UNIMODAL LÉVY PROCESSES ON BOUNDED LIPSCHITZ SETS

by

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DISSERTATION ABSTRACT

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We give asymptotics near the boundary for the distribution of the first exit time of the isotropic α-stable Lévy process on bounded Lipschitz sets in $\mathbb{R}^d$. These asymptotics bear some relation to the existence of limits in the Yaglom sense of α-stable processes. Our approach relies on the uniform integrability of the ratio of Green functions on bounded Lipschitz sets.

We use bounds for the heat remainder to give the first two terms in the small time asymptotic expansion of the trace of the heat kernel of unimodal Lévy processes satisfying some weak scaling conditions on bounded Lipschitz domains.

This dissertation includes previously unpublished co-authored material.

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“Do mo athair agus mo mháthair.”

(To my father and my mother.)

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CHAPTER I

INTRODUCTION

This dissertation explores two distinct properties of the heat kernel of unimodal Lévy processes on bounded Lipschitz sets: the limiting boundary behavior of the heat kernel for isotropic α-stable processes and the small time asymptotic behavior of the trace of the heat kernel for unimodal Lévy processes.

The first of these properties was work co-authored with Prof. Krzysztof Bogdan. We establish how the isotropic α-stable process, conditioned on its survival, acts near the boundary of its domain. This is a generalization of the work done by Bogdan et al. [19] for cones. Our proof relies on similar sharp estimates of the heat kernel on Lipschitz sets and on a formula expressing the survival probability in terms of a Green potential. A noteworthy difference in our approach is that we use the α-harmonicity of the Martin kernel on bounded Lipschitz sets to conclude the uniform integrability of the ratio of Green potentials, which in turn helps us establish the existence of the necessary integrals.

These asymptotics give us some insight into the inherent behavior of the Dirichlet heat kernel on the boundary. An additional implication of this result is the existence of a "Yaglom limit"-like probability measure on our set [33]. That is, the boundary behavior of the heat kernel gives us a measure which represents the probability distribution of the (naturally) rescaled process conditioned on non-extinction.

Such asymptotics are consistent with previous work, and can be regarded as part of the culmination of the study of the Dirichlet fractional Laplacian. This study started with boundary estimates and asymptotics of harmonic functions, these boundary estimates led to Green function estimates and asymptotics, which, in turn, gave the Martin representation of harmonic functions, and finally sharp estimates of the heat kernel.

The study of the Dirichlet fractional Laplacian in this way began with proofs of the boundary Harnack principle for the fractional Laplacian by Bogdan [7] and Song and Wu [52]. Sharp estimates for the Green function of the fractional laplacian were given by Jakubowsi [34]. Boundary asymptotics of ratios of harmonic functions were given by Bogdan et al. [18]. Sharp estimates of the Dirichlet heat kernel are given in [15]. As is the case here, results in this area for Lipschitz sets are often preceded and informed by the results for cones. Related background is given by DeBlassie [29], Kulczycki [40], Burdzy and Kulczycki [25], Méndez-Hernández [43], Bogdan and Jakubowski [17], Michalik [44], Kulczycki and Siudeja [42], and Bogdan and Grzywny [12]. For smooth domains see Kulczycki [41], Song and Chen [28], and Kim et al. [36, 39].

The second property discussed in this dissertation is the approximation of the trace of the heat kernel for unimodal Lévy processes on bounded Lipschitz domains. The trace of the heat kernel has been long studied in analysis and probability. Given a type of stochastic process and a domain, can we estimate or give asymptotics for the trace of that heat kernel? This is related to an older question in analysis [35]: What is the relationship between the spectrum of the Laplacian on a domain and the geometry of the domain? The latter question can, in part, be approached through the theory of stochastic processes using the trace of the heat kernel. Therefore a question about the relationship between the spectrum of the fractional Laplacian and the geometry of D can become a question about the relationship between the trace of the heat kernel and the geometry of D.

We give the first two terms in the small time asymptotic expansion of the trace of the heat kernel of unimodal Lévy processes on bounded Lipschitz domains. This approximation is a bound of the trace which is uniform in t for small values of t. The first component is given by the d-dimensional Lebesgue measure of the domain and the second is given by the $(d-1)$-dimensional Hausdorff measure of the boundary of the domain. Our result fits perfectly with the existing collection of estimates and asymptotics of traces.

The first term of our asymptotic expansion is near trivial to demonstrate. The second term is obtained using properties of the Lipschitz boundary to separate the boundary into good sets, of significant measure, and bad sets, of smaller measure. On both sets we use recent estimates for the heat remainder from [21]. On the good set we show, through a series of applications of the triangle inequality, that the heat remainder on our domain is comparable with the heat remainder on the half-plane, and hence the second term of our asymptotic expansion is the product of the Hausdorff measure of the boundary and a constant only depending on the upper-half space of $\mathbb{R}^d$.

One of the first studies of traces in this way was a two-term estimate for Brownian motion on R-smooth domains by van den Berg [54]. Asymptotics of the Brownian motion case were given for C^1 domains by Brossard and Carmona [23] and Lipschitz domains by Brown [24]. Estimates for isotropic α-stable processes on R-smooth domains were given by Bañuelos and Kulczycki [3], and asymptotics on Lipschitz were given by Bañuelos et al. [4]. Most recently two-term estimates for unimodal Lévy processes on R-smooth domains were given by Bogdan and Siudeja [21]. This leads exactly to our asymptotic expansion.

CHAPTER II

PRELIMINARIES

In this chapter we provide some necessary definitions, notations, and known results from the theory of stochastic processes which will help us in later chapters.

2.1. Definitions

Let $0 < \alpha < 2$, $2 \leq d$, and $\mathbb{R}^d$ be the real d-dimensional Euclidean space. Unless stated otherwise, we will assume that all our random variables are $\mathbb{R}^d$-valued. We summarize the basics of stochastic and Lévy processes in this section, for a more detailed discussion see [1, 47].

A **stochastic process**, $X(\omega) = (X_t(\omega))_{t\in T}$, is a collection of random variables defined on a common probability space, $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a sample space, $\mathcal{F}$ is a σ-algebra, $\mathbb{P}$ is a probability measure. The random variables, $X_t(\omega)$, indexed by some set T, all take values in the same mathematical space $\mathcal{S}$, which must be measurable with respect to some σ-algebra, Σ. Note: we will often suppress the notation and write "$X_t(\omega)$" as just "X_t".

We will be concerned with a specific type of stochastic process: processes whose increments are independent and stationary, otherwise known as a **Lévy process**. These processes can be viewed as the continuous-time analogue of random walks.

Definition 2.1.1. *A stochastic process $X = (X_t)_{t\geq 0}$ is said to be a* **Lévy process** *if it satisfies the following properties:*

1. $X_0 = 0$, *almost surely.*

2. *Independent increments: For any* $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, *the random variables* $X_{t_2} - X_{t_1}$, $X_{t_3} - X_{t_2}$, ..., $X_{t_n} - X_{t_{n-1}}$ *are independent of each other.*

3. *Stationary increments: For any* $s < t$, *the distribution of* $X_t - X_s$ *is equal to the distribution of* X_{t-s}.

4. *Continuity in probability: For any* $\epsilon > 0$ *and* $t \geq 0$ *it is true that*

$$\lim_{h \to 0} \mathbb{P}\left(|X_{t+h} - X_t| > \epsilon\right) = 0.$$

5. *CÀDLÀG: There exists* $\Omega_0 \in \mathcal{F}$ *with* $\mathbb{P}(\Omega_0) = 1$ *such that for every* $\omega \in \Omega_0$ $X_t(\omega)$ *is right-continuous in* $t \geq 0$ *and has left limits in* $t > 0$.

The primary tool in studying the distributions of Lévy processes is the characteristic function, or Fourier transform, of their distributions. Let Y be a random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $\mathbb{R}^d$ with probability law p_Y. The **characteristic function** of Y, $\phi_Y : \mathbb{R}^d \to \mathbb{C}$, is defined by

$$\phi_Y(\xi) = \mathbb{E}\left[e^{i\langle \xi, Y\rangle}\right] = \int_\Omega e^{i\langle \xi, Y(\omega)\rangle} \mathbb{P}(d\omega) = \int_{\mathbb{R}^d} e^{i\langle \xi, y\rangle} p_Y(dy), \tag{2.1}$$

for each $\xi \in \mathbb{R}^d$.

The "Independent increments" property of a Lévy processes, $X = (X_t)_{t \geq 0}$, allows us to conclude that the random variables X_t are infinitely divisible for each $t \geq 0$. That is, for all $n \in \mathbb{N}$ there exist a collection of independent and identically distributed random variables, $Y_1^{(n,t)}, \ldots, Y_n^{(n,t)}$ such that

$$\mathit{distribution}\left(X_t\right) = \mathit{distribution}\left(Y_1^{(n,t)} + \cdots + Y_n^{(n,t)}\right).$$

Because of this "infinite divisibility" property we can apply the Lévy-Khintchine formula to Lévy processes.

Theorem 2.1.1 (Lévy-Khintchine)**.** *Given a Lévy process, $X = (X_t)_{t\geq 0}$, there exists a vector $b \in \mathbb{R}^d$, a positive definite symmetric $d \times d$ matrix A, and a Borel measure ν on $\mathbb{R}\backslash\{0\}$ such that*

$$\phi_{X_t}(\xi) = exp\left[t \cdot \left\{i\langle b, \xi\rangle - \frac{1}{2}\langle \xi, A\xi\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle \xi, y\rangle} - 1 - i\langle \xi, y\rangle \mathbb{1}_{B_1(0)}\right) \nu(dy)\right\}\right], \tag{2.2}$$

for each $\xi \in \mathbb{R}^d$ and $t \geq 0$, where $B_1(0)$ is the ball of radius 1 centered at the origin, $\mathbb{1}$ is the indicator function, and ν satisfies $\int_{\mathbb{R}^d} \left(|y|^2 \wedge 1\right) \nu(dy) < \infty$.

The exponent in (2.2) is often called the **characteristic exponent** or **Lévy exponent** of X, denoted $\psi(x)$. The matrix A is called the **Gaussian covariance matrix** of X and ν is called the **Lévy measure** of X.

We call a measure **isotropic** if it is absolutely continuous on $\mathbb{R}^d\backslash\{0\}$ with respect to Lebesgue measure and it is invariant under linear isometries of $\mathbb{R}^d$. We call a measure **unimodal** if it is isotropic and its density function is radially non-increasing. We say that a Lévy process is **unimodal** if all its density functions are unimodal, see [11, 55]

We can simplify the characteristic exponent of unimodal Lévy processes to

$$\psi(\xi) = \sigma^2 |\xi|^2 + \int_{\mathbb{R}^d} (1 - \cos\langle \xi, y\rangle) \nu(dy), \tag{2.3}$$

where $\nu(dy) = \nu(y)dy = \nu(|y|) dy$ is a unimodal Lévy measure and $\sigma \geq 0$.

For the remainder of this dissertation we assume that all our stochastic processes are unimodal Lévy processes, and that all sets, measures, and functions are Borel.

Let $X = (X_t)_{t\geq 0}$ be a unimodal Lévy process. For $x \in \mathbb{R}^d$, we denote by $\mathbb{P}_x$ and $\mathbb{E}_x$ the probability and the expectation of a process starting from x. We will denote by

$$p_t(x,y) = p_t(x-y) \tag{2.4}$$

the transition density of X. That is, for $A \subset \mathbb{R}^d$, we have

$$\mathbb{P}_x\left(X_t \in A\right) = \int_A p_t(x,y)dy.$$

Let $D \subset \mathbb{R}^d$ be a nonempty open set, then the **first exit time** from D is

$$\tau_D = \inf\left\{t \geq 0 \ : \ X_t \notin D\right\}. \tag{2.5}$$

Note that if D is bounded, then $\tau_D < \infty$ almost surely. The transition density of the process killed upon exiting D (the **heat kernel**) is defined by

$$\mathbb{P}_x\left(X_t \in A,\ \tau_D > t\right) = \int_A p_t^D(x,y)dy. \tag{2.6}$$

The **heat remainder** of the process killed upon exiting D is defined by

$$r_t^D(x,y) = \mathbb{E}_x\left[p_{t-\tau_D}\left(X_{\tau_D}, y\right)\ ;\ \tau_D < t\right] = \int_{\{\tau_D<t\}} p_{t-\tau_D}\big(X_{\tau_D}, y\big)d\mathbb{P}_x \tag{2.7}$$

We can see that

$$p_t(x,y) = p_t^D(x,y) + r_t^D(x,y). \tag{2.8}$$

Note that p_t^D satisfies the **Chapman-Kolmogorov** identity:

$$\int p_s^D(x,y)p_t^D(y,z)dy = p_{t+s}^D(x,z), \qquad s,t>0, \quad x,z \in \mathbb{R}^d, \tag{2.9}$$

see [21, 9, 26]

Let $\eta \geq 0$. The **truncated Green function** of the set D is defined by

$$G_D^\eta(x,y) = \int_0^\eta p_t^D(x,y)dt. \tag{2.10}$$

We will use the term **Green function** and notation $G_D(x,y)$ to refer to the case when $\eta = \infty$. We have $G_D(x,x) = \infty$ if $x \in D$ and we set $G_D(x,x) = 0$ if $x \in D^c$. For integrable or nonnegative functions f, we define the **Green operator** to be

$$\left(G_D f\right)(x) = \int_D G_D(x,y)f(y)dy. \tag{2.11}$$

It is well-known that $G_D(x,y) > 0$ on D. Also $G_D(x,y) = G_D(y,x)$, for $x, y \in \mathbb{R}^d$, and $G_D(x,y) = 0$ is $x \in D^c$ or $y \in D^c$. We can define the **truncated Poisson kernel** of the set D in a similar way:

$$P_D^\eta(x,y) = \int_D G_D^\eta(x,z)\nu(z-y)dz. \tag{2.12}$$

The term **Poisson kernel** and notation $P_D(x,y)$ will refer to the case $\eta = \infty$.

We denote the semigroup on $L^2(D)$ of X killed upon exiting D by $\left\{P_t^D\right\}_{t\geq 0}$. That is, for $x \in D$, $t > 0$ and $f \in L^2(D)$ we define

$$\left(P_t^D f\right)(x) = \mathbb{E}_x\left[f\left(X_t\right) \ ; \ \tau_D > t\right] = \int_D p_t^D(x,y)f(y)dy. \tag{2.13}$$

The **infinitesimal generator** of the killed semigroup $\left\{P_t^D\right\}_{t\geq 0}$ is defined by

$$Lf = \lim_{t\to 0} t^{-1}\left(P_t^D f - f\right). \tag{2.14}$$

Given any unimodal Lévy process X the corresponding infinitesimal generator has a unique representation as an integro-differential operator:

$$(Lf)(x) = \sum_{i=1}^{d} b_i \frac{\partial f}{\partial x_i} + \sigma^2 \Delta f + \int_{\mathbb{R}^d} \left(f(x+y) - f(x) - \sum_{i=1}^{d} y_i \frac{\partial f}{\partial x_i} \mathbb{1}_D(y) \right) \nu(dy), \tag{2.15}$$

where $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the standard Laplace operator.

An open set $D \subset \mathbb{R}^d$ is called **Lipschitz** if we can find $R > 0$ and $\Lambda > 0$ such that for every $Q \in \partial D$ there exists a Lipschitz function $\phi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ with Lipschitz constant not greater than Λ and an orthonormal coordinate system CS_Q such that if $y = (y_1, \ldots, y_{d-1}, y_d)$ in CS_Q coordinates, then

$$D \cap B(Q,R) = \{y \ : \ y_d > \phi_Q(y_1, \ldots, y_{d-1})\} \cap B(Q,R),$$

where $B(Q,R) = \left\{z \in \mathbb{R}^d \ : \ |z - Q| < R\right\}$.

