

$A_\infty$ -STRUCTURES, GENERALIZED KOSZUL PROPERTIES,  
AND COMBINATORIAL TOPOLOGY

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## DISSERTATION ABSTRACT

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Motivated by the Adams spectral sequence for computing stable homotopy groups, Priddy defined a class of algebras called Koszul algebras with nice homological properties. Many important algebras arising naturally in mathematics are Koszul, and the Koszul property is often tied to important structure in the settings which produced the algebras. However, the strong defining conditions for a Koszul algebra imply that such algebras must be quadratic.

A very natural generalization of Koszul algebras called  $\mathcal{K}_2$  algebras was recently introduced by Cassidy and Shelton. Unlike other generalizations of the Koszul property, the class of  $\mathcal{K}_2$  algebras is closed under many standard operations in ring theory. The class of  $\mathcal{K}_2$  algebras includes Artin-Schelter regular algebras of global dimension 4 on three linear generators as well as graded complete intersections.

Our work comprises two distinct projects. Each project was motivated by an aspect of the theory of Koszul algebras which we regard as sufficiently powerful or fundamental to warrant an interpretation for  $\mathcal{K}_2$  algebras.

A very useful theorem due to Backelin and Fröberg states that if  $A$  is a Koszul algebra and  $I$  is a quadratic ideal of  $A$  which is Koszul as a left  $A$ -module, then the factor algebra  $A/I$  is a Koszul algebra. We prove that if  $A$  is Koszul algebra and  ${}_A I$  is a  $\mathcal{K}_2$  module, then  $A/I$  is a  $\mathcal{K}_2$  algebra provided  $A/I$  acts trivially on  $\text{Ext}_A(A/I, k)$ . As an application of our theorem, we show that the class of sequentially Cohen-Macaulay Stanley-Reisner rings are  $\mathcal{K}_2$  algebras and we give examples that suggest the class of  $\mathcal{K}_2$  Stanley-Reisner rings is actually much larger.

Another important recent development in ring theory is the use of  $A_\infty$ -algebras. One can characterize Koszul algebras as those graded algebras whose Yoneda algebra admits only trivial  $A_\infty$ -structure. We show that, in contrast to the situation for Koszul algebras, vanishing of higher

$A_\infty$ -structure on the Yoneda algebra of a  $\mathcal{K}_2$  algebra need not be determined in any obvious way by the degrees of defining relations. We also demonstrate that obvious patterns of vanishing among higher multiplications cannot detect the  $\mathcal{K}_2$  property.

This dissertation includes previously unpublished co-authored material.

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## CHAPTER I

### INTRODUCTION

In this dissertation, we employ techniques from noncommutative ring theory to explore connections between homological algebra, differential homological algebra, and simplicial topology. The central theme of our work is the remarkable notion of Koszul algebra introduced by Priddy in [29]. Motivated by the Adams spectral sequence for computing stable homotopy groups, Priddy defined a class of algebras including the Steenrod algebra and universal enveloping algebras with nice homological properties. A nonnegatively graded, locally finite-dimensional algebra  $A$  over a field  $k$  with  $A/A_+ = k$  is *Koszul* if the bigraded Yoneda algebra  $\text{Ext}_A(k, k)$  is generated by  $\text{Ext}_A^{1,1}(k, k)$ . This strong condition implies that Koszul algebras must have only quadratic relations.

One of the most important aspects of the theory of Koszul algebras is the notion of Koszul duality. Every quadratic algebra  $A$  has a corresponding quadratic dual algebra  $A^!$  which can be described explicitly in terms of the generators and relations of  $A$ . If  $A$  is Koszul,  $\text{Ext}_A(k, k) \cong A^!$ . Moreover,  $A^!$  determines a canonical minimal resolution of the trivial  $A$ -module  $k$ . The book [27] is an excellent reference for the basic theory of Koszul algebras. Many important quadratic algebras arising naturally in mathematics are Koszul, and the Koszul property is often tied to important structure in the settings which produced the algebras (see, for example [2, 5, 26, 28, 32, 36]).

Due to the limiting requirement that Koszul algebras be quadratic, several generalizations of the notion of Koszul algebra appear in the literature. A very natural generalization is the notion of  $\mathcal{K}_2$  algebra introduced recently by Cassidy and Shelton in [7]. They define a graded algebra  $A$  to be  $\mathcal{K}_2$  if  $\text{Ext}_A(k, k)$  is generated algebraically by  $\text{Ext}_A^{1,1}(k, k)$  and  $\text{Ext}_A^{2,*}(k, k)$ . This is the next most restrictive homological condition one can make while allowing the algebra  $A$  to have homogeneous relations of various degrees.

By contrast, most other generalizations of the Koszul property rely to some extent on homological purity. Purity conditions, such as Berger’s  $N$ -Koszul property (see [3]) facilitate the generalization of Koszul duality. However, these classes of algebras are not closed under standard operations such as tensor products and graded Ore extensions. Furthermore, most generalizations based on purity do not allow for commutative or skew-commutative algebras. Cassidy and Shelton [7] prove the class of  $\mathcal{K}_2$  algebras is closed under many standard operations in ring theory: tensor products, regular normal extensions, and graded Ore extensions. The class of  $\mathcal{K}_2$  algebras includes Artin-Schelter regular algebras of global dimension 4 on three linear generators as well as graded complete intersections. While there is not yet a  $\mathcal{K}_2$  analog of the canonical minimal graded free resolution provided by  $A^!$  for Koszul algebras, that problem is not unreasonable. The  $\mathcal{K}_2$  hypothesis guarantees that  $\text{Ext}_A^n(k, k)$  is finite dimensional.

This work represents the combined results of two distinct projects. Each project was motivated by an aspect of the theory of Koszul algebras which we regard as sufficiently powerful or fundamental to warrant an interpretation for  $\mathcal{K}_2$  algebras. The first project, chronologically, sought to understand the structure of “higher-order operations” on the Yoneda algebra of a  $\mathcal{K}_2$  algebra. The results of this project are presented in Chapter IV. The second project sought to establish Koszul-type connections between homological algebra and combinatorics through the study of face rings. The results of our noncommutative approach to a problem of commutative algebra are presented in Chapter III.

Here is a more detailed outline of the contents of this dissertaion.

In Chapter II, we present the relevant background on Yoneda algebras, Koszul algebras and  $\mathcal{K}_2$  algebras needed for our study. Homological properties of graded modules play an important role in later chapters, so we also review the notions of Koszul module and  $\mathcal{K}_2$  module. Chapter II concludes with some new results on *componentwise-linear* (CL) modules over Koszul algebras. In this setting, CL modules may be thought of as the class of  $\mathcal{K}_2$  modules which are “as close to Koszul as possible.”

Chapter III examines the  $\mathcal{K}_2$  property for factor algebras. Backelin and Fröberg [1] proved that if  $A$  is a Koszul algebra and  $I$  is a quadratic ideal of  $A$  which is Koszul as a left  $A$ -module, then the factor algebra  $B = A/I$  is a Koszul algebra. We prove that if  $A$  is Koszul algebra and  ${}_A I$  is a  $\mathcal{K}_2$  module, then  $B$  is a  $\mathcal{K}_2$  algebra provided  $B$  acts trivially on  $\text{Ext}_A(B, k)$ . We also show

the statement is false if the hypothesis that  $B$  acts trivially is removed. This result generalizes the Backelin-Fröberg theorem, since  $B$  necessarily acts trivially on  $\text{Ext}_A(B, k)$  when  ${}_A I$  is Koszul. Though we have not yet proven an analogous theorem when  $A$  is a  $\mathcal{K}_2$  algebra, we believe the approach of Chapter III may be adapted to establish results in that direction.

As an application of our theorem on factor algebras, we consider a class of commutative algebras arising in combinatorial topology. To any simplicial complex  $\Delta$  on  $n$  vertices there is an associated monomial ideal  $I_\Delta$  in the polynomial ring  $k[x_1, \dots, x_n]$  known as the Stanley-Reisner ideal. The monomial generators of  $I_\Delta$  correspond to minimal non-faces of  $\Delta$ . For this reason, the algebra  $k[\Delta] = k[x_1, \dots, x_n]/I_\Delta$  is sometimes called a face ring. Fröberg [11] proved that all quadratic face rings are Koszul. A large number of face rings are not quadratic, yet have nice homological and ring-theoretic properties. By combining the main theorems of Chapters II and III, we show that the class of *sequentially Cohen-Macaulay* face rings are  $\mathcal{K}_2$  algebras and we give examples that suggest the class of  $\mathcal{K}_2$  face rings is actually much larger. When  $k[\Delta]$  is sequentially Cohen-Macaulay, it is known that the homological properties of  $I_\Delta$  are governed by the combinatorial geometry of a related simplicial complex  $\Delta^*$ . This suggests that the  $\mathcal{K}_2$  module property for  $I_\Delta$  can also be expressed in terms of the geometry of  $\Delta^*$ . The problem of combinatorially determining when  $k[\Delta]$  is a  $\mathcal{K}_2$  algebra seems considerably more difficult.

Another important recent development in ring theory is the use of  $A_\infty$ -algebras (cf. [15], [20], [21]). The notion of  $A_\infty$ -algebra was introduced by Stasheff in [35] to study associativity from the point of view of homotopy theory. Later, Kadeishvili [17] showed the data of an  $A_\infty$ -algebra is precisely what is needed to recover a differential graded algebra from its cohomology, up to  $A_\infty$  quasi-isomorphism. The Yoneda algebra  $\text{Ext}_A(k, k)$  of a graded  $k$ -algebra  $A$  is the cohomology algebra of a differential graded algebra. Therefore Kadeishvili's theorem implies that  $\text{Ext}_A(k, k)$  admits a canonical  $A_\infty$ -algebra structure. The goal of Chapter IV is to provide some partial answers to two natural questions.

- What restrictions does the  $\mathcal{K}_2$  condition place on a canonical  $A_\infty$ -structure on the Yoneda algebra?
- Do certain  $A_\infty$ -structures on the Yoneda algebra guarantee the original algebra is  $\mathcal{K}_2$ ?

One can characterize Koszul algebras as those graded algebras whose Yoneda algebra admits only trivial  $A_\infty$ -structure. We give a constructive proof of this fact, as we have not found it explicitly proven in the literature. Though several papers prove, in varying degrees of detail, the existence of an  $A_\infty$ -algebra structure on the cohomology ring of a differential graded algebra, we provide an account which we hope will appeal to readers familiar with the details of Yoneda product calculations. We submit our proof of the triviality of  $A_\infty$ -structure for Koszul algebras as evidence for the value of our perspective.

The analogues of the natural questions above for  $N$ -Koszul algebras with  $N \geq 3$  were considered by He and Lu in [15], where they obtain a result similar to the characterization of Koszul algebras. Their result is aided by the fact that for an  $N$ -Koszul algebra,  $\text{Ext}_A^p(k, k)$  is concentrated in a single internal degree. As we have mentioned, the Yoneda algebra of a  $\mathcal{K}_2$  algebra does not generally satisfy such a strong purity condition. Indeed, the main result of Chapter IV shows, in contrast to the cases of Koszul and  $N$ -Koszul algebras, that vanishing of  $A_\infty$ -structure on the Yoneda algebra of a  $\mathcal{K}_2$  algebra need not be determined in any obvious way by the degrees of defining relations. We also demonstrate that obvious patterns of vanishing among higher multiplications cannot detect the  $\mathcal{K}_2$  property. Despite the somewhat negative conclusions drawn from these examples, we believe that, as in the case of the theorem on factor algebras, there will be positive results in many interesting cases.

Chapter IV contains material which was co-authored.

## CHAPTER II

### YONEDA ALGEBRAS AND GENERALIZED KOSZUL PROPERTIES

#### II.1 Introduction

In this chapter, we set up the notation and the definitions we will use throughout the dissertation. We begin by reviewing some standard facts and techniques from homological algebra in Section II.2.

In Section II.3, we define notions of Koszul algebra and Koszul module that are central to our work. We also define the related notions of  $\mathcal{K}_2$  algebra and  $\mathcal{K}_2$  module and discuss efforts to generalize the notion of Koszul algebra.

Recent progress in the study of monomial commutative algebras highlights the importance of “componentwise linear ideals.” Section II.4 introduces the notion of componentwise linear module and establishes relationships between this notion, Koszulity, and the  $\mathcal{K}_2$  property for modules over Koszul algebras. The results of Section II.4 are applied in the discussion of face rings in Chapter III.

#### II.2 Graded Homological Algebra and Ext

Throughout, let  $k$  denote any field. The term *graded  $k$ -algebra* will mean an  $\mathbb{N}$ -graded, locally finite dimensional, connected  $k$ -algebra generated in degree 1. If  $A$  is a graded  $k$ -algebra, we denote the category of locally finite dimensional,  $\mathbb{Z}$ -graded left (resp. right)  $A$ -modules with degree 0 homomorphisms by  $\mathbf{Gr}\text{-}\mathbf{A}\text{-mod}^f$  (resp.  $\mathbf{Gr}\text{-mod}\text{-}\mathbf{A}^f$ ). The term *graded left (resp. right)  $A$ -module* will refer to an object of the appropriate module category. We do not assume that graded modules are finitely generated or bounded unless we explicitly state otherwise. The *trivial  $A$ -module* is  $k = A/A_+$  where  $A_+ = \bigoplus_{i>0} A_i$ .

If  $M$  is a graded  $A$ -module, we write  $M(d)$  for the shifted graded module with  $M(d)_n = M_{n+d}$  for all  $n \in \mathbb{Z}$ . When it will cause no confusion, we denote  $\text{Hom}_{\mathbf{Gr-A-mod}^f}(M, N)$  and  $\text{Hom}_{\mathbf{Gr-mod-A}^f}(M, N)$  by  $\text{hom}_A(M, N)$ . For  $n \in \mathbb{Z}$ , we define  $\text{Hom}_A^n(M, N) = \text{hom}_A(M, N(-n))$ . One can think of  $\text{Hom}_A^n(M, N)$  as the space of graded homomorphisms  $f : M \rightarrow N$  which lower every homogeneous element's degree by  $n$ . Our primary homological tools are the graded Hom functor  $\text{Hom}_A^*(M, N) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_A^n(M, N)$  and its right derived functors.

The categories  $\mathbf{Gr-A-mod}^f$  and  $\mathbf{Gr-mod-A}^f$  have both enough projectives and enough injectives. The former is immediate because these categories have graded free objects. We note that if a graded module  $M$  is bounded below,  $M$  has a graded projective resolution by bounded below modules. It is less obvious that these categories have enough injectives. See Lemma III.2.1 for a proof that the trivial  $A$ -module has a graded injective resolution. This is not generally true if one works in the subcategory of bounded below modules.

The  $i$ -th right derived functors of  $\text{Hom}_A^*(M, -)$  and  $\text{Hom}_A^*(-, N)$  respectively will be denoted by  $\text{Ext}_A^i(M, -)$  and  $\text{Ext}_A^i(-, N)$ . Recall that  $\text{Ext}_A^i(M, N)$  may be computed by applying the functor  $\text{Hom}_A^*(M, -)$  to a graded injective resolution of  $N$  or by applying  $\text{Hom}_A^*(-, N)$  to a graded projective resolution of  $M$ . We will employ both methods; however, for given  $A$ ,  $M$ , and  $N$  we will always fix a “model” for  $\text{Ext}_A^i(M, N)$ . A third approach - using graded endomorphisms of a graded projective resolution to compute  $\text{Ext}_A^i(k, k)$  - will be considered in Chapter IV.

The *radical* of a graded left (resp. right)  $A$ -module  $P$  is the submodule  $A_+P$  (resp.  $PA_+$ ). The *socle* of a graded  $A$ -module  $I$  is the unique maximal semisimple submodule of  $I$ , denoted  $\text{soc}(I)$ . Equivalently,  $\text{soc}(I)$  is the submodule of  $I$  annihilated by  $A_+$ . We call a graded projective resolution  $P^\bullet \rightarrow M \rightarrow 0$  of a graded  $A$ -module  $M$  *minimal* if  $\text{im } \partial_P^i \subset \text{rad}(P^{i-1})$ . We call a graded injective resolution  $0 \rightarrow N \rightarrow I_\bullet$  of a graded  $A$ -module  $N$  *minimal* if  $\text{soc}(I_j) \subset \text{im } \partial_j^I$ . If  $P_\bullet$  is a minimal graded projective resolution of  $M$ , then  $\text{Ext}_A^i(M, k) = \text{Hom}_A(P^i, k)$ . Similarly, if  $I_\bullet$  is a minimal graded injective resolution of  $N$ , then  $\text{Ext}_A^j(k, N) = \text{Hom}_A(k, I_j)$ . Minimal resolutions exist for bounded below modules by a graded version of Nakayama's Lemma (see Lemma 1.4.1 of [27]).

For each  $i \in \mathbb{N}$ , the  $k$ -vector space  $\text{Ext}_A^i(M, N)$  inherits a  $\mathbb{Z}$ -grading from the graded Hom functor. We call this the *internal* grading. The homogeneous (internal) degree  $j$  component of  $\text{Ext}_A^i(M, N)$  is denoted  $\text{Ext}_A^{i,j}(M, N)$ . The vector space  $\text{Ext}_A(M, N) = \bigoplus_{i \in \mathbb{N}} \text{Ext}_A^i(M, N)$  is



therefore bigraded. If  $L$ ,  $M$ , and  $N$  are graded left  $A$ -modules, the Yoneda composition product gives a bilinear, associative pairing

$$\star : \text{Ext}_A^{i,j}(M, N) \otimes \text{Ext}_A^{k,l}(L, M) \rightarrow \text{Ext}_A^{i+k, j+l}(L, N)$$

If  $P^\bullet \rightarrow L$  and  $Q^\bullet \rightarrow M$  are respective graded projective resolutions of  $L$  and  $M$ , the Yoneda product is computed as follows. If  $f : P^k \rightarrow M$  represents a class in  $\text{Ext}_A^{k,l}(L, M)$  and  $g : Q^i \rightarrow N$  represents a class in  $\text{Ext}_A^{i,j}(M, N)$ , the product  $[g] \star [f]$  is the class  $[gf_i] \in \text{Ext}_A^{i+k, j+l}(L, N)$  where  $f_i$  is determined by successively lifting  $f$  through the projective resolution  $P^\bullet$  as depicted in the following diagram.

$$\begin{array}{ccccccc}
 P^{i+k} & \longrightarrow & \dots & \longrightarrow & P^{k+1} & \longrightarrow & P^k \\
 \downarrow f_i & & & & \downarrow f_1 & & \downarrow f_0 \\
 Q^i & \longrightarrow & \dots & \longrightarrow & Q^1 & \longrightarrow & Q^0 \\
 \downarrow g & & & & & & \searrow \\
 N & & & & & & M
 \end{array}$$

The obvious dual formulation computes the Yoneda product using injective resolutions.

The Yoneda product gives the bigraded vector space  $\text{Ext}_A(M, M)$  the structure of a graded  $k$ -algebra. If  $M = k$ , we call this algebra the *Yoneda algebra* of  $A$ , denoted  $E(A)$ . We denote  $E^{p,q}(A) = \text{Ext}_A^{p,q}(k, k)$  and let  $E^p(A) = \bigoplus_q E^{p,q}(A)$ . The Yoneda product then makes  $\text{Ext}_A(L, k)$  into a graded left  $E(A)$ -module. Similarly,  $\text{Ext}_A(k, N)$  is a graded right  $E(A)$ -module.

### II.3 Koszul and Generalized Koszul Properties

The notion of Koszul algebra was introduced by Priddy in [29], and it has since appeared in varying degrees of generality (see, for example, [2], [25], [27]). At this time, the book [27] is the most (really, the only) comprehensive exposition of the theory of Koszul algebras we have found. There, the authors establish many useful, equivalent formulations of Koszulity for both algebras

and modules. For these reasons, we adopt the setting of [27], with a few minor exceptions which we note.

**Definition II.3.1.** A graded  $k$ -algebra  $A$  is called **Koszul** if the following equivalent conditions hold.

1.  $\text{Ext}_A^{i,j}(k, k) = 0$  for  $i \neq j$
2. The algebra  $E(A)$  is generated by  $E^1(A)$ .

Following [7], we refer to condition 1 as *diagonal purity* and condition 2 as *low-degree generation*. These equivalent conditions are very strong. In particular, Definition II.3.1 implies that the defining relations of the algebra  $A$  must be quadratic (see Corollary I.5.3 of [27]). Berger [3], motivated by the study of Artin-Schelter regular algebras, introduced the notion of  *$N$ -Koszul algebra* in an effort to extend the Koszul theory to algebras with non-quadratic relations. Berger's definition generalizes diagonal purity. While  $N$ -Koszul algebras and Koszul algebras have similarly powerful homological properties, the class of  $N$ -Koszul algebras (for  $N > 2$ ) differs notably in that the class is not closed under many standard ring-theoretic operations such as tensor products and graded Ore extensions. Furthermore, if  $N > 2$ , the class of  $N$ -Koszul algebras does not include any commutative or graded-commutative algebras; the defining relations of an  $N$ -Koszul algebra must have degree  $N$ .

Recently, Cassidy and Shelton [7] introduced a generalization of Koszul algebras that includes  $N$ -Koszul algebras and is closed under tensor products, graded Ore extensions, and regular central extensions. Cassidy and Shelton's definition generalizes low-degree generation.

**Definition II.3.2.** A graded  $k$ -algebra  $A$  is called a  $\mathcal{K}_2$  algebra if it is finitely related and if the algebra  $E(A)$  is generated by  $E^1(A)$  and  $E^2(A)$ .

From Corollary I.5.3 of [27], it is clear that this is the next most restrictive definition one can make while allowing relations of more than one homogeneous degree. In [7], the authors show that, in addition to Koszul and  $N$ -Koszul algebras, the class of  $\mathcal{K}_2$  algebras includes graded complete intersections and global dimension four Artin-Schelter regular algebras on three linear generators. We will be concerned primarily with the Koszul and  $\mathcal{K}_2$  properties of algebras and will not separately consider  $N$ -Koszul algebras.

It will be useful to have analogous notions of Koszul and  $\mathcal{K}_2$  for graded modules. Here we depart slightly from [27].

**Definition II.3.3.** Let  $A$  be a graded  $k$ -algebra and let  $M$  be a graded left  $A$ -module. Let  $D_1(A)$  be the subalgebra of  $E(A)$  generated by  $E^1(A)$  and let  $D_2(A)$  be the subalgebra of  $E(A)$  generated by  $E^1(A) + E^2(A)$ . For  $n = 1, 2$  we call  $M$  a  $\mathcal{K}_n$   $A$ -module if  $\text{Ext}_A(M, k)$  is generated as a left  $D_n(A)$ -module by  $\text{Ext}_A^0(M, k)$ . We call  $M$  a Koszul  $A$ -module if  $M$  is  $\mathcal{K}_1$  and there exists  $d \in \mathbb{Z}$  such that  $M = AM_d$ .

We note that  $A$  is a  $\mathcal{K}_2$  algebra if and only if  ${}_A k$  is a  $\mathcal{K}_2$   $A$ -module and that  $A$  is a  $\mathcal{K}_1$  algebra in the sense of [25] if and only if  ${}_A k$  is a  $\mathcal{K}_1$   $A$ -module. We also note that  $M$  is a Koszul module if and only if there exists a  $d \in \mathbb{Z}$  such that  $M(d)$  is a Koszul module in the sense of [27]. (Obviously, one could define the notion of a  $\mathcal{K}_n$  algebra or module for any  $n > 2$ , however we see no motivation for doing so at this time.)

If  $M$  is a graded  $A$ -module, we say  $M$  has a *linear free resolution* if  $M = AM_d$  for some  $d \in \mathbb{Z}$  and  $M$  admits a resolution  $\cdots \rightarrow P^2 \rightarrow P^1 \rightarrow P^0 \rightarrow M \rightarrow 0$  by graded free  $A$ -modules such that  $P^i$  is generated in degree  $i + d$  for all  $i \geq 0$ . Graded projective modules are graded free (see [3]), thus  $M$  is a Koszul  $A$ -module if and only if  $M$  has a linear free resolution. Furthermore,  $A$  is a Koszul algebra if and only if the trivial module  ${}_A k$  has a linear free resolution.

We now establish a few facts about extensions of  $\mathcal{K}_2$  modules, followed by their Koszul analogs.

