STABILIZATION OF
CHROMATIC
FUNCTORS

by

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We study the Bousfield localization functors known as L_n^f, as described in [MahS] . In particular we would like to understand how they interact with suspension and how they stabilize.

We prove that suitably connected L_n^f-acyclic spaces have suspensions which are built out of a particular type n space, which is an unstable analog of the fact that L_n^f-acyclic spectra are built out of a particular type n spectrum. This theorem follows Dror-Farjoun's proof in the case $n = 1$ with suitable alterations. We also show that L_n^f applied to a space stabilizes in a suitable way to L_n^f applied to the corresponding suspension spectrum.

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CHAPTER I

INTRODUCTION

The notion of localization is an important one in algebraic topology, and can be thought of as analogous to the classical algebraic localization of a module. The general idea is described in [Dw]. Given a category $\mathcal{C}$, and a subcategory $\mathcal{E} \subset \mathcal{C}$, one wants to functorially modify all of the objects and morphisms in the category so that the morphisms in $\mathcal{E}$ become isomorphisms. The modification of an object in this sense is called the *localization* of that object. This is not possible for all pairs $(\mathcal{C}, \mathcal{E})$, but when it is we say the pair *has good localizations.* One situation where this notion is familiar is the localization of the category of R-modules with respect to a multiplicatively closed set.

Given a ring R, and a multiplicative subset $S \subset R$, take $\mathcal{C}$ to be the category of R-modules. A module M is *S-torsion* if for each $x \in M$ there is an element $s \in S$ such that $sx = 0$. Then, take $\mathcal{E}$ to be all of the objects of $\mathcal{C}$, along with all morphisms f such that $\ker(f)$ and $\operatorname{coker}(f)$ are *S-torsion.* The pair $(\mathcal{C}, \mathcal{E})$ has good localizations, and the localization of a module M is its classical algebraic localization $S^{-1}M$. In general, localization can be thought of as a functor from $\mathcal{C}$ to itself.

One can see this idea at work in [Se], where Serre lays the foundation for modern topological localization by studying what he calls classes of abelian groups. For instance, one of the classes of abelian groups described is the class of finite abelian groups with order relatively prime to some prime p. A group homomorphism is considered an isomorphism with respect to this class if both the kernel and cokernel lie in the class. A homomorphism such as this is considered an isomorphism "mod-p." Serre considers $\mathcal{C}$ to be the category of spaces, and $\mathcal{E}$ to be maps between spaces such that on homology such maps induce isomorphisms mod- p. He proves a series of theorems which are reinterpretations of classical theorems in topology, now seen as a class of theorems, one for each class of abelian groups. For instance, he proves a mod-p version of the Whitehead theorem, which states: Given simply connected topological spaces X and Y with finitely generated homology groups and $f : X \to Y$ inducing an isomorphism in π_2, if $f : H_i(X; \mathbb{Z}/p\mathbb{Z}) \to H_i(Y; \mathbb{Z}/p\mathbb{Z})$ is an isomorphism for all $i < n$, then $f : \pi_i(X) \to \pi_i(Y)$ is an isomorphism mod-p for all $i < n$.

The original conceptual idea of localization in the topological category involved inverting maps between spaces which induced isomorphisms in regular homology with coefficients in some group G. Quillen's closed model category structure allows one to do this [Q], but we'll take the approach of Bousfield in [B1]. In fact, Bousfield extends this idea to any generalized homology theory, where a generalized homology theory, h_*, is a suitable functor from the category of spaces to the category of graded groups.

Today, localization in the topological category usually refers to Bousfield localization with respect to some generalized homology theory. In [Rav], Ravenel considers Bousfield localization with respect to several interesting homology theories, among them, the Morava K-theories $K(n)_*$, and the Johnson-Wilson theories $E(n)_*$. (These homology theories and the spectra that represent them are carefully defined in [Rav2]). He lists seven conjectures related to these ideas and their connection to stable homotopy in general. All but one of the conjectures have been proven. The unproven conjecture is often referred to as the "telescope conjecture."

$L_n^f(-)$ is another chromatic localization functor described, for instance, in [Rav3], [B4], [MahS], and [Mil], but I'll use the description from Section 3 of [MahS]. It's worth noting that the superscript f stands for "finite", not some map f. The reference to finite is because L_n^f-acyclic spectra are direct limits of finite L_n^f-acyclic spectra.

Definition I.1. *Let X be a pointed space or a CW-spectrum. X is* type n if $\tilde{K}(n)_*(X) \neq 0$ and $\tilde{K}(i)_*(X) = 0$ for $i < n$.

Definition I.2. *Let X be a pointed space or a CW-spectrum, with $f : \Sigma^d X \to X$ a self-map. f is a v_n* map *if $K(n)_*(f)$ is an isomorphism and $K(i)_*(f) = 0$ for all $i \neq n$.*

Choose, for each $0 \leq i \leq n$, a finite type-i spectrum $F(i)$. It is proven in Theorem 9 of [HSm] that such spectra admit a v_n-self map $\alpha : \Sigma^k F(i) \to F(i)$ for some k. Let, $T(i)$ be the telescope of this self map

$$\operatorname{hocolim}\left(F(i) \to \Sigma^{-k} F(i) \to \Sigma^{-2k} F(i) \to \cdots\right).$$

Then, $L_n^f(-)$ is localization with respect to the homology theory defined by

$$\bigvee_i T(i).$$

It is an easy consequence of [HSm] (see, for example, [MahS], Lemma 2.1) that the resulting functor is independent of the choices of $F(i)$. One statement of the telescope conjecture is that the spectra which have contractible localizations with respect to $E(n)_*$ are the same spectra for which $L_n^f(-)$ is contractible.

I am interested in studying the effect of suspension and stabilization on chromatic localizations. For instance, there is a map

$$\Phi_i : \Sigma^i L_E X \to L_E \Sigma^i X, \tag{I.1}$$

$1 \leq i \leq \infty$, where L_E is localization with respect to some chromatic homology theory, thought of as a functor in the category of spaces (unless $i = \infty$, in which case Φ_i is a map of spectra and we can think of this map as comparing unstable and stable localization). In an effort to draw a connection between the stable and unstable settings, I've proven the following homotopy equivalence of spectra:

Theorem IV.6.

$$\varinjlim_i \Sigma^{-i}\Sigma^\infty L_n^f \Sigma^i X \xrightarrow{\simeq} L_n^f \Sigma^\infty X.$$

A similar result holds for spaces, of the form

$$\varinjlim_i \Omega^i \Sigma^\infty L_n^f \Sigma^i X \xrightarrow{\simeq} \Omega^\infty L_n^f \Sigma^\infty X.$$

The above equivalences are both corollaries of the following lemma: Given a CW-spectrum $X = \{X_0, X_1, X_2, \cdots\}$,

$$L_n^f X \simeq \{L_n^f X_0, L_n^f X_1, L_n^f X_2, \cdots\}.$$

The proof relies on the fact that L_n^f acyclics are direct limits of finite L_n^f acyclics. This result is described in chapter III.

The chromatic homology theories we're dealing with are non-connective, meaning the spectra they are represented by have non-trivial homotopy groups in negative dimension. Therefore, when $i = \infty$, the homotopy groups of the fiber of Φ_i

may not be bounded below, as the domain spectrum is a suspension spectrum, hence connective, but the target spectrum will often not be. Under these circumstances, analyzing Φ_i is quite difficult. However, the situation can be trivial if the homology theory is connective. For instance, when considering localization with respect to regular homology Φ_i is an equivalence for all $1 \leq i \leq \infty$. The difficulty in analyzing Φ_∞ is why we use the colimit in Theorem IV.6. We hoped this would be an easier stabilization of Φ_i to consider.

