

CROSSED PRODUCT  $C^*$ -ALGEBRAS OF CERTAIN NON-SIMPLE  
 $C^*$ -ALGEBRAS AND THE TRACIAL  
QUASI-ROKHLIN PROPERTY

by

JULIAN MICHAEL BUCK

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Julian Buck

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This dissertation has been accepted and approved in partial fulfillment of the requirements for the degree in the Department of Mathematics by:

Christopher Phillips, Chairperson, Mathematics

Daniel Dugger, Member, Mathematics

Huaxin Lin, Member, Mathematics

Marcin Bownik, Member, Mathematics

Van Kolpin, Outside Member, Economics

and Richard Linton, Vice President for Research and Graduate Studies/Dean of the Graduate School for the University of Oregon.

June 14, 2010

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## An Abstract of the Dissertation of

Julian Michael Buck for the degree of Doctor of Philosophy  
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 Title: **CROSSED PRODUCT  $C^*$ -ALGEBRAS OF CERTAIN NON-SIMPLE  $C^*$ -ALGEBRAS  
 AND THE TRACIAL QUASI-ROKHLIN PROPERTY**

Approved: \_\_\_\_\_  
 Dr. N. Christopher Phillips

This dissertation consists of four principal parts. In the first, we introduce the tracial quasi-Rokhlin property for an automorphism  $\alpha$  of a  $C^*$ -algebra  $A$  (which is not assumed to be simple or to contain any projections). We then prove that under suitable assumptions on the algebra  $A$ , the associated crossed product  $C^*$ -algebra  $C^*(\mathbb{Z}, A, \alpha)$  is simple, and the restriction map between the tracial states of  $C^*(\mathbb{Z}, A, \alpha)$  and the  $\alpha$ -invariant tracial states on  $A$  is bijective. In the second part, we introduce a comparison property for minimal dynamical systems (the dynamic comparison property) and demonstrate sufficient conditions on the dynamical system which ensure that it holds. The third part ties these concepts together by demonstrating that given a minimal dynamical system  $(X, h)$  and a suitable simple  $C^*$ -algebra  $A$ , a large class of automorphisms  $\beta$  of the algebra  $C(X, A)$  have the tracial quasi-Rokhlin property, with the dynamic comparison property playing a key role. Finally, we study the structure of the crossed product  $C^*$ -algebra  $B = C^*(\mathbb{Z}, C(X, A), \beta)$  by introducing a subalgebra  $B_{\{y\}}$  of  $B$ , which is shown to be large in a sense that allows properties of  $B_{\{y\}}$  to pass to  $B$ . Several conjectures about the deeper structural properties of  $B_{\{y\}}$  and  $B$  are stated and discussed.

## CURRICULUM VITAE

NAME OF AUTHOR: Julian Michael Buck

PLACE OF BIRTH: Ottawa, Ontario, Canada

DATE OF BIRTH: June 2, 1982

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR  
University of Northern British Columbia, Prince George, BC  
Simon Fraser University, Vancouver, BC

DEGREES AWARDED:

Doctor of Philosophy, University of Oregon, 2010  
Master of Science, University of Oregon, 2006  
Bachelor of Science, University of Northern British Columbia, 2004

AREAS OF SPECIAL INTEREST:

Classification of  $C^*$ -algebras, crossed product and transformation group  $C^*$ -algebras,  
dynamical systems.

PROFESSIONAL EXPERIENCE:

Graduate Teaching Fellow, University of Oregon, 2004-2010

GRANTS, AWARDS AND HONORS:

Frank W. Anderson Award, University of Oregon Department of Mathematics, 2009  
Johnson Research Fellowship, University of Oregon Department of Mathematics, 2008

PUBLICATIONS:

J. Buck and S. Walters, *Connes-Chern characters of hexic and cubic modules*, J. Operator Theory  
57(2007), 35-65.

J. Buck and S. Walters, *Non commutative spheres associated with the hexic transform and their  
K-theory*, J. Operator Theory 58(2007), No. 2, 441-462.

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## CHAPTER I

## INTRODUCTION

The principal subject of this dissertation is the properties of what are known as crossed product  $C^*$ -algebras. Let  $A$  be a  $C^*$ -algebra  $A$  and consider an integer action  $\mathbb{Z} \rightarrow \text{Aut}(A)$  given by fixing an automorphism  $\alpha \in \text{Aut}(A)$  and taking the action to be  $n \mapsto \alpha^n$ . Then the *crossed product  $C^*$ -algebra* of  $A$  by  $\alpha$  is the universal  $C^*$ -algebra  $C^*(\mathbb{Z}, A, \alpha)$  generated by  $A$  and a formal unitary  $u$  satisfying the relation  $uau^* = \alpha(a)$  for all  $a \in A$ . We often refer to  $u$  as the implementing unitary for the crossed product. We may construct the crossed product as the universal  $C^*$ -completion of the skew group ring  $A[\mathbb{Z}]$ , consisting of formal finite power series in  $u$  with coefficients in  $A$  but where the multiplication is twisted by  $\alpha$  according to the rule  $ua = \alpha(a)u$ . A special case of this construction that deserves particular attention is when the algebra  $A$  is the algebra  $C(X)$  of continuous functions  $f: X \rightarrow \mathbb{C}$  for some compact metric space  $X$ , and the automorphism  $\alpha$  is the induced automorphism of a homeomorphism  $h: X \rightarrow X$ , given by  $\alpha(f) = f \circ h^{-1}$ . In this case, the crossed product  $C^*$ -algebra  $C^*(\mathbb{Z}, C(X), \alpha)$  is usually denoted  $C^*(\mathbb{Z}, X, h)$  and is called the *transformation group  $C^*$ -algebra* of  $X$  by  $h$ . The pairing of a compact metric space with a homeomorphism is called a *dynamical system*. Of particular interest in the context of transformation group  $C^*$ -algebras are the dynamical systems where  $h$  is a *minimal homeomorphism*; that is, there are no proper  $h$ -invariant closed subsets of  $X$ . For the case of minimal dynamical systems where the space  $X$  is infinite, the associated transformation group  $C^*$ -algebras are always simple, and under additional assumptions frequently have nice structural properties. This will be discussed in more detail shortly.

The study of  $C^*$ -algebras arising through crossed product constructions has been an area of significant interest in the Elliott classification program for nuclear  $C^*$ -algebras, as in many situations these crossed products are classifiable. Well-known examples such as the irrational

rotation algebras of [46] have been shown to arise naturally as crossed products, and in [8] it is shown that these algebras are simple AT-algebras with real rank zero and are thus classifiable by their  $K$ -theory. Various forms of the Rokhlin property have appeared in the literature and these have been used to establish many structural results about crossed products by automorphisms with these properties. (For example, see [13], [15], [16], and [17].) The tracial Rokhlin property for automorphisms of certain simple  $C^*$ -algebras was first introduced by Osaka and Phillips in [36], where it is shown that crossed products by automorphisms with the tracial Rokhlin property preserve real rank zero, stable rank one, and order on projections being determined by traces. Several versions of the tracial Rokhlin property for actions of finite groups on  $C^*$ -algebras have also appeared, such as those of [42] and [3]. Similar results on the structure of the associated cross products have been obtained in this situation. (For examples see the aforementioned papers, and also [7].) In the best case, it has been shown that tracial rank zero is preserved under crossed products by finite group actions with the tracial Rokhlin property, and hence these crossed products are classifiable by Huaxin Lin's classification theory for  $C^*$ -algebras with tracial rank zero, provided they also satisfy the Universal Coefficient Theorem. (See [22], [20], and [21] for the precise details of this classification theory.)

Perhaps even more successful has been the effort to classify the transformation group  $C^*$ -algebras associated to minimal dynamical systems  $(X, h)$ . The case where the space  $X$  is the Cantor set was analyzed extensively in the work of Giordano, Putnam, and Skau [11], where it is shown that the transformation group  $C^*$ -algebras of two minimal homeomorphisms are isomorphic if and only if the homeomorphisms are strong orbit equivalent. Moreover, it is known that such transformation group  $C^*$ -algebras are AT-algebras with real rank zero. The key results are obtained in Putnam's study [45] of the transformation group  $C^*$ -algebra through certain useful approximating subalgebras, having a particularly tractable structure, resulting from a Rokhlin tower construction. In particular, Putnam's subalgebras were AF-algebras. Putnam's approach was later massively generalized by Qing Lin and Phillips in the long unpublished preprint [29] (see also the survey articles [26] and [27]) to give a careful description of the transformation group  $C^*$ -algebras arising from minimal diffeomorphisms of smooth compact manifolds in terms of a direct limit decomposition. In order to study the properties of their approximating subalgebras, which are much more complicated than Putnam's, Phillips introduced the concept of a recursive subhomogeneous algebra and studied the structure of this class of algebras and their direct limits

in [39], [40], and [41]. Subsequently, Huaxin Lin and Phillips showed in [24] that under suitable  $K$ -theoretic conditions, the crossed product of an infinite compact metric space with finite covering dimension by a minimal homeomorphism has tracial rank zero, and is therefore classifiable.

There is little existing overlap between these two branches of research into crossed products. The tracial Rokhlin property is formulated for a simple  $C^*$ -algebra and requires the existence of many projections, while the  $C^*$ -algebra  $C(X)$  may have few or no non-trivial projections. Also problematic is the so-called “leftover comparison condition” in the definition of the tracial Rokhlin property, which we cannot generally expect to be satisfied in the commutative situation. In fact the tracial Rokhlin property of Osaka and Phillips is only a sensible definition for simple  $C^*$ -algebras with a strong condition on the existence of many projections, such as real rank zero. In the case of finite group actions, Archey has introduced in [3] an analogue of the tracial Rokhlin property which dispenses with projections in favor of positive elements. Unfortunately, the leftover comparison condition in this property is still unsuitable for the situation where the algebra under consideration is  $C(X)$  as it uses Cuntz subequivalence, which is too restrictive for positive elements of  $C(X)$  which are given more or less arbitrarily. Specifically, it roughly requires that the support of one function lie in the support of the other. In this dissertation, we introduce the *tracial quasi-Rokhlin property* for automorphisms of a unital, separable  $C^*$ -algebra  $A$  which is not assumed to be simple. In fact, the  $C^*$ -algebras in which we will be most interested will be of the form  $C(X, A)$ , where  $X$  is an infinite compact metric space having finite covering dimension, and  $A$  is a simple, unital, separable  $C^*$ -algebra with tracial rank zero. By letting  $A = \mathbb{C}$ , this class of algebras includes the algebras  $C(X)$  just discussed.

In Chapter II, we define the tracial quasi-Rokhlin property, and show that if  $\alpha$  is an automorphism of  $A$  and  $A$  has no non-trivial  $\alpha$ -invariant ideals, then the crossed product  $C^*(\mathbb{Z}, A, \alpha)$  is simple. Further, an additional technical assumption about  $A$  (specifically, we assume  $A$  is not a scattered  $C^*$ -algebra) allows us to also show that the restriction mapping  $T(C^*(\mathbb{Z}, A, \alpha)) \rightarrow T_\alpha(A)$ , between the simplex of tracial states on the crossed product and the simplex of  $\alpha$ -invariant tracial states on  $A$ , is a bijection.

In Chapter III we develop a comparison property for minimal, uniquely ergodic dynamical systems  $(X, h, \mu)$  (where  $h$  is a minimal homeomorphism of the compact metric space  $X$  and  $\mu$  is the unique  $h$ -invariant Borel probability measure on  $X$ ) that roughly says an arbitrary closed set with smaller measure than an arbitrary open set can be decomposed into closed subsets, which can then

be moved by powers of  $h$  so that they land in the open set and are pairwise disjoint. We term this the *dynamic comparison property*, and demonstrate that it should hold at a reasonably high level of generality by proving that it is implied by another, more basic dynamical property (the *topological small boundary property*). Based on observations about the tracial quasi-Rokhlin property and the dynamic comparison property, we also suggest possible definitions for a comparison theory of positive elements in dynamical systems.

In Chapter IV we use this condition to show that (with appropriate hypotheses on  $X$  and  $A$ ) certain automorphisms  $\beta$  of the algebra  $C(X, A)$ , which act minimally on the center  $C(X)$ , have the tracial quasi-Rokhlin property. After examining the structure of ideals in  $C(X, A)$  and of its tracial state space, it will follow that the structural theorems of Chapter II apply to the associated crossed product  $C^*$ -algebras  $C^*(\mathbb{Z}, C(X, A), \beta)$ . We also exhibit some examples of known  $C^*$ -algebras which can be realized as crossed product  $C^*$ -algebras of the form  $C^*(\mathbb{Z}, C(X, A), \beta)$  and that are known to have stronger structural properties, which suggests that such properties might hold for these in some generality.

In Chapter V, we introduce the machinery to begin a more detailed study of the structure of the transformation group  $C^*$ -algebras  $C^*(\mathbb{Z}, C(X, A), \beta)$  of the previous chapter. The rough idea is to follow the development of [29] and [24] by approximating the crossed product  $C^*$ -algebra  $B = C^*(\mathbb{Z}, C(X, A), \beta)$  with a subalgebra  $B_{\{y\}} = C^*(\mathbb{Z}, C(X, A), \beta)_{\{y\}}$  (for  $y \in X$ ) that is the appropriate analogue of their approximating subalgebras. We demonstrate that  $B_{\{y\}}$  is a direct limit of certain other subalgebras which generalize the recursive subhomogeneous algebras of [39] by roughly replacing matrix algebras of the form  $C(X_k, M_{n_k})$  with  $C(X_k, M_{n_k}(A))$ . It is our hope that the good behavior of the class of recursive subhomogeneous algebras (particularly in terms of permanence properties for direct limits) is also present in this new class of algebras, and consequently that they can be used to study the approximating subalgebras  $B_{\{y\}}$  and the crossed product  $C^*$ -algebras  $B$ .

Chapter VI investigates the relationship between  $B_{\{y\}}$  and  $B$  by demonstrating that  $B_{\{y\}}$  is a *large* subalgebra of  $B$ , a concept introduced by Phillips in [43] to provide a general formalism for an idea that has already been used for the case of transformation group  $C^*$ -algebras for minimal dynamical systems. By Theorem 4.5 of that paper, it follows that the radius of comparison for  $B$  is no greater than that of  $B_{\{y\}}$ . We conclude by offering some conjectures about the structure of the algebras  $B_{\{y\}}$  and  $B$  that we hope to be true, in analogy with known results for  $C^*(\mathbb{Z}, X, h)$ .

## CHAPTER II

## THE TRACIAL QUASI-ROKHLIN PROPERTY

The following definition is based on Definition 1.1 of [36] and also on the behavior of automorphisms induced by minimal homeomorphisms. Indeed, one of our main applications of it will be to automorphisms related to minimal dynamics.

**Definition II.1.** *Let  $A$  be a separable, unital  $C^*$ -algebra, and let  $\alpha \in \text{Aut}(A)$ . We say that  $\alpha$  has the tracial quasi-Rokhlin property if for every  $\varepsilon > 0$ , every finite set  $F \subset A$ , every  $n \in \mathbb{N}$ , and every positive element  $x \in A$  with  $\|x\| = 1$ , there exist  $c_0, \dots, c_n \in A$  such that:*

1.  $0 \leq c_j \leq 1$  for  $0 \leq j \leq n$ ;
2.  $c_j c_k = 0$  for  $0 \leq j, k \leq n$  and  $j \neq k$ ;
3.  $\|\alpha(c_j) - c_{j+1}\| < \varepsilon$  for  $0 \leq j \leq n-1$ ;
4.  $\|c_j a - a c_j\| < \varepsilon$  for  $0 \leq j \leq n$  and for all  $a \in F$ ;
5. with  $c = \sum_{j=0}^n c_j$ , there exist  $N \in \mathbb{N}$ , positive elements  $e_0, \dots, e_N \in A$ , unitaries  $w_0, \dots, w_N \in A$ , and  $d(0), \dots, d(N) \in \mathbb{Z}$  such that:
  - (a)  $1 - c \leq \sum_{j=0}^N e_j$ ;
  - (b)  $w_j \alpha^{d(j)}(e_j) w_j^* w_k \alpha^{d(k)}(e_k) w_k^* = 0$  for  $0 \leq j, k \leq N$  and  $j \neq k$ ;
  - (c)  $w_j \alpha^{d(j)}(e_j) w_j^* \in \overline{x A x}$  for  $0 \leq j \leq N$ ;
6. with  $c$  as above,  $\|c x c\| > 1 - \varepsilon$ .

The key differences between this definition and Definition 1.1 of [36] are the change from projections to positive elements of norm less than or equal to 1, and the statement of condition

(5) (as compared to condition (3) in Definition 1.1 of [36]). We also make no assumptions about the simplicity of the algebra  $A$ , but it should be noted that this definition is only formulated for cases where the algebra  $A$  is expected to be “ $\alpha$ -simple” (have no non-trivial  $\alpha$ -invariant ideals); it is unclear if this definition is useful without that condition. Condition (6) is an additional requirement, but it is probable that, with certain extra assumptions on  $A$ , condition (6) is implied by condition (5) (this is the case for finite group actions with the tracial Rokhlin property of [42], when  $A$  is stably finite). It is also not clear that condition (5) is actually the most appropriate formulation for the leftover comparison condition in this situation. We postpone further discussion to the end of Chapter III.

**Lemma II.2.** *Let  $A$  be a separable, unital  $C^*$ -algebra, let  $\alpha \in \text{Aut}(A)$ , and let  $u$  be the canonical unitary of the crossed product  $C^*$ -algebra  $C^*(\mathbb{Z}, A, \alpha)$ . Given any  $\varepsilon > 0$  and  $n \in \mathbb{N}$ , let  $c_0, \dots, c_n \in A$  satisfy:*

1.  $0 \leq c_j \leq 1$  for  $0 \leq j \leq n$ ;
2.  $c_j c_k = 0$  for  $0 \leq j, k \leq n$  and  $j \neq k$ ;
3.  $\|\alpha(c_j) - c_{j+1}\| < \varepsilon$  for  $0 \leq j \leq n-1$ .

Then for  $0 \leq j \leq n$  and  $1 \leq k \leq n$ , we have  $\|c_j u^{-k} c_j\| < 3n\varepsilon$  and  $\|c_j u^k c_j\| < 3n\varepsilon$ .

*Proof.* Since  $uau^{-1} = \alpha(a)$  for all  $a \in A$ , we have

$$\|c_j u^{-k} c_j\| = \|u^{-k} \alpha^k(c_j) c_j\| \leq \|\alpha^k(c_j) c_j\|$$

Next, for  $0 \leq i \leq j-1$  we obtain the inequality

$$\begin{aligned} \|\alpha^{k+i}(c_{j-i}) \alpha^i(c_{j-i})\| &\leq \|\alpha^{k+i}(c_{j-i}) \alpha^i(c_{j-i}) - \alpha^{k+i}(c_{j-i}) \alpha^{i+1}(c_{j-i-1})\| \\ &\quad + \|\alpha^{k+i}(c_{j-i}) \alpha^{i+1}(c_{j-i-1}) - \alpha^{k+i+1}(c_{j-i-1}) \alpha^{i+1}(c_{j-i-1})\| \\ &\quad + \|\alpha^{k+i+1}(c_{j-i-1}) \alpha^{i+1}(c_{j-i-1})\| \\ &\leq 2\|c_{j-i} - \alpha(c_{j-i-1})\| + \|\alpha^{k+i+1}(c_{j-i-1}) \alpha^{i+1}(c_{j-i-1})\|. \end{aligned}$$

Repeated application of this inequality gives

$$\begin{aligned}
\|\alpha^k(c_j)c_j\| &\leq \|\alpha^{k+j}(c_0)\alpha^j(c_0)\| + 2\sum_{i=0}^{j-1}\|c_{j-i}-\alpha(c_{j-i-1})\| \\
&< \|\alpha^k(c_0)c_0\| + 2n\varepsilon \\
&= \|\alpha^k(c_0)c_0 - c_k c_0\| + 2n\varepsilon \\
&\leq \|\alpha^k(c_0) - c_k\| + 2n\varepsilon \\
&\leq 2n\varepsilon + \sum_{i=0}^{k-1}\|\alpha^{k-i}(c_i) - \alpha^{k-i-1}(c_{i+1})\| \\
&= 2n\varepsilon + \sum_{i=0}^{k-1}\|\alpha(c_i) - c_{i+1}\| \\
&< 2n\varepsilon + n\varepsilon \\
&= 3n\varepsilon,
\end{aligned}$$

and so we conclude that

$$\|c_j u^{-k} c_j\| \leq \|\alpha^k(c_j)c_j\| < 3n\varepsilon.$$

Similarly, for  $0 \leq i \leq j-1$  we have the inequality

$$\|\alpha^i(c_{j-i})\alpha^{k+i}(c_{j-i})\| \leq 2\|\alpha(c_{j-i-1}) - c_{j-i}\| + \|\alpha^{i+1}(c_{j-i-1}) - \alpha^{k+i+1}(c_{j-i-1})\|,$$

which gives

$$\begin{aligned}
\|c_j u^k c_j\| &= \|c_j \alpha^k(c_j) u^{-k}\| \\
&\leq \|c_j \alpha^k(c_j)\| \\
&\leq \|\alpha^j(c_0)\alpha^{k+j}(c_0)\| + 2\sum_{i=0}^{k-1}\|\alpha(c_{j-i-1}) - c_{j-i-1}\| \\
&< \|c_0 \alpha^k(c_0)\| + 2n\varepsilon \\
&\leq \|\alpha^k(c_0) - c_k\| + 2n\varepsilon \\
&< 3n\varepsilon.
\end{aligned}$$

This completes the proof of the desired inequalities.  $\square$

**Lemma II.3.** *Let  $A$  be a separable, unital  $C^*$ -algebra, let  $\alpha \in \text{Aut}(A)$ , and let  $a \in C^*(\mathbb{Z}, A, \alpha)$  be positive and non-zero. Then for any  $\varepsilon > 0$ , there exist  $N \in \mathbb{N}$  and  $a_j \in A$  for  $-N \leq j \leq N$  such that  $\|a_0\| = 1$  and*

$$\left\| a - \sum_{j=-N}^N a_j u^j \right\| < \varepsilon.$$

*Proof.* Let  $E: C^*(\mathbb{Z}, A, \alpha) \rightarrow A$  be the standard faithful conditional expectation. Set  $b = a^{1/2}$ , which is positive and non-zero. Then as  $E$  is faithful, it follows that

$$E(a) = E(b^2) = E(b^*b) \neq 0,$$

By replacing  $a$  with  $\frac{1}{\|E(a)\|}a$  if necessary, we may assume that  $\|E(a)\| = 1$ . Since  $C_c(\mathbb{Z}, A, \alpha)$  is dense in  $C^*(\mathbb{Z}, A, \alpha)$ , there exist  $N \in \mathbb{N}$  and  $\tilde{b}_j \in A$  for  $-N \leq j \leq N$  such that

$$\left\| (a - E(a)) - \sum_{j=-N}^N \tilde{b}_j u^j \right\| < \frac{1}{2}\varepsilon.$$

Using

$$E(a - E(a)) = E(a) - E(E(a)) = E(a) - E(a) = 0$$

and

$$E\left(\sum_{j=-N}^N \tilde{b}_j u^j\right) = E(\tilde{b}_0),$$

we estimate

$$\|\tilde{b}_0\| = \left\| E(a - E(a)) - E\left(\sum_{j=-N}^N \tilde{b}_j u^j\right) \right\| \leq \left\| (a - E(a)) - \sum_{j=-N}^N \tilde{b}_j u^j \right\| < \frac{1}{2}\varepsilon.$$



Now set  $b_0 = 0$  and  $b_j = \tilde{b}_j$  for  $1 \leq |j| \leq N$ . Then

$$\begin{aligned} \left\| (a - E(a)) - \sum_{j=-N}^N b_j u^j \right\| &= \left\| \tilde{b}_0 + (a - E(a)) - \sum_{j=-N}^N \tilde{b}_j u^j \right\| \\ &\leq \|\tilde{b}_0\| + \left\| (a - E(a)) - \sum_{j=-N}^N \tilde{b}_j u^j \right\| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \\ &= \varepsilon. \end{aligned}$$

By defining  $a_0 = E(a)$  and  $a_j = b_j$  for  $1 \leq |j| \leq N$ , it follows that  $\|a_0\| = 1$  and

$$\left\| a - \sum_{j=-N}^N a_j u^j \right\| = \left\| (a - E(a)) - \sum_{j=-N}^N b_j u^j \right\| < \varepsilon,$$

as required.  $\square$

**Theorem II.4.** *Let  $A$  be a separable, unital  $C^*$ -algebra, let  $\alpha \in \text{Aut}(A)$  have the tracial quasi-Rokhlin property, and suppose that  $A$  has no non-trivial  $\alpha$ -invariant ideals. Then  $C^*(\mathbb{Z}, A, \alpha)$  is simple.*

*Proof.* Let  $J \subset C^*(\mathbb{Z}, A, \alpha)$  be a non-zero ideal, let  $u \in C^*(\mathbb{Z}, A, \alpha)$  be the canonical unitary in the crossed product, set  $\varepsilon = \frac{1}{8}$ , and let  $a \in J$  be non-zero and positive. By Lemma II.3 there exist  $n \in \mathbb{N}$  and  $a_k \in A$  for  $-n \leq k \leq n$  such that  $\|a_0\| = 1$  and

$$\left\| a - \sum_{k=-n}^n a_k u^k \right\| < \frac{1}{4}\varepsilon.$$

For convenience, set  $M = \sum_{k \neq 0} \|a_k\|$ . Define continuous functions  $f, g: [0, 1] \rightarrow [0, 1]$  by

$$f(t) = \begin{cases} 0 & t \leq 1 - \frac{\varepsilon}{8} \\ \frac{16}{\varepsilon}(t-1) + 2 & 1 - \frac{\varepsilon}{8} < t < 1 - \frac{\varepsilon}{16} \\ 1 & t \geq 1 - \frac{\varepsilon}{16} \end{cases}$$

and

$$g(t) = \begin{cases} 0 & t < 1 - \frac{\varepsilon}{16} \\ \frac{16}{\varepsilon}(t-1) + 1 & t \geq 1 - \frac{\varepsilon}{16}. \end{cases}$$

Setting  $q = g(a_0^{1/2})$  and  $r = f(a_0^{1/2})$ , we have the relations  $q, r \geq 0$ ,  $rq = q$ , and  $\|q\| = \|r\| = 1$ .

Now let

$$\varepsilon' = \frac{\varepsilon}{12(M(n+1)^2 + 1)}$$

and  $F = \{a_k : -n \leq k \leq n\}$ . Apply the tracial quasi-Rokhlin property with  $F, \varepsilon', n$ , and  $q$  to obtain  $c_0, \dots, c_n \in A$  such that

1.  $0 \leq c_j \leq 1$  for  $0 \leq j \leq n$ ;
2.  $c_j c_k = 0$  for  $0 \leq j, k \leq n$  and  $j \neq k$ ;
3.  $\|\alpha(c_j) - c_{j+1}\| < \varepsilon'$  for  $0 \leq j \leq n-1$ ;
4.  $\|c_j a_k - a_k c_j\| < \varepsilon'$  for  $0 \leq j \leq n$  and  $-n \leq k \leq n$ ;
5. with  $c = \sum_{j=0}^n c_j$ , we have  $\|cqc\| > 1 - \varepsilon'$ .

Using the mutual orthogonality of the  $c_j$ , we have

$$\begin{aligned} \left\| \sum_{j=0}^n c_j a c_j - \sum_{j=0}^n \sum_{k=-n}^n c_j a_k u^k c_j \right\| &= \left\| \sum_{j=0}^n c_j \left( a - \sum_{k=-n}^n a_k u^k \right) c_j \right\| \\ &\leq \max_{0 \leq j \leq n} \left\| c_j \left( a - \sum_{k=-n}^n a_k u^k \right) c_j \right\| \\ &\leq \left\| a - \sum_{k=-n}^n a_k u^k \right\| \\ &< \frac{1}{4} \varepsilon. \end{aligned}$$

Since the  $c_j$  approximately commute with the  $a_k$ , we obtain

$$\begin{aligned}
\left\| \sum_{j=0}^n \sum_{k=-n}^n c_j a_k u^k c_j - \sum_{j=0}^n \sum_{k=-n}^n a_k c_j u^k c_j \right\| &= \left\| \sum_{j=0}^n \sum_{k=-n}^n (c_j a_k - a_k c_j) u^k c_j \right\| \\
&\leq \sum_{j=0}^n \sum_{k=-n}^n \|c_j a_k - a_k c_j\| \\
&< 2(n+1)^2 \varepsilon' \\
&< \frac{1}{4} \varepsilon.
\end{aligned}$$

Next, applying Lemma II.2 gives

$$\begin{aligned}
\left\| \sum_{j=0}^n \sum_{k=-n}^n a_k c_j u^k c_j - \sum_{j=0}^n a_0 c_j^2 \right\| &= \left\| \sum_{j=0}^n \sum_{k \neq 0} a_k c_j u^k c_j^2 \right\| \\
&\leq \sum_{j=0}^n \sum_{k \neq 0} \|a_k\| \|c_j u^k c_j\| \\
&< 3n(n+1)M\varepsilon' \\
&< \frac{1}{4} \varepsilon.
\end{aligned}$$

Finally, orthogonality of the  $c_j$  gives  $c^2 = \sum_{j=0}^n c_j^2$ , and using this we get the estimate

$$\begin{aligned}
\left\| \sum_{j=0}^n a_0 c_j^2 - c a_0 c \right\| &= \left\| \sum_{j=0}^n (a_0 c_j - c_j a_0) c_j \right\| \\
&\leq \sum_{j=0}^n \|a_0 c_j - c_j a_0\| \\
&< (n+1) \varepsilon' \\
&< \frac{1}{4} \varepsilon.
\end{aligned}$$

Setting  $x = \sum_{j=0}^n c_j a c_j$ , it follows that

$$\|x - c a_0 c\| < \frac{1}{4} \varepsilon + \frac{1}{4} \varepsilon + \frac{1}{4} \varepsilon + \frac{1}{4} \varepsilon = \varepsilon.$$

We next show that  $\|c a_0 c\|$  is sufficiently large. With  $f(t)$  as before, for  $t \in [0, 1]$  we have  $|t f(t) - f(t)| = |t-1| f(t)$ . If  $t \leq 1 - \frac{\varepsilon}{8}$ , then  $f(t) = 0$  and so this quantity is zero. If  $t \geq 1 - \frac{\varepsilon}{8}$ ,

then  $|t - 1| \leq \frac{\varepsilon}{8}$ . Since  $0 \leq f(t) \leq 1$ , this implies  $|t - 1| f(t) \leq \frac{\varepsilon}{8}$  as well. It follows that

$$\left\| a_0^{1/2} r - r \right\| = \sup_{t \in [0,1]} |t f(t) - f(t)| \leq \frac{1}{8} \varepsilon$$

Since  $r q = q$ , we have

$$\left\| a_0^{1/2} q - q \right\| = \left\| a_0^{1/2} r q - r q \right\| \leq \left\| a_0^{1/2} r - r \right\| \|q\| \leq \frac{1}{8} \varepsilon < \varepsilon.$$

This gives

$$\begin{aligned} 1 - \varepsilon &< 1 - \varepsilon' < \|cqc\| \\ &\leq \left\| cqc - ca_0^{1/2} qc \right\| + \left\| ca_0^{1/2} qc \right\| \\ &\leq \left\| q - a_0^{1/2} q \right\| + \left\| ca_0^{1/2} \right\| \\ &< \varepsilon + \left\| ca_0^{1/2} \right\|, \end{aligned}$$

and so  $\left\| ca_0^{1/2} \right\| > 1 - 2\varepsilon$ . Now the  $C^*$ -property, the self-adjointness of  $c$  and  $a_0^{1/2}$ , and  $\varepsilon = \frac{1}{8}$  give

$$\|ca_0c\| = \left\| (ca_0^{1/2})(ca_0^{1/2})^* \right\| = \left\| ca_0^{1/2} \right\|^2 > (1 - 2\varepsilon)^2 = \left(1 - \frac{1}{4}\right)^2 = \frac{9}{16}.$$

Now suppose that  $J \cap A = 0$ . By Theorem 3.1.7 of [35],  $A + J$  is a  $C^*$ -subalgebra of  $C^*(\mathbb{Z}, A, \alpha)$ , and the assumption that  $J \cap A = 0$  implies that the projection map  $\pi: A + J \rightarrow (A + J)/J$  is isometric when restricted to  $A$  (and of course it is norm-reducing in general). Since  $ca_0c \in A$  and  $x \in J$ , it follows that

$$\frac{9}{16} < \|ca_0c\| = \|\pi(ca_0c)\| = \|\pi(ca_0c - x)\| \leq \|ca_0c - x\| < \frac{1}{8},$$

a contradiction. So there must be a non-zero element in  $J \cap A$ . Finally, we claim that  $J \cap A$  is an  $\alpha$ -invariant ideal of  $A$ . To see this, let  $b \in J \cap A$ . Then  $\alpha(b) = ubu^* \in J$  since  $J$  is an ideal, and clearly  $\alpha(b) \in A$ , so  $\alpha(b) \in J \cap A$ . Thus,  $J \cap A$  is a non-zero  $\alpha$ -invariant ideal of  $A$ , which implies that  $J \cap A = A$ . It follows that  $J = C^*(\mathbb{Z}, A, \alpha)$ , and so  $C^*(\mathbb{Z}, A, \alpha)$  is simple.  $\square$

**Lemma II.5.** *Let  $f \in C([0,1])$ . For any  $\varepsilon > 0$ , there is a  $\delta > 0$  (depending on both  $\varepsilon$  and  $f$ ) such that if  $A$  is a unital  $C^*$ -algebra and  $a, b \in A$  satisfy  $0 \leq a, b \leq 1$ , then  $\|ab - ba\| < \delta$  implies  $\|f(b)a - af(b)\| < \varepsilon$ .*

*Proof.* By the Stone-Weierstrass Theorem, there is a polynomial  $p(z) = c_m z^m + \dots + c_1 z + c_0$  such that  $\sup_{x \in [0,1]} \|f(x) - p(x)\| < \frac{1}{3}\varepsilon$ . For any  $n \in \mathbb{N}$ , we have

$$\|b^{n+1}a - ab^{n+1}\| \leq \|b^{n+1}a - b^n ab\| + \|b^n ab - ab^{n+1}\| \leq \|ba - ab\| + \|b^n a - ab^n\|.$$

It follows by induction that  $\|b^n a - ab^n\| \leq n \|ba - ab\|$ . Setting

$$\delta = \frac{\varepsilon}{3m(1 + \sum |c_j|)},$$

we obtain the estimate

$$\begin{aligned} \|f(b)a - af(b)\| &\leq \|f(b)a - p(b)a\| + \|p(b)a - ap(b)\| + \|ap(b) - af(b)\| \\ &\leq 2\|a\| \cdot \sup_{x \in [0,1]} \|f(x) - p(x)\| + \sum_{j=0}^m j |c_j| \|ba - ab\| \\ &< \frac{2}{3}\varepsilon + m\delta \sum_{j=0}^n |c_j| \\ &< \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon \\ &= \varepsilon, \end{aligned}$$

as desired. □

**Lemma II.6.** *Let  $f \in C([0,1])$ . For every  $\varepsilon > 0$ , there is a  $\delta > 0$  (depending on both  $\varepsilon$  and  $f$ ) such that if  $A$  is a unital  $C^*$ -algebra and  $a, b \in A$  satisfy  $0 \leq a, b \leq 1$ , then  $\|a - b\| < \delta$  implies  $\|f(a) - f(b)\| < \varepsilon$ .*

*Proof.* By the Stone-Weierstrass Theorem, there is a polynomial  $p(z) = c_m z^m + \dots + c_1 z + c_0$  such that  $\sup_{x \in [0,1]} \|f(x) - p(x)\| < \frac{1}{3}\varepsilon$ . For any  $n \in \mathbb{N}$ , we have

$$\|a^{n+1} - b^{n+1}\| \leq \|a^{n+1} - ab^n\| + \|ab^n - b^{n+1}\| \leq \|a^n - b^n\| + \|a - b\|.$$

It follows by induction that  $\|a^n - b^n\| \leq n\|a - b\|$ . Setting

$$\delta = \frac{\varepsilon}{3m(1 + \sum |c_j|)},$$

we obtain the estimate

$$\begin{aligned} \|f(a) - f(b)\| &\leq \|f(a) - p(a)\| + \|p(a) - p(b)\| + \|p(b) - f(b)\| \\ &\leq 2 \cdot \sup_{x \in [0,1]} \|f(x) - p(x)\| \sum_{j=0}^m j |c_j| \|a - b\| \\ &< \frac{2}{3}\varepsilon + m\delta \sum_{j=0}^n |c_j| \\ &< \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon \\ &= \varepsilon, \end{aligned}$$

as desired. □

**Definition II.7.** Let  $A$  be a separable, unital  $C^*$ -algebra, and let  $T(A)$  denote the set of tracial states on  $A$ . For  $\alpha \in \text{Aut}(A)$ , we say a trace  $\tau \in T(A)$  is  $\alpha$ -invariant if  $\tau(\alpha(a)) = \tau(a)$  for all  $a \in A$ . For  $\alpha \in \text{Aut}(A)$ , we adopt the notation

$$T_\alpha(A) = \{\tau \in T(A) : \tau \text{ is } \alpha\text{-invariant}\}.$$

**Lemma II.8.** Let  $A$  be a separable, unital  $C^*$ -algebra, let  $\alpha \in \text{Aut}(A)$ , and let  $\tau \in T_\alpha(A)$ . Then the set  $I = \{a \in A : \tau(a^*a) = 0\}$  is an  $\alpha$ -invariant ideal of  $A$ .

*Proof.* The map  $a \mapsto \tau(a^*a)$  is clearly a bounded linear functional  $A \rightarrow \mathbb{C}$ , so the set  $I = \{a \in A : \tau(a^*a) = 0\}$  is closed. In Section 3.4 of [35] it is shown that  $I$  is a closed left ideal of  $A$  (using Theorem 3.3.7 there). As  $\tau(aa^* = \tau(a^*a)$ , it is clear that  $a \in I$  if and only if  $a^* \in I$ . Therefore  $I$  is a closed left ideal of  $A$  that is closed under adjoints. But then for any  $b \in A$  and  $a \in I$ , we have  $b^* \in A$  and  $a^* \in I$ . Since  $I$  is a left ideal of  $A$ , we get  $b^*a^* \in I$ , and since  $I$  is closed under adjoints, it follows that  $ab = (b^*a^*)^* \in I$ . Therefore,  $I$  is an ideal of  $A$ .

Finally, given  $a \in I$ , the  $\alpha$ -invariance of  $\tau$  implies that

$$\tau((\alpha(a))^*(\alpha(a))) = \tau(\alpha(a^*)\alpha(a)) = \tau(\alpha(a^*a)) = \tau(a^*a) = 0,$$

and this gives  $\alpha(a) \in I$ . Therefore,  $I$  is  $\alpha$ -invariant.  $\square$

**Proposition II.9.** *Let  $A$  be a separable, unital  $C^*$ -algebra, let  $\alpha \in \text{Aut}(A)$ , and assume that  $A$  has no  $\alpha$ -invariant ideals. Then given any  $\tau \in T_\alpha(A)$  and any  $y \in A$  with  $\text{sp}(y) = [0, 1]$ , and with  $\mu$  the spectral measure for  $\tau$  on  $C^*(y, 1)$ , there is an open interval  $U \subset [0, 1]$  such that  $U \neq \emptyset$  and  $\mu(U) < \varepsilon$ .*

*Proof.* Since  $A$  has no non-trivial  $\alpha$ -invariant ideals, Lemma II.8 implies that  $\{a \in A: \tau(a^*a) = 0\} = 0$ , and so  $\tau$  is faithful. Let  $V \subset [0, 1]$  be any non-empty open interval, let  $x_0 \in V$ , and define  $f \in C^*(y, 1) \cong C([0, 1])$  by setting  $f(x_0) = 1$ ,  $f(x) = 0$  for  $x \in [0, 1] \setminus V$ , and extending continuously with the Tietze Extension Theorem. Then  $0 \leq f \leq 1$  and  $f \neq 0$ , which imply that

$$\mu(V) \geq \int_0^1 f d\mu = \tau(f) > 0.$$

Hence all non-empty open intervals in  $[0, 1]$  have positive  $\mu$ -measure. For  $n = 2, 3, 4, \dots$  define open intervals  $U_n \subset [0, 1]$  by  $U_n = \left(\frac{1}{n+1}, \frac{1}{n}\right)$ . Then the collection  $(U_n)_{n=2}^\infty$  is pairwise disjoint, and  $\mu(U_n) > 0$  for all  $n \geq 2$  by the previous argument. By pairwise disjointness it follows that

$$\sum_{n=2}^\infty \mu(U_n) = \mu\left(\bigcup_{n=2}^\infty U_n\right) \leq \mu([0, 1]) = 1$$

and so this series converges. Thus for some  $N \in \mathbb{N}$  we must have  $\sum_{n=N}^\infty \mu(U_n) < \varepsilon$ , and so by setting  $U = U_N$  we obtain a non-empty open interval  $U \subset [0, 1]$  with  $\mu(U) < \varepsilon$ .  $\square$

In order for the previous lemma to be useful we must know that our  $C^*$ -algebra  $A$  contains a positive element with spectrum equal to  $[0, 1]$ . We thus introduce the following definition.

