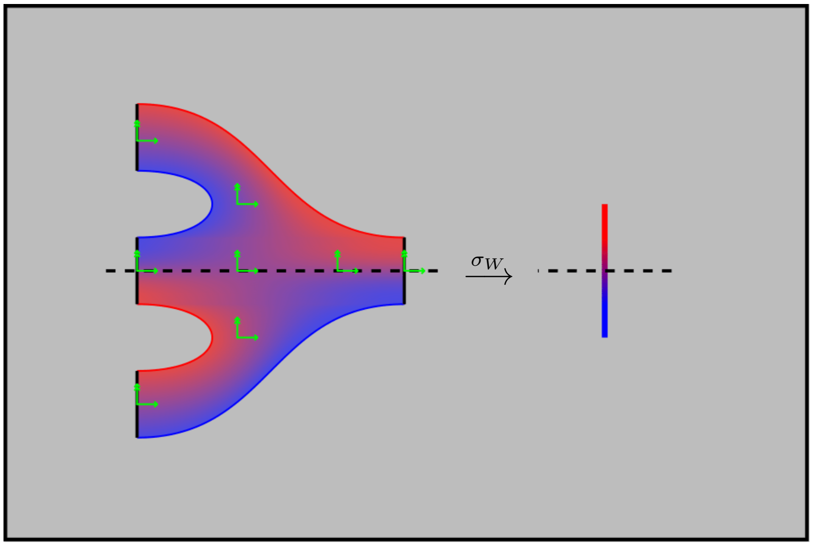


Figure 1.1. The bordism $(W, \sigma_W, \Phi_W) : (M, \sigma_M, \Phi_M) \rightsquigarrow (N, \sigma_N, \Phi_N)$.

On the left is a singular framed bordism W and on the right is the target $D(\mathbb{R}^{1,1})$ of its singular map σ_W. The C_2-action reflects both spaces across the dashed lines. To illustrate σ_W, each point $x \in W$ picks up the color of its image $\sigma_W(x) \in D(\mathbb{R}^{1,1})$. The frame $\Phi_W : \underline{\mathbb{R}^{2,1}} \to TW$ sends basis vectors $e_1, e_2 \in \mathbb{R}^{1,0} \oplus \mathbb{R}^{1,1}$ to the single- and double-headed arrows, respectively.

Note that the bordism W is directed, having an initial manifold M on the left with an *inward* pointing vector field and terminal manifold N on the right with an *outward* pointing vector field. The manifolds M and N have non-empty boundary, and W has an edge $\underset{\rightsquigarrow}{\partial} W = \partial W \setminus (\operatorname{int} M \sqcup \operatorname{int} N)$. The singular map sends $\underset{\rightsquigarrow}{\partial} W$ to $S(\mathbb{R}^{1,1})$. Finally, observe how equivariance manifests for σ_W and Φ_W. Applying the action swaps the colors on both W and $D(\mathbb{R}^{1,1})$ and transforms the frame like the representation $\mathbb{R}^{2,1}$.

1.4 Core Ideas

In non-equivariant framed bordism, we often think of a frame $\Phi : \underline{\mathbb{R}^n} \to TM$ as a collection of n point-wise linearly independent sections. If Φ is instead a V-frame, we can do something similar. Given a one-dimensional subrepresentation $L \subset V$ we can pick a vector $v \in L$. Its image in TM defines an L-**section**.[3] Such sections are

[3]This also works for larger irreps $I \subseteq V$, but it's more sensitive to the choice of basis for I making it harder to use.

not necessarily equivariant as maps $M \to TM$ unless L is a trivial representation. This idea is especially useful for C_2 where we can always decompose a frame into an ordered collection of $\mathbb{R}^{1,0}$- and $\mathbb{R}^{1,1}$-sections.

For a non-equivariant framed manifold $[M, \Phi]$, we can think of $-[M, \Phi]$ as being the same framed manifold but with the first section negated. Really, transforming the sections of Φ by any orientation reversing map $a \in O(n)$ would work. This is a nice perspective because it interprets a product $-1 \cdot [M, \Phi]$ in the homotopy ring purely in terms of modifying the frame Φ. In this case multiplying $[M, \Phi]$ by -1 replaces Φ with the composition

$$\underline{\mathbb{R}^n} \xrightarrow{a} \underline{\mathbb{R}^n} \xrightarrow{\Phi} TM.$$

Informally, we might say that the homotopy class -1 is represented by "a discrete point which is intrinsically oriented by the map $a : \mathbb{R}^n \to \mathbb{R}^n$", and that multiplication $-1 \cdot [M, \Phi]$ twists the frame Φ by a.

This idea goes much further in the equivariant setting. For every subgroup $H \leq G$ and every H-isomorphism of G-reps $\varphi : \operatorname{Res}_H V \to \operatorname{Res}_H W$, there is an element $\theta_{H,\varphi} \in \pi^G_{V-W}$ which is analogously "the orbit G/H intrinsically oriented by the map φ." Then multiplication $\theta_{H,\varphi} \cdot [M, \sigma, \Phi]$ can be interpreted as twisting Φ by φ. Once we've defined certain twisting operators Tw^H_φ this is formalized as

Proposition 1.3. $\theta_{H,\varphi} \cdot [M, \sigma, \Phi] = u \cdot \mathrm{Tw}^H_\varphi(M, \sigma, \Phi)$ *for some unit* $u \in (\pi^G_{\mathbf{0}})^\times$.

In fact, we will actually define $\theta_{H,\varphi} := \mathrm{Tw}_{H,\varphi}(1)$. For $G = C_2$, the units $\pm 1, \pm\epsilon$ and the θ_n's from Theorem 1.1 are all examples of these kinds of elements. Once we have a good grasp on the $\theta_{H,\varphi}$'s and their twisting effect, the techniques in this paper basically boil down to the following program.

1. Start with a singular framed manifold (M, σ, Φ).
2. Produce a cylindrical bordism $W : M \rightsquigarrow M'$ using one of
   a) a G-diffeomorphism of M,
   b) a homotopy of the frame Φ, or
   c) a homotopy of the singular map σ.
3. If c) was used in step 2, apply a kind of excision.
4. Inspect the terminal manifold M' and identify its frame as a twist.

Here is a preview of an example for each case.

a) The relation $\epsilon\eta = \eta$ comes from an automorphism $U(1) \to U(1)$ which sends the frame ∂_θ to $-\partial_\theta$.

b) Constructing the Toda bracket $\langle 1-\epsilon, \eta, 1-\epsilon \rangle$ yields a torus with a twisted frame. We manually untwist that frame until it matches the frame on $U(1) \times U(1)$ associated to $\eta\eta_{\text{free}}$.

c) The product $\rho\eta$ is represented by $[U(1), c_0, \partial_\theta]$ equipped with a singular map c_0 that sends $U(1)$ to $0 \in D(\mathbb{R}^{1,1})$. To show $\rho\eta = 1+\epsilon$, we stretch out the singular map along $D(\mathbb{R}^{1,1})$ and trim the excess. This leaves behind a pair of intervals representing 1 and ϵ.

1.5 Outline

Part I is focused on quickly establishing background necessary to work with framed bordism. It starts in Chapter 2 with a basic review of equivariant homotopy theory. Section 2.1 gives special attention to the forgetful and fixed-point long exact sequences since they are used frequently for computations in the C_2-equivariant setting. Then in Section 3.1 we build up the theory of V-frames on general equivariant vector bundles. Section 3.1.2 develops the twisting operators in this general setting. Section 3.2 introduces the formal definition of singular framed bordism. Sections 3.2.1 to 3.2.3 point out 1) how to turn a framed equivalence of manifolds into a bordism, 2) what Lie groups look like as V-framed manifolds, and 3) how to perform excision on the singular map.

Part II is where the tools developed in Part I come together to start explaining features in $\pi_{p,q}$. Chapter 4 shows how the $\theta_{H,\varphi}$ elements in $\pi_{0,*}$ are intimately related to the twisting operators. We conclude in Chapter 5 by giving geometric witnesses for a number of relations in $\pi_{p,q}$.

Appendix A is basically an extended remark that discusses an alternative formulation of V-frames. It sketches how the twisting operators can be reinterpreted as homotopy classes of paths in equivariant Grassmannians. Appendices B and C are dedicated to detailed proofs of material covered in Sections 3.1.1 and 3.2.2, respectively.

Finally, the reader should be aware of a few notational conventions used through the text. We often compare framed manifolds which have the same underlying topological space but which carry different actions. When necessary we will annotate the action with a subscript. For example, $U(1)$ may carry a complex conjugation action $U(1)_{\mathrm{conj}}$, a free/antipodal action written $U(1)_{\mathrm{free}}$ or $U(1)_{\mathrm{antipodal}}$, as well as a trivial action $U(1)_{\mathrm{triv}}$. With $U(1)$ in particular the conjugation action shows up so frequently that unless surrounding text states otherwise, it is safe to assume $U(1)$ with no subscript is $U(1)_{\mathrm{conj}}$.

The following typographical conventions are used to help delineate smaller blocks of text

Proof. Proofs are terminated by a box. □

Example 1.4. Numbered examples are terminated by a fleuron. ❧

Construction 1.5. Numbered constructions are terminated by a lozenge. ◊

I

TOOLS OF THE TRADE

CHAPTER 2

EQUIVARIANT HOMOTOPY THEORY

Let G be a finite group. The study of G-equivariant topology is concerned with topological spaces X, Y equipped with continuous G-actions and continuous maps $f : X \to Y$ between them that are **equivariant**, meaning $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. Such maps are exactly the G-fixed-points in the mapping space $\mathrm{Hom}(X, Y)$ where G acts on functions by $(g \cdot f)(x) = gf(g^{-1}x)$. For a subgroup $H \leq G$ we write the space of H-fixed-points of X as X^H. So we can write $\mathrm{Hom}_{G-top}(X, Y) = \mathrm{Hom}_{top}(X, Y)^G$. For equivariant maps $f : X \to Y$ we can restrict f to H-fixed-points and write this as $f^H : X^H \to Y^H$.

An equivariant homotopy from f to f' is a homotopy $H : X \times I \to Y$ which is equivariant when G acts trivially on the interval I. Often we'll consider spaces with basepoints, in which case both the actions and homotopies are required to be basepoint preserving.

An important class of G-spaces are representation spheres $S^V = V \sqcup \{\infty\}$, which are formed by taking the one-point compactification of a finite dimensional real orthogonal G-representation V. The basepoint of S^V is ∞. Using representation spheres we can define the V-suspension of a pointed space X to be the smash product $X \wedge S^V$. We denote (based) equivariant homotopy classes of maps $S^V \to X$ by $[S^V, X]^G$. This has a group structure if S^V contains a trivial representation and it is abelian if the trivial representation has dimension at least 2. We can think of this as the Vth homotopy group of X.

Definition 2.1. *The Vth stable G-equivariant homotopy group of X is defined by*

$$\pi_V^G(X) := \varinjlim_{U \subseteq \mathcal{U}} [S^{V \oplus U}, X \wedge S^U]^G$$

where the limit is indexed over finite dimensional subrepresentations U, ordered by inclusion, of a complete G-universe[1] *$\mathcal{U}$.*

We are primarily concerned with understanding these groups in the case $X = S^0$, which we'll abbreviate simply as π_V^G. Note that, due to the stabilization process, these groups are defined even if V is a virtual representation in $RO(G)$, the Grothendieck

[1]An infinite dimensional representation containing at least countably many distinct copies of each isomorphism class of finite dimensional irrep of G.

ring of real orthogonal G-representations. As an abelian group $RO(G)$ is a free $\mathbb{Z}$-module generated by isomorphism classes of the finite dimensional irreducible G-representations. When $G = C_2$, we write $\mathbb{R}^{p,q}$ for the p-dimensional representation on $\mathbb{R}^p$ which negates the last q coordinates ($q \leq p$). The corresponding representation sphere is written $S^{p,q}$ and we use this bigraded notation $\pi_{p,q}$ to simplify writing the $RO(C_2)$-graded homotopy groups. This will be enough for us to get started. A more complete introduction to equivariant homotopy theory is given in May's book [22].

2.1 Long Exact Sequences in $RO(G)$-Graded Homotopy

In $RO(C_2)$-graded homotopy there are two long exact sequences which play a prominent role in computations [5]. The first comes from the forgetful homomorphism

$$\psi : \pi_{p,q} \to \pi_p^s$$

which forgets the C_2-action. The second is associated to the fixed-point homomorphism

$$\phi : \pi_{p,q} \to \pi_{p-q}^s$$

which sends a class $[f]$ to $[f^{C_2}]$. In the spirit of trying to access symmetry groups besides C_2, we will present these exact sequences in slightly more generality than most authors [1, 5, 20]. We'll specialize to C_2 only when we have to.

2.1.1 *The Forgetful Sequence*

Let $H \leq G$ have index 2, so that there is a sign representation L coming from the quotient $G/H \to C_2 \cong O(\mathbb{R}^1)$. Let $\rho_L : S^0 \to S^L$ denote the inclusion of fixed-points. For each G-rep V we can form the cofiber

$$S^V \xhookrightarrow{\rho_L} S^{V\oplus L} \longrightarrow C_{\rho_L} \simeq S^{V\oplus L}/S^V.$$

Mapping to S^W yields the long exact sequence

$$\cdots \longrightarrow [S^{V\oplus\mathbb{R}}, S^W]^G \longrightarrow [S^{V\oplus L}/S^V, S^W]^G \longrightarrow [S^{V\oplus L}, S^W]^G \xrightarrow{\rho_L} [S^V, S^W]^G.$$

This can be simplified by inspecting the term involving $S^{V\oplus L}/S^V$. First, there is a G-homeomorphism

$$S^{V\oplus L}/S^V \cong G/H_+ \wedge S^{V\oplus L}.$$

Then there is a natural isomorphism from G- to H-equivariant maps

$$[G/H_+ \wedge S^{V\oplus L}, S^W]^G \cong [S^{V\oplus L}, S^W]^H.$$

If we fix an H-isomorphism $\varphi : \operatorname{Res}_H L \xrightarrow{\sim} \mathbb{R}$ we can make an identification

$$[S^{V\oplus L}, S^W]^H \cong_\varphi [S^{V\oplus \mathbb{R}}, S^W]^H.$$

All together this lets us rewrite the sequence as

$$\cdots \longrightarrow [S^{V\oplus\mathbb{R}}, S^W]^G \longrightarrow [S^{V\oplus\mathbb{R}}, S^W]^H \longrightarrow [S^{V\oplus L}, S^W]^G \xrightarrow{\rho_L} [S^V, S^W]^G$$

which stabilizes to

$$\cdots \longrightarrow \pi^G_{(V\oplus\mathbb{R})-W} \longrightarrow \pi^H_{(V\oplus\mathbb{R})-W} \longrightarrow \pi^G_{(V\oplus L)-W} \xrightarrow{\rho_L} \pi^G_{V-W} \longrightarrow \cdots . \tag{2.1}$$

This is the **forgetful exact sequence** for the index 2 subgroup H.

This sequence is "forgetful" because the connecting homomorphism

$$\pi^G_{(V\oplus\mathbb{R})-W} \to \pi^H_{(V\oplus\mathbb{R})-W} \tag{2.2}$$

is indeed the map which just restricts the G-action on a G-equivariant class $[f] \in \pi^G_{(V\oplus\mathbb{R})-W}$ to an H-action. We should take a moment to see why this is the case. For simplicity suppose $f : \Sigma S^V \to S^W$. Since (2.1) came from a cofiber sequence, the connecting homomorphism is defined in terms of the composition

$$\begin{array}{ccc} C_{\rho_L} & \longrightarrow & \Sigma S^V \\ & \searrow^{\delta(f)} & \downarrow f \\ & & S^W. \end{array}$$

If we model the cofiber of ρ_L by $G/H_+ \wedge S^{V\oplus\mathbb{R}}$, the connecting map $G/H_+ \wedge S^{V\oplus\mathbb{R}} \to \Sigma S^V$ is projection onto the second factor. It follows that restricting $\delta(f)$ to $\{eH\}_+ \wedge S^{V\oplus\mathbb{R}}$ recovers f, but now only treated as an H-equivariant map due to the smash product with G/H_+. A similar line of reasoning helps to understand how the next homomorphism in the sequence

$$\pi^H_{(V\oplus\mathbb{R})-W} \to \pi^G_{(V\oplus L)-W} \tag{2.3}$$

induces a G-equivariant map from an H-equivariant map. Using the H-isomorphism φ we can think of an H-equivariant map $f : S^{V\oplus\mathbb{R}} \to S^W$ as being an H-equivariant map $f_\varphi : S^{V\oplus L} \to S^W$. That induces a G-equivariant map $G/H_+ \wedge S^{V\oplus L} \to S^W$. Precomposing that with the equator collapsing map $S^{V\oplus L} \to S^{V\oplus L}/S^V \cong G/H_+ \wedge S^{V\oplus L}$ gives a G-equivariant map $S^{V\oplus L} \to S^W$. These ideas of restriction and induction will reappear in the more general guise of twisting operators in Sections 3.1.2 and 4.3.

If $G = C_2$ and $H = \{e\}$ then L is $\mathbb{R}^{1,1}$ and ρ_L is the inclusion $\rho : S^{0,0} \to S^{1,1}$ that we've seen before. This lets us replace H-equivariant homotopy groups with the classical stable stems π_*^s so that the connecting homomorphism is the forgetful map ψ from the beginning of this section. So we can rewrite the forgetful exact sequence using the bigrading:

$$\cdots \longrightarrow \pi_{p+1,q} \xrightarrow{\psi} \pi_{p+1}^s \longrightarrow \pi_{p+1,q+1} \xrightarrow{\rho} \pi_{p,q} \longrightarrow \cdots.$$

2.1.2 The Fixed-Point Sequence

Let U be a G-rep with $U^G = \mathbf{0}$. Then there is another cofiber sequence

$$S(U)_+ \to S^0 \overset{\rho_U}{\hookrightarrow} S^U$$

where the map on the right is inclusion of fixed-points. This is clearer if one thinks first of the cofiber sequence $S(U) \to * \to S^U$ and then adds disjoint basepoints to the first two spaces. The associated long exact sequence that we get from mapping to S^W is

$$\cdots \to [S(U)_+ \wedge S^1, S^W]^G \to [S^U, S^W]^G \xrightarrow{\rho_U} [S^0, S^W]^G \to [S(U)_+, S^W]^G$$

which when stabilized becomes

$$\cdots \longrightarrow \pi_G^{W-\mathbb{R}^1}(S(U)_+) \longrightarrow \pi_{U-W}^G \xrightarrow{\rho_U} \pi_{-W}^G \longrightarrow \pi_G^W(S(U)_+) \longrightarrow \cdots$$

where $\pi_G^V(S(U)_+)$ denotes the Vth stable cohomotopy group of $S(U)_+$. Setting $V = U - W$, the sequence becomes

$$\cdots \longrightarrow \pi_G^{U-(\mathbb{R}^1\oplus V)}(S(U)_+) \longrightarrow \pi_V^G \xrightarrow{\rho_U} \pi_{V-U}^G \longrightarrow \pi_G^{U-V}(S(U)_+) \longrightarrow \cdots.$$

Taking fixed-points commutes with composition of equivariant maps, so given $[f] \in \pi_V^G$ we have $(\rho_U \circ f)^G = (\rho_U)^G \circ f^G$. Since U had a trivial fixed-point space, the stable class of $(\rho_U)^G$ is the identity $1 \in \pi_0^s$. So we can augment the sequence with a triangle

$$\begin{array}{ccccccccc}
\cdots \longrightarrow \pi_G^{U-(\mathbb{R}^1\oplus V)}(S(U)_+) & \longrightarrow & \pi_V^G & \xrightarrow{\rho_U} & \pi_{V-U}^G & \longrightarrow & \pi_G^{U-V}(S(U)_+) & \longrightarrow & \cdots . \\
 & & & \searrow^{\phi} & \downarrow{\scriptstyle \phi} & & & & \\
 & & & & \pi^s_{|V^G|} & & & &
\end{array}$$

Now we'll specialize to $G = C_2$. The only available representations with trivial fixed-points are $U = \mathbb{R}^{k,k}$ for some $k \geq 0$. The inclusion ρ_U is then the kth power of the inclusion $\rho : S^0 \to S^{1,1}$. So the diagram becomes

$$\begin{array}{ccccccccc}
\cdots \to \pi^{k-(p+1),k-q}(S(\mathbb{R}^{k,k})_+) & \to & \pi_{p,q} & \xrightarrow{\rho^k} & \pi_{p-k,q-k} & \to & \pi^{k-p,k-q}(S(\mathbb{R}^{k,k})_+) & \to & \cdots . \\
 & & & \searrow^{\phi} & \downarrow{\scriptstyle \phi} & & & & \\
 & & & & \pi^s_{p-q} & & & &
\end{array} \tag{FP}$$

Bredon [5] describes some facts about ϕ in various ranges:

Proposition 2.2. *The fixed-point map $\phi : \pi_{p,q} \to \pi^s_{p-q}$ is*

a) surjective if $p \geq 2q$,

b) split surjective if $p > 0$ and $q < 0$, and

c) an isomorphism if $p < 0$.

So, in the range $p - k < 0$ the vertical ϕ from (FP) is an isomorphism. Consequently, ρ^k is surjective when the diagonal ϕ is surjective, which is in the range $p \geq 2q$. This means that in the range $2q \leq p < k$ we can terminate the sequence on the right side

$$\cdots \to \pi_{p+1,q} \xrightarrow{\rho^k} \pi_{p+1-k,q-k} \to \pi^{k-(p+1),k-q}(S(\mathbb{R}^{k,k})_+) \to \pi_{p,q} \xrightarrow{\phi} \pi^s_{p-q} \to 0.$$

If we also have $2q \leq p+1 < k$ then we can terminate the sequence on the left as well. This gives a collection of short exact sequences

$$0 \to \pi^{k-(p+1),k-q}(S(\mathbb{R}^{k,k})_+) \to \pi_{p,q} \xrightarrow{\phi} \pi^s_{p-q} \to 0.$$

which are split if additionally $p > 0$ and $q < 0$.

Since the sequence is split, the cohomotopy groups on the left are independent of k once $k \geq p + 2$. In the literature [1, 5, 20] these groups are typically given their own symbol

$$\lambda_{p,q} = \varinjlim_k \pi^{-(p+1)+k,-q+k}(S(\mathbb{R}^{k,k})_+)$$

modulo slight variations in the choice of notation. In particular, Landweber [20, Section 5] relates $\lambda_{p,q}$ to a bordism group of manifolds M^p whose tangent bundle is stably equivalent to q copies of a line bundle over M. Taking the double cover of the classifying map $M \overset{TM}{\to} \mathbb{R}P^\infty$ for such a nearly-framed manifold reinterprets $\lambda_{p,q}$ as the bordism group of C_2-free $\mathbb{R}^{p,q}$-framed manifolds and C_2-free bordisms.

CHAPTER 3

SINGULAR FRAMED BORDISM

In order to have an $RO(G)$-graded version of the Pontryagin-Thom isomorphism, we need an $RO(G)$-graded notion of framed G-manifolds. Let the virtual representation $V - W$ be the formal difference of two real representations V and W. For a manifold M to live in $RO(G)$-degree $V - W$ we need to equip M with two different pieces of data: one for the positive part V and one for the negative part W. The positive datum is a V-frame on TM and the negative datum is an equivariant singular map $M \to D(W)$. We will discuss V-frames first.

Classically, frames are often conceptualized as a collection of linearly independent vector fields. Each of these vector fields can then be dealt with individually. However, introducing a G-action juggles them around in a way that requires a more holistic approach. Another issue that needs to be addressed is how to translate between frames on the tangent and normal bundles of a manifold. This is because the tangent frame controls the $RO(G)$-degree of a manifold, but we need a normal frame to define the Thom collapse map. For simplicity, we will first develop the basic theory of V-frames on abstract vector bundles $E \to X$ before specializing to TM and $\nu(M)$.

3.1 V-Frames

For a representation V, we denote the product bundle over X using an underline, $\underline{V} = X \times V$. Similarly, for a homomorphism $\varphi : V \to U$ we write $\underline{\varphi}$ for $\mathrm{id}_X \times \varphi : \underline{V} \to \underline{U}$.

Definition 3.1. *An* ***equivariant vector bundle*** *$E \to X$ is a vector bundle over a G-space X equipped with a continuous action $G \times E \to E$ which is linear on fibers and covers the action $G \times X \to X$.*

Definition 3.2. *Let V be a representation. A* ***V-frame*** *on a bundle E is an equivariant vector bundle isomorphism $\Phi : \underline{V} \xrightarrow{\sim} E$.*

We generally only care about frames up to homotopy. When we say that two V-frames Φ and Ψ are homotopic, we mean that they are homotopic *through* V-frames. That is, there is a homotopy $H : \underline{V} \times I \to E$ from Φ to Ψ where H_t is a V-frame for each $t \in I$. If there is such a homotopy we may write $\Phi \simeq \Psi$ and say that Φ and Ψ are **homotopic V-frames** on E. We may also write this as $(E, \Phi) \simeq (E, \Psi)$.

If V is equipped with a basis $\{v_i\}$, then a V-frame Φ gives a collection of linearly independent sections $s_i(x) := \Phi(x, v_i)$ of the bundle $E \to X$. However, unless the vector v_i is fixed by G, the sections s_i are *not* equivariant maps $X \to E$. This makes working with the s_i's inconvenient in general. However, if one of the v_i happens to span a 1-dimensional subrepresentation $L \subseteq V$, then it is useful to think of s_i on its own as an **L-section** since it transforms like the representation L. This is especially useful for C_2 since both its irreps are 1-dimensional.

Example 3.3. Let $G = C_2$ with generator τ. Let X be the discrete space $\{a, b\}$ where τ acts by exchanging the two points. The bundle $E \to X$ will just be a pair of lines over X. The C_2-action on E is reflection across the dotted line in Figure 3.1.

The bundle E admits both an $\mathbb{R}^{1,0}$-frame Φ_1 and an $\mathbb{R}^{1,1}$-frame Φ_2 which are also shown in Figure 3.1. The image $\Phi_i(x, e_1)$ of the standard basis vector $e_1 \in \mathbb{R}^1$ is shown as an arrow in each fiber. Note that in $\mathbb{R}^{1,0}$ we have $\tau e_1 = e_1$, but in $\mathbb{R}^{1,1}$ we have $\tau e_1 = -e_1$, so τ acts differently on each frame:

$$\begin{aligned} \tau \cdot \Phi_1(a, e_1) &= \Phi_1(\tau a, \tau e_1) \\ &= \Phi_1(b, e_1) \end{aligned} \qquad\qquad \begin{aligned} \tau \cdot \Phi_2(a, e_1) &= \Phi_2(\tau a, \tau e_1) \\ &= \Phi_2(b, -e_1) \\ &= -\Phi_2(b, e_1). \end{aligned}$$

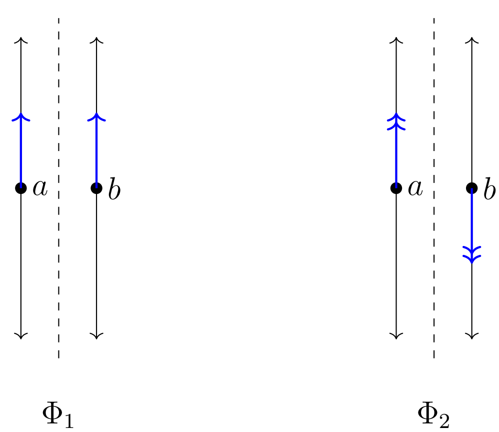


Figure 3.1. The two frames Φ_1 and Φ_2 on E.

