Homotopy Types of Spaces of 2-dimensional Functorial Field Theories

by

Jacob Lebovic

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Dissertation Committee:
Dev Sinha, Chair
Boris Botvinnik, Core Member
Daniel Dugger, Core Member
Victor Ostrik, Core Member
Richard Taylor, Institutional Representative

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DISSERTATION ABSTRACT

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This work studies homotopy types connected to classifying spaces for 2-dimensional functorial field theories valued in $(\infty, 2)$-categories built from the 2-category of Kapranov-Voevodsky 2-vector spaces $2\mathcal{V}$. First, we analyze theories valued directly in a smooth $(\infty, 2)$-enhancement of $2\mathcal{V}$. We find the corresponding classifying space has the homotopy type $\coprod_{n\geq 0} \mathrm{B}^2\mathrm{GL}_1(\mathbb{C})^{\times n}//S_n$, which after group completion produces the suspension spectrum $\mathbb{S}[\mathrm{B}^2\mathrm{GL}_1(\mathbb{C})]$. We also establish a differential model identifying these FFTs with unordered tuples of bundle gerbes with fiberwise connections. Motivated by Baas-Dundas-Rognes' work relating KV 2-vector spaces to iterated K-theory, we introduce an $(\infty, 2)$-category $L_{\mathbb{R}}2\mathcal{DSV}$ based on concordance classes of chain complexes. The core ∞-groupoid $L_{\mathbb{R}}2\mathcal{DSV}^{\times}$ serves as an ∞-groupoid completion of KV 2-vector spaces with "weakly invertible" morphisms. Computing the classifying space for FFTs valued in $L_{\mathbb{R}}2\mathcal{DSV}$, we show its additive group completion recovers the iterated K-theory spectrum $K(ku)$.

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CHAPTER 1

INTRODUCTION

1.1 A Brief Overview of Past Work

Topological Field Theories (TFTs) emerged in the late 1980s from mathematical physics. Work by Atiyah [Ati88] and Segal established axioms that formalized the definition of a TFT. Those axioms defined an n-dimensional TFT as a symmetric monoidal functor:

$$Z : \mathrm{Cob}_n \longrightarrow \mathrm{Vect}_{\mathbb{C}}.$$

The source category, Cob_n, is the category of n-dimensional oriented cobordisms. Its objects are closed, oriented, smooth $(n-1)$-dimensional manifolds Σ. A morphism $M : \Sigma_0 \to \Sigma_1$ in Cob_n is an n-dimensional oriented, compact, smooth manifold M with boundary $\partial M \cong \overline{\Sigma_0} \sqcup \Sigma_1$ (via an orientation-preserving diffeomorphism) where $\overline{\Sigma_0}$ is Σ_0 with reversed orientation. Composition $M_2 \circ M_1$ is defined by gluing $M_1 : \Sigma_0 \to \Sigma_1$ and $M_2 : \Sigma_1 \to \Sigma_2$ along Σ_1. The smooth structure on the glued manifold requires choices (e.g., collars), but the resulting bordism's diffeomorphism class is unaffected. The identity morphism id_Σ for an object Σ is the cylinder $\Sigma \times [0,1]$. Cob_n is a symmetric monoidal category; the monoidal product $\otimes$ is disjoint union $\sqcup$, and the unit object $\mathbf{1}_{\mathrm{Cob}_n}$ is the empty $(n-1)$-manifold $\emptyset_{n-1}$.

The target category, $\mathrm{Vect}_{\mathbb{C}}$, consists of finite-dimensional complex vector spaces with $\mathbb{C}$-linear maps as morphisms. Its symmetric monoidal structure is the standard tensor product $\otimes_{\mathbb{C}}$, with unit object $\mathbf{1}_{\mathrm{Vect}_{\mathbb{C}}} = \mathbb{C}$.

The functor Z must be symmetric monoidal, implying $Z(\Sigma_1 \sqcup \Sigma_2) \cong Z(\Sigma_1) \otimes_{\mathbb{C}} Z(\Sigma_2)$ and $Z(\emptyset_{n-1}) \cong \mathbb{C}$, along with functoriality $Z(M_1 \circ M_2) = Z(M_1) \circ Z(M_2)$ and $Z(\mathrm{id}_\Sigma) = \mathrm{id}_{Z(\Sigma)}$.

An important motivating example is that one dimensional TFTs are determined by their value on a point, hence correspond to finite dimensional vector spaces.

The Atiyah-Segal axioms for TFTs describe assignments to n-manifolds and $(n-1)$-manifolds. The physical principle of locality, however, suggests that a field theory should be decomposable more finely, assigning algebraic data to manifolds of all codimensions, down to points. The classical definition does not satisfy this.

Baez and Dolan, in [BD95], posited that an n-times extended TFT would take values in an n-category. An n-category possesses objects (0-morphisms), 1-morphisms, 2-morphisms, up to n-morphisms. Then a fully extended n-dimensional TFT is a symmetric monoidal functor from an n-category of bordisms (where objects are points, 1-morphisms are 1-dim bordisms, 2-morphisms are 2-dimensional bordisms, etc.) to a target symmetric monoidal n-category. They conjectured the "Cobordism Hypothesis," that such theories are determined by their value on a point.

One issue with this idea is that traditional n-categories are very hard to work with for any large value of n. So proving a general statement in this framework would be very difficult.

Lurie [Lur09] provided a precise mathematical formulation and proof sketch of the Cobordism Hypothesis using the framework of (∞, n)-categories. An (∞, n)-category is a variation of an n-category where all conditions have been relaxed to only hold up to coherent homotopy. For large n, (∞, n)-categories are much easier to work with than traditional n-categories as all the higher coherences can be captured by existing homotopy theoretic framework.

Lurie models (∞, n)-categories explicitly using n-fold complete Segal spaces, extending Rezk's theory for $(\infty, 1)$-categories [Rez01].

Let us recall that a simplicial space $X_\bullet$ is a functor $X : \Delta^{op} \to \mathrm{sSet}$, from the simplex category Δ to the category of spaces (here identified with the category of simplicial sets). In the ∞-category interpretation, we interpret $X_k = X([k])$ as the space of "k-composable sequences of 1-morphisms."

Definition 1.1.1. We say $X_\bullet$ is a Segal space if for $k \geq 0$, $X_k \to X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1$ (homotopy fiber product) is a weak homotopy equivalence. A Segal space $X_\bullet$ is complete if $\delta : X_0 \to X_1^{\simeq}$ (where $X_1^{\simeq}$ is the subspace of invertible 1-morphisms) is a weak homotopy equivalence.

An n-fold simplicial space is a functor $X : (\Delta^{op})^{\times n} \to \mathrm{sSet}$. Then we can define:

Definition 1.1.2. An n-uple Segal space is an n-fold simplicial space $X = X_{\bullet,\ldots,\bullet}$ such that for every $1 \leq i \leq n$, and every $k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n \geq 0$,

$$X_{k_1,\ldots,k_{i-1},\bullet,k_{i+1},\ldots,k_n}$$

is a Segal space.

This not quite the definition we want. For $n = 2$, this would be the ∞-analogue of a double category as opposed to a 2-category. To get the latter, we need to consider an additional condition. We say an n-fold simplicial space $X_{\bullet,\ldots,\bullet}$ is *essentially* *constant* (or Globular) if the map from the constant n-uple simplicial space $X_{0,\dots,0}$ given by the degeneracy maps

$$X_{0,\dots,0} \longrightarrow X$$

is a weak equivalence of n-fold simplicial spaces.

Definition 1.1.3. An n-fold Segal space is an n-uple Segal space $X = X_{\bullet,\dots,\bullet}$ such that for every $1 \leq i \leq n$, and every $k_1, \dots, k_{i-1} \geq 0$, the $(n-i)$-uple simplicial space

$$X_{k_1,\dots,k_{i-1},0,\bullet,\dots,\bullet}$$

is essentially constant.

Lurie presents a definition of an (∞, n)-category of Bordisms using the formalism of n-fold complete Segal spaces. In this definition, the notion of bordism is expressed by cutting up a manifold in different simplicial directions at sets of prescribed regular values.

Definition 1.1.4. Let V be a vector space. For every n-tuple $k_1, \dots, k_n$ of nonnegative integers, we let $(\mathrm{PreBord}^V)_{k_1,\dots,k_n}$ denote the collection of tuples $(M, \{t_0^1 \leq \cdots \leq t_{k_1}^1\}, \dots, \{t_0^n \leq \cdots \leq t_{k_n}^n\})$ where

1. M is a closed submanifold of $V \times \mathbb{R}^n$ of dimension n (not necessarily compact).
2. The projection $M \to \mathbb{R}^n$ is a proper map.
3. For every subset $S \subseteq \{1, \dots, n\}$ and every collection of integers $\{0 \leq j_i \leq k_i\}_{i \in S}$, the projection $M \to \mathbb{R}^S$ does not have $(t_{j_i})_{i \in S}$ as a critical value.
4. The projection map $M \to \mathbb{R}^{\{i+1,\dots,n\}}$ is subversive at every point $x \in M$ whose image in $\mathbb{R}^{\{i\}}$ belongs to the subset $\{t_{i0}, t_{i1}, \dots, t_{ik_i}\}$.

The set $(\mathrm{PreBord}^V)_{k_1,\dots,k_n}$ can be endowed with a topology, in particular as a subspace of an embedding space, so that $\mathrm{PreBord}_n$ becomes an n-fold simplicial space. We let $\mathrm{PreBord}_n$ denote the direct limit, $\mathrm{colim}_V \mathrm{PreBord}_n^V$, as V ranges over the finite dimensional subspaces of $\mathbb{R}^\infty$.

Remark 1.1.1. *Lurie claims this n-fold simplicial space is an n-fold Segal space. However that is not quite true. There is some technical subtlety. The subtlety is discussed and fixed in Section 6.3 of [CS19].*

Definition 1.1.5. The (∞, n)-category of bordisms Bord_n is defined as the completion, in the sense of [Rez01] Section 14, of the n-fold Segal space $\mathrm{PreBord_n}$. By imposing framings on the manifolds in the above definition, we can also defined a framed bordism category Bord_n^{fr}. A fully extended n-dimensional (framed) TFT is a symmetric monoidal functor:

$$Z : \mathrm{Bord}_n^{fr} \longrightarrow \mathcal{C}$$

where the target, $\mathcal{C}$, is a symmetric monoidal (∞, n)-category.

The Cobordism Hypothesis states that if $\mathcal{C}$ is a symmetric monoidal (∞, n)-category with duals, evaluation $Z \mapsto Z(*)$ induces an equivalence:

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \xrightarrow{\simeq} \mathcal{C}^{\times}$$

between the space of symmetric monoidal functors and $\mathcal{C}^{\times}$, the maximal sub-∞-groupoid of $\mathcal{C}$.

We note Lurie does not explicitly define what symmetric monoidal actually means in the case of (∞, n)-categories. This was done later in Section 3 of [CS19], describing symmetric monoidal (∞, n)-categories as Γ-objects in n-fold complete Segal spaces. We will discuss this in the next chapter in a more general context.

While Lurie's framework provides a comprehensive classification of *topological* field theories, many theories relevant to physics and geometry depend explicitly on geometric structures, such as Riemannian metrics. Stolz and Teichner [ST11] have developed a program to study such geometric Functorial Field Theories (FFTs) with a particular emphasis on connecting them to generalized cohomology theories.

The key aspect of the Stolz-Teichner approach is the use of smooth families of geometric objects. This is achieved by using fibered categories over the category of smooth manifolds Man.

Recall that a functor $p : \mathrm{B} \to \mathrm{S}$ between categories is a Grothendieck fibration if for every object $Z \in \mathrm{B}$ and every morphism $f : S' \to S = p(Z)$ in the base category S, there exists a cartesian arrow $\phi : Y \to Z$ in B lying over f (i.e., $p(\phi) = f$). A Cartesian arrow ϕ satisfies a universal property for lifting other maps into Z that factor through f in the base. A category B equipped with such a fibration $p : \mathrm{B} \to \mathrm{S}$ is called a category fibered over S. If one makes a consistent choice of Cartesian liftings (a cleavage), this structure is equivalent to that of a pseudo-functor $\mathrm{S}^{op} \to \mathrm{Cat}$.

In the Stolz-Teichner setting:

- They define categories of bordisms Bord_d that are fibered over Man. An object in the fiber $(\mathrm{Bord}_d)_S$ over a manifold $S \in \mathrm{Man}$ is a smooth family $Y \to S$ of $(d-1)$-manifolds,

- They also incorporate geometric structures. For instance they use $(G, \mathbb{M})$-structures, in the sense of Thurston. In the bordism cateogry they are implemented in such a way that each fiber Y_s is equipped with a $(G, \mathbb{M})$-structure, and this structure varies smoothly with $s \in S$.

  We do not use this framework explicitly, just for motivation. Thus we refer the reader to Definition 2.46 in [ST11] for much more detail.

This use of fibered categories provides a rigorous framework for ensuring that all components of a field theory vary smoothly with respect to any geometric structure. A “field theory over X” (where $X \in \mathrm{Man}$) corresponds to taking X as the base for these fibered categories. Stolz and Teichner illustrate that 0-dimensional FFTs over X are equivalent to smooth functions on X, and 1-dimensional FFTs over X correspond to vector bundles with connections over X.

Grady and Pavlov [GP22] combine the ideas of Lurie and Stolz-Teichner. They create a theory that incorporates both the (∞, d)-categorical framework of Lurie and a framework of geometric structures and smooth families in a manner aligned with the ideas of Stolz-Teichner. They define a notion of smooth functorial field theories and claim to prove a statement analogous to the cobordism hypothesis for these theories. We outline their work in detail in the next chapter.

1.2 Structure of This Work

We outline the structure of this thesis.

Chapter 2: Background: Functorial Field Theories This chapter lays the essential mathematical groundwork from [GP22] and [GP23]. It details the construction of smooth symmetric monoidal (∞, d)-categories, the theory of duals, the definition of geometric structures $\mathcal{S}$ via presheaves on FEmb_d, the bordism (∞, d)-category $\mathfrak{Bord}_d^{\mathcal{S}}$, and the statement and implications of the Geometric Cobordism Hypothesis. The chapter concludes with the 1-dimensional example of FFTs valued in smooth ∞-categorical enhancement of vector spaces.

Chapter 3: 2-Dimensional Field Theories Valued in KV 2-Vector Spaces

This chapter develops a smooth $(\infty, 2)$-category $N^M(2\mathcal{V})$ derived from Kapranov-Voevodsky 2-vector spaces. It then analyzes homotopy type of the core ∞-groupoid of this construction. Then applies this to the computation of a classifying space $\mathrm{BFFT}^{\times}_{2,N^M(2\mathcal{V})}$ for 2-dimensional FFTs. A differential model $2\mathcal{V}^{\times}_{\nabla}$ is also developed.

Chapter 4: ∞-Groupoid Completion of KV 2-Vector Spaces In effort to extract more from information from the 2-category of 2-vector spaces than just what is contained in the core, this chapter constructs an alternative target $(\infty, 2)$-category $L_{\mathbb{R}}2\mathcal{DSV}$ via a concordance construction applied to a 2-category derived from chain complexes, closely related to KV 2-vector spaces. The classifying space $\mathrm{BFFT}^{\times}_{2,L_{\mathbb{R}}2\mathcal{DSV}}$ is computed, and its additive group completion is identified as a field-theoretic model for the iterated K-theory spectrum $K(ku)$.

CHAPTER 2

BACKGROUND: FUNCTORIAL FIELD THEORIES

This chapter provides an overview of some of the main definitions and theorems presented in the papers "Extended field theories are local and have classifying spaces [GP23]" and "The Geometric Cobordism Hypothesis [GP22]" by Grady-Pavlov. The work extends the classical cobordism hypothesis to encompass bordisms equipped with arbitrary geometric structures, utilizing the language of smooth (∞, d)-categories. We claim no originality for anything in this chapter.

2.1 Smooth Symmetric Monoidal (∞, d)-Categories with Duals

The core algebraic structure used is that of smooth symmetric monoidal (∞, d)-categories with duals, realized model-categorically using simplicial presheaves. This is an amalgamation of the ∞-categorical approach of Lurie and the smooth approach of Stolz-Teichner.

Basic Categories and Presheaves

The construction relies on presheaves over a product category involving:

- Δ: The category of finite non-empty ordered sets and order-preserving maps (the simplex category). Its d-fold product $\Delta^{\times d}$ has objects called multisimplices, denoted $\mathbf{m} = ([m_1], \ldots, [m_d])$. This models the (∞, d)-categorical aspect as discussed earlier in Lurie's framework.

- Γ: Segal's category, the opposite of the category of finite pointed sets $\{*, 1, \ldots, n\}$ $(n \geq 0)$. This category encodes the symmetric monoidal structure, modeling a symmetric monoidal (∞, d)-category as an E_∞-object in (∞, d)-categories.

- Cart: The category whose objects are open subsets $U \subset \mathbf{R}^n$ (for any $n \geq 0$) that are diffeomorphic to $\mathbf{R}^n$. Morphisms are smooth maps $f : U \to V$. This category captures the notion of smooth families like in Stolz-Teichner's framework.

The underlying category for the model structures is the category of simplicial presheaves $\mathrm{sPSh}(\mathrm{Cart} \times \Gamma \times \Delta^{\times d})$, which is the functor category $\mathrm{Fun}((\mathrm{Cart} \times \Gamma \times \Delta^{\times d})^{\mathrm{op}}, \mathrm{sSet})$.

Smooth symmetric monoidal (∞, d)-categories should be defined as smooth sheaves of E_∞-objects in d-fold complete Segal spaces. We express this below by defining them to be fibrant objects in a specific model structure obtained by left Bousfield localization of the standard injective model structure on simplicial presheaves, where cofibrations and weak equivalences are defined objectwise, at classes of maps representing each of these conditions.

Definition 2.1.1 (Multiple Injective Model Structure)**.** The multiple injective model structure, [GP23] def. 2.3.2, denoted $\mathrm{C}^\infty\mathrm{Cat}_{\infty,d}^{\otimes,\mathrm{uple}}$, is obtained by localizing the injective model structure on $\mathrm{sPSh}(\mathrm{Cart} \times \Gamma \times \Delta^{\times d})$ at maps that enforce:

- Segal conditions for the $\Delta^{\times d}$ and Γ factors (i.e. Conditions representing the higher composition and symmetric monoidal structures) [GP23] 2.2.4, 2.2.6, 2.2.7
- Completeness condition for the $\Delta^{\times d}$ factor (i.e. Condition requiring the space of objects to be weakly equivalent to the space of invertible 1-morphisms) [GP23] 2.2.5
- Descent (i.e sheaf) conditions for the Cart factor with respect to good open covers [GP23] 2.2.9.