**Definition II.10.** *A  $C^*$ -algebra  $A$  is called scattered if every state on  $A$  is atomic; that is, given any state  $\omega$  on  $A$ , there exist pure states  $(\omega_j)_{j=1}^\infty$  and real numbers  $(t_j)_{j=1}^\infty$ , satisfying  $t_j \geq 0$  for all  $j \geq 1$  and  $\sum_{j=1}^\infty t_j = 1$ , such that*

$$\omega = \sum_{j=1}^\infty t_j \omega_j.$$

By Theorem 2.2 of [18], a  $C^*$ -algebra is scattered if and only if the spectrum of every self-adjoint element of  $A$  is countable. The argument in the fourth fact about scattered  $C^*$ -algebras on page 61 of [1] shows that if  $A$  is unital and not scattered, then there is a positive element  $y \in A$  with  $\text{sp}(y) = [0, 1]$ . For the case in which we have the most interest the algebras involved are not scattered. See Proposition IV.20 for the justification of this claim.

**Proposition II.11.** *Let  $A$  be a separable, unital  $C^*$ -algebra that is not scattered, let  $\alpha \in \text{Aut}(A)$  have the tracial quasi-Rokhlin property, and assume that  $A$  has no non-trivial  $\alpha$ -invariant ideals. Then for every  $\varepsilon > 0$ , every finite set  $F \subset A$ , every  $n \in \mathbb{N}$ , and every  $\tau \in T_\alpha(A)$ , there exist  $c_0, \dots, c_n \in A$  such that*

1.  $0 \leq c_j \leq 1$  for  $0 \leq j \leq n$ ;
2.  $c_j c_k = 0$  for  $0 \leq j, k \leq n$  and  $j \neq k$ ;
3.  $\|\alpha(c_j) - c_{j+1}\| < \varepsilon$  for  $0 \leq j \leq n-1$ ;
4.  $\|ac_j - c_j a\| < \varepsilon$  for  $0 \leq j \leq n$  and for all  $a \in F$ ;
5. with  $c = \sum_{j=0}^n c_j$ , we have  $\tau(1-c) < \varepsilon$ .

*Proof.* Let  $\varepsilon > 0$ ,  $F \subset A$  finite,  $n \in \mathbb{N}$ , and  $\tau \in T_\alpha(A)$  be given. Since  $A$  is not scattered, there is a  $y \in A$  with  $\text{sp}(y) = [0, 1]$ . Let  $\mu$  be the spectral measure for  $\tau$  on  $C^*(y, 1) \cong C([0, 1])$ , so that

$$\tau(f(y)) = \int_0^1 f d\mu$$

for all  $f \in C([0, 1])$ . By Proposition II.9, there is a non-empty open interval  $I \subset [0, 1]$  such that  $\mu(I) < \varepsilon$ . Since  $I$  is an open interval, there exist  $0 < t_0 < t_1 < t_2 < t_3 < t_4 < t_5 < t_6 < 1$  such that  $I = (t_0, t_6)$ . Define continuous functions  $f, g: [0, 1] \rightarrow [0, 1]$  by

$$f(t) = \begin{cases} 0 & 0 \leq t < t_1 \\ \frac{t-t_1}{t_2-t_1} & t_1 \leq t < t_2 \\ 1 & t_2 \leq t < t_4 \\ \frac{t_4-t}{t_5-t_4} & t_4 \leq t < t_5 \\ 0 & t_5 \leq t \leq 1 \end{cases}$$



and

$$g(t) = \begin{cases} 0 & 0 \leq t < t_2 \\ \frac{t-t_2}{t_3-t_2} & t_2 \leq t < t_3 \\ \frac{t_3-t}{t_4-t_3} & t_3 \leq t < t_4 \\ 0 & t_4 \leq t \leq 1 \end{cases}$$

Then  $\text{supp}(f), \text{supp}(g) \subset I$ ,  $fg = g$ , and  $f, g \neq 0$ . Set  $x = g(y)$  and  $b = f(y)$ . Then  $0 \leq x \leq b \leq 1$  and  $xb = bx = x$ . Now for any  $a \in \overline{xAx}$  with  $0 \leq a \leq 1$ , we have  $a = b^{1/2}ab^{1/2} \leq b^{1/2}(\|a\| \cdot 1)b^{1/2} \leq b$ , and so  $\tau(a) \leq \tau(b)$ . It follows that for any  $a \in \overline{xAx}$ , we have

$$\tau(a) \leq \tau(b) = \int_0^1 f \, d\mu \leq \mu(I) < \varepsilon.$$

Now apply the tracial quasi-Rokhlin property with  $\varepsilon, F, n$ , and  $x$ , obtaining  $c_0, \dots, c_n \in A$  such that:

1.  $0 \leq c_j \leq 1$  for  $0 \leq j \leq n$ ;
2.  $c_j c_k = 0$  for  $0 \leq j, k \leq n$  and  $j \neq k$ ;
3.  $\|\alpha(c_j) - c_{j+1}\| < \varepsilon$  for  $0 \leq j \leq n-1$ ;
4.  $\|ac_j - c_j a\| < \varepsilon$  for  $0 \leq j \leq n$  and for all  $a \in F$ ;
5. with  $c = \sum_{j=0}^n c_j$ , there exists  $N \in \mathbb{N}$ , positive elements  $e_0, \dots, e_N \in A$ , unitaries  $w_0, \dots, w_N \in A$ , and  $d(0), \dots, d(N) \in \mathbb{Z}$  such that:
  - (a)  $1 - c \leq \sum_{j=0}^N e_j$ ;
  - (b)  $\alpha^{d(j)}(e_j)\alpha^{d(k)}(e_k) = 0$  for  $0 \leq j, k \leq N$ ;
  - (c)  $j \neq k$ , and  $w_j \alpha^{d(j)}(e_j) w_j^* \in \overline{xAx}$  for  $0 \leq j \leq N$ .

Since each  $w_j \alpha^{d(j)}(e_j) w_j^* \in \overline{xAx}$ , it follows that  $\sum_{j=0}^N w_j \alpha^{d(j)}(e_j) w_j^* \in \overline{xAx}$ , and so

$$\tau \left( \sum_{j=0}^N w_j \alpha^{d(j)}(e_j) w_j^* \right) < \varepsilon$$

Then the linearity and  $\alpha$ -invariance of  $\tau$  imply that

$$\begin{aligned}
\tau(1 - c) &\leq \tau \left( \sum_{j=0}^N e_j \right) \\
&= \sum_{j=0}^N \tau(e_j) \\
&= \sum_{j=0}^N \tau \left( \alpha^{d(j)}(e_j) \right) \\
&= \sum_{j=0}^N \tau \left( w_j^* w_j \alpha^{d(j)}(e_j) \right) \\
&= \sum_{j=0}^N \tau \left( w_j \alpha^{d(j)}(e_j) w_j^* \right) \\
&= \tau \left( \sum_{j=0}^N w_j \alpha^{d(j)}(e_j) w_j^* \right) \\
&< \varepsilon,
\end{aligned}$$

which completes the proof.  $\square$

**Theorem II.12.** *Let  $A$  be a separable, unital  $C^*$ -algebra that is not scattered, let  $\alpha \in \text{Aut}(A)$  have the tracial quasi-Rokhlin property, and suppose that  $A$  has no non-trivial  $\alpha$ -invariant ideals. Then the restriction map  $T(C^*(\mathbb{Z}, A, \alpha)) \rightarrow T_\alpha(A)$  is bijective.*

*Proof.* We first verify that every trace on  $T(C^*(\mathbb{Z}, A, \alpha))$  is  $\alpha$ -invariant when restricted to  $A$ , so that the restriction map indeed has codomain  $T_\alpha(A)$ . For any  $\tau \in T(C^*(\mathbb{Z}, A, \alpha))$  and any  $a \in A$ , we have

$$\tau(\alpha(a)) = \tau(uau^*) = \tau(au^*u) = \tau(a),$$

and so this is in fact the case.

Next, we show that the restriction map is injective. Let  $\tau \in T(C^*(\mathbb{Z}, A, \alpha))$ , let  $\varepsilon > 0$  be given, let  $a \in A$  be non-zero, let  $k \in \mathbb{N} \setminus \{0\}$ , and let  $u \in C^*(\mathbb{Z}, A, \alpha)$  be the canonical unitary. Set  $F = \{a\}$  and choose  $n \in \mathbb{N}$  such that  $n > k$  and

$$\frac{1}{n} < \frac{\varepsilon^2}{16k^2(\|a^*a\| + 1)}.$$

Apply Lemma II.5 with  $f(x) = \sqrt{x}$  to obtain  $\delta_1(\varepsilon) > 0$  such that for all  $b, e \in A$  with  $0 \leq b, e \leq 1$  and  $\|be - eb\| < \delta_1(\varepsilon)$ , we have

$$\left\| b^{1/2}e - eb^{1/2} \right\| < \frac{\varepsilon}{8n}.$$

Similarly, apply Lemma II.6 with the same  $f$  to obtain  $\delta_2(\varepsilon) > 0$  such that for all  $b, e \in A$  with  $0 \leq b, e \leq 1$  and  $\|e - b\| < \delta_2(\varepsilon)$ , we have

$$\left\| e^{1/2} - b^{1/2} \right\| < \frac{\varepsilon}{8nk(\|a\| + 1)}.$$

Define

$$\delta = \min \left\{ \frac{1}{2n^3 + n^2 + 1}, \delta_1(\varepsilon), \delta_2(\varepsilon), \frac{\varepsilon^2}{4(\tau(a^*a) + 1)} \right\}$$

and apply Proposition II.11 with  $\delta, F, n$ , and  $\tau$  (identifying  $\tau$  with its image in  $T_\alpha(A)$  under the restriction map) to obtain  $c_0, \dots, c_n \in A$  such that:

1.  $0 \leq c_j \leq 1$  for  $0 \leq j \leq n$ ;
2.  $c_j c_k = 0$  for  $0 \leq j, k \leq n$  and  $j \neq k$ ;
3.  $\|\alpha(c_j) - c_{j+1}\| < \delta$  for  $0 \leq j \leq n-1$ ;
4.  $\|c_j a - a c_j\| < \delta$  for  $0 \leq j \leq n$ ;
5. with  $c = \sum_{j=0}^n c_j$ , we have  $\tau(1 - c) < \delta$ .

By the choice of  $\delta$ , and since automorphisms commute with continuous functional calculus, we further obtain

$$\left\| \alpha(c_j^{1/2}) - c_{j+1}^{1/2} \right\| < \frac{\varepsilon}{8nk(\|a\| + 1)}$$

for  $0 \leq j \leq n-k$ , and

$$\left\| c_j^{1/2} a - a c_j^{1/2} \right\| < \frac{\varepsilon}{8n}$$

for  $0 \leq j \leq n$ . It is easy to see that  $0 \leq c \leq 1$  and hence also  $0 \leq 1 - c \leq 1$ . Then  $(1 - c)^{1/2}$  is a well-defined positive element of  $A$  that satisfies  $1 - c \leq 1$ . Observing that that continuous functions  $f_0, f_1: [0, 1] \rightarrow [0, 1]$  given by  $f_0(t) = t^2$  and  $f_1(t) = t$  satisfy  $f_0 \leq f_1$ , continuous functional calculus gives  $(1 - c)^2 \leq (1 - c)$ . It follows that  $\tau((1 - c)^2) \leq \tau(1 - c)$  and so the

Cauchy-Schwarz inequality yields

$$\begin{aligned}
|\tau(au^k(1-c))|^2 &\leq \tau((1-c)^*(1-c))\tau((au^k)(au^k)^*) \\
&= \tau((1-c)^2)\tau((au^k)^*(au^k)) \\
&= \tau((1-c)^2)\tau(u^{-k}a^*au^k) \\
&= \tau((1-c)^2)\tau(a^*a) \\
&\leq \tau(1-c)\tau(a^*a) \\
&< \delta\tau(a^*a).
\end{aligned}$$

Hence  $|\tau(au^k(1-c))| < \sqrt{\delta\tau(a^*a)} < \frac{1}{2}\varepsilon$ .

Next, we observe that if  $e, b \in A$  are positive, then  $eb = 0$  implies that  $e^{1/2}b^{1/2} = 0$  as well. Indeed, the  $C^*$ -property gives

$$\|b^{1/2}e\|^2 = \|(b^{1/2}e)^*(b^{1/2}e)\| = \|ebe\| = 0,$$

which implies that  $b^{1/2}e = 0$ . This gives

$$\|e^{1/2}b^{1/2}\|^2 = \|(e^{1/2}b^{1/2})^*(e^{1/2}b^{1/2})\| = \|b^{1/2}eb^{1/2}\| = 0,$$

which implies that  $e^{1/2}b^{1/2} = 0$  as claimed. In particular, for  $0 \leq j \leq n-k$ , we have  $c_j^{1/2}c_{j+k}^{1/2} = 0$ , and so  $\tau(c_{j+k}^{1/2}au^kc_j^{1/2}) = \tau(au^kc_j^{1/2}c_{j+k}^{1/2}) = 0$ . For  $0 \leq j \leq n-k$ , we also have the inequality

$$\begin{aligned}
\|\alpha^k(c_j^{1/2}) - c_{j+k}^{1/2}\| &\leq \sum_{i=0}^{k-1} \|\alpha^{k-i}(c_{j+i}^{1/2}) - \alpha^{k-i-1}(c_{j+i+1}^{1/2})\| \\
&= \sum_{i=0}^{k-1} \|\alpha(c_{j+i}^{1/2}) - c_{j+i+1}^{1/2}\| \\
&< k\delta.
\end{aligned}$$

It follows that for  $0 \leq j \leq n - k$ ,

$$\begin{aligned}
|\tau(au^k c_j)| &= \left| \tau(au^k c_j^{1/2} c_j^{1/2}) \right| \\
&= \left| \tau(a\alpha^k(c_j^{1/2})u^k c_j^{1/2}) \right| \\
&\leq \left| \tau(a\alpha^k(c_j^{1/2})u^k c_j^{1/2}) - \tau(ac_{j+k}^{1/2}u^k c_j^{1/2}) \right| + \left| \tau(ac_{j+k}^{1/2}u^k c_j^{1/2}) \right| \\
&= \left| \tau(a(\alpha^k(c_j^{1/2}) - c_{j+k}^{1/2})u^k c_j^{1/2}) \right| + \left| \tau((ac_{j+k}^{1/2} - c_{j+k}^{1/2}a)u^k c_j^{1/2}) \right| \\
&\leq \|\tau\| \left\| a(\alpha^k(c_j^{1/2}) - c_{j+k}^{1/2})u^k c_j^{1/2} \right\| + \|\tau\| \left\| (ac_{j+k}^{1/2} - c_{j+k}^{1/2}a)u^k c_j^{1/2} \right\| \\
&\leq \|a\| \left\| \alpha^k(c_j^{1/2}) - c_{j+k}^{1/2} \right\| + \|ac_{j+k}^{1/2} - c_{j+k}^{1/2}a\| \\
&< \|a\| k \left( \frac{\varepsilon}{8nk(\|a\| + 1)} \right) + \frac{\varepsilon}{8n} \\
&< \frac{\varepsilon}{4n}.
\end{aligned}$$

For  $0 \leq k \leq n - 1$  the  $\alpha$ -invariance of  $\tau$  implies that

$$|\tau(c_{j+1}) - \tau(c_j)| = |\tau(c_{j+1}) - \tau(\alpha(c_j))| = |\tau(c_{j+1} - \alpha(c_j))| \leq \|c_{j+1} - \alpha(c_j)\| < \delta,$$

and so we obtain

$$\begin{aligned}
\left| (n+1)\tau(c_0) - \sum_{j=0}^n \tau(c_j) \right| &\leq \sum_{j=1}^n |\tau(c_j) - \tau(c_0)| \\
&\leq \sum_{j=1}^n \sum_{i=0}^{j-1} |\tau(c_{j-i}) - \tau(c_{j-i-1})| \\
&< \sum_{j=1}^n j\delta \\
&\leq n^2\delta.
\end{aligned}$$

Now, since  $0 \leq c \leq 1$ , we have  $\sum_{j=0}^n \tau(c_j) \leq 1$ . Combining this with the previous result gives

$$(n+1)\tau(c_0) < n^2\delta + \sum_{j=0}^n \tau(c_j) \leq n^2\delta + 1,$$

and this implies that

$$\tau(c_0) < \frac{n^2\delta + 1}{n+1} < \frac{\frac{1}{2n} + 1}{n+1} < \frac{1}{n}.$$

Further, since  $|\tau(c_j) - \tau(c_0)| < n\delta$  for  $1 \leq j \leq n$  (this follows by iterating one of the previous inequalities with the triangle inequality), we conclude that for  $0 \leq j \leq n$ , we have

$$\tau(c_j) < n\delta + \tau(c_0) < n\delta + \frac{n^2\delta + 1}{n+1} < \frac{(2n^2 + n)\delta + 1}{n+1} < \frac{\frac{1}{n} + 1}{n+1} = \frac{1}{n}.$$

Now  $0 \leq c_j \leq 1$  implies that  $c_j^2 \leq c_j$  by the same functional calculus argument that was used to show  $(1-c)^2 \leq 1-c$ , and consequently  $0 \leq \tau(c_j^2) \leq \tau(c_j)$ . Applying Theorems 3.3.2 and 3.3.7 of [35] gives

$$\begin{aligned} |\tau(au^k c_j)|^2 &\leq \|\tau\| \tau((au^k c_j)^*(au^k c_j)) \\ &= \tau((u^k c_j)^* a^* a (u^k c_j)) \\ &\leq \|a^* a\| \tau((u^k c_j)^*(u^k c_j)) \\ &= \|a^* a\| \tau(c_j^2) \\ &\leq \|a^* a\| \tau(c_j) \\ &< \frac{\|a^* a\|}{n} \\ &< \frac{\varepsilon^2}{16k^2}, \end{aligned}$$

which implies  $|\tau(au^k c_j)| < \frac{\varepsilon}{4k}$ .

Finally, we compute

$$\begin{aligned} |\tau(au^k)| &\leq |\tau(au^k(1-c))| + |\tau(au^k c)| \\ &< \frac{1}{2}\varepsilon + \sum_{j=0}^{n-k} |\tau(au^k c_j)| + \sum_{j=n-k+1}^n |\tau(au^k c_j)| \\ &< \frac{1}{2}\varepsilon + \sum_{j=0}^{n-k} \frac{\varepsilon}{4n} + \sum_{j=n-k+1}^n \frac{\varepsilon}{4k} \\ &\leq \frac{1}{2}\varepsilon + \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon \\ &= \varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  was arbitrary, it follows that  $\tau(au^k) = 0$ . Now if  $k \in \mathbb{Z}$  with  $k < 0$ , then the previous argument implies that  $\tau(a^*u^{-k}) = 0$ , and therefore

$$\tau(au^k) = \tau(u^ka) = \tau((a^*u^{-k})^*) = \overline{\tau(a^*u^{-k})} = 0.$$

Thus for any  $\tau \in T(C^*(\mathbb{Z}, A, \alpha))$ , any non-zero  $a \in A$ , and any  $k \in \mathbb{Z} \setminus \{0\}$ , we have  $\tau(au^k) = 0$ . Let  $E: C^*(\mathbb{Z}, A, \alpha) \rightarrow A$  be the standard conditional expectation. Then for any element  $\sum_{j=-N}^N a_j u^j \in C_c(\mathbb{Z}, A, \alpha)$ , we have

$$\tau \left( \sum_{j=-N}^N a_j u^j \right) = \tau(a_0) = \tau \left( E \left( \sum_{j=-N}^N a_j u^j \right) \right),$$

and so  $\tau = \tau \circ E$  on a dense subset of  $C^*(\mathbb{Z}, A, \alpha)$ . This implies that the restriction map  $T(C^*(\mathbb{Z}, A, \alpha)) \rightarrow T_\alpha(A)$  is injective.

For surjectivity, let  $\tau \in T_\alpha(A)$ , and let  $E$  be the standard conditional expectation introduced above. We claim that  $\tilde{\tau} = \tau \circ E$  is a tracial state on  $C^*(\mathbb{Z}, A, \alpha)$  that satisfies  $\tilde{\tau}|_A = \tau$ . It is clear that  $\tilde{\tau}$  is a positive linear map since both  $\tau$  and  $E$  are positive, and we compute  $\tilde{\tau}(1) = \tau(E(1)) = \tau(1) = 1$ . Let  $a = a_0 u^m$  and  $b = b_0 u^n$  for some  $a_0, b_0 \in A$  and  $m, n \in \mathbb{Z}$ . Then we obtain the formulas

$$ab = a_0 u^m b_0 u^n = a_0 \alpha^m(b_0) u^{m+n}$$

and

$$ba = b_0 u^n a_0 u^m = b_0 \alpha^n(a_0) u^{m+n}$$

If  $m \neq n$ , then  $E(ab) = 0 = E(ba)$ , and consequently  $\tilde{\tau}(ab) = 0 = \tilde{\tau}(ba)$ . So assume that  $m = -n$ , which implies  $E(ab) = a_0 \alpha^{-n}(b_0)$  and  $E(ba) = b_0 \alpha^n(a_0)$ . Using the  $\alpha$ -invariance of  $\tau$  and the trace property, we obtain

$$\tau(a_0 \alpha^{-n}(b_0)) = \tau(\alpha^{-n}(\alpha^n(a_0) b_0)) = \tau(\alpha^n(a_0) b_0) = \tau(b_0 \alpha^n(a_0)),$$

which implies that

$$\tilde{\tau}(ab) = \tau(E(ab)) = \tau(E(ba)) = \tilde{\tau}(ba).$$

Since the dense subset  $C_c(\mathbb{Z}, A, \alpha)$  of  $C^*(\mathbb{Z}, A, \alpha)$  is linearly spanned by elements of the form  $au^n$  for

$a \in A$  and  $n \in \mathbb{Z}$ , it follows that  $\tilde{\tau}$  is a tracial state on  $C^*(\mathbb{Z}, A, \alpha)$ . Since  $E(a) = a$  for all  $a \in A$ , we clearly have  $\tilde{\tau}|_A = \tau$ , which completes the proof that the restriction map  $T(C^*(\mathbb{Z}, A, \alpha)) \rightarrow T_\alpha(A)$  is surjective, and hence a bijection.  $\square$



## CHAPTER III

## COMPARISON IN CERTAIN MINIMAL DYNAMICAL SYSTEMS

Applications of dynamics to  $C^*$ -algebras frequently require the use of techniques from both topology and measure theory. It is therefore crucial that given a dynamical system  $(X, h)$ , there is some degree of control over the interactions between the topological dynamics (given by  $h$ ) and the space  $M_h(X)$  of  $h$ -invariant Borel probability measures on  $X$ . In this chapter, we shall develop a condition which tells us these interactions behave in a reasonably nice way, which will play a crucial role in demonstrating that the tracial quasi-Rokhlin property is satisfied by certain automorphisms related to dynamical systems.

**Notation III.1.** *Throughout, we let  $X$  be an infinite compact metric space with finite covering dimension, and let  $h: X \rightarrow X$  be a minimal homeomorphism. The corresponding minimal dynamical system  $(X, h)$  will frequently be denoted simply by  $X$ , with the homeomorphism  $h$  understood. For  $x \in X$  and  $\varepsilon > 0$ , we will denote the  $\varepsilon$ -ball centered at  $x$  by*

$$B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}.$$

**Lemma III.2.** *Let  $(X, h)$  be as in Notation III.1. If  $U \subset X$  is non-empty and open, then  $X = \bigcup_{n=-\infty}^{\infty} h^n(U)$ . Moreover,  $\mu(U) > 0$  for all  $\mu \in M_h(X)$ .*

*Proof.* Set  $Y = X \setminus \bigcup_{n=-\infty}^{\infty} h^n(U)$ , which is closed. Let  $y \in h(Y)$ , so that  $y = h(y')$  for some  $y' \in Y$ . If  $y \notin Y$ , then we must have  $y \in h^n(U)$  for some  $n \in \mathbb{Z}$ , and we may write  $y = h(x)$  for some  $x \in h^{n-1}(U)$ . But then  $h(y') = h(x)$  implies that  $y' = x$ , a contradiction since  $y' \notin \bigcup_{n=-\infty}^{\infty} h^n(U)$ . Thus  $h(Y) \subset Y$  and now the minimality of  $h$  implies that  $Y = \emptyset$  or  $Y = X$ . But clearly  $\bigcup_{n=-\infty}^{\infty} h^n(U) \neq \emptyset$ , and hence  $Y \neq X$ . Therefore  $Y = \emptyset$  and  $X = \bigcup_{n=-\infty}^{\infty} h^n(U)$ .

Now suppose that  $\mu(U) = 0$  for some  $\mu \in M_h(X)$ . Then the  $h$ -invariance of  $\mu$  implies that

$$1 = \mu(X) = \mu\left(\bigcup_{n=-\infty}^{\infty} h^n(U)\right) \leq \sum_{n=-\infty}^{\infty} \mu(h^n(U)) = \sum_{n=-\infty}^{\infty} \mu(U) = 0,$$

a contradiction.  $\square$

The following version of Urysohn's Lemma (see [48]) will be used frequently without comment in many of the arguments that follow. Note that we take the definition of  $\text{supp}(f)$  to be

$$\text{supp}(f) = \overline{\{x \in X : f(x) \neq 0\}}.$$

**Proposition III.3.** *Let  $X$  be a compact Hausdorff space. Let  $F \subset E \subset X$  with  $F$  closed and  $E$  open. Then there is a continuous function  $f : X \rightarrow [0, 1]$  such that  $f = 1$  on  $F$  and  $\text{supp}(f) \subset E$ .*

**Lemma III.4.** *Let  $(X, h)$  be as in Notation III.1. For any  $\varepsilon > 0$  and any non-empty open set  $U \subset X$ , there is a non-empty open set  $E \subset U$  such that  $\mu(E) < \varepsilon$  for all  $\mu \in M_h(X)$ .*

*Proof.* Let  $x \in U$ , and let  $\delta > 0$  be such that  $B(x, \delta) \subset U$ . Define a sequence  $(E_n)_{n=0}^{\infty}$  of open sets by  $E_n = B(x, \delta/(n+1))$ . Then  $\overline{E_{n+1}} \subset E_n$  for all  $n \in \mathbb{N}$ , and  $\bigcap_{n=0}^{\infty} E_n = \{x\}$ . Choose continuous functions  $f_n : X \rightarrow [0, 1]$  with  $f_n = 1$  on  $\overline{E_{n+1}}$  and  $\text{supp}(f_n) \subset E_n$ . Then  $f_n \geq f_{n+1}$  for all  $n \in \mathbb{N}$ . Now each  $f_n$  defines an affine function  $\widehat{f}_n$  on  $M_h(X)$  by

$$\widehat{f}_n(\mu) = \int_X f_n d\mu.$$

It is easily seen that the minimality of  $h$  implies that  $\mu(\{x\}) = 0$  for all  $\mu \in M_h(X)$ . Applying the Dominated Convergence Theorem, we conclude that

$$\lim_{n \rightarrow \infty} \widehat{f}_n(\mu) = \lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X \lim_{n \rightarrow \infty} f_n d\mu = \mu(\{x\}) = 0.$$

for all  $\mu \in M_h(X)$ . It follows that the monotone decreasing sequence  $(\widehat{f}_n)_{n=1}^{\infty}$  of continuous functions converges pointwise to the continuous affine function  $\widehat{f} = 0$  on the compact set  $M_h(X)$ , and so Dini's Theorem implies that the convergence is uniform. Therefore, there is an  $N \in \mathbb{N}$  such that  $\widehat{f}_N(\mu) < \varepsilon$  for all  $\mu \in M_h(X)$ . Finally, set  $E = E_{N+1}$ . Then  $E \subset U$ , and  $f_N|_{\overline{E}} = 1$  implies

that

$$\mu(E) \leq \int_X f_N d\mu = \widehat{f}_N(\mu) < \varepsilon$$

for all  $\mu \in M_h(X)$ . □

The following definition has been proposed by N. Christopher Phillips [44] as an analogue of a transversality property for manifolds. Its importance in our development will become apparent later.

**Definition III.5.** *Let  $(X, h)$  be as in Notation III.1. A closed subset  $F \subset X$  is said to be topologically  $h$ -small if there is some  $m \in \mathbb{Z}_+$  such that whenever  $d(0), d(1), \dots, d(m)$  are  $m + 1$  distinct elements of  $\mathbb{Z}$ , then  $h^{d(0)}(F) \cap h^{d(1)}(F) \cap \dots \cap h^{d(m)}(F) = \emptyset$ . The smallest such constant  $m$  is called the topological smallness constant. We say  $(X, h)$  has the topological small boundary property if whenever  $F, K \subset X$  are disjoint compact sets, then there exist open sets  $U, V \subset X$  such that  $F \subset U$ ,  $K \subset V$ ,  $\overline{U} \cap \overline{V} = \emptyset$ , and  $\partial U$  is topologically  $h$ -small.*

The next two propositions describe how closed and open sets can be approximated in measure by sets with topologically small boundaries.

**Proposition III.6.** *Suppose that  $(X, h)$  has the topological small boundary property, and let  $\varepsilon > 0$  be given. Then for any closed subset  $F \subset X$  any open subset  $E \subset X$  with  $F \subset E$ , and any  $h$ -invariant Borel probability measure  $\mu$  on  $X$ , there is an open subset  $U \subset X$  such that  $F \subset U \subset \overline{U} \subset E$ ,  $\partial U$  is topologically  $h$ -small, and  $\mu(U) - \mu(F) < \varepsilon$ .*

*Proof.* Using the regularity of  $\mu$  choose an open set  $W_0$  such that  $F \subset W_0$  and  $\mu(W_0) - \mu(F) < \varepsilon$ , and set  $W_1 = W_0 \cap E$ . Then  $F \subset W_1 \subset E$ , and  $\mu(W_1) - \mu(F) \leq \mu(W_0) - \mu(F) < \varepsilon$ . Since  $X$  is locally compact Hausdorff, there is an open set  $W \subset X$  such that  $\overline{W}$  is compact and  $F \subset W \subset \overline{W} \subset W_1$ . Then we also have  $\mu(W) - \mu(F) \leq \mu(W_1) - \mu(F) < \varepsilon$ . Set  $K = X \setminus W$ , which is a compact subset of  $X$  disjoint from  $F$ , and apply the topological small boundary property to  $F$  and  $K$ , obtaining open sets  $U, V \subset X$  such that  $F \subset U$ ,  $K \subset V$ ,  $\overline{U} \cap \overline{V} = \emptyset$ , and  $\partial U$  is topologically  $h$ -small. Since  $K \subset V$ , it follows that  $U \cap (X \setminus W) = \emptyset$  as well, and so  $U \subset W$ . Then  $\overline{U} \subset \overline{W} \subset E$ , and  $\mu(U) - \mu(F) \leq \mu(W) - \mu(F) < \varepsilon$ . □

**Proposition III.7.** *Suppose  $(X, h)$  has the topological small boundary property, and let  $\varepsilon > 0$  be given. Then for any open set  $E \subset X$ , any  $\mu \in M_h(X)$ , and any  $\sigma \geq 0$  with  $\sigma < \mu(E)$ , there is an open set  $U \subset E$  such that  $\overline{U} \subset E$ ,  $\partial U$  is topologically  $h$ -small,  $\mu(E) - \mu(U) < \varepsilon$ , and  $\sigma < \mu(U)$ .*

*Proof.* Set  $\delta = \min \{ \frac{1}{2}\varepsilon, \frac{1}{2}(\mu(E) - \sigma) \}$ , and use the regularity of  $\mu$  to choose a compact set  $F \subset E$  with  $\mu(E) - \mu(F) < \delta$ . Since  $X$  is locally compact Hausdorff, there is an open set  $W$  with  $\overline{W}$  compact satisfying  $F \subset W \subset \overline{W} \subset E$ . Set  $K = X \setminus W$ . Then  $F$  and  $K$  are disjoint compact subsets of  $X$ , so we may apply the small boundary property to obtain open sets  $U, V \subset X$  such that  $F \subset U$ ,  $K \subset V$ ,  $\overline{U} \cap \overline{V} = \emptyset$ , and  $\partial U$  is topologically  $h$ -small. Then  $U \cap (X \setminus W) = \emptyset$ , which implies  $U \subset W$ , and then we immediately have  $\overline{U} \subset \overline{W} \subset E$  as required. Finally,  $F \subset U \subset E$  implies that  $\mu(E) - \mu(U) \leq \mu(E) - \mu(F) < \delta < \varepsilon$ , and that

$$\begin{aligned} \mu(U) - \sigma &= (\mu(E) - \sigma) - (\mu(E) - \mu(U)) \\ &> (\mu(E) - \sigma) - \delta \\ &\geq \frac{1}{2}(\mu(E) - \sigma) \\ &> 0, \end{aligned}$$

which gives  $\sigma < \mu(U)$  as required.  $\square$

The following theorem is the well-known Rokhlin tower construction, where the space  $X$  is decomposed in terms of a closed set  $Y \subset X$  and the “first return times to  $Y$ ” for the points of  $X$ . We show that a Rokhlin tower can be made compatible with some given partition of  $X$  by sets with non-empty interior, in the sense that the interior of each level in the tower is contained in exactly one set of the partition.

**Theorem III.8.** *Let  $(X, h)$  be as in Notation III.1. Let  $Y \subset X$  be a closed set with  $\text{int}(Y) \neq \emptyset$ . For  $y \in Y$ , define  $r(y) = \min \{ m \geq 1 : h^m(y) \in Y \}$ . Then  $\sup_{y \in Y} r(y) < \infty$ , so there are finitely many distinct values  $n(0) < n(1) < \dots < n(l)$  in the range of  $r$ . For  $0 \leq k \leq l$ , set*

$$Y_k = \overline{\{y \in Y : r(y) = n(k)\}} \quad \text{and} \quad Y_k^\circ = \text{int}(\{y \in Y : r(y) = n(k)\}).$$

*Then:*

1. the sets  $h^j(Y_k^\circ)$  are pairwise disjoint for  $0 \leq k \leq l$  and  $0 \leq j \leq n(k) - 1$ ;
2.  $\bigcup_{k=0}^l Y_k = Y$ ;
3.  $\bigcup_{k=0}^l \bigcup_{j=0}^{n(k)-1} h^j(Y_k) = X$ .

Moreover, given any finite partition  $\mathcal{P}$  of  $X$  (consisting of sets with non-empty interior), there exist closed sets  $Z_0, \dots, Z_m \subset Y$  and non-negative integers  $t(0) \leq t(1) \leq \dots \leq t(m)$  such that with  $Z_k^{(0)} = Z_k \setminus \partial Z_k$  (which may be empty) for  $0 \leq k \leq m$ , we have:

1. the sets  $h^j(Z_k^{(0)})$  are pairwise disjoint for  $0 \leq k \leq m$  and  $0 \leq j \leq t(k) - 1$ ;
2.  $\bigcup_{k=0}^m Z_k = Y$ ;
3.  $\bigcup_{k=0}^m \bigcup_{j=0}^{t(k)-1} h^j(Z_k) = X$ ;
4. for  $0 \leq k \leq m$  and  $0 \leq j \leq t(k) - 1$ , the set  $h^j(Z_k^{(0)})$  is contained in exactly one  $P \in \mathcal{P}$ .

*Proof.* The finiteness of  $r(y)$  and all statements concerning the sets  $Y_k$  are shown in [29]. Now suppose we have a finite partition  $\mathcal{P}$  of  $X$  consisting of sets with non-empty interior. For each  $0 \leq k \leq l$ , the set

$$\mathcal{B}_k = \{h^{-j}(h^j(Y_k) \cap P) : 0 \leq j \leq n(k) - 1, P \in \mathcal{P}\}$$

is a cover of  $Y_k$  by a finite collection of sets with non-empty interior. Write  $\mathcal{B}_k = \{B_1, \dots, B_N\}$  for an appropriate choice of  $N \in \mathbb{N}$ . Let  $\mathcal{C}_k$  be the collection of all sets of the form  $D = \bigcap_{i=1}^m C_i$ , where each for each  $i$ , there is a  $j \in \{1, \dots, N\}$  such that either  $C_i = B_j$  or  $C_i = Y_k \setminus B_j$ . Set  $\mathcal{C}^\circ = \bigcup_{k=0}^l \mathcal{C}_k$  and  $\mathcal{C} = \{\overline{D} : D \in \mathcal{C}^\circ\}$ , both of which are finite collections of sets. Write  $\mathcal{C} = \{Z'_0, \dots, Z'_m\}$ , and for  $0 \leq i \leq m$ , set  $t(i) = n(k)$  where  $Z'_i = \overline{D}$  and  $D \in \mathcal{C}_k$ . Without loss of generality, arrange the order of the sets  $Z'_0, \dots, Z'_m$  so that  $t(0) \leq t(1) \leq \dots \leq t(m)$ . Finally, define  $Z_k$  and  $Z_k^{(0)}$  for  $0 \leq k \leq m$  by

$$Z_0 = Z'_0, \quad Z_k = \overline{Z'_k \setminus \bigcup_{j=0}^{k-1} Z'_j}, \quad Z_k^{(0)} = Z_k \setminus \partial Z_k.$$

Then  $Z_0, \dots, Z_m$  is a cover of  $Y$  by closed sets with the desired properties.  $\square$

It is technically important to have some control over the boundary  $\partial Y$  of a closed set  $Y \subset X$  used in the construction of a Rokhlin tower as above. In [29] this is accomplished by restricting to the situation where  $X$  is a compact smooth manifold and  $h$  is a minimal diffeomorphism, then requiring that  $\partial Y$  satisfy a certain transversality condition. Definition III.5 is an attempt to formulate an analogous property for the case of a more general compact metric space. For our purposes, we will find it convenient to use another type of smallness property for closed sets, also

proposed by N. Christopher Phillips. The connection between Definition III.5 and the following one is given by Proposition III.15.

**Definition III.9.** Let  $(X, h)$  be as in Notation III.1. Let  $F \subset X$  be closed and let  $U \subset X$  be open. We write  $F \prec U$  if there exist  $M \in \mathbb{N}$ ,  $U_0, \dots, U_M \subset X$  open, and  $d(0), \dots, d(M) \in \mathbb{Z}$  such that:

1.  $F \subset \bigcup_{j=0}^M U_j$ ;
2.  $h^{d(j)}(U_j) \subset U$  for  $0 \leq j \leq M$ ;
3. the sets  $h^{d(j)}(U_j)$  are pairwise disjoint for  $0 \leq j \leq M$ .

We say the closed set  $F$  is thin if  $F \prec U$  for every non-empty open set  $U \subset X$ .

It is clear that any closed subset of a thin set is thin, and hence the intersection of arbitrarily many thin sets is thin. It is also clear that if  $F$  is thin, then so is  $h^n(F)$  for any  $n \in \mathbb{Z}$ .

**Lemma III.10.** Let  $(X, h)$  be as in Notation III.1. Suppose that  $F \subset X$  is closed and  $U \subset X$  is open with  $F \prec U$ . Then there is an open set  $V \subset X$  such that  $F \subset V$  and  $\bar{V} \prec U$ .

*Proof.* Since  $F \prec U$ , there exist  $M \in \mathbb{N}$ ,  $U_0, \dots, U_M \subset X$  open, and  $d(0), \dots, d(M) \in \mathbb{Z}$  such that  $F \subset \bigcup_{j=0}^M U_j$  and such that the sets  $h^{-d(j)}(U_j)$  are pairwise disjoint subsets of  $U$ . Let  $E = \bigcup_{j=0}^M U_j$ , and use  $X$  locally compact Hausdorff to choose an open set  $V$  with  $\bar{V}$  compact satisfying  $F \subset V \subset \bar{V} \subset E$ . Then  $\bar{V} \prec U$  using the same open sets  $U_j$  and integers  $d(j)$  as for  $F$ .  $\square$

**Lemma III.11.** Let  $(X, h)$  be as in Notation III.1. If  $F \subset X$  is thin, then  $\mu(F) = 0$  for all  $\mu \in M_h(X)$ .

