Covering Maps and other Maps between Higher Dimer Models

by

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DISSERTATION ABSTRACT

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A graph G embedded in the torus $X = [0,1] \times [0,1]$ with opposite edges identified, naturally produces a graph $\tilde{G}$ embedded in the torus $\tilde{X} = [0,m] \times [0,n]$ consisting of mn copies of G. If $\tilde{G}$ is equipped with GL_k edge connections $\tilde{\phi}$, then the main result of this paper is to define connections ϕ on the edges of G so that the multiweb partition function is preserved under the $mn-$fold covering map. The definition of ϕ reveals a correspondence between the representations of deck transformations of a torus and the connections ϕ assigned to various types of edges of G. Reversing the map is explored. As an application of $n-$webs embedded in surfaces, we find a correspondence between $n-$webs and height functions and topographic maps.

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CHAPTER 1

INTRODUCTION

The dimer model studies matchings, or dimers, on a graph. As a generalization of the dimer model, we have the higher dimer model, consisting of $n-$dimers or $n-$webs. In the case of bipartite graphs, the $n-$webs of a graph can be obtained by overlaying n single dimers. One way to overlay dimers on a graph, is by working with covering maps from topology, which is what is novel to this work. This idea was pitched to the author and advisor by Kenyon during his visit to University of Oregon. If a larger graph is an $n-$fold covering space of a smaller graph, we can apply the covering map from the larger graph with single dimers, to induce $n-$dimers or $n-$webs on the smaller graph.

Scalar weights on the edges of a graph allow us to define the weight of an $n-$web, and so we have a measure on the $n-$webs of a graph, as well as a partition function which is the sum of the weights of all the $n-$webs. One can generalize the weight of an $n-$web to the web-trace of an $n-$web when we assign GL_n matrices to the directed edges of G. Given GL_n matrices on a graph which is embedded in a large torus, the main achievement of this work is to define GL_nk matrices on a graph which is embedded in a smaller torus, so that the partition function of the larger graph is preserved by the $k-$sheeted covering map from the larger torus to the smaller torus.

1.1 Current Body of Work

Compact connected surfaces are homeomorphic to a sphere or a connected sum of tori and real projective planes [Hat10]. The universal cover of the sphere and the real projective plane is the sphere. The universal cover of the torus is the plane.

The Covering Map Theorem we achieve here addresses the $n-$fold covering map for a torus, where n is finite and odd. The result is more technical than initially expected. The parity issue of n needing to be odd provides insight, as well as more questions. The completion of this body of work would include the covering map from a sphere to a real projective plane. However, more needs to be developed in this area before we can address this 2-fold covering map.

Kenyon, Okounkov, Sheffield in [KOS06], Kenyon in [Ken14], Douglas, Kenyon, Shi in [DKS24], Kenyon, Ovenhouse in [KO25], and Tata in [Tat24] have developed a collection of Trace Theorems. A web-trace assigns a scalar value to an $n-$web. The Trace Theorems are of the form the sum of some web-traces is equal to a determinant (or Pfaffian). The benefit of these theorems is that traces are generally very difficult to compute, while the determinants (Pfaffians) are quicker to compute. Our Covering Map Theorem relies on Tata's Trace Theorem for a torus. The main difficulty we run into when attempting to move our techniques to the setting of a real projective plane, is that we do not yet have a Trace Theorem for non-orientable surfaces (which include the projective plane), in the case when $n \geq 2$. The situation is hopeful; other authors have outlined how one might obtain this result. That this partition-function-preserving Covering Map Theorem exists, provides motivation for researchers in this field to continue rounding out these Trace Theorems and Covering Map Theorems.

1.2 Webs in the Broader Scientific Community

There is a flurry of recent activity using webs in many fields of science and math. The following is by no means a list of the most important web activity but closer to a random sample of web utilization in various fields. Physicists use webs

to understand quantum conformal field theory and quantum computing[Gal95]. The dimer model is used in statistical mechanics[BD11]. Knot theorists use webs to compute quantum links and tangle invariants [KS25], [Ama25]. Representation theorists work with webs connected to loop groups, crystals, Hecke algebras, and more[HK24]. Combinatorists use webs to study the combinatorics of matchings, standard Young tableaux, and more [Tym12]. Algebraic geometers make use of webs when working with Springer fibers and generalized Schubert calculus [Cum26]. Much of this activity involves webs with boundary. Here we study webs without boundary, though our ideas can be applied to other settings.

1.3 Organization of this Work

Chapter 2 presents preliminary definitions and background results. These include various Trace Theorems with special attention to the method first given by Tata for computing the partition function of a graph embedded in a torus with GL_n matrices assigned to the edges. We also highlight which Trace Theorems are still open.

Chapter 3 contains the main theorems of this, the Covering Map Theorems. We provide the GL_k on each edge that preserves partition functions under the covering map. Furthermore, we discover a neat correspondence between the representations of deck transformations of a torus and the GL_k assigned to various types of edges.

In Chapter 4 we address reversing our Covering Map Theorem. Given a graph embedded in a torus with ϕ assigning GL_k matrices to the graph edges, can we find a covering space and graph that such that ϕ is given by our Covering Map Theorem? We will see that there are an overwhelming number of options to check when

reversing the covering map. The author is still developing more efficient and intuitive methods to reverse the covering map. Chapters 5, and 6 are the preliminary work in this area. Other kinds of covering maps may come from covering spaces with multiple components, some of which may be mapped identically to the space being covered. This motivates the topic of the Overlaying Map of Chapter 5. A more general map, which includes both covering maps and overlaying maps, is the Projection Map, defined and explored in Chapter 6.

Chapter 7 is a fun introductory analysis of the application of $n-$webs to topographic maps. $2-$webs, or double dimers, correspond to the loops on topographic maps. Not surprisingly, there is a height function defined for double dimers on graphs. However, $2n-$webs often contain shapes that are not loops, such as Y-shapes. We address this phenomena, and answer how this can be explained in terms of topographic maps.

Chapter 8 is a discussion of next steps and a list open questions, which include those needed to round out this body of work with Trace Theorems and Covering Map Theorems.

CHAPTER 2

PRELIMINARY DEFINITIONS AND PREVIOUS RESULTS

Preliminary definitions are mostly presented in Section 2.1, with the web-trace definition as the focus of Section 2.2, and a bit more vocabulary in Section 2.3.

In Section 2.4, we present previous results needed for the work in this paper, limiting our attention to planar graphs. In Section 2.5 we extend our attention to previous results on the torus. Section 2.6 summarizes the background results, i.e. the Trace Theorems, on planes, orientable surfaces, and non-orientable surfaces. The existence of a rank n trace theorem on the torus [Tat24], and the current lack of a trace theorem for any rank above 1 on a non-orientable surface (specifically the real projective plane), determines that we focus on the torus for our covering map work.

2.1 Preliminary Definitions

Let G be a finite, bipartite graph with scalar weights on the edges. The following definition is from [DKS24].

Definition 2.1.1. An **n-multiweb** (or $n-$**web**) m of G is a multiset of edges with degree n at each vertex, that is, a mapping $m : E \to \{0, 1, 2, ...\}$ on the undirected edges of G such that for each $v \in V(G)$ we have $\sum_{u \sim v} m(e_{v,u}) = n$.

Let $\Omega_n(G)$ be the set of $n-$webs of G. Let $w(e)$ denote the weight of edge e. We assume nonnegative real weights unless otherwise stated. For $m \in \Omega_n(G)$, let $w(m) = \prod_{e \in E(G)} w(e)^{m_e}$.

Definition 2.1.2. The $n-$**web partition function (without connection)** of G is

$$Z_n(G) := \sum_{m \in \Omega_n(G)} w(m). \tag{2.1}$$

More generally, we can let n vary at each vertex as studied in [KO25].

Definition 2.1.3. For $n : V(G) \to \{1, 2, 3, ...\}$, we can still let $\Omega_n(G)$ denote set of maps $m : E \to \{0, 1, 2, ...\}$ such that for each $v \in V(G)$ we have $\sum_{u \sim v} m(e_{v,u}) = n(v)$. In this case when n varies, the members of the set $\omega_n(G)$ will be referred to as **higher dimers** or sometimes just **multiwebs**.

Multiwebs are sometimes referred to as *web-like subgraphs* in other literature.

Even more generally, rather than scalars $w(e)$, we can instead assign matrices $\phi(e)$ to each edge.

Definition 2.1.4. For $n : V(G) \to \{1, 2, 3, ...\}$, we can assign $\phi(e)$ to each edge e oriented from black to white, with the restriction that $\phi(e_{b,w}) \in \mathrm{Hom}(\mathbb{R}^{n(b)}, \mathbb{R}^{n(w)})$. That is, each $\phi(e_{b,w})$ is an $n(b) \times n(w)$ matrix. The collection ϕ is called a **connection**.

This most general case is studied in [KO25]. We will only use matrices with real entries, but other fields can be considered. In this case,

Definition 2.1.5. the $n-$**web partition function (with connection)** is defined as

$$Z_n(G, \phi) := \sum_{m \in \Omega_n(G)} \mathrm{Tr}(m, \phi) \tag{2.2}$$

where $\mathrm{Tr}(m, \phi)$ is as defined in Definition 2.2.5.

The **web-trace** $\mathrm{Tr}(m, \phi)$ is as defined in [DKS24] and [KO25], and depends on a choice of *linear ordering* $L(G)$ or perhaps *ciliation* around each of the vertices of G, as described in Definitions 2.2.1 and 2.2.2. We restate the web-trace definition here with some caution. Computing the web trace as presented in Section 2.2 is computationally dense, and is not necessary to understand this work, as the computations can be bypassed by subsequent so-called trace theorems, presented in Sections 2.4 and 2.5.

2.2 Web-trace Definition

The definitions of this section are restated from [DKS24] and [KO25]. In Section 2.2.1, we include the formal version of the web-trace definition for reference. In Section 2.2.2, we present an equivalent definition given in terms of edge colorings which is combinatorially more intuitive.

2.2.1 Web-trace Definition with Tensors

Let G be a bipartite graph. Let G have edge connections ϕ where each $\phi(e_{b,w})$ is an $n \times n$ matrix. For each vertex v, select a linear ordering of the half-edges at v. That is for each vertex v, select a linear ordering of the edges incident with v. An edge is incident with two vertices, and the orderings at the two vertices are independent. That is e_{bw} may be the ith edge in the ordering of the edges incident with b, and can be the jth edge in the ordering of the edges incident with w for any i, j.

Definition 2.2.1. The **data of the half-edge orderings** of a graph G is denoted $L(G)$ and is as described above.

Definition 2.2.2. Ciliation is a specific type of data of the half-edge orderings for a graph embedded on an orientable surface, as described below.

When G is embedded in an orientable surface, the half-edges incident to a vertex v inherit a cyclic ordering from the orientation of the surface. When v is a black (white) vertex, our convention is to choose the counterclockwise (clockwise) orientation. This cyclic ordering can be upgraded to a linear ordering when we select the first half-edge at v. Pictorially, we indicate this choice by drawing a *cilium* emanating from the vertex.

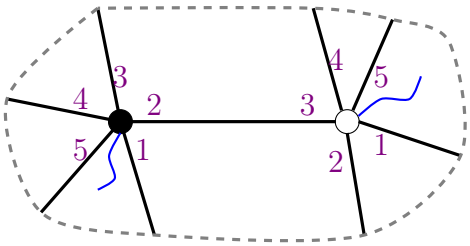


Figure 2.1. Ciliation

2.2.1.1 Web-trace of Proper n-webs

Definition 2.2.3. A **proper** $n-$**web** $m' \in \Omega_n(G)$ is a $n-$web such that the multiplicity of each edge is $m'(e) \in \{0, 1\}$. So m' is an $n-$valent spanning subgraph of G.

The trace of a proper $n-$web $\mathrm{Tr}(m')$ is a scalar function of ϕ and L, is as follows. The data of $L(G)$ restricts to give an ordering of the n half-edges of m' at each vertex, which we denote as $L(m')$. To each black vertex b we assign the vector space V with basis $x_1, x_2, \ldots, x_n$. To each white vertex w we assign the dual vector space V^* with basis $y_1, y_2, \ldots, y_n$. Associate identical copies $V_e = V$ and $V_e^* = V^*$ to each edge e_{bw} of m'. So V_e is associated to the half-edge incident to b and V_e^* is associated the the half-edge incident to w. At each black vertex with adjacent vector spaces $V_1, V_2, \ldots, V_n$ in the given ordering $L(m')$, we define the **codeterminant** to

be the the vector $v_b \in V_1 \otimes \cdots \otimes V_n$ given by

$$v_b := \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)}. \tag{2.3}$$

Similarly, at each white vertex with adjacent vector spaces $V_1^*, V_2^*, \ldots, V_n^*$ in the given ordering $L(m')$, we define the **dual codeterminant** to be the vector $u_w \in V_1^* \otimes V_2^* \otimes \cdots \otimes V_n^*$ given by

$$u_w := \sum_{\sigma \in S_n} (-1)^\sigma y_{\sigma(1)} \otimes y_{\sigma(2)} \otimes \cdots \otimes y_{\sigma(n)}. \tag{2.4}$$

Take the tensor product $\otimes_b v_b \in \otimes_e V_e$ of v_b over all black vertices, and likewise, take the tensor product $\otimes_w u_w \in \otimes_e V_e^*$ of u_w over all white vertices. Now, contract component-wise along the edges as follows. A simple tensor $\otimes_e x_e \in \otimes_e V_e$ and a simple tensor $\otimes_e y_e \in \otimes_e V_e^*$ contract to give $\prod_{e_{bw}} y_{e_{bw}}(\phi(e_{bw}) x_{e_b w})$ or if one prefers $\prod_e \langle y_e | \phi_{bw} | x_e \rangle$. Finally, we sum to get the following definition.

Definition 2.2.4. The **web-trace of a proper $n-$web** m' with connection ϕ is

$$\mathrm{Tr}_L(m', \phi) := \langle \otimes_w u_w | \otimes_e \phi(e_{bw}) | \otimes_b v_b \rangle \,. \tag{2.5}$$

The definition of $\mathrm{Tr}_L(m', \phi)$ does not depend on the choice of basis $x_1, x_2, \ldots, x_n$, as discussed in [DKS24, Sec 3.1].

2.2.1.2 Web-trace of General Multiwebs

Let G be a bipartite graph with connections ϕ, but now let $m \in \Omega_n(G)$ be a general multiweb, where n is allowed to vary at each vertex. That is $n : V(G) \to \mathbb{Z}^+$, and $m : E(G) \to \{0, 1, 2, \ldots\}$ such that $\sum_{u \sim v} m(e_{vu}) = n(v)$. Again, let $L(G)$ be orderings of the half-edges at each vertex. The data $L(G)$ restrict to an ordering $L(m)$ of the half-edges incident to each vertex. For each edge of G, if $m(e) \geq 2$,

then split e into $m(e)$ parallel edges. Let G' be the resulting graph. Then m determines m', a proper $n-$web of G', where n is allowed to vary at each vertex, but $m'(e') \in \{0,1\}$ for each edge e' of G' and $\sum_{u\sim v \text{ in } G'} m(e'_{v,u}) = n(v)$. Assign connections $\phi'(e') := \phi(e)$ if e' was formed by splitting e, or if $e' = e$ since $m(e) = 1$.

To obtain an ordering $L(m')$ for the half-edges of m', select an arbitrary order for the new split half-edges, while still respecting the overall ordering $L(G)$. Finally, we can make the following definition.

Definition 2.2.5. The **webtrace of a general multiweb** is

$$\mathrm{Tr}_L(m,\phi) = \frac{\mathrm{Tr}_{L'}(m',\phi')}{\prod_e m(e)!}, \tag{2.6}$$

where $\mathrm{Tr}_{L'}(m',\phi')$ is as in Equation 2.5. $\mathrm{Tr}_L(m,\phi)$ is indeed independent of the choice of refinement L' as discussed in [DKS24, Sec 3.2].

2.2.2 Web-trace Definition with Half-edge Colorings

The same references, [DKS24] and [KO25] provide a more intuitive and combinatorial definition of the web-trace as follows.

Let m be a multiweb of G. Let $L(G)$ be a linear ordering of the half-edges at each vertex as in Definition 2.2.1 or possibly Definition 2.2.2. At each vertex v choose a coloring of the half-edges of m at v, using the colors $\mathcal{C} = \{1,2,\ldots,n\}$, using each color once so that half-edges with multiplicity get a subset of colors. Observe that if $e_1, e_2, \ldots e_k$ are the edges incident with v, then there are

$$\binom{n}{m(e_1), m(e_2), \ldots, m(e_k)}$$

possible half-edge colorings at v.

Assign a sign c_v to each half-edge coloring at v, as follows. List the colors at v according to $L(G)$ at v, where edges with multiplicity get multiple colors listed in

their natural order. This list according to $L(G)$ is a permutation of $\mathcal{C}$. Let c_v be the sign of that permutation.

A **coloring** c of the multiweb m is the data of a coloring of the half-edges at each vertex v. Given a coloring m, on each edge e of multiplicity $m(e)$ there are two sets of colors, each of size $m(e)$, one associated with the black vertex, and one with the white. If the connection of e is $\phi(e_{bw})$, and is colored with the set S_e at the white vertex, and T_e at the black, then to this edge e of this multiweb m with coloring c we associate the value $\det(\phi(e_{bw})_{S_e,T_e})$ which denotes the determinant of the S_e, T_e-minor of $\phi(e_{bw})$, that is the determinant of the submatrix of $\phi(e)$ with rows S_e and columns T_e, where the rows (columns) of $\phi(e)$ are indexed according to the colors at w (b).

Finally, to each coloring c of m we take the product $\prod_e \det(\phi(e_{bw})_{S_e,T_e})$ and multiply this by the sign $\prod_v(c_v)$.

Proposition 2.2.1. *Then the web-traces is defined as*

$$Tr_L(m,\phi) = \sum_{colorings\ c} \prod_v c_v \prod_e \det(\phi(e_{bw}))_{S_e,T_e}. \tag{2.7}$$

Proposition 3.2 of [DKS24] shows that this coloring definition is equivalent to the formal definition of Equation 2.6.

2.3 More Vocabulary and Gauge Equivalence

When one refers to the $n-$web partition function, it is indeed ambiguous if we are referring to Definition 2.1.2 or Definition 2.1.5, but is often clear from the context, namely whether or not the graph is equipped with connections ϕ on the edges. Recall Definitions 2.1.2and 2.1.5.

Definition 2.3.1. When $n(v) = 1$ for all vertices of G, we are discussing the **dimer model** or the **single dimer model** and the $1-$web partition function (or **single dimer partition function**) has its index set the set of all **single dimers** or **perfect matchings** of G.

Definition 2.3.2. When $n(v) = 2$ for all vertices of G, we are discussing the **double dimer model** and we refer to $2-$webs as **double dimers**.

Definition 2.3.3. When $n(v) = n$ is constant for all vertices, we are discussing the **rank$-n$ dimer model** and the $n-$web partition function has its index set the set of all $n-$webs.

Definition 2.3.4. When $n(v)$ is allowed to vary over the vertices of G, we are discussing the **higher-rank dimer model** and the multiweb partition function sums over the set of multiwebs respecting the function n.

Some other literature refers to $n-$webs and/or multiwebs as **web-like subgraphs**.

Definition 2.3.5. Connections ϕ and ϕ' on the same graph are **GL_n-gauge equivalent** (**SL_n-gauge equivalent**) if there are $A_v \in GL_n$ ($A_v \in SL_n$) such that for all edges e_{bw} we have $\phi'(e_{bw}) = A_w^{-1}\phi(e_{bw})A_b$.

