QUANTUM CLUSTER CHARACTERS

by

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DISSERTATION ABSTRACT

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We define the quantum cluster character assigning an element of a quantum torus to each representation of a valued quiver $(Q, \mathbf{d})$ and investigate its relationship to external and internal mutations of a quantum cluster algebra associated to $(Q, \mathbf{d})$. We will see that the external mutations are related to reflection functors and internal mutations are related to tilting theory. Our main result will show the quantum cluster character gives a cluster monomial in this quantum cluster algebra whenever the representation is rigid, moreover we will see that each non-initial cluster variable can be obtained in this way from the quantum cluster character.

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CHAPTER I

INTRODUCTION

In this dissertation we investigate the relationship between cluster variables in acyclic quantum cluster algebras and the representation theory of finite-dimensional hereditary algebras. Our primary result will show that each quantum cluster variable is a generating function for counting the number of points in Grassmannians of subrepresentations inside an exceptional representation.

Our investigations are motivated by the Quantum Laurent Phenomenon of Berenstein and Zelevinsky [BZ]. This remarkable theorem asserts that each cluster variable, although a priori only a rational function, can be written as a Laurent polynomial in the variables of any given cluster, called the initial cluster. This leads to the following natural problem:

Problem I.0.1. Describe the initial cluster Laurent expansion of each cluster variable in a quantum cluster algebra.

In the classical setting, solutions to this problem have been pursued by many authors. We will give a brief account of the motivations for and history of solving Problem I.0.1.

Lusztig defined canonical bases of quantum groups using the geometry of quiver varieties and perverse sheaves [Lu1]. This basis has a beautiful positivity property that allowed Lusztig to solve classical problems in the theory of totally positive matrices [Lu2]. The difficulty of this approach lies in identifying the canonical basis elements in terms of the standard generators. Thus an attempt was made to avoid this complicated geometry and obtain the canonical basis in a more combinatorial way. Cluster algebras were introduced by Fomin and Zelevinsky [FZ1] in anticipation that quantum cluster monomials would be contained in Lusztig's dual canonical basis for quantized coordinate rings of varieties related to algebraic groups. When the algebraic group has a skew-symmetric Cartan matrix Geiss, Leclerc, and Schröer prove in [GLS] that the coordinate rings of certain unipotent subgroups and unipotent cells have the structure of a skew-symmetric cluster algebra. Furthermore they prove that the cluster monomials of such cluster algebras are elements of the dual of Lusztig's semicanonical basis. This adds weight to the Fomin-Zelevinsky cluster conjecture and motivates a thorough study of the quantum cluster monomials.

After the classical Laurent Phenomenon proved in [FZ1], Fomin and Zelevinsky establish in [FZ2] a simple bijection between the cluster variables and almost positive roots in an associated root system, thus illustrating another instance of the well-known $A-G$ finite-type classification. It has long since been recognized [G],[K],[DR],[H2] that the indecomposable representations of an appropriate quiver/species are also in bijection with a certain root datum, namely the (strictly) positive roots. To properly explain the bijections of Fomin and Zelevinsky and fully understand the role of the negative simple roots, the authors of [BMRRT] introduce the cluster category in which the indecomposable objects are exactly in bijection with the almost positive roots. Moreover, they observe that there is a bijection, compatible with mutations, between the cluster-tilting objects in the cluster category and the clusters of the corresponding cluster algebra.

Extending this, Caldero and Chapoton [CC] introduce cluster characters describing the acyclic initial cluster expansions of all cluster variables/monomials explicitly as generating functions for Euler characteristics of Grassmannians of subrepresentations in the corresponding quiver representations. Following the BMRRT approach, they show in skew-symmetric finite types that the mutation operation for cluster-tilting objects coincides with the seed-mutation of Fomin and Zelevinsky, thus establishing a relationship between rigid objects of the cluster category and cluster monomials. Then in [CK] Caldero and Keller generalize these results to give cluster characters categorifying all cluster algebras with acyclic, skew-symmetric initial exchange matrix.

In both [CC] and [CK] the authors recognize a similarity between multiplication in the cluster algebra and the multiplication in the dual Hall algebra. Building on these observations, Hubery in his preprint [H1] works towards extending the above results to the acyclic skew-symmetrizable case, replacing the category of representations of an acyclic quiver with the hereditary category of finite-dimensional modules over a species. His approach follows a specialization argument which replaces the Euler characteristics of Grassmannians of subrepresentations with certain sums of Hall numbers over a finite field. However, this requires the existence of Hall polynomials, which is still an open conjecture. Nevertheless the computations of [H1] are valid and play a vital role in the proof of our main theorem.

Our study of the skew-symmetrizable case has led to a new realization of the representations of classical objects called "valued quivers". In Section II.1. we introduce a category of representations over a finite field of a valued quiver which restricts to the well-known representation theory of the underlying quiver in the equally valued case. Via an equivalence between the categories of representations of a valued quiver and modules over an associated species, we will transport many well-developed properties of species to valued quivers. We present these definitions and results in Section II.2. Chapter II. continues with classical results from the representation theory of species which will be useful in our study of quantum cluster algebras: reflection functors at source and sink vertices are defined in Section II.3. and we recall necessary results from tilting theory for the category of modules over a species in Section II.4. In Section II.5. we associate a skew-symmetrizable exchange matrix to each local tilting representation, this construction was originally given in [H1].

The main goal of this dissertation is to extend the categorification results above to acyclic, skew-symmetrizable quantum cluster algebras. We will present definitions and necessary background results on quantum cluster algebras in Section III.1. In Section III.2. we define the quantum cluster character assigning to each representation of a valued quiver an element of the initial cluster quantum torus of the corresponding quantum cluster algebra. Chapter III. concludes with a study of the multiplicative and commutation properties of the quantum cluster characters, presented in Sections III.3. and Section III.4. respectively. Our main contribution to this categorification story is the following

Theorem I.0.2. The acyclic initial cluster Laurent expansions of all non-initial cluster variables of a skew-symmetrizable quantum cluster algebra can be obtained by evaluating the quantum cluster character on an indecomposable rigid representation of a valued quiver.

Partial results in this direction were proven in our paper [Ru1] for cluster variables obtained from the initial cluster by sink and source mutations. We present these results in Chapter IV. The main technical results are presented in Theorem IV.1.1 (Grassmannian recursion) proven in Section IV.1. and Theorem IV.0.6 (compatibility with sink-source mutations) which is proven in Section IV.2. The main theorem of [Ru1] is Theorem IV.3.1 (almost acyclic cluster variables are given by the quantum cluster character) presented in Section IV.3. The general case of Theorem I.0.2 was conjectured in [Ru1]. Shortly after this the particular case with equally valued quivers was settled by Qin [Q]. The proof of our main result Theorem I.0.2 occupies Chapters III. and V. The main results leading to the proof are Theorems III.3.4 and III.3.7 (multiplication theorems), Theorems III.4.3 and III.4.4 (commutation theorems), and Theorem V.2.3 (mutation theorem). These results are given in our paper [Ru2].

Our investigations are most naturally split between two perspectives on cluster mutations. We will distinguish these as "internal" mutations and "external" mutations. The internal mutations should be thought of as a recursive process happening inside of a fixed ambient skew-field. This is the flavor of the mutations in the definition of cluster algebras and is the mutation operation investigated by the authors mentioned above. Chapter V. focuses on internal mutations. We follow Hubery's approach but abandon the specialization. Working instead with quantum cluster algebras, we show that the internal mutations are merely a shadow of the classical results from the tilting theory for hereditary categories presented in Section II.4.

The external mutations are algebra isomorphisms between different skew-fields: the skew-fields associated to neighboring clusters. These are best understood as a mutation of the initial cluster: we change our perspective and write the cluster variables as Laurent polynomials in the variables of a neighboring cluster. Chapter IV. focuses on the study of external mutations. Our main result of this chapter establishes a relationship between the mutation of the initial cluster at a sink or source vertex and applying the corresponding reflection functor to the representation in the quantum cluster character.

CHAPTER II

VALUED QUIVERS AND SPECIES

This chapter is dedicated to the study of valued quivers. In Section II.1. we define valued quivers and the category of their representations as well as presenting several definitions and minor results needed later in this dissertation. Section II.2. is dedicated to showing that this category is equivalent to the category of modules over an associated species. Sections II.3. and II.4. present respectively the theory of reflection functors and tilting theory for species, which according to the results of Section II.2. can be applied to valued quiver representations. In Section II.5. we recall Hubery's construction [H1] of a skew-symmetrizable matrix associated to each local tilting representation.

II.1. Valued Quiver Representations

Let $\mathbb{F}$ be a finite field and write $\overline{\mathbb{F}}$ for an algebraic closure of $\mathbb{F}$. For each positive integer k denote by $\mathbb{F}_k$ the degree k extension of $\mathbb{F}$ in $\overline{\mathbb{F}}$. Note that the largest subfield of $\overline{\mathbb{F}}$ contained in both $\mathbb{F}_k$ and $\mathbb{F}_\ell$ is $\mathbb{F}_{gcd(k,\ell)} = \mathbb{F}_k \cap \mathbb{F}_\ell$. If $k|\ell$ we will fix a basis of $\mathbb{F}_\ell$ over $\mathbb{F}_k$ and thus we may freely identify $\mathbb{F}_\ell$ as a vector space over $\mathbb{F}_k$.

Fix an integer n and let $Q = \{Q_0, Q_1, h, t\}$ be a quiver with vertices $Q_0 = \{1, 2, \ldots, n\}$ and arrows Q_1, where we denote by $t(a)$ and $h(a)$ the tail and head of an arrow

$$t(a) \xrightarrow{a} h(a)\,.$$

Let $\mathbf{d} : Q_0 \to \mathbb{Z}_{>0}$ be a collection of valuations associated to the vertices of Q, where we denote the image of a vertex i by d_i. We will call the pair $(Q, \mathbf{d})$ a "valued quiver". Since the valuations will be fixed for all time we will sometimes drop the $\mathbf{d}$ from our notation.

Define a representation $V = (\{V_i\}_{i\in Q_0}, \{\varphi_a\}_{a\in Q_1})$ of $(Q, \mathbf{d})$, or a "($\mathbf{d}$-)valued representation" of Q, by assigning an $\mathbb{F}_{d_i}$-vector space V_i to each vertex $i \in Q_0$ and an $\mathbb{F}_{gcd(d_{h(a)},d_{t(a)})}$-linear map

$$\varphi_a \ : \ V_{t(a)} \longrightarrow V_{h(a)}$$

to each arrow $a \in Q_1$. Notice that $\mathbb{F}_{d_{h(a)}}$ and $\mathbb{F}_{d_{t(a)}}$ are both extensions of $\mathbb{F}_{gcd(d_{h(a)},d_{t(a)})}$ and thus we may view $V_{t(a)}$ and $V_{h(a)}$ as vector spaces over $\mathbb{F}_{gcd(d_{h(a)},d_{t(a)})}$. Let $W = (\{W_i\}_{i\in Q_0}, \{\psi_a\}_{a\in Q_1})$ denote another valued representation of Q and define a morphism

$$\theta \ : \ V \longrightarrow W$$

to be a collection $\theta = \{\theta_i \in \mathrm{Hom}_{\mathbb{F}_{d_i}}(V_i, W_i)\}_{i\in Q_0}$ such that $\theta_{h(a)} \circ \varphi_a = \psi_a \circ \theta_{t(a)}$, i.e. the following diagram commutes, for all $a \in Q_1$:

$$\begin{array}{ccc} V_{t(a)} & \xrightarrow{\varphi_a} & V_{h(a)} \\ \downarrow{\scriptstyle \theta_{t(a)}} & & \downarrow{\scriptstyle \theta_{h(a)}} \\ W_{t(a)} & \xrightarrow{\psi_a} & W_{h(a)} \,. \end{array}$$

Thus we have a category $\mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$ of all finite dimensional valued representations of Q. For valued representations V and W we define their direct sum via

$$V \oplus W = (\{V_i \oplus W_i\}_{i\in Q_0}, \{\varphi_a \oplus \psi_a\}_{a\in Q_1}).$$

Then one easily checks that $\mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$ is an Abelian category where kernels and cokernels are taken vertex-wise.

For each $i \in Q_0$ denote by S_i the simple valued representation associated to vertex i, i.e. we assign the vector space $\mathbb{F}_{d_i}$ to vertex i and the zero vector space to every other vertex. We will write $\mathcal{K}(Q)$ for the Grothendieck group of $\mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$, that is $\mathcal{K}(Q)$ is the free Abelian group generated by the isomorphism classes $[V]$ for $V \in \mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$ subject to the relations $[V] = [U] + [W]$ whenever there is a short exact sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0\,.$$

Let $\alpha_i \in \mathcal{K}(Q)$ denote the class of simple valued representation S_i. Every representation $V \in \mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$ has a finite filtration with simple quotients, so we may write $[V]$ as a linear combination of the α_i. Thus we may identify $[V]$ with the "dimension vector" of V: $\mathbf{v} = (\dim_{\mathbb{F}_{d_i}} V_i)_{i=1}^n \in \mathbb{Z}^n$, where we adopt the convention that for any named representation we will use the same bold face letter to denote its dimension vector. In particular, taking the α_i as a basis we may identify $\mathcal{K}(Q)$ with the free Abelian group $\mathbb{Z}^n$.

Now we introduce some terminology that will be useful for describing valued representations. A valued representation V is called "indecomposable" if $V = U \oplus W$ implies $U = 0$ or $W = 0$. The category $\mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$ is Krull-Schmidt, that is every representation can be written uniquely as a direct sum of indecomposable representations. A representation V will be called "basic" if each indecomposable summand of V appears with multiplicity one. We will call V "rigid" if $\mathrm{Ext}(V, V) = 0$. If V is both rigid and indecomposable then we will say V is "exceptional". Our main concern will be with basic rigid representations. In particular, we are most interested in basic rigid representations V satisfying the following locality condition: the number of vertices in its "support" $\mathrm{supp}(V) = \{i \in Q_0 | V_i \neq 0\}$ is equal to the number of indecomposable summands of V. These representations will be called "local tilting representations". A representation V is called "sincere" if $\mathrm{supp}(V) = Q_0$. A sincere local tilting representation will simply be called a "tilting representation". We justify this terminology in Section II.4. Note that we may identify an insincere representation V with a sincere representation of the full subquiver Q^V of Q with vertices $Q_0^V = \mathrm{supp}(V)$. Then a local tilting representation V may be considered as a tilting representation for Q^V, this explains the adjective local.

For the remainder of this dissertation we assume that the quiver Q is acyclic, that is Q contains no non-trivial paths which begin and end at the same vertex. For $i, j \in Q_0$ denote by n_{ij} the number of arrows connecting vertices i and j and note that these arrows are either all of the form $i \to j$ or all of the form $j \to i$. Define a matrix $B_Q = B_{(Q,\mathbf{d})} = (b_{ij})$ by

$$b_{ij} = \begin{cases} n_{ij} d_j / gcd(d_i, d_j) & \text{if } i \to j \in Q_1; \\ -n_{ij} d_j / gcd(d_i, d_j) & \text{if } j \to i \in Q_1; \\ 0 & \text{otherwise.} \end{cases}$$

Let $d_{ij} = gcd(d_i, d_j)$ and $d^{ij} = lcm(d_i, d_j)$. Notice that since $d_i d_j / d_{ij} = d^{ij}$ we have $d_i b_{ij} = n_{ij} d^{ij} = -d_j b_{ji}$ whenever $i \to j \in Q_1$. Thus we see that DB_Q is skew-symmetric where

$D = diag(d_i)$, in other words B_Q is "skew-symmetrizable". The matrix B_Q will be important in Section II.2. where we associate an $\mathbb{F}$-species to the valued quiver $(Q, \mathbf{d})$ and in Section III.1. where we define a quantum cluster algebra from such a skew-symmetrizable matrix. Note that we may recover Q from the matrix B_Q as the quiver with $gcd(|b_{ij}|, |b_{ji}|)$ arrows from i to j whenever $b_{ij} > 0$. This will be useful in Section III.2. where we use the representation theory of a valued quiver to describe quantum cluster variables.

We now define a symmetrizable Cartan matrix $A_Q = A_{(Q,\mathbf{d})} = (a_{ij})$ associated to the valued quiver $(Q, \mathbf{d})$ via

$$a_{ij} = \begin{cases} 2 & \text{if } i = j; \\ -|b_{ij}| & \text{if } i \neq j; \end{cases}$$

where $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in Q_0$. Let Φ denote the root system associated to A_Q. We will identify the root lattice of Φ with the Grothendieck group $\mathcal{K}(Q)$ of $\mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$ by taking as simple roots the set $\{\alpha_i\}_{i \in Q_0}$. For each $i \in Q_0$ define the simple reflection $s_i : \mathcal{K}(Q) \to \mathcal{K}(Q)$ to be the unique $\mathbb{Z}$-linear map defined on generators by

$$s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i.$$

Then we have the following Theorem mentioned in Chapter I.

Theorem II.1.1 (Gabriel, Dlab-Ringel, Kac, Hubery)**.** The dimension vectors of indecomposable representations of $(Q, \mathbf{d})$ are in one-to-one correspondence with the positive roots of Φ.

Example II.1.2 (Type B_2)**.** Consider the valued quiver $Q = \circ \to \circ$ with $\mathbf{d} = (1, 2)$. We compute the matrix $B_Q = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$ and the associated Cartan matrix $A_Q = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$. The valued quiver $(Q, \mathbf{d})$ has four isomorphism classes of indecomposable representations:

- The simple injective representation $S_1 = \mathbb{F} \to 0$ with dimension vector α_1.
- The injective hull of S_2 is $I_2 = \mathbb{F}^2 \xrightarrow{\sigma} \mathbb{F}_2$, where σ identifies $\mathbb{F}_2$ as a 2-dimensional vector space over $\mathbb{F}$, with dimension vector $2\alpha_1 + \alpha_2 = s_1(\alpha_2)$.
- The projective cover of S_1 is $P_1 = \mathbb{F} \xrightarrow{\iota} \mathbb{F}_2$, where ι is any injective map, with dimension vector $\alpha_1 + \alpha_2 = s_1 s_2(\alpha_1)$.

- The simple projective representation $S_2 = 0 \to \mathbb{F}_2$ with dimension vector $\alpha_2 = s_1 s_2 s_1(\alpha_2)$.

This presentation of the dimension vectors will be explained in Chapter IV. □

We introduce the following notation in preparation for Definition III.2.1. It follows from the results of Section II.2. that $\mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$ is "hereditary", that is for all $V, W \in \mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$ we have $\mathrm{Ext}^i(V, W) = 0$ for every $i \geq 2$. For valued representations $V, W \in \mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$ we define $\langle V, W\rangle = \dim_{\mathbb{F}} \mathrm{Hom}(V, W) - \dim_{\mathbb{F}} \mathrm{Ext}^1(V, W)$. Using the long exact sequence on Ext coming from a short exact sequence of representations, it is easy to see that $\langle V, W\rangle$ only depends on the classes of V and W in $\mathcal{K}(Q)$. Thus we obtain a bilinear form $\langle \cdot, \cdot\rangle : \mathcal{K}(Q) \times \mathcal{K}(Q) \to \mathbb{Z}$ known as the "Ringel-Euler form". Note that by the bilinearity of the Ringel-Euler form, it suffices to compute it on the basis of $\mathcal{K}(Q)$:

$$\langle \alpha_i, \alpha_j\rangle = \begin{cases} d_i & \text{if } i = j, \\ -[d_i b_{ij}]_+ & \text{if } i \neq j, \end{cases} \tag{II.1}$$

where we write $[b]_+ = \max(0, b)$.

We will write $\alpha_i^\vee = \frac{1}{d_i}\alpha_i$ and remark that the skew-symmetrizability of B_Q implies $\langle \alpha_i^\vee, \alpha_j\rangle$ and $\langle \alpha_i, \alpha_j^\vee\rangle$ are integers for all i and j. Then for $\mathbf{e} \in \mathcal{K}(Q)$ define ${}^*\mathbf{e}, \mathbf{e}^* \in \mathcal{K}(Q)$ by

$${}^*\mathbf{e} = \sum_{i=1}^{n} \langle \alpha_i^\vee, \mathbf{e}\rangle \alpha_i, \;\; \mathbf{e}^* = \sum_{i=1}^{n} \langle \mathbf{e}, \alpha_i^\vee\rangle \alpha_i.$$

We define two matrices $B_Q^- = (b_{ij}^-)$ and $B_Q^+ = (b_{ij}^+)$ as follows:

$$b_{ij}^- = \langle \alpha_i^\vee, \alpha_j\rangle = \begin{cases} 1 & \text{if } i = j; \\ -[b_{ij}]_+ & \text{if } i \neq j; \end{cases}$$

$$b_{ij}^+ = \langle \alpha_j, \alpha_i^\vee\rangle = \begin{cases} 1 & \text{if } i = j; \\ -[-b_{ij}]_+ & \text{if } i \neq j. \end{cases}$$

It is clear from the definitions that viewing $\mathbf{e}$ as an element of $\mathbb{Z}^n$ we have

$${}^*\mathbf{e} = B_Q^- \mathbf{e} \text{ and } \mathbf{e}^* = B_Q^+ \mathbf{e}. \tag{II.2}$$

Also note from equation (II.1) and the skew-symmetrizability of B_Q that we have

$$B_Q^+ = D(B_Q^-)^t D^{-1} \text{ and } B_Q^+ - B_Q^- = B_Q.$$

Suppose there exists an $n \times n$ skew-symmetrizable matrix $\Lambda = (\lambda_{ij})$ such that $B_Q^t \Lambda = D$ and write $\Lambda(\cdot,\cdot) : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ for the associated skew-symmetric bilinear form. As in Zelevinsky's Oberwolfach talk on quantum cluster algebras [Z], we may always replace the quiver Q by $\tilde{Q}$, where we attach principal frozen vertices to the valued quiver Q, to guarantee that such a matrix Λ exists. Note that the compatibility condition for B_Q and Λ implies $\Lambda(\mathbf{b}^i, \alpha_j) = d_i \delta_{ij}$ where $\mathbf{b}^i$ denotes the i^{th} column of B_Q.

For a valued representation V define the "socle" $\operatorname{soc} V$ to be the sum of all simple subrepresentations of V and the "radical" $\operatorname{rad} V$ to be the intersection of all maximal subrepresentations of V. We record the following identities for use in Section III.2.

Lemma II.1.3.

1. For any $\mathbf{d} \in \mathcal{K}(Q)$, $\Lambda(\mathbf{b}^i, {}^*\mathbf{d}) = \langle \alpha_i, \mathbf{d} \rangle$ and $\Lambda(\mathbf{d}^*, \mathbf{b}^j) = -\langle \mathbf{d}, \alpha_j \rangle$.

2. For any $\mathbf{b}, \mathbf{d} \in \mathcal{K}(Q)$, $\Lambda(\mathbf{b}^* - {}^*\mathbf{b}, \mathbf{d}^* - {}^*\mathbf{d}) = \langle \mathbf{d}, \mathbf{b} \rangle - \langle \mathbf{b}, \mathbf{d} \rangle$.

3. For any $\mathbf{b}, \mathbf{d}, \mathbf{v}, \mathbf{w} \in \mathcal{K}(Q)$. $\Lambda(-\mathbf{b}^* - {}^*(\mathbf{v} - \mathbf{b}), -\mathbf{d}^* - {}^*(\mathbf{w} - \mathbf{d})) = \Lambda({}^*\mathbf{v}, {}^*\mathbf{w}) - \langle \mathbf{d}, \mathbf{v} - \mathbf{b} \rangle + \langle \mathbf{b}, \mathbf{w} - \mathbf{d} \rangle$.

4. For any injective valued representation I, $[\operatorname{soc} I] = {}^*\mathbf{i}$, where $\mathbf{i} = [I] \in \mathcal{K}(Q)$.

5. For any projective valued representation P, $[P/\operatorname{rad} P] = \mathbf{p}^*$, where $\mathbf{p} = [P] \in \mathcal{K}(Q)$.

Proof. The identities in (1) are a direct consequence of the compatibility of B_Q and Λ. The identity in (2) follows immediately from (II.2) and (1). The identity in (3) can easily be obtained from (1) and (2). There is a unique injective hull and projective cover for each simple representation, the identities in (4) and (5) follow. □

II.2. Modules Over an $\mathbb{F}$-Species

In this section we define the $\mathbb{F}$-species associated to a skew-symmetrizable matrix and show that the category of representations of $(Q, \mathbf{d})$ is equivalent to the well-studied category of modules over the $\mathbb{F}$-species associated to B_Q.

Just as with ordinary quivers it is useful to consider an equivalent category of modules over the path algebra. The analog of the path algebra for valued quivers is the notion of $\mathbb{F}$-species which we define as the tensor algebra of a bimodule over a semisimple algebra. Define the semisimple algebra $\Gamma_0 = \prod_{i=1}^{n} \mathbb{F}_{d_i}$ and let $\Gamma_1 = \bigoplus_{b_{ij}>0} \Gamma_{ij}$ where we set $\Gamma_{ij} := \mathbb{F}_{d_i b_{ij}}$ when $b_{ij} > 0$ and $\Gamma_{ij} = 0$ otherwise. Notice that $\mathbb{F}_{d_i b_{ij}}$ contains both $\mathbb{F}_{d_i}$ and $\mathbb{F}_{d_j}$ and thus we have a Γ_0-Γ_0-bimodule structure on Γ_1 where $\mathbb{F}_{d_k}$ acts by zero on Γ_{ij} if $i \neq k$ and $j \neq k$. Then for $i \geq 2$ we define Γ_i inductively by $\Gamma_i = \Gamma_1 \otimes \Gamma_{i-1}$. Now we may define the "tensor algebra" $\Gamma_Q := T_{\Gamma_0}(\Gamma_1)$ of Γ_1 over Γ_0 as the vector space $\bigoplus_{i=0}^{\infty} \Gamma_i$. The multiplication on Γ_Q is given by concatenation of tensors:

$$\mu \;:\; \Gamma_i \otimes \Gamma_j \longrightarrow \Gamma_{i+j}$$
$$(a_1 \otimes \cdots \otimes a_i) \otimes (b_1 \otimes \cdots \otimes b_j) \longmapsto a_1 \otimes \cdots \otimes a_i \otimes b_1 \otimes \cdots \otimes b_j\,.$$

A module $V = (\{V_i\}, \{\varphi_{ij}^V\})$ over Γ_Q is given by an $\mathbb{F}_{d_i}$-vector space V_i for each vertex i and an $\mathbb{F}_{d_j}$-linear map

$$\varphi_{ij}^V \;:\; V_i \otimes_{\mathbb{F}_{d_i}} \Gamma_{ij} \longrightarrow V_j$$

whenever $b_{ij} > 0$. A morphism of Γ_Q-modules

$$\theta \;:\; V \longrightarrow W$$

is a collection $\{\theta_i \in \mathrm{Hom}_{\mathbb{F}_{d_i}}(V_i, W_i)\}_{i \in Q_0}$ such that $\varphi_{ij}^W(\theta_i \otimes \mathrm{Id}) = \theta_j \varphi_{ij}^V$, i.e. the following diagram commutes, whenever $b_{ij} > 0$:

$$\begin{array}{ccc} V_i \otimes_{\mathbb{F}_{d_i}} \Gamma_{ij} & \xrightarrow{\varphi_{ij}^V} & V_j \\ \downarrow{\scriptstyle \theta_i \otimes \mathrm{Id}} & & \downarrow{\scriptstyle \theta_j} \\ W_i \otimes_{\mathbb{F}_{d_i}} \Gamma_{ij} & \xrightarrow{\varphi_{ij}^W} & W_j\,. \end{array}$$

We will write $\mathrm{Mod}\,\Gamma_Q$ for the category of all finite-dimensional modules over Γ_Q.

Proposition II.2.1. [Ru1] The categories $\mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$ and $\mathrm{Mod}\,\Gamma_Q$ are equivalent.

Thus we may apply all results concerning $\mathbb{F}$-species to valued representations.

Proof. Let $d_{ij} = gcd(d_i, d_j)$ and $d^{ij} = lcm(d_i, d_j)$. Suppose $b_{ij} > 0$. Since $d_{ij} | d_i b_{ij}$ we may identify $\Gamma_{ij} = \mathbb{F}_{d_i b_{ij}}$ as a vector space over $\mathbb{F}_{d^{ij}}$ and write $\Gamma_{ij} \cong \bigoplus_{k=1}^{n_{ij}} \mathbb{F}_{d^{ij}} \cong \bigoplus_{a:i\to j} \mathbb{F}_{d^{ij}}$, where we note that $\frac{d_i b_{ij}}{d^{ij}} = gcd(|b_{ij}|, |b_{ji}|) = n_{ij}$ is the number of arrows $i \to j$ in Q_1. Consider the following isomorphisms:

$$\mathrm{Hom}_{\mathbb{F}_{d_j}}(V_i \otimes_{\mathbb{F}_{d_i}} \Gamma_{ij}, V_j) \cong \bigoplus_{a:i\to j} \mathrm{Hom}_{\mathbb{F}_{d_j}}(V_i \otimes_{\mathbb{F}_{d_i}} \mathbb{F}_{d^{ij}}, V_j),$$

$$V_i \otimes_{\mathbb{F}_{d_i}} \mathbb{F}_{d^{ij}} = V_i \otimes_{\mathbb{F}_{d_i}} (\mathbb{F}_{d_i} \otimes_{\mathbb{F}_{d_{ij}}} \mathbb{F}_{d_j}) \cong V_i \otimes_{\mathbb{F}_{d_{ij}}} \mathbb{F}_{d_j},$$

$$\mathrm{Hom}_{\mathbb{F}_{d_j}}(V_i \otimes_{\mathbb{F}_{d_{ij}}} \mathbb{F}_{d_j}, V_j) \cong \mathrm{Hom}_{\mathbb{F}_{d_{ij}}}(V_i, V_j).$$

Combining these we obtain an isomorphism

$$\mathrm{Hom}_{\mathbb{F}_{d_j}}(V_i \otimes_{\mathbb{F}_{d_i}} \Gamma_{ij}, V_j) \cong \bigoplus_{a:i\to j} \mathrm{Hom}_{\mathbb{F}_{d_j}}(V_i \otimes_{\mathbb{F}_{d_{ij}}} \mathbb{F}_{d_j}, V_j) \cong \bigoplus_{a:i\to j} \mathrm{Hom}_{\mathbb{F}_{d_{ij}}}(V_i, V_j). \tag{II.3}$$

For vertices $i, j \in Q_0$ we will write

$$\omega_{ij} \ : \ \mathrm{Hom}_{\mathbb{F}_{d_j}}(V_i \otimes_{\mathbb{F}_{d_i}} \Gamma_{ij}, V_j) \xrightarrow{\ \sim\ } \bigoplus_{a:i\to j} \mathrm{Hom}_{\mathbb{F}_{d_{ij}}}(V_i, V_j)$$

for this isomorphism. Then we may define functors

$$\begin{aligned} F \ : \ \mathrm{Mod}\,\Gamma_Q &\longleftrightarrow \mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d}) \ : \ G \\ (\{V_i\}, \{\varphi_{ij}^V\}) &\longmapsto (\{V_i\}, \{\omega_{ij}(\varphi_{ij}^V)\}) \\ (\{V_i\}, \{\omega_{ij}^{-1}((\varphi_a)_{a:i\to j})\}) &\longleftarrow\!\shortmid (\{V_i\}, \{\varphi_a\}) \\ \{\theta_i \in \mathrm{Hom}_{\mathbb{F}_{d_i}}(V_i, W_i)\}_{i\in Q_0} &\longleftrightarrow \{\theta_i \in \mathrm{Hom}_{\mathbb{F}_{d_i}}(V_i, W_i)\}_{i\in Q_0}. \end{aligned}$$

Indeed, after the identification $\Gamma_{ij} \cong \bigoplus_{a:i\to j} \mathbb{F}_{d^{ij}}$ we can see via restriction that the commuting squares defining morphisms of Γ_Q-modules induce commuting squares defining morphisms of representations of $(Q, \mathbf{d})$. Similarly we see that morphisms of representations of $(Q, \mathbf{d})$ induce morphisms of Γ_Q-modules. Once defined there is nothing more needed to see that the functors F and G are inverse equivalences. □

Remark II.2.2. One might object that the isomorphism (II.3) is not natural, but the original assignment of the $\mathbb{F}$-species Γ_Q to the skew-symmetrizable matrix B_Q already required choices.