*Proof.* Let  $\varepsilon > 0$  be given, and choose  $N \in \mathbb{N}$  such that  $1/N < \varepsilon$ . Since the action of  $h$  on  $X$  is free, there is a point  $x \in X$  such that  $x, h(x), \dots, h^N(x)$  are distinct. Choose disjoint open neighborhoods  $U_0, \dots, U_N$  of these points, and let  $U = \bigcap_{j=0}^N h^{-j}(U_j)$ , which is an open neighborhood of  $x$  such that  $U, h(U), \dots, h^N(U)$  are pairwise disjoint. Now let  $\mu \in M_h(X)$ . Then using the  $h$ -invariance of  $\mu$ , it follows that

$$(N+1)\mu(U) = \sum_{j=0}^N \mu(h^j(U)) = \mu\left(\bigcup_{j=0}^N h^j(U)\right) \leq \mu(X) = 1,$$

which gives  $\mu(U) < 1/N < \varepsilon$ . Since  $F$  is thin, we have  $F \prec U$ , and so there exist  $M \in \mathbb{N}$ ,  $U_0, \dots, U_M \subset X$  open, and  $d(0), \dots, d(M) \in \mathbb{Z}$  such that  $F \subset \bigcup_{j=0}^M U_j$  and such that the sets  $h^{d(j)}(U_j)$  are pairwise disjoint subsets of  $U$  for  $0 \leq j \leq M$ . Then again using the  $h$ -invariance of  $\mu$ , we have

$$\mu(F) \leq \mu\left(\bigcup_{j=0}^M U_j\right) \leq \sum_{j=0}^M \mu(U_j) = \sum_{j=0}^M \mu(h^{d(j)}(U_j)) = \mu\left(\bigcup_{j=0}^M h^{d(j)}(U_j)\right) \leq \mu(U) < \varepsilon.$$

Since  $\varepsilon > 0$  was arbitrary, it follows that  $\mu(F) = 0$ .  $\square$

**Lemma III.12.** *Let  $(X, h)$  be as in Notation III.1.*

1. *If  $F_1, F_2 \subset X$  are closed and  $V_1, V_2 \subset X$  are open such that  $F_1 \prec V_1$ ,  $F_2 \prec V_2$ , and  $V_1 \cap V_2 = \emptyset$ , then  $F_1 \cup F_2 \prec V_1 \cup V_2$ .*
2. *The union of finitely many thin sets in  $X$  is thin.*

*Proof.* To prove (1), simply observe that since  $V_1 \cap V_2 = \emptyset$ , the union of a pairwise disjoint collection of subsets of  $V_1$  and a pairwise disjoint collection of subsets of  $V_2$  is still pairwise disjoint.

For (2), it is sufficient to prove that the union of two thin sets is thin. Let  $F_1, F_2 \subset X$  be thin closed sets, and let  $U \subset X$  be a non-empty open set. Since  $h$  is minimal there must be distinct points  $x_1, x_2 \in U$ . Let  $V_1 \subset U$  and  $V_2 \subset U$  be disjoint open neighborhoods of  $x_1$  and  $x_2$  respectively. Then  $F_1 \prec V_1$  and  $F_2 \prec V_2$ , and now part 1 implies that  $F_1 \cup F_2 \prec V_1 \cup V_2 \subset U$ , which proves that  $F_1 \cup F_2$  is thin.  $\square$

**Lemma III.13.** *Let  $(X, h)$  be as in Notation III.1. Let  $F \subset X$  be a thin closed set, and let  $U \subset X$  be open. Then there exist  $N \in \mathbb{N}$ ,  $F_0, \dots, F_M \subset X$  closed, and  $d(0), \dots, d(M) \in \mathbb{Z}$  such that:*

1.  $F \subset \bigcup_{j=0}^M F_j$ ;
2.  $h^{d(j)}(F_j) \subset U$  for  $0 \leq j \leq M$ ;
3. *the sets  $h^{d(j)}(F_j)$  are pairwise disjoint for  $0 \leq j \leq M$ .*

*Proof.* Since  $F$  is thin, we have  $F \prec U$ , and so there exist  $N \in \mathbb{N}$ ,  $U_0, \dots, U_M \subset X$  open, and  $d(0), \dots, d(M) \in \mathbb{Z}$  such that  $F \subset \bigcup_{j=0}^M U_j$  and the sets  $h^{d(j)}(U_j)$  are pairwise disjoint subsets of  $U$  for  $0 \leq j \leq M$ . Now temporarily fix  $j \in \{0, \dots, M\}$ . For each  $x \in U_j$ , let  $V_x^{(j)}$  be a neighborhood

of  $x$  such that  $V_x^{(j)} \subset \overline{V_x^{(j)}} \subset U_j$ . Then  $\{V_x^{(j)} : x \in U_j, 0 \leq j \leq M\}$  is an open cover for  $F$ , hence it contains a finite subcover. For  $0 \leq j \leq M$  let  $\mathcal{S}_j$  be the (possibly empty) collection of all sets  $V_x^{(j)}$  that appear in the finite subcover for  $F$ , and set  $F_j = \bigcup_{V \in \mathcal{S}_j} \overline{V}$ . Note that  $F_j = \emptyset$  if the collection  $\mathcal{S}_j$  is empty. Then each  $F_j$  is closed (being the union of finitely many closed sets) and satisfies  $F_j \subset U_j$ . It follows that the sets  $h^{d(j)}(F_j)$  are pairwise disjoint subsets of  $U$  for  $0 \leq j \leq M$ .  $\square$

**Lemma III.14.** *Suppose that  $d_0, \dots, d_m$  are  $m+1$  distinct integers, and that  $n_1, n_2$  are distinct integers (but not necessarily distinct from the  $d_i$ ). Then the set*

$$\{d_i + n_j : 0 \leq i \leq m, j = 1, 2\}$$

*contains at least  $m+2$  distinct integers.*

*Proof.* Without loss of generality, suppose that  $d_0 < d_1 < \dots < d_m$  and  $n_1 < n_2$ . Then we have

$$d_0 + n_1 < d_1 + n_1 < \dots < d_m + n_1 < d_m + n_2,$$

which provides  $m+2$  distinct integers in the set  $\{d_i + n_j : 0 \leq i \leq m, j = 1, 2\}$ .  $\square$

**Proposition III.15.** *Let  $(X, h)$  be as in Notation III.1. If  $F \subset X$  is topologically  $h$ -small, then  $F$  is thin.*

*Proof.* The proof is by induction on the smallness constant  $m$ . First consider the case where the smallness constant is  $m = 1$ . Then given  $j, k \in \mathbb{Z}$  with  $j \neq k$ , we have  $h^j(F) \cap h^k(F) = \emptyset$ . Let  $U \subset X$  be open and non-empty, and let  $V_0 \subset U$  be open and non-empty with  $\overline{V_0} \subset X$ . By Lemma III.2,  $\{h^n(V_0) : n \in \mathbb{Z}\}$  is an open cover for  $F$ , so there exists a finite subcover  $\{h^{-d(0)}(V_0), \dots, h^{-d(M)}(V_0)\}$ . Set  $F_j = F \cap \overline{h^{-d(j)}(V_0)}$ . Then the sets  $h^{d(j)}(F_j)$  are closed, disjoint (since  $h^{d(j)}(F_j) \subset h^{d(j)}(F)$  and these sets are disjoint) and satisfy  $h^{d(j)}(F_j) \subset \overline{V_0} \subset U$ . Since  $X$  is normal, there exist disjoint open sets  $W_0, \dots, W_M \subset X$  such that  $h^{d(j)}(F_j) \subset W_j$ . Finally, for  $0 \leq j \leq M$  set  $U_j = h^{-d(j)}(W_j \cap U)$ . Then  $F \subset \bigcup_{j=0}^M U_j$ , and the sets  $h^{d(j)}(U_j)$  are pairwise disjoint (being subsets of the  $W_j$ ) and contained in  $U$ .

Now let  $m \geq 1$ , and suppose that closed sets which are topologically  $h$ -small with smallness constant  $m$  are thin. Let  $F \subset X$  be topologically  $h$ -small with smallness constant  $m+1$ . For  $j, k \in \mathbb{Z}$  with  $j \neq k$ , define  $F_{j,k} = h^j(F) \cap h^k(F)$ . We claim that the sets  $F_{j,k}$  are topologically



$h$ -small with smallness constant  $m$ . To see this, let  $d_0, \dots, d_m$  be  $m+1$  distinct integers, and let  $j, k \in \mathbb{Z}$  with  $j \neq k$ . By Lemma III.14, the set  $\{d_i + l : 0 \leq i \leq m, l = j, k\}$  contains at least  $m+2$  distinct integers. It follows that

$$h^{d_0}(F_{j,k}) \cap \dots \cap h^{d_m}(F_{j,k}) = \bigcap_{i=0}^m (h^{d_i+j}(F) \cap h^{d_i+k}(F)) = \emptyset,$$

which proves the claim. Now choose disjoint, non-empty open sets  $V_1, V_2 \subset U$ , and choose disjoint, non-empty open sets  $Z_1, Z_2$  with  $\bar{Z}_1 \subset V_1$  and  $\bar{Z}_2 \subset V_2$ . By Lemma III.2, the collection  $\{h^n(Z_1) : n \in \mathbb{Z}\}$  is an open cover for  $F$ , so it contains a finite subcover  $\{h^{-n_0}(Z_1), \dots, h^{-n_K}(Z_1)\}$ . Set  $T = \{(j, k) : 0 \leq j < k \leq K\}$  and for each  $(j, k) \in T$  define  $D_{j,k} = h^{n_j}(F) \cap h^{n_k}(F) \cap \bar{Z}_1$ , which is a closed subset of  $F_{n_j, n_k}$ . By the earlier claim,  $D_{j,k}$  is topologically  $h$ -small with smallness constant  $m$ , and so it is thin by the induction hypothesis. Choose pairwise disjoint open sets  $S_{j,k} \subset Z_2$  for  $(j, k) \in T$ . Since each  $D_{j,k}$  is thin, there exist  $M(j, k) \in \mathbb{N}$ ,  $U_{j,k,0}^{(0)}, \dots, U_{j,k,M(j,k)}^{(0)} \subset X$  open, and  $d_{j,k}(0), \dots, d_{j,k}(M(j, k)) \in \mathbb{Z}$  such that:

1.  $D_{j,k} \subset \bigcup_{i=0}^{M(j,k)} U_{j,k,i}^{(0)}$ ;
2.  $h^{d_{j,k}(i)}(U_{j,k,i}^{(0)}) \subset S_{j,k}$ ;
3. the sets  $h^{d_{j,k}(i)}(U_{j,k,i}^{(0)})$  are pairwise disjoint for  $0 \leq i \leq M(j, k)$ .

Set

$$D = \bigcup_{(j,k) \in T} h^{-n_j}(D_{j,k}) \quad \text{and} \quad W_0 = \bigcup_{(j,k) \in T} h^{-n_j} \left( \bigcup_{i=0}^{M(j,k)} U_{j,k,i}^{(0)} \right).$$

Then  $D$  is closed,  $W_0$  is open, and  $D \subset W_0$ . Choose  $W \subset X$  open such that  $D \subset W \subset \bar{W} \subset W_0$ . For  $0 \leq j \leq K$ , set  $F_j = h^{-n_j}(\bar{Z}_1) \cap (X \setminus W) \cap F$ , which is closed. Let  $x \in F$  and suppose  $x \notin W$ . For some  $j \in \{0, \dots, K\}$ , we have  $x \in h^{-n_j}(Z_1)$ . Then  $x \in F$ ,  $x \in h^{-n_j}(\bar{Z}_1)$ , and  $x \in X \setminus W$ , so  $x \in F_j$ . It follows that  $\{F_0, \dots, F_K, W\}$  covers  $F$ . Next suppose that  $x \in h^{n_j}(F_j) \cap h^{n_k}(F_k)$  for some  $(j, k) \in T$ . Then there are  $x_j \in F_j$  and  $x_k \in F_k$  such that  $h^{n_j}(x_j) = x = h^{n_k}(x_k)$ . Since  $F_j, F_k \subset F$  we certainly have  $x \in h^{n_j}(F) \cap h^{n_k}(F)$ . Moreover,  $x_j = h^{-n_j}(x) \in h^{-n_j}(\bar{Z}_1)$ , which gives  $x \in \bar{Z}_1$ . It follows that  $x \in D_{j,k}$ , and so also  $x_j = h^{-n_j}(x) \in h^{-n_j}(D_{j,k}) \subset W$ . This implies  $x_j \notin F_j$ , a contradiction. Therefore, the sets  $h^{n_j}(F_j)$  are pairwise disjoint. Since  $h^{n_j}(F_j) \subset \bar{Z}_1$ , they are all subsets of  $V_1$ . Using the normality of  $X$ , choose non-empty pairwise disjoint open sets

$U_0^{(0)}, \dots, U_K^{(0)} \subset X$  such that  $h^{n_j}(F_j) \subset U_j^{(0)} \subset V_1$ . For an appropriate  $M \in \mathbb{N}$ , re-index the sets

$$\left\{ h^{-n_0}(U_0^{(0)}), \dots, h^{-n_K}(U_K^{(0)}) \right\} \cup \left\{ h^{-n_j}(U_{j,k,i}^{(0)}) : (j,k) \in T, 0 \leq i \leq M(j,k) \right\}$$

and

$$\{n_0, \dots, n_K\} \cup \{n_j + d_{j,k}(i) : (j,k) \in T, 0 \leq i \leq M(j,k)\}$$

as  $\{U_0, \dots, U_M\}$  and  $\{d(0), \dots, d(M)\}$  respectively. Then  $F \subset \bigcup_{i=0}^M U_j$  and the sets  $h^{d(j)}(U_j)$  are pairwise disjoint subsets of  $U$  for  $0 \leq j \leq M$ . It follows that  $F$  is thin, completing the induction.  $\square$

**Corollary III.16.** *Let  $(X, h)$  be as in Notation III.1. Let  $F \subset X$  be closed and topologically  $h$ -small. Then  $\mu(F) = 0$  for every  $\mu \in M_h(X)$ .*

*Proof.* This follows immediately from Proposition III.15 and Lemma III.11.  $\square$

**Notation III.17.** *From now on, unless stated otherwise, we assume that the minimal homeomorphism  $h$  of Notation III.1 is uniquely ergodic; that is, there is a unique  $h$ -invariant Borel probability measure on  $X$ . Let  $\mu$  denote this measure. Any reference to  $X$  also refers implicitly to the minimal, uniquely ergodic dynamical system  $(X, h, \mu)$ .*

We suspect that most of what follows can be done without the assumption of unique ergodicity, with a corresponding increase in the technicalities of both the proofs and certain definitions.

The essential content of the property given by the following definition is that comparison of measures is sufficient to determine when a closed set can be decomposed and translated disjointly into an open set. The main result of this chapter will be to show that it holds for a reasonably large class of minimal, uniquely ergodic dynamical systems  $(X, h, \mu)$ .

**Definition III.18.** *Let  $(X, h, \mu)$  be as in Notation III.17. We say  $(X, h, \mu)$  has the dynamic comparison property if whenever  $U \subset X$  is open and  $C \subset X$  is closed with  $\mu(C) < \mu(U)$ , then there are  $M \in \mathbb{N}$ , continuous functions  $f_j : X \rightarrow [0, 1]$  for  $0 \leq j \leq M$ , and  $d(0), \dots, d(M) \in \mathbb{Z}$  such that  $\sum_{j=0}^M f_j = 1$  on  $C$ , and such that the sets  $\text{supp}(f_j \circ h^{-d(j)})$  are pairwise disjoint subsets of  $U$  for  $0 \leq j \leq M$ .*

The next lemma gives a condition that implies the dynamic comparison property holds, and is easier to verify because it assumed additional structure for the closed and open sets involved.

**Lemma III.19.** *Let  $(X, h, \mu)$  be as in Notation III.17. Suppose that  $X$  has the property that if whenever  $F \subset X$  is closed with  $\text{int}(F) \neq \emptyset$  and  $\partial F$  topologically  $h$ -small,  $E \subset X$  is open, and there exists an open set  $E_0 \subset E$  with  $\overline{E_0} \subset E$ ,  $\overline{E_0} \cap F = \emptyset$ ,  $\partial E_0$  topologically  $h$ -small, and  $\mu(F) < \mu(E_0)$ , then there exist  $M \in \mathbb{N}$ , continuous functions  $f_j: X \rightarrow [0, 1]$  for  $0 \leq j \leq M$ , and  $d(0), \dots, d(M) \in \mathbb{Z}$  such that  $\sum_{j=0}^M f_j = 1$  on  $F$ , and such that the sets  $\text{supp}(f_j \circ h^{-d(j)})$  are pairwise disjoint subsets of  $E$  for  $0 \leq j \leq M$ . Then  $(X, h)$  has the dynamic comparison property.*

*Proof.* Let  $U \subset X$  be open and let  $C \subset X$  be closed with  $\mu(C) < \mu(U)$ . By Proposition III.7, there is an open set  $U_0 \subset U$  with  $\overline{U_0} \subset U$  and  $\mu(C) < \mu(U_0)$ . First suppose that  $C \subset \overline{U_0}$ . Since  $X$  is a locally compact Hausdorff space, we can choose an open set  $V$ , with  $\overline{V}$  compact, that satisfies  $C \subset V \subset \overline{V} \subset U$ . Now set  $M = 0$  and  $d(0) = 0$ , and choose a continuous function  $f_0: X \rightarrow [0, 1]$  such that  $f_0 = 1$  on  $C$  and  $\text{supp}(f_0) \subset V$ . Then  $\sum_{j=0}^M f_j = f_0 = 1$  on  $C$ , and  $\text{supp}(f_0 \circ h^{-d(0)}) = \text{supp}(f_0) \subset \overline{V} \subset U$  as required.

So we may assume that  $C \cap (X \setminus \overline{U_0}) \neq \emptyset$ . By Proposition III.7 there is an open set  $V \subset U_0$  such that  $\partial V$  is topologically  $h$ -small and  $\mu(C) < \mu(V)$ . Moreover,  $V \subset U_0$  implies that  $\overline{V} \subset \overline{U_0} \subset U$ . Setting  $\delta = \mu(V) - \mu(C)$  and applying Proposition III.6 three times, we obtain open sets  $G_0, G_1, G_2 \subset X$  such that

$$C \cap \overline{V} \subset G_0 \subset \overline{G_0} \subset G_1 \subset \overline{G_1} \subset G_2 \subset \overline{G_2} \subset U_0,$$

with  $\partial G_i$  topologically  $h$ -small for  $i = 0, 1, 2$  (so also  $\mu(\partial G_i) = 0$  for  $i = 0, 1, 2$  by Corollary III.16),  $\mu(G_0) - \mu(C \cap \overline{V}) < \frac{1}{4}\delta$ ,  $\mu(G_1) - \mu(\overline{G_0}) < \frac{1}{4}\delta$ , and  $\mu(G_2) - \mu(\overline{G_1}) < \frac{1}{4}\delta$ .

Set  $F_0 = C \setminus G_0$ ,  $E = U \setminus \overline{G_1}$ , and  $E_1 = V \setminus \overline{G_2}$ . Then:

1.  $F_1$  is closed and non-empty, since  $G_0 \subset U_0$  implies that  $C \cap (X \setminus G_0) \neq \emptyset$ ;
2.  $E_1$  and  $E$  are both open and non-empty, and by construction we have  $E_1 \subset \overline{E_1} \subset E$ ;
3.  $\overline{E_1} \cap F_0 = \emptyset$ ;
4. Observing that  $C \cap \overline{V} \subset G_0$  and  $C \cap \overline{V} \subset C$  imply  $C \cap \overline{V} \subset C \cap G_0$ , and hence  $\mu(C \cap G_0) -$

$\mu(C \cap \bar{V}) \geq 0$ ), it follows that

$$\begin{aligned}
\mu(E_1) - \mu(F_0) &= \mu(V \setminus \bar{G}_2) - \mu(C \setminus G_0) \\
&= \mu(V) - \mu(V \cap \bar{G}_2) - (\mu(C) - \mu(C \cap G_0)) \\
&\geq (\mu(V) - \mu(C)) + \mu(C \cap G_0) \\
&\quad - (\mu(C \cap V) + \mu(G_2 \setminus \bar{G}_1) + \mu(G_1 \setminus \bar{G}_0) + \mu(G_0 \setminus (C \cap \bar{V}))) \\
&\geq \delta - (\mu(G_2 \setminus \bar{G}_1) + \mu(G_1 \setminus \bar{G}_0) + \mu(G_0 \setminus C \cap \bar{V})) \\
&> \delta - \frac{3}{4}\delta \\
&= \frac{1}{4}\delta \\
&> 0.
\end{aligned}$$

Now Proposition III.7 gives an open set  $E_0 \subset E_1$  such that  $\bar{E}_0 \subset E_1$ ,  $\partial E_0$  is topologically  $h$ -small, and  $\mu(E_1) - \mu(E_0) < \frac{1}{16}\delta$ . From  $\bar{E}_1 \cap F_0 = \emptyset$  it follows immediately that  $\bar{E}_0 \cap F_0 = \emptyset$ . By the normality of  $X$  and the regularity of  $\mu$ , there is an open set  $W_0 \subset X$  such that  $F_0 \subset W_0$ ,  $\bar{E}_0 \cap W_0 = \emptyset$ , and  $\mu(W_0) - \mu(F_0) < \frac{1}{16}\delta$ . Next, Proposition III.6 implies that there is an open set  $W \subset X$  such that  $F_0 \subset W \subset \bar{W} \subset W_0$ ,  $\partial W$  is topologically  $h$ -small, and  $\mu(W) > \mu(W_0) - \frac{1}{16}\delta$ . Now set  $F = \bar{W}$ , which satisfies  $\text{int}(F) \neq \emptyset$ ,  $\partial F$  topologically  $h$ -small (which in particular gives  $\mu(F) = \mu(W)$ ), and  $\bar{E}_0 \cap F = \emptyset$ . Finally, we compute

$$\begin{aligned}
\mu(E_0) - \mu(F) &= \mu(E_0) - \mu(W) \\
&> \mu(E_1) - \frac{1}{16}\delta - \mu(W) \\
&> (\mu(F_0) + \frac{1}{4}\delta) - \frac{1}{16}\delta - \mu(W) \\
&> (\mu(W_0) - \frac{1}{16}\delta) + \frac{3}{16}\delta - \mu(W) \\
&= (\mu(W_0) - \mu(W)) + \frac{1}{8}\delta \\
&\geq \frac{1}{8}\delta \\
&> 0,
\end{aligned}$$

where in the next-to-last step we have used the fact that  $W \subset W_0$  implies  $\mu(W_0) - \mu(W) \geq 0$ . It follows that the sets  $F$  and  $E_0$  satisfy the conditions for the property given in the statement

of the Lemma. Therefore, there exist  $M \in \mathbb{N}$ , continuous functions  $f_0, \dots, f_M: X \rightarrow [0, 1]$ , and  $d(0), \dots, d(M) \in \mathbb{Z}$  such that  $\sum_{j=0}^M f_j = 1$  on  $F$ , and such that the sets  $\text{supp}(f_j \circ h^{-d(j)})$  are pairwise disjoint subsets of  $E$  for  $0 \leq j \leq M$ . Choose a continuous function  $f_{M+1}: X \rightarrow [0, 1]$  such that  $f_{M+1} = 1$  on  $\overline{G_1}$  and  $\text{supp}(f_{M+1}) \subset G_2$ , and set  $d(M+1) = 0$ . Now for any  $x \in C$ , either  $x \in F_0$  or  $x \in G_0 \cap C$ . If  $x \in F_0$  then in particular  $x \in F$ , and so  $\sum_{j=0}^{M+1} f_j(x) \geq \sum_{j=0}^M f_j(x) = 1$ . If  $x \in G_0 \cap C$  then in particular  $x \in \overline{G_1}$ , and so  $\sum_{j=0}^{M+1} f_j(x) \geq f_{M+1}(x) = 1$ . It follows that  $\sum_{j=0}^{M+1} f_j(x) \geq 1$  for all  $x \in C$ . From the continuity of the  $f_j$ , there is an open set  $S \subset X$  such that  $C \subset S$  and  $\sum_{j=0}^{M+1} f_j(x) \geq \frac{1}{2}$  for all  $x \in S$ . Choose a continuous function  $f: X \rightarrow [0, 1]$  such that  $f = 1$  on  $C$  and  $\text{supp}(f) \subset S$ . For  $0 \leq j \leq M+1$ , define a continuous function  $g_j: X \rightarrow [0, 1]$  by

$$g_j(x) = \begin{cases} f(x)f_j(x) \left( \sum_{i=0}^{M+1} f_i(x) \right)^{-1} & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

Then for any  $x \in C$ , we have

$$\sum_{j=0}^{M+1} g_j(x) = \left( \sum_{i=0}^{M+1} f_i(x) \right)^{-1} \sum_{j=0}^{M+1} f(x)f_j(x) = \left( \sum_{i=0}^{M+1} f_i(x) \right)^{-1} \sum_{j=0}^{M+1} f_j(x) = 1.$$

Moreover,  $g_j(x) = 0$  for any  $x \in X$  where  $f_j(x) = 0$ , which implies that  $\text{supp}(g_j) \subset \text{supp}(f_j)$ . It follows that  $\text{supp}(g_j \circ h^{-d(j)}) \subset \text{supp}(f_j \circ h^{-d(j)})$  for  $0 \leq j \leq M+1$ . This immediately gives pairwise disjointness of the sets  $\text{supp}(g_j \circ h^{-d(j)})$  for  $0 \leq j \leq M$ , since the sets  $\text{supp}(f_j \circ h^{-d(j)})$  are pairwise disjoint for  $0 \leq j \leq M$ . Further, all of these sets are contained in  $U$  as  $E \subset U$ . Finally,  $\text{supp}(g_{M+1} \circ g^{-d(M+1)}) = \text{supp}(g_{M+1}) \subset \text{supp}(f_{M+1}) = \text{supp}(f_{M+1} \circ h^{-d(M+1)}) \subset G_2 \subset U$ , and  $E \cap G_2 = \emptyset$ . Thus, the sets  $\text{supp}(g_j \circ h^{-d(j)})$  are pairwise disjoint subsets of  $U$  for  $0 \leq j \leq M+1$ . It follows that  $(X, h, \mu)$  has the dynamic comparison property.  $\square$

**Lemma III.20.** *Let  $(X, h, \mu)$  be as in Notation III.17. Suppose that  $F \subset X$  is closed and  $E \subset X$  is open with  $F \cap \overline{E} = \emptyset$  and  $\mu(F) < \mu(E)$ . Then there exist continuous functions  $g_0, g_1: X \rightarrow [0, 1]$  such that  $g_0 = 1$  on  $F$ ,  $\text{supp}(g_0) \subset X \setminus \overline{E}$ ,  $\text{supp}(g_1) \subset E$ , and*

$$\int_X g_1 d\mu > \int_X g_0 d\mu.$$

Moreover, with  $g = g_1 - g_0$ , there exist  $N_0 \in \mathbb{N}$  and  $\sigma > 0$  such that for all  $N \geq N_0$  and  $x \in X$ ,

we have

$$\frac{1}{N} \sum_{j=0}^{N-1} g(h^j(x)) \geq \sigma.$$

*Proof.* Since  $F \cap \overline{E} = \emptyset$ , the normality of  $X$  gives open sets  $V_0, V_1 \subset X$  such that  $F \subset V_0$ ,  $\overline{E} \subset V_1$ , and  $V_0 \cap V_1 = \emptyset$ . Let  $\delta = \mu(E) - \mu(F) > 0$  and use the regularity of  $\mu$  to choose an open set  $W \subset X$  and a compact set  $K \subset X$  such that  $F \subset W$ ,  $K \subset E$ ,  $\mu(W) - \mu(F) < \frac{1}{3}\delta$  and  $\mu(E) - \mu(K) < \frac{1}{3}\delta$ . Set  $W_0 = V_0 \cap W$ , which satisfies  $F \subset W_0$ ,  $W_0 \cap V_1 = \emptyset$ , and  $\mu(W_0) \leq \mu(W)$ . Then this last inequality, the fact that  $W_0 \setminus F$  is open, and Proposition III.2 imply that  $0 < \mu(W_0) - \mu(F) \leq \mu(W) - \mu(F) < \frac{1}{3}\delta$ . Now choose continuous functions  $g_0$  and  $g_1$  such that  $g_0 = 1$  on  $F$ ,  $\text{supp}(g_0) \subset W_0$  (so that  $\text{supp}(g_0)$  is disjoint from  $\overline{E}$ ),  $g_1 = 1$  on  $K$ , and  $\text{supp}(g_1) \subset E$ . Observing that

$$\begin{aligned} \mu(K) - \mu(W_0) &= (\mu(E) - \mu(F)) - (\mu(E) - \mu(K)) - (\mu(W_0) - \mu(F)) \\ &> \delta - \frac{1}{3}\delta - \frac{1}{3}\delta \\ &= \frac{1}{3}\delta \\ &> 0, \end{aligned}$$

we thus obtain

$$\int_X g_0 d\mu = \int_{W_0} g_0 d\mu \leq \mu(W_0) < \mu(K) = \int_X g_1 d\mu \leq \int_X g_1 d\mu.$$

Noting that, by the previous calculation, the function  $g = g_1 - g_0$  satisfies

$$\int_X g d\mu > 0,$$

we define  $\sigma > 0$  by

$$\sigma = \frac{1}{2} \int_X g d\mu.$$

Suppose for a contradiction that no  $N_0 \in \mathbb{N}$  as in the statement lemma exists. Then there exist sequences  $(N_k)_{k=1}^{\infty} \subset \mathbb{N}$  and  $(x_k)_{k=1}^{\infty} \subset X$  such that for all  $k \in \mathbb{N}$  we have

$$\frac{1}{N_k} \sum_{j=0}^{N_k-1} g(h^j(x_k)) \leq \sigma.$$

Passing to subsequences  $(N_{k(l)})_{l=1}^{\infty}$  and  $(x_{k(l)})_{l=1}^{\infty}$  (if necessary) and applying the pointwise ergodic theorem (see the remark after Theorem 1.14 of [54]) yields

$$\int_X g \, d\mu = \lim_{l \rightarrow \infty} \frac{1}{N_{k(l)}} \sum_{j=0}^{N_{k(l)}-1} g(h^j(x_{k(l)})) \leq \sigma,$$

which contradicts the definition of  $\sigma$ . □

**Lemma III.21.** *Let  $(X, h, \mu)$  be as in Notation III.17. Let  $\varepsilon > 0$  be given, and let  $F \subset X$  be thin. Then for any non-empty open set  $U \subset X$  there exist  $M \in \mathbb{N}$ , closed sets  $F_j \subset X$  for  $0 \leq j \leq M$ , open sets  $T_j, V_j, W_j \subset X$  for  $0 \leq j \leq M$ , continuous functions  $f_0, \dots, f_M: X \rightarrow [0, 1]$ , and  $d(0), \dots, d(M) \in \mathbb{Z}$  such that:*

1.  $F \subset \bigcup_{j=0}^M F_j$ ;
2.  $h^{-d(j)}(F_j) \subset T_j \subset \bar{T}_j \subset V_j \subset \bar{V}_j \subset W_j \subset U$  for  $0 \leq j \leq M$ ;
3.  $\sum_{j=0}^M f_j = 1$  on  $\bigcup_{j=0}^M h^{d(j)}(\bar{V}_j)$ ;
4.  $\text{supp}(f_j \circ h^{-d(j)}) \subset W_j$  for  $0 \leq j \leq M$ ;
5. the sets  $W_j$  are pairwise disjoint and  $\sum_{j=0}^M \mu(W_j) < \varepsilon$ .

*Proof.* Since  $U$  is open and non-empty, Lemma III.4 implies there is a non-empty open set  $E \subset U$  with  $\mu(E) < \varepsilon$ . Since  $F$  is thin, we can apply Lemma III.13 to  $F$  and  $E$ , which implies there exist  $M \in \mathbb{N}$ ,  $F_0, \dots, F_M \subset X$  closed, and  $k(0), \dots, k(M) \in \mathbb{Z}$  such that  $F \subset \bigcup_{j=0}^M F_j$  and such that the sets  $h^{k(j)}(F_j)$  are pairwise disjoint subsets of  $E$ . For  $0 \leq j \leq M$ , we set  $d(j) = -k(j)$ . Since  $X$  is normal, we may choose for  $0 \leq j \leq M$  open sets  $W_j$  with  $F_j \subset W_j \subset E$  such that the  $W_j$  are pairwise disjoint. Now we can use the compactness of  $X$  to obtain open sets  $T_j, V_j \subset X$  such that

$$h^{-d(j)}(F_j) \subset T_j \subset \bar{T}_j \subset V_j \subset \bar{V}_j \subset W_j.$$

For  $0 \leq j \leq M$  choose continuous functions  $g_j: X \rightarrow [0, 1]$  such that  $g_j = 1$  on  $h^{d(j)}(\bar{V}_j)$  and  $\text{supp}(g_j) \subset h^{d(j)}(W_j)$ . Then  $\sum_{j=0}^M g_j(x) \geq 1$  for all  $x \in \bigcup_{j=0}^M h^{d(j)}(\bar{V}_j)$ . By the continuity of the  $g_j$ , there is an open set  $Q \subset X$  such that  $\bigcup_{j=0}^M h^{d(j)}(\bar{V}_j) \subset Q$  and  $\sum_{j=0}^M g_j(x) \geq \frac{1}{2}$  for all  $x \in Q$ . Choose a continuous function  $f: X \rightarrow [0, 1]$  such that  $f = 1$  on  $\bigcup_{j=0}^M h^{d(j)}(\bar{V}_j)$  and  $\text{supp}(f) \subset Q$ .

Now, for  $0 \leq j \leq M$ , define continuous functions  $f_j: X \rightarrow [0, 1]$  by

$$f_j(x) = \begin{cases} f(x)g_j(x) \left( \sum_{i=0}^M g_i(x) \right)^{-1} & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$$

Then for any  $x \in \bigcup_{j=0}^M h^{d(j)}(\bar{V}_j)$ , we have

$$\sum_{j=0}^M f_j(x) = \sum_{j=0}^M f(x)g_j(x) = \sum_{j=0}^M g_j(x) = 1.$$

In particular,  $\sum_{j=0}^M f_j = 1$  on  $\bigcup_{j=0}^M h^{d(j)}(T_j)$ . Moreover,  $\text{supp}(f_j) = \text{supp}(g_j) \subset h^{d(j)}(W_j)$ , which implies that  $\text{supp}(f_j \circ h^{-d(j)}) = \text{supp}(g_j \circ h^{-d(j)}) \subset W_j$ . Finally, as the  $W_j$  are pairwise disjoint subsets of  $E$  for  $0 \leq j \leq M$ , it follows that

$$\sum_{j=0}^M \mu(W_j) = \mu \left( \bigcup_{j=0}^M W_j \right) \leq \mu(E) < \varepsilon,$$

which completes the proof.  $\square$

The next proposition is included to contrast the relative ease in which the dynamic comparison property is verified for the special case of the Cantor set compared to the complexity of the proof in more general situations.

**Proposition III.22.** *If  $X$  is the Cantor set and  $(X, h, \mu)$  is as in Notation III.17, then  $(X, h, \mu)$  has the dynamic comparison property.*

*Proof.* This is essentially the content of Lemma 2.5 of [12], although their result is not stated in terms of functions. Since characteristic functions of compact-open subsets of  $X$  are continuous, re-casting it to obtain the dynamic comparison property is straightforward.  $\square$

The situation becomes significantly more complicated once we leave the case where  $X$  is the Cantor set, since we can no longer work with compact-open sets and their characteristic functions. The key technical assumption in the general case is that  $(X, h)$  have the topological small boundary property.



**Lemma III.23.** *Suppose that  $(X, h)$  has the topological small boundary property. Then for any  $N \in \mathbb{N}$ , there exists a closed set  $Y \subset X$  such that  $\text{int}(Y) \neq \emptyset$ ,  $\partial Y$  is topologically  $h$ -small, and the sets  $Y, h(Y), \dots, h^N(Y)$  are pairwise disjoint.*

*Proof.* Since the action of  $h$  on  $X$  is free, for  $y \in X$  the iterates  $y, h(y), \dots, h^N(y)$  are all distinct elements of  $X$ . Choose pairwise disjoint open neighborhoods  $W_0, W_1, \dots, W_N$  of these points, and set  $W = \bigcap_{j=0}^N h^{-j}(W)$ . Then the iterates  $W, h(W), \dots, h^N(W)$  are pairwise disjoint. Let  $F = \{y\}$ , and apply Proposition III.6 with  $F$  and  $W$  to obtain an open set  $U \subset X$  such that  $F \subset U \subset \bar{U} \subset W$  and such that  $\partial U$  is topologically  $h$ -small (we ignore the unneeded measure theoretic conclusion). Setting  $Y = \bar{U}$ , it follows that  $\text{int}(Y) \neq \emptyset$  and  $\partial Y$  is topologically  $h$ -small. Finally, as  $Y \subset W$ , the sets  $Y, h(Y), \dots, h^N(Y)$  are pairwise disjoint.  $\square$

**Lemma III.24.** *Let  $(X, h)$  be as in Notation III.1. Let  $Y \subset X$  be closed with  $\text{int}(Y) \neq \emptyset$  and  $\partial Y$  topologically  $h$ -small. Adopt the notation of Theorem III.8. Then  $\partial(h^j(Y_k))$  is thin for  $0 \leq k \leq l$  and  $0 \leq j \leq n(k) - 1$ .*

*Proof.* By Proposition III.15,  $\partial Y$  is thin. For  $0 \leq j \leq n(k) - 1$ , we have  $\partial h^j(Y_k) = h^j(\partial Y_k)$ , and since translates of thin sets are thin, it suffices to prove that each of the sets  $\partial Y_k$  is thin. But  $\partial Y_k \subset \bigcup_{j=0}^{n(k)-1} h^j(\partial Y)$ , and this set is thin by Lemma III.12, since it is a finite union of translates of thin sets.  $\square$

**Theorem III.25.** *Let  $X$  be an infinite compact metric space with finite covering dimension  $m$ , let  $h: X \rightarrow X$  be a uniquely ergodic minimal homeomorphism, let  $\mu$  be the unique  $h$ -invariant Borel probability measure on  $X$ , and suppose that  $(X, h)$  has the topological small boundary property. Then  $(X, h, \mu)$  has the dynamic comparison property.*

*Proof.* Let  $C \subset X$  be closed and  $U \subset X$  be open such that  $\mu(C) < \mu(U)$ . By Lemma III.19, we may assume that  $\text{int}(C) \neq \emptyset$ ,  $\partial C$  is topologically  $h$ -small, and that there is an open set  $U_0 \subset U$  such that  $\bar{U}_0 \subset U$ ,  $\partial U_0$  is topologically  $h$ -small,  $\bar{U}_0 \cap C = \emptyset$ , and  $\mu(C) < \mu(U_0)$ . Applying Proposition III.20 to  $C$  and  $U_0$ , there exist continuous functions  $g_0, g_1: X \rightarrow [0, 1]$  such that  $g_0 = 1$  on  $C$ ,  $\text{supp}(g_0) \subset X \setminus \bar{U}_0$ ,  $\text{supp}(g_1) \subset U_0$ , and

$$\int_X g_1 d\mu > \int_X g_0 d\mu.$$

Moreover, with  $g = g_1 - g_0$ , there exists  $N_0 \in \mathbb{N}$  and  $\sigma > 0$  such that for all  $N \geq N_0$  and  $x \in X$ , we have

$$\frac{1}{N} \sum_{j=0}^{N-1} g(h^j(x)) \geq \sigma.$$

By Lemma III.23, there exists a closed set  $Y \subset X$  with  $\text{int}(Y) \neq \emptyset$  such that  $\partial Y$  is topologically  $h$ -small, and such that the sets  $Y, h(Y), \dots, h^{N_0}(Y)$  are pairwise disjoint. Following the notation of Theorem III.8, we construct the Rokhlin tower over  $Y$  by first return times to  $Y$ , then apply the second statement of Theorem III.8 with the partition  $\mathcal{P} = \{U_0, C, X \setminus (U_0 \cup C)\}$  of  $X$  by sets with non-empty interior (discarding the third set if it is empty). For convenience, we will use  $Y_0, \dots, Y_l$  and  $n(0) \leq n(1) \leq \dots \leq n(l)$  for the base spaces and first return times in the tower compatible with  $\mathcal{P}$ , and set  $Y_k^{(0)} = Y_k \setminus \partial Y_k$ . (Note that since these  $Y_k$  are the sets  $Z_k$  in Theorem III.8, it may be the case that  $Y_k^{(0)} = \emptyset$ .) We set

$$F = X \setminus \left( \bigcup_{k=0}^l \bigcup_{j=0}^{n(k)-1} h^j(Y_k^{(0)}) \right).$$

For each  $k \in \{0, \dots, l\}$ , the column  $\{h^j(Y_k) : 0 \leq j \leq n(k) - 1\}$  has height at least  $N_0$ . Thus, for any  $x \in Y_k$  we have

$$\frac{1}{n(k)} \sum_{j=0}^{n(k)-1} g(h^j(x)) \geq \sigma > 0.$$

For  $S \subset X$  and  $k \in \{0, \dots, l\}$  define

$$N(S, k) = \{n \in \{0, 1, \dots, n(k) - 1\} : h^n(Y_k) \subset S\}.$$

Letting  $\chi = \chi_{U_0} - \chi_C$ , we observe that  $g_0 = 1$  on  $C$  implies that  $\chi_C \leq g_0$  and  $\text{supp}(g_1) \subset U_0$  implies that  $g_1 \leq \chi_{U_0}$ . Combining these inequalities gives  $g \leq \chi$ , and so

$$0 < \sigma \leq \frac{1}{n(k)} \sum_{j=0}^{n(k)} g(h^j(x)) \leq \frac{1}{n(k)} \sum_{j=0}^{n(k)} \chi(h^j(x)) = \frac{\text{card}(N(U_0, k)) - \text{card}(N(C, k))}{n(k)}.$$

It follows that for  $0 \leq k \leq l$ , we have  $\text{card}(N(U_0, k)) > \text{card}(N(C, k))$  (that is, more levels in the column  $\{h^j(Y_k) : 0 \leq j \leq n(k) - 1\}$  are contained in  $U_0$  than are contained in  $C$ ) and so there is an injective map  $\varphi_k : N(C, k) \rightarrow N(U_0, k)$ . If we order  $N(C, k)$  as  $\{s_k(0), \dots, s_k(L_k)\}$  and

similarly order  $N(U_0, k)$  as  $\{t_k(0), \dots, t_k(L_k), \dots\}$ , then one way to represent the injection  $\varphi_k$  is by  $\varphi_k = (d_k(0), \dots, d_k(L_k)) \in \mathbb{Z}^{L_k}$  where, for  $0 \leq m \leq L_k$ , the integer  $d_k(m)$  satisfies

$$h^{d_k(m)}(h^{s_k(m)}(Y_k)) \subset h^{t_k(m)}(Y_k).$$

Next, we claim that the closed set  $F$  is thin. Since the finite union of thin sets is thin by Lemma III.12, it clearly suffices to prove that  $\partial h^j(Y_k)$  is thin for each  $0 \leq k \leq l$ ,  $0 \leq j \leq n(k) - 1$ . Now,  $\partial C$  and  $\partial U_0$  are both topologically  $h$ -small, hence thin. Since  $\partial(X \setminus (U_0 \cup C)) = \partial(U_0 \cup C) \subset \partial U_0 \cup \partial C$ , it follows that the boundaries of all sets in the partition  $\mathcal{P}$  are thin. As the only processes used in the construction of the Rokhlin tower compatible with this partition are translation by powers of  $h$ , finite unions, and finite intersections, it follows that it is sufficient to prove that the boundaries  $\partial h^j(Y_k)$  in a standard Rokhlin tower (without any condition about compatibility with respect to a partition) are thin. This is true by Lemma III.24, and consequently  $F$  is thin.