We recall the **Ikeda-Watanabe** formula from [21] for the joint distribution of X_{τ_D} and τ_D. For $x \in D$, $0 \leq t_1 \leq t_2$ and $A \subset \left(\overline{D}\right)^c$ we have

$$\mathbb{P}_x\left[X_{\tau_D} \in A,\ t_1 < \tau_D < t_2\right] = \int_D \int_{t_1}^{t_2} p_s^D(x,y)ds \int_A \nu(y-z)dzdy. \tag{2.16}$$

See also [2, Lemma 1], [20], [27, Appendix A], [30, VII.68], or [48, Theorem 2.4].

For the rest of this dissertation we assume that D is an open bounded nonempty Lipschitz set.

2.2. Stable Processes

The isotropic α-stable process is a commonly studied unimodal Lévy process. The process is determined by the jump measure with density function

$$\nu(y) = \frac{2^{\alpha}\Gamma\left((d+\alpha)/2\right)}{\pi^{d/2}\left|\Gamma\left(-\alpha/2\right)\right|}|y|^{-d-\alpha}, \quad y \in \mathbb{R}^d. \tag{2.17}$$

This process has characteristic function $\mathbb{E}_x e^{i\langle \xi, X_t - x\rangle} = e^{-t|\xi|^{\alpha}}$, $\xi \in \mathbb{R}^d$. The transition densities, $p_t(x)$, are smooth real-valued functions on $\mathbb{R}^d$ and satisfy the Fourier transform:

$$\int_{\mathbb{R}^d} p_t(x) e^{i\langle x, \xi\rangle} dx = e^{-t|\xi|^{\alpha}}, \quad \xi \in \mathbb{R}^d. \tag{2.18}$$

Further, the following scaling property is a consequence of (2.18):

$$p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha}x), \quad x \in \mathbb{R}^d,\ t > 0. \tag{2.19}$$

There exists a constant c such that

$$c^{-1}\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right) \leq p_t(x) \leq c\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right), \quad x \in \mathbb{R}^d,\ t > 0.$$

See [6, 13, 22] for the explicit constant. We write $f \approx g$ when the functions $f, g \geq 0$ are **comparable**, i.e. their ratio is uniformly bounded between two constants on the whole domain. Hence

$$p_t(x) \approx t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}, \quad x \in \mathbb{R}^d,\ t > 0. \tag{2.20}$$

Note that throughout this dissertation we use many different constants. The value of these constants is not usually of importance and the same specific constant is rarely required more than once. Hence the letters "c" or "C" are often used generically to refer to a constant, but almost never refer to the same constant more than once. We note that

$$0 \leq p_t^D(x,y) = p_t^D(y,x) \leq p_t(y-x).$$

Since D is Lipschitz, it satisfies the exterior cone condition. Therefore, by Blumenthal's 0-1 law: $\mathbb{P}_x(\tau_D = 0) = 0$ if $x \in D^c$, in particular $p_t^D(x,y) = 0$ when x or y are outside of D. We note that

$$\mathbb{P}_x(\tau_D > t) = \int_D p_t^D(x,y)dy. \tag{2.21}$$

Additionally, the following scaling property follows from (2.19)

$$p_t^D(x,y) = t^{-d/\alpha} p_1^{t^{-1/\alpha}D}\left(t^{-1/\alpha}x, t^{-1/\alpha}y\right), \qquad x, y \in \mathbb{R}^d, \quad t > 0. \tag{2.22}$$

Combining (2.21) and (2.22) we get

$$\begin{aligned} \mathbb{P}_x(\tau_D > t) &= \int_D t^{-d/\alpha} p_1^{t^{-1/\alpha}D}\left(t^{-1/\alpha}x, t^{-1/\alpha}y\right) dy \\ &= \int_{t^{-1/\alpha}D} p_1^{t^{-1/\alpha}D}\left(t^{-1/\alpha}x, u\right) du \\ &= \mathbb{P}_{t^{-1/\alpha}x}\left(\tau_{t^{-1/\alpha}D} > 1\right). \end{aligned} \tag{2.23}$$

We denote the **killing intensity** of X on D by

$$\kappa_D(z) = \int_{D^c} \nu(z-y)dy, \tag{2.24}$$

and the **fractional Laplacian** operator by

$$\left(\Delta^{\alpha/2}u\right)(x) = \lim_{r\to 0}\int_{\mathbb{R}^d\setminus B(0,r)} \left(u(x+y)-u(x)\right)\nu(y)dy, \tag{2.25}$$

where ν is the jump-measure as defined in (2.17).

Definition 2.2.1. *Let $u \geq 0$ be a Borel measurable function on $\mathbb{R}^d$.*

- *We say that $u(x)$ is* **regular α-harmonic** *in an open set $V \subset \mathbb{R}^d$, written $u \in \mathscr{H}^\alpha_{reg}(V)$, if*

$$u(x) = \mathbb{E}_x\left[u\left(X_{\tau_V}\right)\right] < \infty, \qquad \text{for all } x \in V.$$

- *We say that $u(x)$ is* **α-harmonic** *in $V \subset \mathbb{R}^d$, written $u \in \mathscr{H}^\alpha(V)$, if for every bounded $B \subset V$ such that $\overline{B} \subset V$ we have*

$$u(x) = \mathbb{E}_x\left[u\left(X_{\tau_B}\right)\right] < \infty, \qquad \text{for all } x \in B.$$

- *We say that $u(x)$ is* **singular α-harmonic** *in $V \subset \mathbb{R}^d$, written $u \in \mathscr{H}^\alpha_0(V)$, if $u \in \mathscr{H}^\alpha(V)$ and $u(x) = 0$ for every $x \in V^c$.*

Since $\tau_B \leq \tau_V$ for $B \subset V$, it follows by the strong Markov property that regular α-harmonic functions are also α-harmonic. For $x \in \mathbb{R}^d$, the $\mathbb{P}_x$-distribution of X_{τ_V} is called the **α-harmonic measure**, denoted by ω^x_V. This measure is concentrated on V^c and for $u \in \mathscr{H}^\alpha_{reg}(V)$ we have

$$u(x) = \int_{V^c} u(y)\omega^x_V(dy), \qquad x \in V. \tag{2.26}$$

Given an α-harmonic measure on a Lipschitz set we can express it in terms of its density function (the Poisson kernel) [7, Lemma 6]. That is, for every $u \in \mathscr{H}^{\alpha}_{reg}(D)$ we have a representation

$$u(x) = \int_{D^c} P_D(x,y)u(y)dy, \qquad x \in D. \tag{2.27}$$

It can be shown that for each $y \in D$, $G(x,y)$ is α-harmonic in $x \in D\backslash\{y\}$ and regular α-harmonic in $D\backslash B(y,r)$, for every $r > 0$, see [8].

Fix an arbitrary reference point $x_0 \in D$. For $Q \in \partial D$ and $y \in D$ we define the **Martin kernel** on D by

$$M_D^{x_0}(y,Q) = \lim_{x \to Q} \frac{G_D(x,y)}{G_D(x,x_0)}, \qquad x \in D. \tag{2.28}$$

In [8, Lemma 6] it is shown that the Martin kernel exists, the mapping $(y,Q) \mapsto M_D^{x_0}(y,Q)$ is continuous on $D \times \partial D$, and for every $Q \in \partial D$ the function $M_D^{x_0}(\cdot,Q)$ is singular α-harmonic in D.

2.3. Weak Scaling Conditions

Let us return to general unimodal Lévy process setting. We asserted in equation (2.3) that unimodal Lévy processes are characterized by Lévy-Khintchine (characteristic) exponents of the form

$$\psi(\xi) = \sigma^2|\xi|^2 + \int_{\mathbb{R}^d} (1 - \cos\langle \xi, x\rangle)\, \nu(dx),$$

where $\nu(dx) = \nu(x)dx = \nu(|x|)dx$ is a unimodal Lévy measure and $\sigma \geq 0$. Since $\psi(\xi)$ is a radial function, we often let $\xi(r) = \psi(\xi)$ where $\xi \in \mathbb{R}^d$ and $r = |\xi| \geq 0$.

Consider the pure-jump Lévy process $X = (X_t)_{t \geq 0}$ on $\mathbb{R}^d$. That is, let $\sigma = 0$. This process is determined by the Lévy-Khintchine formula:

$$\mathbb{E}e^{i\langle \xi, X_t \rangle} = \int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} p_t(dx) = e^{-t\psi(\xi)}. \tag{2.29}$$

Here $p_t(dx)$ is the distribution of X_t. It turns out that $p_t(dx)$ is also unimodal; therefore we can call the process X (isotropic) unimodal. We wish for $p_t(dx)$ to have bounded and smooth density functions, $p_t(x)$ for $t > 0$. This is a consequence of the Hartman-Wintner condition, see [14, Lemma 1.1]:

$$\lim_{|\xi| \to \infty} \psi(\xi) / \ln(\xi) = \infty. \tag{2.30}$$

The Hartman-Wintner condition itself will be a consequence of our assumption that $\psi(\xi)$ satisfies some weak lower scaling condition, yet to be defined. We always assume that the Lévy-Khintchine exponent, $\psi(\xi)$, is unbounded, that is, $\nu\left(\mathbb{R}^d\right) = \infty$. Clearly $\psi(0) = 0$ and $\psi(u) > 0$ for $u > 0$.

Let X_t^1 be the first coordinate process of X_t. We define the **running maximum** of X_t by

$$m_t = \sup_{0 \leq s \leq t} X_s^1. \tag{2.31}$$

We then define the **local time of** $m_t - X_t^1$ **at** 0, $L^0(t)$, to be the amount of time, up to time t, that $m_t - X_t^1$ spends at 0:

$$L^0(t) = \int_0^t \delta\left(m_s - X_s^1\right) ds, \tag{2.32}$$

where $\delta(\cdot)$ is the Dirac delta function. Consider the right-continuous inverse of $L^0(t)$: $\left(L^0\right)^{-1}(s)$. This is called the **ascending ladder time process** for X_t^1.

Composing X_t^1 with $\left(L^0\right)^{-1}(s)$ gives us the **ascending ladder-height process**:

$$H_s = X^1_{(L^0)^{-1}(s)} = m_{(L^0)^{-1}(s)}. \tag{2.33}$$

The **accumulated potential** of this ascending ladder-height process is defined by

$$V(x) := \mathbb{E} \int_0^\infty 1_{[0,x]}\left(H_s\right) ds = \int_0^\infty \mathbb{P}\left(H_s \le x\right) ds. \tag{2.34}$$

This is a continuous and strictly increasing from $[0, \infty)$ onto $[0, \infty)$. In particular, $\lim_{r\to\infty} V(r) = \infty$ and $V(x)$ is sub-additive:

$$V(x+y) \le V(x) + V(y), \qquad \text{for all } x, y \in \mathbb{R}. \tag{2.35}$$

The relationship between $V(x)$ and $\psi(x)$ is given in [14, Lemma 1.2] by

$$V^2(r) \approx \frac{1}{\psi(1/r)}, \qquad r > 0. \tag{2.36}$$

For more details on the ascending ladder-height process and accumulated potential see [11] and [49].

We are interested in the (relative) power-type behavior of $\psi(r)$ at infinity. We say that $\psi(r)$ satisfies the **weak lower scaling condition at infinity**, $WLSC\left(\underline{\alpha}, \underline{\theta}, \underline{C}\right)$, if there are numbers $\underline{\alpha} > 0$, $\underline{\theta} \ge 0$, and $\underline{C} \in (0, 1]$ such that

$$\psi(\lambda r) \ge \underline{C}\lambda^{\underline{\alpha}}\psi(r), \tag{2.37}$$

for $\lambda \ge 1$, $r > \underline{\theta}$. In general, we write $\psi \in WLSC\left(\underline{\alpha}, \underline{\theta}, \underline{C}\right)$. Or, in short, we write $\psi \in WLSC\left(\underline{\alpha}, \underline{\theta}, \underline{C}\right)$, $\psi \in WLSC\left(\underline{\alpha}, \underline{\theta}\right)$, or $\psi \in WLSC\left(\underline{\alpha}\right)$ depending on how specific we want to be. Further, we say that $\psi(r)$ satisfies the **global** weak lower scaling condition at infinity (**global** $WLSC$) if $\psi \in WLSC(\underline{\alpha}, 0)$. If $\underline{\theta} > 0$, then we can emphasize this by calling the scaling "**local** at infinity".

We say that $\psi(r)$ satisfies the **weak upper scaling condition at infinity**, $WUSC\left(\overline{\alpha}, \overline{\theta}, \overline{C}\right)$, if there are numbers $\overline{\alpha} < 2$, $\overline{\theta} \geq 0$, and $\overline{C} \in [1, \infty)$ such that

$$\psi(\lambda r) \leq \overline{C} \lambda^{\overline{\alpha}} \psi(r), \tag{2.38}$$

for $\lambda \geq 1$, $r > \overline{\theta}$. In general, we write $\psi \in WUSC\left(\overline{\alpha}, \overline{\theta}, \overline{C}\right)$. Or, in short, we write $\psi \in WUSC\left(\overline{\alpha}, \overline{\theta}, \overline{C}\right)$, $\psi \in WUSC\left(\overline{\alpha}, \overline{\theta}\right)$, or $\psi \in WUSC\left(\overline{\alpha}\right)$ depending on how specific we want to be. Further, we say that $\psi(r)$ satisfies the **global** weak upper scaling condition at infinity (**global** $WUSC$) if $\psi \in WUSC(\overline{\alpha}, 0)$. If $\overline{\theta} > 0$, then we can emphasize this by calling the scaling "**local** at infinity".

As pointed out in [14, Remark 1.4], by inflating (or deflating) $\underline{C}$ and $\overline{C}$ we can deflate (or inflate) $\underline{\theta}$ and $\overline{\theta}$ so that $\theta = \underline{\theta} = \overline{\theta} > 0$ in both $WLSC$ and $WUSC$. These scalings are natural conditions on $\psi(r)$ in the unimodal setting and there are many examples of Lévy-Khintchine exponents which satisfy $WLSC$ or $WUSC$. For example, as is shown in [13], every unimodal Lévy process satisfies

$$\psi \in WLSC\left(0, 0, 1/\pi^2\right) \cap WUSC\left(2, 0, \pi^2\right).$$

Another example is $\psi(\xi) = |\xi|^\alpha$, the Lévy-Khintchine exponent of the isotropic α-stable Lévy process in $\mathbb{R}^d$ with $\alpha \in (0, 2)$. This satisfies $WLSC(\alpha, 0, 1)$ and $WUSC(\alpha, 0, 1)$. Alternatively, a non-stable example is $\psi(\xi) = |\xi|^{\alpha_1} + |\xi|^{\alpha_2}$, for which we have $\psi(\xi) \in WLSC\left(\alpha_1, 0, 1\right) \cap WUSC\left(\alpha_2, 0, 1\right)$, where $0 < \alpha_1 < \alpha_2 < 2$. Finally, if $\psi(r)$ is α-regular varying at infinity and $0 < \alpha < 2$, then

$\psi \in WLSC\left(\underline{\alpha}\right) \cap WUSC\left(\overline{\alpha}\right)$, for any $0 < \underline{\alpha} < \alpha < \overline{\alpha} < 2$. See [13] for more details on $WLSC$ and $WUSC$.

By definition, if $\psi \in WLSC\left(\underline{\alpha}, \underline{\theta}\right)$, then there exists some constant $\underline{C}$ such that

$$\frac{V\left(\frac{1}{\lambda r}\right)}{V\left(\frac{1}{r}\right)} \leq \underline{C}\lambda^{-\underline{\alpha}/2},$$

for $\lambda \geq 1$ and $r > \underline{\theta}$. That is,

$$\frac{V\left(\varepsilon s\right)}{V\left(s\right)} \leq \underline{C}\varepsilon^{\underline{\alpha}/2}, \tag{2.39}$$

for $0 < \varepsilon \leq 1$ and $s < 1/\underline{\theta}$. Similarly, if $\psi \in WUSC\left(\overline{\alpha}, \overline{\theta}\right)$ then there exists some constant $\overline{C}$ such that

$$\frac{V\left(s\right)}{V\left(\varepsilon s\right)} \leq \overline{C}\varepsilon^{-\overline{\alpha}/2}, \tag{2.40}$$

for $0 < \varepsilon \leq 1$ and $s < 1/\overline{\theta}$.