**Lemma II.3.4.** *Let  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  be a graded exact sequence of graded left  $A$ -modules.*

1. *If  $L$  and  $N$  are  $\mathcal{K}_2$   $A$ -modules and the natural homomorphism  $\text{Ext}_A^0(M, k) \rightarrow \text{Ext}_A^0(L, k)$  is surjective, then  $M$  is a  $\mathcal{K}_2$   $A$ -module.*
2. *If  $L$  and  $M$  are  $\mathcal{K}_2$   $A$ -modules and  $N$  is generated in degrees strictly greater than the degrees in which  $L$  is generated, then  $N$  is a  $\mathcal{K}_2$   $A$ -module.*

*Proof.* Let  $D(A) = D_2(A)$ . If  $N$  is generated in homogeneous degrees strictly greater than the degrees in which  $L$  is generated, then the natural map  $g_0 : \text{Ext}_A^0(M, k) \rightarrow \text{Ext}_A^0(L, k)$  is surjective. Thus for both (1) and (2), the long exact sequence in cohomology has the form

$$\begin{aligned}
0 \rightarrow \text{Ext}_A^0(N, k) &\xrightarrow{f_0} \text{Ext}_A^0(M, k) \xrightarrow{g_0} \text{Ext}_A^0(L, k) \xrightarrow{0} \text{Ext}_A^1(N, k) \rightarrow \dots \\
&\rightarrow \text{Ext}_A^i(M, k) \rightarrow \text{Ext}_A^i(L, k) \rightarrow \text{Ext}_A^{i+1}(N, k) \rightarrow \dots
\end{aligned}$$

This is a complex of left  $E(A)$ -modules. Since  $L$  is a  $\mathcal{K}_2$  module,  $\text{Ext}_A^i(L, k) = D^i(A) \star \text{Ext}_A^0(L, k) \rightarrow D^i(A) \star \text{Ext}_A^1(N, k) = \text{Ext}_A^{i+1}(N, k)$  is zero for all  $i$ . We therefore obtain a short exact sequence

$$0 \rightarrow \text{Ext}_A(N, k) \rightarrow \text{Ext}_A(M, k) \rightarrow \text{Ext}_A(L, k) \rightarrow 0$$

of left  $E(A)$ -modules. Statement (1) now follows.

To prove (2), let  $h_0 : \text{Ext}_A^0(L, k) \rightarrow \text{Ext}_A^0(M, k)$  be a graded vector space splitting of  $g_0$ . Consideration of internal degrees shows that  $\text{im } f_0 \cap \text{im } h_0 = 0$ , so the map  $h_0$  extends to a splitting of the  $E(A)$ -module homomorphism  $\text{Ext}_A(M, k) \rightarrow \text{Ext}_A(L, k)$ , and (2) follows. □

The strictness in (2) cannot be weakened. See Example III.4.2. The following Lemma should be well known.

**Lemma II.3.5.** *Let  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  be a graded exact sequence of graded left  $A$ -modules.*

1. *If  $L$  and  $N$  are Koszul  $A$ -modules and the natural homomorphism  $\text{Ext}_A^0(M, k) \rightarrow \text{Ext}_A^0(L, k)$  is surjective, then  $M$  is a Koszul  $A$ -module.*
2. *If  $L$  and  $M$  are Koszul  $A$ -modules and the natural homomorphism  $\text{Ext}_A^0(M, k) \rightarrow \text{Ext}_A^0(L, k)$  is surjective, then  $N$  is a Koszul  $A$ -module.*
3. *If  $M$  and  $N$  are Koszul  $A$ -modules and the natural monomorphism  $\text{Ext}_A^0(N, k) \hookrightarrow \text{Ext}_A^0(M, k)$  is an isomorphism, then  $L$  is a Koszul  $A$ -module.*

*Proof.* Since Koszul modules are  $\mathcal{K}_1$  modules, the proof of Lemma II.3.4 shows that in (1) and (2), the long exact sequence in cohomology breaks into short exact sequences. Each conclusion follows immediately by considering degrees.

For (3), suppose  $N$  and  $M$  are generated in homogeneous degree  $d$  and consider the long exact sequence in cohomology

$$\begin{aligned} 0 \rightarrow \text{Ext}_A^0(N, k) \xrightarrow{\cong} \text{Ext}_A^0(M, k) \xrightarrow{0} \text{Ext}_A^0(L, k) \rightarrow \text{Ext}_A^1(N, k) \rightarrow \cdots \\ \rightarrow \text{Ext}_A^i(M, k) \rightarrow \text{Ext}_A^i(L, k) \rightarrow \text{Ext}_A^{i+1}(N, k) \rightarrow \cdots \end{aligned}$$

Since  $\text{Ext}_A^0(L, k) \hookrightarrow \text{Ext}_A^1(N, k)$  and  $N$  is a Koszul  $A$ -module,  $\text{Ext}_A^0(L, k)$  is concentrated in internal degree  $d + 1$ . Since  $M$  is a Koszul  $A$ -module,  $\text{Ext}_A^i(M, k)$  is concentrated in internal degree  $d + i$ , so the map  $\text{Ext}_A^i(M, k) \rightarrow \text{Ext}_A^i(L, k)$  is zero. Thus  $\text{Ext}_A^i(L, k) \hookrightarrow \text{Ext}_A^{i+1}(N, k)$  for all  $i$ . Since  $\text{Ext}_A^{i+1}(N, k)$  is concentrated in internal degree  $d + i + 1$ , the same is true for  $\text{Ext}_A^i(L, k)$ , hence  $L$  is a Koszul  $A$ -module. □

We note that the surjectivity condition in (1) and (2) is satisfied for arbitrary graded modules when  $N$  is generated in homogeneous degrees greater than or equal to the degrees in which  $L$  is generated. We do not expect an analog of Lemma II.3.5(3) for  $\mathcal{K}_1$  or  $\mathcal{K}_2$  modules. For example, submodules of  $\mathcal{K}_1$  modules need not be  $\mathcal{K}_1$ , even with the hypothesis on  $\text{Ext}^0$ . See Example III.4.1 below.

Next we recall the matrix criterion of [7] for a left  $A$ -module  $M$  to be a  $\mathcal{K}_2$  module and state the analogous result for  $\mathcal{K}_1$  modules. We need to introduce some notation. Let  $V$  be a finite dimensional vector space,  $A = T(V)/I$  a graded algebra and  $M$  a graded left  $A$ -module. Let  $Q^\bullet \rightarrow M$  be a minimal resolution of  $M$  by graded projective left  $A$ -modules. Choose homogeneous bases for each  $Q^i$  and let  $M_i$  be the matrix of  $d_i : Q^i \rightarrow Q^{i-1}$  with respect to these bases. Let  $f_i$  be a lift of  $M_i$  to a matrix over  $T(V)_+$  with homogeneous entries. Let  $L(f_i)$  denote  $f_i$  mod  $T(V)_{\geq 2}$ . For  $i \geq 0$ , let  $(f_{i+1}f_i)_{\text{ess}}$  denote the product  $f_{i+1}f_i$  mod  $I'$  where  $I' = V \otimes I + I \otimes V$ . Note that  $f_0 = 0$ .

**Lemma II.3.6** ([7]). *An  $A$ -module  $M$  is a  $\mathcal{K}_2$   $A$ -module if and only if for all  $0 \leq i < \text{pd}_A(M)$ , the matrix  $[(f_{i+1}f_i)_{\text{ess}} \ L(f_{i+1})]$  has linearly independent rows.*

The following Proposition is proven by arguing as in Lemma 4.3 and Theorem 4.4 of [7], replacing the trivial  $A$ -module with  $M$ .

**Proposition II.3.7.** *Let  $A$  be an  $\mathbb{N}$ -graded, connected algebra, and let  $M$  be a graded  $A$ -module. For  $0 \leq i \leq \text{pd}_A(M)$  let  $f_i$  be as above. Then  $\text{Ext}_A^i(M, k) = E^1(A) \star \text{Ext}_A^{i-1}(M, k)$  if and only if  $L(f_i)$  has linearly independent rows. Thus  $M$  is a  $\mathcal{K}_1$   $A$ -module if and only if  $L(f_i)$  has linearly independent rows for all  $1 \leq i \leq \text{pd}_A(M)$ .*

The following characterization of Koszul modules over quadratic algebras will be useful in Section II.4.

**Corollary II.3.8.** *If  $A$  is a quadratic algebra and  $M$  is a graded left  $A$ -module, then  $M$  is a  $\mathcal{K}_2$   $A$ -module if and only if  $M$  is a  $\mathcal{K}_1$   $A$ -module. Furthermore, if  $M$  is generated in a single homogeneous degree, then  $M$  is a  $\mathcal{K}_2$   $A$ -module if and only if  $M$  is a Koszul  $A$ -module.*

*Proof.* It suffices to prove that  $M$  is a  $\mathcal{K}_2$   $A$ -module only if  $M$  is a  $\mathcal{K}_1$   $A$ -module. If  $L(f_i)$  has linearly dependent rows for some  $1 \leq i \leq \text{pd}_A(M)$ , then after changing basis we may assume the first row of  $f_i$  contains no linear entries. This implies that the nonzero entries of the corresponding row of  $f_i f_{i-1}$  have degree at least 3. As  $A$  is quadratic,  $I_{>2} = I'$ . So the first row of  $[(f_i f_{i-1})_{\text{ess}} L(f_i)]$  is zero, and  $M$  is not a  $\mathcal{K}_2$   $A$ -module by Proposition II.3.7. □

## II.4 Componentwise Linear Modules

Let  $M$  be a graded left  $A$ -module. Throughout this section, we additionally assume that  $M$  is bounded below and let  $b \in \mathbb{Z}$  denote the smallest integer such that  $M_b \neq 0$ . If the submodule  $AM_i$  is a Koszul  $A$ -module for all  $i$ , we say that  $M$  is a *componentwise linear*  $A$ -module. Our definition is motivated by the notion of componentwise linear ideal introduced in [16] and studied further in [30]. In this section, we prove that all componentwise linear modules over Koszul algebras are  $\mathcal{K}_2$  (equivalently,  $\mathcal{K}_1$ ) modules. We also characterize which  $\mathcal{K}_2$  modules over Koszul algebras are componentwise linear. Following the notation of [30], let  $M_{\langle j \rangle} = AM_j$  and for  $i \leq j$ , let  $M_{\langle i, j \rangle} = \sum_{t=i}^j AM_t$ .

**Lemma II.4.1.** *If  $A$  is a Koszul algebra and  $M_{\langle j \rangle}$  is a Koszul  $A$ -module for some  $j \in \mathbb{Z}$ , then*

1.  $M_{\langle j \rangle} \cap M_{\langle j+1 \rangle}$  is a Koszul  $A$ -module, and
2.  $M_{\langle j+1 \rangle}$  is a Koszul  $A$ -module if and only if  $M_{\langle j, j+1 \rangle} / M_{\langle j \rangle}$  is.

*Proof.* For  $j \in \mathbb{Z}$  we have the exact sequence  $0 \rightarrow M_{\langle j \rangle} \cap M_{\langle j+1 \rangle} \rightarrow M_{\langle j \rangle} \rightarrow T \rightarrow 0$  where  $T = M_{\langle j \rangle} / M_{\langle j \rangle} \cap M_{\langle j+1 \rangle}$  is a trivial  $A$ -module. Hence  $T$  is a Koszul  $A$ -module. Since  $M_{\langle j \rangle} \cap M_{\langle j+1 \rangle}$  is concentrated in degrees  $\geq j+1$ ,  $\text{Ext}_A^0(T, k) \cong \text{Ext}_A^0(M_{\langle j \rangle}, k)$ . Since  $M_{\langle j \rangle}$  is a Koszul  $A$ -module, Lemma II.3.5(3) implies  $M_{\langle j \rangle} \cap M_{\langle j+1 \rangle}$  is a Koszul  $A$ -module.

To prove (2), we consider the exact sequence

$$0 \rightarrow M_{\langle j \rangle} \cap M_{\langle j+1 \rangle} \rightarrow M_{\langle j+1 \rangle} \rightarrow M_{\langle j, j+1 \rangle} / M_{\langle j \rangle} \rightarrow 0$$

Since all three modules are generated in homogeneous degree  $j+1$ , the map  $\text{Ext}_A^0(M_{\langle j+1 \rangle}, k) \rightarrow \text{Ext}_A^0(M_{\langle j \rangle} \cap M_{\langle j+1 \rangle}, k)$  is surjective. The result follows by Lemma II.3.5(1),(2).  $\square$

**Remark II.4.2.** In our applications of Lemma II.4.1, we require the slightly modified statements that  $M_{\langle d, j \rangle} \cap M_{\langle j+1 \rangle}$  is a Koszul  $A$ -module for all  $d < j$  and  $M_{\langle j+1 \rangle}$  is a Koszul  $A$ -module if and only if  $M_{\langle d, j+1 \rangle} / M_{\langle d, j \rangle}$  is a Koszul  $A$ -module. These are obviously equivalent to (1) and (2) above.

**Lemma II.4.3.** *If  $A$  is a Koszul algebra and  $M$  is a componentwise linear  $A$ -module, then  $M_{\langle d, j \rangle}$  is a  $\mathcal{K}_2$   $A$ -module for all  $d, j \in \mathbb{Z}$  such that  $d \leq j$ .*

*Proof.* By Lemma II.4.1(2),  $M_{\langle d, j \rangle} / M_{\langle d, j-1 \rangle}$  is a Koszul  $A$ -module for all  $j \in \mathbb{Z}$  and all  $d < j$ . We consider the exact sequence

$$0 \rightarrow M_{\langle d, j-1 \rangle} \rightarrow M_{\langle d, j \rangle} \rightarrow M_{\langle d, j \rangle} / M_{\langle d, j-1 \rangle} \rightarrow 0$$

and prove the result by induction on  $j-d$ . If  $j=d$ ,  $M_{\langle d, j \rangle}$  is a Koszul  $A$ -module by assumption. For the induction step, we assume  $M_{\langle d, j-1 \rangle}$  is a  $\mathcal{K}_2$   $A$ -module. Since  $M_{\langle d, j \rangle} / M_{\langle d, j-1 \rangle}$  is a Koszul  $A$ -module, Lemma II.3.4(1) implies that  $M_{\langle d, j \rangle}$  is a  $\mathcal{K}_2$   $A$ -module. The result follows by induction.  $\square$

**Proposition II.4.4.** *Let  $A$  be a Koszul algebra and let  $M$  be a componentwise linear  $A$ -module. If  $b \in \mathbb{Z}$  is minimal such that  $M_b \neq 0$ , then  $M_{\langle b, j \rangle}$  is a  $\mathcal{K}_2$   $A$ -module for all  $j \geq b$ . In particular,  $M$  is a  $\mathcal{K}_2$   $A$ -module.*

*Proof.* The first statement follows immediately from Lemma II.4.3. To prove that  $M$  is  $\mathcal{K}_2$ , we show that  $\text{Ext}_A^{p, b+q}(M, k)$  is  $E(A)$ -generated by  $\text{Ext}_A^{0, b+q-p}(M, k)$  by considering the long exact

sequence in cohomology associated to the exact sequence  $0 \rightarrow M_{\langle b, b+q-p \rangle} \rightarrow M \rightarrow F \rightarrow 0$ . Since  $F$  is generated in degrees  $\geq b+q-p+1$ ,  $\text{Ext}_A^p(F, k)$  is concentrated in internal degrees  $\geq b+q+1$ . Thus the natural map  $\text{Ext}_A^0(M, k) \rightarrow \text{Ext}_A^0(M_{\langle b, b+q-p \rangle}, k)$  is surjective in internal degrees  $\leq b+q-p$  and  $\text{Ext}_A^{p, b+q}(M, k) \hookrightarrow \text{Ext}_A^{p, b+q}(M_{\langle b, b+q-p \rangle}, k)$ . Since the morphisms in the long exact sequence respect the left  $E(A)$ -module structure, the result follows from the assumption that  $M_{\langle b, b+q-p \rangle}$  is a  $\mathcal{K}_2$   $A$ -module. □

Let  $M$  be a bounded below graded left  $A$ -module and recall we let  $b$  be the smallest integer such that  $M_b \neq 0$ . We say  $M$  is a *strongly*  $\mathcal{K}_2$   $A$ -module if  $M_{\langle b, j \rangle}$  is a  $\mathcal{K}_2$   $A$ -module for all  $j \geq b$ . We note that if  $M$  is strongly  $\mathcal{K}_2$  and  $A$  is a quadratic algebra, then  $M_{\langle b \rangle}$  is a Koszul  $A$ -module by Corollary II.3.8. For an example of a  $\mathcal{K}_2$  module over a Koszul algebra that is not strongly  $\mathcal{K}_2$ , see Example III.4.2.

**Lemma II.4.5.** *Let  $A$  be a Koszul algebra and  $M$  a strongly  $\mathcal{K}_2$   $A$ -module. Let  $b$  be the smallest integer such that  $M_b \neq 0$ . Then  $F = M_{\langle b, j \rangle} / M_{\langle b, j-1 \rangle}$  is a Koszul  $A$ -module for all  $j > b$ .*

*Proof.* Let  $j > b$ . Applying Lemma II.3.4(2) to the exact sequence

$$0 \rightarrow M_{\langle b, j-1 \rangle} \rightarrow M_{\langle b, j \rangle} \rightarrow F \rightarrow 0$$

where  $F = M_{\langle b, j \rangle} / M_{\langle b, j-1 \rangle}$  shows that  $F$  is a  $\mathcal{K}_2$   $A$ -module. Since  $F$  is generated in homogeneous degree  $j$ , Corollary II.3.8 implies that  $F$  is a Koszul  $A$ -module. □

We now prove the converse of Proposition II.4.4.

**Proposition II.4.6.** *If  $A$  is a Koszul algebra and if  $M$  is a strongly  $\mathcal{K}_2$   $A$ -module, then  $M$  is componentwise linear.*

*Proof.* Let  $b$  be the smallest integer such that  $M_b \neq 0$ . Since  $A$  is quadratic,  $M_{\langle b \rangle}$  is a Koszul  $A$ -module by Corollary II.3.8. Assume inductively that  $M_{\langle b+t \rangle}$  is a Koszul  $A$ -module for  $t \geq 0$  and consider the exact sequence

$$0 \rightarrow M_{\langle b, b+t \rangle} \cap M_{\langle b+t+1 \rangle} \rightarrow M_{\langle b+t+1 \rangle} \rightarrow M_{\langle b, b+t+1 \rangle} / M_{\langle b, b+t \rangle} \rightarrow 0$$



By the induction hypothesis and by Lemma II.4.1(1),  $M_{\langle b, b+t \rangle} \cap M_{\langle b+t+1 \rangle}$  is a Koszul  $A$ -module. Lemma II.4.5 implies that  $M_{\langle b, b+t+1 \rangle} / M_{\langle b, b+t \rangle}$  is a Koszul  $A$ -module. As all three modules are generated in homogeneous degree  $b + t + 1$ , the result follows by Lemma II.3.5(1).

□

Combining Propositions II.4.4 and II.4.6, we obtain a characterization of componentwise linear modules for Koszul algebras.

**Corollary II.4.7.** *If  $A$  is a Koszul algebra, then  $M$  is a componentwise linear  $A$ -module if and only if  $M$  is a strongly  $\mathcal{K}_2$   $A$ -module.*

## CHAPTER III

### $\mathcal{K}_2$ FACTOR ALGEBRAS OF KOSZUL ALGEBRAS

#### III.1 Introduction

The first part of this chapter is devoted to proving a change-of-rings theorem for  $\mathcal{K}_2$  algebras. Our main result, Theorem III.2.14 may be viewed as both a generalization of a theorem on Koszul algebras and a companion to the change-of-rings theorems of [7].

In [1], Backelin and Fröberg establish the following useful change-of-rings result.

**Theorem III.1.1.** *Let  $A$  be a Koszul algebra and let  $I \subset A$  be a two-sided ideal generated in degree 2 such that  ${}_A I$  is a Koszul left  $A$ -module. Then the algebra  $B = A/I$  is also Koszul.*

In Section III.2 we extend Theorem III.1.1 to ideals which are  $\mathcal{K}_2$  modules. Our theorem requires the additional hypothesis that the natural action of  $B$  on  $\text{Ext}_A(B, k)$  is trivial. This hypothesis is redundant if  ${}_A I$  is Koszul. However, the hypothesis cannot be removed, as we show at the end of the section.

Section III.3 discusses applications of Theorem III.2.14 and the results of Section II.4 to the study of Stanley-Reisner rings. In particular, we show that the class of  $\mathcal{K}_2$  Stanley-Reisner rings includes all sequentially Cohen-Macaulay Stanley-Reisner rings. We also describe the connections between combinatorial geometry and the homological properties of Stanley-Reisner ideals.

The chapter concludes with a collection of illustrative examples referred to throughout Chapters II and III. Of particular interest is Example III.4.3, which refutes a conjecture of Cassidy and Shelton.

### III.2 $\mathcal{K}_2$ Factors of Koszul Algebras

Recall that the *socle* of an  $A$ -module  $M$ , denoted  $\text{soc}(M)$  is the unique maximal semisimple submodule of  $M$ . If  $A$  is a graded  $k$ -algebra, a simple module in  $\mathbf{Gr-A-mod}^f$  is isomorphic to  $k(d)$  for some  $d \in \mathbb{Z}$ . Thus we have  $\text{soc}(M) = M^{A^+} = \{m \in M : am = 0, \forall a \in A_+\}$ . More generally, we define  $M^I = \{m \in M : am = 0, \forall a \in I\}$  for any ideal  $I$  in  $A$ .

Let  $k$  be a field and let  $V$  be a finite dimensional  $k$ -vector space on basis  $X = \{x_1, \dots, x_n\}$ . Let  $A = T(V)/R$  be a graded  $k$ -algebra generated by  $V$ . We identify  $x_i$  with its image in  $A$ . We give the algebra  $A$  the usual  $\mathbb{N}$ -grading by tensor degree with  $\deg(x_i) = 1$  for  $i = 1, \dots, n$ . Let  $\mathfrak{J} \subset A_{\geq 2}$  be a homogeneous ideal and let  $B = A/\mathfrak{J}$ . The algebra  $B$  inherits the  $\mathbb{N}$ -grading of  $A$ .

Let  $(P^\bullet, \partial_P)$  be a minimal resolution of the trivial right  $A$ -module  $k_A$  by graded projective right  $A$ -modules with degree 0 differential. The augmentation map is denoted  $\epsilon : P^0 \rightarrow k$ . We denote the graded dual Hom functor  $\text{Hom}_k(-, k)$  by  $-^*$ .

**Lemma III.2.1.** *The complex  $I_\bullet = (P^\bullet)^*$  is a resolution of  ${}_A k \cong (k_A)^*$  by graded injective left  $A$ -modules. The coaugmentation  $\epsilon^\vee : k \rightarrow I_0$  is given by  $1 \mapsto \epsilon$ .*

*Proof.* Since  $k$  is semisimple, the functor  $-^*$  is exact, hence  $I_\bullet$  is an  $A$ -linear resolution of  $k^*$  with the stated coaugmentation. That  $I_j$  is injective for each  $j$  follows easily from the exactness of  $-^*$  and the fact that  $P^j$  is locally finite-dimensional. □

We denote the induced differential on  $I_\bullet$  by  $\partial^I$ . We assume that  $P^0 = A$ , that  $P^1 = A(-1)^{\oplus n}$ , and that homogeneous  $A$ -bases are chosen for  $P^1$  and  $P^0$  such that the matrix of the differential  $P^1 \rightarrow P^0$  is given by left multiplication by  $(x_1 \cdots x_n)$ .

We take  $H^*(\text{Hom}_A(k, I_\bullet))$  as our model for the bigraded Yoneda Ext-algebra  $E(A) = \bigoplus E^{n,m}(A) = \bigoplus \text{Ext}_A^{n,m}(k, k)$ . We note that  $\text{soc}(I_j)$  is generated by the graded  $k$ -linear duals of an  $A$ -basis for  $P^j$  and thus  $\text{soc}(I_j) \subset \text{im } \partial_j^I$ . Identifying  $\text{Hom}_A(k, I_j)$  with  $\text{soc}(I_j)$ , we conclude that the differential on  $\text{Hom}_A(k, I_\bullet)$  is zero and  $E^j(A) = \text{Hom}_A(k, I_j) \cong \text{soc}(I_j)$ .

We let  $J_\bullet = \text{Hom}_A(B, I_\bullet)$  and we take  $H^*(J_\bullet)$  as our model for  $\text{Ext}_A(B, k)$ . We denote by  $\partial^J$  the differential on  $J_\bullet$  induced by  $\partial^I$ . Clearly,  $g \in \text{soc}(J_p)$  if and only if  $g(1) \in \text{soc}(I_p)$ .