Now, set  $Q = \{k: 0 \leq k \leq l, Y_k^{(0)} \neq \emptyset\}$ ,  $Q' = \{0, \dots, l\} \setminus Q$ , and define

$$\varepsilon = \frac{1}{2} \min \left\{ \mu(Y_k^{(0)}): k \in Q \right\}.$$

The  $\varepsilon > 0$ , and so we may apply Lemma III.21 with  $F$ ,  $U \setminus \bar{U}_0$ , and  $\varepsilon$ . We obtain  $M \in \mathbb{N}$ , and for  $0 \leq i \leq M$  open sets  $T_i, V_i, W_i \subset X$ , closed sets  $F_i \subset X$ , continuous functions  $b_i: X \rightarrow [0, 1]$ , and integers  $r(i)$  such that:

1.  $h^{-r(i)}(F_i) \subset T_i \subset \bar{T}_i \subset V_i \subset \bar{V}_i \subset W_i \subset U \setminus \bar{U}_0$  for  $0 \leq i \leq M$ ;
2.  $\sum_{i=0}^M b_i = 1$  on  $\bigcup_{i=0}^M h^{r(i)}(\bar{V}_i)$ ;
3.  $\text{supp}(b_i \circ h^{-r(i)}) \subset W_i$  for  $0 \leq i \leq M$ ;
4. the sets  $W_i$  are pairwise disjoint and  $\sum_{i=0}^M \mu(W_i) < \varepsilon$ .

By the choice of  $\varepsilon$ , it follows that for  $k \in Q$  and  $0 \leq j \leq n(k) - 1$ ,

$$\begin{aligned}
\mu \left( h^j(Y_k^\circ) \setminus \bigcup_{i=0}^M h^{r(i)}(W_i) \right) &\geq \mu(h^j(Y_k^\circ)) - \mu \left( \bigcup_{i=0}^M h^{r(i)}(W_i) \right) \\
&\geq 2\varepsilon - \sum_{i=0}^M \mu(h^{r(i)}(W_i)) \\
&= \varepsilon - \sum_{i=0}^M \mu(W_i) \\
&> \varepsilon \\
&> 0,
\end{aligned}$$

and so the sets  $h^j(Y_k^\circ) \setminus \bigcup_{i=0}^M h^{r(i)}(W_i)$  are non-empty whenever  $k \in Q$ . It follows that for  $k \in Q$ , each set  $h^j(Y_k) \setminus \bigcup_{i=0}^M h^{r(i)}(V_i)$  is a non-empty closed subset of  $h^j(Y_k)$ . Now for  $k \in Q$  and  $0 \leq m \leq L_k$  choose a continuous function  $f_{m,k}: X \rightarrow [0, 1]$  such that  $f_{m,k} = 1$  on  $h^{s_k(m)}(Y_k) \setminus \bigcup_{i=0}^M h^{r(i)}(V_i)$  and  $\text{supp}(f_{m,k}) \subset h^{s_k(m)}(Y_k) \setminus \bigcup_{i=0}^M h^{r(i)}(\bar{T}_i)$ . Now we have collections of continuous functions

$$\{b_i: 0 \leq i \leq M\} \cup \{f_{m,k}: k \in Q, 0 \leq m \leq L_k\}$$

and associated integers

$$\{r(i): 0 \leq i \leq M\} \cup \{d_k(m): k \in Q, 0 \leq m \leq L_k\}.$$

For any  $x \in C$ , if  $x \in \bigcup_{k \in Q} \bigcup_{m=0}^{L_k} \left( h^{s_k(m)}(Y_k) \setminus \bigcup_{i=0}^M h^{r(i)}(V_i) \right)$ , then  $f_{m,k}(x) \neq 0$  for some  $k \in Q$  and some  $m \in \{0, \dots, L_{i,k}\}$ . Otherwise,  $x \in \bigcup_{i=0}^M h^{r(i)}(V_i)$ , and  $b_i(x) \neq 0$  for some  $0 \leq i \leq M$ . (Notice that if  $x \in \bigcup_{k \in Q} \bigcup_{m=0}^{L_k} h^{s_k(m)}(Y_k)$ , then in fact  $x \in F$ , and so also  $x \in \bigcup_{i=0}^M h^{r(i)}(V_i)$ .) Now re-order the two collections above as  $\{f_j^{(0)}: 0 \leq j \leq K\}$  and  $\{d(j): 0 \leq j \leq K\}$  for an appropriate  $K \in \mathbb{N}$ . Then  $\sum_{j=0}^K f_j^{(0)}(x) > 0$  for all  $x \in C$ . Since  $C$  is compact and the  $f_j^{(0)}$  are continuous, there must be a  $\omega > 0$  such that  $\sum_{j=0}^K f_j^{(0)}(x) \geq \omega$  for all  $x \in C$ . Again using continuity, we can choose an open set  $S \subset X$  such that  $C \subset S$  and  $\sum_{j=0}^K f_j^{(0)}(x) \geq \frac{1}{2}\omega$  for all  $x \in S$ . Choose a continuous function  $f: X \rightarrow [0, 1]$  such that  $f(x) = 1$  for all  $x \in C$ , and  $\text{supp}(f) \subset S$ .

For  $0 \leq j \leq K$  define continuous functions  $f_j: X \rightarrow [0, 1]$  by

$$f_j = \begin{cases} f(x)f_j^{(0)}(x) \left( \sum_{i=0}^K f_i^{(0)}(x) \right)^{-1} & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

Then for any  $x \in C$ ,

$$\sum_{j=0}^K f_j(x) = \left( \sum_{i=0}^K f_i^{(0)}(x) \right)^{-1} \sum_{j=0}^K f(x)f_j^{(0)}(x) = \left( \sum_{i=0}^K f_i^{(0)}(x) \right)^{-1} \sum_{j=0}^K f_j^{(0)}(x) = 1.$$

Moreover,  $\text{supp}(f_j) \subset \text{supp}(f_j^{(0)})$  for  $0 \leq j \leq K$ . If  $f_j^{(0)} = b_i$  for some  $0 \leq i \leq M$ , then

$$\text{supp}(f_j^{(0)} \circ h^{-d(j)}) = \text{supp}(b_i \circ h^{-r(i)}) \subset W_i \subset U \setminus \bar{U}_0$$

and the sets  $W_i$  are pairwise disjoint. Therefore the sets  $\text{supp}(f_j^{(0)} \circ h^{-d(j)})$  are pairwise disjoint for all choices of  $j$  where  $f_j^{(0)} \in \{b_i: 0 \leq i \leq M\}$ . Next, if  $f_j^{(0)} = f_{m,k}$  for some  $k \in Q$  and some  $0 \leq m \leq L_k$ , then

$$\text{supp}(f_j^{(0)} \circ h^{-d(j)}) = \text{supp}(f_{m,k} \circ h^{-d_k(m)}) \subset h^{t_k(m)}(Y_k) \subset U_0.$$

Moreover, the definition of the functions  $f_{m,k}$  implies that

$$\text{supp}(f_{m,k} \circ h^{-d_k(m)}) \subset h^{d(m)} \left( h^{s_k(m)}(Y_k) \setminus \bigcup_{i=0}^M h^{r(i)}(V_i) \right),$$

so that in particular, for  $k \in Q$  the set  $\text{supp}(f_{m,k} \circ h^{-d_k(m)})$  is a subset of  $h^{t_k(m)}(Y_k^{(0)})$  (which is non-empty by the choice of  $k$ ). Since the sets  $h^{t_k(m)}(Y_k^{(0)})$  are pairwise disjoint, the sets  $\text{supp}(f_j^{(0)} \circ h^{-d(j)})$  are pairwise disjoint for all choices of  $j$  where  $f_j^{(0)} \in \{f_{m,k}: k \in Q, 0 \leq m \leq L_k\}$ . Moreover, the sets  $W_i$  are pairwise disjoint from the sets  $h^{t_k(m)}(Y_k^{(0)})$  as  $U \setminus \bar{U}_0$  is certainly disjoint from  $U_0$ . Therefore, the sets  $\text{supp}(f_j^{(0)} \circ h^{-d(j)})$  are pairwise disjoint subsets of  $U$  for all  $0 \leq j \leq K$ . It follows that the sets  $\text{supp}(f_j \circ h^{-d(j)})$  are pairwise disjoint subsets of  $U$  for all  $0 \leq j \leq K$ . This completes the proof.  $\square$

In order for the result of this theorem to be useful, we need to know that we can actually find minimal dynamical systems  $(X, h)$  that have the topological small boundary property. If we

restrict to the case where, in addition to our usual assumptions, we take  $X$  to be a smooth compact connected manifold and  $h$  to be a minimal diffeomorphism of  $X$ , then it is not hard to show that the topological small boundary property holds. We call a closed set  $Y \subset X$  *generic* if  $\partial Y$  is a smooth submanifold of  $X$  such that any finite subfamily of  $\{h^n(\partial Y) : n \in \mathbb{Z}\}$  intersects transversally. In particular, the intersection of any  $\dim(X) + 1$  such sets is empty, so  $\partial Y$  is topologically  $h$ -small with topological smallness constant  $\dim(X)$ . By the main theorem of [28], there exist sufficiently many generic sets  $Y$  so that if  $F, K \subset X$  are disjoint compact sets, then there exist open sets  $U$  and  $V$  with  $\bar{U} \cap \bar{V} = \emptyset$  and such that  $\bar{U}$  is generic. We thus obtain the following existence result.

**Corollary III.26.** *Let  $(X, h)$  be a smooth minimal dynamical system, consisting of a compact connected smooth manifold  $X$  with finite covering dimension and a uniquely ergodic minimal diffeomorphism  $h$ , with unique  $h$ -invariant Borel probability measure  $\mu$ . Then  $(X, h, \mu)$  has the dynamic comparison property.*

*Proof.* By the previous discussion,  $(X, h)$  has the topological small boundary property. Theorem III.25 then implies  $(X, h, \mu)$  has the dynamic comparison property.  $\square$

Before proceeding with our main development, we digress momentarily to make some speculative comments about comparison of positive elements in  $C(X)$ . As mentioned in the introduction, Cuntz subequivalence  $\preceq$  (which will be defined formally in Definition VI.1) is a fairly restrictive form of comparison for positive elements in this situation. Two functions  $f, g \in C(X)$  satisfy  $f \preceq g$  if and only if

$$\{x \in X : f(x) \neq 0\} \subset \{x \in X : g(x) \neq 0\}.$$

The dynamic comparison property suggests that in dynamical systems where it holds, a weaker form of subequivalence of functions could be appropriate. We tentatively propose the following definition.

**Definition III.27.** *Let  $(X, h)$  be as in Notation III.1. Given  $f, g \in C(X)_+$ , we say  $f$  is  $h$ -subequivalent to  $g$ , and write  $f \preceq_h g$ , if there exist  $f_1, \dots, f_M \in C(X)_+$  and  $d(1), \dots, d(M) \in \mathbb{Z}$  such that  $f \preceq \sum_{j=1}^M f_j$  and such that the sets  $\text{supp}(f \circ h^{-d(j)})$  are pairwise disjoint subsets of  $\text{supp}(g)$  for  $1 \leq j \leq M$ .*

**Proposition III.28.** *The relation  $\lesssim_h$  is a partial order on  $C(X)_+$ .*

*Proof.* It is clear that  $\lesssim_h$  is reflexive. (Take  $M = 1$ ,  $f_1 = f$  and  $d(1) = 0$ .) Suppose that  $f \lesssim_h g$  and  $g \lesssim_h k$ . Then there exist  $f_1, \dots, f_M, g_1, \dots, g_N \in C(X)_+$  and  $d(1), \dots, d(M), r(1), \dots, r(N) \in \mathbb{Z}$  such that  $f \lesssim \sum_{i=1}^M f_i$ ,  $g \lesssim \sum_{j=1}^N g_j$ , the sets  $\text{supp}(f_i \circ h^{-d(i)})$  are pairwise disjoint subsets of  $\text{supp}(g)$  for  $1 \leq i \leq M$ , and the sets  $\text{supp}(g_j \circ h^{-r(j)})$  are pairwise disjoint subsets of  $\text{supp}(k)$  for  $1 \leq j \leq N$ . For  $1 \leq i \leq M$  and  $1 \leq j \leq N$ , define  $\varphi_{i,j} \in C(X)_+$  by  $\varphi_{i,j} = f_i(g_j \circ h^{d(i)})$ . We claim that if  $\sum_{i=1}^M \sum_{j=1}^N \varphi_{i,j}(x) = 0$ , then  $f(x) = 0$ . To see this, observe first that

$$\begin{aligned} \sum_{i=1}^M \sum_{j=1}^N \varphi_{i,j} &= \sum_{i=1}^M \sum_{j=1}^N f_i(g_j \circ h^{d(i)}) \\ &= \sum_{i=1}^M f_i \left( \sum_{j=1}^N g_j \circ h^{d(i)} \right) \\ &= \sum_{i=1}^M f_i \left( \sum_{j=1}^N g_j \right) \circ h^{d(i)} \\ &\succcurlyeq \sum_{i=1}^M f_i(g \circ h^{d(i)}). \end{aligned}$$

If  $\sum_{i=1}^M \sum_{j=1}^N \varphi_{i,j}(x) = 0$ , then  $\sum_{i=1}^M f_i(x)g(h^{d(i)}(x)) = 0$  as well. If  $f_i(x) = 0$  for  $1 \leq i \leq M$ , then  $f(x) = 0$  and we are done. If not, then  $g(h^{d(i)}(x)) = 0$  for some  $i$ . Since  $\text{supp}(f_i \circ h^{-d(i)}) \subset \text{supp}(g)$ , it follows that  $f_i \circ h^{-d(i)}(h^{d(i)}(x)) = 0$ , which implies that  $f_i(x) = 0$ . This proves the claim. From the claim we may conclude that  $f \lesssim \sum_{i=1}^M \sum_{j=1}^N \varphi_{i,j}$ . Further,

$$\begin{aligned} \text{supp}(\varphi_{i,j} \circ h^{-(d(i)+r(j))}) &= \text{supp}((f_i \circ h^{-(d(i)+r(j))})(g_j \circ h^{d(i)} \circ h^{-(d(i)+r(j))})) \\ &\subset \text{supp}(g_j \circ h^{-r(j)}), \end{aligned}$$

which implies that the sets  $\text{supp}(\varphi_{i,j} \circ h^{-(d(i)+r(j))})$  are pairwise disjoint subsets of  $\text{supp}(k)$ . It follows that  $f \lesssim_h k$ .  $\square$

It is certainly the case that if  $f \lesssim g$  then  $f \lesssim_h g$ , as  $\text{supp}(f)$  is already a subset of  $\text{supp}(g)$ . If  $(X, h, \mu)$  is as in Notation III.17 and has the dynamic comparison property, then a sufficient condition for  $f \lesssim_h g$  would be that there is an open set  $U \subset \text{supp}(g)$  such that  $\mu(\text{supp}(f)) < \mu(U)$ . Two questions immediately come to mind. The first is whether this definition

is actually useful; that is, can any interesting results be obtained from it. The second is whether it can be generalized to give an appropriate definition of “ $\alpha$ -subequivalence”, where  $\alpha$  is an action of a countable amenable group on a unital  $C^*$ -algebra  $A$ , and what relationship this definition has with the leftover comparison condition in the tracial quasi-Rokhlin property. In the tracial Rokhlin property of [36] for  $\mathbb{Z}$ -actions, this condition is given in terms of Murray-von Neumann subequivalence of projections, while in the projection-free tracial Rokhlin property of [3] for finite group actions, it is given in terms of Cuntz subequivalence of positive elements. Again, we propose a (very tentative) definition.

**Definition III.29.** *Let  $A$  be a separable, unital  $C^*$ -algebra, and let  $\alpha: \Gamma \rightarrow \text{Aut}(A)$  be an action of a countable, amenable group  $\Gamma$  on  $A$ . For  $a, b \in A_+$ , we say  $a$  is  $\alpha$ -subequivalent to  $b$ , and write  $a \lesssim_\alpha b$ , if there exist  $N \in \mathbb{N}$ ,  $\gamma_1, \dots, \gamma_N \in \Gamma$ ,  $a_1, \dots, a_N \in A_+$ , and  $w_1, \dots, w_N \in U(A)$  such that  $a \leq \sum_{j=1}^N a_j$  and the elements  $w_j \alpha_{\gamma_j}(a_j) w_j^*$  are mutually orthogonal positive elements of  $\overline{bAb}$ .*

With this definition available, condition (5) in Definition II.1 could be re-stated as: with  $c = \sum_{j=0}^n c_j$ ,  $1 - c$  is  $\alpha$ -subequivalent to a positive element of  $\overline{xAx}$ . We have not attempted to verify that  $\lesssim_\alpha$  is a partial order on  $A_+$ , and in fact this may not even be true. The computations in the proof of Theorem IV.15 suggest that an additional requirement may be needed regarding the centrality (or perhaps approximate centrality) of the positive elements  $a_1, \dots, a_N$ . Note also that whereas in Definition III.27 we have used  $\lesssim$ , Definition III.29 uses  $\leq$ , mainly for consistency with the tracial quasi-Rokhlin property. It seems possible that we could also use Cuntz subequivalence in this case and not lose any of results about the tracial quasi-Rokhlin property, but this needs to be checked. We do not pursue  $h$ -subequivalence or  $\alpha$ -subequivalence further here, leaving them instead for potential future work.



## CHAPTER IV

AUTOMORPHISMS OF  $C(X, A)$  WITH THE TRACIAL QUASI-ROKHLIN PROPERTY

Our next goal is to study the automorphisms for a sort of noncommutative minimal dynamical system, where the commutative  $C^*$ -algebra  $C(X)$  studied by H. Lin, Q. Lin, and N. C. Phillips is replaced by the algebra of all continuous functions  $f: X \rightarrow A$ , and  $A$  is some abstract  $C^*$ -algebra with sufficiently nice structure. (For any interesting new applications,  $A$  will be a noncommutative  $C^*$ -algebra.) With the dynamic comparison property at our disposal, we prove that automorphisms of such algebras which take the action of a minimal homeomorphism when restricted to the central subalgebra  $C(X)$  satisfy the tracial quasi-Rokhlin property (under some additional technical assumptions). After further consideration of the structure of these algebras, it will follow that our results for crossed products by automorphisms with the tracial quasi-Rokhlin property in Chapter II will apply to their associated transformation group  $C^*$ -algebras. The following definition was first given in [20]. The version presented here is equivalent to the original one by Proposition 3.8 of [20]. Recall that if  $p$  and  $q$  are projections in a  $C^*$ -algebra  $A$ , we say that  $p$  is *Murray-von Neumann subequivalent* to  $q$ , and write  $p \preceq q$ , if there is a partial isometry  $v \in A$  with  $v^*v = p$  and  $vv^* \leq q$ .

**Definition IV.1.** *Let  $A$  be a simple, unital  $C^*$ -algebra. We say that  $A$  has tracial rank zero if for every  $\varepsilon > 0$ , every finite subset  $F \subset A$ , and every nonzero positive element  $x \in A$ , there exists a projection  $p \in A$  and a unital finite-dimensional subalgebra  $D \subset pAp$  such that:*

1.  $\|pa - ap\| < \varepsilon$  for all  $a \in F$ ;
2.  $\text{dist}(pap, D) < \varepsilon$  for all  $a \in F$ ;
3.  $1 - p$  is Murray-von Neumann equivalent to a projection in  $\overline{xAx}$ . (That is, there is a  $v \in A$  such that  $v^*v = 1 - p$  and  $vv^*$  is a projection in  $\overline{xAx}$ .)

A  $C^*$ -algebra with tracial rank zero is thought of as being “approximately finite-dimensional in trace”. (If  $x$  is small enough in a suitable sense, then condition (3) of the definition tells us that  $\tau(1 - p) < \varepsilon$  for all  $\tau \in T(A)$ .) Every AF-algebra (a  $C^*$ -algebra which is a direct limit of finite-dimensional  $C^*$ -algebras) has tracial rank zero, but there are many  $C^*$ -algebras with tracial rank zero which are very far from being AF-algebras. Consequently, tracial rank zero is a rather weak type of approximate finite-dimensionality for a  $C^*$ -algebra  $A$ , that nevertheless is known to imply a great deal about the structure of  $A$ . For our purposes, this definition will be used to ensure that certain  $C^*$ -algebras we will use have tractable structure. It is one of the most important concepts in the classification theory of  $C^*$ -algebras, and our ultimate goals (which is still far from being realized) is to show that the crossed product  $C^*$ -algebras we consider have tracial rank zero under suitable assumptions about their K-theory.

**Notation IV.2.** *Throughout, we take  $(X, h)$  to be as in Notation III.1, and  $A$  to be a simple, unital, separable, infinite-dimensional nuclear  $C^*$ -algebra with tracial rank zero that satisfies the Universal Coefficient Theorem of [47]. Assume in addition that  $A$  is a direct limit of recursive subhomogeneous algebras, in the sense of [39]. Form the algebra  $C(X, A)$ , consisting of all continuous functions  $f: X \rightarrow A$ , with pointwise algebra operations, adjoints given by  $f^*(x) = (f(x))^*$  for all  $x \in X$ , and  $\|f\| = \sup_{x \in X} \|f(x)\|$ . We frequently identify  $C(X, A)$  with  $C(X) \otimes A$  in the canonical way; see [55] for details. For  $f \in C(X)$  and  $a \in A$ , we denote by  $f \otimes a$  the element of  $C(X, A)$  given by  $(f \otimes a)(x) = f(x)a$  for all  $x \in X$ , noting that these elementary tensors in fact span  $C(X, A)$ . We identify  $C(X)$  with the subalgebra of  $C(X, A)$  given by  $\{f \otimes 1: f \in C(X)\}$ , and observe that this is the center  $Z(C(X, A))$  of  $C(X, A)$ .*

We will not elaborate on what it means for a  $C^*$ -algebra to satisfy the Universal Coefficient Theorem, since it is quite complicated and is only necessary for one technical step in our development. It is a technical requirement that is needed to show certain types of  $C^*$ -algebras are classifiable.

We observe some basic facts about the structure of  $C(X, A)$ . Recall that a  $C^*$ -algebra  $A$  is said to have *order on projections over  $A$  determined by traces* if whenever  $p, q \in A$  are projections and  $\tau(p) < \tau(q)$  for all  $\tau \in T(A)$ , then  $p \preceq q$ . This is Blackadar’s Second Fundamental Comparability Question for  $M_\infty(A)$ . (See [4].)

**Proposition IV.3.** *Let  $(X, h)$  and  $A$  be as in Notation IV.2. Then  $C(X, A)$  has cancellation of projections, and order on projections over  $C(X, A)$  is determined by traces.*

*Proof.* Since  $A$  has tracial rank zero and satisfies the Universal Coefficient Theorem, Lin's classification theory (see [22]) implies that  $A$  is a simple infinite-dimensional AH-algebra with no dimension growth. Write  $A \cong \varinjlim A_n$ , where the  $A_n$  are recursive subhomogeneous algebras and the direct system has no dimension growth, and observe that

$$C(X, A) \cong C(X) \otimes A \cong C(X) \otimes \left( \varinjlim A_n \right) \cong \varinjlim C(X) \otimes A_n.$$

Hence  $C(X, A)$  itself is a simple, infinite-dimensional inductive limit of homogeneous algebras with no dimension growth. Now Corollary 1.9 of [40] implies that the associated direct system has strict slow dimension growth. By Theorem 3.7 of [32], it follows that  $C(X, A)$  has cancellation and order on projections over  $C(X, A)$  is determined by traces.  $\square$

This proof used heavy machinery which necessitated the inclusion of hypotheses that are probably not actually needed for the desired result, and a more direct argument should be possible using results of [58] on the homotopy groups for the spaces of projections in certain  $C^*$ -algebras.

**Proposition IV.4.** *Let  $(X, h)$  and  $A$  be as in Notation IV.2, and suppose that  $A$  has a unique tracial state  $\tau$ . Then  $T(C(X, A)) \cong T(C(X)) \cong M(X)$ , the space of Borel probability measures on  $X$ . Given a Borel probability measure  $\mu$  on  $X$ , the induced tracial state  $\lambda_\mu$  on  $C(X, A)$  is given by*

$$\lambda_\mu(f) = \int_X \tau(f(x)) d\mu$$

for all  $f \in C(X, A)$ .

*Proof.* Let  $\lambda \in T(C(X, A))$ , and define  $\omega: C(X) \rightarrow \mathbb{C}$  by  $\omega(f) = \lambda(f \otimes 1)$ . Then  $\omega$  is clearly a tracial state on  $C(X)$ . We claim that  $\lambda = \omega \otimes \tau$ . By the continuity of  $\omega \otimes \tau$ , it suffices to check this on elements of the form  $f \otimes a$ , since these span  $C(X, A)$ . Further, by the linearity of  $\omega$ , it suffices to prove this for  $f \geq 0$ . Fix  $f \in C(X)_+$  and consider the map  $\lambda_f: A \rightarrow \mathbb{C}$  given by  $\lambda_f(a) = \lambda(f \otimes a)$ . Then  $\lambda_f$  is a positive linear functional on  $A$  that is easily seen to satisfy the trace property, but is not necessarily normalized. Therefore,  $\lambda_f$  must be a positive scalar multiple of  $\tau$ . Let  $\omega_f \in [0, \infty)$

be this scalar, so  $\lambda_f(a) = \omega_f\tau(a)$ . Now for any  $f \in C(X)_+$ , we have

$$\omega(f) = \lambda(f \otimes 1) = \lambda_f(1) = \omega_f\tau(1) = \omega_f,$$

and so  $\lambda(f \otimes a) = \omega(f)\tau(a) = (\omega \otimes \tau)(f \otimes a)$  for all  $f \in C(X)_+$  and  $a \in A$ . As discussed, this is sufficient to imply that  $\lambda = \omega \otimes \tau$ .

Finally, the Riesz Representation Theorem yields a Borel probability measure  $\mu$  on  $X$  such that

$$\omega(f) = \int_X f \, d\mu$$

for all  $f \in C(X)$ , from which the given result follows.  $\square$

**Lemma IV.5.** *Let  $(X, h)$  and  $A$  be as in Notation IV.2. Let  $\alpha: X \rightarrow \text{Aut}(A)$  (where  $\alpha(x)$  will be denoted  $\alpha_x$ ) be a map which is continuous in the strong operator topology. (In other words, for each  $a \in A$  the mapping  $x \rightarrow \alpha_x(a)$  is norm-continuous.) Then the map  $\alpha^{-1}: X \rightarrow \text{Aut}(A)$  given by  $\alpha^{-1}(x) = \alpha_x^{-1}$  is continuous in the strong operator topology.*

*Proof.* Let  $\varepsilon > 0$  be given, let  $x \in X$ , and let  $a \in A$ . Then there is a  $b \in A$  such that  $\alpha_x(b) = a$ . By the strong operator continuity of  $\alpha$  at  $x$ , there is a  $\delta > 0$  such that  $d(x, y) < \delta$  implies  $\|\alpha_x(b) - \alpha_y(b)\| < \varepsilon$ . Then for all  $y \in X$  with  $d(x, y) < \delta$ , we have

$$\begin{aligned} \|\alpha_x^{-1}(a) - \alpha_y^{-1}(a)\| &= \|\alpha_y^{-1}(\alpha_x(b)) - \alpha_x^{-1}(\alpha_x(b))\| \\ &= \|b - \alpha_y^{-1}(\alpha_x(b))\| \\ &= \|\alpha_y^{-1}(\alpha_y(b)) - \alpha_y^{-1}(\alpha_x(b))\| \\ &\leq \|\alpha_y(b) - \alpha_x(b)\| \\ &< \varepsilon. \end{aligned}$$

It follows that  $\alpha^{-1}$  is strong operator continuous at  $x$ . Since this holds for all  $x \in X$ ,  $\alpha^{-1}$  is continuous in the strong operator topology.  $\square$

**Proposition IV.6.** *Let  $(X, h)$  and  $A$  be as in Notation IV.2. Let  $\alpha: X \rightarrow \text{Aut}(A)$  be a map which is continuous in the strong operator topology. Define a map  $\beta: C(X, A) \rightarrow C(X, A)$  by  $\beta(f)(x) = \alpha_x(f \circ h^{-1}(x))$  for each  $x \in X$ . Then  $\beta \in \text{Aut}(C(X, A))$ .*

*Proof.* We first verify that  $\beta(f)$  is continuous for  $f \in C(X, A)$ . Let  $\varepsilon > 0$  be given, let  $f \in C(X, A)$ , and let  $x \in X$ . Since  $f \circ h^{-1}(x) \in A$  and  $\alpha$  is continuous in the strong operator topology, there exists  $\delta_1 > 0$  such that  $d(x, y) < \delta_1$  implies  $\|\alpha_x(f \circ h^{-1}(x)) - \alpha_y(f \circ h^{-1}(x))\| < \varepsilon/2$ . Since  $f$  is continuous, there exists  $\delta_2 > 0$  such that  $d(x, y) < \varepsilon/2$  implies  $\|f(x) - f(y)\| < \varepsilon/2$ . Also, since  $h$  is a homeomorphism, there is a  $\delta_3 > 0$  such that  $d(x, y) < \delta_3$  implies  $d(h^{-1}(x), h^{-1}(y)) < \delta_2$ . Now let  $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ . Then for all  $y \in X$  with  $d(x, y) < \delta$ , we have

$$\begin{aligned} \|\beta(f)(x) - \beta(f)(y)\| &= \|\alpha_x(f \circ h^{-1}(x)) - \alpha_y(f \circ h^{-1}(y))\| \\ &\leq \|\alpha_x(f \circ h^{-1}(x)) - \alpha_y(f \circ h^{-1}(x))\| + \|\alpha_x(f \circ h^{-1}(x)) - \alpha_x(f \circ h^{-1}(y))\| \\ &< \frac{\varepsilon}{2} + \|\alpha_x\| \|f \circ h^{-1}(x) - f \circ h^{-1}(y)\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus  $\beta(f)$  is continuous at  $x$ . Since this holds for any  $x \in X$ , it follows that  $\beta(f) \in C(X, A)$ . Therefore  $\beta$  really is a mapping  $C(X, A) \rightarrow C(X, A)$ .

Since the operations on  $C(X, A)$  are given pointwise, each  $\alpha_x$  is an automorphism on  $A$  for  $x \in X$ , and the map  $f \mapsto f \circ h^{-1}$  is an automorphism of  $C(X)$ , it follows easily that for all  $f, g \in C(X, A)$ , we have  $\beta(f + g) = \beta(f) + \beta(g)$ ,  $\beta(fg) = \beta(f)\beta(g)$ , and  $\beta(f^*) = \beta(f)^*$ . This implies that  $\beta$  is a  $*$ -homomorphism.

Next suppose that  $f \in \ker(\beta)$ . Then  $\beta(f)(x) = 0$  for all  $x \in X$ , and so  $\alpha_x(f \circ h^{-1}(x)) = 0$  for all  $x \in X$ . Since each  $\alpha_x$  is an automorphism of  $A$ , this implies that  $f \circ h^{-1}(x) = 0$  for each  $x \in X$ , and hence  $f \circ h^{-1} = 0$ . As  $h$  is a homeomorphism, it follows that  $f = 0$ . Now let  $f \in C(X, A)$ . Define  $g: X \rightarrow A$  by  $g(x) = \alpha_x^{-1}(f \circ h(x))$ . That  $g$  is continuous follows from the same argument that shows  $\beta$  is continuous, using Lemma IV.5. Now for each  $x \in X$ ,  $\beta(g)(x) = \alpha_x(\alpha_x^{-1}((f \circ h) \circ h^{-1}(x))) = f(x)$ , and so  $\beta(g) = f$ . It follows that  $\beta$  is bijective, and hence  $\beta \in \text{Aut}(C(X, A))$ .  $\square$

**Proposition IV.7.** *Let  $(X, h)$  and  $A$  be as in Notation IV.2. Let  $\alpha: X \rightarrow \text{Aut}(A)$  be continuous in the strong operator topology. For  $k \in \mathbb{Z} \setminus \{0\}$ , we define  $\alpha^{(k)}: X \rightarrow \text{Aut}(A)$  by  $\alpha^{(k)}(x) = \alpha_x \circ \alpha_{h^{-1}(x)} \circ \cdots \circ \alpha_{h^{-(k-1)}(x)}$  if  $k \geq 1$  and  $\alpha^{(k)}(x) = \alpha_{h(x)} \circ \cdots \circ \alpha_{h^{|k|}(x)}$  if  $k < 0$ , henceforth denoting  $\alpha^{(k)}(x)$  by  $\alpha_x^{(k)}$ . Then  $\alpha^{(k)}$  is continuous in the strong operator topology. Moreover,*

the map  $\alpha^{-(k)}: X \rightarrow \text{Aut}(A)$ , defined by  $\alpha_x^{-(k)} = \alpha_{h^{-(k-1)}(x)}^{-1} \circ \cdots \circ \alpha_{h^{-1}(x)}^{-1} \circ \alpha_x^{-1}$  for  $k \geq 1$  and  $\alpha_x^{-(k)} = \alpha_{h^{1(k)}(x)}^{-1} \circ \cdots \circ \alpha_{h(x)}^{-1}$  for  $k < 0$ , is continuous in the strong operator topology and satisfies  $\alpha_x^{-(k)} = (\alpha_x^{(k)})^{-1}$  for all  $x \in X$ .

*Proof.* First, assume that  $k \geq 1$ . We proceed by induction on  $k$ . When  $k = 1$  the map  $\alpha^{(1)}: X \rightarrow \text{Aut}(A)$  is simply  $\alpha_x^{(1)} = \alpha_x$ , which is continuous in the strong operator topology by assumption. Suppose that  $\alpha^{(k)}$  is continuous in the strong operator topology for some  $k \geq 1$ . Let  $\varepsilon > 0$  be given, let  $a \in A$ , and let  $x \in X$ . Then there is a  $\delta_1 > 0$  such that  $d(x, y) < \delta_1$  implies  $\|\alpha_x^{(k)}(a) - \alpha_y^{(k)}(a)\| < \frac{1}{2}\varepsilon$ . Further, with  $b = \alpha_x^{(k)}(a)$ , the strong operator continuity of  $\alpha = \alpha^{(1)}$  gives a  $\delta_2 > 0$  such that  $d(x, y) < \delta_2$  implies  $\|\alpha_x(b) - \alpha_y(b)\| < \frac{1}{2}\varepsilon$ . Let  $\delta = \min\{\delta_1, \delta_2\}$ . Then  $d(x, y) < \delta$  implies that

$$\begin{aligned} \|\alpha_x^{(k+1)}(a) - \alpha_y^{(k+1)}(a)\| &\leq \|\alpha_x^{(k+1)}(a) - \alpha_y \circ \alpha_x^{(k)}(a)\| + \|\alpha_y \circ \alpha_x^{(k)}(a) - \alpha_y^{(k+1)}(a)\| \\ &= \|\alpha_x(\alpha_x^{(k)}(a)) - \alpha_y(\alpha_x^{(k)}(a))\| + \|\alpha_y(\alpha_x^{(k)}(a)) - \alpha_y^{(k)}(a)\| \\ &\leq \|\alpha_x(b) - \alpha_y(b)\| + \|\alpha_x^{(k)}(a) - \alpha_y^{(k)}(a)\| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \\ &= \varepsilon. \end{aligned}$$

It follows that  $\alpha^{(k+1)}$  is continuous at  $x$  in the strong operator topology. Since this holds for all  $x \in X$ ,  $\alpha^{(k+1)}$  is continuous in the strong operator topology. By induction,  $\alpha^{(k)}$  is continuous in the strong operator topology for all  $k \geq 1$ . To obtain continuity for all  $k \in \mathbb{Z} \setminus \{0\}$ , note that  $g = h^{-1}$  is also a homeomorphism, and for any  $k \geq 1$  we have

$$\alpha_x^{(-k)} = \alpha_{h^k(x)} \circ \cdots \circ \alpha_{h(x)} = \alpha_{g^{-1}(x)} \circ \cdots \circ \alpha_{g^{-k}(x)}.$$

Applying the above argument to the map  $\gamma^{(k)}: X \rightarrow \text{Aut}(A)$  given by  $\gamma^{(k)}(x) = \alpha_x \circ \alpha_{g^{-1}(x)} \circ \cdots \circ \alpha_{g^{-k}(x)}$  shows that  $\gamma_x^{(k)} = \alpha_x \circ \alpha_x^{(-k)}$  is continuous at  $x$  in the strong operator topology for  $k \geq 1$ . Since  $\alpha_x^{-1}$  is also continuous at  $x$  in the strong operator topology, so is  $\alpha_x^{(-k)} = \alpha_x^{-1} \circ \gamma_x^{(k)}$ . Thus  $\alpha^{(k)}$  is continuous in the strong operator topology for all  $k \in \mathbb{Z}$ .

Finally,  $\alpha^{-1}$  is continuous in the strong operator topology by Lemma IV.5, and so an argument analogous to the one above, with  $\alpha^{-1}$  in place of  $\alpha$ , shows that  $\alpha^{-(k)}$  is continuous

in the strong operator topology for all  $k \in \mathbb{Z}$ . Further, it is easy to see that for any  $x \in X$ ,  $\alpha_x^{(k)} \circ \alpha_x^{-(k)} = \text{id}_A = \alpha_x^{-(k)} \circ \alpha_x^{(k)}$ .  $\square$

**Corollary IV.8.** *Let  $(X, h)$  and  $A$  be as in Notation IV.2, and let  $\beta \in \text{Aut}(C(X, A))$  be the automorphism of Proposition IV.6. For  $n \in \mathbb{Z} \setminus \{0\}$ , the automorphism  $\beta^n \in \text{Aut}(C(X, A))$  is given explicitly by  $\beta^n(f)(x) = \alpha_x^{(n)}(f \circ h^{-n}(x))$  for all  $x \in X$ .*

*Proof.* We consider first the case where  $n \geq 1$ , and proceed by induction on  $n$ . Observe that for all  $x \in X$ , we have

$$\beta^1(f)(x) = \beta(f)(x) = \alpha_x(f \circ h^{-1}(x)) = \alpha_x^{(1)}(f \circ h^{-1}(x))$$

and so the base case holds. Next, suppose that  $\beta^n(f)(x) = \alpha_x^{(n)}(f \circ h^{-n}(x))$  for some  $n \geq 1$ . Then for all  $x \in X$ , we compute

$$\begin{aligned} \beta^{n+1}(f)(x) &= \beta^n(\beta(f))(x) \\ &= \alpha_x^{(n)}((\beta(f)) \circ h^{-n}(x)) \\ &= \alpha_x^{(n)}(\beta(f)(h^{-n}(x))) \\ &= \alpha_x^{(n)}(\alpha_{h^{-n}(x)}(f \circ h^{-1}(h^{-n}(x)))) \\ &= \alpha_x^{(n)} \circ \alpha_{h^{-n}(x)}(f \circ h^{-1-n}(x)) \\ &= \alpha_x^{(n+1)}(f \circ h^{-(n+1)}(x)). \end{aligned}$$

It follows that the result holds for all  $n \geq 1$ . To extend this result to all  $n \in \mathbb{Z} \setminus \{0\}$ , we first observe that  $\psi \in \text{Aut}(C(X, A))$ , given by  $\psi(f)(x) = \alpha_{h(x)}^{-1}(f \circ h(x))$ , satisfies  $\psi \circ \beta(f)(x) = f(x) = \beta \circ \psi(f)(x)$  for all  $f \in C(X, A)$  and  $x \in X$ , and hence  $\psi \circ \beta = \text{id}_{C(X, A)} = \beta \circ \psi$ . This gives  $\psi = \beta^{-1}$ . Further, an induction argument entirely analogous to the one above shows that for  $k \geq 1$ ,  $\psi^k(f)(x) = \alpha_x^{(-k)}(f \circ h^k(x))$  for all  $f \in C(X, A)$  and  $x \in X$ . But  $\psi = \beta^{-1}$  implies that  $\beta^{-k}(f)(x) = \alpha_x^{(-k)}(f \circ h^k(x))$  for  $k \geq 1$ . Letting  $n = -k$ , it follows that  $\beta^n(f)(x) = \alpha_x^{(n)}(f \circ h^{-n}(x))$  for  $n < 0$ .  $\square$

**Definition IV.9.** *Let  $(X, h)$  and  $A$  be as in Notation IV.2. For an open set  $V \subset X$  and a projection  $p \in A$ , the hereditary subalgebra of  $C(X, A)$  determined by  $V$  and  $p_0$ , denoted by*

$\text{Her}(V, p_0)$ , is defined to be the hereditary subalgebra of  $C(X, A)$  generated by all functions  $f \in C(X, A)$  such that  $\text{supp}(f) \subset V$  and  $f \leq 1 \otimes p$ .