Lemma 2.3.1 (Potter-like Bounds)**.** *If $\psi \in WLSC(\underline{\alpha}, \underline{\theta}) \cap WUSC(\overline{\alpha}, \overline{\theta})$, $0 < x < 1/\overline{\theta}$, and $0 < y < 1/\underline{\theta}$, then there exists some constant C such that*

$$\frac{V(x)}{V(y)} \leq C\left[\left(\frac{x}{y}\right)^{\underline{\alpha}/2} \vee \left(\frac{x}{y}\right)^{\overline{\alpha}/2}\right]. \tag{2.41}$$

Proof. Using (2.39) and (2.40) we have

$$\begin{aligned}
\frac{V(x)}{V(y)} \quad &= \quad \begin{cases} \frac{V(ty)}{V(y)}, & \text{if } t = \frac{x}{y} \leq 1, \\[2ex] \frac{V(x)}{V(t^{-1}x)}, & \text{if } t^{-1} = \frac{y}{x} \leq 1. \end{cases} \\
&\leq \quad \begin{cases} \underline{C}t^{\underline{\alpha}/2}, & \text{if } t = \frac{x}{y} \leq 1 \text{ and } y < 1/\underline{\theta}, \\[2ex] \overline{C}t^{\overline{\alpha}/2}, & \text{if } t^{-1} = \frac{y}{x} \leq 1 \text{ and } x < 1/\overline{\theta}. \end{cases}
\end{aligned}$$

Thus

$$\frac{V(x)}{V(y)} \leq C\left[\left(\frac{x}{y}\right)^{\underline{\alpha}/2} \vee \left(\frac{x}{y}\right)^{\overline{\alpha}/2}\right], \qquad \text{for} \qquad x < 1/\overline{\theta}, \;\; y < \underline{\theta}.$$

$\square$

We frequently refer to the inverse function of $V(x)$ on $[0,\infty)$ in this dissertation. To simplify notation, we choose to write

$$T(t) := V^{-1}\left(\sqrt{t}\right). \tag{2.42}$$

This is equivalent to $V^2\left(T(t)\right) = t$. For example, $T(t) = t^{1/\alpha}$ for the isotropic α-stable Lévy process. The scaling properties of $T(t)$ at zero reflect those of $\psi(\xi)$ at infinity. See [21] for further discussion of $T(t)$. A nice consequence of weak scaling conditions is that they imply the Hartman-Wintner condition, mentioned above in (2.30). That is, weak scaling conditions imply that the distribution, $p_t(dx)$, of X_t has a bounded and smooth density function, $p_t(x)$ for $t > 0$.

Throughout the rest of this dissertation we make the following assumptions:

– The Lévy measure ν is unimodal and infinite on $\mathbb{R}^d$ with $d \geq 2$.

– The Lévy-Khintchine exponent satisfies both weak upper and lower scaling conditions

$$0 \neq \psi \in WLSC\left(\underline{\alpha}, \theta\right) \cap WUSC\left(\overline{\alpha}, \theta\right), \tag{2.43}$$

for some constants $0 < \underline{\alpha} \leq \overline{\alpha} < 2$ and $0 \leq \theta \leq \inf_{x \in D}\left(1/\delta_D(x)\right)$.

2.4. Trace Estimates

We define the **trace** of the heat kernel $p_t^D(x, y)$ by

$$Z_D(t) = \int_{\mathbb{R}^d} p_t^D(x, x)dx. \tag{2.44}$$

A classical question in analysis is: what is the relationship between the spectrum of the Laplacian and the geometry of the domain on which it is defined? This problem can be approached through the theory of stochastic processes using the trace of the heat kernel. For example, consider the infinitesimal generator of $\left\{P_t^D\right\}_{t\geq 0}$ for the isotropic α-stable process. In this case, it can be shown that there exists an orthonormal basis of eigenfunctions $\{\varphi_i\}_{i=1}^{\infty}$ for $L^2(D)$ and corresponding eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ of Lf satisfying $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, with $\lambda_i \to \infty$ as $i \to \infty$, see [31]. That is, $\{\varphi_i, \lambda_i\}$ satisfies

$$\left(P_t^D \varphi_i\right)(x) = e^{-\lambda_i t}\varphi_i(x), \qquad x \in D, \quad t > 0. \tag{2.45}$$

Thus

$$p_t^D(x, y) = \sum_{i=1}^{\infty} e^{-\lambda_i t}\varphi_i(x)\varphi_i(y). \tag{2.46}$$

Hence

$$Z_D(t) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \int_D \varphi_i^2(x)dx = \sum_{i=1}^{\infty} e^{-\lambda_i t}. \tag{2.47}$$

Therefore a question about the relationship between the spectrum of the fractional Laplacian and the geometry of D can become a question about the relationship between the trace of the heat kernel and the geometry of D.

CHAPTER III

BOUNDARY BEHAVIOR OF THE HEAT KERNEL OF ALPHA-STABLE PROCESSES ON BOUNDED LIPSCHITZ SETS

3.1. Boundary Behavior of the Heat Kernel

In this chapter we establish the limiting boundary behavior of the heat kernel for isotropic α-stable processes on bounded Lipschitz sets. This has some relationship to Yaglom limits, which refer to the existence of a rescaling of a Lévy process and a non-trivial probability measure such that the rescaling conditioned on non-extinction converges in distribution towards the measure. This project is a generalization of the work done by Bogdan et al. [19] for cones, and was completed under the guidance of Prof. Krzysztof Bogdan at Wrocław University of Science and Technology.

We consider solely the isotropic α-stable processes in this chapter. Let $X = (X_t)_{t\geq 0}$ be the isotropic α-stable Lévy process in $\mathbb{R}^d$.

Theorem 3.1.1. *Let D be an open bounded Lipschitz set in $\mathbb{R}^d$ and let $Q \in \partial D$. The following limit exists*

$$n_{t,Q}(y) = \lim_{x\to Q} \frac{p_t^D(x,y)}{\mathbb{P}_x\left(\tau_D > 1\right)}, \quad x \in D, \quad (t,y) \in [1,\infty) \times D. \tag{3.1}$$

Furthermore it is finite, strictly positive, continuous on t and y, and for $1 \leq t < \infty$, $0 < s < \infty$, $y \in D$ we have

$$n_{1,Q}(y) \approx \mathbb{P}_y\left(\tau_D > 1\right), \tag{3.2}$$

$$n_{t+s,Q}(y) = \int_D n_{t,Q}(z) p_s^D(z,y)dz. \tag{3.3}$$

3.2. Asymptotics of Green Potentials

We wish to prove Theorem 3.1.1. To do this we first establish the asymptotics of Green potentials at the boundary points. This extends what is already known about the asymptotics of Green potentials for cones [19, Lemma 3.5]

Lemma 3.2.1 (Uniform Integrability of the Ratio of Green Functions)**.**
Let f be a measurable function bounded on D and let $Q \in \partial D$. We have

$$\lim_{x\to Q}\int_D \frac{G_D(x,y)}{G_D(x,x_0)}f(y)dy = \int_D \lim_{x\to Q}\frac{G_D(x,y)}{G_D(x,x_0)}f(y)dy < \infty, \qquad x \in D.$$

Proof of Lemma 3.2.1. Let $z_1, z_2 \in D$. We choose

$$\rho = \left(dist\,(z_1,\partial D) \wedge dist\,(z_2,\partial D) \wedge |z_1 - z_2|\right)/3 \tag{3.4}$$

so that $B(z_1,\rho), B(z_2,\rho) \subset D$ and $B(z_1,\rho) \cap B(z_2,\rho) = \emptyset$. We know that $G_D(\cdot,y)$ is regular α-harmonic on $B(z_1,\rho)$ and $B(z_2,\rho)$, see [41]. We also know that $M_D^{x_0}(\cdot,Q)$ is regular α-harmonic on $B(z_1,\rho)$ and $B(z_2,\rho)$, see [8, Lemma 6]. Using the definition of the Martin kernel (2.28) and the α-harmonicity of both the Martin kernel and the Green function we get

$$\begin{aligned}
\int_{B(z_i,\rho)^c}\lim_{x\to Q}\frac{G_D(x,y)}{G_D(x,x_0)}\omega^{z_i}_{B(z_i,\rho)}(dy) &= \int_{B(z_i,\rho)^c} M_D^{x_0}(y,Q)\omega^{z_i}_{B(z_i,\rho)}(dy)\\
&= M_D^{x_0}(z_i,Q)\\
&= \lim_{x\to Q}\frac{G_D(x,z_i)}{G_D(x,x_0)}\\
&= \lim_{x\to Q}\frac{\int_{B(z_i,\rho)^c} G_D(x,y)\omega^{z_i}_{B(z_i,\rho)}(dy)}{G_D(x,x_0)}\\
&= \lim_{x\to Q}\int_{B(z_i,\rho)^c}\frac{G_D(x,y)}{G_D(x,x_0)}\omega^{z_i}_{B(z_i,\rho)}(dy),
\end{aligned}$$

for $i = 1, 2$. This tells us that, with respect to the α-harmonic measure on $B(z_i, \rho)$, the above integral and limit are interchangeable. Vitali's theorem [50, Theorem 16.6] tells us that the collection of functions $\left(\frac{G_D(x,y)}{G_D(x,x_0)}\right)_{x\to Q}$ is uniformly integrable on D with respect to $\omega_{B(z_1,\rho)} + \omega_{B(z_2,\rho)}$.

The α-harmonic measures $\omega^{z_i}_{B(z_i,\rho)}$ have explicit density functions, see e.g. [8]. It can be shown that

$$P_{B(z_i,\rho)}(z_i, y) \geq \frac{c_{d,\alpha,z_1,z_2}}{\left(|y-z_i|^2\right)^{\alpha/2} |y-z_i|^d} \geq \frac{c_{d,\alpha,z_1,z_2}}{\left(|y|+|z_i|\right)^{d+\alpha}} \geq c_{d,\alpha,z_1,z_2},$$

for $y \in B(z_i, \rho)^c$ and $i = 1, 2$. Thus [50, Theorem 16.8] tells us that we also have the uniform integrability of $\left(\frac{G_D(x,y)}{G_D(x,x_0)}\right)_{x\to Q}$ on D with respect to Lebesgue measure. Therefore

$$\lim_{x\to Q} \int_D \frac{G_D(x,y)}{G_D(x,x_0)} f(y)dy = \int_D \lim_{x\to Q} \frac{G_D(x,y)}{G_D(x,x_0)} f(y)dy < \infty,$$

for any bounded measurable $f(y)$. □

3.3. Distributions and Green Potentials

We can also establish the following identity, expressing the distribution of the first exit time in terms of a Green potential:

Lemma 3.3.1. *For $x \in \mathbb{R}^d$, we have*

$$\mathbb{P}_x\left(\tau_D > 1\right) = \left(G_D P_1^D \kappa_D\right)(x). \tag{3.5}$$

Proof of Lemma 3.3.1. Let $x \in D$. Since our domain D is Lipschitz, it follows from [53] that

$$\omega_D^x(\partial D) = \mathbb{P}_x(X_{\tau_D} \in \partial D) = 0,$$

Thus

$$\mathbb{P}_x(X_{\tau_D -} = X_{\tau_D}) = 0.$$

By the Ikeda-Watanabe formula:

$$\begin{aligned}
\mathbb{P}_x(\tau_D > 1) &= \mathbb{P}_x(\tau_D > 1,\ X_{\tau_D -} \in D,\ X_{\tau_D} \in D^c) \\
&= \int_1^\infty \int_{D^c} \int_D p_s^D(x,z)\nu(z-w)\ dz\ dw\ ds \\
&= \int_{\mathbb{R}^d} \int_{D^c} \int_0^\infty p_{t+1}^D(x,z)\nu(z-w)\ dt\ dw\ dz \\
&= \int_{\mathbb{R}^d} \int_{D^c} \int_0^\infty \int_D p_t^D(x,y) p_1^D(y,z)\ dy\ \nu(z-w)\ dt\ dw\ dz \\
&= \int_D \int_0^\infty p_t^D(x,y)\ dt \int_{\mathbb{R}^d} p_1^D(y,z) \int_{D^c} \nu(z-w)\ dw\ dz\ dy \\
&= \int_D G_D(x,y) \int_{\mathbb{R}^d} p_1^D(y,z)\kappa_D(z)\ dz\ dy \\
&= \int_D G_D(x,y) \left(P_1^D \kappa_D\right)(y)\ dy \\
&= \left(G_D P_1^D \kappa_D\right)(x).
\end{aligned}$$

For $x \in D^c$ we can see that $\tau_D = 0$ almost surely with respect to $\mathbb{P}_x$. Therefore the identity (3.5) is established.

□

Next we show that the limit of the ratio of the distribution of the exit time and the Green function exists near the boundary. We will denote this limit by C_1

and we will show that

$$C_1 = \int_D \int_D M_D^{x_0}(y,Q) p_1^D(y,z) \kappa_D(z) dz dy. \tag{3.6}$$

Combining the two lemmas above we obtain the following theorem:

Theorem 3.3.1. *The constant* $0 < C_1 < \infty$ *exists, satisfies equation (3.6), and*

$$\lim_{x \to Q} \frac{\mathbb{P}_x\left(\tau_D > 1\right)}{G_D(x, x_0)} = C_1. \tag{3.7}$$

Proof of Theorem 3.3.1. We first show that $f(y) = \left(P_1^D \kappa_D\right)(y)$ satisfies the assumptions of Lemma 3.2.1, i.e. that f is bounded. Indeed, by [15, Theorem 1] we have the following factorization

$$p_1^D(y,z) \approx \mathbb{P}_y\big(\tau_D > 1\big)\ \mathbb{P}_z\big(\tau_D > 1\big)\ p_1(y,z), \qquad y, z \in D. \tag{3.8}$$

Thus,

$$\begin{aligned} \left(P_1^D \kappa_D\right)(y) &= \int_D p_1^D(y,z)\ \kappa_D(z)\ dz \\ &\approx \mathbb{P}_y\big(\tau_D > 1\big) \int_D \mathbb{P}_z\big(\tau_D > 1\big)\ p_1(y,z)\ \kappa_D(z)\ dz. \end{aligned} \tag{3.9}$$

Since D is bounded, it follows from equation (2.20) that

$$p_1(y,z) \approx 1, \qquad y, z \in D. \tag{3.10}$$

Hence equation (3.9) becomes

$$\left(P_1^D \kappa_D\right)(y) \approx \mathbb{P}_y(\tau_D > 1) \int_D \mathbb{P}_z(\tau_D > 1)\kappa_D(z)\ dz. \tag{3.11}$$

Using equation (3.5) we see that

$$\int_D G_D(x,y)\left(P_1^\Gamma \kappa_\Gamma\right)(y)dy = \left(G_D P_1^\Gamma \kappa_\Gamma\right)(x) = \mathbb{P}_x\left(\tau_D > 1\right) \le 1,$$

for $y \in \mathbb{R}^d$. By [56], $G_D(x,y)$ is strictly positive for all $x, y \in D$. Thus $P_1^\Gamma \kappa_\Gamma$ has to be finite almost everywhere. Hence the integral in equation (3.11) must be finite. Therefore

$$\left(P_1^D \kappa_D\right)(y) \approx \mathbb{P}_y(\tau_D > 1),$$

for $y \in D$. In particular, $\left(P_1^D \kappa_\Gamma\right)(y)$ is bounded on D.