The following Corollary is a consequence of the main result of [DR] obtained using the reflection functors defined in the following section.

Corollary II.2.3. The valued quiver $(Q, \mathbf{d})$ has only finitely many indecomposable representations if and only if the associated Cartan matrix A_Q fits into the Cartan-Killing $A - G$ finite-type classification.

II.3. Functors on $\operatorname{Mod}\Gamma_Q$

In this section we recall several functors acting on $\operatorname{Mod}\Gamma_Q$. Subsection II.3.1. presents the theory of reflection functors for the category of modules over a species developed by Dlab and Ringel [DR]. Subsection II.3.2. defines the Nakayama functor and the Auslander-Reiten translation. By the equivalence of categories in Proposition II.2.1 all constructions and results of this section can be transported to the category $\operatorname{Rep}_{\mathbb{F}}(Q, \mathbf{d})$.

II.3.1. Reflection Functors

A vertex $k \in Q_0$ is a "source" (resp. "sink") if every arrow $a \in Q_1$ incident on vertex k has $t(a) = k$ (resp. $h(a) = k$). Note that we assume the quiver Q is acyclic and so there are no arrows with $t(a) = h(a)$. For a sink or source k of Q denote by $\mu_k Q$ the quiver obtained from Q by reversing all arrows incident on vertex k. We will call a sequence of vertices k_1, k_2, $\ldots$, k_{r+1} in Q "admissible" if the following hold:

- $k_s \neq k_{s+1}$ for each $1 \leq s \leq r$;
- k_1 is a sink or source in Q;
- for each $1 \leq s \leq r-1$, vertex k_{s+1} is a sink or source in the quiver $\mu_{k_s}\mu_{k_{s-1}} \cdots \mu_{k_1} Q$.

Note that k_{r+1} is not required to be a sink or a source in the quiver $\mu_{k_r}\mu_{k_{r-1}} \cdots \mu_{k_1} Q$. This notion of an admissible sequence will be used in Section IV.3.

For $b_{ij} > 0$ define $\Gamma_{ji} := \operatorname{Hom}_{\mathbb{F}_{d_j}}(\Gamma_{ij}, \mathbb{F}_{d_j})$. For a module V over Γ_Q there is a natural isomorphism

$$\operatorname{Hom}_{\mathbb{F}_{d_j}}(\Gamma_{ij}, V_j) \cong V_j \otimes_{\mathbb{F}_{d_j}} \operatorname{Hom}_{\mathbb{F}_{d_j}}(\Gamma_{ij}, \mathbb{F}_{d_j}) = V_j \otimes_{\mathbb{F}_{d_j}} \Gamma_{ji}.$$

We also have the standard adjointness isomorphism

$$\operatorname{Hom}_{\mathbb{F}_{d_j}}(V_i \otimes_{\mathbb{F}_{d_i}} \Gamma_{ij}, V_j) \cong \operatorname{Hom}_{\mathbb{F}_{d_i}}(V_i, \operatorname{Hom}_{\mathbb{F}_{d_j}}(\Gamma_{ij}, V_j)).$$

Combining these we obtain two natural isomorphisms both denoted by $\overline{\cdot}$:

$$\overline{\cdot}\ :\ \operatorname{Hom}_{\mathbb{F}_{d_j}}(V_i \otimes_{\mathbb{F}_{d_i}} \Gamma_{ij}, V_j) \longleftrightarrow \operatorname{Hom}_{\mathbb{F}_{d_i}}(V_i, V_j \otimes_{\mathbb{F}_{d_j}} \Gamma_{ji})\ :\ \overline{\cdot}\,.$$

These isomorphism will be the necessary ingredient for extending the classical notion of reflection functors from representations of a quiver to modules over Γ_Q.

For a sink $k \in Q_0$ we now define the reflection functor $\Sigma_k^+ : \operatorname{Mod}\Gamma_Q \to \operatorname{Mod}\Gamma_{\mu_k Q}$. For a module $V = (V_i, \varphi_{ij}^V)$ we define $\Sigma_k^+(V) = W = (W_i, \varphi_{ij}^W)$ as follows:

- for $i \neq k$ we set $W_i = V_i$ and $\varphi_{ij}^W = \varphi_{ij}^V$;
- set W_k to be the kernel in the diagram

$$0 \longrightarrow W_k \xrightarrow{(\kappa_{kj})} \bigoplus_{j \in Q_0} V_j \otimes_{\mathbb{F}_{d_j}} \Gamma_{jk} \xrightarrow{(\varphi_{jk}^V)} V_k$$

  and when $b_{jk} > 0$ set $\varphi_{kj}^W = \overline{\kappa_{kj}} : W_k \otimes_{\mathbb{F}_{d_k}} \Gamma_{kj} \to W_j$.

To get a functor we need to describe the action of Σ_k^+ on a morphism $\theta : V \to V'$. We define $\Sigma_k^+(\theta) = \omega = (\omega_i)_{i \in Q_0}$ as follows:

- for $i \neq k$ we set $\omega_i = \theta_i$;
- set $\omega_k : W_k \to W_k'$ to be the restriction of

$$\bigoplus_{j \in Q_0}(\theta_j \otimes \mathrm{Id})\ :\ \bigoplus_{j \in Q_0} V_j \otimes_{\mathbb{F}_{d_j}} \Gamma_{jk} \longrightarrow \bigoplus_{j \in Q_0} V_j' \otimes_{\mathbb{F}_{d_j}} \Gamma_{jk}$$

  to W_k and note that, since θ was a morphism of Γ_Q-modules, the image of this restriction will indeed be contained in W_k'.

Similarly, for a source $k \in Q_0$ we define the reflection functor $\Sigma_k^- : \operatorname{Mod}\Gamma_Q \to \operatorname{Mod}\Gamma_{\mu_k Q}$. For a module $V = (V_i, \varphi_{ij}^V)$ we define $\Sigma_k^-(V) = W = (W_i, \varphi_{ij}^W)$ as follows:

- for $j \neq k$ we set $W_j = V_j$ and $\varphi_{ij}^W = \varphi_{ij}^V$;

- set W_k to be the cokernel in the diagram

$$V_k \xrightarrow{(\overline{\varphi_{ki}^V})} \bigoplus_{i\in Q_0} V_i \otimes_{\mathbb{F}_{d_i}} \Gamma_{ik} \xrightarrow{(\pi_{ik})} W_k \longrightarrow 0$$

  and when $b_{ki} > 0$ set $\varphi_{ik}^W = \pi_{ik} : W_i \otimes_{\mathbb{F}_{d_i}} \Gamma_{ik} \to W_k$.

For a morphism $\theta : V \to V'$ we define $\Sigma_k^-(\theta) = \omega = (\omega_i)_{i\in Q_0}$ as follows:

- for $i \neq k$ we set $\omega_i = \theta_i$;

- set $\omega_k : W_k \to W'_k$ to be the map induced by

$$\bigoplus_{i\in Q_0}(\theta_i \otimes \mathrm{Id}) \ : \ \bigoplus_{i\in Q_0} V_i \otimes_{\mathbb{F}_{d_i}} \Gamma_{ik} \longrightarrow \bigoplus_{i\in Q_0} V'_i \otimes_{\mathbb{F}_{d_i}} \Gamma_{ik}.$$

  Indeed, θ being a morphism of Γ_Q-modules implies that we have such a map between the cokernels W_k and W'_k.

We will denote by the same symbols $\Sigma_k^\pm$ these functors transported to $\mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$ via the equivalence in Proposition II.2.1.

Let $\mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})\langle k\rangle$ denote the full subcategory of $\mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$ of all representations of $(Q, \mathbf{d})$ which do not contain S_k as a direct summand. For $k \in Q_0$ a source, it is shown in [DR] that the reflection functors $\Sigma_k^\pm$ restrict to exact equivalences of categories

$$\Sigma_k^- \ : \ \mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})\langle k\rangle \longleftrightarrow \mathrm{Rep}_{\mathbb{F}}(\mu_k Q, \mathbf{d})\langle k\rangle \ : \ \Sigma_k^+ \, .$$

Since it will be clear from context which to use we will often drop the $^\pm$ and simply denote both functors by Σ_k. The following result proved in [DR] will be essential for our study of external mutations presented in Chapter IV.

Proposition II.3.1. [DR, Proposition 2.1] Let $V \in \mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})\langle k\rangle$. Then we have

1. $[\Sigma_k V] = \sigma_k([V])$ where σ_k denotes the simple reflection on $\mathcal{K}(Q)$;

2. $\Sigma_k^2 V = V$.

II.3.2. Other Functors

Let $D : \operatorname{Mod}\Gamma_Q \to \operatorname{Mod}\Gamma_{Q^{op}}$ denote the standard $\mathbb{F}$-linear duality, that is $D = \operatorname{Hom}_{\mathbb{F}}(-, \mathbb{F})$, where Q^{op} denotes the quiver obtained from Q by reversing all arrows. We define the "Nakayama functor" $\nu : \operatorname{Mod}\Gamma_Q \to \operatorname{Mod}\Gamma_Q$ to be the composition $D \operatorname{Hom}_{\Gamma_Q}(-, \Gamma_Q)$. It is well-known that this functor defines an equivalence of categories from the full subcategory of $\operatorname{Mod}\Gamma_Q$ consisting of projective objects to the full subcategory consisting of injective objects.

Our next goal is to define the Auslander-Reiten translation. Let V be a module over Γ_Q and consider the minimal projective resolution

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow V \longrightarrow 0\,.$$

The "Auslander-Reiten translation" $\tau = D\operatorname{Tr} : \operatorname{Mod}\Gamma_Q \to \operatorname{Mod}\Gamma_Q$ is defined on a representation V via the following induced diagram:

$$0 \longrightarrow \operatorname{Hom}(P_0, \Gamma_Q) \longrightarrow \operatorname{Hom}(P_1, \Gamma_Q) \longrightarrow \operatorname{Tr}(V) \longrightarrow 0\,.$$

The Auslander-Reiten translation defines an equivalence between the full subcategories $\operatorname{Mod}_P \Gamma_Q$ and $\operatorname{Mod}_I \Gamma_Q$ consisting of objects with no projective summands and no injective summands respectively. The following "Auslander-Reiten formulas" will be essential in the computations to follow.

Proposition II.3.2. For any representations V, W, there exist isomorphisms:

$$\begin{aligned} D\operatorname{Hom}(V, \tau W) &\cong \operatorname{Ext}^1(W, V), \\ D\operatorname{Ext}^1(V, \tau W) &\cong \operatorname{Hom}(W, V). \end{aligned} \tag{II.4}$$

In particular, we have $\langle \mathbf{v}, \tau\mathbf{w}\rangle = -\langle \mathbf{w}, \mathbf{v}\rangle$.

It follows from results of [APR] that we have a factorization $\tau = \Sigma^+_{i_1} \cdots \Sigma^+_{i_n}$ where $(i_1, \ldots, i_n)$ is a complete source adapted sequence of vertices in Q. Thus τ is sometimes called a "Coxeter functor".

II.4. Tilting Theory for $\mathrm{Mod}\,\Gamma_Q$

The main result of this dissertation uses the classical theory of tilting modules over an $\mathbb{F}$-species. We will present those results necessary to define a mutation operation for local tilting representations. The main result of this dissertation will show that this mutation operation corresponds with the Berenstein-Zelevinsky mutations of quantum seeds that we will present in Section III.1. We freely abuse the equivalence from Proposition II.2.1 to go between representations of $(Q, \mathbf{d})$ and modules over Γ_Q.

We begin with the definition of a tilting module for Γ_Q. For a representation T we write $add(T)$ for the full additive subcategory of $\mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$ generated by the indecomposable summands of T. Then a Γ_Q-module T is called "tilting" if there is a coresolution of Γ_Q of the form

$$0 \longrightarrow \Gamma_Q \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$$

with $T_0, T_1 \in add(T)$. In order to define mutation of local tilting representations we need to recall two well known results from the representation theory of Γ_Q.

Theorem II.4.1 (Happel-Ringel)**.**

1. If V and W are indecomposable modules with $\mathrm{Ext}^1(V, W) = 0$, then any nonzero map $W \to V$ is either a monomorphism or an epimorphism.

2. The dimension vectors of the indecomposable summands of a basic rigid module are linearly independent in the Grothendieck group of Γ_Q.

3. A basic rigid module T is a tilting module if and only if T contains n indecomposable summands.

Note that Theorem II.4.1.3 implies each local tilting representation T of Q is a tilting representation for the full subquiver Q^T of Q where $Q_0^T = \mathrm{supp}(T)$. Note that we will consider the zero representation as a tilting representation for the empty subquiver. A basic rigid representation is called an "almost complete tilting representation" if it contains $n-1$ indecomposable summands. The following theorem describes the possible ways to complete an almost complete tilting representation to a tilting representation.

Theorem II.4.2 (Happel-Unger)**.** Let T be an almost complete tilting module.

1. If T is sincere, then there exist exactly two non-isomorphic complements to T, otherwise there is a unique complement.

2. Suppose T is sincere and write V and W for the complements to T. Suppose $\mathrm{Ext}^1(V, W) \neq 0$, then there is a unique non-split sequence $0 \to W \to E \to V \to 0$. Furthermore $E \in add(T)$ and $\dim_{\mathbb{F}} \mathrm{End}(V) = \dim_{\mathbb{F}} \mathrm{End}(W) = \dim_{\mathbb{F}} \mathrm{Ext}(V, W)$.

Let T be a local tilting representation of $(Q, \mathbf{d})$. We will use Theorem II.4.2 to define a mutation operation on T which will produce another local tilting representation. We will only consider those local tilting representations T which may be obtained by iterating mutations starting from the zero representation. Thus there will be a canonical labeling of the summands of T by the vertices in its support and for $i \in \mathrm{supp}(T)$ we have $\mathrm{End}(T_i) \cong \mathrm{End}(S_i) \cong \mathbb{F}_{d_i}$ by Theorem II.4.2.2.

Given a local tilting representation T and a vertex $k \in Q_0$ we define the mutation $\mu_k(T)$ in direction k as follows:

1. If $k \notin \mathrm{supp}\, T$, then by Theorem II.4.2.1 there exists a unique complement T_k^* so that $\mu_k(T) = T_k^* \oplus T$ is a local tilting representation containing k in its support.

2. If $k \in \mathrm{supp}\, T$, then write $\bar{T} = T/T_k$.

   (a) If $\bar{T}$ is a local tilting representation, i.e. $k \notin \mathrm{supp}\, \bar{T}$, let $\mu_k(T) = \bar{T}$.

   (b) Otherwise $\mathrm{supp}\, \bar{T} = \mathrm{supp}\, T$ and by Theorem II.4.2 there exists a unique compliment $T_k^* \ncong T_k$ so that $\mu_k(T) = T_k^* \oplus \bar{T}$ is a local tilting representation.

Notice that the definitions imply the mutation of local tilting representations is involutive.

Remark II.4.3. It follows from results of [BMRRT] that the mutation operation is transitive on the set of local tilting representations.

Example II.4.4 (Type B_2)**.** We keep the notation from Example II.1.2. In particular the valued quiver $Q = \circ \to \circ$ with $\mathbf{d} = (1, 2)$ has four indecomposable representations S_1, S_2, P_1, and I_2. All of these representations are rigid since each one is either injective or projective. In Figure II.4.1 below we present the mutation graph of local tilting representations where local tilting representations related by the mutation operation just defined are connected by an edge labeled by the mutation. □

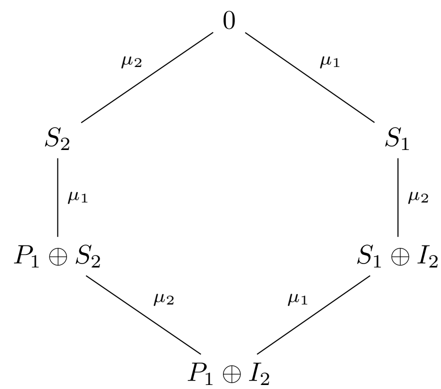

Figure II.4.1: Mutation graph for local tilting representations of the Type B_2 valued quiver.

II.5. Matrices Associated to Local Tilting Representations

Here we define the matrix B_T associated to a local tilting representation T. We will show in Section V.1. that the matrix B_T is skew-symmetrizable and that the mutation of local tilting representations induces the Fomin-Zelevinsky mutations of exchange matrices. This construction was given in [H1], as this work is unpublished we will reproduce Hubery's arguments.

We will require a little preparation before we can define the matrix B_T. For a valued representation V, we will call a morphism $V \to E$ (resp. $E \to V$) a "left $add(\bar{T})$-approximation" (resp. "right $add(\bar{T})$-approximation") of V if

- $E \in add(\bar{T})$ and
- the induced map $\mathrm{Hom}(E, X) \to \mathrm{Hom}(V, X)$ (resp. $\mathrm{Hom}(X, E) \to \mathrm{Hom}(X, V)$) is surjective for any $X \in add(\bar{T})$.

In other words, $V \to E$ (resp. $E \to V$) is a left (resp. right) $add(\bar{T})$-approximation of V if every map to (resp. from) an object of $add(\bar{T})$ factors through $V \to E$ (resp. $E \to V$). A morphism $\phi : V \to E$ (resp. $\varphi : E \to V$) is called "left minimal" (resp. "right minimal") if any endomorphism ψ of E satisfying $\psi \circ \phi = \phi$ (resp. $\varphi \circ \psi = \varphi$) is an isomorphism.

For each vertex k of Q our goal will be to define the entries b_{ik} of the k^{th} column of B_T. First suppose k is in the support of T and, using the notation from the mutation of local tilting representations, suppose $\mathrm{supp}(\bar{T}) = \mathrm{supp}(T)$. Write T_k^* for the other compliment of $\bar{T}$ described by the mutation. Following Theorem II.4.2.2 we assume that there is a unique non-split sequence

$$0 \longrightarrow T_k^* \longrightarrow E \longrightarrow T_k \longrightarrow 0\,. \tag{II.5}$$

Proposition II.5.1. [H1, Lemma 21] The map $T_k^* \to E$ is a minimal left $add(\bar{T})$-approximation of T_k^* and the map $E \to T_k$ is a minimal right $add(\bar{T})$-approximation of T_k.

Proof. By Theorem II.4.2, $E \in add(\bar{T})$. Let $X \in add(\bar{T})$ and note that $\mathrm{Ext}^1(T_k, X) = 0$ since T is rigid. Thus applying $\mathrm{Hom}(-, X)$ to the sequence (II.5) gives the exact sequence

$$\mathrm{Hom}(E, X) \longrightarrow \mathrm{Hom}(T_k^*, X) \longrightarrow 0\,,$$

and from the surjectivity of this map E is a left $add(\bar{T})$-approximation of T_k^*. The approximation $T_k^* \to E$ factors through any other approximation $T_k^* \to F$ and since $T_k^* \to E$ was injective we see that $T_k^* \to F$ must also be injective. Define G by the short exact sequence:

$$0 \longrightarrow T_k^* \longrightarrow F \longrightarrow G \longrightarrow 0\,. \tag{II.6}$$

Since $F \in add(\bar{T})$ and $\bar{T}$ is rigid, applying $\mathrm{Hom}(\bar{T}, -)$ to the sequence (II.6) gives $\mathrm{Ext}^1(\bar{T}, G) = 0$. Since $T_k^* \to F$ is a left $add(\bar{T})$-approximation, the map $\mathrm{Hom}(F, \bar{T}) \to \mathrm{Hom}(T_k^*, \bar{T})$ is surjective and applying $\mathrm{Hom}(-, \bar{T})$ to (II.6) shows that $\mathrm{Ext}^1(G, \bar{T}) = 0$. Then applying $\mathrm{Hom}(G, -)$ to (II.6) implies G and hence $\bar{T} \oplus G$ are rigid. So we must either have $G \in add(\bar{T} \oplus T_k)$ or $G \in add(\bar{T} \oplus T_k^*)$, but F is a non-trivial extension in $\mathrm{Ext}^1(G, T_k^*)$ and thus $G \in add(T)$. Moreover G must contain T_k as a summand. But E is the unique extension in $\mathrm{Ext}^1(T_k, T_k^*)$ and so the sequence (II.5) is a summand of the sequence (II.6). To see minimality of E, suppose we have an endomorphism ψ of E making the diagram

$$\begin{array}{ccc} & & E \\ & \nearrow & \\ T_k^* & & \big\downarrow \psi \\ & \searrow & \\ & & E \end{array}$$

commute. Then the image of ψ is again an $add(\bar{T})$-approximation and from the discussion above we see that E must be a summand, in other words ψ is an isomorphism and E is minimal. The proof for $E \to T_k$ is similar. $\square$

Using the Auslander-Reiten formulas of Proposition II.3.2, the unique extension E from equation II.5 gives rise to a unique morphism $\theta \in \mathrm{Hom}(T_k^*, \tau T_k)$. From θ we get a short exact sequence

$$0 \longrightarrow D \longrightarrow T_k^* \xrightarrow{\theta} \tau T_k \longrightarrow \tau A \oplus I \longrightarrow 0 \tag{II.7}$$

where $D = \ker\theta$, $\tau A \oplus I = \operatorname{coker}\theta$, I is injective, and A and T_k have the same maximal projective summand $P_A = P_{T_k}$. As in the proof of Lemma III.3.3 the sequence (II.7) is equivalent to the following short exact sequences

$$0 \longrightarrow D \longrightarrow T_k^* \longrightarrow C \longrightarrow 0\,, \tag{II.8}$$

$$0 \longrightarrow B \longrightarrow T_k \longrightarrow A \longrightarrow 0\,, \tag{II.9}$$

$$0 \longrightarrow C \longrightarrow \tau B \longrightarrow I \longrightarrow 0\,, \tag{II.10}$$

where B contains no projective summands. The following Proposition will allow us to complete the definition of the k^{th} column of B_T in the case under consideration and shows that we are in a position to apply Theorem III.3.4.

Proposition II.5.2. [H1, Proposition 22]

1. The map $T_k \to A$ is a minimal left $add(\bar{T})$-approximation of T_k.
2. The map $D \to T_k^*$ is a minimal right $add(\bar{T})$-approximation of T_k^*.
3. The objects B and C are indecomposable.
4. The objects $\operatorname{soc} I$ and T have disjoint supports and $\mathrm{Hom}(A, I) = \mathrm{Hom}(D, I) = 0$.

Moreover, we have $\mathrm{End}(B) \cong \mathrm{End}(C) \cong \mathrm{End}(T_k) \cong \mathrm{End}(T_k^*)$.

Proof. Any automorphism ψ of the image C of θ gives rise to another element $\psi\theta$ in $\mathrm{Hom}(T_k^*, \tau T_k)$. The uniqueness of θ implies C must be indecomposable. Applying the functors $\mathrm{Hom}(\bar{T}, -)$ and $\mathrm{Hom}(-, \bar{T})$ to the sequence (II.8) we see that $\mathrm{Ext}^1(\bar{T}, C) = \mathrm{Ext}^1(D, \bar{T}) = 0$ and we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}(\bar{T}, D) \longrightarrow \mathrm{Hom}(\bar{T}, T_k^*) \longrightarrow \mathrm{Hom}(\bar{T}, C) \longrightarrow \mathrm{Ext}^1(\bar{T}, D) \longrightarrow 0\,.$$

Applying the same functors to the sequence (II.9) gives $\mathrm{Ext}^1(\bar{T}, A) = \mathrm{Ext}^1(B, \bar{T}) = 0$ and an exact sequence

$$0 \longrightarrow \mathrm{Hom}(A, \bar{T}) \longrightarrow \mathrm{Hom}(T_k, \bar{T}) \longrightarrow \mathrm{Hom}(B, \bar{T}) \longrightarrow \mathrm{Ext}^1(A, \bar{T}) \longrightarrow 0\,.$$

From the Auslander-Reiten formula we see that $\mathrm{Hom}(\bar{T}, \tau B) = \mathrm{Ext}^1(B, \bar{T}) = 0$. Then applying $\mathrm{Hom}(\bar{T}, -)$ to the sequence (II.10) implies $\mathrm{Hom}(\bar{T}, C) = \mathrm{Hom}(\bar{T}, I) = 0$ and again using the Auslander-Reiten formula we get $\mathrm{Hom}(B, \bar{T}) = \mathrm{Ext}^1(\bar{T}, \tau B) = \mathrm{Ext}^1(\bar{T}, C) = 0$. Thus from the Hom-sequences above we get $\mathrm{Ext}^1(\bar{T}, D) = \mathrm{Ext}^1(A, \bar{T}) = 0$.

Applying $\mathrm{Hom}(-, D)$ to the sequences (II.5), where $E \in add(\bar{T})$, and (II.8) shows that $\mathrm{Ext}^1(T_k^*, D) = 0$ and thus $\mathrm{Ext}^1(D, D) = 0$. Again using sequence (II.5) and that $\mathrm{Ext}^1(\bar{T}, C) = 0$ we see $\mathrm{Ext}^1(T_k^*, C) = 0$. Then applying $\mathrm{Hom}(-, C)$ to (II.8) shows $\mathrm{Ext}^1(D, C) = 0$ and finally applying $\mathrm{Hom}(D, -)$ to (II.8) gives $\mathrm{Ext}^1(D, T_k^*) = 0$. Thus we see that D cannot contain T_k as a summand and since (II.8) is non-split D cannot contain T_k^* as a summand. We conclude that $D \in add(\bar{T})$. A similar computation shows that $A \in add(\bar{T})$.

Since $\mathrm{Hom}(B, \bar{T}) = \mathrm{Hom}(\bar{T}, C) = 0$, the Hom-sequences above imply that $T_k \to A$ is a left $add(\bar{T})$-approximation and $D \to T_k^*$ is a right $add(\bar{T})$-approximation. As in the proof of Proposition II.5.1 the injectivity of $D \to T_k^*$ and the surjectivity of $T_k \to A$ imply they are minimal.

Now since $D \in add(\bar{T})$ we have $\mathrm{Hom}(D, C) = 0$ and applying $\mathrm{Hom}(-, C)$ to (II.8) gives $\mathrm{Hom}(C, C) = \mathrm{Hom}(T_k^*, C)$. Since T_k^* is indecomposable it cannot be the middle term of a split sequence. But $\mathrm{Ext}^1(D, T_k^*) = 0$ and so by Theorem II.4.1 any nonzero map from T_k^* to a summand of D must be surjective. So we must have $\mathrm{Hom}(T_k^*, D) = 0$ and applying $\mathrm{Hom}(T_k^*, -)$ again to the sequence (II.8) gives $\mathrm{Hom}(T_k^*, C) = \mathrm{Hom}(T_k^*, T_k^*)$. Similarly one can show that $\mathrm{Hom}(B, A) = 0$ implying $\mathrm{Hom}(B, B) = \mathrm{Hom}(B, T_k)$ and that $\mathrm{Hom}(A, T_k) = 0$ so that $\mathrm{Hom}(B, T_k) = \mathrm{Hom}(T_k, T_k)$. Then Theorem II.4.2 implies $\mathrm{End}(C) \cong \mathrm{End}(T_k^*) \cong \mathrm{End}(T_k) \cong \mathrm{End}(B)$.

Again we note that $\mathrm{Ext}^1(T_k^*, C) = 0$ so that (II.10) implies $\mathrm{Ext}^1(T_k^*, \tau B) = 0$. Then using the Auslander-Reiten formula and (II.9) we see that $\mathrm{Hom}(T_k^*, \tau B) = \mathrm{Ext}^1(B, T_k^*) = \mathrm{Ext}^1(T_k, T_k^*)$, which according to Theorem II.4.2 is equal to $\mathrm{Hom}(T_k^*, T_k^*) = \mathrm{Hom}(T_k^*, C)$. Thus

$$\langle T_k^*, I\rangle = \langle T_k^*, \tau B\rangle - \langle T_k^*, C\rangle = \dim_{\mathbb{F}} \mathrm{Hom}(T_k^*, \tau B) - \dim_{\mathbb{F}} \mathrm{Hom}(T_k^*, C) = 0.$$

Now $\langle -, I\rangle$ is zero on $add(T_k^* \oplus \bar{T})$, but by Theorem II.4.1 the dimension vectors of the indecomposable summands of $T_k^* \oplus \bar{T}$ form a basis of the Grothendieck group $\mathcal{K}(Q_T)$. Thus, since the Ringel-Euler form is non-degenerate, the support of $\mathrm{soc}\, I$ cannot intersect $\mathrm{supp}(T)$.

