GENERALIZED SELF-INTERSECTION LOCAL TIME FOR A SUPERPROCESS OVER A STOCHASTIC FLOW

by

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This dissertation examines the existence of the self-intersection local time for a superprocess over a stochastic flow in dimensions $d \leq 3$, which through constructive methods, gives a Tanaka like representation. The superprocess over a stochastic flow is a superprocess with dependent spatial motion, and thus Dynkin's proof of existence, which requires multiplicity of the log-Laplace functional, no longer applies. Skoulakis and Adler's method of calculating moments is extended to higher moments, from which existence follows.

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CHAPTER I

INTRODUCTION

Superprocesses, originally studied by Watanabe [29] and Dawson [6],[7] were first shown by Dynkin [9] to have a self-intersection local time (SILT). In particular, Dynkin was able to show existence of the self-intersection local time for super Brownian motion in $\mathbb{R}^d$, $d \leq 7$, provided the SILT is defined over a region that is bounded away from the diagonal. When the region contains any part of the diagonal, through renormalization, the SILT for super Brownian motion has been shown [2] to exist in $d \leq 3$, and further renormalization processes have been found to establish existence in higher dimensions by Rosen [22] and Adler & Lewin [1]. In regards to non-Gaussian superprocesses, the SILT has been shown to exist for certain $\alpha-$stable processes by Adler & Lewin [1], and more recently, encompassing more α values, by Mytnik & Villa [20]. Of important note, as the L^2-limit of an appropriate approximating process, Adler & Lewin have shown the existence of a class of renormalized SILT's (indexed on $\lambda > 0$) for the super Brownian motion in dimensions $d = 4$ and 5 and for the super α-stable processes for $d \in [2\alpha, 3\alpha)$. As one removes Dynkin's restriction of bounding away from the diagonal, a singularity arises from "local double points" (that is $\mu_s \times \mu_t$ where $t = s$) of the process (cf. [2]). The true self-intersection local time should not be concerned with such local double points, and thus a heuristic approach to renormalization is naturally observed in the construction. It should be noted that though this is the method used in [1] and [2], a quite different method for renormalization was developed in [22]. Both methods are legitimate renormalizations, and lead to existence in equivalent dimensions, but for this dissertation, due to the natural occurrence of the term involving local double points, the initial of the two methods will be employed. Moreover, the real beauty of this constructive proof of existence, as seen in [1] & [2], is the aforementioned approximating process is "Tanaka-like" in form. Thus the limit gives a (quite simple) "Tanaka-like" representation for the renormalized SILT.

A major drawback in each of the previous superdiffusions is the requirement of independent spatial motion. Existence as a weak limit of a branching particle system, and uniqueness as the solution to a martingale problem, of the superprocess with dependent spatial motion (SDSM), as a measure-valued Markov process with state space $M(\hat{\mathbb{R}})$,was shown by Wang [28]. It was later shown by Dawson, Li, & Wang [5] to exist uniquely as a process in $M(\mathbb{R})$, and then extended by Ren, Song, & Wang [21] to $M(\mathbb{R}^d)$. G. Skoulakis & R.J. Adler [26] suggested a different model incorporating dependent spatial motion by replacing the space-time white noise of Wang's SDSM with a Brownian flow of homeomorphisms from $\mathbb{R}^d$ to $\mathbb{R}^d$, which was referred to as a Superprocess over a Stochastic Flow (SSF).

As of yet, very little work has been done with regard to the self-intersection local time for superprocesses with dependent spatial motion. Of important note is the work of He [14], in which the existence of the SILT for a superprocess with dependent spatial motion, similar to the model of Wang but discontinuous, is shown to exist in one dimension as a probabilistic limit. Though this was known to be true, since the local time of the superprocess with dependent spatial motion was known to exist in one dimension (cf. [5]), He was able to give a similar "Tanaka-like" representation for the SILT. This dissertation will investigate the existence and further properties of a generalized SILT for the d-dimensional SSF, where the generalization refers to the shift of the support of the Dirac measure away from the origin, to a point $u \in \mathbb{R}^d$. Note that if X_t is a Markov process, then $Y_t \triangleq X_t + u$ is a second, dependent Markov Process. The generalized SILT at u can be realized as the intersection local time of the Markov processes X_t and Y_t.

