Expectational Stability of Stationary Sunspot Equilibria in a Forward-looking Linear Model*

George W. Evans
University of Oregon

Seppo Honkapohja
University of Helsinki

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Abstract

We consider the stability under adaptive learning of the complete set of solutions to the model $x_t = \beta E_t^* x_{t+1}$ when $|\beta| > 1$. In addition to the fundamentals solution, the literature describes both finite-state Markov sunspot solutions and autoregressive solutions depending on an arbitrary martingale difference sequence. We clarify the relationships between these solutions and show that the stability properties of equilibria may depend crucially on the representation used by agents in the learning process. Autoregressive forms of solutions are not learnable, but finite-state Markov sunspot solutions are stable under learning if $\beta < -1$.

JEL classifications: C62, D83, D84, E31, E32

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1 Introduction

We consider a univariate linear model of the form

$$x_t = \beta E_t^* x_{t+1}, \tag{1}$$

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both under rational expectations (RE) and under the assumption that economic agents follow certain natural adaptive learning rules. Here $E_t^* x_{t+1}$ denotes the expectations held by agents, assumed homogeneous. Under RE

$$x_t = \beta E_t x_{t+1}, \tag{2}$$

where $E_t x_{t+1}$ denotes the true conditional expectation given the information set at time t. We are interested in whether rational expectations equilibria (REE) depending on extraneous exogenous random variables (sunspots) can be stable under adaptive learning.

The model (1) does not include intrinsic random shocks, and has been centered, so that there is a steady state at the origin. Both of these assumptions are made for simplicity and could be relaxed. Throughout the paper we will assume that $\beta \neq 0$ and $\beta \neq 1$.

The steady state $x_t = 0$ is an REE (and indeed satisfies perfect foresight). This solution is often called the "fundamentals" solution. As is well known, there are other RE solutions to (1), taking the form

$$x_t = \beta^{-1} x_{t-1} + \varepsilon_t, \tag{3}$$

where ε_t is an arbitrary martingale difference sequence, i.e. a stochastic process satisfying $E_t \varepsilon_{t+1} = 0$. It is easily verified that a process of the form (3) is indeed an RE solution to (1). Conversely, any RE solution to (1) can be written in the form (3), as can be seen by defining $\varepsilon_{t+1} = x_{t+1} - E_t x_{t+1}$. We will refer to (3) as the $AR(1)$ representation of the solution, as we now explain.

When $|\beta| < 1$ the solution $x_t \equiv 0$ is the unique nonexplosive solution, see (Gourieroux, Laffont, and Monfort 1982). The other RE solutions of the form (3) have conditional expectations that, in absolute value, tend to infinity. In the "irregular" case $|\beta| > 1$ there are multiple stationary solutions. In particular, if ε_t is an *iid* process with mean 0 and constant variance (i.e. "white noise"), then (3) is a stationary first-order autoregressive (or $AR(1)$) process.[1] When ε_t takes a different form the process need not be stationary, but for convenience we will continue to refer to (3) as the $AR(1)$ representation of the solution.

[1] If the system begins at $t = 0$ then the solution (3) also requires an initial condition. For an appropriate initial condition the process is stochastically stationary. For other initial conditions there is a transient nonstationarity, but the process is asymptotically stationary.

Considerable attention is given in the literature to finite-state Markov solutions, i.e. solutions taking the form

$$x_t = \bar{x}_i \text{ when } s_t = s_i, \text{ for } i = 1, \ldots, K, \tag{4}$$

where $s_t \in \{s_1, \ldots, s_K\} \subset \mathbb{R}$ follows a finite-state Markov process with fixed transition probabilities $\pi_{ij} = P(s_{t+1} = s_j \mid s_t = s_i)$. Again, the exogenous stationary process s_t is usually called a "sunspot" process and the solutions are called finite Markov stationary sunspot equilibria (SSEs), or K-SSEs. (See (Azariadis and Guesnerie 1986), (Guesnerie 1986) and (Chiappori and Guesnerie 1989).)

Much of this literature considers such solutions more generally in the context of possibly nonlinear models, i.e. in models of the form $x_t = E_t^* F(x_{t+1})$. For example, the existence question is discussed at length in (Guesnerie and Woodford 1992) and (Chiappori, Geoffard, and Guesnerie 1992). It can easily be established that this type of SSE exists in the linear model when $|\beta| > 1$. Clearly, such solutions must have an equivalent representation in the $AR(1)$ form (3).