Finally, since $A, D \in add(\bar{T})$ and $\mathrm{Hom}(\bar{T}, I) = 0$ we see that $\mathrm{Hom}(A, I) = \mathrm{Hom}(D, I) = 0$ and thus the hypotheses of Theorem III.3.4 are satisfied. □

Following Theorem II.4.1 we know that the following elements of $\mathcal{K}(Q)$ form a basis: $\{[T_i]\}_{i\in \mathrm{supp}\, T} \cup \{[P_i]\}_{i\notin \mathrm{supp}\, T}$. We will consider the k^{th} column $\mathbf{b}^k$ of B_T as an element of $\mathcal{K}(Q)$ via $\mathbf{b}^k = \sum\limits_{i\in \mathrm{supp}\, T} b_{ik}[T_i] + \sum\limits_{i\notin \mathrm{supp}\, T} b_{ik}[P_i]$. Then for $k \in \mathrm{supp}(T) = \mathrm{supp}(\bar{T})$ we may define the k^{th} column of B_T via

$$\mathbf{b}^k = [A] + [D] + {}^{*_P}[I] - [E]$$

where we write $\mathbf{e}^{*_P} = \sum\limits_{j\notin \mathrm{supp}\, T} \langle \mathbf{e}, \alpha_j^\vee\rangle [P_j]$ and ${}^{*_P}\mathbf{e} = \sum\limits_{j\notin \mathrm{supp}\, T} \langle \alpha_j^\vee, \mathbf{e}\rangle [P_j]$ for $\mathbf{e} \in \mathcal{K}(Q)$. The k^{th} column of $B_{T_k^*\oplus \bar{T}}$ is by definition $-\mathbf{b}^k$.

We prove the following consistency result which will be necessary to identify B_T as an exchange matrix.

Lemma II.5.3. In the above basis, the elements $[A]+[D]$ and $[E]$ of $\mathcal{K}(Q)$ have disjoint supports.

Proof. We will show that the coefficients of $[T_i]$ in $[A]$ and $[E]$ cannot be simultaneously positive, the same argument will give the result for $[D]$ and $[E]$. We argue for contradiction. Suppose $[T_i]$ appears in both $[A]$ and $[E]$ with positive coefficient. Then we have nonzero maps $\gamma : T_i \to T_k$ and $\zeta : T_k \to T_i$. Since $\mathrm{Ext}^1(T_i, T_k) = \mathrm{Ext}^1(T_k, T_i) = 0$, Theorem II.4.1 implies each of these maps is either a monomorphism or an epimorphism. Notice that this implies one of the compositions $\gamma \circ \zeta$ or $\zeta \circ \gamma$ is nonzero. Since $\mathrm{End}(T_i) \cong \mathbb{F}_{d_i}$ and $\mathrm{End}(T_k) \cong \mathbb{F}_{d_k}$, this implies the nonzero composition is an isomorphism. But then both maps are injective and surjective, i.e. $T_i \cong T_k$, a contradiction. □

Now consider the situation where $\bar{T}$ is a local tilting representation with $k \notin \mathrm{supp}(\bar{T})$.

Write T_k^* for the unique complement of $\bar{T}$. Let I_k denote the injective hull of the simple S_k and write $P_k = \nu^{-1}(I_k)$ for the corresponding projective representation.

Lemma II.5.4. [H1, Proposition 26] There exists a unique nonzero morphism $P_k \to T_k^*$ and a unique morphism $T_k^* \to I_k$.

Proof. Let $\bar{P}_k$ denote the projective representation of $Q_{T_k^* \oplus \bar{T}}$ associated to vertex k. Note that there exists a unique morphism $P_k \to \bar{P}_k$ and since T_k^* has support on $Q_{T_k^* \oplus \bar{T}}$, any morphism $P_k \to T_k^*$ factors through a morphism $\bar{P}_k \to T_k^*$. Now recall that $k \notin \text{supp}(\bar{T})$ and thus $\text{Hom}(\bar{P}_k, \bar{T}) = 0$. Since $T_k^* \oplus \bar{T}$ is a local tilting representation we have a coresolution

$$0 \longrightarrow \bar{P}_k \longrightarrow (T_k^*)^s \longrightarrow F \longrightarrow 0\,,$$

where $s \geq 1$ and $F \in add(\bar{T})$. Since $\text{Hom}(\bar{P}_k, \bar{P}_k)$ is a field, applying the functor $\text{Hom}(\bar{P}_k, -)$ to this coresolution shows $s = 1$ and $\text{Hom}(\bar{P}_k, \bar{P}_k) = \text{Hom}(\bar{P}_k, T_k^*)$, in particular there is a unique (up to scalar) nonzero morphism $\bar{P}_k \to T_k^*$. Composing with the unique morphism $P_k \to \bar{P}_k$ completes the claim. The analogous claim for I_k is dual. □

Lemma II.5.5. [H1, Proposition 26] The endomorphism rings $\text{End}(P_k)$, $\text{End}(\bar{P}_k)$, $\text{End}(T_k^*)$, $\text{End}(\bar{I}_k)$, and $\text{End}(I_k)$ are all isomorphic.

Proof. Since $C \in add(\bar{T})$ we know $\text{Ext}^1(T_k^*, F) = 0$ and according to Theorem II.4.1 any nonzero map from an indecomposable summand of F to T_k^* is injective. But T_k^* is indecomposable and so all such maps must be zero, in particular $\text{Hom}(F, T_k^*) = 0$. Thus applying $\text{Hom}(-, T_k^*)$ to the sequence

$$0 \longrightarrow \bar{P}_k \longrightarrow T_k^* \longrightarrow F \longrightarrow 0$$

gives $\text{Hom}(\bar{P}_k, \bar{P}_k) = \text{Hom}(\bar{P}_k, T_k^*) = \text{Hom}(T_k^*, T_k^*)$.

Since P_k and $\bar{P}_k$ are both projective covers of the simple S_k, albeit as modules over different algebras, they have isomorphic endomorphism rings, i.e. $\text{End}(P_k) \cong \text{End}(S_k) \cong \text{End}(\bar{P}_k)$. The same argument holds for I_k and $\bar{I}_k$ regarding them both as injective envelopes of S_k. □

As in the proof of Lemma II.5.4, write F for the cokernel of the unique map $P_k \to T_k^*$ and let P' denote the kernel. Similarly write G for the kernel of the unique morphism $T_k^* \to I_k$ and let I' be the cokernel.

Proposition II.5.6. [H1, Proposition 26] The map $T_k^* \to F$ is a minimal left $add(\bar{T})$-approximation of T_k^* and $G \to T_k^*$ is a minimal right $add(\bar{T})$-approximation of T_k^*. Moreover, we have $\operatorname{Hom}(P', F) = \operatorname{Hom}(G, I') = 0$ and the supports of $\operatorname{soc} I'$ and $P'/\operatorname{rad} P'$ are disjoint.

Proof. Since $k \notin \operatorname{supp}(\bar{T})$, we have $\operatorname{Hom}(\bar{P}_k, \bar{T}) = \operatorname{Hom}(\bar{T}, \bar{I}_k) = 0$. Then applying $\operatorname{Hom}(-, \bar{T})$ and $\operatorname{Hom}(\bar{T}, -)$ respectively to the defining exact sequences of $T_k^* \to F$ and $G \to T_k^*$ we see that $T_k^* \to F$ is a left $add(\bar{T})$-approximation and $G \to T_k^*$ is a right $add(\bar{T})$-approximation. As in the proof of Proposition II.5.1, these are minimal since $T_k^* \to F$ is surjective and $G \to T_k^*$ is injective.

Since $\bar{P}_k = P_k/P'$ and $I' = I_k/\bar{I}_k$, neither P' nor I' contains vertex k in its support. Recall that P_k can be described in terms of paths beginning at vertex k and I_k can be described in terms of paths ending at vertex k. Thus Q being acyclic implies $[P'/\operatorname{rad} P']$ and $[\operatorname{soc} I']$ have disjoint support.

Now since $k \notin \operatorname{supp}(\bar{T})$ we have $\operatorname{Hom}(P_k, \bar{T}) = \operatorname{Hom}(P', \bar{T}) = \operatorname{Hom}(P', F) = 0$ and $\operatorname{Hom}(\bar{T}, I_k) = \operatorname{Hom}(\bar{T}, I') = \operatorname{Hom}(D, I') = 0$. Thus the hypotheses of Theorem III.3.7 are satisfied. □

As above we will consider the k^{th} column $\mathbf{b}^k$ of B_T as an element of $\mathcal{K}(Q)$. Now when $\bar{T}$ is a local tilting representation with $k \notin \operatorname{supp}(\bar{T})$ we may define the k^{th} column of B_T via

$$\mathbf{b}^k = [F] + [P']^{*_P} - [G] - {}^{*_P}[I'].$$

By definition the k^{th} column of $B_{\bar{T}}$ is $-\mathbf{b}^k$. Using a similar argument as in the proof of Lemma II.5.3 we get the following consistency result.

Lemma II.5.7. Written in the basis $\{[T_i]\}_{i \in \operatorname{supp} T} \cup \{[P_i]\}_{i \notin \operatorname{supp} T}$, the elements $[F]$ and $[G]$ of $\mathcal{K}(Q)$ have disjoint supports.

This completes the definition of the matrix B_T associated to a local tilting representation T. We will further investigate these matrices and their relationship to the mutation of local tilting representations in Section V.1.

CHAPTER III

QUANTUM CLUSTER CHARACTERS

In this chapter we introduce our main objects of study and present necessary background material. Section III.1. recalls definitions and results related to quantum cluster algebras. We define the quantum cluster character in Section III.2. In Section III.3. we prove multiplication theorems for quantum cluster characters and we investigate their quasi-commutation in Section III.4. These will be important in showing that local tilting representations correspond to quantum seeds.

III.1. Quantum Cluster Algebras

In this section we will define quantum cluster algebras and recall some important structure theorems which motivate the main results of this dissertation. We begin with an overview of the construction before getting into the technical details.

As with classical cluster algebras, the generators and relations of a quantum cluster algebra are not given explicitly but rather are constructed inductively. To start the process we need a quasi-commuting collection of initial variables $\mathbf{X} = \{X_1, \ldots, X_m\}$ which we call the "initial cluster". We will record the quasi-commutation via a skew-symmetric matrix $\Lambda = (\lambda_{ij})$ given by $X_i X_j = q^{\lambda_{ij}} X_j X_i$. Fix $n \leq m$ and call this the "rank" of the quantum cluster algebra. For $1 \leq k \leq n$ we may swap out X_k for a new variable X'_k. These "cluster variables" are related by binomial "exchange relations" (III.2) which may be recorded in an $m \times n$ "exchange matrix" $\tilde{B}$ with skew-symmetrizable principal $n \times n$ submatrix B. The remaining variables X_k, $n+1 \leq k \leq m$, are constant across the clusters and we call them "coefficients". We insist that the new cluster $\mu_k \mathbf{X} = \{X_1, \ldots, \hat{X}_k, \ldots, X_m\} \cup \{X'_k\}$ also be quasi-commuting and this forces a compatibility between the commutation matrix Λ and the exchange matrix $\tilde{B}$.

Definition III.1.1 (Compatible Pair, [BZ]). Call a pair $(\tilde{B}, \Lambda)$ *compatible* if $\tilde{B}^t \Lambda = \begin{pmatrix} D & 0 \end{pmatrix}$ where D is an $n \times n$ positive diagonal matrix so that DB is skew-symmetric.

When $(\tilde{B}, \Lambda)$ is compatible we call the collection $\Sigma = (\mathbf{X}, \tilde{B}, \Lambda)$ a " quantum seed".

Write $\mu_k \Lambda$ for the commutation matrix of the new cluster $\mu_k \mathbf{X}$, we will give an explicit formula (III.5) for $\mu_k \Lambda$ below. We also define a mutated exchange matrix $\mu_k \tilde{B}$ given by equation (III.3). Then $\mu_k \Sigma = (\mu_k \mathbf{X}, \mu_k \tilde{B}, \mu_k \Lambda)$ is again a seed which we call the mutation of Σ in direction k. We may then think of $\mu_k \Sigma$ as our initial seed and iterate the process. We visualize the seed mutations via a rooted n-regular tree $\mathbb{T}$. The n edges emanating from a given vertex will be labeled by the set $\{1, \ldots, n\}$. We then label each vertex $t \in \mathbb{T}$ by a seed Σ_t, where the root vertex t_0 is labeled by the initial cluster $(\mathbf{X}, \tilde{B}, \Lambda)$. The labeling of the vertices is not arbitrary but rather we require that $\Sigma_{t'} = \mu_k \Sigma_t$ whenever $t \overset{k}{\text{———}} t'$ in $\mathbb{T}$. The "quantum cluster algebra" $\mathcal{A}_q(\tilde{B}, \Lambda)$ is the algebra generated by all of the cluster variables from all seeds Σ_t subject to the exchange relations.

A result of Fomin and Zelevinsky [FZ4] asserts that the cluster variables of $\mathcal{A}_q(\tilde{B}, \Lambda)$ are completely determined by the cluster variables of the principal coefficients quantum cluster algebra $\mathcal{A}(\tilde{B}_P, \Lambda')$ where $\tilde{B}_P = \begin{pmatrix} B \\ I \end{pmatrix}$ with I the $n \times n$ identity matrix and Λ' is a compatible commutation matrix. Thus we will always assume we are dealing with an exchange matrix having principal coefficients.

Now we will fill in the details. Let q be an indeterminate. The initial cluster of our quantum cluster algebra will form a generating set for an m-dimensional quantum torus

$$\mathcal{T}_{\Lambda,q} = \mathbb{Z}\left[q^{\pm\frac{1}{2}}\right]\left\langle X_1^{\pm 1}, \ldots, X_m^{\pm 1} : X_i X_j = q^{\lambda_{ij}} X_j X_i \right\rangle.$$

The quantum torus has a unique anti-involution, called the "bar-involution", which fixes each X_i and sends $q \mapsto q^{-1}$. Let $\alpha_1, \ldots, \alpha_m$ be the standard basis vectors of $\mathbb{Z}^m$. For each $\mathbf{c} \in \mathbb{Z}^m$, write $\mathbf{c} = \sum_{i=1}^{m} c_i \alpha_i$ and define bar-invariant monomials

$$X^{\mathbf{c}} = q^{-\frac{1}{2}\sum_{i<j} \lambda_{ij} c_i c_j} X_1^{c_1} \cdots X_m^{c_m}.$$

Denoting by $\Lambda(\cdot,\cdot) : \mathbb{Z}^m \times \mathbb{Z}^m \to \mathbb{Z}$ the skew-symmetric bilinear form associated to Λ, we have

$$X^{\mathbf{b}} X^{\mathbf{c}} = q^{\frac{1}{2}\Lambda(\mathbf{b},\mathbf{c})} X^{\mathbf{b}+\mathbf{c}}. \tag{III.1}$$

In Section III.3. we will use a twisted multiplication $* : \mathcal{T}_{\Lambda,q} \times \mathcal{T}_{\Lambda,q} \to \mathcal{T}_{\Lambda,q}$ given by the rule

$$X^{\mathbf{b}} * X^{\mathbf{c}} = X^{\mathbf{b}+\mathbf{c}}.$$

Note that $\mathcal{T}_{\Lambda,q}$ is an Ore-domain, i.e. one has a way to convert between right fractions and left fractions, so we may consider its skew-field of fractions $\mathcal{F}$ where the multiplication is well-defined via this Ore condition. The quantum cluster algebra will be a subalgebra of $\mathcal{F}$.

We need one more piece of notation before we can extract the exchange relations from the exchange matrix $\tilde{B}$. Write $\mathbf{b}_+^k = \sum_{i:b_{ik}>0} b_{ik}\alpha_i$ and $\mathbf{b}_-^k = \sum_{i:b_{ik}<0} -b_{ik}\alpha_i$ for the positive and negative entries of the k^{th} column of $\tilde{B}$, that is if we think of the k^{th} column $\mathbf{b}^k$ of $\tilde{B}$ as an element of $\mathbb{Z}^m$, then $\mathbf{b}^k = \mathbf{b}_+^k - \mathbf{b}_-^k$. Now the mutation of the cluster $\mathbf{X} = \{X_1, \ldots, X_m\}$ in direction k is given by $\mu_k \mathbf{X} = \{X_1, \ldots, \hat{X}_k, \ldots, X_m\} \cup \{X_k'\}$ where

$$X_k' = X^{\mathbf{b}_+^k - \alpha_k} + X^{\mathbf{b}_-^k - \alpha_k} \in \mathcal{F}. \tag{III.2}$$

Note that X_k' has denominator X_k and is bar-invariant.

The Fomin-Zelevinsky [FZ1] mutation of exchange matrices is given by $\mu_k B = (b_{ij}')$ where

$$b_{ij}' = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik}[-b_{kj}]_+ & \text{otherwise}; \end{cases} \tag{III.3}$$

where $[b]_+ = max(0, b)$. We may also describe the mutation of $\tilde{B}$ in direction k via $\mu_k \tilde{B} = E\tilde{B}F$ with $m \times m$ matrix $E = (e_{ij})$ and $n \times n$ matrix $F = (f_{ij})$ given by

$$e_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k; \\ -1 & \text{if } i = j = k; \\ [-b_{ik}]_+ & \text{if } i \neq j = k; \end{cases} \qquad f_{ij} = \begin{cases} \delta_{ij} & \text{if } i \neq k; \\ -1 & \text{if } i = j = k; \\ [b_{kj}]_+ & \text{if } i = k \neq j. \end{cases} \tag{III.4}$$

Then we may compute the commutation matrix $\mu_k\Lambda$ of the cluster $\mu_k\mathbf{X}$ as

$$\mu_k\Lambda = E^t\Lambda E. \tag{III.5}$$

Proposition III.1.2. [BZ] The pair $(\mu_k\tilde{B}, \mu_k\Lambda)$ is compatible.

Thus we see that $\mu_k\Sigma = (\mu_k\mathbf{X}, \mu_k\tilde{B}, \mu_k\Lambda)$ is again a quantum seed and we may label the n-regular tree $\mathbb{T}$ as described above.

Definition III.1.3. The *quantum cluster algebra* $\mathcal{A}_q(\tilde{B}, \Lambda)$ is the $\mathbb{Z}\left[q^{\pm\frac{1}{2}}\right]$-subalgebra of $\mathcal{F}$ generated by the cluster variables from all seeds Σ_t associated to vertices $t \in \mathbb{T}$.

Here we considered mutations as an iterative process occurring inside the skew-field $\mathcal{F}$, we will call these "internal mutations". We investigate the relationship between internal mutations and representations of valued quivers in Chapter V.

It is also interesting to consider an "external mutation" relating cluster algebras with neighboring initial seeds in the exchange graph. Let $\mathcal{F}'$ denote the skew-field of fractions of the quantum torus $\mathcal{T}_{\mu_k\Lambda,q}$ and for $\mathbf{c} \in \mathbb{Z}$ write $(X')^{\mathbf{c}}$ for the bar-invariant monomials in $\mathcal{T}_{\mu_k\Lambda,q}$. Then the external mutations μ_k take the form of bi-rational isomorphisms

$$\begin{aligned} \mu_k \ : \ \mathcal{F} &\longleftrightarrow \mathcal{F}' \ : \ \mu_k \\ X_k &\longmapsto (X')^{\mathbf{b}_+^k - \alpha_k} + (X')^{\mathbf{b}_-^k - \alpha_k} \\ X^{\mathbf{b}_+^k - \alpha_k} + X^{\mathbf{b}_-^k - \alpha_k} &\longleftarrow\!\shortmid X'_k \,. \end{aligned}$$

The following Lemma is immediate but essential.

Lemma III.1.4. For any cluster variable $X \in \mathcal{A}_q(\tilde{B}, \Lambda)$ and $1 \leq k \leq n$, $\mu_k(X)$ is a cluster variable of $\mathcal{A}_q(\mu_k\tilde{B}, \mu_k\Lambda)$.

In Chapter IV. we will need to know the image under the external mutation μ_k of the bar-invariant monomials $(X')^{\mathbf{c}}$. For $n, r \in \mathbb{Z}$, $r \geq 0$, we define the symmetrized quantum binomial coefficients $\begin{bmatrix} n \\ r \end{bmatrix}_q := \frac{(q^n - q^{-n})(q^{n-1} - q^{-n+1})\cdots(q^{n-r+1} - q^{-n+r-1})}{(q^r - q^{-r})\cdots(q - q^{-1})}$, where we note that $\begin{bmatrix} n \\ r \end{bmatrix}_q = 0$ if $r > n \geq 0$. By convention we take $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1$.

Proposition III.1.5. [BZ, Proposition 4.7] Let $\mathbf{c} = \sum_{i=1}^{m} c_i \alpha_i \in \mathbb{Z}^m$. Then we have

$$\mu_k\left((X')^{\mathbf{c}}\right) = \sum_{p \geq 0} \begin{bmatrix} c_k \\ p \end{bmatrix}_{q^{d_k/2}} X^{E\mathbf{c}+p\mathbf{b}^k}. \tag{III.6}$$

Recall that the external mutation μ_k is a map from the skew-field $\mathcal{F}$ containing the quantum cluster algebra $\mathcal{A}(\tilde{B}, \Lambda)$ to the skew-field $\mathcal{F}'$ containing the quantum cluster algebra $\mathcal{A}_q(\mu_k\tilde{B}, \mu_k\Lambda)$. When vertex k is a sink or a source, the valued quiver associated to the pair $(\mu_k\tilde{B}, \mu_k\Lambda)$ is exactly the reflected valued quiver $(\mu_k Q, \mathbf{d})$ defined in Section II.3. Also recall that the reflection functors $\Sigma_k^{\pm}$ associated to a sink or source vertex k convert between representations of $(Q, \mathbf{d})$ and representations of $(\mu_k Q, \mathbf{d})$. In Chapter IV. we show that this similarity is not a mere coincidence but that there is in fact a relationship between external mutations and reflection functors.

The following Theorem of Berenstein and Zelevinsky [BZ] is the first important structural result concerning quantum cluster algebras.

Theorem III.1.6. [BZ, Quantum Laurent Phenomenon] The quantum cluster algebra $\mathcal{A}_q(\tilde{B}, \Lambda)$ is a subalgebra of $\mathcal{T}_{\Lambda_t,q}$ for each vertex $t \in \mathbb{T}$.

This says that although the cluster variables a priori are rational functions in the variables from any seed Σ_t, cancellations inevitably occur and we actually get Laurent polynomials. In particular, each cluster variable is Laurent in the initial cluster. Although the Quantum Laurent Phenomenon guarantees that each cluster monomial is an element of $\mathcal{T}_{\Lambda,q}$ it is a non-trivial task to compute their initial cluster expansions.

We define a valued quiver $\tilde{Q}$ from a compatible pair $(\tilde{B}, \Lambda)$ with principal coefficients as follows. According to the construction in Section II.1. we may associated a valued quiver $(Q, \mathbf{d})$ to the skew-symmetrizable principal $n \times n$ submatrix B of $\tilde{B}$ where the valuation d_i is the i^{th} diagonal entry of the matrix D occurring in the compatibility condition for $(\tilde{B}, \Lambda)$. Then we attach principal vertices $n + i \to i$ for each $1 \leq i \leq n$ and set $d_{n+i} = d_i$ to obtain $\tilde{Q}$. We will only be considering valued representations of $\tilde{Q}$ which are supported on Q and thus we will only refer to the quiver Q in the discussions that follow, however implicitly all equations/constructions involving the Grothendieck group $\mathcal{K}(Q)$ or local tilting representations are happening inside $\mathrm{Rep}_{\mathbb{F}}(\tilde{Q}, \mathbf{d})$. The main result of this dissertation is a description of the acyclic initial cluster expansion of all cluster variables of $\mathcal{A}(\tilde{B},\Lambda)$ using the representation theory of the valued quiver $(Q,\mathbf{d})$. We present the construction in the following section.

Example III.1.7 (Type B_2)**.** Let $\tilde{B} = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$ with compatible $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $D = \tilde{B}^t\Lambda = diag(1,2)$. Then $\mathcal{T}_{\Lambda,q} = \mathbb{Z}[q^{\pm\frac{1}{2}}]\langle X_1^{\pm 1}, X_2^{\pm 1} : X_1X_2 = qX_2X_1\rangle$. The cluster variables of $\mathcal{A}_q(\tilde{B},\Lambda)$ can be labeled X_k, $k \in \mathbb{Z}$, defined recursively by

$$X_{m-1}X_{m+1} = \begin{cases} qX_m^2 + 1 & \text{if } m \text{ is odd} \\ q^{1/2}X_m + 1 & \text{if } m \text{ is even.} \end{cases} \tag{III.7}$$

In Figure III.1.1 we show the exchange graph with vertices labeled by seeds and edges labeled by mutations. Note the similarity to the mutation graph of local tilting representations from Example II.4.4. The cluster variables X_k are given below:

$$X_3 = X^{-\alpha_1+\alpha_2} + X^{-\alpha_1};$$

$$X_4 = X^{-2\alpha_1+\alpha_2} + X^{-\alpha_2} + X^{-2\alpha_1-\alpha_2} + (q^{1/2} + q^{-1/2})X^{-2\alpha_1};$$

$$X_5 = X^{-\alpha_1} + X^{\alpha_1-\alpha_2} + X^{-\alpha_1-\alpha_2};$$

$$X_6 = X^{2\alpha_1-\alpha_2} + X^{-\alpha_2};$$

$$X_7 = X^{\alpha_1} = X_1;$$

$$X_8 = X^{\alpha_2} = X_2;$$

$$X_k = X_{k-6} \text{ for all } k \in \mathbb{Z}.$$

□

III.2. The Quantum Cluster Character X_V

In this section we define the quantum cluster character assigning an element of the quantum torus $\mathcal{T}_{\Lambda,q}$ to each representation V of $(Q,\mathbf{d})$. We will abbreviate $q = |\mathbb{F}|$.

Definition III.2.1. For $V \in \mathrm{Rep}_{\mathbb{F}}(Q,\mathbf{d})$ define the *quantum cluster character* X_V in the quantum

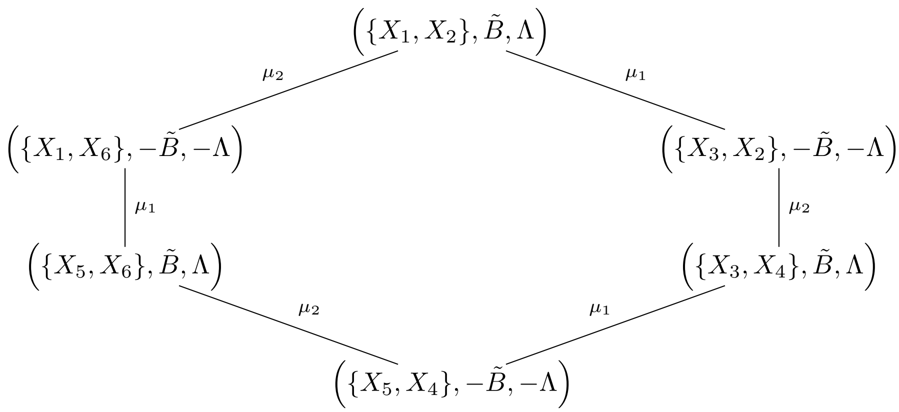

Figure III.1.1: Exchange graph for the quantum cluster algebra of Type B_2.

torus $\mathcal{T}_{\Lambda,q}$ by

$$X_V = \sum_{\mathbf{e} \in \mathcal{K}(Q)} q^{-\frac{1}{2}\langle \mathbf{e}, \mathbf{v}-\mathbf{e}\rangle} |Gr_{\mathbf{e}}(V)| X^{-\mathbf{e}^* - {}^*(\mathbf{v}-\mathbf{e})} \tag{III.8}$$

where $Gr_{\mathbf{e}}(V) = \{U \subset V : [U] = \mathbf{e}\}$ is the Grassmannian of all subobjects of V with dimension vector $\mathbf{e}$.

Example III.2.2 (Type B_2)**.** We keep the notation from Examples II.1.2 and III.1.7.

The representation S_1 has unique subrepresentations with dimension vectors α_1 and 0. So applying the quantum cluster character to S_1 gives:

$$X_{S_1} = X^{-\alpha_1+\alpha_2} + X^{-\alpha_1} = X_3.$$

The representation I_2 has unique subrepresentations with dimension vectors $2\alpha_1 + \alpha_2$, 0 and α_2, and it has $1 + q$ subrepresentations with dimension vector $\alpha_1 + \alpha_2$. So applying the quantum cluster character to I_2 gives:

$$X_{I_2} = X^{-2\alpha_1+\alpha_2} + X^{-\alpha_2} + X^{-2\alpha_1-\alpha_2} + (q^{1/2} + q^{-1/2})X^{-2\alpha_1} = X_4.$$

The representation P_1 has unique subrepresentations with dimension vectors $\alpha_1 + \alpha_2$, 0 and α_2.

So applying the quantum cluster character to P_1 gives:

$$X_{P_1} = X^{-\alpha_1} + X^{\alpha_1-\alpha_2} + X^{-\alpha_1-\alpha_2} = X_5.$$

The representation S_2 has unique subrepresentations with dimension vectors 0 and α_2. So applying the quantum cluster character to S_2 gives:

$$X_{S_2} = X^{2\alpha_1-\alpha_2} + X^{-\alpha_2} = X_6.$$

Notice how these relate the mutation graph of local tilting representations from Example II.4.4 and the exchange graph from Example III.1.7. □

III.3. Quantum Cluster Character Multiplication Theorems

In this section we prove multiplication formulas for products of quantum cluster characters. Our first result is analogous to [H1, Theorem 12], [Q, Proposition 5.4.1], and [DX, Theorem 3.5]. All of their proofs are modeled on that of [H1, Theorem 12] using Hall numbers, we will follow this approach as well.

Define the "Hall number" F^D_{BC} as the number of subobjects $U \subset D$ with $U \cong C$ and $D/U \cong B$. We also write ε^D_{BC} for the size of the "restricted Ext-space" $\mathrm{Ext}^1(B,C)_D \subset \mathrm{Ext}^1(B,C)$ consisting of those short exact sequences with middle term isomorphic to D. Note that F^D_{BC} and ε^D_{BC} are finite since B, C, and D are finite sets. There are two well-known formulas satisfied by these quantities. The first one verifies the associativity of the multiplication in a "Hall algebra" where the Hall numbers are structure constants.