**Lemma IV.10.** *Let  $(X, h)$  and  $A$  be as in Notation IV.2, and let  $\alpha: X \rightarrow \text{Aut}(A)$  be continuous in the strong operator topology. Let  $p_0 \in A$  be a non-zero projection, assume that  $A$  has a unique tracial state  $\tau$ , let  $k \in \mathbb{Z}$ , and let  $\alpha^{(k)}$  be as in Proposition IV.7. Then for any projection  $p \in A$  with the property that  $\tau(p) < \tau(p_0)$ , the function  $q_p: X \rightarrow A$  given by  $q_p(x) = \alpha_x^{(k)}(p)$  is a projection in  $C(X, A)$  that satisfies  $q_p \lesssim 1 \otimes p_0$ .*

*Proof.* It is clear that  $q_p$  is continuous, that  $q_p^* = q_p$ , and that  $q_p^2 = q_p$ . Therefore,  $q_p$  is a projection in  $C(X, A)$ . For any  $x \in X$ ,  $\alpha_x^{(k)} \in \text{Aut}(A)$  implies that  $\tau \circ \alpha_x^{(k)} \in T(A)$ , and therefore  $\tau \circ \alpha_x^{(k)} = \tau$ . Hence for any  $x \in X$ , we have

$$\tau(q_p(x)) = \tau(\alpha_x^{(k)}(p)) = \tau(p) < \tau(p_0) = \tau((1 \otimes p_0)(x)).$$

Now let  $\lambda \in T(C(X, A))$ . Since  $A$  has a unique tracial state, Proposition IV.4 implies that there is a Borel probability measure  $\mu$  on  $X$  such that

$$\lambda(f) = \int_X \tau(f(x)) d\mu$$

for all  $f \in C(X, A)$ . Then the previous inequality gives

$$\lambda(q_p) = \int_X \tau(q_p(x)) d\mu < \int_X \tau((1 \otimes p_0)(x)) d\mu = \lambda(1 \otimes p_0).$$

Since  $\lambda \in T(C(X, A))$  was arbitrary and Proposition IV.3 implies that order on projections over  $C(X, A)$  is determined by traces, we conclude that  $q_p \lesssim 1 \otimes p_0$ .  $\square$

We expect that the assumption that  $A$  has a unique tracial state can eventually be removed, through a more careful analysis of the tracial state space of  $C(X, A)$ . Several of the proofs we give later will thus contain statements such as “for all  $\tau \in T(A)$ ” even though  $T(A)$  will contain only one element  $\tau$ , since it is no more difficult to present them this way and will facilitate adapting them to the more general situation.

**Lemma IV.11.** *Let  $(X, h)$  and  $A$  be as in Notation IV.2. Let  $p, q \in C(X, A)$  be projections with  $p \lesssim q$ . Then there is a unitary  $w \in C(X, A)$  such that  $wpw^* \leq q$ .*



*Proof.* Since  $C(X, A)$  has cancellation by Proposition IV.3, there exists a projection  $e \in C(X, A)$  such that  $e \leq q$  and partial isometries  $s, t \in C(X, A)$  such that  $s^*s = p$ ,  $ss^* = e$ ,  $t^*t = 1 - p$ , and  $tt^* = 1 - e$ . Define  $w = s + t$ . It is straightforward to check that  $s^*t = st^* = ts^* = t^*s = 0$ , from which it follows that  $w^*w = (s^* + t^*)(s + t) = s^*s + t^*t = p + (1 - p) = 1$  and  $ww^* = (s + t)(s^* + t^*) = ss^* + tt^* = e + (1 - e) = 1$ , so  $w$  is unitary. Moreover,

$$\begin{aligned}
wpw^* &= (s + t)p(s^* + t^*) \\
&= sps^* + tpt^* + spt^* + tps^* \\
&= ss^*ss^* + t(1 - t^*t)t^* + ss^*st^* + t(1 - t^*t)s^* \\
&= e^2 + tt^* - tt^*t^* \\
&= e + (1 - e) - (1 - e)^2 \\
&= e,
\end{aligned}$$

as required. □

**Proposition IV.12.** *Let  $(X, h)$  and  $A$  be as in Notation IV.2. Suppose in addition that  $h$  is uniquely ergodic, and let  $(X, h, \mu)$  be as in Notation III.17. Let  $\beta \in \text{Aut}(C(X, A))$  be the automorphism of Proposition IV.6. Suppose that  $(X, h, \mu)$  has the dynamic comparison property, and that  $A$  has a unique tracial state. Then for every non-zero projection  $p_0 \in A$  and every non-empty open set  $V \subset X$ , there exist  $M \in \mathbb{N}$  and  $\varepsilon > 0$  such that whenever  $g_0 \in C(X)$  is positive and satisfies  $\mu(\text{supp}(g_0)) < \varepsilon$ , then there exist for  $0 \leq k \leq M$  positive elements  $a_k \in C(X, A)$ , unitaries  $w_k \in C(X, A)$ , and  $r(k) \in \mathbb{Z}$  such that:*

1.  $\sum_{k=0}^M a_k \geq g_0 \otimes 1$ ;
2. the elements  $\beta^{r(k)}(a_k)$  are mutually orthogonal, and  $\text{supp}(\beta^{r(k)}(a_k)) \subset V$  for each  $k$ ;
3. with  $b_k = w_k \beta^{r(k)}(a_k) w_k^*$ , the  $b_k$  are mutually orthogonal positive elements in  $\text{Her}(V, p_0)$ .

*Proof.* Set  $\delta = \inf_{\tau \in T(A)} \tau(p_0) > 0$ , and choose  $N \in \mathbb{N}$  such that  $N > 1$  and  $1/N < \delta$ . Then by Theorem 1.1 of [58] there exist  $2^N + 1$  mutually orthogonal projections  $q_0, \dots, q_{2^N}$  such that  $q_0 \lesssim q_1 \sim \dots \sim q_{2^N}$  and  $\sum_{j=0}^{2^N} q_j = 1$ . We immediately obtain  $\tau(q_1) = \dots = \tau(q_{2^N})$  for all

$\tau \in T(A)$ . Then for  $1 \leq j \leq 2^N$  and each  $\tau \in T(A)$ , we have

$$1 = \tau(1) = \sum_{i=0}^{2^N} \tau(q_i) \geq \sum_{i=1}^{2^N} \tau(q_i) = 2^N \tau(q_j),$$

and so  $\tau(q_j) \leq 1/2^N$ . This gives  $\tau(q_j) < 1/N < \delta$  for  $1 \leq j \leq 2^N$ . Hence  $\tau(q_j) < \tau(p_0)$  for all  $\tau \in T(A)$ , and since the order on projections in  $A$  is determined by traces, we conclude that  $q_j \lesssim p_0$  for  $1 \leq j \leq 2^N$ . Since  $q_0 \lesssim q_1$ , we actually obtain  $q_j \lesssim p_0$  for  $0 \leq j \leq 2^N$ .

Set  $J = 2^N$ , and let  $\sigma = \mu(V) > 0$ . Choose  $J$  distinct points  $x_0, \dots, x_J \in V$  and for each  $j$  consider the nested sequence of neighborhoods  $(B(x_j, 1/k))_{k=1}^\infty$ . Choose  $K_1 \in \mathbb{N}$  so large that  $B(x_i, 1/K_1) \cap B(x_j, 1/K_1) = \emptyset$  for  $0 \leq i, j \leq J$  and  $i \neq j$  (this can be done since  $X$  is Hausdorff) and choose  $K_2 \in \mathbb{N}$  so large that  $\mu(B(x_j, 1/K_2)) < \sigma/(J+1)$ . This is possible since for  $0 \leq j \leq 2^N$ , the sequence  $(\mu(B(x_j, 1/k)))_{k=1}^\infty$  decreases monotonically to 0. Let  $K = \max\{K_1, K_2\}$ , and for  $0 \leq j \leq 2^N$  set  $V_j = B(x_j, 1/K)$  and  $W_j = B(x_j, 1/(K+1))$ . Then  $W_j \subset \overline{W}_j \subset V_j$ ,  $\mu(V_j) < \sigma/(J+1)$ , and the sets  $V_j$  are pairwise disjoint. Now set  $\varepsilon = \min\{\mu(W_j) : 0 \leq j \leq M\}$ . Let  $g_0 \in C(X)$  be positive such that  $C = \text{supp}(g_0)$  satisfies  $\mu(C) < \varepsilon$ . Then  $\mu(C) < \mu(W_j)$  for  $0 \leq j \leq J$ . By assumption,  $X$  has the dynamic comparison property, and so for each  $0 \leq j \leq J$  there exist  $M_j \in \mathbb{N}$ , continuous functions  $f_{j,i} : X \rightarrow [0, 1]$  for  $0 \leq i \leq M_j$ , and  $r_j(i) \in \mathbb{Z}$  for  $0 \leq i \leq M_j$ , such that  $\sum_{i=0}^{M_j} f_{j,i} = 1$  on  $C$  and such that the sets  $\text{supp}(f_{j,i} \circ h^{-r_j(i)})$  are pairwise disjoint subsets of  $V_j$  for  $0 \leq i \leq M_j$ .

For  $0 \leq j \leq J$  and  $0 \leq i \leq M_j$ , define  $q_{j,i} : X \rightarrow A$  by  $q_{j,i}(x) = \alpha_x^{(r_j(i))}(q_j)$ . Then by Lemma IV.10, each  $q_{j,i}$  is an element of  $C(X, A)$  and  $q_{j,i} \lesssim 1 \otimes p_0$  (since  $\tau(q_j) < \delta$  for all  $\tau \in T(A)$ ). Hence by Lemma IV.11, there exist unitaries  $w_{j,i} \in C(X, A)$  for  $0 \leq j \leq J$ ,  $0 \leq i \leq M_j$  such that  $w_{j,i} q_{j,i} w_{j,i}^* \leq 1 \otimes p_0$ . Now for  $0 \leq j \leq J$  and  $0 \leq i \leq M_j$  set  $a_{j,i} = f_{j,i} \otimes q_j$  and  $b_{j,i} = w_{j,i} \beta^{r_j(i)}(a_{j,i}) w_{j,i}^*$ .

Let  $x \in X$ . If  $x \notin C$ , then  $(g_0 \otimes 1)(x) = 0 \leq \sum_{j=0}^J \sum_{i=0}^{M_j} a_{j,i}(x)$ . If  $x \in C$ , then we compute

$$\sum_{j=0}^J \sum_{i=0}^{M_j} a_{j,i}(x) = \sum_{j=0}^J \sum_{i=0}^{M_j} f_{j,i}(x) q_j = \sum_{j=0}^J q_j \left( \sum_{i=0}^{M_j} f_{j,i}(x) \right) = \sum_{j=0}^J q_j = 1.$$

It follows that  $g_0 \otimes 1 \leq \sum_{j=0}^J \sum_{i=0}^{M_j} a_{j,i}$ . Next, for any  $x \in X$ , we have

$$\begin{aligned} \beta^{r_j(i)}(a_{j,i})(x) &= \alpha_x^{(r_j(i))}((f_{j,i} \circ h^{-r_j(i)}(x))q_j) \\ &= (f_{j,i} \circ h^{-r_j(i)}(x))\alpha_x^{(r_j(i))}(q_j) \\ &= (f_{j,i} \circ h^{-r_j(i)}(x))q_{j,i}(x). \end{aligned}$$

This gives  $\text{supp}(\beta^{r_j(i)}(a_{j,i})) \subset \text{supp}(f_{j,i} \circ h^{-r_j(i)}) \subset V$  and hence the sets  $\text{supp}(\beta^{r_j(i)}(a_{j,i}))$  are pairwise disjoint, implying that the elements  $\beta^{r_j(i)}(a_{j,i})$  are mutually orthogonal. Since  $\text{supp}(b_{j,i}) \subset \text{supp}(\beta^{r_j(i)}(a_{j,i}))$ , it follows immediately that the  $b_{j,i}$  are also mutually orthogonal. Moreover, as  $0 \leq f_{j,i} \leq 1$  and  $w_{j,i}q_{j,i}w_{j,i}^* \leq p_0$ , it follows that  $0 \leq b_{j,i} \leq 1 \otimes p_0$ . Therefore, the  $b_{j,i}$  are mutually orthogonal positive elements in  $\text{Her}(V, p_0)$ . Now simply order the  $a_{j,i}, w_{j,i}, d_j(i)$ , and  $b_{j,i}$  as  $a_k, w_k, d(k)$ , and  $b_k$  for  $0 \leq k \leq M$ , where  $M+1 = \sum_{j=0}^J M_j$ .  $\square$

**Lemma IV.13.** *Let  $E \subset \mathbb{C}$  be open, let  $f: E \rightarrow \mathbb{C}$  be continuous, let  $A$  be a unital  $C^*$ -algebra, and set  $Q = \{b \in A: b \text{ is normal with } \text{sp}(b) \subset E\}$ . Then  $\varphi: Q \rightarrow A$  given by  $\varphi(b) = f(b)$  is norm-continuous.*

*Proof.* This is easily adapted from Lemma 2.5.11 of [19].  $\square$

**Proposition IV.14.** *Let  $(X, h)$  and  $A$  be as in Notation IV.2. Let  $g \in C(X, A)$  be a non-zero positive element with  $\|g\| = 1$ . Then there is an open set  $V \subset \text{supp}(g)$ , a non-zero projection  $p_0 \in A$ , and a unitary  $w \in C(X, A)$  such that  $wfw^* \in \overline{gC(X, A)g}$  for all  $f \in \text{Her}(V, p_0)$ .*

*Proof.* Let  $\varepsilon > 0$  be given, and assume that  $\varepsilon < 1$ . Since  $\|g\| = 1$  and  $X$  is compact, there exists  $x_0 \in \text{supp}(g)$  such that  $\|g(x_0)\| = 1$ . Let  $a = g(x_0)$  (note that  $a \geq 0$  since  $g$  is positive) and define continuous functions  $k_1, k_2: [0, 1] \rightarrow [0, 1]$  by

$$k_1(t) = \begin{cases} \frac{32}{32-\varepsilon}t & 0 \leq t \leq 1 - \frac{\varepsilon}{32} \\ 1 & 1 - \frac{\varepsilon}{32} < t \leq 1 \end{cases}$$

and

$$k_2(t) = \begin{cases} 0 & 0 \leq t \leq 1 - \frac{\varepsilon}{64} \\ \frac{64}{\varepsilon}(t-1) + 1 & 1 - \frac{\varepsilon}{64} < t \leq 1. \end{cases}$$

Setting  $a_1 = k_1(a)$  and  $a_2 = k_2(a)$ , we observe that  $a_2a_1 = a_2$  and

$$\|a - a_1\| = \sup_{t \in [0, \|a\|]} |t - k_1(t)| < \frac{1}{16}\varepsilon.$$

This gives  $\|a_2a - a_2\| = \|a_2a - a_2a_1\| \leq \|a - a_1\| < \frac{1}{16}\varepsilon$ . Since  $A$  is simple, unital, and has tracial rank zero it also has real rank zero by Theorem 3.6.11 of [19], so there is a non-zero projection  $q \in \overline{a_2Aa_2}$ . Then  $a_2a_1 = a_2$  implies that  $qa_1 = q$ . We thus obtain  $\|qa - q\| = \|qa - qa_1\| \leq \|a - a_1\| < \frac{1}{16}\varepsilon$ , and similarly  $\|aq - q\| < \frac{1}{16}\varepsilon$ . Now choose a neighborhood  $U$  of  $x_0$  such that  $\|g(x) - g(x_0)\| < \frac{1}{8}\varepsilon$  for all  $x \in U$ . Using the compactness of  $X$ , choose an open set  $W \subset U$  with  $\overline{W} \subset U$ , and set  $K = \overline{W}$ . Then for all  $x \in K$ ,

$$\begin{aligned} \|qg(x) - q\| &\leq \|qg(x) - qg(x_0)\| + \|qg(x_0) - q\| \\ &\leq \|g(x) - g(x_0)\| + \|qa - q\| \\ &< \frac{1}{8}\varepsilon + \frac{1}{8}\varepsilon \\ &= \frac{1}{4}\varepsilon. \end{aligned}$$

So for all  $x \in K$ , we have

$$\begin{aligned} \|g(x)qg(x) - q\| &\leq \|g(x)qg(x) - g(x)q\| + \|g(x)q - q\| \\ &\leq \|g(x)\| \|qg(x) - q\| + \|g(x)q - q\| \\ &< \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon \\ &= \frac{1}{2}\varepsilon. \end{aligned}$$

Set  $E = (-\infty, 1/2) \cup (1/2, \infty)$ ,  $f = \chi_{(1/2, \infty)}$ , and  $Q = \{b \in A : b \text{ is normal with } \text{sp}(b) \subset E\}$ . Apply Lemma IV.13 to obtain a continuous function  $\varphi : Q \rightarrow A$  such that  $\varphi(b) = \chi_{(1/2, \infty)}(b)$  for all  $b \in Q$ . Next observe that for all  $x \in K$ ,  $\|g(x)qg(x) - q\| < \frac{1}{2}\varepsilon < \frac{1}{2}$  implies that  $g(x)qg(x) \in Q$ .

Thus we may define a function  $\psi: K \rightarrow Q$  by  $\psi(x) = g(x)qg(x)$ . Further, for  $x, y \in K$  we have

$$\begin{aligned} \|\psi(x) - \psi(y)\| &= \|g(x)qg(x) - g(y)qg(y)\| \\ &\leq \|g(x)qg(x) - q\| + \|q - g(y)qg(y)\| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \\ &= \varepsilon, \end{aligned}$$

which implies that  $\psi$  is continuous on  $K$ . Now setting  $p^{(0)} = \varphi \circ \psi$  gives a continuous function  $p^{(0)}: K \rightarrow A$  with  $p^{(0)}(x) = \chi_{(1/2, \infty)}(g(x)qg(x)) \in \overline{g(x)Ag(x)}$  for all  $x \in K$ . Extend  $p^{(0)}$  to a continuous function  $p: X \rightarrow A$  such that  $\text{supp}(p) \subset \text{supp}(g)$ . Choose  $\delta > 0$  so small that  $\delta < 1$  and  $d(x, x_0) < \delta$  implies  $p(x)$  is a projection. Set  $V_0 = B(p(x_0), \delta)$  and  $V = p^{-1}(V_0)$ . Then  $x_0 \in V \subset \overline{V}$ , and  $\|p(x) - p(x_0)\| \leq \frac{1}{2} < 1$  for all  $x \in \overline{V}$  by the continuity of  $p$ . Let  $p_0 = p(x_0)$  and  $F = \overline{V}$ .

Set  $p_F = p|_F$  and let  $e: F \rightarrow A$  be the constant function  $e(x) = p_0$ . Then  $p_F$  and  $e$  are projections in  $C(F, A)$ , and satisfy  $\|p_F(x) - e(x)\| = \|p(x) - p_0\| \leq \delta$  for all  $x \in F$ . This implies that  $\|p_F - e\| < 1$ , and so by Lemma 2.5.1 of [19], there is a unitary  $u \in C(F, A)$  such that  $up_Fu^* = e$  and  $\|1 - u\| \leq \sqrt{2} \|p_F - e\|$ . This norm estimate further implies that  $\|1 - u\| < \sqrt{2}$ , and so  $u \in U_0(C(F, A))$ . (Recall that for a unital  $C^*$ -algebra  $B$ ,  $U_0(B)$  denotes the connected component of  $U(B)$  containing  $1_B$ ). Since the restriction map  $U_0(C(X, A)) \rightarrow U_0(C(F, A))$  is surjective, there is a  $w \in U_0(C(X, A))$  such that  $w|_F = u$ . If  $f \in \text{Her}(V, p_0)$ , then  $\text{supp}(f) \subset F$  and  $f \leq 1 \otimes p_0$ . Then for any  $x \in \text{supp}(f)$ , we have  $w(x)f(x)w(x)^* \leq w(x)p_0w(x)^* = u(x)p_0u(x)^* = p(x)$ . Thus for every  $f \in \text{Her}(V, p_0)$ ,  $\text{supp}(f) \subset F \subset \text{supp}(g)$  and  $f(x) \in \overline{g(x)Ag(x)}$  for all  $x \in X$ .  $\square$

**Theorem IV.15.** *Let  $(X, h)$  and  $A$  be as in Notation IV.2. Assume that  $h$  is uniquely ergodic, and let  $(X, h, \mu)$  be as in Notation III.17. If  $(X, h)$  has the topological small boundary property,  $A$  has a unique tracial state, and  $\beta \in \text{Aut}(C(X, A))$  is the automorphism of Proposition IV.6, then  $\beta$  has the tracial quasi-Rokhlin property.*

*Proof.* First observe that by the choice of  $X$  and  $h$  and the assumption that  $(X, h)$  has the topological small boundary property, Theorem III.25 implies that  $(X, h, \mu)$  has the dynamic comparison property. Let  $\varepsilon > 0$ , let  $F \subset C(X, A)$  be finite, let  $n \in \mathbb{N}$ , and let  $g \in C(X, A)$  be positive with  $\|g\| = 1$ . By Proposition IV.14, there is non-zero projection  $p_0 \in A$ , an open set

$V \subset \text{supp}(g)$ , and a unitary  $u \in C(X, A)$  such that  $ufu^* \in \overline{gC(\overline{X}, A)g}$  for all  $f \in \text{Her}(V, p_0)$ . By Proposition IV.12, there is an  $M \in \mathbb{N}$  and a  $\delta > 0$  such that for any positive element  $g_0 \in C(X)$  with  $\mu(\text{supp}(g_0)) < \delta$ , there exist for  $0 \leq k \leq M$  positive elements  $a_k \in C(X, A)$ , unitaries  $w_k \in C(X, A)$ , and  $r(k) \in \mathbb{Z}$  such that  $\sum_{k=0}^M a_k \geq g_0 \otimes 1$ , the elements  $\beta^{r(k)}(a_k)$  are mutually orthogonal, and such that with  $b_k = w_k \beta^{r(k)}(a_k) w_k^*$ , the  $b_k$  are mutually orthogonal elements of  $\text{Her}(V, p_0)$ . By the continuity of  $g$  and the compactness of  $X$ , there exist  $x_0 \in X$  with  $\|g(x_0)\| = 1$  and an open neighborhood  $G$  of  $x_0$  such that  $\|g(x)\| > 1 - \frac{1}{2}\varepsilon$  for all  $x \in G$ . Choose open neighborhoods  $G_0, G_1, G_2$  of  $x_0$  such that  $G_2 \subset \overline{G_2} \subset G_1 \subset \overline{G_1} \subset G_0 \subset G$ ,  $\mu(G_0) < \delta$ , and  $\|g(x)\| > 1 - \varepsilon$  for all  $x \in G_2$ . Choose continuous functions  $g_0, g_1: X \rightarrow [0, 1]$  such that  $g_1 = 1$  on  $\overline{G_2}$ ,  $\text{supp}(g_1) \subset G_1$ ,  $g_0 = 1$  on  $\overline{G_1}$ , and  $\text{supp}(g_0) \subset G_0$ . Apply Proposition IV.12 with  $g_0$  to obtain the  $a_k, w_k$ , and  $r(k)$  described above. Set  $\sigma = \min\{\frac{1}{2}\mu(G_2), \varepsilon\}$  and choose  $K \in \mathbb{N}$  so large that  $\frac{1}{K} < \frac{1}{8}\sigma$ . Apply Lemma III.23 with  $N = nK$  to obtain a closed set  $Y \subset X$  such that  $\text{int}(Y) \neq \emptyset$ ,  $\partial Y$  is topologically  $h$ -small, and the sets  $Y, h(Y), \dots, h^{nK}(Y)$  are pairwise disjoint. Adopt the notation of Theorem III.8, and let  $M = (l+1) \sum_{k=0}^l n(k)$ . Then:

1. the sets  $h^j(Y_k^\circ)$  are pairwise disjoint for  $0 \leq k \leq l$  and  $0 \leq j \leq n(k) - 1$ ;
2.  $\bigcup_{k=0}^l Y_k = Y$ ;
3.  $\bigcup_{k=0}^l \bigcup_{j=0}^{n(k)-1} h^j(Y_k) = X$ ;
4.  $\partial h^j(Y_k)$  is topologically  $h$ -small for  $0 \leq k \leq l$  and  $0 \leq j \leq n(k) - 1$ ;
5. for  $0 \leq k \leq l$ , there exists an open set  $U_k \subset Y_k^\circ$  such that  $\overline{U_k} \subset Y_k^\circ$ ,  $\partial U_k$  is topologically  $h$ -small, and  $\mu(Y_k^\circ) - \mu(U_k) < \frac{\sigma}{8M}$ ;
6. for  $0 \leq k \leq l$ , there exists an open set  $W_k \subset U_k$  such that  $\overline{W_k} \subset U_k$ ,  $\partial W_k$  is topologically  $h$ -small, and  $\mu(U_k) - \mu(W_k) < \frac{\sigma}{8M}$ .

Properties (1) – (3) follow immediately from Theorem III.8, and property (4) is given by Lemma III.24. For (5), we apply Proposition III.7 to  $Y_k^\circ$  and  $\frac{\sigma}{8M}$  to obtain non-empty open sets  $U_k$  with the given properties, and for (6) we apply Proposition III.7 to  $U_k$  and  $\frac{\sigma}{8M}$  to obtain non-empty open sets  $W_k$  with the given properties. Now for  $0 \leq k \leq l$  set  $s(k) = \max\{m \geq 1: mn \leq n(k) - 1\}$ . Note that  $s(k) \geq K$  by the choice of  $Y$ . For  $0 \leq k \leq l$  and  $0 \leq j \leq s(k)$ , choose continuous functions  $c_{k,j}^{(0)}: X \rightarrow [0, 1]$  such that  $c_{k,j}^{(0)} = 1$  on  $h^{jn}(\overline{W_k})$ , and  $\text{supp}(c_{k,j}^{(0)}) \subset U_k = 0$ . Next set

$c_{k,j} = c_{k,j}^{(0)} \otimes 1$  for  $0 \leq k \leq l$  and  $0 \leq j \leq s(k)$ . Finally, define  $c_0, \dots, c_n \in C(X, A)$  by setting

$$c_0 = \sum_{k=0}^l \sum_{j=0}^{s(k)} c_{k,i}$$

and  $c_{j+1} = \beta(c_j)$  for  $0 \leq j \leq n-1$ . It follows immediately from these definitions that:

1.  $0 \leq c_j \leq 1$  for  $0 \leq j \leq n$ ;
2.  $c_j c_k = 0$  for  $0 \leq j, k \leq n$  and  $j \neq k$ ;
3.  $\|\beta(c_j) - c_{j+1}\| = 0$  for  $0 \leq j \leq n-1$ ;
4.  $\|c_j f - f c_j\| = 0$  for  $0 \leq j \leq n$  and for all  $f \in F$ .

Now set  $c = \sum_{j=0}^n c_j$  and let  $C = \text{supp}(1 - c)$ . Then we have

$$C \subset X \setminus \bigcup_{k=0}^l \bigcup_{j=0}^{s(k)n} h^j(W_k).$$

Also,  $\partial Y_k$  topologically  $h$ -small for  $0 \leq k \leq l$  implies that  $\mu(\partial Y_k) = 0$  by Corollary III.16, and so  $\mu(Y_k) = \mu(Y_k^\circ)$ . Since the  $Y_k^\circ$  are pairwise disjoint, we obtain the inequality

$$\mu(Y) = \mu\left(\bigcup_{k=0}^l Y_k\right) \geq \mu\left(\bigcup_{k=0}^l Y_k^\circ\right) = \sum_{k=0}^l \mu(Y_k^\circ) = \sum_{k=0}^l \mu(Y_k).$$

Further, the  $h$ -invariance of  $\mu$  and the pairwise disjointness of the sets  $h^j(Y)$  for  $0 \leq j \leq nK$  imply that

$$1 \geq \sum_{j=0}^{nK} \mu(h^j(Y)) = \sum_{j=0}^{nK} \mu(Y) = nK\mu(Y)$$

and so we have  $\mu(Y) < 1/(nK)$ . Observing that the sets  $\partial U_k$  and  $\partial W_k$  all have measure zero by

Corollary III.16, it follows that

$$\begin{aligned}
\mu(C) &\leq \mu \left( X \setminus \bigcup_{k=0}^l \bigcup_{j=0}^{s(k)n} h^j(W_k) \right) \\
&\leq \sum_{k=0}^l \sum_{j=s(k)n+1}^{n(k)-1} \mu(h^j(Y_k)) + \sum_{k=0}^l \sum_{j=0}^{s(k)n} (\mu(h^j(U_k \setminus W_k)) + \mu(h^j(Y_k \setminus U_k))) \\
&= \sum_{k=0}^l \sum_{j=s(k)n+1}^{n(k)-1} \mu(Y_k) + \sum_{k=0}^l \sum_{j=0}^{s(k)n} (\mu(U_k \setminus W_k) + \mu(Y_k \setminus U_k)) \\
&\leq (n+1)\mu(Y) + \sum_{k=0}^l \sum_{j=0}^{s(k)n} ((\mu(U_k) - \mu(W_k)) + (\mu(Y_k) - \mu(U_k))) \\
&< \frac{n+1}{nK} + M \left( \frac{\sigma}{8M} + \frac{\sigma}{8M} \right) \\
&< \frac{2}{K} + \frac{1}{4}\sigma \\
&< \frac{1}{2}\sigma.
\end{aligned}$$

Thus  $\mu(C) < \sigma < \mu(G_2)$ , and so by the dynamic comparison property there exist  $N \in \mathbb{N}$ , continuous functions  $f_j^{(0)}: X \rightarrow [0,1]$  for  $0 \leq j \leq N$ , and  $d(0), \dots, d(N) \in \mathbb{Z}$  such that  $\sum_{j=0}^N f_j^{(0)} = 1$  on  $C$ , and such that the sets  $\text{supp}(f_j^{(0)} \circ h^{-d(j)})$  are pairwise disjoint subsets of  $G_1$  for  $0 \leq j \leq N$ . Define continuous functions  $f_j: X \rightarrow A$  by  $f_j = f_j^{(0)} \otimes 1$ . Then  $1 - c \leq \sum_{j=0}^N f_j$ , and for  $0 \leq j \leq N$ , the elements  $\beta^{d(j)}(f_j)$  are mutually orthogonal positive elements in  $\overline{(g_1 \otimes 1)C(X, A)(g_1 \otimes 1)}$ . For  $0 \leq j \leq N$  and  $0 \leq k \leq M$ , define  $e_{j,k} = f_j \beta^{-d(j)}(a_k)$ . Since the  $\beta^{d(j)}(f_j)$  are mutually orthogonal elements of  $\overline{(g_1 \otimes 1)C(X, A)(g_1 \otimes 1)}$ , it follows that  $\sum_{j=0}^N \beta^{d(j)}(f_j \otimes 1) \leq g_0 \otimes 1$ . Moreover, since  $\beta^{d(j)+r(k)}(e_{j,k}) = \beta^{r(k)+d(j)}(f_j) \beta^{r(k)}(a_k)$  and the  $f_j$  are central, the elements  $\beta^{d(j)+r(k)}(e_{j,k})$  are mutually orthogonal. Now let  $u_{j,k} = uw_k$  for  $0 \leq j \leq N, 0 \leq k \leq M$ . Then

$$u_{j,k} \beta^{d(j)+r(k)}(e_{j,k}) u_{j,k}^* = \beta^{d(j)+r(k)}(f_j) u w_k \beta^{r(k)}(a_k) w_k^* u^* = \beta^{d(j)+r(k)}(f_j) u b_k u^*.$$

Since  $\beta^{d(j)+r(k)}(f_j) \in C(X)$  and  $u b_k u^* \in \overline{gC(X, A)g}$ , it follows that  $u_{j,k} e_{j,k} u_{j,k}^* \in \overline{gC(X, A)g}$ .



Finally, we compute

$$\begin{aligned}
\sum_{j=0}^N \sum_{k=0}^M e_{j,k} &= \sum_{j=0}^N \sum_{k=0}^M f_j \beta^{-d(j)}(a_k) = \sum_{j=0}^N f_j \beta^{-d(j)} \left( \sum_{k=0}^M a_k \right) \\
&\geq \sum_{j=0}^N f_j \beta^{-d(j)}(g_0 \otimes 1) \\
&= \sum_{j=0}^N \beta^{-d(j)}(\beta^{d(j)}(f_j)(g_0 \otimes 1)) \\
&= \sum_{j=0}^N \beta^{-d(j)}(\beta^{d(j)}(f_j)) \\
&= \sum_{j=0}^N f_j \\
&\geq 1 - c.
\end{aligned}$$

Now re-order the elements  $e_{j,k}$ ,  $u_{j,k}$ , and  $d(j) + r(k)$  as  $e_i$ ,  $u_i$ , and  $t(i)$  for  $0 \leq i \leq I$ , where  $I = (M+1)(N+1)$ . It follows that  $1 - c \leq \sum_{i=0}^I e_i$ ,  $\beta^{t(i)}(e_i)\beta^{t(j)}(e_j) = 0$  for  $0 \leq i, j \leq I$  and  $i \neq j$ , and  $u_i e_i u_i^* \in \overline{gC(X, A)g}$  for  $0 \leq i \leq I$ . Finally, as  $\mu(G_2) > \mu(C)$ , there is an  $x \in G_2$  such that  $x \notin C$ . Then  $(1 - c)(x) = 0$ , and so  $c(x) = 1$ . It follows that  $\|c(x)g(x)c(x)\| = \|g(x)\| > 1 - \varepsilon$ , which implies that  $\|cgc\| > 1 - \varepsilon$ . Thus,  $\beta$  has the tracial quasi-Rokhlin property.  $\square$

In order to apply our structure theorems from Chapter II to  $C^*(\mathbb{Z}, C(X, A), \beta)$ , we require information about the ideals of  $C(X, A)$ .

**Lemma IV.16.** *Let  $(X, h)$  and  $A$  be as in Notation IV.2. If  $F \subset X$  is closed, then  $I_F = \{f \in C(X, A) : f|_F = 0\}$  is an ideal in  $C(X, A)$ . Moreover, given any ideal  $I \subset C(X, A)$ ,  $I = I_F$  for some closed set  $F \subset X$ .*

*Proof.* For  $F \subset X$  closed, it is obvious that  $I_F$  as given above is an ideal in  $C(X, A)$ . Now let  $I \subset C(X, A)$  be an ideal. Define  $F \subset X$  by  $F = \{x \in X : f(x) = 0 \text{ for all } f \in I\}$ , which is certainly a closed subset of  $X$ . Set  $I_F = \{f \in C(X, A) : f|_F = 0\}$ , which we have already shown is an ideal of  $C(X, A)$ . From the definition of  $F$  it is clear that  $I \subset I_F$ . To prove the converse, let  $x_0 \in X \setminus F$ . We claim that  $\{g(x_0) : g \in I\}$  is dense in  $A$ . To see this, let  $\delta > 0$  be given, and let  $a \in A$ . Since  $x_0 \notin F$ , there is a function  $g_0 \in I$  such that  $g_0(x_0) \neq 0$ . Then the ideal  $\overline{Ag_0(x_0)A}$  is non-zero and so equals  $A$  by the simplicity of  $A$ . It follows that there exist  $b_1, \dots, b_n, c_1, \dots, c_n \in A$  such

that  $\left\|a - \sum_{j=1}^n b_j g_0(x_0) c_j\right\| < \delta$ . Define a function  $g \in C(X, A)$  by  $g = \sum_{j=1}^n (1 \otimes b_j) g_0 (1 \otimes c_j)$ . Then  $f \in I$  as  $g_0 \in I$  and  $1 \otimes b_j, 1 \otimes c_j \in C(X, A)$ , and  $\|g_{x_0} - a\| < \delta$ . Now let  $\varepsilon > 0$  be given and let  $q \in I_F$ . For each  $x \in X$ , choose  $f_x \in I$  such that  $\|f_x(x) - q(x)\| < \frac{1}{4}\varepsilon$ . This can be done by taking  $f_x = 0$  whenever  $x \in F$ , and for  $x \notin F$ ,  $f_x$  can be obtained from the previous claim. Next for each  $x \in X$  choose an open neighborhood  $U_x$  of  $x$  such that  $\|f_x(x) - f_x(y)\| < \frac{1}{4}\varepsilon$  and  $\|q(x) - q(y)\| < \frac{1}{4}\varepsilon$  for all  $y \in U_x$ . We obtain an open cover  $\{U_x : x \in X\}$  of  $X$ , which has a finite subcover  $\{U_{x_1}, \dots, U_{x_N}\}$ . Let  $f_1, \dots, f_N$  be the functions corresponding to the points  $x_1, \dots, x_N$ . Choose a partition of unity  $\varphi_1, \dots, \varphi_N$  subordinate to this cover, let  $g_j = \varphi_j f_j$  for  $1 \leq j \leq N$ , and set  $g = \sum_{j=1}^N g_j$ . Then  $g \in I$ , and for  $1 \leq j \leq N$  and every  $x \in X$  we have

$$\begin{aligned} \|q(x) - f_j(x)\| &\leq \|q(x) - q(x_j)\| + \|q(x_j) - f_j(x_j)\| + \|f_j(x_j) - f_j(x)\| \\ &< \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon \\ &= \frac{3}{4}\varepsilon. \end{aligned}$$

This implies that, for every  $x \in X$ ,

$$\begin{aligned} \|q(x) - g(x)\| &= \left\| q(x) - \sum_{j=1}^N \varphi_j(x) f_j(x) \right\| \\ &= \left\| \sum_{\{j: x \in U_j\}} \varphi_j(x) q(x) - \sum_{\{j: x \in U_j\}} \varphi_j(x) f_j(x) \right\| \\ &= \left\| \sum_{\{j: x \in U_j\}} \varphi_j(x) (q(x) - f_j(x)) \right\| \\ &\leq \sum_{\{j: x \in U_j\}} \|\varphi_j(x) (q(x) - f_j(x))\| \\ &= \sum_{\{j: x \in U_j\}} \varphi_j(x) \|q(x) - f_j(x)\| \\ &\leq \left( \sum_{\{j: x \in U_j\}} \varphi_j(x) \right) \max_{\{j: x \in U_j\}} \{\|q(x) - f_j(x)\|\} \\ &< \frac{3}{4}\varepsilon. \end{aligned}$$

It follows that  $\|q - f\| < \varepsilon$ , and hence  $q \in I$  as  $I$  is closed. Therefore  $I_F \subset I$ , which completes the proof.  $\square$

**Proposition IV.17.** *Let  $(X, h)$  and  $A$  be as in Notation IV.2. Then the  $C^*$ -algebra  $C(X, A)$  has no non-trivial  $\beta$ -invariant ideals.*

*Proof.* Let  $I \subset C(X, A)$  be a non-trivial ideal. By Lemma IV.16, there is a closed set  $F \subset X$  such that  $I = \{f \in C(X, A) : f(x) = 0 \text{ for all } x \in F\}$ . Then  $F \neq \emptyset$  and  $F \neq X$  as  $I$  is non-trivial. Suppose that  $I$  is  $\beta$ -invariant. Then  $\beta(I) \subset I$ , and so for any  $f \in I$ , we have  $\beta(f) \in I$ . Then for any  $x \in F$ ,  $f(x) = 0$  and  $\beta(f)(x) = 0$ . But  $0 = \beta(f)(x) = \alpha_x(f \circ h^{-1}(x))$  implies that  $f \circ h^{-1}(x) = 0$  since  $\alpha_x \in \text{Aut}(A)$ . Thus  $f(x) = 0$  for all  $x \in F \cap h^{-1}(F)$ . The  $\beta$ -invariance of  $I$  further implies that  $\beta^n(f) \in I$  for all  $n \in \mathbb{N}$ , and recalling that  $\beta^n(f)(x) = \alpha_x^{(n)}(f \circ h^{-n}(x))$  (this is Corollary IV.8) and that  $\alpha^{(n)} \in \text{Aut}(A)$ , it follows that for any  $f \in I$ , we have  $f(x) = 0$  for all  $x \in \bigcup_{n=0}^{\infty} h^{-n}(F)$ . By assumption  $F$  is closed and non-empty, and so the minimality of  $h$  gives  $\bigcup_{n=0}^{\infty} h^{-n}(F) = X$ . Thus  $f(x) = 0$  for all  $x \in X$ , which implies  $f = 0$ . It follows that  $I = 0$ , a contradiction. Therefore  $I$  cannot be  $\beta$ -invariant, and the desired result follows.  $\square$

**Corollary IV.18.** *Let  $(X, h, \mu)$ ,  $A$ , and  $\beta$  be as in Theorem IV.15. Then the crossed product  $C^*$ -algebra  $C^*(\mathbb{Z}, C(X, A), \beta)$  is simple.*

*Proof.* By Proposition IV.17,  $C(X, A)$  has no non-trivial  $\beta$ -invariant ideals. Since  $\beta$  has the tracial quasi-Rokhlin property, Theorem II.4 implies that  $C^*(\mathbb{Z}, C(X, A), \beta)$  is simple.  $\square$

**Definition IV.19.** *A topological space  $X$  is topologically scattered if every closed subset  $Y$  of  $X$  contains a point  $y$  that is relatively isolated in  $Y$ .*

It is a standard result (see [38]) that a compact Hausdorff space  $X$  is topologically scattered if and only if every Radon measure on  $X$  is atomic; that is, if and only if for any Radon measure  $\nu$  on  $X$ , there exist point-mass measures  $(\nu_j)_{j=1}^{\infty}$  and real numbers  $(t_j)_{j=1}^{\infty}$ , satisfying  $t_j \geq 0$  for all  $j \geq 1$  and  $\sum_{j=1}^{\infty} t_j = 1$ , such that

$$\nu = \sum_{j=1}^{\infty} t_j \nu_j.$$

Definition II.10 can be thought of as a noncommutative version of this one, with an atomic state playing the role of a “noncommutative atomic Radon measure”.