Remark 2.3.1. As shown in [DKS24], if ϕ and ϕ' are SL_n-gauge equivalent, then $\mathrm{Tr}_L(m, \phi) = \mathrm{Tr}_L(m, \phi')$ and so $Z_n(G, \phi) = Z_n(G, \phi')$ when the same data $L(G)$ is used for both $Z_n(G, \phi)$and $Z_n(G, \phi')$

2.4 Background Results on the Plane

This section is a survey of previous results which we will apply in this work.

Definition 2.1.2 is a special case of Definition 2.1.5 when $\phi = I_n$ and the half-edge orderings are that of a planar embedding with **ciliation** (Definition 2.2.2) [DKS24]. That is,

$$Z_n(G) = Z_n(G, I_n) \tag{2.8}$$

when the half-edge orderings are that of a planar embedding with ciliation. Furthermore, if G is bipartite (as we are always assuming),

$$Z_n(G) = (Z_1(G))^n \tag{2.9}$$

also from [DKS24]. Equation 2.9 does not require any embedding.

The graph G has a weighted bipartite adjacency matrix. We adopt the conventions of orienting each edge from black to white and indexing the rows of the matrix by the white vertices and the columns by the black vertices. Then the (w, b) entry of the weighted bipartite adjacency matrix is $w(e_{b,w})$. The order of the rows and columns will affect the sign of the determinant to be defined shortly, but we will take its absolute value and so the order is inconsequential.

If the graph has a planar embedding, we can further multiply certain entries of the adjacency matrix by -1 according to the **Kasteleyn condition** defined as follows.

Definition 2.4.1. Let G be a bipartite graph with edges directed from black to white. Let G be embedded in some orientable surface. Let some edges be assigned a weight of -1. The graph together with the weighting is said to satisfy the **Kasteleyn condition**, or have a **Kasteleyn weighting**, if around each face, when viewed from the exterior, the number of edges directed clockwise plus the number of edges assigned -1 is an odd sum.

Satisfying this condition is always possible for a graph embedded in an orientable surface [Kup98].

Definition 2.4.2. Given a bipartite graph with scalar edge weights w and a Kasteleyn weighting, let the **Kasteleyn matrix** K be the matrix with rows indexed by white vertices and columns indexed by black vertices given by

$$K_{(w,b)} := \pm w(e_{b,w}) \tag{2.10}$$

In the case of the single dimer model, we consider the partition function Definition 2.1.2 when $n = 1$.

Theorem 2.4.1. *When G is planar, for the single dimer model, [Kas67] showed that*

$$Z_1(G) = |\det K|, \tag{2.11}$$

which is restated in [Kup98] and many other places.

Example 2.4.1. Let G be the complete bipartite graph $K_{2,2}$ with black vertex set $\{v, x\}$, and white vertex set $\{u, w\}$. Let G be equipped with edge weights $w(e_{v,u}) = a$, $w(e_{v,w}) = b$, $w(e_{x,u}) = c$, and $w(e_{x,w}) = d$.

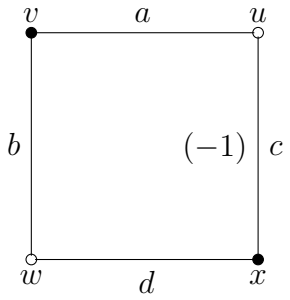


Figure 2.2. $K_{2,2}$

We can compute the single dimer ($n = 1$) partition function of G from Definition 2.1.2, and then also from the determinant of its Kasteleyn matrix as in Theorem 2.4.1.

To compute $Z_1(G)$ from Definition 2.1.2, note that the set of perfect matchings has two members: the pair of horizontal edges, and the pair of vertical edges. Hence $Z_1(G) = ad + bc$.

To compute $Z_1(G)$ from the determinant of its Kasteleyn matrix as in Theorem 2.4.1, embed G in the plane. Recall that all edges are directed from black to white, so when we consider the interior face, there are an even number (two) of edges pointing in the clockwise direction. Thus we must select an edge and assign it a -1. Any will do, so we select $e_{x,u}$. The rows of K are indexed by white vertices. We must select any order, so we choose u, w. For the order of the black vertices and columns we choose v, x. Thus

$$K = \begin{bmatrix} a & -c \\ b & d \end{bmatrix}$$

Again we arrive at

$$Z_1(G) = |\det(K)| = |ad + bc| = ad + bc.$$

In the case of multiple edges between a pair of vertices, we sum the contribution of the edges together for that matrix entry. See Example 2.4.2

Example 2.4.2. Let G denote the following graph with edge weights a, b, c, d.

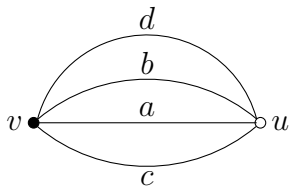


Figure 2.3. Graph with Multiple Edges

The graph is embedded in the plane so we can use a Kasteleyn matrix to compute its partition function. Remember, edges are directed from black to white. Each face has an odd number (one) of edges pointing in the clockwise direction, thus for the Kasteleyn condition we need not multiply any contributions by -1. We compute

$$K = \begin{bmatrix} a+b+c+d \end{bmatrix},$$

and

$$Z_1(G) = |\det(K)| = a+b+c+d. \tag{2.12}$$

Furthermore, applying Equation 2.9 gives

$$Z_n(G) = (a+b+c+d)^n \tag{2.13}$$

which is what one would expect when computing $Z_n(G)$ from Definition 2.1.2.

For $n-$webs, in addition to the edge weights, we can also assign an $n \times n$ connection matrix to each edge as per Definition 2.1.4, that is, an $n \times n$ matrix with real entries. As shown in [KO25] and [DKS24], the connection $\phi(e)$ of an edge e inherits its direction from the direction of the edge e, which by our convention is from black to white. To each black vertex b we assign the vector space V with basis $x_1, x_2, \ldots, x_n$. To each white vertex w we assign the dual vector space V^* with basis $y_1, y_2, \ldots, y_n$. So each connection $\phi(e)$ is a linear transformation from V to V^*.

Once again, if the graph has a planar embedding, we can define a **Kasteleyn matrix** $K(\phi)$ for G as follows. The matrix $K(\phi) = (K_{wb})$ has rows indexed by white vertices, columns indexed by black vertices, and $n \times n$ entries, with

$$K(\phi)_{wb} = \begin{cases} \pm w(e_{bw})\phi(e_{bw}) & b \sim w \\ 0_{n\times n} & \text{otherwise} \end{cases} \tag{2.14}$$

where the $\pm$ signs are chosen according to the Kasteleyn condition as per Definition 2.4.1. So $K(\phi)$ is a $\frac{|V(G)|}{2} \times \frac{|V(G)|}{2}$ matrix with $\text{Mat}_{n\times n}$ entries. We can also interpret $K(\phi)$ as an $\frac{n|V(G)|}{2} \times \frac{n|V(G)|}{2}$ block matrix with $n \times n$ blocks and scalar entries.

Theorem 2.4.2. *When G is a planar bipartite graph, the main Trace Theorem of [DKS24] states that the second equality of*

$$Z_n(G, \phi) = \sum_{m \in \Omega(n)} Tr(m, \phi) = |\det K(\phi)| \tag{2.15}$$

holds, where $\det K(\phi)$ is the determinant of $K(\phi)$ when we interpret $K(\phi)$ as an $\frac{n|V(G)|}{2} \times \frac{n|V(G)|}{2}$ matrix with scalar entries.

We will work out an example in Example 2.4.3.

Remark 2.4.1. In light of Theorem 2.4.2 and similar trace theorems appearing later, in many cases, we opt to not bother with ever computing $\text{Tr}(m, \phi)$ directly from Definition 2.2.5 and prefer to compute the determinant.

Recall that Equations 2.8 and 2.9 show that for a planar graph G,

$$Z_n(G, I_n) = Z_n(G) = (Z_1(G))^n \tag{2.16}$$

where I_n is the $n \times n$ identity matrix, $Z_n(G, I_n)$ is from Definition 2.1.5, $Z_n(G)$ is from Definition 2.1.2, and $Z_1(G)$ is from Definition 2.1.2 when $n = 1$.

A reader may absorb this result by substituting the identity matrix for A, B, C and D in Example 2.4.3. We caution the first equality of Equation 2.16 does not generally hold in when G is not planar, as demonstrated in Example 2.5.2.

Example 2.4.3. Let G denote the same $K_{2,2}$ graph as in Example 2.4.1. This time, in addition to the edge weights w, we also assign $n \times n$ connections to each edge denoted by $\phi(e_{v,u}) = A$, $\phi(e_{v,w}) = B$, $\phi(e_{x,u}) = C$, $\phi(e_{x,w}) = D$ where

A, B, C, D are $n \times n$ matrices. Now the Kasteleyn matrix for G together with ϕ is

$$K = \begin{bmatrix} aA & -cC \\ bB & dD \end{bmatrix},$$

and the $n-$web partition function with connection of G is

$$Z_n(G, \phi) = |\det(K)|,$$

when we interpret K as an $2n \times 2n$ matrix with scalar entries.

For example, if $A = B = C = D = I_2$, then $n = 2$ and

$$K = \begin{bmatrix} a & 0 & -c & 0 \\ 0 & a & 0 & -c \\ b & 0 & d & 0 \\ 0 & b & 0 & d \end{bmatrix},$$

and

$$\begin{aligned} Z_2(G, I_2) &= \det(K) \\ &= a^2d^2 + 2abcd + b^2c^2 \\ &= (ad + bc)^2 \end{aligned} \tag{2.17}$$

This is what we expect when we combine the result of Example 2.4.1 with Equation 2.16, since in Example 2.4.1, $Z_1(G) = ad + bc$.

2.5 Background Results on a Torus

Let X denote the torus, the subset $[0, 1] \times [0, 1]$ of $\mathbb{R}^2$ with opposite edges of the square identified. That is the line segment from $(0, 0)$ to $(1, 0)$ is identified with the line segment from $(0, 1)$ to $(1, 1)$, and likewise the line segment from $(0, 0)$ to $(0, 1)$ is identified with the line segment from $(1, 0)$ to $(1, 1)$.

2.5.1 Torus Computations

Let G be a bipartite graph embedded in the torus $X = S^1 \times S^1$ in such way that each face is contractible, and each edge e is assigned a connection $\phi(e) \in GL_n$, and a reference matching $\mathcal{R}$ is selected.

Definition 2.5.1. For a graph G embedded in a torus, a **reference matching** $\mathcal{R}$ is a perfect matching of G.

One should no longer expect to evaluate the partition function with a single determinant. Instead, [Tat24] provides us with the second equality of Equation 2.18. We will need the Arf invariant of a spin structure of a vector field on the torus, which takes value ± 1 depending on the spin structure. We will provide these for the torus in Equation 2.19.

Theorem 2.5.1. *Trace Theorem on Torus from [Tat24, Corollary 4]*

$$Z_n(G,\phi) = \sum_{m \in \Omega_n(G)} Tr(m,\phi) = \pm \frac{1}{2^g} \sum_{\substack{\eta \text{ spin} \\ \text{structure}}} Arf(\eta) \det(K_\eta(\phi)), \tag{2.18}$$

where g is the genus of the surface, the sum is over spin structures η of distinct homology classes modulo 2 of the surface, $Arf(\eta)$ denotes the Arf invariant of the spin structure (which will provide for the torus in Equation 2.19.), and $K_\eta(\phi)$ is a certain Kasteleyn matrix which we shall detail in Definition 2.5.6.

We now give a detailed description of this computation on a torus. We provide Examples 2.5.1 and 2.5.2 of these computations, but for a more detailed explanation, one should consult [Tat24].

First, we use the reference matching $\mathcal{R}$ to construct a vector field. It will be helpful to think of the torus as a square in $\mathbb{R}^2$ with opposite sides identified when

depicting the vector field and computing the winding numbers. We view the edges from outside the surface and construct the **initial vector field** as follows, with attention to the black and white vertices.

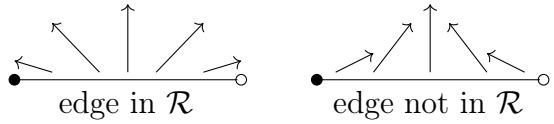


Figure 2.4. Initial Vector Field from $\mathcal{R}$

We can *twist* some of the edges.

Definition 2.5.2. To **twist** an edge means to reflect the arrows of the initial vector field of Figure 2.4 over the edge.

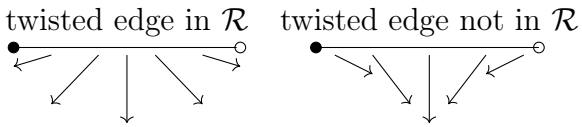


Figure 2.5. Twisted Edges

For a torus, there are four distinct homology classes of $H_1(X, \mathbb{Z}_2)$ which we can represent by $(0,0), (0,1), (1,0), (1,1)$.

Definition 2.5.3. Let $(w_x \mod 2, w_y \mod 2)$ denote the **spin structure of a vector field** where w_x is the winding number around a horizontal path of edges, and w_y is the winding number around a vertical path of edges.

We select four different subsets S_{00}, S_{01}, S_{10}, S_{11} of the edges of G to twist such that:

1. Each subset corresponds to a Kasteleyn orientation. That is, twisting the edges of the subset results in each face having an odd number of edges with fields pointing into the face.

2. Twisting the edges of each subset results in a distinct spin structure as in Definition 2.5.3.

3. Each edge of $\mathcal{R}$ is in all or none of the twisted subsets.

Definition 2.5.4. Let these **twisted subsets** be denoted by $S_{00}, S_{01}, S_{10}, S_{11}$ where the subscripts denote their spin structures.

The genus of the torus is $g = 1$.

Definition 2.5.5. For the torus, we have

$$\text{Arf}((0,1)) = -1 \quad \text{Arf}((0,1)) = 1 \quad \text{Arf}((1,0)) = 1 \quad \text{Arf}((1,1)) = 1. \tag{2.19}$$

Definition 2.5.6. $K_\eta(\phi)$ is given by

$$[K_\eta(\phi)]_{wb} = \begin{cases} \pm w(e_{bw})\phi(e_{bw}) & b \sim w \\ 0_{n\times n} & \text{otherwise} \end{cases}, \tag{2.20}$$

where the $\pm$ is negative if and only if $e_{bw} \in S_\eta$.

If G is equipped with only scalar weights w and no connections ϕ, we can let $\phi = [1]$, and $n = 1$. Likewise, if G is equipped with only GL_n connections ϕ and no weights, then we let $w = 1$. We now have all the pieces of $\pm\frac{1}{2^g}\sum_{\eta \text{ spin structure}} \text{Arf}(\eta)\det(K_\eta(\phi))$ to apply Theorem 2.5.1, and so $Z_n(G,\phi)$ can be computed with respect to a specific reference matching $\mathcal{R}$. For an example of this computation, see Example 2.5.1 and Example 2.5.2.

Remark 2.5.1. For $n = 1$, the single dimer partition function $Z_1(G)$ of Definition 2.1.2 does not depend on embedding or linear ordering data of half-edges. While we compute $Z_1(G)$ as in Example 2.5.1 by using a torus embedding and a set of reference edges, they are not necessary.

Remark 2.5.2. For $n \geq 2$, the partition function $Z_n(G, \phi)$ of Definition 2.1.5 depends on the web-trace $\mathrm{Tr}_L(m, \phi)$ of Definition 2.2.5 which depends on the data $L(G)$ of Definition 2.2.1.

That is $\mathrm{Tr}(m, \phi)$ is not well-defined without the data of $L(G)$, and $Z_n(G, \phi)$ is not well-defined without the data of $L(G)$ for $n \geq 2$.

In the context of the torus, $L(G)$ can be determined by the selection of reference edges $\mathcal{R}$. Therefore, on a torus, $Z_n(G, \phi)$ is dependent on the reference edges $\mathcal{R}$.

If instead the torus is given by $[0, m] \times [0, n]$ with opposite sides identified, then of course the procedure remains the same.

Example 2.5.1. To demonstrate the single dimer partition function computation on a torus, we repeat Example 2.4.2, but with an embedding in a torus. Let a, b, c, d denote the scalar weights of the edges as depicted.

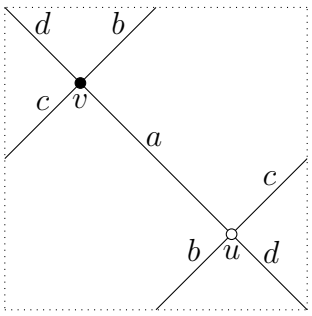


Figure 2.6. Graph Embedded in Torus

We must select a reference matching. Any perfect matching will do, so we choose $\mathcal{R} = \{e_a\}$. We use this to construct our initial vector field.

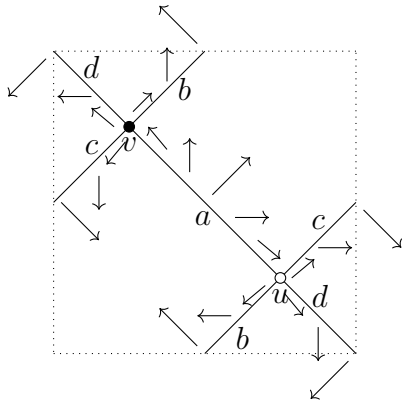


Figure 2.7. Initial Vector Field

We must select four subsets of edges to twist, so that each subset gives a Kasteleyn orientation, gives a different pair of winding numbers modulo 2, and that e_a is in all or none of the subsets. We will see that the subsets $\{e_b\}$, $\{e_c\}$, $\{e_d\}$, and $\{e_b, e_c, e_d\}$ meet these requirements, as we draw their diagrams as in Figure 2.8, confirm their Kasteleyn orientations and compute their winding numbers.