The learning question has also been considered for both forms of solution (3) and (4). In particular, for solutions in the $AR(1)$ form, stability under adaptive learning was considered in (Evans 1989) and discussed further in Section 9.7 of Chapter 9 of (Evans and Honkapohja 2001b). Stability under adaptive learning of Markov SSEs is discussed in (Woodford 1990), (Evans 1989), (Evans and Honkapohja 1994) and Chapter 12 of (Evans and Honkapohja 2001b).[2]

There are some important gaps in this literature, even in the linear case. The relationship between these two types of SSEs has not been explicitly addressed, and existing adaptive learning results are incomplete. Indeed, the adaptive learning results for the two set-ups appear to be at variance, as we show below. In this paper we clarify the relationship between the different solutions and then extend and reconcile the learning results by nesting the two set-ups in a common framework.

[2] For a study of the eductive stability of 2-state Markov SSEs see (Desgranges and Negroni 2001).

2 Two-state Markov SSEs

We now consider finite-state Markov SSEs. For convenience we focus on the 2-state case, which has a prominent role in the literature. Assuming solutions to (2) of the form (4) implies

$$\begin{aligned} \bar{x}_1 &= \beta(\pi_{11}\bar{x}_1 + \pi_{12}\bar{x}_2), \\ \bar{x}_2 &= \beta(\pi_{21}\bar{x}_1 + \pi_{22}\bar{x}_2), \end{aligned}$$

or

$$\Pi\bar{x} = \beta^{-1}\bar{x}, \tag{5}$$

where Π is the matrix with elements π_{ij} and $\bar{x}' = (\bar{x}_1, \bar{x}_2)$. In order for the solution to constitute an SSE we must, of course, have $\bar{x}_1 \neq \bar{x}_2$. It follows that β^{-1} must be an eigenvalue of Π with eigenvector $\bar{x}$. Since the eigenvalues of Π are 1 and $\pi_{11} + \pi_{22} - 1$ we obtain

$$\begin{aligned} \pi_{11} + \pi_{22} - 1 &= \beta^{-1} && (6) \\ (1 - \pi_{22})\bar{x}_1 + (1 - \pi_{11})\bar{x}_2 &= 0. && (7) \end{aligned}$$

It is seen from (5) or (7) that there exists a one-dimensional continuum of $(\bar{x}_1, \bar{x}_2)$ for transition probabilities satisfying (6).[3]

We will refer to (6) as the "resonant frequency condition" since, in our linear model, two state Markov sunspot variables must precisely satisfy the transition probability restriction (6) to be capable of generating equilibrium fluctuations around the steady-state. This condition is well known in the literature on finite state Markov SSEs. In accordance with the literature we will refer to these equilibria as 2-SSEs.

From Section 1 we know that it is also possible to represent such 2-SSEs in the form (3). In a 2-SSE $(\bar{x}_1, \bar{x}_2)$ it is easily seen that x_t can be expressed as a linear (affine) function of s_t. It follows from (3) that ε_t can be written as a linear function of s_t and s_{t-1}, i.e. $\varepsilon_t = a + fs_t + gs_{t-1}$, so that (3) implies that the equations

$$\bar{x}_j = \beta^{-1}\bar{x}_i + a + fs_j + gs_i \text{ for } s_{t-1} = s_i,\ s_t = s_j \tag{8}$$

[3]Corresponding results for the K-state SSEs are given in (Chiappori, Geoffard, and Guesnerie 1992).

for $i, j = 1, 2$ must hold. Equations (8), together with (6)-(7), imply that the corresponding martingale difference sequence ε_t can be written as

$$\varepsilon_t = \bar{a} + \bar{f}s_t + \bar{g}s_{t-1}, \text{ where} \tag{9}$$

$$\bar{f} = \frac{\bar{x}_2 - \bar{x}_1}{s_2 - s_1},\ \bar{g} = \frac{(1 - \pi_{11} - \pi_{22})(\bar{x}_2 - \bar{x}_1)}{s_2 - s_1} \text{ and} \tag{10}$$

$$\bar{a} = \frac{-[(1 - \pi_{22})s_1 + (1 - \pi_{11})s_2](\bar{x}_2 - \bar{x}_1)}{s_2 - s_1}. \tag{11}$$

It can be verified that $E_t\varepsilon_{t+1} = 0$ when ε_t is given by (9)-(11) and (6)-(7) are satisfied. Note, however, that ε_t is not an *iid* process. Note also that the non-uniqueness of $(\bar{x}_1, \bar{x}_2)$ corresponds to a degree of freedom in the parameters $\bar{a}$, $\bar{f}$ and $\bar{g}$.