Finally, since $\left(P_1^D \kappa_\Gamma\right)(y)$ is bounded, we can apply Lemma 3.2.1:

$$\begin{aligned}
\lim_{x\to Q} \frac{\mathbb{P}_x\left(\tau_D > 1\right)}{G_D(x,x_0)} &= \lim_{x\to Q} \frac{\left(G_D P_1^D \kappa_D\right)(x)}{G_D(x,x_0)} \\
&= \lim_{x\to Q} \int_D \frac{G_D(x,y)}{G_D(x,x_0)}\left(P_1^D \kappa_D\right)(y)\ dy \\
&= \int_D M_D^{x_0}(y,Q)\left(P_1^D \kappa_D\right)(y)\ dy \\
&= C_1 < \infty.
\end{aligned} \tag{3.12}$$

□

3.4. Limiting Boundary Behavior

We are now in a position to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. Let us define

$$\mu_x(A) := \frac{\int_A p_1^D(x,y)dy}{\mathbb{P}_x\left(\tau_D > 1\right)}, \qquad x \in D, \quad A \subset \mathbb{R}^d. \tag{3.13}$$

First we note that the family $\{\mu_x \ : \ x \in D\}$ is tight. Indeed, combining the factorization of $p_1^D(x,y)$ in equations (3.8) and (3.10) we get

$$\frac{p_1^D(x,y)}{\mathbb{P}_x\left(\tau_D > 1\right)} \approx \mathbb{P}_y\left(\tau_D > 1\right), \tag{3.14}$$

for $x, y \in D$. The density functions of these $\mu_x(A)$ are bounded by a single integrable function and hence this collection of measures is tight.

Next we wish to prove that the measures μ_x converge weakly to a probability measure μ_Q on D as $x \to Q$. To this end, consider an arbitrary sequence $\{x_n\}$ such that $x_n \to Q$. By tightness there exists a subsequence $\{x_{n_k}\}$ such that $\mu_{x_{n_k}} \Rightarrow \mu_Q$ for some probability measure μ_Q, as $k \to \infty$. We wish to show that this limit is unique.

Let $\phi \in C_c^\infty(D)$ and $u_\phi = -\Delta^{\alpha/2}\phi$, where $\Delta^{\alpha/2}$ is the fractional Laplacian operator defined in equation (2.25). For $x \in \mathbb{R}^d$, we claim that

$$\left(P_1^D \phi\right)(x) = \left(G_D P_1^D u_\phi\right)(x). \tag{3.15}$$

To show this we first remark that the function u_ϕ is bounded and continuous and that $\left(G_D u_\phi\right)(x) = \phi(x)$, see [16, (19)] and [18, (11)]. By equation (2.20) it follows that

$$\left(P_1^D \left|u_\phi\right|\right)(x) = \int_D p_1^D(x,y)\left|u_\phi(y)\right| dy \leq c.$$

It now follows from [18, (74)] that

$$\begin{aligned}\left(G_D P_1^D \left|u_\phi\right|\right)(x) &= \int_D G_D(x,y)\left(P_1^D \left|u_\phi\right|\right)(y)dy \\ &\le c\int_D G_D(x,y)dy < \infty. \qquad (3.16)\end{aligned}$$

As a result of this, we can apply Fubini's theorem:

$$\begin{aligned}\left(G_D P_1^D u_\phi\right)(x) &= \int_D\int_D\int_0^\infty p_t^D(x,y)p_1^D(y,z)u_\phi(z)dtdzdy \\ &= \int_D\int_0^\infty p_{t+1}^D(x,z)u_\phi(z)dtdz \\ &= \int_D\int_0^\infty p_{t+1}^D(z,x)u_\phi(z)dtdz \\ &= \int_D\int_D\int_0^\infty p_t^D(z,y)p_1^D(y,x)u_\phi(z)dtdzdy \\ &= \left(P_1^D G_D u_\phi\right)(x) \\ &= \left(P_1^D \phi\right)(x).\end{aligned}$$

This establishes equation (3.15).

Let us denote $\mu_x(\phi) := \int_D \phi(y)\mu_x(dy)$. Using equation (3.15), Theorem 3.3.1, and Lemma 3.2.1 we get

$$\begin{aligned}\lim_{x\to Q}\mu_x(\phi) &= \lim_{x\to Q}\frac{\left(P_1^D\phi\right)(x)}{\mathbb{P}_x\left(\tau_D>1\right)} \\ &= \lim_{x\to Q}\frac{\left(P_1^D G_D u_\phi\right)(x)}{\mathbb{P}_x\left(\tau_D>1\right)} \\ &= \lim_{x\to Q}\frac{\left(G_D P_1^D u_\phi\right)(x)}{G_D(x,x_0)}\frac{G_D(x,x_0)}{\mathbb{P}_x\left(\tau_D>1\right)} \\ &= \frac{1}{C_1}\int_D M_D^{x_0}(y,Q)\left(P_1^D u_\phi\right)(y)dy. \qquad (3.17)\end{aligned}$$

In particular $\mu_Q(\phi) := \lim_{k\to\infty}\mu_{x_{n_k}}(\phi)$ does not depend on the choice of subsequence. Thus $\mu_x \Rightarrow \mu_Q$ as $x\to Q$.

For any $t > 1$ we can consider $\phi_{t,y}(\cdot) = p^D_{t-1}(\cdot, y) \in C_0(\mathbb{R}^d)$. Using the Chapman-Kolmogorov identity (2.9) we get

$$\begin{aligned}
n_{t,Q}(y) &= \lim_{x\to Q} \frac{p^D_t(x,y)}{\mathbb{P}_x\left(\tau_D > 1\right)} \\
&= \lim_{x\to Q} \frac{\int_D p^D_{t-1}(z,y) p^D_1(x,z)dz}{\mathbb{P}_x\left(\tau_D > 1\right)} \\
&= \lim_{x\to Q} \frac{\left(P^D_1 p^D_{t-1}(\cdot,y)\right)(x)}{\mathbb{P}_x\left(\tau_D > 1\right)} \\
&= \lim_{x\to Q} \mu_x\left(p^D_{t-1}(\cdot,y)\right).
\end{aligned}$$

By (3.17), the existence of $n_{t,Q}(y)$ for $t > 1$ now follows:

$$n_{t,Q}(y) = \mu_Q\left(p^D_{t-1}(\cdot,y)\right).$$

The case $t = 1$ also follows from this, but instead of $p^D_{t-1}(x,y)$ we just need the appropriate identity function on D.

Note that for every $\phi \in C_0(\mathbb{R}^d)$ and every $Q \in \partial D$ we have

$$\mu_Q(\phi) = \lim_{x\to Q} \int_D \frac{p^D_1(x,y)}{\mathbb{P}_x\left(\tau_D > 1\right)} \phi(y)dy = \int_D n_{1,Q}(y)\phi(y)dy.$$

That is, $n_{1,Q}(y)$ is the density function of μ_Q with respect to the Lebesgue measure. Also equation (3.2) follows from equation (3.14), and equation (3.3) follows from the Chapman-Kolmogorov identity (2.9) and the Dominated Convergence Theorem:

$$n_{t+s,Q}(y) = \lim_{x\to Q} \int_D \frac{p^D_t(x,z)}{\mathbb{P}_x\left(\tau_D > 1\right)} p^D_s(z,y)dz = \int_D n_{t,Q}(z)p^D_s(z,y)dz. \tag{3.18}$$

□

Corollary 3.4.1. *Let D be an open bounded Lipschitz set in $\mathbb{R}^d$ and let $Q \in \partial D$. The following limit exists*

$$m_{t,Q}^D(y) = \lim_{x \to Q} \frac{p_t^D(x,y)}{\mathbb{P}_x(\tau_D > t)}, \quad x \in D, \quad (t,y) \in (0,\infty) \times D. \tag{3.19}$$

Furthermore it is finite, strictly positive, and for $0 < t < \infty$, $y \in D$ we have

$$m_{t,Q}^D(y) = t^{-d/\alpha} m_{1,Q/t^{1/\alpha}}^{D/t^{1/\alpha}} \left(y/t^{1/\alpha}\right). \tag{3.20}$$

Proof. For $t \geq 1$, the existence of $m_{t,Q}^D(y)$ follows from Theorem 3.1.1:

$$\begin{aligned} m_{t,Q}^D(y) &= \lim_{x \to Q} \frac{p_t^D(x,y)}{\mathbb{P}_x(\tau_D > t)} \\ &= \lim_{x \to Q} \frac{p_t^D(x,y)}{\mathbb{P}_x(\tau_D > 1)} \cdot \lim_{x \to Q} \frac{\mathbb{P}_x(\tau_D > 1)}{\mathbb{P}_x(\tau_D > t)} \\ &= \frac{n_{t,Q}(y)}{\lim\limits_{x \to \infty} \displaystyle\int_D \frac{p_t^D(x,y)}{\mathbb{P}_x(\tau_D > 1)} dy} \\ &= \frac{n_{t,Q}(y)}{\int_D n_{t,Q}(y) dy}. \end{aligned} \tag{3.21}$$

Using equations (2.22) and (2.23) we get the following rescaling property of $m_{t,Q}^D(y)$ for $t > 0$:

$$\begin{aligned} m_{t,Q}^D(y) &= \lim_{x \to Q} \frac{p_t^D(x,y)}{\mathbb{P}_x(\tau_D > 1)} \\ &= \lim_{x \to Q} \frac{t^{-d/\alpha} p_1^{t^{-1/\alpha} D}\left(t^{-1/\alpha}x, t^{-1/\alpha}y\right)}{\mathbb{P}_{t^{-1/\alpha}x}\left(\tau_{t^{-1/\alpha}D} > 1\right)} \\ &= t^{-d/\alpha} m_{1,t^{-1/\alpha}Q}^{t^{-1/\alpha}D}\left(t^{-1/\alpha}y\right). \end{aligned} \tag{3.22}$$

□

One interpretation of the implications of Theorem 3.1.1 is the following:

Corollary 3.4.2. *Given a bounded Lipschitz set D, $x \in D$, and $0 = Q \in \partial D$ there exists a probability measure μ_Q concentrated on D such that for every Borel set $A \subset \mathbb{R}^d$,*

$$\lim_{t\to\infty} \mathbb{P}_x\left(\frac{X_t}{t^{1/\alpha}} \in A \;\middle|\; \left(\frac{X_s}{t^{1/\alpha}}\right)_{0\le s\le t} \subset D \right) = \mu_Q(A). \tag{3.23}$$

This corollary tells us that, given its survival for a short period of time, the rescaled process $X_t/t^{1/\alpha}$ has a limiting distribution independent of the starting point x.

Proof of Corollary 3.4.2. Suppose that our boundary point, Q, coincides with the origin, that is $Q = 0$. We claim that the probability that the rescaled process belongs to some Borel set A at a time $t > 0$, given that the rescaled process hasn't escaped the domain D yet, can be expressed in terms of the measure $\mu_{t^{-1/\alpha}x}(A)$, irrespective of the starting point x of the process X. Indeed, for any Borel set $A \subset \mathbb{R}^d$ we have

$$\begin{aligned}
\mathbb{P}_x\left(\frac{X_t}{t^{1/\alpha}} \in A \;\middle|\; \left(\frac{X_s}{t^{1/\alpha}}\right)_{0\le s\le t} \subset D \right) &= \frac{\mathbb{P}_x\left(\frac{X_t}{t^{1/\alpha}} \in A,\ \left(\frac{X_s}{t^{1/\alpha}}\right)_{0\le s\le t} \subset D \right)}{\mathbb{P}_x\left(\left(\frac{X_s}{t^{1/\alpha}}\right)_{0\le s\le t} \subset D \right)} \\
&= \frac{\int_{t^{1/\alpha}A} p_t^{t^{1/\alpha}D}(x,y)dy}{\int_{t^{1/\alpha}D} p_t^{t^{1/\alpha}D}(x,y)dy} \\
&= \frac{\int_{t^{1/\alpha}A} t^{-d/\alpha} p_1^D\left(t^{-1/\alpha}x, t^{-1/\alpha}y\right)dy}{\int_{t^{1/\alpha}D} p_1^D\left(t^{-1/\alpha}x, t^{-1/\alpha}y\right)dy} \\
&= \frac{\int_A p_1^D\left(t^{-1/\alpha}x, y\right)dy}{\int_D p_1^D\left(t^{-1/\alpha}x, y\right)dy} \\
&= \mu_{t^{-1/\alpha}x}(A).
\end{aligned}$$

Therefore, as $t \to \infty$, this probability approaches $\mu_0(A) = \mu_Q(A)$.

□

CHAPTER IV

TRACE ASYMPTOTICS FOR UNIMODAL LÉVY PROCESSESS ON LIPSCHITZ DOMAINS

4.1. Trace Asymptotics

In this chapter we provide the first two terms in the small-time asymptotic expansion of the trace of the heat kernel for unimodal Lévy processes on bounded Lipschitz domains. Asymptotics in this form on Lipschitz domains have been established for Brownian motion [24] and for isotropic α-stable processes [4]. The following theorem is the next generalization in this sequence.

Theorem 4.1.1. *Let $D \subset \mathbb{R}^d$ be a bounded open Lipschitz domain. Given any unimodal Lévy process satisfying weak scaling conditions and any $\varepsilon > 0$, there exists some $t_0 > 0$ such that for $0 < t < t_0$ the trace of the heat kernel satisfies*

$$\left| Z_D(t) - p_t(0)|D| + C_{\mathbb{H}}(t)\mathcal{H}^{d-1}(\partial D) \right| \leq c(\varepsilon)T(t)^{1-d}, \tag{4.1}$$

where $c(\varepsilon) \to 0$ as $\varepsilon \to 0$, and

$$C_{\mathbb{H}}(t) = T(t)^{1-d} \int_0^\infty r_{\mathbb{H}}\left(t, (q, 0, ..., 0), (q, 0, ..., 0)\right) dq. \tag{4.2}$$

Here

$$\mathbb{H} = \left\{ (x_1, ..., x_d) \in \mathbb{R}^d : x_1 > 0 \right\} = \mathbb{R}^d_+ \tag{4.3}$$

is the upper half-space of $\mathbb{R}^d$, $|D|$ is the d-dimensional Lebesgue measure of D, and $\mathcal{H}^{d-1}(\partial D)$ is the $(d-1)$-dimensional Hausdorff measure of ∂D.

4.2. Good Sets

In order to prove Theorem 4.1.1 we dissect the Lipschitz domain D in the same way as [4] and [24]; we dissect it into good and bad sets. Let $\varepsilon, r > 0$. We say that $G \subset \partial D$ is (ε, r)-**good** if, for each point $q \in G$, the unit inner-normal at q, $v(q)$, exists, and the boundary of D, near q, is contained in a cone orthogonal to the inner-normal. That is,

$$B(q,r) \cap \partial D \subset \{x : |(x-q) \cdot v(q)| < \varepsilon |x-q|\}.$$

Let $\varphi_\varepsilon \in [0, \pi/2]$ denote the angle, measured from $v(q)$, such that $\cos(\varphi_\varepsilon) = \varepsilon$. We say that $\mathcal{G} \subset D$ is a **good subset** of D if each of its points is contained in the inner-cone of some point in an (ε, r)-good set of D:

$$\mathcal{G} := \bigcup_{q \in G} \Gamma_r(q, \varepsilon), \tag{4.4}$$

where $\Gamma_r(q,\varepsilon)$ is an inner-cone whose axis contains $v(q)$

$$\Gamma_r(q,\varepsilon) := \left\{x : (x-q) \cdot v(q) > \sqrt{1-\varepsilon^2}|x-q|\right\} \cap B(q,r). \tag{4.5}$$

For $x \in \mathbb{R}^d$, let us denote the distance between x and the boundary of D by $\delta_D(x) := dist\,(x, \partial D)$. In [4, Lemma 2.7, Lemma 2.8] it is shown that the collection of points close to the boundary and not contained in the good subset of D has small measure:

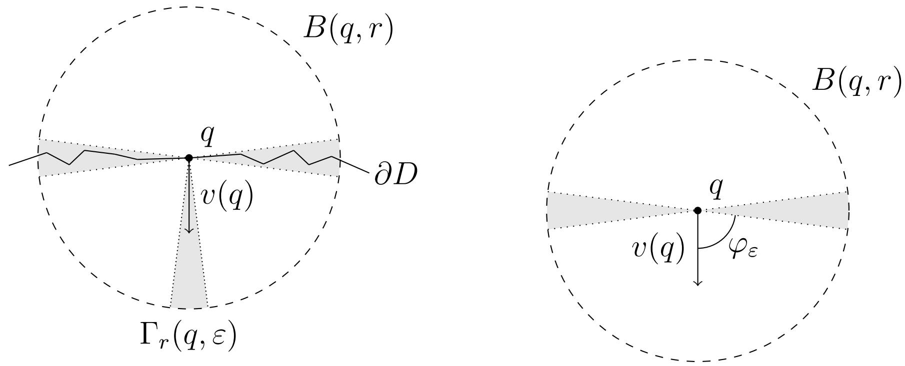

FIGURE 1 The ball on the boundary showing the inner-normal, the inner-cone and the inner-angle.