Using Figure 2.8, we find that $\{e_b\} = S_{10}$, $\{e_c\} = S_{01}$, $\{e_d\} = S_{11}$, and $\{e_b, e_c, e_d\} = S_{00}$. Thus

$$\begin{aligned}
\mathrm{Arf}(\eta_b) = \mathrm{Arf}((1,0)) = 1 \qquad & K(\eta_b) = [a - b + c + d] \\
\mathrm{Arf}(\eta_c) = \mathrm{Arf}((0,1)) = 1 \qquad & K(\eta_c) = [a + b - c + d] \\
\mathrm{Arf}(\eta_d) = \mathrm{Arf}((1,1)) = 1 \qquad & K(\eta_d) = [a + b + c - d] \\
\mathrm{Arf}(\eta_{bcd}) = \mathrm{Arf}((0,0)) = -1 \quad & K(\eta_{bcd}) = [a - b - c - d]
\end{aligned}$$

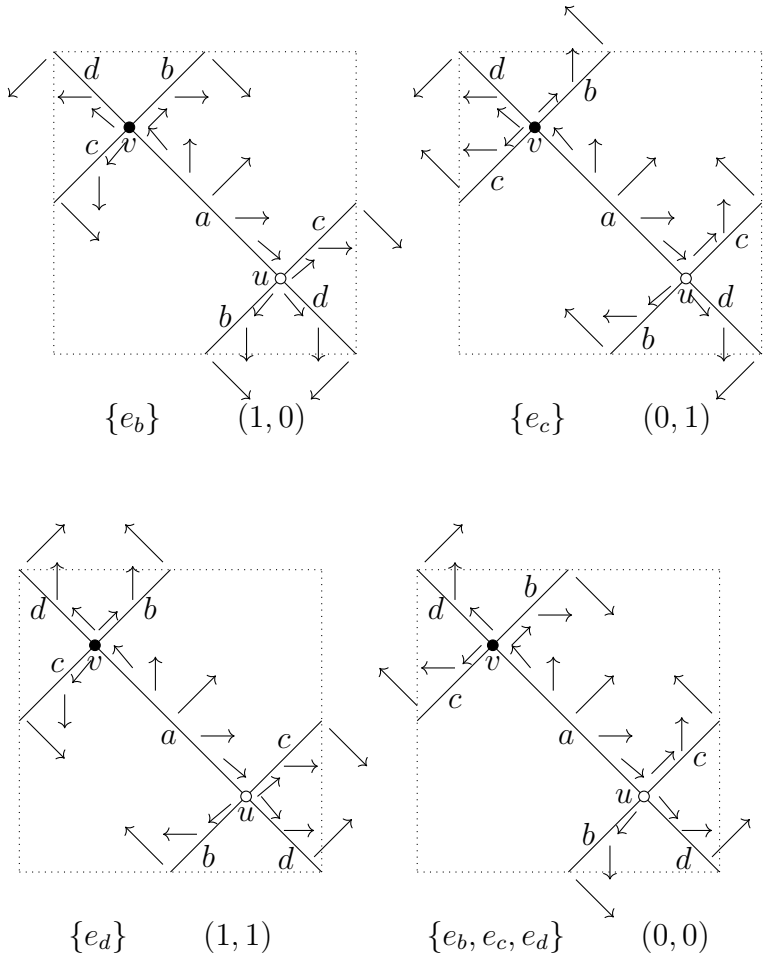


Figure 2.8. Vector Field with Twisted Subsets and their Winding Numbers

The genus of a torus is $g = 1$. Thus

$$\begin{aligned} Z_1(G) &= \pm\frac{1}{2^g}\sum_{\eta}\operatorname{Arf}(\eta)\det K(\eta) \\ &= \pm\frac{1}{2}\left[1(a-b+c+d)+1(a+b-c+d)+1(a+b+c-d)-1(a-b-c-d)\right] \\ &= \pm(a+b+c+d). \end{aligned} \tag{2.21}$$

This graph is the same as that of Example 2.4.2 and so this partition function matches that of Equation 2.12.

Remark 2.5.3. The partition functions without connection (i.e. the single dimer partition functions) are independent of $L(G)$, are independent of $\mathcal{R}$, and are independent the surface in which they are embedded. However, as we will see in our next example, partition functions with connection (so $Z_n(G,\phi)$ for $n \geq 2$) are dependent on the choice of ciliation, i.e. $Z_n(G,\phi)$ depends on the surface and reference matching.

Example 2.5.2. Let us use the same graph and embedding as Example 2.5.1, but in addition to the scalar weights a, b, c, d, also use the generic connection matrices $\phi(e_a), \phi(e_b), \phi(e_c), \phi(e_d)$. To compute $Z(G,\phi)$ with reference matching $\mathcal{R} = \{a\}$ we can use the same twisted subsets. Thus

$$\begin{aligned} Z(G,\phi) = \pm\frac{1}{2}\,(&1\det[a\phi(e_a) - b\phi(e_b) + c\phi(e_c) + d\phi(e_d)] \\ &+ 1\det[a\phi(e_a) + b\phi(e_b) - c\phi(e_c) + d\phi(e_d)] \\ &+ 1\det[a\phi(e_a) + b\phi(e_b) + c\phi(e_c) - d\phi(e_d)] \\ &- 1\det[a\phi(e_a) - b\phi(e_b) - c\phi(e_c) - d\phi(e_d)]) \end{aligned} \tag{2.22}$$

Now if we let $\phi(e_a) = \phi(e_b) = \phi(e_c) = \phi(e_d) = I_2$, then we have

$$\begin{aligned} Z(G,I_2) &= \pm\frac{1}{2}\left((a-b+c+d)^2 + (a+b-c+d)^2 + (a+b+c-d)^2 - (a-b-c-d)^2\right) \\ &= \pm(a^2 + 2ab + 2ac + 2ad + b^2 - 2bc - 2bd + c^2 - 2cd + d^2) \end{aligned} \tag{2.23}$$

Observe that $Z(G,\phi)$ truly depends on the surface in which it is embedded and also the choice of $\mathcal{R}$ reference edges. Contrast Equation 2.23 with Equation 2.16 and also Equation 2.13 of Example 2.4.2 when $n = 2$. If the same graph were embedded in the plane with the same connections, we would have

$$Z(G,I_2) = (a+b+c+d)^2.$$

This illustrates that $Z(G,\phi)$ truly depends on the surface in which it is embedded and also the choice of $\mathcal{R}$ reference edges.

Remark 2.5.4. When n is odd, $Z_n(G, \phi)$ is independent of the choice of reference edges[Tat24, Sec II.A].

Remark 2.5.5. However, $Z_n(G, \phi)$ truly depends on the surface in which G is embedded, even when n is odd[Tat24, Sec II.A].

Example 2.5.3. Let G be embedded in the torus as in Figure 2.6. For example and future reference, if we let $\phi(e_a) = \phi(e_b) = \phi(e_c) = \phi(e_d) = I_9$, this code uses Equation 2.22 to compute $Z_9(G, I_9)$ with this torus embedding, which has both positive and negative terms, and so is not equal to $(a + b + c + d)^9$ as would be the case for a planar embedding. However the absolute values of each monomial of $Z_9(G, I_9)$ with this torus embedding and $(a + b + c + d)^9$ do match. Both have 4^9 terms.

2.6 Summary

The original (Main) Trace Theorem is from [Ken14] and applies to the double dimers on a planar graph with SL_2 connections. Theorem 2.4.2 for planar graphs with $\mathrm{End}(\mathbb{R}^n)$ connections from [DKS24] and Theorem 2.5.1 for graphs embedded in a torus from [Tat24] may be referred to as Trace Theorems. They are of the form the sum of some web-traces is equal to the determinant of some matrix. The sum of the traces is generally difficult to compute, while the determinant is easier. Multiwebs, where $n(v)$ is allowed to vary at each vertex, of a planar graph have a trace theorem found in [KO25]. Once could certainly argue that Theorem 2.4.1 of [Kas67] also falls under the trace theorem umbrella.

When G is embedded in an orientable surface, we have the most general trace theorems from [KO25], [DKS24], and [Tat24]. In the case when G is embedded in the torus, for the single dimer model, [KOS06] tells us the partition function can be

expressed in terms of a characteristic polynomial. When G is embedded in an orientable surface (torus or otherwise), for the rank$-n$ dimer model, [Tat24] gives us Theorem 2.5.1. For the higher rank dimer model (when $n(v)$ varies at each vertex) [Tat24] comments that this case is open.

In the case when G is embedded in a non-orientable surface, for the single dimer model, [Cim09] uses spin structures to compute the partition function. The case when G is embedded in a non-orientable surface, for the rank$-n$ dimer model, this version of a trace theorem is still open. Though [Tat24] speculates the analogous result to hold, and gives suggestions on how to combine their results with that of [Cim09] to get the hoped for outcome.

CHAPTER 3

COVERING MAPS

We are interested in using this body of work to understand covering maps and their relation to web enumeration. Since much has been accomplished on orientable surfaces, we focus on covering maps on tori. We do so in such a way as to preserve partition functions. Now that we have some background, we restate the following from our introduction.

Compact connected surfaces are homeomorphic to a sphere or a connected sum of tori and real projective planes. The universal cover of the sphere and the real projective plane is the sphere. The universal cover of the torus is the plane. The Covering Map Theorem we achieve in Chapter 3 addresses the $n-$fold covering map for a torus, where n is finite.

3.1 Covering Map

Let X denote the 1×1 torus, the subset $[0,1] \times [0,1]$ of $\mathbb{R}^2$ with opposite sides of the square identified. Let $\tilde{X}$ denote the $m \times n$ torus, the subset $[0,m] \times [0,n]$ of $\mathbb{R}^2$ with opposite sides of the rectangle identified, where m and n are positive integers. Let $p : \tilde{X} \to X$ be the $mn-$fold covering map, defined point-wise by $p((x,y)) = (x - \lfloor x \rfloor, y - \lfloor y \rfloor)$.

Let G be a bipartite graph embedded in X, and let $\tilde{G}$ be the graph in $\tilde{X}$ consisting of mn copies of G in the natural way. A point $(x,y) \in \tilde{X}$ is a vertex of $\tilde{G}$ if and only if $p(x,y)$ is a vertex of G. Similarly, a point $(x,y) \in \tilde{X}$ is part of an edge of $\tilde{G}$ if and only if $p((x,y))$ is part of an edge of G. This is equivalent to *enlarging the fundamental domain* as described in [KOS06] and [Ken09].

Suppose the edges of $\tilde{G}$ come equipped with scalar edge weights $\tilde{w}$ and/or GL_k

connections $\tilde{\phi}$. Our goal is to find connections ϕ for the edges of G so that the covering map preserves partition functions.

3.2 Covering Map Theorem: Scalar Version

In this section, suppose $\tilde{G}$ is equipped with scalar edge weights $\tilde{w}$, but is without connections. For each vertex v of G select and fix an ordering of the vertices $p^{-1}(v)$ of $\tilde{G}$,

$$p^{-1}(v) = v_1, v_2, \dots v_{mn}. \tag{3.1}$$

We need these **fixed orderings** now and again at the conclusion of the proof of Theorem 3.2.1. For $e_{bw} \in E(G)$ let $\phi(e_{bw})$ denote the matrix with rows indexed by $w_1, \dots w_{mn}$ as decided in Equation 3.1 and columns indexed by $b_1, \dots, b_{mn}$ as decided in Equation 3.1 with the (i, j) entry of $\phi(e_{bw})$ given $\tilde{w}(e_{b_j,w_i})$. To compute $Z_{mn}(G, \phi)$, we apply Theorem 2.5.1. Recall we must select a collection of reference edges (perfect matching as in Definition 2.5.1) $\mathcal{R}$ of G.

To compute $Z_1(\tilde{G})$, we also apply Theorem 2.5.1. For reference edges $\tilde{\mathcal{R}}$ we use the edges which map to $\mathcal{R}$ via p. That is e is a reference edge of $\tilde{G}$ if and only if $p(e)$ is a reference edge of G. Remember, while $Z_{mn}(G, \phi)$ is dependent on the choice of reference edges, $Z_1(\tilde{G})$ is not.

Theorem 3.2.1 (Covering Map Theorem: Scalar Version)**.** *For mn **odd**, let $p : \tilde{G} \to G$ be the natural $mn-$fold torus covering map. If $\tilde{G}$ has scalar edge weights $\tilde{w}$, define ϕ on the edges of G as described above. Then*

$$Z_{mn}(G, \phi) = Z_1(\tilde{G}), \tag{3.2}$$

and p preserves the partition function.

Proof. We first note that the trace theorem of [Tat24] does indeed apply to $Z_{mn}(G, \phi)$ as each $\phi(e)$ is of the form of a permutation matrix, which is invertible and thus in GL_{mn}. Select a collection of reference edges $\mathcal{R}$, the reference edges of G, and use $\tilde{\mathcal{R}} = p^{-1}(\mathcal{R})$ for the reference edges of $\tilde{G}$.

On G, select four distinct subsets of edges to twist, as described for Definition 2.5.4, denoted $S_{00}, S_{01}, S_{10}, S_{11}$. On $\tilde{G}$, we may select the four twisted subsets $\tilde{S}_{00} = p^{-1}(S_{00})$, $\tilde{S}_{01} = p^{-1}(S_{01})$, $\tilde{S}_{10} = p^{-1}(S_{10})$, and $\tilde{S}_{11} = p^{-1}(S_{11})$. We make two observations.

Observation 3.2.1. If twisting the edges of S_{xy} provides Kasteleyn orientation on G, then twisting the edges of $\tilde{S}_{xy} = p^{-1}(S_{xy})$ provides a Kasteleyn orientation on $\tilde{G}$. A face $\tilde{F}$ of $\tilde{G}$ has an odd number of edges with vector fields pointing into the face if and only if the face $p(\tilde{F})$ of G has an odd number of edges pointing into the face.

Observation 3.2.2. If twisting the edges of S_{xy} results in the winding numbers (w_x, w_y) on G, then twisting the edges of $\tilde{S}_{xy} = p^{-1}(S_{xy})$ results in the winding numbers (mw_x, nw_y) on $\tilde{G}$. This is true since a horizontal loop l on G corresponds to the horizontal loop $p^{-1}(l)$ on $\tilde{G}$ which has the same winding number as taking m laps around l. Likewise for the vertical computation.

Furthermore, since m and n are both odd, we have

$$(w_x \mod 2, w_y \mod 2) = (mw_x \mod 2, nw_y \mod 2).$$

Hence defining $\tilde{S}_{xy} = p^{-1}(S_{xy})$ is justified as the winding numbers of both S_{xy} and $\tilde{S}_{xy}$ do indeed both correspond to the same homology class $(x, y) \in H_1(X, \mathbb{Z}_2) = H_1(\tilde{X}, \mathbb{Z}_2)$.

Thus to justify this theorem, it suffices to show

$$\pm\frac{1}{2^g} \sum_{\substack{\eta \text{ spin} \\ \text{structure of } G}} \text{Arf}(\eta) \det(K_\eta(\phi)) = \pm\frac{1}{2^g} \sum_{\substack{\eta \text{ spin} \\ \text{structure of } \tilde{G}}} \text{Arf}(\eta) \det(K_\eta), \tag{3.3}$$

where the left side of the equation uses values from (G, ϕ) embedded in X, and the right side uses values from $\tilde{G}$ with scalar weights $\tilde{w}$ embedded in $\tilde{X}$, it suffices to show that

$$\det(K_{\eta_{xy}}(\phi)) = \det(K_{\eta_{xy}}). \tag{3.4}$$

Neither $K_{\eta_{xy}}(\phi))$ nor $K_{\eta_{xy}}$ has a specified ordering of the black vertices (columns) and white vertices (rows). For G, $(K_{\eta_{xy}}(\phi))$ select an ordering of the black vertices, say $x, \ldots, z$ and an ordering of the white vertices say $u, \ldots, y$. For $\tilde{G}$, $(K_{\eta_{xy}})$ use the ordering $x_1, \ldots, x_{mn}, \ldots, z_1, \ldots, z_{mn}$ for the columns and $u_1, \ldots, u_{mn}, \ldots, y_1, \ldots, y_{mn}$, where the orders of the subscripts agree with the order fixed at the beginning of this section in Equation 3.1 used to define ϕ.

These orderings together with the definitions of $K_{\eta_{xy}}(\phi)$ and $K_{\eta_{xy}}$ given by Equation 2.20 show that the matrices agree. Specifically, select any i, j, k, l where $0 \leq i, j \leq \frac{V(G)}{2} - 1$, and $1 \leq k, l \leq mn$. Then the $(imn + k, jmn + l)$ entry of $K_{\eta_{xy}}(\phi)$ is the (k, l) entry of $\phi(e_{b,w})$ where b is the $(j + 1)$th vertex in the ordering of the black vertices of G, $x, \ldots, z$; and w is the $(i + 1)$th vertex in the ordering of the white vertices of G, $u \ldots y$. Well, the (k, l) entry of $\phi(e_{b,w})$ is $\tilde{w}(e_{b_l,w_k})$, which is precisely the $(imn + k, jmn + l)$ entry of $K_{\eta_{xy}}$. □

Recall Remarks 2.5.2 and 2.5.1. $Z_{mn}(G, \phi)$ is dependent on the choice of reference edges, while $Z_1(\tilde{G})$ is not. We have inadvertently proven a case of the observation that the web trace definitions are completely invariant for n odd, in [Tat24] and as highlighted in Remark 2.5.4.

Corollary 3.2.1. *If mn is odd and ϕ is as defined for Theorem 3.2.1, then $Z_{mn}(G, \phi)$ is independent of the choice of reference edges.*

Proof. $Z_{mn}(G, \phi)$ is equal to $Z_1(\tilde{G})$ by Theorem 3.2.1 and $Z_1(\tilde{G})$ remains the same for any choice of reference edges by Definition 2.1.2 or Remark 2.5.1. □

3.3 Example of Theorem 3.2.1 Application

Let G be the graph embedded in X from Example 2.5.1, but without weights assigned to the edges. Let $\tilde{G}$ be the graph of Figure 3.1 embedded in $\tilde{X} = [0,3] \times [0,3]$ with opposite sides identified. So $m = n = 3$.

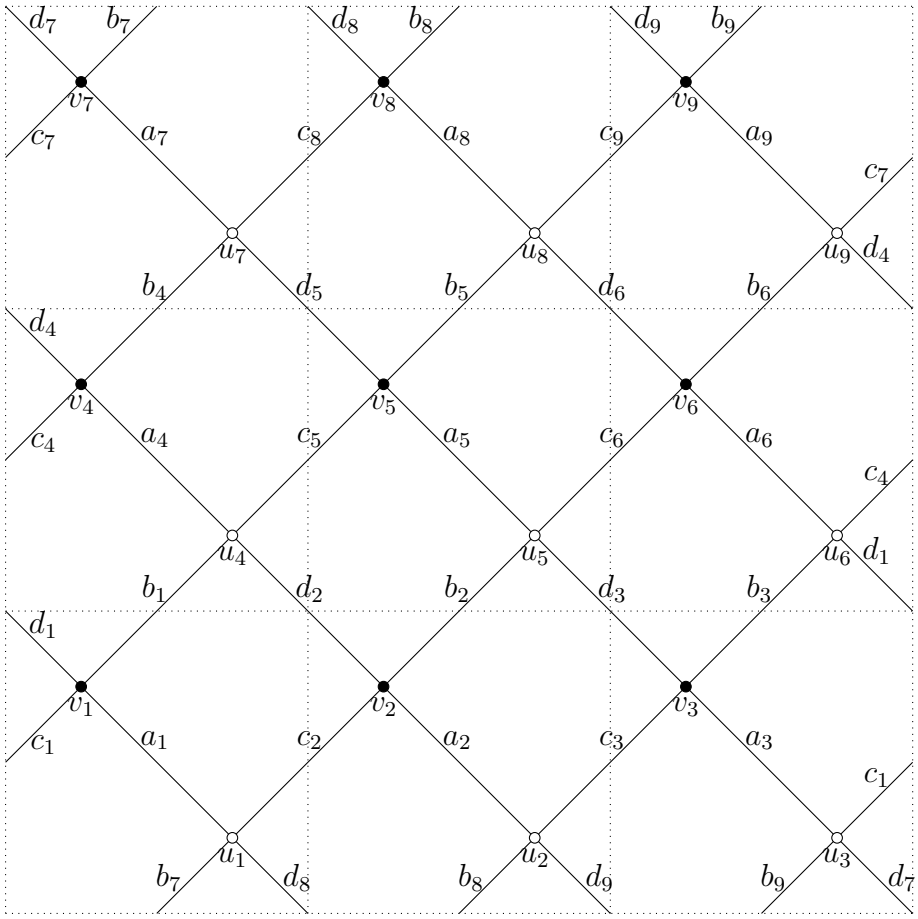


Figure 3.1. Square Lattice on 3×3 Torus

For each edge $e \in V(\tilde{G})$, we define

$$\tilde{w}(e) = \begin{cases} a_i, & \text{if } p(e) = e_a \text{ and } e \text{ is incident with } v_i \\ b_i, & \text{if } p(e) = e_b \text{ and } e \text{ is incident with } v_i \\ c_i, & \text{if } p(e) = e_c \text{ and } e \text{ is incident with } v_i \\ d_i, & \text{if } p(e) = e_d \text{ and } e \text{ is incident with } v_i \end{cases}$$

With this information, we can compute $Z_1(\tilde{G})$.