Lemma III.3.1. For any valued representations B, K, L, and V we have the following "associativity" relation for Hall numbers:

$$\sum_A F^A_{KL} F^V_{AB} = \sum_{A'} F^V_{KA'} F^{A'}_{LB}. \tag{III.9}$$

The second equation, known as "Green's formula", verifies the compatibility of multiplication and comultiplication of the Ringel-Hall algebra. To state Green's formula we introduce the following useful notation:

$$[V,W]^0 := \dim_{\mathbb{F}} \operatorname{Hom}(V,W) \quad \text{and} \quad [V,W]^1 := \dim_{\mathbb{F}} \operatorname{Ext}^1(V,W), \quad \text{for all } V,W \in \operatorname{Rep}_{\mathbb{F}}(Q,\mathbf{d}).$$

Lemma III.3.2. For any valued representations V, W, X, and Y

$$\sum_{E} \varepsilon^{E}_{VW} F^{E}_{XY} = \sum_{A,B,C,D} q^{[V,W]^0-[A,C]^0-[B,D]^0-\langle \mathbf{a},\mathbf{d}\rangle} F^{V}_{AB} F^{W}_{CD} \varepsilon^{X}_{AC} \varepsilon^{Y}_{BD}. \tag{III.10}$$

Let V and W be representations of $(Q,\mathbf{d})$. From a morphism $\theta : W \to \tau V$ we get an exact sequence

$$0 \longrightarrow D \longrightarrow W \xrightarrow{\theta} \tau V \longrightarrow \tau A \oplus I \longrightarrow 0 \tag{III.11}$$

where $D = \ker\theta$, $\tau A \oplus I = \operatorname{coker}\theta$, I is injective, and A and V have the same maximal projective summand $P_A = P_V$. The following notation will be useful in the proof of the theorem:

$$\operatorname{Hom}(W,\tau V)_{DAI} = \{W \xrightarrow{f} \tau V : f \neq 0, \ker f \cong D, \operatorname{coker} f \cong \tau A \oplus I\}.$$

The following Lemma was given in [H1], we reproduce the proof for the convenience of the reader. For a valued representation C we write a_C for the size of the automorphism group of C.

Lemma III.3.3. [H1, Lemma 15] The size of the restricted Hom-space $\operatorname{Hom}(W,\tau V)_{DAI}$ can be computed in terms of Hall numbers, in particular we have

$$|\operatorname{Hom}(W,\tau V)_{DAI}| = \sum_{B,C} a_C F^{W}_{CD} F^{V}_{AB} F^{\tau B}_{IC},$$

where B contains no projective summands.

Proof. Define

$$\mathcal{P}^{Z}_{XY} = \{(s,t) : \; 0 \longrightarrow Y \xrightarrow{s} Z \xrightarrow{t} X \longrightarrow 0 \; \text{ is exact}\}$$

and write $P^{Z}_{XY} = |\mathcal{P}^{Z}_{XY}|$. It is well known that the Hall number F^{Z}_{XY} can be computed as $P^{Z}_{XY}/a_X a_Y$.

For any $\theta \in \mathrm{Hom}(W, \tau V)_{DAI}$ the exact sequence (III.11) can be split into two short exact sequences:

$$0 \longrightarrow D \xrightarrow{d} W \xrightarrow{c} C \longrightarrow 0,$$
$$0 \longrightarrow C \xrightarrow{g} \tau V \xrightarrow{f} \tau A \oplus I \longrightarrow 0, \tag{III.12}$$

such that $gc = \theta$. Thus we get a surjective map $\bigsqcup_C \mathcal{P}^W_{CD} \times \mathcal{P}^{\tau V}_{\tau A \oplus IC} \to \mathrm{Hom}(W, \tau V)_{DAI}$ with fiber over θ isomorphic to $\mathrm{Aut}(D) \times \mathrm{Aut}(C) \times \mathrm{Aut}(\tau A \oplus I)$. Since the fibers are all isomorphic for a fixed C we see that

$$|\mathrm{Hom}(W, \tau V)_{DAI}| = \sum_C P^W_{CD} P^{\tau V}_{\tau A \oplus IC} / a_{\tau A \oplus I} a_C a_D = \sum_C a_C F^W_{CD} F^{\tau V}_{\tau A \oplus IC}.$$

From the surjective map $f : \tau V \to \tau A \oplus I$ we get a surjective map $\varphi : \tau V \to \tau A$. Note that the kernel of this map contains no injective summands since τA cannot have injective summands. Thus we may write $\ker \varphi = \tau B$ for some B containing no projective summands. Since B contains no projective summands, $\mathrm{Hom}(B, P_V) = 0$ and applying the inverse Auslander-Reiten translation τ^{-1} induces a surjective map $V \to A$ with kernel B, where we have used that $P_V = P_A$. Since $\mathrm{im}\, g = \ker f$, we must have $\mathrm{im}\, g \subset \ker \varphi$, in particular g defines an injective map into τB. The discussion thus far can be described by the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & \tau V & \longrightarrow & \tau A \oplus I & \longrightarrow & 0 \\ & & \downarrow g & & \downarrow \mathrm{Id} & & \downarrow \pi_1 & & \\ 0 & \longrightarrow & \tau B & \longrightarrow & \tau V & \xrightarrow{\varphi} & \tau A & \longrightarrow & 0\,. \end{array}$$

Notice that $\tau A \oplus I \cong \tau V / C$ and $\tau A \cong \tau V / \tau B$ so that $I = \ker \pi_1 \cong \tau B / C = \mathrm{coker}\, g$. From this we see that the second sequence in (III.12) above is equivalent to the following exact sequences:

$$0 \longrightarrow B \longrightarrow V \longrightarrow A \longrightarrow 0\,,$$
$$0 \longrightarrow C \longrightarrow \tau B \longrightarrow I \longrightarrow 0\,,$$

and so we have

$$F^{\tau V}_{\tau A \oplus IC} = \sum_B F^V_{AB} F^{\tau B}_{IC}\,.$$

The result follows. ∎

Note that by the Auslander-Reiten formula we have $\mathrm{Ext}^1(V,W) \cong \mathrm{Hom}(W,\tau V)$.

Theorem III.3.4. Assume V and W are representations of $(Q,\mathbf{d})$ with a unique (up to scalar) non-trivial extension $E \in \mathrm{Ext}^1(V,W)$, in particular $\dim_{\mathrm{End}(V)} \mathrm{Ext}^1(V,W) = 1$. Let $\theta \in \mathrm{Hom}(W,\tau V)$ be the equivalent morphism with A, D, I as above. Furthermore assume that $\mathrm{Hom}(A \oplus D, I) = 0 = \mathrm{Ext}^1(A,D)$. Then we have the following multiplication formula:

$$X_V X_W = q^{\frac{1}{2}\Lambda({}^*\mathbf{v},{}^*\mathbf{w})} X_E + q^{\frac{1}{2}\Lambda({}^*\mathbf{v},{}^*\mathbf{w})+\frac{1}{2}\langle \mathbf{v},\mathbf{w}\rangle - \frac{1}{2}\langle \mathbf{a},\mathbf{d}\rangle} X_{D\oplus A} * X^{{}^*\mathbf{i}}. \tag{III.13}$$

When $\dim \mathrm{Ext}^1(V,W) > 1$, there exist similar multiplication formulas with more terms, see [F] and [DS].

Proof. Note that we have

$$|Gr_{\mathbf{e}}(V)| = \sum_{B,C:[C]=\mathbf{e}} F^V_{BC}$$

and thus we may rewrite the quantum cluster character as

$$X_V = \sum_{A,B} q^{-\frac{1}{2}\langle \mathbf{b},\mathbf{a}\rangle} F^V_{AB} X^{-\mathbf{b}^* - {}^*\mathbf{a}}.$$

Then using Lemma II.1.3.3 the product of the quantum cluster characters X_V and X_W becomes:

$$\begin{aligned}
X_V X_W &= \sum_{A,B} q^{-\frac{1}{2}\langle \mathbf{b},\mathbf{a}\rangle} F^V_{AB} X^{-\mathbf{b}^*-{}^*\mathbf{a}} \sum_{C,D} q^{-\frac{1}{2}\langle \mathbf{d},\mathbf{c}\rangle} F^W_{CD} X^{-\mathbf{d}^*-{}^*\mathbf{c}} \\
&= \sum_{A,B,C,D} F^V_{AB} F^W_{CD} q^{-\frac{1}{2}\langle \mathbf{b},\mathbf{a}\rangle} q^{-\frac{1}{2}\langle \mathbf{d},\mathbf{c}\rangle} q^{\frac{1}{2}\Lambda(-\mathbf{b}^*-{}^*\mathbf{a},-\mathbf{d}^*-{}^*\mathbf{c})} X^{-(\mathbf{b}+\mathbf{d})^*-{}^*(\mathbf{a}+\mathbf{c})} \\
&= q^{\frac{1}{2}\Lambda({}^*\mathbf{v},{}^*\mathbf{w})} \sum_{A,B,C,D} F^V_{AB} F^W_{CD} q^{\langle \mathbf{b},\mathbf{c}\rangle} q^{-\frac{1}{2}\langle \mathbf{b}+\mathbf{d},\mathbf{a}+\mathbf{c}\rangle} X^{-(\mathbf{b}+\mathbf{d})^*-{}^*(\mathbf{a}+\mathbf{c})}.
\end{aligned}$$

Our goal is to show that this equals the right hand side of equation (III.13). We accomplish this by cleverly rewriting each term on the right. We begin with the following definitions:

$$\begin{aligned}
\sigma_1 &:= \sum_{E \ncong V\oplus W} \frac{\varepsilon^E_{VW}}{q^{[V,V]^0}-1} X_E, \\
\sigma_2 &:= \sum_{\substack{D,A,I \\ D\ncong W}} \frac{|\mathrm{Hom}(W,\tau V)_{DAI}|}{q^{[V,V]^0}-1} q^{-\frac{1}{2}\langle \mathbf{a},\mathbf{d}\rangle} X_{D\oplus A} * X^{{}^*\mathbf{i}}.
\end{aligned}$$

Since there is a unique nontrivial extension $E \in \operatorname{Ext}^1(V,W)$ and a corresponding unique nonzero morphism $\theta \in \operatorname{Hom}(W,\tau V)$, we see that both of these sums collapse to a single term. Since $\dim_{\operatorname{End}(V)} \operatorname{Ext}^1(V,W) = 1$, we have $\varepsilon_{VW}^E = |\operatorname{Hom}(W,\tau V)_{DAI}| = q^{[V,V]^0} - 1$ and thus the right hand side of equation (III.13) may be written as $q^{\frac{1}{2}\Lambda({}^*\mathbf{v},{}^*\mathbf{w})}\sigma_1 + q^{\frac{1}{2}\Lambda({}^*\mathbf{v},{}^*\mathbf{w})+\frac{1}{2}\langle\mathbf{v},\mathbf{w}\rangle}\sigma_2$. Observe that our computation of $X_V X_W$ above combined with the following Proposition complete the proof.

Proposition III.3.5. We may rewrite σ_1 and σ_2 as

$$\sigma_1 = \sum_{A,B,C,D} \frac{q^{[V,W]^1} - q^{[B,C]^1}}{q^{[V,V]^0} - 1} q^{\langle\mathbf{b},\mathbf{c}\rangle} q^{-\frac{1}{2}\langle\mathbf{b}+\mathbf{d},\mathbf{a}+\mathbf{c}\rangle} F_{AB}^V F_{CD}^W X^{-(\mathbf{b}+\mathbf{d})^* - {}^*(\mathbf{a}+\mathbf{c})},$$

$$\sigma_2 = \sum_{A,B,C,D} \frac{q^{[B,C]^1} - 1}{q^{[V,V]^0} - 1} F_{AB}^V F_{CD}^W q^{-\frac{1}{2}\langle\mathbf{v},\mathbf{w}\rangle} q^{\langle\mathbf{b},\mathbf{c}\rangle} q^{-\frac{1}{2}\langle\mathbf{b}+\mathbf{d},\mathbf{a}+\mathbf{c}\rangle} X^{-(\mathbf{b}+\mathbf{d})^* - {}^*(\mathbf{a}+\mathbf{c})}.$$

Proof. We begin with σ_1. Using Green's formula (III.10) we get

$$\begin{aligned}
\sum_E \varepsilon_{VW}^E X_E &= \sum_{E,X,Y} \varepsilon_{VW}^E q^{-\frac{1}{2}\langle\mathbf{y},\mathbf{x}\rangle} F_{XY}^E X^{-\mathbf{y}^* - {}^*\mathbf{x}} \\
&= \sum_{A,B,C,D,X,Y} q^{[V,W]^0 - [A,C]^0 - [B,D]^0 - \langle\mathbf{a},\mathbf{d}\rangle} q^{-\frac{1}{2}\langle\mathbf{b}+\mathbf{d},\mathbf{a}+\mathbf{c}\rangle} F_{AB}^V F_{CD}^W \varepsilon_{AC}^X \varepsilon_{BD}^Y X^{-\mathbf{y}^* - {}^*\mathbf{x}} \\
&= \sum_{A,B,C,D} q^{[V,W]^1} q^{\langle\mathbf{b},\mathbf{c}\rangle} q^{-\frac{1}{2}\langle\mathbf{b}+\mathbf{d},\mathbf{a}+\mathbf{c}\rangle} F_{AB}^V F_{CD}^W X^{-(\mathbf{b}+\mathbf{d})^* - {}^*(\mathbf{a}+\mathbf{c})},
\end{aligned}$$

where the last equality comes from the identities

$$\sum_X \varepsilon_{AC}^X = |\operatorname{Ext}^1(A,C)| = q^{[A,C]^1},$$

$$\sum_Y \varepsilon_{BD}^Y = |\operatorname{Ext}^1(B,D)| = q^{[B,D]^1}.$$

But the quantum cluster character gives

$$\begin{aligned}
X_{V\oplus W} &= \sum_{X,Y} q^{-\frac{1}{2}\langle\mathbf{y},\mathbf{x}\rangle} F_{XY}^{V\oplus W} X^{-\mathbf{y}^* - {}^*\mathbf{x}} \\
&= \sum_{A,B,C,D} q^{-\frac{1}{2}\langle\mathbf{b}+\mathbf{d},\mathbf{a}+\mathbf{c}\rangle} q^{[B,C]^0} F_{AB}^V F_{CD}^W X^{-(\mathbf{b}+\mathbf{d})^* - {}^*(\mathbf{a}+\mathbf{c})}.
\end{aligned}$$

Since $\varepsilon_{VW}^{V\oplus W} = 1$ we may rewrite σ_1 as

$$\begin{aligned}\sigma_1 &= \sum_{E\ncong V\oplus W} \frac{\varepsilon_{VW}^{E}}{q^{[V,V]^0}-1} X_E \\ &= \frac{\sum_E \varepsilon_{VW}^{E} X_E - X_{V\oplus W}}{q^{[V,V]^0}-1} \\ &= \sum_{A,B,C,D} \frac{q^{[V,W]^1} - q^{[B,C]^1}}{q^{[V,V]^0}-1} q^{\langle \mathbf{b},\mathbf{c}\rangle} q^{-\frac{1}{2}\langle \mathbf{b}+\mathbf{d},\mathbf{a}+\mathbf{c}\rangle} F_{AB}^{V} F_{CD}^{W} X^{-(\mathbf{b}+\mathbf{d})^* - {}^*(\mathbf{a}+\mathbf{c})}.\end{aligned}$$

Now we move to σ_2. Notice that by the Auslander-Reiten formula we have ${}^*\tau\mathbf{b} = -\mathbf{b}^*$. And thus by Lemma III.3.3 we have

$$\begin{aligned}\sigma_2 &= \sum_{\substack{A,D,I,K,L,X,Y\\ D\ncong W}} \frac{|\operatorname{Hom}(W,\tau V)_{DAI}|}{q^{[V,V]^0}-1} q^{-\frac{1}{2}\langle \mathbf{a},\mathbf{d}\rangle} q^{[L,X]^0 - \frac{1}{2}\langle \mathbf{l}+\mathbf{y},\mathbf{k}+\mathbf{x}\rangle} F_{KL}^{A} F_{XY}^{D} X^{-(\mathbf{l}+\mathbf{y})^* - {}^*(\mathbf{k}+\mathbf{x}) + {}^*\mathbf{i}} \\ &= \sum_{\substack{A,B,C,D,I,K,L,X,Y\\ D\ncong W}} \frac{a_C F_{CD}^{W} F_{AB}^{V} F_{IC}^{\tau B}}{q^{[V,V]^0}-1} q^{-\frac{1}{2}\langle \mathbf{a},\mathbf{d}\rangle} q^{[L,X]^0 - \frac{1}{2}\langle \mathbf{l}+\mathbf{y},\mathbf{k}+\mathbf{x}\rangle} F_{KL}^{A} F_{XY}^{D} X^{-(\mathbf{l}+\mathbf{y}+\mathbf{b})^* - {}^*(\mathbf{k}+\mathbf{x}+\mathbf{c})}.\end{aligned}$$

By assumption $\operatorname{Ext}^1(A,D) = 0$ and thus $\operatorname{Ext}^1(L,X) = 0$. So σ_2 becomes

$$\sigma_2 = \sum_{\substack{A,B,C,D,I,K,L,X,Y\\ D\ncong W}} \frac{a_C F_{CD}^{W} F_{AB}^{V} F_{IC}^{\tau B}}{q^{[V,V]^0}-1} q^{-\frac{1}{2}\langle \mathbf{a},\mathbf{d}\rangle} q^{\langle \mathbf{l},\mathbf{x}\rangle - \frac{1}{2}\langle \mathbf{l}+\mathbf{y},\mathbf{k}+\mathbf{x}\rangle} F_{KL}^{A} F_{XY}^{D} X^{-(\mathbf{l}+\mathbf{y}+\mathbf{b})^* - {}^*(\mathbf{k}+\mathbf{x}+\mathbf{c})}.$$

In the case $D = W$, we have $A = V$ and $C = B = I = 0$. Thus removing the condition $D \ncong W$ we get

$$\begin{aligned}\sigma_2 = &\sum_{A,B,C,D,I,K,L,X,Y} \frac{a_C F_{CD}^{W} F_{AB}^{V} F_{IC}^{\tau B}}{q^{[V,V]^0}-1} q^{-\frac{1}{2}\langle \mathbf{a},\mathbf{d}\rangle} q^{\langle \mathbf{l},\mathbf{x}\rangle - \frac{1}{2}\langle \mathbf{l}+\mathbf{y},\mathbf{k}+\mathbf{x}\rangle} F_{KL}^{A} F_{XY}^{D} X^{-(\mathbf{l}+\mathbf{y}+\mathbf{b})^* - {}^*(\mathbf{k}+\mathbf{x}+\mathbf{c})} \\ &- \sum_{K,L,X,Y} \frac{1}{q^{[V,V]^0}-1} F_{KL}^{V} F_{XY}^{W} q^{-\frac{1}{2}\langle \mathbf{v},\mathbf{w}\rangle} q^{\langle \mathbf{l},\mathbf{x}\rangle - \frac{1}{2}\langle \mathbf{l}+\mathbf{y},\mathbf{k}+\mathbf{x}\rangle} X^{-(\mathbf{l}+\mathbf{y})^* - {}^*(\mathbf{k}+\mathbf{x})}.\end{aligned}$$

Now we aim to remove the exponential dependence on A, B, C, D, I, L, and X so that we may apply the associativity of Hall numbers and another simplifying equality. To that end we claim the following identities:

$$\langle \mathbf{k}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{k} \rangle = \langle \mathbf{k}, \mathbf{d} \rangle + \langle \mathbf{a}, \mathbf{k} \rangle,$$
$$\langle \mathbf{v}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{w} \rangle = \langle \mathbf{a}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{d} \rangle.$$

Indeed, recall that by assumption $\mathrm{Hom}(D, I) = \mathrm{Hom}(A, I) = 0$ and so $\mathrm{Hom}(K, I) = \mathrm{Hom}(Y, I) = 0$. Combining these observations with the Auslander-Reiten formulas of Proposition II.3.2 gives

$$\langle \mathbf{k}, \mathbf{w} - \mathbf{d} \rangle = \langle \mathbf{k}, \tau\mathbf{v} - \tau\mathbf{a} - \mathbf{i} \rangle = \langle \mathbf{k}, \tau\mathbf{v} - \tau\mathbf{a} \rangle = -\langle \mathbf{v} - \mathbf{a}, \mathbf{k} \rangle,$$
$$\langle \mathbf{y}, \mathbf{w} - \mathbf{d} \rangle = \langle \mathbf{y}, \tau\mathbf{v} - \tau\mathbf{a} - \mathbf{i} \rangle = \langle \mathbf{y}, \tau\mathbf{v} - \tau\mathbf{a} \rangle = -\langle \mathbf{v} - \mathbf{a}, \mathbf{y} \rangle.$$

These now give the desired identity:

$$\begin{aligned}
&-\frac{1}{2}\langle \mathbf{a}, \mathbf{d} \rangle + \langle \mathbf{l}, \mathbf{x} \rangle - \frac{1}{2}\langle \mathbf{l} + \mathbf{y}, \mathbf{k} + \mathbf{x} \rangle \\
&\quad = -\frac{1}{2}\langle \mathbf{a}, \mathbf{d} \rangle + \langle \mathbf{a} - \mathbf{k}, \mathbf{d} - \mathbf{y} \rangle - \frac{1}{2}\langle \mathbf{a} - \mathbf{k} + \mathbf{y}, \mathbf{k} + \mathbf{d} - \mathbf{y} \rangle \\
&\quad = -\frac{1}{2}\langle \mathbf{a}, \mathbf{y} \rangle - \frac{1}{2}\langle \mathbf{y}, \mathbf{d} \rangle - \frac{1}{2}\langle \mathbf{k}, \mathbf{d} \rangle - \frac{1}{2}\langle \mathbf{a}, \mathbf{k} \rangle + \frac{1}{2}\langle \mathbf{k}, \mathbf{y} \rangle + \frac{1}{2}\langle \mathbf{k}, \mathbf{k} \rangle - \frac{1}{2}\langle \mathbf{y}, \mathbf{k} \rangle + \frac{1}{2}\langle \mathbf{y}, \mathbf{y} \rangle \\
&\quad = -\frac{1}{2}\langle \mathbf{v}, \mathbf{y} \rangle - \frac{1}{2}\langle \mathbf{y}, \mathbf{w} \rangle - \frac{1}{2}\langle \mathbf{k}, \mathbf{w} \rangle - \frac{1}{2}\langle \mathbf{v}, \mathbf{k} \rangle + \frac{1}{2}\langle \mathbf{k}, \mathbf{y} \rangle + \frac{1}{2}\langle \mathbf{k}, \mathbf{k} \rangle - \frac{1}{2}\langle \mathbf{y}, \mathbf{k} \rangle + \frac{1}{2}\langle \mathbf{y}, \mathbf{y} \rangle \\
&\quad = \langle \mathbf{v} - \mathbf{k}, \mathbf{w} - \mathbf{y} \rangle - \frac{1}{2}\langle \mathbf{v} - \mathbf{k} + \mathbf{y}, \mathbf{k} + \mathbf{w} - \mathbf{y} \rangle - \frac{1}{2}\langle \mathbf{v}, \mathbf{w} \rangle.
\end{aligned}$$

Now we may rewrite σ_2 as

$$\begin{aligned}
\sigma_2 = &\sum_{A,B,C,D,I,K,L,X,Y} \frac{a_C F^W_{CD} F^V_{AB} F^{\tau B}_{IC}}{q^{[V,V]^0} - 1} q^{\langle \mathbf{v}-\mathbf{k}, \mathbf{w}-\mathbf{y} \rangle - \frac{1}{2}\langle \mathbf{v}-\mathbf{k}+\mathbf{y}, \mathbf{k}+\mathbf{w}-\mathbf{y} \rangle - \frac{1}{2}\langle \mathbf{v}, \mathbf{w} \rangle} \times \\
&\qquad \times F^A_{KL} F^D_{XY} X^{-(\mathbf{l}+\mathbf{y}+\mathbf{b})^* - {}^*(\mathbf{k}+\mathbf{x}+\mathbf{c})} \\
&- \sum_{K,L,X,Y} \frac{1}{q^{[V,V]^0} - 1} F^V_{KL} F^W_{XY} q^{-\frac{1}{2}\langle \mathbf{v}, \mathbf{w} \rangle} q^{\langle \mathbf{l}, \mathbf{x} \rangle - \frac{1}{2}\langle \mathbf{l}+\mathbf{y}, \mathbf{k}+\mathbf{x} \rangle} X^{-(\mathbf{l}+\mathbf{y})^* - {}^*(\mathbf{k}+\mathbf{x})}
\end{aligned}$$

where the dependence on A only occurs in the product $F^V_{AB} F^A_{KL}$ and the D dependence only occurs in the product $F^W_{CD} F^D_{XY}$. In particular, the associativity of Hall numbers (III.9) applies and σ_2

may be rewritten as

$$\sigma_2 = \sum_{A',B,C,D',I,K,L,X,Y} \frac{a_C F^{D'}_{CX} F^{A'}_{LB} F^{\tau B}_{IC}}{q^{[V,V]^0}-1} q^{\langle \mathbf{v}-\mathbf{k},\mathbf{w}-\mathbf{y}\rangle - \frac{1}{2}\langle \mathbf{v}-\mathbf{k}+\mathbf{y},\mathbf{k}+\mathbf{w}-\mathbf{y}\rangle - \frac{1}{2}\langle \mathbf{v},\mathbf{w}\rangle} \times$$

$$\times F^V_{KA'} F^W_{D'Y} X^{-(\mathbf{a}'+\mathbf{y})^* - {}^*(\mathbf{k}+\mathbf{d}')}$$

$$- \sum_{K,L,X,Y} \frac{1}{q^{[V,V]^0}-1} F^V_{KL} F^W_{XY} q^{-\frac{1}{2}\langle \mathbf{v},\mathbf{w}\rangle} q^{\langle \mathbf{l},\mathbf{x}\rangle - \frac{1}{2}\langle \mathbf{l}+\mathbf{y},\mathbf{k}+\mathbf{x}\rangle} X^{-(\mathbf{l}+\mathbf{y})^* - {}^*(\mathbf{k}+\mathbf{x})}.$$

By Lemma III.3.3 and the Auslander-Reiten formula (II.4) we have

$$\sum_{I,L,X}\sum_{B,C} a_C F^{D'}_{CX} F^{A'}_{LB} F^{\tau B}_{IC} = \sum_{I,L,X} |\operatorname{Hom}(D',\tau A')_{XLI}| = q^{[D',\tau A']^0} = q^{[A',D']^1}.$$

So that σ_2 becomes

$$\begin{aligned}
\sigma_2 = & \sum_{A',D',K,Y} \frac{q^{[A',D']^1}}{q^{[V,V]^0}-1} q^{\langle \mathbf{v}-\mathbf{k},\mathbf{w}-\mathbf{y}\rangle - \frac{1}{2}\langle \mathbf{v}-\mathbf{k}+\mathbf{y},\mathbf{k}+\mathbf{w}-\mathbf{y}\rangle - \frac{1}{2}\langle \mathbf{v},\mathbf{w}\rangle} F^V_{KA'} F^W_{D'Y} X^{-(\mathbf{a}'+\mathbf{y})^* - {}^*(\mathbf{k}+\mathbf{d}')} \\
& - \sum_{K,L,X,Y} \frac{1}{q^{[V,V]^0}-1} F^V_{KL} F^W_{XY} q^{-\frac{1}{2}\langle \mathbf{v},\mathbf{w}\rangle} q^{\langle \mathbf{l},\mathbf{x}\rangle - \frac{1}{2}\langle \mathbf{l}+\mathbf{y},\mathbf{k}+\mathbf{x}\rangle} X^{-(\mathbf{l}+\mathbf{y})^* - {}^*(\mathbf{k}+\mathbf{x})} \\
= & \sum_{A',D',K,Y} \frac{q^{[A',D']^1}}{q^{[V,V]^0}-1} F^V_{KA'} F^W_{D'Y} q^{-\frac{1}{2}\langle \mathbf{v},\mathbf{w}\rangle} q^{\langle \mathbf{a}',\mathbf{d}'\rangle - \frac{1}{2}\langle \mathbf{a}'+\mathbf{y},\mathbf{k}+\mathbf{d}'\rangle} X^{-(\mathbf{a}'+\mathbf{y})^* - {}^*(\mathbf{k}+\mathbf{d}')} \\
& - \sum_{K,L,X,Y} \frac{1}{q^{[V,V]^0}-1} F^V_{KL} F^W_{XY} q^{-\frac{1}{2}\langle \mathbf{v},\mathbf{w}\rangle} q^{\langle \mathbf{l},\mathbf{x}\rangle - \frac{1}{2}\langle \mathbf{l}+\mathbf{y},\mathbf{k}+\mathbf{x}\rangle} X^{-(\mathbf{l}+\mathbf{y})^* - {}^*(\mathbf{k}+\mathbf{x})} \\
= & \sum_{A,B,C,D} \frac{q^{[B,C]^1}-1}{q^{[V,V]^0}-1} F^V_{AB} F^W_{CD} q^{-\frac{1}{2}\langle \mathbf{v},\mathbf{w}\rangle} q^{\langle \mathbf{b},\mathbf{c}\rangle - \frac{1}{2}\langle \mathbf{b}+\mathbf{d},\mathbf{a}+\mathbf{c}\rangle} X^{-(\mathbf{b}+\mathbf{d})^* - {}^*(\mathbf{a}+\mathbf{c})}.
\end{aligned}$$

□

This completes the proof of Proposition III.3.5 and thus of Theorem III.3.4. □

Our final result of this section is analogous to [H1, Theorem 17], [Q, Proposition 5.4.1], and [DX, Theorem 3.8]. We again follow the Hall number approach of [H1, Theorem 17]. Let W and I be valued representations of Q, where I is injective. Write $P = \nu^{-1}(I)$ and note that P is projective with $\operatorname{soc} I \cong P/\operatorname{rad} P$, moreover $\operatorname{End}(I) \cong \operatorname{End}(P)$. From morphisms $\theta : W \to I$ and $\gamma : P \to W$ we get exact sequences

$$0 \longrightarrow G \longrightarrow W \xrightarrow{\theta} I \longrightarrow I' \longrightarrow 0\,,$$
$$0 \longrightarrow P' \longrightarrow P \xrightarrow{\gamma} W \longrightarrow F \longrightarrow 0\,.$$

where $G = \ker\theta$, $I' = \operatorname{coker}\theta$ is injective, $P' = \ker\gamma$ is projective, and $F = \operatorname{coker}\gamma$. We introduce the following notation needed for the proof of the theorem:

$$\operatorname{Hom}(W,I)_{GI'} = \{W \xrightarrow{f} I : f \neq 0, \ker f = G, \operatorname{coker} f = I'\},$$
$$\operatorname{Hom}(P,W)_{P'F} = \{P \xrightarrow{f} W : f \neq 0, \ker f = P', \operatorname{coker} f = F\}.$$