This body of this work is constructed as follows. Beginning with needed background and definitions, we conclude the first chapter with a vital SPDE describing the superprocess over a stochastic flow as a solution to a particular martingale problem. As in the majority of proofs of existence of SILT for superprocesses, higher moment calculations are needed. This is accomplished through the calculation of certain moment formulas for the branching particle system, then limiting to the moments of the superprocess. A similar method was first employed by Skoulakis & Adler [26] for the first and second moments. Though their method worked well for smaller moments, the number of cases to consider for any higher moments of the branching process are too many to make this practical. To get around this difficulty, the moment formulas for the branching process are found at one fixed point in time (for example, with the third moment, instead of calculating $\mathbb{E}\left[\mu_{t_1}^{(n)}(\phi_1)\mu_{t_2}^{(n)}(\phi_2)\mu_{t_3}^{(n)}(\phi_3)\right]$, we would find $\mathbb{E}\left[\mu_t^{(n)}(\phi_1)\mu_t^{(n)}(\phi_2)\mu_t^{(n)}(\phi_3)\right]$), thus greatly reducing the number of cases to consider. By taking limits, the resulting formulae are then used to find the corresponding formulae for the superprocess. Finally, the Markov property is used to extend to moments on varying time parameters. For any $\phi \in C_K^\infty(\mathbb{R}^d)$, taking the $L^2(\mathbb{P})$ norm of

$$\int_0^T dt \int_0^t ds \, \langle \phi, \mu_s \mu_t \rangle \, ,$$

these moment formulae bound the above by $C \|\phi\|_{L^1}$, with the constant C depending only upon T.

Chapter three begins with defining the SILT, which leads to the desire for understanding the expression $\left\langle G_\varepsilon^{\lambda,u}, \mu_s\mu_t \right\rangle$, where $G_\varepsilon^{\lambda,u}$ is a $C_K^\infty(\mathbb{R}^d)$ sequence (in ε), converging in L^1 to the Green's function. Since the SPDE of [26] is of the form $\langle \phi, \mu_t \rangle$, this is employed, along with Itô's Lemma, to construct an Itô formula for $\langle \phi, \mu_s\mu_t \rangle$. This, along with the L^1 bound for the L^2 norm

$$\int_0^T dt \int_0^t ds \, \left\langle G_\varepsilon^{\lambda,u}, \mu_s\mu_t \right\rangle$$

shows L^2 convergence to the desired "Tanaka-like" formula.

I.1 The Branching Particle System And The Superprocess

The SSF is constructed as the weak limit of an $\mathbb{R}^d$ branching particle system. Much of the work that will follow involves using properties of the branching particle system, and thus we will briefly review this construction. This section follows very closely to the work of Skoulakis & Adler [26], and the reader is referenced to this work for further questions. We will let $\overline{\mathbb{R}^d} = \mathbb{R}^d \cup \{\Delta\}$ denote the one-point (Alexandroff) compactification of $\mathbb{R}^d$, where Δ denotes the "cemetery". We extend measurable functions $\phi \in \mathcal{B}(\mathbb{R}^d)$ to $\mathcal{B}(\overline{\mathbb{R}^d})$ by setting $\phi(\Delta) = 0$.

Let $\mathbb{N} = \{1, 2, ...\}$ and set

$$I \triangleq \{\alpha = (\alpha_0, \alpha_1, ..., \alpha_N) : N \geq 0, \alpha_0 \in \mathbb{N}, \alpha_i \in \{0,1\}, 1 \leq i \leq N\},$$

and for any $\alpha = (\alpha_0, ...\alpha_N) \in I$, let $|\alpha| = N$, and $\alpha - i = (\alpha_0, ..., \alpha_{|\alpha|-i})$. In addition, we will write $\alpha \sim_n t$ exactly when $t \in \left[\frac{|\alpha|}{n}, \frac{|\alpha|+1}{n}\right)$. Let $M(n)$ be the number of particles alive at time zero, where the spatial position of each particle is written as $(x_1^n, x_2^n, ..., x_{M(n)}^n)$, and define the initial (atomic) measure by

