APPROXIMATE DIAGONALIZATION OF HOMOMORPHISMS

by

MIN YONG RO

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Student: Min Yong Ro

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This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Huaxin Lin	Chair
Weiyong He	Core Member
N. Christopher Phillips	Core Member
Nicholas Proudfoot	Core Member
Li-Shan Chou	Institutional Representative

and

J. Andrew Berglund	Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

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DISSERTATION ABSTRACT

Min Yong Ro

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In this dissertation, we explore the approximate diagonalization of unital homomorphisms between C^*-algebras. In particular, we prove that unital homomorphisms from commutative C^*-algebras into simple separable unital C^*-algebras with tracial rank at most one are approximately diagonalizable. This is equivalent to the approximate diagonalization of commuting sets of normal matrices.

We also prove limited generalizations of this theorem. Namely, certain injective unital homomorphisms from commutative C^*-algebras into simple separable unital C^*-algebras with rational tracial rank at most one are shown to be approximately diagonalizable. Also unital injective homomorphisms from AH-algebras with unique tracial state into separable simple unital C^*-algebras of tracial rank at most one are proved to be approximately diagonalizable. Counterexamples are provided showing that these results cannot be extended in general.

Finally, we prove that for unital homomorphisms between AF-algebras, approximate diagonalization is equivalent to a combinatorial problem involving sections of lattice points in cones.

CURRICULUM VITAE

NAME OF AUTHOR: Min Yong Ro

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:
University of Oregon, Eugene, OR
Ohio State University, Columbus, OH

DEGREES AWARDED:
Doctor of Philosophy, Mathematics, 2015, University of Oregon
Master of Science, Mathematics, 2010, University of Oregon
Bachelor of Science, Mathematics, 2008, Ohio State University
Bachelor of Science, Economics, 2008, Ohio State University

AREAS OF SPECIAL INTEREST:
C^*-algebras

PROFESSIONAL EXPERIENCE:

Graduate Teaching Fellow, University of Oregon, 2008–2015

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For my parents and my brother.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

One of the most significant theorem of linear algebra is the spectral theorem which is often stated in the following way:

Theorem I.1 (Spectral Theorem)**.** *Let $n \geq 1$ be an integer and let $a \in M_n(\mathbb{C})$ be given. Then a is a normal matrix if and only if there exist $\lambda_i \in \mathbb{C}$ for $i = 1, 2, \ldots, n$ and a unitary matrix $u \in M_n(\mathbb{C})$ such that*

$$uau^* = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}.$$

In this statement, the spectral theorem is apparently a statement regarding the algebraic structure of finite-dimensional operators and thus about C^*-algebras. This connection can be made more apparent by noticing that the following statement implies the spectral theorem:

Theorem I.2. *Let X be a compact Hausdorff space and let $n \geq 1$ be an integer. We denote by $C(X)$ the C^*-algebra of complex-valued continuous functions on X with pointwise operations and supremum norm. We denote by M_n the C^*-algebra of $n \times n$ complex matrices with operator norm. For every unital homomorphism $\phi\colon C(X) \to M_n$, there exist points $\xi_i \in X$ for $i = 1, 2, \ldots, n$ and a unitary $u \in M_n$*

such that

$$u\phi(f)u^* = \begin{pmatrix} f(\xi_1) & 0 & \cdots & 0 \\ 0 & f(\xi_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & f(\xi_n) \end{pmatrix}$$

for all $f \in C(X)$.

Indeed, we can see that Theorem I.1 follows by applying this theorem to the case where $X = \mathrm{sp}(a)$ is the spectrum of a and ϕ is the unital homomorphism induced by continuous functional calculus. In addition to being a C^*-algebraic statement, this theorem can be proved using C^*-algebraic techniques.

Proof. There is the induced injective unital homomorphism $\bar{\phi}\colon C(X)/\ker\phi \to M_n(\mathbb{C})$. By Gelfand's representation theorem, there exists a compact space Y such that $C(X)/\ker\phi \cong C(Y)$. Since $M_n(\mathbb{C})$ is finite-dimensional and ϕ is injective, $C(Y)$ is finite-dimensional and so Y is finite. So there exists an integer $k \geq 1$ such that $C(Y) \cong \mathbb{C}^k$. And so ϕ can be written as the composition of two unital homomorphisms $\alpha\colon C(X) \to \mathbb{C}^k$ and $\beta\colon \mathbb{C}^k \to M_n$.

Let $\pi_j\colon \mathbb{C}^k \to \mathbb{C}$ denote the the jth coordinate projection map for $j = 1, 2, \ldots, k$. Since $\pi_j \circ \alpha$ is a homomorphism from $C(X)$ to $\mathbb{C}$, there exist points ζ_j for $i = 1, 2, \ldots, k$ such that $\pi_j \circ \alpha(f) = f(\zeta_j)$ for all $f \in C(X)$. So

$$\alpha(f) = (f(\zeta_1), f(\zeta_2), \ldots, f(\zeta_k))$$

for all $f \in C(X)$.

Let e_j denote the jth standard basis vector of $\mathbb{C}^k$ for $j = 1, 2, \ldots, k$. Since the projections e_j are mutually orthogonal, the projections $q_j = \beta(e_j)$ are mutually orthogonal and $\sum_{j=1}^{k} q_j = 1$.

Let r_j denote the rank of q_j for $j = 1, 2, \ldots, k$. Let $s_j = \sum_{i=1}^{j} r_i$ for $j = 1, 2, \ldots, k$. Also we set $s_0 = 1$. There exist mutually orthogonal rank one projections p_i for $i = 1, 2, \ldots, n$ such that

$$\sum_{i=s_{j-1}}^{s_j} p_i = q_j$$

for $j = 1, 2, \ldots, k$.

Since

$$\phi(f) = \beta(\alpha(f)) = \sum_{j=1}^{k} f(\zeta_j) q_j,$$

we set $\xi_i = \zeta_j$ for $j = 1, 2, \ldots, k$ and i such that $s_j \leq i \leq s_{j+1}$. So we have

$$\phi(f) = \sum_{i=1}^{n} f(\xi_i) p_i.$$

Finally, there exists a unitary matrix $u \in M_n$ such that $u p_i u^* = e_{i,i}$ for $i = 1, 2, \ldots, n$, where $e_{i,i} \in M_n$ is the matrix with 1 in the i, i position and 0 otherwise. So we have

$$u\phi(f)u^* = \sum_{i=1}^{n} f(\xi_i) u p_i u^* = \begin{pmatrix} f(\xi_1) & 0 & \cdots & 0 \\ 0 & f(\xi_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & f(\xi_n) \end{pmatrix}.$$

□

We will return to this homomorphism perspective in a moment, but considering Theorem I.1, we notice that this formulation lends itself to a natural generalization. In particular, if we replace $\mathbb{C}$ with a complex involutive algebra A, we can consider the algebra $M_n(A)$ of $n \times n$ matrices with entries in A and ask the question: when is every normal matrix in $M_n(A)$ unitarily equivalent to a diagonal matrix? The converse is obvious, since a diagonal matrix is normal if and only if each of its entries is normal. This question is particularly pertinent in the case where A is a C^*-algebra due to the prevalence of amplification as a technique in proofs.

It was with this generalization in mind that Richard Kadison proved the following:

Theorem I.3 (Corollary 3.20 of [12])**.** *Let N be a von Neumann algebra and let $n \geq 1$ be an integer. For any normal matrix $a \in M_n(N)$, there exist $a_i \in N$ for $i = 1, 2, \ldots, n$ such that*

$$uau^* = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_n \end{pmatrix}.$$

To further understanding in the case of general C^*-algebras, Kadison posed the question: for what topological spaces X is every normal matrix over $C(X)$ diagonalizable? In the same year, Karsten Grove and Gert Pedersen gave a full answer to this question:

Theorem I.4 (Theorem 5.6 of [8])**.** *Let X be a compact Hausdorff space and $n \geq 1$ an integer. For every normal matrix $f \in M_n(C(X))$, there exist $f_i \in C(X)$ for*

$i = 1, 2, \ldots, n$ and a unitary matrix $u \in M_n(A)$ such that

$$ufu^* = \begin{pmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & f_n \end{pmatrix}$$

if and only if

1. *X is sub-Stonean,*
2. *$\dim X \leq 2$,*
3. *$H^1(X_0, S_m)$ is trivial for every closed subset $X_0 \subseteq X$ and all m, where S_m denotes the symmetric group on m generators, and*
4. *$H^2(X_0, \mathbb{Z})$ is trivial for every closed subset $X_0 \subseteq X$.*

Theorem I.4 suggests that diagonalization is rare in general. Beyond the restrictive cohomological conditions, X being sub-Stonean corresponds to $C(X)$ being a SAW^*-algebra. While a significant concept, it does not reflect the behavior of more general C^*-algebras. For example, the only compact sub-Stonean spaces for which $C(X)$ is separable are those consisting of finitely many points (Corollary 1.6 of [9]).

With the commutative case as a guide, we should expect diagonalization only in classes of C^*-algebras related to von Neumann algebras, such as the class of AW^*-algebras or SAW^*-algebras. In particular, the proof of Theorem I.3 is based on the abundance of projections in maximal abelian self-adjoint subalgebras in von Neumann algebras, which does not hold in general. In fact, a generalization of Kadison's result has been made by Chris Heunen and Manuel Reyes in [10], where the von Neumann algebra N is replaced with an AW^*-algebra, where there is similar behavior in its maximal abelian self-adjoint subalgebras. Conversely, constructing certain maximal abelian self-adjoint subalgebras with few projections would be sufficient to show that diagonalization does not generally hold in that C^*-algebra.

As an analytic method, when we know that a certain equation cannot be solved exactly, we turn to approximations. Following this principle, since diagonalization seems rare, we consider an approximate version.

Definition I.5. Let A and B be unital C^*-algebras and let $n \geq 1$ be an integer. A unital homomorphism $\phi\colon A \to M_n(B)$ is *approximately diagonalizable* if for every $\varepsilon > 0$ and every finite set $\mathcal{F} \subseteq A$, there exist unital homomorphisms $\phi_i\colon A \to B$ for $i = 1, 2, \ldots, n$ and a unitary $u \in M_n(B)$ such that

$$\left\| u\phi(a)u^* - \begin{pmatrix} \phi_1(a) & 0 & \cdots & 0 \\ 0 & \phi_2(a) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \phi_n(a) \end{pmatrix} \right\| < \varepsilon$$

for all $a \in \mathcal{F}$.

A matrix $a \in M_n(A)$ is *approximately diagonalizable* if for every $\varepsilon > 0$, there exist $a_i \in A$ for $i = 1, 2, \ldots, n$ and a unitary $u \in M_n(A)$ such that

$$\left\| u\phi(a)u^* - \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_n \end{pmatrix} \right\| < \varepsilon.$$

Notice that a normal matrix $a \in M_n(A)$ is approximately diagonalizable if and only if the unital homomorphism induced by continuous functional calculus is approximately diagonalizable.

We see that approximate diagonalization applies far more widely than diagonalization does. Yifeng Xue proves in [30] that if X is a compact metric space such that $\dim(X) \leq 2$ and $\check{H}^2(X, \mathbb{Z}) = 0$, then every self-adjoint matrix over $C(X)$ is approximately diagonalizable. If in addition to the conditions above, $\check{H}^1(X, \mathbb{Z}) = 0$, then every unitary matrix is approximately diagonalizable. Also, Huaxin Lin proves in [21] that if X is locally an absolute retract and Y has $\dim(Y) \leq 2$, then every unital homomorphism from $C(X)$ to $M_n(C(Y))$ is approximately diagonalizable for any integer $n \geq 1$.

On the non-commutative side, Shuang Zhang proves in [31] that projections in a C^*-algebra of real rank zero are diagonalizable and that therefore any matrix that can be approximated by a matrix of finite spectrum. In particular the self-adjoint matrices are approximately diagonalizable. Unfortunately, when K_1 is non-trivial, normal matrices cannot generally be approximated by matrices of finite spectrum.

We point out that the definition of approximate diagonalization was chosen to allow the choice of diagonal entries to rely on ε. This is the notion used in all of the previous work. But there are times when a slightly stronger version holds. To explain, we make the following definitions.

Definition I.6. Let A and B be two unital C^*-algebras. Two unital homomorphisms $\phi\colon A \to B$ and $\psi\colon A \to B$ are *approximately unitarily equivalent* if for every $\varepsilon > 0$ and every finite set $\mathcal{F} \subseteq A$, there exists a unitary $u \in B$ such that

$$\|\phi(a) - u\psi(a)u^*\| < \varepsilon$$

for all $a \in \mathcal{F}$.

So a homomorphism being approximately unitarily equivalent to a diagonal homomorphism is equivalent to being approximately diagonalizable where the diagonal homomorphisms do not depend on the choice of ε.

The main tool for this dissertation comes from the classification of homomorphisms from AH-algebras up to approximate unitary equivalence, which we discuss in Chapter II after reviewing some basic definitions for C^*-algebras and the invariants used in the classification of C^*-algebras. In Chapter III, we review some partially ordered abelian group theory. In Chapter IV, we prove that homomorphisms from commutative C^*-algebras to C^*-algebras of tracial rank at most one are approximately diagonalizable, which implies the approximate diagonalization of normal matrices over those C^*-algebras. We also show that certain homomorphisms from commutative C^*-algebras to C^*-algebras of rational tracial rank at most one are approximately diagonalizable, but that these homomorphisms are not generally approximately diagonalizable. In Chapter V, we show that approximate diagonalization holds generally when the domain has a unique tracial state or when the codomain has divisible K_0. We finally show that for AF-algebras with finitely generated K_0, approximate diagonalization is equivalent to a combinatorial problem involving lattice points in cones.

CHAPTER II

C^*-ALGEBRAS AND THEIR INVARIANTS

Basic C^*-Algebra Theory

For the sake of completeness, we include some of the basic definitions of C^*-algebras. The primary references used for the material in this section are [13] and [3].

Definition II.1.1. A *Banach algebra* is a pair $(A, \|\cdot\|)$ of an associative algebra A and a submultiplicative norm $\|\cdot\|$ on A such that the metric induced by $\|\cdot\|$ is complete.

A Banach algebra A is a *C^*-algebra* if there exists an operation $a \mapsto a^*$ on A such that

1. $(a+b)^* = a^* + b^*$,
2. $(\lambda a)^* = \bar{\lambda} a^*$,
3. $(ab)^* = b^* a^*$,
4. $(a^*)^* = a$, and
5. $\|a^* a\| = \|a\|^2$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$.

A subalgebra B of A is called a *C^*-subalgebra* if $b \in B$ implies $b^* \in B$ and B is a closed set. In other words, B is a C^*-subalgebra if B itself is a C^*-algebra. We say that B is a *unital C^*-subalgebra* if the unit of A is contained in B.

A C^*-algebra is *unital* if it contains a multiplicative identity 1. We note that $1^* = 1$ and $\|1\| = 1$ follow from the properties above.

A C^*-algebra is *simple* if it has no nontrivial closed two-sided ideals.

A C^*-algebra is *separable* if it contains a countable dense subset.

Based on operator theory language, we have the following notions for elements of a C^*-algebra.

Definition II.1.2. Let A be a C^*-algebra. Let $a \in A$. We say that a is *self-adjoint* if $a^* = a$.

We say that a is *normal* if $a^*a = aa^*$.

We say that a is a *projection* if $a = a^* = a^2$.

When A is unital, we say that a is *unitary* if $a^*a = aa^* = 1$.

We say that a is *positive* if there exists $b \in A$ such that $a = b^*b$.

Furthermore, we denote the set of self-adjoint elements of A by A_{sa}, the group of unitaries of A by $U(A)$, and the set of positive elements of A by A_+. For $a \in A_+$, we will write $a \geq 0$.

We define a partial ordering on A_{sa} by $a \leq b$ if and only if $b - a \in A_+$. A C^*-subalgebra B of A is called *hereditary* if for any $a \in A$ and $b \in B$, the inequality $0 \leq a \leq b$ implies $a \in B$.

While an algebraic homomorphism between Banach algberas may not be continuous, an algebraic homomorphism between C^*-algebras that preserves the adjoint operation is even contractive.

Definition II.1.3. Let A and B be C^*-algebras. A function $\phi\colon A \to B$ is a *homomorphism* if

1. $\phi(a + b) = \phi(a) + \phi(b)$,
2. $\phi(\lambda a) = \lambda\phi(a)$,
3. $\phi(a^*) = \phi(a)^*$, and
4. $\phi(ab) = \phi(a)\phi(b)$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$. If in addition A and B are unital, a homomorphism $\phi\colon A \to B$ is *unital* if $\phi(1) = 1$.

Often, homomorphisms are too restrictive. At the same time, linear maps are too general and do not reflect any of the algebraic properties of the C^*-algebra. One of the proper balances is to use positive linear maps and in particular, the positive linear functionals.

Definition II.1.4. Let A and B be C^*-algebras. A linear map $\phi\colon A \to B$ is *positive* if $\phi(A_+) \subseteq B_+$.

If A is a unital C^*-algebra, a positive linear map $\sigma\colon A \to \mathbb{C}$ is a *state* if $\sigma(1) = 1$. A state τ is a *tracial state* if $\tau(ab) = \tau(ba)$ for all $a, b \in A$.

The set of tracial states of A is denoted $T(A)$ and is called the *tracial state space*.

An important method of construction for C^*-algebras is to consider inductive limits (in categorical language, colimits) of well-known C^*-algebras. More precisely, an inductive limit of C^*-algebras is a colimit in the category of C^*-algebras or unital C^*-algebras indexed by the category whose objects are positive integers and for which a morphism from m to n exists if and only if $m \leq n$.

Put concretely, for any sequence of C^*-algebras (A_n) with homomorphisms $\phi_n\colon A_n \to A_{n+1}$, there exists a C^*-algebra A, unique up to homomorphism, and homomorphisms $\phi_{n,\infty}\colon A_n \to A$ such that $\phi_{n,\infty} = \phi_{n+1,\infty} \circ \phi_n$ for all n, and A is the smallest such C^*-algebra in the sense that for any C^*-algebra B and any sequence of homomorphisms $\psi_n\colon A_n \to B$ such that $\psi_n = \psi_{n+1} \circ \phi_n$ for all n, there exists a unique homomorphism from $\psi\colon A \to B$ such that $\psi \circ \phi_{n,\infty} = \psi_n$ for all n.

When a C^*-algebra is isomorphic to an inductive limit of finite-dimensional C^*-algebras, we say that it is an *AF-algebra*. When a C^*-algebra A is isomorphic to an inductive limit of C^*-algebras of the form $pM_n(C(X))p$, where $p \in M_n(C(X))$ is a projection and X is a finite CW-complex, we say that A is an *AH-algebra*. We note that every compact metric space can be written as the inverse limit of finite CW-complexes by using the geometric realizations of the nerves of finite open covers. As a result, every separable, commutative, unital C^*-algebra is a unital AH-algebra. Also by taking finite sets for X, we see that every AF-algebra is an AH-algebra.