Let  $(Q^\bullet, \partial_Q)$  be a minimal resolution of the trivial  $B$ -module  ${}_B k$  by graded projective left  $B$ -modules. We take  $H^*(\text{Hom}_B(Q^\bullet, k)) = \text{Hom}_B(Q^\bullet, k)$  as our model for  $E(B)$ .

The Cartan-Eilenberg change-of-rings spectral sequence

$$E_2^{p,q} = \text{Ext}_B^p(k, \text{Ext}_A^q(B, k)) \Rightarrow \text{Ext}_A^{p+q}(k, k)$$

is the spectral sequence associated with the first-quadrant double cocomplex

$$A^{p,q} = \text{Hom}_B(Q^p, \text{Hom}_A(B, I_q)).$$

This is a spectral sequence of bigraded  $E(A) - E(B)$  bimodules (see Lemmas 6.1 and 6.2 of [7]). The horizontal differential  $d_h$  is precomposition with  $\partial_Q$ . The vertical differential  $d_v$  is composition with  $\partial^J$ . The left  $E(A)$  action on  $E_2^{p,q}$  is induced by the Yoneda product  $\text{Ext}_A^m(k, k) \otimes \text{Ext}_A^n(B, k) \rightarrow \text{Ext}_A^{m+n}(B, k)$ . The right  $E(B)$  action is given by the Yoneda product

$$\text{Ext}_B^m(k, \text{Ext}_A(B, k)) \otimes \text{Ext}_B^n(k, k) \rightarrow \text{Ext}_B^{m+n}(k, \text{Ext}_A(B, k))$$

We will make these module actions more explicit below.

We recall the standard “staircase argument” for double cocomplexes. Let  $(A^{p,q}, d_h, d_v)$  be a first-quadrant double cocomplex. For  $N > 1$  and for  $p, q \geq 0$  let  $\mathcal{S}_N^{p,q}$  denote the subset of  $\prod_{i=0}^{N-1} A^{p+i, q-i}$  consisting of  $N$ -tuples  $(a_1, \dots, a_N)$  such that  $d_v a_1 = 0$  and  $d_h a_i + d_v a_{i+1} = 0$  for all  $1 \leq i \leq N-1$ .

**Lemma III.2.2.** *If  $(a_1, \dots, a_N) \in \mathcal{S}_N^{p,q}$ , then  $a_1$  survives to  $E_N^{p,q}$  and  $d_N([a_1]) = [d_h a_N]$ . Every class in  $E_N^{p,q}$  can be represented by an element of  $\mathcal{S}_N^{p,q}$ , and the representation is unique modulo elements in  $\mathcal{S}_N^{p,q}$  of the form*

- $(d_h \alpha, 0, \dots, 0)$  where  $\alpha \in A^{p-1, q}$ ,
  - $(0, \dots, 0, d_v \beta, d_h \beta, 0, \dots, 0)$  where  $\beta \in A^{p+i-1, q-i}$  and  $d_v \beta$  is in position  $i$  for  $1 \leq i \leq n-1$ ,
- and
- $(0, \dots, 0, \delta)$  where  $\delta \in A^{p+N-1, q-N+1}$ .

By applying Lemma III.2.2 to the double complex  $A^{p,q} = \text{Hom}_B(Q^p, \text{Hom}_A(B, I_q))$ , we can represent equivalence classes of elements in  $E_N^{p,q}$  by diagrams

$$\begin{array}{ccccccc}
Q^{p+N-1} & \xrightarrow{\partial_Q^{p+N-1}} & \cdots & \longrightarrow & Q^{p+2} & \xrightarrow{\partial_Q^{p+2}} & Q^{p+1} & \xrightarrow{\partial_Q^{p+1}} & Q^p \\
\downarrow a_N & & & & \downarrow & & \downarrow a_2 & & \downarrow a_1 \\
J_{q-N+1} & \xrightarrow{\partial_{q-N+2}^J} & \cdots & \longrightarrow & J_{q-2} & \xrightarrow{\partial_{q-1}^J} & J_{q-1} & \xrightarrow{\partial_q^J} & J_q
\end{array}$$

where the squares anticommute and  $\partial_{q+1}^J a_1 = 0$ . If  $(a_1, a_2, \dots, a_N)$  and  $(b_1, b_2, \dots, b_N)$  represent the same class in  $E_N^{p,q}$ , we will write  $(a_1, a_2, \dots, a_N) \sim (b_1, b_2, \dots, b_N)$ .

As in Section 6 of [7], we can describe the  $E(A) - E(B)$  bimodule structure on the spectral sequence at the staircase level by composing diagrams. A class  $[\zeta] \in E^k(B)$  is represented by a  $B$ -linear homomorphism  $\zeta : Q^k \rightarrow k$ . Lifting this map through the complex  $Q^\bullet$  by projectivity, we obtain a commutative diagram of the form

$$\begin{array}{ccccccc}
Q^{k+p+N-1} & \longrightarrow & \cdots & \longrightarrow & Q^{k+p} & \cdots & \longrightarrow & Q^k \\
\downarrow \zeta_{p+N-1} & & & & \downarrow \zeta_p & & & \downarrow \zeta_0 \searrow \zeta \\
Q^{p+N-1} & \longrightarrow & \cdots & \longrightarrow & Q^p & \cdots & \longrightarrow & Q^0 \longrightarrow k
\end{array}$$

Pre-composing with a representative of a class in  $E_N^{p,q}$ , we obtain

$$\begin{array}{ccccccc}
Q^{k+p+N-1} & \longrightarrow & \cdots & \longrightarrow & Q^{k+p+1} & \longrightarrow & Q^{k+p} \\
\downarrow \zeta_{p+N-1} & & & & \downarrow \zeta_{p+1} & & \downarrow \zeta_p \\
Q^{p+N-1} & \longrightarrow & \cdots & \longrightarrow & Q^{p+1} & \longrightarrow & Q^p \\
\downarrow a_N & & & & \downarrow a_2 & & \downarrow a_1 \\
J_{q-N+1} & \longrightarrow & \cdots & \longrightarrow & J_{q-1} & \longrightarrow & J_q
\end{array}$$

If we define  $[(a_1, \dots, a_N)] \star [\zeta] = [(a_1 \zeta_p, \dots, a_N \zeta_{p+N-1})]$  for all  $p, q$ , and  $k \geq 0$ , then  $\star$  gives  $E_N^{*,q}$  a well-defined right  $E(B)$ -module structure. For fixed  $p, q, k$ , we denote the set of all such products by  $E_N^{p,q} \star E^k(B)$ .

Similarly, a class in  $E^m(A)$  is represented by an  $A$ -linear homomorphism  $\gamma : k \rightarrow I_m$  which can be lifted through the complex  $I_\bullet$  by injectivity to obtain a commutative diagram

$$\begin{array}{ccccccc}
k & \longrightarrow & I^0 & \longrightarrow & \cdots & \longrightarrow & I_{q-N+1} & \cdots & \longrightarrow & I_q \\
& \searrow \gamma & \downarrow \gamma_0 & & & & \downarrow \gamma_{q-N+1} & & & \downarrow \gamma_q \\
& & I_m & \longrightarrow & \cdots & \longrightarrow & I_{m+q-N+1} & \cdots & \longrightarrow & I_{m+q}
\end{array}$$

Applying the functor  $\text{Hom}_A(B, -)$  to the last  $N$  terms in this diagram, we obtain the commutative diagram

$$\begin{array}{ccccccc}
J_{q+N-1} & \longrightarrow & \cdots & \longrightarrow & J_{q-1} & \longrightarrow & J_q \\
& & \downarrow \tilde{\gamma}_0 & & \downarrow \tilde{\gamma}_{q-1} & & \downarrow \tilde{\gamma}_q \\
J_{m+q-N+1} & \longrightarrow & \cdots & \longrightarrow & J_{m+q-1} & \longrightarrow & J_{m+q}
\end{array}$$

Post-composing with a representative of  $E_N^{p,q}$ , we get

$$\begin{array}{ccccccc}
Q^{p+N-1} & \longrightarrow & \cdots & \longrightarrow & Q^{p+1} & \longrightarrow & Q^p \\
& & \downarrow a_N & & \downarrow a_2 & & \downarrow a_1 \\
J_{q-N+1} & \longrightarrow & \cdots & \longrightarrow & J_{q-1} & \longrightarrow & J_q \\
& & \downarrow \tilde{\gamma}_0 & & \downarrow \tilde{\gamma}_{q-1} & & \downarrow \tilde{\gamma}_q \\
J_{m+q-N+1} & \longrightarrow & \cdots & \longrightarrow & J_{m+q-1} & \longrightarrow & J_{m+q}
\end{array}$$

If we define  $[\gamma] \star [(a_1, \dots, a_N)] = [(\tilde{\gamma}_q a_1, \dots, \tilde{\gamma}_0 a_N)]$  for all  $p, q$  and  $m \geq 0$ , then  $\star$  gives  $E_N^{p,*}$  a well-defined left  $E(A)$ -module structure. For fixed  $p, q, m$ , we denote the set of all such products by  $E^m(A) \star E_N^{p,q}$ . As  $E(A)$  always acts on the left of  $E_N$  and  $E(B)$  always acts on the right, the meaning of the symbol  $\star$  should always be clear from context.

**Lemma III.2.3.** *If the natural left  $B$ -module structure on  $\text{Ext}_A(B, k)$  is trivial, there results an isomorphism of bigraded bimodules*

$$\bigoplus_{p,q} E_2^{p,q} \cong \bigoplus_{p,q} \text{Ext}_A^q(B, k) \otimes E^p(B)$$

*Proof.* Because we work with locally finite-dimensional modules, for every  $p, q \geq 0$  we have an isomorphism

$$\text{Ext}_A^q(B, k) \otimes \text{Hom}_k(Q^p, k) \rightarrow \text{Hom}_k(Q^p, \text{Ext}_A^q(B, k))$$

given by  $[\zeta] \otimes f \mapsto f_{[\zeta]}$  where  $f_{[\zeta]}(q) = f(q)[\zeta]$ . Since  $\text{Ext}_A(B, k)$  is a trivial  $B$ -module, this isomorphism restricts to an isomorphism

$$\Lambda^{pq} : \text{Ext}_A^q(B, k) \otimes \text{Hom}_B(Q^p, k) \rightarrow \text{Hom}_B(Q^p, \text{Ext}_A^q(B, k))$$

Indeed, if  $b \in B_+$ ,

$$f_{[\zeta]}(bq) = f(bq)[\zeta] = (bf(q))[\zeta] = 0[\zeta] = 0 = b(f_{[\zeta]}(q))$$

so the restricted map is well-defined. Let  $\widetilde{\partial}_Q$  denote the differential induced on the graded vector space  $\text{Hom}_B(Q^\bullet, \text{Ext}_A^q(B, k))$  by  $\partial_Q$ . By the minimality of  $Q^\bullet$ ,  $\text{im } \partial_Q^{p+1} \subset B_+ Q^p$  so  $\widetilde{\partial}_Q(f_{[\zeta]}) = f_{[\zeta]} \partial_Q = 0$ . Thus we have  $\text{Ext}_B(Q^p, \text{Ext}_A^q(B, k)) = \text{Hom}_B(Q^p, \text{Ext}_A^q(B, k))$  and  $\Lambda = \bigoplus \Lambda^{pq}$  yields the desired isomorphism. □

**Remark III.2.4.** It is important to note that the induced  $E(A) - E(B)$  bimodule structure on  $\text{Ext}_A(B, k) \otimes E(B)$  that results from the isomorphism of Lemma III.2.3 is precisely the structure determined by the usual Yoneda product on each tensor component. Our use of the notation  $\star$  for the spectral sequence module structure is therefore consistent with its prior use to indicate the Yoneda products on  $\text{Ext}_A(B, k)$  and  $E(B)$ .

We also note that the hypothesis that  $\text{Ext}_A(B, k)$  is a trivial left  $B$ -module is satisfied in many important cases, including when  $A$  is commutative or graded-commutative.

We prove two lemmas relating to representatives in  $\mathcal{S}_N^{p,q}$ .

**Lemma III.2.5.** *Let  $[\gamma] \in E^1(A)$  and let  $\tilde{\gamma}_0 : J_0 \rightarrow J_1$  be induced by  $\gamma$ . Let  $\phi : Q^p \rightarrow J_0$  be a homomorphism of  $B$ -modules. If the composition  $Q^p \xrightarrow{\phi} J_0 \xrightarrow{\partial_1^J} J_1$  is zero, then  $\phi$  factors through  ${}_B k$  and  $\text{im } \tilde{\gamma}_0 \phi \subset \text{soc}(J_1)$ .*

*Proof.* If  $\phi \partial_1^J = 0$ , then  $\text{im } \phi \subset \ker \partial_1^J = \text{im } \tilde{\epsilon}^\vee$  where  $\tilde{\epsilon}^\vee : {}_B k \rightarrow J_0$  is induced by  $\epsilon^\vee$ . Thus  $\phi$  factors through  ${}_B k$  and  $\text{im } \phi \subset \text{soc}(J_0)$ . Since  $\text{im } \gamma_0 \epsilon^\vee = \text{im } \gamma \subset \text{soc}(I_1)$ , we have  $\text{im } \tilde{\gamma}_0 \phi \subset \text{soc}(J_1)$ .  $\square$

Recall that we fixed an  $A$ -basis for  $P^1$  such that  $P^1 \rightarrow P^0$  is given by left multiplication by  $(x_1 \ x_2 \ \cdots \ x_n)$ . Denote the elements of this  $A$ -basis by  $\varepsilon_1, \dots, \varepsilon_n$ . Denote the  $k$ -linear graded duals of these basis elements by  $\varepsilon_1^*, \dots, \varepsilon_n^*$ . Then  $\text{soc}(I_1)$  is the trivial  $A$ -module generated by  $\{\varepsilon_1^*, \dots, \varepsilon_n^*\}$ .

**Lemma III.2.6.** *Let  $[\gamma] \in E^1(A)$  and let  $(f, g) \in \mathcal{S}_2^{p,1}$ . Let  $\tilde{\gamma}_0 : J_0 \rightarrow J_1$  and  $\tilde{\gamma}_1 : J_1 \rightarrow J_2$  be induced by  $\gamma$ . Then there exists  $h \in A^{p+2,0}$  such that  $(\tilde{\gamma}_1 f, \tilde{\gamma}_0 g, h) \in \mathcal{S}_3^{p,2}$ . Furthermore,  $h$  can be chosen such that, for any vector  $e$  in a  $B$ -basis for  $Q^{p+2}$ ,  $h(e)(1) \in I_0^{A_{\geq 2}}$ .*

*Proof.* Let  $\phi = d_h g = g \partial_Q^{p+2}$ . We have  $\partial_1^J \phi = d_v d_h g = -d_h d_v g = d_h^2 f = 0$ . It suffices to define  $h$  on an arbitrary element  $e$  of a  $B$ -basis for  $Q^{p+2}$ . By Lemma III.2.5,  $\tilde{\gamma}_0 \phi(e) \in \text{soc}(J_1)$  so  $\tilde{\gamma}_0 \phi(e)(1) \in \text{soc}(I_1) \subset \text{im } \partial_1^J$ . Let  $u \in I_0$  such that  $\partial_1^J(u) = \tilde{\gamma}_0 \phi(e)(1)$ . Since  $\partial_1^J(u) = u \partial_P^1$  is annihilated by  $A_+$  and since  $\partial_P^1$  is multiplication by  $(x_1, \dots, x_n)$ , we can choose  $u \in I_0^{A_{\geq 2}}$ . We define an  $A$ -module homomorphism  $h(e) : {}_A B \rightarrow I_0$  by  $h(e)(1) = u$ . This is well-defined since  $\mathfrak{J} \subset A_{\geq 2}$ . Finally, we have

$$(d_v h)(e) = \partial_1^J(h(e)) = \partial_1^J h(e) = \tilde{\gamma}_0 \phi(e) = \tilde{\gamma}_0 d_h g(e) = d_h(\tilde{\gamma}_0 g)(e)$$

so  $(\tilde{\gamma}_1 f, \tilde{\gamma}_0 g, h) \in \mathcal{S}_3^{p,2}$ .  $\square$

**Lemma III.2.7.** *Let  $A$  be a graded algebra. Let  $\mathfrak{J}$  be an ideal of  $A$  and let  $B = A/\mathfrak{J}$ . Assume  $B$  acts trivially on  $\text{Ext}_A(B, k)$ . If  $\text{Ext}_A(\mathfrak{J}, k)$  is generated as a left  $E(A)$ -module by  $\text{Ext}_A^0(\mathfrak{J}, k)$ , then for any  $p$  and for  $q \geq 2$ , the spectral sequence differential  $d_2^{p,q}$  is zero.*

*Proof.* If  $\text{Ext}_A(\mathfrak{J}, k)$  is  $E(A)$ -generated by  $\text{Ext}_A^0(\mathfrak{J}, k)$ , then  $\bigoplus_{q>0} \text{Ext}_A^q(B, k)$  is generated as a left  $E(A)$ -module by  $\text{Ext}_A^1(B, k)$ . By Lemma III.2.3, we have  $E_2^{p,q} \cong \text{Ext}_A^q(B, k) \otimes E^p(B)$ . By Remark



III.2.4, we have  $E_2^{p,q} = E^{q-1}(A) \star E_2^{0,1} \star E^p(B)$  for  $q \geq 1$ . Thus any element of  $E_2^{p,q}$  can be represented as a sum of diagrams of the form

$$\begin{array}{ccc}
Q^{p+1} & \xrightarrow{\partial_Q^{p+1}} & Q^p \\
\downarrow & & \downarrow \\
Q^1 & \xrightarrow{\partial_Q^1} & Q^0 \\
\downarrow a_2 & & \downarrow a_1 \\
J_0 & \xrightarrow{\partial_1^J} & J_1 \\
\downarrow & & \downarrow \\
J_{q-1} & \xrightarrow{\partial_q^J} & J_q
\end{array}$$

which represent Yoneda products in  $E^{q-1}(A) \star E_2^{0,1} \star E^p(B)$ .

Since the spectral sequence differential respects the bimodule structure, it suffices to show  $E^{q-1}(A) \star \text{im}(d_2^{0,1}) = 0$ . Given a pair  $(a_1, a_2) \in \mathcal{S}_2^{0,1}$  representing a class in  $E_2^{0,1}$ , we have  $d_2^{0,1}[a_1] = [d_h a_2]$ . Let  $\phi = a_2 \partial_Q^2$  and  $[\gamma] \in E^{q-1}(A)$ . Since  $d_v d_h(\tilde{\gamma}_0 a_2) = -d_h(\partial_q^J \tilde{\gamma}_0 a_2) = -d_h(\tilde{\gamma}_1 d_v a_2) = d_h(\tilde{\gamma}_1 d_h a_1) = \tilde{\gamma}_1 a_1 \partial_Q^1 \partial_Q^2 = 0$ , we have  $(\tilde{\gamma}_0 \phi, 0) \in \mathcal{S}_2^{2,q-1}$ . But  $\tilde{\gamma}_0 \phi = d_h(\tilde{\gamma}_0 a_2)$  and  $\tilde{\gamma}_0 a_2 \in A^{1,q-1}$ . By Lemma III.2.2,  $(\tilde{\gamma}_0 \phi, 0)$  represents 0 in  $E_2^{2,q-1}$ . Hence  $[\gamma] \star [\phi] = 0$  for all  $[\gamma] \in E^{q-1}(A)$  and all  $[\phi] \in \text{im } d_2^{0,1}$ , so  $d_2^{p,q} = 0$ .

□

Though the differential  $d_2^{p,q}$  vanishes for  $q \geq 2$ , the differential  $d_2^{0,1}$  is likely nonzero. Thus the decomposition  $E_2^{p,q} = E^{q-1}(A) \star E_2^{0,1} \star E^p(B)$  does not immediately translate to the  $E_3$  page. However, if  ${}_A \mathfrak{J}$  is a  $\mathcal{K}_1$   $A$ -module, we obtain a similar result. Recall that  $D_1(A)$  is the subalgebra of  $E(A)$  generated by  $E_1(A)$ .

**Lemma III.2.8.** *Let  $A$  be a graded  $k$ -algebra and let  $\mathfrak{J}$  be a graded ideal such that  ${}_A \mathfrak{J}$  is a  $\mathcal{K}_1$   $A$ -module and  $B = A/\mathfrak{J}$  acts trivially on  $\text{Ext}_A(B, k)$ . Then for any  $p$  and for  $q \geq 2$ , we have  $E_2^{p,q} = E_3^{p,q}$  and  $E_3^{p,q} = D_1^{q-2}(A) \star E_3^{0,2} \star E^p(B)$ .*

*Proof.* By Lemma III.2.7,  $d_2^{p,q}$  and  $d_2^{p-2,q+1}$  are both zero for  $q \geq 2$ , so  $E_2^{p,q} = E_3^{p,q}$  for any  $p$  and for  $q \geq 2$ . Since  ${}_A\mathcal{J}$  is a  $\mathcal{K}_1$  module,  $\text{Ext}_A^q(\mathcal{J}, k) = E^1(A) \star \text{Ext}_A^{q-1}(\mathcal{J}, k)$  for all  $q > 0$ . Thus by induction,

$$E_3^{p,q} = E_2^{p,q} = D_1^{q-2}(A) \star E_2^{0,2} \star E^p(B) = D_1^{q-2}(A) \star E_3^{0,2} \star E^p(B).$$

□

The following technical lemma and its analog for the  $E_N$  page (Lemma III.2.10) show that if  ${}_A\mathcal{J}$  is a  $\mathcal{K}_1$   $A$ -module, the staircase representations of Lemma III.2.2 have a particularly nice form.

**Lemma III.2.9.** *Let  $A$  and  $B$  be graded  $k$ -algebras as in Lemma III.2.8. Then every class in  $E_3^{p,q}$  can be represented as a sum of diagrams of the form*

$$\begin{array}{ccccc}
Q^{p+2} & \xrightarrow{\partial_Q^{p+2}} & Q^{p+1} & \xrightarrow{\partial_Q^{p+1}} & Q^p \\
\downarrow \zeta_{p+2} & & \downarrow \zeta_{p+1} & & \downarrow \zeta_p \\
Q^2 & \xrightarrow{\partial_Q^2} & Q^1 & \xrightarrow{\partial_Q^1} & Q^0 \\
\downarrow h & & \downarrow g & & \downarrow f \\
& & J_0 & \xrightarrow{\partial_1^J} & J_1 \\
& & \downarrow \tilde{\gamma}_0 & & \downarrow \tilde{\gamma}_1 \\
J_0 & \xrightarrow{\partial_1^J} & J_1 & \xrightarrow{\partial_2^J} & J_2 \\
\downarrow \tilde{\gamma}'_0 & & \downarrow \tilde{\gamma}'_1 & & \downarrow \tilde{\gamma}'_2 \\
J_{q-2} & \xrightarrow{\partial_{q-1}^J} & J_{q-1} & \xrightarrow{\partial_q^J} & J_q
\end{array}$$

where the maps  $\zeta_i$  are induced by some  $[\zeta] \in E^p(B)$  and the maps  $\tilde{\gamma}'_k$  are induced by some  $[\gamma'] \in E^{q-2}(A)$ . Furthermore, the map  $h$  may be chosen so that, for any basis vector  $e \in Q^2$ ,  $h_i(e)(1) \in I_0^{A \geq 2}$ .

*Proof.* By Lemma III.2.8, it suffices to prove the statement for  $E_3^{0,2}$ . If  $(a_1, a_2, a_3)$  represents a class in  $E_3^{0,2}$ , then  $(a_1, a_2)$  represents a class in  $E_2^{0,2}$ . Since  $E_2^{0,2} = E^1(A) \star E_2^{0,1}$ , the class  $[a_1] \in E_2^{0,2}$  can be represented as a sum of diagrams of the form

$$\begin{array}{ccc}
Q^1 & \xrightarrow{\partial_Q^1} & Q^0 \\
\downarrow g & & \downarrow f \\
J_0 & \xrightarrow{\partial_1^J} & J_1 \\
\downarrow \tilde{\gamma}_0 & & \downarrow \tilde{\gamma}_1 \\
J_1 & \xrightarrow{\partial_2^J} & J_2
\end{array}$$

where  $(f, g)$  represents a class in  $E_2^{0,1}$  and  $\gamma$  represents a class in  $E^1(A)$ . Let  $\alpha, \beta$ , and  $\delta$  be as in Lemma III.2.2 such that  $(a_1, a_2) = \sum_i ((\tilde{\gamma}_i)_1 f_i, (\tilde{\gamma}_i)_0 g_i) + (d_h \alpha, 0) + (d_v \beta, d_h \beta) + (0, \delta)$ .