The 2-SSE is obtained from the general representation (3) with this choice of ε_t, provided the initial condition is $\bar{x}_1$ or $\bar{x}_2$. For other choices of initial condition, the process (3), with ε_t given by (9)-(11), is an asymptotically stationary solution, tending in the limit to the 2-SSE.

3 E-stability and Adaptive Learning

3.1 E-Stability of $AR(1)$ Sunspot Equilibria

We now take up the question of the local stability under learning of SSEs. Consider first the general representation for SSEs (3). Suppose agents believe that x_t follows an $AR(1)$ process, but are not certain of the coefficients, which they estimate and revise over time. More particularly, suppose u_t is an observable process that satisfies $E_t u_{t+1} = 0$. Suppose that agents believe that x_t follows the process

$$x_t = a + bx_{t-1} + cu_t + \eta_t,$$

where a, b and c are unknown fixed parameters and η_t represents an unobserved white noise disturbance that agents might believe to be present. In an SSE $a = 0$, $b = \beta^{-1}$, $\eta_t \equiv 0$ and c is arbitrary (in the earlier notation $\varepsilon_t = cu_t$). The fundamentals solution is given by $a = b = c = 0$ and $\eta_t \equiv 0$.

The standard way to formulate adaptive learning in this context is "least squares learning," in which agents at time t estimate a, b and c by a least-squares regression of x_i on x_{i-1}, u_i and an intercept, using data $i = 1, \ldots, t-$

1. Note that estimates are updated each period. Assuming that the time t information set is $I_t = \{u_t, u_{t-1}, \ldots, x_{t-1}, x_{t-2}, \ldots\}$, the forecasts $E_t^* x_{t+1}$ are obtained by taking the conditional expectation of $x_{t+1} = a + b(a + bx_{t-1} + cu_t + \eta_t) + cu_{t+1} + \eta_{t+1}$. This yields

$$E_t^* x_{t+1} = a(1+b) + b^2 x_{t-1} + bcu_t, \tag{12}$$

where a, b and c are replaced by their current least-squares estimates a_t, b_t and c_t.

Stability of an REE under least-squares learning is defined in terms of the dynamic system given by the exogenous sunspot process u_t and by (1), where $E_t^* x_{t+1} = a_t(1+b_t) + b_t^2 x_{t-1} + b_t c_t u_t$ and (a_t, b_t, c_t) are updated by least squares. If $(a_t, b_t, c_t) \rightarrow (0,0,0)$ then the fundamentals solution is said to be stable, while if $(a_t, b_t, c_t) \rightarrow (0, \beta^{-1}, c)$ for some c then the class of SSEs is said to be stable. Here local stability is the relevant concept and we omit precise definitions of the appropriate notions of stochastic convergence. For details see (Evans and Honkapohja 2001b).

For a wide range of economic models it has been shown that stability under least-squares learning is governed by expectational stability (E-stability), and in this paper we therefore focus on determining the conditions for the various solutions to be E-stable. E-stability is defined in terms of the mapping from the Perceived Law of Motion (PLM), parameterized here by (a, b, c), to the implied parameters $T(a, b, c)$ of the Actual Law of Motion (ALM). The ALM parameters, corresponding to a given PLM, are here obtained by inserting the corresponding expectation rule (12) into the model (1), yielding

$$T(a, b, c) = (\beta a(1+b), \beta b^2, \beta bc).$$

Note that the fixed points of the T map correspond to the fundamentals solution $(0,0,0)$ and to the continua of SSEs $(0, \beta^{-1}, c)$.