We will consider the AF-algebra $\mathcal{Q}$ particularly. This algebra $\mathcal{Q}$ is defined as the inductive limit of $M_{n!}$ with connecting maps defined by

$$a \mapsto \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a \end{pmatrix},$$

where the latter is a block diagonal matrix consisting of $n+1$ square blocks of size $n!$.

Our main interest will be in tensoring $\mathcal{Q}$ with other C^*-algebras. Generally, tensor products are complicated in C^*-algebras, but in the case where are tensoring with an AF-algebra, we can described the situation concretely. Namely the C^*-algebra $A \otimes \mathcal{Q}$ is isomorphic to the inductive limit of $M_{n!}(A)$ with analogous connecting maps to the ones above.

We will also be considering the Jiang-Su algebra, denoted $\mathcal{Z}$. The Jiang-Su algebra is isomorphic to the inductive limit of certain C^*-subalgebras of $C([0,1], M_n)$ known as dimension drop interval algebras. See [11] for a proper definition and more information. We briefly mention the definition to note that as with $\mathcal{Q}$, the tensor product with $\mathcal{Z}$ is well-defined even without the general theory of tensor products.

Of particular interest are the C^*-algebras for which tensoring with $\mathcal{Z}$ gives us the same C^*-algebra up to isomorphism. We will say that A is *$\mathcal{Z}$-absorbing* if $A \otimes \mathcal{Z} \cong A$.

Critical to understanding C^*-algebras are the various notions of rank, which try to generalize the notion of covering dimension for topological spaces. The most important for this dissertation is the tracial rank, but its connection with real rank and stable rank are worth mentioning.

Definition II.1.5. Let A be a C^*-algebra. For any integer $n \geq 0$, the *real rank of A is at most n*, written $\mathrm{RR}(A) \leq n$, if for every $n+1$ elements $a_1, a_2, \ldots, a_{n+1} \in A_{\mathrm{sa}}$ and $\varepsilon > 0$, there exist $n+1$ elements $b_1, b_2, \ldots, b_{n+1} \in A_{\mathrm{sa}}$ such that $\sum b_k^* b_k$ is invertible and

$$\left\| \sum_{k=1}^{n} (a_k - b_k)^*(a_k - b_k) \right\| < \varepsilon.$$

We write $\mathrm{RR}(A) = n$ if $\mathrm{RR}(A) \leq n$ and $\mathrm{RR}(A) \not\leq n-1$, and say that A has real rank n.

The case $n = 0$ is of particular interest. A C^*-algebras has real rank zero if the invertible self-adjoint elements are dense in the self-adjoint elements. A C^*-algebra having real rank zero is equivalent to the property FS, i.e. self-adjoint elements with finite spectrum are dense in the set of self-adjoint elements. See Theorem 3.2.5 of [13] or Theorem 2.6 of [1]. This is why projections being (simultaneously) diagonalizable in a C^*-algebra of real rank zero implies that self-adjoint matrices are approximately diagonalizable in the same C^*-algebra as noted in Chapter I.

As stated above, real rank is a generalization of covering dimension of a topological space. In particular, if X is a compact metric space, then $\mathrm{RR}(C(X)) = \dim(X)$ (see Corollary 3.2.10 of [13] or Proposition 1.1 of [1]).

Definition II.1.6. Let A be a C^*-algebra. For any integer $n \geq 0$, the *(topological) stable rank of A is at most n*, written $\mathrm{tsr}(A) \leq n$, if for every n elements $a_1, a_2, \ldots, a_n \in A$ and $\varepsilon > 0$, there exists n elements $b_1, b_2, \ldots, b_n \in A$ such that $\sum b_k^* b_k$ is invertible and

$$\left\| \sum_{k=1}^{n} (a_k - b_k)^*(a_k - b_k) \right\| < \varepsilon.$$

We write $\mathrm{tsr}(A) = n$ if $\mathrm{tsr}(A) \leq n$ and $\mathrm{tsr}(A) \not\leq n - 1$, and say that A has (topological) stable rank n.

Notice that a C^*-algebra has stable rank one if the invertible elements are dense in the C^*-algebra. We will be exclusively concerned with the stable rank one case. In particular, C^*-algebras of stable rank one are stably finite in the following sense. See Propositon 3.3.4 of [13].

Definition II.1.7. Let A be a unital C^*-algebra. We say that A is *finite* if $x^*x = 1$ implies $xx^* = 1$ for all $x \in A$. We say that A is *stably finite* if $M_n(A)$ is finite for every integer $n \geq 1$.

Finally, we define the notion of tracial rank.

Definition II.1.8. For every integer n, we denote by $\mathcal{I}_n$ the class of C^*-algebras consisting of unital hereditary C^*-subalgebras of C^*-algebras of the form $C(X) \otimes F$ where X is a finite CW-complex with $\dim(X) \leq n$ and F is a finite-dimensional C^*-algebra.

Let A be a unital simple C^*-algebra. For any integer $n \geq 0$, the *tracial rank of A is at most n*, written $\mathrm{TR}(A) \leq n$ if for any $\varepsilon > 0$, any finite set $\mathcal{F} \subseteq A$ and any nonzero element $a \in A_+$, there exist a nonzero projection $p \in A$ and a unital C^*-subalgebra $B \in \mathcal{I}_n$ of pAp such that

1. $\|px - xp\| < \varepsilon$ for all $x \in \mathcal{F}$,
2. $\inf\{\|pxp - y\| \colon y \in B\} < \varepsilon$ for all $x \in \mathcal{F}$,
3. $1 - p \sim q$ (see Definition II.2.1 below) for some projection $q \in \overline{aAa}$.

We write $\mathrm{TR}(A) = n$ if $\mathrm{TR}(A) \leq n$ and $\mathrm{TR}(A) \nleq n-1$ and we say that A has *tracial rank n*.

We note that if $\mathrm{TR}(A) < \infty$, then $\mathrm{RR}(A) \leq 1$ and $\mathrm{tsr}(A) = 1$ (Theorem 6.9 of [13]).

Elements of K-Theory

From the noncommutative topology viewpoint of C^*-algebras, we consider the K-theory of C^*-algebras, which is closer to topological K-theory than to algebraic K-theory. We will consider a version of algebraic K-theory in Section II.4.

Let A be unital C^*-algebra. We denote by $M_\infty(A)$ the algebraic inductive limit (in other words, the sequential colimit in the category of normed involutive algebras) of $M_n(A)$ with connecting maps

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Definition II.2.1. Two projections p and q in an involutive algebra A are *Murray-von Neumann equivalent*, and we write $p \sim q$, if there exists an element $v \in A$ such that $p = v^*v$ and $q = vv^*$.

We say that two projections p and q in $M_\infty(A)$ are *stably equivalent* if there exist integers $m, n \geq 0$ such that

$$\begin{pmatrix} p & 0 \\ 0 & 1_{M_m(A)} \end{pmatrix} \sim \begin{pmatrix} q & 0 \\ 0 & 1_{M_n(A)} \end{pmatrix}.$$

Let $V(A)$ denote the stable equivalence classes of projections in $M_\infty(A)$. We denote the equivalence class of p by $[p]$. Then $V(A)$ is a semigroup with addition defined by

$$[p \oplus q] = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

We define $K_0(A)$ to be the Grothendieck group of $V(A)$. Further, by considering $V(A)$ as a cone in $K_0(A)$, we can consider $K_0(A)$ as a pre-ordered abelian group (i.e. $K_0(A)$ has a translation-invariant pre-order). When A is stably finite, $K_0(A)$ is a partially ordered abelian group. Also, when A is unital, it is easy to see that $[1_A]$ is an order unit, since by definition of addition, we have $n[1_A] = [1_{M_n(A)}]$. See Definition III.1.1.

K_0 is a functor from the category of unital stably finite C^*-algebras to the category of partially ordered abelian groups with order units. For a unital homomorphism ϕ between unital C^*-algebras, we denote by $K_0(\phi)$ the homomorphism induced by the functor K_0. More generally, K_0 is a functor from the category of C^*-algebras to the category of preordered abelian groups.

For every integer $n \geq 1$, we denote by $U^n(A)$ the group of unitaries in $M_n(A)$. Let $U_0(A)$ denote the connected component of $U(A)$ containing 1_A and let $U_0^n(A)$ denote the connected component of $U^n(A)$ containing $1_{M_n(A)}$. We note that $U_0^n(A)$ is a normal subgroup of $U^n(A)$.

We define $K_1(A) = \varinjlim U^n(A)/U_0^n(A)$ with connecting homomorphisms

$$U^n(A)/U_0^n(A) \to U^{n+1}(A)/U_0^{n+1}(A)$$

defined by

$$u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

We note that $K_1(A)$ is an abelian group. Also when A has stable rank one, the stabilization is unnecessary in the sense that $U(A)/U_0(A) \to K_1(A)$ is an isomorphism. For a unital homomorphism ϕ between unital C^*-algebras, we denote the induced homomorphism

K_1 is a functor from the category of C^*-algebras to the category of abelian groups. For any unital homomorphism ϕ between C^*-algebras, we denote by $K_1(\phi)$ the homomorphism induced by the functor K_1. More generally, K_1 is a functor from the category of C^*-algebras to the category of abelian groups.

Tracial State Spaces

When considered as a subspace of the dual space A^* of bounded linear functionals and equipped with the weak-* topology, $T(A)$ is a compact, convex set. Futhermore, $T(A)$ is a Choquet simplex (see Theorem 3.1.18 of [28]), an infinite-dimensional generalization of a classical simplex. We refer the reader to [26] or Chapter 10 of [6] for more information about Choquet simplices. Some categorical considerations will be necessary for the tracial state space.

Definition II.3.1. Let K_1 and K_2 be convex subsets of real vector spaces V and W. A function $\phi\colon K_1 \to K_2$ is *affine* if for all $\lambda \in \mathbb{R}$ with $0 \leq \lambda \leq 1$, we have

$$\phi(\lambda x + (1-\lambda)y) = \lambda\phi(x) + (1-\lambda)\phi(y)$$

for all $x, y \in K_1$.

Let K be a convex subset of a real vector space V. We say that $x \in K$ is an *extreme point* of K if for all $y, z \in K$ and λ such that $0 < \lambda < 1$ and

$$\lambda y + (1-\lambda)z = x,$$

we have $x = y = z$. The set of extreme points of K is denoted by $\partial_e K$. It is clear that affine functions map extreme points to extreme points.

The category of compact convex sets is the category whose objects are compact convex subsets of real Hausdorff locally convex topological vector spaces and whose morphisms are continuous affine functions. See Chapter 5 of [6] for more details.

There is a contravariant functor $T(\bullet)$ from the category of unital C^*-algebras to the category of compact convex sets that maps a C^*-algebra A to the tracial state space $T(A)$ and that maps a homomorphism $\phi\colon A \to B$ to its pullback $\phi^T\colon T(B) \to T(A)$, which is defined by $\phi^T(\tau) = \tau \circ \phi$ for $\tau \in T(B)$.

Definition II.3.2. A real vector space V which is also a partially ordered abelian group (see Definition III.1.1) is a *partially ordered vector space* if for all $\lambda \in [0, \infty)$ and $x \in V_+$, we have $\lambda x \in V_+$.

There is also a contravariant functor $\mathrm{Aff}(\bullet)$ from the category of compact convex sets to the category of real partially ordered Banach spaces that maps a compact convex set K to the space of real-valued positive continuous affine functions on K denoted $\mathrm{Aff}(K)$, with pointwise operations and supremum norm and which maps a continuous affine function $\phi\colon K_1 \to K_2$ to its pullback from $\mathrm{Aff}(K_2)$ to $\mathrm{Aff}(K_1)$, defined by $f \mapsto f \circ \phi$ for all $f \in \mathrm{Aff}(K_2)$.

We note that for a Choquet simplex K, the restriction from $\mathrm{Aff}(K)$ to $C(\partial_e K, \mathbb{R})$ the real vector space of continuous functions on $\partial_e K$ is an isometric isomorphism. See Corollary 11.15 of [6] for more details.

There are several orderings that one could put on $\mathrm{Aff}(K)$. The one that we will usually use is the pointwise ordering in which $f \le g$ if $f(x) \le g(x)$ for all $x \in K$. When dealing with simple C^*-algebras, we also use the *strict ordering* in which $f \ll g$ if $f(x) < g(x)$ for all $x \in K$.

By composing these functors, one obtains a covariant functor from the category of unital C^*-algebras to the category of real partially ordered Banach spaces. Given a unital homomorphism between unital C^*-algebras $\phi\colon A \to B$, we denote the induced homomorphism by $\phi_\sharp\colon \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$, which is defined by

$$\phi_\sharp(f)(\tau) = f(\tau \circ \phi)$$

for $f \in \mathrm{Aff}(T(A))$ and $\tau \in T(B)$.

In fact, the functor $\mathrm{Aff}(T(\bullet))$ maps the category of unital C^*-algebras to the category of pointed real partially ordered Banach spaces. This category is defined by taking the objects to be pairs (X, x_0) consisting of a real partially ordered Banach space X and a distinguished point $x_0 \in X$. The morphisms from (X, x_0) to (Y, y_0) are positive bounded linear maps $L\colon X \to Y$ such that $L(x_0) = y_0$. We will call these maps *unital*. For $\mathrm{Aff}(K)$, the distinguished element is the constant function 1.

Furthermore, there is a natural transformation $\rho_\bullet$ from $K_0(\bullet)$ to $\mathrm{Aff}(T(\bullet))$. Given an integer $n \geq 1$ and a projection $p = (p_{ij}) \in M_n(A)$, we define

$$\rho_A([p])(\tau) = \sum_{i=1}^{n} \tau(p_{ii}).$$

We will denote the sum by $(\tau \otimes \mathrm{Tr})(p)$. Given another unital C^*-algebra C and a unital homomorphism from C to A, by naturality, we induce a commutative square from this pairing. To consider a pair of morphisms induced from a C^*-algebra homomorphism, we make the following definition.

Definition II.3.3. Let C and A be unital C^*-algebras. Let $\alpha\colon (K_0(C), 1_C) \to (K_0(A), 1_A)$ be a normalized positive group homomorphism and let $\gamma\colon \mathrm{Aff}(T(C)) \to \mathrm{Aff}(T(A))$ be a unital positive continuous linear map. The pair (α, γ) is a *compatible pair* if $\rho_A \circ \alpha = \gamma \circ \rho_C$.

Bivariant KL Groups and the Algebraic K_1 Group

Let A be a C^*-algebra. Following Marius Dadarlat and Terry Loring in [2], we define the K-groups with $\mathbb{Z}/k$ coefficients by $K_i(A; \mathbb{Z}/k) = K_i(A \otimes C_k)$, where C_k is a commutative C^*-algebra with $K_0(C_k) \cong \mathbb{Z}/k$ and $K_1(C_k) = 0$. We also make the convention that $K_i(A; \mathbb{Z}/0) = K_i(A)$. We write

$$\underline{K}(A) = K_0(A) \oplus K_1(A) \oplus \bigoplus_{k=2}^{\infty} (K_0(A; \mathbb{Z}/k) \oplus K_1(A; \mathbb{Z}/k)).$$

Dadarlat and Loring prove in [2] that if C is a C^*-algebra satisfying UCT and A is separable, then

$$KL(C, A) \cong \mathrm{Hom}_\Lambda(\underline{K}(C), \underline{K}(A)),$$

where we mean $\mathbb{Z}/2 \oplus \mathbb{Z}_{\geq 0}$ graded group homomorphisms that preserve certain exact sequences induced by the so-called Bockstein operations. We will take this isomorphism to be our definition of the KL groups. In particular, we will identify $\mathrm{Hom}(K_i(C), K_i(A))$ as a subgroup of $KL(C, A)$.

We also note that a unital homomorphism $\phi\colon C \to A$ induces an element of KL. We will denote this element by $KL(\phi)$.

The only fact that we will need about KL groups is the UCT. To state this fact, we first make some definitions.

Definition II.4.1. Let G be an abelian group. A subgroup H is *pure* if for every integer $n \geq 1$ and every $g \in G$, we have $ng \in H$ implies $g \in H$. An extension

$$0 \to H_1 \to G \to H_2 \to 0$$

is *pure* if H_1 is a pure subgroup of G. Let $\mathrm{Pext}(H_2, H_1)$ denote the subgroup generated by the equivalence classes of pure extensions in $\mathrm{Ext}(H_2, H_1)$. We define $\mathrm{ext}(H_2, H_1) = \mathrm{Ext}(H_2, H_1)/\,\mathrm{Pext}(H_2, H_1)$.

We have a KL version of the UCT (see equation 2.4.9 of [27]) when C satisfies the UCT:

$$0 \to \mathrm{ext}(K_*(C), K_{*+1}(A)) \xrightarrow{\varepsilon} KL(C, A) \xrightarrow{\Gamma} \mathrm{Hom}(K_*(C), K_*(A)) \to 0,$$

Note that we use Γ instead of the more standard γ for the group homomorphism $KL(C,A) \to \mathrm{Hom}(K_*(C), K_*(A))$ due to our use of γ.

Following the notation found in [16], [20], and [22], we make the following definitions.

Definition II.4.2. We denote by $KL_e(C,A)^{++}$ the set of $\kappa \in KL(C,A)$ such that $\kappa(K_0(C)^+ \setminus \{0\}) \subseteq K_0(A) \setminus \{0\}$ and $\kappa([1_C]) = [1_A]$.

Let $\kappa \in KL_e(C,A)^{++}$ and let $\gamma\colon \mathrm{Aff}(T(C)) \to \mathrm{Aff}(T(A))$ be a unital positive continuous linear map. We say that (κ, γ) is a *compatible pair* if the restriction of κ to $K_0(C)$ and γ are compatible.

For any unital C^*-algebra C, let $CU(C)$ denote the normal subgroup generated by the group commutators of $U(C)$. More precisely,

$$CU(C) = \{uvu^*v^* \colon u, v \in U(C)\}$$

and define $CU_0(C) = CU(C) \cap U_0(C)$. We also define

$$\begin{aligned} U^\infty(C) &= \varinjlim U^n(C), \\ U_0^\infty(C) &= \varinjlim U_0^n(C), \\ CU^\infty(C) &= \varinjlim CU^n(C), \text{ and} \\ CU_0^\infty(C) &= \varinjlim CU_0^n(C), \end{aligned}$$

where, as before, we use the connecting homomorphisms

$$u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

Also for every unitary $u \in U(C)$, we denote the equivalence class in $U(C)/\overline{CU(C)}$ of u by $\bar{u}$.

If A is a unital, simple C^*-algebra with $\mathrm{TR}(A) \leq 1$, then the map $U(A)/\overline{CU(A)} \to U^\infty(A)/\overline{CU^\infty(A)}$ is an isomorphism. See Corollary 3.5 of [17].