We can understand Φ_1 and Φ_2 as being determined respectively by an $\mathbb{R}^{1,0}$-section, which is τ-invariant, and an $\mathbb{R}^{1,1}$-section, which globally transforms by a sign under the action of τ. In later 2D figures we will continue to use single- and double-headed arrows to illustrate $\mathbb{R}^{1,0}$- and $\mathbb{R}^{1,1}$-frames, respectively. ❧

3.1.1 Complementary Frames

When working with framed bordism it's easier to deal with tangential frames since the tangent bundle is more intrinsic to a manifold. It is also what controls the $RO(G)$-degree—a V-framed manifold M represents a homotopy class in π_V^G. However, the Thom collapse map

$$S^U \to \mathrm{Th}(\nu(M)) \xrightarrow{\Phi^{-1}} M_+ \wedge S^V \to S^{U-V}$$

is defined using a *normal* frame. So we need to know how to translate between tangential and normal frames.

For example, embed $[U(1), \partial_\theta]$ as the unit circle in the xy-plane of 3-space: $S^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3$. It is well-known that the normal frame on $U(1)$ which represents the Hopf map η_{top} is the one which twists inward by angle θ at the point $e^{i\theta} \in U(1)$. This is illustrated in Figure 3.2.

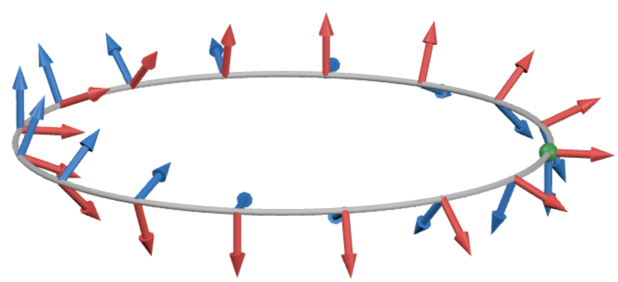


Figure 3.2. The inward-twisting normal frame on S^1. The marked green point is $\theta = 0$, and θ increases in the counterclockwise direction.

What is less obvious is why this normal frame should be the one associated the $U(1)$-invariant tangential frame ∂_θ. It turns out that if we add these normal and tangent frames together, the resulting 3-frame is homotopic to the ambient frame on $\mathbb{R}^3$ restricted to S^1.

The conversion process is sensitive to how we orient the ambient space, *e.g.*, specifying a right- versus left-handed coordinate system would turn Figure 3.2 into $-\eta_{\mathrm{top}}$. Equivariantly, this issue is more sensitive since there are many homotopy-inequivalent ways to decompose the ambient embedding representation into irreps. A detailed account is given in Appendix B. For now, we just need the definition and few basic facts.

Definition 3.4 (Complementary Frames). *Let U, V, and W be representations, let $\varphi : V \oplus U \to W$ be an isomorphism, let $E \to X$ be an equivariant vector bundle, and let $f : E \hookrightarrow \underline{W}$ be an equivariant bundle embedding. For a V-frame $\Phi : \underline{V} \to E$, we say a U-frame $\Psi : \underline{U} \to E^{\perp \underline{W}}$ is **complementary** to Φ with respect to φ and f if there is a homotopy of $(V \oplus U)$-frames $\Phi \oplus \Psi \simeq \underline{\varphi}$ on the bundle $\underline{W}$.*

In the literature, complementary frames as we have defined them are not typically introduced. Instead, most authors use, either explicitly or implicitly, a specific formula to define a frame on the stabilized orthogonal complement $E^{\perp} \oplus V$ [18, 19, 29]. Appendix B shows that the frame their formula defines is in fact complementary in our sense, and furthermore that any two complementary frames (with respect to the same embedding f and isomorphism φ) become homotopic after a stabilization process. Since this means that complementary frames are essentially unique after stabilizing, we can write $\Phi^{\perp}$ for any frame complementary to Φ without too much ambiguity.

The advantage of Definition 3.4 is that it sometimes lets us avoid the stabilization step. This helps keep the dimension of our complementary frames low, which in turn makes it easier to analyze the result of the Thom collapse construction. Other times, the stabilization step is unavoidable.

Example 3.5. The frame ∂_θ on $U(1)_{\text{free}}$ is an $\mathbb{R}^{1,0}$-frame. We can embed $U(1)_{\text{free}}$ into $\mathbb{R}^{2,2}$, but there exists no representation U with an isomorphism $\varphi : \mathbb{R}^{1,0} \oplus U \to \mathbb{R}^{2,2}$. Without such an isomorphism we cannot even begin to look for a complementary frame. We need to enlarge the ambient space to $\mathbb{R}^{3,2}$ at the very least. ❧

However, when an embedding is well-behaved, finding complementary frames without having to stabilize first is almost trivial. We say an embedding $f : E \hookrightarrow \underline{W}$ of a V-framed bundle (E, Φ) is **flat** with respect to an isomorphism $\varphi : V \oplus U \to W$ if there is a commutative diagram

$$\begin{array}{ccc} \underline{V} & \xrightarrow{\underline{\varphi|_V}} & \underline{W}. \\ & {\scriptstyle\Phi}\searrow \quad \nearrow{\scriptstyle f} & \\ & E & \end{array}$$

That is, the embedding takes the frame on E to the existing summand $\underline{V} \subseteq \underline{W}$ determined by φ. The following result tells us that the complement of a flatly embedded framed bundle is exactly what you expect it to be.

Proposition 3.6 (Flat Complements). *If $f : E \hookrightarrow \underline{W}$ is a flat embedding of the V-framed bundle (E, Φ) with respect to $\varphi : V \oplus U \to W$, then $E^\perp = \underline{\varphi}(\underline{U})$ and the intrinsic U-frame $\Psi_{\mathrm{id}} : \underline{U} \to E^\perp$ is complementary to Φ.*

Proof. Since f is flat, the image E coincides with $\underline{\varphi}(\underline{V})$ in $\underline{W}$. Since φ is isometric and an isomorphism, it commutes with taking orthogonal complements. So

$$\begin{aligned} E^{\perp \underline{W}} &= \underline{\varphi}(\underline{V})^{\perp \underline{W}} \\ &= \underline{\varphi}(\underline{V}^{\perp \underline{V \oplus U}}) \\ &= \underline{\varphi}(\underline{U}). \end{aligned}$$

Since $E^\perp$ is the image of $\underline{U}$ under $\underline{\varphi}$ it is already $\underline{U}$-framed by the map $\Psi_{\mathrm{id}} := \underline{\varphi|_U}$. The frame Ψ_{id} is complementary to Φ because $\Phi \oplus \Psi_{\mathrm{id}} = \underline{\varphi|_V} \oplus \underline{\varphi|_U} = \underline{\varphi}$. ☐

Chapter 4 deals with flat embeddings of orbits $G/H \hookrightarrow V$, of disks $D(V) \hookrightarrow V$, and of their product $G/H \times D(V) \hookrightarrow V$. In these cases either the normal bundle or the tangent bundle have rank 0. If a bundle E has rank 0 then it only admits a single frame: the $\mathbf{0}$-frame $\underline{\mathbf{0}} \to E$. In spite of this degeneracy, Proposition 3.6 tells us that for flat embeddings of E the $\mathbf{0}$-frame can either *be* a complementary frame, or *admit* a frame complementary to itself:

- If $V = W$, then the $\mathbf{0}$-frame on $E^\perp$ is complementary to Φ.
- If $V = \mathbf{0}$, then the identity frame $\underline{W} \to E^\perp$ is complementary to $\mathbf{0}$.

Said another way, the complement of the $\mathbf{0}$-frame is the ambient frame, and the complement of the ambient frame is the $\mathbf{0}$-frame. Understanding this makes it easier to analyze the collapse map associated to the embeddings of orbit and disks mentioned in Chapter 4.

3.1.2 Twisting Frames

Let $O(W, V)$ denote the G-space of linear isometries between two representations $W \to V$. The G-action on $O(W, V)$ is conjugation: for $g \in G$, $f \in O(W, V)$, and $w \in W$ we have $(g \cdot f)(w) = gf(g^{-1}w)$. If $W = V$ we simply write $O(V)$. Let $E \to X$ be an equivariant vector bundle and let Φ be a V-frame of E. Given an equivariant map $a : X \to O(W, V)$ we can define the twist of Φ by a to be the composite

$$\begin{array}{ccccc} \underline{W} & \xrightarrow{a} & \underline{V} & \xrightarrow{\Phi} & E \\ (x,w) & \longmapsto & (x, a_x(w)) & \longmapsto & \Phi(x, a_x(w)) \end{array}$$

and denote it $\mathrm{Tw}_a(E, \Phi)$. This is a W-frame and its homotopy class depends only on the homotopy classes of the twisting map a and the original frame Φ. We can also go backwards: given some W-frame Ψ we can define a twisting map

$$\Psi/\Phi : X \longrightarrow O(W, V)$$

$$x \longmapsto \Phi_x^{-1} \circ \Psi_x$$

such that $(E, \Psi) = \mathrm{Tw}_{\Psi/\Phi}(E, \Phi)$.

Example 3.7. When framing the circle, it's tempting to use a normal frame that is itself $U(1)$-left-invariant, like the one illustrated below in Figure 3.3.

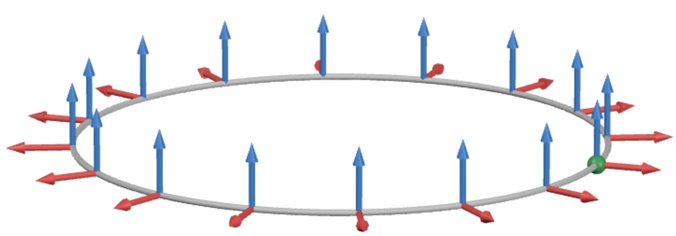


Figure 3.3. The naïve normal frame on S^1.

This frame differs from the frame in Figure 3.2 by a degree 1 twisting map $S^1 \to O(\mathbb{R}^2)$. ❧

In addition to twisting frames on E, it also proves useful to twist frames on the product $G/H \times E \to G/H \times X$ for subgroups $H \leq G$. In this case we can consider the special class of twisting maps which factor through projection to G/H

$$\begin{array}{ccc} G/H \times X & \xrightarrow{a} & O(W, V). \\ & {\scriptstyle pr_1} \searrow \quad \nearrow {\scriptstyle c} & \\ & G/H & \end{array}$$

The second map $c : G/H \to O(W,V)$ can be regarded as the non-equivariant constant map $c_\varphi : \{*\} \to O(W,V)^H$ which picks out an H-isomorphism $\varphi : \operatorname{Res}_H W \to \operatorname{Res}_H V$. These kinds of twists deserve a special notation

$$\mathrm{Tw}^H_\varphi(E,\Phi) := \mathrm{Tw}_{c_\varphi \circ pr_1}(G/H \times E, \mathrm{id}_{G/H} \times \Phi).$$

Even though we think of $\mathrm{Tw}^H_\varphi(E,\Phi)$ as a twist of a frame on E, it is important to remember that total space of $\mathrm{Tw}^H_\varphi(E,\Phi)$ is actually $G/H \times E$ and that its base space is $G/H \times X$.

Here are some basic facts about twists.

Proposition 3.8. *Let $H \leq G$. For $i = 1, 2$, let (E_i, Φ_i) be an equivariant vector bundle over X_i, let Φ_i be a V_i-frame, let $\varphi_i : \operatorname{Res}_H W_i \to \operatorname{Res}_H V_i$ be an H-isomorphism, and define $(E_1,\Phi_1)\cdot(E_2,\Phi_2) = (E_1 \times E_2, \Phi_1 \oplus \Phi_2)$. We will omit the subscript if we only need a single framed bundle to state a property.*

a) If there is a path $\varphi_1 \rightsquigarrow \varphi_2$ in $O(W,V)^H$, then $\mathrm{Tw}^H_{\varphi_1}(E,\Phi) \simeq \mathrm{Tw}^H_{\varphi_2}(E,\Phi)$.

b) When $H = G$, we have $\mathrm{Tw}^G_\varphi(E,\Phi) = \mathrm{Tw}_{\underline{\varphi}}(E,\Phi)$ where $\mathrm{Tw}_{\underline{\varphi}}(E,\Phi)$ is the global twist $\underline{W} \xrightarrow{\varphi} \underline{V} \xrightarrow{\Phi} E$.

c) Global twists Tw^G_ψ satisfy the identities

$$\mathrm{Tw}^G_\psi \circ \mathrm{Tw}^H_\varphi = \mathrm{Tw}^H_{\psi\circ\varphi}$$
$$\mathrm{Tw}^H_\varphi \circ \mathrm{Tw}^G_\psi = \mathrm{Tw}^H_{\varphi\circ\psi}$$

d) Twists are compatible with products in the sense that

$$\mathrm{Tw}^H_{\varphi_1 \oplus \mathrm{id}}((E_1,\Phi_1)\cdot(E_2,\Phi_2)) = \mathrm{Tw}^H_{\varphi_1}(E_1,\Phi_1)\cdot(E_2,\Phi_2)$$
$$\mathrm{Tw}^H_{\mathrm{id}\oplus\varphi_2}((E_1,\Phi_1)\cdot(E_2,\Phi_2)) = (E_1,\Phi_1)\cdot\mathrm{Tw}^H_{\varphi_2}(E_2,\Phi_2)$$

e) If $V_1 = V_2 = V$ and $W_1 = W_2 = W$, then

$$\mathrm{Tw}^H_\varphi(E_1,\Phi_1)\cdot(E_2,\Phi_2) \simeq \mathrm{Tw}^G_{\tau_{W,V}}\left((E_1,\Phi_1)\cdot\mathrm{Tw}^H_{-\varphi}(E_2,\Phi_2)\right)$$

where $\tau_{W,V} : W \oplus V \to V \oplus W$ is the monoidal twist homomorphism.

f) If we additionally have $W = V$, then the previous equivalence becomes

$$\mathrm{Tw}^H_\varphi(E_1,\Phi_1)\cdot(E_2,\Phi_2) \simeq (E_1,\Phi_1)\cdot\mathrm{Tw}^H_\varphi(E_2,\Phi_2).$$

Proof. Most of these are straightforward. The last two equivalences *(e)* and *(f)* follow from *(a)*, *(c)*, and *(d)* using monoidal twist axioms and the fact there are paths between $\tau_{V,V}$, $\mathrm{id}_V \oplus -\mathrm{id}_V$, and $-\mathrm{id}_V \oplus \mathrm{id}_V$ in $O(V \oplus V)^G$. $\square$

Sometimes we will want to compare frames on bundles whose bases might not even be G-homeomorphic. For example ∂_θ is technically a different frame on $U(1)_{\mathrm{conj}}$ than $U(1)_{\mathrm{triv}}$ since on the former it is an $\mathbb{R}^{1,1}$-frame and on the latter it is an $\mathbb{R}^{1,0}$-frame. However, the (non-equivariant) identity map $\mathrm{id}_{U(1)} : U(1)_{\mathrm{conj}} \to U(1)_{\mathrm{triv}}$ is still frame-preserving in some sense. Using twisting operators will let us upgrade this into a genuinely V-frame-preserving equivariant map for either choice of $V = \mathbb{R}^{1,1}$ or $\mathbb{R}^{1,0}$.

For the general setup, we will consider H-vector bundle equivalences $F : E_1 \to E_2$ which cover an H-homeomorphism $F_0 : X_1 \to X_2$. We can then induce a G-vector bundle equivalence $\widetilde{F} : G/H \times E_1 \to G/H \times E_2$ covering a G-homeomorphism $\widetilde{F_0} : G/H \times X_1 \to G/H \times X_2$.

Lemma 3.9. *Let (E_1, Φ_1) and (E_2, Φ_2) be W- and V-framed vector bundles over G-spaces X_1 and X_2 respectively. Furthermore, let $\varphi : \mathrm{Res}_H W \to \mathrm{Res}_H V$ be an H-isomorphism and let $F : E_1 \to E_2$ be an H-vector bundle equivalence respecting φ in the sense that $F \circ \Phi_1 = \Phi_2 \circ (\varphi \times F_0)$. Then there is a unique W-frame-preserving G-equivalence*

$$\widetilde{F} : \mathrm{Tw}^H_{\mathrm{id}_W}(E_1, \Phi_1) \to \mathrm{Tw}^H_{\varphi}(E_2, \Phi_2).$$

which restricts to F on $\{eH\} \times E_1$.

Proof. We know that on $\{eH\} \times E_1$ we want to define $\widetilde{F}$ to be equal to F. On the other components $\{gH\} \times E_1$ equivariance forces $\widetilde{F}$ to be defined according to the diagram

$$\begin{array}{ccc} \{gH\} \times E_1 & \xrightarrow{\widetilde{F}|_{\{gH\}\times E_1}} & \{gH\} \times E_2 \\ {\scriptstyle g^{-1}\cdot}\downarrow & & \uparrow{\scriptstyle g\cdot} \\ \{eH\} \times E_1 & \xrightarrow{F} & \{eH\} \times E_2. \end{array}$$

Verifying that this is well-defined, actually G-equivariant, and W-frame-preserving is routine. $\square$

We can think of this as pushing the W-frame Φ_1 forward onto $\mathrm{Tw}_{H,\varphi}(E_2, \Phi_2)$ using F and φ. Alternatively, we could reformulate this as pulling the V-frame backwards onto $\mathrm{Tw}_{H,\varphi}(E_1, \Phi_1)$.

Corollary 3.10. *The map $\widetilde{F}$ from Lemma 3.9 is also the unique V-frame-preserving G-equivalence*

$$\mathrm{Tw}^H_{\varphi^{-1}}(E_1, \Phi_1) \to \mathrm{Tw}^H_{\mathrm{id}_V}(E_2, \Phi_2).$$

which restricts to F on $\{eH\} \times E_1$.

Example 3.11. Let C_3 act on $W = \mathbb{C}$ by third roots of unity and let it act trivially on $V = \mathbb{R}^2$. The standard identification $\varphi : \mathbb{C} \to \mathbb{R}^2$ is an H-isomorphism for $H = \{e\}$. According to Lemma 3.9 we can push forward a $\mathbb{C}$-frame onto $C_3 \times \mathbb{R}^2$. This is illustrated in Figure 3.4.

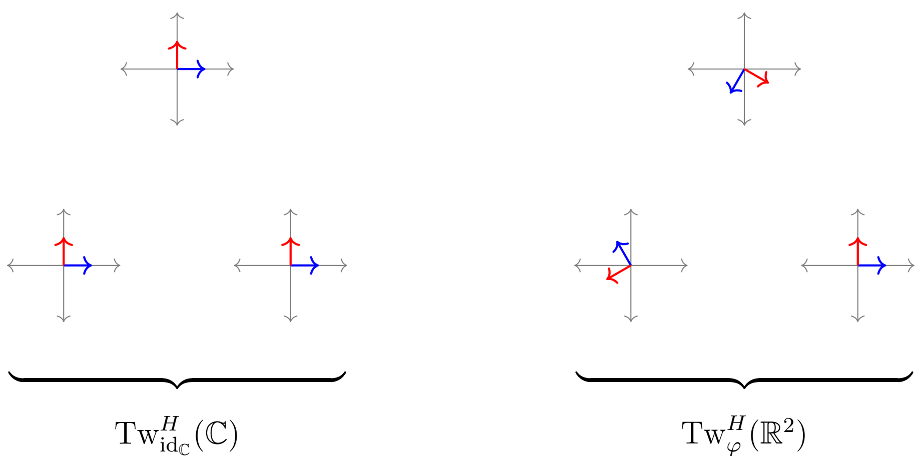


Figure 3.4. Equivalent W-framed bundles over the cyclic C_3-orbit.

On the left side of Figure 3.4 the action is given by a 120° rotation counterclockwise about the center of the three components. Meanwhile, on the right side the action only cyclically permutes the components with no rotation. It is a good exercise to think through the analogous figure for Corollary 3.10 which puts an $\mathbb{R}^2$-frame on $C_3 \times \mathbb{C}$. ❧

There is also a left action of the Weyl group on $F \in \mathrm{Hom}(E_1, E_2)^H$ where $nH \in W_G H$ sends $F \mapsto nFn^{-1}$. This is slightly subtle because n^{-1} on the left uses E_1's

G-action, but n on the right uses E_2's G-action. The map nFn^{-1} respects the H-homomorphism $n\varphi n^{-1}$. How do the extensions $\widetilde{F}$ and $\widetilde{nFn^{-1}}$ coming from Lemma 3.9 compare? It turns out they are related through intertwining with the *right* action of W_GH on G/H. This action sends $gH \mapsto gnH$ and gives us G-vector bundle equivalences $(\cdot n) \times \mathrm{id}_{E_i} : G/H \times E_i \to G/H \times E_i$, $i = 1, 2$. Note that this map only acts on the first factor

Proposition 3.12. *The following square commutes:*

$$\begin{array}{ccc} \mathrm{Tw}^H_{\mathrm{id}_W}(E_1, \Phi_1) & \xrightarrow{\widetilde{nFn^{-1}}} & \mathrm{Tw}^H_{n\varphi n^{-1}}(E_2, \Phi_2) \\ {\scriptstyle (\cdot n)\times \mathrm{id}_E}\downarrow & & \uparrow{\scriptstyle (\cdot n^{-1})\times \mathrm{id}_E} \\ \mathrm{Tw}^H_{\mathrm{id}_W}(E_1, \Phi_1) & \xrightarrow{\widetilde{F}} & \mathrm{Tw}^H_{\varphi}(E_2, \Phi_2). \end{array}$$

Proof. First we do a diagram chase just on $\mathrm{Tw}^H_{\mathrm{id}_W}(E_1, \Phi_1)|_{\{eH\}\times E_1}$.

$$\begin{array}{ccc} (eH, x, v) & \xmapsto{\widetilde{nFn^{-1}}|_{\{eH\}\times E_1} = nFn^{-1}} & (eH, nF_0(n^{-1}x), nF(n^{-1}v)) \\ \downarrow{\scriptstyle (\cdot n)\times \mathrm{id}_{E_1}} & & \uparrow{\scriptstyle (\cdot n^{-1})\times \mathrm{id}_{E_2}} \\ (nH, x, v) & \overset{\widetilde{F}|_{\{nH\}\times E_1}}{\dashrightarrow} & (nH, nF_0(n^{-1}x), nF(n^{-1}v)) \\ {\scriptstyle n^{-1}\cdot}\downarrow & & \uparrow{\scriptstyle n\cdot} \\ (eH, n^{-1}x, n^{-1}v) & \xmapsto{\widetilde{F}|_{\{eH\}\times E_1} = F} & (eH, F_0(n^{-1}x), F(n^{-1}v)) \end{array}$$

The bottom square is the one used to define the extension $\widetilde{F}$ in Lemma 3.9. One should be very careful to track when n is acting on the left versus the right. When acting from the left it acts diagonally on $G/H \times E_i$, but when acting from the right it only acts on the first factor.

Commutativity for the other components $\{gH\} \times E_i$ follows from the fact that all the maps involved are G-equivariant. □

The horizontal maps in Proposition 3.12 are frame preserving since they come from Lemma 3.9. It is easy to see that the vertical map on the left is also frame-preserving. Since each map in the square is a bundle equivalence, it follows that the right-hand map

$$\mathrm{Tw}^H_{\varphi}(E_2, \Phi_2) \xrightarrow{(\cdot n^{-1})\times \mathrm{id}_{E_2}} \mathrm{Tw}^H_{n\varphi n^{-1}}(E_2, \Phi_2) \tag{3.1}$$

is also frame-preserving. This tells us something important. Here we have two ways to push a W-frame onto the V-framed bundle E_2; using either F and φ or nFn^{-1} and $n\varphi n^{-1}$. The map (3.1) tells us there is a W-frame-preserving equivalence between them *which only acts on the factor G/H*. In fact, we could call this is an equivalence between the twisting operators $\mathrm{Tw}_{\varphi}^{H}$ and $\mathrm{Tw}_{n\varphi n^{-1}}^{H}$ themselves.

Example 3.13. Back in Example 3.3 we had a C_2-vector bundle E that admitted an $\mathbb{R}^{1,0}$-frame Φ_1 and an $\mathbb{R}^{1,1}$-frame Φ_2. We can now express these frames in terms of twists

$$(E, \Phi_1) = \mathrm{Tw}_{\mathrm{id}_{\mathbb{R}^{1,0}}}^{H}(\mathbb{R}^{1,0}), \text{ and}$$
$$(E, \Phi_2) = \mathrm{Tw}_{\mathrm{id}_{\mathbb{R}^{1,1}}}^{H}(\mathbb{R}^{1,1})$$

where $H \leq C_2$ is the trivial subgroup.

There are two H-isomorphisms $\pm\varphi : \mathbb{R}^{1,0} \to \mathbb{R}^{1,1}$ which on the level of vector spaces are just $\pm \mathrm{id}_{\mathbb{R}^1}$. By Lemma 3.9 we these two maps let us push forward the $\mathbb{R}^{1,0}$-frame onto $G/H \times \mathbb{R}^{1,1}$ in two different ways, yielding $\mathrm{Tw}_{\varphi}^{H}(\mathbb{R}^{1,1})$ and $\mathrm{Tw}_{-\varphi}^{H}(\mathbb{R}^{1,1})$. These frames are not homotopic, but there is an equivalence between them which covers a nontrivial homeomorphism of the base space $\{a, b\} = G/H$. The generator $\tau \in C_2$ also generates the Weyl group $W_{C_2}H$, so it acts on $\{a, b\}$ (on the right) by exchanging the two points. By Proposition 3.12 we get an associated frame-preserving equivalence

$$\mathrm{Tw}_{\varphi}^{H}(\mathbb{R}^{1,1}) \to \mathrm{Tw}_{\tau\varphi\tau}^{H}(\mathbb{R}^{1,1}).$$

Recall that τ acts trivially before applying φ and by a sign afterwards, so $\tau\varphi\tau = -\varphi$.

Hence, while φ and $-\varphi$ induce non-homotopic $\mathbb{R}^{1,0}$-frames on E, there is still a framed bundle equivalence $\mathrm{Tw}_{\varphi}^{H}(\mathbb{R}^{1,1}) \cong \mathrm{Tw}_{-\varphi}^{H}(\mathbb{R}^{1,1})$. This observation is related to the 2-torsion that appears in $\pi_{0,2q}$. ❧

Remark. There is another perspective on V-frames on E which uses E's classifying map $f : X \xrightarrow{E} B_G O$. If $B_G O$ is modeled as the Grassmannian of $|V|$-planes in a representation W which contains V, then a V-frame can be defined as a null-homotopy of f which terminates at V. From this perspective the twisting operators can be reinterpreted in terms of paths in the Grassmannian. More details on this are given in Appendix A.

3.2 Equivariant Framed Bordism

Now that we have a handle on V-frames we can finally define equivariant singular framed bordism. First, we say a manifold M is V-framed if TM is V-framed. A **singular** framed manifold of a pair (X, A) is a compact V-framed manifold M, potentially with boundary, equipped with an equivariant continuous map $\sigma : (M, \partial M) \to (X, A)$ called the **singular map** since it has no special conditions like smoothness.

Definition 3.14. *Let (M_i, σ_i, Φ_i), $i = 0, 1$ be singular V-framed manifolds of (X, A). A **framed bordism** from M_0 to M_1 is a singular $(V \oplus \mathbb{R}^1)$-framed manifold (W, σ_W, Ψ) with disjoint inclusions $\iota_i : M_i \hookrightarrow \partial W$ such that,*

a) $\sigma_W \circ \iota_i = \sigma_i$,

b) $\Psi \circ \xi_i = d\iota_i \circ \Phi_i$, where $\xi_i : M_i \times V \hookrightarrow W \times (V \oplus \mathbb{R}^1)$

c) $\Psi|_{\underline{\mathbb{R}^1}}$ is inward pointing on $TW|_{M_0}$ and outward pointing on $TW|_{M_1}$, and

d) $\partial W = \iota_0(M_0) \cup \underset{\rightsquigarrow}{\partial} W \cup \iota_1(M_1)$ with $\sigma_W(\underset{\rightsquigarrow}{\partial} W) \subseteq A$ and $\iota_i(M_i) \cap \underset{\rightsquigarrow}{\partial} W = \iota_i(\partial M_i)$.

We use a directed convention for bordisms because it makes keeping track of equivariant orientations easier and because it corresponds more naturally to the directed nature of homotopies. Also note that (M, σ, Φ) and (M, σ', Φ') are bordant via a cylinder if σ and Φ are homotopic to σ' and Φ'.