Lemma III.3.6. The size of the restricted Hom-spaces $\operatorname{Hom}(W,I)_{GI'}$ and $\operatorname{Hom}(P,W)_{P'F}$ can be computed in terms of Hall numbers, in particular we have

$$|\operatorname{Hom}(W,I)_{GI'}| = \sum_A a_A F^W_{AG} F^I_{I'A}, \tag{III.14}$$
$$|\operatorname{Hom}(P,W)_{P'F}| = \sum_B a_B F^W_{FB} F^P_{BP'}. \tag{III.15}$$

Proof. Notice that the θ exact sequence above can be split into the following two short exact sequences:

$$0 \longrightarrow G \longrightarrow W \xrightarrow{c} A \longrightarrow 0\,,$$
$$0 \longrightarrow A \xrightarrow{g} I \longrightarrow I' \longrightarrow 0\,,$$

where $gc = \theta$. Thus we have a surjective map $\bigsqcup_A \mathcal{P}^W_{AG} \times \mathcal{P}^I_{I'A} \to \operatorname{Hom}(W,I)_{GI'}$ with fiber over a morphism θ isomorphic to $\operatorname{Aut}(G) \times \operatorname{Aut}(A) \times \operatorname{Aut}(I')$. Then the identity (III.14) follows from the equality

$$|\operatorname{Hom}(W,I)_{GI'}| = \sum_A P^W_{AG} P^I_{I'A} / a_G a_A a_{I'}.$$

Now notice the γ exact sequence above can be split into the following two short exact sequences:

$$0 \longrightarrow P' \longrightarrow P \xrightarrow{d} B \longrightarrow 0\,,$$
$$0 \longrightarrow B \xrightarrow{h} W \longrightarrow F \longrightarrow 0\,,$$

where $hd = \gamma$. Thus we have a surjective map $\bigsqcup_B \mathcal{P}^P_{BP'} \times \mathcal{P}^W_{FB} \to \operatorname{Hom}(P,W)_{P'F}$ with fiber over a morphism γ isomorphic to $\operatorname{Aut}(P') \times \operatorname{Aut}(B) \times \operatorname{Aut}(F)$. Then the identity (III.15) follows from the equality

$$|\operatorname{Hom}(P,W)_{P'F}| = \sum_B P^P_{BP'} P^W_{FB} / a_{P'} a_B a_F.$$

□

Theorem III.3.7. Let W and I be valued representations of Q with I injective. Write $P = \nu^{-1}(I)$. Assume that there exist unique (up to scalar) morphisms $f \in \operatorname{Hom}(W,I)$ and $g \in \operatorname{Hom}(P,W)$, in particular $\dim_{\operatorname{End}(I)} \operatorname{Hom}(W,I) = \dim_{\operatorname{End}(P)} \operatorname{Hom}(P,W) = 1$. Define F, G, I', P' as above and assume further that $\operatorname{Hom}(P',F) = \operatorname{Hom}(G,I') = 0$. Then we have the following multiplication formula:

$$X_W X^{{}^*\mathbf{i}} = q^{-\frac{1}{2}\Lambda({}^*\mathbf{w},{}^*\mathbf{i})} X_G * X^{{}^*\mathbf{i}'} + q^{-\frac{1}{2}\Lambda({}^*\mathbf{w},{}^*\mathbf{i}) - \frac{1}{2}[I,I]^0} X_F * X^{\mathbf{p}'^*}. \tag{III.16}$$

Proof. We start by computing the product on the left using Lemma II.1.3.1:

$$\begin{aligned} X_W X^{{}^*\mathbf{i}} &= \sum_{X,Y} q^{-\frac{1}{2}\langle \mathbf{y},\mathbf{x}\rangle} F^W_{XY} q^{\frac{1}{2}\Lambda(-\mathbf{y}^* - {}^*\mathbf{x}, {}^*\mathbf{i})} X^{-\mathbf{y}^* - {}^*\mathbf{x} + {}^*\mathbf{i}} \\ &= q^{-\frac{1}{2}\Lambda({}^*\mathbf{w},{}^*\mathbf{i})} \sum_{X,Y} q^{\frac{1}{2}\Lambda({}^*\mathbf{y} - \mathbf{y}^*, {}^*\mathbf{i})} q^{-\frac{1}{2}\langle \mathbf{y},\mathbf{x}\rangle} F^W_{XY} X^{-\mathbf{y}^* - {}^*\mathbf{x} + {}^*\mathbf{i}} \\ &= q^{-\frac{1}{2}\Lambda({}^*\mathbf{w},{}^*\mathbf{i})} \sum_{X,Y} q^{-\frac{1}{2}\langle \mathbf{y},\mathbf{i}\rangle} q^{-\frac{1}{2}\langle \mathbf{y},\mathbf{x}\rangle} F^W_{XY} X^{-\mathbf{y}^* - {}^*\mathbf{x} + {}^*\mathbf{i}}. \end{aligned}$$

Our goal is to show that this is equal to the right hand side of equation (III.16). We again accomplish this by cleverly rewriting each term on the right. We make the following definitions

$$\begin{aligned} \sigma_1 &:= \sum_{\substack{G,I' \\ G \neq W}} \frac{|\operatorname{Hom}(W,I)_{GI'}|}{q^{[I,I]^0} - 1} X_G * X^{{}^*\mathbf{i}'}, \\ \sigma_2 &:= \sum_{\substack{F,P' \\ F \neq W}} \frac{|\operatorname{Hom}(P,W)_{P'F}|}{q^{[I,I]^0} - 1} X_F * X^{\mathbf{p}'^*}. \end{aligned}$$

Since there are unique nonzero morphisms $W \to I$ and $P \to W$ each of these sums collapses to a single term. Note that under the assumption $\dim_{\operatorname{End}(I)} \operatorname{Hom}(W,I) = \dim_{\operatorname{End}(P)} \operatorname{Hom}(P,W) = 1$, we have $|\operatorname{Hom}(W,I)_{GI'}| = |\operatorname{Hom}(P,W)_{P'F}| = q^{[I,I]^0} - 1$ where we have used that $\operatorname{End}(P) \cong \operatorname{End}(I)$. Thus we see that the right hand side of equation (III.16) may be written as $q^{-\frac{1}{2}\Lambda({}^*\mathbf{w},{}^*\mathbf{i})}\sigma_1 + q^{-\frac{1}{2}\Lambda({}^*\mathbf{w},{}^*\mathbf{i})-\frac{1}{2}[I,I]^0}\sigma_2$. Since I is injective, applying $\operatorname{Hom}(-,I)$ to a short exact sequence

$$0 \longrightarrow Y \longrightarrow W \longrightarrow X \longrightarrow 0$$

induces a short exact sequence

$$0 \longrightarrow \operatorname{Hom}(Y,I) \longrightarrow \operatorname{Hom}(W,I) \longrightarrow \operatorname{Hom}(X,I) \longrightarrow 0\,.$$

Since $\dim_{\operatorname{End}(I)} \operatorname{Hom}(W,I) = 1$ we see that either $\operatorname{Hom}(Y,I) \cong \operatorname{Hom}(W,I) \cong \operatorname{Hom}(I,I)$ or $\operatorname{Hom}(Y,I) = 0$. Then $\langle \mathbf{y}, \mathbf{i}\rangle = [Y,I]_0$ either equals $[I,I]_0$ or 0. Now observe that our computation of $X_W X^{{}^*\mathbf{i}}$ above and the following Proposition complete the proof.

Proposition III.3.8. We may rewrite σ_1 and σ_2 as

$$\sigma_1 = \sum_{X,Y:[Y,I]^0=0} q^{-\frac{1}{2}\langle \mathbf{y},\mathbf{x}\rangle} F_{XY}^W X^{-\mathbf{y}^*-{}^*\mathbf{x}+{}^*\mathbf{i}},$$
$$\sigma_2 = \sum_{X,Y:[Y,I]^0=[I,I]^0} q^{-\frac{1}{2}\langle \mathbf{y},\mathbf{x}\rangle} F_{XY}^W X^{-\mathbf{y}^*-{}^*\mathbf{x}+{}^*\mathbf{i}}.$$

Proof. We begin with σ_1. Using Lemma III.3.6 we may rewrite σ_1 as

$$\sigma_1 = \sum_{\substack{A,G,I',X,Y\\ G\neq W}} \frac{a_A F_{AG}^W F_{I'A}^I}{q^{[I,I]^0}-1} q^{-\frac{1}{2}\langle \mathbf{y},\mathbf{x}\rangle} F_{XY}^G X^{-\mathbf{y}^*-{}^*\mathbf{x}+{}^*\mathbf{i}'}$$
$$= \sum_{\substack{A,G,I',X,Y\\ G\neq W}} \frac{a_A F_{AG}^W F_{I'A}^I}{q^{[I,I]^0}-1} q^{-\frac{1}{2}\langle \mathbf{y},\mathbf{x}\rangle} F_{XY}^G X^{-\mathbf{y}^*-{}^*(\mathbf{x}+\mathbf{a})+{}^*\mathbf{i}}.$$

Note that by assumption we have $\operatorname{Hom}(G,I') = 0$ and thus $\operatorname{Hom}(Y,I') = 0$. Since $G \neq W$, we have $A \neq 0$ and $\operatorname{Hom}(A,I) \neq 0$. Thus the induced exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Hom}(A,I) & \longrightarrow & \operatorname{Hom}(W,I) & \longrightarrow & \operatorname{Hom}(G,I) & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & \operatorname{Hom}(Y,I) & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

implies $\mathrm{Hom}(Y, I) = \mathrm{Hom}(G, I) = 0$. So we get the identity

$$\begin{aligned}\langle \mathbf{y}, \mathbf{x} \rangle &= \langle \mathbf{y}, \mathbf{w} - \mathbf{a} - \mathbf{y} \rangle \\ &= \langle \mathbf{y}, \mathbf{w} - \mathbf{y} \rangle - \langle \mathbf{y}, \mathbf{i} - \mathbf{i}' \rangle \\ &= \langle \mathbf{y}, \mathbf{w} - \mathbf{y} \rangle,\end{aligned}$$

and σ_1 becomes

$$\sigma_1 = \sum_{\substack{A,G,I',X,Y \\ G \neq W}} \frac{a_A F^W_{AG} F^I_{I'A}}{q^{[I,I]^0} - 1} q^{-\frac{1}{2}\langle \mathbf{y}, \mathbf{w} - \mathbf{y} \rangle} F^G_{XY} X^{-\mathbf{y}^* - {}^*(\mathbf{x}+\mathbf{a}) + {}^*\mathbf{i}}.$$

In the case $G = W$ we have $A = 0$ and $I = I'$. Therefore we may rewrite σ_1 and then apply the associativity of Hall numbers as follows

$$\begin{aligned}\sigma_1 = &\sum_{A,G,I',X,Y} \frac{a_A F^W_{AG} F^I_{I'A}}{q^{[I,I]^0} - 1} q^{-\frac{1}{2}\langle \mathbf{y}, \mathbf{w} - \mathbf{y} \rangle} F^G_{XY} X^{-\mathbf{y}^* - {}^*(\mathbf{x}+\mathbf{a}) + {}^*\mathbf{i}} \\ &- \sum_{X,Y} \frac{1}{q^{[I,I]^0} - 1} q^{-\frac{1}{2}\langle \mathbf{y}, \mathbf{w} - \mathbf{y} \rangle} F^W_{XY} X^{-\mathbf{y}^* - {}^*\mathbf{x} + {}^*\mathbf{i}} \\ = &\sum_{A,G',I',X,Y} \frac{a_A F^{G'}_{AX} F^I_{I'A}}{q^{[I,I]^0} - 1} q^{-\frac{1}{2}\langle \mathbf{y}, \mathbf{g}' \rangle} F^W_{G'Y} X^{-\mathbf{y}^* - {}^*\mathbf{g}' + {}^*\mathbf{i}} \\ &- \sum_{X,Y} \frac{1}{q^{[I,I]^0} - 1} q^{-\frac{1}{2}\langle \mathbf{y}, \mathbf{x} \rangle} F^W_{XY} X^{-\mathbf{y}^* - {}^*\mathbf{x} + {}^*\mathbf{i}}.\end{aligned}$$

Notice that by Lemma III.3.6 we have

$$\sum_{I',X} \sum_{A} a_A F^{G'}_{AX} F^I_{I'A} = \sum_{I',X} \mathrm{Hom}(G', I)_{XI'} = q^{[G',I]^0},$$

so that σ_1 becomes

$$
\begin{aligned}
\sigma_1 &= \sum_{G',Y} \frac{q^{[G',I]^0}}{q^{[I,I]^0}-1} q^{-\frac{1}{2}\langle \mathbf{y},\mathbf{g}'\rangle} F_{G'Y}^{W} X^{-\mathbf{y}^*-{}^*\mathbf{g}'+{}^*\mathbf{i}} \\
&\qquad - \sum_{X,Y} \frac{1}{q^{[I,I]^0}-1} q^{-\frac{1}{2}\langle \mathbf{y},\mathbf{x}\rangle} F_{XY}^{W} X^{-\mathbf{y}^*-{}^*\mathbf{x}+{}^*\mathbf{i}} \\
&= \sum_{X,Y} \frac{q^{[X,I]^0}-1}{q^{[I,I]^0}-1} q^{-\frac{1}{2}\langle \mathbf{y},\mathbf{x}\rangle} F_{XY}^{W} X^{-\mathbf{y}^*-{}^*\mathbf{x}+{}^*\mathbf{i}} \\
&= \sum_{X,Y:[Y,I]^0=0} q^{-\frac{1}{2}\langle \mathbf{y},\mathbf{x}\rangle} F_{XY}^{W} X^{-\mathbf{y}^*-{}^*\mathbf{x}+{}^*\mathbf{i}}.
\end{aligned}
$$

Turning to σ_2, recall that we have $\mathbf{p}^* = [P/\operatorname{rad} P] = [\operatorname{soc} I] = {}^*\mathbf{i}$. Combining this observation with Lemma III.3.6 we may rewrite σ_2 as

$$
\begin{aligned}
\sigma_2 &= \sum_{\substack{B,F,P',X,Y \\ F\neq W}} \frac{a_B F_{FB}^{W} F_{BP'}^{P}}{q^{[I,I]^0}-1} q^{-\frac{1}{2}\langle \mathbf{y},\mathbf{x}\rangle} F_{XY}^{F} X^{-\mathbf{y}^*-{}^*\mathbf{x}+\mathbf{p}'^*} \\
&= \sum_{\substack{B,F,P',X,Y \\ F\neq W}} \frac{a_B F_{FB}^{W} F_{BP'}^{P}}{q^{[I,I]^0}-1} q^{-\frac{1}{2}\langle \mathbf{y},\mathbf{x}\rangle} F_{XY}^{F} X^{-(\mathbf{y}+\mathbf{b})^*-{}^*\mathbf{x}+{}^*\mathbf{i}}.
\end{aligned}
$$

Note that by assumption we have $\operatorname{Hom}(P',F)=0$ and thus $\operatorname{Hom}(P',X)=0$. Since $F\neq W$, we have $B\neq 0$ and $\operatorname{Hom}(P,B)\neq 0$. Thus the induced exact diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}(P,B) & \longrightarrow & \operatorname{Hom}(P,W) & \longrightarrow & \operatorname{Hom}(P,F) & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & \operatorname{Hom}(P,X) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}
$$

implies $\mathrm{Hom}(P, X) = \mathrm{Hom}(P, F) = 0$. So we get the identity

$$\begin{aligned}\langle \mathbf{y}, \mathbf{x} \rangle &= \langle \mathbf{w} - \mathbf{b} - \mathbf{x}, \mathbf{x} \rangle \\ &= \langle \mathbf{w} - \mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{p} - \mathbf{p}', \mathbf{x} \rangle \\ &= \langle \mathbf{w} - \mathbf{x}, \mathbf{x} \rangle,\end{aligned}$$

and σ_2 becomes

$$\sigma_2 = \sum_{\substack{B,F,P',X,Y \\ F \neq W}} \frac{a_B F^W_{FB} F^P_{BP'}}{q^{[I,I]^0} - 1} q^{-\frac{1}{2}\langle \mathbf{w}-\mathbf{x}, \mathbf{x} \rangle} F^F_{XY} X^{-(\mathbf{y}+\mathbf{b})^* - {}^*\mathbf{x} + {}^*\mathbf{i}}.$$

In the case $F = W$ we have $B = 0$ and $P = P'$. Therefore we may rewrite σ_2 and then apply the associativity of Hall numbers as follows:

$$\begin{aligned}\sigma_2 = &\sum_{B,F,P',X,Y} \frac{a_B F^W_{FB} F^P_{BP'}}{q^{[I,I]^0} - 1} q^{-\frac{1}{2}\langle \mathbf{w}-\mathbf{x}, \mathbf{x} \rangle} F^F_{XY} X^{-(\mathbf{y}+\mathbf{b})^* - {}^*\mathbf{x} + {}^*\mathbf{i}} \\ &- \sum_{X,Y} \frac{1}{q^{[I,I]^0} - 1} q^{-\frac{1}{2}\langle \mathbf{w}-\mathbf{x}, \mathbf{x} \rangle} F^W_{XY} X^{-\mathbf{y}^* - {}^*\mathbf{x} + {}^*\mathbf{i}} \\ = &\sum_{B,F',P',X,Y} \frac{a_B F^{F'}_{YB} F^P_{BP'}}{q^{[I,I]^0} - 1} q^{-\frac{1}{2}\langle \mathbf{f}', \mathbf{x} \rangle} F^W_{XF'} X^{-\mathbf{f}'^* - {}^*\mathbf{x} + {}^*\mathbf{i}} \\ &- \sum_{X,Y} \frac{1}{q^{[I,I]^0} - 1} q^{-\frac{1}{2}\langle \mathbf{y}, \mathbf{x} \rangle} F^W_{XY} X^{-\mathbf{y}^* - {}^*\mathbf{x} + {}^*\mathbf{i}}.\end{aligned}$$

Notice that by Lemma III.3.6 we have

$$\sum_{P',Y} \sum_{B} a_B F^{F'}_{YB} F^P_{BP'} = \sum_{P',Y} \mathrm{Hom}(P, F')_{P'Y} = q^{[P,F']^0},$$

so that σ_2 becomes

$$\begin{aligned}\sigma_2 &= \sum_{F',X} \frac{q^{[P,F']^0}}{q^{[I,I]^0}-1} q^{-\frac{1}{2}\langle \mathbf{f}',\mathbf{x}\rangle} F^W_{XF'} X^{-\mathbf{f}'^*-{}^*\mathbf{x}+{}^*\mathbf{i}} \\ &\quad - \sum_{X,Y} \frac{1}{q^{[I,I]^0}-1} q^{-\frac{1}{2}\langle \mathbf{y},\mathbf{x}\rangle} F^W_{XY} X^{-\mathbf{y}^*-{}^*\mathbf{x}+{}^*\mathbf{i}} \\ &= \sum_{X,Y} \frac{q^{[P,Y]^0}-1}{q^{[I,I]^0}-1} q^{-\frac{1}{2}\langle \mathbf{y},\mathbf{x}\rangle} F^W_{XY} X^{-\mathbf{y}^*-{}^*\mathbf{x}+{}^*\mathbf{i}} \\ &= \sum_{X,Y:[P,Y]^0=[I,I]^0} q^{-\frac{1}{2}\langle \mathbf{y},\mathbf{x}\rangle} F^W_{XY} X^{-\mathbf{y}^*-{}^*\mathbf{x}+{}^*\mathbf{i}} \\ &= \sum_{X,Y:[Y,I]^0=[I,I]^0} q^{-\frac{1}{2}\langle \mathbf{y},\mathbf{x}\rangle} F^W_{XY} X^{-\mathbf{y}^*-{}^*\mathbf{x}+{}^*\mathbf{i}}.\end{aligned}$$

□

This completes the proof of Proposition III.3.8 and thus the proof of Theorem III.3.7. □

III.4. Commutation and Compatibility

Our eventual goal is to conclude that the quantum cluster character applied to exceptional representations of $(Q,\mathbf{d})$ coincides with the initial cluster Laurent expansion of all non-initial cluster variables. Recall that the cluster variables fit into quasi-commuting families and thus in this section we will consider what conditions we need on valued representations V and W so that the quantum cluster characters X_V and X_W quasi-commute. The following Proposition, inspired by [CC, Prop. 3.6] and [Q, Equation (19)], will be the main ingredient.

Proposition III.4.1. Let V and W be representations of $(Q,\mathbf{d})$ with $\mathrm{Ext}^1(V,W)=0$. Then

$$|Gr_{\mathbf{e}}(V\oplus W)| = \sum_{\substack{\mathbf{b},\mathbf{c}\in\mathcal{K}(Q)\\ \mathbf{b}+\mathbf{c}=\mathbf{e}}} q^{\langle \mathbf{b},\mathbf{w}-\mathbf{c}\rangle} |Gr_{\mathbf{b}}(V)||Gr_{\mathbf{c}}(W)|.$$

Proof. Denote by $\pi_1 : V\oplus W\to V$ the natural projections and consider W as a subrepresentations of $V\oplus W$ via the natural inclusion. Fix $B\in Gr_{\mathbf{b}}(V)$ and $C\in Gr_{\mathbf{c}}(W)$ and define

$$Gr_{B,C}(V\oplus W) = \{L\in Gr_{\mathbf{b}+\mathbf{c}}(V\oplus W) : \pi_1(L)=B, L\cap W = C\}.$$

Lemma III.4.2. The following map is an isomorphism:

$$\zeta \;:\; \mathrm{Hom}(B, W/C) \longrightarrow Gr_{B,C}(V \oplus W)$$
$$f \longmapsto L^f := \{b + w \in V \oplus W : b \in B, w \in W, f(b) = p(w)\}$$

where $p : W \to W/C$ is the natural projection.

Proof. Since p is surjective we can find for any $b \in B$ an element $w \in W$ so that $f(b) = p(w)$ and thus $\pi_1(L^f) = B$. Also notice that $L^f \cap W = \ker p = C$. We define a map

$$\eta \;:\; Gr_{B,C}(V \oplus W) \longrightarrow \mathrm{Hom}(B, W/C)$$
$$L \longmapsto f^L$$

where $f^L(b) := p(w)$ for any $w \in W$ such that $b + w \in L$. If $b + w \in L$ and $b + w' \in L$ then $w - w' \in L \cap W = C$ so that $p(w) = p(w')$ and η is well defined. It is easy to see that $\eta \circ \zeta$ and $\zeta \circ \eta$ are identity maps. $\square$

Define a map

$$\varphi \;:\; Gr_{\mathbf{e}}(V \oplus W) \longrightarrow \coprod_{\mathbf{b}+\mathbf{c}=\mathbf{e}} Gr_{\mathbf{b}}(V) \times Gr_{\mathbf{c}}(W)$$
$$L \longmapsto (\pi_1(L), L \cap W)\,.$$

The fiber over a point (B, C) is $Gr_{B,C}(V \oplus W)$, which by Lemma III.4.2 is isomorphic to an affine space with $q^{\dim \mathrm{Hom}(B,W/C)}$ elements. To complete the proof it suffices to show for $B \in Gr_{\mathbf{b}}(V)$ and $C \in Gr_{\mathbf{c}}(W)$ that $\dim \mathrm{Hom}(B, W/C)$ only depends on the dimension vectors of B and C and thus all fibers have the same number of points. To this end we will show that $\mathrm{Ext}(B, W/C) = 0$ so that $\dim \mathrm{Hom}(B, W/C) = \langle \mathbf{b}, \mathbf{w} - \mathbf{c} \rangle$. Consider the following exact sequences:

$$0 \longrightarrow B \longrightarrow V \longrightarrow V/B \longrightarrow 0\,,$$
$$0 \longrightarrow C \longrightarrow W \longrightarrow W/C \longrightarrow 0\,.$$

We apply $\mathrm{Hom}(-, W)$ to the first sequence and $\mathrm{Hom}(B, -)$ to the second sequence to get the following exact diagram taken from the corresponding long exact sequences:

$$\begin{array}{ccccc} \mathrm{Ext}^1(V,W) & \longrightarrow & \mathrm{Ext}^1(B,W) & \longrightarrow & 0 \\ & & \downarrow & & \\ & & \mathrm{Ext}^1(B,W/C) & & \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

Since $\mathrm{Ext}^1(V,W)=0$ we get $\mathrm{Ext}^1(B,W/C)=0$. $\square$

Theorem III.4.3. Let V and W be representations of $(Q,\mathbf{d})$ with $\mathrm{Ext}^1(V,W)=0$, then

$$X_V X_W = q^{\frac{1}{2}\Lambda({}^*\mathbf{v},{}^*\mathbf{w})} X_{V\oplus W}.$$

If in addition $\mathrm{Ext}^1(W,V)=0$, we have

$$X_V X_W = q^{\Lambda({}^*\mathbf{v},{}^*\mathbf{w})} X_W X_V.$$

Proof. Using Lemma II.1.3 and Proposition III.4.1, we have

$$\begin{aligned} X_V X_W &= \sum_{\mathbf{b},\mathbf{c}\in\mathcal{K}(Q)} q^{-\frac{1}{2}\langle \mathbf{b},\mathbf{v}-\mathbf{b}\rangle} |Gr_{\mathbf{b}}(V)| X^{-\mathbf{b}^*-{}^*(\mathbf{c}-\mathbf{b})} \\ &\qquad \cdot q^{-\frac{1}{2}\langle \mathbf{c},\mathbf{w}-\mathbf{c}\rangle} |Gr_{\mathbf{c}}(W)| X^{-\mathbf{c}^*-{}^*(\mathbf{w}-\mathbf{c})} \\ &= \sum_{\mathbf{e}\in\mathcal{K}(Q)} \sum_{\mathbf{b}+\mathbf{c}=\mathbf{e}} q^{\langle \mathbf{b},\mathbf{w}-\mathbf{c}\rangle} |Gr_{\mathbf{b}}(V)||Gr_{\mathbf{c}}(W)| X^{-\mathbf{e}^*-{}^*(\mathbf{v}+\mathbf{w}-\mathbf{e})} \\ &\qquad \cdot q^{-\langle \mathbf{b},\mathbf{w}-\mathbf{c}\rangle} q^{-\frac{1}{2}\langle \mathbf{b},\mathbf{v}-\mathbf{b}\rangle} q^{-\frac{1}{2}\langle \mathbf{c},\mathbf{w}-\mathbf{c}\rangle} q^{\frac{1}{2}\Lambda(-\mathbf{b}^*-{}^*(\mathbf{v}-\mathbf{b}),-\mathbf{c}^*-{}^*(\mathbf{w}-\mathbf{c}))} \\ &= q^{\frac{1}{2}\Lambda({}^*\mathbf{v},{}^*\mathbf{w})} \sum_{\mathbf{e}\in\mathcal{K}(Q)} q^{-\frac{1}{2}\langle \mathbf{e},\mathbf{v}+\mathbf{w}-\mathbf{e}\rangle} |Gr_{\mathbf{e}}(V\oplus W)| X^{-\mathbf{e}^*-{}^*(\mathbf{v}+\mathbf{w}-\mathbf{e})} \\ &= q^{\frac{1}{2}\Lambda({}^*\mathbf{v},{}^*\mathbf{w})} X_{V\oplus W}. \end{aligned}$$

$\square$

Theorem III.4.4. Let V and I be representations of $(Q,\mathbf{d})$ such that I is injective and $\mathrm{supp}(\mathrm{soc}\, I)\cap \mathrm{supp}(V)=\emptyset$, then

$$X_V X^{{}^*\mathbf{i}} = q^{-\frac{1}{2}\Lambda({}^*\mathbf{v},{}^*\mathbf{i})} X_V * X^{{}^*\mathbf{i}}$$

and

$$X_V X^{{}^*\mathbf{i}} = q^{-\Lambda({}^*\mathbf{v},{}^*\mathbf{i})} X^{{}^*\mathbf{i}} X_V.$$

Proof. We compute the following products using equation (III.1) and Lemma II.1.3:

$$\begin{aligned}
X_V X^{{}^*\mathbf{i}} &= \sum_{\mathbf{e}\in\mathcal{K}(Q)} q^{-\frac{1}{2}\langle \mathbf{e},\mathbf{v}-\mathbf{e}\rangle} |Gr_{\mathbf{e}}(V)| q^{\frac{1}{2}\Lambda(-\mathbf{e}^*-{}^*(\mathbf{v}-\mathbf{e}),{}^*\mathbf{i})} X^{-\mathbf{e}^*-{}^*(\mathbf{v}-\mathbf{e})+{}^*\mathbf{i}} \\
&= q^{-\frac{1}{2}\Lambda({}^*\mathbf{v},{}^*\mathbf{i})} \sum_{\mathbf{e}\in\mathcal{K}(Q)} q^{-\frac{1}{2}\langle \mathbf{e},\mathbf{v}-\mathbf{e}\rangle} |Gr_{\mathbf{e}}(V)| q^{\frac{1}{2}\Lambda({}^*\mathbf{e}-\mathbf{e}^*,{}^*\mathbf{i})} X^{-\mathbf{e}^*-{}^*(\mathbf{v}-\mathbf{e})+{}^*\mathbf{i}} \\
&= q^{-\frac{1}{2}\Lambda({}^*\mathbf{v},{}^*\mathbf{i})} \sum_{\mathbf{e}\in\mathcal{K}(Q)} q^{-\frac{1}{2}\langle \mathbf{e},\mathbf{v}-\mathbf{e}\rangle} |Gr_{\mathbf{e}}(V)| q^{-\frac{1}{2}\langle \mathbf{e},\mathbf{i}\rangle} X^{-\mathbf{e}^*-{}^*(\mathbf{v}-\mathbf{e})+{}^*\mathbf{i}}, \\
X^{{}^*\mathbf{i}} X_V &= \sum_{\mathbf{e}\in\mathcal{K}(Q)} q^{-\frac{1}{2}\langle \mathbf{e},\mathbf{v}-\mathbf{e}\rangle} |Gr_{\mathbf{e}}(V)| q^{\frac{1}{2}\Lambda({}^*\mathbf{i},-\mathbf{e}^*-{}^*(\mathbf{v}-\mathbf{e}))} X^{-\mathbf{e}^*-{}^*(\mathbf{v}-\mathbf{e})+{}^*\mathbf{i}} \\
&= q^{-\frac{1}{2}\Lambda({}^*\mathbf{i},{}^*\mathbf{v})} \sum_{\mathbf{e}\in\mathcal{K}(Q)} q^{-\frac{1}{2}\langle \mathbf{e},\mathbf{v}-\mathbf{e}\rangle} |Gr_{\mathbf{e}}(V)| q^{\frac{1}{2}\Lambda({}^*\mathbf{i},{}^*\mathbf{e}-\mathbf{e}^*)} X^{-\mathbf{e}^*-{}^*(\mathbf{v}-\mathbf{e})+{}^*\mathbf{i}} \\
&= q^{-\frac{1}{2}\Lambda({}^*\mathbf{v},{}^*\mathbf{i})} \sum_{\mathbf{e}\in\mathcal{K}(Q)} q^{-\frac{1}{2}\langle \mathbf{e},\mathbf{v}-\mathbf{e}\rangle} |Gr_{\mathbf{e}}(V)| q^{\frac{1}{2}\langle \mathbf{e},\mathbf{i}\rangle} X^{-\mathbf{e}^*-{}^*(\mathbf{v}-\mathbf{e})+{}^*\mathbf{i}}.
\end{aligned}$$

For any $i \in \operatorname{supp}(\operatorname{soc} I)$ there are no morphisms from V to S_i and thus $\operatorname{Hom}(V, I) = 0$. This implies $\langle \mathbf{e}, \mathbf{i}\rangle = \langle \mathbf{v}, \mathbf{i}\rangle = 0$ and the claim follows. □

Remark III.4.5. It is clear that for any injective valued representations I and J, we have

$$X^{{}^*\mathbf{i}} X^{{}^*\mathbf{j}} = q^{\Lambda({}^*\mathbf{i},{}^*\mathbf{j})} X^{{}^*\mathbf{j}} X^{{}^*\mathbf{i}}.$$

In particular, if $I = I_i$ and $J = I_j$ are the injective hulls of the simples S_i and S_j, respectively, then we have

$$X^{{}^*\mathbf{i}} X^{{}^*\mathbf{j}} = q^{\lambda_{ij}} X^{{}^*\mathbf{j}} X^{{}^*\mathbf{i}}.$$

We see from Theorem III.4.3 that a family of valued representations $V_1, \ldots, V_k$ will have quasi-commuting quantum cluster characters $X_{V_1}, \ldots, X_{V_k}$ exactly when $V_1 \oplus \cdots \oplus V_k$ is rigid. Moreover, Theorem III.4.3 implies that for any rigid decomposable valued representation V we may factor the quantum cluster character X_V. This suggests that we should further restrict to indecomposable V_i such that $V_i \ncong V_j$ for $i \neq j$. Since they both give rise to quasi-commuting families we would naturally suspect that there should be a relationship between clusters of $\mathcal{A}_q(\tilde{B}, \Lambda)$

and basic rigid representations of $(Q, \mathbf{d})$. Now the support condition satisfied by local tilting representations combined with Theorem III.4.4 is exactly what is needed to guarantee that we can obtain in this way a full cluster of n mutable variables. We will make these remarks precise in Chapter V. when we explicitly construct a seed of $\mathcal{A}_q(\tilde{B}, \Lambda)$ from each local tilting representation of $(Q, \mathbf{d})$.