We wish to find connections ϕ on the edges of G so that $Z_9(G, \phi) = Z_1(\tilde{G})$.

Applying Theorem 3.2.1, for each edge e of G, we index the columns of $\phi(e)$ by $v_1, v_2, \ldots, v_9$ and the rows by $u_1, u_2, \ldots u_3$. Then

$$\phi(e_a) = \begin{smallmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_9 \end{smallmatrix}, \phi(e_b) = \begin{smallmatrix} 0 & 0 & 0 & 0 & 0 & 0 & b_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_9 \\ b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_6 & 0 & 0 & 0 \end{smallmatrix},$$

$$\phi(e_c) = \begin{smallmatrix} 0 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_7 & 0 & 0 \end{smallmatrix}, \phi(e_d) = \begin{smallmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_7 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_4 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}. \tag{3.5}$$

To compute $Z_9(G, \phi)$, we can to use our same reference edge $\mathcal{R} = \{e_a\}$ and computations from Example 2.5.2, making sure to replace the scalar edge weights with values of 1, we have

$$\begin{aligned} Z_9(G,\phi) = \pm\frac{1}{2}\,(&1\det[\phi(e_a)-\phi(e_b)+\phi(e_c)+\phi(e_d)]+1\det[\phi(e_a)+\phi(e_b)-\phi(e_c)+\phi(e_d)] \\ &+1\det[\phi(e_a)+\phi(e_b)+\phi(e_c)-\phi(e_d)]-1\det[\phi(e_a)-\phi(e_b)-\phi(e_c)-\phi(e_d)])\,. \end{aligned} \tag{3.6}$$

If we let $a_i = a$, $b_i = b$, $c_i = c$, $d_i = d$, then this code (1 in Appendix) computes

$$
\begin{aligned}
Z_9(G,\phi) = {} & a^9 + 3a^6b^3 + 3a^6c^3 + 3a^6d^3 + 27a^5bcd^2 + 54a^4b^2c^2d + 3a^3b^6 \\
& + 21a^3b^3c^3 + 21a^3b^3d^3 + 3a^3c^6 + 21a^3c^3d^3 + 3a^3d^6 + 54a^2b^4cd^2 \\
& + 54a^2bc^4d^2 + 27a^2bcd^5 + 27ab^5c^2d + 27ab^2c^5d + 54ab^2c^2d^4 + b^9 \\
& + 3b^6c^3 + 3b^6d^3 + 3b^3c^6 + 21b^3c^3d^3 + 3b^3d^6 + c^9 + 3c^6d^3 + 3c^3d^6 + d^9,
\end{aligned}
\tag{3.7}
$$

which has 448 terms which are all positive. Contrast this with our earlier (Example 2.5.3) computation of $Z_9(G, I_9)$ on the torus in with the same reference set $\mathcal{R} = \{a\}$ which has many more terms (4^9), and both positive and negative terms.

Remark 3.3.1. On the torus, the covering map and its connection are fundamentally different than overlaying mn copies of G and the identity connection, as $Z_{mn}(G, \phi_{\text{cover map}}) \neq Z_{mn}(G, I_{mn})$. Furthermore, The connections generated by the covering map produce all positive terms on the torus. This is not the case with the identity connections.

To illustrate with clarity:

- $Z_9(G, I_9)$ with a planar embedding has 4^9 terms, all positive.
- $Z_9(G, I_9)$ with a torus embedding has 4^9 terms, some positive, some negative.
- $Z_9(G, \phi_{\text{cover map}})$ with planar embedding has 488 terms, some positive, some negative.
- $Z_9(G, \phi_{\text{cover map}})$ with torus embedding has 488 terms, all positive.

Remark 3.3.2. When moving from the plane to a torus, in some instances, it may be natural to exchange identity connections for covering map connections.

Graphs with planar embeddings have $Z_n(G, I_n) = Z_n(G)$. For a graph G on the torus, is there some connection ϕ with $Z_n(G, \phi) = Z_n(G)$? We are not able to answer this question at this time, and it appears again in the open problems of Chapter 8. The author suspects the answer is sometimes no, but only via

trial and error and without confidence or proof. This ϕ is not necessary to compute $Z_n(G)$ as $Z_n(G) = (Z_1(G))^n$ for bipartite graphs. However, when we begin to consider other kinds of covering maps, specifically covering maps from two disconnected surfaces X_1, X_2 (with graphs G_1, G_2 respectively) to one connected surface X (with graph G), it would be satisfying, though not necessary, to have a ϕ such that $Z_1(G_1)Z_1(G_2) = Z_2(G, \phi)$. I_2 accomplishes this when all the surfaces are planes. ϕ could hypothetically accomplish this when all the surfaces are tori.

3.4 Covering Map Theorem: with Connections

In this section, suppose $\tilde{G}$ is equipped with both scalar edge weights $\tilde{w}$ and also connections $\tilde{\phi} \in GL_k$. As in the scalar version, for each vertex v of G select and fix an ordering of the vertices $p^{-1}(v)$ of $\tilde{G}$,

$$p^{-1}(v) = v_1, v_2, \ldots v_{mn}. \tag{3.8}$$

For $e_{bw} \in E(G)$, let $\phi(e_{bw})$ be the matrix with rows indexed by $w_1, \ldots w_{mn}$ and columns indexed by $b_1, \ldots, b_{mn}$ and the (i, j) entry of $\phi(e_{bw})$ is $\tilde{w}(e_{b_j,w_i})\tilde{\phi}(e_{b_j,w_i})$. To compute the partition function of G select a collection of reference edges (perfect matching). To compute the partition function of $\tilde{G}$, use the edges which map to the reference edges of G. That is e is a reference edge of $\tilde{G}$ if and only if $p(e)$ is a reference edge of G. Remember, both $Z_{kmn}(G, \phi)$ and $Z_k(\tilde{G}, \tilde{\phi})$ are dependent on the choice of reference edges.

Theorem 3.4.1. *For mn **odd**, let $p : \tilde{G} \to G$ be the natural $mn-$fold torus covering map. If $\tilde{G}$ has scalar edge weights $\tilde{w}$ and $k \times k$ connection matrices $\tilde{\phi}$, define ϕ on the edges of G as described above, as well as the reference edges. Then*

$$Z_{kmn}(G, \phi) = Z_k(\tilde{G}, \tilde{\phi}). \tag{3.9}$$

Proof. We first note that the Trace Theorem 2.5.1 of [Tat24] does indeed apply to $Z_{kmn}(G, \phi)$ as for each e, $\phi(e)$ is the shape a permutation matrix (in S_{mn}) with each 1 replaced by a member of GL_k and each 0 replaced by $\mathbf{0}_{k\times k}$. Thus $\phi(e) \in GL_{kmn}$.

The proof is the same as for the previous Theorem 3.2.1, except for this theorem, we now must show that

$$\det(K_{\eta_{xy}}(\phi)) = \det(K_{\eta_{xy}}(\tilde{\phi})). \tag{3.10}$$

Neither $K_{\eta_{xy}}(\phi)$ nor $K_{\eta_{xy}}(\tilde{\phi})$ has a specified ordering of the black vertices (columns) and white vertices (rows). For G, $(K_{\eta_{xy}}(\phi))$ select an ordering of the black vertices, say $x, \ldots, z$ and an ordering of the white vertices say $u, \ldots, y$. For $\tilde{G}$, $(K_{\eta_{xy}}, \tilde{\phi})$ use the ordering $x_1, \ldots, x_{mn}, \ldots, z_1, \ldots, z_{mn}$ for the columns and $u_1, \ldots, u_{mn}, \ldots, y_1, \ldots, y_{mn}$, where the orders of the subscripts agree with the order fixed at the beginning of this section in Equation 3.8 used to define ϕ.

These orderings together with the definitions of $K_{\eta_{xy}}(\phi)$ and $K_{\eta_{xy}}(\tilde{\phi})$ given by Equation 2.20 show that the matrices agree. Specifically, select any i, j, q, l where $0 \leq i, j \leq \frac{V(G)}{2} - 1$, and $1 \leq q, l \leq mn$. Then the $(imn + q, jmn + l)$ entry of $K_{\eta_{xy}}(\phi)$ is the (q, l) entry of $\phi(e_{b,w})$ where b is the $(j + 1)$th vertex in the ordering of the black vertices of G, $x, \ldots, z$; and w is the $(i + 1)$th vertex in the ordering of the white vertices of G, $u \ldots y$. Well, the (q, l) entry of $\phi(e_{b,w})$ is $\tilde{w}(e_{b_l,w_q})\tilde{\phi(e_{b_l,w_q})}$, which is precisely the $(imn + q, jmn + l)$ entry of $K_{\eta_{xy}}(\tilde{\phi})$. □

3.5 Covering Map Corollaries for Periodic Weights or Connections

For notational brevity, we use the Kronecker product

$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \dots \\ a_{21}B & a_{22}B & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}. \tag{3.11}$$

Let $P(+1, m)$ denote the $m \times m$ permutation matrix corresponding to the permutation $(1, 2, 3, ..., m)$. That is

$$P(+1, m) := \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \tag{3.12}$$

Similarly, let $P(-1, m)$ denote the matrix corresponding to the permutation $(m, m-1, m-2, ..., 1)$. Let $P(0, m)$ denote I_m.

Again, we are interested in the situation in which G is a bipartite graph embedded on the 1×1 torus X; The graph $\tilde{G}$ consists of mn copies of G embedded on the $m \times n$ torus $\tilde{X}$ in the natural way; And p is the $mn-$fold covering map from $\tilde{G}$ to G.

Consider G with edges that can be classified into various types. Other types are possible, but for now we restrict to the reasonable possibility that each edge of G is one of the nine types of Figure 3.2.

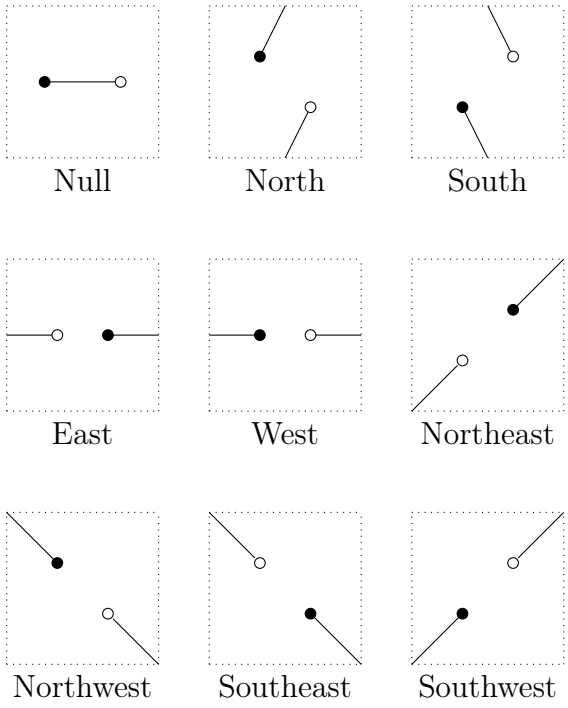


Figure 3.2. Edge Types

Possibilities that are not included here are edges that wrap around the torus one or more times before connecting one vertex to another. We address this in Section 3.6. If one chooses to include these kinds of edges in an embedding on the torus, take care to meet the requirement that each face is contractible.

3.5.1 Periodic Scalar Weights

Suppose $\tilde{G}$ has periodic scalar edge weights $\tilde{w}$ such that $p(e_1) = p(e_2)$ implies $\tilde{w}(e_1) = \tilde{w}(e_2)$. Let $\tilde{w}(\tilde{e})$ denote the scalar weight of any edge in $\tilde{G}$ which p maps to e.

For each edge e, assign (k, l) according to the edge type as follows. Let

$$k = \begin{cases} +1 & \text{if } e \text{ is type North, Northeast, or Northwest} \\ -1 & \text{if } e \text{ is type South, Southeast, or Southwest} \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

Let

$$l = \begin{cases} +1 & \text{if } e \text{ is type East, Northeast, or Southeast} \\ -1 & \text{if } e \text{ is type West, Northwest, or Southwest} \\ 0 & \text{otherwise.} \end{cases} \tag{3.14}$$

Corollary 3.5.1. *For mn odd, let $p : \tilde{G} \to G$ be the natural $mn-$fold torus covering map. Let $\tilde{G}$ have periodic scalar edge weights as described above. For each edge e assign (k, l) as described above.*

If ϕ on the edges of G is given by

$$\phi(e) := P(k, n) \otimes P(l, m) \otimes \tilde{w}(\tilde{e}) \tag{3.15}$$

then $Z_{mn}(G, \phi) = Z_1(\tilde{G})$.

Proof. If for each $v \in V(G)$ we select a particular fixed ordering/labeling of the vertices $p^{-1}(v)$ of $\tilde{G}$, then this corollary is a special case of Theorem 3.2.1. The particular ordering is as follows.

Definition 3.5.1. Let the **standard subscripts** of $\tilde{G}$ be defined as follows. Let (x, y) denote the coordinates of the vertex v in the embedding of G in X. Label a vertex of $p^{-1}(v)$ as $v_{(y'-y)m+(x'-x)+1}$ where (x', y') are the coordinates of the vertex in the embedding of $\tilde{G}$ in $\tilde{X}$.

That is, we assign the standard subscripts $1, 2, \ldots, m$ to the bottom row from left to right, and the subscripts $m+1, m+2, \ldots 2m$ to the second bottom row from left to right, and continue in this manner.

For an edge $e_{vu} \in G$ where v is black and u is white, consider $p^{-1}(e_{vu})$. Each one of these edges is incident with some vertices v_i and u_j in $\tilde{G}$. If the edge is of Type Null, it must be that $j = i$. Since the rows of $\phi(e_{vu})$ are indexed by $u_1, u_2, \ldots u_{mn}$, and the columns by $v_1, v_2, \ldots v_{mn}$, we have $\phi(e_{vu}) = I_{mn} \otimes \tilde{w}(\tilde{e_{vu}})$, which agrees with Equation 3.15.

Similarly, if the edge is of Type North, it must be that $j = i + m \mod mn$. Since the rows of $\phi(e_{vu})$ are indexed by $u_1, u_2, \ldots u_{mn}$, and the columns by $v_1, v_2, \ldots v_{mn}$, we have $\phi(e_{vu}) = P(+1, n) \otimes I_m \otimes \tilde{w}(\tilde{e_{vu}})$.

The remaining cases proceed in a similar manner. □

3.5.2 Periodic Edge Connections

Suppose $\tilde{G}$ has periodic edge connections $\tilde{\phi} \in GL_s$ such that $p(e_1) = p(e_2)$ implies $\tilde{\phi}(e_1) = \tilde{\phi}(e_2)$. Let $\tilde{\phi}(\tilde{e})$ denote the matrix connection of any edge in $\tilde{G}$ which p maps to e. For each edge e, assign (k, l) according to the edge-type as in Equations 3.13 and 3.14.

Corollary 3.5.2. *For mn odd, let $p : \tilde{G} \to G$ be the natural $mn-$fold torus covering map. Let $\tilde{G}$ have periodic edge connections as described above. For each edge e assign (k, l) as described above. If ϕ on the edges of G is given by*

$$\phi(e) := P(k, n) \otimes P(l, m) \otimes \tilde{\phi}(\tilde{e}) \tag{3.16}$$

then $Z_{mns}(G, \phi) = Z_s(\tilde{G}, \tilde{\phi})$.

Proof. If for each vertex v of G we select the particular fixed ordering as given in the proof of Corollary 3.5.1 then this corollary is a special case of Theorem 3.4.1.

□

3.6 Galois Group Observations

First, some notation. Just as we let $P(+1, m)$ denote the $m \times m$ permutation matrix corresponding to the permutation $(1, 2, 3, ..., m)$, we generalize this and let $P(+i, m)$ denote the $m \times m$ permutation matrix corresponding to the permutation taking each element k to $k + i \mod m$. Next, we divide the $m \times n$ torus into mn **regions**. The point (x, y) is said to be in the $(\lfloor x \rfloor + 1, \lfloor y \rfloor + 1)-$**region**.

Recall the Galois correspondence [Hat10, Sec 1.3], which assigns to each covering space $p(\tilde{X}, \tilde{x_0}) \to (X, x_0)$ the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ of $\pi_1(X, x_0)$ induced by p. Recall that the universal cover of a space X is a covering space of every other connected covering space of X. In the case of the torus, the universal cover is $\mathbb{R}^2$. If we keep our notation where $p : \tilde{X} \to X$ is the covering map from the $m \times n$ torus to the 1×1 torus, then $p_*(\pi_1(\tilde{X}, \tilde{x_0})) = m\mathbb{Z} \times n\mathbb{Z}$ is the corresponding subgroup of $\pi_1(X, x_0) = \mathbb{Z} \times \mathbb{Z}$.

The isomorphisms $\tilde{X} \to \tilde{X}$ are called **deck transformations** or *covering transformations*, which form a group $G(\tilde{X})$ under composition. A deck transformation is completely determined by where it sends a single point. In our case when $\tilde{X}$ is the $m \times n$ torus, $G(\tilde{X})$ is $\mathbb{Z}_m \times \mathbb{Z}_n \cong (\mathbb{Z} \times \mathbb{Z})/(m\mathbb{Z} \times m\mathbb{Z})$. Members of this group are of the form $(i \mod m, j \mod n)$ for $i, j \in \mathbb{Z}$.

There is a correspondence between edge types and deck transformations. The deck transformation of an edge e determines the shape of $\phi(e)$. Furthermore, ϕ can only be of the shape $P(+j, n) \otimes P(+i, m)$ with entries determined by $\tilde{w} \cdot \tilde{\phi}$.

Suppose we have our usual setup with $p : \tilde{X} \to X$ an mn fold torus covering map, $p(\tilde{G}) = G$, and $\tilde{G}$ equipped with $\tilde{\phi}(\tilde{e}) \in GL_k$ for each $\tilde{e} \in E(\tilde{G})$. Let the **region** of the $m \times n$ torus $\tilde{X}$ be given by the standard subscripts of that region as in Definition 3.5.1.

Theorem 3.6.1. *If e is the type of edge that connects a black vertex of the (i, j) region to a white vertex of the (k, l) region, then this edge type corresponds to the deck transformation $(k - i, l - j) \in \mathbb{Z}_m \times \mathbb{Z}_n$. Furthermore, $\phi(e)$ is given by $P(+(l - j), n) \otimes P(+(k - i), m)$ with 1s replace by $\tilde{w} \cdot \tilde{\phi}(\tilde{e})$ and 0s replaces by $\boldsymbol{0}_{k\times k}$.*

Proof. Consider the Edge Types of Figure 3.2. Each edge e of G lifts to several edges $\tilde{e}$ of $\tilde{G}$. Each $\tilde{e}$ connects a black vertex of the (i, j) region to a white vertex of the (k, l) region. If we restrict to the nine allowed edge types, we have $k \in \{i - 1, i, i + 1\}$ and $l \in \{j - 1, j, j + 1\}$. This edge corresponds to the deck transformation $(k - i, l - j)$. Furthermore, edge e is $\phi(e)$ $P(+(l - j), n) \otimes P(+(k - i), m)$ with 1s replace by $\tilde{w} \cdot \tilde{\phi}(\tilde{e})$ and 0s replaces by $\mathbf{0}_{k\times k}$ by Theorem 3.4.1.