Bordism classes of singular V-framed manifolds of a pair (X, A) form a group $\Omega_V^{fr}(X, A)$ in the familiar way using disjoint union as addition. The Cartesian product of singular manifolds gives an external product operation

$$\Omega_V^{fr}(X, A) \times \Omega_W^{fr}(Y, B) \to \Omega_{V \oplus W}^{fr}(X \times Y, A \times Y \cup X \times B).$$

Disks $(D(U), S(U))$ have a tautological U-frame $\mathscr{T}$ and can be equipped with the $\mathrm{id}_{D(U)}$ for a singular map. This yields singular framed bordism fundamental class $[D(U), \mathrm{id}, \mathscr{T}] \in \Omega_U^{fr}(D(U), S(U))$. Since there is a G-homeomorphism of pairs

$$(D(U_1), S(U_1)) \times (D(U_2), S(U_2)) \cong (D(U_1 \oplus U_2), S(U_1 \oplus U_2)),$$

we can repeatedly multiply by fundamental classes to get a system of homomorphisms

$$\Omega_V^{fr}(D(W), S(W)) \xrightarrow{\times D(U)} \Omega_{V \oplus U}^{fr}(D(W \oplus U), S(W \oplus U))$$

which stabilize to $RO(G)$-graded groups Ω^{fr}_{V-W}. Together these stable groups form an $RO(G)$-graded ring and it this ring of stable bordism classes that appears in the equivariant Pontryagin-Thom theorem.[1]

Theorem 3.15 ([19])**.** *For finite groups G, the Pontryagin-Thom collapse construction gives an isomorphism of $RO(G)$-graded rings*

$$\pi^G_{V-W} \cong \lim_{U \subset \mathcal{U}} \Omega^{fr}_{V \oplus U}(D(W \oplus U), S(W \oplus U)).$$

We'll generally use brackets $[M, \sigma, \Phi]$ when we're thinking of a singular framed manifold in terms of its stable bordism/homotopy class, and parentheses (M, σ, Φ) if we are thinking of it has an object in its own right. For convenience, we recall the singular version of the collapse map here.

Construction 3.16. Let (M, σ, Φ) be a singular V-framed manifold of a pair (X, A). For M with boundary, it is convenient to define the Thom space of a bundle $E \to M$ to be

$$\mathrm{Th}(E) = \frac{D(E)}{S(E) \cup D(E)|_{\partial M}}$$

which coincides with the usual definition when $\partial M = \emptyset$. Assume we have an embedding $f : M \hookrightarrow U$ where $V \subset U$ and which admits a U'-frame $\Phi^\perp$ complementary to Φ, where $U' = V^{\perp U}$. The associated collapse map is then

$$S^U \longrightarrow \mathrm{Th}(\nu(M)) \xrightarrow{(\Phi^\perp)^{-1}} \mathrm{Th}(\underline{U'}) = (M/\partial M) \wedge S^{U'} \xrightarrow{\sigma \wedge \mathrm{id}} X/A \wedge S^{U'}.$$

Since we will always be using $(X, A) = (D(W), S(W))$ the collapse map specializes to

$$S^U \longrightarrow \mathrm{Th}(\nu(M)) \xrightarrow{(\Phi^\perp)^{-1}} \mathrm{Th}(\underline{U'}) = (M/\partial M) \wedge S^{U'} \xrightarrow{\sigma \wedge \mathrm{id}} S^W \wedge S^{U'} = S^{W \oplus U'}$$

which lies in π^G_{V-W} since $U - (W \oplus U') = (V \oplus U') - (W \oplus U') = V - W$. ◊

[1]This stabilization process makes it so we never have to deal with stable tangent bundles. The bundle stabilization is subsumed by stabilizing the underlying manifold itself.

3.2.1 Isomorphisms Of Singular Framed Manifolds

Part of the data of a framed bordism as given in Definition 3.14 are the inclusions of the initial and terminal manifolds into the boundary of the bordism. Even the cylinder $M \times I$ may represent a nontrivial bordism between non-homotopic V-frames on M if the initial and terminal inclusions of M into the boundary of the cylinder are different. To be precise about this, we will review what exactly makes an isomorphism of V-framed manifolds.

Definition 3.17. *An isomorphism of singular V-framed manifolds (M, σ_M, Φ_M) and (N, σ_N, Φ_N) of (X, A) is an equivariant diffeomorphism $F : M \to N$ which makes the following diagrams commute:*

$$\begin{array}{ccc} \underline{V} & \xrightarrow{\Phi_N} & TN \\ {\scriptstyle \mathrm{id}_V \times F}\uparrow & & \uparrow{\scriptstyle dF} \\ \underline{V} & \xrightarrow[\Phi_M]{} & TM \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} N & & \\ {\scriptstyle F}\uparrow & \searrow^{\sigma_N} & \\ M & \xrightarrow[\sigma_M]{} & X. \end{array}$$

An isomorphism $F : M \to N$ lets us produce a singular framed bordism $N \times I : M \rightsquigarrow N$ where we include $M \hookrightarrow N \times \{0\}$ using F and include $N \hookrightarrow N \times \{1\}$ using id_N.

Example 3.18. It is well-known that $2\eta_{\text{top}} = 0$, or equivalently that $\eta_{\text{top}} = -\eta_{\text{top}}$. This relation can be understood as coming from an isomorphism of $\mathbb{R}^1$-framed manifolds $(U(1), \partial_\theta) \to (U(1), -\partial_\theta)$ which represent η_{top} and $-\eta_{\text{top}}$, respectively. The isomorphism is complex conjugation.

This is not to be confused with treating complex conjugation as a C_2-action on $U(1)$. In fact, in Example 5.1 we treat complex conjugation simultaneously as a C_2-action on $U(1)_{\text{conj}}$ and as a C_2-isomorphism of $\mathbb{R}^{1,1}$-framed manifolds $(U(1)_{\text{conj}}, \partial_\theta) \to (U(1)_{\text{conj}}, -\partial_\theta)$. This gives us a relation in $\pi_{1,1}$, but we'll need ideas from Section 4.2 to understand what the relation is. ❧

Given a singular framed manifold $(M, \sigma, \Phi) \in \Omega^{fr}_V(D(W), D(W))$ and an H-homomorphism $\varphi : \mathrm{Res}_H \underline{U} \to \mathrm{Res}_H \underline{V}$, we can define $\mathrm{Tw}^H_\varphi(M)$ to be the manifold $G/H \times M$ with framed tangent bundle $\mathrm{Tw}^H_\varphi(TM, \Phi)$ and with a singular map given by the composition $G/H \times M \overset{pr_2}{\to} M \overset{\sigma}{\to} D(W)$. This puts $\mathrm{Tw}^H_\varphi(M) \in \Omega^{fr}_U(D(W), S(W))$.

Lemma 3.9 showed us how to turn an H-equivalence of framed bundles into a G-equivalence using twisting operators. We also saw in Proposition 3.12 and Equation (3.1) that the twisting operators intertwine with the right action of $W_G H$ on G/H. Propositions 3.19 and 3.20 below give singular versions of these ideas.

Proposition 3.19. *Let $F : M \to N$ be an H-isomorphism of singular U- and V-framed manifolds M and N of the pair $(D(W), S(W))$. Suppose F respects an H-isomorphism $\varphi : \mathrm{Res}_H U \to \mathrm{Res}_H V$. Then there is an G-isomorphism $\mathrm{Tw}^H_{\mathrm{id}_U}(M) \cong \mathrm{Tw}^H_{\varphi}(N)$ of singular U-framed manifolds of $(D(W), S(W))$.*

Proof. We already know that $\widetilde{F} : \mathrm{Tw}^H_{\mathrm{id}_U}(M) \to \mathrm{Tw}^H_{\varphi}(N)$ is a G-equivalence of framed bundles, so we just need to check that it respects the singular maps. That is, we need to see that the following diagram commutes:

$$\begin{array}{ccccc} G/H \times N & \xrightarrow{pr_2} & N & & \\ {\scriptstyle \widetilde{F}}\big\uparrow & & & \searrow^{\sigma_N} & \\ G/H \times M & \xrightarrow{pr_2} & M & \xrightarrow{\sigma_M} & D(U). \end{array} \tag{3.2}$$

Since $\widetilde{F}|_{\{eH\}\times M} = F$, we know the diagram commutes on the identity component:

$$\begin{array}{ccccc} \{eH\} \times N & \xrightarrow{pr_2} & N & & \\ {\scriptstyle \widetilde{F}_{\{eH\}\times M}}\big\uparrow & & \big\uparrow{\scriptstyle F} & \searrow^{\sigma_N} & \\ \{eH\} \times M & \xrightarrow{pr_2} & M & \xrightarrow{\sigma_M} & D(U). \end{array}$$

On the other components, $\widetilde{F}$ was defined according to the diagram

$$\begin{array}{ccc} \{gH\} \times N & \xleftarrow{g\cdot} & \{eH\} \times N \\ {\scriptstyle \widetilde{F}|_{\{gH\}\times M}}\vdots\uparrow & & \big\uparrow{\scriptstyle F} \\ \{gH\} \times M & \xrightarrow{g^{-1}\cdot} & \{eH\} \times M. \end{array} \tag{3.3}$$

So on the $\{gH\}$-component, the diagram (3.2) looks like

$$\begin{array}{ccccc} \{gH\} \times N & \xrightarrow{pr_2} & N & & \\ {\scriptstyle \widetilde{F}|_{\{gH\}\times M}}\big\uparrow & & {\scriptstyle gFg^{-1}}\big\uparrow & \searrow^{\sigma_N} & \\ \{gH\} \times M & \xrightarrow{pr_2} & M & \xrightarrow{\sigma_M} & D(U). \end{array} \tag{3.4}$$

The square on the left of (3.4) is just a repackaging of (3.3), so it commutes. All that remains is to check commutativity for the triangle on the right of (3.4). This follows from the fact that F commutes with the σ's and that the σ's themselves are both G-equivariant:

$$\sigma_N g F g^{-1} = g \sigma_N F g^{-1}$$

$$= g\sigma_M g^{-1}$$
$$= \sigma_M.$$

□

Proposition 3.20. *Let M be a singular V-framed manifold of $(D(W), S(W))$ and let $\varphi : \mathrm{Res}_H U \to \mathrm{Res}_H V$ be an H-isomorphism. For $n \in W_G H$ the map $(\cdot n^{-1}) \times \mathrm{id}_M$ is a G-isomorphism of singular U-framed manifolds $\mathrm{Tw}^H_\varphi(M) \cong \mathrm{Tw}^H_{n\varphi n^{-1}}(M)$.*

Proof. Like for Proposition 3.19, we know that $(\cdot n^{-1}) \times \mathrm{id}_M$ is a G-equivalence on the level of framed manifolds, so we just need to verify that it also respects the singular maps. This is easier; the diagram

$$\begin{array}{ccccc} G/H \times M & & & & \\ {\scriptstyle (\cdot n^{-1})\times \mathrm{id}_M}\uparrow & \searrow^{pr_2} & & & \\ G/H \times M & \xrightarrow{pr_2} & M & \xrightarrow{\sigma} & D(U) \end{array}$$

clearly commutes. □

3.2.2 Lie Groups as Framed Manifolds

For this section, we'll write Γ for the ambient symmetry group. This lets us use G for a Lie group which is a Γ-manifold. Classically, a Lie group G can be framed in an easy way: pick a basis at the identity T_eG and then left-translate using G's action on itself by left-multiplication to get global sections. The only critical choice is the orientation of the basis.

Perhaps unsurprisingly, we can let $\mathrm{Aut}(G)$ act on G by automorphisms and frame G by its own adjoint representation. Specifically, let $\mathfrak{g} = T_eG$ and define a linear $G \rtimes \mathrm{Aut}(G)$ action on $\mathfrak{g}$ where the factor G acts trivially and $\mathrm{Aut}(G)$ acts by differentials. Then we have the following result, whose proof we defer to Appendix C.

Proposition 3.21. *As a $(G \rtimes \mathrm{Aut}(G))$-manifold, G is naturally $\mathfrak{g}$-framed.*

Remark. Even though the set of unit octonions S^7 is not a Lie group, a version of Proposition 3.21 holds for $S(\mathbb{O})$ and $G_2 = \mathrm{Aut}(\mathbb{O})$ by essentially the same proof, less associativity.

We call this the adjoint $\mathfrak{g}$-frame on G. However, keep in mind that we are also acting by outer automorphisms and left-translation. This is slightly more general that what is typically meant by the adjoint representation where only inner automorphisms are used. As a framed bordism class we will write this as $[G, \mathscr{L}]$ and refer to $\mathscr{L}$ as "the Lie frame".

This proposition can be leveraged in different ways depending on how we let Γ act on G as determined by a homomorphism $h : \Gamma \to G \rtimes \mathrm{Aut}(G)$. However, when working in the $RO(\Gamma)$-graded setting we should be careful to remember how we decompose $\mathrm{Res}_\Gamma\, \mathfrak{g}$ into irreducible Γ-representations. This is analogous to keeping track of orientation in the classical case.

Example 3.22. The isomorphism $C_2 \cong \mathrm{Aut}(U(1))$ puts the complex conjugation action on $U(1)$. Picking the basis vector $\partial_\theta|_{\theta=0}$ at the identity element of $U(1)$ determines an identification $\mathfrak{u}(1) \cong \mathbb{R}^{1,1}$. Observe how the C_2-action is compatible with left-translation by elements $z \in U(1)$. Translating the vector $\partial_\theta|_{\theta=0}$ by $z = e^{i\phi}$ gives $\partial_\theta|_{\theta=\phi}$, and then conjugating gives $-\partial_\theta|_{\theta=-\phi}$. This is equivalent to translating the conjugated vector $-\partial_\theta|_{\theta=0}$ by the conjugated element $\overline{z} = e^{-i\phi}$.

There are three other homomorphisms $C_2 \to U(1) \rtimes \mathrm{Aut}(U(1))$. The trivial homomorphism yields the representative for η_{top}. The remaining two homomorphisms are nontrivial on the $U(1)$-factor, and they both produce representatives for η_{free}. ☙

The Lie frame is compatible with taking fixed-points. If $\Gamma \leq \mathrm{Aut}(G)$, then the frame induced by taking fixed-points of the Lie frame is the same as the Lie frame of the subgroup of fixed-points.

Proposition 3.23. $[G, \mathscr{L}_G]^\Gamma = [G^\Gamma, \mathscr{L}_{G^\Gamma}]$.

That is, we can induce a frame on the fixed-points G^H using the Lie frame on G or we treat G^H as its own Lie group and use its own left-invariant Lie frame. These are the same. The proof of this fact and discussion about some of its easy consequences can also be found in Appendix C.

3.2.3 Excision

One useful technique for a number of constructions is a kind of excision on singular maps. Given a framed manifold (M, Φ) equipped with the singular map $\sigma : (M, \partial M) \to (D(W), S(W))$, we can cut out and ignore any piece of M which σ sends to the boundary $S(W)$. To avoid dealing with smoothing corners we will only prove this for M closed, since that is also the only case where we apply it.

Lemma 3.24. *Let $[M, \sigma, \Phi] \in \Omega_V^{fr}(D(W), S(W))$ be a closed singular V-framed manifold. Suppose that there is a compact invariant submanifold $N \subseteq M$ of codimension 0 such that $\sigma(M \setminus \operatorname{int} N) \subset S(W)$. Then there is an equivariant singular framed bordism from (M, σ, Φ) to $(N, \sigma|_N, \Phi|_N)$.*

Proof. Let $\varphi : M \to [0, 1]$ be a smooth bump function for N so that $\varphi(x) = 1$ if and only if $x \in N$. We can shift and scale φ so that it is never 0. By averaging over G we can also assume that φ is G-invariant. This setup makes it so that the subset

$$W = \{(x, t) \in M \times I \mid t \leq \varphi(x)\}$$

is a smooth G-manifold with boundary $\partial W = \Gamma(\varphi) \sqcup (M \times \{0\})$ the disjoint union of the graph φ and the base of the cylinder. We define the edge of W to be the subset of the boundary

$$\underset{\leadsto}{\partial} W = \Gamma(\varphi) \setminus (\operatorname{int} N \times \{1\}).$$

We then put a frame and singular map on W by restricting the frame $\Phi \oplus \partial_t$ and singular map $\sigma \circ pr_1$ from the cylinder $M \times I$. This makes W a singular framed bordism from (M, σ, Φ) to $(N, \sigma|_N, \Phi|_N)$; the edge is sent to $S(W)$ and the data on W agree with the data on the terminal and initial manifolds $M \times \{0\}$ and $N \times \{1\}$. $\square$

Generally, we must perturb σ before we can use Lemma 3.24 in a meaningful way.

Example 3.25. Let (M, σ, Φ) be any non-equivariant singular framed k-dimensional manifold of $D(\mathbb{R}^n)$ with $k < n$. By smoothing σ and using Sard's theorem, or just by simplicial approximation, we can perturb σ so that some interior point of $D(\mathbb{R}^n)$ has a neighborhood disjoint from the image of M. This allows us to produce a homotopy which sends all of M to the boundary of the disk. We can then set $N = \emptyset$ in Lemma 3.24 which lets us delete all of M. This is why classical negative stems π_{k-n}^s are all 0. ☙

II

CONSTRUCTIONS

CHAPTER 4

THE TOPOLOGICAL 0-STEM

There is an important class of elements which live in the topological 0-stem $\bigoplus_{|V|=0} \pi_V^G$. These are the θ_φ^H's mentioned in the introduction and they are closely related to the twisting operators. We'll start by looking at the $RO(G)$-graded $\mathbf{0}$-stem where the story is well-known.

4.1 The Burnside Ring

Since G is a finite group, each orbit G/H is a discrete 0-dimensional G-manifold and, as such, is naturally $\mathbf{0}$-framed. The isomorphism type of G/H as a ($\mathbf{0}$-framed) G-manifold depends only on the conjugacy class of H. The Burnside ring $A(G)$ is the Grothendieck ring of these orbit classes $[G/H]$. As an abelian group, $A(G)$ is a free $\mathbb{Z}$-module generated by the conjugacy classes of G. Picking a large enough representation V, we can embed $G/H \hookrightarrow V$ and apply the collapse map construction. Since G/H is $\mathbf{0}$-framed, by Proposition 3.6 the complementary frame on the normal bundle $\nu(G/H)$ is the restriction of the ambient frame on V. So we get amap

$$S^V \to \mathrm{Th}(\nu(G/H)) \xrightarrow{(\Phi^\perp)^{-1}} G/H_+ \wedge S^V \to S^V. \tag{4.1}$$

It is Segal's theorem [28] that every class in $\pi_{\mathbf{0}}^G$ can be expressed as $\mathbb{Z}$-linear combination of maps of the form of (4.1). That is, this collapse construction gives an isomorphism $A(G) \cong \pi_{\mathbf{0}}^G$.

We can find other singular representatives for the same map by treating the disk bundle $D(\nu(G/H))$ itself as a singular framed manifold. Restricting the ambient frame on V to $D(\nu(G/H))$ allows us to identify it with $D(\underline{V}) = D(G/H \times V)$. We can then equip $D(\underline{V})$ with projection $\sigma : D(\underline{V}) \to D(V)$ for the singular map. All together this defines an element

$$[D(\underline{V}), \sigma, \mathscr{T}] \in \Omega_V^{fr}(D(V), S(V))$$

where $\mathscr{T}$ denotes the tautological V-frame on $D(\underline{V})$.

If we apply Thom collapse to this new representative we now find that $\nu(D(\underline{V}))$ is rank 0 since $D(\underline{V})$ has the same dimension as V. Note that this is a flat embedding of $D(\underline{V})$ as a V-framed manifold. Hence the $\mathbf{0}$-frame on $\nu(D(\underline{V}))$ is complementary to $\mathscr{T}$. So the collapse map is

$$S^V \to \mathrm{Th}(\nu(D(\underline{V}))) \xrightarrow{(\mathscr{T}^\perp)^{-1}} (D(\underline{V})/S(\underline{V})) \wedge S^0 \xrightarrow{\sigma\wedge\mathrm{id}} S^V \wedge S^0 = S^V.$$

This is the same map as (4.1). However, from the perspective of the collapse construction, the work is now being done by the singular map σ rather than the complementary frame on $\nu(G/H)$. The difference may feel somewhat artificial, but we'll how it can be a useful perspective in Section 4.2. When $H = G$, this reproduces the fundamental class of the pair $(D(V), S(V))$ mentioned in Section 3.2 which represents $1 \in \pi_{\mathbf{0}}^G$.

Example 4.1. We can embed the orbit $C_2/\{e\}$ as $\{\pm 1\} \subseteq \mathbb{R}^{1,1}$. Applying the collapse construction gives a map $S^{1,1} \to S^{1,1}$ which is homotopic to the degree 2 map $z \mapsto z^2$ when we identify $S^{1,1}$ with $U(1)$ equipped with the complex conjugation action. ❧

4.2 Linear Units

While every element of $\pi_{\mathbf{0}}^G$ can be expressed in terms of elements of the Burnside ring, this may not always be the most natural representative. A class of such examples are **linear units**. A linear unit is the one-point compactification of an automorphism $a \in \mathrm{Aut}(V)$. If $a \simeq \mathrm{id}_V$ this coincides with the Burnside element $[G/G]$ and both represent the identity in π_*^G.

Example 4.2. The automorphism $-\mathrm{id} : \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$ produces a linear unit ϵ. As a stable homotopy class this is sometimes defined by the monoidal twist map $t_1 : S^{1,1} \wedge S^{1,1} \to S^{1,1} \wedge S^{1,1}$. It's easy to see that defining ϵ using t_1 gives the same element as using $-\mathrm{id}_{\mathbb{R}^{1,1}}$ because in general $\tau_{V,V} \simeq \mathrm{id}_V \oplus -\mathrm{id}_V \simeq -\mathrm{id}_V \oplus \mathrm{id}_V$. There are four linear units in $\pi_{0,0}$: 1, -1, ϵ, and $-\epsilon$. ❧

We can represent the linear unit associated to the automorphism a by the singular framed manifold

$$[D(V), a, \mathscr{T}] \in \Omega_V^{fr}(D(V), S(V)).$$

To verify this, start with the natural inclusion $D(V) \hookrightarrow V$. This is a flat embedding, so $\mathbf{0}$-frame on the trivial normal bundle complementary to $\mathscr{T}$. Looking at the collapse map

$$S^V \to \mathrm{Th}(\nu(D(V))) \xrightarrow{(\mathscr{T}^\perp)^{-1}} (D(V)/S(V)) \wedge S^0 \xrightarrow{a \wedge \mathrm{id}} S^V \wedge S^0 = S^V, \tag{4.2}$$

we see that it is homotopic to the one-point compactification of a.

We can also represent this linear unit by

$$[D(V), \mathrm{id}, \mathscr{T} \circ \underline{a}] \in \Omega_V^{fr}(D(V), S(V)).$$

This is not as obvious. If we use the natural embedding of $D(V) \hookrightarrow V$, it is *not* a flat embedding of the frame $\mathscr{T}\underline{a}$. This means the $\mathbf{0}$-frame on the normal bundle is *not* complementary and thus is not valid to use in the collapse construction. We can fix this by instead using the embedding $a^{-1} : D(V) \hookrightarrow V$. This embedding *is* flat, and so we can use the $\mathbf{0}$-frame on the normal bundle to get the collapse map

$$S^V \to \mathrm{Th}(\nu(D(V))) \xrightarrow{(\mathscr{T}^{\perp})^{-1}} (D(V)/S(V)) \wedge S^0 \xrightarrow{\mathrm{id} \wedge \mathrm{id}} S^V \wedge S^0 = S^V. \tag{4.3}$$

At first glance this looks like it might be the identity class $S^V \to S^V$, but we should remember that the embedding a^{-1} is how we are identifying a subset of S^V with the manifold $D(V)$. That is, when restricted to the image of the embedding the left-most map of (4.3) is the inverse $(a^{-1})^{-1} = a$ of the embedding map a^{-1}. So the net result of (4.3) is the same as (4.2). In this case the tangent frame is doing to the work by telling us what embedding to choose at the start of the collapse construction.

There is another way to see that $[D(V), \mathrm{id}, \mathscr{T} \circ \underline{a}]$ represents the linear unit associated to a. This time we'll go back to using the natural inclusion $D(V) \hookrightarrow V$. Remember that this is not flat, so to find a complement to $\mathscr{T}\underline{a}$ we will first have to stabilize the embedding $D(V) \hookrightarrow V \hookrightarrow V \oplus V$. We want a frame $(\mathscr{T} \circ \underline{a})^{\perp} : \underline{V} \to \nu_{V \oplus V}(D(V))$ which when added to $\mathscr{T} \circ \underline{a}$ is homotopic to the ambient frame on $V \oplus V$. To help us be a little more precise we'll write $\mathscr{T}_{D(V)}$ for the tautological frame on $D(V)$ and $\mathscr{T}_{V \oplus V}$ for the ambient frame on $V \oplus V$. Remember that these are both technically frames on $TD(V)$ and $T(V \oplus V)$, respectively. We can also restrict $\mathscr{T}_{V \oplus V}$ from $T(V \oplus V)$ to the sub-bundle $\nu_{V \oplus V}(D(V))$ which we will write as $\mathscr{T}_{\nu} : \underline{V} \to \nu_{V \oplus V}(D(V))$. So we're looking for a homotopy of $V \oplus V$-frames

$$(\mathscr{T}_{D(V)} \circ \underline{a}) \oplus (\mathscr{T}_{D(V)} \circ \underline{a})^{\perp} \simeq \mathscr{T}_{V \oplus V}|_{TD(V) \oplus \nu(D(V))}$$

on the bundle $T(V \oplus V)|_{TD(V) \oplus \nu(D(V))}$. The normal frame we should use is $(\mathscr{T}_{D(V)} \circ \underline{a})^{\perp} := \mathscr{T}_{\nu} \circ \underline{a^{-1}} : \underline{V} \to \nu_{V \oplus V}(D(V))$. Let's verify that this normal frame is actually complementary to the tangent frame $\mathscr{T}_{D(V)} \circ \underline{a}$:

$$\begin{aligned}(\mathscr{T}_{D(V)} \circ \underline{a}) \oplus (\mathscr{T}_{D(V)} \circ \underline{a})^{\perp} &= (\mathscr{T}_{D(V)} \circ \underline{a}) \oplus (\mathscr{T}_{\nu} \circ \underline{a^{-1}}) \\ &= (\mathscr{T}_{D(V)} \oplus \mathscr{T}_{\nu}) \circ \underline{(a \oplus a^{-1})} \\ &= \mathscr{T}_{V \oplus V}|_{TD(V) \oplus \nu(D(V)} \circ \underline{(a \oplus a^{-1})}.\end{aligned}$$

For any map $f : V \to V$, we can see that $f \oplus \mathrm{id}_V \simeq \mathrm{id}_V \oplus f$ using the fact $\tau_{V,V} \simeq \mathrm{id}_V \oplus -\mathrm{id}_V$:

$$f \oplus \mathrm{id} = (\mathrm{id} \oplus -\mathrm{id}) \circ (f \oplus \mathrm{id}) \circ (\mathrm{id} \oplus -\mathrm{id})$$

$$
\begin{aligned}
&\simeq \tau_{V\oplus V} \circ (f \oplus \mathrm{id}) \circ (\mathrm{id} \oplus - \mathrm{id}) \\
&= (\mathrm{id} \oplus f) \circ \tau_{V\oplus V} \circ (\mathrm{id} \oplus - \mathrm{id}) \\
&\simeq (\mathrm{id} \oplus f) \circ (\mathrm{id} \oplus - \mathrm{id}) \circ (\mathrm{id} \oplus - \mathrm{id}) \\
&= \mathrm{id} \oplus f.
\end{aligned}
$$

It follows that $(a \oplus a^{-1}) \simeq (\mathrm{id} \oplus \mathrm{id})$, so

$$
\begin{aligned}
(\mathscr{T}_{D(V)} \circ \underline{a}) \oplus (\mathscr{T}_{D(V)} \circ \underline{a})^{\perp} &= \mathscr{T}_{V\oplus V}|_{TD(V)\oplus\nu(D(V)} \circ \underline{(a \oplus a^{-1})} \\
&\simeq \mathscr{T}_{V\oplus V}|_{TD(V)\oplus\nu(D(V)} \circ \underline{\mathrm{id}_{V\oplus V}} \\
&= \mathscr{T}_{V\oplus V}|_{TD(V)\oplus\nu(D(V)}.
\end{aligned}
$$

Now that we know $\mathscr{T}_\nu \circ \underline{a^{-1}}$ is a complementary frame on $\nu_{V\oplus V}$ we can use it in the collapse construction

$$
S^{V\oplus V} \longrightarrow \mathrm{Th}(\nu(D(V)) \xrightarrow{(\mathscr{T}_\nu \circ \underline{a^{-1}})^{-1}} (D(V)/S(V)) \wedge S^V \xrightarrow{\mathrm{id} \wedge \mathrm{id}} S^V \wedge S^V = S^{V\oplus V} \tag{4.4}
$$

which we can see is homotopic to

$$
S^{V\oplus V} \xrightarrow{\mathrm{id} \oplus a} S^{V\oplus V}.
$$

Since, up to homotopy, we can move a across $\oplus$, the above map homotopic to

$$
S^{V\oplus V} \xrightarrow{a \oplus \mathrm{id}} S^{V\oplus V}.
$$

which is just the suspension of the linear unit $S^V \xrightarrow{a} S^V$ as desired. This time the tangent frame did the work by determining a nontrivial complementary frame on the normal bundle. Hopefully this makes it clear why it's easier to work with flat embeddings.