Fibrant objects in the multiple model structure represent (∞, d)-categories where the d composition directions are treated on an equal footing. Further, we define the globular injective model structure, $\mathrm{C}^\infty\mathrm{Cat}_{\infty,d}^{\otimes,\mathrm{glob}}$, [GP23] def. 2.3.7, results from a further localization of $\mathrm{C}^\infty\mathrm{Cat}_{\infty,d}^{\otimes,\mathrm{uple}}$. The globularity condition relates objects across different simplicial levels; for instance, in the $d = 2$ case, it forces degeneracy maps $X_{0,0} \to X_{0,k}$ to be weak equivalences, making "vertical" 1-morphisms invertible, essentially taking us from the ∞-analogue of double categories to the ∞-analogue of 2-categories. A smooth symmetric monoidal (∞, d)-category is defined as a fibrant object in $\mathrm{C}^\infty\mathrm{Cat}_{\infty,d}^{\otimes,\mathrm{glob}}$.

Duals

In order to discuss field theories, we also need to define a notion of dualizability in our (∞, d)-categories. Recall that in a monoidal category, an object X is dualizable if it possesses a “dual object” $X^\vee$, along with an evaluation morphism $\mathrm{ev} : X^\vee \otimes X \to I$ and a coevaluation morphism $\mathrm{coev} : I \to X \otimes X^\vee$ that satisfy the triangle identities. In a monoidal 2-category, an object X is called fully dualizable if it is first 1-dualizable, meaning it has a dual $X^\vee$ and evaluation and coevaluation 1-morphisms ev and coev whose identities hold up to 2-isomorphisms, and then also these 1-morphisms have adjoints. This is generalized to (∞, d)-categories in the style of Lurie. First let’s make the following definition:

Definition 2.1.2 (Homotopy 2-Category)**.** For a d-fold complete Segal space X, the homotopy 2-category $\mathrm{Ho}_2 X$ has objects $X(\mathbf{0})$. For objects x, y, the hom-category $\mathrm{Ho}_2 X(x, y)$ is the homotopy category of the 1-fold Segal space

$$\{x\} \times_{X(\mathbf{0})} X(\mathbf{1}) \times_{X(\mathbf{0})} \{y\}.$$

We can use the above definition to define our desired notion of duals.

Definition 2.1.3 (Adjoints and Duals)**.** We say:

- An (∞, k)-category $\mathcal{C}$ (for $k \geq 2$) admits adjoints for 1-morphisms if every 1-morphism $f : x \to y$ in $\mathrm{Ho}_2\mathcal{C}$ is a left adjoint.
- A smooth symmetric monoidal (∞, d)-category $\mathcal{C}$ has adjoints for k-morphisms, $1 \leq k < d$, if for every $(U, \mathbf{m}) \in \mathrm{Cart} \times \Delta^{\times k}$ the $(\infty, d - k)$-category $\mathcal{C}(U, \langle 1\rangle, \mathbf{m})$ has adjoints for 1-morphisms.
- $\mathcal{C}$ has duals for objects if every object in the monoidal 1-category $\mathrm{Ho}(\mathcal{C})$ has a dual.

Definition 2.1.4 (Model Structure with Duals)**.** $\mathrm{C}^\infty\mathrm{Cat}^{\otimes,\vee}_{\infty,d}$, [GP22] def. 2.3.12, is the left Bousfield localization of $\mathrm{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{glob}}_{\infty,d}$ at a certain set of maps that:

- Enforce existence of adjoints for k-morphisms. [GP22] 2.3.7
- Enforces duals for objects [GP22] 2.3.11.

A smooth symmetric monoidal (∞, d)-category with duals is a fibrant object in this model structure.

Our primary examples of smooth symmetric monoidal (∞, d)-categories will come from bordism categories, which we will define next, and from nerve constructions applied to smooth symmetric monoidal 1-categories and 2-categories, which will appear later in this chapter and in the subsequent chapters.

2.2 Bordism Categories

Like Lurie defined an (∞, d)-category of bordisms, Grady-Pavlov define a smooth (∞, d)-category of bordisms. The structure of the bordisms in this definition is captured somewhat simiarly to Lurie's definition. That is we cut up d-manifolds in different simplicial directions at certain prescribed regular values but, in this case, we do so working in smooth families similar to the ideas of Stolz-Teichner.

First we must define the following category of smooth families:

Definition 2.2.1 (FEmb_d)**.** The category FEmb_d is defined as follows:

- **Objects**: Smooth submersions $p : M \to U$, where U is an object in Cart and p has d-dimensional fibers. M is a smooth manifold.
- **Morphisms**: A morphism from $p : M \to U$ to $q : N \to V$ consists of a pair of smooth maps $(f : M \to N, h : U \to V)$ such that $q \circ f = h \circ p$, and for every $u \in U$, the restriction of f to the fiber $M_u := p^{-1}(u)$, denoted $f_u : M_u \to N_{h(u)}$, is an open embedding of smooth manifolds.

Remark 2.2.1. *This category can be made into a site by defining a covering family to be a family of morphisms* $\{\alpha_i : (M_i \to U_i) \to (M \to U)\}$ *such that the union of the images* $\cup_i f_i(M_i)$ *forms an open cover of* M*.*

We now develop the notion of cuts on smooth families, which are a key part of the notion of bordism in this setting. We start by defining the notion of a single cut, then we define the notion of a cut tuple and a cut grid. These structures are what encode the notion of a manifold being a bordism in this framework. We include these definitions for the sake of completeness, but encourage the reader to look directly at Section 4 of [GP23] where more detail is provided and some examples and visualizations are included.

Cuts

Definition 2.2.2 (Cut)**.** A cut of an object $p \colon M \to U$ in FEmb_d is a triple $(C_<, C_=, C_>)$ of subsets of M such that there exists a smooth map $h \colon M \to \mathbb{R}$, called the height function, whose fiberwise-regular values form an open neighborhood of $0 \in \mathbb{R}$, where we have:

- $C_< = h^{-1}(-\infty, 0)$
- $C_= = h^{-1}(0)$
- $C_> = h^{-1}(0, \infty)$

Additionally, we define $C_\leq = C_< \cup C_=$ and $C_\geq = C_> \cup C_=$. This lets us define an ordering on Cuts as follows: $C \leq C'$ if $C_\leq \subset C'_\leq$.

Cut Tuples

Definition 2.2.3 (Cut tuple)**.** For $d \geq 0$, a simplex $[m] \in \Delta$, and $p\colon M \to U \in \mathrm{FEmb}_d$, a cut $[m]$-tuple C is a collection of cuts $C_j = (C_{<j}, C_{=j}, C_{>j})$ indexed by $j \in [m]$, with $C_0 \leq C_1 \leq \cdots \leq C_m$. For $j \leq j'$ we also define regions:

- $C_{(j,j')} = C_{>j} \cap C_{<j'}$ (the open region between cut j and cut j')
- $C_{[j,j']} = C_{\geq j} \cap C_{\leq j'}$ (the closed region between cut j and cut j')

Cut Grids

Definition 2.2.4 (Cut $\mathbf{m}$-grid)**.** A cut $\mathbf{m}$-grid for an object $p\colon M \to U \in \mathrm{FEmb}_d$ and a multisimplex $\mathbf{m} = ([m_1], \ldots, [m_d]) \in \Delta^{\times d}$, consists of the following data:

- For each $i \in \{1, \ldots, d\}$, a cut $[m_i]$-tuple C^i on $p\colon M \to U$.

This collection of cut tuples must satisfy the transversality property ($\pitchfork$):

- For every subset $S \subset \{1, \ldots, d\}$ and for any function $j\colon S \to \mathbb{Z}$ (assigning an index $j_k \in [m_k]$ for each $k \in S$) such that $0 \leq j_k \leq m_k$ for all $k \in S$, there must exist a smooth map $h_j\colon M \to \mathbb{R}^S$. For each $k \in S$, the map $\pi_k \circ h_j\colon M \to \mathbb{R}$ (where π_k is the k-th projection from $\mathbb{R}^S$) must yield the j_k-th cut $C^k_{j_k}$ in the cut tuple C^k. Furthermore, the fiberwise-regular values of h_j must form an open neighborhood of $0 \in \mathbb{R}^S$.

For a cut $\mathbf{m}$-grid $C = (C^1, \ldots, C^d)$, a subset $S \subset \{1, \ldots, d\}$, and multi-indices j, j' assigning to each $i \in S$ integers j_i, j'_i with $0 \leq j_i \leq j'_i \leq m_i$, we define the region

$$C_{[j,j']_S} := \bigcap_{i \in S} C^i_{[j_i, j'_i]} \subset M$$

where $C^i_{[j_i,j'_i]}$ is the closed region for the i-th cut tuple. If $S = \{1, \dots, d\}$, we simply write $C_{[j,j']}$. The region $C_{(j,j')_S}$ is defined similarly using open regions.

A cut $\mathbf{m}$-grid C for $p\colon M \to U$ is compact if for $S = \{1, \dots, d\}$, the restriction $p|_{C_{[j,j']}}$ is proper for all relevant j, j'.

The Bordism Category Bord_d

Definition 2.2.5 (Bord_d)**.** For $(U, \langle \ell \rangle, \mathbf{m}) \in \mathrm{Cart} \times \Gamma \times \Delta^{\times d}$, the simplicial set $\mathrm{Bord}_d(U, \langle \ell \rangle, \mathbf{m})$ is the nerve of a category:

- Objects: A bordism (M, C, P) where:

  1. M is a d-manifold.
  2. C is a compact cut $\mathbf{m}$-grid for $\mathrm{proj}\colon M \times U \to U$.
  3. $P\colon M \times U \to \langle \ell \rangle$ is a map that partitions $M \times U$ into ℓ disjoint subsets.

  The core of the bordism (M, C, P) is $\mathrm{core}(M, C, P) = C_{[0,\mathbf{m}]} \setminus P^{-1}\{*\}$. Here $C_{[0,\mathbf{m}]}$ denotes the region $\bigcap_{i=1}^d C^i_{[0_i,m_i]}$, where C^i is the i-th cut tuple of the grid C, and $[0_i, m_i]$ refers to the region defined by the initial (0-th) and final (m_i-th) cuts of that tuple.

- Morphisms: $\varphi\colon (M, C, P) \to (M', C', P')$ is a cut-respecting embedding, i.e., $\varphi\colon (M \times U \to U) \to (M' \times U \to U)$ in FEmb_d covering id_U such that $\varphi^* C' = C$, $\varphi^* P' = P$, and $\mathrm{im}(\varphi) \supset \mathrm{core}(M', C', P')$.

- Presheaf Structure Maps:

  - For a map $f\colon U' \to U$ in Cart: objects (M, C, P) over U are pulled back to objects over U' via $M \times U'$, with C and P pulled back along $\mathrm{id}_M \times f$. Morphisms are pulled back similarly.
  - For a map $\alpha\colon \langle \ell \rangle \to \langle \ell' \rangle$ in Γ (corresponding to $g\colon \langle \ell' \rangle \to \langle \ell \rangle$ basepoint-preserving): an object (M, C, P) with $P\colon M \times U \to \langle \ell \rangle$ yields an object $(M, C, g \circ P)$ with $g \circ P\colon M \times U \to \langle \ell' \rangle$.
  - For a map $\sigma\colon \mathbf{m}' \to \mathbf{m}$ in $\Delta^{\times d}$: an object (M, C, P) with cut grid C for $\mathbf{m}$ yields an object $(M, \sigma^* C, P)$ where $\sigma^* C$ is the cut grid for $\mathbf{m}'$ obtained by applying the simplicial operations (removing/duplicating cuts) corresponding to σ to the cut tuples within C.

To account for potential size issues, all manifolds are assumed by Grady-Pavlov to be subsets of $\mathbb{R}$. This is a purely set-theoretic assumption. No compatibility of this with the topological/smooth structures is assumed.

Remark 2.2.2. *The above definition is not fibrant in the globular model structure. This can be fixed either formally by taking a fibrant replacement in the globular model structure, or explicitly by adding additional conditions to the definition of a cut grid as in Definition 4.2.7 of [GP23].*

2.3 Geometric Structures

Grady-Pavlov also introduce a notion of "geometric structure" and define variations of the the above bordism category which include these geometric structures. Their definition of a geometric structure is very general. We will state these definitions for completeness but note that we don't actually work with these general geometric structures in this paper.

Definition 2.3.1 (Geometric Structure)**.** A fiberwise d-dimensional geometric structure $\mathcal{S}$ is a simplicial presheaf on the site FEmb_d

Example 2.3.1. Representable presheaves are the most basic example. That is presheaves of the form $\mathcal{S}(-) = \mathrm{Map}_{\mathrm{FEmb}_d}(-, W \to U)$ for some $W \to U \in \mathrm{FEmb}_d$.

Example 2.3.2. Any smooth manifold M defines a presheaf on FEmb_d by $(W \mapsto U) \mapsto C^\infty_{fc}(W, M)$. The set of fiberwise constant smooth functions considered as a constant simplicial set.

For any geometric structure $\mathcal{S}$, Grady-Pavlov define a bordism category $\mathrm{Bord}_d^{\mathcal{S}}$. See [GP23] Definition 4.4.1. We will not repeat this definition here, but will note in the case of a representable geometric structure associated to an object $p : W \to U$, the bordism categories admit a nice description as categories of embedded bordisms.

Definition 2.3.2 (Embedded Bordism Category E_d^p)**.** Let $p : W \to U \in \mathrm{FEmb}_d$. The value $\mathrm{E}_d^p(V, \langle \ell \rangle, \mathbf{m})$ (for $V \in \mathrm{Cart}$) is the set of equivalence classes $[f, N, C, P]$ where:

1. $f : V \to U$ is a morphism in Cart.

2. $N \subset W \times_U V$ is an open submanifold.
3. C is a compact cut $\mathbf{m}$-grid for the projection $\pi : N \to V$.
4. $P : N \to \langle \ell \rangle$ is a smooth map.

Equivalence $(f, N, C, P) \sim (f', N', C', P')$ means $f = f'$ and there exists an open submanifold $N'' \subset N \cap N'$ containing the cores such that C, C' restrict to the same compact globular cut grid C'' on $N'' \to V$, and P, P' restrict to the same P'' on N''. i.e. We are taking "germs."

Remark 2.3.1. *Again this definition is not globular. This can be fixed in the same ways discussed previously.*

The elements in $\mathrm{E}^p_d(V, \langle \ell \rangle, \mathbf{m})$ should be thought of as V -families of (germs of) cores embedded in W. One can image a family of diced cubes sitting in W with each diced cube having a neighborhood in which the cuts extend.

These embedded bordism categories will be taken as our definition for the bordism categories $\mathrm{Bord}_d^{W \to U}$ associated to representable geometric structures $W \to U$. We will define $\mathrm{Bord}_d^{p: W \to U} := \mathrm{E}^p_d$. One of the main theorems of [GP23] (prop 5.5.1) is the statement that the assignment $\mathcal{S} \mapsto \mathrm{Bord}_d^{\mathcal{S}}$ is homotopy cocontinuous. As any geometric structure $\mathcal{S}$ can written as a homotopy colimit of representables, we could use this homotopy cocontinutiy theorem, combined with the above embedded bordism category definition, as a definition for $\mathrm{Bord}_d^{\mathcal{S}}$.

The 1-dimensional case $\mathrm{Bord}_1^{W \to U}$ is written out explicitly in example 4.5.8 of [GP23]. We reference this so the reader knows where to find an example if they would like to see one. We do not repeat it here as it is best expressed alongside pictures, which are already well drawn in [GP23].

Remark about isotopies

In all of the above discussion we omitted one technical aspect of defining these smooth bordism categories. That is, along with all the other technical components we discussed, we also need to implement isotopies between the bordisms. The presence of isotopies is crucial for the inductive argument in the proof of the cobordism hypothesis to work. Embedded cylinders must be homotopic to identities. Any homotopy deforming a d-dimensional cylinder to a degenerate cylinder through embedding into $\mathbb{R}^d$ necessarily has rank $\leq d$, hence is an isotopy [GP22][Remark 3.2.2].

Isotopies are implemented through a second parametrization of everything over Cart. So, for instance, instead of a tuple (M, C, P), we will have family of tuples (M, C, P) over some base L as L varies. For the sake of not giving even more technical definitions in this section, we will just refer to Section 4.5 of [GP23] for this additional enhancement. Bordism categories with isotopies are denoted as $\mathfrak{Bord}$ as opposed to Bord. In practice, we don't explicitly work with the Bordism category in this paper, only implicitly through the Cobordism Hypothesis, so we suppress these details.

2.4 Functorial Field Theories and the Cobordism Hypothesis

A d-dimensional functorial field theory with geometric structure $\mathcal{S}$ is a smooth symmetric monoidal (∞, d)-functor

$$Z : \mathfrak{Bord}_d^{\mathcal{S}} \to \mathcal{C}$$

for some target smooth symmetric monoidal (∞, d)-category $\mathcal{C}$. If the geometric structure $\mathcal{S}$ is representable, that is $\mathcal{S} = p : W \to U$, we call these framed functorial field theories.

More explicitly,

Definition 2.4.1. The space of d-dimensional functorial field theories with geometric structure $\mathcal{S}$, valued in $\mathcal{C}$,

$$\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_d^{\mathcal{S}}, \mathcal{C})$$

is defined to be the derived mapping simplicial set $\mathbf{R}\mathrm{Map}(\mathfrak{Bord}_d^{\mathcal{S}}, \mathcal{C})$ in $\mathrm{C}^{\infty}\mathrm{Cat}_{\infty,d}^{\otimes,\mathrm{glob}}$. Then a d-dimensional functorial field theory is a vertex in this simplicial set.

Remark: *The above derived mapping simplicial set can be refined to a derived internal hom. However for our purposes it sufficient to think about this simpler case.*

Before we can state the Cobordism Hypothesis, we do need to make a couple more technical definitions in order to define the *core*, i.e. maximal sub-∞-groupoid, of an (∞, d)-category explicitly.