Lemma 4.2.1. *Let $0 < \varepsilon < 1/4$ and $r > 0$. There exists a measurable (ε, r)-good set $G \subset \partial D$ and $s_0(\partial D, G)$ such that for all $s < s_0$*

$$|\{x \in D : \delta_D(x) < s\} \setminus \mathcal{G}| \leq s\varepsilon \left(4 + \mathcal{H}^{d-1}(\partial D)\right). \tag{4.6}$$

Notice that if we take $x \in \mathcal{G}$, then we can find $q(x) \in \partial D$ such that $x \in \Gamma_r(q(x), \varepsilon)$. This allows us to define the **inner** and **outer regions** of $B(q(x), r)$:

$$I_r(q(x)) := \{y : (y - q(x)) \cdot v(q(x)) > \varepsilon|y - q(x)|\} \cap B(q(x), r), \tag{4.7}$$

$$U_r(q(x)) := \{y : (y - q(x)) \cdot v(q(x)) < -\varepsilon|y - q(x)|\} \cap B(q(x), r). \tag{4.8}$$

Note that the following containment always holds

$$\Gamma_r(q(x), \varepsilon) \subset I_r(q(x)) \subset D \subset U_r^c(q(x)).$$

It is shown in [4] that for any $x \in \mathcal{G}$ there exists a half-space $H^*(x)$ containing x, separating the inner and outer regions, and such that the distance between x and the boundary of D is equal to the distance between x and the boundary of the

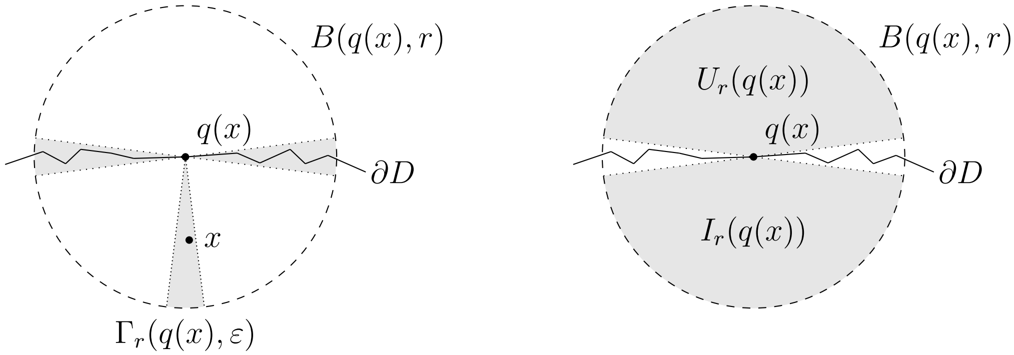

FIGURE 2 The cone corresponding to the point x, the inner and outer regions.

half-space:

$$x \in H^*(x), \qquad \delta_{H^*(x)}(x) = \delta_D(x), \qquad I_r(q(x)) \subseteq H^*(x) \subseteq U_r^c(q(x)). \tag{4.9}$$

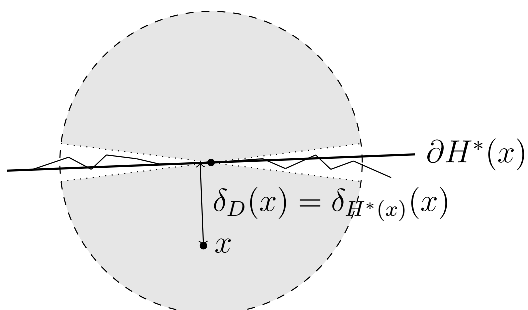

FIGURE 3 The half-space separating the inner and outer regions.

4.3. Establishing the Estimates

We are now in a position to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. Recall that the transition densities of isotropic processes killed upon exiting a domain D are given by the Hunt formula (2.8)

$$p_t^D(x, y) = p_t(y - x) - r_t^D(x, y). \tag{4.10}$$

Restricting to x and integrating over D we get

$$-\int_D r_t^D(x,x)dx = \int_D p_t^D(x,x)\,dx - \int_D p_t(0)\,dx$$
$$= Z_D(t) - p_t(0)|D|. \tag{4.11}$$

This gives us the first term in the asymptotic expansion of the heat kernel. Now, in order to prove Theorem 4.1.1, it is sufficient to show that for an arbitrary $\varepsilon > 0$ there exists a $t_0 > 0$ such that for any $0 < t < t_0$ we have

$$\left|\int_D r_t^D(x,x)\,dx - C_{\mathbb{H}}(t)\mathcal{H}^{d-1}(\partial D)\right| \leq c(\varepsilon)T(t)^{1-d}, \tag{4.12}$$

where $c(\varepsilon) \to 0$ as $\varepsilon \to 0$. In other words, we need to estimate

$$\int_D r_t^D(x,x)\,dx. \tag{4.13}$$

Fix $0 < \varepsilon < 1/4$. Let $G \subset \partial D$ to be the (ε, r)-good set as described above in Lemma 4.2.1. Let $\mathcal{G}$ be the corresponding good subset of D. Divide D into the following domains

$$\begin{aligned} D_1 &= \{x \in D\backslash\mathcal{G} \;:\; \delta_D(x) < s\}\,, \\ D_2 &= \{x \in D \cap \mathcal{G} \;:\; \delta_D(x) < s\}\,, \\ D_3 &= \{x \in D \;:\; \delta_D(x) \geq s\}\,, \end{aligned}$$

We fix s so that it is smaller than the s_0 given in Lemma 4.2.1. For sufficiently small t we can let $s = T(t)/\sqrt{\varepsilon}$.

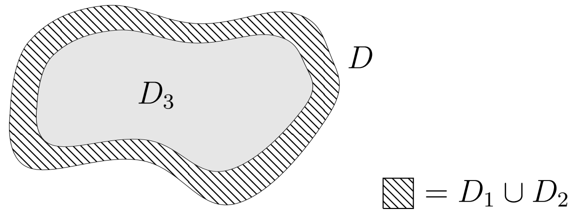

FIGURE 4 The partitioning of D.

The domain D_1: The following estimate for $r_t^D(x, y)$ comes from [21, Lemma 2.4].

Lemma 4.3.1. *Suppose $\psi \in WLSC(\underline{\alpha}, \theta)$ and $T(t) < 1/\theta$. Then*

$$r_t^D(x, y) \leq C \left\{ T(t)^{-d} \wedge \frac{t}{\delta_D^d(x) V^2 \left(\delta_D(x)\right)} \right\}. \tag{4.14}$$

We have assumed that the Lévy-Khintchine exponent satisfies some scaling conditions, see equation (2.43). Thus $\psi \in WLSC\left(\underline{\alpha}, \theta\right)$ and so Lemma 4.3.1 implies that if $T(t) < 1/\theta$, then

$$\begin{aligned} \int_{D_1} r_t^D(x, x)\, dx &\leq C \int_{D_1} T(t)^{-d}\, dx \\ &= CT(t)^{-d} |D_1|. \end{aligned} \tag{4.15}$$

But, by Lemma 4.2.1, we know that the measure of the set of bad points near the boundary is small. Hence if $T(t) < 1/\theta$, then

$$\int_{D_1} r_t^D(x, x)\, dx \leq C(\partial D) \varepsilon s T(t)^{-d} \leq \boxed{C\sqrt{\varepsilon} T(t)^{1-d}}, \tag{4.16}$$

where C is a constant depending on d, $\underline{\alpha}$, and ∂D.

The domain D_3: By assumption $\psi \in WLSC(\underline{\alpha}, \theta)$ and so we can apply Lemma 4.3.1 again: if $T(t) < 1/\theta$, then

$$\int_{D_3} r_t^D(x,x)\,dx \le CT(t)^{-d} \int_{D_3} \left\{ 1 \wedge \frac{T(t)^d}{\delta_D^d(x)} \frac{V^2(T(t))}{V^2(\delta_D(x))} \right\} dx. \tag{4.17}$$

Next, our Potter-like bound in Lemma 2.3.1 tells us that if $T(t) < 1/\theta$, then

$$\int_{D_3} r_t^D(x,x)\,dx \le CT(t)^{-d} \int_{D_3} \left\{ 1 \wedge \frac{T(t)^d}{\delta_D^d(x)} \left(\frac{T(t)^{\underline{\alpha}}}{\delta_D^{\underline{\alpha}}(x)} \vee \frac{T(t)^{\overline{\alpha}}}{\delta_D^{\overline{\alpha}}(x)} \right) \right\} dx. \tag{4.18}$$

By definition, for any $x \in D_3$ we have $\delta_D(x) \ge s = T(t)/\sqrt{\varepsilon}$. Or equivalently $1 \le \frac{\delta_D(x)}{T(t)}\sqrt{\varepsilon}$. Hence

$$\begin{aligned}
\int_{D_3} r_t^D(x,x)\,dx &\le CT(t)^{-d} \int_{D_3} \left\{ 1 \wedge \sqrt{\varepsilon}\frac{T(t)^{d-1}}{\delta_D^{d-1}(x)} \left(\frac{T(t)^{\underline{\alpha}}}{\delta_D^{\underline{\alpha}}(x)} \vee \frac{T(t)^{\overline{\alpha}}}{\delta_D^{\overline{\alpha}}(x)} \right) \right\} dx \\
&\le CT(t)^{-d} \int_{D} \left\{ \left(1 \wedge \sqrt{\varepsilon}\frac{T(t)^{d+\underline{\alpha}-1}}{\delta_D^{d+\underline{\alpha}-1}(x)} \right) + \left(1 \wedge \sqrt{\varepsilon}\frac{T(t)^{d+\overline{\alpha}-1}}{\delta_D^{d+\overline{\alpha}-1}(x)} \right) \right\} dx \\
&= CT(t)^{1-d} \frac{1}{T(t)} \int_{D} \left\{ 1 \wedge \sqrt{\varepsilon} \left(\frac{\delta_D(x)}{T(t)} \right)^{1-d-\underline{\alpha}} \right. \\
&\qquad \left. + 1 \wedge \sqrt{\varepsilon} \left(\frac{\delta_D(x)}{T(t)} \right)^{1-d-\overline{\alpha}} \right\} dx.
\end{aligned} \tag{4.19}$$

We are now in a position to apply the following proposition from [4]:

Proposition 4.3.2. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Suppose that $f : (0, \infty) \to \mathbb{R}$ is continuous and satisfies $f(s) \le c\left(1 \wedge s^{-\beta}\right)$, $s > 0$, for some $\beta > 1$, and suppose that for any $0 < R_1 < R_2 < \infty$, $f(s)$ is Lipschitz on $[R_1, R_2]$. Then*

$$\lim_{\eta \to 0^+} \frac{1}{\eta} \int_D f\left(\frac{\delta_D(x)}{\eta} \right) dx = \mathcal{H}^{d-1}(\partial D) \int_0^\infty f(s)\,ds. \tag{4.20}$$

Letting $\eta = T(t)$ and $f(s) = 1 \wedge \sqrt{\varepsilon} s^{-d-\underline{\alpha}+1}$ and $f(s) = 1 \wedge \sqrt{\varepsilon} s^{-d-\overline{\alpha}+1}$, respectively, we can apply Proposition 4.3.2 to both of the integrals in (4.19). Thus for small values of t we get

$$\int_{D_3} r_t^D(x,x)\,dx \leq C \frac{\mathcal{H}^{d-1}(\partial D)}{T(t)^{d-1}} \int_0^\infty \left\{ \left(1 \wedge \sqrt{\varepsilon} r^{-d-\underline{\alpha}+1}\right) + \left(1 \wedge \sqrt{\varepsilon} r^{-d-\overline{\alpha}+1}\right) \right\} dr.$$

Using substitution this becomes

$$\begin{aligned} \int_{D_3} r_t^D(x,x)\,dx &\leq C(\partial D) T(t)^{1-d} \Big\{ \varepsilon^{\frac{1}{2(d+\underline{\alpha}-1)}} \int_0^\infty \left(1 \wedge r^{-d-\underline{\alpha}+1}\right) dr \\ &\qquad + \varepsilon^{\frac{1}{2(d+\overline{\alpha}-1)}} \int_0^\infty \left(1 \wedge r^{-d-\overline{\alpha}+1}\right) dr \Big\} \\ &\leq \boxed{C T(t)^{1-d} \left(\varepsilon^{\frac{1}{2(d+\underline{\alpha}-1)}} + \varepsilon^{\frac{1}{2(d+\overline{\alpha}-1)}} \right)}. \end{aligned} \tag{4.21}$$

This covers the domains D_1 and D_3.

The domain D_2**:** It remains to show that $r_t^D(x,x)$ is comparable to $r_t^{H^*}(x,x)$ on D_2. Suppose $x \in D_2 \subset \mathcal{G}$. Let $q(x)$ be as above. Then $x \in \Gamma_r(q(x), \varepsilon)$. For the purposes of brevity we will use the following notation $\mathcal{I} := I_r(q(x))$ and $\mathcal{U}^c := U_r^c(q(x))$.

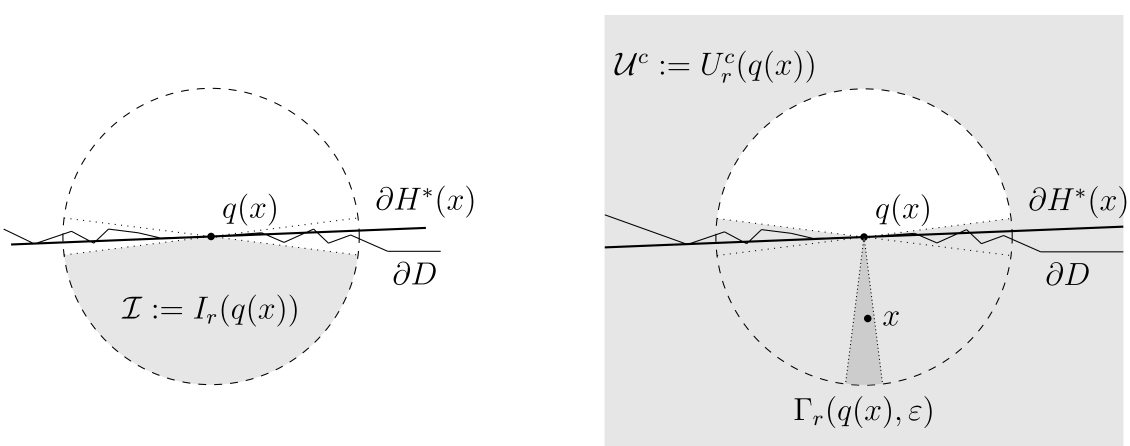

FIGURE 5 Regions used to estimate the heat remainder on D_2.