To summarize, the linear unit associated to the automorphism $a : V \to V$ can be represented by both $[D(V), a, \mathscr{T}]$ and $[D(V), \mathrm{id}, \mathscr{T} \circ \underline{a}]$, but we have to be careful about embeddings and complements when using the second representative. The advantage of the second one is that we can express it as the twist $\mathrm{Tw}_a^G(D(V))$ of the fundamental class $[D(V)]$. This allows us to employ all the theory of Section 3.1.2.

First, recall from Proposition 3.8 that Tw_a^G depends only on the path-component of $a \in O(V)^G = \mathrm{Aut}(V)$. We can compute the path-components of $\mathrm{Aut}(V)$ using Schur's lemma.

Proposition 4.3 (Schur's Lemma)**.** *Let V and W be irreducible G-representations. Then either*

$$\operatorname{Hom}(V, W) = \mathbf{0}$$

or V and W are isomorphic and

$$\operatorname{Hom}(V, W) \cong \mathbb{K}$$

where $\mathbb{K}$ is one of the real associative division algebras $\mathbb{R}, \mathbb{C},$ or $\mathbb{H}$. In the latter case we say V and W are of real-, complex-, or quaternionic- isomorphism type respectively.

For the rest of this discussion we will fix representatives I_i for each isomorphism class of irrep of G. We will order them such that the real-type irreps precede complex, and complex precede quaternionic. We will also fix isometric isomorphisms $\operatorname{End}(I_i) \cong \mathbb{K}$ which in turn give us identifications $\operatorname{Aut}(I_i^{\oplus m_i}) \cong O(m_i), U(m_i)$, or $Sp(m_i)$ in accordance with the type of I_i.

Now if we have a given decomposition of a representation V into irreps

$$V = \bigoplus_{i=1}^{\ell} I_i^{\oplus m_i}$$

it follows that

$$\begin{aligned} \operatorname{Aut}(V) &= \prod_i \operatorname{Aut}(I_i^{\oplus m_i}) \\ &\cong \left(\prod_{\mathbb{R}\text{-type } I_i} O(m_i) \right) \times \left(\prod_{\mathbb{C}\text{-type } I_i} U(m_i) \right) \times \left(\prod_{\mathbb{H}\text{-type } I_i} Sp(m_i) \right). \end{aligned}$$

Since the unitary and symplectic groups are path-connected, only the real-type irreps appearing in V contribute to the number of path-components of $\operatorname{Aut}(V)$. If there are k real-type irreps in V that have multiplicity $m_i \geq 1$, then $\operatorname{Aut}(V)$ has 2^k path-components. We can label these components by a tuple of signs $(\pm, \dots, \pm)$ indexed by the real-type irreps of G. Of course, if I_i has multiplicity $m_i = 0$ in V, then the I_ith sign in the tuple is fixed to be '$+$'; only the k signs associated to the nonzero multiplicities can vary. So it is convenient to assume that V is large enough so that all $m_i \geq 1$.

Now that we know the path-components of $\operatorname{Aut}(V)$ we can list all the linear units. Define ϵ_i to be the linear unit associated to a map $a \in \operatorname{Aut}(V)$ in the path-component labeled

$$(+,\dots,+,\underset{\text{index } i}{-},+,\dots,+).$$

We should order our irreps so that I_0 is the trivial irrep and $\epsilon_0 = -1$. We can understand the effect of multiplication by the ϵ_i using Proposition 3.8. If (M, σ, Φ) is a singular V-framed manifold. Then

$$\begin{aligned}\epsilon_i \cdot [M] &= [\mathrm{Tw}_a^G(D(V)) \times \mathrm{Tw}_{\mathrm{id}}^G(M)] \\ &= [\mathrm{Tw}_{\mathrm{id}}^G(D(V)) \times \mathrm{Tw}_a^G(M)] \\ &= 1 \cdot [\mathrm{Tw}_a^G(M)] \\ &= [\mathrm{Tw}_a^G(M)].\end{aligned}$$

In particular,

$$\begin{aligned}\epsilon_i \cdot \epsilon_j &= [\mathrm{Tw}_a^G(D(V)) \times \mathrm{Tw}_{a'}^G(D(V))] \\ &= [\mathrm{Tw}_{\mathrm{id}}^G(D(V)) \times \mathrm{Tw}_a^G(\mathrm{Tw}_{a'}^G D(V)))] \\ &= [\mathrm{Tw}_{a \circ a'}^G(D(V))]\end{aligned}$$

is the linear unit associated to the path-component labeled by a sign-tuple with '$-$' at positions i and j and '$+$' otherwise. Clearly, the ϵ_i generate a subgroup $(\mathbb{Z}/2)^{\ell_{\mathbb{R}}} \subseteq A(G)^\times$ where $\ell_{\mathbb{R}}$ is the number of real-type irreps of G.

The takeaway is that multiplication $\epsilon_i \cdot [M]$ can be interpreted as doing a global twist of the frame on $[M]$ by a linear map a which when restricted to the $(I_i^{m_i})$-summand of V is in the non-identity component of $\mathrm{Aut}(I_i^{m_i})$, but which is the identity on the other canonical summands of V. For $G = C_2$, our previously defined ϵ would be ϵ_1 coming from the real-type C_2-irrep $\mathbb{R}^{1,1}$. This means that we can represent $\epsilon \cdot [M]$ by negating any single $\mathbb{R}^{1,1}$-section (or doing an odd permutation of the $\mathbb{R}^{1,1}$-sections) on M. Meanwhile -1 effects the same kind of transformation on the $\mathbb{R}^{1,0}$-sections.

Remark. We can think of such a map a as being orientation reversing with respect to the real-type irrep I_i. This is not necessarily the same as being orientation reversing on the underlying vector space. For example, $SO(4)$ acting on $V = \mathbb{R}^4$ by rotations is a real-type representation. We could say that the twist map $\tau_{V,V} : V \oplus V \to V \oplus V$ is orientation reversing with respect to real-type irrep V of $SO(4)$ since $\tau_{V,V}$ is in the non-identity component of $\mathrm{Aut}(V \oplus V) \cong O(2)$. However, $\tau_{V,V}$ is not an orientation reversing map on the underlying vector space $\mathbb{R}^8$.

This begs the question: since $\epsilon_i \in \pi_{\mathbf{0}}^G \cong A(G)$, how does it decompose into a linear combination of orbits $[G/H]$? This can be computed by finding the degrees $(\epsilon_i)^H \in \pi_0^s = \mathbb{Z}$ for various subgroups $H \leq G$ [7, 28]. For example, when $G = C_2$ the degrees for 1, ϵ, and $[C_2]$ are listed in Table 4.1.

α	$\deg(\alpha^{C_2})$	$\deg(\alpha)$
ϵ	1	-1
1	1	1
$[C_2]$	0	2

Table 4.1. Degrees of elements in $\pi_{0,0}$ after taking C_2- and $\{e\}$-fixed-points.

It follows that $\epsilon = 1 - [C_2]$. This computation doesn't provide a lot of geometric insight into *how* these two representatives are the same. We will give an alternative proof of this identity using a bordism between the manifold representatives for $\epsilon + [C_2]$ and 1.

Proposition 4.4. *In $\pi_{0,0}$ there holds the relation $[C_2] + \epsilon = 1$.*

Proof. This relation is witnessed by the bordism shown at the bottom of Figure 4.1. The singular framed manifold representatives we will use are listed in Table 4.2 below.

Homotopy Class	Manifold	Frame	Singular Map
1	$D(\mathbb{R}^{1,1})$	∂_t	$\mathrm{id} : D(\mathbb{R}^{1,1}) \to D(\mathbb{R}^{1,1})$
ϵ	$D(\mathbb{R}^{1,1})$	$-\partial_t$	$\mathrm{id} : D(\mathbb{R}^{1,1}) \to D(\mathbb{R}^{1,1})$
$[C_2]$	$C_2 \times D(\mathbb{R}^{1,1})$	∂_t	$pr_2 : C_2 \times D(\mathbb{R}^{1,1}) \to D(\mathbb{R}^{1,1})$

Table 4.2. Singular representatives for 1, ϵ, and $[C_2]$ in $\Omega^{fr}_{1,1}(D(\mathbb{R}^{1,1}), S(\mathbb{R}^{1,1}))$.

Usually the preferred representative for $[C_2]$ would just be C_2 itself treated as $\mathbf{0}$-framed discrete manifold. Here we have stabilized that representative by multiplying with the fundamental class $[D(\mathbb{R}^{1,1}), \mathrm{id}, \mathscr{T}]$ so that it lives in $\Omega^{fr}_{1,1}(D(\mathbb{R}^{1,1}), S(\mathbb{R}^{1,1}))$, which is the home of ϵ. The conventions used here are the same as in Figure 1.1: the singular map to $D(\mathbb{R}^{1,1})$ is indicated with the red-blue coloring, the dashed lines are the symmetry axes of the C_2-action, and the $\mathbb{R}^{1,0}$- and $\mathbb{R}^{1,1}$-sections are the single- and double-headed green arrows respectively.

We will point out that ϵ appears as the boundary component which is the middle tine of the fork on the left. We can see that this piece of the boundary is ϵ because the $\mathbb{R}^{1,1}$-frame runs in against the gradient of the singular map. The rest of the conditions for a singular framed bordism in Definition 3.14 can be checked by inspection. □

The bordism at the bottom of Figure 4.1 produces an equivariant homotopy $H : S^{1,1} \times I \to S^{1,1}$ which is shown in stages along the top of Figure 4.1. The initial manifold $[C_2] + \epsilon$ represents a map $S^{1,1} \to S^{1,1}$ which factors as

$$S^{1,1} \longrightarrow ((C_2)_+ \wedge S^{1,1}) \vee S^{1,1} \xrightarrow{pr_2 \vee \epsilon} S^{1,1}.$$

This is the stage t_1. Moving across the top of Figure 4.1 shows the intermediate stages as H proceeds to the identity map $\mathrm{id} : S^{1,1} \to S^{1,1}$. These stages correspond to the marked vertical slices of the bordism.

This example helps to show why we need singular maps in the equivariant version of the Pontryagin-Thom isomorphism. In order to have a bordism going from 3 points to 1 point, the negatively oriented point representing ϵ would have to cancel with exactly one of the positively oriented non-fixed-points which together represent $[C_2]$. There is no 1D C_2-manifold that does this. Another way to see this obstruction is in the fact that the homotopy fails to be transverse to $0 \in S^{1,1}$ at the point $(0, t_4) \in S^{1,1} \times I$. At that point both ∂_θ and ∂_t are in the kernel of the differential $dH : T_{(0,t_4)}(S^{1,1} \times I) \to T_0 S^{1,1}$. Equivariance makes it impossible to resolve this transversality issue through a perturbation of H. This is similar to the issue that occurs when trying to define a non-singular representative for the class ρ.

Remark. This example is suggestive of a one-dimensional equivariant framed bordism that looks like "∋" with no singular map but instead having a single non-manifold point. Objects similar to this are the subject of Baas-Sullivan theory, which can be used to construct bordism theories that recover various generalized homology theories. Exploring this connection seems like a potentially fruitful avenue for further research.

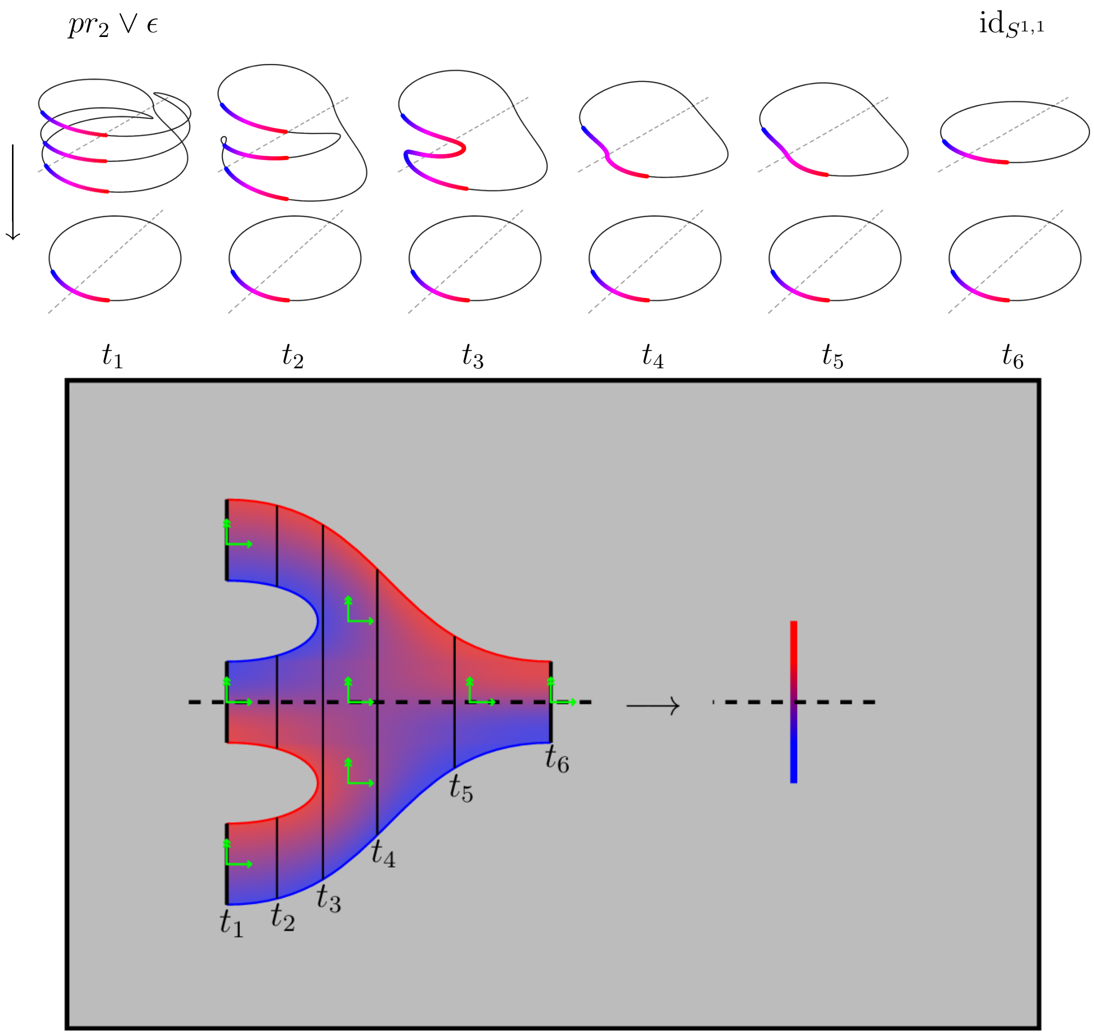


Figure 4.1. A bordism witnessing $[C_2] + \epsilon = 1$ in $\pi_{0,0}$. The vertical slices correspond to the stages of the associated homotopy H, which illustrated in 3D along the top of the figure.

4.3 The Generalized Burnside Ring

We've seen that we can represent the additive generators of the Burnside ring $[G/H] \in \pi^G_{\mathbf{0}}$ by the twists $\mathrm{Tw}^H_{\mathrm{id}_V}(D(V))$, and that these corresponded to maps

$$S^V \to (G/H)_+ \wedge S^V \to S^V.$$

If we replace id_V with non-identity H-isomorphisms $\varphi : \mathrm{Res}_H V \to \mathrm{Res}_H W$ we get a generalized class of elements $\mathrm{Tw}^H_\varphi(D(\underline{V}))$ which represent maps

$$S^V \to (G/H)_+ \wedge S^V \xrightarrow{\varphi} (G/H)_+ \wedge S^W \to S^W$$

in π^G_{V-W}. Note that this is still in the topological 0-stem since V and W have the same dimension. Like with the Burnside ring, these elements can be seen as coming from the normal bundle of G/H in V equipped with a W-frame, or as being the singular manifold $D(G/H \times W)$ equipped with a tangential V-frame and singular map projection to $D(W)$. We denote these elements by

$$\theta_{H,\varphi} := \mathrm{Tw}^H_\varphi(D(W)).$$

For $G = C_2$ we can further simplify the notation by setting $\theta_q = \theta_{\{e\},\varphi}$ where

$$\varphi = \begin{cases} \mathrm{id}_{\mathbb{R}^q} : \mathbb{R}^{q,q} \to \mathbb{R}^{q,0} & , \ q \geq 0 \\ \mathrm{id}_{\mathbb{R}^{-q}} : \mathbb{R}^{-q,0} \to \mathbb{R}^{-q,-q} & , \ q < 0 \end{cases}.$$

The θ's are can be thought of as thickened orbits G/H equipped with non-standard frames. In this sense we can consider the subalgebra they generate as a generalization Burnside ring.

Many of the elements $\theta_{H,\varphi}$ generate infinite cyclic groups. Following ideas discussed by Greenlees and Quigley [13], we can use an isomorphism φ to define an orientation character $o_\varphi : G \to \mathbb{R}^\times$ by $g \mapsto \det(\varphi^{-1} g \varphi g^{-1})$. When the orientation character is trivial, then in the map

$$(G/H)_+ \wedge S^V \xrightarrow{\varphi} (G/H)_+ \wedge S^W$$

each wedge summand

$$\{gH\} \times S^V \xrightarrow{g\varphi g^{-1}} \{gH\} \times S^W$$

has the same degree. In particular, this means that if we forget the group action then $\psi(\theta_{H,\varphi}) = \pm |G/H| \in \pi^s_0$. So $\theta_{H,\varphi}$ generates an infinite cyclic group and is non-nilpotent. If V and W are both special orthogonal G-representations, then o_φ is always trivial, but there may be φ between merely orthogonal G-reps which still have trivial orientation character.

Example 4.5. For the symmetric group $G = S_3$ let V be the dihedral representation and let W be the sum of the one-dimensional trivial and sign representations. The action of S_3 does not preserve orientation on V or W, but it does reverse orientations on them in the same way. This is formalized by the fact that there is an H-isomorphism $\varphi : V \to W$ that has trivial orientation character, where H is the order 2 cyclic subgroup of S_3. ☙

Example 4.6. For $G = C_2$, all the even θ_{2k} have trivial orientation character. These are the source of the 2-periodic copies of ∞ appearing in the groups $\pi_{0,2k}$. Since $\psi(\theta_{2k}) = 2$, it follows from the computations of Araki and Iriye [1, Theorems 3.5 and 7.6] that the θ_{2k}'s are the generators of the summands $\lambda_{0,2k} \cong \mathbb{Z}$ in the $p = 0$ column of Table 1.1. ☙

The $\theta_{H,\varphi}$ which have nontrivial orientation character are more subtle. For example, with $G = C_2$, all the odd θ_{2k+1} have nontrivial orientation character, so it doesn't immediately follow from our above discussion that they should be nontrivial classes in $\pi_{0,2k+1}$. We can show that the odd θ_{2k+1} are nontrivial by combining Example 4.6 with the forgetful exact sequence

$$\cdots \longrightarrow \pi_{0,2k} \xrightarrow{\psi} \pi_0^s \longrightarrow \pi_{0,2k+1} \xrightarrow{\rho} \pi_{-1,2k} \longrightarrow \cdots \tag{4.5}$$

from Section 2.1. We will recall the definition of the middle map of (4.5). First, given an $\{e\}$-equivariant map $f : S^{r,s+2k+1} \to S^{r,s}$ which represents a class $[f] \in \pi_0^s$, we induce a C_2-equivariant map $\widetilde{f} : (C_2)_+ \wedge S^{r,s+2k+1} \xrightarrow{\widetilde{f}} (C_2)_+ \wedge S^{r,s}$. The map $\widetilde{f}$ is then pre-composed with the equator collapsing map $S^{r,s+2k+1} \to (C_2)_+ \wedge S^{r,s+2k+1}$ post-composed with projection to $S^{r,s}$:

$$S^{r,s+2k+1} \to (C_2)_+ \wedge S^{r,s+2k+1} \xrightarrow{\widetilde{f}} (C_2)_+ \wedge S^{r,s} \to S^{r,s}.$$

When the map f is the one-point compactification of $\mathrm{id} : \mathbb{R}^{r,s+2k+1} \to \mathbb{R}^{r,s}$ this coincides with the Thom collapse construction associated to θ_{2k+1}. So $\theta_{2k+1} \in \pi_{0,2k+1}$ is the image of the generator $1 \in \pi_0^s$. Now we need to assume $k \neq 0$. While $\pi_{0,2k}$ contains the $\mathbb{Z}$-summands from Example 4.6, we know that the image of their generators under ψ is 2. Everything else in $\pi_{0,2k}$ is torsion, so $1 \in \pi_0^s$ cannot be in the image of $\psi : \pi_{0,2k} \to \pi^s_0$. Thus, by exactness, 1 is not in the kernel of the middle map of (4.5), so θ_{2k+1} is nontrivial. In fact, since $k \neq 0$ we can combine (4.5) with the split fixed-point short exact sequence:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & \nearrow & \lambda_{0,2k+1} & \overset{0}{\searrow} & & & \\
 & & & & \downarrow & & & & \\
\cdots \longrightarrow & \pi_{0,2k} & \xrightarrow{\psi} & \pi^s_0 & \longrightarrow \pi_{0,2k+1} & \xrightarrow{\rho} & \pi_{-1,2k} & \longrightarrow \cdots . & \\
 & & & & \downarrow{\scriptstyle \rho} & & \wr\| & & \\
 & & & & \pi_{-1,-2k} & \cong & \pi^s_{-1-2k} & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

It is not hard to see that θ_{2k+1} is in the kernel of ρ, so we can pull it back to $\lambda_{0,2k+1}$. This is the natural home for θ_{2k+1} anyways, since it is a C_2-free manifold. So the forgetful exact sequence is actually passing through $\lambda_{0,2k+1}$ here.

The only element in this family that still needs to be addressed is θ_1. This is the case $k = 0$ where the algebraic arguments above break down. Since $\pi_{0,1}$ is 0 in Table 1.1, we should be able to find a null-bordism witnessing that θ_1 is trivial.

Proposition 4.7. *In $\pi_{1,0}$ there holds the relation $\theta_1 = 0$.*

Proof. A framed singular null-bordism of θ_1 is given in Figure 4.2.

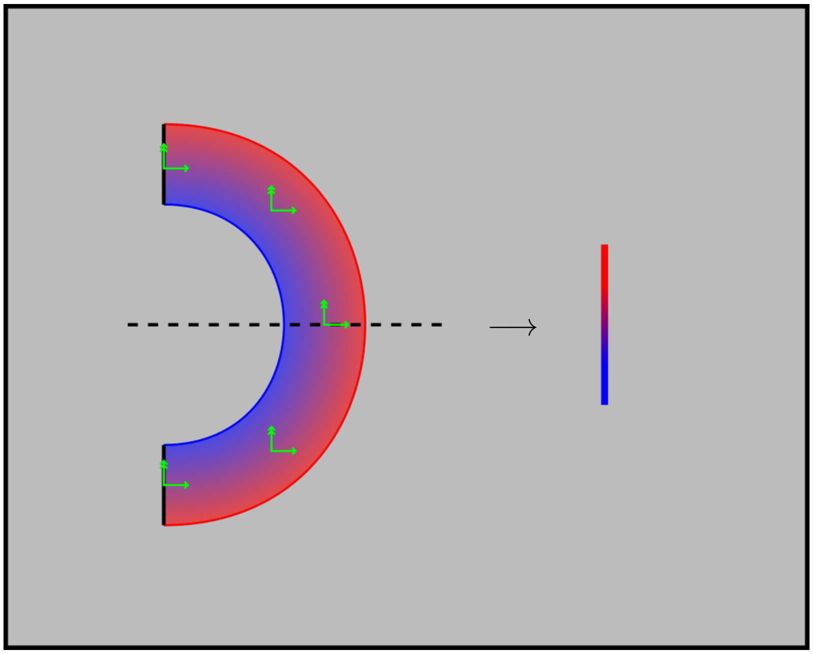


Figure 4.2. A null-bordism of θ_1.

There are a few details to make note of. First, we can see the $\varphi : \mathbb{R}^{1,1} \to \mathbb{R}^{1,0}$ twist in the fact that one component of the boundary the $\mathbb{R}^{1,1}$-frame runs with the gradient of the singular map, where on the other component it runs against the gradient. Second, the target space of the singular map is $D(\mathbb{R}^{1,0})$ which carries the trivial action (hence no dashed line of reflection). This means the colors should be C_2-invariant rather than exchanged red-for-blue like in most of our previous examples. □

In Figure 4.2 we can see that the fixed-point set is one-dimensional and thus must be $\mathbb{R}^{1,0}$-framed. This doesn't work for the other θ_q because, while there is an analogous null-bordism of $C_2 \times D(\mathbb{R}^{q,0})$ given by the cylinder $D(\mathbb{R}^{1,1}) \times D(\mathbb{R}^{q,0})$, it has a q-dimensional fixed-point set. So it can at best be $\mathbb{R}^{q+1,1}$-framed. However, we would need a $\mathbb{R}^{q+1,q}$-frame on the cylinder for it to be a framed null-bordism of θ_q. This gives a little insight on why $q = 1$ is exceptional.

There are a few more relations that hold in the generalized Burnside ring from purely representation theoretic reasons.