Since analyzing acyclics is an important way to compare localization functors, one question that has proven to be worth investigating is whether all (stable or unstable) acyclics with respect to a given homology theory are built out of a "generating space" or spaces. If X is built out of Y, we say X is Y-cellular.

Let $\mathcal{C}_*$ be the category of pointed topological spaces, and let $\mathcal{A}$ be a set of spaces in $\mathcal{C}_*$. The following definition can be found in a variety of places, for instance Definition 5.1 of [Ch].

Definition I.3. *The class of $\mathcal{A}$-cellular spaces is the smallest class of spaces in $\mathcal{C}_*$ such that*

1. *all spaces in $\mathcal{A}$ are $\mathcal{A}$-cellular;*
2. *if X and Y are weakly homotopy equivalent and X is $\mathcal{A}$-cellular, then so is Y;*
3. *if $F : I \to \mathcal{C}_*$ is a diagram such that each F_i is $\mathcal{A}$-cellular, then hocolim$F$ is $\mathcal{A}$-cellular.*

(A similar definition holds for any model category, replacing weakly homotopy equivalent with weakly equivalent). One consequence of this definition is that a contractible space is A-cellular for any A since it is weakly homotopy equivalent to a point, which is the homotopy colimit of the empty diagram. Therefore ΣA is A-cellular as it is the homotopy colimit of the diagram $* \leftarrow A \rightarrow *$.

In Chapter 8 of [DF], Dror Farjoun proves that simply connected rational acyclics are $\mathcal{A}$-cellular, where

$$\mathcal{A} = \{M^2(p) : p \text{ is a prime}\}.$$

In other words, simply connected p-local $E(0)_*$-acyclics are $M^2(p)$-cellular.

In [A1], Adams produced for each prime p a self-map $v_1 : \Sigma^{k_p} M_p \longrightarrow M_p$ of the mod(p) Moore spectrum. Here $k_p = 2p-2$ if p is odd, $k_2 = 8$, and M_p is the cofiber of the degree p map $p : S^0 \longrightarrow S^0$. He showed that this map induced an isomorphism in complex K-theory. The cofiber of this map is called $V(1)$. It's a consequence of [HSm] that p-local K-theory acyclic spectra are $\mathcal{A}$-cellular for $\mathcal{A} = \{\Sigma^k V(1) : k \in \mathbb{Z}\}$.

In [CoN], it is shown that there is a map $\alpha : \Sigma^{2p-2} M^3(p) \longrightarrow M^3(p)$, for odd primes p, where $M^3(p) = S^2 \bigcup_p e^3$. This map also induces an isomorphism in complex K-theory and, in fact, is a particular desuspension of v_1. We'll refer to the cofiber of α as $W(1)$. This space is intended to be an unstable analog of $V(1)$. In fact, $\Sigma^\infty W(1) \simeq \Sigma^2 V(1)$. In Corollary B.5 of [DF], Dror Farjoun proves that simply connected, p-local, K-theory acyclics are, after a suspension, $W(1)$-cellular. This result can be stated in the following way: If X is p-local, simply connected, and $L_1(X) \simeq *$, then ΣX is $W(1)$-cellular.

Dror Farjoun suggested in [DF] that similar techniques could be used to generalize this result to L_n^f-acyclic spaces. To do this, one should define $W(n)$ to be a minimally connected type n space. We do this below.

If X is a finite p-local space of type n, the work in [HSm] guarantees the existence of a v_n map

$$\alpha_n : \Sigma^{d+i} X \to \Sigma^i X$$

for some i and where d is a multiple of $2p^n - 2$. Set $W(-1) = S^1$.

Definition I.4. *For each $n \geq 0$, choose a finite p-local type $n+1$ space $W(n)$ satisfying:*

1. *$W(n) = \operatorname{Cof}\left(\alpha_n : \Sigma^{d_n+i_n} W(n-1) \to \Sigma^{i_n} W(n-1)\right)$, where α_n is a v_n-map.*

2. *i_n is chosen to be as small as possible and d_n is chosen to be as small as possible for the given i_n.*

We'll choose $W(0)$ to be the cofiber of the degree p map from S^1 to itself, so $W(0) = M^2(p)$. Then, $d_1 = 2p - 2$ and $i_1 = 1$ and $W(1) = \operatorname{cof}(\alpha)$. The following result is proved in the next chapter.

Corollary III.6. *If X is p-local and simply connected and $L_n^f(X) \simeq *$, then $\Sigma^M X$ is $W(n)$-cellular, where*

$$M = \sum_{k=1}^{n} i_k + n - 1.$$

CHAPTER II

PRELIMINARIES

In this chapter we'll describe the localization, nullification and cellularization functors. These are functors either from the homotopy category of pointed spaces to itself, or from the homotopy category of CW-spectra to itself. We'll begin by carefully describing localization with respect to a homology theory. See, for instance, [B1].

Definition II.1. *A map $f : A \longrightarrow B$ is an E_*-*equivalence *if it induces an isomorphism $f_* : E_*(A) \cong E_*(B)$.*

Definition II.2. *A space X is E-*local *if, given any E_*-equivalence $f : A \longrightarrow B$, the map $map_*(f, X) : map_*(B, X) \longrightarrow map_*(A, X)$ is an equivalence.*

Theorem II.3 (Bousfield). *There is a functor $L_E(-)$ such that L_EX is E-local, and a natural transformation from the identity functor to $L_E(-)$ which is an E_*-equivalence $\mu : X \longrightarrow L_EX$ for all spaces X satisfying*

1. *for any map, $f : X \longrightarrow Y$ inducing $E_*(X) \cong E_*(Y)$, there is a unique map $r : Y \longrightarrow L_E(X)$ with $r \circ f = \mu$, and*
2. *for any map, $g : X \longrightarrow Y$, where Y is E-local, there is a unique map $s : L_E(X) \longrightarrow Y$ with $s \circ \mu = g$.*

One consequence of these properties is $L_E(L_E(X)) \simeq L_E(X)$, that is, this functor is idempotent. In fact, Dror Farjoun, in [DF], calls functors such as these *coaugmented idempotent functors.* Coaugmented refers to the existence of the map $\mu : X \to L_E(X)$.

Another coaugmented idempotent functor that we'll use is the nullification functor. This is carefully described in [B3], 2.8.

Definition II.4. *Given a space, C, we say X is C-*null *if $map_*(A, X) \simeq *$.*

Definition II.5. *A map $f : A \to B$ is a C-*null equivalence *if it induces an equivalence $map_*(B, Y) \simeq map_*(A, Y)$, for all C-null spaces Y.*

Theorem II.6 (Bousfield). *There is a functor $P_C(-)$ such that $P_C X$ is C-null, and a natural transformation from the identity functor to $P_C(-)$ which is a C-null equivalence $\mu : X \to P_C X$ for all spaces X satisfying*

1. *for any C-null equivalence, $f : A \to B$, there is a unique map $r : B \to P_C(A)$ with $r \circ f = \mu$, and*
2. *given $g : X \to Y$, with Y C-null, there is a unique map $s : P_C(X) \to Y$ with $s \circ \mu = g$.*

Given an arbitrary homology theory, $E_*(-)$, represented by a spectrum E, we can construct P_C so that it is a reasonable approximation to L_E. It's proven in [B6] that there is some infinite cardinal λ so that all E-acyclic spaces are colimits of directed systems of E-acyclic subspaces of cardinality $\leq \lambda$. So, let $C = \bigvee_i X_i$, be an infinite wedge of E-acyclic spaces, one of each homotopy type such that $\#(X_i) \leq \lambda$. Here, $\#(X)$ is the number of cells of the minimal CW-complex structure that X admits. The following proposition is proven in [B6]. Since the proof is short, we include it here.