E-stability of an REE (or a set of REE) is determined by local stability of the REE (or a set of REE) under the ordinary differential equation

$$\frac{d}{d\tau}(a, b, c) = T(a, b, c) - (a, b, c),$$

where τ denotes notional or virtual time. Since the Jacobian of the right-hand side, evaluated at $(0,0,0)$ has one eigenvalue of $\beta - 1$ and two eigenvalues of -1, it follows that the fundamentals solution is E-stable provided $\beta < 1$. On the other hand the set of SSEs are not E-stable. This can be seen from the differential equation for b, which is given by $db/d\tau = \beta b^2 - b$ and which is always locally unstable at $b = \beta^{-1}$. If $\beta > 1$, none of the solutions are E-stable. We summarize these observations in the following:

Proposition 1 *The set of SSEs of the form (3) is not E-stable. The fundamental solution is E-stable if $\beta < 1$ and is not E-stable if $\beta > 1$.*[4]

This way of looking at the full set of REE to (1) thus clearly favors the fundamentals solution. Provided $\beta < 1$, the fundamentals solution is E-stable, and hence locally stable under least-squares learning, while the set of $AR(1)$ SSEs is never E-stable and hence is locally unstable under least-squares learning.

3.2 E-stability of Two-state Markov SSEs

We turn now to the stability of 2-SSEs, i.e. to the 2-state Markov sunspot equilibria given by (5), with $\bar{x}_1 \neq \bar{x}_2$. The exogenous sunspot variable s_t is assumed to be observable at t, with known transition probabilities π_{11}, π_{22}. We assume that agents do not know the values $\bar{x}_1, \bar{x}_2$, taken in the 2-SSE, and that they therefore estimate their values. (The stability results are unaffected if the transition probabilities must also be estimated). A simple and natural adaptive learning rule is state contingent averaging, i.e. $\bar{x}_j$, $j = 1, 2$, is estimated to be the average of the values for x_t obtained when $s_t = s_j$.[5]

(Evans and Honkapohja 1994) and Chapter 12 of (Evans and Honkapohja 2001b) show how E-stability governs local convergence of adaptive learning to finite-state SSEs. Following the E-stability principle we look at the mapping from the PLM to the ALM. The PLM is now

$$x_t = x_j + \eta_t \text{ if } s_t = s_j, \text{ for } j = 1, 2,$$

where in an 2-SSE $x_j = \bar{x}_j$ and $\eta_t \equiv 0$. In state $s_t = s_1$ the expectation corresponding to this PLM is $E_t^* x_{t+1} = \pi_{11} x_1 + (1 - \pi_{11}) x_2$ and in state $s_t = s_2$ we have $E_t^* x_{t+1} = (1 - \pi_{22}) x_1 + \pi_{22} x_2$. Inserting these into (1) yields

$$\begin{aligned} x_t &= \beta \pi_{11} x_1 + \beta (1 - \pi_{11}) x_2 \text{ if } s_t = s_1, \text{ and} \\ x_t &= \beta (1 - \pi_{22}) x_1 + \beta \pi_{22} x_2 \text{ if } s_t = s_2. \end{aligned}$$

[4] These results were first obtained in (Evans 1989).

[5] This formulation was suggested in (Evans and Honkapohja 1994). An alternative and essentially equivalent prodedure (see the following section) would be for agents to estimate a least squares regression of x_t on s_t and an intercept.

The corresponding map from PLM $x = (x_1, x_2)'$ to ALM $T(x)$ is given by

$$T(x) = \beta \Pi x.$$

The fixed points of T, i.e. the equilibria of the differential equation, correspond to REE. These are given by (5) and hence by (6)-(7). Note that this requires that the resonance frequency condition (6) is satisfied. Given this, the equilibrium 2-SSEs form the one-dimensional continuum $(\bar{x}_1, \bar{x}_2)$ specified by (7). The E-stability condition is given by local stability of this set of equilibria under $dx/d\tau = T(x) - x$. The eigenvalues of the associated Jacobian $\beta\Pi - I$ are $\beta - 1$ and $\beta(\pi_{11} + \pi_{22} - 1) - 1$. Imposing condition (6), the eigenvalues at the set of 2-SSEs are $(\pi_{11} + \pi_{22} - 1)^{-1} - 1$ and 0. It follows that the continuum is E-stable if $\pi_{11} + \pi_{22} < 1$ and not E-stable if $\pi_{11} + \pi_{22} > 1$. In terms of the model parameter β we can thus state:

Proposition 2 *The set of 2-SSEs is E-stable if $\beta < -1$ and it is not E-stable if $\beta > 1$.*

Figure 1 illustrates E-stability for the case $\beta < -1$. The result that 2-SSEs are not E-stable when $\beta > 1$ was previously given in (Evans and Honkapohja 1994), but E-stability of the set of 2-SSEs when $\beta < -1$ has not been previously noted.