Definition 2.4.2 (Evaluation ev_S and Constant c_S Functors)**.** For $S \subset \{1, \dots, d\}$, $ev_S : \mathrm{sPSh}(\cdots \times \Delta^{\times d}) \to \mathrm{sPSh}(\cdots \times \Delta^{\{1,\dots,d\}\setminus S})$ evaluates at the zero multisimplex $\mathbf{0}_S$ in directions S. Its left adjoint c_S creates a presheaf constant along the directions in S. This forms a Quillen adjunction $c_S \dashv ev_S$ between the respective model structures.

Definition 2.4.3 (Core $\mathcal{C}^\times$)**.** The core $\mathcal{C}^\times$ of a fibrant $\mathcal{C} \in \mathrm{C}^\infty\mathrm{Cat}_{\infty,d}$ is the derived functor $\mathbb{R}ev_{\{1,\dots,d\}}(\mathcal{C})$. What this construction is doing is just taking the maximal sub ∞-groupoid of your (∞, d)-category. i.e. Only considering the invertible morphisms in your (∞, d)-category.

Now for the main statement:

Theorem 2.4.1. *Geometric Cobordism Hypothesis*

Fix $d \geq 0$ *and let* $\mathcal{C}$ *be a smooth symmetric monoidal* (∞, d)*-category with duals.*

- *We have a weak equivalence of simplicial sets* $\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_d^{\mathbb{R}^d \times U \to U}, \mathcal{C}) \simeq \mathcal{C}^\times(U)$ *for each fixed* $\mathbb{R}^d \times U \to U$*.*
- *Let* $\mathcal{S} \in \mathrm{sPSh}(\mathrm{FEmb}_d)$ *and define a simplicial presheaf* $\mathcal{C}_d^\times$ *on* FEmb_d *by*

  $$\mathcal{C}_d^\times(\mathbb{R}^d \times U \to U) := \mathrm{Fun}^{\otimes}(\mathfrak{Bord}_d^{\mathbb{R}^d \times U \to U}, \mathcal{C}).$$

  Then we have a weak equivalence of simplicial sets

  $$\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_d^{\mathcal{S}}, \mathcal{C}) \simeq \mathrm{Map}_{\mathrm{FEmb}_d}(\mathcal{S}, \mathcal{C}_d^\times).$$

Along with the main theorem, Grady-Pavlov also provide a corollary for computing the spaces $\mathcal{C}_d^\times$.

Corollary 2.4.1.1. *Suppose* $\mathcal{F} \in \mathrm{sPSh}(\mathrm{FEmb}_d)$ *is a simplicial presheaf that satisfies descent on* FEmb_d *and* $f : \mathcal{F} \to \mathcal{C}_d^\times$ *is a morphism of simplicial presheaves. Then* f *is a local weak equivalence if and only if for any* $\mathbb{R}^d \times U \to U \in \mathrm{FEmb}_d$*, the composition* $\mathcal{F}(\mathbb{R}^d \times U \to U) \to \mathcal{C}_d^\times(\mathbb{R}^d \times U \to U) \simeq \mathcal{C}^\times(U)$ *is a local weak equivalence of simplicial presheaves.*

Then they also provide a construction for classifying spaces for concordance classes of such field theories. Two field theories $Z_0, Z_1 : \mathfrak{Bord}_d^M \to \mathcal{C}$ are concordant if there exists a field theory $Z : \mathfrak{Bord}_d^{M \times [0,1]} \to \mathcal{C}$ that restricts to Z_0 and Z_1 at the endpoints of the interval.

Theorem 2.4.2. *We define a classifying space*

$$\mathrm{BFFT}^{\times}_{d,\mathcal{C}} := \operatorname*{hocolim}_{[n]\in\Delta^{op}} \mathrm{Fun}^{\otimes}(\mathfrak{Bord}_d^{\Delta^n}, \mathcal{C})$$

such that if M is a smooth manifold, then the set of homotopy classes of maps $[M, \mathrm{BFFT}^{\times}_{d,\mathcal{C}}]$ is in bijection with the set of concordance classes of field theories $\mathfrak{Bord}_d^M \to \mathcal{C}$.

2.5 One dimensional Field Theories in Vector Spaces

As a motivating example, will use the Geometric Cobordism Hypothesis to compute the classifying space $\mathrm{BFFT}^{\times}_{1,N^R(\mathcal{V})}$ where $N^R(\mathcal{V})$ is a smooth $(\infty,1)$-category of vector spaces to be constructed shortly.

Smooth $(\infty,1)$*-Category of Vector Spaces*

Define a smooth presheaf of categories $\mathcal{V} : \mathrm{Cart}^{op} \to \mathrm{Cat}$ by setting $\mathcal{V}(U)$ to be the category of smooth complex vector bundles over U. We immediately observe that since $U \in \mathrm{Cart}$, all vector bundles over U are trivial. Thus $\mathcal{V}(U)$ is equivalent to the skeletal category with objects non-negative numbers $n \in \mathbb{N}$ and $\mathrm{Hom}(n,m) = C^{\infty}(U, M_{m\times n}(\mathbb{C}))$. We note $\mathcal{V}$ has a bimonoidal structure and duals induced from the usual tensor product, direct sum, and duals of vector spaces. We also note that $\mathcal{V}$ is in fact a sheaf on Cart, it satifies descent because the groups of smooth functions satisfy descent.

Now we need to promote this smooth category to a smooth $(\infty,1)$-category, where we are modeling $(\infty,1)$-categories as complete Segal spaces. For this we need to recall the Rezk nerve of a category.

Definition 2.5.1. Let $\mathcal{C}$ be an ordinary category. Define the Rezk Nerve, a bisimplicial space,

$$N^R : \mathrm{Cat} \to \mathrm{Fun}(\Delta^{op}, \mathrm{sSet})$$

by

$$\mathcal{C} \mapsto ([\ell] \mapsto N(\mathrm{Fun}([\ell], \mathcal{C})^{\times})$$

where N is the ordinary nerve construction, the poset $[\ell]$ is being interpreted as a category, and $\mathrm{Fun}([\ell], \mathcal{C})^{\times}$ is the groupoid of functors $[\ell] \to \mathcal{C}$ and natural isomorphisms. By a theorem of Rezk, this bisimplicial set is a complete Segal space

[Rez01]. We note that when $\mathcal{C}$ is a groupoid, $N^R(\mathcal{C})$ is equivalent to (a fibrant replacement of) the constant simplicial space on $N(\mathcal{C})$.

We can then define the smooth $(\infty, 1)$-category of vector spaces to be the Rezk nerve $N^R(\mathcal{V})$ of the smooth category we defined above. The symmetric monoidal structure on $\mathcal{V}$ also induces the structure of a Γ-object on $N^R(\mathcal{V})$ in the obvious way, so $N^R(\mathcal{V})$ is a smooth symmetric monoidal $(\infty, 1)$-category.

Field Theories

We want to compute the space of field theories $\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_1^-, N^R(\mathcal{V}))$ as a presheaf on FEmb_1. By Corollary 2.4.1.1, to do this it is enough to construct presheaf $\mathcal{F}$ on FEmb_1 and a map $\mathcal{F} \to \mathrm{Fun}^{\otimes}(\mathfrak{Bord}_1^-, N^R(\mathcal{V}))$ such that for each $\mathbb{R} \times U \to U$ composition of this map with the geometric cobordism hypothesis equivalence

$$\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_1^{\mathbb{R}\times U \to U}, N^R(\mathcal{V})) \simeq \mathcal{V}^{\times}(U)$$

gives a weak equivalence. There a couple ways we could do this. The easiest way would be to just take $\mathcal{F} = N^R(\mathcal{V}_{fc}^{\times})$, where $\mathcal{V}_{fc}^{\times}(\mathbb{R} \times U \to U)$ is the groupoid with objects non-negative integers $n \in \mathbb{N}$, and $\mathrm{Aut(n)} = C^{\infty}_{fc}(\mathbb{R} \times U, \mathrm{GL}_n(\mathbb{C}))$ is the group of smooth functions $\mathbb{R} \times U \to \mathrm{GL}_n(\mathbb{C})$ that are fiberwise constant with respect to $\mathbb{R} \times U \to U$. There is an obvious map

$$N^R(\mathcal{V}_{fc}^{\times}) \to \mathrm{Fun}^{\otimes}(\mathfrak{Bord}_1^-, N^R(\mathcal{V}))$$

induced by the cobordism hypothesis equivalence that will give the desired weak equivalence since fiberwise constant smooth functions on $\mathbb{R} \times U$ are the same thing as just smooth functions on U.

However there is also a more geometrically enlightening construction we can do. We can define a presheaf $\mathcal{V}_{\nabla}$ of "fiberwise vector bundles with connection" and then define a map $N^R(\mathcal{V}_{\nabla}) \to \mathrm{Fun}^{\otimes}(\mathfrak{Bord}_1^-, N^R(\mathcal{V}))$ that sends a fiberwise vector bundle with connection to its associated parallel transport functor. The equivalence of vector bundles with connection and smooth 1-dimensional field theories is an established result of Berwick-Evans and Pavlov [BP23]. This particular fiberwise construction is presented in the main geometric cobordism hypothesis paper by Grady and Pavlov.

Definition 2.5.2. We define a presheaf of categories $\mathcal{V}_{\nabla}$ on FEmb_1 by defining $\mathcal{V}_{\nabla}(T \to U)$ as follows:

- Define the objects of our category to be pairs (n, A) where $n \in \mathbb{N}$ and $A \in \Omega^1_U(T; \mathfrak{gl}_n(\mathbb{C}))$. Where $\Omega^k_U(T; -)$ means differential forms on T that are vertical with respect to the $T \to U$.
- Morphisms $(n, A) \to (n, A')$ are a choice of $g \in C^\infty(T; GL_n(\mathbb{C}))$ such that $A' = gAg^{-1} + g^{-1}dg$. i.e. They are gauge transformations.

The map $N^R(\mathcal{V}_\nabla) \to \mathrm{Fun}^\otimes(\mathfrak{Bord}_1^-, N^R(\mathcal{V}))$ is given by sending the data $A \in \Omega^1_U(T; \mathfrak{gl}_n(\mathbb{C}))$ to the field theory $Z : \mathfrak{Bord}_1^{\mathbb{R}\times U \to U} \to N^R(\mathcal{V})$ induced by parallel transport of the fiberwise 1-form A along 1-manifolds γ, that is roughly $Z(*) = n$ and $Z(\gamma) = \mathcal{P}\exp(\int_\gamma A)$. A detailed description is the purpose of the paper [BP23]. One might wonder why $N^R(\mathcal{V}_\nabla)$ would be weakly equivalent to $N^R(\mathcal{V}^\times_{fc})$, as usually we would not have an equivalence between a category of smooth vector bundles and a category of smooth vector bundles with connection. The key point is that in this case we are working with fiberwise connections on bundles defined over a family of 1-manifolds. Any connection on a 1-manifold will always be flat, and we are also working with trivial families over a contractible base so there is no non-trivial topology involved. So in this case we can get an equivalence of fiberwise connections and just fiberwise constant functions.

We will now go ahead and compute the classifying space $\mathrm{BFFT}^\times_{1,N^R(\mathcal{V})}$. By definition this is

$$\operatorname*{hocolim}_{[k]\in\Delta^{op}} \mathrm{Fun}^\otimes(\mathfrak{Bord}_1^{\Delta^k}, N^R(\mathcal{V})).$$

Following the geometric cobordism hypothesis and our above discussion we can identify this with

$$\operatorname*{hocolim}_{[k]\in\Delta^{op}} \mathrm{Map}_{\mathrm{FEmb}_1}(\Delta^k, N^R(\mathcal{V}^\times_{fc})),$$

which can be further identified with

$$\operatorname*{hocolim}_{[k]\in\Delta^{op}} \mathrm{Map}_{\mathrm{Cart}}(\Delta^k, N^R(\mathcal{V}^\times)).$$

But this is just the shape of the smooth simplicial set $N^R(\mathcal{V}^\times)$ which as $\mathcal{V}^\times$ is a smooth groupoid equivalent to the "shape" of the smooth simplicial set $N(\mathcal{V}^\times)$. Recall as in [BBP24], the shape of a smooth simplicial set X is defined as $\mathrm{B}X := \operatorname*{hocolim}_{[k]\in\Delta^{op}} \mathrm{Map}_{\mathrm{Cart}}(\Delta^n, X)$. We can decompose $\mathcal{V}^\times = \coprod_{n\geq 0} \mathbf{B}\mathrm{GL}_n(\mathbb{C})$ so

$$\operatorname*{hocolim}_{[k]\in\Delta^{op}} \mathrm{Map}_{\mathrm{Cart}}(\Delta^k, N(\mathcal{V}^\times)) = \coprod_{n\geq 0} \operatorname*{hocolim}_{[k]\in\Delta^{op}} \mathrm{Map}_{\mathrm{Cart}}(\Delta^k, N(\mathbf{B}\mathrm{GL}_n(\mathbb{C}))).$$

Where

$$\operatorname*{hocolim}_{[k]\in\Delta^{op}} \mathrm{Map}_{\mathrm{Cart}}(\Delta^k, N(\mathbf{B}\mathrm{GL}_n(\mathbb{C}))) = \operatorname*{hocolim}_{[k]\in\Delta^{op}} N(\mathbf{B}C^\infty(\Delta^k, \mathrm{GL}_n(\mathbb{C}))).$$

Which is equivalent to

$$\operatorname*{hocolim}_{[k]\in\Delta^{op}} \mathrm{B}C^\infty(\Delta^k, \mathrm{GL}_n(\mathbb{C})).$$

Which by homotopy cocontinuity of B is just $\mathrm{BGL}_n(\mathbb{C})$. Thus we have

$$\mathrm{BFFT}^\times_{1,N^R(\mathcal{V})} = \coprod_{n\geq 0} \mathrm{BGL}_n(\mathbb{C}).$$

Further, the right-hand side here has the structure of a Γ-space coming from the direct sum of vector bundles. So we could note that this can be group completed to a connective spectrum – in particular, to the connective K-theory spectrum *ku*. So this construction provides a field-theoretic model for connective topological K-theory.

2.6 Interpretation in terms of TFTs

Throughout this paper we compute classifying spaces

$$\mathrm{BFFT}^\times_{d,\mathcal{C}} := \operatorname*{hocolim}_{[n]\in\Delta^{op}} \mathrm{Fun}^{\otimes}(\mathfrak{Bord}_d^{\Delta^n}, \mathcal{C})$$

of [GP23] for a few different values for $\mathcal{C}$. Because of the statement of the geometric cobordism hypothesis, in practice this reduces to computing the spaces

$$\operatorname*{hocolim}_{[n]\in\Delta^{op}} \mathrm{Map}_{\mathrm{Cart}}(\Delta^n, \mathcal{C}^\times).$$

The ordinary cobordism hypothesis, as in [Lur09], could also provide an interpretation of these computations.

Definition 2.6.1. Let $\mathcal{C}$ be a smooth symmetric monoidal (∞, d)-category with duals. Define the smooth space $\mathrm{TFT}_{d,\mathcal{C}} \in \mathrm{sPSh}(\mathrm{Cart})$ by

$$U \mapsto \mathrm{TFT}_{d,\mathcal{C}}(U) := \mathrm{Fun}^{\otimes}_{Lurie}(\mathrm{Bord}_d^{fr}, \mathcal{C}(U))$$

where the right-hand side is the space of topological field theories as in [Lur09] and reviewed in the introductory chapter of this paper.

By the ordinary cobordism hypothesis, we have an equivalence $\mathrm{TFT}_{d,\mathcal{C}}(U) \simeq \mathcal{C}^{\times}(U)$ for each U. Thus the computations of the spaces $\underset{[n]\in\Delta^{op}}{\mathrm{hocolim}}\,\mathrm{Map}_{\mathrm{Cart}}(\Delta^n, \mathcal{C}^{\times})$ could also be interpreted as computing spaces

$$\mathrm{BTFT}_{d,\mathcal{C}} := \underset{[n]\in\Delta^{op}}{\mathrm{hocolim}}\,\mathrm{Map}_{\mathrm{Cart}}(\Delta^n, \mathrm{TFT}_{d,\mathcal{C}}).$$

If M is a smooth manifold, one could potentially interpret maps $M \to \mathrm{BTFT}_{d,\mathcal{C}}$ as classifying some kind of families of topological field theories $Z : \mathrm{Bord}_d^{fr} \to \mathcal{C}(*)$ over M.

We present this potential alternative interpretation for the reader who may be comfortable with the ordinary cobordism hypothesis, but for whatever reason, not comfortable with the geometric cobordism hypothesis.

CHAPTER 3

TWO DIMENSIONAL FIELD THEORIES VALUED IN KV 2-VECTOR SPACES

In this chapter we want to study the homotopy type of the classifying space for 2-dimensional geometric field theories valued in an $(\infty, 2)$-categorical enhancement of the 2-category of Kapranov-Voevodsky 2-vector spaces. This process will be done in a similar fashion to the 1-dimensional case discussed in the previous chapter.

3.1 Kapranov-Voevodsky 2-Vector Spaces

We begin by recalling the definition of Kapranov-Voevodsky 2-vector spaces [KV94]. The 2-category of KV 2-vector spaces is the 2-category with objects being product categories $\mathcal{V}^n$, where $\mathcal{V}$ is the category of finite dimensional complex vector spaces, 1-morphisms are $\mathbb{C}$-linear functors between these objects, and the 2-morphisms are natural transformations between these functors.

Kapranov and Voevodsky proved this 2-category is equivalent to a skeletonized 2-category, def. 5.9 [KV94], which we will denote $2\mathcal{V}$.

Definition 3.1.1. The 2-category $2\mathcal{V}$ of coordinatized complex 2-vector spaces is defined as follows:

- Objects: Non-negative integers $\{n\} \in \mathbb{N}$.
- 1-morphisms: $A : \{m\} \to \{m'\}$ represented by an $m' \times m$ matrix $A = (a_{ij})$, where each entry $a_{ij} \in \mathbb{N}$ is a non-negative integer (dimension).
- 2-morphisms: $S : A \Rightarrow A'$ (where $A = (a_{ij}), A' = (a'_{ij})$) represented by an $m' \times m$ matrix $S = (S_{ij})$, where each entry $S_{ij} : \mathbb{C}^{a_{ij}} \to \mathbb{C}^{a'_{ij}}$ is a $\mathbb{C}$-linear map (represented by an $a'_{ij} \times a_{ij}$ complex matrix).

This 2-category has a bimonoidal structure. The direct sum provides an additive monoidal structure with unit object $\{0\}$.