Proposition II.7 (Bousfield). *With C as described above,*

$$P_C(X) \simeq * \iff L_E(X) \simeq *.$$

Proof. Assume $P_C(X) \simeq *$. First, we'll notice that the map $\mu : X \to P_C(X)$ is an E_*-equivalence by describing the construction of $P_C(X)$, as in [B3]. Let γ be the first limit ordinal with cardinality greater than C, and inductively construct an increasing sequence of CW-complexes

$$X = X(0) \subset X(1) \subset \cdots \subset X(\alpha) \subset X(\alpha+1) \subset \cdots \subset X(\gamma)$$

indexed by the ordinals $\leq \gamma$ as follows. Given $X(\alpha)$, choose a set of maps $\{g : \Sigma^i C \to X(\alpha)\}_{g \in G(i)}$ for each $i \geq 0$ representing all the pointed homotopy classes from $\Sigma^i C$ to $X(\alpha)$, and let $X(\alpha+1)$ be the homotopy pushout of the diagram

$$\begin{tikzcd} \bigvee_{i\geq 0} \bigvee_{g\in G(i)} \Sigma^i C \arrow[r] \arrow[d] & X(\alpha) \\ * & \end{tikzcd}$$

Also, let

$$X(\beta) = \bigcup_{\alpha<\beta} X(\alpha)$$

for each limit ordinal β. Then, $X(\gamma) = P_C(X)$. Notice that in the pushout diagram the vertical map is between E_*-acyclic spaces, so it is an E_*-equivalence. This means that the map $X(\alpha) \to X(\alpha+1)$ is also an E_*-equivalence. Thus, $X \to P_C(X)$ is an E_*-equivalence. Therefore

$$\tilde{E}_*(X) \cong \tilde{E}_*(P_C(X)) \cong 0$$

as we're assuming that $P_C(X)$ is contractible. So, X is E-acyclic and $L_E(X) \simeq *$.

To prove the other implication, we assume that X is E-acyclic. This means that X is the colimit of a directed system of E-acyclic subspaces which are included in the wedge of acyclics C. Let X' be one of these acyclic subspaces. Since X' appears as a wedge summand of C, the null-homotopic map $X' \xrightarrow{X} (1)$ factors through X, which means the map $X \to X(1)$ is the zero map in homotopy. We're using here that X is the colimit of a directed system. This can be repeated for all $X \to X(\alpha)$, so $P_C(X)$ is the direct limit of a sequence of maps all of which are the zero map in homotopy, so $P_C(X) \simeq *$. $\square$

Despite having the same acyclic spaces, the functors won't necessarily be equivalent. However, since $X \to P_C(X)$ is an E_*-equivalence, there is a natural transformation $P_C(X) \to L_E(X)$. The following is Lemma 2.1 in [Tai]. Recall that a group G is *perfect* if its abelianization is trivial.

Theorem II.8 (Tai). *Let X be a* CW-*complex with perfect fundamental group. Then*

$$X^+ \xrightarrow{\simeq} P_C(X) \xrightarrow{\simeq} L_{H\mathbb{Z}}(X).$$

Here X^+ is the Quillen plus construction, defined many places, including in [A3].

The final functor we'll describe here is the cellularization functor. This is described carefully in Chapter 2 of [DF].

Definition II.9. *The class of $\mathcal{A}$-cellular spaces is the smallest class of spaces in $\mathcal{C}_*$ such that*

1. *all spaces in $\mathcal{A}$ are $\mathcal{A}$-cellular;*

2. *if X and Y are weakly homotopy equivalent and X is $\mathcal{A}$-cellular, then so is Y;*

3. *if $F : I \longrightarrow \mathcal{C}_*$ is a diagram such that each F_i is $\mathcal{A}$-cellular, then hocolim$F$ is $\mathcal{A}$-cellular.*

Here, $\mathcal{C}_*$ is either the category of pointed spaces or the category of CW-spectra.

Definition II.10. *A map $f : B \longrightarrow C$ is an $\mathcal{A}$-*cellular equivalence *if it induces an equivalence $map_*(A, B) \simeq map_*(A, C)$, for all $A \in \mathcal{A}$.*

Theorem II.11 (Dror-Farjoun). *There is a functor $C_{\mathcal{A}}(-)$ such that $C_{\mathcal{A}}X$ is $\mathcal{A}$-cellular, and a natural transformation from $C_{\mathcal{A}}(-)$ to the identity functor which is an $\mathcal{A}$-cellular equivalence $\mu : C_{\mathcal{A}}X \longrightarrow X$ for all spaces X satisfying*

1. *for any $\mathcal{A}$-cellular equivalence $f : Y \longrightarrow X$ there is a unique map $s : C_{\mathcal{A}}(X) \longrightarrow Y$ satisfying $f \circ s = \mu$.*
2. *given $g : Y \longrightarrow X$ with Y $\mathcal{A}$-cellular, there is a unique $r : Y \longrightarrow C_{\mathcal{A}}(X)$ with $\mu \circ r = g$.*

Given the map $\mu : C_A X \longrightarrow X$, and the fact that $C_A(C_A(X)) \simeq C_A(X)$, this functor is referred to as an *augmented idempotent functor*. A useful result relating C_A and P_A is proven in [DF], 3.B.2:

Lemma II.12 (Dror-Farjoun). *Let X and A be pointed CW-complexes.*

1. *If X is A-cellular, then $P_A(X) \simeq *$.*
2. *If $P_{\Sigma A}X \simeq *$, then X is A-cellular.*

In this paper, if we want to prove that a space, X, is A-cellular, we'll show that $P_{\Sigma A}(X)$ is contractible. Unfortunately, it is possible for $P_A(X)$ to be contractible without X being A-cellular, and Chacholski provides examples of this in

[Ch], page 35. For instance, $P_{\Omega S^{n+1}} S^n \simeq *$, however S^n is only ΩS^{n+1}-cellular when n is 1,3, or 7. However, if $P_A(X) \simeq *$, this does imply that $P_{\Sigma A}(\Sigma X) \simeq *$, which means that ΣX is A-cellular.

CHAPTER III

GENERATING OBJECTS FOR CHROMATIC HOMOLOGY

As mentioned above, L_n^f-acyclic spectra are direct limits of finite L_n^f acyclics, and in fact, this motivates the "f." Therefore, L_n^f-acyclic spectra are $\mathcal{C}$-cellular if $\mathcal{C}$ is the collection of all finite L_n^f-acyclics.

Let $\mathcal{A} = \{\Sigma^k F_{n+1} : k \in \mathbb{Z}\}$, with F_{n+1} is any finite, p-local, type $(n+1)$ spectrum. The following proposition is an easy consequence of [HSm] and [MahS].

Proposition III.1. *Let X be a p-local L_n^f-acyclic spectrum. Then X is $\mathcal{A}$-cellular.*

Proof. To prove this, one needs the thick subcategory theorem, which is Theorem 7 of [HSm]. A full subcategory of finite p-local CW-spectra is thick if it is closed under retract, cofibration and weak equivalence. If X is in the subcategory, and Y is a retract of X or weakly equivalent to X, then Y is also in the subcategory. Closed under cofibration means if $A \to B \to C$ is a cofibration of spectra, and if any two of the spectra are in the subcategory, then so is the third. The thick subcategory theorem tells us that if a subcategory of finite p-local spectra is thick it must be $\mathcal{C}_r$ for some r, where $\mathcal{C}_r$ is the full subcategory of finite p-local $K(r-1)_*$-acyclics.