Note that if the transition matrix does not satisfy (6) then 2-SSEs do not exist for the given 2-state sunspot process. In this case the only fixed point of the map $T(x)$ is the fundamentals solution $\bar{x}_1 = \bar{x}_2 = 0$. A natural question to ask is whether in this case the fundamentals solution is stable under learning even when agents allow for the possibility that the equilibrium depends on the sunspot. In other words, will the fundamental solution be E-stable if the PLM is $x_t = x_j + \eta_t$ if $s_t = s_j$, for $j = 1, 2$? (In this case we say that the fundamentals solution is "strongly E-stable"). It is easily seen that the required conditions are $\beta < 1$ and $\beta(\pi_{11} + \pi_{22} - 1) - 1$. These conditions are discussed further in the next section.

4 E-stability in a More General Framework

The preceding section appears at first sight to provide incompatible results concerning the adaptive stability of 2-SSEs. Section 3.1 shows the lack of stability under learning of all sunspot equilibria, while Section 3.2 shows the

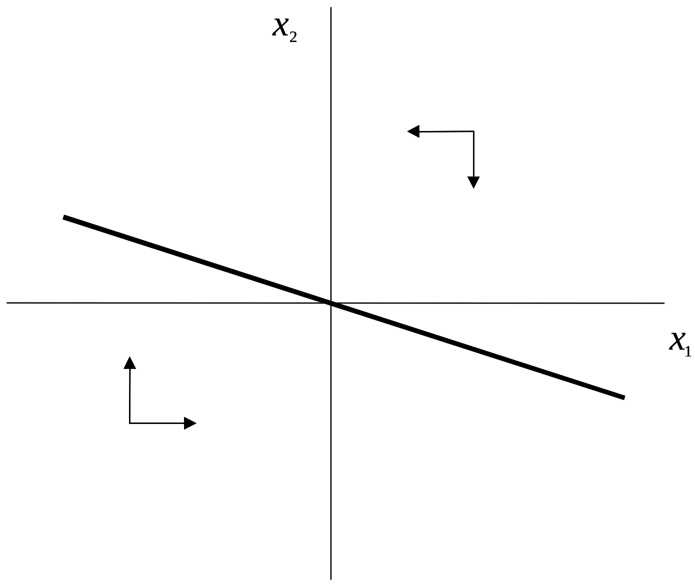

Figure 1: E-stable continuum when $\pi_{11} + \pi_{22} < 1$

possible stability under learning of two-state Markov sunspot equilibria. To understand the relation between these results, we now examine a class of PLMs that nests the 2-SSEs in a way that includes $AR(1)$ representations.

Thus, consider the class of PLMs taking the form

$$x_t = a + bx_{t-1} + fs_t + gs_{t-1} + \eta_t,$$

where η_t is assumed to be white noise. Under RE $\eta_t \equiv 0$ and the coefficients a, b, f, g satisfy certain restrictions. s_t is again an exogenous two-state Markov process, but we do not yet make the assumption that it satisfies the resonant frequency condition (6). We also do not yet impose any condition on β, other than $\beta \neq 0, 1$.

For this PLM the corresponding $E_t^* x_{t+1}$ is obtained from

$$x_{t+1} = a + b(a + bx_{t-1} + fs_t + gs_{t-1} + \eta_t) + fs_{t+1} + gs_t + \eta_{t+1}$$

by taking conditional expectations. This yields

$$E_t^* x_{t+1} = a(1 + b) + b^2 x_{t-1} + fE_t s_{t+1} + (bf + g)s_t + bgs_{t-1}.$$

Expressing $E_t s_{t+1}$ as a linear function of s_t, we have

$$E_t s_{t+1} = (1 - \pi_{22})s_1 + (1 - \pi_{11})s_2 + (\pi_{11} + \pi_{22} - 1)s_t.$$

Inserting these into (1) yields the following map from PLM to ALM:

$$\begin{aligned} & T(a, b, f, g) \\ = \; & \left(\beta a(1 + b) + \beta f[(1 - \pi_{22})s_1 + (1 - \pi_{11})s_2], \beta b^2, \beta f(\pi_{11} + \pi_{22} - 1 + b) + \beta g, \beta b g\right). \end{aligned}$$