For each $\tau \in T(C)$, each $u \in U_0^\infty(C)$, and each piecewise smooth path

$$\zeta \in C([0,1], U_0^\infty(C))$$

with $\zeta(0) = 1$ and $\zeta(1) = u$, we define

$$\Delta(\zeta)(\tau) = \int_0^1 (\mathrm{Tr} \otimes \tau) \left(\frac{d\zeta(t)}{dt} \zeta^{-1}(t) \right) dt.$$

As shown in [29], this induces a continuous homomorphism

$$\bar{\Delta}\colon U_0^\infty(C)/\overline{CU_0^\infty(C)} \to \mathrm{Aff}(T(C))/\overline{\rho_C(K_0(C))},$$

which provides a natural short exact sequence:

$$0 \to \mathrm{Aff}(T(C))/\overline{\rho_C(K_0(C))} \to U^\infty(C)/\overline{CU^\infty(C)} \to K_1(C) \to 0.$$

Since $\mathrm{Aff}(T(C))/\overline{\rho_C(K_0(C))}$ is injective, this short exact sequence splits, though unnaturally. We denote by π_C the quotient map $U^\infty(C)/\overline{CU^\infty(C)} \to K_1(C)$.

Given a unital homomorphism $\phi\colon C \to A$, we denote by $\phi^\ddagger$ the induced continuous homomorphism

$$\phi^\ddagger\colon U^\infty(C)/\overline{CU^\infty(C)} \to U^\infty(A)/\overline{CU^\infty(A)}.$$

Definition II.4.3. Let C and A be unital C^*-algebras. Let $\kappa \in KL_e(C, A)^{++}$, let $\gamma\colon \operatorname{Aff}(T(C)) \to \operatorname{Aff}(T(A))$ be a unital strictly positive continuous linear map, and let $\eta\colon U^\infty(C)/\overline{CU^\infty(C)} \to U^\infty(A)/\overline{CU^\infty(A)}$ be a continuous group homomorphism. We say that (κ, γ, η) is a compatible triple if

1. (κ, γ) is a compatible pair,

2. the restriction of η to $\operatorname{Aff}(T(C))/\overline{\rho_C(K_0(C))}$ is equal to the homomorphism induced from γ, and

3. the restrictions of η and κ to $K_1(C)$ are equal.

Approximate Unitary Equivalent Classes of Homomorphisms

In certain classes of C^*-algebras, the invariants defined above are complete invariants. In other words, homomorphisms of the invariants induce homomorphisms on the C^*-algebras. And consequently, C^*-algebras with isomorphic invariants are isomorphic as C^*-algebras.

The first major result of this kind is due to George Elliott in [5], where it is shown that unital AF-algebras are classified in this sense by their ordered K_0 group with order unit. Elliott conjectured that a large class of simple C^*-algebras can be classified by their K-theory, which gave rise to what is often known as the Elliott program.

A related question is that of classifying homomorphisms up to approximate unitary equivalence from AH-algebras to a class of classifiable C^*-algebras.

Given the natural transformations involved, a unital homomorphism ϕ between C^*-algebras induces a compatible triple $(KL(\phi), \phi_\sharp, \phi^\ddagger)$.

This compatible triple identifies the unital homomorphism ϕ uniquely up to approximate unitary equivalence.

Theorem II.5.1 (Theorem 5.10 of [22])**.** *Let C be a unital AH-algebra and let A be a unital separable simple C^*-algebra with tracial rank at most one. Let $\phi\colon C \to A$ and $\psi\colon C \to A$ be two unital injective homomorphisms. Then ϕ and ψ are approximately unitarily equivalent if and only if*

$$KL(\phi) = KL(\psi),$$
$$\phi_\sharp = \psi_\sharp, \text{ and}$$
$$\phi^\ddagger = \psi^\ddagger.$$

We note that the tracial rank zero case is found as Theorem 3.4 of [15], where the last equation is no longer necessary. Also the same theorem with the additional assumption that C satisfy the UCT is found as Corollary 11.8 of [20].

A more relaxed version of this uniqueness theorem will be needed as well:

Theorem II.5.2. *Let C be a unital AH-algebra and let A be a separable simple unital C^*-algebra with tracial rank at most one. Let $\phi\colon C \to A$ be a unital, injective homomorphism. For every $\varepsilon > 0$ and every finite subset $\mathcal{F} \subseteq C$, there exist $\delta > 0$, a finite subset $\mathcal{P} \subseteq \underline{K}(C)$, a finite subset $\mathcal{U} \subseteq U_\infty(C)$, and a finite subset $\mathcal{G} \subseteq C$, such that for any unital homomorphism $\psi\colon C \to A$, if*

1. $KL(\phi) = KL(\psi)$ *on* $\mathcal{P}$,
2. $\mathrm{dist}(\phi^\ddagger(\bar{z}), \psi^\ddagger(\bar{z})) < \delta$ *for* $z \in \mathcal{U}$, *and*
3. $|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta$ *for* $g \in \mathcal{G}$,

then there exists a unitary $u \in A$ such that

$$\|u\phi(f)u^* - \psi(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$*.*

This is simply Corollary 11.6 of [20] without the condition that C has Property (J). The same proof works in light of Theorem 5.8 and Lemma 5.7(2) of [22].

In addition to the fact that compatible triples determine the approximate unitary equivalence class of a unital homomorphism, every compatible triple arises from a unital homomorphism. More precisely:

Theorem II.5.3 (Theorem 6.10 of [22])**.** *Let* C *be a unital separable AH-algebra and let* A *be a unital infinite-dimensional separable simple* C^**-algebra with tracial rank at most one. For any* $\kappa \in KL_e(C,A)^{++}$*, any unital strictly positive continuous linear map* $\gamma\colon \operatorname{Aff}(T(C)) \to \operatorname{Aff}(T(A))$*, and any continuous group homomorphism* $\eta\colon U^\infty(C)/\overline{CU^\infty(C)} \to U^\infty(A)/\overline{CU^\infty(A)}$ *such that* (κ,γ,η) *is a compatible triple, there exists a unital homomorphism* $\phi\colon C \to A$ *such that*

$$\begin{aligned} KL(\phi) &= \kappa, \\ \phi_\sharp &= \gamma, \ \textit{and} \\ \phi^\ddagger &= \eta. \end{aligned}$$

In a relatively recent development, the class of classifiable C^*-algebras has grown to include simple unital C^*-algebras for which $\mathrm{TR}(A \otimes \mathcal{Q}) \leq 1$. See [19] for more details. The quantity $\mathrm{TR}(A \otimes \mathcal{Q})$ is called the *rational tracial rank* of A. This expanded class of C^*-algebras includes the limits of generalized dimension drop algebras, including the Jiang-Su algebra $\mathcal{Z}$, which plays a major role in the Elliott program (see [18]).

The related question of determining the approximate unitary equivalence classes of homomorphisms from AH-algebras to C^*-algebras of rational tracial rank at most one has also been answered by Huaxin Lin and Zhuang Niu in [24] and independently by Hiroki Matui in [25] for the case of rational tracial rank zero.

Theorem II.5.4 (Corollary 5.4 of [24]). *Let C be a unital AH-algebra and let A be a separable simple unital $\mathcal{Z}$-stable C^*-algebra with rational tracial rank at most one. Let $\phi\colon C \to A$ and $\psi\colon C \to A$ be unital injective homomorphisms. Then ϕ and ψ are approximately unitarily equivalent if and only if*

$$KL(\phi) = KL(\psi),$$
$$\phi_\sharp = \psi_\sharp, \text{ and}$$
$$\phi^\ddagger = \psi^\ddagger.$$

There is also an existence theorem for these homomorphisms, though there is a restriction on the K_1 group of the domain.

Theorem II.5.5 (Theorem 6.10 of [24]). *Let C be a unital AH-algebra such that $K_1(C)$ is free and let A be a separable simple unital $\mathcal{Z}$-stable C^*-algebra with rational tracial rank at most one. For any $\kappa \in KL_e(C,A)^{++}$, any unital strictly positive continuous linear map $\gamma\colon \operatorname{Aff}(T(C)) \to \operatorname{Aff}(T(A))$, and any continuous group homomorphism $\eta\colon U^\infty(C)/\overline{CU^\infty(C)} \to U^\infty(A)/\overline{CU^\infty(A)}$ such that (κ,γ,η) is a compatible triple, there exists a unital homomorphism $\phi\colon C \to A$ such that*

$$KL(\phi) = \kappa,$$
$$\phi_\sharp = \gamma, \text{ and}$$
$$\phi^\ddagger = \eta.$$

Unfortunately, the invariants of these C^*-algebras are not as well-behaved (see Section III.4). This prevents approximate diagonalization generally for C^*-algebras with rational tracial rank one, as we will discuss in Section IV.3.

CHAPTER III

PARTIALLY ORDERED ABELIAN GROUPS

Riesz Interpolation Property

We adopt the language and notation for the material in this section from [6].

Definition III.1.1. An abelian group $(G, +)$ together with a binary relation $\leq$ on G is a *pre-ordered abelian group* if

1. $a \leq a$ (reflexive),
2. $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitive),
3. $a + c \leq b + c$ implies $a \leq b$ (translation invariant),

for all $a, b, c \in G$.

If in addition, $a \leq b$ and $b \leq a$ implies $a = b$, then $(G, +, \leq)$ is a *partially ordered abelian group.*

The set of $g \in G$ such that $g \geq 0$ is called the *positive cone* of G and is denoted G_+. Often, the order structure of a pre-ordered abelian group is defined by designating a cone as its positive cone. If the cone is strict, the resulting group is partially ordered. This is done for example in the case of $K_0(A)$, where the positive cone is the image of $V(A)$.

If G_+ is a cofinal set in G, or equivalently for every $a \in G$ there exist $b, c \in G_+$ such that $a = b - c$, then G is *directed.*

We note that the K_0 group of a unital C^*-algebra is always directed.

Definition III.1.2. A positive element $u \in G_+$ of a partially ordered abelian group G is an *order unit* if for all $g \in G$, there exists an integer $n \geq 1$ such that $-nu \leq g \leq nu$.

Often, we consider partially ordered abelian groups with distinguished order units. For example, for a unital, stably finite C^*-algebra A, we consider not only the directed group $K_0(A)$ but also include the class of the identity $[1_A]$. This means that we also want to preserve these distinguished elements in the homomorphisms we consider.

Definition III.1.3. Let G and H be partially ordered abelian groups. A group homomorphism $\phi\colon G \to H$ is *positive* if $\phi(G_+) \subseteq H_+$. Let $u \in G_+$ and $v \in H_+$ be order units. We say that a positive group homomorphism $\phi\colon G \to H$ is *normalized* if $\phi(u) = v$. To keep track of the order units in consideration, we will also write $\phi\colon (G, u) \to (H, v)$.

Definition III.1.4. A partially ordered group G satisfies the *Riesz interpolation property* and is called an *interpolation group* if for all $x_1, x_2, y_1, y_2 \in G$ such that $x_i \leq y_j$ for $i = 1, 2$ and $j = 1, 2$, there exists $z \in G$ such that $x_i \leq z \leq y_j$ for $i = 1, 2$ and $j = 1, 2$.

Definition III.1.5. A partially ordered abelian group G has *strict interpolation* if for all x_1, x_2, y_1, $y_2 \in G$ such that $x_i < y_j$ for all i, j, there exists $z \in G$ such that $x_i < z < y_j$ for all i, j.

Strict versions of the Riesz decomposition properties follow with proofs analogous to those of Propositions 2.1 and 2.2 of [6].

Proposition III.1.6. *Let G be a partially ordered abelian group. The following are equivalent:*

(a) G has strict interpolation.

(b) If $x, y_1, y_2 \in G$ satisfy $0 < x < y_1 + y_2$, then there exist $x_1, x_2 \in G_+ \setminus \{0\}$ such that $x_1 + x_2 = x$ and $x_i < y_i$ for $i = 1, 2$.

(c) If $x_1, x_2, y_1, y_2 \in G_+ \setminus \{0\}$ satisfy $x_1 + x_2 = y_1 + y_2$, then there exist $z_{i,j} \in G_+ \setminus \{0\}$ for $i, j = 1, 2$ such that $x_i = z_{i,1} + z_{i,2}$ and $y_j = z_{1,j} + z_{2,j}$ for $i, j = 1, 2$.

Proposition III.1.7. *Let G be a partially ordered abelian group with strict interpolation. Then the following hold:*

(a) If $x_1, x_2, \ldots x_n$ and $y_1, y_2, \ldots, y_k$ are in G and satisfy $x_i < y_j$ for all i, j, then there exists $z \in G$ such that $x_i < z < y_j$ for all i, j.

(b) If $x, y_1, y_2, \ldots, y_n \in G_+ \setminus \{0\}$ satisfy $x < y_1 + y_2 + \ldots y_n$, then there exist $x_1, \ldots, x_n \in G_+ \setminus \{0\}$ such that $x = x_1 + \cdots + x_n$ and $x_i < y_i$ for all i.

(c) If $x_1, \ldots, x_n, y_1, \ldots, y_k \in G_+ \setminus \{0\}$, then there exist $z_{i,j} \in G_+ \setminus \{0\}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$ such that $x_i = z_{i,1} + \cdots + z_{i,k}$ and $y_j = z_{1,j} + \cdots + z_{n,j}$.

Considering an ordered abelian group as a $\mathbb{Z}$-module, it is clear by induction that multiplying by a positive element by a positive integer gives a positive element. It is not true that if the multiple of an element by a positive integer is positive, that the original element is positive. To ensure, this we make the following definition.

Definition III.1.8. A partially ordered abelian group G is *unperforated* if for every $x \in G$, if $nx \geq 0$ for some integer $n \geq 1$, then $x \geq 0$.

A partially ordered abelian group G is *weakly unperforated* if for every $x \in G$, if $nx > 0$ for some integer $n \geq 1$, then $x > 0$.

We note that weakly unperforated groups only differ from unperforated groups by allowing the possibility for torsion.

Definition III.1.9. A partially ordered abelian group G is a *dimension group* if G is directed, unperforated, and satisfies the Riesz interpolation property.

The K_0 group of an AF-algebra is a dimension group and, conversely, every countable dimension group is the K_0 group of some AF-algebra. But it turns out that for more general classes of stably finite C^*-algebras, we want to replace "unperforated" with "weakly unperforated." Despite its widespread use, the author was unable to find a name in use for such a group. So, given its close relation to dimension groups, we make the following definition:

Definition III.1.10. A partially ordered abelian group G is a *weak dimension group* if G is directed, weakly unperforated, and satisfies the Riesz interpolation property.

We note that the K_0 group of a separable simple unital C^*-algebra with finite tracial rank is a weak dimension group. See Theorem 6.11 of [14].

Affine Representation of Partially Ordered Abelian Groups

Definition III.2.1. Let (G, u) be a partially ordered abelian group with order unit. A positive homomorphism $\sigma\colon G \to \mathbb{R}$ such that $\sigma(u) = 1$ is called a *state*. The set of states of (G, u) is denoted $S(G, u)$.

We note that $S(G, u)$, just like the tracial state of a C^*-algebra, is a compact convex set. In particular, when G is an interpolation group, $S(G, u)$ is a Choquet simplex. See Theorem 10.17 of [6]. We will call the extreme points of $S(G, u)$ *pure states*.

As with the tracial state space of a C^*-algebra, we have a covariant functor $\mathrm{Aff}(S(\bullet))$ from the category of partially ordered abelian groups with order units to the category of pointed, partially ordered real Banach spaces. Given a normalized positive group homomorphism $\alpha\colon (G, u) \to (H, v)$, the functor maps α to

$\alpha_\rho\colon \operatorname{Aff}(S(G,u)) \to \operatorname{Aff}(S(H,v))$, defined by

$$\alpha_\rho(f)(\tau) = f(\tau \circ \alpha)$$

for all $f \in \operatorname{Aff}(S(G,u))$ and $\tau \in S(H,v)$.

Let $\mathcal{U}$ be the forgetful functor from the category of pointed partially ordered Banach spaces to the category of partially ordered abelian groups with order units.

There is a natural transformation from the identity functor on partially ordered abelian groups with order units to the functor $\mathcal{U}(\operatorname{Aff}(S(\bullet)))$. For a partially ordered abelian group with order unit (G,u), we define $\rho_G\colon (G,u) \to (\operatorname{Aff}(S(G,u)),1)$ by

$$\rho_G(g)(\sigma) = \sigma(g)$$

for $\sigma \in S(G,u)$.

For a stably finite C^*-algebra A, we note that $\rho_{K_0(A)}$ is closely related to the map ρ_A discussed in Section II.3. We will abbreviate $S(K_0(A),[1_A])$ as $SK_0(A)$. A tracial state $\tau \in T(A)$ induces a state $\tau_* \in SK_0(A)$ defined by

$$\tau_*([p]) = (\tau \otimes \operatorname{Tr})(p)$$

for all $p \in M_\infty(A)$ and extending linearly. The map $\tau \mapsto \tau_*$ is an affine continuous map from $T(A)$ to $SK_0(A)$. This induces a continuous positive unital linear map from $\operatorname{Aff}(SK_0(A))$ to $\operatorname{Aff}(T(A))$.

Let C and A be unital C^*-algebras. Given a unital homomorphism from C to A, we have the following commutative diagram:

$$\begin{array}{ccc} K_0(C) & \longrightarrow & K_0(A) \\ \downarrow{\scriptstyle \rho_{K_0(C)}} & & \downarrow{\scriptstyle \rho_{K_0(A)}} \\ \mathrm{Aff}(SK_0(C)) & \longrightarrow & \mathrm{Aff}(SK_0(A)) \\ \downarrow & & \downarrow \\ \mathrm{Aff}(T(C)) & \longrightarrow & \mathrm{Aff}(T(A)). \end{array}$$

$$\rho_C \colon K_0(C) \to \mathrm{Aff}(T(C)), \qquad \rho_A \colon K_0(A) \to \mathrm{Aff}(T(A))$$

A normalized positive group homomorphism $\alpha\colon (K_0(C), [1_C]) \to (K_0(A), [1_A])$ will necessarily induce a commutative square with $\rho_{K_0(A)} \circ \alpha = \alpha_\rho \circ \rho_{K_0(C)}$.

As a result, for $\alpha\colon (K_0(C), [1_C]) \to (K_0(A), [1_A])$, a normalized positive group homomorphism and for $\gamma\colon\ \mathrm{Aff}(T(C)) \to \mathrm{Aff}(T(A))$ a continuous positive unital linear map, if the square

$$\begin{array}{ccc} \mathrm{Aff}(SK_0(C)) & \xrightarrow{\alpha_\rho} & \mathrm{Aff}(SK_0(A)) \\ \downarrow & & \downarrow \\ \mathrm{Aff}(T(C)) & \xrightarrow{\gamma} & \mathrm{Aff}(T(A)) \end{array}$$

commutes, then (α, γ) is a compatible pair. We will consider the following short exact sequence often:

$$0 \to \ker \rho_G \to G \to \rho_G(G) \to 0.$$

Note that this is often not a sequence of partially ordered groups since $\ker \rho_G$ is not necessarily an order ideal, nor even a partially ordered group.

We note that when G is finitely generated, the image $\rho_G(G)$ is a finitely generated torsion-free group. Thus the group is free and we see that this short exact sequence splits.