By Lemma III.2.6, there exist maps  $h_i$  such that  $d_h((\tilde{\gamma}_i)_0 g_i) + d_v h_i = 0$ . Since  $d_h a_2 + d_v a_3 = 0$ , we have

$$\begin{aligned}
-d_v a_3 &= d_h a_2 = d_h \left( \sum_i ((\tilde{\gamma}_i)_0 g_i + d_h \beta + \delta) \right) \\
&= - \sum_i d_v h_i + d_h \delta
\end{aligned}$$

so  $d_v(a_3 - \sum_i h_i) + d_h \delta = 0$ . Since  $d_v \delta = 0$ , Lemma III.2.2 implies that  $(0, \delta, a_3 - \sum_i h_i)$  represents a class in  $E_3^{0,2}$ . Evidently, it represents 0, as do  $(d_h \alpha, 0, 0)$  and  $(d_v \beta, d_h \beta, 0)$ . Thus we have

$$\begin{aligned}
&\sum_i ((\tilde{\gamma}_i)_1 f_i, (\tilde{\gamma}_i)_0 g_i, h_i) + (d_h \alpha, 0, 0) + (d_v \beta, d_h \beta, 0) + \left( 0, \delta, a_3 - \sum_i h_i \right) \\
&= \left( a_1, a_2 - \delta, \sum_i h_i \right) + \left( 0, \delta, a_3 - \sum_i h_i \right) \\
&= (a_1, a_2, a_3)
\end{aligned}$$

and  $(a_1, a_2, a_3) \sim \sum_i ((\tilde{\gamma}_i)_1 f_i, (\tilde{\gamma}_i)_0 g_i, h_i)$  as desired. By Lemma III.2.6, we may assume the  $h_i$  have the property that  $h_i(e)(1) \in I_0^{A \geq 2}$  for each basis vector  $e \in Q^2$ .

□



such that, for any vector  $e$  in a basis for  $Q^{j-1}$ ,  $b_j(e)(1)$  is a linear form. The differential  $d_N^{p,q} = 0$  for all  $q \geq N$ . We have  $E_N^{p,q} = E_{N+1}^{p,q}$  and  $E_{N+1}^{p,q} = D_1^{q-N}(A) \star E_{N+1}^{0,N} \star E^p(B)$  for all  $p \geq 0$  and all  $q \geq N$ .

We recall the following standard facts. See [6] for details.

**Lemma III.2.11.** *Let  $F$  be a filtered cocomplex and let  $E_\bullet$  be the spectral sequence associated with the filtration converging to  $H^*(F)$ . If  $E_\infty^{u,n-u} = 0$  for  $u < p$ , there results an epimorphism  $H^n(F) \rightarrow E_\infty^{p,n-p}$ . If the filtration is convergent and  $E_\infty^{u,n-u} = 0$  for  $u > p$ , there results a monomorphism  $E_\infty^{p,n-p} \rightarrow H^n(F)$ .*

We will apply this result in the cases  $p = 0$  and  $p = n$  to obtain results about  $E_\infty^{0,n}$  and  $E_\infty^{n,0}$  in the case where  $A$  is a Koszul algebra.

**Proposition III.2.12.** *If  $A$  is a quadratic algebra, the differential  $d_2^{0,1} : E_2^{0,1} \rightarrow E_2^{2,0}$  is an isomorphism in internal degrees greater than 2. If, additionally,  $A$  and  $B$  are as in Lemma III.2.10, then for  $p \geq 1$  and in internal degrees greater than  $p + 2$ , the image of  $d_2^{p,1}$  consists of elements of  $E_2^{p+2,0}$  which can be represented as sums in  $E^2(B) \star E^p(B)$ .*

*Proof.* Setting  $p = 0$  and  $n = 1$  in Lemma III.2.11 implies that  $E^1(A) = E^{1,1}(A)$  maps onto  $E_\infty^{0,1}$ , so  $E_\infty^{0,1}$  is concentrated in internal degree 1. For  $N > 2$ , the differentials  $d_N^{0,1}$  are zero, so cocycles of  $d_2^{0,1}$  are permanent, unbounded cocycles and  $E_3^{0,1} = E_\infty^{0,1}$ . Since  $E_3^{0,1} \subset E_2^{0,1} \cong \text{Ext}_A^0(\mathfrak{J}, k)$  and we assume that  $\mathfrak{J}$  contains no elements of degree less than 2,  $\ker d_2^{0,1} = E_3^{0,1} = 0$ .

Setting  $p = n = 2$  in Lemma III.2.11 implies there is a monomorphism  $E_\infty^{2,0} \hookrightarrow E^2(A) = E^{2,2}(A)$ . For  $N > 2$ , the differential  $d_N^{2-N,N-1}$  is zero. Since we also have  $d_N^{2,0} = 0$  for all  $N > 1$ ,  $E_3^{2,0} = E_\infty^{2,0}$ . We conclude that  $E_3^{2,0} = E_2^{2,0} / \text{im } d_2^{0,1}$  is concentrated in internal degree 2. It follows that  $d_2^{0,1}$  is surjective, hence is an isomorphism, in internal degrees greater than 2.

For the second statement, we have  $E_2^{p,1} \cong E_2^{0,1} \star E^p(B)$  and  $E_2^{p+2,0} \cong E_2^{2,0} \star E^p(B)$ . Since  $d_2^{0,1}$  is surjective in internal degrees  $\geq 2$  and since the spectral sequence differential respects the left  $E(B)$ -module structure, the result follows. □

We now prove that the images of the spectral sequence differentials landing in  $E_r^{r,0}$  are contained in a submodule corresponding to a subalgebra of  $D_2(B)$ . The key is that, as a result of

Lemma III.2.6, we can transform the left action of  $E^1(A)$  on  $E_{r-1}^{0,r-2}$  into a left action of  $E^1(B)$  on  $E^{r-1}(B)$  using Proposition II.3.7.

**Proposition III.2.13.** *Let  $A$  be a graded  $k$ -algebra and let  $B = A/\mathfrak{J}$  be a graded factor algebra such that  ${}_A\mathfrak{J}$  is a  $\mathcal{K}_1$   $A$ -module. Assume  $B$  acts trivially on  $\text{Ext}_A(B, k)$ . If  $N > 2$  and  $2 < r \leq N$ , then  $\text{im } d_r^{N-r, r-1} \subset E_2^{1,0} \star E^{N-1}(B)$ .*

*Proof.* For each  $2 \leq r \leq N$ , let  $\{e_i^r\}$  be a homogeneous basis for  $Q^r$ . Let  $M^r$  be the matrix of  $\partial_Q^r$  with respect to these bases. Without loss of generality, we may assume the  $\{e_i^r\}$  are chosen such that the nonzero rows of  $L(M^r)$  are linearly independent.

Let  $(a_1, a_2, \dots, a_r) \in \mathcal{S}_r^{0, r-1}$  represent a class in  $E_r^{0, r-1}$  as in Lemma III.2.2. By Lemma III.2.10, we may assume that for any basis element  $e = e_i^r$  of  $Q^r$ ,  $a_r(e)(1) \in I_0^{A \geq 2}$ . Since  $a_r$  is  $B$ -linear, the minimality of the resolution  $Q^\bullet$  implies that  $d_h a_r(e) = 0$  unless the  $i$ -th row of  $M^r$  contains a linear element. By Proposition II.3.7 and our assumption on the nonzero rows of  $L(M^r)$ , the  $i$ -th row of  $M^r$  contains a linear element if and only if  $[(e_i^r)^*] \in E^r(B)$  is in the subalgebra generated by  $E^1(B) \star E^{r-1}(B)$ .

Therefore, the image of  $d_r^{0, r-1}$  consists of those classes in  $E_r^{r, 0}$  whose  $E_2^{r, 0}$  representatives, under the isomorphism of Lemma III.2.3 above, are in the subalgebra of  $E^r(B)$  generated by  $E^1(B) \star E^{r-1}(B)$ .

Since  $E_2^{1, 0} = E_\infty^{1, 0}$ , we have

$$E^1(B) \star E^{r-1}(B) \cong E_2^{1, 0} \star E^{r-1}(B) = E_r^{1, 0} \star E^{r-1}(B)$$

Thus classes in  $\text{im } d_r^{0, r-1}$  are equal to their  $E_2^{r, 0}$  representatives and the result holds for  $r = N$ . The spectral sequence differential respects the right  $E(B)$ -module structure on  $E_r^{*, r-1}$ , so by Lemma III.2.10,  $\text{im } d_r^{N-r, r-1} = \text{im } d_r^{0, r-1} \star E^{N-r}(B) \subset E_r^{1, 0} \star E^{N-1}(B)$  as desired. □

We are now able to prove our main theorem.

**Theorem III.2.14.** *Let  $A$  be a Koszul algebra and  $\mathfrak{J} \subset A$  an ideal. Assume  $B = A/\mathfrak{J}$  acts trivially on  $\text{Ext}_A(B, k)$ . If  ${}_A\mathfrak{J}$  is a  $\mathcal{K}_2$   $A$ -module, then  $B = A/\mathfrak{J}$  is a  $\mathcal{K}_2$  algebra.*

*Proof.* Since  $A$  is quadratic,  ${}_A\mathfrak{J}$  is a  $\mathcal{K}_1$   $A$ -module by Corollary II.3.8. Let  $N$  be minimal such that a class  $[\zeta] \in E^N(B)$  is not generated by  $E^1(B)$  and  $E^2(B)$ . Without loss of generality, we may assume  $[\zeta]$  is homogeneous in the bigrading on  $E(B)$ . Then  $N > 2$  and the internal degree of  $[\zeta]$  is at least  $N + 1$ . The corresponding class  $\alpha \in E_2^{N,0} \cong E^N(B)$  is a permanent cocycle.

By Proposition III.2.12,  $\alpha \notin \text{im } d_2^{N-2,1}$ , so  $\alpha$  survives to a nonzero class  $[\alpha] \in E_3^{N,0}$ . By Proposition III.2.13,  $[\alpha]$  survives to a nonzero class  $\alpha_\infty \in E_\infty^{N,0}$ . By Lemma III.2.11,  $E_\infty^{N,0} \hookrightarrow E^N(A)$ , so  $E^N(A)$  is not concentrated in internal degree  $N$ . This contradicts the Koszulity of  $A$ .  $\square$

We remark that if  $\mathfrak{J}$  is a Koszul  $A$ -module, then  $B$  acts trivially on  $\text{Ext}_A(B, k)$  by degree considerations. If we further assume that  $\mathfrak{J}$  is generated in degree 2, then  $B$  is a Koszul algebra. This special case of Theorem III.2.14 was proved by Backelin and Froberg in [1].

We also note that Example 9.3 of [7] shows Theorem III.2.14 is false if  $A$  is only assumed to be a  $\mathcal{K}_2$  algebra. The following example shows that the hypothesis that  $B$  acts trivially on  $\text{Ext}_A(B, k)$  cannot be removed from Theorem III.2.14.

**Example III.2.15.** Let  $A = k\langle x, y \rangle / \langle x^2 - xy \rangle$ , and let  $\mathfrak{J} = \langle yx \rangle$  be a two-sided ideal. The algebra  $A$  is isomorphic to the monomial quadratic algebra  $k\langle X, Y \rangle / \langle XY \rangle$ , hence  $A$  is a Koszul algebra (see Corollary 4.3 of [27]). As a left  $A$ -module,  ${}_A\mathfrak{J} = Ayx + Ayx^2$  and

$$0 \rightarrow A(-4) \xrightarrow{\begin{pmatrix} x^2 & -x \end{pmatrix}} A(-2) \oplus A(-3) \xrightarrow{\begin{pmatrix} yx & yx^2 \end{pmatrix}^T} \mathfrak{J} \rightarrow 0$$

is a graded free resolution of  ${}_A\mathfrak{J}$ . The matrix criterion of Proposition II.3.7 shows that  ${}_A\mathfrak{J}$  is a  $\mathcal{K}_1$   $A$ -module. The Hilbert series of  $B = A/\mathfrak{J}$  is easily seen to be  $1 + 2t + 2t^2 + t^3/(1-t)$ . Therefore, the Poincaré series of  ${}_B k$  begins  $1 - 2t + 2t^2 - t^3 - t^4 + \dots$ . The negative coefficient of  $t^4$  implies  $\dim \text{Ext}_B^{i,4}(k, k) \neq 0$  for some  $i \neq 4$ . Thus the quadratic algebra  $B$  is not a Koszul algebra, so  $B$  is not a  $\mathcal{K}_2$  algebra. One can check that the image of  $x$  in  $B$  acts nontrivially on  $\text{Ext}_A(B, k)$  by sending the class in  $\text{Ext}_A^0(B, k)$  corresponding to the generator  $yx$  to the class corresponding to the generator  $yx^2$ .

### III.3 Face Rings

For any  $n \in \mathbb{N}$ , there is a well-known (see Theorem 1.7 of [23]) correspondence between abstract simplicial complexes on the set  $[n] = \{1, \dots, n\}$  and ideals of the commutative polynomial algebra  $k[x_1, \dots, x_n]$  generated by squarefree monomials. The correspondence associates to a simplicial complex  $\Delta$  the Stanley-Reisner ideal  $I_\Delta$  generated by the monomials  $\prod_{i \in \tau} x_i$  for  $\tau \subset [n]$ ,  $\tau \notin \Delta$ . The factor algebra  $k[\Delta] = k[x_1, \dots, x_n]/I_\Delta$  is called the Stanley-Reisner ring, or *face ring*, of the simplicial complex  $\Delta$ . For simplicity, we assume that  $I_\Delta$  contains no linear monomials.

The combinatorics of  $\Delta$  play an important role in determining both the homological and ring-theoretic properties of  $k[\Delta]$ , and vice versa. The study of these connections is an active and deep area of research. However, relatively little seems to be known about the structure of the Yoneda Ext-algebra of  $k[\Delta]$ . A significant result is due to Fröberg.

**Theorem III.3.1** ([11]). *If  $I_\Delta$  can be generated by quadratic monomials, then  $k[\Delta]$  is a Koszul algebra.*

Since Koszul algebras must be quadratic, this theorem completely characterizes Koszul face rings. A major obstacle to proving analogous structural results for the Yoneda algebras of non-quadratic face rings is that these algebras must have defining relations of different homogeneous degrees. Until the recent introduction of  $\mathcal{K}_2$  algebras, most generalized Koszul properties required an algebra to have defining relations of a single homogeneous degree. Determining which face rings are  $\mathcal{K}_2$  algebras is therefore an important and natural problem. In the course of proving some sufficient conditions for a face ring to be a  $\mathcal{K}_2$  algebra, we will show that problem is considerably more subtle than the Koszul case. Our main tool is the following special case of Theorem III.2.14.

**Theorem III.3.2.** *If  $I$  is a squarefree monomial ideal in  $S = k[x_1, \dots, x_n]$  and if  $I$  is a  $\mathcal{K}_2$   $S$ -module, then  $S/I$  is a  $\mathcal{K}_2$  algebra.*

Let  $\Delta$  be an abstract simplicial complex on  $[n]$ . If  $\tau \in \Delta$ , the *link* of  $\tau$  in  $\Delta$  is  $\text{link}_\Delta \tau = \{\sigma \in \Delta \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\}$ . The *Alexander dual* complex of  $\Delta$  is the simplicial complex  $\Delta^* = \{[n] - \tau \mid \tau \notin \Delta\}$ . We denote the subcomplex of  $\Delta$  whose maximal faces are the  $q$ -faces of  $\Delta$  by  $\Delta(q)$ . If all maximal faces of  $\Delta$  have the same dimension, we say that  $\Delta$  is *pure*. We call  $\Delta$  *Cohen-Macaulay over  $k$*  if  $\Delta$  is pure and if for all faces  $\tau \in \Delta$  and all  $i < \dim \text{link}_\Delta \tau$  we have  $\tilde{H}_i(\text{link}_\Delta \tau, k) = 0$ . This definition is motivated by the fundamental result of Reisner (see [31])



that  $\Delta$  is a Cohen-Macaulay complex over  $k$  if and only if  $k[\Delta]$  is a Cohen-Macaulay ring. In [10], Eagon and Reiner characterized the Cohen-Macaulay property in terms of resolutions of  $I_\Delta$ .

**Theorem III.3.3** ([10]). *The simplicial complex  $\Delta^*$  is Cohen-Macaulay if and only if the square-free monomial ideal  $I_\Delta$  has a linear free resolution as a  $k[x_1, \dots, x_n]$ -module.*

In [33], Stanley introduced the more general notion of a sequentially Cohen-Macaulay complex. We call  $\Delta$  *sequentially Cohen-Macaulay over  $k$*  if  $\Delta(q)$  is Cohen-Macaulay for all  $q \in \mathbb{N}$ . In [9], Duval proved that this definition is equivalent to Stanley's. We note that if  $\Delta$  is pure, then  $\Delta$  is sequentially Cohen-Macaulay if and only if  $\Delta$  is Cohen-Macaulay. Herzog and Hibi proved the analog of Theorem III.3.3 for sequentially Cohen-Macaulay complexes.

**Theorem III.3.4** ([16]). *The simplicial complex  $\Delta^*$  is sequentially Cohen-Macaulay if and only if  $I_\Delta$  has a componentwise linear free resolution as a  $k[x_1, \dots, x_n]$ -module.*

In light of Proposition II.4.4 we obtain the following sufficient condition for  $k[\Delta]$  to be a  $\mathcal{K}_2$  algebra.

**Corollary III.3.5.** *If  $\Delta^*$  is sequentially Cohen-Macaulay, then  $k[\Delta]$  is a  $\mathcal{K}_2$  algebra.*

*Proof.* Since commutative factor algebras always act trivially on the appropriate Ext group, the statement follows immediately from Theorem III.3.4, Proposition II.4.4, and Theorem III.3.2. □

Example III.4.2 below shows the sufficient condition of Corollary III.3.5 is not necessary. For an example where  $k[\Delta]$  is a  $\mathcal{K}_2$  algebra and  $I_\Delta$  is not even a  $\mathcal{K}_2$  module, see Example III.4.3. However, we do not know of an example in which  $\Delta^*$  is pure and connected,  $I_\Delta$  is generated by squarefree monomials of degree  $> 2$ , and  $I_\Delta$  is not a  $\mathcal{K}_2$  module but  $k[\Delta]$  is a  $\mathcal{K}_2$  algebra. Indeed, we conjecture that there is no such counterexample.

Since Cohen-Macaulay and sequentially Cohen-Macaulay complexes can be characterized in terms of their combinatorial topology, we hope to establish geometric criteria on  $\Delta^*$  under which  $I_\Delta$  is a  $\mathcal{K}_2$  module.



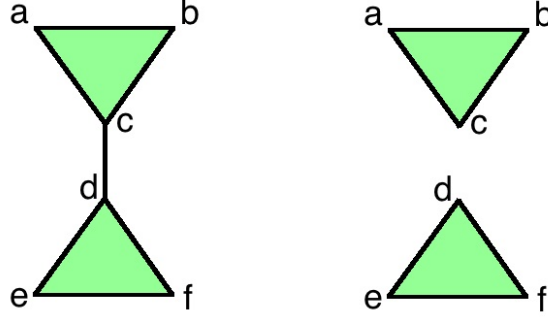


Figure 2: The complex  $\Delta^*$  for  $I$  in Example III.4.2 and  $\Delta^*(2)$ .

By Proposition II.3.7,  ${}_S I$  is a  $\mathcal{K}_2$   $S$ -module. Thus  $S/I$  is a  $\mathcal{K}_2$  algebra by Theorem III.3.2. The subideal  $J = \langle abc, abef \rangle$  is easily seen to be a  $\mathcal{K}_2$   $S$ -module. However, the factor module  $I/J$  is not. The complex

$$0 \rightarrow S(-5) \xrightarrow{\begin{pmatrix} ab \\ \end{pmatrix}} S(-3) \xrightarrow{\begin{pmatrix} def \\ \end{pmatrix}} I/J \rightarrow 0$$

is a minimal free  $S$ -module resolution of  $I/J$ , and it fails the matrix condition of Lemma II.3.6.

**Example III.4.3.** Let  $S = k[a, b, c, d, e]$  and let  $I = \langle abc, cde, abde \rangle$ . Let  $A = S/I$  and let  $B = A/\langle c \rangle$ . Observe that  $B \cong k[a, b, d, e]/\langle abde \rangle$ . By Corollary 9.2 of [7],  $B$  is a  $\mathcal{K}_2$  algebra. The following sequence is the beginning of a minimal projective  $A$ -module resolution of  ${}_A B$ . The resolution is clearly periodic.

$$\dots \xrightarrow{\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \gamma \\ 0 & 0 & \beta \end{pmatrix}} A(-4^2, -5^2) \xrightarrow{\begin{pmatrix} \gamma \\ \beta' \end{pmatrix}} A(-3)^2 \xrightarrow{\alpha} A(-1) \xrightarrow{\begin{pmatrix} c \\ \end{pmatrix}} A \rightarrow B$$

where

$$\alpha = \begin{pmatrix} ab \\ de \end{pmatrix} \quad \beta = \begin{pmatrix} ab & 0 \\ 0 & de \end{pmatrix} \quad \beta' = \begin{pmatrix} de & 0 \\ 0 & ab \end{pmatrix} \quad \gamma = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$$

By the matrix criterion of Lemma II.3.6,  $B$  is a  $\mathcal{K}_2$   $A$ -module. By Theorem 7.4 of [7],  $A$  is a  $\mathcal{K}_2$  algebra.

The following is a minimal projective  $S$ -module resolution of the defining ideal  $I$  of  $A$ .

$$0 \rightarrow S(-5^2) \xrightarrow{\begin{pmatrix} de & 0 & -c \\ 0 & ab & -c \end{pmatrix}} S(-3^2, -4) \xrightarrow{\begin{pmatrix} abc \\ cde \\ abde \end{pmatrix}} I \rightarrow 0$$

As the linearization of the matrix on the left has dependent rows,  $I$  is not a  $\mathcal{K}_2$   $S$ -module. Thus the converse of Theorem III.2.14 is false.

The algebra  $A$  is a noteworthy example in another regard. Let  $J$  be a componentwise linear ideal in a commutative Koszul algebra  $S$ . By Theorem III.2.14,  $S/J$  is a  $\mathcal{K}_2$  algebra. The subideal  $J_{(0,d)}$  is componentwise linear for all  $d \geq 0$ , so  $S/J_{(0,d)}$  is a  $\mathcal{K}_2$  algebra for all  $d \geq 0$ . Cassidy and Shelton conjectured that such a ‘‘tower’’ theorem holds more generally for  $\mathcal{K}_2$  algebras. The algebra  $A$  resolves that conjecture negatively, as we now show.

Let  $C = k[a, b, c, d, e]/\langle abc, cde \rangle$  be the algebra obtained by discarding the highest degree generator of  $I$ .

**Proposition III.4.4.** *The algebra  $C$  is not  $\mathcal{K}_2$ .*

*Proof.* By the usual identifications of  $E^1(C)$  and  $E^2(C)$  respectively with the  $k$ -linear duals of the vector spaces of generators and minimal defining relations of  $C$ , we have  $E^1(C) = \text{Ext}_C^{1,1}(k, k)$  and  $E^2(C) = \text{Ext}_C^{2,2}(k, k) \oplus \text{Ext}_C^{2,3}(k, k)$ . We prove the proposition by showing that  $\text{Ext}_C^{3,5}(k, k) \neq 0$ .

The Hilbert series of the defining ideal  $J = \langle abc, cde \rangle$  is easily seen to be  $2t^3 + 10t^4 + 29t^5 + \dots$  thus the Hilbert series of  $C$  is

$$\frac{1}{(1-t)^5} - (2t^3 + 10t^4 + 29t^5 + \dots) = 1 + 5t + 15t^2 + 33t^3 + 60t^4 + 97t^5 + \dots$$

Inverting the formal power series, we find the Poincare series of  $C$  is

$$1 - 5t + 10t^2 - 8t^3 - 5t^4 + 18t^5 + \dots$$

Since  $J_2 = 0$ , the diagonal subalgebra  $\bigoplus \text{Ext}_C^{i,i}(k, k)$  of  $E(C)$  is isomorphic to  $E(S)$ , which is the exterior algebra on a 5-dimensional vector space (see Proposition 3.1 of [27]). In particular,

$\dim \text{Ext}_C^{2,2}(k, k) = 10$  and  $\dim \text{Ext}_C^{5,5}(k, k) = 1$ . From the minimal  $S$ -module resolution of  $J$  given in Example III.4.1, we see that

$$\text{Ext}_S^1(C, k) \cong \text{Ext}_S^0(J, k) = k(-3) \oplus k(-3), \quad \text{Ext}_S^2(C, k) \cong \text{Ext}_S^1(J, k) = k(-5)$$

and  $\text{Ext}_S^q(C, k) = 0$  for  $q > 2$ .

