GLUING BRIDGELAND'S STABILITY CONDITIONS AND $\mathbb{Z}_2$-EQUIVARIANT SHEAVES ON CURVES

by

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We define and study a gluing procedure for Bridgeland stability conditions in the situation where a triangulated category has a semiorthogonal decomposition. As one application, we construct an open, contractible subset U in the stability manifold of the derived category $\mathcal{D}_{\mathbb{Z}_2}(X)$ of $\mathbb{Z}_2$-equivariant coherent sheaves on a smooth curve X, associated with a degree 2 map $X \to Y$, where Y is another curve. In the case where X is an elliptic curve we construct an open, connected subset in the stability manifold using exceptional collections containing the subset U. We also give a new proof of the constructibility of exceptional collections on $\mathcal{D}_{\mathbb{Z}_2}(X)$. This dissertation contains previously unpublished co-authored material.

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CHAPTER I

INTRODUCTION

I.1 Background

Stability conditions on triangulated categories were introduced by Bridgeland in [5] as a mathematical formalization of Douglas' work on Π-stability of D-branes in [7, 8]. The definition of a stability condition on a triangulated category $\mathcal{D}$ can be thought of as a generalization of the Mumford slope-stability for sheaves on complex projective varieties, where the slope of an object is defined using a function $Z : K_0(\mathcal{D}) \to \mathbb{C}$. The remarkable feature of Bridgeland's theory is that the set of (nice) stability conditions on $\mathcal{D}$ has a structure of complex manifold. Hypothetically this manifold, called the *stability manifold*, has some interesting geometric structures and in the case when $\mathcal{D}$ is the derived category of coherent sheaves on a Calabi-Yau threefold this space should be relevant for mirror symmetry considerations (see [4]). It also has actions by the autoequivalences of $\mathcal{D}$ and the universal cover $\widetilde{\mathrm{GL}}(2, \mathbb{R})$ of the group $\mathrm{GL}^+(2, \mathbb{R})$ of orientation-preserving linear transformations of $\mathbb{R}^2$.

At present, very little is known about stability manifolds in general. The stability manifolds of smooth projective curves were determined in [5], [11], [15]. Some stability conditions have been constructed on projective surfaces as well as a connected component of the stability manifold for a noncompact K3 surface [2], [6]. Macrì gives a method for constructing stability conditions from Ext-exceptional collections in [11], and at the time of writing this method is the only source we have for examples of stability conditions on varieties of dimension greater than 2. It is important, then, that we come up with new techniques for constructing stability conditions.

I.2 Main Results

In this paper we give a method for constructing stability conditions on a triangulated category $\mathcal{D}$, given a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ and stability conditions σ_i on $\mathcal{D}_i$. We say that the stability conditions constructed in this manner are "glued" from the stability conditions on the $\mathcal{D}_i$ since the process is motivated from the gluing of t-structures introduced in [3].

Theorem I.2.1. *Let $(\mathcal{D}_1, \mathcal{D}_2)$ be a semiorthogonal decomposition of a triangulated category $\mathcal{D}$. Suppose (σ_1, σ_2) is a pair of reasonable stability conditions on $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively, with the slicings P_i and central charges Z_i ($i = 1, 2$), and let a be a real number in $(0, 1)$. Assume the following two conditions hold:*

1. $\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(P_1(0,1], P_2(0,1]) = 0$;
2. $\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(P_1(a,a+1], P_2(a,a+1]) = 0$;

Then there exists a reasonable stability σ glued from σ_1 and σ_2. Furthermore,

$$P(0,a] = [P_1(0,a], P_2(0,a]]$$
$$P(a,1] = [P_1(a,1], P_2(a,1]].$$

It should also be noted that this gluing construction is a generalization of the method Macrì gave for Ext-exceptional collections. When restricted to a suitable open subset of $\mathrm{Stab}(\mathcal{D}_1) \times \mathrm{Stab}(\mathcal{D}_2)$, gluing is continuous.

Theorem I.2.2. *Let $U \subset \mathrm{Stab}(\mathcal{D}_1) \times \mathrm{Stab}(\mathcal{D}_2)$ denote the set of pairs of reasonable stabilities ($\sigma_1 = (Z_1, P_1)$ and $\sigma_2 = (Z_2, P_2)$) such that for some $\epsilon > 0$ one has*

$$\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(P_1(-\epsilon, 1], P_2(0, 1+\epsilon)) = 0.$$

Then U is open and the gluing map $\mathrm{gl}: U \to \mathrm{Stab}(\mathcal{D})$ *is continuous.*

Let X be a smooth projective curve and denote by $\mathcal{D}_{\mathbb{Z}_2}(X)$ the derived category of $\mathbb{Z}_2$-equivariant coherent sheaves on X. There is a stability condition on $\mathcal{D}_{\mathbb{Z}_2}(X)$ associated to the Mumford stability for coherent sheaves which we call the *standard stability.* We construct a contractible open subset of stability conditions on $\mathcal{D}_{\mathbb{Z}_2}(X)$ containing the standard stability.

Theorem I.2.3. *Let* $U \subset \mathrm{Stab}_{\mathcal{N}}(\mathcal{D}_{\mathbb{Z}_2}(X))$ *denote the set of locally finite stability conditions* $\sigma = (Z, P)$ *such that*

1. $\mathcal{O}_{\pi^{-1}(y)}$ *is stable of phase* ϕ_σ *for every* $y \in Y \setminus R$*;*

2. $\mathcal{O}_{p_i}, \zeta \otimes \mathcal{O}_{p_i}$ *are semistable with the phases in* $(\phi_\sigma - 1, \phi_\sigma + 1)$ *for all* $i = 1, \ldots, n$.

Then every point in U is obtained from a stability glued along a semiorthogonal decomposition by the action of an element of $\mathbb{R} \times \mathrm{Pic}_{\mathbb{Z}_2}(X)$*, where* $\mathbb{R}$ *acts on* $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D}_{\mathbb{Z}_2}(X))$ *by rotations (shifts of phases). The subset U is open in* $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D}_{\mathbb{Z}_2}(X))$*. Furthermore, U is contractible.*

In the case when X is an elliptic curve, $\mathcal{D}_{\mathbb{Z}_2}(X)$ contains full exceptional collections. Denote by $\Gamma \subset \mathrm{Stab}_{\mathcal{N}}(D_{\mathbb{Z}_2}(X))$ the set of all stability conditions obtained from a stability glued along an Ext-exceptional collection by the action of $\widetilde{\mathrm{GL}}(2, \mathbb{R})$. With U defined as above, we prove

Theorem I.2.4. *The set* $\Gamma \cup U$ *is connected.*

As a byproduct of our methods we obtain some nice results on the structure of stable bundles equipped with $\mathbb{Z}_2$-equivariant structure. It should also be noted that these bundles are exceptional objects of the derived category of $\mathbb{Z}_2$-equivariant sheaves on X.

Theorem I.2.5. *Let V be a stable bundle on X equipped with a $\mathbb{Z}_2$-equivariant structure. Let p_1, p_2, p_3, p_4 be the four points of order at most 2 in the group structure on X. Denote* *by* $(V|_{p_i})^+$ *the subspace of* $V|_{p_i}$ *with positive parity and set*

$$y_i(V) = \chi(V, \mathcal{O}_{p_i}).$$

Then

1. $|dim(V|_{p_i})^+ - \frac{1}{2}\operatorname{rk} V| \leq 1$,
2. *if* $\operatorname{rk} V$ *is odd,* $y_i(V) = \pm\frac{1}{2}, \forall i$,
3. *if* $\operatorname{rk} V$ *is even, there exists* $j \in \{1,2,3,4\}$ *such that* $y_i(V) = \pm\delta_{ij}, \forall i$.

In fact, there is a set of coordinates $\deg, \operatorname{rk}, y_1, y_2, y_3, y_4$ on $K_0(\mathcal{D}_{\mathbb{Z}_2}(X))$ with which we can give a new proof of the known result

Theorem I.2.6. *The action of the braid group* B_6 *by mutations is transitive upon the set of full exceptional collections of sheaves in* $\mathcal{D}_{\mathbb{Z}_2}(X)$*, up to shifts.*

Our method of proof has the advantage of giving insight into the structure of exceptional collections on $\mathcal{D}_{\mathbb{Z}_2}(X)$. In particular, we classify the orbits of the group of autoequivalences upon the set of exceptional collections.

Theorem I.2.7. *Let* A *denote the subgroup of the symmetric group* S_3 *by which the stabilizer of* p_1 *in* $\operatorname{Aut}(X)$ *acts upon the set* $\{p_2, p_3, p_4\}$*. There are* $\frac{246}{|A|}$ *orbits in the action of autoequivalences* $\operatorname{Aut}(\mathcal{D}_{\mathbb{Z}_2}(X))$ *upon the set* $\operatorname{Coll}(\mathcal{D})$ *of full exceptional collections of sheaves on* $\mathcal{D}_{\mathbb{Z}_2}(X)$*.*

The results contained in chapters III and IV originally appeared in the co-authored paper *Gluing Stability Conditions* as a preprint on the Mathematics ArXiv. The sections reproduced here are those that existed in similar form in my research before the paper was published. The final section of that paper has not been included as it contains substantial contributions from the co-author, Alexander Polishchuk.

CHAPTER II

BACKGROUND

Our purpose in this chapter is to introduce the reader to stability conditions and the technical tools we shall need in later chapters. This introduction includes a basic overview of derived categories, t-structures and exceptional collections as needed for the definition of a stability condition and for use in later chapters. We will assume, however, a basic knowledge of algebraic geometry and sheaf theory.

II.1 Derived Categories

A *triangulated category* $\mathcal{D}$ is an additive category equipped with an autoequivalence $[1] : \mathcal{D} \to \mathcal{D}$, called a translation or shift functor, and collection of exact triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

satisfying four axioms. A full subcategory $\mathcal{C} \subset \mathcal{D}$ is a triangulated subcategory if it is closed under the operations of taking cones and the shift functor. The primary example of a triangulated category is the derived category of an abelian category, which we define below. We will need triangulated categories only in the context of certain subcategories of a derived category, so we will not develop all of their properties here. In particular, the reader who is interested in a less abstract context may substitute the term "derived category" for "triangulated category" in the rest of this chapter. Those who are interested in a fuller treatment of the subject triangulated categories can find it in [20].

Definition II.1.1. A *localization of* $\mathcal{C}$ *with respect to* S is a category $S^{-1}\mathcal{C}$, together with a functor $F : \mathcal{C} \to S^{-1}\mathcal{C}$ satisfying the properties

1. $F(s)$ is an isomorphism in $S^{-1}\mathcal{C}$ for all $s \in S$.

2. Any functor $G : \mathcal{C} \to \mathcal{D}$ such that $G(s)$ is an isomorphism for all $s \in S$ factors uniquely through F.

It follows that $S^{-1}\mathcal{C}$ exists, modulo set theoretic problems, and is uniquely defined up to equivalence.

Definition II.1.2. A morphism $f : E \to F$ of (co)chain complexes of elements of an abelian category $\mathcal{A}$ is a *quasi-isomorphism* if the induced morphisms $f : H^n(E) \to H^n(F)$ are isomorphisms for all n. Let Q be the set of all quasi-isomorphisms in the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes of objects of $\mathcal{A}$. The *derived category* $\mathcal{D}(\mathcal{A})$ is then the localization $Q^{-1}\,\mathrm{Ch}(\mathcal{A})$ of the category of chain complexes in $\mathcal{A}$ at the set of quasi-isomorphisms. That is, the objects of $\mathcal{D}(\mathcal{A})$ are the same as those of $\mathrm{Ch}(\mathcal{A})$ and a morphism $f \in \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(E, F)$ is represented by an equivalence class of pairs of morphisms (f', s) of chain complexes,

$$E \xleftarrow{f'} G \xrightarrow{s} F,$$

where s is a quasi-isomorphism. The *bounded derived category* $\mathcal{D}^b(\mathcal{A})$ is the full subcategory of $\mathcal{D}(\mathcal{A})$ of objects E such that the complex of homology groups $H^*(E)$ is bounded.

Remark II.1.3. Let $\mathcal{A}$ be an abelian category. If $X \xrightarrow{f} Y$ is a morphism of chain complexes of objects of $\mathcal{A}$, then the shift functor [1] on $\mathcal{D}(\mathcal{A})$ acts as follows. $X[1]$ is the chain complex with $H^i(X[1]) = H^{i+1}(X)$ and differentials $\partial^i_{X[1]} = \partial^{i+1}_X$, and [1] acts upon the morphism f by -1 so that the induced map on homology $H^{-1}(X[1]) \to H^{-1}(Y[1])$ is given by $-f$. The collection $(X, Y, \mathrm{cone}(f), f, g, \delta)$ is called an *exact triangle* in $\mathcal{D}(\mathcal{A})$, where $g : Y \to \mathrm{cone}(f)$ and $\delta : \mathrm{cone}(f) \to X[1]$ are the induced morphisms. In particular, given any short exact sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

of chain complexes, there is an exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in $\mathcal{D}(\mathcal{A})$, where h is induced from the morphism $\text{cone}(f) \xrightarrow{h'} X[1]$ and the quasi-isomorphism $\text{cone}(f) \to Z$. With the above action of shift functor and these exact triangles, the axioms for a triangulated category are satisfied for $\mathcal{D}(\mathcal{A})$.

Example II.1.4. The objects of the derived category are the same as the objects for the category of chain complexes, but the morphisms between objects are quite different. Here are a few illustrative examples.

1. Let M be an module over a ring R and suppose there exists a projective resolution $M \xrightarrow{f} P^{\cdot}$. Then in $\mathcal{D}$(mod-R), f is an isomorphism. Indeed, f is a quasi-isomorphism in Ch(mod-R) with M viewed as a complex in degree 0.

2. If two morphisms $f, g \in \text{Hom}_{\text{Ch}(\mathcal{A})}(X, Y)$ are homotopic, then $f = g$ in $\mathcal{D}(\mathcal{A})$.

3. A morphism $f : X \to Y$ in $\text{Ch}(\mathcal{A})$ becomes the zero morphism in $\mathcal{D}(\mathcal{A})$ if and only if there is a quasi-isomorphism $s : Y \to Y'$ such that sf is null homotopic. The converse, however, is false.

Notation II.1.5. Given a collection $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n$ of full subcategories in a triangulated category $\mathcal{D}$, we denote by $[\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n]$ (resp. $\langle \mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n \rangle$) the full subcategory (resp. triangulated subcategory) generated by the collection. That is, the smallest full subcategory (resp. triangulated subcategory) containing each $\mathcal{A}_i$. As a consequence, $\mathcal{A} = \langle \mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n \rangle$ is closed under extensions; if $A, A' \in \mathcal{A}$, then given any exact triangle

$$A \to A'' \to A' \to A[1]$$

in $\mathcal{D}$, A'' is also an element of $\mathcal{A}$.

Definition II.1.6. Let $\mathcal{B} \subset \mathcal{D}$ be a triangulated subcategory of $\mathcal{D}$. The *right orthogonal* to $\mathcal{B}$, denoted $\mathcal{B}^{\perp}$, is the full subcategory consisting of all objects $Y \in \mathcal{D}$ such that for every $X \in \mathcal{B}$ we have $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$. It is a triangulated subcategory. The *left orthogonal* is defined by the analogous condition that $\mathrm{Hom}_{\mathcal{D}}(Y, X) = 0$ and is denoted ${}^{\perp}\mathcal{B}$.

Definition II.1.7. A full subcategory $\mathcal{B} \subset \mathcal{D}$ of a triangulated category is *right (left) admissible* if there exist right (left) adjoint functors to the inclusion $\mathcal{B} \to \mathcal{D}$.

Definition II.1.8. A *semiorthogonal decomposition* in $\mathcal{D}$ is given by a collection $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n$ of full triangulated subcategories satisfying the conditions

1. $\mathcal{D} = \langle \mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n \rangle$

2. For every $i < j$, $\mathrm{Hom}_{\mathcal{D}}(\mathcal{A}_j, \mathcal{A}_i) = 0$.

3. For every object $X \in \mathcal{D}$, there exist a series of exact triangles

$$A_i \to X_i \to X_{i-1} \to A_i[1]$$

   with $A_i \in \mathcal{A}_i$, $X_0 = 0$, and $X_n = X$.

In the case $n = 2$, the subcategory $\mathcal{A}_2 = {}^{\perp}\mathcal{A}_1$ is the left orthogonal of $\mathcal{A}_1$ and $\mathcal{A}_1$ is left admissible. Conversely, given a left admissible triangulated subcategory $\mathcal{B}$ there is a semiorthogonal decomposition $(\mathcal{B}, {}^{\perp}\mathcal{B})$.

Proposition II.1.9. *If $\mathcal{D} = \langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ is a semiorthogonal decomposition then both $\mathcal{A}_1$ and $\mathcal{A}_2$ are admissible. In particular, there are also semiorthogonal decompositions $\langle \mathcal{A}_1^{\perp}, \mathcal{A}_1 \rangle$ and $\langle \mathcal{A}_2, {}^{\perp}\mathcal{A}_2 \rangle$ in $\mathcal{D}$.*

II.2 T-Structures

In this section we define t-structures on a triangulated category. The idea is to consider nonobvious abelian categories sitting inside a triangulated category as the heart of a t-structure. We then define two operations for constructing new t-structures, and hence abelian subcategories, from old ones. The first operation is that of tilting at a torsion pair inside an abelian category and the second is the gluing of t-structures along a semiorthogonal decomposition. The latter is a version of the topological gluing construction found in [3]. Historically, the topological gluing was used in the theory of perverse sheaves, where the gluing was related to a stratification of a topological space. Throughout this section $\mathcal{D}$ will denote a triangulated category.

Definition II.2.1. A *t-structure* on $\mathcal{D}$ is two full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, where we write $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n]$, with the properties

1. For each $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$, then $\mathrm{Hom}(X, Y) = 0$.

2. One has $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1}$.

3. For every $Z \in \mathcal{D}$, there exists an exact triangle $X \to Z \to Y \to X[1]$ with $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$.

The *heart* of a t-structure is the full category $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. It is known [3] that the heart is an abelian category. A sequence $0 \to X \to Y \to Z \to 0$ is exact in $\mathcal{A}$ if and only if $X \to Y \to Z \to X[1]$ is an exact triangle in $\mathcal{D}$. For each $n \in \mathbb{Z}$ there exist truncation functors $\tau_{\leq n} : \mathcal{D} \to \mathcal{D}^{\leq n}$ and $\tau_{\geq n} : \mathcal{D} \to \mathcal{D}^{\geq n}$ satisfying the property that for each n and object $0 \neq E \in \mathcal{D}$ there exist exact triangles

$$\tau_{\leq n} E \to E \to \tau_{\geq n+1} \to \tau_{\leq n}[1].$$

A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is *bounded* if

$$\mathcal{D} = \cup_{i,j \in \mathbb{Z}} \mathcal{D}^{\leq i} \cap \mathcal{D}^{\geq j}.$$

Every bounded t-structure is determined by its heart, and in particular in this case $\mathcal{D}^{\leq 0}$

is the union of all $\mathcal{A}[i]$ for $i \geq 0$. A t-structure is *nondegenerate* if the intersections

$$\bigcap_{i\in\mathbb{Z}} \mathcal{D}^{\leq i} \text{ and } \bigcap_{j\in\mathbb{Z}} \mathcal{D}^{\geq j}$$

both contain only the zero object. We sometimes write $\mathcal{D}^{[a,b]} := \mathcal{D}^{\leq b} \cap \mathcal{D}^{\geq a}$.

Remark II.2.2. The motivation for this definition is the standard t-structure on the bounded derived category $\mathcal{D}(\mathcal{A})$. It is obtained by taking $\mathcal{D}(\mathcal{A})^{\leq 0}$ to consist of all objects of $\mathcal{D}(\mathcal{A})$ whose cohomology objects $H^i(X) \in \mathcal{A}$ are zero for all $i > 0$, and similarly taking for $\mathcal{D}(\mathcal{A})^{\geq 0}$ the objects with $H^i(X) = 0$ for all $i < 0$. Hence, in the standard t-structure on $\mathcal{D}(\mathcal{A})$ the heart is $\mathcal{A}$. The standard t-structure on the bounded derived category $\mathcal{D}^b(\mathcal{A})$ is bounded.

Definition II.2.3. Given triangulated categories $\mathcal{C}$ and $\mathcal{D}$ endowed with t-structures $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, a functor $F : \mathcal{C} \to \mathcal{D}$ is *right (left) t-exact* if $F(\mathcal{C}^{\leq 0}) \subset \mathcal{D}^{\leq 0}$ (resp. $F(\mathcal{C}^{\geq 0}) \subset \mathcal{D}^{\geq 0}$). We say that F is *t-exact* if it is left and right t-exact.

Let X be a topological space with open subset $U \subset X$ and complement $F = X \backslash U$. Denote by $\mathcal{O}$ a sheaf of rings on X and for any subset $A \subset X$ let $M(A)$ denote the abelian category of $\mathcal{O}$-modules on A. Denote by $\mathcal{D}$, $\mathcal{D}_U$ and $\mathcal{D}_F$ the derived categories of $\mathcal{O}$-modules on X, U and F, respectively. The following morphisms of sheaves induce functors with the stated exactness properties with respect to the standard t-structures on each category: inclusion from U, $j_! : \mathcal{D}_U \to \mathcal{D}$ is t-exact; restriction to U, $j^* : \mathcal{D} \to \mathcal{D}_U$ is t-exact; direct image from U, $j_* : \mathcal{D}_U \to \mathcal{D}$ is left t-exact; restriction to F, $i^* : \mathcal{D} \to \mathcal{D}_F$ is t-exact; direct image from F, $i_* : \mathcal{D}_F \to \mathcal{D}$ is t-exact; sections with support in F, $i^! : \mathcal{D} \to \mathcal{D}_F$ is left t-exact.

Theorem II.2.4 (Topological Gluing). *Given X, U, F as above, suppose that the following conditions are satisfied:*

1. *i_* has left and right adjoint functors, namely i^* and $i^!$.*

2. j^* *has left and right adjoint functors, namely* $j_!$ *and* j_*.

3. $j^*i_* = 0$. *This implies that for every* $A \in \mathcal{D}_F$ *and* $B \in \mathcal{D}_U$,

$$\mathrm{Hom}(j_!B, i_*A) = \mathrm{Hom}(i_*A, j_*B) = 0.$$

4. $\mathcal{D} = \langle i_*\mathcal{D}_F, j_!\mathcal{D}_U\rangle = \langle j_*\mathcal{D}_U, i_*\mathcal{D}_F\rangle$ *are semiorthogonal decompositions in* $\mathcal{D}$.

5. i_* *and* $j_!$ *are fully faithful functors.*

Suppose as well that there are t-structures $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$ *and* $(\mathcal{D}_F^{\leq 0}, \mathcal{D}_F^{\geq 0})$ *on* $\mathcal{D}_U$ *and* $\mathcal{D}_F$, *respectively. Then there is a t-structure on* $\mathcal{D}$ *defined by the properties*

$$\mathcal{D}^{\leq 0} = \{E \in \mathcal{D} \mid j^*E \in D_U^{\leq 0} \text{ and } i^*E \in \mathcal{D}_F^{\leq 0}\}$$
$$\mathcal{D}^{\geq 0} = \{E \in \mathcal{D} \mid j^*E \in D_U^{\geq 0} \text{ and } i^!E \in \mathcal{D}_F^{\geq 0}\}.$$

We say that this t-structure on $\mathcal{D}$ *is obtained by gluing.*

The topological gluing was first used in the construction of perverse sheaves, which are the objects in the heart of the t-structure obtained by gluing the standard t-structure on $\mathcal{D}_U$ to a shift of the standard t-structure on $\mathcal{D}_F$. Semiorthogonal decompositions provide a natural framework for generalization of this result beyond the topological setting. We present an abstract form of the topological gluing as well as two special cases below.

Let $\mathcal{D}$ be a triangulated category equipped with a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2\rangle$. For $E \in \mathcal{D}$ there exist objects $E_1 \in \mathcal{D}_1$ and $E_2 \in \mathcal{D}_2$ forming an exact triangle $E_2 \to E \to E_1 \to E_2[1]$ and these objects depend functorially on E. Namely, $E_2 = \rho_2(E)$, where ρ_2 is the right adjoint functor to the inclusion $\mathcal{D}_2 \to \mathcal{D}$, and $E_1 = \lambda_1(E)$, where λ_1 is the left adjoint functor to the inclusion $\mathcal{D}_1 \to \mathcal{D}$. If $\mathcal{D}_1$ is admissible, denote by ρ_1 the right adjoint functor to the inclusion $\mathcal{D}_1 \to \mathcal{D}$.

Proposition II.2.5 (Theorem 3.1.1 in [18]). *Assume we have a semiorthogonal decomposition* $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2\rangle$ *and t-structures* $(\mathcal{D}_i^{\leq 0}, \mathcal{D}_i^{\geq 0})$. *Then there are t-structures* $(\mathcal{D}_\lambda^{\leq 0}, \mathcal{D}_\lambda^{\geq 0}), (\mathcal{D}_\rho^{\leq 0}, \mathcal{D}_\rho^{\geq 0})$

on $\mathcal{D}$ for which

$$\mathcal{D}_{\lambda}^{\leq 0} = \{E \in \mathcal{D} \mid \lambda_1 E \in \mathcal{D}_1^{\leq 0} \text{ and } \lambda_2 E \in \mathcal{D}_2^{\leq 0}\}$$
$$\mathcal{D}_{\lambda}^{\geq 0} = \{E \in \mathcal{D} \mid \rho_1 E \in \mathcal{D}_1^{\geq 0} \text{ and } \lambda_2 E \in \mathcal{D}_2^{\geq 0}\}$$
$$\mathcal{D}_{\rho}^{\leq 0} = \{E \in \mathcal{D} | \ \lambda_1 E \in \mathcal{D}_1^{\leq 0} \text{ and } \rho_2 E \in \mathcal{D}_2^{\leq 0}\}$$
$$\mathcal{D}_{\rho}^{\geq 0} = \{E \in \mathcal{D} | \ \rho_1 E \in \mathcal{D}_1^{\geq 0} \text{ and } \rho_2 E \in \mathcal{D}_2^{\geq 0}\}$$

This lemma will be used in the proof of the corollary.