Proposition 4.8. *For $nH \in W_G H$, we have $\theta_{H,\varphi} = \theta_{H,n\varphi n^{-1}}$.*

Proof. This is a special case of Proposition 3.20. □

Proposition 4.9. *If $\varphi, \psi : \mathrm{Res}_H V \to \mathrm{Res}_H W$ are homotopic H-isomorphisms, then there holds the relation $\theta_{H,\varphi} = \theta_{H,\psi}$ in π^G_{V-W}.*

Proof. This follows directly from Proposition 3.8(a). □

Proposition 4.10. *Let ϵ_i and ϵ_j be linear units associated to irreps I_i and I_j. If I_i and I_j are H-isomorphic, then $\epsilon_i\theta_{H,\varphi} = \epsilon_j\theta_{H,\varphi}$.*

Proof. Assume that V is large enough to contain both I_i and I_j. We know multiplication by the ϵ_i is the same as global twist by an automorphism $a_i : V \to V$ which is orientation reversing with respect to the the I_i-canonical summand of V. So we would like to show

$$\theta_{H,a_i\circ\varphi} \cong \theta_{H,a_j\circ\varphi}$$

However, a_i and a_j are in the same path component of $O(V)^H$ because I_i and I_j are H-isomorphic. Then we can apply Proposition 4.9 from above. □

With these we can actually prove that the θ_{2k+1} are 2-torsion directly, without appealing to the computation $\lambda_{0,2k+1} \cong \mathbb{Z}/2$. Let H be the trivial subgroup of C_2. First, using $\varphi : \mathbb{R}^{q,q} \to \mathbb{R}^{q,0}$ if $q \geq 0$ and $\varphi : \mathbb{R}^{-q,0} \to \mathbb{R}^{-q,-q}$ if $q < 0$, we have

$$\theta_q = \theta_{H,\varphi} = \theta_{H,\tau\varphi\tau}.$$

For either definition of φ, it's easy to check that $\tau\varphi\tau = -\varphi$. So $\theta_{H,\varphi} = \theta_{\{e\},-\varphi}$. Note that $\theta_{H,\varphi}$ and $\theta_{H,-\varphi}$ have the same underlying manifold but that on the latter, all the sections of the frame have been negated compared to the former. Recall that negating any single $\mathbb{R}^{1,0}$-section corresponds to multiplication by -1, and negating $\mathbb{R}^{1,1}$-sections corresponds to multiplication by ϵ. So we have

$$\theta_q = \theta_{H,\varphi} = \theta_{H,-\varphi} = \begin{Bmatrix} \epsilon^q & \text{if } q > 0 \\ 1 & \text{if } q = 0 \\ (-1)^q & \text{if } q < 0 \end{Bmatrix} \cdot \theta_{H,\varphi}.$$

We also know that $\mathbb{R}^{1,1}$ and $\mathbb{R}^{1,0}$ are isomorphic as H-representations, so $\epsilon\theta_{H,\varphi} = -\theta_{H,\varphi}$. This allows us to unify the three cases above into a single equation

$$\theta_q = \theta_{H,\varphi} = \theta_{H,-\varphi} = (-1)^q \cdot \theta_{H,\varphi} = (-1)^q\theta_q.$$

If q is odd, this proves $2\theta_q = 0$.

Lastly, we can say a little about how the generalized Burnside ring acts on other framed manifolds.

Proposition 4.11. *For M a singular V-framed manifold and an H-isomorphism $\varphi : \operatorname{Res}_H W \to \operatorname{Res}_H V$ we have $\theta_{H,\varphi} \cdot [M] = u \cdot \mathrm{Tw}^H_\varphi(M)$ for some $u \in (\pi^G_{\mathbf{0}})^\times$.*

Proof. From Proposition 3.8 we have

$$\begin{aligned}\theta_{H,\varphi} \cdot [M] &= [\mathrm{Tw}^H_\varphi(D(V)) \times M] \\ &= [\mathrm{Tw}^G_{\tau_{W,V}}\left(D(V) \times \mathrm{Tw}^H_{-\varphi}(M)\right)].\end{aligned}$$

This is nearly the twist $\mathrm{Tw}^H_\varphi(M)$ times the fundamental class $[D(V)]$. Unfortunately there are two small obstacles. First is the minus sign on φ. If we know the multiplicities of irreps in V, we could say what effect that minus sign has in terms of the ϵ_i's. The bigger issue is the twist $\mathrm{Tw}^H_{\tau_{W,V}}$. It involves permuting some number of non-isomorphic irreps past each other, which incurs a penalty of multiplying by some unit.[1] *Which* unit is a matter of convention [9, 10]. Regardless, we can sweep these issues into a combined error term $u \in (\pi^G_{\mathbf{0}})$ and write $\mathrm{Tw}^G_{\tau_{W,V}}\left([D(V)] \cdot \mathrm{Tw}^H_{-\varphi}([M])\right) = u \cdot \mathrm{Tw}_{H,\varphi}([M])$. □

Proposition 4.11 can be combined with Proposition 3.19 obtain the following statement.

Proposition 4.12. *Let $F : M \to N$ be an H-equivalence of V- and W-framed manifolds respecting an H-isomorphism $\varphi : \operatorname{Res}_H V \to \operatorname{Res}_H W$, then $[G/H] \cdot [M] = u\theta_{H,\varphi} \cdot [N]$ for some $u \in \pi^G_{\mathbf{0}}$.*

The main advantage of Proposition 4.12 over Proposition 3.19 is that it lets us work entirely within the homotopy ring without explicitly naming the twisting operators. Unfortunately, it comes at the cost of having the convention-sensitive unit u show up.

Example 4.13. The elements $\eta \in \pi_{1,1}$ and $\eta_{\mathrm{top}} \in \pi_{1,0}$ are both represented by $[U(1), \partial_\theta]$, but with different C_2-actions. By Proposition 4.12 we have two relations

$$\begin{aligned}u\theta_{-1}\eta = [C_2]\eta_{\mathrm{top}} = (1-\epsilon)\eta_{\mathrm{top}}, \\ (1-\epsilon)\eta = [C_2]\eta = u'\theta_1\eta_{\mathrm{top}} = 0\eta_{\mathrm{top}} = 0.\end{aligned}$$

The second relation happens to prove $\eta = \epsilon\eta$ regardless of the actual value of u'. We'll see an easier way to prove this in Example 5.1. ❧

[1] Not necessarily even a linear unit

Remark. We could consider the algebra of the $\theta_{H,\varphi}$'s only up to isomorphism rather than the weaker relation of bordism. We will call this the representation-theoretic generalized Burnside ring. By

1) carefully picking and ordering a preferred set of subgroups H_i for each conjugacy class of G,
2) specifying decompositions of their products $G/H_i \times G/H_j \cong \sum_\ell m_{ij}^\ell G/H_\ell$,
3) fixing representatives I_i for each of G's classes of irreps, and
4) enumerating (up to homotopy) the H_i-isomorphisms between them,

it should be possible to completely compute the representation-theoretic generalized Burnside ring. For C_2, this process doesn't involve making too many choices and the divided power structure $\theta_{q_1}\theta_{q_2} = 2\theta_{q_1+q_2}$ becomes apparent. The only new relation that shows up when we weaken the equivalence relation from isomorphism to bordism is $\theta_1 = 0$. In a sense, that is the only genuinely topological relation to occur in the generalized Burnside ring of C_2. We could think of θ_1 as being in the kernel as we pass from the representation-theoretic generalized Burnside ring to the bordism-theoretic generalized Burnside ring. It would be interesting to see what this kernel is for other finite groups G, and if computing it is tractible in general.

CHAPTER 5

VARIOUS RELATIONS IN $RO(C_2)$-GRADED HOMOTOPY

Now that we have a good tool belt we can start constructing witnesses for a variety of interesting relations in $\pi_{*,*}$.

5.1 Relations based on Automorphisms

First we'll consider relations which come from cylindrical bordisms where we use distinct inclusions of the same manifold into the initial and terminal boundaries.

5.1.1 Automorphisms of the Circle

Example 5.1. In Example 3.18 we saw that the complex conjugation map is an isomorphism of framed manifolds $(U(1), \partial_\theta) \to (U(1)_{\text{triv}}, -\partial_\theta)$. This isomorphism gave us a cylindrical bordism between the two framed manifolds, so we have $[U(1)_{\text{triv}}, \partial_\theta] = [U(1)_{\text{triv}}, -\partial_\theta]$ in the bordism ring. On $U(1)_{\text{triv}}$ the frame ∂_θ is an $\mathbb{R}^{1,0}$-frame. In Section 4.2 we showed that negating an $\mathbb{R}^{1,0}$-section is equivalent to multiplication by -1, so we conclude

$$\begin{aligned}\eta_{\text{top}} &= [U(1)_{\text{triv}}, \partial_\theta] \\ &= [U(1)_{\text{triv}}, -\partial_\theta] \\ &= -1 \cdot [U(1)_{\text{triv}}, \partial_\theta] \\ &= -\eta_{\text{top}}.\end{aligned}$$

We can still treat complex conjugation as a framed isomorphism of C_2-manifolds $(U(1)_{\text{conj}}, \partial_\theta) \to (U(1)_{\text{conj}}, -\partial_\theta)$ since it commutes with the action. The same arguments apply, but now ∂_θ is an $\mathbb{R}^{1,1}$-frame. To negate an $\mathbb{R}^{1,1}$-frame we need to multiply by the linear unit ϵ rather than -1. So the new relation is

$$\begin{aligned}\eta &= [U(1), \partial_\theta] \\ &= [U(1), -\partial_\theta] \\ &= \epsilon \cdot [U(1), \partial_\theta] \\ &= \epsilon\eta.\end{aligned}$$

Rearranging this yields $(1 - \epsilon)\eta = 0$.

We could also equip $U(1)$ with the antipodal action for which ∂_θ again becomes an $\mathbb{R}^{1,0}$-frame. The complex conjugation map is still a framed isomorphism to $(U(1)_{\text{free}}, -\partial_\theta)$ so we get the identity $\eta_{\text{free}} = -\eta_{\text{free}}$ in the same way as for η_{top}. ❧

Since $1-\epsilon = [C_2]$, there should also be a null-bordism of $(C_2, \mathbf{0}) \times (U(1), \partial_\theta)$. This could of course be done by concatenating the pitchfork shaped bordism in Proposition 4.4 with the cylindrical bordism associated to $\eta = \epsilon\eta$. However, it's actually easier to show $[C_2] \cdot \eta = 0$ directly; this is done in Figure 5.1. The C_2-action is given by reflection across the gray plane.

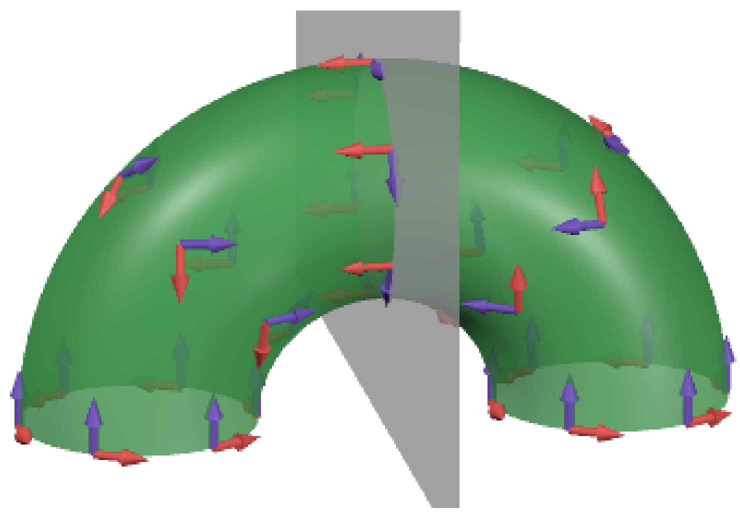


Figure 5.1. A null-bordism of $C_2 \times U(1)$.

5.1.2 Automorphisms of the 3-Sphere and 3-Torus

Example 5.2. Let $M = (Sp(1), \mathscr{L})$ be the unit sphere of quaternions with its Lie frame. The C_2-action on M is inherited from the automorphism of $\mathbb{H}$ which negates the imaginary units j and k.[1]

At the identity element 1 of $Sp(1)$ there is a canonical tangent basis $\mathbf{i}, \mathbf{j}, \mathbf{k} \in T_1M$. The Lie frame is obtained by translating these vectors to all of M, so we can actually think of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as the associated global sections themselves. Note that the C_2-action we have described on M is also the inner automorphism $q \mapsto -iqi$ given by i-conjugation. Since the C_2-action on M is a Lie group automorphism, we can infer from Proposition 3.21 that $\mathbf{i}$ is an $\mathbb{R}^{1,0}$-section and that $\mathbf{j}, \mathbf{k}$ are $\mathbb{R}^{1,1}$-

[1] This is the Galois symmetry of $\mathbb{H}$ associated to adjoining a new imaginary to $\mathbb{C}$ in the Cayley-Dickson construction.

sections. This tells us that $[M]$ is $\mathbb{R}^{3,2}$-framed. Since it is known non-equivariantly that $[Sp(1), \mathscr{L}] = \nu_{\text{top}}$, we can conclude that $[M]$ represents the generator $\nu \in \pi_{3,2}$ using the forgetful homomorphism.

Now consider $Sp(1)$ equipped with the frame given by the ordered set of sections $-\mathbf{i}, -\mathbf{j}, \mathbf{k}$. Despite having the same underlying space, since the frame is different, this is a distinct object; we will call it N. We can see that we have negated the $\mathbb{R}^{1,0}$-section $\mathbf{i}$, which can be achieved by multiplying by -1, and the $\mathbb{R}^{1,1}$-section $\mathbf{j}$, which can be achieved by multiplying by ϵ. So N is a representative for $-\epsilon\nu$ since $N = -\epsilon \cdot M$.

As elements of $\text{Aut}(Sp(1))$, i-conjugation and k-conjugation commute, so conjugating by k can be seen as a C_2-equivariant map $M \to N$. Again by Proposition 3.21, it's easy to see that k-conjugating takes $\mathbf{i}, \mathbf{j}$ to $-\mathbf{i}, -\mathbf{j}$ while leaving $\mathbf{k}$ unchanged. That means k-conjugation maps M's frame to N's, so M and N are isomorphic $\mathbb{R}^{3,2}$-framed manifolds. Hence $\nu = -\epsilon\nu$. ❧

There are many more automorphism-based identities which come from comparing two different actions on the same underlying framed manifold using Proposition 4.12. We've chosen to highlight the following two since they will be useful in Section 5.3.2.

Example 5.3. Let $H \leq C_2$ be the trivial group. The identity map is an H-equivariant framed isomorphism between $(Sp(1), \mathscr{L})$ with the standard C_2-action and $(Sp(1), \mathscr{L})$ with the trivial C_2-action. These are $\mathbb{R}^{3,2}$- and $\mathbb{R}^{3,0}$-framed, respectively. Lemma 3.9 then lets us induce a C_2-equivalence of $\mathbb{R}^{3,2}$-framed manifolds

$$\text{Tw}^H_{\text{id}_{\mathbb{R}^{3,2}}}(Sp(1)_{standard}, \mathscr{L}) \cong \text{Tw}^H_{\varphi}(Sp(1)_{trivial}, \mathscr{L})$$

where $\varphi : \mathbb{R}^{3,2} \to \mathbb{R}^{3,0}$ is the set-level identity map. Proposition 4.12 reformulates this equivalence into the relation $[C_2]\nu = u\theta_2\nu_{\text{top}}$ for an appropriate unit u.

The same argument gives a framed equivalence between the 3-torus T^3 with an action of complex conjugation on the first two circle factors and T^3 with the trivial action. This corresponding relation is $[C_2]\eta^2\eta_{\text{top}} = u'\theta_2\eta^3_{\text{top}}$ for some other unit u'. ❧

5.2 Relations based on Excision

Recall that Lemma 3.24 lets us delete the parts of a manifold that the singular map sends to the boundary. Most of the time when we're dealing with a singular map, it's a constant map c_0 which came from multiplying by the representative $[\{*\}, c_0, \mathbf{0}] \in \Omega^{fr}_{0,0}(D(\mathbb{R}^{1,1}), S(\mathbb{R}^{1,1}))$ for ρ. Since c_0 sends nothing to the boundary, we will have to apply various perturbations first.

5.2.1 *Excision on the Circle*

Proving the relation $\rho\eta = 1 + \epsilon$ gives a good taste of how we go about using excision in a fruitful way. To give an explicit bordism witnessing $\rho\eta = 1 + \epsilon$ we will need the manifold representatives listed in Table 5.1 below.

Homotopy Class	Manifold	Frame	Singular Map
1	$D(\mathbb{R}^{1,1})$	∂_t	$\mathrm{id} : D(\mathbb{R}^{1,1}) \to D(\mathbb{R}^{1,1})$
ϵ	$D(\mathbb{R}^{1,1})$	$-\partial_t$	$\mathrm{id} : D(\mathbb{R}^{1,1}) \to D(\mathbb{R}^{1,1})$
η	$U(1)$	∂_θ	$U(1) \to *$
ρ	$\{0\}$	$\mathbf{0}$	$\iota : \{0\} \to D(\mathbb{R}^{1,1})$

Table 5.1. Singular representatives used to show the relation $\rho\eta = 1 + \epsilon$.

Note that, except for ρ, the frames are all $\mathbb{R}^{1,1}$-frames. We can then form a representative for $\rho\eta$ by the Cartesian product of $\{0\}$ and $U(1)$ (and their singular maps). Ultimately, this is just $(U(1), \partial_\theta)$ equipped with the constant singular map to zero $c_0 : U(1) \to D(\mathbb{R}^{1,1})$.

A bordism

$$W : (U(1), c_0, \partial_\theta) \rightsquigarrow (D(\mathbb{R}^{1,1}), \mathrm{id}, \partial_t) \sqcup (D(\mathbb{R}^{1,1}), \mathrm{id}, -\partial_t)$$

given by the annulus shown in Figure 5.2 below.

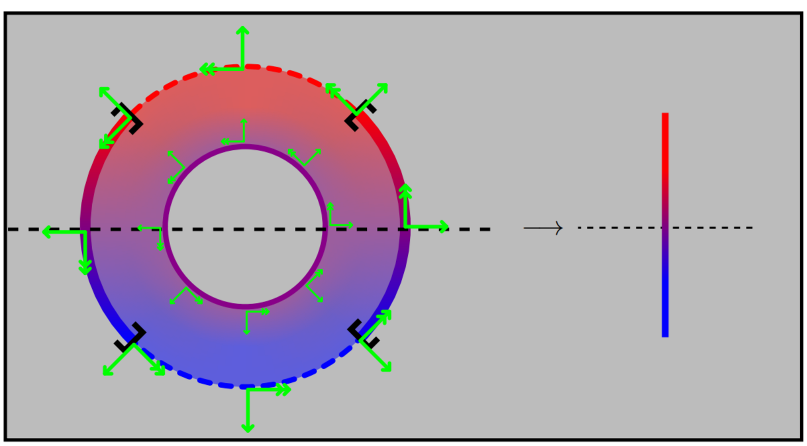


Figure 5.2. A singular bordism witnessing $\rho\eta = 1 + \epsilon$.

We will note the important features. Let $M_0 = U(1)$ denote the initial manifold and let $M_1 = D(\mathbb{R}^{1,1}) \sqcup D(\mathbb{R}^{1,1})$ denote the terminal manifold.

- The initial boundary M_0 has an *inward* pointing invariant vector field.
- The terminal boundary M_1 has an *outward* pointing invariant vector field.
- W is $(\mathbb{R}^{1,1} \oplus \mathbb{R}^{1,0})$-framed such that restricting the frame to the first summand yields an $\mathbb{R}^{1,1}$-frame on ∂W which coincides with the $\mathbb{R}^{1,1}$-frames on M_0 and M_1.
- The singular map on W coincides the ones on the terminal and initial manifolds.
- The edge of the bordism $\underset{\rightsquigarrow}{\partial} W$, illustrated as the dashed part of ∂W, is sent to the boundary of $D(\mathbb{R}^{1,1})$.
- Everything is equivariant.

This bordism can be found using excision. Initially $\rho\eta$ is just the circle with singular map c_0. This singular map is equivariantly homotopic to a map which stretches the lower and upper semi-circles of $U(1)$ so that neighborhoods of i and $-i$ map to 1 and -1 respectively. Applying excision to remove those neighborhoods is how we produced this bordism.

Figure 5.3 shows the underlying manifold of the same bordism but where the singular map is approximately represented by height instead of coloration.

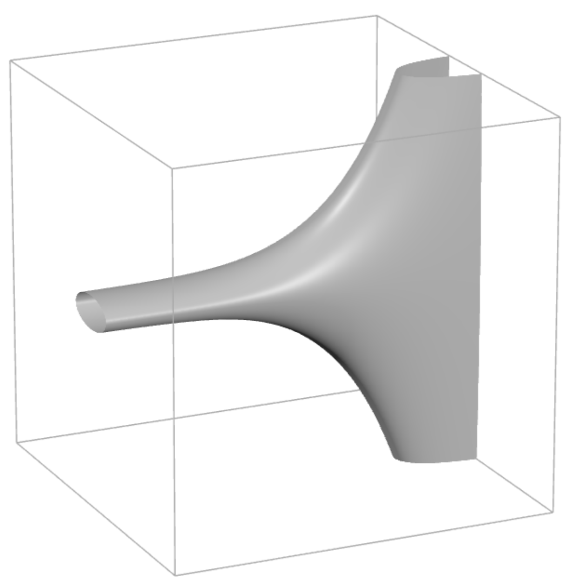


Figure 5.3. An alternative view of the bordism in Figure 5.2.

Forgetful Exactness

If we think about the forgetful exact sequence

$$\cdots \longrightarrow \pi_{p+1,q+1} \xrightarrow{\rho} \pi_{p,q} \xrightarrow{\psi} \pi_p^s \longrightarrow \pi_{p,q} \longrightarrow \cdots$$

we can notice that $\rho\eta = 1+\epsilon$ is an instance of exactness occurring at $\pi_{0,0}$ since $1+\epsilon$ is in the kernel of the forgetful map ψ:

$$\begin{array}{ccccc} \pi_{1,1} & \xrightarrow{\rho} & \pi_{0,0} & \xrightarrow{\psi} & \pi_0^s \\ \eta & \longmapsto & 1+\epsilon & \longmapsto & 1+(-1)=0. \end{array}$$

We might additionally notice that, non-equivariantly, the upper semicircle of $U(1)$ actually *is* the null-bordism of $1+(-1)$. These observations can be generalized into a construction which executes this piece of exactness in the forgetful sequence.

Construction 5.4. For simplicity, we'll only describe the construction for closed non-singular manifolds. Let $[M,\Phi_M] \in \Omega^{fr}_{p,q}(*)$ be such a manifold and assume that it is in the kernel of the forgetful map. So there is some (non-equivariant) $\mathbb{R}^{p+1}$-framed null-bordism

$$(W,\Phi_W) : (M,\Phi_M) \rightsquigarrow \emptyset.$$

Keep in mind that at the boundary the last $\mathbb{R}^1$-section of Φ_W, which we will label ℓ, is an *inward* pointing vector field. We can pick a collar neighborhood of $M \times I \hookrightarrow W$ so that $\ell|_{M\times I} = \partial_t$, where t is the parameter for the interval I. We will want to use the interval $I = [-1,1]$.

We will now construct a new $\mathbb{R}^{p+1,q+1}$-framed manifold $\widetilde{W}$ as a quotient of $C_2 \times W$. Let $H = \{e\}$ be the trivial subgroup of C_2 and $\varphi : \mathbb{R}^{p+1,q+1} \to \mathbb{R}^{p+1,0}$ be the identity map considered as an H-isomorphism. Equipping W with the trivial C_2-action allows us to form the twist $\mathrm{Tw}^H_\varphi(W,\Phi_W)$. This is an $\mathbb{R}^{p+1,q+1}$-framed manifold with base space $C_2 \times W$, and on the identity component the twisted frame agrees with Φ_W. The effect of the twist causes the last $q+1$ sections on the non-identity component $\{\tau\} \times W$ to be negated compared to those on $\{e\} \times W$.

Finally, we glue the two components of $C_2 \times W$ along their collar neighborhoods. This is where we have to be careful. Currently $C_2 \times W$ does not remember the action of C_2 on M. We fix this by using a gluing map from the collar neighborhood on the identity component to the τ-component

$$f : \{e\} \times M \times I \longrightarrow \{\tau\} \times M \times I$$
$$(e, x, t) \longmapsto (\tau, \tau \cdot x, -t).$$

The gluing map f is frame-preserving and equivariant, so the frame and action on $\mathrm{Tw}^H_\varphi(W, \Phi_W)$ descend to the quotient

$$(\widetilde{W}, \Phi_{\widetilde{W}}) := \mathrm{Tw}^H_\varphi(W, \Phi_W)/(e, x, t) \sim f(e, x, t)$$

which is now a closed $\mathbb{R}^{p+1,q+1}$-framed manifold. A (non-framed) illustration of this type of gluing is given in Figure 5.4 below.

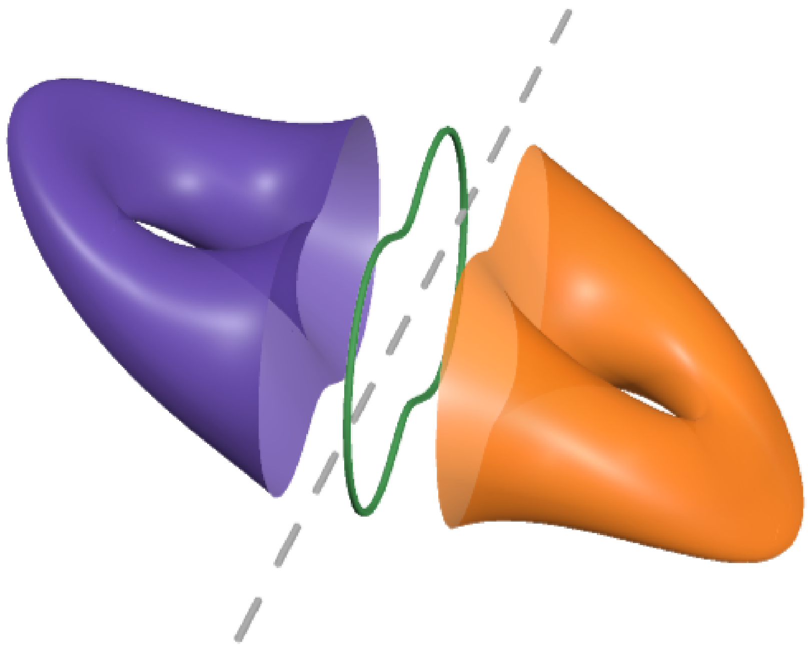


Figure 5.4. Equivariantly gluing $C_2 \times W$ to itself using the C_2-action on ∂W. The C_2-action on the whole picture is rotation about the dashed line.

The common image of the two non-equivariant collar neighborhoods is now an embedded copy of $M \times D(\mathbb{R}^{1,1})$ where M now carries its original C_2-action. Outside of this equator, the quotient still decomposes unambiguously into an identity component above $M \times D(\mathbb{R}^{1,1})$ and a τ-component below. So we can define an equivariant singular map $\sigma : \widetilde{W} \to D(\mathbb{R}^{1,1})$ by

$$\sigma(w) = \begin{cases} 1 & \text{for points above the equator} \\ pr_2(w) & \text{for points } w \in M \times D(\mathbb{R}^{1,1}) \text{ in the equator,} \\ -1 & \text{for points below the equator.} \end{cases}$$

There are two ways we can modify the singular framed manifold $[\widetilde{W}, \sigma, \Phi_{\widetilde{W}}]$ so that it decomposes into pieces we already know.

On one hand, we can excise everything that σ sends to the boundary which gives the singular framed manifold

$$[M \times D(\mathbb{R}^{1,1}), pr_2, \Phi_M \oplus \partial_t] \in \Omega^{fr}_{p+1,q+1}(D(\mathbb{R}^{1,1}), S(\mathbb{R}^{1,1})).$$

This is just $[M, \Phi_M] \cdot 1$ where 1 is represented by the fundamental class of $D(\mathbb{R}^{1,1})$. On the other hand, the singular map σ is equivariantly homotopic to the constant map c_0 to 0 just by contracting the interval $D(\mathbb{R}^{1,1})$ to the origin. This gives the singular framed manifold

$$[\widetilde{W}, c_0, \Phi_{\widetilde{W}}] \in \Omega^{fr}_{p+1,q+1}(D(\mathbb{R}^{1,1}), S(\mathbb{R}^{1,1}))$$

which is the product $\rho \cdot [\widetilde{W}, \Phi_{\widetilde{W}}]$. So we get that $[M, \Phi_M]$, which was in the kernel of ψ, is in the image of ρ: $\rho \cdot [\widetilde{W}, \Phi_{\widetilde{W}}] = [M, \Phi_M]$.