Consider the first quadrant spectral sequence  $\text{Ext}_C^p(k, \text{Ext}_S^q(C, k)) \Rightarrow \text{Ext}_S^{p+q}(k, k)$ . By Lemma III.2.3,  $E_2^{p,q} \cong E^p(C) \otimes \text{Ext}_S^q(C, k)$ . Every element in  $\text{Ext}_C^{4,5}(k, k)$  represents a permanent cocycle in  $E_2^{4,0}$ . Since the target of the spectral sequence is the Yoneda algebra of a Koszul algebra, each of these cocycles must be eventually bounded. The differentials which could bound these cocycles are

$$E_2^{2,1} \rightarrow E_2^{4,0} \quad E_3^{1,2} \rightarrow E_3^{4,0} \quad \text{and} \quad E_4^{0,3} \rightarrow E_4^{4,0}$$

Since  $\text{Ext}_S^3(C, k) = 0$ ,  $E_4^{0,3} = 0$ . Since  $E_3^{1,2} \subset E_2^{1,2} \cong E^1(C) \otimes \text{Ext}_S^2(C, k)$ , the vector space  $E_3^{1,2}$  is concentrated in internal degree 6, so it cannot bound a cocycle with internal degree 5. Thus  $E_2^{4,0} \cong E_2^{2,1} \cong E^2(C) \otimes \text{Ext}_S^1(C, k)$  in internal degree 5. From the calculations above, we have  $\dim \text{Ext}_C^{4,5}(k, k) = 20$ . Since  $\dim \text{Ext}_C^{5,5}(k, k) = 1$ , it follows from the Poincare series that  $\dim \text{Ext}_C^{3,5}(k, k) = 1$ .

□

## CHAPTER IV

### $A_\infty$ -ALGEBRA STRUCTURES ASSOCIATED WITH $\mathcal{K}_2$ ALGEBRAS

#### IV.1 Introduction

In this chapter, we explore another important recent development in ring theory: the use of  $A_\infty$ -algebras (cf. [15], [20], [21]). The material in this chapter was first developed and authored jointly with Pete Goetz in the manuscript [8], accepted for publication. A preprint of the paper is available on the arXiv preprint server.

The notion of an  $A_\infty$ -algebra was first defined by Stasheff in [35].

**Definition IV.1.1.** An  $A_\infty$ -algebra over a field  $k$  is a  $\mathbb{Z}$ -graded vector space  $E = \bigoplus_{p \in \mathbb{Z}} E^p$  together with graded  $k$ -linear maps

$$m_n : E^{\otimes n} \rightarrow E, \quad n \geq 1$$

of degree  $2 - n$  which satisfy the Stasheff identities

$$\mathbf{SI}(n) \quad \sum (-1)^{r+st} m_u(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0.$$

where the sum runs over all decompositions  $n = r + s + t$  such that  $r, t \geq 0$ ,  $s \geq 1$ , and  $u = r + 1 + t$ .

The linear maps  $m_n$  for  $n \geq 3$  are called *higher multiplications*. The  $A_\infty$ -structure  $\{m_i\}$  on a graded  $k$ -vector space  $E$  determines a  $k$ -linear map  $\bigoplus m_i : T(E)_+ \rightarrow E$ . This map induces the structure of a differential coalgebra on  $T(E)$  (see [17]).

We note that the identity  $\mathbf{SI}(1)$  is  $m_1 m_1 = 0$  and the degree of  $m_1$  is 1 so  $m_1$  is a differential on  $E$ . The identity  $\mathbf{SI}(2)$  can be written as

$$m_1 m_2 = m_2(m_1 \otimes 1 + 1 \otimes m_1).$$

Thus the differential  $m_1$  is a graded derivation with respect to  $m_2$ . Hence a differential graded algebra is an  $A_\infty$ -algebra with  $m_n = 0$  for all  $n \geq 3$ . The map  $m_2$  plays the role of multiplication in an  $A_\infty$ -algebra, but in general  $m_2$  is not associative. However, it is clear from **SI(3)** that  $m_2$  is associative if  $m_1$  or  $m_3$  is zero.

An important result in the theory of  $A_\infty$ -algebras is the following theorem of Kadeishvili.

**Theorem IV.1.2** ([17]). *Let  $(E, \{m_i^E\})$  be an  $A_\infty$ -algebra. The cohomology  $H^*E$  with respect to  $m_1^E$  admits an  $A_\infty$ -algebra structure  $\{m_i\}$  such that*

1.  $m_1 = 0$  and  $m_2$  is induced by  $m_2^E$ , and
2. there is a quasi-isomorphism of  $A_\infty$ -algebras  $H^*E \rightarrow E$  lifting the identity of  $H^*E$ .

Moreover, this structure is unique up to (non unique) isomorphism of  $A_\infty$ -algebras.

We refer the interested reader to [18] for definitions of *morphism*, *quasi-isomorphism*, and *isomorphism* of  $A_\infty$ -algebras. We will not use these notions in this paper. We call an  $A_\infty$ -algebra structure on  $H^*E$  *canonical* if it belongs to the  $A_\infty$ -isomorphism class of structures provided by the theorem.

Let  $A$  be a  $k$ -algebra. We recall that the Yoneda algebra of  $A$ ,  $E(A)$ , is the cohomology algebra of a differential graded algebra (see Section IV.4). Therefore Kadeishvili's theorem implies that  $E(A)$  admits a canonical  $A_\infty$ -algebra structure. We remark that  $E(A)$  is bigraded. The grading on the maps  $m_n$  refers to the cohomological degree. By the construction described in Section IV.4, we may assume the internal degree of  $m_n$  is 0 for all  $n$ .

The following theorem was the main motivation for this chapter.

**Theorem IV.1.3.** *Let  $A$  be a  $k$ -algebra which is finitely generated in degree 1. Let  $E(A)$  be the Yoneda algebra of  $A$ . Then  $A$  is a Koszul algebra if and only if every canonical  $A_\infty$ -algebra structure on  $E(A)$  has  $m_n = 0$  for all  $n \neq 2$ .*

One direction of this theorem is clear. Suppose that  $A$  is a Koszul algebra. Then the Yoneda algebra is concentrated on the diagonal  $\oplus E^{p,p}(A)$ . It follows, purely for degree reasons, that all multiplications other than  $m_2$  are zero.

The converse is implied by work of May and Gugenheim [14] and Stasheff [34]. It can also be proved using the fact that  $E(A)$  is  $A_\infty$ -generated by  $E^1(A)$ . This fact appears in Keller's paper

[19] without proof. We give a constructive proof that  $E(A)$  is  $A_\infty$ -generated by  $E^1(A)$  in Chapter V.

The goal of this chapter is to provide some partial answers to the following questions.

- What restrictions does the  $\mathcal{K}_2$  condition place on a canonical  $A_\infty$ -structure on the Yoneda algebra?
- Do certain  $A_\infty$ -structures on the Yoneda algebra guarantee the original algebra is  $\mathcal{K}_2$ ?

The analogues of these questions for  $N$ -Koszul algebras with  $N \geq 3$  were considered by He and Lu in [15], where they obtain a result similar to Theorem IV.1.3. They prove an algebra  $A$  is  $N$ -Koszul if and only if  $E(A)$  is a reduced  $(2, N)$   $A_\infty$ -algebra and is  $A_\infty$ -generated by  $E^1$ . In a  $(2, N)$   $A_\infty$ -algebra, all multiplications other than  $m_2$  and  $m_N$  are zero. Their result is aided by the fact that for an  $N$ -Koszul algebra,  $E^p(A)$  is concentrated in a single internal degree. We remark yet again that the Yoneda algebra of a  $\mathcal{K}_2$  algebra does not generally satisfy such a strong purity condition.

Our main results are the following.

**Theorem IV.1.4.** *For each  $n \in \mathbb{N}$  there exists a  $\mathcal{K}_2$  algebra  $B$  such that*

1. *The defining relations of  $B$  are quadratic and cubic, and*
2.  *$E(B)$  has a canonical  $A_\infty$ -algebra structure such that  $m_i$  is nonzero for all  $3 \leq i \leq n + 3$ .*

This shows, in contrast to the cases of Koszul and  $N$ -Koszul algebras, that vanishing of higher multiplications on the Yoneda algebra of a  $\mathcal{K}_2$  algebra need not be determined in any obvious way by the degrees of defining relations. Recently, Green and Marcos [12] defined the notion of  $2$ - $d$ -Koszul algebras. We remark that the algebra  $B$  of Theorem IV.1.4 is a  $2$ - $d$ -Koszul algebra.

**Theorem IV.1.5.** *There exist  $k$ -algebras  $A^1$  and  $A^2$  such that*

1.  *$A^1$  is  $\mathcal{K}_2$  and  $A^2$  is not  $\mathcal{K}_2$ ,*
2.  *$A^1$  and  $A^2$  have the same Poincaré series, and*
3.  *$E(A^1)$  and  $E(A^2)$  admit canonical  $A_\infty$ -structures which are not distinguished by  $m_n$  for  $n \geq 3$ .*



These examples demonstrate that obvious vanishing patterns among higher multiplications cannot detect the  $\mathcal{K}_2$  property.

We are now ready to outline the chapter. In Section IV.2 we introduce the  $k$ -algebra  $B$  and prove some basic facts about a canonical graded  $k$ -basis for  $B$ . Section IV.3 is the technical heart of the chapter. We describe a minimal projective resolution of the trivial  $B$ -module and prove that  $B$  is a  $\mathcal{K}_2$  algebra. In Section IV.4 we prove some general results on using a minimal resolution of the trivial module to compute  $A_\infty$ -algebra structures on Yoneda algebras. In Section IV.5 we compute part of an  $A_\infty$ -algebra structure on  $E(B)$  and finish the proof of Theorem IV.1.4. Finally in Section IV.6 we prove Theorem IV.1.5.

## IV.2 The Algebra $B$

We begin this section by defining an algebra  $B$ . We exhibit a canonical  $k$ -basis for  $B$  and use it to compute left annihilator ideals of certain elements of  $B$ .

Let  $k$  be a field and fix  $n \in \mathbb{N}$ . Let  $k^*$  denote the set of nonzero elements of  $k$ . Let  $V$  be a  $k$ -vector space with basis  $X = \{a_i, b_i, c_i\}_{0 \leq i \leq n}$ . It will be convenient to define elements  $b_s$  of  $V$  to be 0 for  $s > n$ . Likewise we define  $c_t$  to be 0 in  $V$  for  $t < 0$ . Let  $T(V)$  be the tensor algebra on  $V$ , graded by tensor degree. Let  $R \subset V^{\otimes 2} \oplus V^{\otimes 3}$  be the set of tensors

$$\{a_n b_n c_n, c_0 a_0\} \cup \{a_i b_i c_i + c_{i+1} a_{i+1} b_{i+1}, b_{i+1} c_{i+1} a_{i+1}, c_i c_{i+1}, b_{i+1} a_i\}_{0 \leq i < n}.$$

We define  $B$  to be the  $k$ -algebra  $T(V)/I$  where  $I$  is the ideal of  $T(V)$  generated by  $R$ . The ideal  $I$  is homogeneous, so  $B$  inherits a grading from  $T(V)$ . We have the canonical quotient map  $\pi_B : T(V) \rightarrow B$  of graded  $k$ -algebras.

The basis  $X$  generates a free semigroup  $\langle X \rangle$  with 1 under the multiplication in  $T(V)$ . We call elements of  $\langle X \rangle$  *pure tensors*. Let  $\langle X \rangle_d$  denote the set of pure tensors of degree  $d$ . We call an element  $x \in B_+$  a *monomial* if  $\pi_B^{-1}(x)$  contains a pure tensor.

We need a monomial  $k$ -basis for  $B$ . Let  $R' \subset \langle X \rangle$  be the set

$$\{a_n b_n c_n, c_0 a_0\} \cup \{c_{i+1} a_{i+1} b_{i+1}, b_{i+1} c_{i+1} a_{i+1}, c_i c_{i+1}, b_{i+1} a_i\}_{0 \leq i < n}$$

and let

$$\langle R' \rangle = \{AWB \mid A, B \in \langle X \rangle, W \in R'\}.$$

The following proposition is a straightforward application of Bergman's Diamond Lemma [4]. We order pure tensors by degree-lexicographic order with the basis elements ordered  $a_i < b_i < c_i < a_{i+1} < b_{i+1} < c_{i+1}$  for  $0 \leq i \leq n-1$ .

**Proposition IV.2.1.** *The image under  $\pi_B$  of  $\langle X \rangle - \langle R' \rangle$ , the set of tensors which do not contain any element of  $R'$ , gives a monomial  $k$ -basis for  $B$ .*

**Remark IV.2.2.** In Section 5 of [7], Cassidy and Shelton give a simple algorithm for determining if a monomial algebra is  $\mathcal{K}_2$ . The algorithm can be used to show the algebra  $T(V)/\langle R' \rangle$  is  $\mathcal{K}_2$ . Proposition IV.2.1 implies that  $R$  is an *essential Gröbner basis* for  $I$  (see [24]). By Theorem 3.13 in [24], the fact that  $T(V)/\langle R' \rangle$  is  $\mathcal{K}_2$  implies  $B$  is  $\mathcal{K}_2$ . Though this method of proving  $B$  is  $\mathcal{K}_2$  is quite easy, we do not present the details here. A minimal resolution of the trivial module is central to our calculation of  $A_\infty$ -structure, so we prove that  $B$  is  $\mathcal{K}_2$  using the matrix criterion of Theorem IV.3.9.

The basis provided by Proposition IV.2.1 determines a vector space splitting  $\rho_B : B \rightarrow T(V)$  such that  $T(V) = \rho_B(B) \oplus I$  by mapping a basis element to its pre-image in  $\langle X \rangle - \langle R' \rangle$ . If  $x \in B$ , we write  $\hat{x}$  for the image of  $x$  under  $\rho_B$ . If  $w \in T(V)$ , there exist unique  $w^c \in \rho_B(B)$  and  $w^r \in I$  such that  $w = w^c + w^r$ . We call  $w^c$  the *canonical form* of  $w$ , and if  $w = w^c$  we say  $w$  is in *canonical form*. We say  $w$  is *reducible* if it is not in canonical form.

As a consequence of Proposition IV.2.1, we show that left annihilators of monomials are monomial left ideals.

**Proposition IV.2.3.** *Let  $x \in B$  be a nonzero monomial of degree  $d'$  and let  $w \in l.\text{ann}_B(x)$  be homogeneous of degree  $d$ . If  $\hat{w} = \sum_{t=1}^k \alpha_t w_t$  where the  $w_t \in \langle X \rangle_d$  are distinct and  $\alpha_t \in k^*$ , then  $\pi_B(w_t \hat{x}) = 0$  for  $1 \leq t \leq k$ .*

*Proof.* Let  $\hat{x} = x_1 \cdots x_{d'} \in \langle X \rangle_{d'}$  and  $w_t = w_{t,1} \cdots w_{t,d} \in \langle X \rangle_d$ . It suffices to consider the case where  $\pi_B(w_t \hat{x})$  is nonzero for every  $t$ . Since  $\hat{w} \hat{x} \in I$ ,  $w_t \hat{x}$  is reducible for some  $t$ . Reordering if necessary, there is an index  $j$  such that  $w_t \hat{x}$  is reducible for  $1 \leq t \leq j$  and in canonical form for  $j < t \leq k$ .

Fix an index  $t$  such that  $1 \leq t \leq j$ . Since  $\pi_B(w_t \hat{x}) \neq 0$ , Proposition IV.2.1 implies that  $w_t \hat{x}$  must contain  $c_i a_i b_i$  for some  $1 \leq i \leq n$ . Since  $\hat{w}$  and  $\hat{x}$  are in canonical form, either  $w_{t,d-1} w_{t,d} x_1 = c_i a_i b_i$  or  $w_{t,d} x_1 x_2 = c_i a_i b_i$ . In the first case, define  $y_t = \alpha_t w_{t,1} \cdots w_{t,d-2} (a_{i-1} b_{i-1} c_{i-1}) x_2 \cdots x_d$  and in the second case,  $y_t = \alpha_t w_{t,1} \cdots w_{t,d-1} (a_{i-1} b_{i-1} c_{i-1}) x_3 \cdots x_d$ .

Clearly,  $\alpha_t w_t \hat{x} + y_t \in I$  so  $z = \sum_{t=1}^j -y_t + \sum_{t=j+1}^k \alpha_t w_t \hat{x} \in I$ . We claim that  $z$  is in canonical form, and for this it suffices to check that  $y_t$  is in canonical form. In the case  $x_1 = b_i$ , Proposition IV.2.1 implies  $y_t$  is reducible only if  $w_{t,d-2} = c_{i-1}$  or  $w_{t,d-2} = b_i$  or  $x_2 = a_{i-1}$  or  $x_2 = c_i$ . If  $x_2 = c_i$ , then  $\pi_B(y_t) = 0$  which implies  $\pi_B(w_t \hat{x}) = \pi_B(w_t \hat{x} + y_t) = 0$ , a contradiction. The other three cases all lead to the contradiction that  $\hat{w}$  or  $\hat{x}$  is reducible. The case  $x_1 x_2 = a_i b_i$  is the same as the case  $x_1 = b_i$ , with a shift in index. We conclude  $z$  is in canonical form. Since  $z \in I$  we know  $z = 0$  in  $T(V)$ . We note that all of the pure tensors appearing in  $z = \sum_{t=1}^j -y_t + \sum_{t=j+1}^k \alpha_t w_t \hat{x}$  are distinct. Therefore  $\alpha_t = 0$  for all  $1 \leq t \leq k$ , which is a contradiction.  $\square$

Henceforth, we identify the basis vectors  $a_i, b_i, c_i$  with their images in  $B$ . The following lemma is immediate from Proposition IV.2.3 and the presentation of the algebra.

**Lemma IV.2.4.** *For  $0 \leq i \leq n$ , the element  $b_i \in B$  is left regular. For  $1 \leq i \leq n-1$ ,*

$$l.\text{ann}_B(b_i c_i) = B c_i a_i, \quad l.\text{ann}_B(a_i) = B b_i c_i + B b_{i+1}, \quad \text{and}$$

$$l.\text{ann}_B(c_i a_i) = B b_i + B c_{i-1}.$$

Furthermore

$$l.\text{ann}_B(b_n c_n) = B a_n, \quad l.\text{ann}_B(c_n) = B a_n b_n + B c_{n-1}, \quad \text{and}$$

$$l.\text{ann}_B(a_0) = l.\text{ann}_B(a_0 b_0) = B b_1 + B c_0.$$

### IV.3 A Minimal Resolution of ${}_B k$

In this section we construct a minimal resolution of the trivial  $B$ -module  ${}_B k$ . We then use a criterion of Cassidy and Shelton to prove  $B$  is a  $\mathcal{K}_2$  algebra.

For  $d \in \mathbb{Z}$  let  $B(d)$  denote the  $B$ -module  $B$  with grading shifted by  $d$ , that is  $B(d)_k = B_{k+d}$ . If  $\bar{d} = (d_1, \dots, d_r) \in \mathbb{Z}^r$  we define

$$B(\bar{d}) = B(d_1, d_2, \dots, d_r) = B(d_1) \oplus B(d_2) \oplus \dots \oplus B(d_r).$$

If  $Q = B(d_1, \dots, d_r)$  and  $Q' = B(D_1, \dots, D_{r'})$  are graded free left  $B$ -modules and  $M = (m_{i,j})$  is an  $r \times r'$  matrix of homogeneous elements of  $B$  such that  $\deg m_{i,j} = D_j - d_i$ , then right multiplication by  $M$  defines a degree 0 homomorphism  $f : Q \rightarrow Q'$ . We denote this homomorphism  $Q \xrightarrow{M} Q'$ , and for convenience refer to both the matrix and the homomorphism it defines as  $M$ . A graded free resolution

$$\dots \rightarrow Q_n \xrightarrow{M_n} Q_{n-1} \rightarrow \dots \rightarrow Q_0$$

of the  $B$ -module  $N$  is *minimal* if  $\text{im}(M_n) \subseteq B_+ Q_{n-1}$  for every  $n \geq 1$ . Equivalently, each entry of the matrix representation of  $M_n$  is an element of  $B_+$ .

**Lemma IV.3.1.** *For  $1 \leq i \leq n-1$  the sequence*

$$B(-6, -5) \xrightarrow{\begin{pmatrix} b_i c_i \\ b_{i+1} \end{pmatrix}} B(-4) \xrightarrow{\begin{pmatrix} a_i & c_{i+1} a_{i+1} \end{pmatrix}} B(-3, -2) \xrightarrow{\begin{pmatrix} b_i c_i \\ b_{i+1} \end{pmatrix}} B(-1)$$

*of graded free  $B$ -modules is exact at  $B(-4)$  and  $B(-3, -2)$ .*

*Proof.* The sequence is clearly a complex. Exactness at  $B(-4)$  is clear from Lemma IV.2.4. To prove exactness at  $B(-3, -2)$ , let  $w, x \in B$  and suppose  $\pi_B(\hat{w}b_i c_i + \hat{x}b_{i+1}) = 0$ . Let  $\hat{w} = w_1 + w_2$  and  $\hat{x} = x_1 + x_2$  where  $w_1, w_2, x_1$ , and  $x_2 \in T(V)$  (all in canonical form) are such that  $w_1 b_i c_i$  and  $x_1 b_{i+1}$  are in canonical form, and all pure tensors in  $w_2 b_i c_i$  and  $x_2 b_{i+1}$  are reducible.

By Proposition IV.2.1,  $w_2 = w' c_i a_i$  and  $x_2 = x' c_{i+1} a_{i+1}$  for some  $w', x' \in T(V)$ . So  $\pi_B(w_2 b_i c_i) = \pi_B(w' c_i a_i b_i c_i) = \pi_B(-w' a_{i-1} b_{i-1} c_{i-1} c_i) = 0$ . Since  $x_2$  is in canonical form, no pure tensor in  $x'$  can end in  $b_{i+1}$  or  $c_i$ . Now consider

$$\begin{aligned} 0 &= \pi_B(w_1 b_i c_i + x_1 b_{i+1} + x_2 b_{i+1}) \\ &= \pi_B(w_1 b_i c_i + x_1 b_{i+1} - x' a_i b_i c_i). \end{aligned}$$

Thus  $z = w_1 b_i c_i + x_1 b_{i+1} - x' a_i b_i c_i$  is in  $I$ . Since  $z$  is also in canonical form we know  $z = 0$ . Therefore  $x_1 b_{i+1} = x' a_i b_i c_i - w_1 b_i c_i$  in  $T(V)$  and it follows that  $x_1 = 0$ .

We have  $\pi_B(\hat{w} b_i c_i - x' a_i b_i c_i) = 0$ , thus  $\pi_B(\hat{w} - x' a_i) \in \text{l.ann}_B(b_i c_i)$ . By Lemma IV.2.4,  $w - \pi_B(x') a_i \in B c_i a_i$ , so there exists  $z \in B$  such that  $w = z c_i a_i + \pi_B(x') a_i$ . Therefore

$$\begin{aligned} (w \ x) &= (z c_i a_i + \pi_B(x') a_i \ \pi_B(x') c_{i+1} a_{i+1}) \\ &= (z c_i + \pi_B(x')) (a_i \ c_{i+1} a_{i+1}). \end{aligned}$$

This proves exactness at  $B(-3, -2)$ . □

**Lemma IV.3.2.** *The periodic complex*

$$\cdots B(-6) \xrightarrow{(b_n c_n)} B(-4) \xrightarrow{(a_n)} B(-3) \xrightarrow{(b_n c_n)} B(-1)$$

is a minimal graded free resolution of  $Ba_n$ .

If  $1 \leq i \leq n - 1$ , the periodic complex

$$\begin{aligned} \cdots \rightarrow B(-3t-4) \xrightarrow{\begin{pmatrix} a_i & c_{i+1} a_{i+1} \end{pmatrix}} B(-3t-3, -3t-2) \xrightarrow{\begin{pmatrix} b_i c_i \\ b_{i+1} \end{pmatrix}} \\ \cdots \rightarrow B(-4) \xrightarrow{\begin{pmatrix} a_i & c_{i+1} a_{i+1} \end{pmatrix}} B(-3, -2) \xrightarrow{\begin{pmatrix} b_i c_i \\ b_{i+1} \end{pmatrix}} B(-1) \end{aligned}$$

of free left  $B$ -modules is a minimal graded free resolution of  $Ba_i$ .