Lemma II.2.6. *Let $\mathcal{D}$ be a triangulated category. Suppose H and H' are hearts of bounded t-structures on $\mathcal{D}$. If $H \subseteq H'$, then $H = H'$.*

Corollary II.2.7. *If the functor $\rho_1|_{\mathcal{D}_2} : \mathcal{D}_2 \to \mathcal{D}_1$ is right t-exact with respect to the t-structures on $\mathcal{D}_1$ and $\mathcal{D}_2$ then there is a t-structure on $\mathcal{D}$ for which*

$$\mathcal{D}^{[a,b]} = \{X \in \mathcal{D} : \rho_1(X) \in \mathcal{D}_1^{[a,b]},\ \rho_2(X) \in \mathcal{D}_2^{[a,b]}\}.$$

Proof. We show that under the assumption $\rho_1|_{\mathcal{D}_2} : \mathcal{D}_2 \to \mathcal{D}_1$ is right t-exact that the heart H_ρ of the t-structure $(\mathcal{D}_{\rho}^{\leq 0}, \mathcal{D}_{\rho}^{\geq 0})$ is of the form

$$H_\rho = H := \{E \in \mathcal{D} : \rho_i E \in H_i\}$$

where H_i is the heart of the t-structure on $\mathcal{D}_i$.

Suppose that $E \in H_\rho$. To show that $E \in H$ we need only show $\rho_1 E \in \mathcal{D}_2^{\leq 0}$. Apply ρ_1 to the exact triangle

$$\rho_2 E \to E \to \lambda_1 E \to \rho_2 E[1]$$

and observe that $\rho_1 \rho_2 E \in \mathcal{D}_1^{\leq 0}$ and $\rho_1 \lambda_1 E \cong \lambda_1 E \in \mathcal{D}_1^{\leq 0}$. Thus, $\rho_1 E \in \mathcal{D}_1^{\leq 0}$ and $E \in H$. Also, it is easy to check that H defines the heart of a bounded t-structure on $\mathcal{D}$. Since $H_\rho \subseteq H$, the corollary is a consequence of the lemma above. $\square$

Corollary II.2.8. *Suppose the hearts H_i of the t-structures $(\mathcal{D}_i^{\leq 0}, \mathcal{D}_i^{\geq 0})$ on $\mathcal{D}_i$ (where $i = 1, 2$) satisfy the condition $\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(H_1, H_2) = 0$. Then there is a t-structure on $\mathcal{D}$ with the heart*

$$H = \{X \in \mathcal{D} \mid \lambda_1(X) \in H_1, \rho_2(X) \in H_2\}. \tag{II.1}$$

With respect to this t-structure on $\mathcal{D}$ the functors $\lambda_1 : \mathcal{D} \to \mathcal{D}_1$ and $\rho_2 : \mathcal{D} \to \mathcal{D}_2$ are t-exact.

Note that in the situation of the above Corollary we have $H_1 \subset H$ and $H_2 \subset H$. Furthermore, every object $E \in H$ fits into an exact sequence in H

$$0 \to \rho_2(E) \to E \to \lambda_1(E) \to 0,$$

where $\rho_2(E) \in H_2$ and $\lambda_1(E) \in H_1$. Therefore, we also have

$$H = [H_2, H_1].$$

Definition II.2.9. A *torsion pair* (see [9]) in an abelian category $\mathcal{C}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ in $\mathcal{C}$ such that $\mathrm{Hom}(T, F) = 0$ for every $T \in \mathcal{T}, F \in \mathcal{F}$ and every object $C \in \mathcal{C}$ fits into a short exact sequence

$$0 \to T \to C \to F \to 0.$$

A torsion pair $(\mathcal{T}, \mathcal{F})$ in a triangulated category $\mathcal{D}$ defines a nondegenerate t-structure with heart

$$\mathcal{C}^t = \{E \in \mathcal{D} | H^i(E) = 0 \text{ for } i \neq 0, -1, H^0(E) \in \mathcal{T}, H^{-1}(E) \in \mathcal{F}\}.$$

That is,

$$\mathcal{C}^t = [\mathcal{F}[1], \mathcal{T}].$$

The process of passing from $\mathcal{C}$ to $\mathcal{C}^t$ is called *tilting*, and we say that $\mathcal{C}^t$ is a *tilt* of $\mathcal{C}$.

Remark II.2.10. The pair (H_2, H_1) from Corollary II.2.8 is a torsion pair in H.

Remark II.2.11. Recall that a bounded t-structure on a triangulated category is determined by its heart, which is an abelian category. The tilting procedure is then seen as a method for constructing new abelian categories inside a triangulated category. Moreover, the tilted category determines a new bounded t-structure.

II.3 Exceptional Collections

Definition II.3.1. Let K be a field. A triangulated category $\mathcal{D}$ is *K-linear* if for every pair of objects E, F the set $\mathrm{Hom}_{\mathcal{D}}(E, F)$ has the structure of a vector space over K. $\mathcal{D}$ is of *finite type* if in addition the graded vector space $\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(E, F[i])$ is finite-dimensional for every pair E, F of objects.

Given $\mathcal{D}$ a K-linear triangulated category. We denote by

$$\mathrm{Hom}^\bullet(E, F) = \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}^k_{\mathcal{D}}(E, F)[-k]$$

the graded complex of K-vector spaces with trivial differential. An object E of $\mathcal{D}$ is an *exceptional object* if it satisfies the condition

$$\mathrm{Hom}^\bullet_{\mathcal{D}}(E, E) = K.$$

An ordered collection $(E_1, E_2, \cdots, E_n)$ of exceptional objects of $\mathcal{D}$ is an *exceptional collection* if it satisfies the condition

$$\mathrm{Hom}^\bullet_{\mathcal{D}}(E_j, E_i) = 0 \text{ for all } i < j.$$

A *full exceptional collection* is an exceptional collection for which $\mathcal{D} = \langle E_1, E_2, \cdots, E_n\rangle$.

An exceptional collection is *Ext-exceptional* if it satisfies the condition

$$\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(E_i, E_j) = 0, \quad \text{for } i < j.$$

An exceptional collection consisting of two objects is called an *exceptional pair.*

Example II.3.2. Exceptional collections on projective spaces. For any $n \geq 1$ and integers $k, m_1, \cdots, m_n$, the n-tuple $(O(k)[m_1],\ O(k+1)[m_2],\ \cdots,\ O(k+n)[m_n])$ is a full exceptional collection of sheaves on $\mathcal{D}(\mathrm{Coh}(\mathbb{P}^n))$. If in addition $m_1 > m_2 > \cdots > m_n$, then it is an Ext-exceptional collection.

Definition II.3.3. Let (E, F) be an exceptional pair in $\mathcal{D}$. We define objects $L_E F$ and $R_F E$ by the exact triangles in $\mathcal{D}$:

$$L_E F \to \mathrm{Hom}^\bullet(E, F) \otimes E \to F \to L_E F[1]$$
$$E \to \mathrm{Hom}^\bullet(E, F)^* \otimes F \to R_F E \to E[1]$$

where $V[k] \otimes E$, with V a vector space, denotes the object that is $\dim V$ copies of $E[k]$ and dualizing $\mathrm{Hom}^\bullet(E, F)^*$ changes the sign on the grading. We say that $L_E F$ is the *left mutation* of F through E and that $R_F E$ is the *right mutation* of E through F. A left mutation of the pair (E, F) is the pair $(L_E F, E)$ and a right mutation of that pair is $(F, R_F E)$.

A (left or right) mutation of an exceptional collection $\sigma = (E_1, \cdots, E_n)$ is defined to be a mutation of an adjacent pair of objects in this collection:

$$\begin{aligned} R_i \cdot \sigma &= (E_1, \cdots, E_{i-1}, E_{i+1}, R_{E_{i+1}} E_i, E_{i+2}, \cdots, E_n) \\ L_i \cdot \sigma &= (E_1, \cdots, E_{i-2}, L_{E_i} E_{i+1}, E_i, E_{i+2}, \cdots, E_n) \end{aligned}$$

If the action of this braid group upon the set of all full exceptional collections on $\mathcal{D}$, up to the action of shifting each exceptional object of a collection, has a single orbit then this set is said to be *constructible.* We also say in this case that $\mathcal{D}$ is constructible.

Remark II.3.4. It is known that the sets of full exceptional collections on $\mathcal{D}(\mathrm{Coh}(\mathbb{P}^1))$ and $\mathcal{D}(\mathrm{Coh}(\mathbb{P}^2))$ are constructible, but for any $n > 2$ this is currently an open question.

Lemma II.3.5. *A mutation of an exceptional collection σ is an exceptional collection. If σ is full, then so is the mutated collection.*

Proposition II.3.6. *Let $\sigma = (E_1, \cdots, E_n)$ be an exceptional collection.*

1. *R_i and L_i are mutually inverse; $R_iL_i \cdot \sigma = L_iR_i \cdot \sigma = \sigma$;*

2. *the R_i (respectively L_i) induce actions of the braid group B_n on n strings;*

$$R_iR_{i+1}R_i \cdot \sigma = R_{i+1}R_iR_{i+1} \cdot \sigma, \qquad L_iL_{i+1}L_i \cdot \sigma = L_{i+1}L_iL_{i+1} \cdot \sigma;$$

3. *The actions of the groups $\mathrm{Aut}(\mathcal{D})$ and B_n on exceptional collections $\{E_1, \cdots E_n\}$ commute.*

If there exist full exceptional collections in a triangulated category $\mathcal{D}$ then we are able to construct many interesting t-structures from these collections. In particular, there is a specialization of Lemma II.2.8 to the case of Ext-exceptional collections.

Proposition II.3.7. *Suppose $\{E_1, \cdots, E_n\}$ is a full Ext-exceptional collection on $\mathcal{D}$. Then there exists a bounded t-structure on $\mathcal{D}$ with heart $H = [E_1, \cdots, E_n]$, glued from the t-structures on the categories $\langle E_i \rangle$ with hearts $[E_i]$.*

II.4 Stability Conditions

Definition II.4.1. A *stability condition* $(Z, \mathcal{P})$ on a triangulated category $\mathcal{D}$ consists of a group homomorphism $Z : K(\mathcal{D}) \to \mathbb{C}$ called the *central charge* and full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$, satisfying the following axioms:

1. if $E \in \mathcal{P}(\phi)$ then $Z(E) = m(E)\exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$,

2. for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$,

3. if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ then $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$,

4. for each nonzero object $E \in \mathcal{D}$ there is a finite sequence of real numbers $\phi_1 > \phi_2 > \cdots > \phi_n$ and for each $1 \leq i \leq n$ an exact triangle

$$E_{i-1} \to E_i \to A_i \to E_{i-1}[1]$$

   with $A_j \in \mathcal{P}(\phi_j)$, $E_0 = 0$ and $E_n = E$.

The data $\mathcal{P} := \{\mathcal{P}(\phi), \phi \in \mathbb{R}\}$ is called a *slicing*. Given an interval $I \subset \mathbb{R}$, we write $\mathcal{P}I$ for the full extension-closed category generated by all $\mathcal{P}(\phi)$ for $\phi \in I$. That is, $\mathcal{P}I = [\cup_{\phi \in I} \mathcal{P}(\phi)]$. If $E \in \mathcal{P}(\phi)$ then the number $m(E)$ is called the *mass* of E.

Lemma II.4.2. *Suppose $(Z, \mathcal{P})$ is a stability condition on a triangulated category $\mathcal{D}$. For each $\phi \in \mathbb{R}$ the pair of full subcategories $(\mathcal{P}(>\phi), \mathcal{P}(\leq \phi+1))$ is a bounded nondegenerate t-structure on $\mathcal{D}$ with heart $\mathcal{P}(\phi, \phi+1]$.*

Definition II.4.3. Suppose that $\mathcal{D}$ is K-linear and of finite type. The Euler form, a bilinear form on $K(\mathcal{D})$, is then defined by the formula

$$\chi(E, F) = \sum_i (-1)^i \dim_k \mathrm{Hom}_{\mathcal{D}}(E, F[i]),$$

and the free abelian group $\mathcal{N}(\mathcal{D}) = K(\mathcal{D})/K(\mathcal{D})^{\perp}$ is called the *numerical Grothendieck group* of $\mathcal{D}$. If this group has finite rank the category $\mathcal{D}$ is said to be *numerically finite.* A stability condition $\sigma = (Z, \mathcal{P})$ is *numerical* if the central charge $Z : K(\mathcal{D}) \to \mathbb{C}$ factors through the quotient group $\mathcal{N}(\mathcal{D})$.

Definition II.4.4. A *stability function* on an abelian category $\mathcal{A}$ is a group homomorphism $Z : K(\mathcal{A}) \to \mathbb{C}$ such that for all $0 \neq E \in \mathcal{A}$ the complex number $Z(E)$ lies in the strict upper halfplane $H = \{r \exp(i\pi\phi) : r > 0 \text{ and } 0 < \phi \leq 1\} \subset \mathbb{C}$. Given a stability function $Z : K(\mathcal{A}) \to \mathbb{C}$, the *phase* of an object $0 \neq E \in \mathcal{A}$ is defined to be

$$\phi(E) = \frac{1}{\pi} \arg Z(E) \in (0, 1].$$

The object E is said to be *semistable* (with respect to Z) if every subobject $0 \neq A \subset E$ satisfies $\phi(A) \leq \phi(E)$. If strict inequality holds we say that E is *stable* (with respect to Z). Equivalently, E is semistable if $\phi(E) \leq \phi(B)$ for every nonzero quotient $E \twoheadrightarrow B$.

Remark II.4.5. Given a stability condition $(Z, \mathcal{P})$, Z is a stability function on $\mathcal{P}(0, 1]$. In fact, Z is a stability function on $\mathcal{P}(2n, 2n + 1]$ for all $n \in \mathbb{Z}$.

Definition II.4.6. Let Z be a stability function on $\mathcal{A}$. A *Harder-Narasimhan filtration* (HN-filtration) of an object $0 \neq E \in \mathcal{A}$ is a finite chain of subobjects

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

whose factors $F_j = E_j / E_{j-1}$ are semistable objects of $\mathcal{A}$ with respect to Z with phases

$$\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n).$$

The stability function Z is said to have the *Harder-Narasimhan property* (HN-property) if every nonzero object of $\mathcal{A}$ has a Harder-Narasimhan filtration.

Proposition II.4.7. *To give a stability condition on a triangulated category $\mathcal{D}$ is equivalent to giving a bounded t-structure on $\mathcal{D}$ and a stability function on its heart with the HN-property.*

Definition II.4.8. A slicing $\mathcal{P}$ of a triangulated category $\mathcal{D}$ is *locally-finite* if there exists a real number $\nu > 0$ such that for all $t \in \mathbb{R}$ the category $\mathcal{P}((t - \nu, t + \nu)) \subset \mathcal{D}$ is of finite length. A stability condition $(Z, \mathcal{P})$ is locally finite if the corresponding slicing $\mathcal{P}$ is. We denote by $\mathrm{Stab}(\mathcal{D})$ the set of all locally-finite stability conditions on $\mathcal{D}$, and by $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D})$ the subset of $\mathrm{Stab}(\mathcal{D})$ consisting of numerical stability conditions.

Local finiteness is a "niceness" condition imposed upon stability conditions in order to obtain the main result of this section. For now, we note merely that local finiteness is a kind of smoothness result in that it prevents "too much" of a category from being placed into a small interval of phases in a slicing.

Theorem II.4.9 (Theorem 1.2 of [5]). *Let $\mathcal{D}$ be a triangulated category. For each connected component, $\Sigma \subset \mathrm{Stab}(\mathcal{D})$ there is a linear subspace $V(\Sigma) \subset \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$ with a well-defined linear topology and a local homeomorphism $\Sigma \to V(\Sigma)$ which maps a stability condition $(Z, \mathcal{P})$ to its central charge Z.*

Corollary II.4.10. *If $\mathcal{D}$ is numerically finite then $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D})$ is a finite-dimensional complex manifold.*

As a consequence, we call $\mathrm{Stab}(\mathcal{D})$ the *stability manifold* of $\mathcal{D}$. Given any $\sigma \in \mathrm{Stab}(\mathcal{D})$, let us set

$$W_\sigma := \{U \in \mathrm{Hom}(K_0(\mathcal{D}), \mathbb{C}) : ||U||_\sigma < \infty\}.$$

The linear subspaces $W_\sigma \subset \mathrm{Hom}(K_0(\mathcal{D}), \mathbb{C})$ do not change as σ varies over a connected component C of $\mathrm{Stab}(\mathcal{D})$. Furthermore, the natural projection $C \to W_\sigma$ is the local homeomorphism from the theorem above. The theorem on the structure of $\mathrm{Stab}(\mathcal{D})$ implies that in a neighborhood of $\sigma \in \mathrm{Stab}_{\mathcal{N}}(\mathcal{D})$ the space $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D})$ is modeled on the linear space $W_\sigma^{\mathcal{N}} = W_\sigma \cap \mathrm{Hom}(\mathcal{N}(\mathcal{D}), \mathbb{C})$. A numerical stability condition σ is called *full* if $W_\sigma^{\mathcal{N}} = \mathrm{Hom}(\mathcal{N}(\mathcal{D}), \mathbb{C})$ (see [6]).

We turn next to describing the topology on $\mathrm{Stab}(\mathcal{D})$. The function

$$d(\mathcal{P}, \mathcal{Q}) = \inf\{\epsilon \in \mathbb{R}_{\geq 0} : \mathcal{Q}(\phi) \subset \mathcal{P}[\phi - \epsilon, \phi + \epsilon] \text{ for all } \phi \in \mathbb{R}\}$$

defines a generalized metric on the set of all slicings on $\mathcal{D}$. That is, $d(\cdot, \cdot)$ satisfies all of the axioms for a metric except that it takes values in $[0, \infty] := \mathbb{R}_{\geq 0} \cup \{\infty\}$. For each

$\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$, define a function

$$\|\cdot\|_\sigma : \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C}) \to [0, \infty]$$

by sending a linear map $U : K(\mathcal{D}) \otimes \mathbb{C} \to \mathbb{C}$ to

$$\|U\|_\sigma = \sup \left\{ \frac{|U(E)|}{|Z(E)|} : E \text{ semistable in } \sigma \right\}.$$

This function has the properties of a norm on the complex vector space $\mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$ except that it may not be finite. Let $\epsilon \in (0, \frac{1}{4})$. The topology on $\mathrm{Stab}(\mathcal{D})$ is the one defined by the basis of sets of the form

$$B_\epsilon(\sigma) = \{\tau = (W, \mathcal{Q}) : ||W - Z||_\sigma < \sin(\pi\epsilon) \text{ and } d(\mathcal{P}, \mathcal{Q}) < \epsilon\} \subset \mathrm{Stab}(\mathcal{D}).$$

The set $B_\epsilon(\sigma)$ contains all stability conditions which for all $\phi \in \mathbb{R}$ and $E \in \mathcal{P}(\phi)$, the phase of $W(E)$ differs from ϕ by less than ϵ, and moreover if E is not semistable in $\mathcal{Q}$ then $E \in \mathcal{Q}(\phi - \epsilon, \phi + \epsilon)$. Thus, we have bounds on the phases of the semistable factors of E in τ. These generalized metrics on slicings and stability functions can be extended to give a generalized metric on $\mathrm{Stab}(\mathcal{D})$ in a natural way.

The proof of the theorem is done by first showing local injectivity of the maps $\Sigma \to V(\Sigma)$. Given this, a local deformation result is then proved in order to construct stabilities $\tau = (W, \mathcal{Q})$ in a sufficiently small neighborhood of a fixed $\sigma = (Z, \mathcal{P})$ from their central charge W. We include the statements of these two results for reference.

Lemma II.4.11 (Lemma 6.4 of [5]). *Suppose $\sigma = (Z, \mathcal{P})$ and $\tau = (Z, \mathcal{Q})$ are stability conditions on $\mathcal{D}$ with the same central charge Z. Suppose also that $d(\sigma, \tau) < 1$. Then $\sigma = \tau$.*

Theorem II.4.12 (Theorem 7.1 of [5]). *Let $\sigma = (Z, \mathcal{P})$ be a locally-finite stability condition on a triangulated category $\mathcal{D}$. Then there is an $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$ and*

$W : K(\mathcal{D}) \to \mathbb{C}$ is a group homomorphism satisfying

$$|W(E) - Z(E)| < \sin(\pi\epsilon)|Z(E)|$$

for all $E \in \mathcal{D}$ semistable in σ, then there is a locally-finite stability condition $\tau = (W, \mathcal{Q})$ on $\mathcal{D}$ with $d(\mathcal{P}, \mathcal{Q}) < \epsilon$.

The stability manifold has both left and right group actions.

Lemma II.4.13. *The generalized metric space* $\mathrm{Stab}(\mathcal{D})$ *carries a right action of the group* $\widetilde{\mathrm{GL}}(2, \mathbb{R})$*, the universal covering space of* $\mathrm{GL}^+(2, \mathbb{R})$*, and a left action by isometries of the group* $\mathrm{Aut}(\mathcal{D})$ *of exact autoequivalences of* $\mathcal{D}$*. These two actions commute.*

In particular, the actions are as follows. The group $\widetilde{\mathrm{GL}}(2, \mathbb{R})$ can be thought of as the set of pairs (A, f) where $f : \mathbb{R} \to \mathbb{R}$ is an increasing map with $f(\phi + 1) = f(\phi) + 1$, and $A : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation-preserving linear isomorphism, such that the induced maps on $\mathcal{S}^1 = \mathbb{R}/2Z = \mathbb{R}^2/\mathbb{R}_{>0}$ are the same. Given a stability condition $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$ and a pair $(A, f) \in \widetilde{\mathrm{GL}}(2, \mathbb{R})$, define a new stability condition $\sigma' = (Z', \mathcal{P}')$ by setting

$$Z' = A^{-1} \circ Z \quad \text{and} \quad \mathcal{P}'(\phi) = \mathcal{P}(f(\phi)).$$

This action does not change the semistable objects, but their phases and masses are changed.

For the action by autoequivalences, given $\Psi \in \mathrm{Aut}(\mathcal{D})$ observe that Ψ induces an automorphism ψ of $K(\mathcal{D})$. Define $\Phi(\sigma)$ to be the stability condition $(Z \circ \psi^{-1}, \mathcal{P}')$ where $\mathcal{P}'(\phi) = \Phi(\mathcal{P}(\phi))$.

II.5 Examples of Stability Conditions

In this section we collect a few examples of stability conditions known from the literature. If $\mathcal{D}(X) = \mathcal{D}(\mathrm{Coh}(X))$ is the bounded derived category of coherent sheaves on a variety X, we will often write $\mathrm{Stab}(X)$ for the stability manifold $\mathrm{Stab}(\mathcal{D}(X))$.

Example II.5.1 (Standard Stability)**.** Let X be a smooth curve. The abelian category $\mathrm{Coh}(X)$ of coherent sheaves on X is the heart of a bounded nondegenerate t-structure on $\mathcal{D}(X)$. Consider the stability function

$$Z_{st}(E) = -\deg(E) + i\,\mathrm{rk}(E).$$

We show that Z_{st} has the Harder-Narasimhan property on $\mathrm{Coh}(X)$, thereby defining a stability condition $\sigma_{st} = (Z_{st}, \mathcal{P}_{st})$ where $\mathcal{P}_{st}(0,1] = \mathrm{Coh}(X)$. We call σ_{st} the *standard stability.*

Given a coherent sheaf E, there exists a short exact sequence

$$0 \to T \to E \to B \to 0 \tag{II.2}$$

where T is torsion and B is a vector bundle. For any torsion sheaf T, $Z_{st}(T)$ lies on the negative real ray so $Z_{st}(T)$ has phase 1. Furthermore, all nonzero subobjects of T are torsion so in fact T is semistable. Moreover, all semistable vector bundles will have phases in $(0,1)$ so to give a Harder-Narasimhan filtration for E it suffices to construct a Harder-Narasimhan filtration for B.

A *maximal destabilizing quotient* (with respect to a stability function Z) of an object $0 \neq E \in \mathcal{A}$ of an abelian category $\mathcal{A}$ is a nonzero quotient $E \twoheadrightarrow F$ such that any nonzero quotient $E \twoheadrightarrow F'$ satisfies $\phi(F') \geq \phi(F)$, with equality only if $E \twoheadrightarrow F'$ factors through $E \twoheadrightarrow F$. In [5], it is shown that maximal destabilizing quotients always exist for a stability function if there exist no infinite sequences of destabilizing quotients.

This condition on Z_{st} is easily checked to hold for vector bundles E. Indeed, consider a maximal destabilizing quotient $B \twoheadrightarrow A_0$ for the vector bundle B in II.2. If $B_1 = \ker(B \twoheadrightarrow A_0)$ is semistable, then

$$0 \to B_1 \to B \to A_0 \to 0$$

is a Harder-Narasimhan filtration for B. If not, let $B_1 \twoheadrightarrow A_1$ be a maximal destabilizing quotient for B_1 and repeat the above step. Since B has finite rank and $\mathrm{rk}(B_1) < \mathrm{rk}(B)$, this procedure must terminate after finitely many steps, giving a Harder-Narasimhan filtration for B.

The stability manifolds $\mathrm{Stab}(X)$, where X is a projective curve, are all known (see [15], [11], [5]).

Theorem II.5.2. *Let X be a smooth projective curve.*

1. *If $X = \mathbb{P}^1$ then* $\mathrm{Stab}(X) \cong \mathbb{C}^2$.
2. *If the genus of X is at least* 1*, then* $\mathrm{Stab}(X) \cong \widetilde{\mathrm{GL}}(2, \mathbb{R})$.

The genus $g(X) \geq 1$ case was done by showing that the $\widetilde{\mathrm{GL}}(2, \mathbb{R})$-action was free and that all stability conditions on X were contained in the $\widetilde{\mathrm{GL}}(2, \mathbb{R})$-orbit of the standard stability. The case of $X = \mathbb{P}^1$ was more difficult, due to the presence of exceptional objects, since it is possible to construct stability conditions from certain full exceptional collections. Indeed, in [11] Macrì gave a construction for stability conditions whose hearts are generated by an Ext-exceptional collection. In particular, using a version of Corollary II.2.8 generalized to semiorthogonal decompositions with n full subcategories we can restate his result as follows.

Proposition II.5.3. *Let $\{E_1, E_2, \cdots, E_n\}$ be a full Ext-exceptional collection in a triangulated category $\mathcal{D}$. For each $j \in \{1, \cdots, n\}$, choose real numbers $\phi_j \in (0, 1]$ and $m_j \in \mathbb{R}_{>0}$. Then there exists a stability condition $\sigma = (Z, \mathcal{P})$ with heart $[E_1, \cdots, E_n]$ and central charge*

$$Z(E_j) = m_j \exp(\pi i \phi_j).$$

Furthermore, the objects E_j are stable in σ.