In Chapter V, we will need to consider groups for which $\ker \rho_G = 0$. We therefore make the definition.

Definition III.2.2. A partially ordered group G is *archimedian* if for every $x, y \in G$ such that $nx \leq y$ for all positive integers $n \geq 1$, we have $x \leq 0$.

The notion of archimedian is used differently in different contexts. As suggested previously, being archimedian is equivalent to ρ_G being an injective positive group homomorphism. See Theorem 7.7 of [6] for more details. Since we can identify G as an ordered subgroup of $\mathrm{Aff}(S(G, u))$, we see that the order structure of an archimedian group is given by its states in the sense that $g \geq 0$ if and only if $\tau(g) \geq 0$ for all $\tau \in S(G, u)$.

Finally we note that for any partially ordered abelian group G, the group $\rho_G(G)$ is an archimedian group.

Simple Partially Ordered Abelian Groups

Definition III.3.1. A subgroup H of a partially ordered abelian group G is an *order ideal* if $H_+ = G_+ \cap H$ is directed and for all $g \in G$ and $h \in H$, if $0 \leq g \leq h$, then $g \in H$.

A partially ordered abelian group G is *simple* if the only order ideals are 0 and G.

Note that the K_0 group of a stably finite, simple C^*-algebra is a simple ordered group. See Proposition 3.3.7 of [13].

We note a useful concrete characterization of simple partially ordered groups is that every nonzero positive element is an order unit. This follows from the fact that an order ideal generated by a single positive element is the whole group if and only if the element is an order unit. This makes positive elements of simple ordered groups well-behaved. For example, since for any order unit v, we have $\sigma(v) > 0$ for every state $\sigma \in S(G, u)$, it follows that for a simple ordered group, $\sigma(g) > 0$ for every nonzero positive $g \in G_+$. Also if in addition to being simple, G is weakly unperforated, then the converse holds: if $\sigma(g) > 0$ for all $\sigma \in S(G, u)$, then g is a nonzero positive element of G.

Definition III.3.2. Let S be a partially ordered set with a least element 0. An element $x \in S$ is called an *atom* of S if there is no element $y \in S$ for which $0 < y < x$.

In the K_0 group of commutative C^*-algebras, we will see that atoms play an important role in generating $\rho_G(G)$. In contrast, when a partially ordered abelian group G is simple, G_+ has either one or no atoms. The former only occurs when G is cyclic (see Lemma 14.2 of [6]). As a result, we have to treat $\mathbb{Z}$ separately. For example, an interpolation group without atoms will satisfy strict interpolation. As a result, noncyclic simple interpolation groups satisfy strict interpolation.

Simple dimension groups can be constructed from triples consisting of an abelian group G, a Choquet simplex K, and a group homomorphism $\psi \colon G \to \mathrm{Aff}(K)$ with dense range. The ordering on G is determined by $g \geq 0$ if either $g = 0$ or $\psi(g) \gg 0$. In fact, these triples completely characterize noncyclic simple dimension groups. See Chapter 14 of [6] for the proof and more details. For our purposes, we mention this as a natural source of examples. In particular, we will often look at dense subgroups of $\mathrm{Aff}(K)$, where K is a classical simplex, as examples of simple dimension groups.

Tensor Products of Partially Ordered Abelian Groups

The K_0 groups of C^*-algebras with rational tracial rank one are weakly unperforated, but will not necessarily have the Riesz interpolation property. They will have the following weaker version of the Riesz interpolation property:

Definition III.4.1. A partially ordered abelian group G has the *rational Riesz interpolation property* if for any $x_1, x_2, y_1, y_2 \in G$ with $x_i \leq y_j$ for $i = 1, 2$ and $j = 1, 2$, there exist $z \in G$ and integers $m, n \geq 1$ such that

$$mx_i \leq nz \leq my_i$$

for $i = 1, 2$ and $j = 1, 2$.

See Section 5 of [23] for more details.

A useful characterization of rational Riesz interpolation is available if we consider tensor products. The tensor product of two partially ordered abelian groups G and H can be made into a partially ordered abelian group by taking the positive cone

$$(G \otimes H)^+ = \left\{ \sum_{i=1}^{n} g_i \otimes h_i \colon g_i \in G_+ \text{ and } h_i \in H_+, \text{ for } n \geq 1 \text{ and } i = 1, 2, \ldots, n \right\}.$$

See Section 2 of [7] for more details.

It is clear that if $u \in G_+$ and $v \in H_+$ are order units, then $u \otimes v$ is an order unit for $G \otimes H$.

If $\sigma_1 \in S(G, u)$ and $\sigma_2 \in S(H, v)$, then $\sigma_1 \otimes \sigma_2$, defined by

$$(\sigma_1 \otimes \sigma_2)(g \otimes h) = \sigma_1(g)\sigma_2(h),$$

is a state on $S(G \otimes H, u \otimes v)$.

When G and H are partially ordered abelian groups, the pure states of $G \otimes H$ are the pure tensors of pure states of G with the pure states of H. To be precise:

$$\partial_e S(G \otimes H, u \otimes v) = \{\sigma_1 \otimes \sigma_2 \colon \sigma_1 \in \partial_e S(G, u) \text{ and } \sigma_2 \in \partial_e S(H, v)\}.$$

This is shown as Lemma 4.1 of [7].

So when G and H are simple weak dimension groups, we have a nice computational characterization of the positive elements of $G \otimes H$. Namely, a pure tensor $g \otimes h$ is positive if and only if either $g \otimes h = 0$ or $(\sigma_1 \otimes \sigma_2)(g \otimes h) > 0$ for all $\sigma_1 \in \partial_e S(G, u)$ and $\sigma_2 \in \partial_e S(H, v)$.

It is shown in Proposition 5.7 of [23] that a countable weakly unperforated simple partially ordered abelian group G has the rational Riesz interpolation property if and only if $G \otimes \mathbb{Q}$ is an interpolation group.

As an example, we consider $\mathbb{Z}^2$ with the strict ordering. The group $\mathbb{Z}^2$ is simple since every non-zero element is an order unit and unperforated since $\mathbb{Z}$ is unperforated. Also $\mathbb{Z}^2$ is not an interpolation group since we have

$$\begin{aligned}
&(1,0) \ll (2,2),\\
&(1,0) \ll (2,3),\\
&(0,1) \ll (2,2), \text{ and}\\
&(0,1) \ll (2,3),
\end{aligned}$$

but there is no element $(z_1, z_2) \in \mathbb{Z}^2$ such that

$$(1,0) \ll (z_1, z_2) \ll (2,3) \text{ and}$$
$$(0,1) \ll (z_1, z_2) \ll (2,2)$$

since this would require $1 < z_1 < 2$ and $1 < z_2 < 2$.

As an abelian group, we have $\mathbb{Z}^2 \otimes \mathbb{Q} \cong \mathbb{Q}^2$ with an isomorphism satisfying $(x,y) \otimes r \mapsto (xr, yr)$. We claim that when $\mathbb{Q}^2$ has the strict ordering, this map is an isomorphism of partially ordered groups. To check that the map is positive, by definition, it suffices to check for pure tensors. If $(x,y) \gg 0$, then $x > 0$ and $y > 0$ and so if $r > 0$, then $xr > 0$ and $yr > 0$ and so $(x,y) \otimes r \mapsto (xr, yr) \gg 0$. Take $(a,b) \in \mathbb{Q}^2$ and suppose that $a > 0$ and $b > 0$. Then there exists an integer $n \geq 1$ such that na and nb are positive integers. So the inverse maps (a,b) to $(na, nb) \otimes 1/n$, which is a pure tensor with $(na, nb) > 0$ and $1/n > 0$.

Since $\mathbb{Q}^2$ with the strict ordering is a simple dimension group, $\mathbb{Z}^2$ with the strict ordering is a simple partially ordered group with rational Riesz interpolation.

CHAPTER IV

APPROXIMATE DIAGONALIZATION OF NORMAL MATRICES

Ordered K_0 Groups of Commutative C^*-Algebras

The main obstruction to approximate diagonalization, as we will see shortly, is the ordered K_0 group. The other invariants can either be extended from other invariants as with the trace maps and the algebraic K_1 group, or can be decomposed in a rather trivial manner.

Let X be a compact metric space. Since X can be written as an inverse limit of finite CW-complexes, $K_0(C(X))$, as an abelian group, can be written as the inductive limit of finitely generated abelian groups. So it is a relatively straightforward matter to define homomorphisms from $K_0(C(X))$. But the ordering of $K_0(C(X))$ is not easily determined. For example, the group may have perforation.

Fortunately, if the target of the homomorphism is a simple weakly unperforated group, then the order structure on $K_0(C(X))$ can be managed and we can define the positive group homomorphisms we need. There are a few properties of $K_0(C(X))$ that contribute to this relatively good behavior of homomorphisms, which we describe now.

By the Riesz Representation Theorem, $T(C(X))$ can be identified with the set of regular Borel probability measures. Given $g \in K_0(C(X))^+$, there exists an integer $n \geq 1$ and a projection-valued continuous function $p\colon X \to M_n(\mathbb{C})$ so that $[p] = g$ and we have

$$\rho_{C(X)}(g)(\tau) = \int_X \mathrm{Tr}(p)\, d\mu_\tau,$$

where μ_τ is the measure induced by the Riesz Representation Theorem and τ. Also, since the extreme points of $T(C(X))$ are given by Dirac point masses and since

$$\mathrm{Aff}(T(C(X))) \cong C(\partial_e T(C(X)))_{\mathrm{sa}} \cong C(X)_{\mathrm{sa}},$$

we see that on the Dirac point mass δ_x and with g and p as above,

$$\rho_{C(X)}(g)(\delta_x) = \int_X \mathrm{Tr}(p)\, d\delta_x = \mathrm{Tr}(p(x)) \in \mathbb{Z},$$

since the trace of a projection is equal to its rank. As a result, the range of $\rho_{C(X)}$ is isomorphic to $C(X, \mathbb{Z})$. Consequently, the short exact sequence:

$$0 \to \ker \rho_{C(X)} \to K_0(C(X)) \to C(X, \mathbb{Z}) \to 0$$

splits. In fact, $C(X, \mathbb{Z})$ is a free abelian group, but one can consider an explicit splitting map from $C(X, \mathbb{Z})$ to $K_0(C(X))$, where a function f is mapped to the vector bundle such that the restriction to any connected subset is trivial and has rank $f(x)$ at each point x. Furthermore, we will only being applying this to the case where X has finitely many connected components, where it is apparent that $C(X, \mathbb{Z})$ is a finitely generated free abelian group.

The implied distinguished order unit of $K_0(C(X))$ is the constant function 1. When X has finitely many connected components, this means that the distinguished order unit can be written as the sum of the atoms of $K_0(C(X))^+$ without any repetition. This is particular useful when applying the Riesz decomposition property.

We now prove the main results about partially ordered abelian groups that we will need for approximate diagonalization of homomorphisms from commutative C^*-algebras.

Lemma IV.1.1. *Let G be a partially ordered abelian group such that*

$$G = \ker \rho_G \oplus \rho_G(G)$$

and G_+ has finitely many atoms $x_1, x_2, \ldots, x_k$, which generate $\rho_G(G)$, and so that $u = \sum_{j=1}^{k} x_j$ is an order unit.

Let $n \geq 1$ be an integer and let H be a simple, non-cyclic weak dimension group with order units v_i for $i = 1, 2, \ldots, n$.

For any normalized positive group homomorphism $\alpha \colon (G, u) \to (H, \sum_{i=1}^{n} v_i)$ such that $\ker \alpha \cap \rho_G(G) = 0$, there exist normalized positive group homomorphisms $\alpha_i \colon (G, u) \to (H, v_i)$ for $i = 1, 2, \ldots, n$ such that $\ker \alpha_i \cap \rho_G(G) = 0$ for all i, $\ker \rho_G \subseteq \ker \alpha_i$ for $i > 1$, and

$$\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

Proof. We have

$$\alpha(x_1) + \alpha(x_2) + \cdots + \alpha(x_k) = \alpha(u) = v_1 + v_2 + \cdots + v_n,$$

with $\alpha(x_j) > 0$ since $\ker\alpha \cap \rho_G(G) = 0$. Therefore, by strict decomposition, there exist nonzero $z_{i,j} \in H_+$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$ such that

$$\sum_{i=1}^{n} z_{i,j} = \alpha(x_j) \text{ and}$$
$$\sum_{j=1}^{k} z_{i,j} = v_i.$$

We define $\alpha_i\colon G \to H$ by setting $\alpha_i(x_j) = z_{i,j}$ for all i and j, and by setting

$$\alpha_i(g) = \begin{cases} \alpha(g) & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}$$

for $g \in \ker\rho_G$. Since the set of atoms is $\mathbb{Z}$-independent (Lemma 3.10 of [6]), α_i is a group homomorphism for all i. By construction, $\ker\rho_G \subseteq \ker\alpha_i$ for $i > 1$.

Since

$$\sum_{i=1}^{n} \alpha_i = \alpha_1 = \alpha$$

on $\ker\rho_G$ and

$$\sum_{i=1}^{n} \alpha_i(x_j) = \sum_{i=1}^{n} z_{i,j} = \alpha(x_j),$$

we have

$$\sum_{i=1}^{n} \alpha_i = \alpha.$$

Let x be a nonzero, positive element of G. There exist integers $m_j \geq 0$ for $j = 1, 2, \ldots, k$, at least one of which is nonzero, and $g \in \ker\rho_G$ such that

$$x = g + \sum_{j=1}^{k} m_j x_j.$$

Take $\tau \in S(H, \sum v_i)$. Since $\tau \circ \alpha_i$ is a scalar multiple of a state in $S(G, u)$, we have $\tau(\alpha_i(g)) = 0$ for all i and so

$$\tau(\alpha_i(x)) = \sum_{j=1}^{n} m_j \tau(z_{i,j}) > 0,$$

since at least one m_j is nonzero and $\tau(z_{i,j}) > 0$ for all i and j. So we have $\alpha_i(x) > 0$, and so α_i is a positive group homomorphism for all i. Also

$$\alpha_i(u) = \alpha_i\left(\sum_{j=1}^{k} x_j\right) = \sum_{j=1}^{k} \alpha_i(x_j) = \sum_{j=1}^{k} z_{i,j} = v_i.$$

So $\alpha_i \colon (G, u) \to (H, v_i)$ is a normalized positive group homomorphism for all i. □

Lemma IV.1.2. *Let G_1 and G_2 be partially ordered abelian groups such that for $s = 1, 2$,*

$$G_s = \ker \rho_{G_s} \oplus \rho_{G_s}(G_s),$$

G_1^+ and G_2^+ have finitely many atoms $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_m$, which generate $\rho_{G_1}(G_1)$ and $\rho_{G_2}(G_2)$, respectively, and such that $u_1 = \sum_{j=1}^{k} x_j$ and $u_2 = \sum_{t=1}^{m} y_t$ are order units.

Let $n \geq 1$ be an integer and let H be a simple non-cyclic weak dimension group with order units v_i for $i = 1, 2, \ldots, n$.

Let $\alpha \colon (G_1, u_1) \to (G_2, u_2)$ be a normalized positive group homomorphism such that $\ker \alpha \cap \rho_{G_1}(G_1) = 0$ and

$$(\alpha \circ \rho_{G_1})(G_1) \subseteq \rho_{G_2}(G_2).$$

For $s = 1, 2$, *let* $\beta_s\colon (G_s, u_s) \to (H, \sum_{i=1}^n v_i)$ *be normalized positive group homomorphisms such that* $\ker \beta_s \cap \rho_{G_s}(G_s) = 0$. *Further, assume* $\beta_1 = \alpha \circ \beta_2$.

If there exist $\beta_{1,i}\colon (G_1, u_1) \to (H, v_i)$ *for* $i = 1, 2, \dots, n$ *such that* $\ker \beta_{1,i} \cap \rho_{G_1}(G_1) = 0$ *for all* i, $\ker \rho_{G_1} \subseteq \ker \beta_{1,i}$ *for* $i > 1$,

$$\sum_{i=1}^n \beta_{1,i} = \beta_1,$$

then there exist $\beta_{2,i}\colon (G_2, u_2) \to (H, v_i)$ *for* $i = 1, 2, \dots, n$ *such that* $\ker \beta_{2,i} \cap \rho_{G_2}(G_2) = 0$ *for all* i, $\ker \rho_{G_2} \subseteq \ker \beta_{2,i}$ *for* $i > 1$, $\beta_{1,i} = \beta_{2,i} \circ \alpha$ *for all* i, *and*

$$\sum_{i=1}^n \beta_{2,i} = \beta_2.$$

Proof. Since α is a positive homomorphism, $\alpha(u_1) = u_2$, and $(\alpha \circ \rho_{G_1})(G_1) \subseteq \rho_{G_2}(G_2)$, for each $j = 1, 2, \dots, k$ there exists a non-empty subset $S_j \subseteq \{1, 2, \dots, m\}$ such that

$$\alpha(x_j) = \sum_{t \in S(j)} y_t.$$

Furthermore, $S_i \cap S_j = \varnothing$ if $i \neq j$ and $\bigcup_{j=1}^k S_j = \{1, 2, \dots, m\}$. So we have

$$\sum_{t \in S_j} \beta_2(y_t) = \beta_2(\alpha(x_j)) = \beta_1(x_j) = \sum_{i=1}^n \beta_{1,i}(x_j).$$

Since $\beta_2(y_t) > 0$ and $\beta_{1,i}(x_j) > 0$ for all i, t and j, by strict decomposition, there exist nonzero $z_{i,t} \in H_+$ for $t \in S_j$ and $i = 1, 2, \ldots, n$ so that

$$\sum_{t \in S_j} z_{i,t} = \beta_{1,i}(x_j) \text{ and}$$
$$\sum_{i=1}^{n} z_{i,t} = \beta_2(y_t).$$

We define $\beta_{2,i} \colon G_2 \to H$ by setting $\beta_{2,i}(y_t) = z_{i,t}$ for all i, t and by setting

$$\beta_{2,i}(g) = \begin{cases} \beta_2(g) & \text{if } i = 1 \\ 0 & \text{if } i \neq 1. \end{cases}$$

We see that $\beta_{2,i}$ is well-defined since the sets S_j partition $\{1, 2, \ldots, m\}$ and $\beta_{2,i}$ is a group homomorphism since atoms are $\mathbb{Z}$-independent and the infinitesimals of G_2 split. Since S_j is non-empty and $z_{i,t} > 0$ for all i and t, $\ker \beta_{2,i} \cap \rho_{G_2} = 0$ for all i. By construction, $\ker \rho_{G_2} \subseteq \ker \beta_{2,i}$ for $i > 1$.