- On Objects: $\{n\} \oplus \{m\} = n + m$.
- On 1-morphisms: For $A = (a_{ij}) : \{m\} \to \{m'\}$ and $B = (b_{kl}) : \{n\} \to \{n'\}$, the map $A \oplus B : \{m + n\} \to \{m' + n'\}$ is given by the $(m' + n') \times (m + n)$

block diagonal matrix of integers:

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

- On 2-morphisms: For $S : A \Rightarrow A'$ and $T : B \Rightarrow B'$, the map $S \oplus T : A \oplus B \Rightarrow A' \oplus B'$ is given by the $(m' + n') \times (m + n)$ block diagonal matrix of linear maps:

$$S \oplus T = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$$

The tensor product provides a multiplicative monoidal structure with unit object $\{1\}$ (Definition 5.22 [KV94]).

- On Objects: $\{m\} \otimes \{n\} = mn$.
- On 1-morphisms: For $A = (a_{ij}) : \{m\} \to \{m'\}$ and $B = (b_{kl}) : \{n\} \to \{n'\}$, the map $A \otimes B : \{mn\} \to \{m'n'\}$ is represented by the $(m'n') \times (mn)$ matrix of non-negative integers given by the Kronecker product of A and B.
- On 2-morphisms: For $S = (S_{ij}) : A \Rightarrow A'$ and $T = (T_{kl}) : B \Rightarrow B'$, the map $S \otimes T : A \otimes B \Rightarrow A' \otimes B'$ is represented by the Kronecker product of S and T.

The compatibility of these two monoidal structures to give a bimonoidal structure follows from the compatibility between block sums and Kronecker product of matrices. That is $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$.

Duals

We describe the dualizing structure in $2\mathcal{V}$. For this it is convienient to work in the non-coordinatized format. Working in the non-coordinatized format, the monoidal structure $\otimes$ can be defined formally by $\mathcal{V}^m \otimes \mathcal{V}^n := \mathcal{V}^{mn}$. It can also be defined as the Deligne tensor product of abelian categories, which is equivalent to the above formal definition in our case, but we don't need that interpretation here. The dual of $\mathcal{V}^n$ is itself and the evaluation map $ev : \mathcal{V}^{n^2} \to \mathcal{V}$ is the categorified trace, $(E_{ij}) \mapsto \bigoplus_i E_{ii}$. The coevaluation map $coev : \mathcal{V} \to \mathcal{V}^{n^2}$ is the diagonal embedding, $E \mapsto \mathrm{diag}(E, ..., E)$.

Now let $F : \mathcal{V}^n \to \mathcal{V}^m$ be a 1-morphism represented by the $m \times n$ matrix $M = (M_{ij})$ where each M_{ij} is a finite-dimensional vector space over $\mathbb{C}$. The right dual 1-morphism $G : \mathcal{V}^m \to \mathcal{V}^n$ is represented by the $n \times m$ matrix $N = (N_{jk})$ defined by

$$N_{jk} = (M_{kj})^* = \mathrm{Hom}_{\mathbb{C}}(M_{kj}, \mathbb{C}).$$

This matrix N is the "dual transpose" of M.

The unit $\eta : \mathrm{Id}^{(n)} \Rightarrow G \circ F$ is a 2-morphism represented by an $n \times n$ matrix of linear maps $A^\eta = (A^\eta_{kj})$. The source 1-morphism $\mathrm{Id}^{(n)}$ is represented by the matrix $I^{(n)}$ with entries $(I^{(n)})_{kj} = \delta_{kj}\mathbb{C}$. The target 1-morphism $G \circ F$ is represented by the matrix $P = N \cdot M$ with entries $P_{kj} = \bigoplus_{l=1}^m (N_{kl} \otimes M_{lj}) = \bigoplus_{l=1}^m ((M_{lk})^* \otimes M_{lj})$. The component map $A^\eta_{kj} : (I^{(n)})_{kj} \to P_{kj}$ is defined as:

$$A^\eta_{kj} = \begin{cases} 0 & \text{if } k \neq j \\ \sum_{l=1}^m \eta_{M_{lk}} & \text{if } k = j \end{cases}$$

Here, $\eta_V : \mathbb{C} \to V^* \otimes V$ is the standard coevaluation map for a finite-dimensional vector space V. The notation $\sum_{l=1}^m \eta_{M_{lk}}$ represents the map from $\mathbb{C}$ to the direct sum $\bigoplus_{l=1}^m ((M_{lk})^* \otimes M_{lk})$ whose l-th component map is $\eta_{M_{lk}}$.

The counit $\epsilon : F \circ G \Rightarrow \mathrm{Id}^{(m)}$ is a 2-morphism represented by an $m \times m$ matrix of linear maps $B^\epsilon = (B^\epsilon_{ik})$. The source 1-morphism $F \circ G$ is represented by the matrix $Q = M \cdot N$ with entries $Q_{ik} = \bigoplus_{l=1}^n (M_{il} \otimes N_{lk}) = \bigoplus_{l=1}^n (M_{il} \otimes (M_{kl})^*)$. The target 1-morphism $\mathrm{Id}^{(m)}$ is represented by the matrix $I^{(m)}$ with entries $(I^{(m)})_{ik} = \delta_{ik}\mathbb{C}$. The component map $B^\epsilon_{ik} : Q_{ik} \to (I^{(m)})_{ik}$ is defined as:

$$B^\epsilon_{ik} = \begin{cases} 0 & \text{if } i \neq k \\ \sum_{l=1}^n \epsilon_{M_{il}} & \text{if } i = k \end{cases}$$

Here, $\epsilon_V : V \otimes V^* \to \mathbb{C}$ is the standard evaluation map for a finite-dimensional vector space V. The notation $\sum_{l=1}^n \epsilon_{M_{il}}$ represents the map from the direct sum $\bigoplus_{l=1}^n (M_{il} \otimes (M_{il})^*) \to \mathbb{C}$ which applies $\epsilon_{M_{il}}$ to the l-th component and sums the results.

The unit η (represented by A^η) and counit ϵ (represented by B^ϵ) satisfy the zigzag identities.

The Core 2-groupoid

We aim to study functorial field theories valued in this 2-category. Now since the Cobordism Hypothesis tells us that the space of field theories is equivalent to the core of its target, we want to study the core groupoid of $2\mathcal{V}$, which we denote $2\mathcal{V}^\times$. Clearly a 1-morphism $n \to m$ in $2\mathcal{V}$ can only be invertible in the case $n = m$. The invertible elements in $M_n(\mathbb{N})$ are then exactly the permutation matrices. We will denote the group of $n \times n$ permutation matrices by S_n. For 2-morphisms, we need the T_{ij} to also be invertible matrices. So the core groupoid $2\mathcal{V}^\times$ is given as follows:

Definition 3.1.2. $2\mathcal{V}^\times$

- $Ob(2\mathcal{V}^\times) = \mathbb{N}$
- A 1-morphism $n \to n$ is a permutation matrix $(a_{ij}) \in S_n$
- A 2-morphism $(a_{ij}) \to (a_{ij})$ is a matrix (T_{ij}) where each $T_{ij} \in \mathrm{GL}_{a_{ij}}(\mathbb{C})$

We note that as our (a_{ij}) are permutation matrices, a_{ij} will always be 0 or 1. So our 2-morphisms are really permutation matrices where we replace the 1's with elements of $\mathrm{GL}_1(\mathbb{C})$.

Now that we have defined the core 2-groupoid $2\mathcal{V}^\times$ of the 2-category $2\mathcal{V}$, we will prove that we can reduce studying this 2-groupoid to studying the 2-groupoid $\mathbf{B}^2\mathrm{GL}_1(\mathbb{C})$. Let us denote by $S_n(\mathcal{V})$ the Hom groupoid on the element n in $2\mathcal{V}^\times$. Each $S_n(\mathcal{V})$ is a monoidal groupoid in the obvious way and we will consider the associated 2-groupoids $\mathbf{B}S_n(\mathcal{V})$. Clearly we have $2\mathcal{V}^\times = \coprod_{n\geq 0} \mathbf{B}S_n(\mathcal{V})$. What we claim is that $\mathbf{B}S_n(\mathcal{V})$ can be realized as a certain "S_n-shaped" colimit of $\mathbf{B}^2\mathrm{GL}_1(\mathbb{C})^{\times n}$.

First we must recall the definition of the 2-categorical Grothendieck construction, as in Def. 7.1 of [CCG10].

Definition 3.1.3. Let $\mathcal{F} : I^{op} \to 2\mathrm{Cat}$ be a functor for I a small category. Then the 2-categorical Grothendieck construction on $\mathcal{F}$, denoted $\int_I \mathcal{F}$, is the 2-category whose objects are pairs $(x, i) \in Ob(\mathcal{F}(i)) \times Ob(I)$. A 1-morphism $(u, a) : (y, j) \to (x, i)$ in $\int_I \mathcal{F}$ is pair of 1-morphisms $a : j \to i$ in I and $u : y \to \mathcal{F}(a)(x)$ in $\mathcal{F}(j)$. A 2-morphism $(u, a) \Rightarrow (u', a')$ exists if $a = a'$ and in such case is determined by a 2-morphism $\phi : u \Rightarrow u'$ in $\mathcal{F}(j)$. For each triplet of objects $(z, k), (y, j), (x, k)$ of $\int_I \mathcal{F}$, composition works as follows:

$$\begin{tikzcd}
(z,k) \arrow[r, bend left, "{(v,b)}"{name=U1}] \arrow[r, bend right, "{(v',b)}"'{name=D1}] & (y,j) \arrow[r, bend left, "{(u,a)}"{name=U2}] \arrow[r, bend right, "{(u',a)}"'{name=D2}] & (x,i) \arrow[r, phantom, "\overset{\circ}{\mapsto}"] & (z,k) \arrow[r, bend left, "{\mathcal{F}(b)(u)\circ v, a\circ b}"{name=U3}] \arrow[r, bend right, "{\mathcal{F}(b)(u')\circ v', a\circ b}"'{name=D3}] & (x,i)
\arrow[from=U1, to=D1, Rightarrow, "{(\psi,b)}"]
\arrow[from=U2, to=D2, Rightarrow, "{(\phi,a)}"]
\arrow[from=U3, to=D3, Rightarrow, "{\mathcal{F}(b)(\phi)\circ\psi, a\circ b}"]
\end{tikzcd}$$

Then we can formulate the following theorem:

Theorem 3.1.1. *Let* 2Cat *be the category of 2-categories and pseudofunctors. Define a functor* $\mathcal{D} : \mathbf{B}S_n^{op} \to 2\mathrm{Cat}$, *given on objects by* $\mathcal{D}(*) = \mathbf{B}^2\mathrm{GL}_1(\mathbb{C})^{\times n}$ *and on morphisms by sending a permutation* $P \in S_n$ *to the permutation functor* $\mathbf{B}^2\mathrm{GL}_1(\mathbb{C})^{\times n} \to \mathbf{B}^2\mathrm{GL}_1(\mathbb{C})^{\times n}$ *corresponding to* P.

Then $\mathbf{B}S_n(\mathcal{V})$ *is equivalent to the 2-categorical Grothendieck construction applied to this functor. i.e.* $\mathbf{B}S_n(\mathcal{V}) = \int_{\mathbf{B}S_n} \mathcal{D}$

Proof. We consider the above 2-categorical Grothendieck construction in our specific case. Since both $\mathbf{B}S_n$ and $\mathbf{B}^2\mathrm{GL}_1(\mathbb{C})^{\times n}$ has a single object, $\int_{\mathbf{B}S_n} \mathcal{D}$ will have a single object. Since $\mathbf{B}^2\mathrm{GL}_1(\mathbb{C})^{\times n}$ has only one 1-morphism, the 1-morphisms in $\int_{\mathbf{B}S_n} \mathcal{D}$ will correspond exactly to 1-morphisms in $\mathbf{B}S_n$, i.e. permutations $P \in S_n$. 2-morphisms in $\int_{\mathbf{B}S_n} \mathcal{D}$ then correspond to pairs (g, P) where $P \in S_n$ and $g \in \mathrm{GL}_1(\mathbb{C})^{\times n}$. We observe this almost identitical to our definition of $\mathbf{B}S_n(\mathcal{V})$. We just need to check that the composition structure of these 2-categories are indeed the same. Based on the above diagram, composition of 1-morphisms in $\int_{\mathbf{B}S_n} \mathcal{D}$ will be given by composition of 1-morphisms in $\mathbf{B}S_n$, i.e. the group operation on S_n. This is exactly the same as composition in $\mathbf{B}S_n(\mathcal{V})$. Composition of 2-morphisms will be given by the group operation in $\mathrm{GL}_1(\mathbb{C})^n$, which is also exactly the same as composition in $\mathbf{B}S_n(\mathcal{V})$. Thus we have an equivalence $\mathbf{B}S_n(\mathcal{V}) \cong \int_{\mathbf{B}S_n} \mathcal{D}$. □

With this theorem we will be able to reduce proving things about $2\mathcal{V}^\times$ to proving things about $\mathbf{B}^2\mathrm{GL}_1(\mathbb{C})$. The following fact about the 2-categorical Grothendieck construction will come in handy in a later section.

Theorem 3.1.2. *Thm 7.3 [CCG10]*

To every functor $\mathcal{F} : I^{op} \to 2\mathrm{Cat}$ *there exists a natural weak equivalence of simplicial sets*

$$\mathrm{hocolim}_I N^D(\mathcal{F}) \to N^D\Big(\int_I \mathcal{F}\Big)$$

where N^D *is the Duskin nerve and* $N^D(\mathcal{F})$ *means the diagram of simplicial sets associated to the functor* F.

Smooth Enhancement

To properly work with functorial field theories, we also need to promote our 2-category into a "smooth 2-category." Recall the category Cart. This is the category whose objects are open subsets of $\mathbb{R}^n$ for various n and whose morphisms are smooth maps. A smooth 2-category will be a 2-category parametrized over Cart, or equivalently a sheaf of categories on Cart. By abuse of notation, we will denote the smooth categories in the same way as the ordinary categories on which they are based.

Definition 3.1.4. We define a smooth category $2\mathcal{V} : \mathrm{Cart}^{op} \to 2\mathrm{Cat}$, by defining, for $U \in \mathrm{Cart}$, $2\mathcal{V}(U)$ to have the same objects and 1-morphisms as the 2-category $2\mathcal{V}$ but for 2-morphisms we replace the copies of $M_-(\mathbb{C})$ with $C^\infty(U, M_-(\mathbb{C}))$.

The fact that this preshef satsifies descent for the standard topology on Cart follows immediately from the fact that the presheaves of smooth functions satisfies descent.

In the context of the Geometric Cobordism Hypothesis, we also need to consider a version of this that is a presheaf on FEmb_2.

Definition 3.1.5. We define a presheaf of 2-categories $2\mathcal{V}^\times_{fc}$ on FEmb_2 by sending $T \to U$ to the 2-groupoid $2\mathcal{V}^\times_{fc}(T \to U)$, which has the same objects and 1-morphisms as the 2-groupoid $2\mathcal{V}^\times$ but for 2-morphisms we replace the copies of $\mathrm{GL}_1(\mathbb{C})$ with $C^\infty_{fc}(T, \mathrm{GL}_1(\mathbb{C}))$. Here $C^\infty_{fc}(T, \mathrm{GL}_1(\mathbb{C}))$ is the space of $\mathrm{GL}_1(\mathbb{C})$-valued smooth functions on T that are fiberwise constant with respect to the submersion $T \to U$.

This will again satisfy descent in the appropriate topology because the group of smooth functions satifies descent.

$(\infty, 2)$-*Enhancement*

In order to properly formulate field theories valued in Kapranov-Voevodsky 2-vector spaces in the framework of Grady-Pavlov, we need not just a 2-category but rather an $(\infty, 2)$-category of 2-vector spaces. This is not much of an issue. The solution is clearly to just take “the nerve” of our 2-category. However the problem is, which nerve? There are many different nerve constructions for 2-categories. A good reference for the many different ones is [MOR22]. In our case, the model of $(\infty, 2)$-categories we will use is 2-fold complete Segal spaces. So we need a nerve construction that takes a 2-category and outputs a 2-fold complete Segal space. Moser made such a construction in [Mos24].

First, let us define the standard Duskin nerve of a 2-category. Recall the ordinary nerve of a category has n-simplices given by $N(\mathcal{C})_n = \mathrm{Fun}([n], \mathcal{C})$. The Duskin nerve of a 2-category is defined similarly, but with the category $[n]$ replaced by a 2-category $\mathcal{O}_2(n)$. The 2-categories $\mathcal{O}_2(n)$ are 2-truncated versions of Street's orientals [Str87]. $\mathcal{O}_2(n)$ should be thought of as the free 2-category over a n-simplex. Explicitly:

Definition 3.1.6. $\mathcal{O}_2(n)$

- Objects: The set $\{0, 1, \ldots, n\}$.
- Morphisms: For objects x, x', the hom-category $\mathcal{O}_2(n)(x, x')$ is the poset of subsets $I \subseteq \{y \mid x \leq y \leq x'\}$ such that $x, x' \in I$, ordered by inclusion. This is non-empty only if $x \leq x'$. A morphism corresponds to a path from x to x'.
- 2-Morphisms: A unique 2-morphism $I \Rightarrow J$ exists if $I \subseteq J$. This represents filling in intermediate vertices in a path.

The Duskin nerve $N^D(\mathcal{C}) \in \mathrm{sSet}$ of a 2-category is then defined as the simplicial set whose n-simplicies are given by the set of 2-functors $N^D(\mathcal{C})_n = \mathrm{Fun}(\mathcal{O}_2(n), \mathcal{C})$. An explicit description of this data is given in [Lur25] which we present below:

Definition 3.1.7. An element of $N^D(\mathcal{C})_n$ consists of the following data:

- A collection of objects $\{X_i\}_{0 \leq i \leq n}$ of the 2-category $\mathcal{C}$.
- A collection of 1-morphisms $\{f_{j,i} : X_i \to X_j\}_{0 \leq i \leq j \leq n}$ in the 2-category $\mathcal{C}$.
- A collection of 2-morphisms $\{\mu_{k,j,i} : f_{k,j} \circ f_{j,i} \Rightarrow f_{k,i}\}_{0 \leq i \leq j \leq k \leq n}$ in the 2-category $\mathcal{C}$.