This proposition and the results on gluing of t-structures in [3] are the motivation for the gluing of stability conditions given in the following chapter.

CHAPTER III

GLUING STABILITY CONDITIONS

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In this chapter we present our method for the gluing of stability conditions along semiorthogonal decompositions. Under certain assumptions, the gluing is then shown to be continuous. In section III.1 we introduce the notion of a *reasonable stability*, which is a strengthening of the locally finite condition imposed by Bridgeland in order to prove that $\mathrm{Stab}(\mathcal{D})$ has a manifold structure [5]. This assumption is fairly modest, however, for we show that it is satisfied by all stability conditions constructed in a variety of known cases. In section III.2 we give the gluing construction itself and prove its basic properties. This construction does not address the question of existence, so in section III.3 we give conditions under which the glued stability function will satisfy the Harder-Narasimhan property on the glued heart. Finally, in section III.4 we give conditions under which the gluing map is continuous.

III.1 Reasonable Stability Conditions

Throughout this section $\mathcal{D}$ denotes a triangulated category. We single out a particular class of elements of $\widetilde{\mathrm{GL}}(2,\mathbb{R})$ whose actions on stability conditions will be of use in what follows. For a real number a let us denote by $R_a : \mathrm{Stab}(\mathcal{D}) \to \mathrm{Stab}(\mathcal{D})$ the operation of shifting the phase by a. This is part of the $\widetilde{\mathrm{GL}}(2,\mathbb{R})$-action on $\mathrm{Stab}(\mathcal{D})$. More explicitly, for $\sigma = (Z, P)$ one has $R_a\sigma = (r_{-\pi a} \circ Z, P')$, where $P'(t) = P(t + a)$, $r_{-\pi a}$ is the rotation in $\mathbb{C} = \mathbb{R}^2$ through the angle $-\pi a$. We refer to the transformations R_a as rotations. For a complex number ξ, we denote by $\Re\xi$ and $\Im\xi$, respectively, the real and imaginary parts of ξ.

Definition III.1.1. A stability condition $\sigma = (Z, P)$ on $\mathcal{D}$ is called *reasonable* if

$$\inf_{E \text{ semistable}, E \neq 0} |Z(E)| > 0$$

where E runs over all nonzero σ-semistable objects.

Lemma III.1.2. *Let $\sigma = (Z, P)$ be a stability condition on $\mathcal{D}$.*

1. *If σ is reasonable then for every $0 < \eta < 1$ one has*

$$\inf_{t \in \mathbb{R}, E \in P(t, t+\eta) \setminus 0} |Z(E)| > 0;$$

2. *σ is reasonable if and only if for every t and every $0 < \eta < 1$ the point 0 is an isolated point of $Z(P(t, t+\eta))$;*

3. *If σ is reasonable then every category $P(t, t+\eta)$ for $0 < \eta < 1$ is of finite length, hence, σ is locally finite;*

4. *If the image of Z in $\mathbb{C}$ is discrete then σ is reasonable.*

Proof. (1) Let

$$c = \inf_{E \text{ semistable}, E \neq 0} |Z(E)| > 0.$$

Given an object $E \in \mathbb{P}(t, t+\eta)$ let E_i be the HN-factors of E. Then all numbers $Z(E_i)$ (and $Z(E)$) lie in the cone $C(t, t+\eta)$ of complex numbers with phases between t and $t+\eta$. Let $h : \mathbb{C} \to \mathbb{R}$ denote the scalar product with the unit vector of phase $t + \eta/2$. Then we have $\cos(\pi\eta/2)|z| \leq h(z) \leq |z|$ for all $z \in C(t, t+\eta)$. Hence,

$$|Z(E)| \geq h(Z(E)) = \sum_i h(Z(E_i)) \geq \cos(\pi\eta/2)c.$$

(2) The "only if" part follows from (1). Conversely, assuming that 0 is an isolated point of $Z(P(0,3/4))$ and of $Z(P(1/2,5/4))$ we see that there is a universal lower bound for $|Z(E)|$, where E is semistable of the phase in $(0,1]$. This implies that σ is reasonable.
(3) This is similar to Lemma 4.4 of [6]. The point is that if $h:\mathbb{C}\to R$ denotes the scalar product with the unit vector of phase $t+\eta/2$ then $h(A)>c>0$ for a fixed constant c, where A is a nonzero object of $P(t,t+\eta)$. Since h is an additive function with respect to strict short exact sequences, the assertion follows.
(4) This is clear. □

Proposition III.1.3. *Let Σ be a connected component of* $\mathrm{Stab}(\mathcal{D})$ *containing some reasonable stability condition. Then every $\sigma\in\Sigma$ is reasonable.*

Proof. Let $\sigma=(Z,P)$, $\sigma'=(Z',P')$ be points of Σ. Assume first that σ' is reasonable, and $\sigma'\in B_\epsilon(\sigma)$, where $\epsilon<1/4$. Then for every σ-semistable object E of phase t we have $|Z'(E)-Z(E)|<\sin(\pi\epsilon)|Z(E)|$ and $E\in P'(t-\epsilon,t+\epsilon)$. Hence, by Lemma III.1.2(1), there exists a constant $c>0$ independent of E such that $|Z'(E)|>c$. Therefore,

$$|Z(E)|>(1+\sin(\pi\epsilon))^{-1}|Z'(E)|>(1+\sin(\pi\epsilon))^{-1}c,$$

so σ is reasonable. This shows that the set of reasonable stabilities is closed. Conversely, assume that σ is reasonable and $\sigma'\in B_\epsilon(\sigma)$, where ϵ is sufficiently small. Given a σ'-semistable object E of phase t we have $E\in P(t-\epsilon,t+\epsilon)$. Let (E_i) be the HN-factors of E with respect to σ. Then $E_i\in P(t-\epsilon,t+\epsilon)\subset P'(t-2\epsilon,t+2\epsilon)$. Let us denote by $h:\mathbb{C}\to\mathbb{R}$ the scalar product with the unit vector of phase t. Then

$$|Z'(E)|=h(Z'(E))=\sum_i h(Z'(E_i))\geq\frac{1}{2}\sum_i|Z'(E_i)|$$

provided ϵ is small enough. But $|Z'(E_i)|>(1-\sin(\pi\epsilon))|Z(E_i)|$, which is bounded below by a positive constant depending only on ϵ. Hence, σ' is reasonable, so the set of reasonable stabilities is open. □

Corollary III.1.4. *If* $\Sigma \subset \mathrm{Stab}(\mathcal{D})$ *is a connected component containing some stability condition such that the corresponding central charge has discrete image, then every* $\sigma \in \Sigma$ *is reasonable.*

Remark III.1.5. This corollary implies that all (locally finite) stability conditions constructed in [2], [5], [6] and [11] are reasonable. It should also be noted that at the time of writing, these papers together contain essentially all known examples of stability conditions. This gives some evidence for viewing our reasonable condition to be a natural one to consider.

III.2 Gluing Construction

A stability condition is determined by a stability function on the heart of a bounded t-structure. In this section we give a construction for gluing stability conditions satisfying certain conditions. We start by recalling the gluing construction of Corollary II.2.8 from the previous chapter.

Corollary III.2.1. *Assume we have a semiorthogonal decomposition* $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ *and t-structures* $(\mathcal{D}_i^{\leq 0}, \mathcal{D}_i^{\geq 0})$ *with the hearts* H_i *on* $\mathcal{D}_i$ *(where* $i = 1, 2$*), such that* $\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(H_1, H_2) = 0$. *Then there is a t-structure on* $\mathcal{D}$ *with the heart*

$$H = \{X \in \mathcal{D} \mid \lambda_1(X) \in H_1, \rho_2(X) \in H_2\}. \tag{III.1}$$

With respect to this t-structure on $\mathcal{D}$ *the functors* $\lambda_1 : \mathcal{D} \to \mathcal{D}_1$ *and* $\rho_2 : \mathcal{D} \to \mathcal{D}_2$ *are t-exact. Furthermore, every object* $E \in H$ *fits into an exact sequence in* H

$$0 \to \rho_2(E) \to E \to \lambda_1(E) \to 0, \tag{III.2}$$

where $\rho_2(E) \in H_2$ *and* $\lambda_1(E) \in H_1$. *Therefore, we also have*

$$H = [H_2, H_1]. \tag{III.3}$$

Assume now that the hearts H_1 and H_2 are equipped with stability functions $Z_i : K_0(H_i) \to \mathbb{C}$. Then the formula

$$Z(X) = Z_1(\lambda_1(X)) + Z_2(\rho_2(X)) \tag{III.4}$$

defines a stability function on the glued heart H.

Definition III.2.2. Suppose we have stability conditions $\sigma_1 = (Z_1, P_1)$ on $\mathcal{D}_1$ and $\sigma_2 = (Z_2, P_2)$ on $\mathcal{D}_2$, such that the corresponding hearts $H_1 = P_1(0,1]$ and $H_2 = P_2(0,1]$ satisfy $\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(H_1, H_2) = 0$. Then we say that a stability condition $\sigma = (Z, P)$ on $\mathcal{D}$ is *glued from* σ_1 and σ_2 if Z is given by (III.4), and the heart $H = P(0,1]$ is given by (II.1) (or equivalently, by (III.3)).

Note that this glued stability condition is uniquely determined by σ_1 and σ_2. It exists if and only if the Harder-Narasimhan property for the stability function Z on the glued heart H is satisfied. Before addressing when this function Z has the HN-property, we observe the following easy properties of glued stability conditions.

Proposition III.2.3. *1. A stability condition $\sigma = (Z, P)$ on $\mathcal{D}$ is glued from $\sigma_1 = (Z_1, P_1)$ on $\mathcal{D}_1$ and $\sigma_2 = (Z_2, P_2)$ on $\mathcal{D}_2$ if and only if $Z_i = Z|_{\mathcal{D}_i}$ for $i = 1, 2$, $\mathrm{Hom}^{\leq 0}(H_1, H_2) = 0$ and $H_i \subset H$ for $i = 1, 2$, where $H = P(0,1]$, $H_i = P_i(0,1]$.*

2. Let σ be a stability condition on $\mathcal{D}$ with the central charge Z and the heart H. Assume that H is glued from the hearts $H_1 \subset \mathcal{D}_1$ and $H_2 \subset \mathcal{D}_2$, where $\mathrm{Hom}^{\leq 0}(H_1, H_2) = 0$, so that (II.1) *holds. Then for $i = 1, 2$ there exists a stability condition σ_i on $\mathcal{D}_i$ with the heart H_i and the central charge $Z_i = Z|_{\mathcal{D}_i}$, so that σ is glued from σ_1 and σ_2.*

3. If $\sigma = (Z, P)$ is glued from $\sigma_1 = (Z_1, P_1)$ and $\sigma_2 = (Z_2, P_2)$ then for every $\phi \in \mathbb{R}$ one has $P_1(\phi) \subset P(\phi)$ and $P_2(\phi) \subset P(\phi)$.

Proof. (1) Let us observe that for every $E \in \mathcal{D}$ one has the equality $[E] = [\rho_2(E)] + [\lambda_1(E)]$ in $K_0(\mathcal{D})$, so the definition (III.4) is equivalent to the condition $Z|_{\mathcal{D}_i} = Z_i$ for $i = 1, 2$. It remains to note also that the embeddings $H_1, H_2 \subset H$ imply that $\langle H_1, H_2 \rangle \subset H$. Since both are hearts of nondegenerate t-structures this is equivalent to the equality (III.3).

(2) The subcategory $H_1 \subset H$ (resp., $H_2 \subset H$) is exactly the kernel of the exact functor $\rho_2 : H \to H_2$ (resp., $\lambda_1 : H \to H_1$). It follows that these subcategories are closed under passing to subobjects and quotient-objects in H. This easily implies that the Harder-Narasimhan property holds for $Z|_{H_i}$ on H_i, $i = 1, 2$, so we obtain the stability conditions on $\mathcal{D}_1$ and $\mathcal{D}_2$. The fact that σ is glued from these stabilities follows from definition.

(3) It is enough to check this in the case when $\phi \in (0, 1]$. Then this follows immediately from the fact that H_1 and H_2 are stable under subobjects and quotient-objects in H. □

In the case of semiorthogonal decompositions associated with a full Ext-exceptional collection $(E_1, \ldots, E_n)$ the above gluing procedure was considered by Macrì in [11]. Namely, we can consider the semiorthogonal decomposition $\mathcal{D} = \langle \langle E_1 \rangle, \ldots, \langle E_n \rangle \rangle$, and equip $\langle E_i \rangle$ with the t-structure for which E_i belongs to the heart. Then our orthogonality condition on the hearts reduces to the condition that the collection is Ext-exceptional, i.e., $\mathrm{Hom}^{\le 0}(E_i, E_j) = 0$ for $i < j$, and the glued heart is $H = [E_1, \cdots, E_n]$. We say that a stability condition $\sigma = (Z, P)$ on $\mathcal{D}$ is *glued from an* Ext*-exceptional collection* $(E_1, \cdots, E_n)$ if $P(0, 1] = H$.

III.3 Harder-Narasimhan Property and Gluing of Stability Conditions

In this section we show how to check the Harder-Narasimhan property for the glued stability function under different sets of additional assumptions. First, we recall the following basic criterion.

Proposition III.3.1. *([5], Prop. 2.4) Suppose $\mathcal{A}$ is an abelian category with a stability function $Z : K_0(\mathcal{A}) \to \mathbb{C}$ satisfying the chain conditions*

1. *there are no infinite sequences of subobjects in $\mathcal{A}$*

$$\cdots \subset E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1$$

with $\phi(E_{j+1}) > \phi(E_j)$ *for all* j*,*

2. *there are no infinite sequences of quotients in* $\mathcal{A}$

$$E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots \twoheadrightarrow E_j \twoheadrightarrow E_{j+1} \twoheadrightarrow \cdots$$

with $\phi(E_j) > \phi(E_{j+1})$ *for all* j*.*

Then Z *has the Harder-Narasimhan property on* $\mathcal{A}$*.*

The following Lemma is a more precise version of Proposition 5.0.1 of [1].

Lemma III.3.2. *(a) Let* Z *be a stability function on an abelian category* $\mathcal{A}$*. Assume that* 0 *is an isolated point of* $\Im Z(\mathcal{A}) \subset \mathbb{R}_{\geq 0}$*, and that the category* $\mathcal{A}_0 = \{A \in \mathcal{A} \mid \Im Z(A) = 0\}$ *is Noetherian. Then* Z *satisfies the Harder-Narasimhan property on* $\mathcal{A}$ *if and only if* $\mathcal{A}$ *is Noetherian.*
(b) Let $\sigma = (Z, P)$ *be a stability condition on* $\mathcal{D}$ *with Noetherian heart* $P(0,1]$*. Assume that* 0 *is an isolated point of* $\Im Z(P(0,1)) \subset \mathbb{R}_{\geq 0}$*. Then the category* $P(0,1)$ *is of finite length. Also,* σ *is reasonable if and only if* 0 *is an isolated point of* $Z(P(1)) \subset \mathbb{R}_{\leq 0}$*.*

Proof. (a) Assume first that $\mathcal{A}$ is Noetherian. Then condition (2) of Proposition III.3.1 is automatic. To check condition (1) we observe that if $E \to F$ is a destabilizing inclusion in $\mathcal{A}$ then $\Im Z(E) < \Im Z(F)$. Indeed, we have either $\Im Z(F/E) > 0$ or $\Re Z(F/E) < 0$. But in the latter case the phase of $Z(E)$ would be smaller than that of $Z(F)$. Thus, if we have a chain

$$\cdots \subset E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1 \tag{III.5}$$

of destabilizing inclusions in $\mathcal{A}$ then the sequence $(\Im Z(E_j))$ is strictly decreasing. But this implies that $\Im Z(E_j/E_{j+1})$ tends to 0 which is a contradiction. Conversely, assume Z satisfies the Harder-Narasimhan property. To check that $\mathcal{A}$ is Noetherian we have to check that every sequences of quotients in $\mathcal{A}$

$$E_1 \twoheadrightarrow E_2 \twoheadrightarrow E_3 \twoheadrightarrow \cdots \tag{III.6}$$

stabilizes. Note that in this situation the sequence $(\Im Z(E_i))$ is decreasing, so it has to stabilize. Without loss of generality we can assume that the sequence $(\Im Z(E_i))$ is constant. Then the kernel K_i of $E_1 \to E_i$ belongs to $\mathcal{A}_0$. Since Z satisfies the Harder-Narasimhan property, there exists a maximal subobject $F \subset E_1$ such that $F \in \mathcal{A}_0$. Then the kernels K_i form an increasing chain of subobjects in F. Since $\mathcal{A}_0$ is Noetherian, this sequence stabilizes, so the original sequence (E_i) also stabilizes. It remains to check that in this situation $\mathcal{A}_{>0}$ is Artinian. But a sequence of inclusions (III.5) with $\Im Z(E_j/E_{j+1}) > 0$ is impossible since $\Im Z(E_j/E_{j+1})$ would tend to zero.

(b) To see that $P(0,1)$ is of finite length we observe that any increasing chain of admissible inclusions in $P(0,1)$ stabilizes since $\mathcal{A} = P(0,1]$ is Noetherian. Also, if we have a chain (III.5) of admissible proper inclusions in $P(0,1)$ then the sequence $\Im Z(E_j)$ is strictly decreasing, which is impossible. Under our assumptions $|Z(E)|$ is bounded below by some positive constant, where E runs through nonzero semistable objects in $P(0,1)$. Thus, σ is reasonable if and only if

$$\inf_{E \in P(1)\setminus\{0\}} |Z(E)| > 0.$$

$\square$

Proposition III.3.3. *Let $(\mathcal{D}_1, \mathcal{D}_2)$ be a semiorthogonal decomposition of a triangulated category $\mathcal{D}$, and let $\sigma_1 = (Z_1, H_1)$ and $\sigma_2 = (Z_2, H_2)$ be a pair of locally finite stability conditions on $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. Assume that $\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(H_1, H_2) = 0$, and let H be the heart in $\mathcal{D}$ glued from H_1 and H_2. As before, consider the stability function $Z = Z_1\lambda_1 + Z_2\rho_2$ on H. Assume in addition that one of the following two conditions hold:*

(a) 0 is an isolated point of $\Im Z_i(H_i) \subset \mathbb{R}_{\geq 0}$ for $i = 1, 2$;

(b) $\mathrm{Hom}_{\mathcal{D}}^{\leq 1}(H_1, P_2(0,1)) = 0$.

Then Z has the Harder-Narasimhan property on H. Furthermore, in case (a) the category $P(0,1)$ for the glued stability condition $\sigma = (Z, P)$ is of finite length. In case (b) the stability condition σ is locally finite.

Proof. First, assume that (a) holds. Then it is easy to see that 0 is an isolated point of

$\Im Z(H) \subset \mathbb{R}_{\geq 0}$. Also, by Lemma III.3.2(a), both categories H_1 and H_2 are Noetherian (the condition on $\mathcal{A}_0$ in this Lemma follows from the assumption that σ_i's are locally finite). Using the exact functors $\lambda_1 : H \to H_1$ and $\rho_2 : H \to H_2$ we easily deduce that H is Noetherian. Now the assertion follows by applying Lemma III.3.2(a) again.

(b) In this case for every $t \in (0, 1]$ let us define the subcategory $P(t) \subset H$ by

$$P(t) := \{E \in H \mid \lambda_1(E) \in P_1(t), \rho_2(E) \in P_2(t)\}.$$

Note that each object of $P(t)$ is an extension of an object in $P_2(t)$ by an object in $P_1(t)$. It is enough for every $E \in H$ to construct the HN-filtration with respect to this slicing. We start with the canonical extension

$$0 \to E_2 \to E \to E_1 \to 0$$

where $E_2 = \rho_2(E) \in H_2$ and $E_1 = \lambda_1(E) \in H_1$. Consider also the canonical exact sequences

$$0 \to A_i \to E_i \to B_i \to 0$$

with $A_i \in P_i(1)$ and $B_i \in P_i(0, 1)$ for $i = 1, 2$. Since $\mathrm{Ext}^1(E_1, B_2) = 0$ by assumption, we get a splitting $E \to B_2$ which gives rise to an exact sequence

$$0 \to A_2 \to E \to B_2 \oplus E_1 \to 0$$

Let $E(1) \subset E$ be the preimage of $A_1 \subset E_1 \subset B_2 \oplus E_1$. Then $E(1)$ is an extension of A_1 by A_2, so $E(1) \in P(1)$. Also, $E/E(1) \simeq B_1 \oplus B_2$, so we get the required filtration by using the HN-filtrations on B_1 and B_2. The obtained glued stability has the property that $\lambda_1(P(a, b)) \subset P_1(a, b)$ and $\rho_2(P(a, b)) \subset P_2(a, b)$. This easily implies that it is locally finite. $\square$

Remark III.3.4. We do not know how to check local finiteness of the glued stability condition in Proposition III.3.3(a) without imposing additional assumptions.

If we work with reasonable stability conditions, we can prove the existence of the glued stability conditions under a slightly stronger orthogonality assumption.

Theorem III.3.5. *Let $(\mathcal{D}_1, \mathcal{D}_2)$ be a semiorthogonal decomposition of a triangulated category $\mathcal{D}$. Suppose (σ_1, σ_2) is a pair of reasonable stability conditions on $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively, with the slicings P_i and central charges Z_i ($i = 1, 2$), and let a be a real number in $(0, 1)$. Assume the following two conditions hold:*

1. $\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(P_1(0,1], P_2(0,1]) = 0$;
2. $\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(P_1(a,a+1], P_2(a,a+1]) = 0$;

Then there exists a reasonable stability σ glued from σ_1 and σ_2. Furthermore,

$$P(0,a] = [P_1(0,a], P_2(0,a]]$$
$$P(a,1] = [P_1(a,1], P_2(a,1]].$$

Lemma III.3.6. *Let $\mathcal{A}$ be an abelian category equipped with a torsion pair $(\mathcal{T}, \mathcal{F})$. Suppose Z is a stability function on $\mathcal{A}$ such that for any nonzero $T \in \mathcal{T}$ and $F \in \mathcal{F}$ one has $\phi Z(T) > \phi Z(F)$ (recall that ϕz denotes the phase of z). Let $Z|_{\mathcal{T}}$ and $Z|_{\mathcal{F}}$ be the stability functions on the exact categories $\mathcal{T}$ and $\mathcal{F}$ induced by Z. Then every $Z|_{\mathcal{T}}$-semistable object of $\mathcal{T}$ (resp., $Z|_{\mathcal{F}}$-semistable object of $\mathcal{F}$) is Z-semistable as an object of $\mathcal{A}$.*

Proof. We consider only the case of a $Z|_{\mathcal{T}}$-semistable object $T \in \mathcal{T}$ (the second case is similar). Suppose T is not Z-semistable as an object of $\mathcal{A}$. Then there exists a subobject $A \subset T$ such that $\phi Z(A) > \phi Z(T)$. Consider the canonical exact sequence

$$0 \to T(A) \to A \to F(A) \to 0$$

with $T(A) \in \mathcal{T}$, $F(A) \in \mathcal{F}$. By the assumption either $\phi Z(T(A)) > \phi Z(F(A))$ or one of the objects $T(A)$, $F(A)$ is zero. Note that $T(A) \neq 0$, since otherwise A would be an object of F, so the inequality $\phi Z(A) > \phi Z(T)$ would be impossible. It follows that $\phi Z(T(A)) \geq \phi Z(A) > \phi Z(T)$. Thus, we found a destabilizing subobject $T(A) \subset T$ (the quotient is automatically in $\mathcal{T}$ since $\mathcal{T}$ is always closed under quotients). □

Proof of Theorem III.3.5. Let $H \subset \mathcal{D}$ be the heart glued from $P_1(0,1]$ and $P_2(0,1]$ and let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ denote the corresponding t-structure. Using the second condition we can construct a t-structure on $\mathcal{D}$ with the heart

$$H_a = [P_1(a, a+1], P_2(a, a+1]].$$

One immediately checks that $H \subset [H_a, H_a[-1]]$ and $H_a \subset [H[1], H] = \mathcal{D}^{[-1,0]}$. Now for every $E \in H$ consider the canonical triangle

$$A \to E \to B \to A[1]$$

with $A \in H_a$ and $B \in H_a[-1]$. We claim that A and B belong to H. Indeed, we have $A \in H_a \subset \mathcal{D}^{\leq 0}$. On the other hand, A is an extension of E by $B[-1] \in H_a[-2]$, so $A \in \mathcal{D}^{\geq 0}$. Hence, $A \in H$. Similarly, $B \in H_a[-1] \subset \mathcal{D}^{\geq 0}$, and also $B \in \mathcal{D}^{\leq 0}$ as an extension of $A[1] \in H_a[1]$ by E. Therefore, if we set

$$P(0,a] = \{E \in \mathcal{D} \mid \lambda_1(E) \in P_1(0,a], \rho_2(E) \in P_2(0,a]\},$$

$$P(a,1] = \{E \in \mathcal{D} \mid \lambda_1(E) \in P_1(a,1], \rho_2(E) \in P_2(a,1]\}, \tag{III.7}$$

then $(P(a,1], P(0,a])$ is a torsion pair in H. Next, let Z be the glued central charge given by (III.4). Note that both $P(0,a]$ and $P(a,1]$ have the structure of an exact category (as full subcategories of H), so we can define Z-semistable objects in $P(0,a]$ and $P(a,1]$. Finally, we claim that every object of $P(0,a]$ (resp., $P(a,1]$) admits a HN-filtration. Indeed, since σ_1 and σ_2 are reasonable, the categories $P_1(0,a]$ and $P_2(0,a]$ are of finite length (see Lemma III.1.2(3)). Therefore, $P(0,a]$ is also of finite length. Now the version of III.3.1 for exact categories implies the Harder-Narasimhan property for $P(0,a]$. The same argument works also for $P(a,1]$. Applying Lemma III.3.6 to the torsion pair $(P(a,1], P(0,a])$ in $P(0,1]$ we obtain HN-filtrations for every object in H. Thus, (Z,H) defines a stability condition σ on $\mathcal{D}$. It follows from the definition of $P(0,a]$ and $P(a,1]$ that 0 is an isolated point of $Z(P(0,a])$ and of $Z(P(a,1])$. This immediately implies that σ is reasonable. $\square$

Remark III.3.7. It may not be easy in general to determine for a particular pair of stabilities σ_1, σ_2 with $\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(P_1(>0), P_2(\leq 1) = 0$ whether there exists $a \in (0,1)$ such that

$$\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(P_1(>a), P_2(\leq a+1) = 0.$$

However, in the following two cases this is automatic.
1. If there exists $\phi > 0$ such that $P_2(0,\phi] = \{0\}$ then any $a \in (0,\phi]$ works, since in this case $P_2(\leq a+1) = P_2(\leq 1)$. For instance, this condition is satisfied when $P_2(0,1]$ is of finite length and has finite number of simple objects.
2. If there exists $\phi < 1$ such that $P_1(\phi,1] = \{0\}$ then any $a \in (\phi,1]$ works, since in this case $P_1(>-a) = P_1(>0)$. For example, this condition holds when $P_1(0,1]$ is of finite length with finite number of simple objects and $P_1(1) = \{0\}$.