Notice that

$$H^*(x) \subseteq \mathcal{U}^c \quad \text{and} \quad \mathcal{I} \subseteq D.$$

Hence

$$\left| r_t^D(x,x) - r_t^{H^*(x)}(x,x) \right| \;\leq\; r_t^{\mathcal{I}}(x,x) - r_t^{\mathcal{U}^c}(x,x). \tag{4.22}$$

We can use the following proposition to estimate the heat remainder on the above inner and outer regions:

Proposition 4.3.3. *Let $v(q) \in \mathbb{R}^d$ be a unit vector. Assume that $0 < \varepsilon < 1/4$ and $r > 0$. If $x \in \Gamma_{2s}(v(q), \varepsilon)$ and $s = T(t)/\sqrt{\varepsilon} < r/4$, then*

$$0 \leq r_t^{\mathcal{I}}(x,x) - r_t^{\mathcal{U}^c}(x,x) \leq \frac{\left(\varepsilon^{1-\underline{\alpha}/2} + \varepsilon^{1-\overline{\alpha}/2}\right) \vee \sqrt{\varepsilon}}{T(t)^d} \left(1 \wedge \frac{T(t)^{d-1}}{\delta_{\mathcal{I}}^{d-1}(x)} \frac{V^2(T(t))}{V^2\left(\delta_{\mathcal{I}}(x)\right)} \right). \tag{4.23}$$

We postpone the proof of this proposition until Section 4.4.

From equation (4.22) and Proposition 4.3.3 we get

$$\begin{aligned}
\int_{D_2} \left| r_t^D(x,x) - r_t^{H^*}(x,x) \right| dx \;&\leq\; \int_{D_2} \left(r_t^{\mathcal{I}}(x,x) - r_t^{\mathcal{U}^c}(x,x) \right) dx \\
&\leq\; \frac{C(\varepsilon)}{T(t)^d} \int_{D_2} \left(1 \wedge \frac{T(t)^{d-1}}{\delta_{\mathcal{I}}^{d-1}(x)} \frac{V^2(T(t))}{V^2\left(\delta_{\mathcal{I}}(x)\right)} \right) dx. \qquad (4.24)
\end{aligned}$$

Notice that since $x \in \Gamma_{2s}(v(q), \varepsilon)$, $\partial D \cap B(q,r) \subset B(q,r) \backslash \mathcal{I}$, and $\varepsilon < 1/4$ we have

$$\begin{aligned}
\delta_{\mathcal{I}}(x) \;&\geq\; |x-q| \sin\left(2\varphi_\varepsilon - \pi/2\right) = -|x-q| \cos\left(2\varphi_\varepsilon\right) \\
&=\; \left(1 - 2\varepsilon^2\right) |x-q| \geq \left(1 - 2\varepsilon^2\right) \delta_D(x) > \frac{7}{8} \delta_D(x). \qquad (4.25)
\end{aligned}$$

Hence our bound for the difference between the heat remainders becomes

$$\int_{D_2} \left| r_t^D(x,x) - r_t^{H^*}(x,x) \right| dx \leq \frac{C(\varepsilon)}{T(t)^d} \int_{D_2} \left(1 \wedge \frac{T(t)^{d-1}}{\delta_D^{d-1}(x)} \frac{V^2(T(t))}{V^2\left(\delta_D(x)\right)} \right) dx. \tag{4.26}$$

We can use our Potter-like bounds, (2.41), again: if $T(t) < 1/\theta$, then

$$\begin{aligned}\int_{D_2} \left| r_t^D(x,x) - r_t^{H^*}(x,x) \right| dx &\leq \frac{C(\varepsilon)}{T(t)^d} \int_{D_2} \left\{ 1 \wedge \frac{T(t)^{d-1}}{\delta_D^{d-1}(x)} \left(\frac{T(t)^{\underline{\alpha}}}{\delta_D^{\underline{\alpha}}(x)} \vee \frac{T(t)^{\overline{\alpha}}}{\delta_D^{\overline{\alpha}}(x)} \right) \right\} dx \\ &\leq \frac{C(\varepsilon)}{T(t)^d} \int_{D_2} \left\{ 1 \wedge \left(\frac{T(t)}{\delta_D(x)} \right)^{d+\underline{\alpha}-1} \right. \\ &\qquad \left. + 1 \wedge \left(\frac{T(t)}{\delta_D(x)} \right)^{d+\overline{\alpha}-1} \right\} dx. \end{aligned} \tag{4.27}$$

Letting $\eta = T(t)$, as above, we can apply Proposition 4.3.2. This says that for sufficiently small t we have

$$\begin{aligned}\int_{D_2} \left| r_t^D(x,x) - r_t^{H^*}(x,x) \right| dx &\leq \frac{C(\varepsilon)}{T(t)^{d-1}} \mathcal{H}^{d-1}\left(\partial D\right) \int_0^\infty \left\{ \left(1 \wedge r^{-d-\underline{\alpha}+1} \right) \right. \\ &\qquad \left. + \left(1 \wedge r^{-d-\overline{\alpha}+1} \right) \right\} dr \\ &\leq \boxed{C(\varepsilon) T(t)^{1-d}}. \end{aligned} \tag{4.28}$$

Finally, it remains to show that

$$\left| \int_{D_2} r_t^{H^*(x)}(x,x)\, dx - \frac{\mathcal{H}^{d-1}(\partial D)}{T(t)^{d-1}} \int_0^\infty r_t^{\mathbb{H}}\left((q,0,...,0), (q,0,...,0) \right) dq \right| \leq c(\varepsilon) T(t).$$

To do this we apply Proposition 4.3.2 to $\int_{D_2} r_t^{H^*(x)}(x,x)dx$. Note that, by construction, the diagonal of the heat remainder of H^* can be expressed in terms of the diagonal of the heat remainder of $\mathbb{H}$:

$$\begin{aligned} r_t^{H^*(x)}(x,x) &= r_t^{H^*(x)}\left(\left(\delta_{H^*(x)}(x),0,...,0\right),\left(\delta_{H^*(x)}(x),0,...,0\right)\right) &(4.29)\\ &= r_t^{H^*(x)}\left(\left(\delta_D(x),0,...,0\right),\left(\delta_D(x),0,...,0\right)\right) &(4.30)\\ &= r_t^{\mathbb{H}}\left(\left(\delta_D(x),0,...,0\right),\left(\delta_D(x),0,...,0\right)\right) &(4.31)\\ &=: r_t^{\mathbb{H}}\left(\delta_D(x)\right). &(4.32)\end{aligned}$$

The heat remainder of $H^*(x)$ on D_2 satisfies

$$\int_{D_2} r_t^{H^*(x)}(x,x)\,dx = \int_D r_t^{\mathbb{H}}\left(\delta_D(x)\right)dx - \int_{D_1\cup D_3} r_t^{\mathbb{H}}\left(\delta_D(x)\right)dx \tag{4.33}$$

and the argument we used in equations (4.16) and (4.21) can be applied to the heat remainder of $\mathbb{H}$ on $D_1 \cup D_3$:

$$\int_{D_1\cup D_3} r_t^{\mathbb{H}}\left(\delta_D(x)\right)dx \le c(\varepsilon)T(t)^{1-d}, \tag{4.34}$$

where $c(\varepsilon) \to 0$, as $\varepsilon \to 0$.

The bound for the heat remainder in Lemma 4.3.1 tells us that

$$r_t^{\mathbb{H}}(\delta_D(x)) \le CT(t)^{-d}\left(1 \wedge \frac{T(t)^d}{\delta_D^d}\frac{V^2(T(t))}{V^2\left(\delta_D(x)\right)}\right). \tag{4.35}$$

To which we can apply our Potter-like bounds from Lemma 2.3.1:

$$r_t^{\mathbb{H}}(\delta_D(x)) \le \frac{C}{T(t)^d}\left\{1 \wedge \left(\frac{T(t)}{\delta_D(x)}\right)^{d+\underline{\alpha}} + 1 \wedge \left(\frac{T(t)}{\delta_D(x)}\right)^{d+\overline{\alpha}}\right\}. \tag{4.36}$$

We wish to show that $r_t^{\mathbb{H}}(\delta_D(x))$ satisfies the assumptions of Proposition 4.3.2. Hence we must show that $r_t^{\mathbb{H}}(t, \delta_D(x))$ is Lipschitz. Firstly, the following bound for the gradient of $p_t(x)$ is provided by [32]:

Lemma 4.3.4. *Let $\psi \in WLSC(\underline{\alpha}, \theta)$. For $T(t) < 1/\theta$ we have*

$$|\nabla_x p_t(x)| \leq \frac{c}{T(t)} \min\left\{p_t(0), \frac{t}{|x|^d V^2(|x|)}\right\}. \tag{4.37}$$

Next we show that the heat remained of D is a Lipschitz function in one of its spacial components

Lemma 4.3.5. *Let $D \subset \mathbb{R}^d$ be an open nonempty set. Fix $\varepsilon > 0$. For any $y \in D$ and w, $z \in D$ with $\delta_D(w) > \varepsilon$, $\delta_D(z) > \varepsilon$, there exists $c(\varepsilon, t)$ such that*

$$\left|r_t^D(w,y) - r_t^D(z,y)\right| \leq c(\varepsilon, t)\,|w - z|\,. \tag{4.38}$$

Proof. The mean value theorem and Lemma 4.3.4 tells us that there exists some $0 \leq l \leq 1$ such that

$$\begin{aligned}
|p_t(w) - p_t(z)| &\leq |\nabla_x p_t(lw + (1-l)w)|\,|w - z| \\
&\leq \frac{c}{T(t)} \min\left\{p_t(0), \frac{t}{|lw + (1-l)z|^d V^2\left(|lw + (1-l)z|\right)}\right\} |w - z| \\
&\leq \frac{c}{T(t)} \min\left\{p_t(0), \frac{t}{(|w| \wedge |z|)^d V^2\left(|w| \wedge |z|\right)}\right\} |w - z|. \qquad (4.39)
\end{aligned}$$

Recall the definition of the heat remainder (2.7)

$$r_t^D(x,y) = \mathbb{E}_y\left[\tau_D < t;\ p_{t-\tau_D}\left(X_{\tau_D} - x\right)\right].$$

This allows us to rewrite equation (4.39) as follows

$$\begin{aligned}
\left|r_t^D(w,y) - r_t^D(z,y)\right| &\leq \mathbb{E}_y\left[\tau_D < t;\ p_{t-\tau_D}\left(X_{\tau_D} - w\right) - p_{t-\tau_D}\left(X_{\tau_D} - z\right)\right] \\
&\leq c\mathbb{E}_y\Bigg[\tau_D < t;\ \frac{|w-z|}{T(t-\tau_D)} \min\Bigg\{p_{t-\tau_D}(0), \\
&\qquad \frac{t-\tau_D}{\left(|X_{\tau_D} - w| \wedge |X_{\tau_D} - z|\right)^d V^2\left(|X_{\tau_D} - w| \wedge |X_{\tau_D} - z|\right)}\Bigg\}\Bigg] \\
&\leq c\mathbb{E}_y\Bigg[\tau_D < t;\ \frac{|w-z|}{T(t-\tau_D)} \min\Bigg\{p_{t-\tau_D}(0), \\
&\qquad \frac{t-\tau_D}{\left(|\delta_D(w)| \wedge |\delta_D(z)|\right)^d V^2\left(|\delta_D(w)| \wedge |\delta_D(z)|\right)}\Bigg\}\Bigg]
\end{aligned}$$

Hence our assumption that both $\delta_D(w)$ and $\delta_D(z)$ are larger than ε implies

$$\begin{aligned}
\left|r_t^D(w,y) - r_t^D(z,y)\right| &\leq c\frac{|w-z|}{\left(|\delta_D(w)| \wedge |\delta_D(z)|\right)^d V^2\left(|\delta_D(w)| \wedge |\delta_D(z)|\right)} \\
&\qquad \times \mathbb{E}^y\left[\tau_D < t;\ \frac{t-\tau_D}{T(t-\tau_D)}\right] \\
&\leq c(\varepsilon,t)|w-z|. \qquad (4.40)
\end{aligned}$$

□

We can now show that $r_t^{\mathbb{H}}\left(\delta_D(x)\right)$ is a Lipschitz function:

Lemma 4.3.6. *Let $D \subset \mathbb{R}^d$ be an open nonempty set. Fix $\varepsilon > 0$. For any $y \in D$ and $w,\ z \in D$ with $\delta_D(w) > \varepsilon$, $\delta_D(z) > \varepsilon$, there exists $c(\varepsilon,t)$ such that*

$$\left|r_t^D(w,w) - r_t^D(z,z)\right| \leq c(\varepsilon,t)\left|w-z\right|. \qquad (4.41)$$

Proof. Using Lemma 4.3.5 and the symmetry of the heat remainder, that is $r_t^D(w,z) = r_t^D(z,w)$, we get

$$\begin{aligned} \left| r_t^D(w,w) - r_t^D(z,z) \right| &\leq \left| r_t^D(w,w) - r_t^D(z,w) \right| + \left| r_t^D(w,z) - r_t^D(z,z) \right| && (4.42) \\ &\leq c(\varepsilon,t) \left| w - z \right|. && (4.43) \end{aligned}$$

□

Lemma 4.3.6 tells us that $r_t^{\mathbb{H}}(\delta_D(x))$ is a Lipschitz function, thus it satisfies the assumptions of Proposition 4.3.2. Therefore we can apply Proposition 4.3.2 to $r_t^{\mathbb{H}}(\delta_D(x))$. For small t, we have

$$\left| \int_D r_t^{\mathbb{H}}(\delta_D(x))\, dx - C_{\mathbb{H}}(t)\mathcal{H}^{d-1}(\partial D) \right| \leq \boxed{\varepsilon T(t)^{1-d}}. \tag{4.44}$$

This completes the proof of Theorem 4.1.1.

□

4.4. Heat Remainders of the Inner and Outer Regions

All that remains is to prove Proposition 4.3.3.

Proof of Proposition 4.3.3. We wish to show that

$$0 \leq r_t^{\mathcal{I}}(x,x) - r_t^{\mathcal{U}^c}(x,x) \leq \frac{\left(\varepsilon^{1-\underline{\alpha}/2} + \varepsilon^{1-\overline{\alpha}/2}\right) \vee \sqrt{\varepsilon}}{T(t)^d} \left(1 \wedge \frac{T(t)^{d-1}}{\delta_{\mathcal{I}}^{d-1}(x)} \frac{V^2(T(t))}{V^2\left(\delta_{\mathcal{I}}(x)\right)}\right). \tag{4.45}$$

In order to establish this inequality we combine different aspects of similar proofs given in Proposition 3.2 of [21] and Proposition 3.1 of [4].

Firstly, by definition, we have

$$r_t^{\mathcal{I}}(x,x) - r_t^{\mathcal{U}^c}(x,x) = p_t^{\mathcal{U}^c}(x,x) - p_t^{\mathcal{I}}(x,x) \tag{4.46}$$

$$= \mathbb{E}^x\left[\tau_{\mathcal{I}} < t,\ X(\tau_{\mathcal{I}}) \in \mathcal{U}^c \backslash \mathcal{I};\ p_{t-\tau_{\mathcal{I}}}^{\mathcal{U}^c}(X(\tau_{\mathcal{I}}), x)\right]. \tag{4.47}$$

The space-time Ikeda-Watanabe formula (2.16) then tells us that

$$r_t^{\mathcal{I}}(x,x) - r_t^{\mathcal{U}^c}(x,x) = \int_{\mathcal{I}} \int_0^t p_l^{\mathcal{I}}(x,y) \int_{\mathcal{U}^c \backslash \mathcal{I}} \nu(y-z)\, p_{t-l}^{\mathcal{U}^c}(x,z)\, dz\, dl\, dy. \tag{4.48}$$

Without loss of generality we can assume that $q = 0$ and $v(0) = (1, 0, \ldots, 0)$. Let

$$I = \{y\ :\ y \cdot v(0) > \varepsilon |y|\}, \tag{4.49}$$

$$U = \{y\ :\ y \cdot v(0) < -\varepsilon |y|\}, \tag{4.50}$$

$$\Gamma(0,\varepsilon) = \left\{y\ :\ y \cdot v(0) > \sqrt{1-\varepsilon^2}|y|\right\}. \tag{4.51}$$

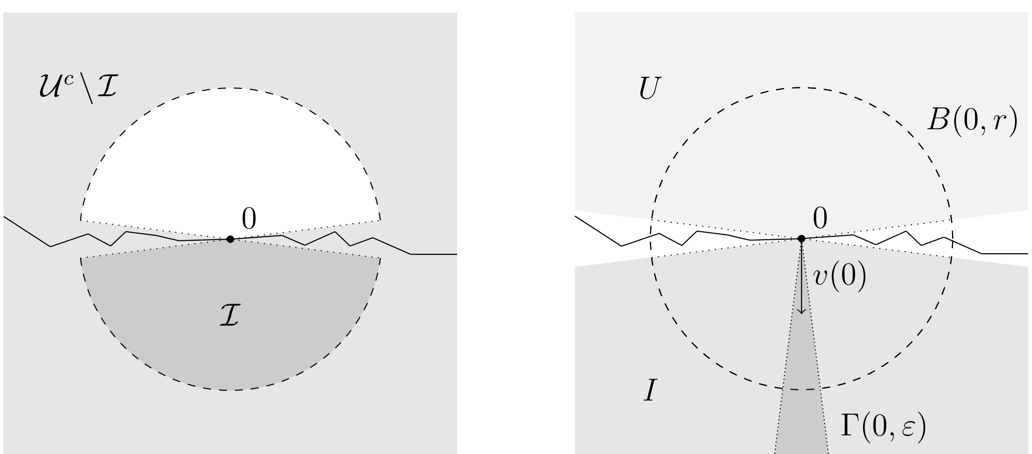

FIGURE 6 Interior and exterior regions of the domain.