RE solutions are given by the fixed points of this map (together with $\eta_t \equiv 0$). There are three classes of solutions as specified by the following proposition:

Proposition 3 *The REE can be divided into three classes:*
(I) $a = b = f = g = 0$. This is the "fundamentals" solution.
(II) When (6) holds, there are solutions of the form $b = g = 0$, with f arbitrary and $a = \frac{(1-\pi_{22})s_1 + (1-\pi_{11})s_2}{\pi_{11}+\pi_{22}-2} f$. For $f \neq 0$ we obtain the 2-SSEs.[6]
(III) Setting $b = \beta^{-1}$, f arbitrary, $g = -(\pi_{11} + \pi_{22} - 1)f$ and $a = -[(1 - \pi_{22})s_1 + (1 - \pi_{11})s_2]f$ yields the set of $AR(1)$ solutions generated by choices of ε_t as a linear combination of s_t, s_{t-1} and an intercept.[7]

Consider now E-stability for these three sets of RE solutions. For E-stability we require local stability of the solution set under the differential equation

$$\frac{d}{d\tau}(a, b, f, g) = T(a, b, f, g) - (a, b, f, g).$$

The relevant condition is that all roots of the Jacobian $DT - I$ have negative real parts. For the fundamentals solution (I) this leads to the conditions $\beta < 1$ and $\beta(\pi_{11} + \pi_{22} - 1) < 1$. These conditions are never satisfied if $\beta > 1$, are always satisfied if $|\beta| < 1$ and may be satisfied if $\beta < -1$. Next, we consider the $AR(1)$ solution set (III). The differential equation for b is autonomous and is always unstable at $b = \beta^{-1}$. Thus this solution set is never E-stable.

Lastly, consider the solution set (II). These exist only when $|\beta| > 1$ and the resonant frequency condition (6) holds. The subsystem in b, g is

[6] $f = 0$ yields the fundamentals solution as a special case.

[7] Imposing (6) and an appropriate initial condition gives the $AR(1)$ representation of the 2-SSEs.

autonomous and $b = g = 0$ is always locally stable. It can then be verified that the condition for local stability condition of this solution set is that is that $\beta < 1$. Thus the set of 2-SSEs is not E-stable if $\beta > 1$ but is E-stable if $\beta < -1$.

We collect these arguments in the following proposition:

Proposition 4 *(a) The fundamental solution (I) is E-stable if $\beta < 1$ and $\beta(\pi_{11} + \pi_{22} - 1) < 1$ and is not E-stable if $\beta > 1$ or $\beta(\pi_{11} + \pi_{22} - 1) > 1$.*
(b) The solution set (II) of 2-SSEs, which exists only when $|\beta| > 1$ and (6) holds, is E-stable if $\beta < -1$ and it is not E-stable if $\beta > 1$.
(c) The solution set (III) is never E-stable whether or not condition (6) holds.

To summarize, when $|\beta| < 1$, stationary sunspot equilibria do not exist. The fundamentals solution is E-stable, and therefore stable under adaptive learning.[8] There exist explosive sunspot equilibria, taking the $AR(1)$ form, but these are not E-stable. When $\beta > 1$ SSEs do exist, but no solution or solution set is E-stable. When $\beta < -1$ but the resonant frequency condition does not hold, SSEs and asymptotically stationary sunspot equilibria do exist, taking the $AR(1)$ from, but they are not E-stable. Finally, when $\beta < -1$ and the resonant frequency condition (6) holds, 2-SSEs exist and are E-stable. There are two ways to represent the same solution, but the $AR(1)$ representation is not E-stable. Adaptive learning will, however, locally converge to the set of 2-SSEs if the conditions $\beta < -1$ and $\pi_{11}+\pi_{22}-1 = \beta^{-1}$ are satisfied. Note that in this case the continuum of E-stable 2-SSEs includes sunspot equilibria arbitrarily close to the fundamental solution, as illustrated in Figure 1.