*Proof.* The resolution of  $Ba_n$  is immediate from Lemma IV.2.4.

Let  $1 \leq i \leq n - 1$ . By Lemma IV.2.4,  $b_i c_i$  and  $b_{i+1}$  generate the left annihilator of  $a_i$ . Exactness in higher degrees follows from Lemma IV.3.1 and a degree shift. □

The complexes of Lemma IV.3.2 will be direct summands of our minimal resolution of  $Bk$ . We denote these complexes by  $P_\bullet^i$  where  $P_1^i = B(-1)$  for  $1 \leq i \leq n$ . The other summands of our resolution are built inductively. The following lemma provides the base cases for induction.

**Lemma IV.3.3.**

1. If  $w, x \in B$  and  $wc_i a_i b_i + xc_{i-1} = 0$  for some  $1 \leq i \leq n$ , then there exist  $w', w'', x' \in B$  such that

$$w = w' + w''c_{i-1} \text{ and } x = w'a_{i-1}b_{i-1} + x'c_{i-2}.$$

2. The complexes

$$B(-4, -3) \xrightarrow{\begin{pmatrix} c_i a_i & a_{i-1} b_{i-1} \\ 0 & c_{i-2} \end{pmatrix}} B(-2, -2) \xrightarrow{\begin{pmatrix} b_i \\ c_{i-1} \end{pmatrix}} B(-1)$$

are exact at  $B(-2, -2)$  for  $1 \leq i \leq n$ .

3. The complex

$$B(-4, -3) \xrightarrow{\begin{pmatrix} c_n & a_{n-1} b_{n-1} \\ 0 & c_{n-2} \end{pmatrix}} B(-3, -2) \xrightarrow{\begin{pmatrix} a_n b_n \\ c_{n-1} \end{pmatrix}} B(-1)$$

is exact at  $B(-3, -2)$ .

*Proof.* For (1), we assume  $w, x \in B$  and  $wc_i a_i b_i + xc_{i-1} = 0$  for some  $i, 1 \leq i \leq n$ . Since  $c_i a_i b_i = -a_{i-1} b_{i-1} c_{i-1}$ , we have

$$\pi_B(\hat{x}c_{i-1} - \hat{w}a_{i-1}b_{i-1}c_{i-1}) = 0.$$

Let  $\hat{w} = w_1 + w_2$  and  $\hat{x} = x_1 + x_2$  where  $w_1, w_2, x_1$ , and  $x_2 \in T(V)$  are such that  $w_1 a_{i-1} b_{i-1} c_{i-1}$  and  $x_1 c_{i-1}$  are in canonical form, and all pure tensors in  $w_2 a_{i-1} b_{i-1} c_{i-1}$  and  $x_2 c_{i-1}$  are reducible.

By Proposition IV.2.1, there exist  $y', y'',$  and  $z' \in T(V)$  so that  $w_2 = y' b_i + y'' c_{i-1}$  and  $x_2 = z' c_{i-2}$ . (If  $i = 1$ , then  $z' = 0$ .) Therefore we have  $\pi_B(w_2 a_{i-1} b_{i-1} c_{i-1}) = 0$  and  $\pi_B(x_2 c_{i-1}) = 0$ .

We have  $\pi_B(x_1 c_{i-1} - w_1 a_{i-1} b_{i-1} c_{i-1}) = 0$ , and since all terms are in canonical form,  $x_1 = w_1 a_{i-1} b_{i-1}$ . Since  $b_i a_{i-1} = 0$ , we may write  $x_1 = (w_1 + y' b_i) a_{i-1} b_{i-1}$ . The result follows by setting  $w' = \pi_B(w_1 + y' b_i)$ ,  $w'' = \pi_B(y'')$ , and  $x' = \pi_B(z')$ .

For (2), suppose  $w, x \in B$  such that  $wb_i + xc_{i-1} = 0$ . Consider  $\hat{x}c_{i-1}$  and notice that no pure tensor in its canonical form can end in  $b_i$ . So every pure tensor in  $\hat{w}b_i$  is reducible. By Proposition IV.2.1, there exists  $w' \in B$  such that  $w = w'c_i a_i$ . By (1), there exist  $w'', w''', x' \in B$  such that

$$w' = w'' + w'''c_{i-1} \text{ and } x = w''a_{i-1}b_{i-1} + x'c_{i-2}.$$

Now note that  $x = w'a_{i-1}b_{i-1} + (x' + w'''a_{i-2}b_{i-2})c_{i-2}$ . The result follows.

For (3), suppose  $w, x \in B$  such that  $wa_n b_n + xc_{n-1} = 0$ . Part (2) implies  $wa_n = w'c_n a_n$  and  $x = w'a_{n-1}b_{n-1} + x'c_{n-2}$  for some  $w', x' \in B$ . From the proof of (2),  $\hat{w}'c_n a_n$  may be assumed to be in canonical form. Let  $\hat{w} = w_1 + w_2$  where  $w_1 a_n$  is in canonical form and all pure tensors of  $w_2 a_n$  are reducible. By Proposition IV.2.1, there exists  $y' \in B_n$  such that  $w_2 = y' b_n c_n$ . Thus  $\pi_B(w_2 a_n) = 0$  and we have  $\pi_B(w_1 a_n - \hat{w}'c_n a_n) = 0$ . Since both terms are in canonical form,  $w_1 = \hat{w}'c_n$ . So

$$w = \pi_B(\hat{w}' + y'b_n)c_n \text{ and } x = \pi_B(\hat{w}' + y'b_n)a_{n-1}b_{n-1} + \pi_B(x')c_{n-2}.$$

The result follows. □

The following lemma is clear by combining Lemma IV.2.4 with Lemma IV.3.3(2).

**Lemma IV.3.4.** *The periodic complex*

$$\begin{aligned} \cdots \rightarrow B(-3t-4) \xrightarrow{\begin{pmatrix} a_0 b_0 & c_1 a_1 \end{pmatrix}} B(-3t-2, -3t-2) \xrightarrow{\begin{pmatrix} c_0 \\ b_1 \end{pmatrix}} \\ \cdots \rightarrow B(-4) \xrightarrow{\begin{pmatrix} a_0 b_0 & c_1 a_1 \end{pmatrix}} B(-2, -2) \xrightarrow{\begin{pmatrix} c_0 \\ b_1 \end{pmatrix}} B(-1) \end{aligned}$$

of free left  $B$ -modules is a minimal graded free resolution of  $Ba_0$ .

We denote the complex of Lemma IV.3.4 by  $P_\bullet^0$  where  $P_1^0 = B(-1)$ .

To simplify the exposition from this point, we introduce an operation which we use to inductively build matrices in our minimal resolution of  $Bk$ . If  $M = (m_{i,j})$  is an  $a \times b$  matrix and  $N = (n_{i,j})$  is a  $c \times d$  matrix, we define the *star product*  $M \star N$  to be the  $(a+c-1) \times (b+d)$  matrix

$$\begin{pmatrix} m_{1,1} & \cdots & m_{1,b} & & & \\ \vdots & & \vdots & & 0 & \\ m_{a,1} & \cdots & m_{a,b} & n_{1,1} & \cdots & n_{1,d} \\ & & 0 & \vdots & & \vdots \\ & & & n_{c,1} & \cdots & n_{c,d} \end{pmatrix}.$$

We note that this product is associative and we define

$$M \star_{j=1}^p N_j = M \star N_1 \star \cdots \star N_p.$$

For  $1 \leq i \leq n$  let  $\Gamma_i = \begin{pmatrix} a_i b_i \\ c_{i-1} \end{pmatrix}$  and let  $T_i = \begin{pmatrix} b_i \\ c_{i-1} \end{pmatrix}$ . Let  $\Gamma_0 = (a_0 b_0)$ . It will be convenient to define  $M \star \Gamma_i = M$  for  $i < 0$  and  $(\ ) \star \Gamma_i = \Gamma_i$ , where  $(\ )$  denotes the empty matrix.

The next lemma is the key to making inductive arguments in later proofs.

**Lemma IV.3.5.** *Let  $r_1, r_2, r_3$  be positive integers and  $\bar{d}_1 \in \mathbb{Z}^{r_1}$ ,  $\bar{d}_2 \in \mathbb{Z}^{r_2}$ ,  $\bar{d}_3 \in \mathbb{Z}^{r_3}$ . Let  $d_{j,k}$  be the  $k$ th component of  $\bar{d}_j$ . Let  $i$  be a positive integer. Let  $M$  and  $N$  be matrices of homogeneous elements of  $B$  such that  $B(\bar{d}_3) \xrightarrow{N \star \Gamma_i} B(\bar{d}_2) \xrightarrow{M} B(\bar{d}_1)$  is an exact sequence of degree zero homomorphisms.*

1. *If  $i = 1$ , then the sequence*

$$B(\bar{d}_3) \xrightarrow{N \star \Gamma_1} B(\bar{d}_2) \xrightarrow{M \star \Gamma_0} B(\bar{d}_1) \oplus B(d_{2,r_2} + 2)$$

*is exact at  $B(\bar{d}_2)$ .*

2. *If  $i = 2$ , then the sequence*

$$B(\bar{d}_3) \xrightarrow{N \star \Gamma_2 \star \Gamma_0} B(\bar{d}_2) \oplus B(d_{3,r_3} + 2) \xrightarrow{M \star \Gamma_1} B(\bar{d}_1) \oplus B(d_{3,r_3} + 3)$$

*is exact at  $B(\bar{d}_2) \oplus B(d_{3,r_3})$ .*



3. If  $i > 2$ , then for all  $t \in \mathbb{Z}$  the sequence

$$B(\bar{d}_3) \oplus B(t-2) \xrightarrow{N \star \Gamma_i \star \Gamma_{i-2}} B(\bar{d}_2) \oplus B(t-1) \xrightarrow{M \star \Gamma_{i-1}} B(\bar{d}_1) \oplus B(t)$$

is exact at  $B(\bar{d}_2) \oplus B(t-1)$ .

4. If  $i > 2$  and

$$B(\bar{d}_3) \xrightarrow{N \star T_i} B(\bar{d}_2) \xrightarrow{M} B(\bar{d}_1)$$

is exact at  $B(\bar{d}_2)$ , then the sequence

$$B(\bar{d}_3) \oplus B(t-2) \xrightarrow{N \star T_i \star \Gamma_{i-2}} B(\bar{d}_2) \oplus B(t-1) \xrightarrow{M \star \Gamma_{i-1}} B(\bar{d}_1) \oplus B(t)$$

is exact at  $B(\bar{d}_2) \oplus B(t-1)$ .

*Proof.* The statements (1) and (2) follow immediately from Lemma IV.2.4. So we assume  $i > 2$ .

For (3), the hypothesis implies that the rows of  $N' = N \star \Gamma_i \star \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  generate the kernel of  $\begin{pmatrix} M \\ 0 \end{pmatrix}$ . To compute the kernel of  $M \star \Gamma_{i-1}$ , it suffices to determine which elements of the submodule generated by the rows of  $N'$  also left annihilate the last column of  $M \star \Gamma_{i-1}$ .

Since  $b_i a_{i-1} = 0$ , the last two rows of  $N'$  are the only rows not in the kernel of  $M \star \Gamma_{i-1}$ . A row  $(0 \cdots 0 \ w c_{i-1} \ x)$  is in the kernel of  $M \star \Gamma_{i-1}$  if and only if  $w c_{i-1} a_{i-1} b_{i-1} + x c_{i-2} = 0$ . By Lemma IV.3.3(1), this occurs if and only if  $(0 \cdots 0 \ w c_{i-1} \ x)$  is a  $B$ -linear combination of  $(0 \cdots 0 \ c_{i-1} \ a_{i-2} b_{i-2})$  and  $(0 \cdots 0 \ 0 \ c_{i-3})$ , which are the last two rows of  $N \star \Gamma_i \star \Gamma_{i-2}$ . Since all other rows of  $N'$  and  $N \star \Gamma_i \star \Gamma_{i-2}$  are equal, the proof of (3) is complete.

The proof of (4) is the same as the proof of (3).

□

To help keep track of the ranks and degree shifts of free modules which appear in later resolutions we introduce the following notations. If  $i, j \in \mathbb{Z}$  and  $i \leq j$ , we denote  $(i, i+1, \dots, j-1, j)$

by  $[i, j]$ . If  $i$  is even, for  $j \in \mathbb{N}$ , let

$$\bar{d}_i(j) = \begin{cases} [-\lfloor 3j/2 \rfloor, -j] & j \leq i \\ [-\lfloor 3j/2 \rfloor, -\lfloor 3j/2 \rfloor + i/2] & j > i \end{cases}.$$

If  $i$  is odd, for  $j \in \mathbb{N}$ , let

$$\bar{d}_i(j) = \begin{cases} [-\lfloor 3j/2 \rfloor, -j] & j \leq i \\ [-\lfloor 3j/2 \rfloor, -\lfloor 3j/2 \rfloor + \lfloor i/2 \rfloor] & j > i \end{cases}.$$

Figure 3 illustrates the combinatorics of the vector  $\bar{d}_i(j)$ .

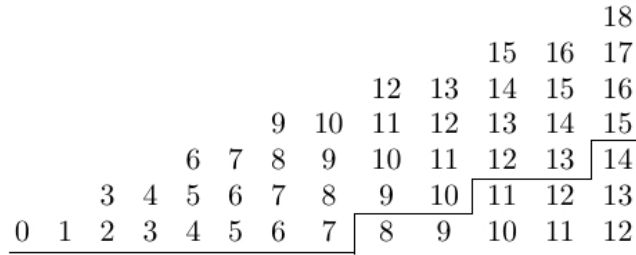


Figure 3: The degree vector  $\bar{d}_i(j)$  consists of negatives of values above the line in column  $j$ . In this figure  $i = 7$ .

We need a bit more notation. Let  $0 \leq p \leq n$  such that  $n \equiv p \pmod{2}$ . Let

$$R_p = \Gamma_n \star_{j=1}^{(n-p)/2} \Gamma_{n-2j} \text{ and } U_{p-1} = \binom{c_n}{c_n} \star_{j=0}^{(n-p)/2} \Gamma_{n-2j-1}.$$

**Lemma IV.3.6.**

1. If  $n$  is even, the eventually periodic sequence

$$\cdots \xrightarrow{U_1} B(\bar{d}_{n+1}(n+2)) \xrightarrow{R_0} B(\bar{d}_{n+1}(n+1)) \xrightarrow{U_1} B(\bar{d}_{n+1}(n)) \xrightarrow{R_2} \cdots$$

$$\xrightarrow{U_{n-3}} B(\bar{d}_{n+1}(4)) \xrightarrow{R_{n-2}} B(\bar{d}_{n+1}(3)) \xrightarrow{U_{n-1}} B(\bar{d}_{n+1}(2)) \xrightarrow{R_n} B(\bar{d}_{n+1}(1))$$

is a minimal graded free resolution of  $Bc_n$ .

2. If  $n$  is odd, the eventually periodic sequence

$$\begin{aligned} \cdots \xrightarrow{R_1} B(\bar{d}_{n+1}(n+2)) \xrightarrow{U_0} B(\bar{d}_{n+1}(n+1)) \xrightarrow{R_1} B(\bar{d}_{n+1}(n)) \xrightarrow{U_2} \cdots \\ \xrightarrow{U_{n-3}} B(\bar{d}_{n+1}(4)) \xrightarrow{R_{n-2}} B(\bar{d}_{n+1}(3)) \xrightarrow{U_{n-1}} B(\bar{d}_{n+1}(2)) \xrightarrow{R_n} B(\bar{d}_{n+1}(1)) \end{aligned}$$

is a minimal graded free resolution of  $Bc_n$ .

*Proof.* It is clear that the homomorphisms are degree zero by inspecting the degrees of entries in  $U_{p-1}$  and  $R_p$ . We note that  $R_n = \Gamma_n$ . Since  $\text{l.ann}_B(c_n) = Ba_n b_n + Bc_{n-1}$  we see the resolutions have the correct first terms to be minimal resolutions of  $Bc_n$ .

We will prove (1). If  $n = 0$ , the result follows from Lemma 2.3. So we assume  $n > 0$  and  $n$  is even. Exactness at  $B(\bar{d}_{n+1}(2))$  is Lemma 3.3 (3). For exactness at  $B(\bar{d}_{n+1}(3))$ , we apply Lemma 3.5 with  $i = n$ ,  $M = \begin{pmatrix} c_n \end{pmatrix}$  and  $N$  the empty matrix. Similarly, exactness at  $B(\bar{d}_{n+1}(4))$  follows from exactness at  $B(\bar{d}_{n+1}(2))$  by applying Lemma 3.5 with  $i = n - 1$ ,  $M = R_n$ , and  $N = (c_n)$ . We remark that exactness at  $B(\bar{d}_{n+1}(n+1))$  follows from exactness at  $B(\bar{d}_{n+1}(n-1))$  by applying Lemma 3.5 (2). Exactness at  $B(\bar{d}_{n+1}(n+2))$  follows from exactness at  $B(\bar{d}_{n+1}(n))$  by applying Lemma 3.5 (1). By induction, we conclude the sequence in (1) is exact.

The proof of (2) is similar and is omitted. □

The resolution of Lemma IV.3.6 is the second piece of our resolution of  $Bk$ . We denote this complex by  $C_\bullet$  where  $C_1 = B(-1)$ .

For integers  $i, p, n$  such that  $1 \leq i \leq n$ ,  $0 \leq p \leq i$ , and  $i \equiv p \pmod{2}$ , define

$$S_{i,p-1} = \begin{pmatrix} c_i a_i \end{pmatrix} \star_{j=0}^{(i-p)/2} \Gamma_{i-2j-1} \text{ and } T_{i,p} = T_i \star_{j=1}^{(i-p)/2} \Gamma_{i-2j},$$

where  $T_i = \begin{pmatrix} b_i \\ c_{i-1} \end{pmatrix}$ .

**Lemma IV.3.7.**

1. If  $1 \leq i \leq n$  and  $i$  is even, then the eventually periodic sequence

$$\begin{aligned} & \cdots \xrightarrow{S_{i,1}} B(-3i/2 - 1, \bar{d}_i(i+1)) \xrightarrow{T_{i,0}} B(\bar{d}_i(i)) \xrightarrow{S_{i,1}} \\ & \cdots \xrightarrow{S_{i,i-3}} B(-4, \bar{d}_i(3)) \xrightarrow{T_{i,i-2}} B(\bar{d}_i(2)) \xrightarrow{S_{i,i-1}} B(-1, \bar{d}_i(1)) \end{aligned}$$

is a minimal graded free resolution of  $Bb_i + Bc_{i-1}$ .

2. If  $1 \leq i \leq n$  and  $i$  is odd, the eventually periodic sequence

$$\begin{aligned} & \cdots \xrightarrow{T_{i,1}} B(\bar{d}_i(i+1)) \xrightarrow{S_{i,0}} B(-(3i-1)/2, \bar{d}_i(i)) \xrightarrow{T_{i,1}} B(\bar{d}_i(i-1)) \xrightarrow{S_{i,2}} \\ & \cdots \xrightarrow{S_{i,i-3}} B(-4, \bar{d}_i(3)) \xrightarrow{T_{i,i-2}} B(\bar{d}_i(2)) \xrightarrow{S_{i,i-1}} B(-1, \bar{d}_i(1)) \end{aligned}$$

is a minimal graded free resolution of  $Bb_i + Bc_{i-1}$ .

*Proof.* We will prove (2). The case  $i = 1$  follows immediately from Lemma 3.4. Suppose that  $i > 2$  and  $i$  is odd. Lemma 3.3 (2) shows the first map  $S_{i,i-1}$  is the start of a minimal resolution of  $Bb_i + Bc_{i-1}$ . Exactness in higher degrees follows by induction and the remark in the paragraph following the proof of Lemma 3.5.

The proof of (1) is similar and is omitted. □

For  $1 \leq i \leq n$ , we denote the complexes of Lemma IV.3.7 by  $Q_\bullet^i$  with  $Q_1^i = B(-1, -1)$ . Denote by  $Q_\bullet^0$  the complex  $0 \rightarrow B(-1)$  which is a minimal graded free resolution of  $Bb_0$  by Lemma IV.2.4. We take  $Q_1^0 = B(-1)$ .

Define the chain complex  $\widetilde{Q}_\bullet$  by

$$\widetilde{Q}_\bullet = P_\bullet^0 \oplus P_\bullet^1 \oplus \cdots \oplus P_\bullet^n \oplus C_\bullet \oplus Q_\bullet^n \oplus \cdots \oplus Q_\bullet^1 \oplus Q_\bullet^0.$$

Let  $M_1 = (a_0 \ a_1 \ \cdots \ a_n \ c_n \ b_n \ \cdots \ c_0 \ b_0)^T$ . Let  $M_d$  be the matrix of  $\widetilde{Q}_d \rightarrow \widetilde{Q}_{d-1}$ . Denote by  $\hat{M}_d$  the matrix with entries in  $T(V)$  such that  $(\hat{M}_d)_{i,j} = \rho_B((M_d)_{i,j})$ .

**Theorem IV.3.8.** *The complex  $\widetilde{Q}_\bullet \xrightarrow{M_1} B$  is a minimal graded free resolution of the trivial left  $B$ -module  ${}_B k$*

*Proof.* By Lemmas IV.3.2, IV.3.4, IV.3.6, and IV.3.7, it is enough to check exactness at  $B$  and at  $\tilde{Q}_1$ . The image of  $M_1$  in  $B$  is clearly the augmentation ideal  $B_+$ . The complex is exact at  $\tilde{Q}_1$  since the entries of  $\hat{M}_2\hat{M}_1$  give the set  $R$  of relations of  $B$ . □

Next we show the algebra  $B$  is a  $\mathcal{K}_2$  algebra using the criterion established by Cassidy and Shelton in [7]. Put  $I' = V \otimes I + I \otimes V$ . An element in  $I$  is an *essential relation* if its image is nonzero in  $I/I'$ . For each  $d \geq 2$ , let  $L_d$  be the image of  $\hat{M}_d$  modulo the ideal  $T(V)_{\geq 2}$ . Let  $E_d$  be the image of  $\hat{M}_d\hat{M}_{d-1}$  modulo  $I'$ . Finally, let  $[L_d : E_d]$  be the matrix obtained by concatenating  $L_d$  and  $E_d$ .

**Theorem IV.3.9** ([7]). *The algebra  $B$  is a  $\mathcal{K}_2$  algebra if and only if for all  $d > 2$ ,  $\tilde{Q}_d$  is a finitely generated  $B$ -module and the rows of  $[L_d : E_d]$  are independent over  $k$ .*

**Theorem IV.3.10.** *For any  $n \in \mathbb{N}$ ,  $B$  is a  $\mathcal{K}_2$  algebra.*

*Proof.* Let  $d > 2$ . It is clear that  $\tilde{Q}_d$  is finitely generated. To see that the rows of  $[L_d : E_d]$  are independent, it suffices to check the condition on the blocks of  $\hat{M}_d$  and  $\hat{M}_{d-1}$ . The blocks  $U_p$  and  $T_{i,p}$  have exactly one linear term in each row and no two are in the same column except for the upper left corner of  $T_{i,p}$  which is  $\begin{pmatrix} b_i & c_{i-1} \end{pmatrix}^T$  and  $b_i, c_{i-1}$  are independent over  $k$ . So the condition holds for these blocks.

The blocks  $(a_n)$ ,  $(a_i \ c_{i+1}a_{i+1})$ , and  $(b_1 \ c_0)^T$  contain linear terms in each row and the rows are independent, so they satisfy the condition. The block  $(c_1a_1 \ a_0b_0)$  does not contain a linear term, but the corresponding block of  $E_d$  is the essential relation  $c_1a_1b_1 + a_0b_0c_0$ . The blocks  $(b_i c_i \ b_{i+1})^T$  have a linear term in the second row, and the first row of the corresponding block of  $E_d$  contains the essential relation  $b_i c_i a_i$ .

The blocks  $S_{i,p}$  and  $R_p$  have one linear term in each row except the first, and no two are in the same column, so it is enough to check that the first row of the corresponding block of  $E_d$  is nonzero. A direct calculation shows that respectively the rows contain the essential relations  $c_i a_i b_i + a_{i-1} b_{i-1} c_{i-1}$  and  $a_n b_n c_n$ . □

#### IV.4 $A_\infty$ -Algebra Structures from Resolutions

Let  $A$  be a  $k$ -algebra and let  $(Q_\bullet, d_\bullet)$  be a minimal graded projective resolution of  ${}_A k$  by  $A$ -modules with  $Q_0 = A$ . For  $n \in \mathbb{Z}$ , a *degree  $n$  endomorphism* of  $(Q_\bullet, d_\bullet)$  is a collection of degree zero homomorphisms of graded  $A$ -modules  $\{f_j : Q_j \rightarrow Q_{j+n} \mid j \in \mathbb{Z}\}$ . Note that  $f_j = 0$  for  $j < \max\{0, -n\}$ .