Notice that

$$\mathcal{U}^c \backslash \mathcal{I} = B^c(0,r) \cup (U^c \backslash I) \qquad \text{and} \qquad \mathcal{I} \subset I.$$

By the construction of these domains we can see that equation (4.48) can be broken up as follows

$$\begin{aligned} r_t^{\mathcal{I}}(x,x) - r_t^{\mathcal{U}^c}(x,x) &\leq \int_{\mathcal{I}} \int_0^t p_l^{\mathcal{I}}(x,y) \int_{(U^c \backslash I) \cap B(0,r)} \nu(y-z)\, p_{t-l}^{\mathcal{U}^c}(x,z)\, dz\, dl\, dy \\ &\quad + \int_{\mathcal{I}} \int_0^t p_l^{\mathcal{I}}(x,y) \int_{B^c(0,r)} \nu(y-z)\, p_{t-l}^{\mathcal{U}^c}(x,z)\, dz\, dl\, dy \\ &= A_t(x) + B_t(x). \end{aligned} \tag{4.52}$$

$\boldsymbol{A_t(x)}$ **:** Lemma 1.5 in [14] gives a bound for the heat kernel under certain scaling conditions:

Lemma 4.4.1. *Suppose $\psi \in WLSC(\underline{\alpha}, \theta)$ and $T(t) < 1/\theta$. Then there exists a constant C such that*

$$p_t(x-z) \leq C \left(T^{-d}(t) \wedge \frac{t}{|x-z|^d V^2(|x-z|)} \right). \tag{4.53}$$

Notice that if $x \in \Gamma(0,\varepsilon)$ and $z \in U^c \backslash I = \{y : -\varepsilon|y| < y \cdot v(0) < \varepsilon|y|\}$, then

$$|x-z| \geq |x| \sin\left(2\varphi_\varepsilon - \frac{\pi}{2}\right) = |x| \left(1 - 2\cos^2(\varphi_\varepsilon)\right) = |x|(1-2\varepsilon^2). \tag{4.54}$$

Thus Lemma 4.4.1 and the monotonicity of $V(r)$ imply that

$$\begin{aligned} p_{t-l}(x-z) &\leq C\frac{1}{|x-z|^d}\frac{t}{V^2(|x-z|)} \\ &\leq C\frac{1}{(1-2\varepsilon^2)^d\,|x|^d}\frac{t}{V^2\left((1-2\varepsilon^2)|x|\right)}. \end{aligned} \tag{4.55}$$

By assumption $\psi \in WUSC\left(\overline{\alpha},\theta\right)$ and $\varepsilon < 1/4$, hence

$$p_{t-l}(x-z) \;\leq\; C\left(1-2\varepsilon^2\right)^{-d-\overline{\alpha}}\frac{1}{|x|^d}\frac{t}{V^2\left(|x|\right)} \leq C\frac{1}{|x|^d}\frac{t}{V^2\left(|x|\right)}. \tag{4.56}$$

We can now apply this bound directly to $A_t(x)$:

$$\begin{aligned} A_t(x) &\leq \int_{\mathcal{I}}\int_0^t p_l^{\mathcal{I}}(x,y)\int_{(U^c\setminus I)\cap B(0,r)}\nu(y-z)\,p_{t-l}(x,z)\,dz\,dl\,dy \\ &\leq \frac{C}{|x|^d}\frac{t}{V^2(|x|)}\int_{\mathcal{I}}\int_0^t p_l^{\mathcal{I}}(x,y)\int_{(U^c\setminus I)\cap B(0,r)}\nu(y-z)\,dz\,dl\,dy \\ &\leq \frac{C}{|x|^d}\frac{t}{V^2(|x|)}\int_{\mathcal{I}}\int_0^{V^2(1/\theta)} p_l^{\mathcal{I}}(x,y)\,dl\int_{(U^c\setminus I)\cap B(0,r)}\nu(y-z)\,dz\,dy \\ &= \frac{C}{|x|^d}\frac{t}{V^2(|x|)}\int_{(U^c\setminus I)\cap B(0,r)}\int_{\mathcal{I}} G_{\mathcal{I}}^{V^2(1/\theta)}(x,y)\,\nu(y-z)\,dy\,dz \qquad (4.57) \\ &= \frac{C}{|x|^d}\frac{t}{V^2(|x|)}\int_{(U^c\setminus I)\cap B(0,r)} P_{\mathcal{I}}^{V^2(1/\theta)}(x,z)\,dz, \qquad (4.58) \end{aligned}$$

where in the last two equations we have used definitions of the truncated Green function and the truncated Poisson kernel, (2.10) and (2.12) respectively. We can then apply the bound for truncated Poisson kernels given in Lemma 2.9 of [21]:

$$A_t(x) \;\leq\; \frac{C}{|x|^d}\frac{t}{V^2(|x|)}\int_{(U^c\setminus I)\cap B(0,r)}\frac{c_\theta}{|x-z|^d}\frac{V(\delta_{\mathcal{I}}(x))}{V(\delta_{\mathcal{I}^c}(z))}\,dz. \tag{4.59}$$

Our Potter-like bounds in Lemma 2.3.1 tell us that

$$\begin{aligned}
\int\limits_{(U^c\setminus I)\cap B(0,r)} \frac{1}{|x-z|^d}\frac{V(\delta_{\mathcal{I}}(x))}{V(\delta_{\mathcal{I}^c}(z))}\,dz &\leq \int\limits_{(U^c\setminus I)\cap B(0,r)} \frac{1}{|x-z|^d}\left\{\left(\frac{\delta_{\mathcal{I}}(x)}{\delta_{\mathcal{I}^c}(z)}\right)^{\underline{\alpha}/2} \vee \left(\frac{\delta_{\mathcal{I}}(x)}{\delta_{\mathcal{I}^c}(z)}\right)^{\overline{\alpha}/2}\right\} dz \\
&\leq \delta_{\mathcal{I}}^{\underline{\alpha}/2}(x) \int\limits_{(U^c\setminus I)\cap B(0,r)} \frac{dz}{\delta_{\mathcal{I}^c}^{\underline{\alpha}/2}(z)|x-z|^d} \\
&\qquad + \delta_{\mathcal{I}}^{\overline{\alpha}/2}(x) \int\limits_{(U^c\setminus I)\cap B(0,r)} \frac{dz}{\delta_{\mathcal{I}^c}^{\overline{\alpha}/2}(z)|x-z|^d}.
\end{aligned}$$

In Lemma 3.2 of [4] it is shown that:

Lemma 4.4.2. *For any $\varepsilon \in (0, 1/4)$, $w \in \Gamma(0,\varepsilon)$, $M \in (0,\infty]$ we have*

$$\int\limits_{(U^c\setminus I)\cap B(0,M)} \frac{dz}{\delta_{I^c}^{\alpha/2}(z)|z-w|^\gamma} \leq \begin{cases} c_\gamma \varepsilon^{1-\alpha/2}|w|^{d-\alpha/2-\gamma} & \textit{for } \gamma > d-\alpha/2, \\ c_\gamma \varepsilon^{1-\alpha/2} M^{d-\alpha/2-\gamma} & \textit{for } 0<\gamma<d-\alpha/2. \end{cases} \tag{4.60}$$

Notice that for $z \in (U^c\setminus I)\cap B(0,r)$ we must have $\delta_{I^c}(z) = \delta_{\mathcal{I}^c}(z)$. Thus for $\gamma = d$ we get:

$$\begin{aligned}
\int\limits_{(U^c\setminus I)\cap B(0,r)} \frac{1}{|x-z|^d}\frac{V(\delta_{\mathcal{I}}(x))}{V(\delta_{\mathcal{I}^c}(z))}\,dz &\leq C\left\{\frac{\delta_{\mathcal{I}}^{\underline{\alpha}/2}(x)\varepsilon^{1-\underline{\alpha}/2}}{|x|^{\underline{\alpha}/2}} + \frac{\delta_{\mathcal{I}}^{\overline{\alpha}/2}(x)\varepsilon^{1-\overline{\alpha}/2}}{|x|^{\overline{\alpha}/2}}\right\} \\
&\leq C\left\{\frac{\delta_{\mathcal{I}}^{\underline{\alpha}/2}(x)\varepsilon^{1-\underline{\alpha}/2}}{\delta_{\mathcal{I}}^{\underline{\alpha}/2}(x)} + \frac{\delta_{\mathcal{I}}^{\overline{\alpha}/2}(x)\varepsilon^{1-\overline{\alpha}/2}}{\delta_{\mathcal{I}}^{\overline{\alpha}/2}(x)}\right\} \\
&\leq C\left\{\varepsilon^{1-\underline{\alpha}/2} + \varepsilon^{1-\overline{\alpha}/2}\right\}.
\end{aligned} \tag{4.61}$$

This gives us one bound for $A_t(x)$:

$$A_t(x) \leq \boxed{C\left(\varepsilon^{1-\underline{\alpha}/2} + \varepsilon^{1-\overline{\alpha}/2}\right)\frac{1}{|x|^d}\frac{V^2(T(t))}{V^2(|x|)}}. \tag{4.62}$$

Let us now consider $A_t(x)$ from another perspective. We divide $A_t(x)$ into the following subregions:

$$\begin{aligned} A_t(x) &= \int_{\mathcal{I}}\int_0^{t/2} p_l^{\mathcal{I}}(x,y)\int_{(U^c\setminus I)\cap B(0,r)} \nu(y-z)\,p_{t-l}^{\mathcal{U}^c}(x,z)\,dz\,dl\,dy \\ &\quad+\int_{\mathcal{I}}\int_{t/2}^{t} p_l^{\mathcal{I}}(x,y)\int_{(U^c\setminus I)\cap B(0,r)\cap\{|x-z|\le T\}} \nu(y-z)\,p_{t-l}^{\mathcal{U}^c}(x,z)\,dz\,dl\,dy \\ &\quad+\int_{\mathcal{I}}\int_{t/2}^{t} p_l^{\mathcal{I}}(x,y)\int_{(U^c\setminus I)\cap B(0,r)\cap\{|x-z|> T\}} \nu(y-z)\,p_{t-l}^{\mathcal{U}^c}(x,z)\,dz\,dl\,dy \\ &\quad= \mathbf{I}+\mathbf{II}+\mathbf{III}. \end{aligned} \tag{4.63}$$

Short jump time: I. For $l \in [0,t/2]$ we can use the bound for the heat kernel given in equation (4.53) of Lemma 4.4.1:

$$p_{t-l}^{\mathcal{U}^c}(x,z) \le p_{t-l}(x,z) \le CT(t-l)^{-d}. \tag{4.64}$$

The monotonicity of $T(t)$ then implies

$$p_{t-l}^{\mathcal{U}^c}(x,z) \le CT\,(t/2)^{-d}\,. \tag{4.65}$$

The scaling of $\psi(\xi)$ at infinity implies the scaling of $T(t)$ at 0, as is shown in Lemma 2.1 of [21]. Hence

$$p_{t-l}^{\mathcal{U}^c}(x,z) \le C\,(1/2)^{-d/\underline{\alpha}}\,T(t)^{-d} = CT(t)^{-d}. \tag{4.66}$$

Thus

$$\mathbf{I} \le CT(t)^{-d} \int_{\mathcal{I}} \int_0^{t/2} p_l^{\mathcal{I}}(x,y) \int_{(U^c \setminus I) \cap B(0,r)} \nu(y-z)\, dz\, dl\, dy \tag{4.67}$$

$$\le CT(t)^{-d} \int_{(U^c \setminus I) \cap B(0,r)} P_{\mathcal{I}}^{V^2(1/\theta)}(x,z)\, dz. \tag{4.68}$$

It now follows from the calculations in (4.58), (4.59), (4.60), and (4.61) that

$$\mathbf{I} \le \boxed{C\left(\varepsilon^{1-\underline{\alpha}/2} + \varepsilon^{1-\overline{\alpha}/2}\right) T(t)^{-d}}. \tag{4.69}$$

Long exit time and short jumps: II. The following bound for the heat kernel is given in Lemma 2.6 of [21]:

Lemma 4.4.3. *There exists a constant c_θ such that if $T(t) < 1/\theta \vee |x-y|$, then*

$$p_t^D(x,y) \le c_\theta \left(\frac{V(\delta_D(x))}{V(T)} \wedge 1 \right) \left(\frac{V(\delta_D(y))}{V(T)} \wedge 1 \right) \left(\frac{t}{|x-y|^d V^2(|x-y|)} \wedge T(t)^{-d} \right). \tag{4.70}$$

Let $S := (U^c \setminus I) \cap B(0,r) \cap \{|x-z| \le T\}$. For $l \in [t/2, t)$ we can use the bounds from Lemma 4.4.1 and Lemma 4.4.3 to get

$$\begin{aligned}
\mathbf{II} &= \int_{\mathcal{I}} \int_{t/2}^{t} p_l^{\mathcal{I}}(x,y) \int_S \nu(y-z) p_{t-l}^{\mathcal{U}^c}(x,z)\, dz\, dl\, dy \\
&\le C \int_{\mathcal{I}} \int_{t/2}^{t} T(t)^{-d} \frac{V(\delta_{\mathcal{I}}(y))}{V(T(t))} \int_S \frac{1}{|y-z|^d V^2(|y-z|)} p_{t-l}^{\mathcal{U}^c}(x,z)\, dz\, dl\, dy \\
&= CT(t)^{-d} \int_{\mathcal{I}} \int_S \frac{V(\delta_{\mathcal{I}}(y))}{V(T(t))} \frac{1}{|y-z|^d V^2(|y-z|)} \int_{t/2}^{t} p_{t-l}^{\mathcal{U}^c}(x,z)\, dl\, dz\, dy \\
&\le CT(t)^{-d} \int_S \int_{\mathcal{I}} \frac{V(\delta_{\mathcal{I}}(y))}{V(T(t))} \frac{1}{|y-z|^d V^2(|y-z|)} G_{\mathcal{U}^c}^{t/2}(x,z)\, dy\, dz. \qquad (4.71)
\end{aligned}$$

It follows from bounds given in [14] and [21] that

$$\mathbf{II} \leq C\frac{T(t)^{-d}}{V(T(t))}\int_S\int_{\mathcal{I}}\frac{V(\delta_{\mathcal{I}}(y))}{|y-z|^d V^2(|y-z|)}\frac{V(|x|)V(\delta_{\mathcal{U}^c}(z))}{|x-z|^d}\,dy\,dz. \tag{4.72}$$