These data are required to satisfy the following conditions:

- For $0 \leq i \leq n$, the 1-morphism $f_{i,i} : X_i \to X_i$ is the identity 1-morphism id_{X_i}.
- For $0 \leq i \leq j \leq n$, the 2-morphisms

  $$\mu_{j,j,i} : f_{j,j} \circ f_{j,i} \Rightarrow f_{j,i} \qquad \mu_{j,i,i} : f_{j,i} \circ f_{i,i} \Rightarrow f_{j,i} \tag{3.1}$$

  are the left unit constraints $\lambda_{f_{j,i}}$ and the right unit constraints $\rho_{f_{j,i}}$, respectively.

- For $0 \leq i \leq j \leq k \leq \ell \leq n$, we have a commutative diagram

$$\begin{tikzcd}
f_{\ell,k} \circ (f_{k,j} \circ f_{j,i}) \arrow[rr, Rightarrow, "\alpha_{f_{\ell,k}, f_{k,j}, f_{j,i}}"] \arrow[d, Rightarrow, "id_{f_{\ell,k}} \circ \mu_{k,j,i}"'] & & (f_{\ell,k} \circ f_{k,j}) \circ f_{j,i} \arrow[d, Rightarrow, "\mu_{\ell,k,j} \circ id_{f_{j,i}}"] \\
f_{\ell,k} \circ f_{k,i} \arrow[dr, Rightarrow, "\mu_{\ell,k,i}"'] & & f_{\ell,j} \circ f_{j,i} \arrow[dl, Rightarrow, "\mu_{\ell,j,i}"] \\
& f_{\ell,i} &
\end{tikzcd} \tag{3.2}$$

  in the category $\mathrm{Hom}_{\mathcal{C}}(X_i, X_\ell)$.

If $\mathcal{C}$ is a 2-groupoid, then the Duskin nerve is a 2-truncated Kan complex. And if $\mathcal{C}$ is a $(2,1)$-category, then the Duskin nerve is a quasi-category.

The Duskin nerve gives us a simplicial set from a 2-category. But what we really need is a 2-fold complete Segal space from a 2-category. In [Mos24], Moser gives a nerve construction, that we will call N^M, which does exactly that. Moser's nerve is built from variations of the 2-truncated orientals discussed above. We will not give the explicit definition here, rather leave it for the appendix of this paper as it is fairly intricate. However, we will cite its main property as a theorem.

Theorem 3.1.3 ([Mos24])**.** *There exists a functor* $N^M : 2\mathrm{Cat} \to \mathrm{sSet}^{(\Delta\times\Delta)^{op}}$ *that sends a 2-category to a 2-fold complete Segal space.*

When $\mathcal{C}$ is a 2-groupoid, the Moser nerve is equivalent to a Reedy fibrant replacement of the constant bisimplicial space on the Duskin nerve. We will construct this replacement explicitly. Let

$$d : \Delta \to \Delta \times \Delta \times \Delta$$

be the three–fold diagonal in the simplex category. Precomposition with d defines a "restriction to the diagonal" functor

$$d^* : \mathrm{sSet}^{(\Delta\times\Delta)^{op}} \to \mathrm{sSet}$$

It admits a right adjoint

$$d_* : \mathrm{sSet} \to \mathrm{sSet}^{(\Delta\times\Delta)^{op}}.$$

On objects the right adjoint is given explicitly by

$$d_*(X)_{m,n} = \mathrm{Map}_{\mathrm{sSet}}(\Delta[m]\times\Delta[n], X), \qquad m,n\geq 0.$$

Proposition 3.1.1. *Let X be a Kan complex. The bisimplicial space*

$$d_*(X)_{m,n} \;=\; \mathrm{Map}_{\mathrm{sSet}}\big(\Delta[m]\times\Delta[n],\, X\big), \qquad m,n\geq 0,$$

is a two-fold complete Segal space.

Proof. For every pair (m,n) the mapping space $d_*(X)_{m,n}$ is Kan, because mapping into a Kan complex preserves fibrancy. Hence $d_*(X)$ is Reedy fibrant. Fix n. Cutting $\Delta[m]$ into its m edges gives a map

$$d_*(X)_{m,n} \;\to\; d_*(X)_{1,n} \times_{d_*(X)_{0,n}} \cdots \times_{d_*(X)_{0,n}} d_*(X)_{1,n}.$$

The inclusion of the edge–spine in $\Delta[m]$ is inner anodyne; mapping it into X yields a weak equivalence of Kan complexes, so this horizontal Segal map is a weak equivalence. Interchanging m and n gives the same argument, so the vertical Segal maps are also weak equivalences. Write $d_*(X)_{0,\bullet}$ for the object space and $d_*(X)_{\bullet}^{\mathrm{equiv}} \subseteq d_*(X)_{1,\bullet}$ for the simplicial subset of homotopy-invertible 1-morphisms. Because every edge in a mapping space into a Kan complex is homotopy invertible, we have $d_*(X)_{\bullet}^{\mathrm{equiv}} = d_*(X)_{1,\bullet}$. The degeneracy map $s_0 : d_*(X)_{0,\bullet} \to d_*(X)_{1,\bullet}$ is therefore a weak equivalence, satisfying the completeness axiom. Thus $d_*(X)$ is a two-fold complete Segal space. □

Proposition 3.1.2. *If $\mathcal{C}$ is a 2-groupoid, then $N^M(\mathcal{C})$ is weakly equivalent to the construction $d_*N^D(\mathcal{C})$ from above.*

We prove this proposition in appendix 1 after we give the explicit definition of the Moser nerve.

∞-*Categorical Monoidal Structure*

We also need to comment on the the (bi-)monoidal structures. A symmetric monoidal structure on an (∞, d)-category is captured in terms of a Γ-space structure – that is, a symmetric monoidal (∞, d)-category is a Γ-object in (∞, d)-categories. The nerves of symmetric monoidal 2-categories have obvious Γ-space structures induced from the 2-categorical symmetric monoidal structures. That is, for a symmetric monoidal 2-category $\mathcal{C}$, we can define a Γ-space structure by $\langle \ell \rangle \mapsto N^M(\mathcal{C})^{\times \ell}$. Then for a map $\alpha : \langle k \rangle \to \langle \ell \rangle$ in Γ, the corresponding functor $N^M(\mathcal{C})(\alpha) : (N^M(\mathcal{C}))^{\times \ell} \to (N^M(\mathcal{C}))^{\times k}$ is constructed from the symmetric monoidal structure of $N^M(\mathcal{C})$. Specifically, for an ℓ-tuple of objects $(X_1, \ldots, X_\ell)$ in $(N^M(\mathcal{C}))^{\times \ell}$, the j-th component (for $j \in \{1, \ldots, k\}$) of its image $N^M(\mathcal{C})(\alpha)(X_1, \ldots, X_\ell)$ is given by the tensor product:

$$\bigotimes_{s \in \alpha^{-1}(j) \setminus \{0\}} X_s$$

If the set $\alpha^{-1}(j) \setminus \{0\}$ is empty, this component is the unit object I of $N^M(\mathcal{C})$. This construction extends naturally to morphisms and higher morphisms within C. The symmetric braiding ensures that the order of products for different preimages $\alpha^{-1}(j)$ does not matter up to coherent isomorphism, satisfying the requirements for a Γ-object. In the case C is bimonoidal, we can produce two appropriately compatible Γ-structures on $N^M(\mathcal{C})$ this way.

3.2 The Homotopy Type of the Space of Field Theories

Let $2\mathcal{V}$ be the 2-category of Kapranov-Voevodsky 2-Vector spaces. Using the Geometric Cobordism Hypothesis and the corollary to it given in section 1 of this paper, we will examine the space of field theories $\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_2^{\mathbb{R}^2 \times U \to U}, N^M(2\mathcal{V}))$. We note as every object and 1-morphism in $2\mathcal{V}$ is already dualizable, we do not need to take ^{fd}. Based on the earlier corollary, we want to “guess” a candidate to be weakly equivalent to this space of field theories. Recall in Definition 3.1.5 we defined a version of $2\mathcal{V}^{\times}$ which was a sheaf on FEmb_2 that we denoted $2\mathcal{V}^{\times}_{fc}$. Our “guess” will be the corresponding infinity groupoid $N^D(2\mathcal{V}^{\times}_{fc})$.

Proposition 3.2.1. *There is a weak equivalence of simplicial presheaves*

$$N^D(2\mathcal{V}^{\times}_{fc}) \simeq \mathrm{Fun}^{\otimes}(\mathfrak{Bord}_2^{-}, N^M(2\mathcal{V}))$$

in $\mathrm{sPSh}(\mathrm{FEmb}_2)$.

Proof. We construct a map of simplicial presheaves

$$N^D(2\mathcal{V}^{\times}_{fc}) \to \mathrm{Fun}^{\otimes}(\mathfrak{Bord}_2^{-}, N^M(2\mathcal{V}))$$

in $\mathrm{sPSh}(\mathrm{FEmb}_2)$ such that for each $\mathbb{R}^2 \times U \to U$ the composition

$$N^D(2\mathcal{V}^\times_{fc})(\mathbb{R}^2 \times U \to U) \to \mathrm{Fun}^\otimes(\mathfrak{Bord}_2^{\mathbb{R}^2\times U\to U}, N^M(2\mathcal{V})) \simeq N^D(2\mathcal{V}^\times)(U)$$

is a weak equivalence of simplicial sets.

As objects of $2\mathcal{V}^\times_{fc}(\mathbb{R}^2 \times U \to U)$ are the same as objects of $2\mathcal{V}^\times(U)$, the construction of this map follows from the Cobordism Hypothesis statement itself. We send the object n to the field theory Z_n corresponding to n under the equivalance

$$\mathrm{Fun}^\otimes(\mathfrak{Bord}_2^{\mathbb{R}^2\times U\to U}, N^M(2\mathcal{V})) \simeq N^D(2\mathcal{V}^\times)(U).$$

This field theory Z_n is the basic field theory, that evaluates to n on a point. That is, the structure of Z_n on morphisms is determined entirely from identities and the maps which endow $n \in 2\mathcal{V}$ with the structure of a fully dualizable object.

The fact that the composition is a weak equivalence then simply follows from the fact that the group of fiberwise constant functions on $\mathbb{R}^2 \times U \to U$ is isomorphic to the group of functions on U. □

So by the Geometric Cobordism Hypothesis, we see that the homotopy type of the sheaf of field theories valued in $N^M(2\mathcal{V})$ will just be determined by the homotopy type of the sheaf $N^D(2\mathcal{V}^\times_{fc})$.

Theorem 3.2.1. *The classifying space* $\mathrm{BFFT}^\times_{2,N^M(2\mathcal{V})}$ *is weakly equivalent to the homotopy quotient space* $\coprod_{n\geq 0} \mathrm{B}^2\mathrm{GL}_1(\mathbb{C})^{\times n}//S_n$.

Proof. By definition we have that

$$\mathrm{BFFT}^\times_{2,N^M(2\mathcal{V})} := \operatorname*{hocolim}_{[n]\in\Delta^{op}} \mathrm{Fun}^\otimes(\mathfrak{Bord}_2^{\Delta^n}, N^M(2\mathcal{V}))^\times.$$

The Geometric Cobordism Hypothesis combined with Proposition 3.2.1 we proved above tells us

$$\mathrm{Fun}^\otimes(\mathfrak{Bord}_2^{\Delta^n}, N^M(2\mathcal{V}))^\times \simeq \mathrm{Map}_{\mathrm{sPSh}(\mathrm{FEmb}_2)}(\Delta^n, N^D(2\mathcal{V}^\times_{fc})).$$

So we have

$$\mathrm{BFFT}^\times_{2,N^M(2\mathcal{V})} := \operatorname*{hocolim}_{[n]\in\Delta^{op}} \mathrm{Map}_{\mathrm{sPSh}(\mathrm{FEmb}_2)}(\Delta^n, N^D(2\mathcal{V}^\times_{fc})).$$

Let $\mathrm{FEmbCart}_2$ be the sub-site of FEmb_2 consisting of just trivial submersions $\mathbb{R}^2 \times U \to U$. Categories of sheaves on these two sites are equivalent as sheaves are determined locally, [GP22] Prop. 3.3.3, so we can observe

$$\mathrm{Map}_{\mathrm{sPSh}(\mathrm{FEmb}_2)}(\Delta^n, N^D(2\mathcal{V}^\times_{fc})) \simeq \mathrm{Map}_{\mathrm{sPSh}(\mathrm{FEmbCart}_2)}(\Delta^n, N^D(2\mathcal{V}^\times_{fc}))$$

and use the fact $2\mathcal{V}^{\times}_{fc}(\mathbb{R}^2 \times U \to U) \simeq 2\mathcal{V}^{\times}(U)$ to see

$$\mathrm{Map}_{\mathrm{sPSh(FEmbCart_2)}}(\Delta^n, N^D(2\mathcal{V}^{\times}_{fc})) \simeq \mathrm{Map}_{\mathrm{sPSh(Cart)}}(\Delta^n, N^D(2\mathcal{V}^{\times})).$$

So the homotopy colimit on the right hand side is just the shape of the smooth simplicial set $U \mapsto N^D(2\mathcal{V}^{\times}(U))$. i.e.

$$\mathrm{BFFT}^{\times}_{2,N^M(2\mathcal{V})} \simeq \operatorname*{hocolim}_{[n]\in\Delta^{op}} \mathrm{Map}_{\mathrm{sPSh(Cart)}}(\Delta^n, N^D(2\mathcal{V}^{\times})).$$

We review a smooth enhancement of the discussion in the first section of this chapter. We note that $2\mathcal{V}^{\times}(U) = \coprod_{n\geq 0} \mathbf{B}S_n(\mathcal{V})(U)$ and $\mathbf{B}S_n(\mathcal{V})(U)$ is equivalent to a Grothendieck construction $\mathbf{B}S_n(\mathcal{V})(U) \simeq \int_{\mathbf{B}S_n} \mathcal{D}(U)$ where $\mathcal{D}(U) : \mathbf{B}S_n^{op} \to$ 2Cat is a diagram given on objects by $\mathcal{D}(U)(*) = \mathbf{B}^2 C^{\infty}(U, \mathrm{GL}_1(\mathbb{C}))^{\times n}$ and on morphisms by sending a permutation $P \in S_n$ to the corresponding permutation functor. In the earlier core groupoid section, we quoted Theorem 7.3 from [CCG10], which tells us

$$N^D(\mathbf{B}S_n(\mathcal{V})(U)) \simeq N^D(\int_{\mathbf{B}S_n} \mathcal{D}(U)) \simeq \mathrm{hocolim}_{\mathbf{B}S_n} N^D(\mathcal{D}(U)).$$

Now the shape of the smooth simplicial set $U \mapsto N^D(\mathbf{B}^2 C^{\infty}(U, \mathrm{GL}_1(\mathbb{C}))^{\times n})$ is just the space

$$\operatorname*{hocolim}_{[k]\in\Delta^{op}} \mathrm{B}^2 C^{\infty}(\Delta^k, \mathrm{GL}_1(\mathbb{C}))^{\times n} \simeq \mathrm{B}^2\mathrm{GL}_1(\mathbb{C})^{\times n}.$$

So taking shapes, we are looking for the homotopy colimit of the diagram of spaces $\mathbf{B}S_n^{op} \to$ Spaces sending the single object to the space $\mathrm{B}^2\mathrm{GL}_1(\mathbb{C})^{\times n}$. The homotopy colimit of such a diagram is precisely the homotopy quotient

$$\mathrm{B}^2\mathrm{GL}_1(\mathbb{C})^{\times n}//S_n := (\mathrm{E}S_n \times \mathrm{B}^2\mathrm{GL}_1(\mathbb{C})^{\times n})/S_n.$$

□

The space $\coprod_{n\geq 0} \mathrm{B}^2\mathrm{GL}_1(\mathbb{C})^{\times n}//S_n$ is a Γ-space with respect to the block sum structure. That is the Γ-structure induced from the $\oplus$-structure on $2\mathcal{V}$. We can use the equivalence of spaces above to transfer this Γ-structure to $\mathrm{BFFT}^{\times}_{2,N^M(2\mathcal{V})}$. This transferred Γ-structure corresponds to a direct sum of field theories. We will not describe this too explicitly, but essentially given two field theories $Z, Z' : \mathfrak{Bord}_2^X \to N^M(2\mathcal{V})$, the direct sum field theory $Z \oplus Z'$ is given by taking the naive direct sum on connected objects and then extending to disjoint unions of connected components in the way necessary to force $Z \oplus Z'$ to be symmetric monoidal.

Corollary 3.2.1.1. *The connective spectrum obtained from the group completion of the* Γ*-space* $\mathrm{BFFT}^{\times}_{2,N^M(2\mathcal{V})}$*, where the* Γ*-structure is as above, is equivalent to* $\mathbb{S}[\mathrm{B}^2\mathrm{GL}_1(\mathbb{C})]$.

Proof. It is proven in Proposition 3.6 of [Seg74] that for any space X the group completion of the Γ-space $\coprod_{n\geq 0}(\mathrm{E}S_n \times X^{\times n})/S_n$ is the suspension spectrum $\mathbb{S}[X]$.

$\square$

3.3 A Differential Model

In studying the space of 1-dimensional field theories $\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_1^{T\to U}, \mathcal{V})$, we were able to build a differential form model for this space by identifying field theories with parallel transport functors. In the 2-dimensional case, we would also like to build a differential form model using the same idea.

A Differential Form Construction

To do this first let us define $\mathbf{B}^2_{\nabla}\mathrm{GL}_1(\mathbb{C})$. This is a smooth 2-groupoid $M \mapsto \mathbf{B}^2_{\nabla}\mathrm{GL}_1(\mathbb{C})(M)$ where $\mathbf{B}^2_{\nabla}\mathrm{GL}_1(\mathbb{C})(M)$ is defined as follows:

Definition 3.3.1. $\mathbf{B}^2_{\nabla}\mathrm{GL}_1(\mathbb{C})(M)$

- The objects are 2-forms $B \in \Omega^2(M; \mathfrak{gl}_1(\mathbb{C}))$.
- A 1-morphism $A : B \to B'$ is a 1-form $A \in \Omega^1(M; \mathfrak{gl}_1(\mathbb{C}))$ such that $\mathrm{d}A = B' - B$, composition is given by addition of 1-forms.
- A 2-morphism $g : A \Rightarrow A'$ is a smooth function $g \in C^{\infty}(M, \mathrm{GL}_1(\mathbb{C}))$ such that $A' = gAg^{-1} + g^{-1}dg$, composition is given by the group operation.