$\Diamond$

Remark. There are many parallelizable C_2-free manifolds that admit invariant parallelizations, *i.e.*, they can be $\mathbb{R}^{p,0}$-framed. For example, any Lie group equipped with the left-translation action by one of its order 2 elements. For such a manifold M, as long as $[M] \in \pi^s_p$ is nontrivial, the equivariant classes $[M_{triv}]$ and $[M_{free}]$ are distinct elements in $\pi_{p,0}$. The linear combination $[M_{triv}] + \epsilon[M_{free}]$ is in the kernel of ψ, and in fact there is a canonical non-equivariant null-bordism given by the cylinder on M. Performing the above construction then produces a framed equivariant $U(1)$-bundle $\widetilde{W}$ over M/C_2 which can also be understood as the change of fiber $U(1) \times_{C_2} M_{free}$ of the principal C_2-bundle $M \to M/C_2$. The class $[\widetilde{W}] \in \pi_{p+1,1}$ is nontrivial since its fixed-points are $[M] \in \pi^s_p$. This can be seen as a shift of the image of Kahn-Priddy homomorphism from the $\pi_{p,0}$ line to $\pi_{p+1,1}$.

5.2.2 How to Split Off Fixed-Points

In Section 5.2.1 we saw how to use excision to compute $\rho\eta = 1 + \epsilon$, and the process ultimately amounted to cutting out everything except a neighborhood of the fixed-points. This was somewhat ad-hoc, but if we use the frame more carefully there is a more systematic alternative.

Construction 5.5. We'll again assume $[M, \Phi] \in \Omega^{fr}_{p,q}(*)$ for simplicity. In particular that means we have $0 \leq q \leq p$. Over the fixed-point submanifold M^{C_2}, the frame Φ

splits into a pair of frames

$$\Phi|_{M^{C_2}} = \Phi_{\text{triv}} \oplus \Phi_{\text{sign}}$$

where $\Phi_{\text{triv}} : \underline{\mathbb{R}^{p-q,0}} \to TM^{C_2}$ and $\Phi_{\text{sign}} : \underline{\mathbb{R}^{q,q}} \to \nu_M(M^{C_2})$. We can find an equivariant tubular neighborhood [17] $\mathcal{N}$ of M which we identify with $D(\nu_M(M^{C_2}))$. Using the frame Φ_{sign} we can further identify $\mathcal{N}$ with $D(\underline{\mathbb{R}^{q,q}}) = M^{C_2} \times D(\mathbb{R}^{q,q})$. Note that the product frame on $(M^{C_2}, \Phi_{\text{triv}}) \times (D(\mathbb{R}^{q,q}), \mathscr{T})$ is the same frame as $\Phi|_{\mathcal{N}}$ under this identification.

We could equip $\mathcal{N} = M^{C_2} \times D(\mathbb{R}^{q,q})$ with a singular map that projects to $D(\mathbb{R}^{q,q})$, but we need a way to extend it to the rest of M. This actually isn't too hard. First, we multiply by ρ^q so that $\rho^q \cdot [M] \in \Omega^{fr}_{p,q}(D(\mathbb{R}^{q,q}), S(\mathbb{R}^{q,q}))$. This equips $[M]$ with the constant singular map c_0 to $0 \in D(\mathbb{R}^{q,q})$. We'll now describe a specific perturbation of c_0 so that nearby M^{C_2} it looks like projection to $D(\mathbb{R}^{q,q})$, but outside $\mathcal{N}$ it remains 0.

Define $f : D(\mathbb{R}^{q,q}) \to D(\mathbb{R}^{q,q})$ by

$$f(v) = \begin{cases} 3v & \text{if } 0 \leq \|v\| < \frac{1}{3} \\ \frac{v}{\|v\|} & \text{if } \frac{1}{3} \leq \|v\| < \frac{2}{3} \\ 3(1 - \|v\|)\frac{v}{\|v\|} & \text{if } \frac{2}{3} \leq \|v\| \leq 1. \end{cases}$$

and then define a homotopy from the constant 0 map to f by $H_t(v) = tf(v)$. Near the origin H lifts the 0 map to look like a scaled version of the identity map, but between radii 1/3 and 1 it stops and then returns to 0 on the boundary of $D(\mathbb{R}^{q,q})$. This is illustrated in Figure 5.5.

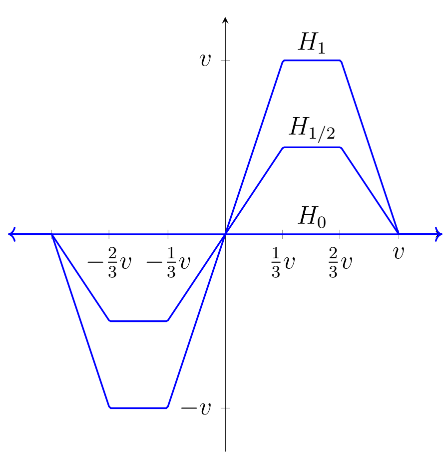


Figure 5.5. A graph indicating stages of the homotopy H along a unit vector v.

Now we can define a homotopy $\widetilde{H}$ of c_0 on all of M. Set

$$\widetilde{H}_t(x) = \begin{cases} H_t(pr_2(x)) & \text{for } x \in \mathcal{N} = M^{C_2} \times D(\mathbb{R}^{q,q}), \\ 0 & \text{otherwise.} \end{cases}$$

On $\partial\mathcal{N}$ the homotopy is 0 for all t, so this function is continuous. It is easy to check that f, and therefore $\widetilde{H}$, are both equivariant. The terminal singular map $\widetilde{H}_1$ sends the middle third of $\mathcal{N}$ to the boundary of $D(\mathbb{R}^{q,q})$, so we can excise it. This leaves behind the region outside radius $\frac{2}{3}$ and inside radius $\frac{1}{3}$ of $\mathcal{N}$. We will call the outer region F since it is a free C_2-manifold as it has no fixed-points. The inner region is (after rescaling the frame) $[M^{C_2} \times D(\mathbb{R}^{q,q}), pr_2, \Phi_{\text{triv}} \oplus \mathscr{T}] = [M, \Phi]^{C_2} \cdot 1$ where 1 is represented by the fundamental class of $D(\mathbb{R}^{q,q})$. So in $\pi_{p-q,0}$ we have $\rho^q \cdot [M] = [F] \sqcup [M]^{C_2}$. $\Diamond$

The above construction is a geometric manifestation of the splitting we saw for the fixed-point exact sequence in Section 2.1. We reproduce the sequence for ρ^k here:

$$\begin{array}{ccccccc} \cdots \to \pi^{k-(p+1),k-q}(S(\mathbb{R}^{k,k})_+) & \to & \pi_{p,q} & \xrightarrow{\rho^k} & \pi_{p-k,q-k} & \to & \pi^{k-p,k-q}(S(\mathbb{R}^{k,k})_+) \to \cdots . \\ & & & \searrow^{\phi} & \downarrow \phi & & \\ & & & & \pi^s_{p-q} & & \end{array}$$

When $2q \leq p+1 < k$ this became a split exact sequence

$$\begin{array}{ccccccc} 0 \to \pi^{k-(p+1),k-q}(S(\mathbb{R}^{k,k})_+) & \to & \pi_{p,q} & \xrightarrow{\rho^k} & \pi_{p-k,q-k} & \to & 0. \\ & & & \searrow^{\phi} & \| \phi & & \\ & & & & \pi^s_{p-q} & & \end{array} \tag{5.1}$$

At the start of Construction 5.5 we only assumed $0 \leq k = q < p$. So we are not yet in the splitting range and the group $\pi^{k-(p+1),k-q}(S(\mathbb{R}^{k,k})_+)$ has not necessarily stabilized to $\lambda_{p,q}$. However, if we instead look at the fixed-point sequence for ρ^ℓ when $\ell \geq p-q+2$, then $\pi_{p-q,0}$ *is* in the splitting range. We can combine the non-split sequence for ρ^q at $\pi_{p,q}$ with the split sequence for ρ^ℓ and $\pi_{p-q,0}$:

$$
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & [M] & & \lambda_{p-q,0} & \ni & [F] & \\
 & \pitchfork & & \downarrow & & & \\
\cdots \longrightarrow \pi^{q-(p+1),0}(S(\mathbb{R}^{q,q})_+) \longrightarrow & \pi_{p,q} & \xrightarrow{\rho^q} & \pi_{p-q,0} & \longrightarrow & \pi^{q-p,0}(S(\mathbb{R}^{q,q})_+) & \longrightarrow \cdots . \\
 & {\scriptstyle \phi}\downarrow & \swarrow{\scriptstyle \phi} & \downarrow{\scriptstyle \rho^\ell} & & & \\
 & \pi^s_{p-q} & \underset{\phi}{\overset{\cong}{\longleftarrow}} & \pi_{p-q-\ell,-\ell} & \ni & [M^{C_2}] & \\
 & & & \downarrow & & & \\
 & & & 0 & & &
\end{array}
$$

Since $[F]$ is free, it is in the kernel of ϕ and ρ^ℓ so we can pull it up to $\lambda_{p-q,0}$. On the other hand, the manifold $[M^{C_2}]$ is its own fixed-point set and can be pushed down to $\pi^s_{p-q} \cong \pi_{p-q-\ell,-\ell}$. So while $[M]$ may not originally be in the splitting range, our construction shows how to split it geometrically once we push it into slitting range by applying ρ^q.

5.2.3 Splitting Fixed-Points Off the Circle

The class η, when represented by $[U(1), \partial_\theta]$, has two fixed points $e^{i\pi}$ and e^{i0}. Figure 5.6 shows the output of Construction 5.5 applied to $\rho\eta$. Height indicates the resulting singular map $U(1) \to D(\mathbb{R}^{1,1})$. When we use excision to remove the extrema, marked in red, four components are left. Two of them contain the fixed-points and these are each equivalent to the singular framed manifold $[D(\mathbb{R}^{1,1}), \mathrm{id}, \mathscr{T}]$ which represents $1 \in \pi_{0,0}$.

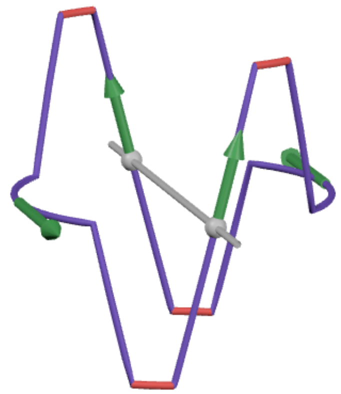


Figure 5.6. Using excision to cut $\rho\eta$ into 2 and $\epsilon[C_2]$. The C_2-action is a 180° rotation about the gray segment.

The other two components are exchanged by the C_2-action. Together they are equivalent to the singular manifold $C_2 \times D(\mathbb{R}^{1,1}) \overset{pr_2}{\to} D(\mathbb{R}^{1,1})$. The frame, however, runs in the opposite direction than the tautological frame on $D(\mathbb{R}^{1,1})$ does; from high to low instead of low to high. Since negating an $\mathbb{R}^{1,1}$-frame is equivalent to multiplication by ϵ we can identify this as the framed manifold $\epsilon \cdot [C_2] = [C_2 \times D(\mathbb{R}^{1,1}), pr_2, -\mathscr{T}]$. Altogether, we have a decomposition of $\rho\eta$ into a fixed manifold and a free manifold $\rho\eta = 2 + \epsilon[C_2]$. This also provides another proof that $\rho\eta = 1 + \epsilon$ since

$$\begin{aligned}\rho\eta &= 2 + \epsilon[C_2] \\ &= 2 + \epsilon(1 - \epsilon) \\ &= 2 + \epsilon - 1 \\ &= 1 + \epsilon.\end{aligned}$$

5.2.4 Splitting Fixed-Points Off the Torus

Copies of π_k^s split off from $\pi_{2k,k}$ for $k > 0$. These come from Bredon's doubling construction [4]. From the perspective of framed bordism, this construction takes a non-equivariant framed manifold (M^k, Φ) and sends it to $M \times M$ equipped with the C_2-action which swaps the two factors. If Φ is given by sections s_i, the $\mathbb{R}^{2k,k}$-frame on $M \times M$ is given by k many $\mathbb{R}^{1,0}$-sections

$$\frac{1}{\sqrt{2}}(s_i \circ pr_1 + s_i \circ pr_2)$$

and another k many $\mathbb{R}^{1,1}$-sections

$$\frac{1}{\sqrt{2}}(-s_i \circ pr_1 + s_i \circ pr_2).$$

We denote this manifold $Sq(M, \Phi)$ or simply $Sq(M)$. Clearly taking fixed-points recovers the non-equivariant manifold (M, Φ) as the diagonal of $M \times M$. These doubles are what generate the copies of $\pi^s_{p-q} \hookrightarrow \pi_{p,q}$ along the line $q = \frac{1}{2}$. Applying Construction 5.5 to these Bredon squares should split them into their fixed diagonal and a free manifold.

Let's carry out this program for $Sq(U(1)) \in \pi_{2,1}$. We will represent the torus $U(1) \times U(1)$ as a square with edges identified so that the action is literally reflection across the diagonal. The product $\rho \cdot Sq(U(1))$ is then equipped with a singular map to $D(\mathbb{R}^{1,1})$ which is just the constant map to 0.

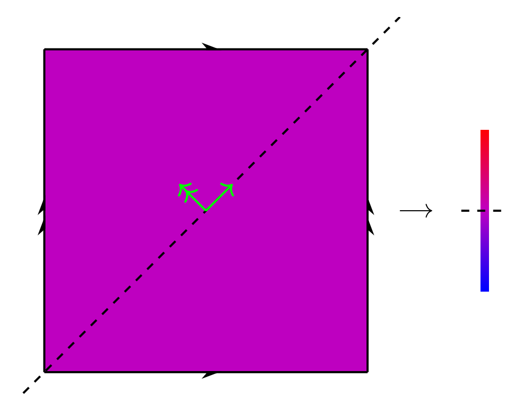


Figure 5.7. The singular manifold $\rho \cdot Sq(U(1))$.

Then we identify a neighborhood of the fixed diagonal with $U(1) \times D(\mathbb{R}^{1,1})$ and perturb the singular map by a homotopy so that some parts of the torus map to the boundary of $D(\mathbb{R}^{1,1})$.

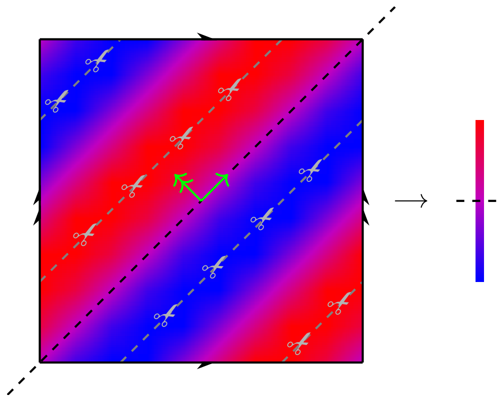


Figure 5.8. Perturbing the singular map of $\rho \cdot Sq(U(1))$ to prepare it for excision.

Next we cut along the points which now map to $S(\mathbb{R}^{1,1})$. This results in two components, one containing the fixed purple diagonal and the other containing the shifted purple diagonal which is not fixed. These two components are both the product $D(\mathbb{R}^{1,1}) \times U(1)$ equipped with the singular map which projects to $D(\mathbb{R}^{1,1})$. However, they carry different actions and frames.

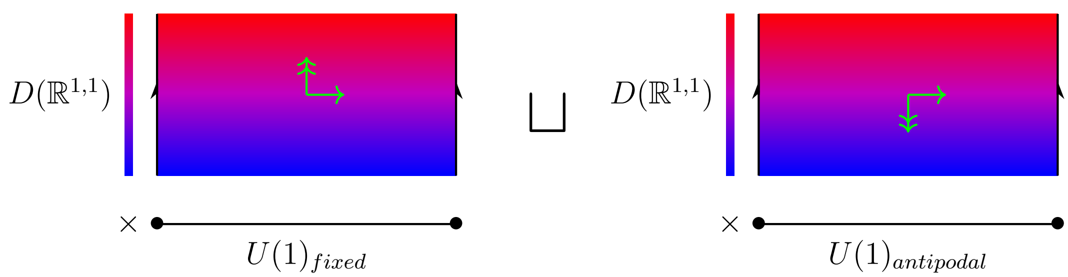


Figure 5.9. The results of excision.

On each component the frame respects the decomposition of the underlying manifold into a Cartesian product. So we can identify these as products of framed manifolds

$$(U(1)_{fixed}, \partial_\theta) \times (D(\mathbb{R}^{1,1}), \partial_t) \quad \text{and} \quad (U(1)_{antipodal}, \partial_\theta) \times (D(\mathbb{R}^{1,1}), -\partial_t).$$

In stable homotopy these are $\eta_{\text{top}} \cdot 1$ and $\eta_{\text{free}} \cdot \epsilon$. This proves $\rho \cdot Sq(\eta) = \eta_{\text{top}} + \epsilon\eta_{\text{free}}$.

5.3 Other Relations

5.3.1 A Toda Bracket

Toda brackets have a simple interpretation in framed bordism. Suppose we have three singular framed manifolds L, M, N and singular framed bordisms $X : \emptyset \rightsquigarrow L \times M$ and $Y : M \times N \rightsquigarrow \emptyset$. Then there are two bordisms of the triple product

$$X \times N : \emptyset \rightsquigarrow L \times M \times N \quad \text{and} \quad L \times Y : L \times M \times N \rightsquigarrow \emptyset.$$

These can be identified along their common boundary $L \times M \times N$ to obtain a new singular framed manifold in the Toda bracket $\langle [L], [M], [N] \rangle$. Here we see the indeterminacy of the Toda bracket both in the choice of manifold representatives L, M, N and the choice of bordisms X, Y. In spite of this, we will refer to the resulting manifold we get from a particular instance of this construction as "the Toda bracket." Lastly, if L, M, N are in $RO(G)$-degree a, b, c respectively, then $\langle [L], [M], [N] \rangle \subseteq \pi^G_{a+b+c+\mathbb{R}^1_{\text{triv}}}$. This is because the inward and outward vector fields on the boundary of bordisms are $\mathbb{R}^1_{\text{triv}}$-sections by definition.

We will show that $\eta\eta_{\text{free}} \in \langle 1 - \epsilon, \eta, 1 - \epsilon \rangle$. To construct the Toda bracket we'll need representatives for η and $1 - \epsilon$. For η choose $[U(1)_{\text{conj}}, \mathscr{L}]$ and for $1 - \epsilon$ choose $[C_2]$. Next we'll describe explicit bordisms witnessing $(1 - \epsilon) \cdot \eta = 0$. For easier notation we'll think of the framed manifold $[C_2]$ as the set $\{\pm\}$.

We'll start with the terminal half of the Toda bracket $Y : U(1)_{\text{conj}} \times C_2 \rightsquigarrow \emptyset$. The data of Y consists of

- the underlying manifold $U(1)_{\text{triv}} \times D(\mathbb{R}^{1,1})$„
- a frame, given by an $\mathbb{R}^{1,1}$- and $\mathbb{R}^{1,0}$-section

$$s_1(\theta, t) = \cos\left(\frac{\pi}{2}(t+1)\right)\partial_\theta + \sin\left(\frac{\pi}{2}(t+1)\right)\partial_t$$
$$s_2(\theta, t) = -\sin\left(\frac{\pi}{2}(t+1)\right)\partial_\theta + \cos\left(\frac{\pi}{2}(t+1)\right)\partial_t,$$

- and the embedding of the boundary $\iota : U(1)_{\text{conj}} \times C_2 \to U(1)_{\text{triv}} \times D(\mathbb{R}^{1,1})$, given by

$$(e^{i\theta}, +) \mapsto (e^{i\theta}, -1)$$
$$(e^{i\theta}, -) \mapsto (e^{-i\theta}, 1).$$

This is a cylinder there the frame gradually twists by π radians as we move from one base to the other. The C_2-action is reflection about the midsection. Since $U(1) \times C_2$ is the initial manifold of Y we have an inward pointing $\mathbb{R}^{1,0}$-section on the boundary.

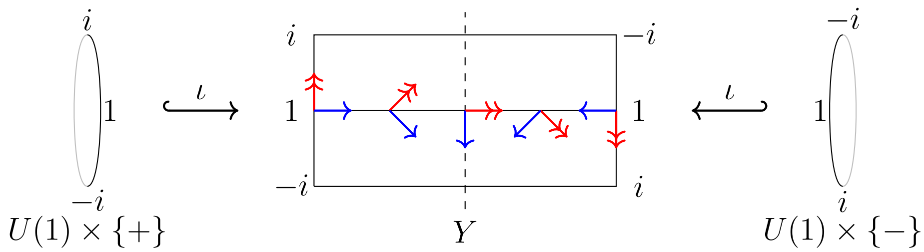


Figure 5.10. A chart showing the front of the null-bordism Y. The boundary is annotated with the image of $U(1)$ under the inclusion ι.

Constructing the Toda bracket will require gluing together four such cylinders with slight variations, so we'll introduce a shorthand for drawing them. Ultimately, we just need to keep track of how the frame twists along the cylinder and which copy of $U(1)$ gets sent to which boundary component. This information can be compactly expressed using the annotated interval shown in Figure 5.11.

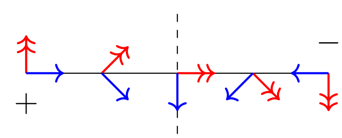


Figure 5.11. A decorated interval shorthand for the bordism Y in Figure 5.10. The signs indicate which copy of $U(1)$ gets sent to which boundary component of Y.

We also need a null-bordism $X : \emptyset \rightsquigarrow C_2 \times U(1)$. We'll use the one shown in Figure 5.12.

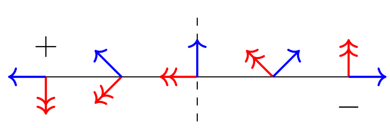


Figure 5.12. The bordism $X : \emptyset \rightsquigarrow C_2 \times U(1)$. In contrast to being the *initial* manifold of Y, $C_2 \times U(1)$ appears as the *terminal* manifold of X. Consequently, the $\mathbb{R}^{1,0}$-section at the boundary is outward pointing instead of inward pointing.

Next we will glue

$$X \times C_2 \bigcup_{C_2 \times U(1) \times C_2} C_2 \times Y.$$

To do this, it will be convenient to write X_+ for $X \times \{+\}$, ${}_-Y$ for $\{-\} \times Y$, etc. There are two ways to represent the C_2-action on $X \times C_2$ and $C_2 \times Y$. In order to draw the gluing flat on the page, we need the less intuitive representation on the right of Figure 5.13.

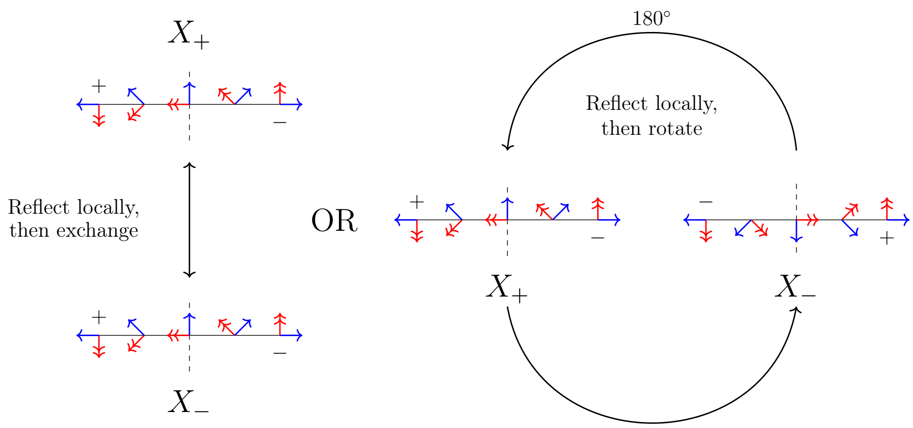


Figure 5.13. Two ways of thinking of the C_2-action on $X \times C_2$. In either case we start with a reflection locally on each component, indicated by the dashed lines. On the left, we follow the reflections by an exchange. On the right, we follow the reflections by a rotation.

Finally, we glue all the pieces together as illustrated in Figure 5.14. I invite the reader to inspect the figure to see that the boundaries and their frames match, that the vectors transform as expected, and that the action respects the gluing.

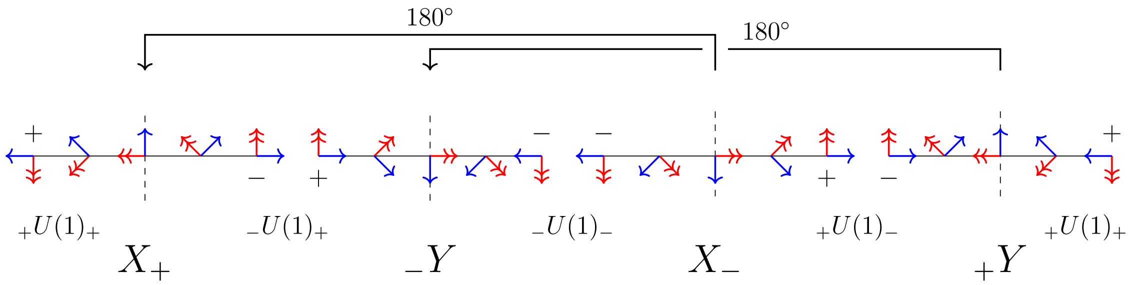


Figure 5.14. The bordisms $X \times C_2$ and $C_2 \times Y$ prepped to be glued along the triple product $C_2 \times U(1) \times C_2$. The C_2-action reflects each interval, but also swaps corresponding pairs of intervals by a 180° rotation. The left edge of this figure is identified with the right edge.

This finishes the construction of the Toda bracket $\langle [C_2], [U(1)], [C_2] \rangle$, but we still need to determine which element of $\pi_{2,1}$ it represents. First, recall that each of the four intervals shown in Figure 5.14 were shorthand for cylinders. We have glued them together in such a way as to obtain a torus. We'll parameterize the vertical and horizontal directions of the torus by angles u and v respectively. Observe that, perhaps surprisingly, the reflect-locally-then-pairwise-180° action actually sends $(u, v) \mapsto (-u, v + \pi)$ as seen in Figure 5.15.

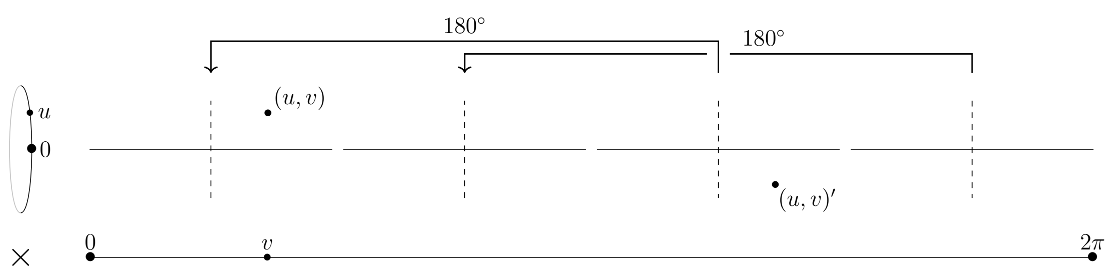


Figure 5.15. The Toda bracket parameterized by angles u and v. An example point (u, v) is shown with its C_2-conjugate $(u, v)'$.

This parameterization identifies the underlying C_2-manifold of the Toda bracket as $U(1)_{\text{conj}} \times U(1)_{\text{free}}$. Having made this identification, we'll now focus on the frame. The product $U(1)_{\text{conj}} \times U(1)_{\text{free}}$ carries a Lie group frame given by the ordered pair of vector fields (∂_u, ∂_v). With respect to the C_2-action, ∂_u is an $\mathbb{R}^{1,1}$-section, ∂_v is an $\mathbb{R}^{1,0}$-section, and together they make an $\mathbb{R}^{2,1}$-frame.

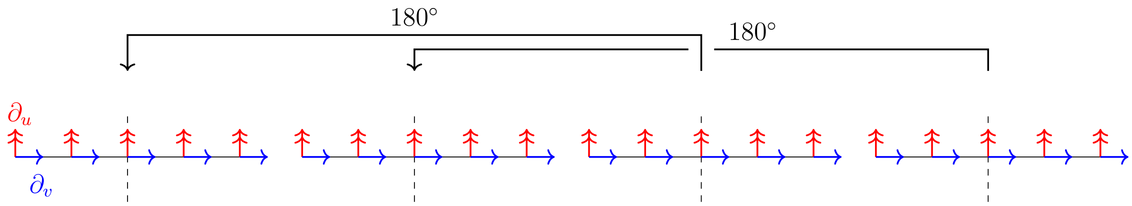


Figure 5.16. The Lie frame on $U(1)_{\text{conj}} \times U(1)_{\text{free}}$.