By construction $\delta_{\mathcal{I}}(y), \delta_{\mathcal{I}}(z) \leq |y-z|$ and so

$$\begin{aligned}
\mathbf{II} &\leq CT(t)^{-d}\frac{V(|x|)}{V(T(t))}\int_S\int_{\mathcal{I}}\frac{1}{|y-z|^d V(|y-z|)}\frac{V(\delta_{\mathcal{U}^c}(z))}{|x-z|^d}\,dy\,dz \\
&\leq CT(t)^{-d}\frac{V(|x|)}{V(T(t))}\int_S\int_{\mathcal{I}}\frac{\delta_{\mathcal{I}}^{\underline{\alpha}/2}(z)}{|y-z|^{d+\underline{\alpha}/2}V(\delta_{\mathcal{I}}(z))}\frac{V(\delta_{\mathcal{U}^c}(z))}{|x-z|^d}\,dy\,dz \\
&\leq CT(t)^{-d}\frac{V(|x|)}{V(T(t))}\int_S\frac{\delta_{\mathcal{I}}^{\underline{\alpha}/2}(z)}{|x-z|^d}\frac{V(\delta_{\mathcal{U}^c}(z))}{V(\delta_{\mathcal{I}}(z))}\int_{\mathcal{I}}\frac{1}{|y-z|^{d+\underline{\alpha}/2}}\,dy\,dz. \qquad (4.73)
\end{aligned}$$

We have seen in (4.54) that $|x-z| > (1-2\varepsilon^2)|x|$. Thus for these short jumps we have $(1-2\varepsilon^2)|x| < T(t)$ and hence $V(|x|) < cV(T(t))$, for some constant c. Therefore

$$\begin{aligned}
\mathbf{II} &\leq CT(t)^{-d}\int_S\frac{\delta_{\mathcal{I}}^{\underline{\alpha}/2}(z)}{|x-z|^d}\frac{V(\delta_{\mathcal{U}^c}(z))}{V(\delta_{\mathcal{I}}(z))}\int_{\mathcal{I}}\frac{1}{|y-z|^{d+\underline{\alpha}/2}}\,dy\,dz \\
&\leq CT(t)^{-d}\int_S\frac{\delta_{\mathcal{I}}^{\underline{\alpha}/2}(z)}{|x-z|^d}\frac{V(\delta_{\mathcal{U}^c}(z))}{V(\delta_{\mathcal{I}}(z))}\int_{B(z,\delta_{\mathcal{I}}(z))^c}\frac{1}{|y-z|^{d+\underline{\alpha}/2}}\,dy\,dz. \qquad (4.74)
\end{aligned}$$

Changing to polar coordinates we get

$$\begin{aligned}
\mathbf{II} &\leq CT(t)^{-d}\int_S\frac{\delta_{\mathcal{I}}^{\underline{\alpha}/2}(z)}{|x-z|^d}\frac{V(\delta_{\mathcal{U}}^c(z))}{V(\delta_{\mathcal{I}}(z))}\int_{\delta_{\mathcal{I}^c}(z)}^{\infty}\frac{1}{r^{d+\underline{\alpha}/2}}r^{d-1}\,dr\,dz \\
&= CT(t)^{-d}\int_S\frac{\delta_{\mathcal{I}}^{\underline{\alpha}/2}(z)}{|x-z|^d}\frac{V(\delta_{\mathcal{U}^c}(z))}{V(\delta_{\mathcal{I}}(z))}\frac{1}{\delta_{\mathcal{I}}^{\underline{\alpha}/2}(z)}\,dz \\
&= CT(t)^{-d}\int_S\frac{1}{|x-z|^d}\frac{V(\delta_{\mathcal{U}^c}(z))}{V(\delta_{\mathcal{I}}(z))}\,dz. \qquad (4.75)
\end{aligned}$$

Lemma 4.4.4. *For any* $\varepsilon \in (0,1/4)$, $x \in \Gamma(0,\varepsilon)$, $r \in (0,\infty)$ *we have*

$$\int_{(U^c\setminus I)\cap B(0,r)} \frac{1}{|x-z|^d}\frac{\delta_{\mathcal{U}^c}^{\alpha/2}(z)}{\delta_{\mathcal{I}}^{\alpha/2}(z)}\,dz \le c\varepsilon^{1-\alpha/2}. \tag{4.76}$$

Proof. Let us use polar coordinates $(\rho, \varphi_1, ..., \varphi_d)$, with center $q = 0$ and principal axis $v(0) = (1, 0, ..., 0)$. We prove this lemma for the case $d \ge 3$, the case with $d = 2$ is essentially the same but with different restrictions on the angle. As above, we let $\varphi_\varepsilon \in [0, \pi/2]$ be the angle such that $\cos(\varphi_\varepsilon) = \varepsilon$. Then

$$\begin{aligned} U^c\setminus I &= \{(\rho,\varphi_1,...,\varphi_{d-1}) : \varphi_1 \in (\varphi_\varepsilon, \pi-\varphi_\varepsilon)\}, \\ \delta_{\mathcal{I}}(z) &= \rho\sin(\varphi_1-\varphi_\varepsilon), \\ \delta_{\mathcal{U}^c}(z) &= \rho\sin(\varphi_\varepsilon+\varphi_1), \end{aligned}$$

for $z \in U^c\setminus I$.

Let $V_1 = (U^c\setminus I) \cap B(0,|x|)$ and $V_2 = (U^c\setminus I) \cap B^c(0,|x|) \cap B(0,r)$. Recall, $(1-2\varepsilon^2)|x|, (1-2\varepsilon^2)|z| \le |x-z|$ and notice that for $z \in V_1$ we have $|x-z| \le 2|x|$, thus $|x-z| \simeq |x|$ for $z \in V_1$. Similarly, if $z \in V_2$, then $|x-z| \simeq |z|$. Thus

$$\begin{aligned} \int_{V_1} \frac{1}{|x-z|^d}\frac{\delta_{\mathcal{U}^c}^{\alpha/2}(z))}{\delta_{\mathcal{I}}^{\alpha/2}(z))}dz &\le \frac{c}{|x|^d}\int_{V_1}\frac{\delta_{\mathcal{U}^c}^{\alpha/2}(z)}{\delta_{\mathcal{I}}^{\alpha/2}(z)}dz \\ &\le \frac{c}{|x|^d}\int_0^{|x|}\int_{\varphi_\varepsilon}^{\pi-\varphi_\varepsilon}\frac{\rho^{\alpha/2}\sin(\varphi_\varepsilon+\varphi_1)\,\rho^{d-1}\sin^{d-2}(\varphi_1)}{\rho^{\alpha/2}\sin^{\alpha/2}(\varphi_1-\varphi_\varepsilon)}d\varphi_1 d\rho \\ &\le \frac{c}{|x|^d}\int_0^{|x|}\rho^{d-1}d\rho\int_{\varphi_\varepsilon}^{\pi-\varphi_\varepsilon}\frac{1}{\sin^{\alpha/2}(\varphi_1-\varphi_\varepsilon)}d\varphi_1 \\ &\le c\int_0^{\pi-2\varphi_\varepsilon}\frac{1}{\varphi^{\alpha/2}}d\varphi \\ &\le c\varepsilon^{1-\alpha/2}. \end{aligned} \tag{4.77}$$

The last inequality in (4.77) follows from the fact that for $\varepsilon \in (0, 1/4)$ we have $\sin(\pi - 2\varphi_\varepsilon) \simeq 2\sin(\pi/2 - \varphi_\varepsilon)$, so $\pi - 2\varphi_\varepsilon \leq c\varepsilon$. On the remaining domain we have

$$\begin{aligned}
\int_{V_2} \frac{1}{|x-z|^d} \frac{\delta_{\mathcal{U}^c}^{\alpha/2}(z))}{\delta_{\mathcal{I}}^{\alpha/2}(z))} dz &\leq \int_{V_2} \frac{\delta_{\mathcal{U}^c}^{\alpha/2}(z)}{|z|^d \delta_{\mathcal{I}}^{\alpha/2}(z)} dz \\
&\leq \int_{|x|}^{r} \int_{\varphi_\varepsilon}^{\pi - \varphi_\varepsilon} \frac{\rho^{\alpha/2} \sin\left(\varphi_\varepsilon + \varphi_1\right) \rho^{d-1} \sin^{d-2}\left(\varphi_1\right)}{\rho^{d+\alpha/2} \sin^{\alpha/2}\left(\varphi_1 - \varphi_\varepsilon\right)} d\varphi_1 d\rho \\
&\leq \int_{|x|}^{r} \rho^{-1} d\rho \int_0^{\pi - 2\varphi_\varepsilon} \frac{1}{\varphi^{\alpha/2}} d\varphi \\
&\leq c\varepsilon^{1-\alpha/2}.
\end{aligned}$$

□

It now follows from equation (4.75) and Lemma 4.4.4 that

$$\mathbf{II} \leq \boxed{C\left(\varepsilon^{1-\underline{\alpha}/2} + \varepsilon^{1-\overline{\alpha}/2}\right) T(t)^{-d}}. \tag{4.78}$$

Long exit time and large jumps: III. We now suppose that $|x-z| > T$. Let $Q := (U^c \backslash I) \cap B(0,r) \cap \{|x-z| > T\}$. Again using the bound from Lemma 4.4.3 we get

$$\begin{aligned}
\mathbf{III} &\leq C \int_{\mathcal{I}} \int_{t/2}^{t} p_l^{\mathcal{I}}(x,y) \int_Q \nu(y-z) T(t-l)^{-d} \frac{V(\delta_{\mathcal{U}^c}(z))}{V(T(t-l))} \\
&\qquad \times \left(1 \wedge \frac{T(t-l)^d V^2(T(t-l))}{|x-z|^d V^2(|x-z|)}\right) dz\, dl\, dy \\
&\leq CT(t)^{-d} \int_Q P_{\mathcal{I}}^{V^2(1/\theta)}(x,z) \frac{V(\delta_{\mathcal{U}^c}(z))}{V(T(t))} \left(1 \wedge \frac{T(t)^d V^2(T(t))}{|x-z|^d V^2(|x-z|)}\right) dz. \tag{4.79}
\end{aligned}$$

We can use the Poisson kernel bound from Lemma 2.9 in [21]:

$$\begin{aligned}
\mathbf{III} &\le \frac{C}{T(t)^d}\int_Q \frac{V(|x|)}{V(\delta_{\mathcal{I}}(z))}\frac{1}{|x-z|^d}\frac{V(\delta_{\mathcal{U}^c}(z))}{V(T(t))}\left(1\wedge\frac{T(t)^dV^2(T(t))}{|x-z|^dV^2(|x-z|)}\right)dz\\
&\le CV(T(t))\int_Q \frac{V(|x|)}{|x-z|^{2d}V^2(|x-z|)}\frac{V(\delta_{\mathcal{U}^c}(z))}{V(\delta_{\mathcal{I}}(z))}\,dz\\
&\le CV(T(t))\int_Q \frac{1}{|x-z|^{2d}V(|x-z|)}\frac{V(\delta_{\mathcal{U}^c}(z))}{V(\delta_{\mathcal{I}}(z))}\,dz\\
&\le C\frac{V(T(t))}{(T(t))^dV(T(t))}\int_Q \frac{V(\delta_{\mathcal{U}^c}(z))}{|x-z|^dV(\delta_{\mathcal{I}}(z))}\,dz. \qquad (4.80)
\end{aligned}$$

Finally, applying Lemma 4.4.4 we get

$$\mathbf{III} \le \boxed{C\left(\varepsilon^{1-\underline{\alpha}/2}+\varepsilon^{1-\overline{\alpha}/2}\right)T(t)^{-d}}. \qquad (4.81)$$

Therefore

$$A_t(x) \le \boxed{C\left(\varepsilon^{1-\underline{\alpha}/2}+\varepsilon^{1-\overline{\alpha}/2}\right)\left(T(t)^{-d}\wedge\frac{1}{|x|^d}\frac{V^2(T(t))}{V^2(|x|)}\right)}. \qquad (4.82)$$

$\boldsymbol{B_t(x)}$: It remains to find a bound for

$$B_t(x) \le \int_{\mathcal{I}}\int_0^t p_l^{\mathcal{I}}(x,y)\int_{B^c(0,r)}\nu(y-z)p_{t-l}(x,z)\,dz\,dl\,dy. \qquad (4.83)$$

By assumption $x\in\Gamma_{2s}(v(q),\varepsilon)$, $s<r/4$, and $z\in B^c(0,r)$. Thus $|x-z|>r/2>2s$. Combining this with the bound for the heat kernel in Lemma 4.4.1, we get:

$$
\begin{aligned}
p_{t-l}(x,z) &\leq C\left(T(t-l)^{-d} \wedge \frac{1}{|x-z|^d}\frac{t-l}{V^2(|x-z|)}\right) && (4.84)\\
&\leq C\left(T(t-l)^{-d} \wedge \frac{1}{s^d}\frac{t-l}{V^2(s)}\right). && (4.85)
\end{aligned}
$$

Thus

$$
\begin{aligned}
B_t(x) &\leq C\left(\frac{1}{T(t-l)^d} \wedge \frac{1}{s^d}\frac{V^2(T(t))}{V^2(s)}\right)\int_{\mathcal{I}}\int_0^t p_l^{\mathcal{I}}(x,y)\int_{B^c(0,r)}\nu(y-z)dzdldy\\
&\leq C\left(T(t-l)^{-d} \wedge \frac{1}{s^d}\frac{V^2(T(t))}{V^2(s)}\right)\mathbb{P}_x\left(\tau_{\mathcal{I}}<t, |X_{\tau_{\mathcal{I}}}|>r\right)\\
&\leq C\frac{1}{s^d}\frac{V^2(T(t))}{V^2(s)}. && (4.86)
\end{aligned}
$$

We can chose $s = T(t)/\sqrt{\varepsilon}$ so that

$$
B_t(x) \leq C\frac{(\sqrt{\varepsilon})^d}{T(t)^d}\frac{V^2(T(t))}{V^2\left(\frac{T(t)}{\sqrt{\varepsilon}}\right)}. \tag{4.87}
$$

Since $x \in \Gamma_{2s}(v(q),\varepsilon)$ this choice of s also tells us that $|x| < 2s = 2T(t)/\sqrt{\varepsilon}$. Hence

$$
B_t(x) \leq C\frac{(\sqrt{\varepsilon})^\beta}{T(t)^\beta}\frac{1}{|x|^{d-\beta}}\frac{V^2(T(t))}{V^2\left(\frac{T(t)}{\sqrt{\varepsilon}}\right)} \leq C\frac{(\sqrt{\varepsilon})^\beta}{T(t)^\beta}\frac{1}{|x|^{d-\beta}}\frac{V^2(T(t))}{V^2(|x|)}. \tag{4.88}
$$

Letting $\beta = d$ and $\beta = 1$ in (4.88) gives us

$$
\begin{aligned}
B_t(x) &\leq C\left(\frac{(\sqrt{\varepsilon})^d}{T(t)^d} \wedge \frac{\sqrt{\varepsilon}}{T(t)}\frac{1}{|x|^{d-1}}\frac{V^2(T(t))}{V^2(|x|)}\right)\\
&\leq \boxed{C\sqrt{\varepsilon}T(t)^{-d}\left(1 \wedge \frac{T(t)^{d-1}}{|x|^{d-1}}\frac{V^2(T(t))}{V^2(|x|)}\right)}. && (4.89)
\end{aligned}
$$

Therefore, combining our bound for $A_t(x)$ in (4.82) and $B_t(x)$ in (4.89) we get

$$r_t^{\mathcal{I}}(x,x) - r_t^{\mathcal{U}^c}(x,x) \leq \boxed{\frac{C}{T(t)^d}\left\{\left(\varepsilon^{1-\underline{\alpha}/2} + \varepsilon^{1-\overline{\alpha}/2}\right) \vee \sqrt{\varepsilon}\right\}\left(1 \wedge \frac{T(t)^{d-1}}{|x|^{d-1}}\frac{V^2(T(t))}{V^2(|x|)}\right)}.$$

This completes the proof of Proposition 4.3.3.

$\square$

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