As before,
$$\sum_{i=1}^{n} \beta_{2,i} = \beta_{2,1} = \beta_2$$
on $\ker \rho_{G_2}$ and
$$\sum_{i=1}^{n} \beta_{2,i}(y_t) = \sum_{i=1}^{n} z_{i,t} = \beta_2(y_t)$$
for all t. So
$$\sum_{i=1}^{n} \beta_{2,i} = \beta_2.$$

Notice that for all $\sigma \in S(G_2, u_2)$, we have $\sigma \circ \alpha \in S(G_1, u_1)$, so if $g \in \ker \rho_{G_1}$, then $(\sigma \circ \alpha)(g) = 0$ for all $\sigma \in S(G_2, u_2)$. So $\alpha(g) \in \ker \rho_{G_2}$. So

$$\beta_{1,i} = 0 = \beta_{2,i} \circ \alpha$$

on $\ker \rho_{G_1}$ when $i > 1$, and

$$\beta_{1,1} = \beta_1 = \beta_2 \circ \alpha = \beta_{2,1} \circ \alpha.$$

Also

$$\beta_{2,i}(\alpha(x_j)) = \sum_{t \in S_j} \beta_{2,i}(y_t) = \sum_{t \in S_j} z_{i,t} = \beta_{1,i}(x_j)$$

for all i, j. Thus $\beta_{1,i} = \beta_{2,i} \circ \alpha$ for all i. Further, since $\alpha(u_1) = u_2$, we have

$$\beta_{2,i}(u_2) = \beta_{1,i}(u_1) = v_i.$$

Let x be a nonzero, positive element of G_2. So there exist integers $r_t \geq 0$, at least one of which is nonzero, for $t = 1, 2, \ldots, m$ and $g \in \ker \rho_{G_2}$ so that

$$x = g + \sum_{t=1}^{m} r_t y_t.$$

Take $\tau \in S(H, \sum_{i=1}^{n} v_i)$. Since $\tau \circ \beta_{2,i}$ is a scalar multiple of a state in $S(G, u)$, we see $\tau(\beta_{2,i}(g)) = 0$ and so

$$\tau(\beta_{2,i}(x)) = \sum_{t=1}^{m} r_t \tau(z_{i,t}) > 0,$$

since at least one r_t is positive and $\tau(z_{i,t}) > 0$ for all i and t.

So the maps $\beta_{2,i}\colon (G_2, u_2) \to (H, v_i)$ are normalized positive group homomorphisms with the required properties. □

Matrices over C^*-Algebras with Tracial Rank One

First, to construct the necessary trace maps, we prove some elementary facts about projections in a C^*-algebra of stable rank one that the author could not find in the literature.

Lemma IV.2.1. *Let A be a unital C^*-algebra with stable rank one, let $p \in A$ a projection, and let $g \in K_0(A)^+$ satisfy $g \leq [p]$. There exists a projection $q \in A$ such that $q \leq p$ and $[q] = g$.*

Proof. There exist projections $q_1, r \in M_\infty(A)$ such that $[q_1 \oplus r] = [p]$. Since A has cancellation there exist an integer $n \geq 1$ and element $v \in M_n(A)$ such that $v^*v = q_1 \oplus r$ and $vv^* = p \oplus 0$. So

$$v(q_1 \oplus 0)v^* \leq v(q_1 \oplus r)v^* = p \oplus 0.$$

Notice that

$$v(q_1 \oplus 0)(q_1 \oplus 0)v^* = v(q_1 \oplus 0)v^* \text{ and}$$
$$(q_1 \oplus 0)v^*v(q_1 \oplus 0) = q_1 \oplus 0.$$

So $[v(q_1 \oplus 0)v^*] = g$.

Also since $v(q_1 \oplus 0)v^*$ is in the hereditary subalgebra generated by $p \oplus 0$, there exists a projection $q \in A$ such that $v(q_1 \oplus 0)v^* = q \oplus 0$. So q is the projection with the properties that we want. □

Lemma IV.2.2. *Let A be a unital C^*-algebra with stable rank one. For any projection $p \in A$ and elements $g_1, g_2, \ldots, g_n \in K_0(A)^+$ such that*

$$g_1 + g_2 + \cdots + g_n = [p],$$

there exist mutually orthogonal projections $q_1, q_2, \ldots, q_n \in A$ such that

$$q_1 + q_2 + \cdots + q_n = p$$

and $[q_i] = g_i$ for all $i = 1, 2, \ldots, n$.

Proof. We proceed by induction on n. If $n = 2$, then by the previous lemma, there exists $q_1 \leq p$ with $[q_1] = p$. Let $q_2 = p - q_1$. Notice that

$$q_1 q_2 = q_1(p - q_1) = q_1 p - q_1 = q_1 - q_1 = 0,$$
$$[q_2] = [p] - [q_1] = [p] - g_1 = g_2, \text{ and}$$
$$q_1 + q_2 = q_1 + p - q_1 = p,$$

and so the result is true for $n = 2$.

Suppose the result holds for n. We wish to show it holds for $n+1$. By Lemma IV.2.1, there exists a projection q_1 in A such that $[q_1] = g_1$ and $q_1 \leq p$. By the induction hypothesis, and since

$$[p - q_1] = g_2 + g_3 + \cdots + g_{n+1},$$

there exist mutually orthogonal projections $q_2, q_3, \ldots, q_{n+1}$ such that $[q_i] = g_i$ for $i = 2, 3, \ldots, n+1$ and

$$p - q_1 = q_2 + q_3 + \cdots + q_{n+1}.$$

So the result holds for $n+1$. Thus by induction, the result holds for all n. □

Lemma IV.2.3. *Let C be a unital nuclear stably finite C^*-algebra and let A be a unital separable stably finite C^*-algebras. Assume we are given:*

1. a normalized positive group homomorphism $\alpha\colon (K_0(C), [1_C]) \to (K_0(A), n \cdot [1_A])$,

2. a strictly positive unital linear map $\gamma\colon \operatorname{Aff}(T(C)) \to \operatorname{Aff}(T(A))$,

3. an element $\kappa \in KL_e(C, M_n(A))^{++}$ such that κ restricted to $K_0(C)$ is α, and

4. a group homomorphism $\eta\colon U^\infty(C)/\overline{CU^\infty(C)} \to U(A)/\overline{CU(A)}$ such that the triple (κ, γ, η) is compatible.

Suppose there exist normalized positive group homomorphisms $\alpha_i\colon (K_0(C), [1_C]) \to (K_0(A), [1_A])$ and strictly positive linear maps $\gamma_i\colon \operatorname{Aff}(C) \to \operatorname{Aff}(A)$ for $i = 1, 2, \ldots, n$ such that the pairs (α_i, γ_i) are compatible for $i = 1, 2, \ldots, n$ and

$$\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \text{ and}$$

$$\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n.$$

Then there exist elements $\kappa_i \in KL_e(C, A)^{++}$ and continuous group homomorphisms $\eta_i\colon U^\infty(C)/\overline{CU^\infty(C)} \to U(A)/\overline{CU(A)}$ for $i = 1, 2, \ldots, n$ such *that the triple* $(\kappa_i, \gamma_i, \eta_i)$ *is compatible,*

$$\kappa = \kappa_1 + \kappa_2 + \cdots + \kappa_n, \text{ and}$$

$$\eta = \eta_1 + \eta_2 + \cdots + \eta_n.$$

We note that the restrictions on C and A are only to ensure that the invariants exist as written. With the appropriate modification, this lemma likely holds in greater generality.

Proof. Let $\beta \colon K_1(C) \to K_1(A)$ be the restriction of κ to $K_1(C)$. We define group homomorphisms $\beta_i \colon K_1(C) \to K_1(A)$ by

$$\beta_i = \begin{cases} \beta & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}$$

for $i = 1, 2, \ldots, n$. So

$$\sum_{i=1}^{n} \beta_i = \beta_1 = \beta.$$

For $1 < i \leq n$, by the UCT, there exist $\kappa_i \in KL(C, A)$ such that $\Gamma(\kappa_i) = (\alpha_i, \beta_i)$. We set

$$\kappa_1 = \kappa - \sum_{i=2}^{n} \kappa_i.$$

Notice that

$$\Gamma(\kappa_1) = (\alpha, \beta) - \sum_{i=2}^{n} (\alpha_i, \beta_i) = (\alpha_1, \beta_1).$$

Since α_i is a positive, normalized group homomorphism, compatible with γ_i, it follows that $\kappa_i \in KL_e(C, A)^{++}$ is compatible with γ_i, and by construction,

$$\kappa_1 + \kappa_2 + \cdots + \kappa_n = \kappa.$$

The compatible pair (κ_i, γ_i) induce the group homomorphism

$$\eta_i^0 \colon \operatorname{Aff}(T(C))/\rho_C(K_0(C)) \to \operatorname{Aff}(T(A))/\rho_A(K_0(A)).$$

Recall that for any unital C^*-algebra B, we have a split exact sequence

$$0 \to \operatorname{Aff}(T(B))/\overline{\rho_B(K_0(B))} \to U^\infty(B)/\overline{CU^\infty(B)} \xrightarrow{\pi_B} K_1(B) \to 0.$$

So we extend η_i^0 to a homorphism

$$\eta_i \colon U^\infty(C)/\overline{CU^\infty(C)} \to U(A)/\overline{CU(A)}$$

by setting

$$\eta_i(u) = \begin{cases} \eta(u) & \text{if } i = 1 \\ 0 & \text{if } i \neq 1. \end{cases}$$

for $u \in K_1(C)$. By naturality, we have

$$\pi_A \circ \eta_1 = \beta \circ \pi_C = \beta_1 \circ \pi_C,$$

and so (κ_1, η_1) is a compatible pair. Since $\beta_i = 0 = \eta_i$ on $K_1(C)$ for $i = 2, 3, \ldots, n$, (κ_i, η_i) is a compatible pair for $i = 2, 3, \ldots, n$. By construction, (γ_i, η_i) is a compatible pair for $i = 1, 2, \ldots, n$. We see that the triple $(\kappa_i, \gamma_i, \eta_i)$ is compatible for $i = 1, 2, \dots, n$. Since η_i restrict to β_i on $K_1(C)$ and η_i is induced from γ_i on $\mathrm{Aff}(T(C))/\rho_C(K_0(C))$, we have

$$\eta_1 + \eta_2 + \cdots + \eta_n = \eta$$

on $U^\infty(C)/\overline{CU^\infty(C)}$. □

Theorem IV.2.4. *Let X be a compact metric space with finitely many connected components. Let A be a separable simple unital C^*-algebra with tracial rank at most one. Let $n \geq 1$ be an integer. Any unital injective homomorphism $\phi\colon C(X) \to M_n(A)$ is approximately unitarily equivalent to a diagonal homomorphism.*

Proof. Since X has finitely many connected components, $C(X, \mathbb{Z})$ is generated by the atoms of $K_0(C(X))_+$, which are the characteristic functions of the connected components of X. We denote the characteristic functions of the connected components of X by χ_j for $j = 1, 2, \dots, k$ and so

$$K_0(C(X)) = C(X, \mathbb{Z}) \oplus \ker \rho_{C(X)}.$$

Also, since ϕ is injective, $\ker K_0(\phi) \cap C(X, \mathbb{Z}) = 0$. So by Lemma IV.1.1, there exist normalized group homomorphisms $\alpha_i\colon (K_0(C(X)), [1_{C(X)}]) \to (K_0(A), [1_A])$ such that $\ker \alpha_i \cap C(X, \mathbb{Z}) = 0$ for all i, $\ker \alpha_i = \ker \rho_{C(X)}$ for $i > 1$ and

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = K_0(\phi).$$

Since A has stable rank one, by Lemma IV.2.2, there exist non-zero, mutually orthogonal projections $p_{i,j} \in M_n(A)$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, k$ such that

$[p_{i,j}] = \alpha_i(\chi_j)$ and

$$\sum_{i=1}^{n} p_{i,j} = \phi(\chi_j).$$

We define $\gamma_i \colon C(X)_{\mathrm{sa}} \to \mathrm{Aff}(T(A))$ such that

$$\gamma_i(f)(\tau) = \sum_{j=1}^{k} \tau(p_{i,j}\phi(f)p_{i,j}).$$

Since the projections $p_{i,j}$ are non-zero and mutually orthogonal, γ_i is a positive linear map with $\ker \gamma_i = 0$. For all $\tau \in T(A)$ and j_0, we have

$$\gamma_i(\chi_{j_0})(\tau) = \sum_{j=1}^{k} \tau(p_{i,j}\phi(\chi_{j_0})p_{i,j}) = \tau(p_{i,j_0}) = \tau(\rho_A(\alpha_i(\chi_{j_0})))$$

and so (α_i, γ_i) is a compatible pair for $i = 1, 2, \ldots, n$.

By Lemma IV.2.3, there exist $\kappa_i \in KL_e(C(X), A)^{++}$ such that

$$\kappa_1 + \kappa_2 + \cdots + \kappa_n = KL(\phi)$$

and group homomorphisms $\eta_i \colon U^\infty(C(X))/\overline{CU^\infty(C(X))} \to U(A)/\overline{CU(A)}$ such that

$$\eta_1 + \eta_2 + \cdots + \eta_n = \phi^\ddagger$$

and $(\kappa_i, \gamma_i, \eta_i)$ are compatible triples for $i = 1, 2, \ldots, n$.

So by II.5.3, there exist unital homomorphisms $\psi_i\colon C(X) \to A$ for $i = 1, 2, \ldots, n$ such that

$$KL(\psi_i) = \kappa_i,$$
$$\tau(\psi_i(f)) = \gamma_i(f)(\tau), \text{ and}$$
$$\psi_i^{\ddagger} = \eta_i.$$

Let

$$\psi = \begin{pmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \phi_n \end{pmatrix}.$$

We can see that

$$KL(\psi) = \sum_{i=1}^{n} KL(\phi_i) = \sum_{i=1}^{n} \kappa_i = KL(\phi),$$
$$\tau(\psi(f)) = \sum_{i=1}^{n} \tau(\phi_i(f)) = \sum_{i=1}^{n} \gamma_i(f)(\tau) = \tau(\phi(f)), \text{ and}$$
$$\psi^{\ddagger} = \sum_{i=1}^{n} \phi_i^{\ddagger} = \sum_{i=1}^{n} \eta_i = \phi^{\ddagger}.$$

So by Theorem II.5.1, ϕ and ψ are approximately unitarily equivalent. □

Theorem IV.2.5. *Let X be a compact metric space. Let A be a separable simple unital C^*-algebra with tracial rank at most one. Let $n \geq 1$ be an integer. Any unital injective homomorphism $\phi\colon C(X) \to M_n(A)$ is approximately diagonalizable.*

Proof. Let $\varepsilon > 0$ and let $\mathcal{F} \subseteq C(X)$ be a finite subset. By Theorem II.5.2, there exist $\delta > 0$, a finite subset $\mathcal{F} \subseteq C(X)$, a finite subset $\mathcal{P} \subseteq \underline{K}(C(X))$, and a finite subset $\mathcal{U} \subseteq U^\infty(C(X))$ such that for any unital homomorphism $\psi\colon C(X) \to M_n(A)$, if

1. $KL(\phi) = KL(\psi)$ on $\mathcal{P}$,
2. $\mathrm{dist}(\phi^\ddagger(\bar{z}), \psi^\ddagger(\bar{z})) < \delta$ for $z \in \mathcal{U}$, and
3. $|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta$ for $g \in \mathcal{G}$,

then there exists a unitary $u \in A$ such that

$$\|u\phi(f)u^* - \psi(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$. Recall that for every unitary $u \in U(C)$, we denote the equivalence class in $U(C)/\overline{CU(C)}$ of u by $\bar{u}$.

Since X is a compact metric space, there exist finite simplicial complexes X_m for $m \in \mathbb{Z}_{\geq 0}$ and unital homomorphisms $s_m\colon C(X_m) \to C(X_{m+1})$ such that $C(X) \cong \varinjlim C(X_m)$. Let $s_{m,\infty}\colon C(X_m) \to C(X)$ denote the homomorphisms induced by the inductive limit. Let $k(m)$ denote the number of connected components of X_m and let χ_m^j the characteristic functions of the connected components of X_m for $j = 1, 2, \ldots, k(m)$. We may assume that $s_m(\chi_m^j) \neq 0$ for all j.

Since $\mathcal{G}$ is finite, there exist an integer M and a finite set $\mathcal{G}' \subseteq C(X_M)_{\mathrm{sa}}$ such that for every $g \in \mathcal{G}$, there exists $g' \in \mathcal{G}'$ such that $\|g - s_{M,\infty}(g')\| < \delta/2$.

Furthermore, by taking a possibly larger value of M, there exists a finite set $\mathcal{U}' \subseteq U^\infty(C(X_M))/CU^\infty(C(X_M))$ such that for every $u \in \mathcal{U}$, there exists $u_0 \in \mathcal{U}'$ such that $\mathrm{dist}(\bar{u}, s_{M,\infty}^\ddagger(\bar{u}_0)) < \delta/2$.

We proceed in the exact same fashion as in the proof of Theorem IV.2.4. Since X_M has finitely many connected components, $C(X_M, \mathbb{Z})$ is generated by the atoms of $K_0(C(X_M))_+$ and so

$$K_0(C(X_M)) = C(X_M, \mathbb{Z}) \oplus \ker \rho_{C(X_M)}.$$

In addition we see that since ϕ is injective, $\ker K_0(\phi) \cap C(X, \mathbb{Z}) = 0$. So by Lemma IV.1.1, there exist normalized group homomorphisms $\alpha_{i,M} \colon (K_0(C(X_M)), 1_{C(X_M)}) \to (K_0(A), 1_A)$ such that $\ker \alpha_{i,M} \cap C(X_M, \mathbb{Z}) = 0$ for all i, $\ker \alpha_{i,M} = \ker \rho_{C(X_M)}$ when $i > 1$ and

$$K_0(\phi \circ s_{M,\infty}) = \sum_{i=1}^{n} \alpha_{i,M}.$$

Since A has stable rank one, by Lemma IV.2.2, there exist non-zero mutually orthogonal projections $p_{i,j}^M \in M_n(A)$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k(M)$ such that $[p_{i,j}^M] = \alpha_{i,M}(\chi_M^j)$ and

$$\phi(s_{M,\infty}(\chi_M^j)) = \sum_{i=1}^{n} p_{i,j}^M.$$

We define $\gamma_{i,M} \colon C(X_M)_{\mathrm{sa}} \to \mathrm{Aff}(T(A))$ by

$$\gamma_{i,M}(f)(\tau) = \sum_{j=1}^{k(M)} \tau(p_{i,j}^M \phi \circ s_{M,\infty}(f) p_{i,j}^M).$$

Since the projections $p_{i,j}^M$ are non-zero and mutually orthogonal, $\gamma_{i,M}$ is a positive, linear map with $\ker \gamma_{i,M} = \ker s_{M,\infty}$. For all $\tau \in T(A)$ and j_0, we have

$$\gamma_{i,M}(\chi_M^{j_0})(\tau) = \sum_{j=1}^{k(M)} \tau(p_{i,j}^M \phi(s_{M,\infty}(\chi_M^{j_0})) p_{i,j}^M) = \tau(p_{i,j_0}^M) = \tau(\rho_A(\alpha_i(\chi_M^{j_0}))).$$

So $(\alpha_{i,M}, \gamma_{i,M})$ is a compatible pair for $i = 1, 2, \ldots, n$.