This above 2-groupoid is a common appearance in the study of the differential geometry of bundle gerbes. That is, this 2-groupoid is equivalent to a 2-groupoid of trivial bundle gerbes with connection – similar to how in the 1-dimensional case, our groupoid of differential form data was equivalent to a groupoid of trivial vector bundles with connection.

Recall our earlier Theorem 3.1.1 that allowed us to describe $\mathbf{B}S_n(\mathcal{V})$ in terms of a 2-categorical Grothendieck construction on a $\mathbf{B}S_n$-shaped functor taking values in $\mathbf{B}^2\mathrm{GL}_1(\mathbb{C})^{\times n}$. We will apply the same construction, but now considering a $\mathbf{B}S_n$-shaped functor taking values in $\mathbf{B}^2_\nabla\mathrm{GL}_1(\mathbb{C})(M)^{\times n}$, in order to figure out the proper definition for $\mathbf{B}_\nabla S_n(\mathcal{V})(M)$.

Definition 3.3.2. Let $\mathcal{D}_\nabla(M) : \mathbf{B}S_n^{op} \to 2\mathrm{Cat}$ be the functor defined by

$$\mathcal{D}_\nabla(M)(*) = \mathbf{B}^2_\nabla\mathrm{GL}_1(\mathbb{C})(M)^{\times n}$$

on objects. On 1-morphisms $P \in S_n$, we define

$$\mathcal{D}_\nabla(M)(P) : \mathbf{B}^2_\nabla\mathrm{GL}_1(\mathbb{C})(M)^{\times n} \to \mathbf{B}^2_\nabla\mathrm{GL}_1(\mathbb{C})(M)^{\times n}$$

to be the permutation functor associated to the permutation P.

We can then define $\mathbf{B}_\nabla S_n(\mathcal{V})(M)$ to be the 2-categorical Grothendieck construction applied to this functor. That is

$$\mathbf{B}_\nabla S_n(\mathcal{V})(M) := \int_{\mathbf{B}S_n} \mathcal{D}_\nabla(M).$$

We know from the theorem in the first section that this should be the proper definition, but we still need to break it down and describe this 2-category explicitly. From the general construction, the objects of this category must just be objects of $\mathbf{B}^2_\nabla\mathrm{GL}_1(\mathbb{C})(M)^{\times n}$, that is n-tuples of 2-forms (B_i). A 1-morphism $(B_i) \to (B'_i)$ is then a pair $((A_i), P)$ where $P \in S_n$ and (A_i) is an n-tuple of 1-forms such that $\mathrm{d}A_i = B'_{P(i)} - B_i$. A 2-morphism $((g_i), P) : ((A_i), P) \Rightarrow ((A'_i), P)$ is an n-tuple of smooth functions (g_i) such that $A'_i = g_i A_i g_i^{-1} + g_i^{-1} dg_i$.

We now define our general 2-groupoid:

Definition 3.3.3. We define a smooth 2-groupoid $2\mathcal{V}^\times_\nabla : \mathrm{Cart}^{op} \to 2\mathrm{Cat}$, by defining, for $U \in \mathrm{Cart}$, $2\mathcal{V}^\times_\nabla(U) := \coprod_{n\geq 0} \mathbf{B}_\nabla S_n(\mathcal{V})(U)$.

For purposes of the Geometric Cobordism Hypothesis, we also need to define $2\mathcal{V}^\times_\nabla$ as a presheaf on FEmb_2. We will define $2\mathcal{V}^\times_\nabla(T \to U)$ to be very similar to our previous definition, but for differential forms we use $\Omega^k_U(T; \mathfrak{gl}_1(\mathbb{C}))$, i.e. we use differential forms on T that are vertical with respect to the submersion $T \to U$. Explicitly:

Definition 3.3.4. $2\mathcal{V}^\times_\nabla(T \to U)$

- Objects are $(n, (B_i))$ where $n \in \mathbb{N}$ and (B_i) is an n-tuple of elements of $\Omega^2_U(T; \mathfrak{gl}_1(\mathbb{C}))$

- 1-morphisms are $(n, P, (A_i)) : (n, (B_i)) \to (n, (B'_i))$ where $P \in S_n$, $A_i \in \Omega^1_U(T; \mathfrak{gl}_1(\mathbb{C}))$, and $\mathrm{d}A_i = B'_{P(i)} - B_i$
- 2-morphisms are $((g_i), P) : ((A_i), P) \Rightarrow ((A'_i), P)$ where $g_i \in C^\infty(T, \mathrm{GL}_1(\mathbb{C}))$ and such that $A'_i = g_i A_i g_i^{-1} + g_i^{-1} dg_i$

We note this construction has a bimonoidal structure, which are induced from the bimonoidal structure on $2\mathcal{V}$ along a slight generalization of the monoidal structures for line bundles with connection.

The direct sum structure combines objects and morphisms by concatenation.

- For $B = (n, (B_i))$ and $C = (m, (C_j))$:

  $$B \oplus C := (n + m, (B_1, \dots, B_n, C_1, \dots, C_m)).$$

- For $\alpha = (P, (A_i)) : B \to B'$ and $\beta = (Q, (D_j)) : C \to C'$:

  $$\alpha \oplus \beta := (P \oplus Q, (A_1, \dots, A_n, D_1, \dots, D_m)) : B \oplus C \to B' \oplus C'$$

  where $P \oplus Q \in S_{n+m}$ is the standard block permutation.

- For $\gamma = ((g_i)) : \alpha \Rightarrow \alpha'$ and $\delta = ((h_j)) : \beta \Rightarrow \beta'$:

  $$\gamma \oplus \delta := ((g_1, \dots, g_n, h_1, \dots, h_m)) : \alpha \oplus \beta \Rightarrow \alpha' \oplus \beta'.$$

- The identity is the object with rank $n = 0$ and the empty tuple, $I_\oplus := (0, ())$.
- For objects $X = (n, (B_i))$ and $Y = (m, (C_j))$, the braiding $\beta_{X,Y} : X \oplus Y \to Y \oplus X$ is the 1-morphism:

  $$\beta_{X,Y} = (\sigma_{n,m}, (0)_{k=1}^{n+m})$$

  where $\sigma_{n,m} \in S_{n+m}$ swaps the first n with the last m elements, and (0) is the tuple of zero 1-forms.

The tensor product structure combines objects and morphisms pairwise, using addition for forms and multiplication for $\mathrm{GL}_1(\mathbb{C})$ functions, reflecting the tensor product of line bundles with connections.

- For $B = (n, (B_i))$ and $C = (m, (C_j))$:

  $$B \otimes C := (nm, (B_i + C_j)_{\substack{1 \le i \le n \\ 1 \le j \le m}})$$

  (using lexicographical order for index pair (i, j)).

- For $\alpha = (P, (A_i)) : B \to B'$ and $\beta = (Q, (D_j)) : C \to C'$:

  $$\alpha \otimes \beta := (P \otimes Q, (A_i + D_j)_{\substack{1 \le i \le n \\ 1 \le j \le m}}) : B \otimes C \to B' \otimes C'$$

  where $P \otimes Q \in S_{nm}$ is the permutation acting as $(i, j) \mapsto (P(i), Q(j))$.

- For $\gamma = ((g_i)) : \alpha \Rightarrow \alpha'$ and $\delta = ((h_j)) : \beta \Rightarrow \beta'$:

  $$\gamma \otimes \delta := ((g_i h_j)_{\substack{1 \le i \le n \\ 1 \le j \le m}}) : \alpha \otimes \beta \Rightarrow \alpha' \otimes \beta'.$$

- The identity is the object with rank $n = 1$ and the empty tuple, $I_\otimes := (1, (0))$.

- For objects $X = (n, (B_i))$ and $Y = (m, (C_j))$, the braiding $\beta_{X,Y} : X \otimes Y \to Y \otimes X$ is the 1-morphism:

  $$\beta_{X,Y} = (\sigma'_{n,m}, (0)_{k=1}^{nm})$$

  where $\sigma'_{n,m} \in S_{nm}$ is the permutation mapping index (i, j) to index (j, i), and (0) is the tuple of zero 1-forms.

As a Model For Field Theories

We begin by summarizing Pavlov-Stolz-Teichner's proof method for the bundle gerbe case. This will motivate our case.

Theorem 3.3.1. *[Pavlov-Stolz-Teichner]*

There is a weak equivalence of simplicial presheaves

$$N^D(\mathbf{B}^2_\nabla \mathrm{GL}_1(\mathbb{C})) \simeq \mathrm{Fun}^\otimes(\mathfrak{Bord}_2^-, d_* N^D(\mathbf{B}^2 \mathrm{GL}_1(\mathbb{C})))$$

in sPSh(FEmb$_2$).

Proof. This is once again done by constructing a map of simplicial presheaves

$$N^D(\mathbf{B}^2_\nabla \mathrm{GL}_1(\mathbb{C})) \to \mathrm{Fun}^\otimes(\mathfrak{Bord}_2^-, d_* N^D(\mathbf{B}^2 \mathrm{GL}_1(\mathbb{C})))$$

in sPSh(FEmb$_2$) such that for each $\mathbb{R}^2 \times U \to U$ the composition with the cobordism hypothesis equivalence

$$\mathrm{Fun}^\otimes(\mathfrak{Bord}_2^{\mathbb{R}^2\times U\to U}, d_* N^D(\mathbf{B}^2 \mathrm{GL}_1(\mathbb{C}))) \simeq N^D(\mathbf{B}^2 \mathrm{GL}_1(\mathbb{C}))(U)$$

is a weak equivalence of simplicial presheaves. The map

$$N^D(\mathbf{B}^2_\nabla \mathrm{GL}_1(\mathbb{C})) \to \mathrm{Fun}^\otimes(\mathfrak{Bord}_2^-, d_* N^D(\mathbf{B}^2 \mathrm{GL}_1(\mathbb{C})))$$

is constructed by 2-dimensional parallel transport. That is a 2-form B is sent to the field theory Z_B defined by sending a surface Σ to $\exp(\int_\Sigma B) \in \mathrm{GL}_1(\mathbb{C})$. Since the 2-forms B are abelian, this simple construction does in fact give a functor, that is

$$\exp(\int_{\Sigma_1 \sqcup \Sigma_2} B) = \exp(\int_{\Sigma_1} B) \exp(\int_{\Sigma_2} B).$$

This is in contrast to the 1-dimensional case where we worked with with non-abelian forms and path-ordered exponentials.

With this map defined, we need to show the composition is a weak equivalence. We note that

$$N^D(\mathbf{B}^2 \mathrm{GL}_1(\mathbb{C}))(U) = N^D(\mathbf{B}^2 C^\infty(U, \mathrm{GL}_1(\mathbb{C})))$$

is a simplicial abelian group that can be identified, under the Dold-Kan correspondence, with the chain complex concentrated in degree 2 on $C^\infty(U, \mathrm{GL}_1(\mathbb{C}))$. We also note $N^D(\mathbf{B}^2_\nabla \mathrm{GL}_1(\mathbb{C}))(\mathbb{R}^2 \times U \to U)$ is a simplicial abelian group that can be identified, under the Dold-Kan correspondence, with the fiberwise Deligne complex

$$\Omega^2_U(\mathbb{R}^2 \times U; \mathfrak{gl}_1(\mathbb{C})) \xleftarrow{\mathrm{d}} \Omega^1_U(\mathbb{R}^2 \times U; \mathfrak{gl}_1(\mathbb{C})) \xleftarrow{\mathrm{dlog}} C^\infty(\mathbb{R}^2 \times U, \mathrm{GL}_1(\mathbb{C})).$$

Here we have Ω^2 in degree 0. Since we are working with differential forms that are vertical with respect to a submersion with two dimensional fibers, there will be no higher degree forms in this complex. Then as $U \in$ Cart we have exactness of this complex by the Poincaré lemma except in degree 0. The kernel of dlog will consist of functions with a vanishing vertical differential, which are exactly the fiberwise constant functions. Thus this fiberwise Deligne complex is quasi-isomorphic to the complex concentrated on $C^\infty_{fc}(\mathbb{R}^2 \times U, \mathrm{GL}_1(\mathbb{C}))$ in degree 2. As $C^\infty_{fc}(\mathbb{R}^2 \times U, \mathrm{GL}_1(\mathbb{C}))$

itself is isomorphic to $C^\infty(U, \mathrm{GL}_1(\mathbb{C}))$, this quasi-isomorphism of chain complexes induces the desired weak equivalence of simplicial abelian groups under the Dold-Kan correspondence. $\square$

Now we generalize this idea to our case. To mimic the previous proof we would construct a map

$$N^D(2\mathcal{V}^\times_\nabla) \to \mathrm{Fun}^\otimes(\mathfrak{Bord}_2^-, N^M(2\mathcal{V}))$$

from a version of parallel transport. However we already know that

$$N^D(2\mathcal{V}^\times_{fc}) \simeq \mathrm{Fun}^\otimes(\mathfrak{Bord}_2^-, N^M(2\mathcal{V}))$$

by Proposition 3.2.1. So it is easier to just prove that $N^D(2\mathcal{V}^\times_\nabla)$ is weakly equivalent to $N^D(2\mathcal{V}^\times_{fc})$ without reference to field theories.

Theorem 3.3.2. *There is a weak equivalence of simplicial presheaves* $N^D(2\mathcal{V}^\times_\nabla) \simeq N^D(2\mathcal{V}^\times_{fc})$.

Proof. This statement can be reduced to proving $N^D(\mathbf{B}^2_\nabla\mathrm{GL}_1(\mathbb{C}))$ is weakly equivalent to $N^D(\mathbf{B}^2\mathrm{GL}_1(\mathbb{C}))$ considered as a presheaf on FEmb_2. Recall that both $2\mathcal{V}^\times$ and $2\mathcal{V}^\times_\nabla$ have decompositions $2\mathcal{V}^\times = \coprod_{n\geq 0}\mathbf{B}S_n(\mathcal{V})$ and $2\mathcal{V}^\times_\nabla = \coprod_{n\geq 0}\mathbf{B}_\nabla S_n(\mathcal{V})$ respectively, where the components are given by Grothendieck constructions $\mathbf{B}S_n(\mathcal{V}) \simeq \int_{\mathbf{B}S_n}\mathcal{D}$ and $\mathbf{B}_\nabla S_n(\mathcal{V}) \simeq \int_{\mathbf{B}S_n}\mathcal{D}_\nabla$ of the diagrams $\mathcal{D}, \mathcal{D}_\nabla : \mathbf{B}S_n^{op} \to 2\mathrm{Cat}$ discussed earlier in the paper. Taking Duskin nerves gives us that $N^D(\mathbf{B}S_n(\mathcal{V})) \simeq \mathrm{hocolim}_{\mathbf{B}S_n} N^D(\mathcal{D})$ and $N^D(\mathbf{B}_\nabla S_n(\mathcal{V})) \simeq \mathrm{hocolim}_{\mathbf{B}S_n} N^D(\mathcal{D}_\nabla)$. So proving a weak equivalence $N^D(2\mathcal{V}^\times_\nabla) \simeq N^D(2\mathcal{V}^\times_{fc})$ is equivalent to proving a weak equivalence of diagrams $N^D(\mathcal{D}_\nabla) \simeq N^D(\mathcal{D})$. By definition of these diagrams, this reduces to proving a weak equivalence $N^D(\mathbf{B}^2_\nabla\mathrm{GL}_1(\mathbb{C})) \simeq N^D(\mathbf{B}^2\mathrm{GL}_1(\mathbb{C}))$. This is exactly the weak equivalence proven above in the result of Pavlov-Stolz-Teichner.

$\square$

This result can be interpreted as identifying smooth 2-dimensional field theories valued in $2\mathcal{V}$ with unordered tuples of bundle gerbes with fiberwise connections. That is, by the Geometric Cobordism Hypothesis,

$$\mathrm{Fun}^\otimes(\mathfrak{Bord}_2^M, 2\mathcal{V}) \simeq \mathrm{Map}_{\mathrm{FEmb}_2}(M, 2\mathcal{V}^\times_\nabla)$$

for any smooth manifold M (nerves suppressed).

3.4 A connection to Topological Hochschild Homology

We deduced that the homotopy type of the classifying space for field theories valued in KV 2-vector spaces is $\coprod_{n\geq 0} \mathrm{B}^2\mathrm{GL}_1(\mathbb{C})^{\times n}//S_n$, which after group completion produces the connective spectrum $\mathbb{S}[\mathrm{B}^2\mathrm{GL}_1(\mathbb{C})]$. We will observe that the spectrum $\mathbb{S}[\mathrm{B}^2\mathrm{GL}_1(\mathbb{C})]$ is closely related to the Topological Hochschild Homology spectrum of periodic K-theory $THH(KU)$.

Consider the spectrum $\mathbb{S}[\mathrm{BGL}_1(\mathbb{C})]$. This is an E_∞-ring, the free E_∞-ring on the space $\mathrm{BGL}_1(\mathbb{C})$, and we can form the Topological Hochschild Homology spectrum $\mathrm{THH}(\mathbb{S}[\mathrm{BGL}_1(\mathbb{C})])$. This can be computed fairly explicitly as

$$\mathrm{THH}(\mathbb{S}[\mathrm{BGL}_1(\mathbb{C})]) \simeq \mathbb{S}[\mathrm{B}^{cyc}\mathrm{BGL}_1(\mathbb{C})]$$

where B^{cyc} denotes the cyclic bar construction. As $\mathrm{BGL}_1(\mathbb{C})$ is a topological abelian group, we get a homeomorphism

$$\mathrm{B}^{cyc}\mathrm{BGL}_1(\mathbb{C}) \cong \mathrm{BGL}_1(\mathbb{C}) \times \mathrm{B}^2\mathrm{GL}_1(\mathbb{C})$$

which induces on topological hochschild homology

$$\mathrm{THH}(\mathbb{S}[\mathrm{BGL}_1(\mathbb{C})]) \simeq \mathbb{S}[\mathrm{BGL}_1(\mathbb{C})] \otimes_{\mathbb{S}} \mathbb{S}[\mathrm{B}^2\mathrm{GL}_1(\mathbb{C})].$$

Now recall the Snaith theorem [Sna81] (also [Sto20] Thm 5.18), which tells us there exists a map of ring spectra $\mathbb{S}[\mathrm{BGL}_1(\mathbb{C})] \to ku$ which becomes an equivalence after inversion of the Bott element. That is, $\mathbb{S}[\mathrm{BGL}_1(\mathbb{C})][\beta^{-1}] \simeq KU$. In [Sto20], thm 5.21, it is proven that THH commutes with this inversion and we have a result

$$\mathrm{THH}(KU) \simeq KU \otimes_{\mathbb{S}} \mathbb{S}[\mathrm{B}^2\mathrm{GL}_1(\mathbb{C})] = KU[\mathrm{B}^2\mathrm{GL}_1(\mathbb{C})].$$

As KU is non-connective, so is $\mathrm{THH}(KU)$. So we will not be able to produce this spectrum as the group completion of some E_∞ space. However we could possibly produce its connective cover $\tau_{\geq 0}KU[\mathrm{B}^2\mathrm{GL}_1(\mathbb{C})] = ku[\mathrm{B}^2\mathrm{GL}_1(\mathbb{C})]$.