III.4 Continuity of Gluing

Let us recall the following basic result.

Lemma III.4.1. *(Lemma 6.4 of [5]) Suppose $\sigma = (Z,P)$ and $\tau = (Z,Q)$ are stability conditions on $\mathcal{D}$ with the same central charge Z. Suppose also that $d(P,Q) < 1$. Then $\sigma = \tau$.*

We start with the observation that the condition $d(P,Q) < 1$ in the above Lemma can be weakened and use this to give a nice criterion for determining when two stability conditions are close (part (b) of the following Proposition).

Proposition III.4.2. *Let $\sigma_1 = (Z_1, P_1)$ and $\sigma_2 = (Z_2, P_2)$ be stability conditions on $\mathcal{D}$.*

(a) Assume that

1. $Z_1 = Z_2$ *and*

2. $P_1(0,1] \subset P_2(-1,2]$.

Then $\sigma_1 = \sigma_2$.

(b) Assume that σ_1 *is locally finite. There exists* $\epsilon_0 > 0$ *such that if for some* $0 < \epsilon < \epsilon_0$ *one has*

1. $||Z_1 - Z_2||_{\sigma_1} < \sin(\pi\epsilon)$ *and*

2. $P_2(0,1] \subset P_1(-1+\epsilon, 2-\epsilon]$,

then $\sigma_2 \in B_\epsilon(\sigma_1)$.

Proof. (a) First, using properties of t-structures we can easily deduce that $P_2(0,1] \subset P_1(-1,2]$. Now given $E \in P_1(0,1]$, there is an exact triangle

$$F \to E \to G \to F[1]$$

with $F \in P_2(1,2]$ and $G \in P_2(-1,1]$. Observe that $F \in P_1(>0)$ and $G \in P_1(\leq 2)$. Since F is an extension of E by $G[-1]$, we derive that $F \in P_1(0,1]$. But the intersection $P_1(0,1] \cap P_2(1,2]$ is trivial (since $Z_1 = Z_2$), so $F = 0$. This proves that $E \in P_2(-1,1]$.

Next, consider an exact triangle

$$F \to E \to G \to F[1]$$

with $F \in P_2(0,1]$ and $G \in P_2(-1,0]$. Observe that $F \in P_1(>-1)$ and $G \in P_1(\leq 1]$. Since G is an extension of $F[1]$ by E, we get $G \in P_2(-1,0] \cap P_1(0,1] = \{0\}$. Therefore, $P_1(0,1] \subset P_2(0,1]$. Since these are both hearts of bounded t-structures, they have to be equal, so $\sigma_1 = \sigma_2$.

(b) Let $\sigma = (Z_2, P)$ be the unique stability in $B_\epsilon(\sigma_1)$ lifting the central charge Z_2 —it exists by our assumption that $||Z_2 - Z_1||_{\sigma_1} < \sin(\pi\epsilon)$ (using Theorem 7.1 of [5]). Then

$$P_2(0,1] \subset P_1(-1+\epsilon, 2-\epsilon] \subset P(-1,2].$$

By part (a), this implies that $\sigma = \sigma_2$. □

Now we can show that the gluing construction of Theorem III.3.5 is continuous.

Theorem III.4.3. *Let $(\mathcal{D}_1, \mathcal{D}_2)$ be a semiorthogonal decomposition in a triangulated category $\mathcal{D}$. For a real number $a \in (0,1)$ let $S(a) \subset \mathrm{Stab}(\mathcal{D}_1) \times \mathrm{Stab}(\mathcal{D}_2)$ denote the subset of (σ_1, σ_2) such that σ_1 and σ_2 are reasonable stability conditions satisfying*

1. $\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(P_1(0,1], P_2(0,1]) = 0$,
2. $\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(P_1(a,a+1], P_2(a,a+1]) = 0$.

Let $\mathrm{gl} : S(a) \longrightarrow \mathrm{Stab}(\mathcal{D})$ *be the map associating to (σ_1, σ_2) the corresponding glued stability condition σ on $\mathcal{D}$ (see Theorem III.3.5). Then the map* gl *is continuous on $S(a)$.*

Proof. Let $\sigma_i = (Z_i, P_i)$, $\sigma_i' = (Z_i', P_i')$ be stabilities on $\mathcal{D}_i$ for $i = 1, 2$, such that (σ_1, σ_2) and (σ_1', σ_2') are points of $S(a)$, and let us denote by $\sigma = (Z, P)$ and $\sigma' = (Z', P')$ the corresponding glued stability conditions. Assume that $\sigma_i' \in B_\delta(\sigma_i)$ for $i = 1, 2$. Then for $\epsilon \geq \delta$ we have

$$\begin{aligned} P(0,1] = [P_1(0,1], P_2(0,1]] &\subset [P_1'(-\epsilon, 1+\epsilon], P_2'(-\epsilon, 1+\epsilon]] \\ &\subset P'(-\epsilon, 1+\epsilon]. \end{aligned}$$

Thus, we can deduce the required continuity from Proposition III.4.2(b), once we show that $||Z - Z'||_\sigma \leq \sin(\pi\epsilon)$ provided δ is small enough. Let $\phi \in (0,1]$ and $E \in P(\phi)$. We have to prove that

$$|Z(E) - Z'(E)| \leq |Z(E)| \sin(\pi\epsilon).$$

Assume first that $\phi \in (a, 1]$. Let $h : \mathbb{C} \to \mathbb{R}$ denote the scalar product with the unit vector of phase $\frac{a+1}{2}$. Then there exists a positive constant c (depending only on a) such that

$$h(z) \leq |z| \leq c \cdot h(z),$$

for all nonzero complex numbers z with phase θ, where $a \leq \theta \leq 1$.

Let $F_1, \cdots, F_n$ (resp., $G_1, \cdots, G_m$) be the HN-factors of $\lambda_1(E)$ (resp., $\rho_2(E)$) with respect to σ_1 (resp., σ_2). Then we have

$$\begin{aligned} |Z(E) - Z'(E)| &\leq |Z_1(\lambda_1 E) - Z_1'(\lambda_1 E)| + |Z_2(\rho_2 E) - Z_2'(\rho_2 E)| \\ &\leq \sum_{i=1}^{n} |Z_1(F_i) - Z_2'(F_i)| + \sum_{j=1}^{m} |Z_2(G_j) - Z_2'(G_j)| \\ &\leq \sin(\pi\delta) \left[\sum_{i=1}^{n} |Z_1(F_i)| + \sum_{j=1}^{m} |Z_2(G_j)| \right]. \end{aligned}$$

Recall that by (III.7), we have $\lambda_1(E) \in P_1(a, 1]$ and $\rho_2(E) \in P_2(a, 1]$. Hence, all the numbers $Z_1(F_i)$ and $Z_2(G_j)$ have phases between a and 1, so we derive

$$\begin{aligned} |Z(E) - Z'(E)| &\leq c \sin(\pi\delta)[\textstyle\sum_{i=1}^{n} h(Z_1(F_i)) + \sum_{j=1}^{m} h(Z_2(G_j))] \\ &= c \sin(\pi\delta) h(Z(E)) \leq c \sin(\pi\delta) |Z(E)|. \end{aligned}$$

So δ must be chosen to satisfy the relation $c \sin(\pi\delta) < \sin(\pi\epsilon)$. A similar argument covers the case of objects $F \in P(0, a]$ and imposes a second condition that $c' \sin(\pi\delta) < \sin(\pi\epsilon)$ for some positive constant c', depending only on a. Given δ satisfying both conditions, it follows that

$$||Z - Z'||_\sigma \leq \sin(\pi\epsilon).$$

□

The following Corollary describes an open subset of pairs of stabilities that can be glued, obtained by imposing a stronger orthogonality assumption on (σ_1, σ_2).

Corollary III.4.4. *Let $U \subset \mathrm{Stab}(\mathcal{D}_1) \times \mathrm{Stab}(\mathcal{D}_2)$ denote the set of pairs of reasonable stabilities ($\sigma_1 = (Z_1, P_1)$ and $\sigma_2 = (Z_2, P_2)$) such that for some $\epsilon > 0$ one has*

$$\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(P_1(-\epsilon, 1], P_2(0, 1+\epsilon)) = 0.$$

Then U is open and the gluing map $\mathrm{gl} : U \to \mathrm{Stab}(\mathcal{D})$ is continuous.

Proof. Note that our assumption on (σ_1, σ_2) is equivalent to

$$\mathrm{Hom}_{\mathcal{D}}(P_1(-\epsilon, +\infty), P_2(-\infty, 1+\epsilon)) = 0.$$

For each $\epsilon > 0$ let us denote by T_ϵ the set of pairs (σ_1, σ_2) satisfying this condition. Note that $U = \cup_{\epsilon > 0} T_\epsilon$. Now to check that U is open suppose we have $(\sigma_1, \sigma_2) \in T_\epsilon$. Given a pair $(\sigma_1' = (Z_1', P_1'), \sigma_2' = (Z_2', P_2'))$, such that $\sigma_i' \in B_\delta(\sigma_i)$, for $i = 1, 2$, where $0 < \delta < \epsilon$, we have $P_1'(> -\epsilon + \delta) \subset P_1(> -\epsilon)$ and $P_2'(< 1 + \epsilon - \delta) \subset P_1(< 1 + \epsilon)$. Hence, (σ_1', σ_2') belongs to $T_{\epsilon - \delta}$. It remains to apply Theorem III.4.3. $\square$

On the other hand, in the situation when $\mathcal{D}_1$ is generated by an exceptional object, we have the following result that will be used later.

Corollary III.4.5. *Let $(\mathcal{D}_1, \mathcal{D}_2)$ be a semiorthogonal decomposition in a triangulated category $\mathcal{D}$.*

(i) Assume that $\mathcal{D}_1$ is generated by an exceptional object E_1, and $H_2 \subset \mathcal{D}_2$ is a heart of some bounded t-structure on $\mathcal{D}_2$, such that $\mathrm{Hom}_{\mathcal{D}}^{\leq -1}(E_1, H_2) = 0$. Let $S_2 \subset \mathrm{Stab}(\mathcal{D}_2)$ denote the set of reasonable stability conditions $\sigma_2 = (Z, P)$ with $P(0, 1] = H_2$. On the other hand, let $R_1 \subset \mathrm{Stab}(\mathcal{D}_1)$ denote the set of stability conditions such that the phase of E_1 is < 0. Then there is continuous gluing map $R_1 \times S_2 \to \mathrm{Stab}(\mathcal{D})$.

(ii) Similarly, assume that $\mathcal{D}_2$ is generated by an exceptional object E_2, and $H_1 \subset \mathcal{D}_1$ is a heart of some bounded t-structure on $\mathcal{D}_2$, such that $\mathrm{Hom}_{\mathcal{D}}^{\leq -1}(H_1, E_2) = 0$. Let $S_1 \subset$ $\mathrm{Stab}(\mathcal{D}_1)$ *denote the set of reasonable stability conditions with the heart H_1, and let $R_2 \subset$*

$\mathrm{Stab}(\mathcal{D}_2)$ *denote the set of stability conditions such that the phase of* E_2 *is* > 1. *Then there is continuous gluing map* $S_1 \times R_2 \to \mathrm{Stab}(\mathcal{D})$.

Proof. We will only consider (i) since the proof of (ii) is analogous. Let $R_1(\epsilon) \subset \mathrm{Stab}(\mathcal{D}_1)$ denote the set of stability conditions such that the phase of E_1 is $< -\epsilon$. It is enough to check that for every $\epsilon > 0$ one has $R_1(\epsilon) \times S_2 \subset S(1-\epsilon)$, where $S(1-\epsilon) \subset \mathrm{Stab}(\mathcal{D}_1) \times \mathrm{Stab}(\mathcal{D}_2)$ is the subset considered in Theorem III.4.3 for $a = 1-\epsilon$. Note that $P_1(0,1] = \langle E_1[n] \rangle$, where n is determined by the condition that the phase of E_1 is in the interval $(-n, -n+1]$. Hence, $n \geq 1$, so the condition $\mathrm{Hom}^{\leq 0}(P_1(0,1], H_2) = 0$ is satisfied. Similarly, $P_1(-\epsilon, 1-\epsilon] = \langle E_1[m] \rangle$, where $m \geq 1$. Hence, $\mathrm{Hom}^{\leq 0}(P_1(-\epsilon, 1-\epsilon], P_2(\leq 1)) = 0$ which implies the condition (2) of Theorem III.4.3 for $a = 1-\epsilon$. □

CHAPTER IV

STABILITIES ON CURVES WITH $\mathbb{Z}_2$-EQUIVARIANT STRUCTURE

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IV.1 Semiorthogonal Decompositions Associated with Double Coverings

Let $\pi : X \to Y$ be a double covering of smooth projective varieties X and Y, ramified along a smooth divisor R in Y. Then we have an action of $\mathbb{Z}_2$ on X such that the nontrivial element acts by the corresponding involution $\tau : X \to X$. Let us denote by $\mathcal{D}_{\mathbb{Z}_2}(X)$ the corresponding bounded derived category of $\mathbb{Z}_2$-equivariant coherent sheaves on X. We denote by ζ the nontrivial character of $\mathbb{Z}_2$. Note that τ-invariant stability conditions on $\mathcal{D}(X)$ correspond to stability conditions on $\mathcal{D}_{\mathbb{Z}_2}(X)$ that are invariant under the autoequivalence $F \mapsto \zeta \otimes F$ (see [12] or [18]). Below we will show how to construct stability conditions on $\mathcal{D}_{\mathbb{Z}_2}(X)$ starting from a pair of stability conditions on $\mathcal{D}(Y)$ and on $\mathcal{D}(R)$, satisfying certain assumptions.

Let us denote by $i : R \to X$ (resp., $j : R \to Y$) the closed embedding of the ramification divisor into X (resp., Y). For every sheaf F on R we equip i_*F with the trivial $\mathbb{Z}_2$-equivariant structure. This gives a functor $i_* : \mathcal{D}(R) \to \mathcal{D}_{\mathbb{Z}_2}(X)$. On the other hand, for a coherent sheaf F on Y we have a natural $\mathbb{Z}_2$-equivariant structure on π^*F, so we obtain a functor $\pi^* : \mathcal{D}(Y) \to \mathcal{D}_{\mathbb{Z}_2}(X)$.

Theorem IV.1.1. *The functors $i_* : \mathcal{D}(R) \to \mathcal{D}_{\mathbb{Z}_2}(X)$ and $\pi^* : \mathcal{D}(Y) \to \mathcal{D}_{\mathbb{Z}_2}(X)$ are fully* *faithful. We have two canonical semiorthogonal decompositions of* $\mathcal{D}_{\mathbb{Z}_2}(X)$*:*

$$\mathcal{D}_{\mathbb{Z}_2}(X) = \langle \pi^*\mathcal{D}(Y), i_*\mathcal{D}(R)\rangle = \langle \zeta \otimes i_*\mathcal{D}(R), \pi^*\mathcal{D}(Y)\rangle$$

Proof. The case where X and Y are curves was considered in Theorem 1.2 of [17], and the proof in our case is very similar. The fact that π^* is fully faithful follows immediately from the equality $(\pi_*\mathcal{O}_X)^{\mathbb{Z}_2} = \mathcal{O}_Y$ and the projection formula. Similarly, to prove that i_* is fully faithful it suffices to check $(Li^*i_*F)^{\mathbb{Z}_2} = F$. We have a canonical exact triangle

$$F \otimes N^\vee[1] \to Li^*i_*F \to F \to \dots$$

compatible with $\mathbb{Z}_2$-action, where $N^\vee = \mathcal{O}_X(-R)|_R$ is the conormal bundle. It remains to observe that $\mathbb{Z}_2$ acts on $N^\vee$ by multiplication with -1.

Now let $F \in \mathcal{D}(Y)$ and $G \in \mathcal{D}(R)$ be some objects. Then we have

$$\mathrm{Hom}_{\mathbb{Z}_2}(\pi^*(F), \zeta \otimes i_*(G)) \simeq \mathrm{Hom}_{\mathbb{Z}_2}(Lj^*F, \zeta \otimes G) = 0$$

which gives one of the required orthogonality conditions. On the other hand, by Serre duality, denoting $d = \dim X$, we get

$$\mathrm{Hom}_{\mathbb{Z}_2}(i_*(G), \pi^*(F))^* \simeq \mathrm{Hom}_{\mathbb{Z}_2}(\pi^*(F), \omega_X \otimes i_*(G)[d]) \simeq \mathrm{Hom}_{\mathbb{Z}_2}(Lj^*F, i^*\omega_X \otimes G[d]).$$

Note that $\mathbb{Z}_2$ acts nontrivially on $i^*\omega_X \simeq \omega_Y \otimes N^\vee$, so the above Hom-space vanishes.

Finally, we have to check that for every $F \in \mathcal{D}_{\mathbb{Z}_2}(X)$ such that $\mathrm{Hom}_{\mathbb{Z}_2}(i_*\mathcal{D}(R), F) = 0$ or $\mathrm{Hom}_{\mathbb{Z}_2}(F, \zeta \otimes i_*\mathcal{D}(R)) = 0$, lies in the essential image of $\pi^* : \mathcal{D}(Y) \to \mathcal{D}_{\mathbb{Z}_2}(X)$. Note that by Serre duality, these two orthogonality conditions are equivalent. Assume that $\mathrm{Hom}_{\mathbb{Z}_2}(F, \zeta \otimes i_*\mathcal{D}(R)) = 0$. Then $\mathbb{Z}_2$ acts trivially on i^*F. Now the assertion follows from the main theorem of [19]. $\square$

We can use the above Theorem as a setup for gluing stability conditions. The situation seems to be especially nice when either $\mathcal{D}(R)$ or $\mathcal{D}(Y)$ admits an exceptional collection. The former possibility occurs when X and Y are curves and will be considered below. The latter possibility happens if, say, Y is a projective space. In particular, we derive the following result.

Proposition IV.1.2. *Let $\pi : X \to \mathbb{P}^n$ be a smooth double covering ramified along a smooth hypersurface $j : R \hookrightarrow \mathbb{P}^n$. Assume we are given a reasonable stability $\sigma^R = (Z^R, P^R)$ on $\mathcal{D}(R)$, an* Ext*-exceptional collection $(E_0, \dots, E_n)$ on $\mathbb{P}^n$, and a set of vectors $v_0, \dots, v_n$ in the upper half-plane such that $j^*E_i \in P^R(> 1)$ for $i = 0, \dots, n$. Then there exists a reasonable stability $\sigma = (Z, P)$ on $\mathcal{D}_{\mathbb{Z}_2}(X)$ with*

$$P(0, 1] = [i_* P^R(0, 1], \pi^* E_0, \dots, \pi^* E_n],$$

$$Z(E) = v_0 x_0 (R\pi_*(E(R))^{\mathbb{Z}_2}) + \dots + v_n x_n (R\pi_*(E(R))^{\mathbb{Z}_2}) - Z^R((i^* E \otimes N)^{\mathbb{Z}_2}),$$

where $x_0, \dots, x_n : K_0(\mathbb{P}^n) \to \mathbb{Z}$ are the coordinates dual to the basis $([E_i])$.

Proof. This stability is obtained by gluing with respect to the semiorthogonal decomposition

$$\mathcal{D}_{\mathbb{Z}_2}(X) = \langle \pi^* \mathcal{D}(Y), i_* \mathcal{D}(R) \rangle. \tag{IV.1}$$

It exists by Theorem III.3.5, where $a < 1$ should be taken bigger than all of the phases of the vectors v_i (see Remark after Theorem III.3.5). To get the formula for the central charge we note that for $E \in D_{\mathbb{Z}_2}(X)$ one has

$$\rho_2(E) = i^!(E)^{\mathbb{Z}_2} \simeq (i^* E \otimes N)^{\mathbb{Z}_2}[-1],$$

$$\lambda_1(E) = R\pi_*(E(R))^{\mathbb{Z}_2}.$$

$\square$

For example, if $X \to \mathbb{P}^3$ is a double covering ramified along a smooth surface

$S \subset \mathbb{P}^3$ then we can consider stabilities on S constructed in [2]. Choosing an appropriate Ext-exceptional collection on $\mathbb{P}^3$ and using the above result we get examples of stabilities on $\mathcal{D}_{\mathbb{Z}_2}(X)$.

IV.2 Double Coverings of Curves

In section we will consider the case when X and Y are curves. In this case the ramification divisor R consists of points $p_1, \dots, p_n$, and the category $\mathcal{D}(R)$ is generated by the orthogonal exceptional objects $\mathcal{O}_{p_1}, \dots, \mathcal{O}_{p_n}$. Recall that the category $\mathcal{D}(X)$ has a standard stability condition σ_{st} with $Z_{st} = -\deg + i\,\mathrm{rk}$ and $P_{st}(0,1] = \mathrm{Coh}(X)$. There is an induced stability condition on $\mathcal{D}_{\mathbb{Z}_2}(X)$ with the heart $\mathrm{Coh}_{\mathbb{Z}_2}(X)$ that we still denote by σ_{st} (see [12]).

Lemma IV.2.1. *Let E be an endosimple object of the category $\mathcal{D}_{\mathbb{Z}_2}(X)$ (i.e., $\mathrm{Hom}(E,E) = k$). Then for some $n \in \mathbb{Z}$ the object $E[n]$ is one of the following types:*

1. *a vector bundle;*
2. *the sheaf $\mathcal{O}_{\pi^{-1}(y)}$ for $y \in Y$;*
3. *the sheaf $\zeta \otimes \mathcal{O}_{2p_i}$ for some $i \in \{1, \dots, n\}$;*
4. *the sheaf $\mathcal{O}_{p_i}$ for some i;*
5. *the sheaf $\zeta \otimes \mathcal{O}_{p_i}$ for some i.*

Proof. The category $\mathrm{Coh}_{\mathbb{Z}_2}(X)$ has cohomological dimension 1, so every indecomposable object in $\mathcal{D}_{\mathbb{Z}_2}(X)$ has only one nonzero cohomology. Thus, we can assume that E is a $\mathbb{Z}_2$-equivariant coherent sheaf. Furthermore, since the torsion part of such a sheaf splits as a direct summand, it is enough to consider the case when E is an indecomposable torsion sheaf. Then the support of E is either $\pi^{-1}(y)$, where $y \in Y \setminus R$, or $\{p_i\}$ for some $i \in \{1, \dots, n\}$. In the former case $E \simeq \pi^* E'$, where E' is an endosimple sheaf on Y supported at y, so $E' \simeq \mathcal{O}_y$. In the latter case there exists m such that $E \simeq \mathcal{O}_{mp_i}$ or

$E \simeq \zeta \otimes \mathcal{O}_{mp_i}$. It remains to observe that for $m \geq 3$ the sheaf $\mathcal{O}_{mp_i}$ is not endosimple, since we can construct its nonscalar endomorphism as the composition of natural maps

$$\mathcal{O}_{mp_i} \to \mathcal{O}_{(m-2)p_i} \to \mathcal{O}_{mp_i}.$$

□

We are going to construct explicitly some stability conditions on $\mathcal{D}_{\mathbb{Z}_2}(X)$. For this we will use a slight variation of the semiorthogonal decompositions considered in Theorem IV.1.1. Namely, for every partition of $\{1, \dots, n\}$ into two disjoint subset I and J we have

$$\mathcal{D}_{\mathbb{Z}_2}(X) = \langle\langle \zeta \otimes \mathcal{O}_{p_j} \mid j \in J\rangle, \pi^*\mathcal{D}(Y), \langle \mathcal{O}_{p_i} \mid i \in I\rangle\rangle. \qquad \text{(IV.2)}$$

For a subset $I \subset \{1, \dots, n\}$ let us denote by $\mathcal{D}(I) \subset \mathcal{D}_{\mathbb{Z}_2}(X)$ the full triangulated subcategory generated by $\pi^*\mathcal{D}(Y)$ and $\mathcal{O}_{p_i}$ with $i \in I$.