For example, a lifted Type East edge connects a black vertex of region (i, j) to a white vertex region $(i + 1 \mod m, j)$ in the $m \times n$ torus. The edge corresponds to the deck transformation $(1, 0)$. The connection of a Type East edge is $\phi(e_{\text{east}}) = P(0, n) \otimes P(+1, m) = I_n \otimes P(+1, m)$ with 1s replace by $\tilde{w} \cdot \tilde{\phi}(\tilde{e})$ and 0s replaces by $\mathbf{0}_{k\times k}$.

□

3.6.1 Irregular Multiwebs with Connections where $n(v)$ Varies

Should we be in the situation where $\tilde{G}$ comes equipped with connections $\tilde{\phi}(e_{bw})$ that are $n(w) \times n(b)$, the trace theorem of [Tat24] on a torus does not apply. However, it is quite possible this more general version is true, and no counterexample

has been provided. Should this version of a trace theorem later be shown to hold, then we expect everything in these theorems to apply, and we can extend Theorem 3.4.1 to this situation.

Conjecture 3.6.1. *For mn odd, let $p : \tilde{G} \to G$ be the natural $mn-$fold torus covering map. Suppose $n : V(\tilde{G}) \to \{1, 2, 3, \dots\}$, for each $\tilde{e}_{bw} \in E(\tilde{G})$ we have $\phi(\tilde{e}_{bw}) \in \hom(\mathbb{R}^{n(b)}, \mathbb{R}^{n(w)})$, that is, $\phi(\tilde{e}_{bw})$ is an $n(w) \times n(b)$ matrix. For each $v \in V(G)$ select and fix and ordering $v_1, v_2, \dots v_{mn}$ of the vertices of $p^{-1}(v)$. If ϕ on the edges of G is given by*

$$\phi(e_{b,w})_{i,j} := \begin{cases} \tilde{\phi}(\tilde{e}_{b_j,w_i}), & \text{if } b_j \sim w_i \text{ in } \tilde{G}, \\ \boldsymbol{0}_{n(w_i)\times n(b_j)} & \text{if } b_j \nsim w_i \text{ in } \tilde{G}, \end{cases} \tag{3.17}$$

then

$$Z(G, \phi) = Z(\tilde{G}, \tilde{\phi}) \tag{3.18}$$

3.7 Examples

3.7.1 Application of Corollary 3.5.1

Let G be the graph embedded in the torus X as depicted in Figure 3.3,

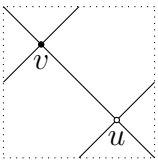


Figure 3.3. Square Lattice on Small Torus

and let $\tilde{G}$ in $\tilde{X}$ with scalar edge weights as depicted in Figure 3.4,

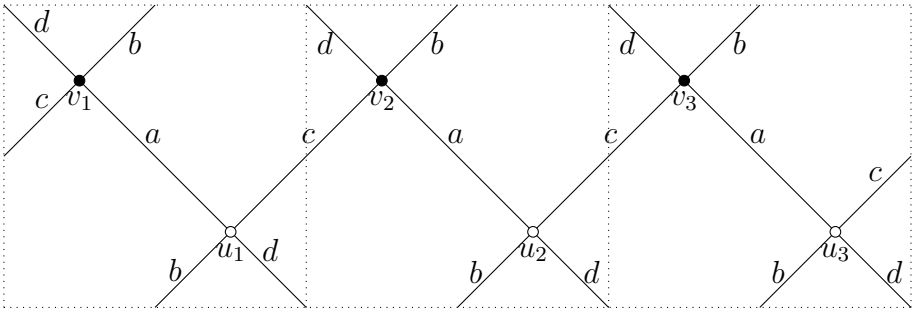


Figure 3.4. Square Lattice on 3×1 Torus

so $m = 3, n = 1$.

Applying Theorem 3.2.1 or Corollary 3.5.1, we get connections on the edges of G :

$$\phi(e_a) = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}, \phi(e_b) = \begin{bmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix},$$
$$\phi(e_c) = \begin{bmatrix} 0 & c & 0 \\ 0 & 0 & c \\ c & 0 & 0 \end{bmatrix}, \phi(e_d) = \begin{bmatrix} 0 & d & 0 \\ 0 & 0 & d \\ d & 0 & 0 \end{bmatrix}. \tag{3.19}$$

Without loss of generality, we again decide to make our reference set $\mathcal{R}$ in G equal to $\{e_a\}$, and we again can use the twisted subsets $S_{00} = \{e_b, e_c, e_d\}$, $S_{01} = \{e_c\}$, $S_{10} = \{e_b\}$, and $S_{11} = \{e_d\}$. Again, we have

$$\begin{aligned} Z(G,\phi) = \pm\frac{1}{2}\,(&1\det[\phi(e_a)-\phi(e_b)+\phi(e_c)+\phi(e_d)]+1\det[\phi(e_a)+\phi(e_b)-\phi(e_c)+\phi(e_d)] \\ &+1\det[\phi(e_a)+\phi(e_b)+\phi(e_c)-\phi(e_d)]-1\det[\phi(e_a)-\phi(e_b)-\phi(e_c)-\phi(e_d)])\,. \end{aligned} \tag{3.20}$$

After some computing (2 of Appendix) we obtain,

$$Z(G,\phi) = a^3 + 3a^2b + 3ab^2 + b^3 + c^3 + 3c^2d + 3cd^2 + d^3. \tag{3.21}$$

If we let $a = b = c = d = 1$, then we get $Z_3(G, \phi) = 16 = Z_1(\tilde{G})$.

Viewing this example in light of Theorem 3.6.1, notice the edge e_a is Type Null and so corresponds to the deck transformation $(0, 0)$ and so $\phi(e_a)$ is from $P(0, n) \otimes P(0, m) = P(0, 1) \otimes P(0, 3) = I_1 \otimes I_3 = I_3$. The edge e_b is Type North and so corresponds to the deck transformation $(0, 1)$ and so $\phi(e_b)$ is from $P(1, n) \otimes P(0, m) = P(1, 1) \otimes P(0, 3) = I_1 \otimes I_3 = I_3$. The edge e_c is Type West and so corresponds to the deck transformation $(-1, 0)$ and so $\phi(e_c)$ is from $P(0, n) \otimes P(-1, m) = P(0, 1) \otimes P(-1, 3) = I_1 \otimes P(-1, 3)$. The edge e_d is Type Northwest and so corresponds to the deck transformation $(-1, 1)$ and so $\phi(e_d)$ is from $P(1, n) \otimes P(-1, m) = P(1, 1) \otimes P(-1, 3) = I_1 \otimes P(-1, 3) = P(-1, 3)$.

3.7.2 Application of Corollary 3.5.2

We use the same G and $\tilde{G}$ from Example 3.7.1, including the scalar edge weights of $\tilde{G}$, however we now assign connections $\tilde{\phi}(\tilde{e}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ for each $\tilde{e} \in E(\tilde{G})$. Applying Corollary 3.5.2 we obtain

$$\phi(e_x) = \begin{cases} I_3 \otimes I_2 \otimes x & \text{for } x = a, b \\ P(-1, 3) \otimes I_2 \otimes x & \text{for } x = c, d \end{cases} \tag{3.22}$$

Using the same computations as Equation 3.20, and this code (3 of Appendix) we have

$$\begin{aligned} Z_6(G, \phi) = {} & a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 2a^3c^3 + 6a^3c^2d + 6a^3cd^2 + 2a^3d^3 + 15a^2b^4 \\ & - 6a^2bc^3 - 18a^2bc^2d - 18a^2bcd^2 - 6a^2bd^3 + 6ab^5 + 6ab^2c^3 + 18ab^2c^2d \\ & + 18ab^2cd^2 + 6ab^2d^3 + b^6 - 2b^3c^3 - 6b^3c^2d - 6b^3cd^2 - 2b^3d^3 + c^6 - 6c^5d \\ & + 15c^4d^2 - 20c^3d^3 + 15c^2d^4 - 6cd^5 + d^6 \\ = {} & Z_2(\tilde{G}, I_2) \end{aligned} \tag{3.23}$$

If we let $a = b = c = d = 1$, then $Z_6(G, \phi) = 64$.

3.7.3 Hexagonal Lattice from Square Lattice

If we are instead interested in the hexagonal lattice, we can let $w(e_d) = 0$.

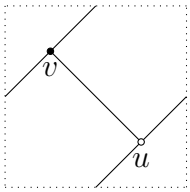


Figure 3.5. Hexagonal Lattice on Small Torus

If $\tilde{G}$ is embedded in the 3×1 torus and consists of 3 copies of G, then we can use the result of Example 3.7.1, substituting $d = 0$, and find

$$\begin{aligned} Z_3(G, \phi) &= a^3 + 3a^2b + 3ab^2 + b^3 + c^3 \\ &= Z_1(\tilde{G}). \end{aligned} \tag{3.24}$$

If $\tilde{G}$ is embedded in the 3×3 torus and consists of 9 copies of G, then we can use the result of Example 3.3 and Equation 3.7, substituting $d = 0$ to obtain

$$\begin{aligned} Z_9(G, \phi) &= a^9 + 3a^6b^3 + 3a^6c^3 + 3a^3b^6 + 21a^3b^3c^3 + 3a^3c^6 + b^9 + 3b^6c^3 + 3b^3c^6 + c^9 \\ &= Z_1(\tilde{G}). \end{aligned} \tag{3.25}$$

CHAPTER 4

REVERSING THE COVERING MAP

Given a graph with connection (G, ϕ) embedded in the torus X so that the faces of G are contractible, we wish to know if there exists some $\tilde{G}$ with or without connection $\tilde{\phi}$ embedded in some larger torus $\tilde{X}$ which covers G so that ϕ agrees with that of Theorem 3.4.1.

Suppose we have G embedded in the torus X with contractible faces. For any edge e of G, its edge-type is either from Figure 3.2, or in the case of the edge wrapping around the torus multiple times, the edge corresponds to a deck transformation (k, l), as described in Section 3.6. For example, let us assume e is of type Northeast. This corresponds to the deck transformation $(1, 1)$. So if $\phi(e)$ is the result of a covering map where each set of vertices $p^{-1}(v)$ are ordered with the *standard subscripts* of Definition 3.5.1, then we know it is of the form

$$P(1, n) \otimes P(1, m) \otimes (\text{some } GL_k \text{ matrices}). \tag{4.1}$$

The GL_k matrices will tell us exactly what the original connections $\tilde{\phi}(e_i)$ are for each $e_i \in p^{-1}(e)$. If we can find m, n odd that are consistent for every edge of G, then we have found that $\tilde{X}$ is the $m \times n$ torus, p is to corresponding $mn-$fold covering map, and all the connections $\tilde{\phi}(e_i)$ can be read off from the corresponding GL_k matrices in $\phi(e)$. For an example, see Example 4.0.1

However, these are not the only possibilities. As per our covering map connection definition, it may be that we selected and fixed some other order for $p^{-1}(v)$ for any or all of the vertices v of G, which allows for other shapes. So there are very many possibilities to check! No matter the choices for $p^{-1}(v)$, the connection $\phi(e)$

of an edge e is still of the form

$$\text{(some permutation matrix)} \otimes \text{(some permutation matrix)} \otimes \text{(some } GL_k \text{ matrices)}. \tag{4.2}$$

For this reason, we introduce *Permutation Equivalence*, a special kind of SL_n gauge equivalence (Definition 2.3.5), after the example. Later, it will be important to keep in mind from Remark 2.3.1 that if ϕ and ϕ' are SL_n-gauge equivalent, then $\mathrm{Tr}_L(m, \phi) = \mathrm{Tr}_L(m, \phi')$; however this equality does require that the data L (Definition 2.2.1) is the same on both sides.

4.0.1 Example: reversing covering map

Let G be the graph with vertices u, v, w, x and connections ϕ_1 as depicted in Figure 4.1.

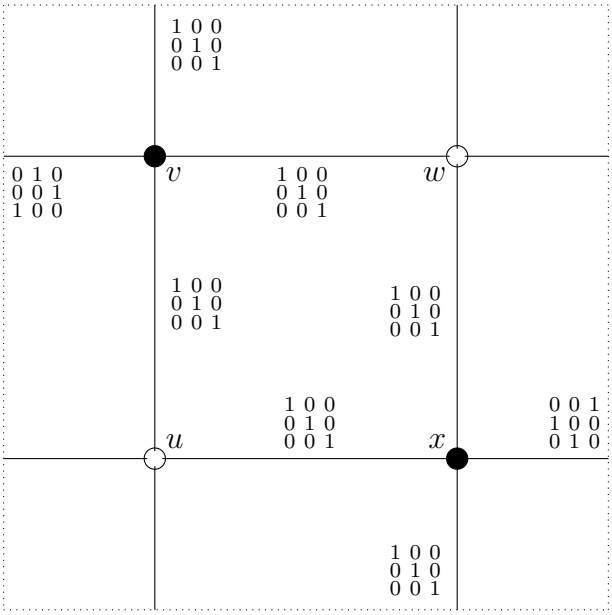


Figure 4.1. (G, ϕ_1) embedded in torus

Observe the shape of each connection $\phi(e)$ and the edge type all agree with

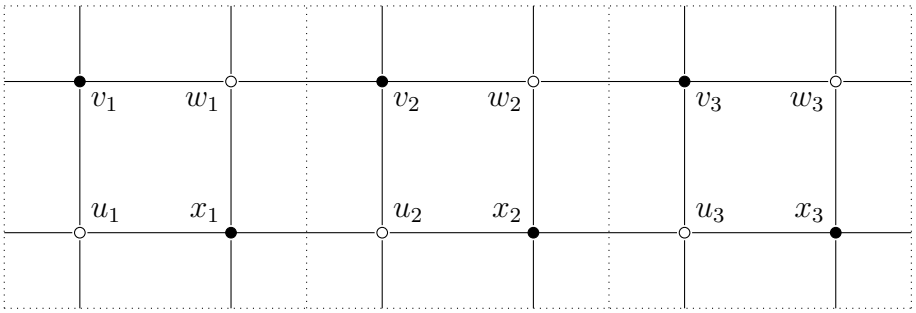


Figure 4.2. $\tilde{G}$ which covers (G, ϕ_1)

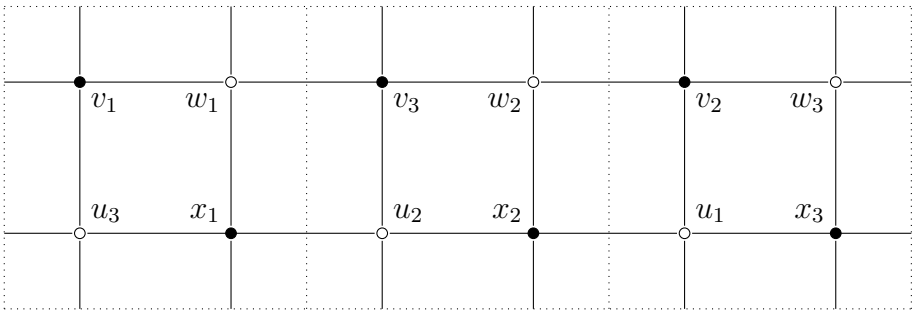


Figure 4.3. another labeling of $\tilde{G}$

$m = 3$, $n = 1$, for $\tilde{G}$ with scalar weights of 1 on each edge. So we can determine that the graph $\tilde{G}$ of Figure 4.2 covers G.

The alternate labeling in Figure 4.3 of the same graph $\tilde{G}$ induces different connections ϕ_2 on G as shown in Figure 4.4 which we will soon see is *Permutation Equivalent* to (G, ϕ_1).

4.1 Permutation Equivalence

Recall from Section 2.2 and Definition 2.2.5 that when we define the web-trace of an $n-$web with connection, $\mathrm{Tr}(m, \phi)$, implicit in the definition is the assignment of basis vectors $e_1, e_2, \ldots e_n$ of an $n-$dimensional vector space V (or its dual V^*) to each vertex. Connections ϕ determine a map between vector spaces $\phi : V \to V^*$.

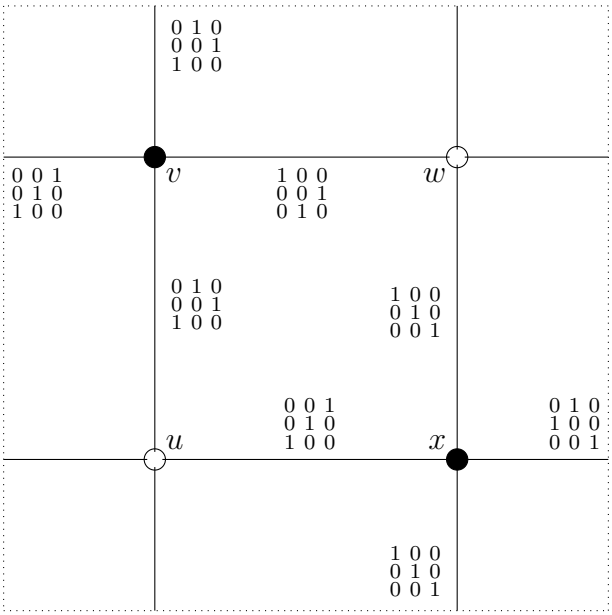


Figure 4.4. (G, ϕ_2)

So permuting the basis elements assigned to a vertex is equivalent to permuting the rows or columns of the connections incident with that vertex. We define an equivalence class accordingly.

We say a graph with connection (G_1, ϕ_1) is **Permutation Equivalent** to (G_2, ϕ_2) if $G_1 = G_2$ and there exists a collection of permutations on the bases of the vertices of G_1 which permutes the rows and to columns from ϕ_1 into ϕ_2.

For example, let $G_1 = C_4$ with black vertices x_1, x_2 and white vertices w_1, w_2

and connections

$$\phi_1(e_{b_1,w_1}) = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}, \phi_1(e_{b_2,w_1}) = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix},$$
$$\phi_1(e_{b_2,w_2}) = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}, \phi_1(e_{b_1,w_2}) = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}. \tag{4.3}$$

Since our connections are rank 3, each vertex is associated with a 3-dimensional vector space, and 3 basis elements. Let us say that the basis elements for x_1 are v_1, v_2, v_3. Permuting these, say from their current order v_1, v_2, v_3 to v_2, v_1, v_3 permutes the columns of the connections of the incident edges. With just this one permutation, we can have $G_2 = G_1$, and

$$\phi_2(e_{b_1,w_1}) = \begin{bmatrix} 0 & a_1 & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & a_3 \end{bmatrix}, \phi_2(e_{b_2,w_1}) = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix},$$
$$\phi_2(e_{b_2,w_2}) = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}, \phi_2(e_{b_1,w_2}) = \begin{bmatrix} 0 & d_1 & 0 \\ d_2 & 0 & 0 \\ 0 & 0 & d_3 \end{bmatrix}. \tag{4.4}$$

Thus

$$(G_1, \phi_1) \equiv_{\text{Permutation}} (G_2, \phi_2). \tag{4.5}$$

Permuting basis elements of a white vertex would permute rows of the connections of the incident edges. A series of permutations of bases would also result in Permutation Equivalence.

Observation 4.1.1. If (G_1, ϕ_1) and (G_2, ϕ_2) are Permutation Equivalent then they are gauge equivalent and by Remark 2.3.1, $\mathrm{Tr}_L(m, \phi_1) = \mathrm{Tr}_L(m, \phi_2)$ for each m. However,

$$Z(G_1, \phi_1) =^{?} \pm Z(G_2, \phi_2). \tag{4.6}$$

does **not** always hold when the data $L(G_1) \neq L(G_2)$.

To bypass this issue, care must be taken to ensure some Trace Theorem applies.

Observation 4.1.2. The following circumstances ensure that some Trace Theorem applies and so Equation 4.6 holds.