The Toda frame in Figure 5.14 and the Lie frame in Figure 5.16 differ by an equivariant twisting map

$$a : U(1)_{\text{conj}} \times U(1)_{\text{free}} \to O(\mathbb{R}^{2,1}).$$

The difference between the two frames only depends on the coordinate v, so a factors through a map $a' : U(1)_{\text{free}} \to O(\mathbb{R}^{2,1})$. There are a pair of conjugate points $p = (0, \pi/2)$ and $q = (0, 3\pi/2)$ where both frames agree: $a'(p) = a'(q) = \mathrm{id}_{\mathbb{R}^{2,1}}$. So we can factor a' through the quotient $a'' : U(1)_{\text{free}}/(p \sim q) \to O(\mathbb{R}^{2,1})$. Furthermore, there is a C_2-homeomorphism

$$U(1)_{\text{free}}/(p \sim q) \cong (C_2/\{e\})_+ \wedge S^{1,0}$$

which lets us treat a'' as a non-equivariant map $S^1 \to O(\mathbb{R}^2)$. In Figure 5.14 this corresponds to the behavior of the frame on the middle two components. As we go from p to q we see that, relative to the Lie frame, the Toda frame first twists clockwise by π and then counterclockwise by π. This is a null-homotopic map $S^1 \to O(\mathbb{R}^2)$. We conclude that the Toda frame and the Lie frame on $U(1)_{\text{conj}} \times U(1)_{\text{free}}$ are homotopic. Since our Toda bracket is equivalent to the product of framed manifolds $[U(1)_{\text{conj}}, \partial_u] \cdot [U(1)_{\text{free}}, \partial_v]$, it represents $\eta\eta_{\text{free}} \in \pi_{2,1}$.

We could have used a different representative for $1 - \epsilon$ rather than the orbit $[C_2]$. The other natural representative consists of two fixed points: a positively oriented point representing 1 and a point representing $-\epsilon$ which is negatively oriented with respect to $\mathbb{R}^{1,0}$ *and* $\mathbb{R}^{1,1}$. The null-bordism for these representatives is the cylinder $U(1)_{\text{conj}} \times D(\mathbb{R}^{1,0})$. Repeating the exercise above with this setup results in the manifold $U(1)_{\text{conj}} \times U(1)_{\text{triv}}$, which instead represents $\eta\eta_{\text{top}}$.

In fact, one can see that $\eta\eta_{\text{free}} = \eta\eta_{\text{top}}$ by comparing the forgetful and fixed-point homomorphisms ψ and ϕ. In Table 1.1 we find $\pi_{2,1} = \mathbb{Z}/2 \oplus \pi_1^s \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. The copy of π_1^s comes from the doubling construction, so it is generated by $Sq(\eta_{\text{top}})$. This is represented by $U(1) \times U(1)$ with the Lie group frame and the action which swaps the two $U(1)$ factors. Clearly $\psi(Sq(\eta_{\text{top}})) = \eta_{\text{top}}^2$ and $\phi(Sq(\eta_{\text{top}})) = \eta_{\text{top}}$. On the other hand, we have $\psi(\eta\eta_{\text{free}}) = \eta_{\text{top}}^2$ and $\phi(\eta\eta_{\text{free}}) = 0$. Hence $\eta\eta_{\text{free}}$ is nontrivial and distinct from $Sq(\eta_{\text{top}})$, and thus must generate the other $\mathbb{Z}/2$ summand. This information, summarized in Table 5.2 is enough to determine any element of $\pi_{2,1}$ using ψ and ϕ.

$\alpha \in \pi_{2,1}$	$\psi(\alpha)$	$\phi(\alpha)$
0	0	0
$\eta\eta_{\text{free}}$	η_{top}^2	0
$Sq(\eta)$	η_{top}^2	η_{top}
$\eta\eta_{\text{free}} + Sq(\eta)$	0	η_{top}

Table 5.2. The fixed and forgetful maps on $\pi_{2,1}$.

Since $\psi(\eta\eta_{\text{top}}) = \eta_{\text{top}}^2$ and $\phi(\eta\eta_{\text{top}}) = \pm 2\eta_{\text{top}} = 0$, it follows $\eta\eta_{\text{top}} = \eta\eta_{\text{free}}$.

5.3.2 Order of the Quaternionic Hopf Map

In classical stable homotopy it is known that $24\nu_{\text{top}} = 0$. There is a folklore bordism interpretation of this fact in terms of a K3 surface with 24 punctures [26]. Furthermore, half a K3 surface with 12 punctures witnesses the relation $12\nu_{\text{top}} = \eta_{\text{top}}^3$. It's not immediately obvious that the order of the C_2-equivariant ν is also 24. Both η and σ have different orders than η_{top} and σ_{top}, and clearly $12\nu \neq \eta^3$ since they are not even in the same $RO(C_2)$-graded stem. However, we now have enough relations proved purely in terms of singular framed bordism to show that $24\nu = 0$.

Across Examples 5.2 and 5.3 and Propositions 4.4 and 4.12 we have shown that

(1) $\nu = -\epsilon\nu$,

(2) $1 - \epsilon = [C_2]$,

(3) $[C_2]\nu = u\theta_2\nu_{\text{top}}$,

(4) $u'\theta_2\eta_{\text{top}}^3 = [C_2]\eta^2\eta_{\text{top}}$, and

(5) $[C_2]\eta = 0$,

where u and u' are the convention-sensitive units coming from in Proposition 4.12. Equipped with these relations, and the non-equivariant half K3 surface bordism which we take as given, we can chain together a bordism which witness the 24-torsion of $\nu \in \pi_{3,2}$:

$$
\begin{array}{ccc}
24\nu & & 12u\theta_2\nu_{\mathrm{top}} \\
\Big\| & \nearrow & \Big\| \\
12(\nu+\nu) & & u\theta_2(12\nu_{\mathrm{top}}) \\
\Big\downarrow{\scriptstyle (1)} & & \Big\downarrow{\scriptstyle \frac{1}{2}\mathrm{K3}} \\
12(\nu-\epsilon\nu) & {\scriptstyle (3)} & u\theta_2(\eta_{\mathrm{top}}^3) \\
\Big\| & & \Big\downarrow{\scriptstyle (4)} \\
12(1-\epsilon)\nu & & u(u')^{-1}[C_2]\eta^2\eta_{\mathrm{top}} \\
\Big\downarrow{\scriptstyle (2)} & & \Big\| \\
12[C_2]\nu & \nearrow & u(u')^{-1}([C_2]\eta)\eta\eta_{\mathrm{top}} \\
 & & \Big\downarrow{\scriptstyle (5)} \\
 & & \emptyset.
\end{array}
$$

APPENDIX A

HOMOTOPICAL FRAMING

This appendix outlines a second, homotopical perspective on V-frames. This perspective fits better into the picture of generic "B-structures on a manifold" [21], where "B" happens to be "V-frame" in our case.

Non-equivariantly, if a bundle $E \to X$ is classified by a map $f : X \to BO$ then f is null-homotopic if and only if E is trivializable. Fixing the terminal point b_0, we can say a **homotopy framing** of E is a choice of a (homotopy class of a) null-homotopy H from f to b_0. A homotopy framing can also be thought of as a lift of the classifying map to the space of based paths

$$\begin{array}{ccc} & & PBO \simeq EO \\ & \overset{H}{\nearrow} & \downarrow \\ X & \xrightarrow{\ f\ } & BO. \end{array}$$

What the most general form of the Pontryagin-Thom isomorphism really does is identify the bordism ring of manifolds equipped with such lifts with the homotopy groups of $MPBO$, the Thom spectrum of PBO. Since PBO is contractible, the associated Thom spectrum is equivalent to the sphere spectrum which is how we get to stable homotopy groups.

This idea works just as well when we replace BO with its equivariant analogue B_GO. However, when constructing the path-space fibration over B_GO we have to be more careful about picking the basepoint. We will be modeling B_GO using the Grassmannian of planes in a G-representation W, where picking a basepoint is the same as picking a subrepresentation of W. We get different path-space fibrations depending on the isomorphism class of this subrepresentation.

A.1 Equivariant Grassmannians

In Section 3.1.2 we said $O(V, W)$ denotes the G-space of linear isometries between two G-representations $V \to W$. That was convenient then since we were only considering V and W of the same dimension. Now we will consider $|V| \leq |W|$, so we will let $O(V, W)$ denote the space of linear isometric embeddings $V \to W$. Recall that the G-action on $O(V, W)$ is given by $(g \cdot f)(v) = gf(g^{-1}v)$ for $g \in G$, $f \in O(V, W)$, and $v \in V$. When $|V| = |W|$ this agrees with the old definition, so we can still write $O(V) := O(V, V)$. Note that $O(V)$ has a free and transitive right action on $O(V, W)$

given by pre-composition. Furthermore, this $O(V)$-action is compatible with the G-action on both $O(V)$ and $O(V,W)$ in the sense that $g \cdot (f \circ a) = (g \cdot f) \circ (g \cdot a)$ for $a \in O(V)$. This makes $O(V,W)$ an $(O(V); O(V) \rtimes G)$-principal bundle in the sense of May [22, Chapter VII]. For $H \leq G$, the fixed-point space $O(V,W)^H$ is the set of H-isomorphisms $V \to W$. We define the **Grassmannian of subreps of** W **isomorphic to** V to be

$$\mathrm{Gr}_V(W) := O(V,W)^G / O(V)^G = (O(V,W)/O(V))^G.$$

For $H \leq G$ note that

$$\mathrm{Gr}_{\mathrm{Res}_H V}(\mathrm{Res}_H W) = \mathrm{Gr}_V(W)^H = O(V,W)^H / O(V)^H = (O(V,W)/O(V))^H$$

When H is the trivial subgroup we write

$$\mathrm{Gr}_{|V|}(W) := \mathrm{Gr}_{\mathrm{Res}_H V}(\mathrm{Res}_H W).$$

As a manifold $\mathrm{Gr}_{|V|}(W)$ is the familiar Grassmannian of $|V|$-planes in W, but it carries a G-action coming from representation W. We can turn $\mathrm{Gr}_{|V|}(W)$ into the classifying space of rank $|V|$ equivariant vector bundles by taking the limit over W indexed by a complete G-universe $\mathcal{U}$:

$$B_G O = \mathrm{Gr}_{|V|}(\mathcal{U}) = \varinjlim_{W \subset \mathcal{U}} \mathrm{Gr}_{|V|}(W).$$

If we pick a basepoint of $\mathrm{Gr}_{|V|}(\mathcal{U})$ it is necessarily fixed, and fixed-points are precisely the $|V|$-dimensional subrepresentations of $\mathcal{U}$. If the basepoint we pick is a subrepresentation isomorphic to V then we write the corresponding path-space as $P_V B_G O$. The space $P_V B_G O$ models the universal bundle $E_V O$ over the classifying space $B_V O$. Another way of modeling $E_V O$ is using the V-frame bundle

$$\mathrm{Fr}_V(\mathcal{U}) = O(V, \mathcal{U}).$$

Changing the isomorphism class of V does not change the topology of $E_V O$, but it does change its G-action. There is an intermediate fiber bundle $\gamma_{|V|}$

$$E_V O \to \gamma_{|V|} \to B_G O$$

which is the tautological bundle of $|V|$-planes over the infinite Grassmannian.

As introduced in Section 3.1, our notion of a V-frame can be though of as a lift of the classifying map of E to the frame bundle

$$\begin{array}{ccc} & & \mathrm{Fr}_V(\mathcal{U}) \\ & \overset{\Phi}{\nearrow} & \downarrow \\ X & \xrightarrow{f} & BO. \end{array}$$

To distinguish lifts to $P_V B_G O$ from lifts to $\mathrm{Fr}_V(\mathcal{U})$, we'll call the former homotopy-V-frames and the latter simply V-frames.

Recall that the basepoint of $P_V B_G O$ was some subrepresentation of $\mathcal{U}$ which is isomorphic to V. Fixing this isomorphism also gives us a basepoint of $\varphi \in \mathrm{Fr}_V(\mathcal{U}) = O(V, \mathcal{U})$. There is a canonical contraction of $P_V B_G O$ to the constant path with image $\varphi(V) \in B_G O$. Using φ as the initial lift, we can lift this contraction using the homotopy lifting property of $\mathrm{Fr}_V(\mathcal{U}) \to B_G O$ to get a map $P_V B_G O \to \mathrm{Fr}_V(\mathcal{U})$. This way we can recover V-frames from homotopy-V-frames.

Finding frame complements is much easier using homotopy-V-frames. We will model $B_G O$ at a finite stage using a Grassmannian $\mathrm{Gr}_{|V|}(W)$ and pick a basepoint $b_0 = V \subset W$. Suppose the classifying map $f : X \overset{E}{\to} \mathrm{Gr}_{|V|}(W)$ comes from an embedding $E \hookrightarrow \underline{W}$. If $H : f \rightsquigarrow b_0$ is a null-homotopy, then we can compose H with the map $-^{\perp} : \mathrm{Gr}_{|V|}(W) \to \mathrm{Gr}_{|W|-|V|}(W)$ to get a homotopy-$V^{\perp}$-frame $E^{\perp}$. If f and H are smooth, there is actually a natural connection on the tautological bundle $\gamma_k \to \mathrm{Gr}_k(W)$ which can be used to parallel-transport the ambient frames on $V, V^{\perp} \subseteq W$ to E and $E^{\perp}$. This idea is actually what motivated Definition 3.4. It can be seen in action in Example B.8.

A.2 Revisiting Twists

There is a natural action of the fixed loop space $(\Omega_V B_G O)^G$ on homotopy-V-frames. Given a null-homotopy of a classifying map $H : f \rightsquigarrow V$ we can concatenate H with a loop $\alpha : V \rightsquigarrow V$ in $B_G O$ to get a new null-homotopy $H * \alpha$. Up to homotopy, this only depends on the homotopy class of α so we can think of it as an action of $\pi_1((B_G O)^G, V)$. As discussed in Section 4.2, Schur's lemma decomposes $O(V, W)$ into a product of real, unitary, and symplectic groups which in turn decomposes $\mathrm{Gr}_V(W) = \mathrm{Gr}_{|V|}(W)^G$ into a product of real, complex, and quaternionic Grassmannians. Since complex and quaternionic Grassmannians are simply connected, we find that $\pi_1((B_G O)^G) = (\pi_1(BO))^{\ell_{\mathbb{R}}} = (\mathbb{Z}/2)^{\ell_{\mathbb{R}}}$ where $\ell_{\mathbb{R}}$ is the number of real-type irrep isomorphism classes of G. Using the correspondence between homotopy-V-frames and V-frames, it is not hard to see that a generator $\alpha_i \in \pi_1((B_G O)^G)$ associated to the ith real-type representation I_i effects homotopy-V-frames in the same way that the linear unit ϵ_i affects V-frames.

There is a similar path-based interpretation for the twisting operators Tw_φ^H. From the homotopy-V-frame perspective applying Tw_φ^H can be understood as concatenating an equivariant null homotopy $H : G/H \times X \times I \to B_GO$ with an equivariant map $G/H \times I \to B_GO$ which begins and terminates at G-reps $V, W \in B_GO$. This contains the same data as a non-equivariant path $I \to (B_GO)^H$ from V to W, which also determines the twisting H-isomorphism φ up to homotopy.

In full generality, we can concatenate an equivariant null-homotopy of $f : X \to B_GO$ with an equivariant map $\alpha : X \times I \to B_GO$ such that α_0 and α_1 are constant maps to subrepresentations $V, W \in B_GO$. This is the homotopy-V-frame-theoretic version of twisting a V-frame into a W-frame using an equivariant map $X \to O(W, V)$.

APPENDIX B

DETAILS ON FRAME CONVERSION

This section is dedicated to showing that our notion of complementary frames coincides with the standard definition in the literature [18, 29]. We also establish that complementary frames satisfy a number of expected properties.

B.1 Basic Properties of Complementary Frames

For convenience, we restate the definition here.

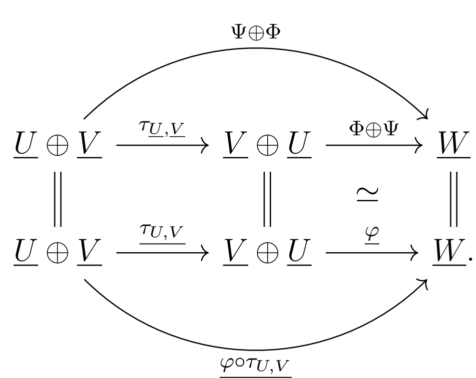
Definition 3.4 (Complementary Frames)**.** *Let U, V, and W be representations, let $\varphi : V \oplus U \to W$ be an isomorphism, let $E \to X$ be an equivariant vector bundle, and let $f : E \hookrightarrow \underline{W}$ be an equivariant bundle embedding. For a V-frame $\Phi : \underline{V} \to E$, we say a U-frame $\Psi : \underline{U} \to E^{\perp \underline{W}}$ is* ***complementary*** *to Φ with respect to φ and f if there is a homotopy of $(V \oplus U)$-frames $\Phi \oplus \Psi \simeq \underline{\varphi}$ on the bundle $\underline{W}$.*

Remark. A frame complementary to Φ with respect to f and φ is not guaranteed to exist. For example, the unit circle $S^1 \subset \mathbb{R}^2$ admits only the inward or outward pointing normal frames. Neither of these are complementary to the Lie frame ∂_θ on TS^1.

Taking complements twice recovers the original frame. But we do need to take a little care about which isomorphisms they are taken with respect to.

Proposition B.1 (Double Complements)**.** *If a frame Ψ complementary to Φ exists, then Φ is a complementary frame to Ψ with respect to $\varphi \circ \tau_{U,V}$ where $\tau_{U,V}$ is the natural isomorphism $\tau_{U,V} : U \oplus V \to V \oplus U$.*

Proof. The equivalence $\Psi \oplus \Phi \simeq \underline{\varphi \circ \tau_{U,V}}$ follows from the homotopy-commutative diagram,



□

Although we speak of Ψ as being complementary to Φ "with respect to φ", the notion of complement actually only depends on φ up to homotopy. So a complement is really defined relative to a choice of connected path-component of $O(V \oplus U, W)^G$.

Proposition B.2. *If there is a path between φ and φ' in $O(V \oplus U, W)^G$, then Ψ is a complement to Φ w.r.t. φ if and only if it is a complement w.r.t. φ'.*

If we enlarge the ambient embedding space from $\underline{W}$ to $\underline{W} \oplus \underline{W'}$, we can stabilize frames in a natural way.

Proposition B.3 (Complements Stabilize)**.** *Let V, U, W, φ, E, f, and Φ be as in Definition 3.4 and let Ψ be a U-frame complementary to Φ. Let W' be another representation and denote the inclusion $\underline{W} \hookrightarrow \underline{W \oplus W'}$ by ι. Then*

$$\Psi \oplus \mathrm{id}_{\underline{W'}} : U \oplus W' \to E^{\perp \underline{W} \oplus \underline{W'}}$$

is a $(U \oplus W')$-frame complementary to Φ with respect to the embedding $\iota \circ f : E \hookrightarrow \underline{W \oplus W'}$ and the isomorphism $\varphi \oplus \mathrm{id}_{W'} : V \oplus (U \oplus W') \to W \oplus W'$.

Proof. This is mostly an exercise is associativity:

$$\Phi \oplus (\Psi \oplus \mathrm{id}_{\underline{W'}}) = (\Phi \oplus \Psi) \oplus \mathrm{id}_{\underline{W'}} \cong \underline{\varphi} \oplus \mathrm{id}_{\underline{W'}} = \underline{\varphi \oplus \mathrm{id}_{W'}}.$$

□

Two complements Ψ and Ψ' of Φ in $\underline{W}$ (with respect to the same φ) might not necessarily be homotopic frames of $E^{\perp}$. But they do become equivalent after stabilizing sufficiently. In particular, if Φ is a V-frame, then stabilizing the complements Ψ and Ψ' by V does the trick.

Proposition B.4 (Uniqueness after Stabilizing)**.** *Let Ψ and Ψ' be U-frames complementary to Φ in $\underline{W}$ with respect to φ. Then $\Psi \oplus \mathrm{id}_{\underline{V}}$ and $\Psi' \oplus \mathrm{id}_{\underline{V}}$ are equivalent $(U \oplus V)$-frames of $E^{\perp \underline{W} \oplus \underline{V}}$.*

Proof. We need to show that $\Psi \oplus \mathrm{id}_{\underline{V}}$ and $\Psi' \oplus \mathrm{id}_{\underline{V}}$ are homotopic frames on $E^{\perp \underline{W} \oplus \underline{V}} = E^{\perp \underline{W}} \oplus \underline{V}$. To simplify, write $E^{\perp}$ for $E^{\perp \underline{W}}$. If we compose both frames by the bundle equivalences indicated in the diagram

$$\underline{V} \oplus \underline{U} \xrightarrow{\tau_{\underline{V},\underline{U}}} \underline{U} \oplus \underline{V} \underset{\Psi' \oplus \mathrm{id}_{\underline{V}}}{\overset{\Psi \oplus \mathrm{id}_{\underline{V}}}{\rightrightarrows}} E^{\perp} \oplus \underline{V} \xrightarrow{\mathrm{id}_{E^{\perp}} \oplus \Phi} E^{\perp} \oplus E \xrightarrow{\tau_{E^{\perp},E}} E \oplus E^{\perp}$$

then we obtain $\Phi \oplus \Psi$ along the top and $\Phi \oplus \Psi'$ along the bottom. These are both homotopic to $\underline{\varphi}$ by assumption. Since $\Psi \oplus \mathrm{id}_{\underline{V}}$ and $\Psi' \oplus \mathrm{id}_{\underline{V}}$ become homotopic after being composed with the same bundle equivalences, they must have originally been homotopic. $\square$

B.2 The Canonical Complement

Now that we know some basic facts about complementary frames, we should actually show how to produce one. The following construction defines a frame which is common in the literature [18]. We will call this the **canonical complement**. Our goal is to show that the canonical complement is complementary in the sense of Definition 3.4. This is primarily to justify that Definition 3.4 is correct, but it also provides some geometric insight about the canonical complement that is not obvious from its definition.

Construction B.5 (Canonical Complement)**.** Let E be a bundle with V-frame Φ. Given an embedding $f : E \to \underline{W}$ we may not be able to find a frame complementary to Φ. In fact, V might not even appear as a subrepresentation of W. However, there is always a canonical W-frame $\Phi^{\perp}$ of $E^{\perp \underline{W} \oplus \underline{V}}$ defined by

$$\underline{W} = E^{\perp \underline{W}} \oplus E \xrightarrow{\mathrm{id} \oplus \Phi^{-1}} E^{\perp \underline{W}} \oplus \underline{V} = E^{\perp \underline{W} \oplus \underline{V}} \ .$$

This frame is complementary to Φ in $\underline{W} \oplus \underline{V}$ with respect to the map

$$(\mathrm{id}_W \oplus - \mathrm{id}_V) \circ \tau_{V,W} : V \oplus W \to W \oplus V.$$

We will describe an explicit homotopy from $\Phi \oplus \Phi^{\perp}$ to $\underline{(\mathrm{id}_W \oplus - \mathrm{id}_V) \circ \tau_{V,W}}$. The core idea is to, simultaneously in each linear fiber, pick two orthogonal hyperplanes of dimension $|V|$ and rotate one onto the other. Once we've identified the two V-

hyperplanes, this rotation can be expressed succinctly using the tensor product $\mathbb{R}^2 \otimes V$:

$$\begin{array}{ccc} V \oplus V & \xrightarrow{\quad \text{id} \quad} & V \oplus V \\ \wr\| & R(0)\otimes \text{id}_V & \wr\| \\ \mathbb{R}^2 \otimes V & \simeq & \mathbb{R}^2 \otimes V \\ \wr\| & R(-\pi/2)\otimes \text{id}_V & \wr\| \\ V \oplus V & \xrightarrow[(\text{id}_V \oplus -\text{id}_V)\circ \tau_{V,V}]{} & V \oplus V \end{array} \tag{B.1}$$

where

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

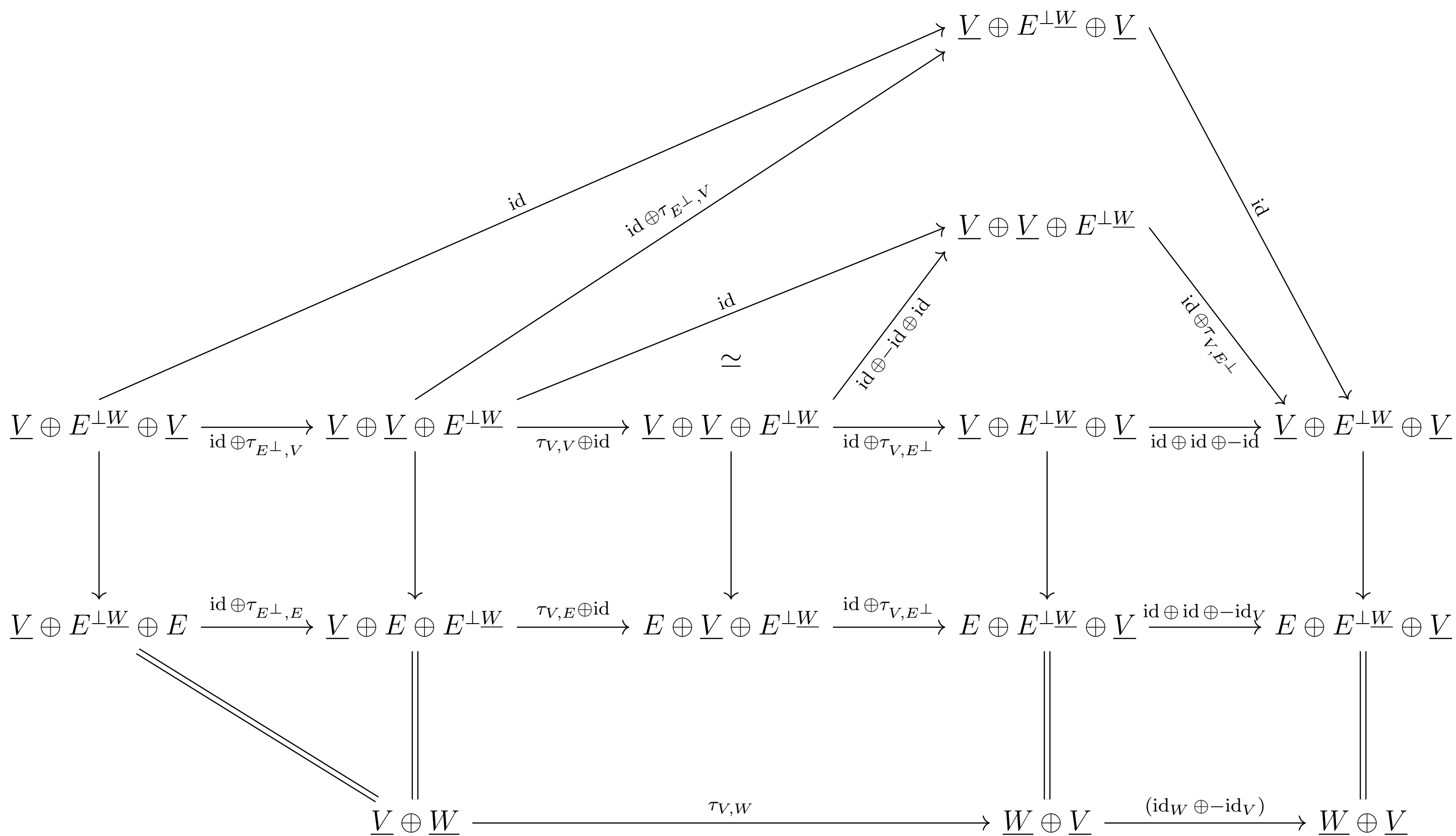
We'll perform this rotation fiberwise within the bundle $\underline{W} \oplus \underline{V}$. In each fiber one copy of V appears naturally from the second summand and the second copy appears in W coming from the image of $f \circ \Phi$.