Lemma IV.2.2. *For $I \subset \{1, \dots, n\}$ set $\mathrm{Coh}(I) := \mathrm{Coh}_{\mathbb{Z}_2}(X) \cap \mathcal{D}(I)$. Then $\mathrm{Coh}(I)$ is the heart of a t-structure on $\mathcal{D}(I)$. The natural exact functor $\mathrm{Coh}(I) \to \mathrm{Coh}_{\mathbb{Z}_2}(X)$ gives an equivalence of $\mathrm{Coh}(I)$ with the full subcategory of $\mathrm{Coh}_{\mathbb{Z}_2}(X)$ consisting of all successive extensions of sheaves in $\pi^*\,\mathrm{Coh}(Y)$ and equivariant sheaves supported on $\{p_i \mid i \in I\}$. The category $\mathrm{Coh}(I)$ is Noetherian.*

Proof. Note that an object $E \in \mathcal{D}_{\mathbb{Z}_2}(X)$ belongs to $\mathcal{D}(I)$ if and only if $\mathrm{Hom}^*(\mathcal{O}_{p_i}, E) = 0$ for each $i \notin I$. Since the category $\mathrm{Coh}_{\mathbb{Z}_2}(X)$ has cohomological dimension 1, we have $E \simeq \oplus H^iE[-i]$, where $H^iE \in \mathrm{Coh}_{\mathbb{Z}_2}(X)$. Therefore, $E \in \mathcal{D}(I)$ if and only if $H^iE \in \mathcal{D}(I)$ for every i. This immediately implies that the standard t-structure restricts to a t-structure on $\mathcal{D}(I)$ with $\mathrm{Coh}(I)$ as the heart. We have an exact embedding $\mathrm{Coh}(I) \to \mathrm{Coh}_{\mathbb{Z}_2}(X)$, so $\mathrm{Coh}(I)$ is Noetherian. Let $\mathcal{F} \in \mathrm{Coh}(I)$. Then the torsion part (resp., torsion-free part) of $\mathcal{F}$ is also in $\mathrm{Coh}(I)$. Assume first that $\mathcal{F}$ is an indecomposable torsion sheaf with the support at p_i for $i \notin I$. Then the condition $\mathrm{Hom}^*(\mathcal{O}_{p_i}, \mathcal{F}) = 0$ easily implies that $\mathcal{F} \simeq \mathcal{O}_{2np_i}$. On the other hand, if $\mathcal{F}$ is a vector bundle then we have $\mathrm{Hom}(\mathcal{F}, \zeta \otimes \mathcal{O}_{p_i}) = 0$

for $i \notin I$, which implies that the fiber of $\mathcal{F}$ at p_i has trivial $\mathbb{Z}_2$-action for $i \notin I$. Therefore, making appropriate elementary transformations at p_i for $i \in I$ we can represent $\mathcal{F}$ as an extension of a sheaf supported at $\{p_i \mid i \in I\}$ by the pull-back of a vector bundle from Y (cf. proof of Theorem 1.8 of [17]). $\square$

Given a partition of $\{1, \dots, n\}$ into three disjoint subsets I^0, I^+ and I^- we obtain from (IV.2) a semiorthogonal decomposition

$$\mathcal{D}_{\mathbb{Z}_2}(X) = \langle\langle \zeta \otimes \mathcal{O}_{p_i} \mid i \in I^- \rangle, \mathcal{D}(I^0), \langle \mathcal{O}_{p_i}, i \in I^+ \rangle\rangle. \tag{IV.3}$$

Proposition IV.2.3. *Fix a partition $\{1, \dots, n\} = I^0 \sqcup I^+ \sqcup I^-$ and a collection of positive integers (n_i) for $i \notin I^0$.*

(a) Let $Z : \mathcal{N}(\mathcal{D}_{\mathbb{Z}_2}(X)) \to \mathbb{C}$ be a homomorphism, such that

1. *$\Im Z(\mathcal{O}_X) > 0$, and $Z(\mathcal{O}_{\pi^{-1}(y)}) \in \mathbb{R}_{<0}$ for any point $y \in Y$;*

2. *$Z(\mathcal{O}_{p_i}[-n_i]) \in \mathfrak{h}'$ for $i \in I^+$, and $Z(\zeta \otimes \mathcal{O}_{p_i}[n_i]) \in \mathfrak{h}'$ for $i \in I^-$;*

3. *$Z(\mathcal{O}_{p_i}) \in \mathbb{R}_{<0}$ and $Z(\zeta \otimes \mathcal{O}_{p_i}) \in \mathbb{R}_{<0}$ for $i \in I^0$,*

where $\mathfrak{h}' \subset \mathbb{C}$ denotes the union of the upper half-plane with $\mathbb{R}_{<0}$. Then there exists a reasonable stability condition σ with the central charge Z and the heart

$$H(I^+, I^-; \mathbf{n}) = [[\zeta \otimes \mathcal{O}_{p_i}[n_i] \mid i \in I^-], \mathrm{Coh}(I^0), [\mathcal{O}_{p_i}[-n_i], i \in I^+]], \tag{IV.4}$$

which is glued with respect to the semiorthogonal decomposition (IV.3). *All the objects $\mathcal{O}_{\pi^{-1}(y)}$ for $y \in Y$ are σ-semistable (of phase 1). The objects $\mathcal{O}_{\pi^{-1}(y)}$ for $y \in Y \setminus \{p_i \mid i \in I^0\}$, as well as $\mathcal{O}_{p_i}$ for $i \in I^0 \cup I^+$ and $\zeta \otimes \mathcal{O}_{p_i}$ for $i \in I^0 \cup I^-$, are σ-stable.*

(b) Assume in addition that $n_i = 1$ for all $i \notin I^0$. Then all the objects $\mathcal{O}_{p_i}$ and $\zeta \otimes \mathcal{O}_{p_i}$ for $i \in \{1, \dots, n\}$ are σ-stable.

Proof. (a) Using the orthogonalities

$$\mathrm{Hom}^{\leq 0}(\mathrm{Coh}(I^0), \mathcal{O}_{p_i}[-n_i]) = \mathrm{Hom}^{\leq 0}(\zeta \otimes \mathcal{O}_{p_j}[n_j], \mathrm{Coh}(I^0)) = \mathrm{Hom}^{\leq 0}(\zeta \otimes \mathcal{O}_{p_j}[n_j], \mathcal{O}_{p_i}[-n_i])$$

for $i \in I^+$, $j \in I^-$, we get the glued heart $H = H(I^+, I^-; \mathbf{n})$ given by (IV.4). Note that the restriction of Z to $\mathcal{N}(\pi^*\mathcal{D}(Y))$ is determined by $Z(\mathcal{O}_X)$ and by $Z(\mathcal{O}_{\pi^{-1}(y)})$ for a point $y \in Y$. Thus, $\Im Z(\pi^* F) = c\,\mathrm{rk}(F)$ for some positive constant c. Since $\mathrm{Coh}(I^0)$ is generated by extensions from $\pi^* \mathrm{Coh}(Y)$ and $\mathcal{O}_{p_i}$ and $\zeta \otimes \mathcal{O}_{p_i}$ for $i \in I^0$, we deduce that Z is a stability function on H. It is also easy to see that 0 is an isolated point of $\Im Z(H)$. Since H is glued from Noetherian hearts, it is also Noetherian, so Lemma III.3.2(a) implies that the Harder-Narasimhan property is satisfied for Z. Thus, we have a stability condition $\sigma = (Z, P)$ with $P(0,1] = H$. By Proposition III.2.3(2), it is glued from the induced stability on $\mathcal{D}(I^0)$ and the exceptional objects $\zeta \otimes \mathcal{O}_{p_i}[n_i]$, $i \in I^-$ and $\mathcal{O}_{p_i}[-n_i]$, $i \in I^+$. The fact that σ is reasonable follows from Lemma III.3.2(b). Note that $P(1) \subset H$ consists of successive extensions of sheaves of the form $\mathcal{O}_{\pi^{-1}(y)}$, $y \in Y$, and of $\mathcal{O}_{p_i}$ and $\zeta \otimes \mathcal{O}_{p_i}$ for $i \in I^0$. The simple objects in $P(1)$ are the sheaves $\mathcal{O}_{\pi^{-1}(y)}$, $y \in Y \setminus \{p_i \mid i \in I^0\}$, and $\mathcal{O}_{p_i}$ and $\zeta \otimes \mathcal{O}_{p_i}$ for $i \in I^0$, so all these objects are σ-stable. On the other hand, Proposition III.2.3(iii) implies that the above exceptional objects in the heart corresponding to $i \in I^+ \cup I^-$, are σ-stable.

(b) Let us denote

$$\mathcal{C}^+ := [\mathcal{O}_{p_i} \mid i \in I^+] \subset \mathrm{Coh}_{\mathbb{Z}_2}(X),$$

$$\mathcal{C}^- := [\zeta \otimes \mathcal{O}_{p_i} \mid i \in I^-] \subset \mathrm{Coh}_{\mathbb{Z}_2}(X).$$

From the definition of H one can easily deduce that for every object $C \in H$ one has

$$H^{-1}C \in \mathcal{C}^-; \quad H^1 C \in \mathcal{C}^+; H^0 C \simeq H^0(F_{-1} \to F_0 \to F_1), \text{where } F_0 \in \mathrm{Coh}(I^0), F_{-1} \in \mathcal{C}^-, F_1 \in \mathcal{C}^+.$$

The last condition easily implies that $\mathrm{Hom}(\mathcal{C}^+, H^0C) = \mathrm{Hom}(H^0C, \mathcal{C}^-) = 0$.

Now let us fix $i \in I^+$ and consider the object $E = \zeta \otimes \mathcal{O}_{p_i}$. Note that $\zeta \otimes \mathcal{O}_{p_i}$ belongs to H, as an extension of $\mathcal{O}_{2p_i}$ by $\mathcal{O}_{p_i}[-1]$. Suppose we have a short exact sequence

$$0 \to A \to E \to B \to 0$$

in H with nonzero A and B. Since $H^2 A = H^{-2} B = 0$, we derive that $H^1 B = H^{-1} A = 0$ and there is an exact sequence

$$0 \to H^{-1}B \to H^0 A \to E \to H^0 B \to H^1 A \to 0 \qquad \text{(IV.5)}$$

in $\mathrm{Coh}_{\mathbb{Z}_2}(X)$. Note that since E is a simple object of $\mathrm{Coh}_{\mathbb{Z}_2}(X)$ we have one of the following two cases: (i) $H^0 B \to H^1 A$ is an isomorphism; (ii) $H^{-1}B \to H^0 A$ is an isomorphism. In the first case we obtain that $H^0 B \in \mathcal{C}^+$ which implies that $H^0 B = 0$. Hence, in this case $B \in \mathcal{C}^-[1]$, so $\mathrm{Hom}(E, B) = 0$ which is a contradiction. Now let us consider case (ii). We have $H^0 A \in \mathcal{C}^-$, hence $H^0 A = 0$. It follows that $A = H^1 A[-1]$, and $B = H^0 B$ is an extension of $H^1 A$ by E. Since $\mathrm{Hom}(H^1 A, B) = 0$, this extension cannot split on any direct summands of $H^1 A$, which implies that $A \simeq \mathcal{O}_{p_i}[-1]$ and $B \simeq \mathcal{O}_{2p_i}$. Since $Z(\mathcal{O}_{p_i}[-1])$ has smaller phase then $Z(E)$, this shows that $\zeta \otimes \mathcal{O}_{p_i}$ is stable. Similarly one proves that all the objects $\mathcal{O}_{p_i}$ for $i \in I^-$ are stable. □

In the case when all n_i's are equal to 1, we denote the heart $H(I^+, I^-, \mathbf{n})$ considered in the above Proposition simply by $H(I^+, I^-)$.

We have the following partial characterization of stability conditions constructed above.

Lemma IV.2.4. *Let $\sigma = (Z, P)$ be a stability condition such that $\mathcal{O}_{\pi^{-1}(y)} \in P(1)$ for all $y \in Y \setminus R$.*

(a) Assume that $\mathcal{O}_{2p_i} \in P(1)$ for all i, and for every i one of the following three conditions holds:

1. *both $\mathcal{O}_{p_i}$ and $\zeta \otimes \mathcal{O}_{p_i}$ are σ-semistable of phase 1;*

2. $\mathcal{O}_{p_i}$ *is* σ*-semistable of phase* >1*;*

3. $\zeta \otimes \mathcal{O}_{p_i}$ *is* σ*-semistable of phase* ≤ 0.

Assume in addition that for every line bundle L *on* Y *one has* $\pi^*L \in P(0,1]$*. Then* σ *coincides with one of the stability conditions constructed in Proposition IV.2.3. The latter condition is uniquely determined by* Z *and by the phases of* $\mathcal{O}_{p_i}$ *and* $\zeta \otimes \mathcal{O}_{p_i}$ *for* $i \in \{1, \dots, n\}$*.*

(b) Now assume that σ *is locally finite, and for all* $i \in \{1, \dots, n\}$ *one has* $\mathcal{O}_{p_i} \in P[1,2)$ *and* $\zeta \otimes \mathcal{O}_{p_i} \in P(0,1]$*. Assume in addition that either all objects* $\mathcal{O}_{\pi^{-1}(y)}$ *for* $y \in Y \setminus R$ *are stable, or* $\Im Z(V) > 0$ *for every* $\mathbb{Z}_2$*-equivariant vector bundle* V*. Then* σ *coincides with one of stability conditions constructed in Proposition IV.2.3 with* $I^- = \emptyset$*,*

$$I^+ = \{i \mid \Im Z(\mathcal{O}_{p_i}) < 0\},$$

and all n_i*'s equal to* 1*.*

Proof. (a) Let I^0, I^+ and I^- be the subsets of i such that conditions (1), (2) and (3) hold, respectively. Note that since we have nonzero maps $\mathcal{O}_{p_i} \to \zeta \otimes \mathcal{O}_{p_i}[1]$, the conditions (2) and (3) (and therefore, the subsets I^0, I^+ and I^-) are mutually disjoint. For each $i \in I^+$ (resp., $i \in I^-$) there is a unique $n_i > 0$ such that $\phi(\mathcal{O}_{p_i}) - n_i \in (0,1]$ (resp., $\phi(\mathcal{O}_{p_i}) + n_i \in (0,1]$). Then Z satisfies the conditions of Proposition IV.2.3, so it remains to check that $H = H(I^+, I^-; \mathbf{n}) \subset P(0,1]$. Note that by definition, we have $\mathcal{O}_{\pi^{-1}(y)} \in P(0,1]$ for all $y \in Y$; $\mathcal{O}_{p_i}, \zeta \otimes \mathcal{O}_{p_i} \in P(1)$ for $i \in I^0$; $\mathcal{O}_{p_i}[-n_i] \in P(0,1]$ for $i \in I^+$ and $\zeta \otimes \mathcal{O}_{p_i}[n_i] \in P(0,1]$ for $i \in I^-$. It remains to show that $\pi^*V \in P(0,1]$ for every vector bundle V on Y. But such a vector bundle can be presented as an extension of line bundles, so this follows from our assumption.

(b) It is enough the check that $P(0,1] \subset H = H(I^+, \emptyset)$ (where I^0 is the complement to I^+). First, we observe that in this case all equivariant vector bundles are in H, as extensions of direct sums of sheaves of the form $\zeta \otimes \mathcal{O}_{p_i}$ by a sheaf in $\pi^* \operatorname{Coh}(Y)$. Let E

be a σ-stable object in $P(0,1)$. Note that E is endosimple. Let us consider possibilities for E listed in Lemma IV.2.1. Since $Z(\mathcal{O}_{\pi^{-1}(y)}) = Z(\zeta \otimes \mathcal{O}_{2p_i}) \in \mathbb{R}_{<0}$ and $E \in P(0,1)$, we obtain that for some $m \in \mathbb{Z}$, $E[m]$ is either a vector bundle, or isomorphic to $\mathcal{O}_{p_i}[-1]$, or to $\zeta \otimes \mathcal{O}_{p_i}$. In the last two cases our assumptions on σ imply that $m = 0$, so $E \in H$. If $E[m]$ is a vector bundle then using the condition $E \in P(0,1)$ we get

$$\mathrm{Hom}^{\leq -1}(E, \mathcal{O}_{\pi^{-1}(y)}) = \mathrm{Hom}^{\leq 0}(\mathcal{O}_{\pi}^{-1}(y), E) = 0. \tag{IV.6}$$

This implies that $m = 0$, so $E \in H$. Next, let E be a σ-stable object in $P(1)$. We can assume that E is not isomorphic to $\mathcal{O}_{\pi^{-1}(y)}$ for $y \in Y \setminus R$ since these objects are in H. Assume that $E[m]$ is a vector bundle. Note that this case cannot occur if $\Im Z(V) > 0$ for all equivariant vector bundles, so we can assume that the objects $\mathcal{O}_{\pi^{-1}(y)}$ for $y \in Y \setminus R$ are stable. Then the vanishing (IV.6) still holds, so we deduce again that $m = 0$. The case when $E[m]$ is either $\mathcal{O}_{p_i}$, or $\zeta \otimes \mathcal{O}_{p_i}$ (where $i \in I^0$) is also clear. Note that for $i \in I^0$ we have $\mathcal{O}_{p_i}, \zeta \otimes \mathcal{O}_{p_i} \in P(1)$. Hence, for such i the objects $\mathcal{O}_{2p_i}$ and $\zeta \otimes \mathcal{O}_{2p_i}$ are not σ-stable. Now assume that $E[m] \simeq \mathcal{O}_{2p_i}$, where $i \in I^+$. Since $\mathcal{O}_{2p_i} \in P(0,2)$ as an extension of $\mathcal{O}_{p_i}$ by $\zeta \otimes \mathcal{O}_{p_i}$, this implies that $m = 0$, so $E \in H$. Finally, we observe that for $i \in I^+$ the object $\zeta \otimes \mathcal{O}_{2p_i}$ is not semistable since it is an extension of $\zeta \otimes \mathcal{O}_{p_i}$ by $\mathcal{O}_{p_i}$, where $\phi_{\min}(\zeta \otimes \mathcal{O}_{p_i}) < 1$ and $\phi_{\max}(\mathcal{O}_{p_i}) > 1$. □

Note that the classes $[\mathcal{O}_X]$, $[\mathcal{O}_{\pi^{-1}(y)}]$, and $[\mathcal{O}_{p_i}]$, $i \in \{1, \ldots, n\}$, form a basis in $\mathcal{N}(\mathcal{D}_{\mathbb{Z}_2}(X))$. Thus, we can define a norm on the vector space $\mathrm{Hom}(\mathcal{N}(\mathcal{D}_{\mathbb{Z}_2}(X)), \mathbb{C})$ by setting

$$||Z|| = \max(|Z(\mathcal{O}_X)|, \max_E |Z(E)|),$$

where E runs over all endosimple torsion sheaves in $\mathrm{Coh}_{\mathbb{Z}_2}(X)$ (see Lemma IV.2.1). It is also convenient to set for $Z \in \mathrm{Hom}(\mathcal{N}(\mathcal{D}_{\mathbb{Z}_2}(X)), \mathbb{C})$

$$v_Z := Z(\mathcal{O}_{\pi^{-1}(y)}) \in \mathbb{C}.$$

Let us define an open subset $\overline{U} \subset \mathrm{Hom}(\mathcal{N}(\mathcal{D}_{\mathbb{Z}_2}(X)), \mathbb{C})$ as the set of central charges Z satisfying the following assumptions:

1. for every $\mathbb{Z}_2$-equivariant line bundle L on X one has $\det(Z(L), v_Z) > 0$;

2. for every $i = 1, \ldots, n$ one has $Z(\mathcal{O}_{p_i}) \notin \mathbb{R}_{\leq 0} \cdot v_Z$, $Z(\zeta \otimes \mathcal{O}_{p_i}) \notin \mathbb{R}_{\leq 0} \cdot v_Z$.

Note that in the first condition it is enough to consider representatives in the cosets for the subgroup $\pi^* \mathrm{Pic}(Y) \subset \mathrm{Pic}_{\mathbb{Z}_2}(X)$, so there is only finite number of inequalities to check (hence, $\overline{U}$ is open). Also, this condition implies that $\det(Z(V), v_Z) > 0$ for every equivariant vector bundle Von X, since they can be obtained from line bundles by successive extensions.

Lemma IV.2.5. *1. Let $Z : \mathcal{N}(\mathcal{D}_{\mathbb{Z}_2}(X)) \to \mathbb{C}$ be a homomorphism such that $\Im Z(\mathcal{O}_X) > 0$, $Z(\mathcal{O}_{\pi^{-1}(y)}) \in \mathbb{R}_{<0}$, and for every $i = 1, \ldots, n$ one has $Z(\mathcal{O}_{p_i}) \neq 0$ and $\Im Z(\mathcal{O}_{p_i}) \leq 0$. Then there exists a constant $r > 0$ such that for every $Z' \in \mathrm{Hom}(\mathcal{N}(\mathcal{D}_{\mathbb{Z}_2}(X)), \mathbb{C})$ and every endosimple object $E \in \mathcal{D}_{\mathbb{Z}_2}(X)$ one has*

$$|Z'(E)| \leq r \cdot ||Z'|| \cdot |Z(E)|.$$

2. The above conclusion also holds for $Z \in \overline{U}$.

Proof. (1) Our conditions on Z imply that $Z(E) \neq 0$ for every endosimple torsion $\mathbb{Z}_2$-equivariant coherent sheaf E. Therefore, we can set

$$r_1 = \max_E(|Z(E)|^{-1}),$$

where E runs over all endosimple torsion sheaves. If E is such a sheaf then $|Z'(E)| \leq ||Z'||$, so the required inequality holds for E provided $r \geq r_1$. Now assume that E is a $\mathbb{Z}_2$-equivariant vector bundle on X. Then there exists an exact sequence of the form

$$0 \to \pi^* E' \to E \to \oplus_i \zeta \otimes \mathcal{O}_{p_i}^{m_i} \to 0,$$

where $0 \leq m_i \leq \mathrm{rk}(E)$. Then

$$|Z'(E)| \leq |Z'(\pi^* E')| + n\,\mathrm{rk}(E) \cdot ||Z'||.$$

Note that

$$[\pi^* E'] = \mathrm{rk}(E)[\mathcal{O}_X] + \deg(E')[\mathcal{O}_{\pi^{-1}(y)}] \tag{IV.7}$$

in $\mathcal{N}(\mathcal{D}_{\mathbb{Z}_2}(X))$. Thus, we obtain

$$|Z'(E)| \leq ||Z'|| \cdot [(n+1)\,\mathrm{rk}(E) + \deg(E')]. \tag{IV.8}$$

On the other hand, from the above exact sequence we get

$$\Im Z(E) = \Im Z(\pi^* E') + \sum_i m_i \cdot \Im Z(\zeta \otimes \mathcal{O}_{p_i}).$$

Since $\Im Z(\zeta \otimes \mathcal{O}_{p_i}) \geq 0$ and $\Im Z(\pi^* E') = \Im Z(\mathcal{O}_X) \cdot \mathrm{rk}(E)$, we deduce that

$$\mathrm{rk}(E) \leq \frac{|Z(E)|}{\Im Z(\mathcal{O}_X)}.$$

Also, from (IV.7) we get

$$|\deg(E') Z(\mathcal{O}_{\pi^{-1}(y)})| \leq |Z(\pi^* E')| + \mathrm{rk}(E)|Z(\mathcal{O}_X)| \leq |Z(E)| + (n+1)\,\mathrm{rk}(E) \cdot ||Z||.$$

Using our estimate for $\mathrm{rk}(E)$ we get that

$$\deg(E') \leq |Z(\mathcal{O}_{\pi^{-1}(y)})|^{-1} \cdot [1 + (n+1)\frac{||Z||}{\Im Z(\mathcal{O}_X)}] \cdot |Z(E)|.$$

Therefore, from (IV.8) we obtain

$$|Z'(E)| \leq r_2 ||Z'|| \cdot |Z(E)|,$$

where

$$r_2 = \frac{n+1}{\Im Z(\mathcal{O}_X)} + |Z(\mathcal{O}_{\pi^{-1}(y)})|^{-1} \cdot [1 + (n+1)\frac{||Z||}{\Im Z(\mathcal{O}_X)}].$$

It remains to set $r = \max(r_1, r_2)$.

(2) The subset $\overline{U} \subset \mathrm{Hom}(\mathcal{N}(\mathcal{D}_{\mathbb{Z}_2}(X)), \mathbb{C})$ is stable under composition with rotations of $\mathbb{C}$ and with automorphisms of $\mathcal{N}(\mathcal{D}_{\mathbb{Z}_2}(X))$ given by tensoring with an equivariant line bundle L. Also, the norms $||\cdot||$ and $Z' \mapsto ||Z' \circ (\otimes L)||$ on the finite-dimensional vector space $\mathrm{Hom}(\mathcal{N}(\mathcal{D}_{\mathbb{Z}_2}(X)), \mathbb{C})$ are equivalent, while composing with a rotation of $\mathbb{C}$ does not change the norms. Therefore, we can modify Z using these operations before checking the required inequalities. Rotating Z we can assume that $v_Z \in \mathbb{R}_{<0}$. Next, let $I \subset \{1, \dots, n\}$ be the set of i such that $\Im Z(\mathcal{O}_{p_i}) > 0$. Taking $L = \mathcal{O}(\sum_{i \in I} p_i)$ we will have

$$L \otimes \mathcal{O}_{p_i} \simeq \begin{cases} \zeta \otimes \mathcal{O}_{p_i}, & i \in I \\ \mathcal{O}_{p_i}, & i \notin I. \end{cases}$$

Therefore composing Z with tensoring by L we get the situation considered in (1). $\square$

Recall that for every point $\sigma \in \mathrm{Stab}_{\mathcal{N}}(\mathcal{D})$ a neighborhood of σ in $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D})$ is homeomorphic to a neighborhood of the corresponding central charge in the linear subspace $W_\sigma^{\mathcal{N}} \subset \mathrm{Hom}(\mathcal{N}(\mathcal{D}), \mathbb{C})$. When $W_\sigma^{\mathcal{N}} = \mathrm{Hom}(\mathcal{N}(\mathcal{D}), \mathbb{C})$, we say that the corresponding stability condition σ is *full*. The above Lemma implies that every stability condition with the central charge in the set $\overline{U}$ is full.

Theorem IV.2.6. *Let $U \subset \mathrm{Stab}_{\mathcal{N}}(\mathcal{D}_{\mathbb{Z}_2}(X))$ denote the set of locally finite stability conditions $\sigma = (Z, P)$ such that*

1. *$\mathcal{O}_{\pi^{-1}(y)}$ is stable of phase ϕ_σ for every $y \in Y \setminus R$;*
2. *$\mathcal{O}_{p_i}, \zeta \otimes \mathcal{O}_{p_i}$ are semistable with the phases in $(\phi_\sigma - 1, \phi_\sigma + 1)$ for all $i = 1, \dots, n$.*

Then every point in U is obtained from one of the stability conditions described in Proposition IV.2.3 with $I^- = \emptyset$ and all $n_i = 1$ by the action of an element of $\mathbb{R} \times \mathrm{Pic}_{\mathbb{Z}_2}(X)$, *where $\mathbb{R}$ acts on $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D}_{\mathbb{Z}_2}(X))$ by rotations (shifts of phases). The subset U is open in $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D}_{\mathbb{Z}_2}(X))$. The natural map $U \to \overline{U}$ is a universal covering of $\overline{U}$, and $\overline{U} = U/\mathbb{Z}$, where $1 \in \mathbb{Z}$ acts on the stability space by shifting phases by 2. Furthermore, U is contractible.*

Proof. **Step 1**. If $\sigma = (Z, P) \in U$ then σ is obtained from one of the stability conditions described in Proposition IV.2.3 with $I^- = \emptyset$ and all $n_i = 1$ by the action of an element of $\mathbb{R} \times \mathrm{Pic}_{\mathbb{Z}_2}(X)$. Indeed, by rotating σ we can assume that $\phi_\sigma = 1$. Now using tensoring with an appropriate equivariant line bundle we can assume that $\Im Z(\mathcal{O}_{p_i}) \leq 0$ for all i. It remains to apply Lemma IV.2.4(b).

Note that this step implies that for $\sigma = (Z, P) \in U$ one has $Z \in \overline{U}$.

Step 2. Let U' be the preimage of $\overline{U}$ in $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D}_{\mathbb{Z}_2}(X))$. Then the projection $U' \to \overline{U}$ is a covering map. This is checked exactly as in Proposition 8.3 of [6] using Lemma IV.2.5(b).