First, since a retract can be obtained as an infinite direct limit of the retraction followed by the inclusion, the class of $\mathcal{A}$-cellular spectra is closed under retracts. Secondly, the cofiber of a map $f : A \to B$ of spectra can be obtained as the homotopy colimit of the diagram $* \leftarrow A \to B$, so given a cofiber sequence of spectra, if any two of the spectra are $\mathcal{A}$-cellular so is the third. We already have that the class of $\mathcal{A}$-cellular spectra is closed under weak equivalence. Therefore, the subcategory of finite p-local $\mathcal{A}$-cellular spectra is a thick subcategory, so it is $\mathcal{C}_r$ for some r. However, $F_{n+1} \in \mathcal{C}_{n+1} \setminus \mathcal{C}_{n+2}$, so $r \leq n+1$. Also, there are no type n spectra in $\mathcal{A}$ so $r = n+1$. So we have that the subcategory of finite p-local $\mathcal{A}$-cellular spectra is exactly the subcategory of finite p-local $K(n)_*$-acyclic spectra.

So, given any p-local L_n^f-acyclic spectrum, it's built out of finite p-local L_n^f-acyclics. A finite p-local L_n^f-acyclic is a finite p-local $K(n)_*$-acyclic, and these are finite p-local $\mathcal{A}$-cellular spectra. It follows easily from the universal properties of the cellularization functor that if A is B-cellular and B is C-cellular then A is C-cellular. Therefore, p-local L_n^f-acyclic spectra are $\mathcal{A}$-cellular. $\square$

In the next section we prove analogous results for unstable L_n^f-acyclics.

III.1 Generating Spaces for Unstable Acyclics

The main goal in this section is to prove Corollary III.6, describing spaces, X, with $L_n^f(X) \simeq *$. First, we need a definition. Let X and A be pointed spaces, and $g : \Sigma^d A \to A$ a map of pointed spaces.

Definition III.2.

$$T_g X = \mathit{hocolim}\left(\mathit{map}_*(A, X) \to \mathit{map}_*(\Sigma^d A, X) \to \mathit{map}_*(\Sigma^{2d} A, X) \to \cdots\right),$$

where the maps defining the colimit are induced by $\Sigma^{di} g$, for $i \geq 0$.

Since mapping out of finite complexes commutes with homotopy colimits, we have

$$\Omega^d T_g \simeq \text{hocolim}\left(\text{map}_*(\Sigma^d A, X) \to \text{map}_*(\Sigma^{2d} A, X) \to \cdots\right) \simeq T_g X.$$

So we see that $T_g X$ is an infinite loop space.

Recall that $W(n)$ is defined as the homotopy cofiber of a v_n map

$$\alpha_n : \Sigma^{d_n + i_n} W(n-1) \to \Sigma^{i_n} W(n-1).$$

So we can consider $T_{\alpha_n} X$. In fact, we'll show that this space is L_n^f-local. To prove this, we'll construct a L_n^f-local spectrum, Y, such that $T_{\alpha_n} X \simeq \Omega^\infty Y$. This, combined with the fact that $\Omega^\infty(-)$ takes local spectra to local spaces, will give the desired result.

Lemma III.3. *$T_{\alpha_n} X$ is an L_n^f-local space.*

Proof. We use the construction $\Phi_v(X)$ from [K], 3.1, where X is a space and v is a self map of spaces $\Sigma^d B \to B$. Here $\Phi_v(X)_0 =$Map$_*(B, X)$ and $\Phi_v(X)_{di-k} = \Omega^k \text{Map}_*(B, X)$ for $i \geq 1$ and $0 \leq k < d$. If $k \neq 0$, the structure maps are the identity.

$$\begin{tikzcd}
\Phi_v(X)_{di-k} \arrow[r] \arrow[d, equal] & \Omega\Phi_v(X)_{di-(k-1)} \arrow[d, equal] \\
\Omega^k \text{Map}_*(B, X) \arrow[r] & \Omega\Omega^{k-1}\text{Map}_*(B, X)
\end{tikzcd}$$

When $k = 0$ and $i \geq 0$ the structure maps are induced by the self map v.

$$\begin{tikzcd}
\Phi_v(X)_{di} \arrow[r] \arrow[d, equal] & \Omega\Phi_v(X)_{d(i+1)-(d-1)} \arrow[d, equal] \\
\text{Map}_*(B, X) \arrow[r] & \Omega\Omega^{d-1}\text{Map}_*(B, X)
\end{tikzcd}$$

We're again using the fact that $\Omega^d \text{ Map }_*(B,X) \simeq \text{ Map }_*(\Sigma^d B, X)$. So, when $k=0$ and $i \geq 0$ our structure maps are of the form

$$\text{Map}_*(v,X) : \text{ Map }_*(B,X) \to \text{ Map }_*(\Sigma^d B, X).$$

Kuhn proves, in [K] Theorem 4.2, that when v is a v_n-self map and B is a finite, type n space, $\Phi_v(X)$ is $T(n)$-local. Since every L_n^f-equivalence is a $T(n)$-equivalence, it follows that $\Phi_v(X)$ is L_n^f-local.

If we can show $\Omega^\infty \Phi_v(X) \simeq T_v X$, this will imply that $T_v X$ is an L_n^f-local space. Taking $B = \Sigma^{i_n} W(n-1)$ and $v = \alpha_n$ we will have that $T_{\alpha_n} X$ is L_n^f-local.

Given an arbitrary spectrum, $X = \{X_1, X_2, ...\}$, $\Omega^\infty(X) \simeq \text{hocolim}\Omega^i X_i$. Applied to our situation, this yields

$$\Omega^\infty \Phi_v(X) \simeq \text{hocolim}(\Omega^d \text{ Map }_*(B,X) \to \Omega^{2d} \text{ Map }_*(B,X) \to \cdots)$$

I've omitted from the colimit the maps that are simply the identity. But the colimit above is equivalent to

$$\text{hocolim}(\text{ Map }_*(\Sigma^d B, X) \to \text{ Map }_*(\Sigma^{2d} B, X) \to \text{ Map }_*(\Sigma^{3d} B, X) \to \cdots) \simeq T_v X.$$

This shows that $T_{\alpha_n} X$ is an L_n^f-local space. ☐

Corollary III.6 follows from two theorems. The first of these demonstrates a connection between L_n^f-acyclics and T_{α_n}-acyclics. Recall the definition of $W(n)$:

$$W(n) = \text{Cof}\left(\Sigma^{d_n + i_n} W(n-1) \xrightarrow{\alpha_n} \Sigma^{i_n} W(n-1)\right).$$

Theorem III.4. *If X is N-connected, where $N = \sum_{j=1}^{n-1} d_j + \sum_{j=1}^{n} i_j + n + 1$, and $L_n^f(\Omega^k X) \simeq *$ for all $k \leq N$ then $T_{\alpha_n} X \simeq *$.*

We'll use the following fact several times in the proof of this theorem. If X is N-connected and Y is a finite cell complex with top cell in dimension $i < N$, then $\mathrm{map}_*(Y, X)$ is path-connected.