Ultimately, we will use the tensor twist (B.1) in the form of a triangle with $E^{\perp}$ appended:

$$\begin{array}{ccc} \underline{V} \oplus \underline{V} \oplus E^{\perp \underline{W}} & \xrightarrow{\qquad \text{id} \qquad} & \underline{V} \oplus \underline{V} \oplus E^{\perp \underline{W}} \\ {\scriptstyle \tau_{V,V} \oplus \text{id}_{E^{\perp \underline{W}}}} \searrow & \simeq & \nearrow {\scriptstyle \text{id}_V \oplus -\text{id}_V \oplus \text{id}_{E^{\perp \underline{W}}}} \\ & \underline{V} \oplus \underline{V} \oplus E^{\perp \underline{W}} & \end{array} \tag{B.2}$$

This triangle appears in the center of the homotopy-commutative diagram on the next page, which shows that

$$\Phi \oplus \Phi^{\perp} \simeq (\text{id}_W \oplus - \text{id}_V) \circ \tau_{V,W}.$$



All the vertical arrows along the horizontal midline of the diagram are applications of Φ to the obvious summand. We can think of this diagram as a ship facing towards the left. Going right across bottom of the keel is $(\mathrm{id}_W \oplus - \mathrm{id}_V) \circ \tau_{V,W}$. Going up the prow and across the top of the rigging and down the stern is $\Phi \oplus \Phi^\perp = \Phi \oplus \mathrm{id}_{E^{\perp \underline{W}}} \oplus \Phi^{-1}$.

To summarize: If we have a V-frame Φ of E and an embedding $E \hookrightarrow \underline{W}$, there may not be a complement of Φ. If we enlarge the target to $\underline{W} \oplus \underline{V}$ we can always construct the canonical complement $\Phi^\perp$, which is a W-frame complementary to Φ in $\underline{W} \oplus \underline{V}$ with respect to $(\mathrm{id}_W \oplus -\mathrm{id}_V) \circ \tau_{V,W} : V \oplus W \to W \oplus V$. ◊

Suppose that we had already been able to find a complement in $\underline{W}$ without enlarging the ambient space. That is, we already had some decomposition $\varphi : V \oplus U \to W$ and a U-frame Ψ complementary to Φ in $\underline{W}$ with respect to φ. How does Ψ compare to the canonical complement $\Phi^\perp$? By Proposition B.3 we know that $\Psi \oplus \mathrm{id}_V$ is a $(U \oplus V)$-frame complementary to Φ in $\underline{W} \oplus \underline{V}$ with respect to $\varphi \oplus \mathrm{id}_V$. In order to compare it to the canonical complement $\Phi^\perp$ we need to think of $\Psi \oplus \mathrm{id}_{\underline{V}}$ as a W-frame. We can convert the $(U \oplus V)$-frame $\Psi \oplus \mathrm{id}_V$ into a W-frame Ψ' using φ:

$$\begin{array}{ccc} \underline{W} & \overset{\Psi'}{\dashrightarrow} & E^{\perp \underline{W} \oplus \underline{V}} = E^{\perp \underline{W}} \oplus \underline{V} \\ {\scriptstyle \underline{\varphi^{-1}}}\downarrow & & \uparrow {\scriptstyle \Psi \oplus \mathrm{id}_{\underline{V}}} \\ \underline{V} \oplus \underline{U} & \xrightarrow{\tau_{\underline{V},\underline{U}}} & \underline{U} \oplus \underline{V} \end{array}$$

The W-frame Ψ' is complementary to Φ in $\underline{W} \oplus \underline{V}$ with respect to the isomorphism $\varphi' : V \oplus W \to W \oplus V$ given by

$$\begin{array}{ccc} V \oplus W & \overset{\varphi'}{\dashrightarrow} & W \oplus V \\ {\scriptstyle \mathrm{id}_V \oplus \varphi^{-1}}\downarrow & & \uparrow {\scriptstyle \varphi \oplus \mathrm{id}_V} \\ V \oplus V \oplus U & \xrightarrow{\mathrm{id}_V \oplus \tau_{V,U}} & V \oplus U \oplus V. \end{array}$$

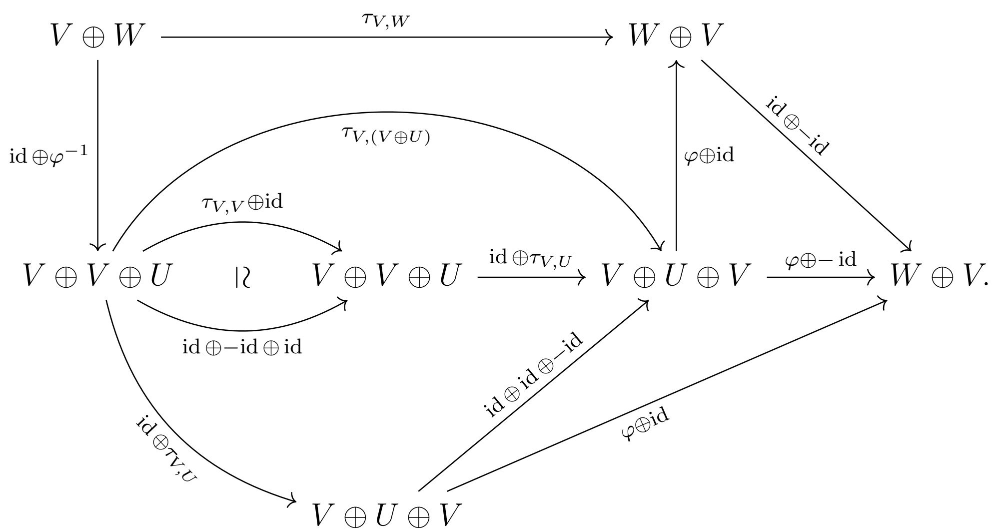
So we have two W-frames complementary to Φ in $\underline{W} \oplus \underline{V}$ but which are complements taken with respect to different isomorphisms $V \oplus W \to W \oplus V$:

- $\Phi^\perp$ is a complement with respect to $(\mathrm{id}_W \oplus -\mathrm{id}_V) \circ \tau_{V,W}$, and
- Ψ' is a complement respect to $(\varphi \oplus \mathrm{id}_V) \circ (\mathrm{id}_V \oplus \tau_{V,U}) \circ (\mathrm{id}_V \oplus \varphi^{-1})$.

It turns out that this difference is not a problem because these two decompositions of the ambient representation $W \oplus V$ into $V \oplus W$ are homotopic.

Proposition B.6. *The maps* $(\mathrm{id}_W \oplus -\mathrm{id}_V) \circ \tau_{V,W}$ *and* $(\varphi \oplus \mathrm{id}_V) \circ (\mathrm{id}_V \oplus \tau_{V,U}) \circ (\mathrm{id}_V \oplus \varphi^{-1})$ *are homotopic through isomorphisms* $V \oplus W \to W \oplus V$.

Proof. In (B.1) we saw that $\mathrm{id}_{V \oplus V} \simeq (\mathrm{id}_V \oplus -\mathrm{id}_V) \circ \tau_{V,V}$; equivalently, $\tau_{V,V} \simeq \mathrm{id}_V \oplus -\mathrm{id}_V$. We can use this second form for another diagram chase:



The squares and triangle on the left side all follow from basic properties of the twist maps τ. The triangles on the right are obvious. The one bi-gon on the left comes from (B.1). Going across the top is $(\mathrm{id}_W \oplus -\mathrm{id}_V) \circ \tau_{V,W}$, while going down and along the bottom produces $(\varphi \oplus \mathrm{id}_V) \circ (\mathrm{id}_V \oplus \tau_{V,U}) \circ (\mathrm{id}_V \oplus \varphi^{-1})$. □

Since the two isomorphisms of Proposition B.6 are homotopic, Proposition B.2 tells us that both Ψ' and $\Phi^\perp$ are complements to Φ with respect to either one. When we stabilize both frames by V one more time, by Proposition B.4, $\Psi' \oplus \mathrm{id}_{\underline{V}}$ and $\Phi^\perp \oplus \mathrm{id}_{\underline{V}}$ are equivalent $(W \oplus V)$-frames of $E^{\perp \underline{W} \oplus \underline{V} \oplus \underline{V}}$.

Proposition B.7. *If Ψ is complementary to Φ, then Ψ and $\Phi^\perp$ stabilize to the same frame. More precisely,*

$$\Psi' \oplus \mathrm{id}_V \simeq \Phi^\perp \oplus \mathrm{id}_V \,.$$

where $\Psi' = (\Psi \oplus \mathrm{id}_{\underline{V}}) \circ \tau_{\underline{V},\underline{U}} \circ \underline{\varphi^{-1}}$.

Keep in mind that Ψ' in Proposition B.7 is just Ψ stabilized and treated as a W-frame rather than a $(U \oplus V)$-frame. Since we now know that, after stabilizing, the canonical complement always exists, and furthermore, that any other complement is equivalent to it after stabilizing again, we are justified in writing $\Phi^\perp$ unambiguously.

Example B.8 (Normal frame for $U(1)$)**.** Here we show how to obtain the once-inward-twisting normal frame associated to the Lie frame on $U(1)$.

Embed $U(1) \hookrightarrow \mathbb{R}^{2,1} = W$ as the unit circle with coordinate θ and equip $U(1)$ with the tangential Lie V-frame $\Phi = \partial_\theta$ where $V = \mathbb{R}^{1,1}$. As mentioned before, there is no complementary frame to ∂_θ in $\mathbb{R}^2$. So we need to enlarge the ambient space to be $\mathbb{R}^{2,1} \oplus \mathbb{R}^{1,1} = W \oplus V$.

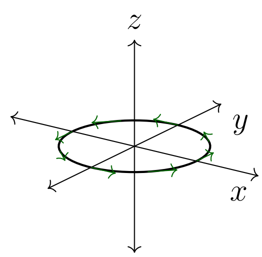


Figure B.1. The tangential Lie frame on the circle embedded in $\mathbb{R}^{2,1} \oplus \mathbb{R}^{1,0}$.

To find a W-frame $\Phi^\perp$ complementary to Φ, we need to pick what decomposition of $\varphi : V \oplus W \to \mathbb{R}^{2,1} \oplus \mathbb{R}^{1,1}$ we want to use. For simplicity, we'll use $\tau_{V,W}$ instead of than $(\mathrm{id}_W \oplus -\mathrm{id}_V) \circ \tau_{V,W}$ so that the representation V framing the tangent bundle is identified with the z-axis. There should be a homotopy $\Phi \oplus \Phi^\perp \simeq \underline{\tau_{V,W}}$. This is the fiber-wise rotation which terminates by pointing all the tangent vectors up along the z-axis.

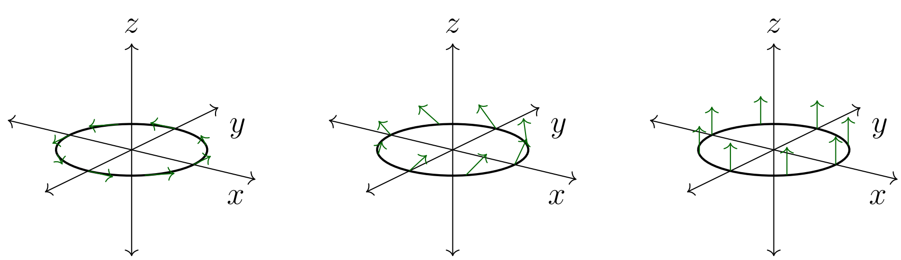


Figure B.2. Rotating the tangent frame to point along the stabilizing copy of $\mathbb{R}^{1,1}$.

To get the frame $\Phi^\perp$ we start by picking a basis for W, and then run the rotation backwards so that it takes vectors in W to the normal bundle of $U(1)$.

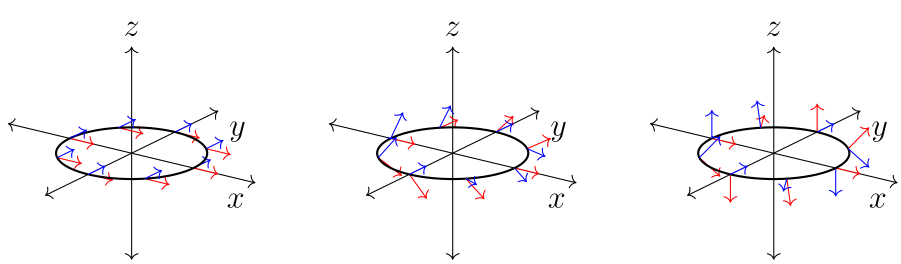


Figure B.3. Vectors in W coming along for the ride as the z-axis rotates back down to Φ.

By tracing the tips of the red or blue arrows in the normal frame we find a circle Hopf linked to the original S^1. These become the Hopf linked fibers of $\eta \in \pi_3(S^2)$ after applying the collapse map.

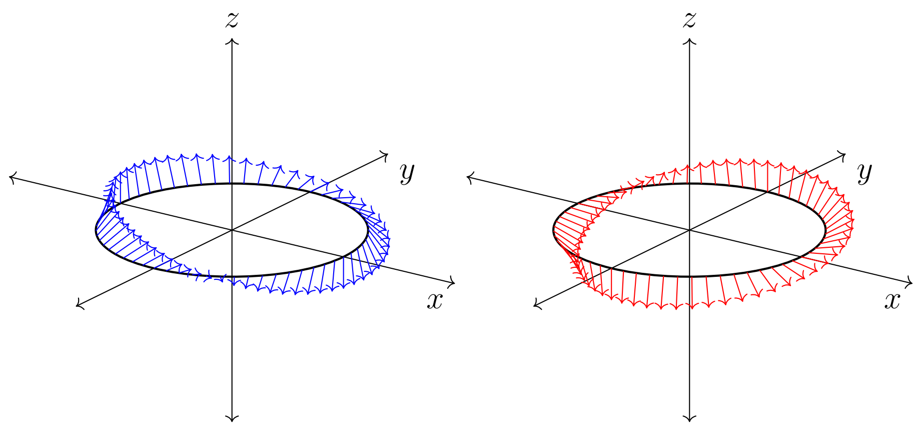


Figure B.4. Isolated views of the two sections of the normal bundle.

Depending on how one chooses to identify $W \oplus V$ with $\mathbb{R}^{3,2}$ this normal frame could yield η or $\epsilon\eta = -\eta$. It is a matter of how we order the $\mathbb{R}^{1,1}$-axes z and y. Using $\tau_{V,W}$ for our decomposition is essentially putting z-first which was convenient for our illustration. However, since W already contained the y-axis the intuitive decision might be to put y-first. This is why $-\,\mathrm{id}_V$ appears in at the end Construction B.5. ❧

B.3 Frames and Fixed-Points

It is also useful to know how taking complements interacts with taking fixed-points. This works as one expects. For $H \leq G$, a V-frame on E naturally induces a V^H-frame on $E^H \to X^H$.

Proposition B.9. *Let $\Phi : \underline{V} \to E$ be a V-frame of an equivariant vector bundle $E \to X$. Then $\Phi^H : (\underline{V})^H \to E^H$ is a V^H-frame of the $W_G H$-equivariant vector bundle $E^H \to X^H$.*

This leads to the question of whether it matters if we take fixed-points before or after finding a complementary frame; *i.e.,* does $(\Phi^\perp)^H = (\Phi^H)^\perp$? The answer is yes, as long as we remember that $\Phi^\perp$ is only well-defined up to stabilization.

Proposition B.10 (Fixed-Points of Complements)**.** *Let Ψ be complementary to Φ. Then Ψ^H is complementary to Φ^H.*

Proof. Since Ψ and Φ are complementary, there is a G-equivariant homotopy through $(V \oplus U)$-frames

$$\alpha : I \times (\underline{V} \oplus \underline{U}) \to \underline{W}$$

from $\Phi \oplus \Psi$ to $\underline{\varphi}$. Applying the fixed-point functor then yields a $W_G H$-equivariant map of $W_G H$-spaces

$$\alpha^H : I \times (\underline{V} \oplus \underline{U})^H \to \underline{W}^H$$

from $(\Phi \oplus \Psi)^H$ to $\underline{\varphi}^H$. It is clear that $(\Phi \oplus \Psi)^H = \Phi^H \oplus \Psi^H$, that $\underline{\varphi}^H = \underline{\varphi^H}$, and that a_t^H is still a bundle equivalence for each $t \in I$. So Ψ^H is a U^H-frame complementary to the V^H-frame Φ^H of E^H in $\underline{W}^H$ with respect to φ^H. □

Remark. Although these facts are straightforward, it is worth remembering that a homotopy from $\Phi \oplus \Psi$ to φ can rotate the images of $\underline{V}$ and $\underline{U}$ inside $\underline{W}$ in nontrivial ways, like in Construction B.5. In spite of this, when we look at fibers over X^H, the fixed-point hyperplanes $\underline{V}^H$ and $\underline{U}^H$ are confined to rotating around within $\underline{W}^H$.

Another possible point of confusion is that there is another way to induce framed bundles over X^H. Instead of taking H-fixed-points on the whole bundle, we could merely restrict the bundle from X to X^H. Then

$$\Phi|_{X^H} : \underline{V}_{X^H} \to E|_{X^H}$$

is a $(\mathrm{Res}_{N_G H} V)$-frame of the $N_G H$-equivariant vector bundle $E|_{X^H}$. In this case the base is H-fixed, but the fibers might still have a nontrivial H-action. The analogous statement to Proposition B.10 holds for these restricted bundles. We do not use this alternative construction, but it's good to be aware of it.

APPENDIX C

DETAILS ON LIE GROUP FRAMES

Here we will provide a detailed proof of Proposition 3.21.

Proposition 3.21. *As a* $(G \rtimes \mathrm{Aut}(G))$*-manifold,* G *is naturally* $\mathfrak{g}$*-framed.*

At the end of the day this is just an expanded perspective on the familiar adjoint action of G on T_eG given by the derivative of conjugation. However, for the purposes of working with G as V-framed manifold we would like to additionally verify that the action works globally and in a way which is compatible with left-translation and outer automorphisms of G.

Proof. We denote the left-translation map by $L_g : G \to G : h \mapsto gh$ and define $\mathfrak{g}$ to be the space of G-left-invariant vector fields

$$\mathfrak{g} = \{X \in \Gamma(TG) \mid dL_g \circ X = X \circ L_g \text{ for all } g \in G\}.$$

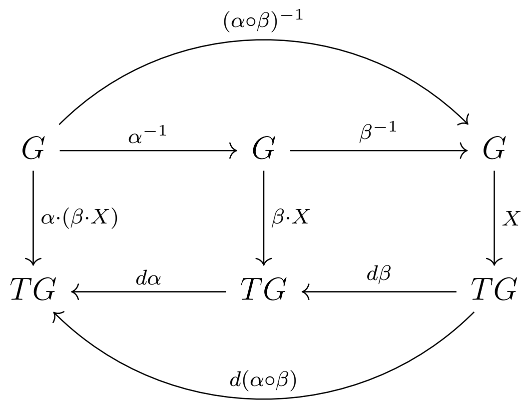
For each $X \in \mathfrak{g}$ and $g \in G$, we can express left-invariance with a commutative diagram that we will call the **invariance square for** g:

$$\begin{array}{ccc} G & \xrightarrow{X} & TG \\ \downarrow{\scriptstyle L_g} & & \downarrow{\scriptstyle dL_g} \\ G & \xrightarrow{X} & TG \end{array}$$

We define an action of $\alpha \in \mathrm{Aut}(G)$ on $\mathfrak{g}$ by $\alpha \cdot X = d\alpha \circ X \circ \alpha^{-1}$. We call the associated diagram the **defining square for** $\alpha\cdot$:

$$\begin{array}{ccc} G & \overset{\alpha\cdot X}{\dashrightarrow} & TG \\ \downarrow{\scriptstyle \alpha^{-1}} & & \uparrow{\scriptstyle d\alpha} \\ G & \xrightarrow{X} & TG \end{array}$$

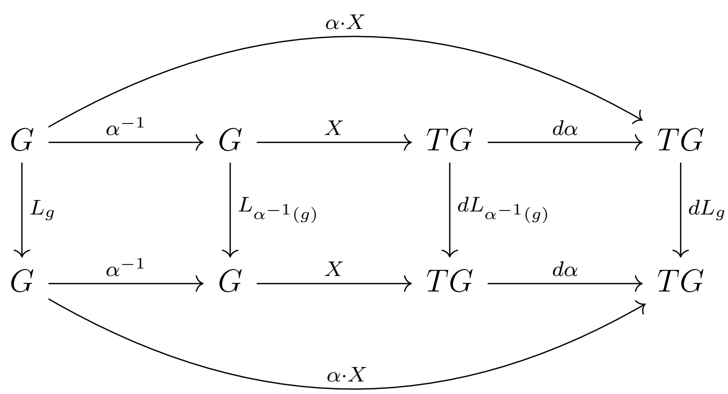
The compatibility condition for the action is proved by pasting the defining squares for $\alpha\cdot$ and $\beta\cdot$ together



and recognizing that the boundary is the defining square for $(\alpha \circ \beta)\cdot$. To verify that $\alpha \cdot X$ is indeed still a G-invariant vector field, we will need one more square. Since α is a homomorphism we have $\alpha(gh) = \alpha(g)\alpha(h)$ for all $g, h \in G$. If we view h as a free variable then we can interpret this as equality of maps $\alpha \circ L_g = L_{\alpha(g)} \circ \alpha$, which is represented by the **homomorphism square for α and g**

$$\begin{array}{ccc} G & \xrightarrow{L_g} & G \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ G & \xrightarrow{L_{\alpha(g)}} & G \end{array}$$

Then G-left-invariance for $\alpha \cdot X$ is witnessed by the boundary of the diagram



The top and bottom squares are the defining squares for $\alpha\cdot$. In the center, from left to right, we have: the homomorphism square for α^{-1} and g, the invariance square for $\alpha^{-1}(g)$, and the differential of the homomorphism square for α and $\alpha^{-1}(g)$.

So $\mathrm{Aut}(G)$ does in fact act on $\mathfrak{g}$ globally, not just at T_eG. We can also define a similar action of G on $\mathfrak{g}$ by $g \cdot X = dL_g \circ X \circ L_g^{-1}$, although this action is trivial.[1] These two actions can be combined into a single action of $G \rtimes \mathrm{Aut}(G)$ on $\mathfrak{g}$.

Now we define a frame by evaluation

$$\Phi : \underline{\mathfrak{g}} \to TG : (h, X) \mapsto X(h).$$

We just need to show that this frame is $(G \rtimes \mathrm{Aut}(G))$-equivariant. That is, for all $(g, \alpha) \in G \rtimes \mathrm{Aut}(G)$ and $(h, X) \in \underline{\mathfrak{g}}$, we need $(g, \alpha) \cdot \Phi(h, X)$ equal to $\Phi((g, \alpha) \cdot (h, X))$. This can be shown with the aide of the following commutative diagram

$$\begin{array}{ccccc} G & \xrightarrow{\alpha} & G & \xrightarrow{L_g} & G \\ \downarrow{\scriptstyle X} & & \downarrow{\scriptstyle \alpha\cdot X} & & \downarrow{\scriptstyle \alpha\cdot X} \\ TG & \xrightarrow{d\alpha} & TG & \xrightarrow{dL_g} & TG \end{array}$$

which is made from the defining square of $\alpha\cdot$ and the invariance square for g.

Now, if we trace an element $h \in G$ from the top left corner, down, and all the way to the right we get $(dL_g \circ d\alpha)(X(h))$. This is $(g, \alpha) \cdot \Phi(h, X)$. On the other hand, the right-most vertical arrow is $\alpha \cdot X$, but it is also $(g, \alpha) \cdot X$ since g acts trivially on X. Evaluating $(g, \alpha) \cdot X$ on $g\alpha(h)$ is the definition of $\Phi((g, \alpha) \cdot (h, X))$. Since $g\alpha(h)$ is the image of h across the top row, we can conclude $(g, \alpha) \cdot \Phi(h, X) = \Phi((g, \alpha) \cdot (h, X))$ because the diagram is commutative. □

The other fact deferred from Section 3.2.2 was Proposition 3.23 regarding the case when Γ acts on G by automorphisms.

Proposition 3.23. $[G, \mathscr{L}_G]^\Gamma = [G^\Gamma, \mathscr{L}_{G^\Gamma}]$.

On the level of tangential frames this is obvious. The slightly subtle fact is that restricting the frame to fixed-points is compatible with taking complementary frames so that the two homotopy classes $[G\mathscr{L}_G]^\Gamma$ and $[G^\Gamma, \mathscr{L}_{G^\Gamma}]$ obtained by the Thom collapse construction coincide. This is an immediate consequence of Proposition B.10.

Example C.1. Figure C.1 shows $U(1)_{\mathrm{conj}}$ embedded in $\mathbb{R}^{3,2}$ with the normal frame complementary to its Lie frame. The action is 180° rotation about the purple line

[1]We should emphasize that this action is trivial on the representation $\mathfrak{g}$, but is very much nontrivial on the base space G which is carrying the $\mathfrak{g}$-frame.

which is $\mathbb{R}^{1,0} = (\mathbb{R}^{3,2})^{C_2}$. The two fixed points, which together form the subgroup $O(1) = U(1)^{C_2}$, are marked in green.

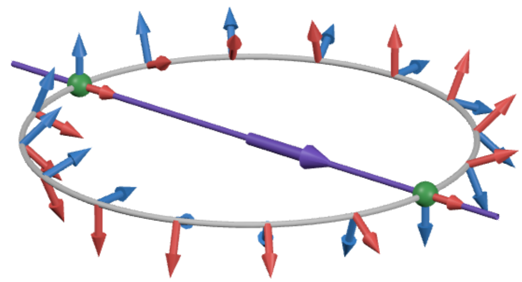


Figure C.1. Restricting the Lie frame on $U(1)$ to its fixed point subgroup $O(1)$.

Since $O(1)$ is discrete its Lie frame is the **0**-frame. So the complementary normal frame should come from the ambient frame on $\mathbb{R}^1$. We see that, in spite of the twisting that occurs over $U(1)$, things line up just right so that the induced frame on $O(1)$ (red arrows at the green points) agrees with the ambient frame. ❧

Example C.2. The class $[SU(5)] \in \pi_{24,14}$, which is trivial in classical homotopy, is no less than 3-torsion when equipped with the complex conjugation action. This is because $[SU(5)]^{C_2} = [SO(5)] \in \pi^s_{10}$ is 3-torsion [23]. Since $[SU(5)]$ is trivial under the forgetful homomorphism, we can use Construction 5.4 to pull it back along the forgetful exact sequence to get another nontrivial element $[\widetilde{SU(5)}] \in \pi_{25,15}$. ❧

If G is semi-simple we can find some maximal abelian subgroup $A \leq G$. Since A is maximal abelian, if we let it act on G by conjugation we get $A = G^A$. So $[G]^A = [G^A] = [A] \in \pi^s_0$. The left-invariant frame on a finite subgroup is the **0**-frame, its complementary frame is the ambient frame, and so applying the Thom collapse construction produces a degree $|A|$ map in π^s_0. Since the image of $[G] \in \pi^A_{\mathrm{Res}_A \mathfrak{g}}$ under a fixed-point map is a nonzero integer we conclude that $[G]$ is non-nilpotent and generates an infinite cyclic subgroup. These kinds of elements survive when passing to the reduced $RO(G)$-graded homotopy ring [2].

Example C.3. This means even E_8 represents an interesting $RO(A)$-graded homotopy class for suitable abelian groups A, such as $(C_2)^9$ and $(C_6)^3$ [6, 8]. By contrast, $[E_8, \mathscr{L}]$ is very much trivial in π^s_{248} [23, 24]. ❧

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