Step 3. U is open in $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D}_{\mathbb{Z}_2}(X))$. Let $\sigma_0 = (Z_0, P_0) \in U$. We have to prove that any stability $\sigma = (Z, P)$, sufficiently close to σ_0, is still in U. Using rotations it is enough to consider the case when $Z(\mathcal{O}_{\pi^{-1}(y)}) \in \mathbb{R}_{<0}$. By Step 1 we can assume that σ_0 is a stability arising in Proposition IV.2.3 with $I^- = \emptyset$ and all n_i's equal to 1. For a $\mathbb{Z}_2$-equivariant line bundle L and a stability condition $\sigma' = (Z', P')$ we denote by $\sigma' \otimes L$ the stability condition with central charge $E \mapsto Z'(E \otimes L^{-1})$ and the heart $P'(0,1] \otimes L$. It is enough to check that $\sigma = \sigma' \otimes L$, where σ' is one of the stability conditions from Proposition IV.2.3 (with $I^- = \emptyset$ and $n_i = 1$). Let us set $L = \mathcal{O}_X(\sum_{i \in I(+)} p_i)$, where $I(+) = \{i \mid \Im Z(\mathcal{O}_{p_i}) > 0\}$. We claim that the central charge $Z'(E) := Z(E \otimes L)$ satisfies the assumptions of Proposition IV.2.3 with $I^+ = \{i \mid \Im Z(\mathcal{O}_{p_i}) \neq 0\}$, $I^- = \emptyset$ and all $n_i = 1$. Indeed, first, note that $Z'(\mathcal{O}_{\pi^{-1}(y)}) = Z(\mathcal{O}_{\pi^{-1}(y)}) \in \mathbb{R}_{<0}$, and $Z'(\mathcal{O}_X) = Z(L)$ is in the upper-half plane, provided σ is close enough to σ_0. Next, using the fact that

$$\mathcal{O}_{p_i} \otimes L \simeq \begin{cases} \mathcal{O}_{p_i}, & i \notin I(+), \\ \zeta \otimes \mathcal{O}_{p_i}, & i \in I(+) \end{cases}$$

one checks the remaining assumptions. Therefore, by Proposition IV.2.3, there exists a stability condition σ' with the central charge Z' and the heart $H(I^+, \emptyset)$. Now we claim that $\sigma = \sigma' \otimes L$. Since the corresponding central charges are the same, by Proposition III.4.2(a), it remains to check that $H(I^+, \emptyset) \otimes L \subset P(-1, 2]$. It is easy to see that

$$\begin{aligned} H(I^+, \emptyset) \otimes L = \qquad & [\mathcal{O}_X(\textstyle\sum_{i \in I(+)} p_i) \otimes \pi^* \operatorname{Coh}(Y), \\ & [\zeta \otimes \mathcal{O}_{p_i} \mid i \notin I(+)], [\mathcal{O}_{p_i} \mid i \notin I(-)], \\ & [\zeta \otimes \mathcal{O}_{p_i}[-1] \mid i \in I(+)], [\mathcal{O}_{p_i}[-1] \mid i \in I(-)]], \end{aligned} \tag{IV.9}$$

where $I(-) = \{i \mid \Im Z(\mathcal{O}_{p_i}) < 0\}$. Hence,

$$H(I^+, \emptyset) \otimes L \subset T_0 := [P_0(0,1], [\mathcal{O}_{p_i}, \mathcal{O}_{p_i}[-1], \zeta \otimes \mathcal{O}_{p_i}[-1] \mid i = 1, \dots, n]].$$

Furthermore, we have $\mathcal{O}_{p_i} \in P_0[1,2)$ and $\zeta \otimes \mathcal{O}_{p_i} \in P_0(0,1]$. Hence, we have $T_0 \subset P_0(-1+\epsilon, 2-\epsilon)$ for some $\epsilon > 0$ depending only on σ_0. Thus, for $d(P, P_0) < \epsilon$ we obtain

$$H(I^+, \emptyset) \otimes L \subset P_0(-1+\epsilon, 2-\epsilon) \subset P(-1, 2]$$

as required.

Step 4. U is closed in U'. More precisely, we claim that U coincides with the set of $\sigma \in U'$ such that $\mathcal{O}_{\pi^{-1}(y)}$ is semistable of phase ϕ_σ for every $y \in Y \setminus R$, and for every $i \in \{1, \dots, n\}$ the objects $\mathcal{O}_{p_i}$ and $\zeta \otimes \mathcal{O}_{p_i}$ are semistable with the phases in $[\phi_\sigma - 1, \phi_\sigma + 1]$. (recall that the set of stability conditions such that a given object E is semistable is closed). Indeed, given such $\sigma = (Z, P)$, by rotating it and using tensoring with an equivariant line bundle we can assume that $\phi_\sigma = 1$, and $\Im Z(\mathcal{O}_{p_i}) \leq 0$ for all i. Note that the condition $Z \in \overline{U}$ implies that the phase of $\mathcal{O}_{p_i}$ (resp., $\zeta \otimes \mathcal{O}_{p_i}$) is in $[1,2)$ (resp., in $(0,1]$) for every i, and $\Im Z(V) > 0$ for every $\mathbb{Z}_2$-equivariant vector bundle V. Hence, by Lemma IV.2.4(b), σ is obtained by the construction of Proposition IV.2.3, which implies that $\mathcal{O}_{\pi^{-1}(y)}$ is stable for every $y \in Y \setminus R$. It remains to note that for $\sigma \in U'$ the phases of $Z(\mathcal{O}_{p_i})$ and of

$Z(\zeta \otimes \mathcal{O}_{p_i})$ never equal $\phi_\sigma \pm 1$.

Combining Steps 2, 3 and 4 we obtain that $U \to \overline{U}$ is a covering map.

Step 5. Assume $\sigma_1, \sigma_2 \in U$ have the same central charge Z. Then σ_2 is obtained from σ_1 by a shift of phase in $2\mathbb{Z}$. Indeed, applying such a shift we can assume that $\phi_{\sigma_1} = \phi_{\sigma_2}$. Furthermore, applying a rotation and tensoring with a line bundle, we reduce to the situation $\phi_{\sigma_1} = 1$ and $\Im Z(\mathcal{O}_{p_i}) \leq 0$ for all i. By Lemma IV.2.4(b), in this case the hearts of σ_1 and σ_2 are the same.

Step 6. It remains to show that U is contractible. We have a free action of $\mathbb{R}$ on U by the shift of phase, so it is enough to consider the section of this action consisting of $\sigma \in U$ with $\phi_\sigma = 1$. In other words, we have to consider the subset of $\overline{U}$ consisting of Z with $v_Z = Z(\mathcal{O}_{\pi^{-1}(y)}) \in \mathbb{R}_{<0}$. A homomorphism Z in this subset is determined by the following contractible data:

1. $v_Z \in \mathbb{R}_{<0}$;
2. for every $i \in \{1, \ldots, n\}$, $Z(\mathcal{O}_{p_i}) \in \mathbb{C} \setminus (\mathbb{R}_{\geq 0} \cup (v_Z + R_{\leq 0}))$;
3. $Z(\mathcal{O}_X)$ in some half-plane of the form $\Im z > c$.

$\square$

CHAPTER V

CONSTRUCTIBILITY OF EXCEPTIONAL COLLECTIONS ON $\mathcal{D}_{\mathbb{Z}_2}(X)$

This chapter is primarily devoted to a new proof of the constructibility of the bounded derived category $\mathcal{D}_{Z_2}(X)$ of Z_2-equivariant coherent sheaves on an elliptic curve X. The proof given has the advantages of using little powerful machinery, and its method of proof allows us insight into the orbits of the action of $\mathrm{Aut}(\mathcal{D}_{\mathbb{Z}_2}(X))$ upon the set $\mathrm{Coll}(\mathcal{D}_{\mathbb{Z}_2}(X))$ of full exceptional collections of sheaves on $\mathcal{D}_{\mathbb{Z}_2}(X)$. In the first section we define a set of coordinates upon $K_0(\mathcal{D}_{Z_2}(X))$ that will be essential to the proof of constructibility. The second section is devoted to the proof itself, and in the final section we classify the orbits of the action by autoequivalences. Throughout this chapter $\mathcal{D} = \mathcal{D}_{\mathbb{Z}_2}(X)$ denotes the bounded derived category of $\mathbb{Z}_2$-equivariant coherent sheaves on an elliptic curve X over a field K. $\mathcal{D}$ is a K-linear category of finite type. The points $p_1, p_2, p_3, p_4 \in X$ are exactly the points of order 2 under the group action on X, with p_1 the identity. Given a line bundle L, denote by Λ_L the autoequivalence of tensoring with L. Denote by $\hat{X}$ the dual curve to X. It is well known that an elliptic curve is isomorphic to its dual. In particular, the map $p \mapsto O(p_1 - p)$ gives an isomorphism. The *Fourier-Mukai Transform* F associated to this isomorphism is a functor (see [14])

$$F(K) = p_{2*}(p_1^* K \otimes \mathcal{P}),$$

where $p_1 : X \times \hat{X} \to X$ and $p_2 : X \times \hat{X} \to \hat{X}$ are the projections and $\mathcal{P}$ is a certain line bundle.

It will be necessary to understand the structure of $\mathrm{Aut}(\mathcal{D})$ for the proof of constructibility of $\mathcal{D}$ in the second section. Denote by $\mathrm{Pic}_{\mathbb{Z}_2}(X)$ the group of all line bundles on X endowed with Z_2-equivariant structure. Let $\mathrm{Pic}_0(X)$ be the subgroup of $\mathrm{Pic}_{\mathbb{Z}_2}(X)$ consisting of line bundles of degree 0. Consider the group $\mathrm{Aut}(X)$ of $\mathbb{Z}_2$-equivariant automorphisms of X as a subgroup of $\mathrm{Aut}(\mathcal{D})$ of $\mathcal{D}$ by extending each $g \in \mathrm{Aut}(X)$ in the natural way. The Fourier-Mukai Transform F defined above and $\Lambda_{O(p_1)}$ generate a subgroup of $\mathrm{Aut}(\mathcal{D})$ isomorphic to a central extension of $\mathrm{SL}(2,\mathbb{Z})$ by $\mathbb{Z}$ [16]. In particular, this extension is isomorphic to B_3, the braid group on 3 strands [10]. The shift functor [1] is an element of this subgroup, for we have $(F\Lambda_{O(p_1)})^3 = [1]$.

Theorem V.0.7 (Thm. 5.1, 6.3 of [10]). *There is a short exact sequence*

$$1 \to \mathrm{Pic}_0(X) \rtimes \mathrm{Aut}(X) \to \mathrm{Aut}(\mathcal{D}) \to B_3 \to 1.$$

V.1 An Useful Basis for $K_0(\mathcal{D})$

For $E \in \mathcal{D}$ and $i = 1, 2, 3, 4$ define functions $y_i(E) := \chi(E, \mathcal{O}_{p_i}) - \frac{1}{2}\operatorname{rk} E$. Recall that for coherent sheaves E, F on X, $\chi(E,F) = \dim_K \mathrm{Hom}_{\mathcal{D}}(E,F) - \dim_K \mathrm{Ext}^1_{\mathcal{D}}(E,F)$.

Proposition V.1.1. *The functions* $\operatorname{rk}, \deg, y_i \in \mathrm{Hom}_K(K_0(\mathcal{D}), K)$*, for* $i = 1,2,3,4$ *define a system of coordinates on* $K_0(\mathcal{D}) \otimes \mathbb{Q}$ *with the property that*

$$\chi(E,F) = \frac{1}{2}(\operatorname{rk} E \deg F - \deg E \operatorname{rk} F) + \sum_{i=1}^{4} y_i(E) y_i(F). \tag{V.1}$$

Proof. It suffices to show that for some basis $(E_1, E_2, \cdots, E_6)$ of $K_0(\mathcal{D}) \otimes K$, the vectors $(\operatorname{rk} E_i, \deg E_i, y_1(E_i), \cdots, y_4(E_i))$ are linearly independent and satisfy the equation V.1. First, we compute these vectors for the basis $(\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_{p_1}, \mathcal{O}_{p_2}, \mathcal{O}_{p_3}, \mathcal{O}_{p_4})$. Since

$\deg \mathcal{O}_X(1) = 2$ and

$$\begin{aligned} y_i(\mathcal{O}_X) &= \chi(\mathcal{O}_X, \mathcal{O}_{p_i}) - \frac{1}{2} = \frac{1}{2}, \quad i = 1,2,3,4 \\ y_i(\mathcal{O}_X(1)) &= \chi(\mathcal{O}_X(1), \mathcal{O}_{p_i}) - \frac{1}{2} = \frac{1}{2}, \quad i = 1,2,3,4 \\ y_i(\mathcal{O}_{p_j}) &= \chi(\mathcal{O}_{p_j}, \mathcal{O}_{p_i}) = \delta_{ij}, \quad 1 \le i,j \le 4 \end{aligned}$$

it is easy to see that the vectors are linearly independent. The Euler form $\chi(E,F)$ is zero on all pairs in the exceptional collection except for the following:

$$\begin{aligned} \chi(\mathcal{O}_X, \mathcal{O}_{p_i}) &= 1, \quad i = 1,2,3,4 \\ \chi(\mathcal{O}_X(1), \mathcal{O}_{p_i}) &= 1, \quad i = 1,2,3,4 \\ \chi(\mathcal{O}_X, \mathcal{O}_X(1)) &= 2. \end{aligned}$$

Computing, we check that the desired values are given:

$$\begin{aligned} \frac{1}{2}(\operatorname{rk}\mathcal{O}_X \deg \mathcal{O}_{p_j} - \deg \mathcal{O}_X \operatorname{rk}\mathcal{O}_{p_j}) + \sum_{i=1}^{4} y_i(\mathcal{O}_X) y_i(\mathcal{O}_{p_j}) &= \frac{1}{2} + y_j(\mathcal{O}_X) &&= 1, \\ \frac{1}{2}(\operatorname{rk}\mathcal{O}_{p_j} \deg \mathcal{O}_X - \deg \mathcal{O}_{p_j} \operatorname{rk}\mathcal{O}_X) + \sum_{i=1}^{4} y_i(\mathcal{O}_{p_j}) y_i(\mathcal{O}_X) &= -\frac{1}{2} + y_j(\mathcal{O}_X) &&= 0, \\ \frac{1}{2}(\operatorname{rk}\mathcal{O}_X(1) \deg \mathcal{O}_{p_j} - \deg \mathcal{O}_X(1) \operatorname{rk}\mathcal{O}_{p_j}) + \sum_{i=1}^{4} y_i(\mathcal{O}_X(1)) y_i(\mathcal{O}_{p_j}) &= \frac{1}{2} + y_j(\mathcal{O}_X(1)) &&= 1, \\ \frac{1}{2}(\operatorname{rk}\mathcal{O}_{p_j} \deg \mathcal{O}_X(1) - \deg \mathcal{O}_{p_j} \operatorname{rk}\mathcal{O}_X(1)) + \sum_{i=1}^{4} y_i(\mathcal{O}_{p_j}) y_i(\mathcal{O}_X(1)) &= -\frac{1}{2} + y_j(\mathcal{O}_X(1)) &&= 0, \\ \frac{1}{2}(\operatorname{rk}\mathcal{O}_X \deg \mathcal{O}_X(1) - \deg \mathcal{O}_X \operatorname{rk}\mathcal{O}_X(1)) + \sum_{i=1}^{4} y_i(\mathcal{O}_X) y_i(\mathcal{O}_X(1)) &= 1 + \sum_{i=1}^{4} \frac{1}{4} &&= 2, \\ \frac{1}{2}(\operatorname{rk}\mathcal{O}_X(1) \deg \mathcal{O}_X - \deg \mathcal{O}_X(1) \operatorname{rk}\mathcal{O}_X) + \sum_{i=1}^{4} y_i(\mathcal{O}_X(1)) y_i(\mathcal{O}_X) &= -1 + \sum_{i=1}^{4} \frac{1}{4} &&= 0. \end{aligned}$$

This completes the proof. ∎

Given an object E, we denote by $\xi(E) = (\operatorname{rk} E_i, \deg E_i, y_1(E_i), \cdots, y_4(E_i))$ the coordinate vector of E with respect to this basis.

Remark V.1.2. Every exceptional object $E \in \mathcal{D}$ satisfies the relations

$$\begin{aligned} y_i(E) &\equiv \frac{1}{2}\operatorname{rk} E \mod \mathbb{Z},\ \forall i \\ \sum_{i=1}^{4} y_i(E)^2 &= 1, \\ \sum_{i=1}^{4} y_i(E) &\equiv \deg E \mod 2\mathbb{Z}. \end{aligned} \tag{V.2}$$

Indeed, these are the immediate consequences of $\chi(\cdot,\cdot)$ taking values in $\mathbb{Z}$ and applying V.1 to the pairs $\chi(E, \mathcal{O}_{p_i}), \chi(E,E)$ and $\chi(\mathcal{O}_X, E)$, respectively. Note also that the equation $\sum_{i=1}^4 y_i(E)^2 = 1$ bounds $|y_i(E)| \leq 1$ for $i = 1,2,3,4$.

Theorem V.1.3. *Let V be an exceptional bundle on X equipped with a $\mathbb{Z}_2$-equivariant structure. Denote by $(V|_{p_i})^+$ the subspace of $V|_{p_i}$ with positive parity. Then*

1. $|dim(V|_{p_i})^+ - \frac{1}{2}\operatorname{rk} V| \leq 1$,
2. *if* $\operatorname{rk} V$ *is odd,* $y_i(V) = \pm\frac{1}{2}, \forall i$,
3. *if* $\operatorname{rk} V$ *is even, there exists* $j \in \{1,2,3,4\}$ *such that* $y_i(V) = \pm\delta_{ij}, \forall i$.

Proof. Apply V.2 to the equality $\chi(V, \mathcal{O}_{p_i}) = dim(V|_{p_i})^+$. □

Lemma V.1.4. *Let E be an exceptional $\mathbb{Z}_2$-equivariant sheaf. Consider E as an object of the category $\mathcal{D}(X)$ of coherent sheaves on X by means of the forgetful functor $\mathcal{D}_{\mathbb{Z}_2}(X) \to \mathcal{D}(X)$. Then E is an endosimple object of $\mathcal{D}(X)$.*

Proof. Recall that if ω_X is the canonical sheaf on X, then as K-vector spaces

$$\operatorname{Hom}_{\mathcal{D}(X)}(E,E) = \operatorname{Hom}_{\mathcal{D}_{\mathbb{Z}_2}(X)}(E,E) \oplus \operatorname{Hom}_{\mathcal{D}_{\mathbb{Z}_2}(X)}(E, E\otimes\omega_X).$$

By assumption, $\operatorname{Hom}_{\mathcal{D}_{\mathbb{Z}_2}(X)}(E,E) = K$ and by Serre Duality

$$\operatorname{Hom}_{\mathcal{D}_{\mathbb{Z}_2}(X)}(E, E\otimes\omega_X) \cong \operatorname{Ext}^1_{\mathcal{D}_{\mathbb{Z}_2}(X)}(E,E) = 0,$$

so $\mathrm{Hom}_{\mathcal{D}(X)}(E,E)=K$. $\square$

Theorem V.1.5 (see [16], Cor. 14.8). *The degree and rank of an endosimple bundle on $\mathcal{D}(X)$ are relatively prime.*

Proposition V.1.6. *The actions of the Fourier-Mukai Transform F and $\Lambda_{\mathcal{O}(p_i)}$ are transitive upon the degree and rank of exceptional objects.*

Proof. F acts on the rank and degree of a sheaf E by

$$\operatorname{rk} F(E) = \deg E$$
$$\deg F(E) = -\operatorname{rk} E$$

(see [16]). It follows that $F[1]$ and $\Lambda_{\mathcal{O}(p_i)}$ act upon $(\deg E, \operatorname{rk} E)$, respectively, by the matrices $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, with their standard action on $\mathbb{Z}^2$. These matrices generate the group $\mathrm{SL}(2,\mathbb{Z})$, which acts transitively upon the subset of $\mathbb{Z}^2$ of pairs of relatively prime integers. Since the degree and rank of an exceptional object of $\mathcal{D}_{\mathbb{Z}_2}(X)$ are relatively prime, this completes the proof. $\square$

Lemma V.1.7. $\mathrm{Aut}(\mathcal{D})$ *acts transitively upon the set of exceptional objects of $\mathcal{D}$.*

Proof. The actions of the Fourier-Mukai Transform F and $\Lambda_{\mathcal{O}(p_i)}$ are transitive upon the rank and degree of exceptional objects so it suffices to construct autoequivalences acting transitively upon exceptional objects of rank 0. Furthermore, all exceptional objects of $\mathcal{D}$ are shifts of exceptional sheaves so it suffices to prove that the action of autoequivalences on exceptional torsion sheaves is transitive.

Direct computation shows that $\mathcal{O}_{p_i}$ and $\zeta\mathcal{O}_{p_i}$ are exceptional objects. We prove that all exceptional torsion sheaves are of this form. Exceptional objects are endosimple, so by Lemma IV.2.1 it suffices to show that $\mathcal{O}_{2p_i}$ and $\zeta\mathcal{O}_{2p_i}$ are not exceptional for any i. In fact, we need only consider $\mathcal{O}_{2p_i}$ as the character ζ acts by an involution on the set of exceptional objects. From the short exact exact sequence

$$0 \to \zeta\mathcal{O}_{p_i} \to \mathcal{O}_{2p_i} \to \mathcal{O}_{p_i} \to 0$$

we have $[\mathcal{O}_{2p_i}] = [\mathcal{O}_{p_i}] + [\zeta\mathcal{O}_{p_i}]$. Hence by V.1,

$$\chi(\mathcal{O}_{2p_i}, \mathcal{O}_{2p_i}) = \chi(\mathcal{O}_{p_i} + \zeta\mathcal{O}_{p_i}, \mathcal{O}_{p_i} + \zeta\mathcal{O}_{p_i}) = 0.$$

Thus $\mathcal{O}_{2p_i}$ is not exceptional. Observe that this shows there are exactly 8 exceptional sheaves of rank 0 and degree 1 - the sheaves $\mathcal{O}_{p_i}$ and $\zeta\mathcal{O}_{p_i}$ for $i = 1, 2, 3, 4$.

It remains to construct an autoequivalence sending $\mathcal{O}_{p_i}$ to $\mathcal{O}_{p_j}$ for each $1 \leq i \neq j \leq 4$. Acting by the Fourier-Mukai transform, $F(\mathcal{O}_{p_i})$ is a line bundle of degree 0. Tensor with the dual of this line bundle and denote by $\Phi_i(E) = \Lambda_{F(\mathcal{O}_{p_i})^{-1}} \circ F$. Then the composition $\Phi_j^{-1} \circ \Phi_i$ is the desired autoequivalence mapping $\mathcal{O}_{p_i}$ to $\mathcal{O}_{p_j}$. □

Corollary V.1.8. *For each pair of relatively prime integers r, d with $r \geq 0$, there exist exactly 8 exceptional $\mathbb{Z}_2$-equivariant sheaves of rank r and degree d.*

Proof. In the proof of Lemma V.1.7 this was shown for $r = 0$ and $d = 1$. Since every exceptional $\mathbb{Z}_2$-equivariant bundle E has relatively prime rank and degree, apply transitivity of $\operatorname{Aut}(\mathcal{D})$ on the set of exceptional objects to complete the proof. □

Corollary V.1.9. *An exceptional $\mathbb{Z}_2$-equivariant sheaf E is uniquely determined by the vector $\xi(E) = (\operatorname{rk} E, \deg E, y_1(E), y_2(E), y_3(E), y_4(E))$.*

Proof. It suffices to prove this for E torsion. Using V.2, if E has support at p_i then

$$\xi(E) = (0, 1, \pm\delta_{1i}, \pm\delta_{2i}, \pm\delta_{3i}, \pm\delta_{4i}).$$

There are eight possible vectors $\xi(E)$ and each of them corresponds to a distinct exceptional torsion sheaf. □

V.2 Proof of Constructibility

This section is devoted to a new proof of the constructibility of $\mathcal{D} = \mathcal{D}_{\mathbb{Z}_2}(X)$, for X an elliptic curve (see [13] for an earlier original proof). For convenience, in this section we denote by H the line bundle $\mathcal{O}_X(1)$.

Theorem V.2.1 ([10]). *The action of the braid group B_6 is transitive upon the set of full exceptional collections of sheaves in $\mathcal{D}$, up to shifts.*

This theorem was first proved using tilting sheaves to construct an equivalence of categories between $\mathcal{D}$ and a certain derived category of a quiver algebra. Our proof will be built up from a series of lemmas that use the basis for $K_0(\mathcal{D})$ defined in the previous section to impose strong conditions on the objects of exceptional collections on $\mathcal{D}$.

Lemma V.2.2. *If a full exceptional collection of sheaves is of the form $\{E_1, \cdots, E_5, \mathcal{O}_{p_i}\}$ then one of the following holds:*

1. *For each i, $\operatorname{rk} E_i$ is either 0 or 1.*

2. *For each i, $\operatorname{rk} E_i$ is either 1 or 2 and there exists an unique j such that $\operatorname{rk} E_j = 2$.*

Proof. From the orthogonality relations on the collection and V.1,

$$-\frac{1}{2}\operatorname{rk} E_i + y_1(E_i) = \chi(\mathcal{O}_{p_1}, E_i) = 0.$$

Recalling that $|y_1(E_i)| \leq 1$, we have that $\operatorname{rk} E_i$ is either $0, 1$ or 2. For each object E_i with $\operatorname{rk} E_i = 2$ for some i, by V.1.3 we have $y_1(E_i) = 1$ and $y_k(E_i) = 0$ for $k = 2, 3, 4$.

Suppose E_i, E_j are two bundles of rank 2 in the collection, and without loss of generality assume $i < j$. Then

$$0 = \chi(E_j, E_i) = \deg E_i - \deg E_j + 1.$$

On the other hand,

$$\deg E_i \equiv \sum_{k=1}^{4} y_k(E_i) \equiv \sum_{k=1}^{4} y_k(E_j) \mod 2\mathbb{Z}.$$

This is a contradiction, so there is at most one bundle of rank 2 in the collection.