**Proposition IV.20.** *Given any infinite compact metric space  $X$  that has a minimal homeomorphism  $h: X \rightarrow X$  and any simple, separable, unital  $C^*$ -algebra  $A$ , the  $C^*$ -algebra  $C(X, A)$  is not scattered.*

*Proof.* First note that as  $X$  has a minimal homeomorphism, it cannot be topologically scattered. Indeed if we take  $Y = X$ , then for  $X$  to be topologically scattered it must contain at least one isolated point  $y$ , which is impossible since the  $h$ -orbit of every  $x \in X$  is dense in  $X$ . Therefore  $X$  has a non-atomic radon measure  $\nu$ . Define a state  $\varphi_\nu$  on  $C(X)$  by

$$\psi_\nu(f) = \int_X f d\nu.$$

We claim that  $\psi_\nu$  is a non-atomic state. If it were atomic, we could write  $\psi_\nu = \sum_{i=1}^{\infty} \delta_i \varphi_i$  for some sequence of pure states  $(\varphi_i)_{i=1}^{\infty}$  and some sequence of nonnegative real numbers  $(\delta_i)_{i=1}^{\infty}$  such that  $\sum_{i=1}^{\infty} \delta_i = 1$ . By the Riesz Representation Theorem, we would obtain  $\nu = \sum_{i=1}^{\infty} \nu_i$  for some sequence of point-mass measures  $\nu_i$ , a contradiction. Now let  $\omega$  be any non-zero state on  $A$ , and suppose the state  $\psi_\nu \otimes \omega$  is atomic. By Theorem IV.4.14 of [49], we may write  $\psi_\nu \otimes \omega = \sum_{i=1}^{\infty} t_i (\varphi_i \otimes \omega_i)$  for some sequences of pure states  $(\varphi_i)_{i=1}^{\infty}$  on  $C(X)$  and  $(\omega_i)_{i=1}^{\infty}$  on  $A$ , and for some sequence of nonnegative real numbers  $(t_i)_{i=1}^{\infty}$  such that  $\sum_{i=1}^{\infty} t_i = 1$ . Then for any  $f \in C(X)$ , we have

$$(\psi_\nu \otimes \omega)(f \otimes 1) = \sum_{i=1}^{\infty} t_i \varphi_i(f)$$

which implies that  $\psi_\nu = \sum_{i=1}^{\infty} t_i \varphi_i$ , a contradiction to  $\psi_\nu$  being non-atomic.  $\square$

**Corollary IV.21.** *Let  $(X, h, \mu)$ ,  $A$ , and  $\beta$  be as in Theorem IV.15. Then the restriction map  $T(C^*(\mathbb{Z}, C(X, A), \beta)) \rightarrow T_\beta(C(X, A))$  is a bijection.*

*Proof.* By Proposition IV.20,  $C(X, A)$  is not a scattered  $C^*$ -algebra, and by Proposition IV.17,  $C(X, A)$  has no  $\beta$ -invariant ideals. Since  $\beta$  has the tracial quasi-Rokhlin property, the result follows from Theorem II.12.  $\square$

We summarize the results of this chapter for crossed product  $C^*$ -algebras by automorphisms with the tracial quasi-Rokhlin property.

**Theorem IV.22.** *Let  $X$  be an infinite compact metric space with finite covering dimension, let  $h: X \rightarrow X$  be a uniquely ergodic minimal homeomorphism with unique  $h$ -invariant Borel probability measure  $\mu$ , and let  $A$  be a simple, separable, unital  $C^*$ -algebra with tracial rank zero and satisfying the Universal Coefficient Theorem. Let  $\alpha: X \rightarrow \text{Aut}(A)$  be a strong operator continuous map, and let  $\beta \in \text{Aut}(C(X, A))$  be defined as in Proposition IV.6. Suppose that  $(X, h, \mu)$  has the topological*

small boundary property, and that  $A$  has a unique tracial state. Then the crossed product  $C^*$ -algebra  $C^*(\mathbb{Z}, C(X, A), \beta)$  is simple and has a unique tracial state.

We conclude by presenting some examples of crossed product  $C^*$ -algebras of the form  $C^*(\mathbb{Z}, C(X, A), \beta)$  that have good structure properties. All of these results are already known, but they suggest that algebras of this form (that is, those described by Theorem IV.22) could have these properties more generally.

**Example IV.23.** If  $A = \mathbb{C}$ , then  $C^*(\mathbb{Z}, C(X, A), \beta)$  is just  $C^*(\mathbb{Z}, X, h)$ , whose structure has been extensively studied in [29] and [24] (among other places), as discussed in the Introduction. (Note that any results about  $C(X, A)$  which depended on  $A$  being infinite-dimensional, specifically Proposition IV.3, are well-known for the commutative case). In particular, if the map  $\rho_{C^*(\mathbb{Z}, X, h)}: K_0(C^*(\mathbb{Z}, X, h)) \rightarrow \text{Aff}(T(C^*(\mathbb{Z}, X, h)))$  (where  $\text{Aff}(\Delta)$  denotes the space of real-valued affine functions on  $\Delta$ ) given by

$$\rho_{C^*(\mathbb{Z}, X, h)}([\eta])(\tau) = \tau(\eta)$$

has dense range, then  $C^*(\mathbb{Z}, X, h)$  has tracial rank zero. If  $X$  is a compact smooth manifold and  $h$  is a minimal diffeomorphism, then it is possible to give an explicit direct limit decomposition for  $C^*(\mathbb{Z}, X, h)$  as a direct limit of recursive subhomogeneous algebras.

Let  $\theta, \eta \in \mathbb{R} \setminus \mathbb{Q}$ , let  $X = S^1$ , let  $A = A_\theta$ , and let  $h: X \rightarrow X$  be given by  $h(\zeta) = e^{-2\pi i \eta \zeta}$ . Let  $f, g \in C(S^1, S^1)$  and let  $\lambda \in \text{Aut}(A_\theta)$ . We identify  $A_\theta$  with  $C^*(u, v)$ , where  $vu = e^{2\pi i \theta} uv$ . Define a mapping  $\alpha: S^1 \rightarrow \text{Aut}(A_\theta)$  by  $\alpha(\zeta) = \alpha_\zeta$ , where

$$\alpha_\zeta(u) = f(\zeta)\lambda(u), \quad \alpha_\zeta(v) = g(\zeta)\lambda(v).$$

To see that  $\alpha$  is continuous in the strong operator topology, let  $\varepsilon > 0$  be given. Choose  $\delta > 0$  such that  $\|f(\zeta_1) - f(\zeta_2)\| < \varepsilon$  and  $\|g(\zeta_1) - g(\zeta_2)\| < \varepsilon$  whenever  $d(\zeta_1, \zeta_2) < \delta$ . Then

$$\begin{aligned} \|\alpha_{\zeta_1}(u) - \alpha_{\zeta_2}(u)\| &= \|f(\zeta_1)\lambda(u) - f(\zeta_2)\lambda(u)\| \\ &\leq \|f(\zeta_1) - f(\zeta_2)\| \|\lambda(u)\| \\ &= \|f(\zeta_1) - f(\zeta_2)\| \\ &< \varepsilon, \end{aligned}$$

and similarly  $\|\alpha_{\zeta_1}(v) - \alpha_{\zeta_2}(v)\| \leq \|g(\zeta_1) - g(\zeta_2)\| < \varepsilon$ . This checks pointwise norm continuity on the generators of  $A_\theta$ , and it follows that  $\alpha$  is strong operator continuous. By Theorem IV.15,  $\beta$  has the tracial quasi-Rokhlin property. Let us identify  $C(S^1)$  with  $C^*(z)$ , where  $z$  is the image (under the Gelfand transform) of the function  $z(\zeta) = \zeta$ . Then we have the further identification

$$C(S^1, A_\theta) \cong C(S^1) \otimes A_\theta \cong C^*(z) \otimes C^*(u, v) \cong C^*(1 \otimes u, 1 \otimes v, z \otimes 1),$$

where the relations are given by (writing  $u, v$ , and  $z$  instead of  $1 \otimes u, 1 \otimes v$ , and  $z \otimes 1$ )

$$uz = zu, \quad vz = zv, \quad vu = e^{2\pi i\theta} uv.$$

Using functional calculus, we may then write  $\beta$  explicitly as

$$\beta(z) = e^{2\pi i\eta} z, \quad \beta(u) = f(z)\lambda(u), \quad \beta(v) = g(z)\lambda(v).$$

Making specific choices of  $f$ ,  $g$ , and  $\lambda$  allows us to say even more.

**Example IV.24.** Let  $\eta = \theta$  (so that  $h(\zeta) = e^{2\pi i\theta}\zeta$ ), let  $f$  and  $g$  be given by  $f(\zeta) = 1$  and  $g(\zeta) = \zeta$ , and let  $\lambda = \text{id}_A$  be the identity automorphism of  $A$ . Then  $\alpha_\zeta$  is given by  $\alpha_\zeta(u) = u$ ,  $\alpha_\zeta(v) = \zeta v$ .

It follows that  $\beta$  is given by

$$\beta(z) = e^{2\pi i\theta} z, \quad \beta(u) = u, \quad \beta(v) = zv.$$

Letting  $w$  denote the canonical unitary in the transformation group  $C^*$ -algebra  $C^*(\mathbb{Z}, C(S^1, A_\theta), \beta)$ , we can identify this algebra with  $C^*(u, v, z, w)$ , subject to the relations

$$\begin{aligned} vu &= e^{2\pi i\theta} uv, & uz &= zu, & vz &= zv \\ wz &= e^{2\pi i\theta} zw, & wu &= uw, & wv &= zwv. \end{aligned}$$

This gives an isomorphism between  $C^*(\mathbb{Z}, C(S^1, A_\theta), \beta)$  and the  $C^*$ -algebra  $A_\theta^{5,3}$  of [33]. Proposition 4.1 of [37] then implies that  $C^*(\mathbb{Z}, C(S^1, A_\theta), \beta)$  is isomorphic to a transformation group  $C^*$ -algebra  $C^*(\mathbb{Z}, C^*(\mathbb{Z}, S^1 \times S^1, \phi), \gamma)$ , where  $\phi$  is a smooth minimal Furstenberg transformation and  $\gamma$  has the tracial Rokhlin property. By Corollary 4.2 of [37],

$C^*(\mathbb{Z}, C(S^1, A_\theta), \beta)$  has stable rank one, real rank zero, a unique tracial state, and order on projections is determined by traces.

**Example IV.25.** We can also obtain the  $C^*$ -algebra  $A_\theta^{5,6}$  of [33] as a crossed product  $C^*$ -algebra  $C^*(\mathbb{Z}, C(S^1, A_\theta), \beta)$  (with analogous structural conclusions using [37]). This time, take  $\eta = \theta$ ,  $f(\zeta) = \zeta$ ,  $g(\zeta) = 1$ , and let  $\lambda$  be given by  $\lambda(u) = u$  and  $\lambda(v) = uv$ . Then  $\alpha_\zeta$  is given by  $\alpha_\zeta(u) = \zeta u$ ,  $\alpha_\zeta(v) = v$  and  $\beta$  is given by

$$\beta(z) = e^{2\pi i\theta} z, \quad \beta(u) = zu, \quad \beta(v) = uv.$$

Again letting  $w$  denote the canonical unitary in  $C^*(\mathbb{Z}, C(S^1, A_\theta), \beta)$ , we can identify this crossed product  $C^*$ -algebra with  $C^*(u, v, w, z)$  subject to the relations

$$\begin{aligned} vu &= e^{2\pi i\theta} uv, & uz &= zu, & vz &= zv \\ wz &= e^{2\pi i\theta} zw, & wu &= zuw, & wv &= uvw. \end{aligned}$$

which is easily seen to be the same set of generators and relations as for  $A_\theta^{5,6}$ .

## CHAPTER V

RECURSIVE STRUCTURE FOR CERTAIN SUBALGEBRAS OF  $C^*(\mathbb{Z}, C(X, A), \beta)$ 

In order to obtain a more complete description for the structure of the crossed product  $C^*$ -algebra  $C^*(\mathbb{Z}, C(X, A), \beta)$ , we begin an adaption of the extensive theory developed in [29] and subsequent work. Specifically, for  $Y \subset X$  we introduce a class of subalgebras  $B_Y$  of  $C^*(\mathbb{Z}, C(X, A), \beta)$  that will play an analogous role to the algebras  $A(Y)$  of [29], and show that, under appropriate conditions on  $Y$ , they have a tractable recursive structure. For a point  $y \in X$ , we will be especially interested in the relationship between the approximating subalgebra  $B_{\{y\}}$  and the entire crossed product  $C^*$ -algebra, which will be explored in Chapter VI. We start by introducing the formalism for a generalization of the recursive subhomogeneous algebras introduced in [39] that were crucial for the analysis in [29] and [24].

**Definition V.1.** *Let  $A, B, C$  be unital  $C^*$ -algebras, and let  $\varphi: A \rightarrow C$  and  $\psi: B \rightarrow C$  be unital homomorphisms. Then the associated pullback  $C^*$ -algebra  $A \oplus_{C, \varphi, \psi} B$  is defined by*

$$A \oplus_{C, \varphi, \psi} B = \{(a, b) \in A \oplus B : \varphi(a) = \psi(b)\}.$$

We frequently write  $A \oplus_C B$  when the maps  $\varphi$  and  $\psi$  are understood.

**Definition V.2.** *Let  $A$  be a simple, unital  $C^*$ -algebra. The class of recursive  $A$ -subhomogeneous algebras is the smallest class  $\mathcal{R}$  of  $C^*$ -algebras that is closed under isomorphism such that:*

1. *If  $X$  is a compact Hausdorff space and  $n \geq 1$ , then  $C(X, M_n(A)) \in \mathcal{R}$ .*
2. *If  $B \in \mathcal{R}$ ,  $X$  is compact Hausdorff,  $n \geq 1$ ,  $X^{(0)} \subset X$  is closed,  $\varphi: B \rightarrow C(X^{(0)}, M_n(A))$  is a unital homomorphism, and  $\rho: C(X, M_n(A)) \rightarrow C(X^{(0)}, M_n(A))$  is the restriction*



homomorphism, then the pullback

$$B \oplus_{C(X^{(0)}, M_n(A))} C(X, M_n(A)) = \{(b, f) \in B \oplus C(X, M_n(A)) : \varphi(b) = \rho(f)\}$$

is in  $\mathcal{R}$ .

Taking  $A = \mathbb{C}$  in this definition gives the usual definition for the class of recursive subhomogeneous algebras (see [39]).

**Definition V.3.** We adopt the following standard notation for recursive  $A$ -subhomogeneous algebras. The definition implies that any recursive  $A$ -subhomogeneous algebra  $R$  can be written in the form

$$R \cong \left[ \cdots \left[ \left[ C_0 \oplus_{C_1^{(0)}} C_1 \right] \oplus_{C_2^{(0)}} \cdots \right] \oplus_{C_l^{(0)}} C_l \right]$$

with  $C_k = C(X_k, M_{n(k)}(A))$  for compact Hausdorff spaces  $X_k$  and positive integers  $n(k)$ , and with  $C_k^{(0)} = C(X_k^{(0)}, M_{n(k)}(A))$  for compact subsets  $X_k^{(0)} \subset X_k$  (possibly empty), where the maps  $\rho_k: C_k \rightarrow C_k^{(0)}$  are always the restriction maps. An expression of this type for  $R$  will be referred to as a decomposition of  $R$ , and the notation that appears here will be referred to as the standard notation for a decomposition. We associate to this decomposition:

1. its length  $l$ ;
2. the  $k$ -th stage algebra

$$R^{(k)} = \left[ \cdots \left[ \left[ C_0 \oplus_{C_1^{(0)}} C_1 \right] \oplus_{C_2^{(0)}} C_2 \right] \cdots \right] \oplus_{C_k^{(0)}} C_k;$$

3. its base spaces  $X_0, \dots, X_l$  and total space  $X = \coprod_{k=0}^l X_k$ ;
4. its matrix sizes  $n(0), \dots, n(l)$  and matrix size function  $m: X \rightarrow \mathbb{Z}_{\geq 0}$  defined by  $m(x) = n(k)$  when  $x \in X_k$  (this is called the matrix size of  $R$  at  $x$ );
5. its minimum matrix size  $\min_k n(k)$  and maximum matrix size  $\max_k n(k)$ ;
6. its topological dimension  $\dim(X) = \max_k \dim(X_k)$  and topological dimension function  $d: X \rightarrow \mathbb{Z}_{\geq 0}$ , defined by  $d(x) = \dim(X_k)$  for  $x \in X_k$  (this is called the topological dimension of  $R$  at  $x$ );

7. its standard representation  $\sigma = \sigma_R: R \rightarrow \bigoplus_{k=0}^l C(X_k, M_{n(k)}(A))$ , defined by forgetting the restriction to a subalgebra in each of the fibered products in the decomposition;
8. the associated evaluation maps  $\text{ev}_x: R \rightarrow M_{n(k)}(A)$ , defined to be the restriction of the usual evaluation map on  $\bigoplus_{k=0}^l C(X_k, M_{n(k)}(A))$  to  $R$  (where  $R$  is identified with a subalgebra of this algebra through the standard representation  $\sigma_R$ ).

**Definition V.4.** Adopt Notation IV.2, let  $\beta$  be the automorphism of Proposition IV.6, and write  $B = C^*(\mathbb{Z}, C(X, A), \beta)$ . For  $Y \subset X$  closed, we define

$$B_Y = C^*(\mathbb{Z}, C(X, A), \beta)_Y = C^*(C(X, A), uC_0(X \setminus Y, A)) \subset C^*(\mathbb{Z}, C(X, A), \beta)$$

where we identify  $C_0(X \setminus Y, A)$  with the subalgebra of  $C(X, A)$  consisting of all continuous functions  $f: X \rightarrow A$  that vanish on  $Y$ .

**Proposition V.5.** Let  $y_0 \in X$ , and let  $Y_1 \supset Y_2 \supset \dots$  be closed subsets of  $X$  such that  $\bigcap_{n=1}^{\infty} Y_n = \{y_0\}$ . Then  $B_{\{y_0\}} = \overline{\bigcup_{n=1}^{\infty} B_{Y_n}} = \varinjlim B_{Y_n}$ .

*Proof.* Let  $\varepsilon > 0$  be given and let  $f \in C_0(X \setminus \{y_0\}, A)$ . Since  $f(y_0) = 0$ , there is a  $\delta > 0$  such that  $\|f(x)\| < \frac{1}{2}\varepsilon$  for all  $x \in B(y_0, \delta)$ . The compactness of the  $Y_n$  and the inclusions  $Y_{n+1} \subset Y_n$  imply that  $\infty > \text{diam}(Y_1) \geq \text{diam}(Y_2) \geq \dots$ , and moreover  $\text{diam}(Y_n) \rightarrow \text{diam}(\{y_0\}) = 0$ . Hence there is an  $N \in \mathbb{N}$  such that  $\text{diam}(Y_n) < \frac{1}{3}\delta$  for  $n \geq N$ . Let  $V$  be an open set such that  $Y_N \subset V$  and  $\text{diam}(V) < \frac{2}{3}\delta$ . Since  $y_0 \in V$ , we must have  $V \subset B(y_0, \delta)$ . Now choose a continuous function  $g_0: X \rightarrow [0, 1]$  such that  $g_0 = 0$  on  $Y_N$  and  $g_0 = 1$  on  $X \setminus V$ , and set  $g = g_0 f$ . Then  $g \in C_0(X \setminus Y_n, A)$  for  $n \geq N$ ,  $g(x) = f(x)$  for all  $x \in X \setminus V$ , and  $x \in V$  implies that  $\|f(x) - g(x)\| \leq \|f(x)\| (1 - g_0(x)) \leq \|f(x)\| < \frac{1}{2}\varepsilon$ . It follows that  $\|f - g\| < \varepsilon$ , and so  $f \in C_0(X \setminus Y_n, A)$  for  $n \geq N$ . Then  $uf \in B_{Y_n}$ , which implies the result since these elements, along with the elements of  $C(X, A)$ , generate  $B_{\{y_0\}}$ . Note that  $1 \in C(X, A) \subset B_{Y_n}$  so the inclusion maps  $B_{Y_n} \rightarrow B_{Y_{n+1}}$  are unital, and clearly injective.  $\square$

The results that follow for the remainder of this chapter are mostly adapted from Section 1 of [29]. Some of the proofs there go through nearly or entirely unchanged, while others require more substantial adjustment to handle the fact that  $C(X, A)$  is not in general a commutative  $C^*$ -algebra.

**Notation V.6.** Let  $Y \subset X$  be closed with  $\text{int}(Y) \neq \emptyset$  and  $\mu(Y) = 0$ . Following Theorem III.8, construct the Rokhlin tower over  $Y$  by first return times to  $Y$ , obtaining non-negative integers  $n(0) < n(1) < \dots < n(l)$  and sets

$$Y_k = \overline{\{y \in Y : r(y) = n(k)\}} \quad \text{and} \quad Y_k^\circ = \text{int}(\{y \in Y : r(y) = n(k)\})$$

such that:

1. the sets  $h^j(Y_k^\circ)$  are pairwise disjoint for  $0 \leq k \leq l$  and  $1 \leq j \leq n(k)$ ;
2.  $\bigcup_{k=0}^l h^{n(k)}(Y_k) = Y$ ;
3.  $\bigcup_{k=0}^l \bigcup_{j=1}^{n(k)} h^j(Y_k) = X$ .

For  $m \geq 0$ , we set

$$G_m = C_0\left(\left(X \setminus \bigcup_{j=0}^m h^{-j}(Y)\right), A\right).$$

We observe that  $G_m = 0$  for  $m \geq n(l) - 1$  since

$$\bigcup_{j=0}^{n(l)-1} h^{-j}(Y) = h^{-n(l)}\left(\bigcup_{j=1}^{n(l)} h^j(Y)\right) = h^{-n(l)}\left(\bigcup_{k=0}^l \bigcup_{j=1}^{n(k)} h^j(Y_k)\right) = h^{-n(l)}(X) = X.$$

Note that we have departed slightly from the notation of Theorem III.8 by effectively taking the base of the tower to be  $h(Y)$  rather than  $Y$ , a choice that will prove more convenient for our present purposes.

**Proposition V.7.** Following Notation V.6 and Definition V.4, we have the Banach space topological direct sum

$$B_Y = \bigoplus_{j=1}^{n(l)-1} G_{j-1} u^{-j} \oplus C(X, A) \oplus \bigoplus_{j=1}^{n(l)-1} u^j G_{j-1}.$$

*Proof.* Let

$$G = \bigoplus_{j=1}^{n(l)-1} G_{j-1} u^{-j} \oplus C(X, A) \oplus \bigoplus_{j=1}^{n(l)-1} u^j G_{j-1}.$$

Note that this is clearly an algebraic direct sum, and that each summand is a closed subspace of  $C^*(\mathbb{Z}, C(X, A), \beta)$ . Let  $E: C^*(\mathbb{Z}, C(X, A), \beta) \rightarrow C(X, A)$  be the canonical conditional expectation,

and for  $1 \leq j \leq n(l)$  define maps  $\pi_j$  and  $\rho_j$  on  $G$  by  $\pi_j(a) = E(au^j)u^{-j}$  and  $\rho_j(a) = u^j E(u^{-j}a)$ . Then  $\pi_j$  and  $\rho_j$  are continuous projections from  $G$  to the summands  $G_{j-1}u^{-j}$  and  $u^j G_{j-1}$  respectively. Defining  $\pi(a) = E(a)$  gives a continuous projection from  $G$  to the summand  $C(X, A)$ , and together with the  $\pi_j$  and  $\rho_j$  this implies  $G$  is a Banach space topological direct sum.

Next, we verify that  $G$  is a  $C^*$ -subalgebra of  $C^*(\mathbb{Z}, C(X, A), \beta)$ . First, it is clear that  $G$  is closed under addition. Also, for any  $j$  with  $1 \leq j \leq n(l) - 1$ , we have  $[u^j G_{j-1}]^* = G_{j-1}u^{-j}$  and  $[G_{j-1}u^{-j}]^* = u^j G_{j-1}$ , which shows that  $G$  is closed under adjoints. Now let  $f \in G_{j-1}$  and  $g \in G_{k-1}$  with  $1 \leq j, k \leq n(l) - 1$ . We claim that  $\beta^{-k}(f)g \in G_{j+k-1}$ . To see this, let  $x \in \bigcup_{i=0}^{j+k-1} h^{-i}(Y)$ . Then either  $x \in \bigcup_{i=0}^{k-1} h^{-i}(Y)$ , in which case  $g(x) = 0$ , or  $x \in \bigcup_{i=k}^{j+k-1} h^{-i}(Y)$ , in which case  $h^k(x) \in \bigcup_{i=k}^{j+k-1} h^{k-i}(Y) = \bigcup_{r=0}^{j-1} h^{-r}(Y)$ , which implies  $f \circ h^k(x) = 0$ . This proves the claim. It follows immediately that  $(u^j f)(u^k g) = u^{j+k} \beta^{-k}(f)g \in u^{j+k} G_{j+k-1}$ . Next, the previous calculation shows that  $(u^k g^*)(u^j f^*) \in u^{j+k} G_{j+k-1}$ , and the adjoint calculation then gives  $(fu^{-j})(gu^{-k}) = [(u^j f^*)(u^k g^*)]^* \in G_{j+k-1}u^{-(j+k)}$ . If  $j > k$ , then we further compute  $(u^j f)(gu^{-k}) = u^j f u^{-k} \beta^k(g) = u^{j-k} \beta^k(f) \beta^k(g)$  and observe that  $\beta^k(f) \beta^k(g) \in G_{j-k-1}$  since for any  $x \in \bigcup_{i=0}^{j-k-1} h^{-i}(Y)$ , we have  $h^{-k}(x) \in \bigcup_{i=0}^{j-k-1} h^{-(i+k)}(Y) = \bigcup_{r=0}^{j-1} h^{-r}(Y)$  and so  $f \circ h^{-k}(x) = 0$ . Finally, for  $j > k$  the previous calculation and the adjoint calculation together give  $(u^k g)(fu^{-j}) = [(u^j f^*)(gu^{-k})]^* \in G_{j-k-1}u^{-(j-k)}$ . From these four cases it follows that  $G$  is closed under multiplication. Hence  $G$  is a  $C^*$ -subalgebra of  $C^*(\mathbb{Z}, C(X, A), \beta)$  which certainly contains  $C(X, A)$  and  $uG_0 = uC_0(X \setminus Y, A)$ , and hence contains  $B_Y$  as well.

To see that  $G$  is contained in  $B_Y$ , it suffices to prove that for any  $k$  with  $0 \leq k \leq n(l) - 1$  and any  $f \in G_{k-1}$ , we have  $u^k f \in B_Y$ . By the Cohen factorization theorem (see Theorem 2.9.24 of [6]), there exist  $f_0, \dots, f_{k-1} \in G_{k-1}$  such that  $f = \prod_{j=0}^{k-1} f_j$ . Then we may write

$$u^k f = (u\beta^{k-1}(f_{k-1}))(u\beta^{k-2}(f_{k-2})) \cdots (u\beta(f_1))(uf_0).$$

For any  $x \in Y$  and any  $0 \leq i \leq k - 1$ ,  $h^{-i}(x) \in \bigcup_{j=0}^{k-1} h^{-j}(Y)$ , and so  $\beta^i(f_i)(x) = 0$ . Therefore  $\beta^i(f_i) \in C_0(X \setminus Y, A)$  for  $0 \leq i \leq k - 1$ , which implies that  $u\beta^i(f_i) \in B_{\{y\}}$  for  $0 \leq i \leq k - 1$ . It follows that  $u^k f \in B_{\{y\}}$  as required.  $\square$

**Notation V.8.** Adopting Notation IV.2 and Notation V.6, we set

$$C_Y = \bigoplus_{k=0}^l C(Y_k, M_{n(k)}(A))$$

and define a unitary  $s_Y \in C_Y$  by  $s_Y = (s_0, s_1, \dots, s_l)$ , where for  $0 \leq k \leq l$ ,  $s_k \in C(Y_{n(k)}, M_{n(k)}(A))$  is given by

$$s_k = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

**Theorem V.9.** Using the notation of Proposition V.7 and V.8, for  $0 \leq k \leq l$  define a map  $\sigma_k: B_Y \rightarrow C(Y_k, M_{n(k)}(A))$  by:

1. for  $f \in C(X, A)$ ,  $\sigma_k(f) = \text{diag}(\beta^{-1}(f)|_{Y_k}, \dots, \beta^{-n(k)}(f)|_{Y_k})$ ;
2. for  $g \in G_{j-1}$ ,  $\sigma_k(u^j f) = s_k^j \sigma_k(f)$  and  $\sigma_k(f u^{-j}) = \sigma_k(f) s_k^{-j}$ .

(Note that  $\sigma_k$  is well-defined due to the Banach space direct sum decomposition of Proposition V.7). Further define a map  $\sigma: B_Y \rightarrow C_Y$  by  $\sigma(f) = (\sigma_0(f), \sigma_1(f), \dots, \sigma_l(f))$ . Then  $\sigma$  is an injective  $*$ -homomorphism.

Before proving the theorem, it is helpful to state as a lemma an explicit matrix form for the products  $s_k^r \sigma_k(f)$ ,  $\sigma_k(f) s_k^{-r}$ , and  $s_k^r \sigma_k(f) s_k^{-r}$ . These calculations will be used repeatedly in the proof of the theorem, usually without comment.

**Lemma V.10.** If  $f \in G_{r-1}$ ,  $0 \leq k \leq l$ , and  $r < n(k)$ , then:

1. The diagonal entries of  $\sigma_k(f)$  corresponding to the positions  $n(k) - (r - 1), \dots, n(k) - 1, n(k)$  are all zero;
2. We have

$$[s_k^r \sigma_k(f)]_{uv} = \begin{cases} \beta^{-u}(f)|_{Y_k} & \text{if } r + 1 \leq v \leq n(k) \text{ and } u = v - r, \\ 0 & \text{otherwise} \end{cases}$$

or explicitly

$$s_k^r \sigma_k(f) = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \cdots & \cdots & \vdots \\ 0 & & & & \vdots \\ \beta^{-1}(f)|_{Y_k} & 0 & \cdots & \cdots & \vdots \\ 0 & \ddots & 0 & 0 & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \beta^{-(n(k)-r)}(f)|_{Y_k} & 0 \end{bmatrix}.$$

3. We have

$$[\sigma_k(f)s_k^{-r}]_{uv} = \begin{cases} \beta^{-u}(f)|_{Y_k} & \text{if } r+1 \leq v \leq n(k) \text{ and } u = v-r, \\ 0 & \text{otherwise} \end{cases}$$

or explicitly,

$$\sigma_k(f)s_k^{-r} = \begin{bmatrix} 0 & \cdots & \beta^{-1}(f)|_{Y_k} & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & \ddots & 0 & 0 & \beta^{-(n(k)-r)}(f)|_{Y_k} \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}.$$

4. Conjugation by  $s_k^r$  gives

$$s_k^r \sigma_k(f) s_k^{-r} = \text{diag}(0, \dots, 0, \beta^{-1}(f)|_{Y_k}, \dots, \beta^{-(n(k)-r)}(f)|_{Y_k}),$$

where the first  $r$  diagonal entries are all zero.

*Proof.* We first prove part (1). Recall that  $h^{n(k)}(Y_k) \subset Y$  for each  $k$ , and so also  $h^{n(k)-j}(Y_k) \subset h^{-j}(Y)$ . As  $f \in G_{r-1}$ , for  $0 \leq j \leq r-1$  and  $x \in h^{n(k)-j}(Y_k)$ , we have  $f(x) = 0$ , which proves the claim.

Next, a straightforward matrix calculation shows that for  $1 \leq r \leq n(k)$ ,  $s_k^r = [a_{uv}]$  where

$$a_{uv} = \begin{cases} 1 & \text{if } 1 \leq u \leq r \text{ and } v = n(k) - u \text{ or } r + 1 \leq u \leq n(k) \text{ and } v = u - r \\ 0 & \text{otherwise.} \end{cases}$$

By part (1), we have

$$\sigma_k(f) = \text{diag}(\beta^{-1}(f)|_{Y_k}, \dots, \beta^{-(n(k)-r)}(f)|_{Y_k}, 0, \dots, 0),$$

where the last  $r$  diagonal entries are zero. A routine matrix multiplication now shows that  $s_k^r \sigma_k(f)$  has the form given in part (2). The formula for  $\sigma_k(f) s_k^{-r}$  in part (3) is easily obtained from the one for  $s_k^r \sigma_k(f)$  by replacing  $f$  with  $f^*$  and using  $\sigma_k(f) s_k^{-r} = (s_k^r \sigma_k(f^*))^*$ . (This equality is easily verified, and this will be done in the proof of Theorem V.9.) Finally, part (4) follows immediately from parts (2) and (3).  $\square$

We now prove Theorem V.9.

*Proof.* To prove that  $\sigma$  is a  $*$ -homomorphism, it suffices to prove that each  $\sigma_k$  is a  $*$ -homomorphism. Linearity of these maps is clear. For  $f \in C(X, A)$ , the equality  $\sigma_k(f^*) = \sigma_k(f)^*$  follows immediately from the fact that  $\beta$  and all of its powers are automorphisms. Further, for  $f \in G_{j-1}$  we have

$$\sigma_k((u^j f)^*) = \sigma_k(f^* u^{-j}) = \sigma_k(f^*) s_k^{-j} = (s_k^j \sigma_k(f))^* = \sigma_k(u^j f)^*$$

and

$$\sigma_k((f u^{-j})^*) = \sigma_k(u^j f^*) = s_k^j \sigma_k(f^*) = (\sigma_k(f) s_k^{-j})^* = \sigma_k(f u^{-j})^*.$$

It follows that each  $\sigma_k$  preserves adjoints. Next, it follows from the part (1) of Lemma V.10 that if  $f \in G_{r-1}$  and  $r \geq n(k)$ , then  $\sigma_k(f) = 0$ . Now, we further claim that for  $0 \leq k \leq l$ , we have the equalities

1.  $s_k^{-r} \sigma_k(f) s_k^r \sigma_k(g) = \sigma_k(\beta^{-r}(f)g)$

2.  $s_k^r \sigma_k(g) \sigma_k^{-r} = \sigma_k(\beta^r(g))$

whenever  $f \in C(X, A)$  and  $g \in G_{r-1}$ . First, part (1) of Lemma V.10 implies that the last  $r$  diagonal entries of both  $\sigma_k(g)$  and  $\sigma_k(\beta^{-r}(f)g)$  are zero. Further,  $\beta^r(g)(x) = 0$  for  $x \in \bigcup_{j=0}^{r-1} h^{r-j}(Y)$ , which

implies that  $\beta^{r-j}(g)$  is zero on  $Y_k$  for  $1 \leq j \leq r$ . Hence the first  $r$  diagonal entries of  $\sigma_k(\beta^r(g))$  are also zero. If  $r \geq n(k)$ , then both sides of equations (1) and (2) are zero. If  $r < n(k)$ , then we readily compute

$$\begin{aligned} s_k^{-r} \sigma_k(f) s_k^r \sigma_k(g) &= \text{diag}(\beta^{-(r+1)}(f) \beta^{-1}(g)|_{Y_k}, \dots, \beta^{-n(k)}(f) \beta^{-(n(k)-r)}(g)|_{Y_k}, 0, \dots, 0) \\ &= \text{diag}(\beta^{-r}(\beta^{-1}(f)g)|_{Y_k}, \dots, \beta^{-n(k)}(f \beta^r(g))|_{Y_k}, 0, \dots, 0) \\ &= \sigma_k(\beta^{-r}(f)g) \end{aligned}$$

and

$$\begin{aligned} \sigma_k^r \sigma_k(g) \sigma_k^{-r} &= \text{diag}(0, \dots, 0, \beta^{-1}(g)|_{Y_k}, \dots, \beta^{-(n(k)-r)}(g)|_{Y_k}) \\ &= \sigma_k(\beta^r(g)), \end{aligned}$$

which establishes the claim. We now use equations (1) and (2) to prove that each  $\sigma_k$  is multiplicative. Using the direct sum decomposition of  $B_Y$ , there are several cases to consider. Let  $f \in G_{j-1}$ ,  $g \in G_{r-1}$ , and  $j > r$ . Then (making frequent use of equations (1) and (2) where appropriate) we have the four equalities

$$\begin{aligned} \sigma_k(u^j f) \sigma_k(u^r g) &= s_k^j \sigma_k(f) s_k^r \sigma_k(g) \\ &= s_k^j s_k^r \sigma_k(\beta^{-r}(f)g) \\ &= \sigma_k(u^{j+r} \beta^{-r} g) \\ &= \sigma_k(u^j (u^r \beta^{-r}(f) u^{-r}) u^r g) \\ &= \sigma_k((u^j f)(u^r g)); \end{aligned}$$

$$\begin{aligned} \sigma_k(u^r g) \sigma_k(u^j f) &= s_k^r \sigma_k(g) s_k^j \sigma_k(f) \\ &= s_k^r s_k^j \sigma_k(\beta^{-j}(g)f) \\ &= \sigma_k(u^{r+j} \beta^{-j}(g)f) \\ &= \sigma_k(u^r (u^j \beta^{-j}(g) u^{-j}) u^j f) \\ &= \sigma_k((u^r g)(u^j f)); \end{aligned}$$



$$\begin{aligned}
\sigma_k(u^j f)\sigma_k(gu^{-r}) &= s_k^j \sigma_k(f)\sigma_k(g)s_k^{-r} \\
&= \sigma_k(\beta^j(f))s_k^{j-r} \sigma_k(\beta^r(g)) \\
&= \sigma_k(\beta^j(f)u^{j-r}\beta^r(g)) \\
&= \sigma_k(u^j(u^{-j}\beta^j(f)u^j)(u^{-r}\beta^r(g)u^r)u^{-r}) \\
&= \sigma_k((u^j f)(gu^{-r}));
\end{aligned}$$

and

$$\begin{aligned}
\sigma_k(gu^{-r})\sigma_k(u^j f) &= \sigma_k(g)s_k^{-r} \sigma_k(f) \\
&= s_k^{-r} \sigma_k(\beta^r(g))s_k^j \sigma_k(f) \\
&= s_k^{j-r} \sigma_k(\beta^{r-j}(g)f) \\
&= \sigma_k(u^{j-r}\beta^{r-j}(g)f) \\
&= \sigma_k(u^{-r}\beta^r(g)u^j f) \\
&= \sigma_k((gu^{-r})(u^j f)).
\end{aligned}$$

These equalities establish that  $\sigma_k$  is multiplicative for the most difficult cases. If  $f, g \in C(X, A)$ , then  $\sigma_k(f)\sigma_k(g) = \sigma_k(fg)$  is clear since the left-hand side is just a product of diagonal matrices.

If  $f \in C(X, A)$  and  $g \in G_{r-1}$ , then

$$\begin{aligned}
\sigma_k(f)\sigma_k(u^r g) &= \sigma_k(f)s_k^r(g) \\
&= s_k^r \sigma_k(\beta^{-r}(f)g) \\
&= \sigma_k(u^{-r}\beta^{-r}(f)g) \\
&= \sigma_k(f(u^r g)).
\end{aligned}$$

The arguments for the equalities  $\sigma_k(u^r g)\sigma_k(f) = \sigma_k((u^r g)f)$ ,  $\sigma_k(f)\sigma_k(gu^{-r}) = \sigma_k(f(gu^{-r}))$ , and  $\sigma_k(gu^{-r})\sigma_k(f) = \sigma_k((gu^{-r})f)$  are similar to the previous one. It follows that  $\sigma_k$  is multiplicative for  $0 \leq k \leq l$ . We have thus established that for  $0 \leq k \leq l$ ,  $\sigma_k$  is a  $*$ -homomorphism, and hence so is  $\sigma$ . It remains to show that  $\sigma$  is injective. Let  $f \in B_Y$ , and using Proposition V.7, find

$f_0 \in C(X, A)$  and  $f_j, g_j \in G_{j-1}$  for  $1 \leq j \leq n(l) - 1$  such that

$$f = f_0 + \sum_{j=1}^{n(l)-1} u^j f_j + \sum_{j=1}^{n(l)-1} g_j u^{-j}.$$

Suppose that  $\sigma(f) = 0$ , which is equivalent to  $\sigma_k(f) = 0$  for  $0 \leq k \leq l$ , and fix some  $k \in \{0, \dots, l\}$ .