We inductively apply Lemma IV.1.2 to construct normalized positive group homomorphisms $\alpha_{i,m}\colon K_0(C(X_m)) \to K_0(A)$ for $i = 1, 2, \ldots, n$ and $m \geq M$ so that $K_0(\phi \circ s_{m,\infty}) = \sum_{i=1}^{n} \alpha_{i,m}$ with $\alpha_{i,m} = \alpha_{i,m+1} \circ s_m$, and $\ker \alpha_{i,m} \cap C(X_m, \mathbb{Z}) = 0$ for all i with $\ker \alpha_{i,m} = \ker \rho_{C(X_m)}$ when $i > 1$.

As before, there exist non-zero mutually orthogonal projections $p^m_{i,j} \in M_n(A)$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k(m)$ such that $[p^m_{i,j}] = \alpha_{i,m}(\chi^j_m)$ and

$$\phi \circ s_{m,\infty}(\chi^j_m) = \sum_{i=1}^{n} p^m_{i,j}.$$

We see that $\gamma_{i,m}$ is a positive unital linear map with $\ker \gamma_{i,m} = \ker s_{m,\infty}$. The pair $\alpha_{i,m}, \gamma_{i,m})$ is compatible by a computation identical to the case where $m = M$.

Let α_i be the homomorphism induced by the inductive limit and the homomorphisms $\alpha_{i,m}$ and let γ_i be the linear map induced by the inductive limit and the linear maps $\gamma_{i,m}$. Since

$$K_0(\phi \circ s_{m,\infty}) = \alpha_{1,m} + \alpha_{2,m} + \cdots + \alpha_{n,m},$$

by the uniqueness maps induced by the inductive limit, we have

$$K_0(\phi) = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

Since $\ker \alpha_{i,m} \cap C(X_m, \mathbb{Z}) = 0$, it follows that $\ker \alpha_i \cap C(X, \mathbb{Z}) = 0$ for all i. Also γ_i is injective, since $\ker \gamma_{i,m} = \ker s_{m,\infty}$. And since $(\alpha_{i,m}, \gamma_{i,m})$ is a compatible pair, we have that (α_i, γ_i) is a compatible pair.

By Lemma IV.2.3, there exist $\kappa_i \in KL_e(C(X), A)^{++}$ such that

$$\kappa_1 + \kappa_2 + \cdots + \kappa_n = KL(\phi)$$

and group homomorphisms $\eta_i \colon U^\infty(C(X))/\overline{CU^\infty(C(X))} \to U(A)/\overline{CU(A)}$ such that

$$\eta_1 + \eta_2 + \cdots + \eta_n = \phi^\ddagger,$$

and such that $(\kappa_i, \gamma_i, \eta_i)$ is a compatible triple for $i = 1, 2, \ldots, n$.

We note that

$$\sum_{i=1}^{n} \eta_i \circ s^\ddagger_{M,\infty} = (\phi \circ s_{M,\infty})^\ddagger$$

on $U^\infty(C(X_M))/\overline{CU(C(X_M))}$.

By Theorem 4.5 of [22], there exist unital injective homomorphisms $\phi_i \colon C(X) \to A$ such that

$$\begin{aligned} KL(\phi_i) &= \kappa_i, \\ \tau(\phi_i(f)) &= \gamma_i(f)(\tau), \text{ and} \\ \phi_i^\ddagger &= \eta_i. \end{aligned}$$

for all $f \in C(X)_{\mathrm{sa}}$ and $\tau \in T(A)$. Let

$$\psi = \begin{pmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \phi_n \end{pmatrix}.$$

So

$$KL(\psi) = \sum_{i=1}^{n} KL(\phi_i) = \sum_{i=1}^{n} \kappa_i = KL(\phi).$$

In particular, this holds for $\mathcal{P}$.

Let $f \in \mathcal{G}$ and $\tau \in T(M_n(A))$. There exists $f' \in \mathcal{G}'$ so that $\|f - s_{M,\infty}(f')\| < \delta/2$. Note that

$$\begin{aligned}
\tau(\psi(s_{M,\infty}(f'))) &= \sum_{i=1}^{n} \gamma_i(s_{M,\infty}(f'))(\tau) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{k(M)} \tau(p_{i,j}^M \phi(s_{M,\infty}(f')) p_{i,j}^M) \\
&= \sum_{j=1}^{k(M)} \tau(\phi(s_{M,\infty}(\chi_M^j))\phi(s_{M,\infty}(f'))\phi(s_{M,\infty}(\chi_M^j))) \\
&= \tau(\phi(s_{M,\infty}(f'))).
\end{aligned}$$

Consequently,

$$\begin{aligned}
|\tau(\phi(f)) - \tau(\psi(f))| &\leq |\tau(\phi(f)) - \tau(\phi(s_{M,\infty}(f')))| \\
&\quad + |\tau(\phi(s_{M,\infty}(f'))) \quad - \tau(\psi(s_{M,\infty}(f')))| \\
&\quad + |\tau(\psi(s_{M,\infty}(f'))) - \tau(\psi(f))| \\
&< \|\tau \circ \phi\| (\delta/2) + \|\tau \circ \psi\| (\delta/2) \\
&= \delta.
\end{aligned}$$

Let $u \in \mathcal{U}$. There exists $u_0 \in \mathcal{U}'$ such that $\operatorname{dist}(\bar{u}, s^{\ddagger}_{M,\infty}(\bar{u}_0)) < \delta/2$. So we have

$$\begin{aligned}
\operatorname{dist}(\phi^{\ddagger}(\bar{u}), \psi^{\ddagger}(\bar{u})) &\leq \operatorname{dist}(\phi^{\ddagger}(\bar{u}), (\phi \circ s_{M,\infty})(\bar{u})) \\
&\quad + \operatorname{dist}((\phi \circ s_{M,\infty})^{\ddagger}(\bar{u}), (\psi \circ s_{M,\infty})^{\ddagger}(\bar{u})) \\
&\quad + \operatorname{dist}((\psi \circ s_{M,\infty})^{\ddagger}(\bar{u}), \psi^{\ddagger}(\bar{u})) \\
&\leq \delta/2 + 0 + \delta/2 \\
&= \delta.
\end{aligned}$$

Therefore, there exists a unitary $u \in M_n(A)$ such that for all $f \in \mathcal{F}$,

$$\left\| u\phi(f)u^* - \begin{pmatrix} \phi_1(f) & 0 & \cdots & 0 \\ 0 & \phi_2(f) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \phi_n(f) \end{pmatrix} \right\| < \varepsilon.$$

□

Corollary IV.2.6. *Let X be a compact metric space and let A be a simple separable unital C^*-algebra with tracial rank at most one. Let $n \geq 1$ be an integer. Any unital homomorphism $\phi\colon C(X) \to M_n(A)$ is approximately diagonalizable.*

Proof. There exists a metric space Y such that $C(Y) \cong C(X)/\ker\phi$. Let $\psi\colon C(Y) \to M_n(A)$ denote the induced injective homomorphism and let $\pi\colon C(X) \to C(Y)$ denote the canonical quotient. By Theorem IV.2.5, there exist unital homomorphisms $\psi_n\colon C(Y) \to M_n(A)$ and a unitary $u \in M_n(A)$ such that

$$\left\| u\psi(g)u^* - \begin{pmatrix} \psi_1(g) & 0 & \cdots & 0 \\ 0 & \psi_2(g) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \psi_n(g) \end{pmatrix} \right\| < \varepsilon$$

for all $g \in \pi(F)$. So for all $f \in \mathcal{F}$,

$$\left\| u\psi(\pi(f))u^* - \begin{pmatrix} \psi_1(\pi(f)) & 0 & \cdots & 0 \\ 0 & \psi_2(\pi(f)) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \psi_n(\pi(f)) \end{pmatrix} \right\| < \varepsilon.$$

So by setting $\phi_i = \psi_i \circ \pi$, we obtain the result. □

Corollary IV.2.7. *Let A be a simple separable unital C^*-algebra with tracial rank at most one. For any integer $n \geq 1$, every normal matrix $a \in M_n(A)$ is approximately diagonalizable. Furthermore, when $\mathrm{sp}(a)$ has finitely many connected components, a is approximately unitarily equivalent to a diagonal matrix.*

Matrices over C^*-Algebras with Rational Tracial Rank One

Since we have a classification of homomorphisms from AH-algebras to C^*-algebras with rational tracial rank one, we can use similar methods to prove approximate diagonalization results. Without the Riesz interpolation, however, approximate diagonalization won't hold in general.

Theorem IV.3.1. *Let X be a compact, connected, metric space such that $K_1(C(X))$ is free. Let A be a simple separable unital $\mathcal{Z}$-stable C^*-algebra with rational tracial rank at most one. Let $n \geq 1$ be an integer. Any injective unital homomorphism $\phi\colon C(X) \to M_n(A)$ is approximately unitarily equivalent to a diagonal homomorphism.*

Proof. Since X is connected, $C(X, \mathbb{Z}) \cong \mathbb{Z}$ and so

$$K_0(C(X)) = \mathbb{Z} \oplus \ker \rho_{C(X)}.$$

Also $[1_{C(X)}] = (1, 0)$ in this decomposition. We define normalized group homomorphisms $\alpha_i\colon (K_0(C(X)), 1_{C(X)}) \to (K_0(A), 1_A)$ by $\alpha_i(1_{C(X)}) = 1_A$ on $C(X, \mathbb{Z})$ and

$$\alpha_i = \begin{cases} K_0(\phi) & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}$$

on $\ker \rho_{C(X)}$. One can readily see that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = K_0(\phi).$$

We define $\gamma_i\colon C(X)_{\mathrm{sa}} \to \mathrm{Aff}(T(A))$ by

$$\gamma_i(f)(\tau) = \tau(\phi(f))$$

for $i = 1, 2, \ldots, n$. Since $\rho_{C(X)}(C(X))$ is cyclic and γ_i is unital, (α_i, γ_i) is a compatible pair for $i = 1, 2, \ldots, n$.

By Lemma IV.2.3, there exist elements $\kappa_i \in KL_e(C(X), A)^{++}$ such that

$$\kappa_1 + \kappa_2 + \cdots + \kappa_n = KL(\phi)$$

and group homomorphisms $\eta_i \colon U^\infty(C(X))/\overline{CU^\infty(C(X))} \to U(A)/\overline{CU(A)}$ such that

$$\eta_1 + \eta_2 + \cdots + \eta_n = \phi^\ddagger$$

and $(\kappa_i, \gamma_i, \eta_i)$ is a compatible triple for $i = 1, 2, \ldots, n$.

So by Theorem II.5.5, there exist unital homomorphisms $\psi_i \colon C(X) \to A$ for $i = 1, 2, \ldots, n$ such that

$$\begin{aligned} KL(\psi_i) &= \kappa_i, \\ \tau(\psi_i(f)) &= \gamma_i(f)(\tau), \text{ and} \\ \psi_i^\ddagger &= \eta_i. \end{aligned}$$

Let

$$\psi = \begin{pmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \phi_n \end{pmatrix}.$$

We can see that

$$KL(\psi) = \sum_{i=1}^{n} KL(\phi_i) = \sum_{i=1}^{n} \kappa_i = KL(\phi),$$
$$\tau(\psi(f)) = \sum_{i=1}^{n} \tau(\phi_i(f)) = \sum_{i=1}^{n} \gamma_i(f)(\tau) = \tau(\phi(f)),$$
$$\text{and } \psi^{\ddagger} = \sum_{i=1}^{n} \phi_i^{\ddagger} = \sum_{i=1}^{n} \eta_i = \phi^{\ddagger}.$$

So by Theorem II.5.4, ϕ and ψ are approximately unitarily equivalent. □

Corollary IV.3.2. *Let A be a simple, separable, unital, $\mathcal{Z}$-stable C^*-algebra with rational tracial rank at most one. For any integer $n \geq 1$, every normal matrix $a \in M_n(A)$ is approximately diagonalizable if $\mathrm{sp}(a)$ is connected and $K_1(C(\mathrm{sp}(a))$ is free. In particular, self-adjoint matrices with connected spectra are approximately diagonalizable.*

Unfortunately, the condition that X is connected is essential to Theorem IV.3.1.

To construct a concrete counterexample, by Theorem 6.8 of [23], there exists a simple, separable, unital C^*-algebra A_0 with $\mathrm{TR}(A \otimes \mathcal{Q}) \leq 1$ such that $K_0(A) \cong \mathbb{Z}^2$ with the strict ordering and with order unit $(2,2)$, $K_1(A) = 0$, and $T(A) \cong [0,1]$.

As we've seen in Section III.4, the group $\mathbb{Z}^2$ with the strict ordering tensored with $\mathbb{Q}$ is isomorphic to $\mathbb{Q}^2$ with the strict ordering. Since $\mathbb{Q}^2$ with the strict ordering is a simple dimension group, it follows that $\mathbb{Z}^2$ satisfies the rational Riesz property.

There exists a projection $p \in M_2(A_0)$ such that $[p] = (3,1)$ in $K_0(A_0)$. We note that $[p]$ is an atom, since if $g \in G_+$ is a nonzero positive element with $g < [p]$, then, writing $g = (x, y)$, we have $y > 0$ and $(3 - x, 1 - y) \gg (0, 0)$. So $1 - y > 0$, or $0 < y < 1$, which is impossible.

So we see that not even projections are necessarily approximately diagonalizable.

We see that this example fails when the K_0 group of the codomain is not an interpolation group. When the the K_0 group of the codomain is a weak dimension group, approximate diagonalization holds generally.

Theorem IV.3.3. *Let X be a compact metric space such that $K_1(C(X))$ is free. Let A be a separable simple unital $\mathcal{Z}$-stable C^*-algebra with rational tracial rank at most one such that $K_0(A) = \mathbb{Z}$. Let $n \geq 1$ be an integer. Any unital injective homomorphism $\phi\colon C(X) \to M_n(A)$ is approximately diagonalizable.*

Proof. Let $x_0 \in K_0(A)_+$ denote the unique atom of $K_0(A)_+$. There exists a unique positive group isomorphism $\theta\colon K_0(A) \to \mathbb{Z}$ such that $\theta(x_0) = 1$. Since ϕ is injective, we have $K_0(\phi)([\chi]) \neq 0$ for any projection $\chi \in C(X)$. We see that X has no more than $m = \theta(n[1_A])$ connected components. Otherwise, there would exist mutually orthogonal projections χ_i for $i = 1, 2, \ldots, m + 1$ such that

$$\sum_{i=1}^{m+1} \chi_i = 1_{C(X)},$$

and we would have

$$m = n[1_A] = K_0(\phi)([1_{C(X)}]) = \sum_{i=1}^{m+1} K_0(\phi)(\chi_i) \geq m + 1,$$

a contradiction. So $C(X)$ has finitely many connected components. Enumerate the connected components of X by X_j for $j = 1, 2, \ldots, k$. Let χ_j denote the characteristic function of X_j for $j = 1, 2, \ldots, k$. Note that $C(X) = \bigoplus_{j=1}^{k} C(X_j)$.

Since ϕ is unital, we have

$$\sum_{j=1}^{k} K_0(\phi)([\chi_j]) = n[1_A].$$

So by the Riesz interpolation property, there exists $z_{i,j} \in K_0(A)_+$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$ such that

$$\sum_{i=1}^{n} z_{i,j} = [\chi_j] \text{ and}$$
$$\sum_{j=1}^{k} z_{i,j} = [1_A]$$

For $i = 1, 2, \ldots, n$, we define

$$N(i) = \{j \colon z_{i,j} \neq 0\} \text{ and}$$
$$N'(i) = \{j \colon z_{i,j} = 0\}.$$

We also denote $C_i = \bigoplus_{j \in N(i)} C(X_j)$ and identify C_i as a subalgebra of $C(X)$ for $i = 1, 2, \ldots, n$. Since $\sum_j z_{i,j} = [1_A]$, for each j, there exists i such that $z_{i,j} \neq 0$ or equivalently $i \in N(j)$. Let

$$m(j) = \inf\{i \colon j \in N(i)\}.$$

Let $\alpha_i\colon K_0(C_i) \to K_0(A)$ be defined by We define $\alpha_i\colon G \to H$ by setting $\alpha_i(x_j) = z_{i,j}$ for all i and j, and by setting

$$\alpha_i(g) = \begin{cases} K_0(\phi)(g) & \text{if } i = m(j) \\ 0 & \text{if } i \neq m(j) \end{cases}$$

for $g \in \ker \rho_{C(X_j)} \subseteq \ker \rho_{C(X)}$. We note that α_i is a strictly positive, normalized group homomorphism for $i = 1, 2, \ldots, n$. By extending α_i to group homomorphisms $\bar{\alpha}_i\colon K_0(C(X)) \to K_0(A)$ by setting $\bar{\alpha}_i(g) = 0$ for $g \in \bigoplus j \in N'(i)C(X_j)$, we see

$$K_0(\phi) = \bar{\alpha}_1 + \bar{\alpha}_2 + \cdots \bar{\alpha}_n.$$

Since A has stable rank one, by Lemma IV.2.2, there exist non-zero, mutually orthogonal projections $p_{i,j} \in M_n(A)$ for $i = 1, 2, \ldots, n$ and $j \in N(i)$ such that $[p_{i,j}] = \alpha_i(\chi_j)$ and

$$\sum_{i=1}^{n} p_{i,j} = \phi(\chi_j).$$

We define $\gamma_i\colon (C_i)_{\mathrm{sa}} \to \mathrm{Aff}(T(A))$ such that

$$\gamma_i(f)(\tau) = \sum_{j=1}^{k} \tau(p_{i,j}\phi(f)p_{i,j}).$$

Since the projections $p_{i,j}$ are non-zero and mutually orthogonal, γ_i is a positive linear map with $\ker \gamma_i = 0$. For all $\tau \in T(A)$ and $j_0 \in N(i)$, we have

$$\gamma_i(\chi_{j_0})(\tau) = \sum_{j=1}^{k} \tau(p_{i,j}\phi(\chi_{j_0})p_{i,j}) = \tau(p_{i,j_0}) = \tau(\rho_A(\alpha_i(\chi_{j_0})))$$

and so (α_i, γ_i) is a compatible pair for $i = 1, 2, \ldots, n$.

By Lemma IV.2.3, there exist elements $\kappa_i \in KL_e(C_i, A)^{++}$ such that

$$\kappa_1 + \kappa_2 + \cdots + \kappa_n = KL(\phi)$$

and group homomorphisms $\eta_i \colon U^\infty(C(X))/\overline{CU^\infty(C(X))} \to U(A)/\overline{CU(A)}$ such that

$$\eta_1 + \eta_2 + \cdots + \eta_n = (\phi)^\ddagger$$

and $(\kappa_i, \gamma_i, \eta_i)$ is a compatible triple for $j = 1, 2, \ldots, k$ and $i = 1, 2, \ldots, n$.