Proof. Given the below cofibration for $k \geq 2$,

$$\Sigma^{k-2} W(0) \to S^k \xrightarrow{p} S^k$$

we can map it into X yielding a fibration,

$$\Omega^k X \to \Omega^k X \to X^{\Sigma^{k-2} W(0)}$$

Notice that the top cell in $\Sigma^{k-2} W(0)$ is in dimension k so the space on the right is connected when $k \leq N$ because of the connectivity of X. In [DF](1.H.1), Dror Farjoun proves that if $F \to E \to B$ is a fibration with connected base and $\tilde{h}_*(F) = \tilde{h}_*(E) = 0$, then $\tilde{h}_*(B) = 0$, for any homology theory $h_*(-)$. Since, by hypothesis, $L_n^f(\Omega^k) \simeq *$, we see that $L_n^f(X^{\Sigma^{k-2} W(0)}) \simeq *$ for all k in the above range.

Similarly, we have cofibrations

$$\begin{array}{ccccc}
\Sigma^{a_1} W(1) & \to & \Sigma^{a_0} W(0) & \to & \Sigma^{a_0 - d_1} W(0) \\
\Sigma^{a_2} W(2) & \to & \Sigma^{a_1} W(1) & \to & \Sigma^{a_1 - d_2} W(1) \\
 & & \vdots & & \\
\Sigma^{a_{n-1}} W(n-1) & \to & \Sigma^{a_{n-2}} W(n-2) & \to & \Sigma^{a_{n-2} - d_{n-1}} W(n-2)
\end{array}$$

where $a_0 = k - 2$, $a_j = k - 2 - j - (i_1 + i_2 + \ldots + i_j) - (d_1 + d_2 + \ldots + d_j)$. These cofibrations are simply shifted versions of the ones that define the $W(j)$. One can check that each space on the left has top cell in dimension k. Now, we can map these cofibrations into X yielding fibrations, each one having a connected base for the same reason as above. Then, using successive applications of the Dror Farjoun result listed above, we see that:

$$L_n^f(X^{\Sigma^{a_1}W(1)}) \simeq *$$

for all $2 + i_1 + d_1 \leq k \leq N$, which means:

$$L_n^f(X^{\Sigma^{a_2}W(2)}) \simeq *$$

for all $4 + i_1 + i_2 + d_1 + d_2 \leq k \leq N$. Continuing this, we eventually get:

$$L_n^f(X^{\Sigma^{a_{n-1}}W(n-1)}) \simeq *$$

for all $N - i_n \leq k \leq N$. So, letting $k = N$ we get,

$$L_n^f(X^{\Sigma^{i_n}W(n-1)}) \simeq *$$

Since $T_{\alpha_n}X$ is L_n^f-local, any map $X^{\Sigma^{i_n}W(n-1)} \longrightarrow T_{\alpha_n}X$ factors through $L_n^f(X^{\Sigma^{i_n}W(n-1)})$, which we've just shown is contractible.

Now consider the natural maps arising from the telescope $T_{\alpha_n}X$:

$$b_0 : X^{\Sigma^{i_n}W(n-1)} \longrightarrow T_{\alpha_n}X$$

$$b_k : X^{\Sigma^{i_n+kd_n}W(n-1)} \longrightarrow T_{\alpha_n}X$$

We've just shown that b_0 is null-homotopic. In fact, b_k is null homotopic for all k, since b_k is identified with $\Omega^{kd_n}b_0$ under the identification of $\Omega^{d_n}T_{\alpha_n}X$ with $T_{\alpha_n}X$. These maps come from including the spaces from which the telescope is built into the telescope itself. It's a general fact that applying the colimit functor to the map of diagrams

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[drr] & X_1 \arrow[r] \arrow[dr] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[dl] & X_4 \arrow[r] \arrow[dll] & \cdots \arrow[dlll] \\
 & & \mathrm{colim} X_i & & &
\end{tikzcd}$$

where the vertical maps are the normal maps of the constituent spaces into the colimit, yields the identity map $\mathrm{id} : \mathrm{colim} X_i \to \mathrm{colim} X_i$. Applying this to our situation, where the b_k are all null homotopic, we see that the identity map $\mathrm{id} : T_{\alpha_n} \to T_{\alpha_n}$ is null homotopic. Therefore, $T_{\alpha_n} \simeq *$.

$\square$

Since the ultimate goal of this chapter is to prove a connection between L_n^f-acyclics and $W(n)$-cellular spaces, and the above theorem demonstrates a connection between L_n^f-acyclics and T_{α_n}-acyclics, we now need to demonstrate a connection between T_{α_n}-acyclics and $W(n)$-cellular spaces.

Theorem III.5. *Let X be $\Sigma^{i_n}W(n-1)$-cellular. Then $P_{\Sigma W(n)}\Sigma X \simeq *$ if and only if there exists an $M \geq 1$ with $T_{\alpha_n}\Sigma^M X \simeq *$.*

Proof. Assume $T_{\alpha_n}\Sigma^M X \simeq *$, for some $M \geq 1$. To prove this, we'll take the following steps. First, we'll prove the equivalence

$$(P_{\Sigma W(n)}\Sigma^M X)^{\Sigma^{d_n+i_n}W(n-1)} \simeq T_{\alpha_n}\Sigma^M X,$$

proving that the space on the left is contractible. Then, we'll show that $P_{\Sigma W(n)}\Sigma^M X$ is both $\Sigma^{i_n+1}W(n-1)$-cellular and $\Sigma^{i_n+1}W(n-1)$-null, which means it's contractible.

If $M = 1$, we're done at this point. If $M > 1$, we use a lemma of Bousfield's to show that $P_{\Sigma W(n)}\Sigma X \simeq *$ implies $P_{\Sigma W(n)}\Sigma^M X \simeq *$, for $M > 1$.

In [DF] Theorem A.10, Dror Farjoun proves that given a self map of a finite complex $\omega : \Sigma^q W \longrightarrow W$ there is a weak equivalence

$$(P_{\Sigma C} X)^{\Sigma^2 W} \simeq \Omega^2 T_\omega P_{\Sigma C} X \simeq \Omega^2 T_\omega X.$$

Here, C =cof(ω) and T_ω is the associated telescope, as in Definition III.2. We will apply this to the self map defining $W(n)$, namely $\alpha_n : \Sigma^{i_n+d_n} W(n-1) \longrightarrow \Sigma^{i_n} W(n-1)$. This yields

$$(P_{\Sigma W(n)} X)^{\Sigma^{i_n+2} W(n-1)} \simeq \Omega^2 T_{\alpha_n} X.$$

Looping this equivalence $d_n - 2$ times gives us

$$(P_{\Sigma W(n)} X)^{\Sigma^{d_n+i_n} W(n-1)} \simeq \Omega^{d_n} T_{\alpha_n} X.$$

Recall the fact, discussed in the construction of T_{α_n}, that there is an equivalence $\Omega^{d_n} T_{\alpha_n} Z \simeq T_{\alpha_n} Z$. So we have an equivalence

$$(P_{\Sigma W(n)} X)^{\Sigma^{d_n+i_n} W(n-1)} \simeq T_{\alpha_n} X.$$

Replacing X with $\Sigma^M X$ yields

$$(P_{\Sigma W(n)} \Sigma^M X)^{\Sigma^{d_n+i_n} W(n-1)} \simeq T_{\alpha_n} \Sigma^M X.$$

But we're assuming that $T_{\alpha_n}\Sigma^M X$ is contractible. Therefore, we have that

$$(P_{\Sigma W(n)}\Sigma^M X)^{\Sigma^{d_n+i_n}W(n-1)} \simeq *.$$

Consider the cofibration

$$\Sigma^{d_n+i_n+1}W(n-1) \to \Sigma^{i_n+1}W(n-1) \to \Sigma W(n)).$$

Setting $(P_{\Sigma W(n)}\Sigma^M X) = Z$, we then get a fibration

$$Z^{\Sigma W(n)} \to Z^{\Sigma^{i_n+1}W(n-1)} \to Z^{\Sigma^{d_n+i_n+1}W(n-1)}.$$