CHAPTER IV

EXTERNAL MUTATIONS

We begin this chapter with an account of the ideas which led us to the main results. In [CZ], Caldero and Zelevinsky deduce formulas for the Euler characteristics of Grassmannians in indecomposable representations of the Kronecker quiver with two vertices and two arrows, these are given by a product of two binomial coefficients. The work presented in this chapter grew out of a desire to extend these formulas to describe cluster variables in the quantum Kronecker cluster algebra associated to the pair $\tilde{B} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. A natural analog to replace the Euler characteristic is the number of points in the Grassmannians of subrepresentations over a finite field. Szanto computed these counts in [Sz]. His approach develops a recursion relating the Grassmannian of subrepresentations in an indecomposable representation with the Grassmannian of its image under a reflection functor. This recursion can then be solved explicitly using a q-analog of a classical binomial identity. Using Berenstein-Zelevinsky relations in the quantum Kronecker cluster algebra one can use Szanto's formulas to show that the quantum cluster character applied to rigid objects gives cluster variables in this case. This fact was discovered independently by Phillip Lampe in [Lam] and so we omit this proof.

Szanto's recursion relies only on the well-established properties of simple projective and simple injective representations and the corresponding reflection functors at the sink and source vertices. Section IV.1. is dedicated to extending the Szanto recursion to this more general setting. We remark that this recursion relates the Grassmannians in representations of quivers with changed orientation. Thus it is most natural to suspect that they would be related to external mutations. The remarkable main result of Section IV.2. establishes that the Szanto recursion on quiver Grassmannians does match the external mutation given by mutating the initial cluster at a sink or source vertex. More specifically, mutating the initial cluster at a sink or source vertex transforms the quantum cluster character of an indecomposable representation to the quantum cluster character of its reflection.

In this chapter we will be considering the quantum cluster character applied to representations of different quivers. To avoid cluttering our notation we will not refer to the specific quiver corresponding to the initial cluster. However, the underlying quiver is inherent in the representation and so this should not lead to any confusion. Thus we may state our main result as follows.

Theorem IV.0.6. Suppose vertex k is a sink or a source in Q. Then for any $V \in \mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})\langle k \rangle$, $\mu_k X_V = X_{\Sigma_k V}$. In particular, if X_V is a cluster variable in $\mathcal{A}_q(\tilde{B}, \Lambda)$ then $X_{\Sigma_k V}$ is a cluster variable in $\mathcal{A}_q(\mu_k \tilde{B}, \mu_k \Lambda)$.

Thus any cluster variable obtained from the initial cluster by sink or source mutations can be given explicitly in terms of Grassmannians of subrepresentations. From this observation we obtain the following consequences.

Corollary IV.0.7. If the valued quiver Q only has two vertices, then each cluster variable of $\mathcal{A}_{|\mathbb{F}|}(\tilde{B}, \Lambda)$ is of the form X_V for some exceptional representation $V \in \mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$.

Corollary IV.0.8. If the valued quiver Q is an orientation of a finite-type Dynkin diagram, then each cluster variable of $\mathcal{A}_{|\mathbb{F}|}(\tilde{B}, \Lambda)$ is of the form X_V for some exceptional representation $V \in \mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$.

One naturally conjectures that all non-initial cluster variables are given by the quantum cluster character. This fact will be established in Chapter V.

IV.1. Recursion on Quiver Grassmannians

Suppose for the remainder of this section that vertex k is a source in the quiver Q. We will prove a recursion for the Grassmannians of subrepresentations of $\mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$. It will be necessary to specify in which category the Grassmannian occurs. We will write Gr^Q for Grassmannians in $\mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$. We will also need to consider Grassmannians of $\mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})\langle k \rangle$, denoted $Gr^{Q\langle k \rangle}$. We will use the following convention for Grassmannians of $V \in \mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})\langle k\rangle$:

$$Gr_{\mathbf{e}}^{Q\langle k\rangle}(V) = \{0 \to W \to V \to V/W \to 0 : [W] = \mathbf{e}; W, V/W \in \mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})\langle k\rangle\}.$$

Theorem IV.1.1. Let $V \in \mathrm{Rep}_{\mathbb{F}}(\mu_k Q, \mathbf{d})\langle k\rangle$ and $\mathbf{e} \in \mathcal{K}(Q)$. Then we have the following recursion for the number of points in $Gr_{\mathbf{e}}^{\mu_k Q}(V)$:

$$|Gr_{\mathbf{e}}^{\mu_k Q}(V)| = \sum_{c \geq 0} q^{d_k c(\sigma_k(\mathbf{e})_k + c)} |Gr_{c\alpha_k}^{Q}((m_k - \sigma_k(\mathbf{e})_k - e_k)S_k)||Gr_{\sigma_k(\mathbf{e})+c\alpha_k}^{Q}(\Sigma_k V)| \tag{IV.1}$$

where $\mathbf{v} = [V]$ denotes the dimension vector of V.

Proof. The main content of the proof is contained in the following lemmas.

Lemma IV.1.2. For $V \in \mathrm{Rep}_{\mathbb{F}}(\mu_k Q, \mathbf{d})\langle k\rangle$ and $\mathbf{e} \in \mathcal{K}(Q)$ we have

$$|Gr_{\mathbf{e}}^{\mu_k Q}(V)| = \sum_{a \geq 0} |Gr_{a\alpha_k}^{\mu_k Q}((m_k - e_k + a)S_k')||Gr_{\mathbf{e}-a\alpha_k}^{\mu_k Q\langle k\rangle}(V)|.$$

Proof. Write S_k' for the simple projective representation of $\mu_k Q$. Consider the map $\zeta : Gr_{\mathbf{e}}^{\mu_k Q}(V) \to \coprod_{a \geq 0} Gr_{\mathbf{e}-a\alpha_k}^{\mu_k Q\langle k\rangle}(V)$ given by $U \oplus aS_k' \mapsto U$. This map is clearly surjective. Suppose $f : U \hookrightarrow V$ is an element of $Gr_{\mathbf{e}-a\alpha_k}^{\mu_k Q\langle k\rangle}(V)$. The fibers of ζ are given by $\zeta^{-1}(U) = \{(f, g) : U \oplus aS_k' \hookrightarrow V\}$. Since $U \in Gr_{\mathbf{e}-a\alpha_k}^{\mu_k Q\langle k\rangle}(V)$ and S_k' is projective, the images of such f and g must be disjoint and the fiber over U may be written as $\zeta^{-1}(U) = \{g : aS_k' \hookrightarrow V/U\} = Gr_{a\alpha_k}^{\mu_k Q}((m_k - e_k + a)S_k')$. Since the fiber only depends on the dimension vector of U, the result follows. ☐

Lemma IV.1.3. For $V \in \mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})\langle k\rangle$ and $\mathbf{e} \in \mathcal{K}(Q)$ we have

$$|Gr_{\mathbf{e}}^{Q}(V)| = \sum_{d \geq 0} |Gr_{d\alpha_k}^{Q}((e_k + d)S_k)||Gr_{\mathbf{e}+d\alpha_k}^{Q\langle k\rangle}(V)|.$$

Proof. Let Q^* denote the quiver obtained from Q by reversing all the arrows. Note that vertex k is a sink in Q^*. We will use the same notation for the linear duality functor $(-)^* = \mathrm{Hom}(-, \mathbb{F})$:

$\mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d}) \to \mathrm{Rep}_{\mathbb{F}}(Q^*, \mathbf{d})$. The following equalities are immediate:

$$
\begin{aligned}
|Gr_{\mathbf{e}}^{Q}(V)| = |Gr_{\mathbf{v}-\mathbf{e}}^{Q^*}(V^*)| &= \sum_{d \geq 0} |Gr_{d\alpha_k}^{Q^*}((e_k + d)S_k^*)||Gr_{\mathbf{v}-\mathbf{e}-d\alpha_k}^{Q^*\langle k \rangle}(V^*)| \\
&= \sum_{d \geq 0} |Gr_{e_k\alpha_k}^{Q}((e_k + d)S_k)||Gr_{\mathbf{e}+d\alpha_k}^{Q\langle k \rangle}(V)| \\
&= \sum_{d \geq 0} |Gr_{d\alpha_k}^{Q}((e_k + d)S_k)||Gr_{\mathbf{e}+d\alpha_k}^{Q\langle k \rangle}(V)|.
\end{aligned}
$$

where the second equality follows from Lemma IV.1.2. □

The following result is well-known, we include a sketch of the proof for completeness.

Lemma IV.1.4. Suppose $V, W \in \mathrm{Vect}_{\mathbb{F}}$ and $\ell \in \mathbb{Z}_{>0}$. Then

$$
|Gr_\ell(V \oplus W)| = \sum_{a+b=\ell} q^{a(w-b)} |Gr_a(V)||Gr_b(W)|.
$$

Proof. Write $\pi_1 : V \oplus W \to V$ for the natural projection and consider W as a subspace of $V \oplus W$ via the natural inclusion. Consider the map $\varphi : Gr_\ell(V \oplus W) \to \coprod_{a+b=\ell} Gr_a(V) \times Gr_b(W)$ given by $U \mapsto (\pi_1 U, U \cap W)$. This map is clearly surjective and the fiber over any point $(A, B) \in Gr_a(V) \times Gr_b(W)$ is isomorphic to an affine space of dimension $a(w - b)$. The result follows. □

Putting the preceding three lemmas together we get our recursion. Let $a = c + d$ and consider the following:

$$\begin{aligned}
&\sum_{c\geq 0} q^{d_k c(\sigma_k(\mathbf{e})_k+c)}|Gr^Q_{c\alpha_k}((m_k-\sigma_k(\mathbf{e})_k-e_k)S_k)||Gr^Q_{\sigma_k(\mathbf{e})+c\alpha_k}(\Sigma_k V)|\\
&=\sum_{c\geq 0}\sum_{d\geq 0} q^{d_k c(\sigma_k(\mathbf{e})_k+c)}|Gr^Q_{c\alpha_k}((m_k-\sigma_k(\mathbf{e})_k-e_k)S_k)||Gr^Q_{d\alpha_k}((\sigma_k(\mathbf{e})_k+(c+d))S_k)|\times\\
&\qquad\times|Gr^{Q\langle k\rangle}_{\sigma_k(\mathbf{e})+(c+d)\alpha_k}(\Sigma_k V)|\\
&=\sum_{a\geq 0}\sum_{c\geq 0} q^{d_k c(\sigma_k(\mathbf{e})_k+c)}|Gr^Q_{c\alpha_k}((m_k-\sigma_k(\mathbf{e})_k-e_k)S_k)||Gr^Q_{(a-c)\alpha_k}((\sigma_k(\mathbf{e})_k+a)S_k)|\times\\
&\qquad\times|Gr^{Q\langle k\rangle}_{\sigma_k(\mathbf{e})+a\alpha_k}(\Sigma_k V)|\\
&=\sum_{a\geq 0}|Gr^Q_{a\alpha_k}((m_k-e_k+a)S_k)||Gr^{Q\langle k\rangle}_{\sigma_k(\mathbf{e})+a\alpha_k}(\Sigma_k V)|\\
&=\sum_{a\geq 0}|Gr^{\mu_k Q}_{a\alpha_k}((m_k-e_k+a)S'_k)||Gr^{\mu_k Q\langle k\rangle}_{\mathbf{e}-a\alpha_k}(V)|\\
&=|Gr^{\mu_k Q}_{\mathbf{e}}(V)|.
\end{aligned}$$

To see the second to last equality note that each of $Gr^Q_{a\alpha_k}((m_k-e_k+a)S_k)$ and $Gr^{\mu_k Q}_{a\alpha_k}((m_k-e_k+a)S'_k)$ is just the classical Grassmannian of vector subspaces $Gr_a(\mathbb{F}_{d_k}^{m_k-e_k+a})$ and $Gr^{Q\langle k\rangle}_{\sigma_k(\mathbf{e})+a\alpha_k}(\Sigma_k V) = Gr^{\mu_k Q\langle k\rangle}_{\mathbf{e}-a\alpha_k}(V)$ under the exact equivalence of categories Σ_k. □

IV.2. Relationship to the Quantum Cluster Character

We now show that the recursion on the Grassmannians just obtained matches the recursion in the quantum cluster algebra obtained by mutating the initial cluster in direction k.

Proof of Theorem IV.0.6. We will prove Theorem IV.0.6 in the case when vertex k is a source in Q. This will immediately imply that the theorem holds for mutations of the initial cluster in a sink direction. Indeed, assume the result holds when k is a source in $\mu_k Q$ and suppose $V \in \mathrm{Rep}_{\mathbb{F}}(Q,\mathbf{d})\langle k\rangle$. If we begin with $X_{\Sigma_k V}$ in $\mathcal{A}_q(\mu_k\tilde{B},\mu_k\Lambda)$ and mutate the initial cluster in direction k we will get $\mu_k X_{\Sigma_k V} = X_{\Sigma_k\Sigma_k V} = X_V$ in the quantum cluster algebra $\mathcal{A}_q(\tilde{B},\Lambda)$. But the mutation of clusters is involutive so starting with X_V and mutating the initial cluster in direction k gives $\mu_k X_V = \mu_k\mu_k X_{\Sigma_k V} = X_{\Sigma_k V}$.

Suppose $V \in \mathrm{Rep}_{\mathbb{F}}(Q,\mathbf{d})\langle k\rangle$. Our plan is as follows: first we expand X_V via the quantum cluster character (III.8) to get an element of $\mathcal{A}_q(\tilde{B},\Lambda)$. Then we mutate the initial cluster in direction k (apply the external mutation μ_k) to get an element of the quantum cluster algebra $\mathcal{A}_q(\mu_k\tilde{B}, \mu_k\Lambda)$ which we then show to be $X_{\Sigma_k V}$. This result holds regardless of whether or not X_V is a cluster variable. We record the main technical details in the following Lemma.

Lemma IV.2.1. Write $B' = \mu_k B$, $\mathbf{c} = \sum c_i\alpha_i = {}^*\mathbf{e} - \mathbf{e}^* - {}^*\mathbf{v}$, and $\mathbf{f} = \sum f_i\alpha_i = \sigma_k\mathbf{e} + r\alpha_k$. Then we have the following identities:

(a) $\sigma_k = F$;

(b) $EB^- = (B')^- F$;

(c) $E\mathbf{c} + r\mathbf{b}^k = -B'(\sigma_k\mathbf{e} + r\alpha_k) - (B')^-\sigma_k\mathbf{v}$;

(d) $c_k = \sigma_k(\mathbf{v})_k - \sigma_k(\mathbf{f})_k - f_k$.

Proof. We write $\sigma^k = (\sigma^k_{ij})$ in matrix form as

$$\sigma^k_{ij} = \begin{cases} 1 & \text{if } i = j \neq k; \\ -1 & \text{if } i = j = k; \\ |b_{kj}| & \text{if } i = k \neq j. \end{cases}$$

Since vertex k is a source we have $|b_{kj}| = [b_{kj}]_+ = b_{kj}$ and identity (a) follows.

To see the identity (b), we compute the ij-entry of the matrix on each side. Using the definitions of E and B^- this is given on the left by

$$\sum_{\ell=1}^{n} e_{i\ell}b^-_{\ell j} = \begin{cases} b^-_{ij} + [-b_{ik}]_+ b^-_{kj} & \text{if } i \neq k; \\ -b^-_{kj} & \text{if } i = k. \end{cases}$$

From the definitions of $(B')^-$ and F on the right we get

$$\sum_{\ell=1}^{n} b'^-_{i\ell} f_{\ell j} = \begin{cases} b'^-_{ij} + b'^-_{ik}[b_{kj}]_+ & \text{if } j \neq k; \\ -b'^-_{ik} & \text{if } j = k. \end{cases}$$

Using that k is a source in Q we see that each of these is equal to

$$\begin{cases} -1 & \text{if } i=j=k; \\ 1-[-b_{ik}]_+[b_{ki}]_+ & \text{if } i=j\neq k; \\ -b_{ik} & \text{if } i\neq k=j; \\ b_{kj} & \text{if } i=k\neq j; \\ -[b_{ij}]_+ - [-b_{ik}]_+[b_{kj}]_+ & \text{if } i\neq k\neq j. \end{cases}$$

The identity (c), can be seen as follows:

$$\begin{aligned} E\mathbf{c}+r\mathbf{b}^k &= -EB\mathbf{e}-EB^-\mathbf{v}+r\mathbf{b}^k \\ &= -EBFF\mathbf{e}-(B')^-F\mathbf{v}+r\mathbf{b}^k \\ &= -B'F\mathbf{e}-(B')^-\sigma_k\mathbf{v}+r\mathbf{b}^k \\ &= -B'(\sigma_k\mathbf{e}+r\alpha_k)-(B')^-\sigma_k\mathbf{v}. \end{aligned}$$

Notice that the k^{th} row of B^- is the negative of the k^{th} row of F and thus $(-B^-\mathbf{v})_k = (F\mathbf{v})_k = \sigma_k(\mathbf{v})_k$. Also the k^{th} entry of $-B\mathbf{e} = -B(\sigma_k\mathbf{f}+r\alpha_k)$ is equal to the k^{th} entry of $-B\sigma_k\mathbf{f}$. But $-B\sigma_k = -BF = -EEBF = -EB'$ whose k^{th} row is the same as the k^{th} row of $-B$. With the exception of the k^{th} entry this is the same as the k^{th} row of $-F$ since vertex k is a source. So the k^{th} entry of $-B\mathbf{e}$ is the same as the k^{th} entry of $-F\mathbf{f}-f_k\alpha_k = -\sigma_k\mathbf{f}-f_k\alpha_k$. The identity (d) follows. □

We now expand X_V in terms of the initial seed $(\{X_1,\ldots,X_k',\ldots,X_n\},Q)$. To simplify notation we will write $A_{\mathbf{e}}^Q(V) = q^{-\frac{1}{2}\langle\mathbf{e},\mathbf{v}-\mathbf{e}\rangle}|Gr_{\mathbf{e}}^Q(V)|$. Renormalizing in the recursion (IV.1) we get:

$$A_{\mathbf{e}}^{\mu_k Q}(V) = \sum_{c\geq 0}\begin{bmatrix} m_k-\sigma_k(\mathbf{e})_k-e_k \\ c\end{bmatrix}_{q^{d_k/2}} A^Q_{\sigma_k(\mathbf{e})+c\alpha_k}(\Sigma_k V).$$

The quantum cluster character applied to V is given by

$$X_V = \sum_{\mathbf{e}} q^{-\frac{1}{2}\langle\mathbf{e},\mathbf{v}-\mathbf{e}\rangle}|Gr_{\mathbf{e}}^Q(V)|X^{({}^*\mathbf{e}-\mathbf{e}^*-{}^*\mathbf{v})} = \sum_{\mathbf{e}} A_{\mathbf{e}}^Q(V)X^{({}^*\mathbf{e}-\mathbf{e}^*-{}^*\mathbf{v})}.$$

We apply the external mutation μ_k given by equation (III.6) with $\mathbf{c} = {}^*\mathbf{e} - \mathbf{e}^* - {}^*\mathbf{v}$ to get $\mu_k X_V$ in terms of the seed $(\{X_1, \ldots, X_k, \ldots, X_n\}, \mu_k Q)$:

$$\mu_k X_V = \mu_k \left(\sum_{\mathbf{e}} A^Q_{\mathbf{e}}(V) X^{({}^*\mathbf{e} - \mathbf{e}^* - {}^*\mathbf{v})} \right) = \sum_{\mathbf{e}} \sum_{r \geq 0} \begin{bmatrix} c_k \\ r \end{bmatrix}_{q^{d_k/2}} A^Q_{\mathbf{e}}(V) X^{(E\mathbf{c} + r\mathbf{b}^k)}.$$

Using identity (c) of Lemma IV.2.1 we get

$$\mu_k X_V = \sum_{\mathbf{e}} \sum_{r \geq 0} \begin{bmatrix} c_k \\ r \end{bmatrix}_{q^{d_k/2}} A^Q_{\mathbf{e}}(V) X^{(-B'(\sigma_k \mathbf{e} + r\alpha_k) - (B')^- \sigma_k \mathbf{v})}$$

Now the substitution $\mathbf{f} = \sigma_k \mathbf{e} + r\alpha_k$ and identity (d) from Lemma IV.2.1 gives

$$\begin{aligned} \mu_k X_V &= \sum_{\mathbf{f}} \sum_{r \geq 0} \begin{bmatrix} \sigma_k(\mathbf{v})_k - \sigma_k(\mathbf{f})_k - f_k \\ r \end{bmatrix}_{q^{d_k/2}} A^Q_{\sigma_k(\mathbf{f}) + r\alpha_k}(\Sigma_k \Sigma_k V) X^{(-B'(\sigma_k \mathbf{e} + r\alpha_k) - (B')^- \sigma_k \mathbf{v})} \\ &= \sum_{\mathbf{f}} \sum_{r \geq 0} \begin{bmatrix} \sigma_k(\mathbf{v})_k - \sigma_k(\mathbf{f})_k - f_k \\ r \end{bmatrix}_{q^{d_k/2}} A^Q_{\sigma_k(\mathbf{f}) + r\alpha_k}(\Sigma_k \Sigma_k V) X^{({}^*\mathbf{f} - \mathbf{f}^* - {}^*\sigma_k \mathbf{v})}. \end{aligned}$$

Then we may apply the recursion (IV.1) to get

$$\mu_k X_V = \sum_{\mathbf{f}} A^{\mu_k Q}_{\mathbf{f}}(\Sigma_k V) X^{({}^*\mathbf{f} - \mathbf{f}^* - {}^*\sigma_k(\mathbf{v}))} = X_{\Sigma_k V}.$$

This completes the proof of Theorem IV.0.6. □

IV.3. Consequences

In order to state our next result we introduce some new notation for describing a cluster variable. Define $X^Q_{[a_0]} := X^{\alpha_{a_0}}$ in $\mathcal{A}_q(\tilde{B}, \Lambda)$, so the ordered tuple $(X_{[1]}, \ldots, X_{[n]})$ forms the initial cluster of $\mathcal{A}_q(\tilde{B}, \Lambda)$. Now recursively define

$$X^Q_{[a_0; a_1, a_2, \ldots, a_r]} = \mu_{a_r} X^{\mu_{a_r} Q}_{[a_0; a_1, a_2, \ldots, a_{r-1}]}$$

where $X^{\mu_{a_r} Q}_{[a_0; a_1, a_2, \ldots, a_{r-1}]}$ is in $\mathcal{A}_q(\mu_{a_r} \tilde{B}, \mu_{a_r} \Lambda)$ and the birational isomorphism μ_{a_r} pulls it back to $\mathcal{A}_q(\tilde{B}, \Lambda)$. Alternatively one could start with the initial ordered seed $(\mathbf{X}, \tilde{B}, \Lambda)$ of $\mathcal{A}_q(\tilde{B}, \Lambda)$ and perform the internal mutations in direction a_1, then a_2, etc. to obtain the ordered seed $\mu_{a_r} \cdots \mu_{a_1}(\mathbf{X}, \tilde{B}, \Lambda)$ then $X^Q_{[a_0; a_1, a_2, \ldots, a_r]}$ is the $a_0{}^{th}$ variable in the cluster $\mu_{a_r} \cdots \mu_{a_1} \mathbf{X}$. When it is clear from context we will drop the Q from the notation. Here are some simple observations that follow from this notation:

1. If $a_i = a_{i+1}$ for some $i > 0$, then $X^Q_{[a_0;a_1,a_2,\ldots,a_r]} = X^Q_{[a_0;a_1,\ldots,a_{i-1},a_{i+2},\ldots,a_r]}$.
2. If $a_0 \neq a_r$, then $X^Q_{[a_0;a_1,a_2,\ldots,a_r]} = X^Q_{[a_0;a_1,a_2,\ldots,a_{r-1}]}$.
3. If we mutate the seed $(\{X^Q_{[1;a_1,\ldots,a_r]}, \ldots, X^Q_{[n;a_1,\ldots,a_r]}\}, \tilde{B}, \Lambda)$ in direction t we get the new seed $(\{X^Q_{[1;a_1,\ldots,a_r,t]}, \ldots, X^Q_{[n;a_1,\ldots,a_r,t]}\}, \mu_t\tilde{B}, \mu_t\Lambda)$.
4. If we start with $X^Q_{[a_0;a_1,a_2,\ldots,a_{r-1}]}$ and mutate the *initial* seed in direction t then we get $X^{\mu_t Q}_{[a_0;t,a_1,a_2,\ldots,a_{r-1}]}$.

Theorem IV.3.1. Suppose $k_1, k_2, \ldots, k_{r+1}$ is an admissible sequence of vertices in Q. Let $V \in \mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$ be the unique indecomposable representation of Q with $[V] = \sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_r}(\alpha_{k_{r+1}})$. Then in the cluster algebra $\mathcal{A}_{|\mathbb{F}|}(\tilde{B}, \Lambda)$ we have $X^Q_{[k_{r+1};k_1,k_2,\ldots,k_{r+1}]} = X_V$.

Proof. The following Lemma states that the first mutation from the initial cluster is always given by the quantum cluster character applied to a simple representation.

Lemma IV.3.2. Let Q be a valued quiver. Inside the quantum cluster algebra $\mathcal{A}_{|\mathbb{F}|}(\tilde{B}, \Lambda)$, we have $X^Q_{[k;k]} = X_{S_k}$ where S_k is the simple representation associated to vertex k in Q.

Proof. First note that S_k has only two subrepresentations 0 and S_k. So we have

$$X_{S_k} = X^{-\mathbf{0}^* - {}^*(\alpha_k - \mathbf{0})} + X^{-\alpha_k^* - {}^*(\alpha_k - \alpha_k)} = X^{-\alpha_k + \sum_{\ell=1}^{n}[b_{\ell k}]_+\alpha_\ell} + X^{-\alpha_k + \sum_{\ell=1}^{n}[-b_{\ell k}]_+\alpha_\ell}.$$

But the last expression is just the exchange relation defining $X_{[k;k]}$. □

Suppose the seed $(\mathbf{X}, \tilde{B}, \Lambda)$, can be transformed into the seed $(\mathbf{X}', \tilde{B}', \Lambda')$, by a sequence of mutations in directions $k_1, k_2, \ldots, k_{r+1}$ such that the corresponding sequence of vertices is admissible in Q, i.e. $Q' = \mu_{k_{r+1}}\mu_{k_r}\cdots\mu_{k_1}Q$.