So by Theorem II.5.5, there exist unital homomorphisms $\psi_i \colon C(X) \to A$ for $i = 1, 2, \ldots, n$ such that

$$\begin{aligned} KL(\psi_i) &= \kappa_i, \\ \tau(\psi_i(f)) &= \gamma_i(f)(\tau), \text{ and} \\ \psi_i^\ddagger &= \eta_i. \end{aligned}$$

We set $\psi_i = 0$ for $i = 1, 2, \ldots, n$ and $j \in N'(i)$. Let

$$\psi = \begin{pmatrix} \psi_1 & 0 & \cdots & 0 \\ 0 & \psi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \psi_n \end{pmatrix}.$$

We can see that

$$\begin{aligned}
KL(\psi) &= \sum_{i=1}^{n} KL(\psi_i) = KL(\phi), \\
\tau(\psi(f)) &= \sum_{i=1}^{n} \gamma_i(f)(\tau) = \tau(\phi(f)), \\
\text{and } \psi^{\ddagger} &= \sum_{i=1}^{n} \sum_{j=1}^{k} \eta_i^j = \phi^{\ddagger}.
\end{aligned}$$

So by Theorem II.5.4, ϕ and ψ are approximately unitarily equivalent. □

In particular, self-adjoint matrices over the Jiang-Su algebra are approximately diagonalizable.

CHAPTER V

APPROXIMATE DIAGONALIZATION OF HOMOMORPHISMS

Approximate Diagonalization When Domain Has Unique Trace

As noted in Section IV.1, the K_0 groups of commutative C^*-algebras have nice properties that enable general approximate diagonalization results. Unfortunately, the K_0 groups of general AH-algebras are more diverse. In particular, the infinitesimals do not always split and the natural choice of order unit is not as amenable to the use of the Riesz decomposition property.

But this diversity does lead to a well-behaved class of AH-algebras for our purposes which is nearly disjoint from the commutative C^*-algebras, namely those AH-algebras with unique tracial state. Since the extreme tracial states of a commutative C^*-algebra correspond to the points of the space, as shown in Section IV.1, the only commutative C^*-algebra with a unique tracial state is $\mathbb{C}$. In contrast, many AH-algebras have unique tracial state. Every UHF-algebra, for example, has a unique tracial state.

Lemma V.1.1. *Let (G,u) and (H,v) be partially ordered abelian groups with order units. Suppose that G has a unique state, that H is simple, and that there is a normalized positive group homomorphism from (G,u) to (H,v). Then for any integer $n \geq 1$ and any normalized positive group homomorphism $\alpha\colon (G,u) \to (H,nv)$, there exist normalized positive group homomorphisms $\alpha_i\colon (G,u) \to (H,v)$ such that*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = \alpha.$$

Proof. Let $\beta\colon (G,u) \to (H,v)$ be a normalized positive group homomorphism. Let $\sigma \in S(G,u)$ denote the unique state. Since G has a unique state, $\operatorname{Aff}(S(G,u)) \cong \mathbb{R}$ and so $\{\rho_G(u)\}$ is a basis for $\operatorname{Aff}(S(G,u))$. So for any positive group homomorphism from G to H, the induced map from $\operatorname{Aff}(S(G,u))$ to $\operatorname{Aff}(S(H,v))$ is determined by where u is mapped. In particular, we have

$$\alpha_\rho = n\beta_\rho,$$

since $\alpha(u) = n\beta(u) = nv$. Now we define $\alpha_i\colon G \to H$ by

$$\alpha_i = \begin{cases} \alpha - (n-1)\beta & \text{if } i = 1 \\ \beta & \text{if } i \neq 1 \end{cases}$$

for $i = 1, 2, \ldots, n$. It is clear that α_i is a group homomorphism and

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = \alpha.$$

By construction, $\alpha_i\colon (G,u) \to (H,v)$ is a normalized positive group homomorphism for $i = 2, 3, \ldots, n$. It suffices to show α_1 is normalized and positive. Note that

$$\alpha_1(u) = \alpha(u) - (n-1)\beta(u) = nv - (n-1)v = v.$$

So $\alpha_1\colon (G,u) \to (H,v)$ is normalized.

Let $g \in G_+$. Since g is positive, $\sigma(g) \geq 0$. We first assume that $\sigma(g) > 0$. Note that for any $\tau \in S(H,v)$, $\tau \circ \beta = \sigma$ since $\tau \circ \beta \in S(G,u)$ and σ is the unique state of G. So for any $\tau \in S(H, v)$, we have

$$\begin{aligned}\tau(\alpha_1(g)) &= \tau(\alpha(g) - (n-1)\beta(g)) \\ &= \tau(\alpha(g)) - (n-1)\tau(\beta(g)) \\ &= \alpha_\rho(\tau)(g) - (n-1)\beta_\rho(\tau)(g) \\ &= \beta_\rho(\tau)(g) \\ &= \tau(\beta(g)) \\ &= \sigma(g) > 0.\end{aligned}$$

Now suppose that $g \in G_+ \cap \ker \sigma$. As we've seen just now, for any $\tau \in S(H, v)$, we have $\tau \circ \beta = \sigma$. So it follows that $\tau(\beta(g)) = 0$. So $\beta(g) \in H_+$ with $\beta(g) \in \ker \rho_H$. Since H is simple, it follows that $\beta(g) = 0$. Similarly, we have $\tau \circ \alpha = n\sigma$ and so it follows that $\tau(\alpha(g)) = 0$. So $\alpha(g) \in H_+ \cap \ker \rho_H$ and so by simplicity, $\alpha(g) = 0$. Hence $\alpha_1(g) = 0$. So α_1 is a positive homomorphism. □

Theorem V.1.2. *Let C be a separable, unital AH-algebra with a unique tracial state and let A be a separable, simple, unital C^*-algebra with tracial rank at most one. Suppose there exists a unital homomorphism from C to A. Let $n \geq 1$ be an integer. Any unital, injective homomorphism $\phi\colon C \to M_n(A)$ is approximately unitarily equivalent to a diagonal homomorphism.*

Proof. Let $\theta\colon C \to A$ be a unital homomorphism. The induced map $K_0(\theta)$ is a normalized positive group homomorphism. Also since C is exact and has a unique tracial state, $K_0(C)$ has a unique state. By Lemma V.1.1, there exist normalized positive group homomorphisms $\alpha_i\colon (K_0(C), 1_C) \to (K_0(A), 1_A)$ such that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = K_0(\phi).$$

Let $\sigma \in T(C)$ be the unique tracial state. Since C has a unique tracial state,

$$\operatorname{Aff}(T(C)) \cong \operatorname{Aff}(S(K_0(C)), [1_C]) \cong \mathbb{R},$$

and so there exists a unique unital linear map $\gamma\colon \operatorname{Aff}(T(C)) \to \operatorname{Aff}(T(A))$, which is strictly positive unital and continuous.

As noted in Section III.2, to show compatibility it suffices to show that the square

$$\begin{array}{ccc} \operatorname{Aff}(SK_0(C)) & \xrightarrow{(\alpha_i)_\rho} & \operatorname{Aff}(SK_0(A)) \\ \downarrow & & \downarrow \\ \operatorname{Aff}(T(C)) & \xrightarrow{\gamma_i} & \operatorname{Aff}(T(A)) \end{array}$$

commutes for each i, but by uniqueness we have $(\alpha_i)_\rho = K_0(\theta)_\rho$ and $\gamma_i = \theta_\sharp$. So the diagram does commute.

By Lemma IV.2.3, there exist $\kappa_i \in KL_e(C(X), A)^{++}$ such that

$$\kappa_1 + \kappa_2 + \cdots + \kappa_n = KL(\phi)$$

and group homomorphisms $\eta_i\colon U^\infty(C(X))/\overline{CU^\infty(C(X))} \to U(A)/\overline{CU(A)}$ such that

$$\eta_1 + \eta_2 + \cdots + \eta_n = \phi^\ddagger,$$

and the triple $(\kappa_i, \gamma_i, \eta_i)$ is compatible for $i = 1, 2, \ldots, n$.

So by II.5.3, there exist unital homomorphisms $\psi_i\colon C(X) \to A$ for $i = 1, 2, \ldots, n$ such that

$$KL(\psi_i) = \kappa_i,$$
$$\tau(\psi_i(f)) = \gamma_i(f)(\tau), \text{and } \psi_i^\ddagger \qquad = \eta_i.$$

Let

$$\psi = \begin{pmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \phi_n \end{pmatrix}.$$

We can see that

$$KL(\psi) = \sum_{i=1}^{n} KL(\phi_i) = \sum_{i=1}^{n} \kappa_i = KL(\phi),$$
$$\tau(\psi(f)) = \sum_{i=1}^{n} \tau(\phi_i(f)) = \sum_{i=1}^{n} \gamma_i(f)(\tau) = \tau(\phi(f)),$$
$$\text{and } \psi^\ddagger = \sum_{i=1}^{n} \phi_i^\ddagger = \sum_{i=1}^{n} \eta_i = \phi^\ddagger.$$

So by Theorem II.5.4, ϕ and ψ are approximately unitarily equivalent. □

Approximate Diagonalization When the Codomain Has Torsion-Free Divisible K_0

Theorem V.2.1. *Let C be a separable unital AH-algebra and let A be a separable simple unital C^*-algebra with tracial rank at most one such that $K_0(A)$ is torsion-free and divisible. Let $n \geq 1$ be an integer. Any unital injective homomorphism $\phi\colon C \to M_n(A)$ is approximately unitarily equivalent to a diagonal homomorphism.*

Proof. Since, as a torsion-free divisible group, $K_0(A)$ is a rational vector space, we can define group homomorphisms $\alpha_i\colon K_0(C) \to K_0(A)$ by

$$\alpha_i(g) = \frac{1}{n} K_0(\phi)(g)$$

for $i = 1, 2, \ldots, n$ and $g \in K_0(C)$.

It is clear that

$$\alpha_i(1_C) = \frac{1}{n} K_0(\phi)(1_C) = \frac{1}{n} \cdot n \cdot 1_A = 1_A,$$

and so α_i is normalized. It is clear that α_i is strictly positive.

We define $\gamma_i\colon \operatorname{Aff}(T(C)) \to \operatorname{Aff}(T(A))$ by taking

$$\gamma_i = \frac{1}{n} \phi_\sharp,$$

which, as a scalar multiple of an induced map, is strictly positive unital and linear. By construction, we have

$$\gamma_1 + \gamma_2 + \cdots + \gamma_n = \gamma.$$

As we saw in the computation showing α_i is positive above,

$$\rho_A(\alpha_i([p]))(\tau) = \frac{1}{n} \tau(\phi(p)) = \gamma_i(\rho_C([p]))(\tau)$$

for all $p \in M_\infty(A)$ and $\tau \in T(A)$. So (α_i, γ_i) is a compatible pair for $i = 1, 2, \ldots, n$.

By Lemma IV.2.3, there exist $\kappa_i \in KL_e(C(X), A)^{++}$ such that

$$\kappa_1 + \kappa_2 + \cdots + \kappa_n = KL(\phi),$$

and group homomorphisms $\eta_i \colon U^\infty(C(X))/\overline{CU^\infty(C(X))} \to U(A)/\overline{CU(A)}$ such that

$$\eta_1 + \eta_2 + \cdots + \eta_n = \phi^\ddagger,$$

and $(\kappa_i, \gamma_i, \eta_i)$ is a compatible triple for $i = 1, 2, \ldots, n$.

So by Theorem II.5.3, there exist unital homomorphisms $\psi_i \colon C(X) \to A$ for $i = 1, 2, \ldots, n$ such that

$$KL(\psi_i) = \kappa_i,$$
$$\tau(\psi_i(f)) = \gamma_i(f)(\tau),$$
$$\psi_i^\ddagger = \eta_i.$$

Let

$$\psi = \begin{pmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \phi_n \end{pmatrix}.$$

We can see that

$$KL(\psi) = \sum_{i=1}^{n} KL(\phi_i) = \sum_{i=1}^{n} \kappa_i = KL(\phi),$$
$$\tau(\psi(f)) = \sum_{i=1}^{n} \tau(\phi_i(f)) = \sum_{i=1}^{n} \gamma_i(f)(\tau) = \tau(\phi(f)),$$
$$\text{and } \psi^\ddagger = \sum_{i=1}^{n} \phi_i^\ddagger = \sum_{i=1}^{n} \eta_i = \phi^\ddagger.$$

So by Theorem II.5.1, ϕ and ψ are approximately unitarily equivalent. □

Counterexample to Approximate Diagonalization

In this section, we will be considering two paricular AF-algebras to demonstrate that approximate diagonalization does not hold generally and to illustrate the ideas in the next section.

Let G_0 be the subgroup of $\mathbb{R}^2$ generated by $(1, 0)$, $(0, 1)$, and $(\sqrt{2}, \sqrt{3})$ with order induced from the strict ordering on $\mathbb{R}^2$. Take $u = (1, 1)$.

Let H_0 be the subgroup of $\mathbb{R}$ generated by 1 and $\sqrt{2} + \sqrt{3}$ with order induced from the usual ordering of $\mathbb{R}$. Let $v \in H_0^+$ be an arbitrary order unit. We will see that approximate diagonalization will depend on the choice of v.

We note that G_0 and H_0 are countable simple dimension groups by Theorem 14.16 of [6]. By the Effros-Handelman-Shen Theorem (see Theorem 2.2 of [4]), there exist unital simple AF-algebras C_0 and B_0 such that

$$(K_0(C_0), K_0(C_0)_+, [1_{C_0}]) \cong (G_0, G_0^+, (1, 1)) \text{ and}$$
$$(K_0(A_0), K_0(A_0)_+, [1_{A_0}]) \cong (H_0, H_0^+, v).$$

For this and the next section, we wish to consider appropriate group homomorphisms as elements of a group. Since positive group homomorphisms do not form a group and the group of all group homomorphisms does not take into account the order structure, we make the following definition.

Definition V.3.1. Let (G, u) and (H, v) be partially ordered groups with order units. We denote by $\mathrm{Hom}_c(G, H)$ the set of group homomorphisms $\alpha\colon G \to H$ such that there exists a continuous linear map $\beta\colon \operatorname{Aff}(S(G, u)) \to \operatorname{Aff}(S(H, v))$ such that $\rho_H \circ \alpha = \beta \circ \rho_G$.

We note that $\mathrm{Hom}_c(G, H)$ is a subgroup of $\mathrm{Hom}(G, H)$ and that every positive group homomorphism from G to H is an element of $\mathrm{Hom}_c(G, H)$.

We also note that in general, different group homomorphisms can induce the same linear map. But if the domain of the homomorphism is archimedean, then ρ_G is injective and so different group homomorphisms in $\mathrm{Hom}_c(G, H)$ will induce different linear maps from $\mathrm{Aff}(S(G, u))$ to $\mathrm{Aff}(S(H, v))$.

Returning to our specific example, we note that by Elliott's classification of AF-algebras (see Theorem 1.3.3 of [27]) and since we are considering finitely generated groups, we see that approximate diagonalization is equivalent to decomposing normalized positive group homomorphisms as the sum of normalized positive group homomorphisms on the K_0 groups.

The next few propositions show that positive group homomorphisms from G_0 to H_0 are associated with rational approximations of an irrational number. Furthermore, the choice of order unit v associates normalization with a bound for the denominator of these approximations. So approximate diagonalization is equivalent to the condition that certain rational approximations can be written as the sum of other rational approximations, which is not often the case. This will be made precise in Proposition V.3.5.

Proposition V.3.2. *There is a one-to-one correspondence* $\Delta\colon \mathbb{Z}^2 \to \mathrm{Hom}_c(G_0, H_0)$ *such that* $(x, y) \mapsto \alpha$*, where* $\alpha((1,0)) = y + x(\sqrt{2}+\sqrt{3})$ *and* $\alpha((0,1)) = y - x(\sqrt{2}+\sqrt{3})$*. Furthermore,* $\Delta(x, y)(u) \in 2\mathbb{Z}$ *for all* $x, y \in \mathbb{Z}$*.*

Proof. First we need to prove that Δ is well-defined. Given $x, y \in \mathbb{Z}$, we have a linear map $\beta\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $\beta((1,0)) = y + x(\sqrt{2}+\sqrt{3})$ and $\beta((0,1)) = y - x(\sqrt{2}+\sqrt{3})$. The restriction of β to G is a group homomorphism from G to H

if a set of generators of G maps to H. By construction $(1,0)$ and $(0,1)$ map to H. Finally, consider

$$\begin{aligned}\beta((\sqrt{2},\sqrt{3})) &= \sqrt{2}(y + x(\sqrt{2}+\sqrt{3})) + \sqrt{3}(y - x(\sqrt{2}+\sqrt{3})) \\ &= y\sqrt{2} + 2x + x\sqrt{6} + y\sqrt{3} - x\sqrt{6} - 3x \\ &= -x + y(\sqrt{2}+\sqrt{3}) \in H.\end{aligned}$$

So the restriction of β to G is in $\mathrm{Hom}_c(G,H)$.

It is clear that Δ is injective. Now fix $\alpha \in \mathrm{Hom}_c(G,H)$. There exist integers x_1, x_2, y_1, y_2 such that $\alpha((1,0)) = y_1 + x_1(\sqrt{2}+\sqrt{3})$ and $\alpha((0,1)) = y_2 + x_2(\sqrt{2}+\sqrt{3})$. There exists a linear map $\beta\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that the restriction of β to G is α. Consider

$$\begin{aligned}\alpha((\sqrt{2},\sqrt{3})) &= \beta((\sqrt{2},\sqrt{3})) \\ &= \sqrt{2}\beta((1,0)) + \sqrt{3}\beta((0,1)) \\ &= \sqrt{2}\alpha((1,0)) + \sqrt{3}\alpha((0,1)) \\ &= \sqrt{2}(y_1 + x_1(\sqrt{2}+\sqrt{3})) + \sqrt{3}(y_2 + x_2(\sqrt{2}+\sqrt{3})) \\ &= y_1\sqrt{2} + 2x_1 + x_1\sqrt{6} + y_2\sqrt{3} + x_2\sqrt{6} + 3x_2 \\ &= 2x_1 + 3x_2 + y_1\sqrt{2} + y_2\sqrt{3} + (x_1+x_2)\sqrt{6}.\end{aligned}$$

So $y_1 = y_2$ and $x_1 = -x_2$. So Δ is surjective.

Fix $x, y \in \mathbb{Z}$. Notice

$$\begin{aligned}\Delta(x,y)((1,1)) &= \Delta(x,y)((1,0)) + \Delta(x,y)((0,1)) \\ &= y + x(\sqrt{2}+\sqrt{3}) + y - x(\sqrt{2}+\sqrt{3}) \\ &= 2y.\end{aligned}$$

So $\Delta(x,y) \in 2\mathbb{Z}$. □

Proposition V.3.3. *For $(x,y) \in \mathbb{Z}^2$, the homomorphism $\Delta(x,y)$ is positive if and only if $y \geq x(\sqrt{2}+\sqrt{3})$ and $y \geq -x(\sqrt{2}+\sqrt{3})$.*

Proof. Let $\alpha = \Delta(x,y)$. We see that $y \geq x(\sqrt{2}+\sqrt{3})$ and $y \geq -x(\sqrt{2}+\sqrt{3})$ if and only if $\alpha((1,0)) \geq 0$ and $\alpha((0,1)) \geq 0$. So if α is positive, then $y \geq x(\sqrt{2}+\sqrt{3})$ and $y \geq -x(\sqrt{2}+\sqrt{3})$.