1. The surface is the plane/sphere. (Theorem 2.4.2)
2. The surface is orientable, and all connections are invertible, and the set of reference edges is the same for both G_1 and G_2. (Theorem 2.5.1)
3. The surface is not orientable, but each connection is 1×1. [Cim09]

When we can apply some Trace Theorem, the Permutation Equivalence is just permuting the rows and columns of $K(\phi_1)$ in order to compute the partition functions.

Corollary 4.1.1. *If (G_2, ϕ_2) is Permutation Equivalent to some (G_1, ϕ_1), where (G_1, ϕ_1) matches the description of Equation 4.1, then (G_2, ϕ_2) is also the result of a covering map. Furthermore, the map can be reversed by permuting the basis vectors for each vertex and thus permuting the rows and columns of its connections, and then following the procedure described by Equation 4.1.*

Proof. Permuting the basis vectors of each vertex v from ϕ_2 to ϕ_1 tells us exactly how to permute the fixed ordering of $p^{-1}(v)$ in order to obtain the ordering prescribed by the *standard subscripts* of Definition 3.5.1. Then we may apply the procedure of Equation 4.1 obtaining m, n, and $\tilde{\phi}$. □

4.2 Reverse Cover Map from Particular Multiwebs

Suppose $p : \tilde{G} \to G$ is a $n-$fold covering map, where $\tilde{G}$ is a graph with scalar edge weights, without connections and G is equipped with edge connections ϕ such that $Z_1(\tilde{G}) = Z_n(G, \phi)$. Observe p induces a map from the $1-$webs of $\tilde{G}$ to $n-$webs of G. Are there $n-$webs of G which are not in the range of the induced map? The answer can be yes, which is demonstrated in the following example.

4.2.1 Example: covering map is not $n-$web surjective

Recall Example 3.7.1, where we found

$$Z(G, \phi) = a^3 + 3a^2b + 3ab^2 + b^3 + c^3 + 3c^2d + 3cd^2 + d^3. \tag{4.7}$$

Each term corresponds to a 3-web of G. However, this is not a complete list of weights of the 3-webs of G. For example, there is a 3-web of G which contributes a^2c, but is missing from this list.

CHAPTER 5

OVERLAY MAP FOR PLANES ONLY

Consider covering maps from a disconnected space onto a single space. For example, we can map from two copies of $\mathbb{R}^2$ onto a single copy of $\mathbb{R}^2$ via the identity map on each. We refer to this kind of a covering map as the *overlay map*.

For a graph embedded in each copy of the plane in the covering space, there is a natural way to define connections on the graph in the image of the overlay map so that the partition function is preserved. However, the obvious analogue for the torus fails. We will describe and demonstrate all of this here, highlighting where the failure occurs.

5.1 Overlay Map Motivating Planar Example

Suppose we have the graph $G_1 = C_4$ embedded in $\mathbb{R}^2$ with black vertices v_1, x_1, white vertices u_1, w_1 and edge weights $w(e_{v_1,u_1}) = a_1$, $w(e_{v_1,w_1}) = b_1$, $w(e_{x_1,w_1}) = c_1$, and $w(e_{x_1,u_1}) = d_1$. Also suppose we have the graph $G_2 = C_4$ embedded identically in another copy of $\mathbb{R}^2$, where all the subscripts of the vertices and edge weights are now 2. Finally, suppose we have the graph $G = C_4$ embedded identically in another copy of $\mathbb{R}^2$, where all the vertices have no subscripts, as in Figure 5.1. The edges of G do not have scalar weights, but instead have connections ϕ, where $\phi(e_{b,w})$ is the 2 by 2 matrix where the diagonal (i,i) entry is $w(e_{b_i,w_i})$ and the (i,j) entry is 0 when $i \neq j$. For example, $\phi(e_{x,u}) := \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$. If $p : \mathbb{R}^2 \dot{\cup} \mathbb{R}^2 \to \mathbb{R}^2$, then p restricted to $G_1 \dot{\cup} G_2$ induces a map from the 1-webs of $G_1 \dot{\cup} G_2$ to the 2-webs of G.

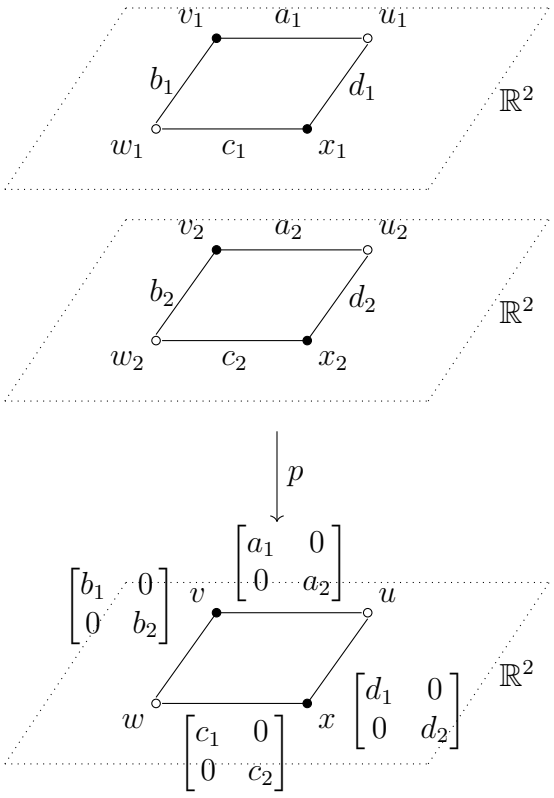


Figure 5.1. Overlay Map

Observe that

$$\begin{aligned} Z_1(G_1 \dot{\cup} G_2) &= Z_1(G_1) \cdot Z_1(G_2) \\ &= \pm \det\left(\begin{bmatrix} a_1 & d_1 \\ b_1 & -c_1 \end{bmatrix}\right) \det\left(\begin{bmatrix} a_2 & d_2 \\ b_2 & -c_2 \end{bmatrix}\right) \\ &= \pm \det\left(\begin{bmatrix} a_1 & d_1 & 0 & 0 \\ b_1 & -c_1 & 0 & 0 \\ 0 & 0 & a_2 & d_2 \\ 0 & 0 & b_2 & -c_2 \end{bmatrix}\right) \\ &= \pm \det\left(\begin{bmatrix} a_1 & 0 & d_1 & 0 \\ 0 & a_2 & 0 & d_2 \\ b_1 & 0 & -c_1 & 0 \\ 0 & b_2 & 0 & -c_2 \end{bmatrix}\right) \\ &= \pm Z_2(G, \phi). \end{aligned} \tag{5.1}$$

Note that the second to last equality is obtained by permuting the rows and columns of the large matrix, and that we can use any Kasteleyn weighting on G and let G_1 and G_2 inherit that same weighting.

5.2 Overlay of Graphs with Connection Map Defined

Let $\{G_i\}$ for $1 \leq i \leq m$ be m copies of the same graph G. Suppose each G_i comes equipped with connections ϕ_i. Let $p : \dot{\cup}_{i=1}^{m} G_i \to G$ denote the map where p maps identically from each G_i to G.

Definition 5.2.1. For each $e_{b,w} \in E(G)$, define the **overlay connection** $\phi(e_{b,w})$

by,

$$[\phi(e_{b,w})]_{j,k} := \begin{cases} \phi(e_{b_k,w_j}) & \text{if } j = k, \\ \mathbf{0}_{n(w_j)\times n(b_k)} & \text{if } j \neq k. \end{cases} \tag{5.2}$$

If one were hoping that the $\{G_i\}$ had edges with scalar weights rather than connections, then one could have each connection be a 1×1 matrix with the scalar weight as its single entry.

Proposition 5.2.1. *Let $(X_i = \mathbb{R}^2, G_i)$ for $1 \leq i \leq m$ be m copies of the same graph G embedded in the same way on m copies of the space $X = \mathbb{R}^2$. Suppose each G_i comes equipped with connections ϕ_i. Let $p : \dot{\cup}_{i=1}^m X_i \to X$ denote the overlay map where p maps identically from each X_i to X. Let ϕ be defined as in Equations 5.2. Then p preserves the partition function. That is*

$$Z(\dot{\cup}_{i=1}^m (G_i, \phi_i)) = Z(G, \phi). \tag{5.3}$$

Proof. The proof follows the format of the example of Section 5.1. Select any Kasteleyn weighting of G, and let each G_i inherit the same weighting. Then

$$\begin{aligned} Z(\dot{\cup}_{i=1}^m (G_i, \phi_i)) &= \prod_{i=1}^m Z(G_i, \phi_i) \\ &= \pm \prod_{i=1}^m \det(K(\phi_i)) \\ &= \pm \det \left(\begin{bmatrix} K(\phi_1) & 0 & \dots & 0 \\ 0 & K(\phi_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & K(\phi_m) \end{bmatrix} \right) \end{aligned} \tag{5.4}$$

which after permuting rows and columns is equal to

$$\pm \det(K(\phi)) = \pm Z(G, \phi) \tag{5.5}$$

□

5.3 Overlay Fails on Torus Example

The overlay map on graphs embedded in the torus does not generally preserve partition functions. Since we are doing computations on the torus, recall we will need a set of referenced edges $\mathcal{R}$ (Definition 2.5.1) and a collection of twisted subsets (Definition 2.5.4).

For example, let X be the torus with the graph G embedded as in Figure 5.2.

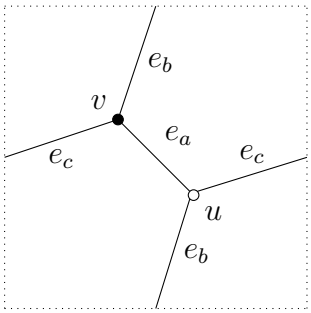


Figure 5.2. G Embedded in Torus

Let G_1 be the same graph where the vertices and edges have subscripts of 1, embedded identically embedded in another copy of the torus X_1. Let G_2 be the same graph where the vertices and edges have subscripts of 2, embedded identically embedded in another copy of the torus X_2. Supposed G_1, G_2 come equipped with scalar weights $w(e_{a1}) = a_1$, $w(e_{a2}) = a_2$, $w(e_{b1}) = b_1$, $w(e_{b2}) = b_2$, $w(e_{c1}) = c_1$, $w(e_{c2}) = c_2$. Let $p : X_1 \dot{\cup} X_2 \to X$ be the overlay map of spaces, which maps identically from the two copies of the torus onto the single copy of the torus. The restriction of p to $G_1 \dot{\cup} G_2$ maps to G with connections ϕ as defined in Equation 5.2. So

$$\phi(e_a) = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \phi(e_b) = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}, \phi(e_c) = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}. \tag{5.6}$$

Since (G, ϕ) is of even rank (rank 2) embedded on the torus, the computation of $Z_2(G, \phi)$ truly depends on the choice of reference edge $\mathcal{R}$ as discussed in Remark 2.5.2. If we have different options for $Z_2(G, \phi)$, they cannot all be equal to $Z(G_1 \cup G_2)$. We continue with our example anyway, to see where the equality fails.

We select our reference edge $\mathcal{R} = \{e_a\}$ and try twisted subsets $\emptyset$, $\{b\}$, $\{c\}$, $\{b, c\}$. We compute the winding numbers of each, using Figure 5.3.

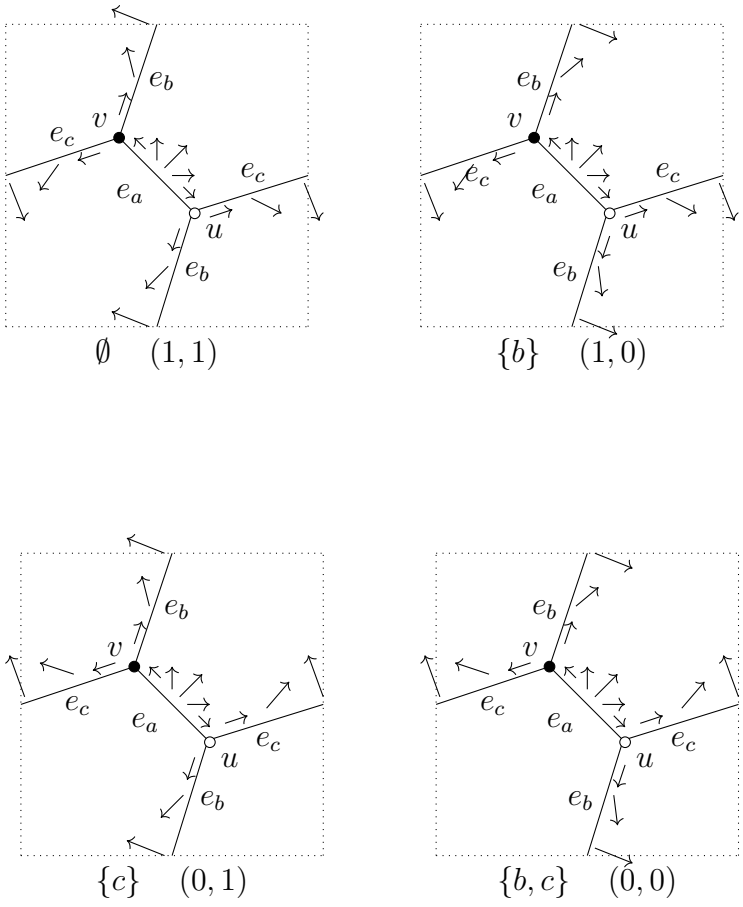


Figure 5.3. Winding Numbers of Example

Let G_1 and G_2 inherit the same reference edge, and use the same spin structures. The only face of G is a single hexagon, so no Kasteleyn weights of (-1) are

needed. So

$$\begin{aligned} Z(G_1) &= \pm\frac{1}{2}(\det[a_1+b_1+c_1]+\det[a_1-b_1+c_1]+\det[a_1+b_1-c_1]-\det[a_1-b_1-c_1]) \\ &= \pm(a_1+b_1+c_1). \end{aligned} \tag{5.7}$$

Hence

$$\begin{aligned} Z(G_1 \dot\cup G_2) &= Z(G_1)Z(G_2) \\ &= \pm\frac{1}{4}\Big(\det[a_1+b_1+c_1]+\det[a_1-b_1+c_1]+\det[a_1+b_1-c_1]-\det[a_1-b_1-c_1]\Big) \\ &\quad\cdot\Big(\det[a_2+b_2+c_2]+\det[a_2-b_2+c_2]+\det[a_2+b_2-c_2]-\det[a_2-b_2-c_2]\Big) \end{aligned} \tag{5.8}$$

Of course this is equal to $\pm(a_1+b_1+c_1)(a_2+b_2+c_2)$, but we leave the above equation as it is to compare with the following computation.

$$\begin{aligned} Z_2(G,\phi) &= \pm\frac{1}{2}\left(\det\begin{bmatrix} a_1+b_1+c_1 & 0 \\ 0 & a_2+b_2+c_2 \end{bmatrix} + \det\begin{bmatrix} a_1-b_1+c_1 & 0 \\ 0 & a_2-b_2+c_2 \end{bmatrix}\right. \\ &\quad \left. + \det\begin{bmatrix} a_1+b_1-c_1 & 0 \\ 0 & a_2+b_2-c_2 \end{bmatrix} - \det\begin{bmatrix} a_1-b_1-c_1 & 0 \\ 0 & a_2-b_2-c_2 \end{bmatrix}\right) \\ &= \pm\frac{1}{2}\left(\det[a_1+b_1+c_1]\det[a_2+b_2+c_2]+\det[a_1-b_1+c_1]\det[a_2-b_2+c_2]\right. \\ &\quad \left. +\det[a_1+b_1-c_1]\det[a_2+b_2-c_2]-\det[a_1-b_1-c_1]\det[a_2-b_2-c_2]\right) \end{aligned} \tag{5.9}$$

However, it turns out this value is not equal to the value of Equation 5.8. If we let $a_1 = a_2 = a$, $b_1 = b_2 = b$, and $c_1 = c_2 = c$, then rather than $(a+b+c)^2$ we instead get $a^2+2ab+2ac+b^2-2bc+c^2$. Different choices of a reference edge also seems to give us a mix of positive and negative terms.

The core of the issue is that the partition function for a torus comes from *summing* determinants, thus the last equality in Equation 5.4 is no longer helpful.

5.3.1 Odd-fold Overlay also Fails on Torus

One might wonder if the issue is that 2 is even. What if we overlay 3 copies onto the torus?

Let us use the same graph G of Figure 5.2, but this time we overlay 3 copies, so we have

$$\phi(e_a) = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}, \phi(e_b) = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}, \phi(e_c) = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}. \quad (5.10)$$

For our purposes and simplicity of computations it will suffice to let $a_1 = a_2 = a$, $b_1 = b_2 = b$, and $c_1 = c_2 = c$, so

$$\begin{aligned} Z_3(G,\phi) = \pm\frac{1}{2}\Bigg(& \det \begin{bmatrix} a+b+c & 0 & 0 \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{bmatrix} \\ & + \det \begin{bmatrix} a-b+c & 0 & 0 \\ 0 & a-b+c & 0 \\ 0 & 0 & a-b+c \end{bmatrix} \\ & + \det \begin{bmatrix} a+b-c & 0 & 0 \\ 0 & a+b-c & 0 \\ 0 & 0 & a+b-c \end{bmatrix} \\ & - \det \begin{bmatrix} a-b-c & 0 & 0 \\ 0 & a-b-c & 0 \\ 0 & 0 & a-b-c \end{bmatrix} \Bigg) \\ = \pm\frac{1}{2}\left((a+b+c)^3 + (a-b+c)^3 + (a+b-c)^3 - (a-b-c)^3\right) \\ = \pm(a^3 + 3a^2b + 3a^2c + 3ab^2 - 6abc + 3ac^2 + b^3 + 3b^2c + 3bc^2 + 1c^3) \end{aligned} \quad (5.11)$$

which has a negative term. However,

$$\begin{aligned} Z(G_1 \dot{\cup} G_2 \dot{\cup} G_3) &= Z(G_1)Z(G_2)Z(G_3) \\ &= (a+b+c)^3 \end{aligned} \tag{5.12}$$

has all positive terms. Thus the overlay map does not preserve partition functions on this torus example either.

CHAPTER 6

PROJECTION MAP

Let $q : G' \to G$ be any graph homomorphism, so vertices map to vertices and q preserves adjacency.

This set up differs from the overlay map. The overlay map is $n-$fold, while in this new map, we can have many vertices mapping to a single vertex, no vertices mapping to some other vertex, and similar for the edges.

For each vertex $v \in V(G)$ select and fix an ordering of the vertices of its preimage $q^{-1}(v)$, which may possibly be empty. Let this ordering be denoted by $v_1, v_2, ..., v_m$.

Definition 6.0.1. For each edge $e_{v,u} \in E(G)$, the define the **projection connection** $\phi(e_{v,u})$ by

$$[\phi(e_{v,u})]_{i,j} := \phi'(e_{v_j,u_i}), \tag{6.1}$$

where the columns are indexed by $v_1, v_2, ..., v_n$, and the rows are indexed by u_1, u_2, ..., u_m.

We will call such a map a **projection**. Since some of the preimages $q^{-1}(v)$ may be empty, some of the connections may be zero dimensional, and so we can think of this as no connection and/or a scalar weight of 0 assigned to that edge, or even just deleting that edge of G.

Definition 6.0.2. Alternatively stated, we call q a **projection** if G, G' come equipped with ϕ, ϕ' respectively, $q : G' \to G$ is graph homomorphism, and if for each $v \in V(G)$ there exists an ordering of the vertices $q^{-1}(v)$ so that ϕ and ϕ' satisfy Equation 6.1.