First we recall the definition of the tensor (smash) product of Γ-spaces as in section 1 of [SCH99]. Given two Γ-spaces X and Y, the tensor product is defined by

$$(X \otimes_\Gamma Y)(\langle n\rangle) := \operatorname*{colim}_{\langle k\rangle\wedge\langle \ell\rangle\to\langle n\rangle} X(\langle k\rangle) \wedge Y(\langle \ell\rangle)$$

where $\wedge$ is the smash product of pointed sets/spaces. When the Γ-spaces are "very special" and correspond to connective spectra, this models the smash product of spectra.

Proposition 3.4.1. *Let* $\mathrm{BFFT}^{\times}_{1,N^R(\mathcal{V})}$ *and* $\mathrm{BFFT}^{\times}_{2,N^M(2\mathcal{V})}$ *be the classifying spaces from Section 2.5 and Section 3.2 together with their additive* Γ*-structure. Then* $\Omega^\infty\tau_{\geq 0}\mathrm{THH}(KU)$ *is equivalent to the group completion of the* Γ*-space*

$$\mathrm{BFFT}^{\times}_{1,N^R(\mathcal{V})} \otimes_\Gamma \mathrm{BFFT}^{\times}_{2,N^M(2\mathcal{V})}.$$

Proof. As we noted earlier the tensor/smash product of Γ-spaces models the tensor/smash product of connective spectra. So since we know the group completion of the first classifying space is $\Omega^\infty ku$ and the group completion of the second classifying space is $\Omega^\infty\mathbb{S}[\mathrm{B}^2\mathrm{GL}_1(\mathbb{C})]$, the result is clear. □

CHAPTER 4

∞-GROUPOID COMPLETION OF KV 2-VECTOR SPACES

In the previous chapter, we studied the homotopy type of the space of field theories valued in Kapranov-Voevodsky 2-vector spaces. However because of the cobordism hypothesis, this homotopy type was just determined by the homotopy type of the core of the 2-category of KV 2-vector spaces. While in the 1-dimensional case taking the core does not make a significant difference in the homotopy type, in this 2-dimensional case it makes a huge difference. The core $2\mathcal{V}^\times$ of $2\mathcal{V}$ is much less complicated than $2\mathcal{V}$ itself. The 1-morphisms end up just corresponding to permutation matrices, which means the 2-morphisms all just end corresponding to matrices of 1-dimensional linear maps. So, ultimately, the space of field theories ended up with a relatively simple homotopy type.

In this section we aim to construct a variation with a more complicated homotopy type. We will do this motivated by the work of Baas, Dundas, and Rognes [BDR04] on developing a notion of a 2-vector bundle based on KV 2-vector spaces. They develop a theory of 2-vector bundles that are classified by a space they denote $B\mathrm{GL}_\infty(\mathcal{V})$, which after group completion they prove is equivalent to the iterated K-theory spectrum $K(ku)$. We aim to construct some $(\infty, 2)$-category $\mathcal{C}$, closely related to KV 2-vector spaces, such that the group completion of the classifying space $BFFT^\times_{2,\mathcal{C}}$ is equivalent to $K(ku)$. Some motivation behind this construction is the long standing problem of trying to describe ellipitc cohomology in terms of 2-dimensional field theories. $K(ku)$ is not ellipitc cohomology, though it does sit at the same chromatic level as elliptic cohomology. The target category we construct is fairly artificial, but it does provide a statement somewhat in line with this idea.

4.1 KV 2-Vector Spaces with Weakly Invertible Morphisms

In developing their theory of 2-vector bundles, Baas, Dundas, and Rognes consider a certain 2-category that sits in between $2\mathcal{V}^\times$ and $2\mathcal{V}$. Recall in our skeletonized definition of $2\mathcal{V}$, 1-morphisms are given by $\mathbb{N}$-valued matrices. That is, we have $1\mathrm{Mor}_{2\mathcal{V}}(n, n) = M_n(\mathbb{N})$. When taking the core of $2\mathcal{V}$ we end up taking the invertible elements of this monoid, and end up with $1\mathrm{Mor}_{2\mathcal{V}^\times}(n, n) = M_n(\mathbb{N})^\times = S_n$. Baas, Dundas, and Rognes consider a variation of $2\mathcal{V}$ where we only consider "weakly-invertible" morphisms. That is, instead of considering just truly invertible elements of $M_n(\mathbb{N})$, we consider elements of $M_n(\mathbb{N})$ that become invertible when considered over $\mathbb{Z}$. Let us make a definition.

Definition 4.1.1. We define a 2-catgeory $2\mathcal{V}^w$ as follows:

- The set of objects is $\mathbb{N}$
- The only 1-morphisms are $n \to n$ and are given by elements of $\mathrm{GL}_n(\mathbb{N}) := M_n(\mathbb{N}) \cap \mathrm{GL}_n(\mathbb{Z})$
- A 2-morphism $(a_{ij}) \to (a_{ij})$ is a matrix (T_{ij}) where each $T_{ij} \in \mathrm{GL}_{a_{ij}}(\mathbb{C})$

We note that $2\mathcal{V}^\times \hookrightarrow 2\mathcal{V}^w \hookrightarrow 2\mathcal{V}$. So this 2-category of KV 2-vector spaces with weakly invertible 1-morphisms sits in between the 2-category of 2-vector spaces and its core 2-groupoid. We also note this inherits the bimonoidal structure from $2\mathcal{V}$.

Proposition 4.1.1. *Consider a smooth enhancement of* $2\mathcal{V}^w$*, where for* $U \in \mathrm{Cart}$ *we define* $2\mathcal{V}^w(U)$ *to have the same objects and 1-morphisms as* $2\mathcal{V}^w$*, but for 2-morphisms we consider matrices where each* $T_{ij} \in C^\infty(U, \mathrm{GL}_{a_{ij}}(\mathbb{C}))$*. Then the classifying space* $B2\mathcal{V}^w := \operatorname*{hocolim}_{[n]\in\Delta^{op}} N^D(2\mathcal{V}^w(\Delta^n))$ *is weakly equivalent to the classifying space* $B\mathrm{GL}_\infty(\mathcal{V})$ *defined by Baas, Dundas, and Rognes.*

Proof. The classifying space $B\mathrm{GL}_\infty(\mathcal{V})$ from [BDR04] is shown to be equivalent to the geometric realization of the nerve of a topological version of the 2-category described above in example 3.3 of [BBK12]. To translate this result from the setting of topological categories to the setting of smooth categories we are working in, all we need to note is the shape construction $\operatorname*{hocolim}_{[n]\in\Delta^{op}} C^\infty(\Delta^n, \mathrm{GL}_m(\mathbb{C}))$ is the same thing as the singular simplicial set of $\mathrm{GL}_m(\mathbb{C})$ ([GP23] remark 7.1.5). □

4.2 ∞-Groupoid Completion of $2\mathcal{V}^w$

The failure for $2\mathcal{V}^w$ to be a 2-groupoid is essentially due to the fact that the direct sum of vector spaces is not invertible. So if we want to construct a groupoid completion of $2\mathcal{V}^w$, we have to develop a varation that is built off some variation of $\mathcal{V}$ where the direct sum is invertible. This concept has been well-studied in the past in various formulations. The main idea is that first we need to introduce a notion of "negative" vector spaces. This can be done by passing from vector spaces to two-term chain complexes of vector spaces. However the direct sum of such things is still not genuinely invertible. To make the direct sum invertible, you must pass to concordance classes. So we need to build a version of $2\mathcal{V}^w$ that is based on concordance classes of two-term complexes of vector spaces, instead of just vector spaces.

Let $\mathcal{DSV}$ be the smooth category $U \mapsto \mathcal{DSV}(U)$ where $\mathcal{DSV}(U)$ is defined to be the category of $\mathbb{Z}/2$-graded chain complexes of complex vector bundles over U with chain homotopy classes as morphisms. As with our previous definitions, we will define a (partially) skeletonized version of this.

Definition 4.2.1. $\mathcal{DSV}(U)$

- The objects are tuples (n_0, n_1, d_0, d_1) where $n_0, n_1 \in \mathbb{N}$, $d_0 \in C^\infty(U, M_{n_1 \times n_0}(\mathbb{C}))$, and $d_1 \in C^\infty(U, M_{n_0 \times n_1}(\mathbb{C}))$ such that $d_1 d_0 = 0$ and $d_0 d_1 = 0$.
- The morphisms from $C = (n_0, n_1, d_0, d_1)$ to $C' = (n_0', n_1', d_0', d_1')$ are chain homotopy classes of chain maps $[f]$.
  - A representative is a map $f = (f_0, f_1)$, where $f_0 \in C^\infty(U, M_{n_0' \times n_0}(\mathbb{C}))$ and $f_1 \in C^\infty(U, M_{n_1' \times n_1}(\mathbb{C}))$, such that $d_0' f_0 = f_1 d_0$ and $d_1' f_1 = f_0 d_1$.
  - For two such representatives f, f' we have $[f] = [f']$ if there exists $h' = (h_0', h_1')$ with $h_0' \in C^\infty(U, M_{n_1' \times n_0}(\mathbb{C}))$ and $h_1' \in C^\infty(U, M_{n_0' \times n_1}(\mathbb{C}))$ such that $f_0 - f_0' = d_1' h_1' + h_0' d_0$ and $f_1 - f_1' = d_0' h_0' + h_1' d_1$.

The category is bimonoidal with structures induced from the usual direct sum and tensor product of chain complexes. Now this is not yet quite what we want. What we want is for the monoidal structure $\oplus$ to be invertible, but that is not true for chain complexes. However it is true for concordance classes of chain complexes. So we need to form the concordance space of this groupoid to get what we want.

Definition 4.2.2. [BLM23] Let $\mathcal{F}$ be a smooth simplicial presheaf. We define the concordance simplicial presheaf $L_\mathbb{R}\mathcal{F}$ to be the smooth simplicial presheaf given by

$$U \mapsto \operatorname*{hocolim}_{[n] \in \Delta^{op}} \mathcal{F}(U \times \Delta^n).$$

Here we slightly abuse notation, $\mathcal{F}(U \times \Delta^n) := \mathrm{Map}_{\mathrm{Cart}}(U \times \Delta^n, \mathcal{F})$ as $\Delta^n \notin \mathrm{Cart}$. To explain what this is doing, suppose a and a' are simplices of $\mathcal{F}(U)$, b is a simplex of $\mathcal{F}(U \times \Delta^1)$, and we have equivalences $d_0(b) \cong a$ and $d_1(b) \cong a'$, where d_0 and d_1 are the face maps. Then a and a' are equivalent in $L_{\mathbb{R}}\mathcal{F}(U)$. It is the same for $\mathcal{F}(U \times \Delta^n)$ and the relevant higher face maps. So what this construction is doing is adding in equivalences between simplices that are concordant.

Remark Whenever we write $L_{\mathbb{R}}\mathcal{C}$ for some smooth category or 2-category, we mean $L_{\mathbb{R}}N^-(\mathcal{C})$ for the appropriate nerve construction N^- applied to $\mathcal{C}$.

Theorem 4.2.1. *[BLM23] Theorem 3.20*

There is a weak equivalence of infinite loop spaces

$$L_{\mathbb{R}}\mathcal{DSV}^{\times}(*) \simeq \Omega^{\infty}ku.$$

Moreover, this weak equivalence is compatible with the multiplicative structures.

There is a faithful map $\mathcal{V} \to \mathcal{DSV}$. So what the above result is saying is that if we take cores and pass to concordance, $L_{\mathbb{R}}\mathcal{V}^{\times} \to L_{\mathbb{R}}\mathcal{DSV}^{\times}$, this map will become an equivalence after group completion.

Definition 4.2.3. $2\mathcal{DSV}$

Define $M_{m\times n}(\mathcal{DSV})$ to be the smooth category whose objects are $m \times n$ matrices of objects in $\mathcal{DSV}$ and whose morphisms are $m \times n$ matrices of morphisms in $\mathcal{DSV}$. Then define the smooth 2-category $2\mathcal{DSV}$ to be the 2-category with objects $\mathbb{N}$ and Hom-categories $\mathrm{Hom}(n, m) = M_{m\times n}(\mathcal{DSV})$.

We note this category has a bimonoidal structure and duals, both induced from the the bimonoidal structure and duals in $\mathcal{DSV}$ the same way they are from $\mathcal{V}$ in $2\mathcal{V}$.

Denote by $L_{\mathbb{R}}2\mathcal{DSV}$ the concordance $(\infty, 2)$-category given by

$$L_{\mathbb{R}}2\mathcal{DSV}(U) := \operatorname*{hocolim}_{[n]\in\Delta^{op}} N^M(2\mathcal{DSV}(U \times \Delta^n)).$$

We note this construction is well-defined as an $(\infty, 2)$-category because both completeness and the Segal condition are expressed in terms of finite limits, which commute with filtered colimits in $\mathrm{sPSh}(\Delta^2)$. If we consider the natural fully-faithful map $2\mathcal{V} \to 2\mathcal{DSV}$, passing to concordances as above we get a map $L_{\mathbb{R}}2\mathcal{V} \to L_{\mathbb{R}}2\mathcal{DSV}$, and hence also a map $N^M(2\mathcal{V}) \to L_{\mathbb{R}}2\mathcal{DSV}$. By nature of the concordance construction, this map sends weakly invertible 1-morphisms as defined above to invertible 1-morphisms in the target. So in particular we get an induced map $N^M(2\mathcal{V}^w) \to L_{\mathbb{R}}2\mathcal{DSV}^{\times}$ presenting $L_{\mathbb{R}}2\mathcal{DSV}^{\times}$ as an ∞-groupoid completion of $N^M(2\mathcal{V}^w)$.

Proposition 4.2.1. *$L_{\mathbb{R}}2\mathcal{DSV}^{\times}(*)$ is weakly equivalent to $\coprod_{n\geq 0}\mathrm{BGL}_n(ku)$.*

Proof. Consider the sub 2-category $\coprod_{n\geq 0}\mathbf{B}M_n(\mathcal{DSV})$ of $2\mathcal{DSV}$. Note that $L_{\mathbb{R}}2\mathcal{DSV}^{\times}$ will be contained within $L_{\mathbb{R}}\coprod_{n\geq 0}\mathbf{B}M_n(\mathcal{DSV})$. Recall $M_n(\mathcal{DSV})$ as just a category is equivalent to $\mathcal{DSV}^{n^2}$, the notation M_n simply indicates the monoidal structure. So we can observe $L_{\mathbb{R}}M_n(\mathcal{DSV})$ is a monoidal $(\infty,1)$-category, we will denote $M_n(L_{\mathbb{R}}\mathcal{DSV})$, whose underlying $(\infty,1)$-category is just $L_{\mathbb{R}}\mathcal{DSV}^{n^2}$. Next we can observe that $L_{\mathbb{R}}\mathbf{B}M_n(\mathcal{DSV}) \simeq \mathrm{B}M_n(L_{\mathbb{R}}\mathcal{DSV})$ where B denotes the delooping of a monoidal $(\infty,1)$-category to an $(\infty,2)$-category.

This delooping construction requires some explanation. A monoidal $(\infty,1)$-category is by definition an E_1-object in $(\infty,1)$-categories. E_1-objects in a category can be modeled as simplicial objects in that category satisfying the Segal conditions. So if we are modeling $(\infty,1)$-categories as complete Segal spaces, then monoidal $(\infty,1)$-categories are simplicial objects in the category of complete Segal spaces satisfying the Segal condition. In other words, they are 2-fold Segal spaces complete in one simplicial direction. Apply the completion functor of [Rez01], Section 14, in the second simplicial direction, we obtain a 2-fold complete Segal space. Thus our desired delooping procedure is simply this completion functor. This makes sense because, as noted in [Rez01] Remark 14.1, the completion functor is just a generalization of the traditional classifying space construction.

Now by definition of the core construction we must have an equivalence

$$(\mathrm{B}M_n(L_{\mathbb{R}}\mathcal{DSV}))^{\times} \simeq (\mathrm{B}M_n(L_{\mathbb{R}}\mathcal{DSV}^{\times}))^{\times}.$$

As $L_{\mathbb{R}}\mathcal{DSV}^{\times}(*) \simeq \Omega^{\infty}(ku)$, we see $M_n(L_{\mathbb{R}}\mathcal{DSV}^{\times}(*))$ models $\Omega^{\infty}M_n(ku)$ where $M_n(ku)$ is the matrix E_1-ring as defined in Appendix 2 of this paper. Then the sub ∞-groupoid of $\otimes$-invertible objects in $M_n(L_{\mathbb{R}}\mathcal{DSV}^{\times}(*))$ models the space of homotopy invertible elements in $M_n(ku)$, denoted $\mathrm{GL}_n(ku)$. So we arrive at our desired statement, $(\mathrm{B}M_n(L_{\mathbb{R}}\mathcal{DSV}))^{\times} \simeq \mathrm{BGL}_n(ku)$. $\square$

4.3 Field Theories valued in $L_{\mathbb{R}}2\mathcal{DSV}$

Like earlier in the paper, to work with field theories we need to make one more enhancement of our definition to a presheaf on not just Cart but on $\mathrm{FEmb}_2 \times \mathrm{Cart}$.

For $T \to U \in \mathrm{FEmb}_2$, define $\mathcal{DSV}_{fc}(T \to U)$ to be the same category as definition 4.2.1, but with $C^{\infty}_{fc}(T, M_{-}(\mathbb{C}))$ everywhere we had $C^{\infty}(U, M_{-}(\mathbb{C}))$ before. That is, we have the same definition as before, but take smooth functions on T which are fiberwise constant with respect to $T \to U$, instead of just smooth functions on U. Then in a completely analogous way as before, we can define $2\mathcal{DSV}_{fc}$.