Suppose instead that $\operatorname{rk} E_j = 0$ and $\operatorname{rk} E_i = 2$. The vanishing of at least one of $\chi(E_i, E_j)$ or $\chi(E_j, E_i)$ imposes the condition that $y_1(E_j) = \pm \deg E_j = \pm 1$. In particular, E_j has support at p_1 and so must be $\zeta\mathcal{O}_{p_1}$ since $\mathcal{O}_{p_1}$ is already in the exceptional collection. The groups $\operatorname{Ext}^1(\mathcal{O}_{p_1}, \zeta\mathcal{O}_{p_1})$ and $\operatorname{Ext}^1(\zeta\mathcal{O}_{p_1}, \mathcal{O}_{p_1})$ are both nonzero, however, so it is not possible for both sheaves to be elements of the same exceptional collection. Therefore, the rank of E_j cannot be 0. □

Lemma V.2.3. *For each pair E, F of exceptional objects of $\mathcal{D}$ there exists at most one integer k such that*

$$\operatorname{Hom}^k_{\mathcal{D}}(E, F) \neq 0.$$

Proof. Applying an autoequivalence, we may assume that $F \cong \mathcal{O}_{p_1}$. The statement of the Lemma is invariant under action by shifts, so without loss of generality E is a sheaf. Hence E is either torsion or a vector bundle. If torsion, the Lemma is obvious. If E is a bundle then for $k \neq 0, 1$, $\operatorname{Hom}_{\mathcal{D}}(E, \mathcal{O}_{p_1}) = 0$. The Lemma is now a consequence of Serre Duality:

$$\operatorname{Ext}^1_{\mathcal{D}}(E, \mathcal{O}_{p_1}) \cong \operatorname{Hom}_{\mathcal{D}}(\zeta \otimes \mathcal{O}_{p_1}, E) = 0.$$

□

Note that if E, F are exceptional sheaves then $\chi(E, F) > 0$ if and only if $\operatorname{Hom}^\bullet(E, F) = \operatorname{Hom}(E, F)$ and $\chi(E, F) < 0$ if and only if $\operatorname{Hom}^\bullet(E, F) = \operatorname{Ext}^1(E, F)[-1]$.

Lemma V.2.4. *Let $\{E_1, \cdots, E_5, \mathcal{O}_{p_1}\}$ be as in the previous lemma. If $\operatorname{rk} E_l = 2$ for some l, then there exists a sequence of mutations of the collection such that the mutated collection consists only of line bundles and torsion sheaves.*

Proof. Without loss of generality $\deg E_l = 1$. Indeed, tensor the collection with a suitable line bundle L such that $\mathcal{O}_{p_1} \otimes L \cong \mathcal{O}_{p_1}$ and $d(E_l \otimes L) = 1$. Using V.1, the orthogonality relations on the exceptional collection impose the conditions

$$\begin{aligned} \deg E_i &= \frac{1}{2} - y_1(E_i) & \qquad & i < l \\ \deg E_j &= \frac{1}{2} + y_1(E_j) & & j > l \\ y_1(E_k) &= \frac{1}{2} & & 1 \leq k \neq l \leq 5. \end{aligned}$$

Taken together, $\deg E_i = 0$ for $i < l$ and $\deg E_j = 1$ for $j > l$. If $l \neq 5$, $\chi(E_l, E_{l+1}) = \frac{1}{2}(2-1) + \frac{1}{2} = 1$ so by Lemma V.2.3 $\mathrm{Hom}_{\mathcal{D}}(E_l, E_{l+1}) = K$. Hence there is an exact sequence

$$0 \to R_{E_{l+1}} E_l[-1] \to E_l \to E_{l+1} \to 0,$$

where $\mathrm{rk}\, R_{E_{l+1}} E_l = 1$. If $l = 5$, $\chi(E_4, E_5) = 1$ then, as above, there is an exact sequence

$$0 \to E_4 \to E_5 \to L_{E_4} E_5[1] \to 0$$

with $\mathrm{rk}\, L_{E_4} E_5 = 1$. In either case, there exists a mutation of the collection in which there are no sheaves of rank greater than 1. $\square$

Notation V.2.5. Let $E = \{E_1, \cdots, E_n\}$ be an exceptional collection with n objects. Given an element $b \in B_n$ of the braid group on n strands, we denote by $b \cdot E$ the mutation of E obtained from acting by b. Given an autoequivalence $g \in \mathrm{Aut}(\mathcal{D})$, we set $g \cdot E = \{g(E_1), \cdots, g(E_n)\}$.

Lemma V.2.6. *Let S denote the standard collection $\{\mathcal{O}_X, H, \mathcal{O}_{p_1}, \mathcal{O}_{p_2}, \mathcal{O}_{p_3}, \mathcal{O}_{p_4}\}$. For every autoequivalence $g \in \mathrm{Aut}(\mathcal{D})$ on S there is a corresponding braid group element $b_g \in B_6$ such that*

$$g \cdot S = b_g \cdot S,$$

up to the action of the shift functor.

Proof. We prove the lemma by showing that it is satisfied for a set of generators of the finitely presented group $\mathrm{Aut}(\mathcal{D})$. Recall that $\mathrm{Aut}(\mathcal{D})$ is generated by the action of the Fourier-Mukai Transform, tensoring with $\mathcal{O}(p_1)$, the shift functor, $\mathrm{Aut}(X)$ and $\mathrm{Pic}_0(X)$. It is easy to check that the functors $\Lambda_{\mathcal{O}(p_i)}$ each act on the exceptional collection S by a series of mutations. These functors generate $\mathrm{Pic}_{\mathbb{Z}_2}(X)$, of which $\mathrm{Pic}_0(X)$ is a subgroup. If $\mathrm{Aut}(X)$ is nontrivial, it acts by permutations of the torsion sheaves while fixing $\mathcal{O}_X$ and H. These actions are clearly represented by braid group actions. Finally, we show that the Fourier-Mukai Transform F acts on S by mutations.

$$F(S) = \{\zeta\mathcal{O}_{p_1}, V, \mathcal{O}_X, H(-p_3 - p_4), H(-p_2 - p_4), H(-p_2 - p_3)\},$$

where V is a bundle of rank 2 fitting into the exact sequence $0 \to H(-p_2 - p_3 - p_4) \to V \to \mathcal{O}_X \to 0$. Apply the mutations

$$\begin{aligned} R_{H(-p_3-p_4)} R_{\mathcal{O}_X} V &= \mathcal{O}_{p_2} \\ L_{\mathcal{O}_{p_2}} H(-p_2 - p_4) &= \mathcal{O}(p_4) \\ L_{\mathcal{O}_{p_2}} H(-p_2 - p_3) &= \mathcal{O}(p_3). \end{aligned}$$

The resulting collection is $\{\zeta\mathcal{O}_{p_1}, \mathcal{O}_X, H(-p_3-p_4), \mathcal{O}(p_4), \mathcal{O}(p_3), \mathcal{O}_{p_2}\}$. After the additional mutations

$$\begin{aligned} R_{\mathcal{O}_X} \zeta\mathcal{O}_{p_1} &= \mathcal{O}(p_1) \\ R_{\mathcal{O}(p_4)} H(-p_3 - p_4) &= \mathcal{O}_{p_3} \\ L_{\mathcal{O}_{p_3}} \mathcal{O}(p_3) &= H, \end{aligned}$$

the collection has the form $\{\mathcal{O}_X, \mathcal{O}(p_1), \mathcal{O}(p_4), H, \mathcal{O}_{p_3}, \mathcal{O}_{p_2}\}$. This is easily seen to be in the B_6-orbit of S. $\square$

Lemma V.2.7. *Every full exceptional collection on $\mathcal{D}$ of the form $\{E_1, \cdots, E_5, \mathcal{O}_{p_1}\}$ is* *in the orbit of the standard collection S under the braid group action by mutations.*

Proof. By Lemma V.2.4, may assume that the collection consists of line bundles L_i and torsion sheaves T_j. Mutate the line bundles left past the torsion sheaves so that the collection has the form

$$C = \{L_0, L_1, \cdots, L_n, T_1, \cdots, T_m\}.$$

Acting by the autoequivalence $\Lambda_{L_0^{-1}}$, so that $L_0 = O_X$. If we can show that the resulting collection $\Lambda_{L_0^{-1}} \cdot C$ is in the orbit of the standard collection S under the action of mutations, then as the actions by autoequivalences and mutations commute, Lemma V.2.6 implies that C is in the orbit of S as well. Thus, we may assume $L_0 = \mathcal{O}_X$.

The orthogonality relations, $\chi(T, \mathcal{O}_X) = 0$, hold for every torsion sheaf T in the collection. Thus, each of the torsion sheaves is of the form $\mathcal{O}_{p_j}$ for some j. If for some j, $\mathcal{O}_{p_j}$ is in the collection, then $\chi(\mathcal{O}_{p_j},\ L_i) = 0$. This implies that $y_j(L_i) = \frac{1}{2}$ for each $i = 1, \cdots, n$. In particular, this holds for $j = 1$. The orthogonality relations

$$-\frac{1}{2}\deg L_i + \frac{1}{2}\sum_{k=1}^{4} y_k(L_i) = -\frac{1}{2}\deg L_i + \frac{1}{4} + \frac{1}{2}\sum_{k=2}^{4} y_k(L_i) = \chi(L_i, \mathcal{O}_X) = 0$$

imply that $\deg L_i$ is either $-1, 0, 1$, or 2. Moreover, $\deg L_i = 2$ exactly when $y_k(L_i) = \frac{1}{2}$, $\forall k$ and $\deg L_i = -1$ exactly when $y_k = -\frac{1}{2}$ for $k = 2, 3, 4$. In the former case $L_i = H$, and in the latter $L_i = H(-p_2 - p_3 - p_4)$.

We shall prove that there exists a series of mutations by which the number of line bundles in the collection is reduced to two. For this, observe first that if $\deg L_i = \deg L_j - 1$ for $i < j$, then

$$-\frac{1}{2} + \sum_{k=1}^{4} y_k(L_i) y_k(L_j) = 0.$$

This holds exactly when there exists an unique integer k such that $y_k(L_i) \neq y_k(L_j)$, in which case L_i is uniquely determined as the line bundle $L_i = L_j \otimes \mathcal{O}(-p_k)$. Then, the right mutation $R_{L_j}(L_i) = \mathcal{O}_{p_k}$ is torsion. Likewise, if any L_j is of degree 1 then by the same argument $R_{L_j}(\mathcal{O}_X)$ is torsion. Mutating all of the line bundles left past this torsion sheaf and tensoring the resulting collection with L_1^{-1}, we obtain a new collection of the form

$$\{\mathcal{O}_X, L_1, \cdots, L_{n-1}, \mathcal{O}_{p_k}, T_1, \cdots, T_m\}$$

in which the number of line bundles is reduced by one. We may therefore assume that no line bundle in the collection is of degree 1.

Given a line bundle L_i of degree 2 recall that this implies $L_i = H$. In this case, mutate the other bundles right past H so that the collection has the form $\{\mathcal{O}_X, H, E_1, E_2, E_3, \mathcal{O}_{p_1}\}$. Since the triangulated category generated by $\mathcal{O}_X$ and H is $\pi^*\mathcal{D}(\mathbb{P}^1)$, the left orthogonal is

$$^{\perp}\langle \mathcal{O}_X, H\rangle = {}^{\perp}\pi^*\mathcal{D}(\mathbb{P}^1) = \langle \mathcal{O}_{p_1}, \mathcal{O}_{p_2}, \mathcal{O}_{p_3}, \mathcal{O}_{p_4}\rangle.$$

Therefore the sheaves E_1, E_2, E_3 are all torsion.

It remains to consider the case where $\deg L_i \in \{-1,\ 0\}$, $\forall i$. If no bundle has degree -1, the relations

$$y_1(L_i) = \frac{1}{2}$$
$$\frac{1}{2}\sum_{k=1}^{4} y_k(L_i) = \chi(L_i,\ \mathcal{O}_X) = 0$$

have at most three solutions, $H_{2,3} = H(-p_2 - p_3), H_{2,4} = H(-p_2 - p_4)$, and $H_{3,4} = H(-p_3 - p_4)$, and therefore the collection has at least two torsion sheaves. Without loss of generality $\mathcal{O}_{p_2}$ is also in the collection. But then for $j = 3, 4$,

$$\chi(\mathcal{O}_{p_2}, H_{2,j}) = -\frac{1}{2} + y_2(H_{2,j}) = -1,$$

contradicting the assumption that the collection has the form $\{\mathcal{O}_X, L_1, \cdots, L_{n-1}, T_1, \cdots, T_m\}$. A similar computation shows that $\chi(\mathcal{O}_{p_3}, H_{3,i}) = -1$ for $i = 2, 4$. Thus, no exceptional collection can have only line bundles of degree 0 and torsion sheaves.

Recall that the line bundle of degree -1 in the collection was necessarily $L :=$

$H(-p_2 - p_3 - p_4)$, and recall from the proof of Lemma V.2.6 that mutating L past $\mathcal{O}_X$ produces a bundle V of rank 2, up to shifts. Hence Lemma V.2.2 implies that this collection has exactly one torsion sheaf, $\mathcal{O}_{p_1}$. Other than $\mathcal{O}_X$ and L, the only other line bundles that can exist in this collection are the $H_{i,j}$. Consequently, the collection has the form $\{\mathcal{O}_X, L, H_{2,3}, H_{2,4}, H_{3,4}, \mathcal{O}_{p_1}\}$, up to a permutation of the orthogonal bundles $H_{i,j}$, and each of the right mutations $R_{H_{i,j}} L = \mathcal{O}_{p_k}$ is torsion.

This proves that we may reduce the number of line bundles down to 2 through a series of mutations. Mutating, if necessary, so the line bundles are listed first and as $\mathcal{O}_X$ is a member of the collection, the four torsion sheaves are exactly the $\mathcal{O}_{p_i}, i = 1, 2, 3, 4$. If the second line bundle is L, then

$$\pi^* \mathcal{D}(\mathbb{P}^1) = \langle \mathcal{O}_{p_i}, i = 1, 2, 3, 4 \rangle^{\perp} = \langle \mathcal{O}_X, L \rangle,$$

implies that $L = H$. Hence this collection is the standard collection S, completing the proof. □

Lemma V.2.8. *Given any full exceptional collection $E = \{E_1, E_2, \cdots, E_6\}$, and autoequivalence $g \in \mathcal{D}$, there exists a braid group element $b_g \in B_6$ such that*

$$g \cdot E = b_g \cdot E.$$

Proof. Choose any $h \in \mathrm{Aut}(\mathcal{D})$ such that $h(E_6) = \mathcal{O}_{p_1}$. There exists a $b \in B_6$ such that $h \cdot E = b \cdot S$ by Lemma V.2.7. Recall that the actions by mutations and autoequivalences on exceptional collections commute, so

$$E = h^{-1} \cdot (b \cdot S) = b \cdot (h^{-1} \cdot S) = b b_{h^{-1}} \cdot S.$$

Given any $g \in \mathrm{Aut}(\mathcal{D})$, there exists a $b_g \in B_6$ such that $g \cdot S = b_g \cdot S$. by V.2.6. Thus,

$$\begin{aligned} g \cdot E \quad = g \cdot (bb_{h^{-1}} \cdot S) \quad &= bb_{h^{-1}} \cdot (g \cdot S) \\ &= bb_{h^{-1}} \cdot (b_g \cdot S) \\ &= bb_{h^{-1}} b_g \cdot ((b_{h^{-1}})^{-1} b^{-1} \cdot E) \end{aligned}$$

□

Theorem V.2.9. *The action of the braid group B_6 is transitive upon the set of full exceptional collections of sheaves in $\mathcal{D}$, up to shifts.*

Proof. Given a full exceptional collection $E = \{E_1, E_2, \cdots, E_6\}$, apply an autoequivalence g with the property that $g(E_6) = \mathcal{O}_{p_1}$. By Lemma V.2.8 there exists a braid group element b_g with the property that $g \cdot E = b_g \cdot E$. By Lemma V.2.7 there exists a braid group element b such that $b \cdot (b_g \cdot E) = S$, the standard exceptional collection, up to shifts. □

V.3 Orbits of Exceptional Collections Under $\mathrm{Aut}(\mathcal{D})$

In this section we determine representatives for each orbit of the action of $\mathrm{Aut}(\mathcal{D})$ on the set of all full exceptional collections of sheaves on $\mathcal{D}$ and count the number of orbits. This result will be used in the following chapter in constructing glued stability conditions on $\mathrm{Stab}(\mathcal{D})$.

Theorem V.3.1. *Let A denote the subgroup of the symmetric group S_3 by which the stabilizer of p_1 in $\mathrm{Aut}(X)$ acts upon the set $\{p_2, p_3, p_4\}$. There are $\frac{246}{|A|}$ orbits in the action of $\mathrm{Aut}(\mathcal{D})$ upon the set $\mathrm{Coll}(\mathcal{D})$ of full exceptional collections of sheaves on $\mathcal{D}$.*

Proof. We begin by recalling the short exact sequence V.0.7

$$1 \to \mathrm{Pic}_0(X) \rtimes \mathrm{Aut}(X) \to \mathrm{Aut}(\mathcal{D}) \to B_3 \to 1.$$

Since the degree and rank of an exceptional object are relatively prime, it follows that the induced action of B_3 is transitive upon the set of all pairs of integers $(\operatorname{rk} E, \deg E)$ for E exceptional. Given a full exceptional collection $E = \{E_1, E_2, \cdots, E_6\}$ on $\mathcal{D}$, acting by a suitable element of $\operatorname{Aut}(\mathcal{D})$ we may assume that $E_6 = O_{p_1}$. By Lemma V.2.2, the rank of each E_i is then either $0, 1$ or 2. There are at most 4 torsion sheaves in the collection and some E_i has rank 2 exactly when there are also four line bundles, so there exist at least two line bundles in the collection. Tensoring the collection with a suitable line bundle, we may assume that the first line bundle in the collection is $\mathcal{O}_X$. Since the only elements of the B_3 subgroup that act trivially on degree and rank are the shifts $[n]$ for n even, the only autoequivalences that fix both $\mathcal{O}_X$ and $\mathcal{O}_{p_1}$ are elements of $\operatorname{Aut}(X)$.

The action of $\operatorname{Aut}(X)$ on exceptional collections is by permutations on the set of 2-torsion points $\{p_1, p_2, p_3, p_4\}$. Furthermore, assuming $\mathcal{O}_{p_1}$ is fixed by the action we need only consider the stabilizer subgroup of p_1 in $\operatorname{Aut}(X)$. Let A denote the subgroup of the symmetric group S_3 by which the stabilizer of p_1 in $\operatorname{Aut}(X)$ acts upon the set $\{p_2, p_3, p_4\}$. In particular, if X has an action by $\sqrt{-1}$ then $A \cong \mathbb{Z}_2$ and if X has an action by $\sqrt[3]{-1}$ then $A \cong \mathbb{Z}_3$. Otherwise, A is trivial.

The actions of mutations and autoequivalences on exceptional collections commute, so there is an induced action of mutations on the equivalence classes of exceptional collections up to autoequivalence. We make use of this induced action in order to simplify the classification of orbits, which is further broken down into cases based on the number of torsion sheaves in the collection and presence, if any, of bundles of rank greater than 1. Lemma V.2.2 on the structure of exceptional collections of the form $\{E_1, \cdots, E_5, \mathcal{O}_{p_1}\}$ is useful for restricting the number of cases we have to consider in the classification. Finally, we observe a few conditions imposed by V.1. Suppose $E_i = O_X$, E_j is torsion with support at $\{p_a\}$ and that E_k is any line bundle. Then,

$$\begin{aligned} E_j &\cong \zeta \mathcal{O}_{p_a}, \quad j < i \\ E_j &\cong \mathcal{O}_{p_a}, \quad j > i, \\ y_a(E_k) &= \frac{1}{2}, \quad j < i \text{ or } k < j, \\ y_a(E_k) &= -\frac{1}{2}, \quad i < j < k. \end{aligned} \tag{V.3}$$

Note also that $\mathcal{O}_X(1) \cong \mathcal{O}_X(2p_i)$ for each $i = 1, 2, 3, 4$.

Case 1: Four torsion sheaves. There are two line bundles, $\mathcal{O}_X$ and L. Mutating each to the left and tensoring with a suitable line bundle, the collection has the form

$$\{\mathcal{O}_X, L, \mathcal{O}_{p_2}, \mathcal{O}_{p_3}, \mathcal{O}_{p_4}, \mathcal{O}_{p_1}\}.$$

From V.3, $y_i(L) = \frac{1}{2}$ for $i = 1, 2, 3, 4$. Since $\chi(L, \mathcal{O}_X) = 0$, from V.1 we get that $\deg L = 2$. Hence $L \cong \mathcal{O}_X(1)$, and the collection is the standard collection. All other exceptional collections in this case are obtained by mutations of the line bundles past the torsion sheaves. There are 10 ways to place the two line bundles between the torsion sheaves so that $\mathcal{O}_X$ appears before L and $\mathcal{O}_{p_1}$ is the last object in the collection. Up to reordering the torsion sheaves, there are $\frac{60}{|A|}$ orbits in this case.

Case 2: Three torsion sheaves. There are three line bundles, $\mathcal{O}_X, L_1, L_2$. We assume that the torsion sheaves have support at p_1, p_2, p_3, and mutate the line bundles left past the torsion sheaves as in the previous case so that that collection has the form

$$\{\mathcal{O}_X, L_1, L_2, \mathcal{O}_{p_2}, \mathcal{O}_{p_3}, \mathcal{O}_{p_1}\}.$$

This implies by V.1 that the L_i satisfy the relation

$$\deg L_i = \frac{3}{2} + y_4(L_i),$$

so $L_i \cong \mathcal{O}_X(1)$ if $y_4(L_i) = \frac{1}{2}$ and $L_i \cong \mathcal{O}_X(p_4)$ if $y_4(L_i) = -\frac{1}{2}$. Since $\mathrm{Hom}(\mathcal{O}_X(p_4), \mathcal{O}_X(1)) = K$ we must have $L_1 = \mathcal{O}_X(p_4), L_2 = \mathcal{O}_X(1)$. All other exceptional collections in this case are obtained from $\{\mathcal{O}_X, \mathcal{O}_X(p_4), \mathcal{O}_X(1), \mathcal{O}_{p_2}, \mathcal{O}_{p_3}, \mathcal{O}_{p_1}\}$ by mutations and permutations of p_2, p_3, p_4. There are 10 ways to place three line bundles between the torsion sheaves so that $\mathcal{O}_{p_1}$ is the last object, and so up to permuting the points p_2, p_3, p_4, there are $\frac{60}{|A|}$ orbits in this case.

Case 3: Two torsion sheaves. There are four line bundles, $\mathcal{O}_X, L_1, L_2, L_3$. Suppose the torsion sheaves have support at p_1 and p_2. Observe that if $i < j$ and there are no torsion sheaves between L_i and L_j, then $\deg L_i \leq \deg L_j$. Indeed,

$$0 = \chi(L_j, L_i) = \frac{1}{2}(\deg L_i - \deg L_j) + \sum_{k=1}^{4} y_k(L_i) y_k(L_j)$$

and both $y_1(L_i) = y_1(L_j)$ and $y_2(L_i) = y_2(L_j)$, so $\sum_{k=1}^4 y_k(L_i)y_k(L_j) \geq 0$. Mutate all the line bundles left past the torsion sheaves as in the previous cases so that the collection has the form

$$\{\mathcal{O}_X, L_1, L_2, L_3, \mathcal{O}_{p_2}, \mathcal{O}_{p_1}\}.$$

For each i, $y_1(L_i) = y_2(L_i) = \frac{1}{2}$ so it suffices to determine the possible values of $y_3(L_i), y_4(L_i)$ for each i. Let $A_i \subset \{3, 4\}$ be the subset for which $y_a(L_i) = -\frac{1}{2}$ for each $a \in A_i$. Then $L_i \cong \mathcal{O}_X(1)(-\sum_{a \in A} p_a)$, and $A_i \subset A_j$ if and only if $\mathrm{Hom}(L_i, L_j) = 0$. Also, $\mathrm{Ext}^1(\mathcal{O}_X(1), \mathcal{O}_X(1)(-p_3 - p_4) = K$. Thus, we have that up to reordering the orthogonal line bundles $\mathcal{O}_X(p_3)$ and $\mathcal{O}_X(p_4)$, there are two possibilities:

$$\{\mathcal{O}_X, \mathcal{O}_X(1)(-p_3 - p_4), \mathcal{O}_X(p_3), \mathcal{O}_X(p_4), \mathcal{O}_{p_2}, \mathcal{O}_{p_1}\}$$
$$\{\mathcal{O}_X, \mathcal{O}_X(p_3), \mathcal{O}_X(p_4), \mathcal{O}_X(1), \mathcal{O}_{p_2}, \mathcal{O}_{p_1}\}.$$

In each of these two, there are five possible positions for the $\mathcal{O}_{p_2}$ to be mutated to, and therefore $\frac{60}{|A|}$ orbits in this case after we consider permutations of the three points. We include here the eight other exceptional collections in this case obtained by mutations from either of the two collections listed above as these will make it easier to check the argument in the next case.

$$\{\zeta\mathcal{O}_{p_2}, \mathcal{O}_X, \mathcal{O}_X(p_3), \mathcal{O}_X(p_4), \mathcal{O}_X(1), \mathcal{O}_{p_1}\}$$
$$\{\zeta\mathcal{O}_{p_2}, \mathcal{O}_X, \mathcal{O}_X(1)(-p_3-p_4), \mathcal{O}_X(p_3), \mathcal{O}_X(1), \mathcal{O}_{p_1}\}$$
$$\{\mathcal{O}_X, \mathcal{O}_{p_2}, \mathcal{O}_X(1)(-p_2-p_3), \mathcal{O}_X(1)(-p_2-p_4), \mathcal{O}_X(p_2), \mathcal{O}_{p_1}\}$$
$$\{\mathcal{O}_X, \mathcal{O}_{p_2}, \mathcal{O}_X(1)(-p_2-p_3-p_4), \mathcal{O}_X(1)(-p_2-p_3), \mathcal{O}_X(p_2), \mathcal{O}_{p_1}\}$$
$$\{\mathcal{O}_X, \mathcal{O}_X(1)(-p_3-p_4), \mathcal{O}_{p_2}, \mathcal{O}_X(1)(-p_2-p_3), \mathcal{O}_X(p_2), \mathcal{O}_{p_1}\}$$
$$\{\mathcal{O}_X, \mathcal{O}_X(p_3), \mathcal{O}_{p_2}, \mathcal{O}_X(1)(-p_2-p_4), \mathcal{O}_X(p_2), \mathcal{O}_{p_1}\}$$
$$\{\mathcal{O}_X, \mathcal{O}_X(1)(-p_3-p_4), \mathcal{O}_X(p_3), \mathcal{O}_{p_2}, \mathcal{O}_X(p_2), \mathcal{O}_{p_1}\}$$
$$\{\mathcal{O}_X, \mathcal{O}_X(p_3), \mathcal{O}_X(p_4), \mathcal{O}_{p_2}, \mathcal{O}_X(p_2), \mathcal{O}_{p_1}\}$$

Case 4: One torsion sheaf and no bundle of rank 2. There are five line bundles, $\mathcal{O}_X, L_1, L_2, L_3, L_4$. We claim that all possible exceptional collections in this form can be obtained from an exceptional collection with four line bundles by mutating a torsion sheaf either left or right past a line bundle. Indeed, in Lemma V.2.7 we proved that if there are more than two bundles in the collection then there exists a sequence of mutations reducing the number of bundles by one.