Let  $U = \text{End}_A(Q_\bullet)$  be the differential graded endomorphism algebra of  $(Q_\bullet, d_\bullet)$  with multiplication given by composition. We denote the maps in  $U$  of degree  $-n$  by  $U^n$  for all  $n \in \mathbb{Z}$ . The differential  $\partial$  on  $U$  is given on homogeneous elements by  $\partial(f) = df - (-1)^{|f|}fd$ , where  $|f|$  denotes the degree of  $f$ . With respect to the endomorphism degree,  $\deg(\partial) = 1$ . We let  $B^n$  and  $Z^n$  respectively denote the set of coboundaries and the set of cocycles in  $U^n$ .

**Lemma IV.4.1.** *Let  $g \in Z^n$ . If  $\text{im}(g_n) \subset (Q_0)_+$ , then there exists  $f \in U^{n-1}$  with  $f_j = 0$  for all  $j < n$  such that  $\partial(f) = g$ .*

*Proof.* We define  $f$  inductively. Put  $f_j = 0$  for all  $j < n$ . Since  $Q_\bullet$  is a resolution,  $\text{im}(d_1) = (Q_0)_+$ . Hence  $\text{im}(g_n) \subset \text{im}(d_1)$ . By graded projectivity of  $Q_n$ , there exists a degree zero homomorphism of graded  $A$ -modules  $f_n : Q_n \rightarrow Q_1$  such that  $g_n = d_1 f_n = d_1 f_n - (-1)^{n-1} f_{n-1} d_n$ .

Fix  $j > n$  and assume that for all  $k < j$ ,  $f_k$  is defined and  $df_k = (-1)^{n-1} f_{k-1} d + g_k$ . Then

$$\begin{aligned} d[(-1)^{n-1} f_{j-1} d + g_j] &= (-1)^{n-1} (df_{j-1} d) + dg_j \\ &= (-1)^{n-1} ((-1)^{n-1} f_{j-2} d + g_{j-1}) d + dg_j \\ &= (-1)^{n-1} g_{j-1} d + dg_j \\ &= \partial(g)_j \\ &= 0. \end{aligned}$$

Since  $Q_\bullet$  is a resolution,  $\text{im}((-1)^{n-1} f_{j-1} d + g_j) \subseteq \text{im}(d_{j+1-n})$ . Thus, by graded projectivity of  $Q_j$ , there exists a degree zero homomorphism of graded  $A$ -modules  $f_j : Q_j \rightarrow Q_{j+1-n}$  such that  $df_j = (-1)^{n-1} f_{j-1} d + g_j$ .

By induction, there exists  $f \in U^{n-1}$  such that  $\partial(f) = g$ .

□

The minimality of the resolution implies that  $\text{Hom}_A(Q_\bullet, k)$  is equal to its cohomology. We take  $H^*(\text{Hom}_A(Q_\bullet, k)) = \text{Hom}_A(Q_\bullet, k)$  as our model for the Yoneda algebra  $E(A)$ . We abuse notation slightly and write  $E(A) = \text{Hom}_A(Q_\bullet, k)$ . It will be convenient to view  $E(A)$  as a differential graded algebra with trivial differential.

Let  $\epsilon : Q_0 \rightarrow k$  be the augmentation homomorphism. For every  $n \in \mathbb{Z}$  define a map  $\phi^n : U^n \rightarrow \text{Hom}_A^n(Q_\bullet, k)$  by  $\phi^n(f) = \epsilon f_n$ . We remark that the map  $\phi = \bigoplus_n \phi^n$  induces a surjective homomorphism  $\Phi : H^*(U) \rightarrow E(A)$ . Using Lemma IV.4.1, it is easy to prove that  $\Phi$  is also injective.

We compute the structure of an  $A_\infty$ -algebra on  $E(A)$  by specifying the data of a strong deformation retraction (SDR-data) from  $U$  to  $E(A)$ . More precisely, we choose maps  $i : E(A) \rightarrow U$ ,  $p : U \rightarrow E(A)$ , and  $G : U \rightarrow U$  such that  $i$  and  $p$  are degree 0 morphisms of complexes and  $G$  is a homogeneous  $k$ -linear map of degree  $-1$  such that  $pi = 1_{E(A)}$  and  $1_U - ip = \partial G - G\partial$ .

We define a family of homogenous  $k$ -linear maps  $\{\lambda_j : U^{\otimes j} \rightarrow U\}_{j \geq 2}$  with  $\deg(\lambda_j) = 2 - j$  as follows. There is no map  $\lambda_1$ , but we define the formal symbol  $G\lambda_1$  to mean  $-1_U$ . The map  $\lambda_2$  is the multiplication on  $U$ . For  $n \geq 3$ , the map  $\lambda_j$  is defined recursively by

$$\lambda_j = \lambda_2 \sum_{\substack{s+t=j \\ s,t \geq 1}} (-1)^{s+1} (G\lambda_s \otimes G\lambda_t).$$

The following theorem due to Merkulov provides an  $A_\infty$ -structure on  $E(A)$ . Merkulov's theorem applies to the subcomplex  $i(E(A))$  of  $U$  and gives structure maps  $m_j = ip\lambda_j$ . Since  $pi = 1_{E(A)}$ , we may restate the theorem for  $E(A)$  as follows.

**Theorem IV.4.2** ([22]). *Let  $m_1 = 0$  and for  $j \geq 2$ , let  $m_j = p\lambda_j i$ . Then  $(E(A), m_1, m_2, m_3, \dots)$  is an  $A_\infty$ -algebra satisfying the conditions of Theorem IV.1.2.*

There are many choices for  $i$  and  $G$ . We describe a method for defining  $i$  and  $G$  based on a minimal projective resolution  $(Q_\bullet, d_\bullet)$  of the trivial module  ${}_A k$ . Our  $G$  will additionally satisfy the *side conditions*  $G^2 = 0$ ,  $Gi = 0$ , and  $pG = 0$ . As observed in [13], initial SDR-data can always be altered to satisfy these conditions, but they are corollaries of our construction. We define  $p = \phi$ .

Next, we define the inclusion map  $i$ . Assume that a homogeneous  $A$ -basis  $\{u_n^\ell\}$  has been fixed for each  $Q_n$  and let  $\{\mu_n^\ell\}$  be the graded dual basis for  $E^n(A)$ .

We choose  $i : E(A) \rightarrow U$  as follows. For  $n \geq 0$  let  $\mu \in E^n(A)$  be a dual basis vector. By graded projectivity, there exists a sequence of degree zero  $A$ -module homomorphisms  $\{i(\mu)_k\}_{k \geq n}$  so that the diagram

$$\begin{array}{ccccc}
 & Q_{n+1} & \longrightarrow & Q_n & & \\
 & \downarrow & & \downarrow & \searrow & \mu \\
 \cdots & i(\mu)_{n+1} & & i(\mu)_n & & \\
 & Q_1 & \longrightarrow & Q_0 & \xrightarrow{\epsilon} & k
 \end{array}$$

has commuting squares when  $n$  is even and anticommuting squares when  $n$  is odd. This ensures that  $i(\mu)$  is contained in  $Z^n$ . We choose the map  $i(\mu)_n$  to be represented by a matrix with scalar entries in the given  $A$ -basis. We extend  $i$  to  $E(A)$  by  $k$ -linearity, and we note that  $pi = 1_{E(A)}$ .

For each integer  $n$  define the  $k$ -vector space  $H^n$  by  $H^n = i(E^n(A))$  if  $n \geq 0$  and  $H^n = 0$  if  $n < 0$ .

**Proposition IV.4.3.** *For every  $n \in \mathbb{Z}$ ,  $Z^n = B^n \oplus H^n$ .*

*Proof.* Let  $n \in \mathbb{Z}$  and let  $g \in B^n$ . Then there exists  $f \in U^{n-1}$  such that  $\partial(f) = g$ . The minimality condition ensures that  $\text{im}(g_n) = \text{im}(\partial(f)_n)$  is contained in  $A_+Q_0 = (Q_0)_+$ . Thus,  $g_n$  cannot be represented by a matrix with nonzero scalar entries. We conclude that  $B^n \cap H^n = 0$ .

Now, let  $f \in Z^n$ . By definition,  $ip(f) \in H^n$ . To prove  $Z^n = B^n \oplus H^n$ , it suffices to show there exists  $g \in U^{n-1}$  such that  $\partial(g) = f - ip(f)$ . This follows from Lemma IV.4.1. □

Finally, we define a homogeneous  $k$ -linear map  $G : U \rightarrow U$  of degree  $-1$ . There will be many choices of  $G$ , but the main properties we want  $G$  to satisfy are  $\partial G|_{B^n} = 1_{B^n}$  and  $G(B^n) \cap Z^{n-1} = 0$ .

We start by defining  $G$  on the  $k$ -linear space  $B^n$ . Let  $b$  denote a basis element of  $B^n$ . The hypotheses of Lemma IV.4.1 hold for the map  $b$  so there exists a map  $f \in U^{n-1}$  such that  $\partial(f) = b$ . Define  $G(b) = f$ . Extending by  $k$ -linearity, we have the desired  $G$  defined on  $B^n$ .

For each integer  $n$ , let  $L^n$  denote the  $k$ -vector space  $G(B^{n+1})$ .



**Proposition IV.4.4.** *For every  $n \in \mathbb{Z}$ ,  $U^n = L^n \oplus Z^n$ .*

*Proof.* Let  $n \in \mathbb{Z}$ . First suppose  $h \in L^n \cap Z^n$ . Then  $\partial h = 0$  and  $h = G(h')$  for some  $h' \in B^{n+1}$ . So  $h' = \partial G(h') = \partial h = 0$ . Hence  $h = G(h') = G(0) = 0$ .

Now let  $f \in U^n$ . By definition  $G\partial(f) \in L^n$ . Furthermore we have  $\partial(f - G\partial(f)) = \partial(f) - \partial G\partial(f) = \partial(f) - \partial(f) = 0$ . So  $f - G\partial(f) \in Z^n$ . Since  $f = G\partial(f) + (f - G\partial(f))$ , we conclude that  $U^n = L^n \oplus Z^n$ . □

Proposition IV.4.3 and Proposition IV.4.4 allow us to extend  $G$  to a  $k$ -linear map on all of  $U$  by defining  $G$  to be zero on  $H^n \oplus L^n$ . It is straightforward to check that  $1_U - ip = \partial G + G\partial$  and the side conditions hold.

#### IV.5 $A_\infty$ -Algebra Structure on $E(B)$

We return to the setting of Sections 2 and 3. We wish to compute only a few nonzero multiplications of an  $A_\infty$ -structure on  $E(B)$ , not the entire structure. Therefore, we define  $i$  explicitly only on certain linearly independent elements of  $E(B)$ . Similarly, we explicitly define  $G$  only on certain elements of  $U = \text{End}_B(\tilde{Q}_\bullet)$ . We then apply the results of Section 4 to extend the definition of  $i$  to all of  $E(B)$  and the definition of  $G$  to all of  $U$ .

It is standard to identify  $E^1(B)$  with the  $k$ -linear graded dual space  $V^*$ . For  $0 \leq \ell \leq n$ , we denote the dual basis vectors of  $a_\ell, b_\ell, c_\ell \in V$  by  $\alpha_\ell, \beta_\ell$ , and  $\gamma_\ell$ , respectively. We suppress the  $m_2$  notation for products in  $E(B)$ . For example, we denote  $m_2(\beta_1 \otimes \alpha_0)$  by  $\beta_1\alpha_0$ .

The diagrams below define  $i : E(B) \rightarrow U$  on  $\alpha_n, \beta_\ell\alpha_{\ell-1}, \beta_0$ , and  $\gamma_0 \cdots \gamma_n$ . Recall that we have chosen homogeneous  $A$ -bases and given the corresponding matrix representations for the resolution  $(\tilde{Q}_\bullet, d_\bullet)$ . We write  $e_{i,j}$  for the standard matrix unit. We include just enough of the endomorphism to see the general pattern. Finally, to make cleaner statements it will be useful to define

$$j = \begin{cases} \frac{3n}{2} + 1 & \text{if } n \text{ is even} \\ \frac{5n+3}{2} & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad k = \begin{cases} \frac{5n}{2} + 2 & \text{if } n \text{ is even} \\ \frac{3n+3}{2} & \text{if } n \text{ is odd} \end{cases}.$$

We note that the  $(j+1, k)$ -entry of  $M_{n+1}$  is  $c_0$  and that this is the lower rightmost position in the block  $U_1$  when  $n$  is even. When  $n$  is odd, the  $(j+1, k)$ -entry is the lower rightmost position in

$R_1$ . The upper leftmost entry of  $U_{p-1}$  or  $R_p$  is always the  $(n+2, 2n+2)$ -entry of  $M_d$  when  $d \geq 3$  is odd. It is the  $(2n+2, n+2)$ -entry of  $M_d$  when  $d$  is even.

$$i(a_0)_t = (-1)^{t+1} e_{1,2} \text{ for all } t > 1$$

$i(\alpha_n)$  for  $n > 0$  (see Lemma IV.3.2 and Lemma IV.3.6):

$$\begin{array}{ccccccc} & \tilde{Q}_4 & \longrightarrow & \tilde{Q}_3 & \longrightarrow & \tilde{Q}_2 & \longrightarrow & \tilde{Q}_1 \\ \cdots & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & -b_n e_{2n+1, n+2} & & e_{n+1, 2n+2} & & -b_n e_{2n+1, n+2} & & e_{n+1, 1} \\ & \tilde{Q}_3 & \longrightarrow & \tilde{Q}_2 & \longrightarrow & \tilde{Q}_1 & \longrightarrow & \tilde{Q}_0 \end{array}$$

$i(\beta_\ell \alpha_{\ell-1})$ ,  $1 \leq \ell \leq n$  (see Lemma IV.3.4 for  $\ell = 1$  and Lemma IV.3.2 for  $\ell > 1$ ):

$$\begin{array}{ccccccc} & \tilde{Q}_5 & \longrightarrow & \tilde{Q}_4 & \longrightarrow & \tilde{Q}_3 & \longrightarrow & \tilde{Q}_2 \\ \cdots & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & c_\ell e_{\ell, \ell+1} & & e_{2\ell, 2\ell+1} & & c_\ell e_{\ell, \ell+1} & & e_{2\ell, 1} \\ & \tilde{Q}_3 & \longrightarrow & \tilde{Q}_2 & \longrightarrow & \tilde{Q}_1 & \longrightarrow & \tilde{Q}_0 \end{array}$$

$i(\beta_0)$ :

$$\begin{array}{ccccccc} & \tilde{Q}_3 & \longrightarrow & \tilde{Q}_2 & \longrightarrow & \tilde{Q}_1 \\ \cdots & \downarrow & & \downarrow & & \downarrow \\ & 0 & & 0 & & e_{3n+3, 1} \\ & \tilde{Q}_2 & \longrightarrow & \tilde{Q}_1 & \longrightarrow & \tilde{Q}_0 \end{array}$$

$i(\gamma_0 \cdots \gamma_n)$  (see Lemma IV.3.6):

$$\begin{array}{ccccccc} & \tilde{Q}_{n+3} & \longrightarrow & \tilde{Q}_{n+2} & \longrightarrow & \tilde{Q}_{n+1} \\ \cdots & \downarrow & & \downarrow & & \downarrow \\ & 0 & & (-1)^{n+1} a_0 e_{k, 3n+3} & & e_{j+1, 1} \\ & \tilde{Q}_2 & \longrightarrow & \tilde{Q}_1 & \longrightarrow & \tilde{Q}_0 \end{array}$$

We henceforth suppress the inclusion map  $i$ . From the definitions, it is easy to check that  $\lambda_2(\beta_1 \alpha_0 \otimes \beta_0) = 0$  and  $\lambda_2(\beta_\ell \alpha_{\ell-1} \otimes \beta_{\ell-1} \alpha_{\ell-2}) = 0$  for  $2 \leq \ell \leq n$ . It is also easy to see that  $\lambda_2(\beta_0 \otimes \gamma_0 \cdots \gamma_n) = (-1)^{n+1} a_0 e_{k, 1}$ .

Next, we define values of the homotopy  $G$  on certain elements. It is important to observe that the elements on which we define  $G$  are linearly independent coboundaries in  $U$ . Each morphism below is in a different graded component of  $U$ , so linear independence is clear. To check that  $G$  is well-defined, it suffices to observe that  $\partial G = 1$  on the element in question. Recall that we define

$$\lambda_\ell = \lambda_2 \sum_{\substack{s+t=\ell \\ s,t \geq 1}} (-1)^{s+1} (G\lambda_s \otimes G\lambda_t).$$

First, let  $G\lambda_2(\beta_0 \otimes \gamma_0 \cdots \gamma_n) =$

$$\begin{array}{ccccc} \tilde{Q}_{n+4} & \longrightarrow & \tilde{Q}_{n+3} & \longrightarrow & \tilde{Q}_{n+2} \\ \cdots & & \downarrow (-1)^{n+1}e_{k,1} & & \downarrow (-1)^{n+1}e_{k,1} \\ & & \tilde{Q}_3 & \longrightarrow & \tilde{Q}_2 & \longrightarrow & \tilde{Q}_1 \\ & & \downarrow a_1 e_{j,2} & & \downarrow +e_{j+1,1} & & \downarrow (-1)^{n+1}e_{k,1} \end{array}$$

Next, since  $\lambda_2(\beta_1\alpha_0 \otimes \beta_0) = 0$ , we have

$$\lambda_3(\beta_1\alpha_0 \otimes \beta_0 \otimes \gamma_0 \cdots \gamma_n) = -\lambda_2(\beta_1\alpha_0 \otimes G\lambda_2(\beta_0 \otimes \gamma_0 \cdots \gamma_n)).$$

We choose  $G\lambda_3(\beta_1\alpha_0 \otimes \beta_0 \otimes \gamma_0 \cdots \gamma_n) =$

$$\begin{array}{ccccc} \tilde{Q}_{n+5} & \longrightarrow & \tilde{Q}_{n+4} & \longrightarrow & \tilde{Q}_{n+3} \\ \cdots & & \downarrow -e_{j,2} & & \downarrow (-1)^{n+1}a_2 e_{k-1,4} & & \downarrow -e_{j,2} \\ & & \tilde{Q}_3 & \longrightarrow & \tilde{Q}_2 & \longrightarrow & \tilde{Q}_1 \end{array}$$

For  $2 \leq \ell \leq n$ , since  $\lambda_2(\beta_\ell\alpha_{\ell-1} \otimes \beta_{\ell-1}\alpha_{\ell-2}) = 0$ , we have by induction,

$$\begin{aligned} \lambda_{\ell+2}(\beta_\ell\alpha_{\ell-1} \otimes \cdots \otimes \beta_1\alpha_0 \otimes \beta_0 \otimes \gamma_0 \cdots \gamma_n) = \\ -\lambda_2(\beta_\ell\alpha_{\ell-1} \otimes G\lambda_{\ell+1}(\beta_{\ell-1}\alpha_{\ell-2} \otimes \cdots \otimes \beta_1\alpha_0 \otimes \beta_0 \otimes \gamma_0 \cdots \gamma_n)) \end{aligned}$$

where  $G\lambda_{\ell+2}(\beta_\ell\alpha_{\ell-1} \otimes \cdots \otimes \beta_1\alpha_0 \otimes \beta_0 \otimes \gamma_0 \cdots \gamma_n)$  is given by the following diagram.

$$\begin{array}{ccccc}
\tilde{Q}_{n+\ell+4} & \longrightarrow & \tilde{Q}_{n+\ell+3} & \longrightarrow & \tilde{Q}_{n+\ell+2} \\
\cdots & & \downarrow X(\ell)e_{x(\ell),\ell+1} & & \downarrow Y(\ell)a_{\ell+1}e_{y(\ell),2(\ell+1)} & & \downarrow X(\ell)e_{x(\ell),\ell+1} \\
\tilde{Q}_3 & \longrightarrow & \tilde{Q}_2 & \longrightarrow & \tilde{Q}_1
\end{array}$$

The matrix indices are  $x(\ell) = \frac{1}{2}(k+j+(-1)^\ell(k-j)) - \left\lfloor \frac{\ell}{2} \right\rfloor$  and  $y(\ell) = x(\ell+1)$ , and the signs, which repeat with period 4, are determined by  $Y(\ell) = (-1)^{\left\lfloor \frac{2\ell+1+(-1)^\ell}{4} \right\rfloor}$  and  $X(\ell) = Y(\ell+1)$ .

**Theorem IV.5.1.** *The algebra  $E(B)$  admits a canonical  $A_\infty$ -structure for which  $m_{j+3}$  is nonzero for all  $0 \leq j \leq n$ .*

*Proof.* By Proposition IV.4.2,  $m_{n+3} = p\lambda_{n+3}i$ . We will show that

$$m_{n+3}(\alpha_n \otimes \beta_n \alpha_{n-1} \otimes \cdots \otimes \beta_1 \alpha_0 \otimes \beta_0 \otimes \gamma_0 \cdots \gamma_n) \neq 0.$$

Since  $pG = 0$ , we have

$$\begin{aligned}
p\lambda_{n+3} &= p\lambda_2 \sum_{\substack{s+t=n+3 \\ s,t \geq 1}} (-1)^{s+1} (G\lambda_s \otimes G\lambda_t) \\
&= p\lambda_2 (G\lambda_1 \otimes G\lambda_{n+2}) \\
&= -p\lambda_2 (1 \otimes G\lambda_{n+2}).
\end{aligned}$$

By the definitions of  $G$  above, we see that

$$\begin{aligned}
&-p\lambda_2(\alpha_n \otimes G\lambda_{n+2}(\beta_n \alpha_{n-1} \otimes \cdots \otimes \beta_1 \alpha_0 \otimes \beta_0 \otimes \gamma_0 \cdots \gamma_n)) \\
&= (-1)^{X(n)+1} \mu_{2n+2}^{x(n)}
\end{aligned}$$

where  $\mu_{2n+2}^{x(n)}$  is a graded dual basis element in  $E^{2n+2}(A)$ . Hence

$$m_{n+3}(\alpha_n \otimes \beta_n \alpha_{n-1} \otimes \cdots \otimes \beta_1 \alpha_0 \otimes \beta_0 \otimes \gamma_0 \cdots \gamma_n) \neq 0.$$

By choosing a cocycle  $i(\alpha_j)$ , it is straightforward to check that with the above definitions,  $m_{j+3}(\alpha_j \otimes \beta_j \alpha_{j-1} \otimes \cdots \otimes \beta_1 \alpha_0 \otimes \beta_0 \otimes \gamma_0 \cdots \gamma_n) \neq 0$  for all  $0 \leq j \leq n$ .

□

## IV.6 Detecting the $\mathcal{K}_2$ Condition

In this section we present two monomial  $k$ -algebras whose Yoneda algebras admit very similar canonical  $A_\infty$ -structures. Only one of the algebras is  $\mathcal{K}_2$ . These examples illustrate that the  $\mathcal{K}_2$  property is not detected by any obvious vanishing patterns among higher multiplications.

Let  $V$  be the  $k$ -vector space spanned by  $\{x, y, z, w\}$ . Let  $W^1 \subset T(V)$  be the subspace on  $k$ -basis  $\{y^2zx, zx^2, y^2w\}$ . Let  $W^2 \subset T(V)$  be the  $k$ -linear subspace on basis  $\{y^2z, zx^2, y^2w^2\}$ . Let  $A^1 = T(V)/\langle W^1 \rangle$  and  $A^2 = T(V)/\langle W^2 \rangle$ . The algebra  $A^2$  was studied in [7].