Observe, a projection $q : G' \to G$ induces a map from the multiwebs (Definition 2.1.3) of G to multiwebs of G'. A projection, does not in general preserve partition functions, as shown in the following example.

6.1 Example with Different Partition Functions

For example, if q is a projection from $G' = C_6$ to $G = C_4$, ϕ' is the connection on C_6, and of ϕ is the *projection connection* given by Definition 6.0.1, then we have examples in which

$$Z(G', \phi') \neq Z(G, \phi). \tag{6.2}$$

This inequality is due to the Kasteleyn matrices having different Kasteleyn weights. Let C_6 have black vertices v'_1, v'_2, x' and white vertices u'_1, u'_2, w', with $\phi(e'_{v_1,u_1}) = [a]$, $\phi(e'_{v_2,u_1}) = [b]$, $\phi(e'_{v_2,u_2}) = [c]$, $\phi(e'_{x,u_2}) = [d]$, $\phi(e'_{x,w}) = [e]$, and $\phi(e'_{v_1,w}) = [f]$, as depicted in Figure 6.1. Let C_4 have black vertices v, x and vertices u, w, and edges as shown in Figure 6.1.

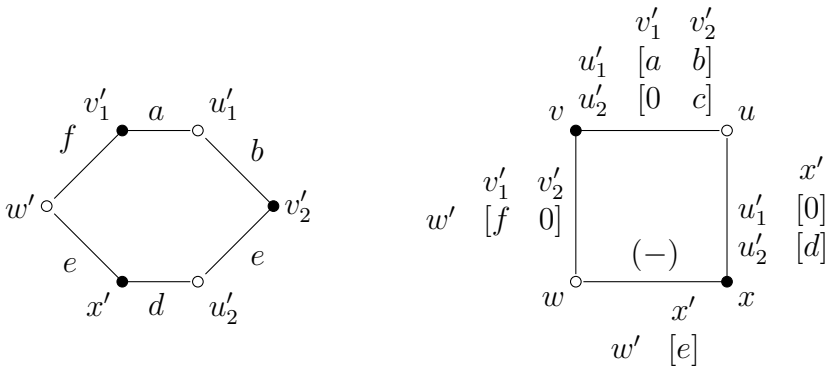


Figure 6.1. Projection Map: C_6 and C_4

Then the projection $q : C_6 \to C_4$ with $q(v'_1) = q(v'_2) = v$, $q(x') = x$, $q(u'_1) = q(u'_2) = u$ and $q(w') = w$ induces the connections $\phi(e_{v,u}) = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$

with columns indexed by v_1', v_2' and rows indexed by u_1', u_2'; $\phi(e_{v,w}) = \begin{bmatrix} f & 0 \end{bmatrix}$, with columns indexed by v_1', v_2' and row indexed by w'; $\phi(e_{x,u}) = \begin{bmatrix} 0 \\ d \end{bmatrix}$ with column indexed by v' and rows indexed by u_1', u_2'; $\phi(e_{x,w}) = \begin{bmatrix} e \end{bmatrix}$, with column indexed by x' and row indexed by w'.

However,

$$\begin{aligned} Z(C_6, \phi') &= \pm \det \begin{bmatrix} a & b & 0 \\ 0 & c & d \\ f & 0 & e \end{bmatrix} \\ &\neq \pm \det \begin{bmatrix} a & b & 0 \\ 0 & c & d \\ f & 0 & -e \end{bmatrix} \\ &= Z(C_4, \phi). \end{aligned} \tag{6.3}$$

But notice that implicit in the computations of $Z(C_6, \phi)$ and $Z(C_4, \phi')$ are planar embeddings. A planar embedding of C_4 together with q induces an embedding of C_6 which is not planar. We can avoid this issue by requiring that q be injective.

Lemma 6.1.1. *If G is a finite graph embedded on a surface with a Kasteleyn weighting, and G' is a subgraph of G, then the embedding and weighting induced on G' is also Kasteleyn.*

Proof. The proof is inductive. Delete one edge of G at a time until G' is obtained. Deleting one edge at a time gives us a sequence of graphs $G = G_1, G_2, \ldots, G_z = G'$. We suppose that G_i has a Kasteleyn orientation induced by G, and we wish to show that G_{i+1} also has a Kasteleyn orientation induced by G.

The the lone edge $e \in E(G_i \setminus G_{i+1})$ either separates two distinct faces of G_i or borders the same face of G_i.

Suppose $e \in E(G_i \setminus G_{i+1})$ separates two distinct faces F_1, F_2 of G_i. Since G_i has a Kasteleyn orientation, each of F_1, F_2 has an odd sum of edges pointed clockwise plus edges with a Kasteleyn (-1) weight. Deleting e subtracts an odd number from the sum for one of F_1, F_2 and and even number from the sum of the other. Thus the new face $F_1 \cup F_2$ has sum (odd-even)+(odd-odd)=odd.

Suppose $e \in E(G_i \setminus G_{i+1})$ is incident with the same face F on both sides of e. Since G_i has a Kasteleyn orientation, F has an odd sum of edges pointed clockwise plus edges with a Kasteleyn weight (-1). Deleting e will change this sum by an even number. Thus the face with an altered perimeter in G_{i+1} still has an odd sum □

Lemma 6.1.2. *If $q : G' \to G$ is injective, a Kasteleyn weighting of G induces a Kasteleyn weighting of G'*

Proof. View G' as the subgraph $q(G')$ of G. An edge e of G' is assigned a (-1) Kasteleyn weight if and only if $q(e)$ has a (-1) Kasteleyn weight in G. This is indeed a Kasteleyn weighting by Lemma 6.1.1. □

Theorem 6.1.1. *Let $q : (G', \phi') \to (G, \phi)$ be a projection of planar graphs, where q is injective on each component of G', and ϕ, ϕ' satisfy Definition 6.0.1. Then*

$$Z(G, \phi) = \pm Z(G', \phi').$$

Proof. Select and fix and ordering of the vertices of G. For each $v \in V(G)$, recall we selected and fixed and ordering of the vertices $q^{-1}(v)$ to define ϕ. Use both these orderings to index the rows and columns of $K(\phi), K(\phi')$. Now by Lemma 6.1.2, we can use any Kasteleyn weighting of G to also provide a Kasteleyn weighting of G', resulting in $K(\phi) = K(\phi')$. □

Corollary 6.1.1. *Given a collection of graphs with edge connections* (G_1, ϕ_1), (G_2, ϕ_2), $\ldots$ (G_k, ϕ_k), *if* $q : (G', \phi') \to (G, \phi)$ *is a projection of planar graphs where* $G' = G_1 \cup G_2 \cup \ldots \cup G_k$, *where* q *is injective on each* G_i, *then*

$$Z(G, \phi) = \pm \prod_{i=1}^{k} Z(G_i, \phi_i). \tag{6.4}$$

Proof. By Theorem 6.1.1, we need only show $Z(G', \phi') = \prod_{i=1}^{k} Z(G_i, \phi_i)$.

Order all the vertices of G' so that if $v_i \in V(G_i)$ and $u_j \in V(G_j)$ with $i < j$, then v_i comes before u_j in the ordering. Then

$$K(\phi') = \begin{bmatrix} K(\phi_1) & 0 & \cdots & 0 \\ 0 & K(\phi_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K(\phi_k) \end{bmatrix}. \tag{6.5}$$

Thus

$$\begin{aligned} Z(G', \phi') &= \pm \det(K(\phi')) \\ &= \pm \prod_{i=1}^{k} \det(K(\phi_i)) \\ &= \pm \prod_{i=1}^{k} Z(G_i, \phi_i). \end{aligned} \tag{6.6}$$

□

CHAPTER 7

HEIGHT FUNCTIONS

7.1 Height Function for Single Dimers

We restate several definitions from and [KOS06] in this section, as well as introduce the concept of contour or elevation lines for height functions on graphs.

Definition 7.1.1. A **flow** is a function ω on the oriented edges of a graph G such that $\omega(e_{v,u}) = -\omega(e_{u,v})$.

Definition 7.1.2. The **divergence** d^* of the flow ω is the function on the vertices of G defined by

$$d^*_\omega(v) := \sum_{u \sim v} \omega(e_{v,u}) \tag{7.1}$$

where the sum is over vertices adjacent to v.

Definition 7.1.3. A **divergence-free flow** is a flow in which the divergence is zero for all vertices.

Definition 7.1.4. Given a divergence-flow and a fixed face f_0 of a planar graph, we define a **height function** on all the faces of G; $h(f_0) := 0$, and for any other face f, $h(f)$ is the net flow crossing (from left to right) of a path in the dual graph from f_0 to f.

We now define a height function h_M for each perfect matching M of G. If G is a bipartite graph, we can orient every edge from black to white. Fix a flow ω_0 with divergence 1 at all black vertices and divergence -1 at all white vertices. For example, ω_0 could be determined by some fixed reference matching M_0 by assigning weight 1 to every edge in the matching (agreeing with the direction of the orientation) and 0 to all others.

Definition 7.1.5. Given a matching M, the **flow ω_M determined by** M is the flow determined by assigning weight 1 to every edge in the matching (agreeing with the direction of the orientation) and 0 to all others.

The flow ω_M determined by M has divergence 1 at all black vertices and divergence -1 at all white vertices. If the flow determined by M_0 is denoted by ω_0 and the flow determined by M is ω_M, then $\omega_M - \omega_0$ is a divergence-free flow.

Definition 7.1.6. Given a fixed flow ω_0 with divergence 1 at all black vertices and divergence -1 at all white vertices, and given a fixed face f_0, and given a matching M we define the **height function h_M determined by** M. Let ω_M be the flow determined by M as in Definition 7.1.5. Then h_M is the height function determined by the divergence-free flow $\omega_M - \omega_0$ and the value $h_M(f_0) = 0$ as in Definition 7.1.4.

For the height function to be well-defined, it is necessary for the graph to be planar and the flow to be divergence-free. While h_{M_1}, h_{M_2} both depend on a reference flow (or reference matching determining a flow), $h_{M_1} - h_{M_2}$ does not.

7.1.1 Single Dimer Example

For example, consider the graph G as shown in Figure 7.1, along with the reference matching M_0.

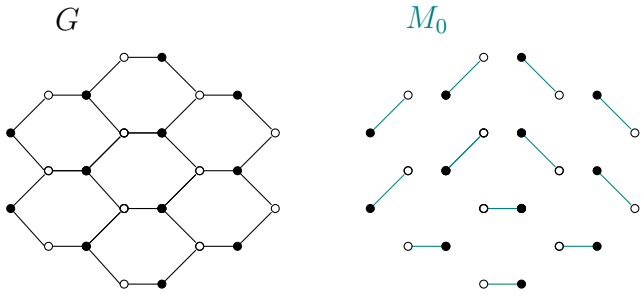


Figure 7.1. Graph and Reference Matching

Let M_1 and M_2 be the matchings as in Figure 7.2.

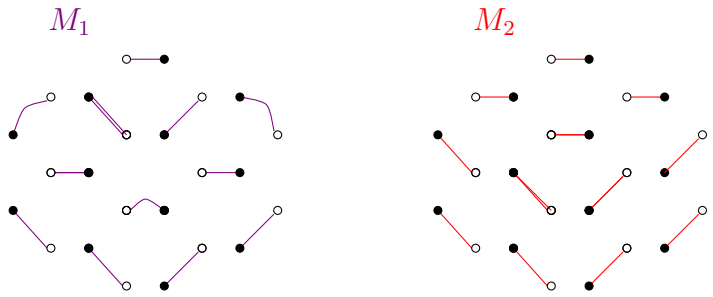


Figure 7.2. Single Dimers M_1 and M_2

The matchings M_0, M_1, M_2 determine the flows $\omega_{M_1} - \omega_0$ and $\omega_{M_2} - \omega_0$, depicted in Figure 7.3. These flows, together with assigning a height of 0 to the exterior face determine two different height functions on the faces of G, also shown in Figure 7.3.

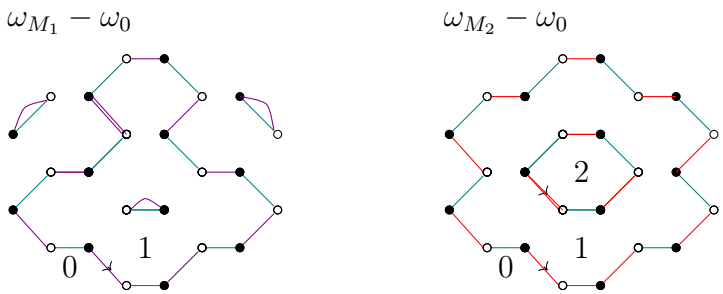


Figure 7.3. Height Functions from Single Dimers M_1 and M_2

Definition 7.1.7. An **elevation line** or **contour line** at height h of a graph with a height function on the faces is the set of edges of the graph for which one face incident to the edge has height greater than h and the other face has height less than h.

Using the height functions from Figure 7.3, we see the faces have heights with values $0, 1, 2$, so we decide to draw contour lines at heights 0.5 and 1.5, shown in Figure 7.4.

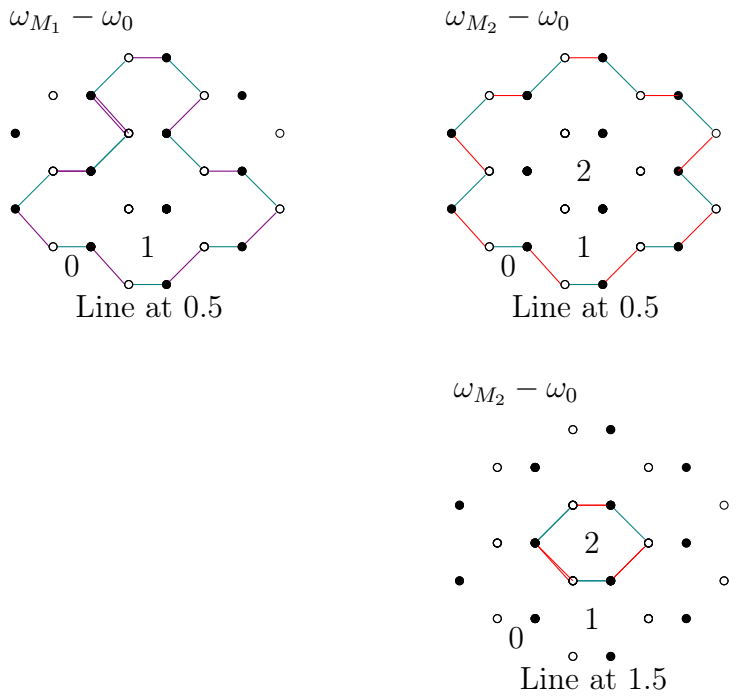


Figure 7.4. Elevation Lines from Pairs of Single Dimers

Observation 7.1.1. Pairs of single dimers, overlay to make doubled edges and loops. The loops correspond to contour lines of topographic maps. Pairs of doubled edges consist of one edge from M and one edge from M_0 and cancel one another in $\omega_M - \omega_0$.

We could use our height model to emulate real world situations. Notice the similarities between the elevation lines lines produced by $\omega_{M_2} - \omega_0$ of Figure 7.4 and the elevation lines in the topographic map in Figure 7.5

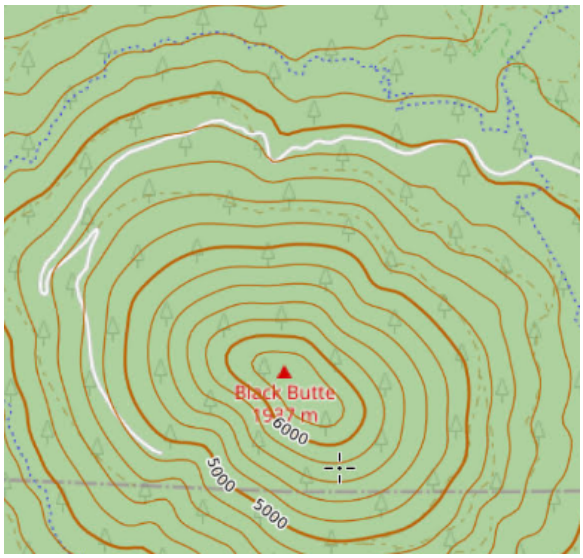


Image from caltopo.com [cal26].

Figure 7.5. Topographic Map of Black Butte

7.2 Height Function for Higher Rank Dimers

Recall Definition 2.1.1 of $n-$**webs** and Definition 2.1.3 of **multiwebs** from Section 2.1. We will say that higher rank dimers (multiwebs) differ from rank-n dimers ($n-$webs) in that the sum of the edge multiplicities around a vertex,

$$\sum_{u\sim v} m(e_{v,u}) = n(v), \tag{7.2}$$

need not have $n(v)$ the same for each vertex. We view $n-$webs as a special case of multiwebs, and so discuss height functions for both of them at once.

Let G be a bipartite graph with edges oriented from black to white vertices.

Definition 7.2.1. Given $n : V(G) \to \{1, 2, 3, \dots\}$, which may vary or may be constant at each vertex, we say an $n-$**multiflow** is a flow on the edges of G so that the divergence at each black vertex is $n(v)$ and the divergence at each white vertex is -$n(v)$, keeping in mind that $n(v)$ may vary at each vertex.

Definition 7.2.2. Given $n : V(G) \to \{1, 2, 3, \dots\}$, and a multiweb M which respects the function n, we define the **multiflow ω_M determined by** M to be the $n-$multiflow given by assigning weight $\omega_M(e_{v,u}) := M(e_{v,u})$ to each positively oriented edge $e_{v,u}$, and thus $\omega_M(e_{u,v}) = -M(e_{v,u})$ for each negatively oriented edge $e_{u,v}$.

Given $n : V(G) \to \{1, 2, 3, \dots\}$, fix a reference $n-$multiflow ω_0 on the edges of G. For example, ω_0 could be the $n-$multiflow determined by some multiweb M_0. Let ω_M be the multiflow determined by some other multiweb M. So long as M_0 and M both respect n, we have that $\omega_M - \omega_0$ is divergence-free. Once again, as in Definition 7.1.4, this divergence-free flow together with a reference face f_0 of G determines a height function.

Definition 7.2.3. Given $n : V(G) \to \{1, 2, 3, \dots\}$, a fixed $n-$multiflow ω_0 (as in Definition 7.2.1), and a fixed face f_0, and given a multiweb M which respects n, we define the **raw height function h_M^{raw} determined by** M. Let ω_M be the $n-$multiflow determined by M as in Definition 7.2.2. Then h_M^{raw} is the height function determined by the divergence-free flow $\omega_M - \omega_0$ and the value $h_M^{\text{raw}}(f_0) = 0$ as in Definition 7.1.4.

When considering $n-$webs only when n is constant, we define the **normalized height function** as

$$h_M^{\text{norm}}(f) := \frac{h_M^{\text{raw}}(f)}{n}. \tag{7.3}$$

The normalized height function definition is similar to what is referred to as the *height function* in work on a different kind of dimer model, the Multinomial Dimer Model [KW25, Remark 2.7]. We arrived at the definition of h_M^{raw} independently in our attempt to link $n-$webs to height functions, only to discover similarities with

the [KW25] afterward. This is encouraging to learn that this line of thought can be useful in many areas.

For our raw height model from pairs of $n-$webs, it makes sense to select contour lines at heights of $h^{\text{raw}}(f) \pm 0.5$ where $h^{\text{raw}}(f)$ is the height of a face f and $h^{\text{raw}}(f)$ takes on integer values. For the normalized height function, it makes sense to select contour lines at heights of $h^{\text{norm}}(f) \pm \frac{0.5}{n}$.