The cluster variable $X'_{k_{r+1}}$ in $\mathcal{A}_q(\tilde{B}', \Lambda')$ may be considered as the cluster variable $X^{\mu_{k_{r+1}}Q'}_{[k_{r+1};k_{r+1}]}$ in $\mathcal{A}_q(\mu_{k_{r+1}}\tilde{B}', \mu_{k_{r+1}}\Lambda')$. By Lemma IV.3.2 we can write $X_{[k_{r+1};k_{r+1}]} = X_{S_{k_{r+1}}}$ for $S_{k_{r+1}} \in \mathrm{Rep}_{\mathbb{F}}(\mu_{k_{r+1}}Q', \mathbf{d})$. Now assume for some $i \in [1, r]$ that we have

$$X_{[k_{r+1};k_{i+1},\ldots,k_{r+1}]} = X_{\Sigma_{k_{i+1}}\cdots\Sigma_{k_r}(S_{k_{r+1}})}$$

inside the quantum cluster algebra $\mathcal{A}_q(\mu_{k_{i+1}}\cdots\mu_{k_{r+1}}\tilde{B}', \mu_{k_{i+1}}\cdots\mu_{k_{r+1}}\Lambda')$ where $\Sigma_{k_{i+1}}\cdots\Sigma_{k_r}(S_{k_{r+1}}) \in \mathrm{Rep}_{\mathbb{F}}(\mu_{k_{i+1}}\cdots\mu_{k_{r+1}}Q', \mathbf{d})$. Notice that this representation is indecomposable and, since the sequence of vertices was admissible, it does not contain S_{k_i} as a direct summand. Thus mutating the initial cluster in direction k_i gives

$$X_{[k_{r+1};k_i,k_{i+1},\ldots,k_{r+1}]} = X_{\Sigma_{k_i}\Sigma_{k_{i+1}}\cdots\Sigma_{k_r}(S_{k_{r+1}})}$$

in the quantum cluster algebra $\mathcal{A}_q(\mu_{k_i}\mu_{k_{i+1}}\cdots\mu_{k_{r+1}}\tilde{B}', \mu_{k_i}\mu_{k_{i+1}}\cdots\mu_{k_{r+1}}\Lambda')$. By induction we have the following equality inside the quantum cluster algebra $\mathcal{A}_q(\mu_{k_1}\cdots\mu_{k_{r+1}}\tilde{B}', \mu_{k_1}\cdots\mu_{k_{r+1}}\Lambda') = \mathcal{A}_q(\tilde{B}, \Lambda)$:

$$X_{[k_{r+1};k_1,k_2,\ldots,k_{r+1}]} = X_{\Sigma_{k_1}\cdots\Sigma_{k_r}(S_{k_{r+1}})}$$

where $\Sigma_{k_1}\cdots\Sigma_{k_r}(S_{k_{r+1}}) \in \mathrm{Rep}_{\mathbb{F}}(\mu_{k_1}\cdots\mu_{k_{r+1}}Q') = \mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$. This completes the proof. □

We will call a quiver Q "almost acyclic" if there exists i so that the valued quiver $\mu_i Q$ is acyclic. We will call a seed/cluster of of $\mathcal{A}_{|\mathbb{F}|}(\tilde{B}, \Lambda)$ almost acyclic if the corresponding quiver is almost acyclic.

Corollary IV.3.3. Suppose the valued quiver Q is acyclic. Then each cluster variable of $\mathcal{A}_{|\mathbb{F}|}(\tilde{B}, \Lambda)$ in an almost acyclic cluster is of the form X_V for some indecomposable $V \in \mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$.

Proof. In [CK, Corollary 4], the authors show that all acyclic clusters are connected by sink and source mutations. The result then follows from Theorem IV.3.1. □

Corollary IV.0.7 immediately follows.

Proof of Corollary IV.0.7. For rank 2 cluster algebras, the valued quiver associated to any cluster is acyclic. So the result follows from Corollary IV.3.3. □

We also get Corollary IV.0.8.

Proof of Corollary IV.0.8. Let $k_1, k_2, \ldots, k_n$ be an admissible ordering of Q and $C = \Sigma_{k_1}\Sigma_{k_2}\cdots\Sigma_{k_n}$ be the corresponding Coxeter functor. For each $t \in [1, n]$ write $P_{k_t} = \Sigma_{k_1}\Sigma_{k_2}\cdots\Sigma_{k_{t-1}}S_{k_t}$ for the corresponding projective representation where $S_{k_t} \in \mathrm{Rep}_{\mathbb{F}}(\mu_{k_t}\mu_{k_{t+1}}\cdots\mu_{k_{t_n}}Q, \mathbf{d})$ is the simple representation associated to vertex k_t. By [DR, Propositions 1.9 and 2.6], every indecomposable representation of Q is of the form $C^r P_{k_t}$ for some r.

From Theorem IV.3.1 we see that each $X_{C^r P_{k_t}}$ is a cluster variable in $\mathcal{A}_{|\mathbb{F}|}(\tilde{B}, \Lambda)$. Following [FZ2] there is a one-to-one correspondence between non-initial cluster variables of $\mathcal{A}_{|\mathbb{F}|}(\tilde{B}, \Lambda)$ and positive roots in the root system associated to Q. Since there is a one-to-one correspondence between positive roots and indecomposable representations of Q, we see that every cluster variable must be obtained in this way. □

In Chapter V. we will extend these results to show that all cluster variables in a quantum cluster algebra with acyclic initial exchange matrix are given by the quantum cluster character.

IV.4. Open Problems

We have shown that the mutation of the initial cluster at a sink or source vertex coincides with the action of the reflection functor on representations, however our understanding of these external mutations is restricted to sink and source mutations. One would naturally hope to remove this restriction and understand external mutations in arbitrary directions. In the classical skew-symmetric setting this was accomplished by Derksen, Weyman, and Zelevinsky [DWZ1], [DWZ2] through a study of the representation theory of a new object called "quiver with potential". They introduce "mutation functors" at arbitrary vertices and show that applying a mutation functor to a representation corresponds to applying the external mutation to the cluster character. It would be natural to ask for a quantum skew-symmetrizable analog and define valued quivers with potential. Recent attempts in this direction reveal some obstructions to extending the DWZ story to valued quivers [Lab].

CHAPTER V

INTERNAL MUTATIONS

This chapter is devoted to the proof that all non-initial cluster variables of $\mathcal{A}_q(\tilde{B},\Lambda)$ are given by the quantum cluster character applied to exceptional representations of the associated valued quiver $(Q,\mathbf{d})$. The results of this chapter rely on realizing the internal mutation of cluster variables as a shadow of the mutation of local tilting representations defined in Section II.5. The results of Section V.1. are derived from the arguments of [H1, Main Theorem].

V.1. Mutations of Exchange Matrices

Here we show that the Fomin-Zelevinsky mutation of exchange matrices associated to local tilting representations coincides with the mutation operation for local tilting representations defined in Section II.5.

Since the valued quiver Q has no oriented cycles, according to [Ri2, Section 2.4] the Ringel-Euler form is non-degenerate. Also by Theorem II.4.1 the classes of indecomposable summands of a local tilting representation T are linearly independent in the Grothendieck group $\mathcal{K}(Q)$. Since there are as many non-isomorphic summands of T as simple modules for Q_T, their isomorphism classes form a basis of the Grothendieck group $\mathcal{K}(Q_T)\subset\mathcal{K}(Q)$. Notice that the isomorphism classes of projective objects P_i for $i\notin\operatorname{supp}T$ are linearly independent in $\mathcal{K}(Q)$ and the Ringel-Euler form is non-degenerate when restricted to their span. Moreover, if we consider the isomorphism classes of summands of T and these projective representations we obtain a basis for all of $\mathcal{K}(Q)$. Define the natural projections $\pi_T:\mathcal{K}(Q)\to\mathcal{K}(Q_T)$ and $\pi_T^c:\mathcal{K}(Q)\to\operatorname{span}\{P_i : i\notin\operatorname{supp}T\}$.

Definition V.1.1.

1. Since the Ringel-Euler form is non-degenerate on $\mathcal{K}(Q_T)$, for $i\in\operatorname{supp}T$ we may define left

and right duals $\lambda_i, \rho_i \in \mathcal{K}(Q)$ and λ_i^T, ρ_i^T in $\mathcal{K}(Q_T)$ satisfying:

$$\pi_T(\lambda_i) = \lambda_i^T \text{ and } \pi_T(\rho_i) = \rho_i^T,$$
$$\langle \lambda_i^T, T_j \rangle = \delta_{ij} d_i \text{ and } \langle T_j, \rho_i^T \rangle = \delta_{ij} d_i \text{ for all } j \in \operatorname{supp} T.$$

2. For $i \notin \operatorname{supp} T$ we define left and right duals $\lambda_i, \rho_i \in \mathcal{K}(Q)$, $\lambda_i^T, \rho_i^T \in \mathcal{K}(Q_T)$, and $\lambda_i^c, \rho_i^c \in \operatorname{span}\{P_i : i \notin \operatorname{supp} T\}$ satisfying the following identities:

$$\pi_T(\lambda_i) = \lambda_i^T \text{ and } \pi_T(\rho_i) = \rho_i^T,$$
$$\pi_T^c(\lambda_i) = \lambda_i^c \text{ and } \pi_T^c(\rho_i) = \rho_i^c,$$
$$\langle \lambda_i^T, T_j \rangle = \langle S_i, T_j \rangle \text{ and } \langle T_j, \rho_i^T \rangle = \langle T_j, S_i \rangle \text{ for all } j \in \operatorname{supp} T,$$
$$\langle \lambda_i^c, P_j \rangle = \delta_{ij} d_i \text{ and } \langle P_j, \rho_i^c \rangle = \delta_{ij} d_i \text{ for all } j \notin \operatorname{supp} T.$$

Write $\lambda_i = \sum_{j \in \operatorname{supp} T} \ell_{ij}[T_j] + \sum_{j \notin \operatorname{supp} T} \ell_{ij}[P_j]$ and $\rho_i = \sum_{j \in \operatorname{supp} T} r_{ij}[T_j] + \sum_{j \notin \operatorname{supp} T} r_{ij}[P_j]$. Note that it still remains to define ℓ_{ij} and r_{ij} for $i \in \operatorname{supp} T$ and $j \notin \operatorname{supp} T$. We begin with the following Lemma which motivates that definition.

Lemma V.1.2.

1. The matrices $L^T = (\ell_{ij})$ and $R^T = (r_{ij})$ with rows and columns labeled by $\operatorname{supp} T$ are related by $L^T D^T = (D^T R^T)^t$ where $D^T = (d_{ij})$ is the diagonal matrix given by $d_{ij} = \delta_{ij} d_i$ for $i, j \in \operatorname{supp} T$.

2. The matrices $L^c = (\ell_{ij})$ and $R^c = (r_{ij})$ with rows and columns labeled by $i, j \notin \operatorname{supp} T$ are related by $L^c D^c = (D^c R^c)^t$ where $D^c = (d_{ij})$ is the diagonal matrix given by $d_{ij} = \delta_{ij} d_i$ for $i, j \notin \operatorname{supp} T$.

Proof. By pairing λ_i^T and ρ_j^T for $i, j \in \operatorname{supp} T$ we see that $\ell_{ij}^T d_j = r_{ji}^T d_i$. Similarly, by pairing λ_i^c and ρ_j^c for $i, j \notin \operatorname{supp} T$ we see that $\ell_{ij}^c d_j = r_{ji}^c d_i$. The result follows. □

Thus we complete the definition of the matrices $L = (\ell_{ij})$ and $R = (r_{ij})$ by declaring that $LD = (DR)^t$. Note that this uniquely defines $\pi_T^c(\lambda_i)$ and $\pi_T^c(\rho_i)$ for $i \in \operatorname{supp} T$.

Suppose $k \in \operatorname{supp}(T) = \operatorname{supp}(\bar{T})$. We will write $\lambda_i^{T_k}$, $\rho_i^{T_k}$, $\lambda_i^{T_k^*}$ and $\rho_i^{T_k^*}$ for the duals with respect to the $T_k \oplus \bar{T}$-basis and the $T_k^* \oplus \bar{T}$-basis respectively. Recall the matrix $B_T = (b_{ij}^{T_k})$ from Section II.5. The following Lemma is based upon [H1, Lemmas 24 & 25].

Lemma V.1.3. We may compute $\pi_T(\lambda_k^{T_k})$, $\pi_T(\lambda_k^{T_k^*})$, $\pi_T(\rho_k^{T_k})$, and $\pi_T(\rho_k^{T_k^*})$ as follows:

$$\pi_T(\lambda_k^{T_k}) = -\pi_T(\lambda_k^{T_k^*}) = [T_k] - [A] \text{ and } -\pi_T(\rho_k^{T_k}) = \pi_T(\rho_k^{T_k^*}) = [T_k^*] - [D].$$

We may also compute $\pi_T^c(\lambda_k^{T_k})$, $\pi_T^c(\lambda_k^{T_k^*})$, $\pi_T^c(\rho_k^{T_k})$, and $\pi_T^c(\rho_k^{T_k^*})$ as follows:

$$\pi_T^c(\lambda_k^{T_k}) = -\pi_T^c(\lambda_k^{T_k^*}) = [B]^{*_P} \text{ and } \pi_T^c(\rho_k^{T_k}) = -\pi_T^c(\rho_k^{T_k^*}) = -{}^{*_P}[C],$$

where we write $\mathbf{e}^{*_P} = \sum_{j \notin \operatorname{supp} T} \langle \mathbf{e}, \alpha_j^\vee \rangle [P_j]$ and ${}^{*_P}\mathbf{e} = \sum_{j \notin \operatorname{supp} T} \langle \alpha_j^\vee, \mathbf{e} \rangle [P_j]$. Moreover, for $i \in \operatorname{supp} T$ we have

$$\lambda_i^{T_k^*} = \lambda_i^{T_k} + [-b_{ik}^{T_k}]_+ d_i/d_k \lambda_k^{T_k} \text{ and } \rho_i^{T_k^*} = \rho_i^{T_k} + [-b_{ik}^{T_k}]_+ d_i/d_k \rho_k^{T_k},$$

and for $i \notin \operatorname{supp} T$ we have

$$\lambda_i^{T_k^*} = \lambda_i^{T_k} \text{ and } \rho_i^{T_k^*} = \rho_i^{T_k}.$$

Proof. According to the proof of Proposition II.5.2 we have $\langle A, T_k \rangle = 0$ and $\langle A, T_j \rangle = \langle T_k, T_j \rangle$ for all $j \in \operatorname{supp}(T)$, $j \neq k$. Since $A \in add(\bar{T})$ we see that $\pi_T(\lambda_k^{T_k}) = [T_k] - [A]$. Similarly one sees that $\pi_T(\rho_k^{T_k^*}) = [T_k^*] - [D]$. Now notice that

$$\langle \pi_T(\lambda_k^{T_k}) + \pi_T(\lambda_k^{T_k^*}), [T_j] \rangle = 0 \text{ and}$$
$$\langle \pi_T(\lambda_k^{T_k}) + \pi_T(\lambda_k^{T_k^*}), [T_k] \rangle = d_k + \langle \pi_T(\lambda_k^{T_k^*}), [E] - [T_k^*] \rangle = d_k - d_k = 0,$$

so that $\pi_T(\lambda_k^{T_k}) + \pi_T(\lambda_k^{T_k^*}) = 0$ by the non-degeneracy of the Ringel-Euler form on $\mathcal{K}(Q_T)$. Similarly we compute

$$\langle [T_j], \pi_T(\rho_k^{T_k}) + \pi_T(\rho_k^{T_k^*})\rangle = 0 \text{ and}$$
$$\langle [T_k], \pi_T(\rho_k^{T_k}) + \pi_T(\rho_k^{T_k^*})\rangle = d_k + \langle [E] - [T_k^*], \pi_T(\rho_k^{T_k^*})\rangle = d_k - d_k = 0,$$

so that $\pi_T(\rho_k^{T_k}) + \pi_T(\rho_k^{T_k^*}) = 0$. Assume $i, j \in \operatorname{supp} T$ and $i, j \neq k$. Then we may obtain identities relating $\pi_T(\lambda_i^{T_k})$ and $\pi_T(\lambda_i^{T_k^*})$ as follows:

$$\langle \pi_T(\lambda_i^{T_k}) - \pi_T(\lambda_i^{T_k^*}), [T_j]\rangle = 0 \text{ and}$$
$$\langle \pi_T(\lambda_i^{T_k}) - \pi_T(\lambda_i^{T_k^*}), [T_k]\rangle = -\langle \pi_T(\lambda_i^{T_k^*}), [E] - [T_k^*]\rangle = -[-b_{ik}^{T_k}]_+ d_i,$$

so that $\pi_T(\lambda_i^{T_k}) - \pi_T(\lambda_i^{T_k^*}) = -[-b_{ik}^{T_k}]_+ d_i/d_k \pi_T(\lambda_k^{T_k})$. We remark that this implies the same identity holds when we replace π_T by π_T^c. Indeed, for $j \notin \operatorname{supp} T$ we may compute the pairing $\langle -, \pi_T^c(\rho_j^{T_k})\rangle$ on both sides of the desired identity as follows:

$$\begin{aligned}
\langle \pi_T^c(\lambda_i^{T_k}) - \pi_T^c(\lambda_i^{T_k^*}), \pi_T^c(\rho_j^{T_k})\rangle &= \ell_{ij}^{T_k} d_j - \ell_{ij}^{T_k^*} d_j \\
&= r_{ji}^{T_k} d_i - r_{ji}^{T_k^*} d_i \\
&= \langle \pi_T(\lambda_i^{T_k}) - \pi_T(\lambda_i^{T_k^*}), \pi_T(\rho_j^{T_k})\rangle, \\
\langle -[-b_{ik}^{T_k}]_+ d_i/d_k \pi_T^c(\lambda_k^{T_k}), \pi_T^c(\rho_j^{T_k})\rangle &= -[-b_{ik}^{T_k}]_+ d_i/d_k \ell_{kj}^{T_k} d_j \\
&= -[-b_{ik}^{T_k}]_+ d_i/d_k r_{jk}^{T_k} d_k \\
&= \langle -[-b_{ik}^{T_k}]_+ d_i/d_k \pi_T(\lambda_k^{T_k}), \pi_T(\rho_j^{T_k})\rangle.
\end{aligned}$$

Now notice that the right hand sides of these are equal for all $j \notin \operatorname{supp} T$ by the previously obtained identity. Since the Ringel-Euler form is non-degenerate this implies $\pi_T^c(\lambda_i^{T_k}) - \pi_T^c(\lambda_i^{T_k^*}) = -[-b_{ik}^{T_k}]_+ d_i/d_k \pi_T^c(\lambda_k^{T_k})$. Similarly we obtain identities relating $\pi_T(\rho_i^{T_k})$ and $\pi_T(\rho_i^{T_k^*})$:

$$\langle [T_j], \pi_T(\rho_i^{T_k}) - \pi_T(\rho_i^{T_k^*})\rangle = 0 \text{ and}$$
$$\langle [T_k], \pi_T(\rho_i^{T_k}) - \pi_T(\rho_i^{T_k^*})\rangle = -\langle [E] - [T_k^*], \pi_T(\rho_i^{T_k^*})\rangle = -[-b_{ik}^{T_k}]_+ d_i,$$

so that $\pi_T(\rho_i^{T_k}) - \pi_T(\rho_i^{T_k^*}) = -[-b_{ik}^{T_k}]_+ d_i/d_k \pi_T(\rho_k^{T_k})$ and $\pi_T^c(\rho_i^{T_k}) - \pi_T^c(\rho_i^{T_k^*}) = -[-b_{ik}^{T_k}]_+ d_i/d_k \pi_T^c(\rho_k^{T_k})$

as above. Notice that for $i \notin \operatorname{supp} T$ we have

$$\langle \pi_T(\lambda_i^{T_k}) - \pi_T(\lambda_i^{T_k^*}), [T_j] \rangle = 0 \text{ and}$$
$$\langle \pi_T(\lambda_i^{T_k}) - \pi_T(\lambda_i^{T_k^*}), [T_k] \rangle = \langle S_i, [T_k] \rangle - \langle S_i, [E] - [T_k^*] \rangle = 0,$$

so that the non-degeneracy of the Ringel-Euler form implies $\pi_T(\lambda_i^{T_k}) - \pi_T(\lambda_i^{T_k^*}) = 0$. Similarly one shows that $\pi_T(\rho_i^{T_k}) - \pi_T(\rho_i^{T_k^*}) = 0$. Note that we may write these identities in the $\bar{T} \oplus T_k$-basis and equate coefficients of $[T_k]$ to get $\ell_{ik}^{T_k} = -\ell_{ik}^{T_k^*}$ and $r_{ik}^{T_k} = -r_{ik}^{T_k^*}$, or equivalently $r_{ki}^{T_k} = -r_{ki}^{T_k^*}$ and $\ell_{ki}^{T_k} = -\ell_{ki}^{T_k^*}$. In other words, we have $\pi_T^c(\rho_k^{T_k}) = -\pi_T^c(\rho_k^{T_k^*})$ and $\pi_T^c(\lambda_k^{T_k}) = -\pi_T^c(\lambda_k^{T_k^*})$. For $i \notin \operatorname{supp} T$ we have the following:

$$\langle \pi_T(\lambda_i^{T_k^*}), \pi_T(\rho_k^{T_k^*}) \rangle = \langle \pi_T(\lambda_i^{T_k^*}), [T_k^*] - [D] \rangle = \langle S_i, [T_k^*] - [D] \rangle = \langle S_i, [C] \rangle.$$

On the other hand this is equal to $\ell_{ik}^{T_k^*} d_k = r_{ki}^{T_k^*} d_i$ and thus $\pi_T^c(\rho_k^{T_k^*}) = {}^{*_P}[C]$. A similar computation with $\langle \pi_T(\lambda_k^{T_k}), \pi_T(\rho_i^{T_k}) \rangle$ gives $\langle [B], S_i \rangle = r_{ik}^{T_k} d_k = \ell_{ki}^{T_k} d_i$ and $\pi_T^c(\lambda_k^{T_k}) = [B]^{*_P}$. Finally we note that $\pi_T^c(\lambda_i^{T_k})$ and $\pi_T^c(\lambda_i^{T_k^*})$ have the same defining equations. Therefore we have $\pi_T^c(\lambda_i^{T_k}) = \pi_T^c(\lambda_i^{T_k^*})$ and similarly $\pi_T^c(\rho_i^{T_k}) = \pi_T^c(\rho_i^{T_k^*})$. □

The following Proposition is based upon [H1, Proposition 23].

Proposition V.1.4. Thinking of the k^{th} column $\mathbf{b}^k$ of B_T as representing an element of $\mathcal{K}(Q)$ written in the $\bar{T} \oplus T_k$-basis, we may write

$$\mathbf{b}^k = \rho_k^{T_k} - \lambda_k^{T_k}.$$

Similarly, thinking of the k^{th} column $\mathbf{b}^k$ of $B_{T_k^* \oplus \bar{T}}$ as representing an element of $\mathcal{K}(Q)$ written in the $T_k^* \oplus \bar{T}$-basis, we may write

$$\mathbf{b}^k = \rho_k^{T_k^*} - \lambda_k^{T_k^*}.$$

Proof. According to Lemma V.1.3 we may write

$$\begin{aligned}\rho_k^{T_k} - \lambda_k^{T_k} &= [D] - [T_k^*] - {}^{*_P}[C] - [T_k] + [A] - [B]^{*_P} \\ &= [A] + [D] - [E] + {}^{*_P}[\tau B] - {}^{*_P}[C] \\ &= [A] + [D] + {}^{*_P}[I] - [E],\end{aligned}$$

but this is exactly $\mathbf{b}^k$. From above we see that $\rho_k^{T_k^*} = -\rho_k^{T_k}$ and $\lambda_k^{T_k^*} = -\lambda_k^{T_k}$, the result for $B_{T_k^* \oplus \bar{T}}$ follows. □

Let $\bar{T}$ be a local tilting representation with $k \notin \operatorname{supp}(\bar{T})$ and write $T = \bar{T} \oplus T_k^*$ where T_k^* is the unique compliment to $\bar{T}$. We will write $\lambda_i^{T_k^*}$, $\rho_i^{T_k^*}$, λ_i^k and ρ_i^k for the duals with respect to the $\bar{T} \oplus T_k^*$-basis and the $\bar{T}$-basis respectively. We again recall the matrix $B_T = (b_{ij}^{T_k^*})$ defined in Section II.5. The following Lemma is based upon [H1, Lemma 28].

Lemma V.1.5. We may compute $\pi_T(\lambda_k^{T_k^*})$, $\pi_{\bar{T}}(\lambda_k^k)$, $\pi_T(\rho_k^{T_k^*})$, and $\pi_{\bar{T}}(\rho_k^k)$ as follows:

$$\pi_T(\lambda_k^{T_k^*}) = [T_k^*] - [F] \text{ and } \pi_T(\rho_k^{T_k^*}) = [T_k^*] - [G],$$
$$\pi_{\bar{T}}(\lambda_k^k) = -[\operatorname{rad} \bar{P}_k] \text{ and } \pi_{\bar{T}}(\rho_k^k) = -[\bar{I}_k/S_k].$$

We may also compute $\pi_T^c(\lambda_k^{T_k^*})$, $\pi_{\bar{T}}^c(\lambda_k^k)$, $\pi_T^c(\rho_k^{T_k^*})$, and $\pi_{\bar{T}}^c(\rho_k^k)$ as follows:

$$\pi_T^c(\lambda_k^{T_k^*}) = -[P']^{*_P} \text{ and } \pi_T^c(\rho_k^{T_k^*}) = -{}^{*_P}[I'],$$
$$\pi_{\bar{T}}^c(\lambda_k^k) = [P_k] - {}^{*_P}[I'] \text{ and } \pi_{\bar{T}}^c(\rho_k^k) = [P_k] - [P']^{*_P}.$$

Moreover, for $i \in \operatorname{supp} \bar{T}$ we have

$$\pi_{\bar{T}}(\lambda_i^k) = \pi_T(\lambda_i^{T_k^*}) + [-b_{ik}^{T_k^*}]_+ d_i/d_k \pi_T(\lambda_k^{T_k^*}) \quad \text{and} \quad \pi_{\bar{T}}(\rho_i^k) = \pi_T(\rho_i^{T_k^*}) + [b_{ik}^{T_k^*}]_+ d_i/d_k \pi_T(\rho_k^{T_k^*}),$$
$$\pi_{\bar{T}}^c(\lambda_i^k) = \pi_T^c(\lambda_i^{T_k^*}) + [-b_{ik}^{T_k^*}]_+ d_i/d_k \pi_T^c(\lambda_k^{T_k^*}) \quad \text{and} \quad \pi_{\bar{T}}^c(\rho_i^k) = \pi_T^c(\rho_i^{T_k^*}) + [b_{ik}^{T_k^*}]_+ d_i/d_k \pi_T^c(\rho_k^{T_k^*}),$$

and for $i \notin \operatorname{supp} T$ we have

$$\pi_{\bar{T}}(\lambda_i^k) = \pi_T(\lambda_i^{T_k^*}) + [-b_{ik}^{T_k^*}]_+ d_i/d_k \pi_T(\lambda_k^{T_k^*}) \quad \text{and} \quad \pi_{\bar{T}}(\rho_i^k) = \pi_T(\rho_i^{T_k^*}) + [b_{ik}^{T_k^*}]_+ d_i/d_k \pi_T(\rho_k^{T_k^*}),$$
$$\pi^c_{\bar{T}}(\lambda_i^k) = \pi^c_T(\lambda_i^{T_k^*}) - [b_{ik}^{T_k^*}]_+ d_i/d_k \pi^c_{\bar{T}}(\lambda_k^k) \quad \text{and} \quad \pi^c_{\bar{T}}(\rho_i^k) = \pi^c_T(\rho_i^{T_k^*}) - [-b_{ik}^{T_k^*}]_+ d_i/d_k \pi^c_{\bar{T}}(\rho_k^k).$$

Proof. Applying the functor $\operatorname{Hom}(-, T_j)$ for $j \in \operatorname{supp}(\bar{T})$ to the sequence

$$0 \longrightarrow \bar{P}_k \longrightarrow T_k^* \longrightarrow F \longrightarrow 0$$

shows that $\operatorname{Hom}(F, T_j) \cong \operatorname{Hom}(T_k^*, T_j)$ and thus $\langle F, T_j \rangle = \langle T_k^*, T_j \rangle$. Similarly we may apply $\operatorname{Hom}(T_j, -)$ to the sequence

$$0 \longrightarrow G \longrightarrow T_k^* \longrightarrow \bar{I}_k \longrightarrow 0$$

to see that $\langle T_j, G \rangle = \langle T_j, T_k^* \rangle$. From the proof of Proposition II.5.2 we have that $\operatorname{Hom}(F, T_k^*) = 0$ and by a similar argument $\operatorname{Hom}(T_k^*, G) = 0$. Since $F, G \in \mathit{add}(\bar{T})$ we may identify $\pi_T(\lambda_k^{T_k^*}) = [T_k^*] - [F]$ and $\pi_T(\rho_k^{T_k^*}) = [T_k^*] - [G]$.

Recall that we have the following short exact sequences defining P' and F:

$$0 \longrightarrow \bar{P}_k \longrightarrow T_k^* \longrightarrow F \longrightarrow 0\,,$$
$$0 \longrightarrow P' \longrightarrow P_k \longrightarrow \bar{P}_k \longrightarrow 0\,.$$

Applying $\operatorname{Hom}(P_i, -)$ to the first sequence shows that $\operatorname{Hom}(P_i, \bar{P}_k) = 0$ for $i \notin \operatorname{supp} T$ and thus applying the same functor to the second sequence gives $\operatorname{Hom}(P_i, P') \cong \operatorname{Hom}(P_i, P_k)$ for $i \notin \operatorname{supp} T$. By noting that $\operatorname{Hom}(P_k, P') = 0$ we see that $\pi^c_{\bar{T}}(\rho_k^k) = [P_k] - [P'] = [P_k] - [P']^{*_P}$. Since $k \notin \operatorname{supp}(\bar{T})$ we have $\operatorname{Hom}(\bar{P}_k, \bar{T}) = 0$, in particular $\langle \bar{P}_k, \bar{T} \rangle = 0$. Thus we see that $-\langle \operatorname{rad} \bar{P}_k, T_j \rangle = \langle S_k, T_j \rangle$ for $j \in \operatorname{supp}(\bar{T})$ and hence $\pi_{\bar{T}}(\lambda_k^k) = -[\operatorname{rad} \bar{P}_k] \in \mathcal{K}(Q_{\bar{T}})$.