Consider each of the possible exceptional collections obtained from the 10 exceptional collections listed in Case 3 by mutating the torsion sheaf $\zeta\mathcal{O}_{p_2}$ or $\mathcal{O}_{p_2}$ left or right past a line bundle. Up to permuting the points p_2, p_3, p_4, the resulting exceptional collections are exactly

$$\{\mathcal{O}_X, \mathcal{O}_X(p_2), \mathcal{O}_X(p_3), \mathcal{O}_X(p_4), \mathcal{O}_X(1), \mathcal{O}_{p_1}\}$$
$$\{\mathcal{O}_X, \mathcal{O}_X(p_2), \mathcal{O}_X(1)(-p_3-p_4), \mathcal{O}_X(p_3), \mathcal{O}_X(1), \mathcal{O}_{p_1}\}$$
$$\{\mathcal{O}_X, \mathcal{O}_X(1)(-p_2-p_3), \mathcal{O}_X(p_3), \mathcal{O}_X(1)(-p_2-p_4), \mathcal{O}_X(p_2), \mathcal{O}_{p_1}\}$$
$$\{\mathcal{O}_X, \mathcal{O}_X(1)(-p_2-p_3-p_4), \mathcal{O}_X(1)(-p_3-p_4), \mathcal{O}_X(1)(-p_2-p_3), \mathcal{O}_X(p_2), \mathcal{O}_{p_1}\}$$
$$\{\mathcal{O}_X, \mathcal{O}_X(1)(-p_3-p_4), \mathcal{O}_X(1)(-p_2-p_3), \mathcal{O}_X(p_3), \mathcal{O}_X(p_2), \mathcal{O}_{p_1}\}$$
$$\{\mathcal{O}_X, \mathcal{O}_X(1)(-p_3-p_4), \mathcal{O}_X(p_3), \mathcal{O}_X(p_2), \mathcal{O}_X(1), \mathcal{O}_{p_1}\}$$

Thus, there are $\frac{36}{|A|}$ orbits in this case.

Case 5: One torsion sheaf and one bundle V of rank 2. There are four line bundles, $\mathcal{O}_X, L_1, L_2, L_3$. Suppose that the bundle V of rank 2 is the first object in the exceptional collection. Since $\operatorname{rk} V = 2$ and $\mathcal{O}_{p_1}$ is in the collection, by V.1.3 $y_i(V) = \delta_{1i}$. Then, from V.1 and $\chi(\mathcal{O}_X, V) = \chi(L_i, \mathcal{O}_X) = 0$, we have $\deg V = -1$ and $\deg L_i = 0, i = 1, 2, 3$. There are only four line bundles of degree 0 with $y_1(L) = \frac{1}{2}$, and all four are orthogonal. Thus, if V is the first object then the exceptional collection is

$$\{V, \mathcal{O}_X, \mathcal{O}_X(1)(-p_2 - p_3), \mathcal{O}_X(1)(-p_2 - p_4), \mathcal{O}_X(1)(-p_3 - p_4), \mathcal{O}_{p_1}\}.$$

By Lemma V.2.2 we know that the mutation of a line bundle past V is still a line bundle. Also, if V is not the first object in the exceptional collection then, from V.1 we get $\deg V = 1$. Since $y_1(V) = 1$ by Proposition V.1.3, V is uniquely determined as an exceptional object by the condition of whether $\mathcal{O}_X$ appears to the left or to the right of V in the collection. Moreover, if L_i is to the left of V in the collection then $\deg L_i = 0$ and if L_i is to the right of V in the collection then $\deg L_i = 1$. Thus, the other possibilities are quickly determined:

$$\{\mathcal{O}_X, V, \mathcal{O}_X(p_2), \mathcal{O}_X(p_3), \mathcal{O}_X(p_4), \mathcal{O}_{p_1}\}$$
$$\{\mathcal{O}_X, \mathcal{O}_X(1)(-p_2 - p_3), V, \mathcal{O}_X(p_2), \mathcal{O}_X(p_3), \mathcal{O}_{p_1}\}$$
$$\{\mathcal{O}_X, \mathcal{O}_X(1)(-p_2 - p_3), \mathcal{O}_X(1)(-p_2 - p_4), V, \mathcal{O}_X(p_4), \mathcal{O}_{p_1}\}$$
$$\{\mathcal{O}_X, \mathcal{O}_X(1)(-p_2 - p_3), \mathcal{O}_X(1)(-p_2 - p_4), \mathcal{O}_X(1)(-p_2 - p_4), V, \mathcal{O}_{p_1}\}$$

In all, there are $\frac{30}{|A|}$ orbits in this case and the proof of the theorem is complete. $\square$

CHAPTER VI

ANALYSIS OF $\mathrm{Stab}(\mathcal{D}_{\mathbb{Z}_2}(X))$

This chapter is primarily an analysis of the stability manifold of $\mathcal{D} = \mathcal{D}_{\mathbb{Z}_2}(X)$ for X an elliptic curve, although some of our results have natural extensions to the case of hyperelliptic curves. It is crucial in what follows that these curves possess a natural morphism $X \to \mathbb{P}^1$ because $\mathcal{D}(\mathbb{P}^1)$ is the only derived category of a smooth curve that is generated by a full exceptional collection. Building off the results of the previous two chapters, we shall construct a large open and connected set of stability conditions in $\mathrm{Stab}(\mathcal{D})$ that contains all stability conditions on $\mathcal{D}$ glued from exceptional collections as well as the $\widetilde{\mathrm{GL}}(2,\mathbb{R})$-orbit of the standard stability. In Chapter IV we considered a degree 2 morphism of curves $X \to Y$ in order to construct stability conditions on the category $\mathcal{D}_{\mathbb{Z}_2}(X)$. If we suppose that the base is $Y = \mathbb{P}^1$ then we can extend our results.

Proposition VI.0.2. *Let X be a smooth curve with a degree 2 morphism $X \to \mathbb{P}^1$. Let p_1, p_2, $\cdots$, p_n be the ramification points of the morphism. Consider a stability $\sigma = (Z, P) \in U \subset \mathrm{Stab}_{\mathcal{N}}(\mathcal{D}_{\mathbb{Z}_2}(X))$, where U is as in Theorem IV.2.6. Assume that for every $i = 1, \dots, n$ the vectors $Z(\mathcal{O}_{p_i})$ and $Z(\mathcal{O}_{2p_i})$ are linearly independent over $\mathbb{R}$. Then some rotation of σ is glued from an exceptional collection.*

Proof. By Theorem IV.2.6, it is enough to check the same statement for a stability σ arising from the construction of Proposition IV.2.6 with $I^+ = \{1, \dots, n\}$, $I^- = I^0 = \emptyset$ and $n_i = 1$. We claim that in this situation for any sufficiently small $a > 0$ the rotated stability $R_a\sigma = (Z_a, P_a)$ is glued from an exceptional collection. Indeed, if a is small enough then we still have $\Im Z_a(\mathcal{O}_{p_i}) < 0$ for all $i = 1, \dots, n$. There is a unique $N \in \mathbb{Z}$

such that $\Im Z_a(\pi^*\mathcal{O}(N)) < 0$ and $\Im Z_a(\pi^*\mathcal{O}(N+1)) > 0$. Consider the following full Ext-exceptional collection on $\mathcal{D}_{\mathbb{Z}_2}(X)$:

$$(\pi^*\mathcal{O}(N)[1], \pi^*\mathcal{O}(N+1), \mathcal{O}_{p_1}[-1], \ldots, \mathcal{O}_{p_n}[-1]). \tag{VI.1}$$

There exists a glued stability condition with the heart generated by this exceptional collection and with the central charge Z_a. To see that $R_a\sigma$ coincides with this stability condition, by Proposition III.4.2(a), it is enough to check that all the objects of our exceptional collection lie in $P_a(-1,2] = P(-1-a, 2-a]$. Recall that

$$P(0,1] = [\pi^* \operatorname{Coh}(\mathbb{P}^1), [\zeta \otimes \mathcal{O}_{p_i}, \mathcal{O}_{p_i}[-1] \mid i = 1, \ldots, n]]$$

Thus, all the objects of the collection (VI.1), except for $\pi^*\mathcal{O}(N)[1]$, lie in $P(0,1] \subset P(-1-a, 2-a]$. Note that by our assumptions, the phases of $\mathcal{O}_{p_i}[-1]$ are in $(0,1)$. Also, it is easy to see that $\pi^*\mathcal{O}(m) \in P(0,1)$ for every $m \in \mathbb{Z}$. The exact sequence

$$0 \to \pi^*\mathcal{O}(m-1) \to \pi^*\mathcal{O}(m) \to \mathcal{O}_{\pi^{-1}(y)} \to 0$$

in $P(0,1]$ shows that $\phi_{\max}(\pi^*\mathcal{O}(m-1)) \leq \phi_{\max}(\pi^*\mathcal{O}(m))$.

Now let us consider the exact sequence

$$0 \to F \to \pi^*\mathcal{O}(N) \to G \to 0$$

in $P(0,1]$, where F is the maximal σ-destabilizing subobject in $\pi^*\mathcal{O}(N)$. The corresponding long exact cohomology sequence in $\operatorname{Coh}_{\mathbb{Z}_2}(X)$ takes form

$$0 \to H^0F \to \pi^*\mathcal{O}(N) \to H^0G \to H^1F \to 0,$$

so either $H^0F = 0$ or H^0F is a line bundle. In the former case we have $F = H^1F[-1] \in [\mathcal{O}_{p_i}[-1] \mid i = 1, \ldots, n]$. In the latter case we have $H^0F \simeq \pi^*\mathcal{O}(m)(-\sum_{j\in J} p_j)$ for some

$m \in \mathbb{Z}$ and $J \subset \{1, \ldots, n\}$. Hence, in the derived category H^0F can be viewed as an extension of $\pi^*\mathcal{O}(m)$ by $\oplus_{j\in J}\mathcal{O}_{p_j}[-1]$. Therefore, the phase of F is bounded above by the maximum of the phases of $Z(\mathcal{O}_{p_i}[-1])$, $i = 1, \ldots, n$ and of $Z(\pi^*\mathcal{O}(m))$. Note that we have a nonzero map from $\pi^*\mathcal{O}(m)(-\sum_{i=1}^n 2p_i) \simeq \pi^*\mathcal{O}(m-n)$ to $\pi^*\mathcal{O}(N)$, so $m \leq N+n$. By making a small enough we can assume that $N \leq 0$, so in this case we deduce that $\pi^*\mathcal{O}(N) \in P(0,\phi)$, where $\phi < 1$ is the maximum of the phases of $Z(\mathcal{O}_{p_i}[-1])$, $i = 1, \ldots, n$ and of $Z(\pi^*\mathcal{O}(n))$. If in addition $a < 1-\phi$ then we get $\pi^*\mathcal{O}(N)[1] \in P(1, 2-a] \subset P(-1-a, 2-a]$ as required. $\square$

Since the subset U of Theorem IV.2.6 contains the standard stability, note that this implies that all stabilities in the orbit of the standard stability condition are limit points of stabilities glued from exceptional collections, up to the action of $\widetilde{\mathrm{GL}}(2,\mathbb{R})$.

For the rest of this chapter we assume that X is an elliptic curve, $\mathcal{D}$ denotes the bounded derived category of $\mathbb{Z}_2$-equivariant coherent sheaves on X, and p_1, p_2, p_3, p_4 are the four ramification points of the morphism $X \to \mathbb{P}^1$.

Notation VI.0.3. Let $\mathcal{E} = \{E_1, E_2, \cdots, E_6\}$ be an exceptional collection of sheaves in $\mathcal{D}$. Define $\Gamma_{\mathcal{E}}$ to be the subset of $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D})$ consisting of all stability conditions σ for which there exist $g \in \widetilde{\mathrm{GL}}(2,\mathbb{R})$ and integers $n_1, n_2, \cdots, n_6$ such that $\sigma \cdot g$ is glued from the exceptional collection $\{E_1[n_1], E_2[n_2], \cdots, E_6[n_6]\}$. Set

$$\Gamma = \bigcup_{\mathcal{E}\in\mathrm{Coll}(\mathcal{D})} \Gamma_{\mathcal{E}}.$$

Lemma VI.0.4. *For any exceptional collection of sheaves $\mathcal{E}$, let $\Gamma'_{\mathcal{E}}$ be the $\widetilde{\mathrm{GL}}(2,\mathbb{R})$-orbit of the subset of all $\sigma = (Z, \mathcal{P}) \in \Gamma_{\mathcal{E}}$ glued along any* Ext*-exceptional collection of the form*

$$\{E_1[n_1], E_2[n_2], \cdots, E_6[n_6]\},$$

satisfying the condition that $\phi(E_i[n_i]) \in (0,1)$, for each i. Then $\Gamma'_{\mathcal{E}} = \Gamma_{\mathcal{E}}$.

Proof. Fix $\sigma = (Z, \mathcal{P}) \in \Gamma_{\mathcal{E}}$ and let $\phi = \min_i(\phi(E_i[n_i]))$. Since $\mathcal{P}(0,1]$ is generated by the exceptional collection, it follows that $\mathcal{P}(0,\phi) = \{0\}$. The central charge of the rotated stability $R_{\frac{\phi}{2}}\sigma$ sends each of the exceptional objects $E_i[n_i]$ into the strict upper halfplane. Hence $\sigma \in \Gamma'_{\mathcal{E}}$. □

Lemma VI.0.5. *For any exceptional collection of sheaves $\mathcal{E}$, $\Gamma_{\mathcal{E}}$ is open and connected.*

Proof. This is proved in [11] as Lemma 3.19. □

When $\mathcal{E}$ and $\mathcal{F}$ are exceptional collections related by a mutation we can construct a sequence of stability conditions in $\Gamma_{\mathcal{F}}$ approaching $\Gamma_{\mathcal{E}}$.

Lemma VI.0.6. *Suppose $\mathcal{E} = \{E_1, E_2, \cdot, E_6\}$ is a full exceptional collection of sheaves. Fix i, and set $\mathcal{F} = L_i\mathcal{E}$, a left mutation of $\mathcal{E}$ at the i^{th} position. There exist integers $n_1, \cdots, n_6, m_1, \cdots, m_6$ depending only on $\mathcal{E}$ and $\mathcal{F}$ such that for any $\epsilon > 0$ there exist stability conditions $\sigma_{\mathcal{E}}$ and $\sigma_{\mathcal{F}}$ with hearts*

$$H_{\mathcal{E}} := \langle E_1[n_1], E_2[n_2], \cdots, E_6[n_6]\rangle$$
$$H_{\mathcal{F}} := \langle F_1[m_1], F_2[m_2], \cdots, F_6[m_6]\rangle$$

satisfying $d(\sigma_{\mathcal{E}}, \sigma_{\mathcal{F}}) < \epsilon$.

Proof. Fix $\epsilon > 0$ and denote by L the object $L_{E_i}E_{i+1}$. Choose integers n_j for $j = 1, 2, \cdots, 6$ satisfying

$$n_j >> n_{j+1}, \quad j < i \text{ and } i < j,$$
$$n_i = 0,$$
$$n_{i+1} = \begin{cases} 0 & \text{if } \mathrm{Hom}_{\mathcal{D}}(E_i, E_{i+1}) = 0 \\ -1 & \text{if } \mathrm{Hom}_{\mathcal{D}}(E_i, E_{i+1}) \neq 0 \end{cases}$$

The collection $\{E_1[n_1], E_2[n_2], \cdots, E_6[n_6]\}$ is clearly Ext-exceptional. We claim that there exist integers $m_1, \cdots, m_6$ such that the collection

$$\{F_1[m_1], F_2[m_2], \cdots, F_6[m_6]\}$$

is Ext-exceptional and the containment relations

$$H_{\mathcal{F}} \subset [H_{\mathcal{E}} \cup H_{\mathcal{E}}[1]] \qquad H_{\mathcal{E}} \subset [H_{\mathcal{F}} \cup H_{\mathcal{F}}[-1]]$$

are satisfied.

If $\chi(E_i, E_{i+1}) = 0$, then $L \cong E_{i+1}[-1]$ and the collection

$$\{F_1[n_1], \cdots, F_{i-1}[n_{i-1}], E_{i+1}, E_i, F_{i+1}[n_{i+1}], \cdots, F_6[n_6]\}$$

is Ext-exceptional. Moreover, $H_{\mathcal{F}} = H_{\mathcal{E}}$ since $F_j \cong E_j$ for $j \neq i, i+1$.

If $\chi(E_i, E_{i+1}) > 0$, there is an exact triangle

$$E_{i+1}[-1] \to L \to \mathrm{Hom}_{\mathcal{D}}(E_i, E_{i+1}) \otimes E_i \to E_{i+1}.$$

This implies that $\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(L[1], E_i) = 0$, so the collection

$$\{F_1[n_1], \cdots, F_{i-1}[n_{i-1}], L[1], E_i, F_{i+1}[n_{i+1}], \cdots, F_6[n_6]\}$$

is Ext-exceptional. Recalling that $H_{\mathcal{E}}$ and $H_{\mathcal{F}}$ are generated from the respective exceptional collections, it is an easy consequence of the exact triangle above that $H_{\mathcal{F}} \subset [H_{\mathcal{E}} \cup H_{\mathcal{E}}[1]]$ and $H_{\mathcal{E}} \subset [H_{\mathcal{F}} \cup H_{\mathcal{F}}[-1]]$.

If $\chi(E_i, E_{i+1}) < 0$, there is an exact triangle

$$E_{i+1}[-1] \to L \to \mathrm{Ext}^1_{\mathcal{D}}(E_i, E_{i+1}) \otimes E_i[-1] \to E_{i+1}.$$

This implies that $\mathrm{Hom}_{\mathcal{D}}^{-1}(L, E_i) \cong \mathrm{Hom}_{\mathcal{D}}(E_i, E_i) \neq 0$, and so the collection

$$\{F_1[n_1], \cdots, F_{i-1}[n_{i-1}], F_i[2] = (L_{E_i}E_{i+1})[2], F_{i+1} = E_i, F_{i+2}[n_{i+2}], \cdots, F_6[n_6]\}$$

is Ext-exceptional. From the exact triangle we again get that the containment relation $H_{\mathcal{F}} \subset [H_E \cup H_E[1]]$ and $H_E \subset [H_F \cup H_F[-1]]$ is satisfied.

Define stability functions $Z_{\mathcal{E}}, Z_{\mathcal{F},\delta}$ for $0 < \delta \leq \frac{1}{4}$ on the hearts $H_{\mathcal{E}}$ and $H_{\mathcal{F}}$ by

$$\begin{aligned} Z_{\mathcal{E}}(E_j[n_j]) = Z_{\mathcal{F},\delta}(F_j[n_j]) &= -1 + \sqrt{-1}, \quad j < i-1 \text{ or } i+1 < j \\ Z_{\mathcal{E}}(E_i[n_i]) &= \exp(\pi\sqrt{-1}(1-\delta)) \\ Z_{\mathcal{F},\delta}(F_{i+1}[n_i]) &= -1, \\ Z_{\mathcal{E}}(E_{i+1}) &= -2, \\ Z_{\mathcal{F},\delta}(L) &= (2 \pm 1)\exp(\pi\sqrt{-1}\delta) \end{aligned}$$

where the sign in $Z_{\mathcal{F},\delta}(L)$ is chosen to be positive if $\mathrm{Hom}_{\mathcal{D}}(E_i, E_{i+1}) = 0$ and negative if $\mathrm{Hom}_{\mathcal{D}}(E_i, E_{i+1}) \neq 0$. With these choices, the two stability functions agree at $F_j, j \neq i, i+1$, have the same mass at $L = F_i[m_i]$ and $F_{i+1}[m_{i+1}] = E_i[n_i]$, but differ at these both by an angle of phase δ. Since the hearts $H_{\mathcal{E}}$ and $H_{\mathcal{F}}$ are both of finite length, $Z_{\mathcal{E}}$ and $Z_{\mathcal{F},\delta}$ have the HN-property on the respective hearts and there exist locally finite stability conditions $\sigma_{\mathcal{E}} = (Z_{\mathcal{E}}, P_{\mathcal{E}})$ and $\sigma_{\mathcal{F},\delta} = (Z_{\mathcal{F},\delta}, P_{\mathcal{F},\delta})$. By construction, $P_{\mathcal{F},\delta}(0,1] \subset P_{\mathcal{E}}[0,1]$ and $P_{\mathcal{E}}(0,1] \subset P_{\mathcal{F},\delta}(0, 1+\delta]$. If E_i and E_{i+1} are orthogonal, that is if

$$\mathrm{Hom}_{\mathcal{D}}^{\bullet}(E_i, E_{i+1}) = 0,$$

then it is easy to see that we may redefine $Z_{\mathcal{E}}$ so that $Z_{\mathcal{E}} = Z_{\mathcal{F},\delta}$ and the result follows trivially. We assume in what follows that the pair (E_i, E_{i+1}) is not orthogonal.

Let $\mathcal{A}_{i-1} = \langle E_1, \cdots, E_{i-1}\rangle$, $\mathcal{A}_{i,i+1} = \langle E_i, E_{i+1}\rangle$, and $\mathcal{A}_6 = \langle E_{i+2}, \cdots, E_6\rangle$. Since $\mathcal{A}_{i,i+1} = \langle F_i, F_{i+1}\rangle$, both $\sigma_{\mathcal{E}}$ and $\sigma_{\mathcal{F},\delta}$ are obtained by gluing along the semiorthogonal decomposition

$$\langle \mathcal{A}_{i-1}, \mathcal{A}_{i,i+1}, \mathcal{A}_6 \rangle.$$

Denote by $\sigma_{\mathcal{E},i}$ and $\sigma_{\mathcal{F},\delta,i}$, respectively, the restrictions of these stabilities to $\mathcal{A}_{i,i+1}$. The category $\mathcal{A}_{i,i+1}$ is naturally equivalent to the bounded derived category $\mathcal{D}(\mathbb{P}^1)$ of coherent sheaves on $\mathbb{P}^1$ by an autoequivalence sending $E_i \to \mathcal{O}_{\mathbb{P}^1}$ and $E_{i+1}[n_{i+1}] \to \mathcal{O}_{\mathbb{P}^1}(1)[-1]$. Under this autoequivalence, $F_i[n_i]$ is sent to $\mathcal{O}_{\mathbb{P}^1}(-1)[1]$ since $L_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}(1)[-1]) \cong \mathcal{O}_{\mathbb{P}^1}(-1)[1]$. The naturally induced stability conditions on $\mathcal{D}(\mathbb{P}^1)$, which we also denote by $\sigma_{\mathcal{E},i}$ and $\sigma_{\mathcal{F},\delta,i}$ satisfy:

$$\phi_{\sigma_{\mathcal{E},i}}(\mathcal{O}_{\mathbb{P}^1}(1)) < \phi_{\sigma_{\mathcal{E},i}}(\mathcal{O}_{\mathbb{P}^1}) < \phi_{\sigma_{\mathcal{E},i}}(\mathcal{O}_{\mathbb{P}^1}(1)) + 1,$$

$$\phi_{\sigma_{\mathcal{F},\delta,i}}(\mathcal{O}_{\mathbb{P}^1}) < \phi_{\sigma_{\mathcal{F},\delta,i}}(\mathcal{O}_{\mathbb{P}^1}(-1)) < \phi_{\sigma_{\mathcal{F},\delta,i}}(\mathcal{O}_{\mathbb{P}^1}) + 1.$$

By Proposition 4.4 of [11], for each $n \in \mathbb{N}$, $\mathcal{O}_{\mathbb{P}^1}(n)$ is semistable in $\sigma_{\mathcal{E},i}$ and $\sigma_{\mathcal{F},\delta,i}$. Noting that the central charges of these two stabilities differ by a rotation through an angle of phase δ, it follows that

$$\sigma_{\mathcal{E},i} = R_\delta \sigma_{\mathcal{F},\delta,i},$$

where, we recall, R_δ denotes the element of $\widetilde{\mathrm{GL}}(2,\mathbb{R})$ that acts by rotations through an angle of phase δ. This implies that $d(\sigma_{\mathcal{E},i}, \sigma_{\mathcal{F},\delta,i}) = \delta$.

Finally by continuity of gluing, for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d(\sigma_{\mathcal{E}}, \sigma_{\mathcal{F},\delta}) < \epsilon.$$

□

Proposition VI.0.7. Γ *is open and connected.*

Proof. It is an immediate consequence of Lemma VI.0.5 that Γ is open. In the Lemma above we constructed for a fixed $\sigma \in \Gamma_{\mathcal{E}}$ and each $n \in \mathbb{N}$ a stability condition $\sigma_n \in \Gamma_{\mathcal{F}}$ such that $d(\sigma, \sigma_n) < \frac{1}{n}$. Since $\Gamma_{\mathcal{E}}$ is open, for some n, $\sigma_n \in \Gamma_{\mathcal{E}} \cap \Gamma_{\mathcal{F}}$. This proves that Γ

is connected. □

Theorem VI.0.8. *Let* $U \subset \mathrm{Stab}_{\mathcal{N}}(D_{\mathbb{Z}_2}(X))$ *be the subset defined in Theorem IV.2.6. The set* $\Gamma \cup U$ *is open and connected.*

Proof. Both Γ and U are open and connected and by Proposition VI.0.2, $\Gamma \cap U$ is nonempty. Moreover, $\Gamma \cap U$ is dense in U since for any $\epsilon > 0$ and $\sigma \in U$ there exists a $\sigma' \in \Gamma \cap U$ satisfying the assumptions of Proposition VI.0.2 with $d(\sigma, \sigma') < \epsilon$. Therefore $\Gamma \cap U$ is connected, hence $\Gamma \cup U$ is also connected. It is clear that the union is open. □

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