Then

$$\sigma_k(f_0) + \sum_{j=1}^{n(l)-1} s_k^j \sigma_k(f_j) + \sum_{j=1}^{n(l)-1} \sigma_k(g_j) s_k^{-j} = 0.$$

Since  $\sigma_k(f_j) = \sigma_k(g_j) = 0$  for  $j \geq n(k)$ , this reduces to

$$\sigma_k(f_0) + \sum_{j=1}^{n(k)-1} s_k^j \sigma_k(f_j) + \sum_{j=1}^{n(k)-1} \sigma_k(g_j) s_k^{-j} = 0.$$

Using Lemma V.10, it follows that this equation takes the matrix form

$$\begin{bmatrix} \beta^{-1}(f_0)|_{Y_k} & \beta^{-1}(g_1)|_{Y_k} & \cdots & \beta^{-1}(g_{n(k)-1})|_{Y_k} \\ \beta^{-1}(f_1)|_{Y_k} & \beta^{-2}(f_0)|_{Y_k} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \beta^{-1}(f_{n(k)-1}) & \beta^{-(n(k)-1)}(f_1)|_{Y_k} & \cdots & \beta^{-n(k)}(f_0)|_{Y_k} \end{bmatrix} = 0.$$

This implies  $f_0 = 0$  on  $\bigcup_{r=1}^{n(k)} h^r(Y_k)$  and  $f_j = g_j = 0$  on  $\bigcup_{r=1}^{n(k)-j} h^r(Y_k)$  for  $1 \leq j \leq n(k) - 1$ . Since  $k$  is arbitrary and  $\bigcup_{k=0}^l \bigcup_{r=1}^{n(k)} h^r(Y_k) = X$  and  $h^{n(k)-j}(Y_k) \subset h^{-j}(Y)$ , we conclude that  $f_0 = 0$  and  $f_j = g_j = 0$  for  $1 \leq j \leq n(l) - 1$ . It follows that  $f = 0$  and so  $\sigma$  is injective.  $\square$

**Lemma V.11.** *Identify  $C(Y_k, M_{n(k)}(A))$  with  $M_{n(k)}(C(Y_k, A))$  in the obvious way. Define maps  $p_k^{(m)}: C(Y_k, M_{n(k)}(A)) \rightarrow C(Y_k, M_{n(k)}(A))$  by  $p_k^{(m)}(b)_{m+j,j} = b_{m+j,j}$  for  $1 \leq j \leq n(k) - m$  (if  $m \geq 0$ ) and for  $-m+1 \leq j \leq n(k)$  (if  $m \leq 0$ ), and  $p_k^{(m)}(b)_{i,j} = 0$  for all other pairs  $(i, j)$ . (Thus,  $p_k^{(m)}$  is the projection map on the  $m$ th subdiagonal.) Write*

$$P_m = \bigoplus_{k=0}^l p_k^{(m)}(C(Y_k, M_{n(k)}(A))).$$



*Proof.* Suppose first that  $(b_0, \dots, b_l) \in \sigma(B_{\{Y\}})$ . Then  $(b_0, \dots, b_l) = \sigma(u^m f)$  for some  $m \geq 0$  and  $f \in G_{m-1}$  (or  $C(X, A)$  in the case  $m = 0$ ). Let  $x \in Y(k, t_1, \dots, t_r)$  with  $n(t_1) + \dots + n(t_r) = n(k)$ . The  $m$ th subdiagonal of  $b_k(x)$  is given by

$$(\beta^{-1}(f)(x), \dots, \beta^{-(n(k)-m)}(f)(x)),$$

while the  $m$ th subdiagonal of the block diagonal matrix

$$\text{diag}(b_{t_1}(x), \beta^{-n(t_1)}(b_{t_2})(x), \dots, \beta^{-(n(t_1)+\dots+n(t_{r-1}))}(b_{t_r})(x))$$

is given by

$$\begin{aligned} &(\beta^{-1}(f)(x), \dots, \beta^{-(n(t_1)-m)}(f)(x), 0, \dots, 0, \beta^{-(n(t_1)+1)}(f)(x), \dots, \beta^{-(n(t_1)+n(t_2)-m)}(f)(x), \\ &0, \dots, 0, \dots, \beta^{-(n(t_1)+\dots+n(t_{r-1})+1)}(f)(x), \dots, \beta^{-(n(t_1)+\dots+n(t_r)-m)}(f)(x)). \end{aligned}$$

Each sequence of zeros in this second expression has length  $m$  (if  $m = 0$  the first and second expression are clearly equal, so assume that  $m \geq 1$ ) and the corresponding entries in the first expression have the form

$$\beta^{-(n(t_1)+\dots+n(t_i)-j)}(f)(x) = \alpha_x^{-(n(t_1)+\dots+n(t_i)-j)}(f \circ h^{n(t_1)+\dots+n(t_i)-j}(x)),$$

where  $0 \leq j \leq m-1$  and  $1 \leq i \leq r-1$ . But  $x \in Y(k, t_1, \dots, t_r)$  implies that  $h^{n(t_1)+\dots+n(t_i)}(x) \in Y_{i+1} \subset Y$ , which further implies that all such entries are 0 as  $f \in G_{m-1}$  vanishes on  $\bigcup_{j=0}^{m-1} h^{-j}(Y)$ . It follows that the two expressions are equal.

For the converse, let

$$(b_0, \dots, b_l) \in \bigoplus_{k=0}^l C(Y_k, M_{n(k)}(A))$$

and assume that  $(b_0, \dots, b_l)$  satisfies the given relations. By Corollary V.11 and by taking adjoints, it suffices to prove that for each  $m \geq 0$ , we have  $(b_0, \dots, b(l)) \in \sigma(B_Y)$  under the additional assumption that  $b_k \in p_k^{(m)}(C(Y_k, M_{n(k)}(A)))$ . Define continuous functions  $f_k^{(j)}: h^j(Y_k) \rightarrow A$  by

requiring that the  $m$ th subdiagonal of  $b_k$  be given by

$$\left( \beta^{-1}(f_k^{(1)}), \beta^{-2}(f_k^{(2)}), \dots, \beta^{-(n(k)-m)}(f_k^{(n(k)-m)}) \right),$$

and by setting  $f_k^{(j)} = 0$  for  $0 \leq k \leq l$  and  $n(k) - m + 1 \leq j \leq n(k)$ . We claim there is a continuous function  $f: X \rightarrow A$  such that  $f|_{h^j(Y_k)} = f_k^{(j)}$  for  $0 \leq k \leq l$  and  $1 \leq j \leq n(k)$ . Moreover, if  $m \geq 1$  then  $f \in G_{m-1}$ . If such a function  $f$  exists, then by construction we have  $u^m f \in B_Y$  and  $\sigma(u^m f) = (b_0, \dots, b_l)$  as required.

Suppose that  $f$  exists and is continuous, and that  $m \geq 1$ . Then the condition  $f|_{h^j(Y_k)} = f_k^{(j)}$  implies that  $f = 0$  on  $h^j(Y_k)$  for  $0 \leq k \leq l$  and  $n(k) - m + 1 \leq j \leq n(k)$ . But then  $f = 0$  on

$$\bigcup_{k=0}^l \bigcup_{j=n(k)-m+1}^{n(k)} h^j(Y_k) = \bigcup_{k=0}^l \bigcup_{j=-m+1}^0 h^{n(k)+j}(Y_k) = \bigcup_{j=0}^{m-1} h^{-j} \left( \bigcup_{k=0}^l h^{n(k)}(Y_k) \right) = \bigcup_{j=0}^{m-1} h^{-j}(Y),$$

which implies that  $f \in G_{m-1}$ . So it suffices to prove that  $f$  exists and is continuous. Since the sets  $h^j(Y_k)$  give a cover of  $X$  by closed sets, it suffices to prove that  $f$  is well-defined on the intersections  $h^{j_1}(Y_{k_1}) \cap h^{j_2}(Y_{k_2})$ . To do this, we need to show that if  $x \in h^{j_1}(Y_{k_1}) \cap h^{j_2}(Y_{k_2})$ , then  $f_{k_1}^{(j_1)}(x) = f_{k_2}^{(j_2)}(x)$ . First suppose that  $j_1 = j_2 = j$ , and assume without loss of generality that  $k_1 < k_2$ . Then  $h^{-j}(x) \in Y_{k_1} \cap Y_{k_2}$ , and moreover  $h^{-j}(x) \in Y_{k_2} \setminus Y_{k_2}^\circ$  as  $n(k_1) < n(k_2)$  is a return time for  $h^{-j}(x)$ . Let  $n(t_1)$  be the first return time to  $Y$  of the point  $h^{n(k_1)-j}(x)$ . If  $n(k_1) + n(t_1) < n(k_2)$ , let  $n(t_2)$  be the first return time to  $Y$  of  $h^{n(k_1)+n(t_1)-j}(x)$ . Since  $h^{-j}(x) \in Y_{k_2}$ , we must have  $h^{n(k_2)-j}(x) \in Y$ . Proceeding inductively, we obtain a sequence  $n(k_1), n(k_1) + n(t_1), n(k_1) + n(t_1) + n(t_2), \dots$  of increasing return times to  $Y$  for the point  $h^{-j}(x)$ , that is bounded above by  $n(k_2)$ . So there must be an  $r$  and a return time  $n(t_r)$  such that

$$n(k_1) + n(t_1) + n(t_2) + \dots + n(t_r) = n(k_2).$$

Then we have

$$h^{-j}(x) \in (Y_{k_2} \setminus Y_{k_2}^\circ) \cap Y_{k_1} \cap h^{-n(k_1)}(Y_{t_1}) \cap h^{-(n(k_1)+n(t_1))}(Y_{t_2}) \cap \dots \cap h^{-(n(k_1)+n(t_1)+\dots+n(t_{r-1}))}(Y_{t_r}).$$





using the same reasoning as in the previous case. On the other hand,  $f_{k_1}^{(j_1)}(x) = 0$  since  $f_{k_1}^{(j)} = 0$  by definition for any  $j > n(k_1) - m$ . So we again have the equality  $f_{k_2}^{(j_2)}(x) = f_{k_1}^{(j_1)}(x)$ .

Finally, we handle the case where  $j_1 < j_2$  but  $n(k_1) + j_1 > n(k_2) + j_2$ . Set  $x_1 = h^{-j_1}(x) \in Y_{k_1}$ . Proceeding as before, construct a sequence  $t_1, t_2, \dots, t_r$  such that  $n(t_1), n(t_1) + n(t_2), \dots, n(t_1) + \dots + n(t_r)$  are successive first return times of  $x_1$  to  $Y$ , and such that

$$n(t_1) + n(t_2) + \dots + n(t_r) = n(k_1).$$

We claim that  $r \geq 2$ . To see this, observe that

$$h^{n(k_2) - (j_2 - j_1)}(x_1) = h^{n(k_2) - j_2}(x) = h^{n(k_2)}(x_2) \in h^{n(k_2)}(Y_{k_2}) \subset Y,$$

which implies that  $n(k_2) - (j_2 - j_1)$  is a return time of  $x_1$  to  $Y$ , and  $n(k_1) > n(k_2 - j_2 + j_1)$  by assumption. Choose  $i$  such that

$$n(t_1) + \dots + n(t_{i-1}) < j_1 \leq n(t_1) + \dots + n(t_i),$$

and set  $k_3 = t_i$ ,  $j_3 = j_1 - [n(t_1) + \dots + n(t_{i-1})]$ , and  $x_3 = h^{n(t_1) + \dots + n(t_{i-1})}(x_1)$ . Then  $x_3 \in Y_{k_3}$  and  $h^{j_3}(x_3) = h^{j_1}(x_1) = x$ , which imply that

$$x \in h^{j_3}(Y_{k_3}) \cap h^{j_1}(Y_{k_1}) \cap h^{j_2}(Y_{k_2}).$$

By construction, we have  $j_3 < j_1$  and  $n(k_3) - j_3 < n(t_i) - j_1 < n(k_1) - j_1$ . Now the cases we have already done imply that  $f_{k_1}^{(j_1)}(x) = f_{k_3}^{(j_3)}(x)$ , and so we may replace  $j_1$  and  $k_1$  with  $j_3$  and  $k_3$  in the argument for  $f_{k_1}^{(j_1)}(x) = f_{k_2}^{(j_2)}(x)$ . But  $n(k_3) < n(k_1)$  by the observation that  $n(k_1) = n(t_1) + \dots + n(t_r)$  with  $r \geq 2$ , so  $n(k_3) + n(k_2) < n(k_1) + n(k_3)$ . The result follows by a finite descent argument.  $\square$

We now have the necessary machinery to give a decomposition of  $B_Y$  as a recursive  $A$ -subhomogeneous algebra.

**Theorem V.13.** *Let  $Y \subset X$  be closed with  $\text{int}(Y) \neq \emptyset$ , and adopt Notation V.6 and the notation of Theorem V.9. Then the homomorphism  $\sigma: B_Y \rightarrow C_Y$  induces an isomorphism of  $B_Y$  with the recursive  $A$ -subhomogeneous algebra defined, in the notation of Definition V.3, as follows:*



1.  $l$  and  $n(0), n(1), \dots, n(l)$  are as in Notation V.6;
2.  $X_k = Y_k$  for  $0 \leq k \leq l$ ;
3.  $X_k^{(0)} = Y_k \cap \bigcup_{j=0}^{k-1} Y_j$ ;
4. For  $x \in X_k^{(0)}$  and  $(b_0, b_1, \dots, b_{k-1})$  in the image of the  $k-1$  stage algebra  $B_Y^{(k-1)}$  (in  $\bigoplus_{j=0}^{k-1} C(Y_j, M_{n(j)}(A))$ ), whenever  $x \in Y(k, t_1, \dots, t_r)$  with  $n(t_1) + n(t_2) + \dots + n(t_r) = n(k)$ , then  $\varphi_k(b_0, b_1, \dots, b_{k-1})(x)$  is given by the block diagonal matrix

$$\varphi_k(b_0, b_1, \dots, b_{k-1})(x) = \begin{bmatrix} b_{t_1}(x) & & & & \\ & \beta^{-n(t_1)}(b_{t_2})(x) & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \beta^{-[n(t_1)+\dots+n(t_{r-1})]}(b_{t_r})(x) \end{bmatrix};$$

5.  $\rho_k$  is the restriction map.

The topological dimension of this decomposition is  $\dim(X)$ , and the standard representation of  $\sigma(B_Y)$  is the inclusion map in  $C_Y$ .

*Proof.* We prove by induction on  $k$  that the homomorphism  $\varphi_k: B_Y^{(k-1)} \rightarrow C(Y_k^{(0)}, M_{n(k)}(A))$  given by the formula in (4) is well defined. As we shall see this is the key element of the proof. For the base case, we prove that  $\varphi_1$  is well-defined. Let  $x \in Y_1^{(0)} = Y_1 \cap Y_0$ . Let  $t_0, t_1, \dots, t_{r-1}$  be the successive return times of  $x$  to  $Y_0$ , and let  $t_r$  be the first return time of  $x$  to  $Y_1$ , and require that  $t_0 = 0$ . Then we certainly have  $t_1 = n(0)$  and  $t_r = n(1)$ . Since  $n(0) < n(1)$ , it follows that  $r \geq 2$ . Also, for  $i < r$  each  $h^{t_i}(x)$  is in  $Y_0$  and its first return time to  $Y_0$  is  $t_{i+1} - t_i$ , which is always strictly less than  $n(1)$ , and so must be  $n(0)$ . Then the recursion relations  $t_0 = 0, t_1 = n(0), t_{i+1} - t_i = n(0)$  imply that  $t_i = in(0)$  for  $0 \leq i \leq r$ . In particular, we obtain  $n(1) = t_r = rn(0)$ . Now, if  $Y_1^{(0)} = \emptyset$  then  $\varphi_1$  is trivially well-defined. If  $Y_1^{(0)} \neq \emptyset$  then  $x \in Y_1 \setminus Y_1^\circ$  (since if we had  $x \in Y_1^\circ$ , we could not have  $x \in Y_0$ ), and so we may write  $Y_1^{(0)}$  as

$$Y_1^{(0)} = (Y_1 \setminus Y_1^\circ) \cap Y_0 \cap h^{-n(0)}(Y_0) \cap h^{-2n(0)}(Y_0) \cap \dots \cap h^{-(r-1)n(0)}(Y_0).$$

Then  $\varphi_1(b)$  is well-defined by the formula

$$\varphi_1(b)(x) = \left( b(x), \beta^{-n(0)}(b)(x), \dots, \beta^{-(r-1)n(0)}(b)(x) \right).$$

Now suppose that  $\varphi_1, \varphi_2, \dots, \varphi_{k-1}$  are well-defined. Then  $B_Y^{(k-1)}$  is a recursive  $A$ -subhomogeneous algebra, and its elements are exactly the sequences  $(b_0, \dots, b_{k-1})$  satisfying the conditions of Lemma V.12 up to  $l = k - 1$ . We define a homomorphism  $\varphi_k: B_Y^{(k-1)} \rightarrow C(Y_k^{(0)}, M_{n(k)}(A))$  by the formula in condition (4). Once we have shown this is well-defined, the induction will be complete, and it will follow that  $B_Y^{(k)}$  is a recursive  $A$ -subhomogeneous algebra, whose elements are exactly the sequences  $(b_0, \dots, b_k)$  satisfying the conditions of Lemma V.12 up to  $l = k$ . Let  $S$  be the set of all sequences  $(t_1, t_2, \dots, t_r)$  such that  $r \geq 2$  and  $n(t_1) + n(t_2) + \dots + n(t_r) = n(k)$ . Since  $r \geq 2$ , it follows that  $t_i < k$  for every possible  $t_i$ . For a sequence  $\sigma = (t_1, \dots, t_r) \in S$ , define

$$Y_k^{(\sigma)} = (Y_k \setminus Y_k^\circ) \cap Y_{t_1} \cap h^{-n(t_1)}(Y_{t_2}) \cap \dots \cap h^{-[n(t_1)+n(t_2)+\dots+n(t_{r-1})]}(Y_{t_r}).$$

By an argument analogous to that done for the base case of the induction, we observe that  $Y_k^{(0)} = \bigcup_{\sigma \in S} Y_k^{(\sigma)}$ . To show that  $\varphi_k$  is well-defined, it is therefore sufficient to prove that given  $\sigma, \tau \in S$  and  $x \in Y_k^{(\sigma)} \cap Y_k^{(\tau)}$ , the corresponding formulas of condition (4) agree at the point  $x$ . For  $b \in B_Y^{(k-1)}$ , denote these expressions by  $\varphi_k^{(\sigma)}(b)(x)$  and  $\varphi_k^{(\tau)}(b)(x)$  respectively. For  $\sigma = (t_1, t_2, \dots, t_r) \in S$ , denote by  $R(\sigma)$  the set of successive return times associated with  $\sigma$ :

$$R(\sigma) = \{0, n(t_1), n(t_1) + n(t_2), \dots, n(t_1) + \dots + n(t_{r-1}), n(k)\}.$$

For  $\sigma, \tau \in S$  and  $x \in Y_k^{(\sigma)} \cap Y_k^{(\tau)}$ , let  $\rho = (t_1, t_2, \dots, t_r) \in S$  be the sequence such that  $n(t_1)$  is the first return time of  $x$ ,  $n(t_2)$  is the first return time of  $h^{n(t_1)}(x)$ , and so on. Then  $x \in Y_k^{(\rho)}$  and  $R(\rho)$  is contained in both  $R(\sigma)$  and  $R(\tau)$ . It is therefore sufficient to prove that if  $\sigma, \tau \in S$  and  $x \in Y_k^{(\sigma)} \cap Y_k^{(\tau)}$ , then  $\varphi_k^{(\sigma)}(b)(x) = \varphi_k^{(\tau)}(b)(x)$  under the additional simplification that  $R(\sigma) \subset R(\tau)$ .

So finally, assume that  $\sigma, \tau \in S$  with  $R(\sigma) \subset R(\tau)$  and that  $x \in Y_k^{(\sigma)} \cap Y_k^{(\tau)}$ . Writing  $\tau = (t_1, t_2, \dots, t_r)$ , we have

$$R(\tau) = \{0, n(t_1), n(t_1) + n(t_2), \dots, n(t_1) + \dots + n(t_r)\}.$$

Since  $R(\sigma) \subset R(\tau)$ , there exist  $0 = j(0) < j(1) < j(2) < \dots < j(m)$  such that

$$R(\sigma) = \{0, n(t_1) + \dots + n(t_{j(1)}), n(t_1) + \dots + n(t_{j(2)}), \dots, n(t_1) + \dots + n(t_{j(m)})\}$$

and  $n(t_1) + n(t_2) + \dots + n(t_{j(m)}) = n(k)$ . Then  $\sigma = (s_1, s_2, \dots, s(m))$  where

$$n(s_i) = n(t_{j(i-1)+1}) + n(t_{j(i-1)+2}) + \dots + n(t_{j(i)}).$$

Now  $\varphi_k^{(\sigma)}(b)(x)$  is given by the block diagonal matrix

$$\varphi_k^{(\sigma)}(b)(x) = \begin{bmatrix} b_{s_1}(x) & & & & \\ & \beta^{-n(s_1)}(b_{s_2})(x) & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \beta^{-[n(s_1)+\dots+n(s_{m-1})]}(b_{s_m})(x) \end{bmatrix}.$$

We apply the induction hypothesis to the individual blocks in this matrix. For  $1 \leq i \leq m$  it follows that whenever

$$y \in (Y_{s_i} \setminus Y_{s_i}^o) \cap Y_{t_{j(i-1)+1}} \cap h^{-n(t_{j(i-1)+1})}(Y_{t_{j(i-1)+2}}) \cap \dots \cap h^{-[n(t_{j(i-1)+1})+\dots+n(t_{j(i)-1})]}(Y_{t_{j(i)}}),$$

then  $b_{s_i}(y)$  is given by the block diagonal matrix

$$b_{s_i}(y) = \begin{bmatrix} b_{t_{j(i-1)+1}}(y) & & & & \\ & \beta^{-n(t_{j(i-1)+1})}(b_{t_{j(i-1)+2}})(y) & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \beta^{-[n(t_{j(i-1)+1})+\dots+n(t_{j(i)-1})]}(b_{t_{j(i)}})(y) \end{bmatrix}.$$

By evaluating  $b_{s_i}(y)$  at  $y = x$  for  $i = 1$  and at  $y = h^{n(s_1)+\dots+n(s_{i-1})}(x)$  for  $i \geq 2$ , and noting that  $n(s_1) + \dots + n(s_{i-1}) = n(t_1) + \dots + n(t_{j(i-1)})$ , it follows that  $\varphi_k^{(\sigma)}(b)(x) = \varphi_k^{(\tau)}(b)(x)$  as required.

This completes the induction.

To complete the proof, we need only show that the topological dimension of the recursive  $A$ -subhomogeneous decomposition is in fact  $\dim(X)$ . Since the sets  $Y_k$  are closed subsets of  $X$ , they must all satisfy  $\dim(Y_k) \leq \dim(X)$  by Theorems 1.1.2 and 1.7.7 of [10]. On the other hand,

the finite collection  $\{h^j(Y_k) : 0 \leq k \leq l, 1 \leq j \leq n(k)\}$  covers  $X$ , and so Theorems 1.5.3 and 1.7.7 of [10] imply that  $\dim(Y_k) = \dim(X)$  for at least one value of  $k$ .  $\square$

**Corollary V.14.** *For any  $y \in X$ ,  $B_{\{y\}}$  is a direct limit of recursive  $A$ -subhomogeneous algebras.*

*Proof.* Given  $y \in X$ , choose a sequence  $(Y_n)_{n=1}^{\infty}$  of closed subsets of  $X$  with  $\text{int}(Y_n) \neq \emptyset$  for all  $n$ ,  $Y_{n+1} \subset Y_n$  for  $n \geq 1$ , and  $\bigcap_{n=1}^{\infty} Y_n = \{y\}$ . Then the result follows immediately by applying Theorems V.13 and V.5.  $\square$

## CHAPTER VI

THE RELATIONSHIP BETWEEN  $C^*(\mathbb{Z}, C(X, A), \beta)_{\{y\}}$  AND  $C^*(\mathbb{Z}, C(X, A), \beta)$ 

For the approximating subalgebra  $C^*(\mathbb{Z}, C(X, A), \beta)_{\{y\}}$  of  $C^*(\mathbb{Z}, C(X, A), \beta)$  to be useful, it must be in some sense “big enough” so that various properties it satisfies can pass to the entire crossed product  $C^*$ -algebra. Giving a useful definition of this idea and showing that it is satisfied by our subalgebra will be the main content of this chapter. In order to carefully state this definition, we require some discussion of Cuntz subequivalence and the Cuntz semigroup, ideas that have been mentioned occasionally but for which careful exposition was not required until now. The following definition first appeared in [5].

**Definition VI.1.** *Let  $A$  be a  $C^*$ -algebra, and let  $M_\infty(A)$  denote the set  $\bigcup_{n=1}^\infty M_n(A)$ , which we may interpret more formally as the algebraic direct limit of the system  $(M_n(A))_{n=1}^\infty$  where the maps  $\varphi_n: M_n(A) \rightarrow M_{n+1}(A)$  are the usual embedding maps  $\varphi_n(a) = \text{diag}(a, 0)$ . For  $a, b \in M_\infty(A)$ , we write  $a \oplus b$  for the element  $\text{diag}(a, b)$  of  $M_\infty(A)$ .*

1. *Given  $a, b \in M_\infty(A)_+$ , we say that  $a$  is Cuntz subequivalent to  $b$ , and write  $a \preceq b$ , if there exists a sequence  $(y_n)_{n=1}^\infty \subset M_\infty(A)$  such that  $y_n b y_n^* \rightarrow a$ .*
2. *Given  $a, b \in M_\infty(A)_+$ , we say that  $a$  and  $b$  are Cuntz equivalent, and write  $a \sim b$ , if  $a \preceq b$  and  $b \preceq a$ . It is easy to check that  $\sim$  is an equivalence relation on  $M_\infty(A)$ , and for  $a \in M_\infty(A)_+$  we write  $\langle a \rangle$  for its equivalence class under  $\sim$ .*
3. *The Cuntz semigroup of  $A$  is the set*

$$W(A) = M_\infty(A)_+ / \sim$$

with the commutative semigroup operation  $\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle$ . It is has a partial order  $\leq$  given by  $\langle a \rangle \leq \langle b \rangle$  if and only if  $a \preceq b$ .

**Definition VI.2.** Let  $A$  be a  $C^*$ -algebra, let  $a \in A_+$ , and let  $\varepsilon > 0$ . Let  $f: [0, \infty) \rightarrow [0, \infty)$  be the continuous uncton

$$f(t) = \begin{cases} 0 & 0 \leq t \leq \varepsilon \\ t - \varepsilon & \varepsilon < t. \end{cases}$$

Then, using continuous functional calculus, define  $(a - \varepsilon)_+ = f(a)$ .

We summarize some of the known results about Cuntz subequivalence that will be necessary for our purposes. Proofs can be found in Section 2 of [9], Section 2 of [14], and Section 1 of [43], although some of them were originally given elsewhere.

**Lemma VI.3.** Let  $A$  be a  $C^*$ -algebra.

1. Let  $c \in a$  and let  $\alpha > 0$ . Then  $(c^*c - \alpha)_+ \sim (cc^* - \alpha)_+$ .
2. Let  $a, b \in A$  be positive. Then the following are equivalent:
  - (a)  $a \preceq b$ ;
  - (b)  $(a - \varepsilon)_+ \preceq b$  for all  $\varepsilon > 0$ ;
  - (c) for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that  $(a - \varepsilon)_+ \preceq (b - \delta)_+$ .
3. Let  $a, b \in A$  satisfy  $0 \leq a \leq b$ , and let  $\varepsilon > 0$ . Then  $(a - \varepsilon)_+ \preceq (b - \varepsilon)_+$ .
4. If  $a \in A$  is positive and  $b \in \overline{aAa}$  is positive, then  $b \preceq a$ .
5. If  $a, b \in A$  are positive and  $u \in U(A)$ , then  $a \sim b$  if and only if  $uau^* \sim b$ .
6. If  $a, b \in A$  are positive and there is an  $x \in A$  such that  $x^*x = a$  and  $xx^* = b$ , then  $a \sim b$ .

The next definition is adapted from Definition 2.2 of [43]. The only difference is that normalized quasitraces have been replaced with tracial states; for nuclear  $C^*$ -algebras, the definitions coincide.

**Definition VI.4.** Let  $C$  be a simple, separable, unital, nuclear, stably finite, infinite-dimensional  $C^*$ -algebra. A subalgebra  $D \subset C$  is said to be large in  $C$  if:

1.  $D$  contains the identity of  $C$ ;
2.  $D$  is simple;
3. The restriction map  $T(C) \rightarrow T(D)$  is surjective;
4. For every  $\varepsilon > 0$ ,  $m \in \mathbb{N}$ ,  $a_1, \dots, a_m \in C$ , and  $b \in D_+ \setminus \{0\}$ , there exist  $c_1, \dots, c_m \in C$  and  $g \in D$  such that:
  - (a)  $0 \leq g \leq 1$ ;
  - (b)  $\|c_j - a_j\| < \varepsilon$  for  $1 \leq j \leq m$ ;
  - (c)  $(1 - g)c_j, c_j(1 - g) \in B$  for  $1 \leq j \leq m$ ;
  - (d)  $g \precsim b$  relative to the subalgebra  $D$ .

**Notation VI.5.** Throughout this chapter, we let  $(X, h, \mu)$ ,  $A$ , and  $\beta$  be as in the hypotheses of Theorem IV.15, set  $B = C^*(\mathbb{Z}, C(X, A), \beta)$  and let  $u$  be the canonical unitary for  $B$ . For  $Y \subset X$  closed, we let  $B_Y = C^*(\mathbb{Z}, C(X, A), \beta)$  be as in Definition V.4. Denote by  $C(X, A)[\mathbb{Z}]$  the dense subalgebra of  $B$  given by all sums of the form  $\sum_{k \in T} a_k u^k$ , where  $T \subset \mathbb{Z}$  is finite and  $a_k \in C(X, A)$  for all  $k \in T$ . Let  $E: B \rightarrow C(X, A)$  be the standard canonical expectation, which is given explicitly on  $C(X, A)[\mathbb{Z}]$  by  $E(\sum_{k \in T} a_k u^k) = a_0$ .

Our goal is to show that for  $y \in X$ , the algebra  $B_{\{y\}} = C^*(\mathbb{Z}, C(X, A), \beta)_{\{y\}}$  is a large subalgebra of  $B = C^*(\mathbb{Z}, C(X, A), \beta)$ . Condition (1) of the definition follows immediately from the definition of  $B_{\{y\}}$ . We prove conditions (2) and (3) in the following propositions. For condition (3) we actually show more, namely that the restriction map between the tracial state spaces is bijective. Moreover, we identify these tracial states with the  $\beta$ -invariant tracial states on the algebra  $C(X, A)$ .

**Proposition VI.6.** *Adopt Notation VI.5. Then for any  $y \in X$ ,  $B_{\{y\}}$  is simple.*

*Proof.* Let  $I \subset B_{\{y\}}$  be a non-zero ideal. Then  $I \cap C(X, A)$  is an ideal in  $C(X, A)$ , so by Proposition IV.16 there is a closed set  $F \subset X$  such that

$$I \cap C(X, A) = \{f \in C(X, A) : f|_F = 0\},$$

and  $F$  is given explicitly by  $F = \{x \in X : f(x) = 0 \text{ for all } f \in I\}$ . We first claim that  $F \neq X$ . Using Proposition V.5, we may write  $B_{\{y\}} = \varinjlim B_{Y_n}$  for some sequence  $Y_1 \supset Y_2 \supset \dots$  with  $\text{int}(Y_n) \neq \emptyset$  and  $\bigcap_{n=1}^{\infty} Y_n = \{y\}$ . Then there is an  $N$  such that  $I \cap B_{Y_N} \neq \emptyset$ . Let  $a \in I \cap B_{Y_N}$  with  $a \geq 0$  and  $a \neq 0$ , and adopt Notation V.6 with  $Y = Y_N$ . Then Proposition V.7 implies there are  $f_0 \in C(X, A)$  and  $f_{-j}, f_j \in G_{j-1}$  for  $1 \leq j \leq n(l) - 1$  such that  $a = f_0 + \sum_{j=1}^{n(l)-1} (f_{-j}u^{-j} + u^j f_j)$ . In fact, using the relation  $u^j f_j = \beta^j(f_j)u^j$ , we may write  $a$  as  $a = \sum_{j=-(n(l)-1)}^{n(l)-1} g_j u^j$  where each  $g_j \in C(X, A)$  and  $g_0 \geq 0$  is non-zero. Let  $x \in X$  be a point where  $g_0(x) \neq 0$ , and choose a neighborhood  $V$  of  $x$  such that the sets  $h^j(V)$  are pairwise disjoint for  $-(n(l)-1) \leq j \leq n(l)-1$ . Choose a continuous function  $g : X \rightarrow [0, 1]$  such that  $g(x) = 1$  and  $\text{supp}(g) \subset V$ . Then  $g \in B_{\{y\}}$ , and so  $gag \in I$ . Moreover,

$$gag = \sum_{j=-(n(l)-1)}^{n(l)-1} gg_j u^j g = \sum_{j=-(n(l)-1)}^{n(l)-1} g_j g \beta^j(g) u^j = g_0 g^2,$$

which implies that  $gag \in C(X, A)$ . Therefore  $gag \in I \cap C(X, A)$  and  $(gag)(x) = (g_0 g^2)(x) \neq 0$ . It follows that  $F \neq X$ .

Next, we claim that  $F \subset \{h^n(y) : n \in \mathbb{Z}\}$ . Suppose not, and that  $x_0 \in F \setminus \{h^n(y) : n \in \mathbb{Z}\}$ . Let  $f \in I \cap C(X, A)$ , and for each  $n \geq 1$ , choose a continuous function  $g_n : X \rightarrow [0, 1]$  such that  $g_n(h^{-n}(x_0)) = 1$  (note this implies  $\beta(g_n)(h^{-(n-1)}(x_0)) = 1$  for  $n \geq 1$ ) and  $g_n(y) = 0$ . Then  $ug_n^{1/2}, g_n^{1/2}u^{-1} \in B_{\{y\}}$ , so that  $ug_n^{1/2}fg_n^{1/2}u^{-1} \in I \cap C(X, A)$ . Also, we may write  $ug_n^{1/2}fg_n^{1/2}u^{-1} = ufg_n u^{-1} = \beta(f)\beta(g_n)$ . Since this is an element of  $I \cap C(X, A)$ , we must have  $\beta(f)\beta(g_n)(x) = 0$  for every  $x \in F$ . In particular,  $\beta(f)\beta(g_1)(x_0) = 0$ , which implies that  $\beta(f)(x_0) = 0$  as  $\beta(g_1)(x_0) = 1$ . Since this holds for every  $f \in I \cap C(X, A)$ , it follows that  $h^{-1}(x_0) \in F$ . Assuming that  $x_0, h^{-1}(x_0), \dots, h^n(x_0) \in F$  for  $n \geq 0$ , we obtain  $\beta(f)\beta(g_{n+1})(h^{-n}(x_0)) = 0$ , which implies that  $\beta(f)(h^{-n}(x_0)) = 0$ . Since this holds for every  $f \in I \cap C(X, A)$ , it follows that  $h^{-(n+1)}(x_0) \in F$  as well. By induction,  $F$  thus contains the entire forward orbit  $\{h^n(x_0) : n \geq 0\}$ , which is dense in  $X$  by minimality and compactness. Since  $F$  is closed, it follows that  $F = X$ , a contradiction. Therefore, we must have  $F \subset \{h^n(y) : n \in \mathbb{Z}\}$  as claimed.

If  $F \neq \emptyset$  and  $x \in F$ , then  $x = h^n(y)$  for some  $n \in \mathbb{Z}$ . First suppose that  $n \leq 0$ . For each  $k \geq 1$ , choose a continuous function  $g_k : X \rightarrow [0, 1]$  such that  $g_k(h^{n-k}(y)) = 1$  and  $g_k(y) = 0$ . As in the previous argument, for any  $f \in I \cap C(X, A)$  we have  $ug_k^{1/2}fg_k^{1/2}u^{-1} = \beta(f)\beta(g_k) \in I \cap C(X, A)$ . This implies that  $\beta(f)\beta(g_1)(h^n(y)) = 0$  since  $h^n(y) \in F$ . Then  $\beta(g_k)(h^n(y)) = 1$



implies that  $\beta(f)(h^n(y)) = 0$ , and this must hold for every  $f \in I \cap C(X, A)$ , so we obtain  $h^{n-1}(y) \in F$ . Assuming we have  $h^n(y), h^{n-1}(y), \dots, h^{n-k}(y) \in F$ ,  $\beta(f)\beta(g_{k+1})(h^{n-k}(y)) = 0$  implies that  $\beta(f)(h^{n-k}(y)) = 0$  for every  $f \in I \cap C(X, A)$ , which gives  $h^{n-(k+1)}(y) \in F$ . By induction,  $F$  contains the entire backwards orbit  $\{h^{n-k}(y) : k \geq 0\} = \{h^k(x) : k \leq 0\}$ , which implies that  $F = X$ , a contradiction.

Finally, suppose that  $n \geq 1$ . For  $k \geq 0$  choose a continuous function  $g_k : X \rightarrow [0, 1]$  such that  $g_k(h^{n+k}(y)) = 1$  and  $g_k(y) = 0$ . Then for any  $f \in I \cap C(X, A)$ , we have  $g_k^{1/2} u^{-1} f u g_k^{1/2} = g_k^{1/2} \beta^{-1}(f) g_k^{1/2} = g_k \beta^{-1}(f) \in I \cap C(X, A)$ . This gives  $g_0 \beta^{-1}(f)(h^n(y)) = 0$  and so  $\beta^{-1}(f)(h^n(y)) = 0$  for every  $f \in I \cap C(X, A)$ . It follows that  $h^{n+1}(y) \in F$ . Assuming that  $h^n(y), h^{n+1}(y), \dots, h^{n+k}(y) \in F$ ,  $g_k \beta^{-1}(f)(h^{n+k}(y)) = 0$  implies  $\beta^{-1}(f)(h^{n+k}(y)) = 0$  for every  $f \in I \cap C(X, A)$ , and so  $h^{n+(k+1)}(y) \in F$ . By induction,  $F$  contains the entire forward orbit  $\{h^{n+k}(y) : k \geq 0\} = \{h^k(x) : k \geq 0\}$ , which implies  $F = X$ , again a contradiction. Therefore, we must have  $F = \emptyset$ , which implies that  $I \cap C(X, A) = C(X, A)$  and hence  $I = B_{\{y\}}$ .

□

**Proposition VI.7.** *Adopt Notation VI.5. Then the restriction map  $T(B) \rightarrow T(B_{\{y\}})$  is a bijection.*

*Proof.* Recall that from Definition II.7, the set  $T_\beta(C(X, A))$  is the space of  $\beta$ -invariant tracial states on  $C(X, A)$ . By Corollary IV.21, there is a bijection between  $T(B)$  and  $T_\beta(C(X, A))$ . We first prove that the restriction map  $T(B_{\{y\}}) \rightarrow T_\beta(C(X, A))$  is injective. To see this, it suffices to prove that, given any  $\tau \in T(B_{\{y\}})$ , we have  $\tau(\beta(f)) = \tau(f)$  for every  $f \in C(X, A)$ . We may assume that  $f \geq 0$  and  $\|f\| = 1$ . Let  $\varepsilon > 0$  be given, and choose  $N \in \mathbb{N}$  such that  $1/N < \frac{1}{2}\varepsilon$ . Let  $V_0, V_1, \dots, V_N$  be pairwise disjoint neighborhoods of the distinct points  $y, h(y), \dots, h^N(y)$  respectively, and set  $V = \bigcap_{j=0}^N h^{-j}(V_j)$ , which is a neighborhood of  $y$  whose first  $N+1$  iterates  $V, h(V), \dots, h^N(V)$  are pairwise disjoint. Choose open sets  $W_0, W_1 \subset X$  such that  $y \in W_0 \subset \overline{W_0} \subset W_1 \subset \overline{W_1} \subset V$ . Choose continuous functions  $g_0^{(0)}, g_1^{(0)} : X \rightarrow [0, 1]$  such that  $g_0^{(0)} = 1$  on  $X \setminus W_1$ ,  $g_1^{(0)} = 1$  on  $\overline{W_1}$ ,  $\text{supp}(g_0^{(0)}) \subset X \setminus \overline{W_0}$ , and  $\text{supp}(g_1^{(0)}) \subset V$ . Note that  $(g_0^{(0)} + g_1^{(0)})(x) \neq 0$  for all  $x \in X$ . Now define  $g_0 = g_0^{(0)}(g_0^{(0)} + g_1^{(0)})^{-1}$  and  $g_1 = g_1^{(0)}(g_0^{(0)} + g_1^{(0)})^{-1}$ , and set  $f_0 = g_0 f$  and  $f_1 = g_1 f$ . Then  $f = f_0 + f_1$ , where  $f_0 \in C_0(X \setminus \{y\}, A)$  and  $\beta^{-j}(f_1) \in C_0(X \setminus \{y\}, A)$  for  $1 \leq j \leq N$ . The second observation follows from the fact that  $y \in \text{supp}(f_1) \subset \text{supp}(g_1) \subset V$ , and the sets  $\text{supp}(\beta^{-j}(f_1))$  are pairwise disjoint for  $0 \leq j \leq N$  (being subsets of the sets  $h^j(V)$  for  $0 \leq j \leq N$ ).