In Section 5 of [7], the authors give an algorithm for producing a minimal projective resolution of the trivial module for a monomial  $k$ -algebra. Applying the algorithm, we obtain minimal projective resolutions of  ${}_{A^1}k$  and  ${}_{A^2}k$  of the form

$$0 \rightarrow A^t(-5) \xrightarrow{M_3^t} A^t(-3, -3, -4) \xrightarrow{M_2^t} A^t(-1)^{\oplus 4} \xrightarrow{M_1^t} A^t \rightarrow k \rightarrow 0$$

for  $t \in \{1, 2\}$  where

$$M_3^1 = M_3^2 = \begin{pmatrix} 0 & y^2 & 0 \end{pmatrix}$$

$$M_2^1 = \begin{pmatrix} 0 & 0 & 0 & y^2 \\ zx & 0 & 0 & 0 \\ y^2z & 0 & 0 & 0 \end{pmatrix} \quad M_2^2 = \begin{pmatrix} 0 & 0 & y^2 & 0 \\ zx & 0 & 0 & 0 \\ 0 & 0 & 0 & y^2w \end{pmatrix}$$

$$M_1^1 = M_1^2 = \begin{pmatrix} x & y & z & w \end{pmatrix}^T.$$

Theorem IV.3.9 shows that  $A^1$  is a  $\mathcal{K}_2$  algebra and  $A^2$  is not, because  $y^2zx$  is an essential relation in  $A^1$  but not in  $A^2$ . We note that this implies all Yoneda products on  $E(A^2)$  are zero.

We make the identifications  $E^1(A^1) = V^* = E^1(A^2)$ ,  $E^2(A^1) = (W^1)^*$ , and  $E^2(A^2) = (W^2)^*$ . We choose  $k$ -bases for  $E(A^1)$  and  $E(A^2)$  as follows. Let  $X, Y, Z$ , and  $W$  denote the  $k$ -linear duals of  $x, y, z$ , and  $w$  respectively. Let  $R_1^1, R_2^1$ , and  $R_3^1$  be the  $k$ -linear duals of the

tensors  $y^2w, zx^2, y^2zx \in W^1$ , respectively. Let  $R_1^2, R_2^2$ , and  $R_3^2$  be the  $k$ -linear duals of the tensors  $y^2z, zx^2, y^2w^2 \in W^2$ , respectively. For  $t \in \{1, 2\}$ , let  $\alpha^t$  be a nonzero vector in  $E^{3,5}(B^t)$ . We denote the  $A_\infty$ -structure map on  $E(A^t)^{\otimes i}$  by  $m_i^t$ .

A standard calculation in the Yoneda algebra  $E(A^1)$  determines the Yoneda product  $m_2^1(R_3^1 \otimes X) = c\alpha^1$  for some  $c \in k^*$ . It is straightforward to check that Yoneda products of all other pairs of our specified basis vectors are zero. We note that, for degree reasons, the only higher multiplications on  $E = E(A^t)$  which could be nonzero are the following.

$$\begin{aligned} m_4^t &: (E^1)^{\otimes 4} \rightarrow E^{2,4} & m_3^t &: (E^1)^{\otimes 3} \rightarrow E^{2,3} \\ m_3^t &: E^1 \otimes E^1 \otimes E^{2,3} \rightarrow E^{3,5} & m_3^t &: E^1 \otimes E^{2,3} \otimes E^1 \rightarrow E^{3,5} \\ m_3^t &: E^{2,3} \otimes E^1 \otimes E^1 \rightarrow E^{3,5} \end{aligned}$$

We recall from the introduction that the  $A_\infty$ -structure  $\{m_i\}$  on a graded  $k$ -vector space  $E$  determines a  $k$ -linear map  $\oplus m_i : T(E)_+ \rightarrow E$ .

**Proposition IV.6.1.** *There exists a canonical  $A_\infty$ -structure on  $E(A^1)$  such that the map  $\oplus m_i^1$  vanishes on all monomials of  $T(E(A^1))_+$  except*

$$\{YYZX, ZXX, YYW, YYR_2^1, R_3^1X\}.$$

*There exists a canonical  $A_\infty$ -structure on  $E(A^2)$  such that the map  $\oplus m_i^2$  vanishes on all monomials of  $T(E(A^2))_+$  except*

$$\{YYWW, ZXX, YYZ, YYR_2^2, R_1^2XX\}.$$

We remark that these canonical structures are not the only canonical  $A_\infty$ -structures on  $E(A^1)$  and  $E(A^2)$ . However, it can be shown that the map  $\oplus m_i^t$  associated to any other canonical structure on  $E(A^t)$  is non-vanishing on the vectors listed in the proposition.

It is well known (see [20], [19]) that there is a canonical  $A_\infty$ -structure on the Yoneda algebra of a graded algebra  $A = T(V)/I$  such that the restriction of  $m_n$  to  $E^1(A)^{\otimes n}$  is dual to the natural inclusion of degree  $n$  essential relations  $(I/I)_n \hookrightarrow V^{\otimes n}$ . If we choose such an  $A_\infty$ -structure for  $E(A^1)$ , the Stasheff identity **SI**(5) determines the remaining structure on  $E(A^1)$ , giving the first part of the proposition. We prove the proposition for  $E(A^2)$ .

*Proof.* Let  $Q_\bullet$  be the minimal projective resolution of  ${}_{A^2}k$  given above, with  $Q_0 = A^2$ . Let  $e_{i,j}$  denote the  $(i,j)$  matrix unit. We define maps  $i$  and  $G$  as in Section 4. When defining  $G$ , we suppress the inclusion maps  $i$ . We recall that  $\lambda_2$  is multiplication in  $\text{End}_{A^2}(Q_\bullet)$  and  $\lambda_3 = \lambda_2(G\lambda_2 \otimes 1 - 1 \otimes G\lambda_2)$ . We define

$$\begin{array}{ccc}
\begin{array}{c} Q_2 \longrightarrow Q_1 \\ \downarrow -ze_{2,1} \quad \downarrow e_{1,1} \\ Q_1 \longrightarrow Q_0 \end{array} & i(X) = & \begin{array}{c} Q_1 \\ \downarrow e_{2,1} \\ Q_0 \end{array} & i(Y) = & \begin{array}{c} Q_2 \longrightarrow Q_1 \\ \downarrow -ye_{1,2} \quad \downarrow e_{3,1} \\ Q_1 \longrightarrow Q_0 \end{array} & i(Z) =
\end{array}$$

$$\begin{array}{ccc}
\begin{array}{c} Q_2 \longrightarrow Q_1 \\ \downarrow -y^2e_{3,4} \quad \downarrow e_{4,1} \\ Q_1 \longrightarrow Q_0 \end{array} & i(W) = & \begin{array}{c} Q_2 \\ \downarrow e_{1,1} \\ Q_0 \end{array} & i(R_1^2) = & \begin{array}{c} Q_3 \longrightarrow Q_2 \\ \downarrow ye_{1,2} \quad \downarrow e_{2,1} \\ Q_1 \longrightarrow Q_0 \end{array} & i(R_2^2) =
\end{array}$$

$$\begin{array}{ccc}
\begin{array}{c} Q_2 \\ \downarrow e_{3,1} \\ Q_0 \end{array} & i(R_3^2) = & \begin{array}{c} Q_3 \longrightarrow Q_2 \\ \downarrow e_{1,1} \quad \downarrow -e_{2,3} \\ Q_2 \longrightarrow Q_1 \end{array} & G\lambda_2(XX) = & \begin{array}{c} Q_2 \\ \downarrow -e_{1,2} \\ Q_1 \end{array} & G\lambda_2(YZ) =
\end{array}$$

$$\begin{array}{ccc}
\begin{array}{c} Q_3 \\ \downarrow e_{1,2} \\ Q_1 \end{array} & G\lambda_2(YR_2^2) = & \begin{array}{c} Q_2 \\ \downarrow -ye_{3,2} \\ Q_1 \end{array} & G\lambda_2(WW) = & \begin{array}{c} Q_2 \\ \downarrow e_{3,2} \\ Q_1 \end{array} & G\lambda_3(YWW) =
\end{array}$$

We recall that  $m_i = p\lambda_i$ . With the definitions above, it is straightforward to check that  $m_3(ZXX)$ ,  $m_3(YYZ)$ ,  $m_3(YR_2^2)$  and  $m_3(R_1^2XX)$  are nonzero. Recalling that  $p\lambda_4(YYWW) = -p\lambda_2(YG\lambda_3(YWW))$ , we see that  $m_4(YYWW)$  is nonzero.

It remains to check that all other higher multiplications are zero. This is easily done by inspection.

□

## CHAPTER V

### $A_\infty$ -ALGEBRA STRUCTURES ASSOCIATED WITH KOSZUL ALGEBRAS

In this chapter we prove that if the Yoneda algebra  $E(A)$  of a graded  $k$ -algebra of the form  $A = T(V)/I$  is given the canonical  $A_\infty$ -algebra structure described in Chapter IV, then  $E(A)$  is  $A_\infty$ -generated by  $E^1(A)$ . The characterization of Koszul algebras as those whose Yoneda algebra admits only trivial  $A$ -infinity structures follows immediately. These results are known, but we were unable to find a proof in the literature. Additionally, we feel that our approach nicely demonstrates the computational advantages of the construction described in Section IV.4. See that section for the relevant definitions. We prove the following.

**Theorem V.0.2.** *For every  $n \geq 1$ ,  $E^{n+1}(A) \subset \sum_i m_{i+1}(E^1(A) \otimes \cdots \otimes E^1(A) \otimes E^n(A))$ .*

The following useful fact motivates our approach to proving the theorem. We omit the straightforward proof.

**Lemma V.0.3.** *If  $a_1, \dots, a_t \in U^1$  and  $a_{t+1} \in U^n$ , then  $g = \lambda_2(a_1 \otimes G\lambda_t(a_2 \otimes \cdots \otimes a_{t+1})) \in U^{n+1}$  and  $g_{n+1} = \pm \lambda_2(a_1 \otimes G\lambda_2(a_2 \otimes G\lambda_{t-1}(a_3 \otimes \cdots \otimes a_{t+1})))_{n+1}$ . Therefore, if  $a_1, \dots, a_{t+1}$  represent classes in  $E(A)$ ,*

$$\begin{aligned} & m_{t+1}([a_1] \otimes \cdots \otimes [a_{t+1}]) \\ &= \pm p \lambda_2(a_1 \otimes G\lambda_2(a_2 \otimes G\lambda_2(\cdots \otimes a_{t-1} \otimes G\lambda_2(a_t \otimes a_{t+1}) \cdots))) \end{aligned}$$

Before proving the Theorem V.0.2, we fix some notation. Let  $V$  be a finite dimensional  $k$ -vector space on basis  $X = \{x_1, \dots, x_g\}$ . Let  $T(V)$  be the tensor algebra on  $V$ , graded by tensor degree. Let  $I \subset T(V)_{\geq 2}$  be a graded ideal and let  $A = T(V)/I$ . Let  $\pi : T(V) \rightarrow A$  be the natural quotient map.



Let  $(Q_\bullet, \partial_\bullet)$  be a minimal graded projective resolution of  ${}_A k$ . As  ${}_A k$  is bounded below, we assume the same is true of each  $Q_i$ . For each  $i \geq 0$ , let  $\{e_i^j\}$  be a homogeneous  $A$ -basis for  $Q_i$ . Let  $M_i$  be the matrix of  $\partial_i$  with respect to the fixed bases. We may further assume that bases have been chosen so that  $M_1 = (x_1 \cdots x_n)^T$ . We note that the entries of  $M_i$  are homogeneous of positive degree and that  $M_i$  is row-finite. Let  $\hat{M}_i$  be a lift of  $M_i$  to a matrix with homogeneous entries in  $T(V)_+$ . We require that 0 lift to 0 so that  $\hat{M}_i$  remains row-finite.

As  $Q_\bullet$  is minimal,  $E^n(A) = \text{Hom}_A(Q_n, k)$  for all  $n$ . Let  $\epsilon_n^j$  be the dual basis vector to  $e_n^j$ . For each  $\epsilon_n^j$  we define the matrix  $P(n, j)$  with homogeneous entries in  $A$  by the commutative diagram

$$\begin{array}{ccccc}
 Q_{n+1} & \xrightarrow{M_{n+1}} & Q_n & & \\
 \downarrow P(n, j) & & \downarrow e_j & \searrow \epsilon_n^j & \\
 Q_1 & \xrightarrow{M_1} & Q_0 & \longrightarrow & k
 \end{array}$$

where  $e_j$  is the matrix of the standard projection onto the  $j$ -th coordinate. Since  $M_1$  is linear, there exists a unique homogeneous lift  $\widehat{P(n, j)}$  of  $P(n, j)$  to a matrix with entries in  $T(V)$  such that  $\widehat{P(n, j)}\hat{M}_1 = \hat{M}_{n+1}e_j$ .

We define  $i(\epsilon_n^j)_n$  and  $i(\epsilon_n^j)_{n+1}$  to be the  $A$ -module homomorphisms given respectively by right multiplication by  $e_j$  and  $P(n, j)$ . The remaining component maps of  $i(\epsilon_n^j)$  may be defined by the graded projectivity of the resolution as described in Chapter IV. We will not use any higher components in our proof, so their explicit definitions do not concern us. Extending the definition of  $i$  by  $k$ -linearity, we have defined  $i : E(A) \rightarrow U$ . Henceforth, we suppress  $i$ .

If  $M$  is a matrix with homogeneous entries in  $A$ , let  $s(M)$  be the image of  $M$  modulo  $A_+$ . We think of  $s(M)$  as the scalar part of  $M$  and, since the entries of  $M$  are homogeneous, we canonically identify  $s(M)$  with a scalar matrix over  $A$ . We define  $r(M) = M - s(M)$  and note that  $\widehat{r(M)} = r(\hat{M})$ .

We now describe the homotopy  $G$ . Recall that by our construction,  $G$  is nonzero only on coboundaries and that  $Z^n = B^n \oplus H^n$  where  $H^n = i(E^n(A))$ . Thus  $r(P(n, j))e_i$  is the  $n+1$  component of the ‘‘boundary part’’ of  $\lambda_2(\epsilon_1^i \otimes \epsilon_n^j)$ .

We define  $G(n, j, i)$  with homogeneous entries in  $A$  by the commutative diagram

$$\begin{array}{ccc}
Q_{n+1} & \xrightarrow{r(P(n, j))} & Q_1 \\
G(n, j, i) \downarrow & & \downarrow e_i \\
Q_1 & \xrightarrow{M_1} & Q_0
\end{array}$$

where  $G(n, j, i)$  is the matrix of  $G\lambda_2(\epsilon_1^i \otimes \epsilon_n^j)$ . As above, since  $M_1$  is linear, there exists a unique homogeneous lift  $\widehat{G(n, j, i)}$  of  $G(n, j, i)$  to a matrix with entries in  $T(V)$  such that  $\widehat{G(n, j, i)}\widehat{M}_1 = r(\widehat{P(n, j)})e_i$ .

We have the following analog of Lemma 4.3 of [7].

**Lemma V.0.4.** *Define scalars  $\mu_l$  by  $m_3(\epsilon_1^{i_1} \otimes \epsilon_1^{i_2} \otimes \epsilon_n^j) = \sum \mu_l \epsilon_{n+1}^l$ . Then  $\mu_l \neq 0$  if and only if the  $(l, i_1)$ -entry of  $G(n, j, i_2)$  is a unit. Furthermore, there exist  $i_1, i_2$ , and  $j$  such that  $\mu_l \neq 0$  if and only if some entry in row  $l$  of  $\widehat{M}_{n+1}$  is (nonzero) quadratic.*

*Proof.* By Lemma V.0.3,  $m_3(\epsilon_1^{i_1} \otimes \epsilon_1^{i_2} \otimes \epsilon_n^j) = \pm p\lambda_2(\epsilon_1^{i_1} \otimes G\lambda_2(\epsilon_1^{i_2} \otimes \epsilon_n^j))$ . The statements follow from an easy diagram chase. □

For all  $t > 2$ , define matrices  $G(n, j, i_t, \dots, i_2)$  inductively by the above procedure, but replace  $r(P(n, j))$  with  $r(G(n, j, i_t, \dots, i_3))$ . We state the general version of Lemma V.0.4 and omit the proof.

**Lemma V.0.5.** *Define scalars  $\mu_l$  by  $m_{t+1}(\epsilon_1^{i_1} \otimes \dots \otimes \epsilon_1^{i_t} \otimes \epsilon_n^j) = \sum \mu_l \epsilon_{n+1}^l$ . Then  $\mu_l \neq 0$  if and only if the  $(l, i_1)$ -entry of  $G(n, j, i_t, \dots, i_2)$  is a unit. Furthermore, there exist  $i_1, \dots, i_t$ , and  $j$  such that  $\mu_l \neq 0$  if and only if some entry in row  $l$  of  $\widehat{M}_{n+1}$  is (nonzero) of degree  $t$ .*

Now the proof of Theorem V.0.2 is straightforward.

*Proof of Theorem V.0.2.* Let  $W \subset E^n(A)$  be the subspace generated by  $\sum_i m_{i+1}(E^1(A) \otimes \dots \otimes E^1(A) \otimes E^n(A))$ . Choose a homogeneous  $k$ -basis for  $W$  and extend to a basis for  $E^n(A)$ . This corresponds to a homogeneous basis for  $Q_n$ . If we assume that the matrices in the resolution  $Q_\bullet$  are calculated with respect to this basis, Lemma V.0.5 implies that the rows of  $M_{n+1}$  corresponding to basis elements not in  $W$  consist of all zeroes. Thus  $W = E^n(A)$ . □

**Corollary V.0.6.** *A graded  $k$ -algebra  $A$  is a Koszul algebra if and only if any  $A_\infty$ -algebra structure on  $E(A)$  is trivial.*

*Proof.* If  $A$  is Koszul, then  $E^n(A) = E^{n,n}(A)$  for all  $n \geq 0$ . For  $i > 2$ , the higher multiplication  $m_i$  lowers cohomology degree by  $2 - i$ , but preserves internal degree. Thus  $m_i = 0$  by degree considerations for  $i > 2$ .

Conversely, if all higher multiplications vanish in any  $A_\infty$ -algebra structure on  $E(A)$ , then in particular they all vanish in the canonical structure described in Chapter IV. Thus by Theorem V.0.2,  $E^n(A)$  is generated algebraically by  $E^1(A)$  using only  $m_2$ . It follows that  $E^n(A) = E^{n,n}(A)$  and  $A$  is Koszul.

□

## REFERENCES CITED

- [1] Jürgen Backelin and Ralf Fröberg, *Koszul algebras, Veronese subrings and rings with linear resolutions*, Rev. Roumaine Math. Pures Appl. **30** (1985), no. 2, 85–97. MR MR789425 (87c:16002)
- [2] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–527. MR MR1322847 (96k:17010)
- [3] Roland Berger, *Koszulity for nonquadratic algebras*, J. Algebra **239** (2001), no. 2, 705–734. MR MR1832913 (2002d:16034)
- [4] George M. Bergman, *The diamond lemma for ring theory*, Adv. in Math. **29** (1978), no. 2, 178–218. MR MR506890 (81b:16001)
- [5] I. N. Bernšteĭn, I. M. Gelfand, and S. I. Gelfand, *Algebraic vector bundles on  $\mathbf{P}^n$  and problems of linear algebra*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, 66–67. MR 509387 (80c:14010a)
- [6] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., 1956. MR 0077480 (17,1040e)
- [7] Thomas Cassidy and Brad Shelton, *Generalizing the notion of Koszul algebra*, Math. Z. **260** (2008), no. 1, 93–114. MR MR2413345 (2009e:16047)
- [8] Andrew Conner and Pete Goetz,  *$A_\infty$ -algebra structures associated to  $\mathcal{K}_2$ -algebras*, Journal of Algebra (2011).
- [9] Art M. Duval, *Algebraic shifting and sequentially Cohen-Macaulay simplicial complexes*, Electron. J. Combin. **3** (1996), no. 1, Research Paper 21, approx. 14 pp. (electronic). MR 1399398 (98b:06009)
- [10] John A. Eagon and Victor Reiner, *Resolutions of Stanley-Reisner rings and Alexander duality*, J. Pure Appl. Algebra **130** (1998), no. 3, 265–275. MR 1633767 (99h:13017)
- [11] Ralph Fröberg, *Determination of a class of Poincaré series*, Math. Scand. **37** (1975), no. 1, 29–39. MR 0404254 (53 #8057)
- [12] Edward L. Green and E.N. Marcos,  *$d$ -Koszul Algebras,  $2$ -d Determined Algebras and  $2$ -d Koszul Algebras*, arXiv:0812.3408v2.
- [13] V. K. A. M. Gugenheim and L. A. Lambe, *Perturbation theory in differential homological algebra. I*, Illinois J. Math. **33** (1989), no. 4, 566–582. MR MR1007895 (91e:55023)
- [14] V. K. A. M. Gugenheim and J. Peter May, *On the theory and applications of differential torsion products*, American Mathematical Society, Providence, R.I., 1974, Memoirs of the American Mathematical Society, No. 142. MR MR0394720 (52 #15519)
- [15] Ji-Wei He and Di-Ming Lu, *Higher Koszul algebras and  $A$ -infinity algebras*, J. Algebra **293** (2005), no. 2, 335–362. MR MR2172343 (2006m:16030)

- [16] Jürgen Herzog and Takayuki Hibi, *Componentwise linear ideals*, Nagoya Math. J. **153** (1999), 141–153. MR 1684555 (2000i:13019)
- [17] T. V. Kadeišvili, *On the theory of homology of fiber spaces*, Uspekhi Mat. Nauk **35** (1980), no. 3(213), 183–188, International Topology Conference (Moscow State Univ., Moscow, 1979). MR MR580645 (82a:57041a)
- [18] Bernhard Keller, *Introduction to  $A$ -infinity algebras and modules*, Homology Homotopy Appl. **3** (2001), no. 1, 1–35. MR MR1854636 (2004a:18008a)
- [19] ———,  *$A$ -infinity algebras in representation theory*, Representations of algebra. Vol. I, II, Beijing Norm. Univ. Press, Beijing, 2002, pp. 74–86. MR MR2067371 (2005b:16021)
- [20] D.-M. Lu, J. H. Palmieri, Q.-S. Wu, and J. J. Zhang, *Regular algebras of dimension 4 and their  $A_\infty$ -Ext-algebras*, Duke Math. J. **137** (2007), no. 3, 537–584. MR MR2309153 (2008d:16022)
- [21] ———,  *$A$ -infinity structure on Ext-algebras*, J. Pure Appl. Algebra **213** (2009), no. 11, 2017–2037. MR MR2533303
- [22] S. A. Merkulov, *Strong homotopy algebras of a Kähler manifold*, Internat. Math. Res. Notices (1999), no. 3, 153–164. MR MR1672242 (2000h:32026)
- [23] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005. MR 2110098 (2006d:13001)
- [24] Christopher Phan, *Generalized Koszul properties for augmented algebras*, J. Algebra **321** (2009), no. 5, 1522–1537. MR MR2494406 (2010a:16017)
- [25] ———, *Koszul and generalized Koszul properties for noncommutative graded algebras*, Ph.D. Thesis, University of Oregon (2009).
- [26] A. Polishchuk, *Koszul configurations of points in projective spaces*, J. Algebra **298** (2006), no. 1, 273–283. MR MR2215128
- [27] Alexander Polishchuk and Leonid Positselski, *Quadratic algebras*, University Lecture Series, vol. 37, American Mathematical Society, Providence, RI, 2005. MR MR2177131
- [28] Leonid Positselski and Alexander Vishik, *Koszul duality and Galois cohomology*, Math. Res. Lett. **2** (1995), no. 6, 771–781. MR 1362968 (97b:12008)
- [29] Stewart B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39–60. MR MR0265437 (42 #346)
- [30] Victor Reiner and Dumitru Ioan Stamate, *Koszul incidence algebras, affine semigroups, and Stanley-Reisner ideals*, Adv. Math. **224** (2010), no. 6, 2312–2345. MR 2652208
- [31] Gerald Allen Reisner, *Cohen-Macaulay quotients of polynomial rings*, Advances in Math. **21** (1976), no. 1, 30–49. MR 0407036 (53 #10819)
- [32] Brad Shelton and Sergey Yuzvinsky, *Koszul algebras from graphs and hyperplane arrangements*, J. London Math. Soc. (2) **56** (1997), no. 3, 477–490. MR 1610447 (99c:16044)
- [33] Richard P. Stanley, *Combinatorics and commutative algebra*, second ed., Progress in Mathematics, vol. 41, Birkhäuser Boston Inc., Boston, MA, 1996. MR 1453579 (98h:05001)
- [34] James Stasheff,  *$H$ -spaces from a homotopy point of view*, Lecture Notes in Mathematics, Vol. 161, Springer-Verlag, Berlin, 1970. MR MR0270372 (42 #5261)

- [35] James Dillon Stasheff, *Homotopy associativity of H-spaces. I, II*, Trans. Amer. Math. Soc. **108** (1963), 275-292; *ibid.* **108** (1963), 293-312. MR MR0158400 (28 #1623)
- [36] John Tate and Michel van den Bergh, *Homological properties of Sklyanin algebras*, Invent. Math. **124** (1996), no. 1-3, 619-647. MR 1369430 (98c:16057)