Since $Z^{\Sigma^{d_n+i_n}W(n-1)} \simeq *$ and $\Omega Z^{\Sigma^{d_n+i_n}W(n-1)} \simeq Z^{\Sigma^{d_n+i_n+1}W(n-1)}$, we have that the fibration has contractible base. The fiber is contractible since Z is by definition $\Sigma W(n)$-null. Therefore, $Z^{\Sigma^{i_n+1}W(n-1)} \simeq *$. But X is $\Sigma^{i_n}W(n-1)$-cellular so, if $M \geq 1$, $\Sigma^M X$ is $\Sigma^{i_n+1}W(n-1)$-cellular. $\Sigma W(n)$ is also $\Sigma^{i_n+1}W(n-1)$-cellular since the collection of $\Sigma^{i_n+1}W(n-1)$-cellular spaces is closed under cofibrations.

If we examine how Z is constructed, we see that Z is also $\Sigma^{i_n+1}W(n-1)$-cellular as it is built from $\Sigma^M X$ and $\Sigma W(n)$ as a homotopy colimit. Recall $(P_{\Sigma W(n)}\Sigma^M X) = Z$. Thus $Z = \text{hocolim} Z(i)$, where the maps $Z(i) \to Z(i+1)$ are induced by the homotopy pushout

$$\begin{tikzcd}
\bigvee_i \bigvee_f \Sigma^i\Sigma W(n) \arrow[r] \arrow[d] & Z(i) \arrow[d] \\
* \arrow[r] & Z(i+1)
\end{tikzcd}$$

The inner wedge is taken over all $f \in [\Sigma^{i+1}W(n), Z(i)]$. Since $Z(0) = \Sigma^M X$, $Z(0)$ is $\Sigma^{i_n+1}W(n-1)$-cellular. If $Z(i)$ is $\Sigma^{i_n+1}W(n-1)$-cellular then $Z(i+1)$ is

$\Sigma^{i_n+1}W(n-1)$-cellular since it's the homotopy pushout of a diagram wherein each of the spaces in the diagram is $\Sigma^{i_n+1}W(n-1)$-cellular. Recall any contractible space is A-cellular for any A.

Finally, $Z = \text{hocolim} Z(i)$ and each $Z(i)$ is $\Sigma^{i_n+1}W(n-1)$-cellular so Z is $\Sigma^{i_n+1}W(n-1)$-cellular.

Dror-Farjoun's Lemma II.12, (i), tells us that if Z is $\Sigma^{i_n+1}W(n-1)$-cellular then $P_{\Sigma^{i_n+1}W(n-1)}Z$ is contractible. But, we already proved that $Z^{\Sigma^{i_n+1}W(n-1)} \simeq *$ which means that Z is already $\Sigma^{i_n+1}W(n-1)$-null. Therefore, $P_{\Sigma^{i_n+1}W(n-1)}Z \simeq Z$. So, $Z \simeq *$.

Assume $M > 1$. In [B1], Theorem 9.10, Bousfield shows that

$$P_X(\Sigma W) \simeq *$$

if, and only if both

$$P_X(\Sigma^k W) \simeq * \quad \text{and} \quad P_X(K(\mathbb{Z}/p, n+1)) \simeq *,$$

where W is a p-torsion CW-complex with bottom cell in dimension n, $k \geq 1$, and X is any CW-complex. In Lemma 7.4 of the same article, he proves that $P_W(K(\mathbb{Z}/p, j)) \simeq *$ if W is a p-torsion CW-complex with bottom cell in dimension n and $j \geq n$. His results are more general than this, but this is what we need. In our situation, ΣX is $\Sigma^{i_n+1}W(n-1)$cellular, so it is a p-torsion CW-complex with bottom cell in dimension no smaller than $(\sum_{j=1}^{n} i_j)+3$. We have that $(P_{\Sigma W(n)}\Sigma^M X) \simeq *$. By the above Lemma, $P_{W(n)}(K(\mathbb{Z}/p, (\sum_{j=1}^{n} i_j)+4)) \simeq *$, so $P_{\Sigma W(n)}\Sigma X$ is contractible, as desired.

In the other direction, if we assume $P_{\Sigma W(n)}\Sigma X \simeq *$, then [DF], Proposition

A.9 gives us $T_{\alpha_n}\Sigma X \simeq T_{\alpha_n}P_{\Sigma W(n)}\Sigma X \simeq *$. □

Recall Dror-Farjoun's Lemma $II.12$, (ii), which states that $P_{\Sigma A}X \simeq *$ implies that X is A-cellular. Thus, in light of the following corollary, $P_{\Sigma W(n)}\Sigma X \simeq *$ implies that ΣX is $W(n)$-cellular.

Set

$$M = \sum_{j=1}^{n} i_j + n - 1$$

and

$$N = \sum_{j=1}^{n-1} d_j + \sum_{j=1}^{n} i_j + n + 1.$$

Corollary III.6. *If $L_n^f X \simeq *$ and X is p-local and simply connected, then*

$$P_{\Sigma W(n)}\Sigma^M X \simeq *$$

.

Proof. Thompson proves, in [Th] that $\tilde{E}_* X = 0 \Rightarrow \tilde{E}_*(\Omega^k\Sigma^k X) = 0$ for $k > 0$, where E_* is any homology theory. We also have that $E_*\Sigma^k X = 0$, for all $k > 0$. Therefore, if X is E_*-acyclic, we can arrange it so that $E_*(\Omega^k\Sigma^N X) = 0$ for all $0 \leq k \leq N$. Also, $L_n^f X \simeq *$ implies that $L_i^f X \simeq *$ for all $i \leq n$, so we have that $L_i^f \Omega^k \Sigma^N \simeq *$ for all $i \leq n$. $\Sigma^N X$ is certainly N-connected so Theorem III.4 applies yielding $T_{\alpha_i}\Sigma^N X \simeq *$ for all $i \leq n$.

Since X is an L_1^f-acyclic, Corollary B.5 of [DF] tells us that ΣX is $W(1)$-cellular. Since $i_1 = 1$, this is equivalent to $\Sigma^{i_1} X$ is $W(1)$-cellular. Therefore, $\Sigma^{i_1+i_2} X$ is $\Sigma^{i_2} W(1)$-cellular. As mentioned above, $T_{\alpha_2}\Sigma^N X \simeq *$, and $N > i_1 + i_2$, so we may apply Theorem III.5 to $\Sigma^{i_1+i_2} X$, which tells us that $\Sigma^{i_1+i_2+1} X$ is $W(2)$-cellular. Therefore, $\Sigma^{i_1+i_2+i_3+1} X$ is $\Sigma^{i_3} W(2)$-cellular, and another application of Theorem

III.5 implies that $\Sigma^{i_1+i_2+i_3+2}X$ is $W(3)$-cellular. Repeated applications, if necessary, of Theorem III.5 show that $\Sigma^M X$ is $W(n)$-cellular, as desired. $\square$

CHAPTER IV

STABLE AND UNSTABLE LOCALIZATION

As mentioned in the introduction, much of this work came from an attempt to understand something about the map $\Phi_{i,X} : \Sigma^i L_E X \to L_E \Sigma^i X$, when L_E is localization with respect to some chromatic homology theory. This map exists by Property 2.2 above, applied to the E_*-isomorphism $\Sigma^i \mu : \Sigma^i X \to \Sigma^i L_E X$, where μ is the E_*-localization of X. We consider the following diagram:

$$\begin{tikzcd} L_E X \arrow[r] & \Omega L_E \Sigma X \arrow[r] & \Omega^2 L_E \Sigma^2 X \arrow[r] & \cdots \end{tikzcd}$$

The i^{th} map in this diagram is $\Omega^{i-1} L_E \Sigma^{i-1} X \to \Omega^i L_E \Sigma^i X$, which is $\Omega^{i-1}(-)$ applied to the adjoint of $\Phi_{1,\Sigma^{i-1}X} : \Sigma L_E \Sigma^{i-1} X \to L_E \Sigma^i X$. Theorem IV.3 describes the homotopy colimit of this diagram, when $L_E = L_n^f$.