For  $1 \leq k \leq N$ , set  $v_k = u\beta^{-1}(f^{1/2})u^{k-1}\beta^{-k}(g_1^{1/2})$ . We first claim that  $v_k \in B_{\{y\}}$ . To see this, write  $q = \beta^{-k}(g_1^{1/2})^{\frac{1}{k+1}}$ , and observe that  $\beta^j(q) = \beta^{j-k}(g_1^{1/2})^{\frac{1}{k+1}} \in C_0(X \setminus \{y\}, A)$  for  $0 \leq j \leq k-1$ . Then we write

$$\begin{aligned} u\beta^{-1}(f^{1/2})u^{k-1}\beta^{-k}(g_1^{1/2}) &= u\beta^{-1}(f^{1/2})u^{k-1}q^{k+1} \\ &= u\beta^{-1}(f^{1/2})\beta^{k-1}(q)u^{k-1}q^k \\ &= [u\beta^{-1}(f^{1/2})\beta^{k-1}(q)] \cdot [u\beta^{k-2}(q)] \cdots [u\beta(q)] \cdot [uq^2]. \end{aligned}$$

Since  $\beta^{k-1}(q), \dots, \beta(q), q \in C_0(X \setminus \{y\}, A)$ , it follows that each term in this product is an element of  $B_{\{y\}}$ , and so  $v_k \in B_{\{y\}}$ . Next, we compute

$$\begin{aligned} v_k^*v_k &= (u\beta^{-1}(f^{1/2})u^{k-1}\beta^{-k}(g_1^{1/2}))^*(u\beta^{-1}(f^{1/2})u^{k-1}\beta^{-k}(g_1^{1/2})) \\ &= \beta^{-k}(g_1^{1/2})(u^{k-1})^*\beta^{-1}(f^{1/2})u^*u\beta^{-1}(f^{1/2})u^{k-1}\beta^{-k}(g_1^{1/2}) \\ &= \beta^{-k}(g_1^{1/2})(u^{k-1})^*\beta^{-1}(f)u^{k-1}\beta^{-k}(g_1^{1/2}) \\ &= \beta^{-k}(g_1^{1/2})\beta^{-k}(f)\beta^{-k}(g_1^{1/2}) \\ &= \beta^{-k}(g_1f) \\ &= \beta^{-k}(f_1) \end{aligned}$$

and

$$\begin{aligned} v_kv_k^* &= (u\beta^{-1}(f^{1/2})u^{k-1}\beta^{-k}(g_1^{1/2}))(u\beta^{-1}(f^{1/2})u^{k-1}\beta^{-k}(g_1^{1/2}))^* \\ &= u\beta^{-1}(f^{1/2})u^{k-1}\beta^{-k}(g_1)(u^{k-1})^*\beta^{-1}(f^{1/2})u^* \\ &= u\beta^{-1}(f_1^{1/2})\beta^{-1}(g_1)\beta^{-1}(f^{1/2})u^* \\ &= f^{1/2}u\beta^{-1}(g_1)u^*f^{1/2} \\ &= f^{1/2}g_1f^{1/2} \\ &= g_1f \\ &= f_1. \end{aligned}$$

Now, it follows that for  $1 \leq k \leq N$  we have

$$\tau(\beta^{-k}(f_1)) = \tau(v_k^* v_k) = \tau(v_k v_k^*) = \tau(f_1).$$

Since the supports of the  $\beta^{-k}(f_1)$  are disjoint for  $0 \leq k \leq N$ , we further have

$$N\tau(f_1) = \sum_{k=1}^N \tau(\beta^{-k}(f_1)) = \tau\left(\sum_{k=1}^N \beta^{-k}(f_1)\right) \leq \tau\left(\sum_{k=0}^N \beta^{-k}(f_1)\right) \leq \left\|\sum_{k=0}^N \beta^{-k}(f_1)\right\| = \|f_1\| = 1.$$

It follows that  $\tau(\beta^{-k}(f_1)) < 1/N < \frac{1}{2}\varepsilon$  for  $0 \leq k \leq N$ .

Next, choose a continuous function  $\varphi: X \rightarrow [0, 1]$  such that  $\varphi(y) = 0$  and  $\varphi = 1$  on  $\text{supp}(f_0)$ . Then  $f_0\varphi = \varphi f_0 = f_0$  and  $\varphi \in C_0(X \setminus \{y\}, A)$ , so both  $uf_0$  and  $u\varphi$  are elements of  $B_{\{y\}}$ . It follows that

$$\tau(\beta(f_0)) = \tau(uf_0 u^*) = \tau(uf_0 \varphi u^*) = \tau((uf_0)(u\varphi)^*) = \tau((u\varphi)^*(uf_0)) = \tau(\varphi f_0) = \tau(f_0).$$

Now finally, we have

$$\begin{aligned} |\tau(\beta(f)) - \tau(f)| &= |\tau(\beta(f_1)) + \tau(\beta(f_0)) - \tau(f_1) - \tau(f_0)| \\ &\leq |\tau(\beta(f_1))| + |\tau(f_1)| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \\ &= \varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  was arbitrary, this implies that  $\tau(\beta(f)) = \tau(f)$ . Hence any trace on  $B_{\{y\}}$ , when restricted to  $C(X, A)$ , induces a  $\beta$ -invariant trace on  $C(X, A)$ . This establishes the injectivity of the restriction map  $T(B_{\{y\}}) \rightarrow T_\beta(C(X, A))$ .

For surjectivity, it suffices to prove that the extension of any trace on  $C(X, A)$  to  $B_{\{y\}}$  is unique. To show this, it is sufficient to prove that for any closed set  $Y \subset X$  with  $\text{int}(Y) \neq \emptyset$ , any trace on  $B_Y$  is determined by its restriction to  $C(X, A)$ . Given such a set  $Y$ , adopt Notation V.6, and let  $g \in B_Y$ . Then by Proposition V.7, there are  $g_0 \in C(X, A)$  and  $g_j, g_{-j} \in G_{j-1}$  for  $1 \leq j \leq n(l) - 1$  such that  $g = g_0 + \sum_{j=0}^{n(l)-1} (u^j g_j + g_{-j} u^{-j})$ . For each  $x \in X$ , choose a neighborhood  $U_x$  of  $x$  such that the sets  $h^j(U_x)$  are pairwise disjoint for  $-(n(l)-1) \leq j \leq n(l)-1$ .

Then  $\{U_x : x \in X\}$  is an open cover of  $X$ , and hence contains a finite subcover  $\{U_1, \dots, U_M\}$ . Let  $\{\varphi_i\}_{i=1}^M$  be a partition of unity subordinate to this cover. Then for  $1 \leq i \leq M$ , we have  $\beta^j(\varphi_i)\beta^k(\varphi_i) = 0$  for  $-(n(l) - 1) \leq j, k \leq n(l) - 1$  and  $j \neq k$ , and the same relation holds with  $\varphi_i^{1/2}$  in place of  $\varphi_i$ . Next, we set  $a = \varphi_i^{1/2}$ ,  $b = u^j \varphi_i^{1/2} g_j$ , and  $c = g_{-j} \varphi_i^{1/2} u^{-j}$ . Then  $a \in Z(C(X, A))$  and so in particular  $a \in B_{\{y\}}$ . By Proposition V.7, we have  $u^j g_j \in B_{\{y\}}$  and  $g_{-j} u^{-j} \in B_{\{y\}}$ . Since  $\varphi_i^{1/2}$  commutes with both  $g_j$  and  $g_{-j}$ , we may write  $b = u^j g_j \varphi_i^{1/2}$  and  $c = \varphi_i^{1/2} g_{-j} u^{-j}$ , from which it follows that  $b, c \in B_{\{y\}}$ . Using the trace property for  $\tau$  on  $B_{\{y\}}$ , we obtain  $\tau(ab) = \tau(ba)$  and  $\tau(ac) = \tau(ca)$ . Then for  $1 \leq i \leq M$  and  $1 \leq j \leq n(l) - 1$  we have

$$\begin{aligned}
\tau(u^j \varphi_i g_j) &= \tau(u^j \varphi_i^{1/2} \varphi_i^{1/2} g_j) \\
&= \tau(u^j \varphi_i^{1/2} g_j \varphi_i^{1/2}) \\
&= \tau(\varphi_i^{1/2} u^j \varphi_i^{1/2} g_j) \\
&= \tau(\varphi_i^{1/2} \beta^j(\varphi_i^{1/2}) u^j g_j) \\
&= 0,
\end{aligned}$$

which implies that  $\tau(u^j g_j) = \sum_{i=1}^M \tau(u^j \varphi_i g_j) = 0$ . Similarly,

$$\begin{aligned}
\tau(g_{-j} \varphi_i u^{-j}) &= \tau(g_{-j} \varphi_i^{1/2} \varphi_i^{1/2} u^{-j}) \\
&= \tau(g_{-j} \varphi_i^{1/2} \varphi_i^{1/2} u^{-j}) \\
&= \tau(\varphi_i^{1/2} g_{-j} \varphi_i^{1/2} u^{-j}) \\
&= \tau(g_{-j} \varphi_i^{1/2} u^{-j} \varphi_i^{1/2}) \\
&= \tau(g_{-j} \varphi_i^{1/2} \beta^{-j}(\varphi_i^{1/2}) u^{-j}) \\
&= 0,
\end{aligned}$$

which implies that  $\tau(g_{-j} u^{-j}) = \sum_{i=1}^M \tau(g_{-j} \varphi_i u^{-j}) = 0$ . It follows that  $\tau(g) = \tau(g_0)$ , as required.  $\square$

As it currently stands, this might seem uninteresting because Corollary IV.21, which is used in the proof, requires both that  $h$  be uniquely ergodic and that  $A$  have a unique tracial state. This implies that  $C(X, A)$  has a unique  $\beta$ -invariant tracial state (namely  $\mu \otimes \tau$  with  $\mu$  the unique  $h$ -invariant Borel probability measure on  $X$ , and  $\tau$  the unique tracial state of  $A$ ). However, we

expect our results to hold in a far greater degree of generality than what has been proven here (both the assumption of unique ergodicity on  $h$  and that  $A$  has a unique tracial state should ultimately not be required), and so we prove Proposition VI.7 in its stated form as it would apply to this more general situation without any change in the argument.

The next three lemmas will allow us to replace an arbitrary non-zero positive element of  $B_{\{y\}}$  with a non-zero positive element of  $C(X)$  in part (4d) of Definition VI.4. The first two are analogues of Lemmas 3.3 and 3.4 of [43], and both proofs are adapted from there with little change.

**Lemma VI.8.** *Adopt Notation VI.5, let  $a \in C(X, A)[\mathbb{Z}]$ , and let  $\varepsilon > 0$ . Then there is an  $f \in C(X)$  such that*

$$0 \leq f \leq 1, \quad fa^*af \in C(X, A), \quad \text{and} \quad \|fa^*af\| \geq \|E(a^*a)\| - \varepsilon.$$

*Proof.* Set  $b = a^*a$ . If  $E(b) \leq \varepsilon$ , we can take  $f = 0$ , so assume that  $E(b) > \varepsilon$ . Then there are  $N \in \mathbb{N}$  and  $b_k \in C(X, A)$  for  $-N \leq k \leq N$  such that  $b = \sum_{k=-N}^N b_k u^k$ . Moreover,  $E(b) > \varepsilon$  implies  $b_0$  is a non-zero positive element of  $C(X, A)$ . Define

$$U = \{x \in X : \|b_0(x)\| > \|E(b)\| - \varepsilon\},$$

which is a non-empty open subset of  $X$ . Using the freeness of the action of  $h$  on  $X$ , choose a non-empty open set  $W \subset U$  such that the sets  $h^k(W)$  are pairwise disjoint for  $-N \leq k \leq N$ , and fix some  $x_0 \in W$ . Choose  $f: X \rightarrow [0, 1]$  such that  $f(x_0) = 1$  and  $\text{supp}(f) \subset W$ . Then with  $T = \{-N, \dots, N\} \setminus \{0\}$ , we have

$$fa^*af = fbf = fb_0f + \sum_{k \in T} fb_k u^k f = fb_0f + \sum_{k \in T} fb_0 \beta^k(f) u^k$$

Since the sets  $\text{supp}(\beta^k(f))$  are disjoint for  $-N \leq k \leq N$ , it follows that  $fb_k \beta^k(f) = b_k f \beta^k(f) = 0$  for  $k \in T$ . Thus  $fa^*af = fb_0f \in C(X, A)$  as required. Finally,

$$\|fa^*af\| = \|fb_0f\| \geq \|f(x_0)b_0(x_0)f(x_0)\| = \|b_0(x_0)\| > \|E(a^*a)\| - \varepsilon,$$

which completes the proof. □

**Lemma VI.9.** *Adopt Notation VI.5, let  $y \in Y$ , and let  $a \in (B_{\{y\}})_+ \setminus \{0\}$ . Then there is a  $b \in C(X, A)_+ \setminus \{0\}$  with  $b \lesssim a$  relative to the subalgebra  $B_{\{y\}}$ .*

*Proof.* Without loss of generality, assume that  $\|a\| \leq 1$ . Since  $a$  is non-zero and  $E$  is faithful, we have  $E(a) > 0$ . Set  $\varepsilon = \frac{1}{6} \|E(a)\|$ . Since  $B_{\{y\}} \cap C(X, A)[\mathbb{Z}]$  is dense in  $B_{\{y\}}$ , there is a  $c \in B_{\{y\}} \cap C(X, A)[\mathbb{Z}]$  such that  $\|c - a^{1/2}\| < \varepsilon$ . Then  $\|cc^* - a\| < 2\varepsilon$  and  $\|c^*c - a\| < 2\varepsilon$ . Apply Lemma VI.8 with  $c$  and  $\varepsilon$ , obtaining  $f \in C(X)$  such that

$$0 \leq f \leq 1, \quad fc^*cf \in C(X, A), \quad \text{and} \quad \|fc^*cf\| \geq \|E(c^*c)\| - \varepsilon.$$

The third property gives  $\|fc^*cf\| > \|E(a)\| - 3\varepsilon = 3\varepsilon$ , and so  $(fc^*cf - 2\varepsilon)_+$  is a nonzero positive element of  $C(X, A)$ . By Lemma VI.3(1), it follows that  $(fc^*cf - 2\varepsilon)_+ \sim (cf^2c^* - 2\varepsilon)_+$ . Since  $f^2 \leq 1$ , we have  $cf^2c^* \leq cc^*$ , and combining this with Lemma VI.3(3) gives  $(cf^2c^* - 2\varepsilon)_+ \lesssim (cc^* - 2\varepsilon)_+$ . Finally,  $\|cc^* - a\| < 2\varepsilon$  implies that  $(cc^* - 2\varepsilon)_+ \lesssim a$ . Putting these statements together, we conclude that  $(fc^*cf - 2\varepsilon)_+ \lesssim a$ . This gives the desired positive element of  $C(X, A)$ .  $\square$

**Lemma VI.10.** *Adopt Notation VI.5, let  $y \in Y$ , and let  $b \in C(X, A)_+ \setminus \{0\}$ . Then there is an  $f \in C(X)_+ \setminus \{0\}$  with  $f \lesssim b$  relative to the subalgebra  $B_{\{y\}}$ .*

*Proof.* Without loss of generality, assume that  $\|b\| = 1$ . Choose a point  $x_0 \in X \setminus \{y\}$  such that  $\|b(x_0)\| = 1$  and an open set  $V_0 \subset \text{supp}(b)$  such that  $x_0 \in V_0$  and  $y \notin V_0$ . Choose a continuous function  $b_0: X \rightarrow [0, 1]$  such that  $b_0(x_0) = 1$  and  $\text{supp}(b_0) \subset V_0$ . Set  $e = b_0b$ , and observe that  $e \leq b$ . By Proposition IV.14, there exist an open set  $V \subset \text{supp}(e)$ , a non-zero projection  $p \in A$ , and a unitary  $w \in C(X, A)$  such that  $waw^* \in \overline{eC(X, A)e}$  for all  $a \in \text{Her}(V, p)$ . Notice that  $y \notin \bar{V}$  by construction.

By Proposition IV.12, there exist  $M \in \mathbb{N}$  and  $\varepsilon > 0$  such that whenever  $g \in C(X)$  is positive with  $\mu(\text{supp}(g)) < \varepsilon$ , then there exist, for  $0 \leq k \leq M$ , positive elements  $a_k \in C(X, A)$ , unitaries  $w_k \in C(X, A)$ , and  $r(k) \in \mathbb{Z}$  such that:

1.  $g = g \otimes 1 \leq \sum_{k=0}^M a_k$ ;
2. the elements  $\beta^{r(k)}(a_k)$  are mutually orthogonal, and  $\text{supp}(\beta^{r(k)}(a_k)) \subset V$  for each  $k$ ;
3. with  $b_k = w_k \beta^{r(k)}(a_k) w_k^*$ , the  $b_k$  are mutually orthogonal positive elements in  $\text{Her}(V, p)$ .

Choose a point  $x_0 \in V$  and an open set  $W \subset V$  such that  $x_0 \in W$  and  $\mu(W) < \varepsilon$ . Choose a continuous function  $g: X \rightarrow [0, 1]$  such that  $g(x_0) = 1$  and  $\text{supp}(g) \subset W$ . Then  $\mu(\text{supp}(g)) < \varepsilon$ , and so Proposition IV.12 yields positive elements  $a_k \in C(X, A)$ , unitaries  $w_k \in C(X, A)$ , and  $r(k) \in \mathbb{Z}$  with the aforementioned properties. Let  $N = \max \{|r(k)| : 0 \leq k \leq M\}$ . For  $0 \leq k \leq M$  and  $-N \leq j \leq N$ , let  $U_{j,k}^{(0)}$  be an open neighborhood of  $y$  such that

$$\mu(U_{j,k}^{(0)}) < \frac{\mu(W)}{2(2N+1)(M+1)+1}.$$

For  $0 \leq k \leq M$  and  $-N \leq j \leq N$ , choose an open neighborhood  $U_{j,k}$  of  $y$  such that  $\bar{U}_{j,k} \subset U_{j,k}^{(0)}$ , and set

$$U^{(0)} = \bigcup_{k=0}^M \bigcup_{j=-N}^N h^{-j}(U_{j,k}^{(0)}) \quad \text{and} \quad U = \bigcup_{k=0}^M \bigcup_{j=-N}^N h^{-j}(U_{j,k}).$$

Then  $\bar{U} \subset U^{(0)}$ , and

$$\mu(U^{(0)}) \leq \sum_{k=0}^M \sum_{j=-N}^N \mu(U_{j,k}^{(0)}) < (M+1)(2N+1) \left( \frac{\mu(W)}{2(2N+1)(M+1)+1} \right) < \frac{1}{2}\mu(W).$$

It follows that  $\mu(W \setminus \bar{U}) > 0$ . Now choose  $x_1 \in W \setminus \bar{U}$  and an open neighborhood  $W_1$  of  $x_1$  such that  $W_1 \subset W$  and  $W_1 \cap \bar{U} = \emptyset$ . Choose a continuous function  $f_1: X \rightarrow [0, 1]$  such that  $f_1(x_1) = 1$  and  $\text{supp}(f_1) \subset W_1$ . Set  $f = f_1 g$ , and for  $0 \leq k \leq M$ , set  $s_k = f_1 a_k$  and  $t_k = w_k \beta^{r(k)}(s_k) w_k^*$ . Finally, set

$$S = \bigcap_{k=0}^M \bigcap_{j=-N}^N U_{j,k},$$

which is an open neighborhood of  $y$ . Then we claim that:

1.  $f \leq \sum_{k=0}^M s_k$ ;
2. the elements  $\beta^{r(k)}(s_k)$  are mutually orthogonal, and  $\text{supp}(\beta^{r(k)}(s_k)) \subset V$  for each  $k$ ;
3. with  $t_k = w_k \beta^{r(k)}(s_k) w_k^*$ , the  $t_k$  are mutually orthogonal positive elements in  $\text{Her}(V, p)$ ;
4. for  $0 \leq k \leq M$  and  $|j| \leq |r(k)|$ , we have  $\beta^j(s_k)(x) = 0$  for all  $x \in S$ .

The first three statements follow immediately. To prove property (4), suppose  $|j| \leq |r(k)|$ . Then

$$\text{supp}(\beta^j(s_k)) \subset \text{supp}(f_1 \circ h^{-j}) \subset h^j(W_1).$$

If  $x \in S$ , then  $x \in U_{j,k}$  and hence  $h^{-j}(x) \in h^{-j}(U_{j,k})$ . This implies  $h^{-j}(x) \in U$ , and so  $h^{-j}(x) \notin W_1$ . Thus  $x \notin h^j(W_1)$ , and consequently  $x \notin \text{supp}(\beta^j(s_k))$ . This verifies (4).

Next, we claim that  $\beta^{r(k)}(s_k) \sim s_k$  in  $B_{\{y\}}$  for  $0 \leq k \leq M$ . If we write  $v_k = u^{r(k)}s_k^{1/2}$ , then  $v_kv_k^* = \beta^{r(k)}(s_k)$  and  $v_k^*v_k = s_k$ . So it suffices to prove that  $v_k \in B_{\{y\}}$ . First assume that  $r(k) > 0$ . Since  $\text{supp}(s_k^{1/2}) = \text{supp}(s_k)$ , we have  $\beta^j(s_k^{1/2})(x) = 0$  for all  $j$  such that  $0 \leq j \leq r(k)$  and all  $x \in S$ . Choose an open neighborhood  $S_0$  of  $y$  such that  $\bar{S}_0 \subset S$  and  $\bar{S}_0 \cap \text{supp}(\beta^j(s_k^{1/2})) = \emptyset$  for  $0 \leq j \leq r(k)$ . Choose a continuous function  $\varphi: X \rightarrow [0, 1]$  such that  $\varphi = 1$  on  $\text{supp}(s_k^{1/2})$  and  $\text{supp}(\varphi) \subset X \setminus \bigcup_{j=0}^{r(k)} h^{-j}(S_0)$ . Then  $\varphi s_k^{1/2} = s_k^{1/2}$ , and  $\psi = \varphi^{1/r(k)}$  is continuous. Now, we may write

$$u^{r(k)}s_k^{1/2} = u^{r(k)}\varphi s_k^{1/2} = \left(u\beta^{r(k)-1}(\psi)\right) \left(u\beta^{r(k)-2}(\psi)\right) \cdots \left(u\beta(\psi)\right) (u\psi) s_k^{1/2}.$$

Now  $\beta^j(\psi)(y) = 0$  for  $0 \leq j \leq r(k) - 1$ , since  $\text{supp}(\psi) = \text{supp}(\varphi)$ . Thus  $u\beta^j(\psi) \in B_{\{y\}}$  for  $0 \leq j \leq r(k)$ , and  $s_k^{1/2} \in B_{\{y\}}$ . It follows that  $u^{r(k)}s_k^{1/2} \in B_{\{y\}}$ .

Now if  $r(k) < 0$ , we can write

$$u^{r(k)}s_k^{1/2} = \beta^{r(k)}(s_k^{1/2})u^{r(k)} = \left(u^{-r(k)}\beta^{r(k)}(s_k^{1/2})\right)^*.$$

Let  $d(k) = -r(k)$  and  $e_k = \beta^{r(k)}(s_k)$ . Then  $d(k) > 0$ , and  $\beta^j(e_k^{1/2}) = \beta^{j-d(k)}(s_k^{1/2})$ . For all  $j$  such that  $0 \leq j \leq d(k)$ , we have  $-N \leq j - d(k) \leq 0$ . For any  $i$  with  $-N \leq i \leq 0$ , we have  $\beta^i(s_k^{1/2})(x) = 0$  for all  $x \in S$ . This implies that  $\beta^j(e_k^{1/2})(x) = 0$  for all  $j$  with  $0 \leq j \leq d(k)$  and  $x \in S$ . Applying the previous argument with  $d(k)$  in place of  $r(k)$  and  $e_k^{1/2}$  in place of  $s_k^{1/2}$ , we obtain  $u^{d(k)}e_k^{1/2} \in B_{\{y\}}$ , and this in turn gives  $u^{-r(k)}\beta^{r(k)}(s_k^{1/2}) \in B_{\{y\}}$ . Since  $B_{\{y\}}$  is closed under adjoints, it follows that  $u^{r(k)}s_k^{1/2} \in B_{\{y\}}$ . This completes the proof that  $v_k \in B_{\{y\}}$  for  $0 \leq k \leq M$ .

Finally, we have  $w_k \in B_{\{y\}}$  for  $0 \leq k \leq M$ , and so  $z_k = w_kv_k \in B_{\{y\}}$ . Then  $z_k z_k^* = w_k \beta^{r(k)}(s_k) w_k^* = t_k$  and  $z_k^* z_k = v_k^* v_k = s_k$ . By part (6) of Lemma VI.3, it follows that  $t_k \sim s_k$  with equivalence in  $B_{\{y\}}$ . Further,  $w \in B_{\{y\}}$ , and so part (5) of Lemma VI.3 implies that  $wt_k w^* \sim t_k \sim s_k$  relative to  $B_{\{y\}}$ . Moreover, the elements  $wt_k w^*$  are orthogonal, and  $\sum_{k=0}^M wt_k w^* \in \overline{eC(X, A)e}$  since each  $t_k$  is an element of  $\text{Her}(V, p)$ . Part (5) of Lemma VI.3 then implies that  $\sum_{k=0}^M wt_k w^* \lesssim e$  relative to  $B_{\{y\}}$ . We conclude that  $f \lesssim e$  relative to  $B_{\{y\}}$ . Since  $e \leq b$ , we have  $f \lesssim b$  relative to  $B_{\{y\}}$ .  $\square$



**Theorem VI.11.** *Adopt Notation VI.5, and let  $y \in X$ . Then  $B_{\{y\}}$  is a large subalgebra of  $B$ .*

*Proof.* As previously mentioned, condition (1) follows immediately from the definition of  $B_{\{y\}}$ , while conditions (2) and (3) are given by Propositions VI.6 and VI.7 respectively. It remains to prove that condition (4) holds. Let  $\varepsilon > 0$ ,  $m \in \mathbb{N}$ ,  $a_1, \dots, a_m \in B$ , and  $b \in (B_{\{y\}})_+ \setminus \{0\}$  be given. Choose  $N \in \mathbb{N}$  such that, for  $1 \leq k \leq m$ , there exist  $c_{jk} \in C(X, A)$  for  $-N \leq j \leq N$  with

$$\left\| a_k - \sum_{j=-N}^N c_{jk} u^j \right\| < \varepsilon.$$

For  $1 \leq k \leq m$ , set

$$c_k = \sum_{j=-N}^N c_{jk} u^j.$$

Then  $\|a_k - c_k\| < \varepsilon$  for  $1 \leq k \leq m$ , which is condition (4b).

Next, use the simplicity of  $B_{\{y\}}$  and Lemma 1.9 of [43] to find nonzero orthogonal positive elements  $y_j \in B_{\{y\}}$  for  $-N \leq j \leq N$  such that  $y_j \sim y_l$  for all  $j, l \in \{-N, \dots, N\}$  and such that  $\sum_{j=-N}^N y_j \in \overline{bB_{\{y\}}b}$ . Apply Lemmas VI.9 and VI.10 to obtain  $z_j \in C(X)_+ \setminus \{0\}$  such that  $z_j \lesssim y_j$  for  $-N \leq j \leq N$ . Apply Lemma 3.5 of [43] to obtain open sets  $V_j \subset X$  for  $-N \leq j \leq N$  such that  $h^j(y) \in V_j$  and such that whenever  $f \in C(X)$  satisfies  $\text{supp}(f) \subset V_j$ , then  $f \lesssim z_j$ . Choose an open set  $W \subset X$  such that  $y \in W$ , such that the sets  $h^j(W)$  are pairwise disjoint for  $-N \leq j \leq N$ , and such that  $h^j(W) \subset V_j$  for  $-N \leq j \leq N$ . Choose a continuous function  $g_0: X \rightarrow [0, 1]$  such that  $g_0(y) = 1$  and  $\text{supp}(g_0) \subset W$ . Finally, set

$$g = \sum_{j=-N}^N \beta^j(g_0)$$

Then  $0 \leq g \leq 1$ , which verifies condition (4a), and  $g \lesssim b$  relative to  $B_{\{y\}}$ , which verifies condition (4d).

To complete the proof, we need to verify condition (4c); that is, show that  $(1-g)c_k \in B_{\{y\}}$  and  $c_k(1-g) \in B_{\{y\}}$  for  $1 \leq k \leq m$ . Since

$$c_k = \sum_{j=-N}^N c_{jk} u^j = \sum_{j=-N}^N u^j \beta^{-j}(c_{jk}),$$

it is sufficient to verify that  $u^j(1-g) \in B_{\{y\}}$  and  $(1-g)u^j \in B_{\{y\}}$  for  $-N \leq j \leq N$ . First assume

that  $0 \leq j \leq N$ . When  $j = 0$  this is immediate. Now suppose that  $0 < j \leq N$ . Since  $g_0(y) = 1$ , it follows that  $u(1 - g_0) \in B_{\{y\}}$ . Observe that  $\beta^i(g_0)\beta^j(g_0) = 0$  for  $-N \leq i, j \leq N$  and  $i \neq j$  by the disjointness of the sets  $h^i(W)$  and  $h^j(W)$ . This implies that we can write

$$1 - g = 1 - \sum_{j=-N}^N \beta^j(g_0) = \prod_{j=-N}^N (1 - \beta^j(g_0)).$$

Then we have

$$[u(1 - g_0)]^j = u^j (1 - \beta^{j-1}(g_0)) (1 - \beta^{j-2}(g_0)) \cdots (1 - \beta(g_0)) (1 - g_0).$$

Set  $T_j = \{-N, \dots, -1\} \cup \{j, \dots, N\}$ . Then we can write

$$u^j(1 - g) = [u(1 - g_0)]^j \prod_{i \in T_j} (1 - \beta^i(g_0)).$$

Since  $u(1 - g_0) \in B_{\{y\}}$ , we have  $[u(1 - g_0)]^j \in B_{\{y\}}$ , and certainly  $\prod_{i \in T_j} (1 - \beta^i(g_0)) \in B_{\{y\}}$ . It follows that  $u^j(1 - g) \in B_{\{y\}}$ . Analogously, we may write

$$[u(1 - g_0)]^j = (1 - \beta^{-1}(g_0)) (1 - \beta^{-2}(g_0)) \cdots (1 - \beta^{-j}(g_0)) u^j$$

and set  $T'_j = \{-N, \dots, -j + 1\} \cup 0, \dots, N$ . Then we have

$$(1 - g)u^j = \left( \prod_{i \in T'_j} (1 - \beta^i(g_0)) \right) [u(1 - g_0)]^j,$$

and so  $(1 - g)u^j \in B_{\{y\}}$ . Finally, if  $-N \leq j < 0$ , then we may write  $(1 - g)u^j = (u^{-j}(1 - g))^*$  and  $u^j(1 - g) = ((1 - g)u^{-j})^*$ . Using the previous argument and the fact that  $B_{\{y\}}$  is closed under adjoints, it follows that  $(1 - g)u^j, u^j(1 - g) \in B_{\{y\}}$  for  $-N \leq j < 0$ . This completes the verification of condition (4c), and completes the proof.  $\square$

The following definition is a simplified form of a more general definition, introduced in [50], where the tracial state space  $T(A)$  is replaced by the set  $QT(A)$  of normalized quasitraces on  $A$ . Since our  $C^*$ -algebras of interest are nuclear, these two sets are equal in our situation.

**Definition VI.12.** Let  $A$  be a stably finite unital nuclear  $C^*$ -algebra. For  $\tau \in T(A)$ , define  $d_\tau: M_\infty(A)_+ \rightarrow [0, \infty)$  by

$$d_\tau(a) = \lim_{n \rightarrow \infty} \tau(a^{1/n})$$

for all  $a \in M_\infty(A)_+$ .

1. For  $r \in [0, \infty)$ , we say that  $A$  has  $r$ -comparison if whenever  $a, b \in M_\infty(A)$  satisfy  $d_\tau(a) < r + d_\tau(b)$  for all  $\tau \in T(A)$ , then  $a \preceq b$ .
2. The radius of comparison of  $A$ , denoted  $\text{rc}(A)$ , is the number

$$\text{rc}(A) = \inf \{r \in [0, \infty) : A \text{ has } r\text{-comparison}\}.$$

If this set is empty ( $A$  does not have  $r$ -comparison for any  $r \geq 0$ ), then we define  $\text{rc}(A) = \infty$ .

3. If  $\text{rc}(A) = 0$ , we say that  $A$  has strict comparison of positive elements.

**Proposition VI.13.** For  $y \in X$ , we have  $\text{rc}(B) \leq \text{rc}(B_{\{y\}})$ .

*Proof.* We have already seen that  $B_{\{y\}}$  is large in  $B$  by Theorem VI.11. Since  $B$  is nuclear, Definition VI.4 is equivalent to Definition 2.2 of [43]. Therefore Lemma 2.4 of [43] implies that  $B_{\{y\}}$  is also quasitracially large in the sense of Definition 2.1 there. Now the stated result follows by Theorem 4.5 of [43].  $\square$

We conclude by presenting classification results for  $B_{\{y\}}$  and  $B$  that we have not yet been able to prove. The first of these, at the very least, seems reasonably accessible and can be combined with our results to produce useful new ones.

**Conjecture VI.14.** If  $Y \subset X$  is closed with  $\text{int}(Y) \neq \emptyset$ , then  $B_Y = C^*(\mathbb{Z}, C(X, A), \beta)_Y$  has strict comparison of positive elements.

If this result holds, then we obtain strong information about the structure of the Cuntz semigroups for  $B_{\{y\}}$  and  $B$ .

**Theorem VI.15.** Suppose that  $y \in X$ , and that Conjecture VI.14 holds. Then  $B_{\{y\}}$  has strict comparison of positive elements. Consequently,  $B$  has strict comparison of positive elements as well.

*Proof.* By Corollary V.14 and Proposition VI.6,  $B_{\{y\}}$  is a simple direct limit of a unital direct system  $(A_n, \phi_n)$ , where each  $A_n$  is a recursive  $A$ -subhomogeneous algebra of the form  $A_n = B_{Y_n}$  for some  $Y_n \subset X$  closed with  $\text{int}(Y_n) \neq \emptyset$ . If the result of Conjecture VI.14 holds, then each  $A_n$  has strict comparison of positive elements, so that  $\text{rc}(A_n) = 0$  for all  $j$ . Then

$$\liminf_{n \rightarrow \infty} \text{rc}(A_n) = 0,$$

and Theorem 5.3 of [52] implies that  $B_{\{y\}}$  has strict comparison of positive elements. Now Proposition VI.13 implies that  $B$  has strict comparison of positive elements as well.  $\square$

It seems likely that a direct proof of Theorem VI.15 can be given, so that  $B_{\{y\}}$  has strict comparison of positive elements, even if it turns out that  $B_Y$  does not have strict comparison of positive elements for more general sets  $Y$ . If such a direct argument does exist, it is also possible that it can be adapted to show that  $B_Y$  has strict comparison of positive elements when  $Y$  is a finite set consisting of points with disjoint orbits.

The interest in the Cuntz semigroup lies in its usefulness as an invariant in the classification theory of simple, separable, nuclear  $C^*$ -algebras; in particular, it can distinguish between certain  $C^*$ -algebras with the same Elliott invariant. However, it can be considerably more difficult to compute. (See [51] for a discussion of its importance to classification theory and an example that justifies the claim about its computability.) Strict comparison of positive elements allows us to identify the Cuntz semigroup of a  $C^*$ -algebra with a more tractable set. More precisely, let  $A$  be a simple, unital, nuclear, stably finite  $C^*$ -algebra, let  $V(A)$  be its Murray-von Neumann semigroup of projections in  $M_\infty(A)$  (this is a subsemigroup of  $W(A)$ ), and let  $\text{LAff}_b(T(A))_{++}$  denote the set of lower semicontinuous real affine functions on  $T(A)$  that are bounded and strictly positive. Then we define a map  $\iota: W(A) \rightarrow \text{LAff}_b(T(A))_{++}$  by  $\iota(\langle a \rangle)(\tau) = d_\tau(a)$ , where  $d_\tau(a)$  is defined in Definition VI.12. Then if  $A$  has strict comparison of positive elements, the map

$$\text{id} \sqcup \iota: V(A) \sqcup W(A) \rightarrow V(A) \sqcup \text{LAff}_b(T(A))_{++}$$

is a semigroup order embedding by Theorem 5.6 of [52]. Thus in the case where  $A$  has strict comparison of positive elements,  $W(A)$  is identified in a structure-preserving way with a subset of  $\text{LAff}_b(T(A))_{++}$ . An even more powerful result that we hope is true would be a generalization

of Theorem 0.2 of [53]. Let  $\mathcal{Z}$  denote the *Jiang-Su Algebra*, which is a simple, separable, unital, infinite-dimensional, nuclear  $C^*$ -algebra having the same K-theory as the complex numbers  $\mathbb{C}$ , and is strongly self-absorbing (in particular,  $\mathcal{Z} \otimes \mathcal{Z} \cong \mathcal{Z}$ ). A  $C^*$ -algebra  $A$  is called  $\mathcal{Z}$ -stable if there is an isomorphism  $A \otimes \mathcal{Z} \cong A$ . The property of  $\mathcal{Z}$ -stability appears to be intimately connected to the question of whether or not a simple, separable, nuclear  $C^*$ -algebra is classified by its Elliott invariant. Again, see [51] for a discussion of this.

**Conjecture VI.16.** *The crossed product  $C^*$ -algebras  $C^*(\mathbb{Z}, C(X, A), \beta)$  are  $\mathcal{Z}$ -stable; that is, there is an isomorphism*

$$C^*(\mathbb{Z}, C(X, A), \beta) \otimes \mathcal{Z} \cong C^*(\mathbb{Z}, C(X, A), \beta).$$

Whether Conjecture VI.16 is true or not is much less certain than Conjecture VI.14. Winter [56] believes that the problem is likely to be very difficult. In particular, one must show that for each  $x \in X$ , the crossed product  $C^*(\mathbb{Z}, A, \alpha_x)$  is  $\mathcal{Z}$ -stable. To proceed in the same manner as [53], we must also be able to obtain information about the decomposition rank of the algebras  $B_{\{y\}}$  and  $B_{\{y_1, y_2\}}$  (where  $y_1 \neq y_2$ ). It is far from clear that this is possible, and a worthwhile question in its own right.

**Conjecture VI.17.** *For  $Y \subset X$ , with  $Y = \{y\}$  or  $Y = \{x, y\}$  where  $x \neq y$ , the  $C^*$ -algebra  $B_Y$  has finite decomposition rank in the sense of [57]. The formal definition of decomposition rank is quite technical, but it should be thought of as a version of noncommutative covering dimension; in particular,  $\text{dr}(C(X)) = \dim(X)$ .*

The desired result for the structure of the crossed product  $C^*$ -algebras  $C^*(\mathbb{Z}, C(X, A), \beta)$  is an analogue of the main theorem from [24]. In order to carefully state it, we require some additional machinery.

**Definition VI.18.** *For a compact convex set  $\Delta$ , let  $\text{Aff}(\Delta)$  denote the space of all continuous affine functions  $f: \Delta \rightarrow \mathbb{R}$ . For a  $C^*$ -algebra  $A$ , let  $V(A)$  be its Murray von-Neumann semigroup, and let  $K_0(A)$  be the Grothendieck group of  $V(A)$ . Define a map*

$$\rho_A: K_0(A) \rightarrow \text{Aff}(T(A))$$

by  $\rho_A([\eta])(\tau) = \tau(\eta)$ .

**Conjecture VI.19.** *Suppose that the map  $\rho_B: K_0(B) \rightarrow \text{Aff}(T(B))$  of Definition VI.18 has dense range. Then  $B = C^*(\mathbb{Z}, C(X, A), \beta)$  is a simple unital  $C^*$ -algebra with tracial rank zero that satisfies the Universal Coefficient Theorem (compare with Theorem 4.6 of [24]).*

An affirmative answer to this conjecture would provide a large new collection of classifiable  $C^*$ -algebras, arising as the crossed product  $C^*$ -algebras of algebras which are neither commutative, nor simple, nor necessarily containing many projections. Previous classification work on crossed products has frequently assumed at least one of these conditions on the underlying  $C^*$ -algebra. As we have seen, the tracial quasi-Rokhlin property was formulated specifically for the study of such  $C^*$ -algebras and their associated crossed products.

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