Remark 7.2.1. To allow for multiwebs where $n(v)$ can vary at each vertex, we will make use of the raw height function and drop the superscript "raw".

7.2.1 3-web Example

To illustrate our definitions, consider the graph G of Figure 7.6, and let $n(v) = 3$ for all $v \in V(G)$. M_0 is our reference $3-$web, and M is another $3-$web. Together they determine the divergence free flow $\omega_M - \omega_0$ which in turn, together with assigning a height of 0 to the exterior face, determines the heights of the faces.

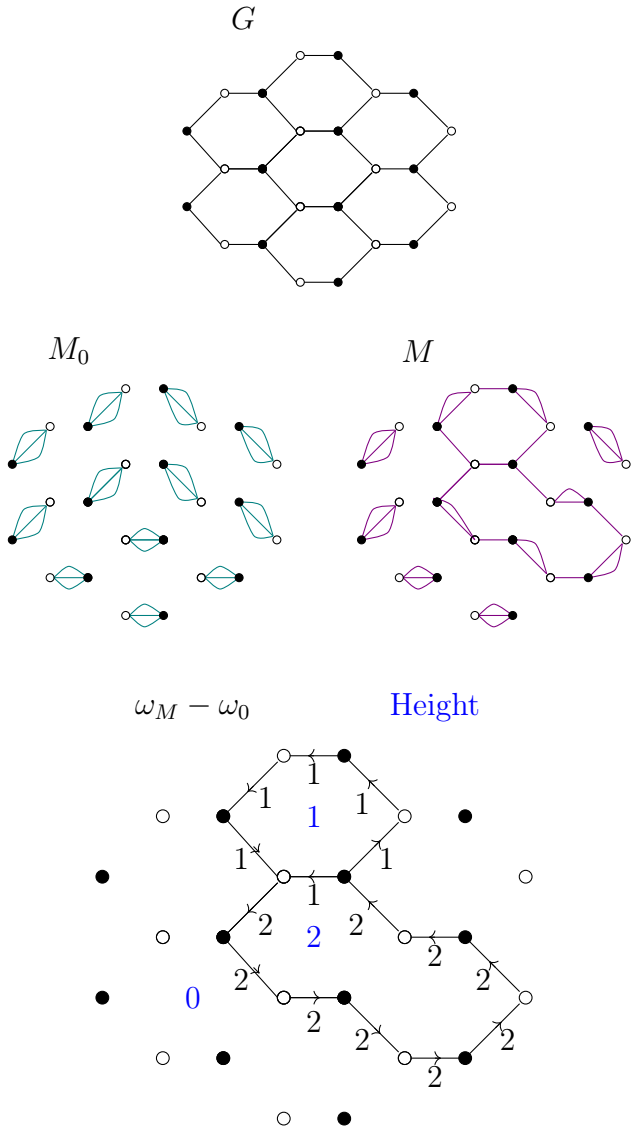


Figure 7.6. Height Function from a Pair of 3-webs

7.2.2 Y-shapes on Topographic Maps

We indulge in a diversion for the purpose of building intuition about the meaning of the height function from multiwebs.

Recall Observation 7.1.1. Pairs of single dimers unite to a collection of loops and doubled edges. The doubled edges cancel one another in $\omega_M - \omega_0$, and the loops

correspond to contour lines of a topographic map. However, the union $M \cup M_0$ of two $n-$webs is a $2n-$web which may have components that are not loops, as depicted by $\omega_M - \omega_0$ in Figure 7.6. Edges of M and M_0 at the same location cancel one another as before, but we are left with some non-loop components. For example, we have a vertex which is incident with three edges, all of which have multiplicity 1 in $M \cup M_0$.

Definition 7.2.4. Let M and M_0 be multiwebs on a graph G. We say that there is a **Y-shape at** v if there exist three or more edges e incident with v with $|M(e) - M_0(e)| \geq 1$.

The height functions that arise from the double dimer model (pairs of single dimers) are free of Y-shapes. One might believe that the elevation lines of topographic maps consist entirely of loops void of any Y-shapes.

Since the example in Figure 7.6 has a Y-shape, we decide draw the contour lines, and determine we must find them at heights $h = 0.5$ and $h = 1.5$, as done in Figure 7.7

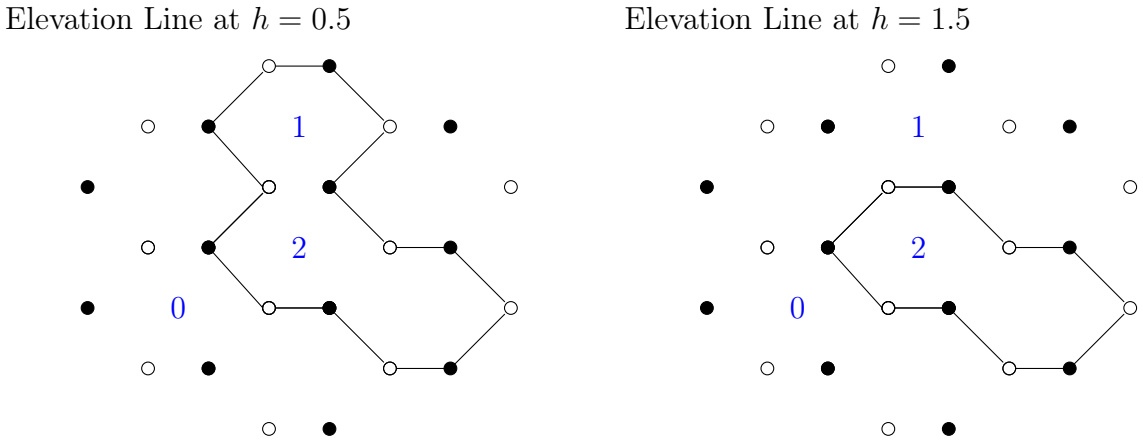


Figure 7.7. Elevation Lines from Pair of 3-webs

We do see that contour lines at specific heights are still loops. The Y-shapes come from overlaying loops. When we use pairs of 1-webs, the height can only change in increments of 0 or 1 between adjacent faces. But when we use 3-webs, our height can change in increments of 0, 1, 2, or 3 between adjacent faces, allowing for Y-shapes when we overlay contour lines of different elevations. Is this specific to our discrete model? It would be, if heights of some continuous model changed continuously from an aerial view. But there are real world examples of cliffs and overhangs as in Figure 7.8.

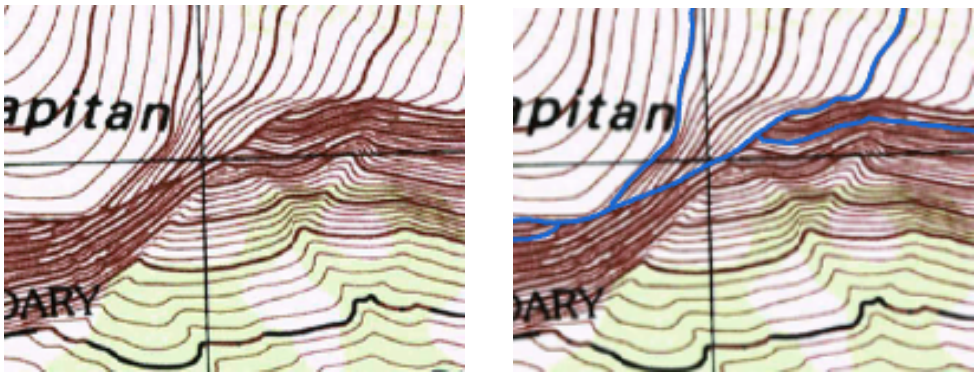


Image from commons.wikimedia.org [wik26].

Figure 7.8. Topographic Map of El Capitan with Y-shapes

Observation 7.2.1. When we allow heights to change in increments other than 0 or 1, then we can have a situation in which 3 or more faces around a single vertex, have distinct heights. This results in multiple contour lines over lapping and a Y-shape at the vertex.

7.2.3 Multiweb Example

Let M_0 be a reference multiweb M_0 of a graph G. Recall from Definition 2.1.3 that each vertex has $n_{M_0}(v)$ equal to the sum of the multiplicities of the edges in

M_0 which are incident to v. Let M be another multiweb for which $n_M(v) = n_{M_0}(v)$ for each vertex v of G. For example, we can let G, M_0, and M be as illustrated in Figure 7.9. Then the flow $\omega_M - \omega_0$ is divergence free and produces a well-defined height function on the faces of the graph, also as shown in Figure 7.9

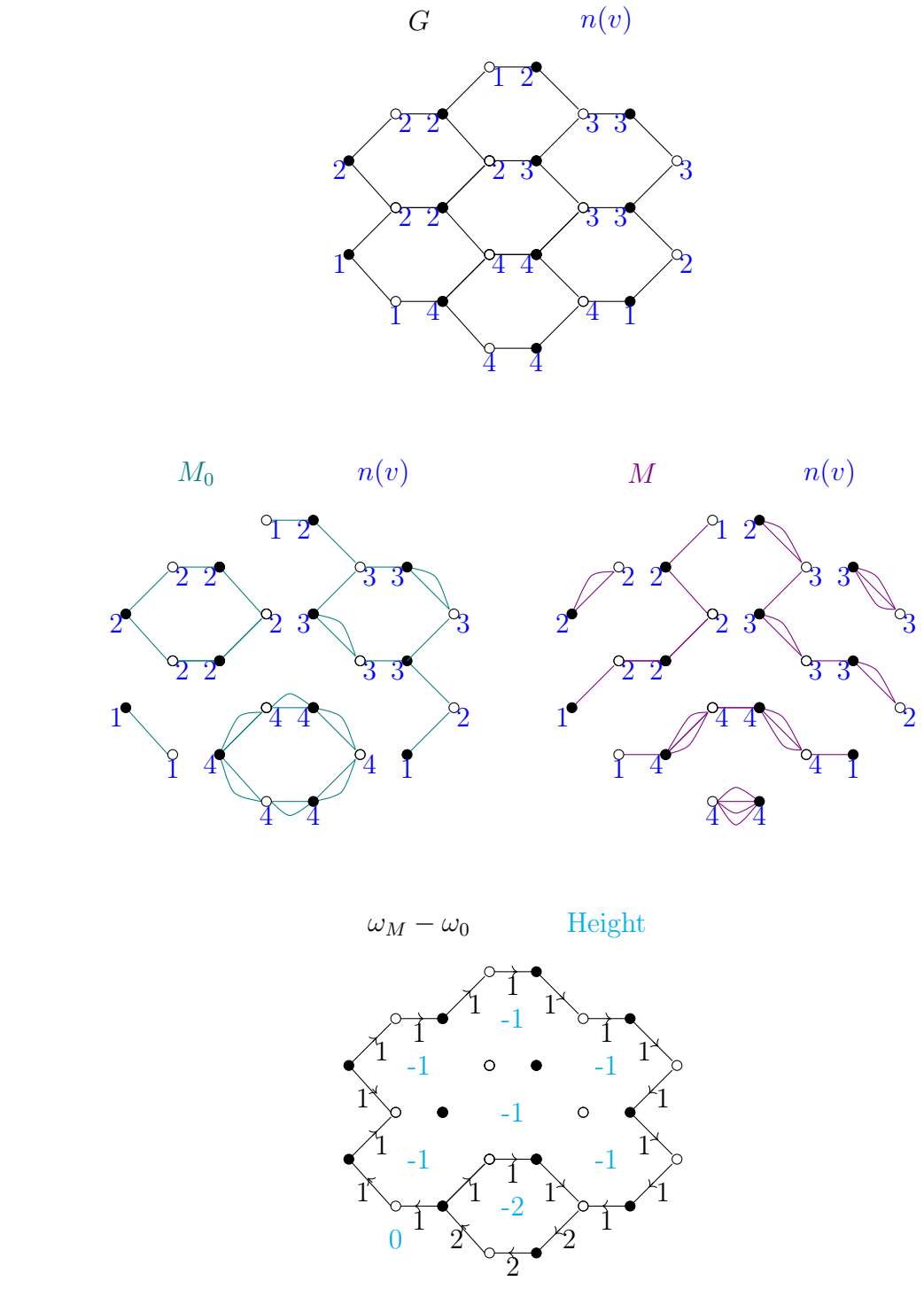


Figure 7.9. Height Function from Pair of Multiwebs

Since the heights of the faces are $-2, -1, 0$ we determine to draw contour lines at heights -1.5 and -0.5, as in Figure 7.10

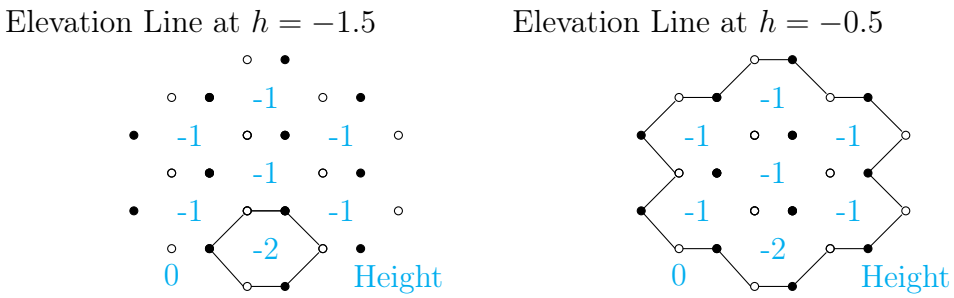


Figure 7.10. Elevation Lines from Pair of multiwebs

7.3 Multiweb Correspondence to Height Functions

Given a reference multiweb M_0 and any other multiweb M for which $n_M(v) = n_{M_0}(v)$ for every vertex v, the flow $\omega_M - \omega_0$ is divergence-free produces a well-defined height function on the faces of the graph. Given a reference multiweb M_0 and an integer valued height function h on the faces of the graph, can we always find a multiweb M such that $\omega_M - \omega_0$ has height function h?

To address this question we define the **height change of an edge** $\Delta h(e)$ to be

$$\Delta h(e) := \omega_M(e_{b,w}) - \omega_0(e_{b,w}). \tag{7.4}$$

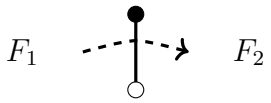


Figure 7.11. Height Change of an Edge

In Figure 7.11 we have $\Delta h(e) = h(F_2) - h(F_1)$.

Theorem 7.3.1. *Let ω_0 be a flow induced by a reference multiweb M_0, and let h be an integer-valued height function on the faces of G.*

There exists a multiweb M with $n_M(v) = n_{M_0}(v)$ for all $v \in V(G)$ such that $\omega_M - \omega_0$ produces h if and only if the height change of each edge is greater than or equal to the negative of the multiplicity of the edge in M_0;

$$-m_{M_0}(e) \leq \Delta h(e). \tag{7.5}$$

Proof. If there exists a multiweb M that agrees with the multiplicities and $\omega_M - \omega_0$ provides h, then for each edge,

$$\begin{aligned} \Delta h(e) &= \omega_M(e_{b,w}) - \omega_0(e_{b,w}) \\ &= m_M(e) - m_{M_0}(e) \end{aligned} \tag{7.6}$$

So

$$-m_{M_0} = \Delta h(e) - m_M(e) \tag{7.7}$$

and Inequality 7.5 holds.

If the height change satisfies Inequality 7.5, then let $m_M(e) = \Delta h(e) + m_{M_0}(e)$ which is a nonnegative integer. We have left to confirm that $n_{M_0}(v) = n_M(v)$ at each vertex v.

Select any vertex v. Let t be the number of faces the vertex touches. Select one of these faces. If the vertex is black (white), select a tiny path counterclockwise (clockwise) around the vertex, so that the path begins and ends on the same face, and crosses each of the t edges incident with v exactly once, and does not cross any other edges. Let $e_1, e_2, ..., e_t$ denote these edges. Since the tiny path begins and

ends on the same face, we know that $\sum_{i=1}^{t} \Delta h(e_i) = 0$. Thus

$$\begin{aligned} 0 &= \sum_{i=1}^{t} \Delta h(e_i) \\ &= \sum_{i=1}^{t} \omega_M(e_{i_{b,w}}) - \omega_0(e_{_{b,w}}) \\ &= \sum_{i=1}^{t} m_M(e_i) - m_{M_0}(e_i) \\ &= \sum_{i=1}^{t} m_M(e_i) - \sum_{i=1}^{t} m_{M_0}(e_i). \end{aligned} \tag{7.8}$$

Therefore,

$$\sum_{i=1}^{t} m_M(e_i) = \sum_{i=1}^{t} m_{M_0}(e_i)$$

and the multiplicities for M_0 and M agree at each vertex. □

CHAPTER 8

NEXT DIRECTIONS AND OPEN PROBLEMS

The completion of this body of work would include the covering map from a sphere to a real projective plane. However, more needs to be developed in this area before we can address this 2-fold covering map.

We have a collection of Trace Theorems of the form the sum of some traces is equal to a determinant (or Pfaffian). The benefit of these theorems is that traces are generally very difficult to compute, while the determinants (Pfaffians) are quicker to compute. Our Covering Map Theorem relies on Tata's Trace Theorem for a torus[Tat24, Cor 4]. The main difficulty we run into when attempting to move our techniques to the setting of a real projective plane, is that we do not yet have a Trace Theorem for non-orientable surfaces (which include the projective plane), in the case when $n \geq 2$. The situation is hopeful; [Tat24, Sec 5] speculates how one might obtain this result. That this partition-function-preserving Covering Map Theorem exists, provides motivation for researchers in this field to continue rounding out these Trace Theorems and Covering Map Theorems.

8.1 Open Problems

1. What can be said about the Covering Map Theorem for mn even? Can we address a 2-fold covering map on a torus? If so, do these methods extend to all mn even?

2. Is there a Trace Theorem for a graph with connections embedded in a non-orientable surface?

3. If Question 2 is solved, could one then have a Covering Map Theorem for a

graph with connections embedded in a non-orientable surface? Could we have more general theory for covering maps on any 2-manifold? If so, can we classify all covering maps of 2-manifolds?

4. Is there a Trace Theorem for a graph embedded in an orientable surface with $k \times j$ connections?

5. In the situation where $\tilde{G}$ comes equipped with connections $\tilde{\phi}(e_{bw})$ that are $n(b) \times n(w)$, the Trace Theorem of [Tat24] on a torus does not apply, hence Question 4. However, it is quite possible this more general version is true. Should this version of a Trace Theorem later be shown to hold, do we also have a covering map theorem for this situation?

6. Must we rely on Trace Theorems for Covering Map Theorems? Can we develop any Covering Map Theorem direction from the definitions?

7. Given a bipartite graph embedded on the plane, we have $Z_n(G, I_n) = Z_n(G)$. Given a graph embedded on the torus, does there exist connections such that $Z_n(G, \phi) = Z_n(G)$. (The identity matrix does not work). Given a graph embedded in some non-planar surface does there exist connections such that $Z_n(G, \phi) = Z_n(G)$? If not in general, are there situations in which we can accomplish this?

8. If we can solve Question 7, then can we extend the results to provide connections when overlaying tori so that the partition function is preserved?

APPENDIX

REPOSITORY FOR CODE USED

Here is the github repository[Boe26]: https://github.com/megboes/UO-Thesis.git

1. In Example 3.7 we use:

   https://github.com/megboes/UO-Thesis/blob/main/3_by_3_example.ipynb

2. In Example 3.7.1 we use:

   https://github.com/megboes/UO-Thesis/blob/main/3_by_1_example.ipynb

3. In Section 3.7.2 we use:

   https://github.com/megboes/UO-Thesis/blob/main/3_by_1_example_with_connections.ipynb

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