The reconciliation of Propositions 1 and 2 clearly lies in the fact that the 2-SSEs have two distinct representations, as the solution set (II) and as members of the solution set (III). One way to view our results is that the learning processes are attempting to learn different things for the different representations: for the $AR(1)$ form of SSEs the learning rule corresponds to least squares estimation of the coefficients, while for the two-state Markov representation of 2-SSEs the learning scheme in effect estimates the support

[8] We remark that the conditions in part (a) of the proposition are "strong E-stability" conditions for the fundamental solution, i.e. the stability conditions required when the PLM allows for the presense of sunspots. The weaker condition $\beta < 1$ is sufficient for E-stability of the fundamental solution when the PLM does not include a possible dependence on a sunspot.

of the distribution. From this viewpoint it is perhaps not surprising that the stability of an SSE can depend on the specific learning scheme, though the results here obtained are certainly not obvious.

There is another way to interpret our results. The 2-SSEs are a kind of “common factor” solution similar to those discussed in (Evans and Honkapohja 1986). Solutions of type (III) are of the form

$$x_t = -[(1-\pi_{22})s_1 + (1-\pi_{11})s_2]f + \beta^{-1}x_{t-1} + fs_t - (\pi_{11}+\pi_{22}-1)fs_{t-1}.$$

Under the resonant frequency condition (6) we obtain

$$(1-\beta^{-1}L)x_t = -f[(1-\pi_{22})s_1 + (1-\pi_{11})s_2] + f(1-\beta^{-1}L)s_t,$$

where L is the lag operator defined by $Lx_t = x_{t-1}$. The two sides of this equation have the lag polynomial $1-\beta^{-1}L$ as a common factor. For an appropriate initial condition on x_t the stochastic process is stationary and we can multiply through by $(1-\beta^{-1}L)^{-1}$ to cancel this common factor and obtain

$$x_t = \frac{-f[(1-\pi_{22})s_1 + (1-\pi_{11})s_2]}{1-\beta^{-1}} + fs_t,$$

which is indeed the form of the 2-SSEs.

It has been observed in previous work (see Chapters 8 and 9 of (Evans and Honkapohja 2001b) and the references cited in (Evans and Honkapohja 1999)) that, under least squares learning, common factor solutions can be stable even when the larger set of solutions in which they are located is unstable.[9] From this viewpoint, as well, the apparently contradictory results of the previous section are not surprising. Convergence to a 2-SSE can arise only when the PLM is parameterized in such a way that it includes the common factor representation of the 2-SSE.

5 Conclusion

We have considered the simplest dynamic expectations model that permits stationary sunspot equilibria. In the linear one-step forward looking model

[9] And this result is found even when the common factor solutions also have an autoregressive form with coefficients that are estimated.

$x_t = \beta E_t^* x_{t+1}$, SSEs exist when $|\beta| > 1$. These can take the form of $AR(1)$ solutions or of finite-state Markov processes. The latter must satisfy a "resonant frequency" restriction on the transition probabilities. We have shown that two-state Markov SSEs also have an AR(1) representation and that their stability properties depend on the representation used by agents in the learning process.

In line with the previous literature we have found that none of the SSEs are stable under adaptive learning when $\beta > 1$. However, when $\beta < -1$, two-state Markov SSEs are stable under adaptive learning, even though the whole class of SSEs taking the $AR(1)$ form is not. Our two main conclusions are thus that stability under adaptive learning of SSEs requires $\beta < -1$, and that in this case the set of two-state Markov SSEs are stable when learning is conditioned directly on the exogenous two-state Markov process.

One might wonder how sensitive are the results to the assumed linearity of the model. In a companion paper (Evans and Honkapohja 2001a) we show how our results carry over to the corresponding nonlinear model $x_t = E_t^* F(x_{t+1})$ in a neighborhood of a steady state $\hat{x}$. In particular, if $F'(\hat{x}) > 1$ there are no nearby E-stable SSEs. In contrast, if $F'(\hat{x}) < -1$ there exist E-stable SSEs in every neighborhood of the steady state.

The linear and nonlinear models do differ in the following respect. In the linear case two-state Markov SSEs must satisfy exactly the resonant frequency condition $\beta^{-1} = \pi_{11} + \pi_{22} - 1$ and they then form a continuum in the states of the SSE. In the nonlinear model there is again a continuum of two-state Markov SSEs near the steady state and these exist provided the transition probabilities are sufficiently close to satisfying the resonant frequency condition. Within a neighborhood of the steady state this continuum is indexed by the deviation from the resonant frequency condition, with the states of the SSE uniquely determined once transition probabilities are given. Therefore, when properly interpreted, our results for an approximating linear model provide a satisfactory guide to the local results on E-stable SSEs in the corresponding nonlinear model.

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