Let $\beta\colon \mathbb{R}^2 \to \mathbb{R}$ denote a linear map such that β restricted to G is α. If $\alpha((1,0)) \geq 0$ and $\alpha((0,1)) \geq 0$, then for all $(a,b) \in G_0^+$, we have $a \geq 0$ and $b \geq 0$. So

$$\begin{aligned}\alpha((a,b)) &= \beta((a,b)) \\ &= a\beta((1,0)) + b\beta((0,1)) \\ &= a\alpha((1,0)) + b\alpha((0,1)) \geq 0.\end{aligned}$$

So α is positive. □

For every $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the largest integer not exceeding x.

Proposition V.3.4. *For every integer $n \geq 1$ and every $v \in 2\mathbb{Z} \cap H_0$, if*

$$n\left\lfloor \frac{v}{2(\sqrt{2}+\sqrt{3})} \right\rfloor = \left\lfloor \frac{nv}{2(\sqrt{2}+\sqrt{3})} \right\rfloor,$$

then for every normalized positive group homomorphism $\alpha\colon (G_0, u) \to (H_0, nv)$, *there exist normalized positive group homomorphisms* $\alpha_i\colon (G_0, u) \to (H_0, v)$ *for* $i = 1, 2, \ldots, n$ *such that*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = \alpha.$$

Conversely, if

$$n\left\lfloor \frac{v}{2(\sqrt{2}+\sqrt{3})} \right\rfloor < \left\lfloor \frac{nv}{2(\sqrt{2}+\sqrt{3})} \right\rfloor,$$

then there exists a normalized positive group homomorphism $\alpha\colon (G_0, u) \to (H_0, nv)$ *such that* α *cannot be written as the sum of* n *normalized positive group homomorphisms from* (G_0, u) *to* (H_0, v).

Proof. First, assume

$$n\left\lfloor \frac{v}{2(\sqrt{2}+\sqrt{3})} \right\rfloor = \left\lfloor \frac{nv}{2(\sqrt{2}+\sqrt{3})} \right\rfloor.$$

By Proposition V.3.2, α is determined by $\alpha((1,0)) = y + x(\sqrt{2}+\sqrt{3})$. From the proof of the proposition, we have $2y = nv$ and from Proposition V.3.3, we have

$$|x| \le \frac{nv}{2(\sqrt{2}+\sqrt{3})}.$$

Since $x \in \mathbb{Z}$, we have

$$|x| \le \left\lfloor \frac{nv}{2(\sqrt{2}+\sqrt{3})} \right\rfloor = n\left\lfloor \frac{v}{2(\sqrt{2}+\sqrt{3})} \right\rfloor.$$

So there exist integers $x_1, x_2, \ldots, x_n$ such that $x = x_1 + x_2 + \cdots + x_n$ and

$$|x_i| \le \left\lfloor \frac{v}{2(\sqrt{2}+\sqrt{3})} \right\rfloor$$

for all i. So there exists $\alpha_i \in \mathrm{Hom}_c(G, H)$ such that $\alpha_i((1,0)) = v/2 + x_i(\sqrt{2}+\sqrt{3})$. Furthermore, α_i is a positive group homomorphism by Proposition V.3.3 satisfying $\alpha_i(u) = v$. Since

$$\begin{aligned}\alpha_1((1,0)) + \alpha_2((1,0)) + \cdots + \alpha_n((1,0)) &= \sum_{i=1}^{n}(v/2 + x_i(\sqrt{2}+\sqrt{3})) \\ &= nv/2 + x(\sqrt{2}+\sqrt{3}) \\ &= \alpha((1,0)),\end{aligned}$$

we have $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

Now suppose that

$$n\left\lfloor \frac{v}{2(\sqrt{2}+\sqrt{3})} \right\rfloor < \left\lfloor \frac{nv}{2(\sqrt{2}+\sqrt{3})} \right\rfloor.$$

There exists a normalized positive group homomorphism $\alpha\colon (G,u) \to (H, nv)$ such that

$$\alpha((1,0)) = nv/2 + \left\lfloor \frac{nv}{2(\sqrt{2}+\sqrt{3})} \right\rfloor (\sqrt{2}+\sqrt{3}).$$

Any normalized positive group homomorphism $\beta\colon (G,u) \to (H,v)$ has $\beta((1,0)) = v/2 + z(\sqrt{2}+\sqrt{3})$ with $z \in \mathbb{Z}$ and

$$|z| \leq \left\lfloor \frac{v}{2(\sqrt{2}+\sqrt{3})} \right\rfloor.$$

The sum of n such integers will be at most

$$n\left\lfloor \frac{v}{2(\sqrt{2}+\sqrt{3})} \right\rfloor.$$

So the sum of n normalized positive group homomorphisms cannot equal α. □

Proposition V.3.5. *Let $n \geq 1$ be an integer. Every unital homomorphism $\phi\colon C_0 \to M_n(A_0)$ is approximately diagonalizable if and only if*

$$n\left\lfloor \frac{[1_{A_0}]}{2(\sqrt{2}+\sqrt{3})} \right\rfloor = \left\lfloor \frac{n[1_{A_0}]}{2(\sqrt{2}+\sqrt{3})} \right\rfloor,$$

where we make the identification $K_0(A_0) \subseteq \mathbb{R}$.

Proof. Since $K_0(\phi)\colon (G_0, u_0) \to (H_0, n[1_{A_0}])$ is a normalized group homomorphism, by Proposition V.3.4, there exists $\alpha_i\colon (G_0, u_0) \to (H_0, [1_{A_0}])$ such that

$$K_0(\phi) = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

By Proposition 1.3.4(iii) of [27], there exist unital homomorphisms $\phi_i\colon C_0 \to A_0$ such that $K_0(\phi_i) = \alpha_i$ for $i = 1, 2, \ldots, n$ and by Proposition 1.3.4(i) of [27], ϕ and

$$\begin{pmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \phi_n \end{pmatrix}.$$

are approximately unitarily equivalent. So ϕ is approximately diagonalizable.

Conversely, suppose

$$n\left\lfloor \frac{[1_{A_0}]}{2(\sqrt{2}+\sqrt{3})} \right\rfloor < \left\lfloor \frac{n[1_{A_0}]}{2(\sqrt{2}+\sqrt{3})} \right\rfloor.$$

Let $\alpha\colon (G, u) \to (H, nv)$ be the normalized group homomorphism such that

$$\alpha((1,0)) = \frac{n[1_{A_0}]}{2} + \left\lfloor \frac{n[1_{A_0}]}{2(\sqrt{2}+\sqrt{3})} \right\rfloor (\sqrt{2}+\sqrt{3}).$$

By Proposition 1.3.4(iii) of [27], there exists a unital homomorphism $\phi\colon C_0 \to M_n(A_0)$ such that $K_0(\alpha) = \phi$. Note that there exists a projection $p \in C_0$ such that $[p] = (1,0)$. Suppose towards a contradiction that ϕ is approximately diagonalizable. Then there exist unital homomorphisms $\phi_i\colon C_0 \to A_0$ for $i = 1, 2, \ldots, n$ and a unitary $u \in M_n(A_0)$ such that

$$\left\| u\phi(p)u^* - \begin{pmatrix} \phi_1(p) & 0 & \cdots & 0 \\ 0 & \phi_2(p) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \phi_n(p) \end{pmatrix} \right\| < 1.$$

So we have

$$K_0(\phi)((1,0)) = K_0(\phi_1)((1,0)) + K_0(\phi_2)((1,0)) + \cdots + K_0(\phi_n)((1,0)),$$

but

$$\begin{aligned} \sum_{i=1}^{n} K_0(\phi_i)((1,0)) &\leq n \left\lfloor \frac{[1_{A_0}]}{2(\sqrt{2}+\sqrt{3})} \right\rfloor (\sqrt{2}+\sqrt{3}) \\ &< K_0(\phi)((1,0)), \end{aligned}$$

a contradiction. □

Homomorphisms between AF-algebras and Lattice Points

First we recall the notion of Minkowski sum and introduce some notation for taking the n-fold sum of a set.

Definition V.4.1. Let M, N be subsets of a vector space. We define $M + N$ to be the set

$$M + N = \{x + y \colon x \in M, y \in N\}.$$

For any subset S of a vector space, we denote by $\Sigma^n S$ the n-fold sum of elements in S, or, more precisely,

$$\Sigma^1 S = S \text{ and}$$
$$\Sigma^{n+1} S = \{x + y \colon x \in \Sigma^n S, y \in S\}.$$

Lemma V.4.2. *Let G be an archimedian dimension group with finitely many pure states. Let H be a finitely generated simple dimension group.*

There exists a finite dimensional real vector space V, a discrete group $\mathcal{L} \subseteq V$, and a bijection

$$\theta \colon \mathcal{L} \to \mathrm{Hom}_c(G, H).$$

Moreover, there exists a cone $\mathcal{L}^+ \subseteq \mathcal{L}$ such that $\theta(\mathcal{L}^+) = \mathrm{Hom}_+(G, H)$.

Furthermore, for each pair of order units $u \in G_+$ and $v \in H_+$, there exists an affine subspace $S \subseteq V$ such that $\theta(\mathcal{L}^+(G, H) \cap S)$ is the set of normalized positive group homomorphisms from (G, u) to (H, v)

Proof. Since, by Theorem 7.9 of [6], the rational span of $\rho_G(G_+)$ is dense in $\mathrm{Aff}(S(G, u))$, there exists a set $B \subseteq G_+$ such that $\rho_G(B)$ is a basis of $\mathrm{Aff}(S(G, u))$.

Since G has finitely many pure states,

$$\mathrm{Aff}(S(G, u)) \cong C(\partial_e S(G, u), \mathbb{R}) \cong \mathbb{R}^s,$$

where $s \geq 1$ is the number of pure states of G. So B contains s elements and we label the elements $B = \{b_1, b_2, \ldots, b_s\}$.

Since H is a finitely generated, torsion-free abelian group, there exists some integer k such that $H \cong \mathbb{Z}^k$ as groups. Fix a group isomorphism $\zeta\colon H \to \mathbb{Z}^k$. We set $V = \mathbb{R}^{sk}$.

Let

$$\mathcal{L} = \{(\zeta \circ \alpha(b_1), \zeta \circ \alpha(b_2), \ldots, \zeta \circ \alpha(b_s)) \in \mathbb{Z}^{sk} \colon \alpha \in \mathrm{Hom}_c(G, H)\}.$$

There is the group homomorphism

$$\theta(\alpha) = (\zeta \circ \alpha(b_1), \zeta \circ \alpha(b_2), \ldots, \zeta \circ \alpha(b_s)).$$

By construction, θ is onto. We note that $\alpha \in \ker\theta$ if and only if $\alpha(b_i)$ for $i = 1, 2, \ldots, s$. Since $\rho_G(B)$ is a basis for $\mathrm{Aff}(S(G, u))$, we have that the map from $S(G, u)$ to $S(H, v)$ is 0 and so $\alpha = 0$. So θ is an isomorphism.

We note that $\mathcal{L}$ is a subgroup of $\mathbb{Z}^{sk}$. So $\mathcal{L}$ is a discrete group.

Since H is finitely generated, H has finitely many pure states and so $\mathrm{Aff}(S(H, v)) \cong C(\partial_e S(H, v), \mathbb{R})$ is finite-dimensional. Consider $\rho_H \circ \zeta^{-1}\colon \mathbb{Z}^k \to \mathrm{Aff}(S(H, v))$. By tensoring with the identity on $\mathbb{R}$, we obtain a linear map $\lambda\colon \mathbb{R}^k \to \mathrm{Aff}(S(H, v))$. We denote by $\lambda_m\colon \mathbb{R}^k \to \mathbb{R}$ the linear functionals by composing λ with evaluation at a pure state for $m = 1, 2, \ldots, t$. So we see that $h \geq 0$ if and only if either $h = 0$ or $\lambda_m(\zeta(h)) > 0$ for $m = 1, 2, \ldots, t$.

Since G is archimedian and H is simple, we see that $\alpha \in \mathrm{Hom}_c(G, H)$ is a positive group homomorphism if and only if the induced linear map

$\beta\colon \operatorname{Aff}(S(G,u)) \to \operatorname{Aff}(S(H,v))$ satisfies $(\beta(\sigma_i))(\tau_m) > 0$ for the pure states $\sigma_i \in S(G,u)$ and $\tau_m \in S(H,v)$ where $i = 1,2,\ldots,s$ and $m = 1,2,\ldots,t$.

Since $\rho_G(B)$ is a basis for $\operatorname{Aff}(S(G,u))$, there exist $a_{i,j} \in \mathbb{R}$ such that

$$\sigma_i = \sum_{j=1}^{s} a_{i,j}\rho_H(b_j).$$

So $\alpha \in \operatorname{Hom}_c(G,H)$ is positive if and only if

$$\sum_{j=1}^{s} a_{i,j}\tau_m(\alpha(b_i)) > 0$$

for all $i,j = 1,2,\ldots,s$ and $m = 1,2,\ldots,t$.

We define linear functionals $\mu_i\colon \mathbb{R}^s \to \mathbb{R}$ for $i = 1,2,\ldots,s$ and $(x_1,x_2,\ldots,x_s) \in \mathbb{R}^s$ by

$$\mu_i((x_1,x_2,\ldots,x_s)) = \sum_{j=1}^{s} a_{i,j}x_j.$$

Define the linear functionals $\omega_{i,m}\colon V \to \mathbb{R}$ for $i = 1,2,\ldots,s$, $m = 1,2,\ldots,t$ and $(v_1,v_2,\ldots,v_s) \in V$ where each $v_j \in \mathbb{R}^k$ by

$$\omega_i((v_1,v_2,\ldots,v_s)) = \alpha_i(\lambda_m(v_1),\lambda_m(v_2),\ldots,\lambda_m(v_s)).$$

By construction, we have that $\alpha \in \mathcal{L}^+$ if and only if $\omega_{i,m}(\theta(\alpha)) > 0$ for $i = 1,2,\ldots,s$, $m = 1,2,\ldots,t$ if and only if α is a nonzero positive homomorphism.

Finally we note that $u = \sum_{i=1}^{s} c_{i,j}b_j$. So we set $\mathcal{L}^+_{u,v}$ to be

$$\mathcal{L}^+_{u,v}(G,H) = \{(v_1,v_2,\ldots,v_s) \in \mathcal{L}^+(G,H)\colon \sum_{i=1}^{s} c_{i,j}\lambda(v_i)\} = \rho_H(v).$$

And we see that $\theta(\alpha) \in \mathcal{L}^+_{u,v}$ if and only if $\alpha(u) = v$. $\square$

This results in the following:

Lemma V.4.3. *Let G be an archimedian dimension group with finitely many pure states. Let H be a finitely generated simple dimension group. Let u and v be order units of G and H.*

Let $n \geq 1$ be an integer. There exists a finite dimensional real vector space V, a discrete group $\mathcal{L} \subseteq V$, a cone $\mathcal{L}^+ \subseteq \mathcal{L}$ and an affine subspaces $S_1, S_2 \subseteq V$ such that every normalized positive group homomorphism $\alpha\colon (G, u) \to (H, nv)$ can be written as the sum of n normalized positive group $\alpha_i\colon (G, u) \to (H, v)$ if and only if $\Sigma^n \mathcal{L}^+ \cap S_1 = S_2$

Proof. Take V, $\mathcal{L}$, $\mathcal{L}^+$ to be the same as in V.4.2 and S_1 to be the affine subspace associated with the order units $u \in G_+$ and $v \in H_+$. Take S_2 to be the affine subspace associated with the order units $u \in G_+$ and $v \in H_+$. Let $\theta\colon \mathcal{L} \to \mathrm{Hom}_c(G, H)$ be the isomorphism in V.4.2. We see that $\theta(\Sigma^n \mathcal{L}^+ \cap S_1)$ is the set of sums of n normalized positive group homomorphisms from (G, u) to (H, v). $\square$

Theorem V.4.4. *Let C be a unital separable AF-algebra with finitely many pure tracial states such that $K_0(C)$ is archimedian. Let $n \geq 1$ be an integer. Let A be a unital separable simple AF-algebra such that $K_0(A)$ is finitely generated. There exists a finite dimensional real vector space V, a discrete group $\mathcal{L} \subseteq V$, a cone $\mathcal{L}^+ \subseteq \mathcal{L}$ and an affine subspaces $S_1, S_2 \subseteq V$ such that every unital injective homomorphism $\phi\colon C \to M_n(A)$ is approximately unitarily equivalent to a diagonal homomorphism if and only if $\Sigma^n \mathcal{L}^+ \cap S_1 = S_2$.*

Proof. Take $V, \mathcal{L}, \mathcal{L}^+, S_1, S_2$ as those in Lemma V.4.3. Suppose that $\Sigma^n \mathcal{L}^+ \cap S_1 = S_2$. By Lemma V.4.3, there exists normalized positive group homomorphisms

$\alpha_i\colon K_0(C) \to K_0(A)$ such that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = K_0(\phi).$$

By Proposition 1.3.4(iii) of [27], there exists a unital homomorphism $\psi_i\colon C \to A$ such that $K_0(\psi_i) = \alpha_i$ for $i = 1, 2, \ldots, n$.

Let

$$\psi = \begin{pmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \phi_n \end{pmatrix}.$$

Since

$$K_0(\psi) = \sum_{i=1}^{n} K_0(\psi_i) = K_0(\phi),$$

by Proposition 1.3.4(i) of [27], ϕ and ψ are approximately unitarily equivalent.

Suppose that $\Sigma^n \mathcal{L}^+ \cap S_1 \neq S_2$. By Lemma V.4.3, there exists a normalized positive group homomorphism $\alpha\colon (K_0(C), [1_C]) \to (K_0(A), n[1_A])$ that cannot be written as the sum of n normalized positive group homomorphisms from $(K_0(C), [1_C])$ to $(K_0(A), [1_A])$.

By Proposition 1.3.4(iii) of [27], there exists a unital homomorphism $\phi\colon C \to M_n(A)$ such that $K_0(\phi) = \alpha$. Suppose, toward a contradiction, that there exist unital homomorphisms $\psi_i\colon C \to A$ such that

$$\psi = \begin{pmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \phi_n \end{pmatrix}$$

is approximately unitarily equivalent to ϕ. Then by Proposition 1.3.4(i) of [27], we have

$$\alpha = K_0(\psi) = \sum_{i=1}^{n} K_0(\psi_i),$$

which contradicts the fact that α cannot be written as the sum of n normalized positive group homomorphisms. □

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