A stable version of this diagram that we also consider is

$$\begin{tikzcd} \Sigma^\infty L_E X \arrow[r] & \Sigma^{-1}\Sigma^\infty L_E \Sigma X \arrow[r] & \Sigma^{-2}\Sigma^\infty L_E \Sigma^2 X \arrow[r] & \cdots \end{tikzcd}$$

To understand the maps in this diagram, begin by considering the map of spectra of the form $\Sigma\Sigma^\infty L_E \Sigma^i X \to \Sigma^\infty L_E \Sigma^{i+1} X$, defined by maps on the constituent spaces of the form $\Sigma^{n+1} L_E \Sigma^i X \to \Sigma^n L_E \Sigma^{i+1} X$, for $n \geq 0$. The unstable maps are simply $\Sigma^n \Phi_{1,\Sigma^i X}$, for $n \geq 0$. Desuspending this map of spectra i times, produces a map of the form $\Sigma^{1-i}\Sigma^{\infty}L_E\Sigma^iX \to \Sigma^{-i}\Sigma^{\infty}L_E\Sigma^{i+1}X$, which is the i^{th} map in the above diagram. Theorem IV.6 describes the homotopy colimit of this diagram, when $L_E = L_n^f$.

It turns out that both of these Theorems will follow from a more general Lemma. In the proof of the Lemma, we'll use the E_*-colocalization functor. The following can be found in section 1 of [B5].

Definition IV.1. *A spectrum X is E_*-*colocal *if $[X,A] \xrightarrow{f_*} [X,B]$ is an isomorphism for every E_*-equivalence $f : A \to B$.*

In Prop. 1.5 of [B5], it is proven that each spectrum X has an E_*-colocalization ${}^EX$, and that there exists a map ${}^EX \to X$. In fact, this functor is another example of an augmented localization functor.

Let X be any CW-spectrum $X = \{X_0, X_1, X_2, \cdots\}$, with structure maps $s_k : \Sigma X_k \to X_{k+1}$.

Lemma IV.2. $L_n^fX \simeq \{L_n^fX_0, L_n^fX_1, L_n^fX_2, \cdots\}$.

Proof. Let $L'(X)$ be a spectrum with $L'(X)_k = L_n^fX_k$, and structure maps given by

$$\Sigma L_n^f(X_k)) \xrightarrow{\Phi_{1,X_k}} L_n^f(\Sigma X_k) \xrightarrow{L_n^f(s_k)} L_n^f(X_{k+1}).$$

We'll show that $L'(X)$ is a L_n^f-local spectrum, then we'll exhibit an L_n^f-isomorphism from X to $L'(X)$ which implies that the Lemma holds.

To show that $L'(X)$ is L_n^f-local requires that we show $[A, L'(X)]_* = 0$ for any L_n^f-acyclic spectrum A. This will require two steps.

First, we'll show that $\Sigma A \simeq \text{hocolim}C_s$, where C_s is a sequence of spectra such that C_0 and $\text{Cof}\,(C_s \to C_{s+1})$ are equivalent to wedges of finite L_n^f-acyclics.

Then, we will show that $[B, L'(X)]_* = 0$ for any finite L_n^f-acyclic B. Therefore, $[C_0, L'(X)] = 0$, and $[C_s, L'(X)] = 0 \Rightarrow [C_{s+1}, L'(X)] = 0$. This will allow us to argue, inductively, that $[C_s, L'(X)] = 0$ for all s, hence

$$[\Sigma A, L'(X)] \cong \varinjlim_i [C_i, L'(X)] = 0,$$

which implies $[A, L'(X)] = 0$. We will be using the fact that mapping out of a cofibration of spectra is exact.

Let A be an arbitrary L_n^f-acyclic spectrum. We want to consider the F_{n+1}-colocalization of A where F_{n+1} is any finite type-$(n+1)$ spectrum. Since F_{n+1} is type-$(n+1)$, it is $K(n)_*$-acyclic. Since F_{n+1} is finite, it is also a L_n^f-acyclic. Then, the colocalization construction (see Prop. 5 in [B5]) proceeds as follows. Take $B_0 = A$, and then inductively construct a countable sequence of CW-spectra

$$A = B_0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow \cdots$$

where $B_\gamma = \text{hocolim} B_s$ and where $B_s \longrightarrow B_{s+1}$ is given by the homotopy pushout square

$$\begin{tikzcd}
\bigvee_{i\in\mathbb{Z}} \bigvee_f \Sigma^i F_{n+1} \arrow[r] \arrow[d] & B_s \arrow[d] \\
* \arrow[r] & B_{s+1}
\end{tikzcd}$$

in which f ranges over all cellular functions $\Sigma^i F_{n+1} \longrightarrow B_s$ of degree 0. Since $B_0 = A$ one has maps $A \longrightarrow B_s$ for all s. Set $C_s = \text{hocolim} A \longrightarrow B_s$. Then, the F_{n+1}-colocalization of A is

$$\Sigma^{-1} \text{hocolim} C_s.$$

And one has a homotopy cofiber sequence

$$\Sigma^{-1}\mathrm{hocolim} C_s \to A \to B_\gamma.$$

Furthermore, one can see that C_0 is a wedge of finite L_n^f-acyclics. Also, since $\mathrm{cof}\,(B_s \to B_{s+1})$ is a wedge of finite L_n^f-acyclics and $\mathrm{cof}\left(A \xrightarrow{\mathrm{id}} A\right) \simeq *$, it follows that $\mathrm{cof}\,(C_s \to C_{s+1})$ is a wedge of finite L_n^f-acyclics.

Given a homotopy pushout square

$$Z = \mathrm{hocolim}\,(X_1 \leftarrow X_2 \to X_3)$$

with all X_i E_*-acyclic for some E, then Z is E_*-acyclic. Therefore, a consequence of this construction is that since $B_0 = A$ is L_n^f-acyclic and F_{n+1} is L_n^f-acyclic, then B_1 is L_n^f-acyclic. Hence, all B_s are L_n^f-acyclic. Hence B_γ is L_n^f-acyclic. As explained in Prop. 3.3 of [MahS], B_γ is also L_n^f-local, hence $B_\gamma \simeq *$. This gives that A is equivalent to its colocalization, or $\Sigma A \simeq \mathrm{hocolim} C_s$, where $\mathrm{cof}\,(C_s \to C_{s+1})$ is a wedge of finite L_n^f-acyclics, as desired.