Similarly we have the following short exact sequences defining G and I':

$$0 \longrightarrow G \longrightarrow T_k^* \longrightarrow \bar{I}_k \longrightarrow 0\,,$$
$$0 \longrightarrow \bar{I}_k \longrightarrow I_k \longrightarrow I' \longrightarrow 0\,.$$

Applying the functor $\operatorname{Hom}(-, I_j)$ we see that $\operatorname{Hom}(I_k, I_j) \cong \operatorname{Hom}(I', I_j)$ for all $j \notin \operatorname{supp} T$. Recall that the inverse Nakayama functor ν^{-1} is an equivalence of categories from the full subcategory of

$\mathrm{Rep}_{\mathbb{F}}(Q, \mathbf{d})$ consisting of injective objects to the full subcategory consisting of projective objects. Thus we see that

$$\begin{aligned}\mathrm{Hom}(P_k, P_j) &= \mathrm{Hom}(\nu^{-1}I_k, \nu^{-1}I_j) \\ &\cong \mathrm{Hom}(\nu^{-1}I', \nu^{-1}I_j) = \mathrm{Hom}(\nu^{-1}I', P_j)\end{aligned}$$

and $\mathrm{Hom}(\nu^{-1}I', P_k) = \mathrm{Hom}(\nu^{-1}I', \nu^{-1}I_k) = 0$. Thus we see that $\pi^c_{\bar{T}}(\lambda^k_k) = [P_k] - [\nu^{-1}I'] = [P_k] - {}^{*_P}[I']$. Since $k \notin \mathrm{supp}(\bar{T})$ we have $\mathrm{Hom}(\bar{T}, \bar{I}_k) = 0$, in particular $\langle \bar{T}, \bar{I}_k\rangle = 0$. Thus we see that $-\langle T_j, \bar{I}_k/S_k\rangle = \langle T_j, S_k\rangle$ for $j \in \mathrm{supp}(\bar{T})$ and hence $\pi_{\bar{T}}(\rho^k_k) = -[\bar{I}_k/S_k] \in \mathcal{K}(Q_{\bar{T}})$. To obtain the remaining duals $\pi^c_T(\rho^{T^*_k}_k)$ and $\pi^c_T(\lambda^{T^*_k}_k)$ we assume $i \notin \mathrm{supp}\, T$ and compute

$$\langle \pi_T(\lambda^{T^*_k}_i), \pi_T(\rho^{T^*_k}_k)\rangle = \langle \pi_T(\lambda^{T^*_k}_i), [T^*_k] - [G]\rangle = \langle S_i, [T^*_k] - [G]\rangle = \langle S_i, [I_k] - [I']\rangle = -\langle S_i, [I']\rangle.$$

On the other hand this is equal to $\ell^{T^*_k}_{ik}\, d_k = r^{T^*_k}_{ki}\, d_i$ and thus $\pi^c_T(\rho^{T^*_k}_k) = -{}^{*_P}[I']$. A similar computation with $\langle \pi_T(\lambda^{T^*_k}_k), \pi_T(\rho^{T^*_k}_i)\rangle$ gives $-\langle [P'], S_i\rangle = r^{T^*_k}_{ik}\, d_k = \ell^{T^*_k}_{ki}\, d_i$ and $\pi^c_T(\lambda^{T^*_k}_k) = -[P']^{*_P}$.

For $i \in \mathrm{supp}\, \bar{T}$ we may compare $\pi_T(\lambda^{T^*_k}_i)$ and $\pi_{\bar{T}}(\lambda^k_i)$ as follows:

$$\begin{aligned}&\langle \pi_T(\lambda^{T^*_k}_i) - \pi_{\bar{T}}(\lambda^k_i), [T_j]\rangle = 0 \text{ and} \\ &\langle \pi_T(\lambda^{T^*_k}_i) - \pi_{\bar{T}}(\lambda^k_i), [T^*_k]\rangle = -\langle \pi_{\bar{T}}(\lambda^k_i), [T^*_k]\rangle = -\langle \pi_{\bar{T}}(\lambda^k_i), [G]\rangle = -[-b^{T^*_k}_{ik}]_+ d_i,\end{aligned}$$

so that $\pi_T(\lambda^{T^*_k}_i) - \pi_{\bar{T}}(\lambda^k_i) = -[-b^{T^*_k}_{ik}]_+ d_i/d_k \pi_T(\lambda^{T^*_k}_k)$. Similarly we may compare $\pi_T(\rho^{T^*_k}_i)$ and $\pi_{\bar{T}}(\rho^k_i)$ as follows:

$$\begin{aligned}&\langle [T_j], \pi_T(\rho^{T^*_k}_i) - \pi_{\bar{T}}(\rho^k_i)\rangle = 0 \text{ and} \\ &\langle [T^*_k], \pi_T(\rho^{T^*_k}_i) - \pi_{\bar{T}}(\rho^k_i)\rangle = -\langle [T^*_k], \pi_{\bar{T}}(\rho^k_i)\rangle = -\langle [F], \pi_{\bar{T}}(\rho^k_i)\rangle = -[b^{T^*_k}_{ik}]_+ d_i,\end{aligned}$$

so that $\pi_T(\rho^{T^*_k}_i) - \pi_{\bar{T}}(\rho^k_i) = -[b^{T^*_k}_{ik}]_+ d_i/d_k \pi_T(\rho^{T^*_k}_k)$. Now we suppose $i \notin \mathrm{supp}\, T$. Then we may compare $\pi_T(\lambda^{T^*_k}_i)$ and $\pi_{\bar{T}}(\lambda^k_i)$ as follows:

$$\begin{aligned}&\langle \pi_T(\lambda^{T^*_k}_i) - \pi_{\bar{T}}(\lambda^k_i), [T_j]\rangle = 0 \text{ and} \\ &\langle \pi_T(\lambda^{T^*_k}_i) - \pi_{\bar{T}}(\lambda^k_i), [T^*_k]\rangle = \langle S_i, [T^*_k]\rangle - \langle \pi_{\bar{T}}(\lambda^k_i), [G]\rangle = \langle S_i, [T^*_k] - [G]\rangle = \langle S_i, I_k - I'\rangle = -[-b^{T^*_k}_{ik}]_+ d_i,\end{aligned}$$

so that $\pi_T(\lambda_i^{T_k^*}) - \pi_{\bar{T}}(\lambda_i^k) = -[-b_{ik}^{T_k^*}]_+ d_i/d_k \pi_T(\lambda_k^{T_k^*})$. For $j \in \operatorname{supp} \bar{T}$ we consider the coefficient of $[T_j]$ in this identity to get

$$\begin{aligned}
\ell_{ij}^{T_k^*} - \ell_{ij}^k &= -[-b_{ik}^{T_k^*}]_+ d_i/d_k \ell_{kj}^{T_k^*} \\
\ell_{ij}^{T_k^*} d_j - \ell_{ij}^k d_j &= -[-b_{ik}^{T_k^*}]_+ d_i/d_k \ell_{kj}^{T_k^*} d_j \\
r_{ji}^{T_k^*} d_i - r_{ji}^k d_i &= r_{ki}^{T_k^*} d_i/d_k \ell_{kj}^{T_k^*} d_j \\
r_{ji}^{T_k^*} - r_{ji}^k &= \ell_{kj}^{T_k^*} d_j/d_k r_{ki}^{T_k^*},
\end{aligned}$$

which we may recognize as giving the coefficient of $[P_i]$ in the identity $\pi_T^c(\rho_j^{T_k^*}) - \pi_{\bar{T}}^c(\rho_j^k) = -[b_{jk}]_+ d_j/d_k \pi_T^c(\rho_k^{T_k^*})$. Similarly we may compare $\pi_T(\rho_i^{T_k^*})$ and $\pi_{\bar{T}}(\rho_i^k)$ as follows:

$$\langle [T_j], \pi_T(\rho_i^{T_k^*}) - \pi_{\bar{T}}(\rho_i^k) \rangle = 0 \text{ and}$$
$$\langle [T_k^*], \pi_T(\rho_i^{T_k^*}) - \pi_{\bar{T}}(\rho_i^k) \rangle = \langle [T_k^*], S_i \rangle - \langle [F], \pi_{\bar{T}}(\rho_i^k) \rangle = \langle [T_k^*] - [F], S_i \rangle = \langle P_k - P', S_i \rangle = -[b_{ik}^{T_k^*}]_+ d_i,$$

so that $\pi_T(\rho_i^{T_k^*}) - \pi_{\bar{T}}(\rho_i^k) = -[b_{ik}^{T_k^*}]_+ d_i/d_k \pi_T(\rho_k^{T_k^*})$. For $j \in \operatorname{supp} \bar{T}$ we again consider the coefficient of $[T_j]$ in this identity to get

$$\begin{aligned}
r_{ij}^{T_k^*} - r_{ij}^k &= -[b_{ik}^{T_k^*}]_+ d_i/d_k r_{kj}^{T_k^*} \\
r_{ij}^{T_k^*} d_j - r_{ij}^k d_j &= -[b_{ik}^{T_k^*}]_+ d_i/d_k r_{kj}^{T_k^*} d_j \\
\ell_{ji}^{T_k^*} d_i - \ell_{ji}^k d_i &= \ell_{ki}^{T_k^*} d_i/d_k r_{kj}^{T_k^*} d_j \\
\ell_{ji}^{T_k^*} - \ell_{ji}^k &= r_{kj}^{T_k^*} d_j/d_k \ell_{ki}^{T_k^*},
\end{aligned}$$

which we may recognize as giving the coefficient of $[P_i]$ in the identity $\pi_T^c(\lambda_j^{T_k^*}) - \pi_{\bar{T}}^c(\lambda_j^k) = -[-b_{jk}]_+ d_j/d_k \pi_T^c(\lambda_k^{T_k^*})$. We may also compare $\pi_T^c(\lambda_i^{T_k^*})$ and $\pi_{\bar{T}}^c(\lambda_i^k)$ as follows:

$$\langle \pi_T^c(\lambda_i^{T_k^*}) - \pi_{\bar{T}}^c(\lambda_i^k), [P_j] \rangle = 0 \text{ and}$$
$$\langle \pi_T^c(\lambda_i^{T_k^*}) - \pi_{\bar{T}}^c(\lambda_i^k), [P_k] \rangle = \langle \pi_T^c(\lambda_i^{T_k^*}), [P_k] \rangle = \langle \pi_T^c(\lambda_i^{T_k^*}), [P'] \rangle = [b_{ik}^{T_k^*}]_+ d_i,$$

so that $\pi_T^c(\lambda_i^{T_k^*}) - \pi_{\bar{T}}^c(\lambda_i^k) = [b_{ik}^{T_k^*}]_+ d_i/d_k \pi_{\bar{T}}^c(\lambda_k^k)$. Finally we may compare $\pi_T^c(\rho_i^{T_k^*})$ and $\pi_{\bar{T}}^c(\rho_i^k)$ as follows:

$$\langle [P_j], \pi^c_T(\rho_i^{T_k^*}) - \pi^c_{\bar{T}}(\rho_i^k)\rangle = 0 \text{ and}$$
$$\langle [P_k], \pi^c_T(\rho_i^{T_k^*}) - \pi^c_{\bar{T}}(\rho_i^k)\rangle = \langle [P_k], \pi^c_T(\rho_i^{T_k^*})\rangle = \langle [\nu^{-1}I'], \pi^c_T(\rho_i^{T_k^*})\rangle = [-b_{ik}^{T_k^*}]_+ d_i,$$

so that $\pi^c_T(\rho_i^{T_k^*}) - \pi^c_{\bar{T}}(\rho_i^k) = [-b_{ik}^{T_k^*}]_+ d_i/d_k \pi^c_{\bar{T}}(\rho_k^k)$. □

The following Proposition is based upon [H1, Proposition 27].

Proposition V.1.6. Thinking of the k^{th} column $\mathbf{b}^k$ of B_T as representing an element of $\mathcal{K}(Q)$ written in the $\bar{T} \oplus T_k^*$-basis, we may write

$$\mathbf{b}^k = \rho_k^{T_k^*} - \lambda_k^{T_k^*}.$$

Similarly, thinking of the k^{th} column $\mathbf{b}^k$ of $B_{\bar{T}}$ as representing an element of $\mathcal{K}(Q)$ written in the $\bar{T}$-basis, we may write

$$\mathbf{b}^k = \rho_k^k - \lambda_k^k.$$

Proof. According to Lemma V.1.5 we may write

$$\begin{aligned}\rho_k^{T_k} - \lambda_k^{T_k} &= [T_k^*] - [G] - {}^{*_P}[I'] - [T_k^*] + [F] + [P']^{*_P}\\ &= [F] + [P']^{*_P} - [G] - {}^{*_P}[I'],\end{aligned}$$

but this is exactly the k^{th} column of B_T. Similarly we have

$$\begin{aligned}\rho_k^k - \lambda_k^k &= -[\bar{I}_k/S_k] + [P_k] - [P']^{*_P} + [\text{rad}\,\bar{P}_k] - [P_k] + {}^{*_P}[I']\\ &= -[\bar{I}_k] + [S_k] - [P']^{*_P} + [\bar{P}_k] - [S_k] + {}^{*_P}[I']\\ &= -[T_k^*] + [G] - [P']^{*_P} + [T_k^*] - [F] + {}^{*_P}[I']\\ &= [G] + {}^{*_P}[I'] - [F] - [P']^{*_P},\end{aligned}$$

but this is exactly the k^{th} column of $B_{\bar{T}}$. □

We are finally ready to show that mutation of tilting pairs corresponds to mutation of exchange matrices.

Theorem V.1.7. Suppose $\mu_k(T) = T'$. Then B_T and $B_{T'}$ are related by Fomin-Zelevinsky matrix mutation in direction k.

Proof. Note that we have shown for any vertex $i \in Q_0$ we have $\mathbf{b}^i = \rho_i^{T_k} - \lambda_i^{T_k}$ for $k \in \operatorname{supp} \bar{T}$ and $\mathbf{b}^i = \rho_i^k - \lambda_i^k$ when $k \notin \operatorname{supp}(T)$. In particular, this implies that the matrices B_T and $B_{T'}$ are skew-symmetrizable, i.e. DB_T and $DB_{T'}$ are skew-symmetric. We will have two cases to consider.

Case 1: Suppose we have $k \in \operatorname{supp}(T) = \operatorname{supp}(\bar{T})$. Write $T_k^* \not\cong T_k$ for the unique compliment of $\bar{T}$. We will show that $B_{\bar{T}\oplus T_k^*} = EB_TF$ where the matrices E and F are given by (III.4). We will consider the j^{th} column $\mathbf{b}^j = B_T^j$ of B_T as an element of $\mathcal{K}(Q)$ via $\mathbf{b}^j = \sum\limits_{i\in\operatorname{supp} T} b_{ij}^{T_k}[T_i] + \sum\limits_{i\notin\operatorname{supp} T} b_{ij}^{T_k}[P_i]$. First note that when $j \notin \operatorname{supp} T$ we have $b_{jk}^{T_k} \geq 0$ and so

$$\begin{aligned}(B_TF)^j &= \mathbf{b}^j + [b_{kj}^{T_k}]_+\mathbf{b}^k \\ &= \mathbf{b}^j + [-b_{jk}^{T_k}]_+ d_j/d_k\mathbf{b}^k \\ &= \rho_j^{T_k} - \lambda_j^{T_k} \\ &= \rho_j^{T_k^*} - \lambda_j^{T_k^*} \\ &= B_{\bar{T}\oplus T_k^*}^j.\end{aligned}$$

Then since $\mathbf{b}_-^k := \sum\limits_{i\in\operatorname{supp} T}[-b_{ik}^{T_k}]_+[T_i] + \sum\limits_{i\notin\operatorname{supp} T}[-b_{ik}^{T_k}]_+[P_i]$ is $[E]$ the short exact sequence

$$0 \longrightarrow T_k^* \longrightarrow E \longrightarrow T_k \longrightarrow 0$$

implies that the change from the $\bar{T}\oplus T_k$-basis to the $\bar{T}\oplus T_k^*$-basis is given by left multiplication by the matrix E from equation (III.4). Hence the j^{th} columns of $B_{\bar{T}\oplus T_k}$ and $B_{\bar{T}\oplus T_k^*}$ are related by the Fomin-Zelevinsky matrix mutation. Now we suppose $j \in \operatorname{supp} T$ and make the following similar computation:

$$\begin{aligned}(B_T F)^j &= \mathbf{b}^j + [-b_{jk}^{T_k}]_+ d_j/d_k \mathbf{b}^k \\ &= \rho_j^{T_k} - \lambda_j^{T_k} + [-b_{jk}^{T_k}]_+ d_j/d_k(\rho_k^{T_k} - \lambda_k^{T_k}) \\ &= \rho_j^{T_k^*} - \lambda_j^{T_k^*} \\ &= B^j_{\bar{T}\oplus T_k^*}.\end{aligned}$$

Again note that the change from the $\bar{T} \oplus T_k$-basis to the $\bar{T} \oplus T_k^*$-basis is given by left multiplication by the matrix E from equation (III.4). This completes the proof in the case $k \in \operatorname{supp}(T) = \operatorname{supp}(\bar{T})$.

Case 2: Suppose $k \notin \operatorname{supp}(T)$. Then T has a unique complement T_k^* such that $T_k^* \oplus T$ is a local tilting representation and $\operatorname{supp}(T_k^* \oplus T) = \operatorname{supp}(T) \cup \{k\}$. Note that by the definition of B_T and $B_{\bar{T}}$ their k^{th} columns are related by the Fomin-Zelevinsky matrix mutation. Thus our goal is to show for $i, j \neq k$ that $b_{ij}^k = b_{ij}^{T_k^*} + \delta_{ij}^{T_k^*}$ for $\delta_{ij}^{T_k^*} = [b_{ik}^{T_k^*}]_+ b_{kj}^{T_k^*} + b_{ik}^{T_k^*}[-b_{kj}^{T_k^*}]_+$. We have three sub-cases to consider depending on the positions of i and j.

1. Suppose $j \in \operatorname{supp}\bar{T}$ and $i \neq k$. Then according to Lemma V.1.5 we have

$$\begin{aligned}b_{ij}^k &= r_{ji}^k - \ell_{ji}^k \\ &= r_{ji}^{T_k^*} + [b_{jk}^{T_k^*}]_+ d_j/d_k r_{ki}^{T_k^*} - \ell_{ji}^{T_k^*} - [-b_{jk}^{T_k^*}]_+ d_j/d_k \ell_{ki}^{T_k^*} \\ &= b_{ij}^{T_k^*} + [-b_{kj}^{T_k^*}]_+ r_{ki}^{T_k^*} - [b_{kj}^{T_k^*}]_+ \ell_{ki}^{T_k^*} \\ &= b_{ij}^{T_k^*} - [-b_{kj}^{T_k^*}]_+[-b_{ik}^{T_k^*}]_+ + [b_{kj}^{T_k^*}]_+[b_{ik}^{T_k^*}]_+ \\ &= b_{ij}^{T_k^*} + [b_{ik}^{T_k^*}]_+[b_{kj}^{T_k^*}]_+ - [b_{ik}^{T_k^*}]_+[-b_{kj}^{T_k^*}]_+ + [b_{ik}^{T_k^*}]_+[-b_{kj}^{T_k^*}]_+ - [-b_{ik}^{T_k^*}]_+[-b_{kj}^{T_k^*}]_+ \\ &= b_{ij}^{T_k^*} + [b_{ik}^{T_k^*}]_+([b_{kj}^{T_k^*}]_+ - [-b_{kj}^{T_k^*}]_+) + ([b_{ik}^{T_k^*}]_+ - [-b_{ik}^{T_k^*}]_+)[-b_{kj}^{T_k^*}]_+ \\ &= b_{ij}^{T_k^*} + [b_{ik}^{T_k^*}]_+ b_{kj}^{T_k^*} + b_{ik}^{T_k^*}[-b_{kj}^{T_k^*}]_+.\end{aligned}$$

2. Suppose $j \notin \operatorname{supp} T$ and $i \in \operatorname{supp}\bar{T}$. Following Lemma V.1.5 we have

$$b_{ij}^k = r_{ji}^k - \ell_{ji}^k = r_{ji}^{T_k^*} + [b_{jk}^{T_k^*}]_+ d_j/d_k r_{ki}^{T_k^*} - \ell_{ji}^{T_k^*} - [-b_{jk}^{T_k^*}]_+ d_j/d_k \ell_{ki}^{T_k^*}$$

and the proof proceeds as in case (1).

3. Finally suppose $i, j \notin \operatorname{supp} T$. Then Lemma V.1.5 gives

$$\begin{aligned}
b_{ij}^k &= r_{ji}^k - \ell_{ji}^k \\
&= r_{ji}^{T_k^*} - [-b_{jk}^{T_k^*}]_+ d_j/d_k r_{ki}^k - \ell_{ji}^{T_k^*} + [b_{jk}^{T_k^*}]_+ d_j/d_k \ell_{ki}^k \\
&= b_{ij}^{T_k^*} - r_{ki}^k [b_{kj}^{T_k^*}]_+ + \ell_{ki}^k [-b_{kj}^{T_k^*}]_+ \\
&= b_{ij}^{T_k^*} + [b_{ik}^{T_k^*}]_+ [b_{kj}^{T_k^*}]_+ - [-b_{ik}^{T_k^*}]_+ [-b_{kj}^{T_k^*}]_+ \\
&= b_{ij}^{T_k^*} + [b_{ik}^{T_k^*}]_+ [b_{kj}^{T_k^*}]_+ - [b_{ik}^{T_k^*}]_+ [-b_{kj}^{T_k^*}]_+ + [b_{ik}^{T_k^*}]_+ [-b_{kj}^{T_k^*}]_+ - [-b_{ik}^{T_k^*}]_+ [-b_{kj}^{T_k^*}]_+ \\
&= b_{ij}^{T_k^*} + [b_{ik}^{T_k^*}]_+ ([b_{kj}^{T_k^*}]_+ - [-b_{kj}^{T_k^*}]_+) + ([b_{ik}^{T_k^*}]_+ - [-b_{ik}^{T_k^*}]_+)[-b_{kj}^{T_k^*}]_+ \\
&= b_{ij}^{T_k^*} + [b_{ik}^{T_k^*}]_+ b_{kj}^{T_k^*} + b_{ik}^{T_k^*} [-b_{kj}^{T_k^*}]_+ .
\end{aligned}$$

This completes the proof. □

V.2. Quantum Seeds Associated to Local Tilting Representations

In this section we assign a quantum seed $\Sigma_T = (\mathbf{X}_T, B_T, \Lambda_T)$ to each local tilting representation T. Combining with Theorem V.1.7 we will complete the proof that all cluster variables are given by the quantum cluster character applied to an exceptional representation of $(Q, \mathbf{d})$.

Recall that starting with a principal compatible pair $(\tilde{B}, \Lambda)$ we construct the quantum cluster algebra $\mathcal{A}_{|\mathbb{F}|}(\tilde{B}, \Lambda)$ with initial cluster $\{X_1, \ldots, X_n\}$ and a valued quiver $(Q, \mathbf{d})$ with principal frozen vertices. For a local tilting representation T we define the cluster $\mathbf{X}_T = \{X'_1, \ldots, X'_n\}$ as follows:

$$X'_i = \begin{cases} X_i & \text{if } i \notin \operatorname{supp} T, \\ X_{T_i} & \text{if } i \in \operatorname{supp} T. \end{cases}$$

Recall that the exchange matrix B_T was defined in Section II.5. We will only consider those local tilting representations which may be obtained from the trivial local tilting representation $T_0 = 0$ by a sequence of mutations.

Proposition V.2.1. The matrix B_{T_0} is the initial exchange matrix $\tilde{B}$.

Proof. Notice that $\langle P_i, S_j \rangle = \delta_{ij} d_i$ so that $\rho_j = S_j$. On the other hand we may write

$$r_{ji} d_i = \langle \rho_j, S_i \rangle = \langle S_j, S_i \rangle = -[d_j b_{ji}]_+ = -[-d_i b_{ij}]_+,$$

so that $r_{ji} = -[-b_{ij}]_+$ and $\ell_{ji} = r_{ij}d_j/d_i = -[b_{ij}]_+$. Following Proposition V.1.6 we may compute the ij-entry of B_{T_0} as $b_{ij}^{T_0} = r_{ji} - \ell_{ji} = [b_{ij}]_+ - [-b_{ij}]_+ = b_{ij}$. □

Recall that the matrix Λ_T should record the commutation exponents of the cluster $\mathbf{X}_T$. In Section III.4. we have already computed the commutation for the cluster $\mathbf{X}_T$ and thus we have defined Λ_T intrinsically in terms of the local tilting representation T. To give Λ_T explicitly, we write I_i for the injective hull of the simple valued representation S_i, and write $\mathbf{i}_i$ for its dimension vector. Also we will write $\mathbf{t}_i$ for the dimension vector of the summand T_i of T.

Proposition V.2.2. Let T be a local tilting representation. Then $\Lambda_T = (\lambda'_{ij})$ is given by

$$\lambda'_{ij} = \begin{cases} \Lambda({}^*\mathbf{i}_i, {}^*\mathbf{i}_j) & \text{if } i, j \notin \operatorname{supp} T, \\ -\Lambda({}^*\mathbf{i}_i, {}^*\mathbf{t}_j) & \text{if } i \notin \operatorname{supp} T \text{ and } j \in \operatorname{supp} T, \\ -\Lambda({}^*\mathbf{t}_i, {}^*\mathbf{i}_j) & \text{if } i \in \operatorname{supp} T \text{ and } j \notin \operatorname{supp} T, \\ \Lambda({}^*\mathbf{t}_i, {}^*\mathbf{t}_j) & \text{if } i, j \in \operatorname{supp} T. \end{cases}$$

Following Remark III.4.5 we have $\Lambda_{T_0} = \Lambda$ so that $\Sigma_{T_0} = (\mathbf{X}_{T_0}, B_{T_0}, \Lambda_{T_0})$ forms the initial seed for the quantum cluster algebra $\mathcal{A}_{|\mathbb{F}|}(\tilde{B}, \Lambda)$. We now obtain the main technical result of this dissertation.

Theorem V.2.3. Suppose $\mu_k(T) = T'$. Then the quantum seeds Σ_T and $\Sigma_{T'}$ are related by the quantum seed mutation in direction k.

We have the following immediate Corollary.

Corollary V.2.4. The quantum cluster character $V \mapsto X_V$ defines a bijection from exceptional representations V of $(Q, \mathbf{d})$ to non-initial quantum cluster variables of the quantum cluster algebra $\mathcal{A}_{|\mathbb{F}|}(\tilde{B}, \Lambda)$.

This completes the proof of Theorem I.0.2 from Chapter I.

Proof. Note that the multiplication formulas of Section III.3. exactly correspond to the Berenstein-Zelevinsky quantum cluster exchange relations relating $\mathbf{X}_T$ and $\mathbf{X}_{T'}$. To complete the proof we remark that the compatibility of the pair $(\tilde{B},\Lambda)$ guarantees that each cluster consists of a quasi-commuting collection of cluster variables and that the commutation matrices of neighboring clusters are related by the Berenstein-Zelevinsky mutation rule. □

Since the quantum seed Σ_{T_0} identifies with the initial quantum seed of $\mathcal{A}_{|\mathbb{F}|}(\tilde{B},\Lambda)$ we finally complete our goal of generalizing the classical cluster characters of Caldero and Chapoton to the quantum cluster algebra setting. Note that since there is a unique isomorphism class for each exceptional valued representation, the isomorphism classes of rigid objects in the Grothendieck group $\mathcal{K}(Q)$ are independent of the choice of ground field $\mathbb{F}$. This corollary together with the specialization argument of [Q, Proposition 2.2.3] immediately implies the following

Corollary V.2.5. Let V be a rigid representation in $\mathrm{Rep}_{\mathbb{F}}(Q,\mathbf{d})$. Then for any $\mathbf{e}\in\mathcal{K}(Q)$ the Grassmannian $Gr_{\mathbf{e}}(V)$ has a counting polynomial $P_{\mathbf{e}}^{\mathbf{v}}(q)$ such that $|Gr_{\mathbf{e}}(V)| = P_{\mathbf{e}}^{\mathbf{v}}(|\mathbb{F}|)$.

Proof. In [Q, Proposition 2.2.3] Qin shows that there is a support preserving surjection from $\mathcal{A}_q(\tilde{B},\Lambda)$, where q is an indeterminate, to $\mathcal{A}_{|\mathbb{F}|}(\tilde{B},\Lambda)$. The quantum Laurent phenomenon [BZ] asserts that the structure constants of the initial cluster expansion of any cluster monomial live in $\mathbb{Z}[q^{\pm\frac{1}{2}}]$. Thus for each rigid valued representation V there is a Laurent polynomial $L_{\mathbf{e}}^{\mathbf{v}}(q)$ in $q^{\frac{1}{2}}$ which specializes to $L_{\mathbf{e}}^{\mathbf{v}}(|\mathbb{F}|) = |\mathbb{F}|^{-\frac{1}{2}\langle\mathbf{e},\mathbf{v}-\mathbf{e}\rangle}|Gr_{\mathbf{e}}(V)|$. Now since $|Gr_{\mathbf{e}}(V)|$ is an integer for all finite fields $\mathbb{F}$, we see that $P_{\mathbf{e}}^{\mathbf{v}}(q) = q^{\frac{1}{2}\langle\mathbf{e},\mathbf{v}-\mathbf{e}\rangle}L_{\mathbf{e}}^{\mathbf{v}}(q)$ is an honest polynomial in q and $P_{\mathbf{e}}^{\mathbf{v}}(|\mathbb{F}|) = |Gr_{\mathbf{e}}(V)|$. □

We conjecture that these polynomials always have positive coefficients.

Conjecture V.2.6. For any acyclic valued quiver $(Q,\mathbf{d})$ and any rigid representation V, the polynomial $P_{\mathbf{e}}^{\mathbf{v}}(q)\in\mathbb{Z}[q]$ has nonnegative integer coefficients.

Qin [Q] has settled this conjecture for acyclic equally valued quivers. The positivity of these counting polynomials in skew-symmetrizable rank 2 Grassmannians follows from the results of our recent combinatorial description of noncommutative rank 2 cluster variables in [Ru3]. In a recent preprint [E], Efimov shows in the acyclic equally valued case that in addition the counting polynomials are unimodular, i.e. they are a shifted sum of bar-invariant q-numbers. We conjecture that this property always holds.

Conjecture V.2.7. In the hypotheses of Conjecture V.2.6, the polynomial $P_{\mathbf{e}}^{\mathbf{v}}(q) \in \mathbb{Z}[q]$ is unimodular.

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