We then define a concordance category $L_{\mathbb{R}}2\mathcal{DSV}_{fc}$ as a presheaf on FEmb_2 by letting

$$L_{\mathbb{R}}2\mathcal{DSV}_{fc}(T \to U) := \underset{[n]\in\Delta^{op}}{\operatorname{hocolim}} \operatorname{Map}_{\mathrm{FEmb}_2}(\Delta^n \times (T \to U), N^M(2\mathcal{DSV}_{fc})).$$

Proposition 4.3.1. *There is a weak equivalence of presheaves*

$$L_{\mathbb{R}}2\mathcal{DSV}^{\times}_{fc} \simeq \operatorname{Fun}^{\otimes}(\mathfrak{Bord}_2^{-}, L_{\mathbb{R}}2\mathcal{DSV})$$

in $\mathrm{sPSh}(\mathrm{FEmb}_2)$.

The proof of this proceeds exactly as in the Proposition 3.2.1. It is just a slight enhancement of the cobordism hypothesis statement from a statement about simplicial presheaves to simplicial presheaves on FEmb_2.

Proposition 4.3.2. *The group completion of the additive* Γ*-space* $\mathrm{BFFT}^{\times}_{2,L_{\mathbb{R}}2\mathcal{DSV}}$ *is the spectrum* $K(ku)$.

Proof. By definition we have that

$$\mathrm{BFFT}^{\times}_{2,L_{\mathbb{R}}2\mathcal{DSV}} := \underset{[n]\in\Delta^{op}}{\operatorname{hocolim}} \operatorname{Fun}^{\otimes}(\mathfrak{Bord}_2^{\Delta^n}, L_{\mathbb{R}}2\mathcal{DSV})^{\times}.$$

The Geometric Cobordism Hypothesis combined with the proposition above tells us

$$\operatorname{Fun}^{\otimes}(\mathfrak{Bord}_2^{\Delta^n}, L_{\mathbb{R}}2\mathcal{DSV})^{\times} \simeq \operatorname{Map}_{\mathrm{sPSh}(\mathrm{FEmb}_2)}(\Delta^n, L_{\mathbb{R}}2\mathcal{DSV}^{\times}_{fc}).$$

So we have that

$$\mathrm{BFFT}^{\times}_{2,L_{\mathbb{R}}2\mathcal{DSV}} := \underset{[n]\in\Delta^{op}}{\operatorname{hocolim}} \operatorname{Map}_{\mathrm{sPSh}(\mathrm{FEmb}_2)}(\Delta^n, L_{\mathbb{R}}2\mathcal{DSV}^{\times}_{fc}).$$

Then we can identify

$$\operatorname{Map}_{\mathrm{sPSh}(\mathrm{FEmb}_2)}(\Delta^n, L_{\mathbb{R}}2\mathcal{DSV}^{\times}_{fc}) \simeq \operatorname{Map}_{\mathrm{sPSh}(\mathrm{FEmbCart}_2)}(\Delta^n, L_{\mathbb{R}}2\mathcal{DSV}^{\times}_{fc})$$

and use the fact $L_{\mathbb{R}}2\mathcal{DSV}^{\times}_{fc}(\mathbb{R}^2 \times U \to U) \simeq L_{\mathbb{R}}2\mathcal{DSV}^{\times}(U)$ to see

$$\operatorname{Map}_{\mathrm{sPSh}(\mathrm{FEmbCart}_2)}(\Delta^n, L_{\mathbb{R}}2\mathcal{DSV}^{\times}_{fc}) \simeq \operatorname{Map}_{\mathrm{sPSh}(\mathrm{Cart})}(\Delta^n, L_{\mathbb{R}}2\mathcal{DSV}^{\times}).$$

Thus we have

$$\mathrm{BFFT}^{\times}_{2,L_{\mathbb{R}}2\mathcal{DSV}} \simeq \mathrm{B}L_{\mathbb{R}}2\mathcal{DSV}^{\times}.$$

Note as the sheaf $L_{\mathbb{R}}2\mathcal{DSV}^{\times}$ is already concordance invariant by definition, $\mathrm{B}L_{\mathbb{R}}2\mathcal{DSV}^{\times}$ is just $L_{\mathbb{R}}2\mathcal{DSV}^{\times}(*)$. The desired claim then follows from proposition 4.2.1. $\square$

APPENDIX A

THE BISIMPLICIAL NERVE CONSTRUCTION

A.1

This appendix reviews the bisimplicial nerve construction of Moser. It is taken from [Mos24] and [MOR22].

The Nerve Functor N^M

The nerve functor, denoted N^M (originally $\mathbb{NH}^{\simeq}$ in Moser's work), maps a 2-category $\mathcal{D}$ to a bisimplicial space $N^M\mathcal{D}$.

For indices $i, j, k \geq 0$, corresponding to the object $([i], [j], [k])$ in $\Delta \times \Delta \times \Delta$, the component space is defined as the set of 2-functors:

$$(N^M\mathcal{D})_{i,j,k} := \mathrm{Fun}(\mathcal{O}_j^{\sim} \otimes_{\mathrm{ic}} (\mathcal{O}_i^{\sim} \otimes_{\mathrm{ps}} \tilde{\mathcal{O}}_k), \mathcal{D}).$$

The components $\mathcal{O}_n^{\sim}$ and $\tilde{\mathcal{O}}_n$ are variations of Street's orientals [Str87], while $\otimes_{\mathrm{ic}}$ and $\otimes_{\mathrm{ps}}$ are versions of the Gray tensor product of 2-categories [GRA76]. We define these as follows:

$\mathcal{O}_2(n)$: **The 2-truncated n-oriental**

This 2-category models the basic pasting diagram of an n-simplex.

- Objects: The set $\{0, 1, \ldots, n\}$.
- Morphisms: For objects x, x', the hom-category $\mathcal{O}_2(n)(x, x')$ is the poset of subsets $I \subseteq \{y \mid x \leq y \leq x'\}$ such that $x, x' \in I$, ordered by inclusion. (This is non-empty only if $x \leq x'$). A morphism corresponds to a path from x to x'.
- 2-Morphisms: A unique 2-morphism $I \Rightarrow J$ exists if $I \subseteq J$. This represents filling in intermediate vertices in a path.

$\mathcal{O}_n^{\sim}$: **Orientals with Invertible 2-Morphisms**

This 2-category is obtained from $\mathcal{O}_2(n)$ by formally making every generating 2-morphism (which corresponds to filling a triangle $x < x' < x''$) invertible.

$\tilde{\mathcal{O}}_n$: Orientals with Adjoint Equivalences

This 2-category is obtained from $\mathcal{O}_n^{\sim}$ by formally promoting every generating 1-morphism (corresponding to an edge $x \to x'$) into an adjoint equivalence.

Tensor Products Involved

The formula uses two specific tensor products from the family of Gray tensor products for 2-categories. These tensor products ($\otimes$) are characterized by their corresponding internal hom 2-categories $[\mathcal{A}, \mathcal{B}]$, which differ primarily in the nature of the 1-cells (natural transformations) they admit between 2-functors $\mathcal{A} \to \mathcal{B}$:

- $\otimes_{\mathrm{ps}}$: The pseudo Gray tensor product. This tensor product is related via adjunction to the internal hom 2-category $[\mathcal{A}, \mathcal{B}]_{\mathrm{ps}}$ whose 1-cells are *pseudonatural transformations*. These transformations relax the strict naturality condition by introducing coherent, structural 2-isomorphisms for each 1-morphism in the source 2-category $\mathcal{A}$.

- $\otimes_{\mathrm{ic}}$: The icon tensor product. This tensor product is related via adjunction to the internal hom 2-category $[\mathcal{A}, \mathcal{B}]_{\mathrm{ic}}$ whose 1-cells are *icons*. Icons are a specific kind of lax natural transformation where the 1-morphism component associated with each object of $\mathcal{A}$ is required to be an identity 1-morphism. The non-trivial structure lies in the 2-morphism components associated with the 1-morphisms of $\mathcal{A}$. [MOR22]Remark 3.10

That is, we could equivalently write

$$(N^M \mathcal{D})_{i,j,k} = \mathrm{Fun}(\mathcal{O}_i^{\sim}, [\tilde{\mathcal{O}}_k, [\mathcal{O}_j^{\sim}, \mathcal{D}]_{\mathrm{ic}}]_{\mathrm{ps}})$$

Example A.1.1. There is a canonical map which is a biequivalence:

$$\mathcal{O}_j^{\sim} \otimes_{\mathrm{ic}} \mathcal{O}_i^{\sim} \to [j] \otimes_{\mathrm{ic}} [i] \cong [i|j, \dots, j]$$

where $[i|j_1, \dots, j_i]$ is the 2-category with objects $\{0, ..., i\}$ and non-empty Hom categories

$$\mathrm{Hom}(a, b) = [j_{a+1}] \times ... \times [j_b]$$

for each pair $0 \leq a \leq b \leq i$.

The key result established by Moser demonstrates the good homotopical behavior of this nerve:

Theorem A.1.1. *[MOR22] Theorem 3.7*

The functor $N^M : 2\mathrm{Cat} \to \mathrm{sPSh}(\Delta^{\times 2})$ *sends a 2-category to a 2-fold complete Segal space.*

Proposition A.1.1. *If* $\mathcal{C}$ *is a 2-groupoid, then* $N^M(\mathcal{C})$ *is weakly equivalent to the diagonal embedding of the Duskin nerve,* $d_* N^D(\mathcal{C})$*, as discussed in chapter 2 of this paper.*

Proof. If $X_{\bullet,\bullet}$ is a 2-fold Segal space modeling an $(\infty, 2)$-category in which all 1 and 2 morphisms are invertible, i.e. an ∞-groupoid, then we must have $X_{p,q} \simeq X_{0,0}$ for all p, q. In other words $X_{\bullet,\bullet}$ is the diagonal embedding of the Kan complex $X_{0,0}$. So in our case we just need to observe $N^M(\mathcal{C})_{0,0} \simeq N^D(\mathcal{C})$. By definition $N^M(\mathcal{C})_{i,0,0} = \mathrm{Fun}(\mathcal{O}_0^{\sim} \otimes_{\mathrm{ic}} (\mathcal{O}_i^{\sim} \otimes_{\mathrm{ps}} \tilde{\mathcal{O}}_0), \mathcal{C})$. So simplifying, $N^M(\mathcal{C})_{i,0,0} = \mathrm{Fun}(\mathcal{O}_i^{\sim}, \mathcal{C})$. Now if our 2-category $\mathcal{C}$ is a 2-groupoid, then every morphism is invertible, so by definition of $\mathcal{O}_i^{\sim}$, we have $N^M(\mathcal{C})_{i,0,0} = \mathrm{Fun}(\mathcal{O}_2(i), \mathcal{C})$. By 5.2 in [MOR22], this is exactly the Duskin nerve. $\square$

Example A.1.2. Let $\mathcal{A}$ be a 2-category. We describe the low-dimensional simplicies of $N^M\mathcal{A}$.

Space of Objects: $(m, k) = (0, 0)$

We describe the space $(N^M\mathcal{A})_{0,0}$.

1. A 0-simplex is the data of an object $A \in \mathcal{A}$.
2. A 1-simplex is the data of an adjoint equivalence $A \xrightarrow{\simeq} C$ in $\mathcal{A}$.
3. A 2-simplex is the data of three adjoint equivalences $(\theta : A \xrightarrow{\simeq} E, \varphi : A \xrightarrow{\simeq} C, \psi : C \xrightarrow{\simeq} E)$ and an invertible 2-morphism $\mu : \theta \cong \psi\varphi$.

Space of Morphisms: $(m, k) = (1, 0)$

We describe the space $(N^M\mathcal{A})_{1,0}$.

1. A 0-simplex is the data of a morphism $f : A \to B$ in $\mathcal{A}$.
2. A 1-simplex is the data of two morphisms $(f : A \to B, g : C \to D)$, two adjoint equivalences $(\varphi_0 : A \xrightarrow{\simeq} C, \varphi_1 : B \xrightarrow{\simeq} D)$, and an invertible 2-morphism $\varphi : g\varphi_0 \cong \varphi_1 f$.
3. A 2-simplex corresponds to data involving two such 1-simplices composed horizontally, related by invertible 2-morphisms. Specifically, it involves data (f, g, h) and $(\varphi_0, \psi_0, \theta_0)$, $(\varphi_1, \psi_1, \theta_1)$ related by invertible 2-cells $\mu_0 : \psi_0\varphi_0 \cong \theta_0$ and $\mu_1 : \psi_1\varphi_1 \cong \theta_1$, satisfying a pasting condition involving the invertible squares φ, ψ, θ from the 1-simplex data.

Space $(m, k) = (0, 1)$

We describe the space $(N^M\mathcal{A})_{0,1}$.

1. A 0-simplex is the data of an adjoint equivalence $u : A \xrightarrow{\simeq} A'$ in $\mathcal{A}$.
2. A 1-simplex is the data of two adjoint equivalences $(u : A \xrightarrow{\simeq} A', w : C \xrightarrow{\simeq} C')$, two other adjoint equivalences $(\varphi : A \xrightarrow{\simeq} C, \varphi' : A' \xrightarrow{\simeq} C')$, and an invertible 2-morphism $\widetilde{\varphi} : w\varphi \cong \varphi' u$.
3. A 2-simplex corresponds to data involving two such 1-simplices composed vertically, related by invertible 2-morphisms. Specifically, it involves data (u, w, y) and (φ, ψ, θ), $(\varphi', \psi', \theta')$ related by invertible 2-cells $\mu : \psi\varphi \cong \theta$ and $\mu' : \psi'\varphi' \cong \theta'$, satisfying a pasting condition involving the invertible squares $\widetilde{\varphi}, \widetilde{\psi}, \widetilde{\theta}$ from the 1-simplex data.

This space captures the structure of adjoint equivalences and the homotopies relating them. It is homotopically equivalent to the space of objects $(N^M\mathcal{A})_{0,0}$.

Space of 2-Morphisms (Squares): $(m, k) = (1, 1)$

We describe the space $(N^M\mathcal{A})_{1,1}$.

1. A 0-simplex is the data of a 2-morphism $\alpha : vf \Rightarrow f'u$ in $\mathcal{A}$, where $u : A \xrightarrow{\simeq} A'$ and $v : B \xrightarrow{\simeq} B'$ are adjoint equivalences, and $f : A \to B$, $f' : A' \to B'$ are 1-morphisms.

2. A 1-simplex relates two such 0-simplices (α, u, v, f, f') and (β, w, x, g, g') via four adjoint equivalences ($\varphi_0 : A \overset{\simeq}{\to} C, \varphi_1 : B \overset{\simeq}{\to} D, \varphi'_0 : A' \overset{\simeq}{\to} C', \varphi'_1 : B' \overset{\simeq}{\to} D'$) and four invertible 2-morphisms ($\varphi : g\varphi_0 \cong \varphi_1 f, \varphi' : g'\varphi'_0 \cong \varphi'_1 f', \widetilde{\varphi_0} : w\varphi_0 \cong \varphi'_0 u, \widetilde{\varphi_1} : x\varphi_1 \cong \varphi'_1 v$) which satisfy a pasting condition relating $\beta \circ \widetilde{\varphi_0} \circ \varphi$ and $\varphi' \circ \widetilde{\varphi_1} \circ \alpha$.

3. A 2-simplex relates compositions of such 1-simplices via invertible 2-morphisms.

APPENDIX B

ALGEBRAIC K-THEORY OF CONNECTIVE RING SPECTRA

B.1

This appendix provides an overview of one standard construction of the algebraic K-theory spectrum $K(R)$ associated to a connective ring spectrum R. This approach generalizes the classical group completion construction for algebraic K-theory of ordinary rings. We develop this generally in the modern style of [Lur17] but refer back to [Elm+97] for connection to the traditional approach.

Matrix Ring Spectra

Definition A.1. Let R^n denote the wedge sum of n copies of R:

$$R^n := \bigvee_{k=1}^{n} R.$$

The $n \times n$ matrix ring spectrum $M_n(R)$ is defined as the internal hom of R^n to itself in the stable $(\infty, 1)$-category of R-module spectra (in the sense of chapter 7 of [Lur17]):

$$M_n(R) := \mathrm{Hom}_R(R^n, R^n).$$

Equivalently, this is the derived internal hom in any model category which presents the $(\infty, 1)$-category of R-module spectra. Composition of endomorphisms provides the multiplicative structure of the ring spectrum $M_n(R)$.

Remark The internal Hom interacts with finite coproducts and finite products as follows:

1. $\mathrm{Hom}_R(\bigvee_{k=1}^n X_k, Y) \simeq \prod_{k=1}^n \mathrm{Hom}_R(X_k, Y)$
2. $\mathrm{Hom}_R(X, \prod_{j=1}^n Y_j) \simeq \prod_{j=1}^n \mathrm{Hom}_R(X, Y_j)$

So since finite coproducts and products are equivalent, and $\mathrm{Hom}_R(R, R) \simeq R$, we can have that $M_n(R) \simeq R^{n\times n}$ as spectra.

General Linear Groups

For a ring spectrum A, the space of homotopy units $\mathrm{GL}_1(A)$, is defined as the homotopy pullback of the diagram

$$\pi_0(\Omega^\infty A)^\times \to \pi_0(\Omega^\infty A) \leftarrow \Omega^\infty A$$

where $\Omega^\infty A$ is underlying infinite loop space of A, $\pi_0(A)$ is the ring of path components, and $(\pi_0 A)^\times$ is its group of units. We can then define

$$\mathrm{GL}_n(R) := \mathrm{GL}_1(M_n(R)).$$

K-theory

Now that we have defined $\mathrm{GL}_n(R)$, we can form the classifying space

$$\mathrm{BGL}(R) := \mathrm{hocolim}_n \mathrm{BGL}_n(R).$$

One example of a model category that presents the stable infinity category we are working with is the category of R-modules as defined in Chapter 4 of [Elm+97]. So our $\Omega^\infty M_n(R)$ is weakly equivalent to their $\tilde{M}_n(R)$ defined in Section 7 of Chapter 6. And our $\mathrm{GL}_n(R)$ is weakly equivalent to their $\tilde{\mathrm{GL}}_n(R)$. So using the results of [Elm+97] Chapter 7, we can see our $\mathrm{BGL}(R)$ is an E_∞-space and we can group complete this space to an infinite loop space

$$\Omega^\infty K(R) := \Omega B(\mathrm{BGL}(R)).$$

The Algebraic K-theory spectrum of R, $K(R)$, is the connective spectrum corresponding to this infinite loop space.

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