At this point, if we knew that $[Y, L'(X)] = 0$ whenever Y was a wedge of finite acyclics, we'd have $[C_0, L'(X)] = 0$. Then, whenever $[C_s, L'(X)] = 0$ we have that $[C_{s+1}, L'(X)] = 0$ since mapping out a cofibration is exact and the cofiber of the map $C_s \to C_{s+1}$ is a wedge of finite acyclis. This would give us that $[C_s, L'(X)] = 0$ for all s, hence $[\Sigma A, L'(X)] = 0$ which gives $[A, L'(X)] = 0$. So, what's left is to show that $[Y, L'(X)] = 0$ whenever Y is a wedge of finite acyclics.

If B is a finite L_n^f-acyclic spectrum, then we may assume $B \simeq \Sigma^{-k}\Sigma^\infty Z$ with Z a finite L_n^f-acyclic CW-complex. We'd like to know that $[\Sigma^{-k}\Sigma^\infty Z, L'(X)]_r = 0$ for all r. This is equivalent to $[\Sigma^\infty Z, L'(X)]_{r-k}$, which, it is proven in [A2] Proposition

2.8, is isomorophic to

$$\varinjlim_{m}[\Sigma^{m+r-k}Z, L_n^f X_m] \simeq 0.$$

The final equivalence is a result of there being no non-trivial maps from a L_n^f-acyclic space to a L_n^f-local one. So, we have that finite acyclics can't map non-trivially into $L'(X)$. Therefore, $[C_s, L'(X)] = 0$ for all $i \geq 0$, which gives $[A, L'(X)] = 0$. So, $L'(X)$ is L_n^f-local.

Clearly, there is a map from X to $L'(X)$ built out of the maps on the underlying spaces $\mu_k : X_k \rightarrow L_n^f X_k$, which is a L_n^f-equivalence since the maps on each space are such. This induces a L_n^f-isomorphism from $L'(X)$ to $L_n^f X$. Therefore $L'(X) \simeq L_n^f X$.

□

The above Theorem fails for localization with respect to integral homology. If we take $Y = \{Y_0, Y_1, Y_2, \cdots\}$ to be the spectrum representing $K(n)_*$, then Y is a non-contractible $L_{H\mathbb{Z}}$-acyclic spectrum. If $Y\langle 1\rangle = \{Y_0\langle 1\rangle, Y_1\langle 1\rangle, Y_2\langle 1\rangle, \cdots\}$ where $Y_k\langle 1\rangle$ is the simply connected cover of Y_k, then one can show that $Y\langle 1\rangle \simeq Y$. But the constituent spaces of $Y\langle 1\rangle$ are simply connected, hence $L_{H\mathbb{Z}}$-local. But, since Y is a homology acyclic, it's localization should be contractible, which Y is not.

Theorem IV.3. *With L_n^f as defined above, we have*

$$\Omega^\infty L_n^f \Sigma^\infty X \simeq \varinjlim_{i} \Omega^i L_n^f \Sigma^i X$$

Proof. Applying the above lemma yields an equivalence $L_n^f \Sigma^\infty X \simeq L'(X)$.

Therefore, $\Omega^\infty L'(X) \simeq \varinjlim_{i} \Omega^i L_n^f \Sigma^i X$, must also be homotopy equivalent to

$\Omega^\infty L_n^f \Sigma^\infty X$, yielding

$$\Omega^\infty L_n^f \Sigma^\infty X \simeq \varinjlim_i \Omega^i L_n^f \Sigma^i X.$$

□

Before proving the next theorem, we need a technical lemma about homotopy colimits of spectra.

Definition IV.4. *Let $Y(i)$ be a sequence of CW-spectra, with maps $\phi_i : Y(i) \to Y(i{+}1)$ induced by maps on the underlying spaces of the form $\phi_{i,j} : Y(i)_j \to Y(i{+}1)_j$ which commute, up to homotopy, with the structure maps.*

Thus, we have the following commutative square of spaces, for each $i, j \geq 0$:

$$\begin{tikzcd}
\Sigma Y(i)_j \arrow[r, "\Sigma(i)_j"] \arrow[d, "\phi_{i,j}"] & Y(i)_{j+1} \arrow[d, "\phi_{i,j+1}"] \\
\Sigma Y(i+1)_j \arrow[r, "\Sigma(i+1)_j"] & Y(i+1)_{j+1}
\end{tikzcd}$$

Let Y be the CW-spectrum with j^{th} space $\text{hocolim} Y(i)_j$.

Lemma IV.5. $\text{hocolim} Y(i) \simeq Y$

Proof. We have maps $\psi_{i,j}j : Y(i)_j \to \text{hocolim} Y(i)_j$ into the colimit for each $i, j \geq 0$. These induce a map of spectra $\psi : \text{hocolim} Y(i) \to Y$. We'll show that this map induces an isomorphism of homotopy groups. We have the following diagram:

$$\begin{tikzcd}
\pi_n(\text{hocolim} Y(i)) \arrow[r, "\psi_*"] & \pi_n(Y) \\
\varinjlim_i \pi_n(Y(i)) \arrow[u, "\cong"'] & \varinjlim_j \pi_{n+j}(\text{hocolim} Y(i)_j) \arrow[u, "\cong"'] \\
\varinjlim_i \varinjlim_j \pi_{n+j}(Y(i)_j) \arrow[u, "\cong"'] \arrow[r, "\cong"] & \varinjlim_j \varinjlim_i \pi_{n+j}(Y(i)_j) \arrow[u, "\cong"']
\end{tikzcd}$$

To see that this diagram is commutative, we notice that the isomorphism

$$\varinjlim_j \varinjlim_i \pi_{n+j}(Y(i)_j) \to \varinjlim_j \pi_{n+j}(\mathrm{hocolim} Y(i)_j)$$

is induced by

$$\psi_{i,j} : Y(i)_j \to \mathrm{hocolim} Y(i)_j.$$

Therefore, ψ induces an ismorphism of homotopy groups and we have an equivalence of spectra $\mathrm{hocolim} Y(i) \simeq Y$. □

Theorem IV.6.

$$L_n^f \Sigma^\infty X \simeq \lim_{i \to \infty} \Sigma^{-i} \Sigma^\infty L_n^f \Sigma^i X.$$

Proof. Let $\Sigma^{-i}\Sigma^\infty L_n^f \Sigma^i X$ be modeled by the spectrum $Y(i)$, where $Y(i)_k = L_n^f \Sigma^k X$ for $0 \leq k \leq i-1$, and $Y(i)_k = \Sigma^{k-i} L_n^f \Sigma^i X$ for $k \geq i$ are . The structure maps, $\Sigma Y(i)_k \to Y(i)_{k+1}$, for $k \leq i-1$, are given by $\Phi_1(\Sigma^k X) : \Sigma L_n^f \Sigma^k X \to L_n^f \Sigma^{k+1} X$. For $k \geq i$, the structure maps are the identity. Clearly, this model for $\Sigma^{-i}\Sigma^\infty L_n^f \Sigma^i X$ is equivalent to the standard one.

One has maps $\zeta(i) : Y(i) \to Y(i+1)$, defined on the constituent spaces via the identity if $k \leq i$ and via $\Sigma^{k-(i+1)}\Phi_1(\Sigma^i X)$ if $k \geq i+1$. It's immediate that these maps commute with the structure maps. Therefore, if we fix a k, and consider the sequence $Y(1)_k \to Y(2)_k \to \cdots$, eventually the maps are identity maps. This gives, under these circumstances, $\varinjlim_i Y(i) \simeq Z$, where $Z_k \simeq \varinjlim_i Y(i)_k \simeq L_n^f \Sigma^k X$. Now, the previous lemma can be applied to see that

$$L_n^f \Sigma^\infty X \simeq Z \simeq \lim_{i \to \infty} \Sigma^{-i} \Sigma^\infty L_n^f \Sigma^i X.$$

□

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