$$\mu_0^{(n)} \triangleq \sum_{i=1}^{M(n)} \delta_{x_i^n}.$$

For each $n \in \mathbb{N}$, $\{B^{\alpha,n} : \alpha_0 \leq M(n), |\alpha| = 0\}$ is defined to be a family of independent $\mathbb{R}^d$ Brownian motions, stopped at time $t = n^{-1}$, with $B^{\alpha,0} = x_{\alpha_0}^n$. A recursive definition then gives a tree: for each $k \in \mathbb{N}$, let $\{B^{\alpha,n} : \alpha_0 \leq M(n), |\alpha| = k\}$ be a collection of $\mathbb{R}^d$ valued Brownian motions, stopped at time $t = (|\alpha| + 1)n^{-1}$, and conditionally independent given the σ-field generated by $\{B^{\alpha,n} : \alpha_0 \leq M(n), |\alpha| < k\}$ and for which

$$B_t^{\alpha,n} = B_t^{\alpha-1,n}, \qquad t \leq |\alpha|\, n^{-1}$$

In regards to branching, for $n \in \mathbb{N}$ let $\{N^{\alpha,n} : \alpha_0 \leq M(n)\}$ be a family of iid copies of N_n, where N_n is an $\mathbb{N}$−valued random variable such that

$$\mathbb{P}(N_n = k) = \begin{cases} \frac{1}{2}, & k = 2 \\ \frac{1}{2}, & k = 0 \end{cases}.$$

Note that it is implicit in the above that the branching is assumed to be binary, and that for each $n \in \mathbb{N}$,

$$\mathbb{E}N_n = 1,$$

$$\mathbb{E}N_n^2 - (\mathbb{E}N_n)^2 = 1,$$

and

$$\mathbb{E}N_n^q = 2^{q-1}, \quad q \in \mathbb{N}.$$

Moreover, it is assumed that the families $\{B^{\alpha,n} : \alpha_0 \leq M(n)\}$ and $\{N^{\alpha,n} : \alpha_0 \leq M(n)\}$ are independent.

The final component is that of the stochastic flow. Let $b : \mathbb{R}^d \to \mathbb{R}^d$ and $c : \mathbb{R}^d \to M(d,m)$, where $M(d,m)$ is the space of $d \times m$ matrices, $m \in \mathbb{N}$, satisfying the following:

(i) the global Lipschitz condition

$$|b(x) - b(y)| + |c(x) - c(y)| \leq C\,|x - y|\,,$$

for any $x, y \in \mathbb{R}^d$;

(ii) the linear growth condition,

$$|b(x)| + |c(x)| \leq C(1 + |x|),$$

for any $x \in \mathbb{R}^d$;

(iii) for all $i = 1, 2, ..., d$, $j = 1, 2, ..., m$ b_i and c_{ij} are bounded with bounded and continuous first and second partial derivatives.

Assume that $t \mapsto F^n_{s,t}(x)$ is the solution of the stochastic differential equation

$$dY_t = c(Y_t)dW^n_t, \quad Y_s = x$$

for all $t \geq s$ and $x \in \mathbb{R}^d$, where W^n is a $\mathbb{R}^m$-valued Brownian motion, independent of the families $\{B^{\alpha,n}\}$ and $\{N^{\alpha,n}\}$. This defines a unique Brownian flow of homeomorphisms from $\mathbb{R}^d \to \mathbb{R}^d$ [26].

Set $a_n = n^{-1}$ and $k_n = kn^{-1}$. Then the tree of Brownian motions over the flow is given by the family of processes $Y^{\alpha,n}$, defined by: Let $\alpha \sim_n k_n$ for some $k \in \mathbb{N}$. Over the time interval $[0, k_n + a_n]$, $Y^{\alpha,n}$ is defined to be the solution of the d-dimensional stochastic differential equation:

$$\begin{aligned} dY_t = b(Y_t)dB^{\alpha,n}_t + c(Y_t)dW^n_t&, \\ Y_0 = x^n_{\alpha_0}&. \end{aligned}$$

Note that existence and strong uniqueness of the aforementioned solution is ensured due to the assumed conditions on b and c. Now set $Y^{\alpha,n}_t = Y^{\alpha,n}_{k_n+a_n}$ for $t > k_n + a_n$ and note that due to construction,

$$Y^{\alpha,n}_t = Y^{\alpha-1,n}_t$$

for $0 \leq t \leq k_n$, $k \in \mathbb{N}$.

We now define the stopping times $\tau^{\alpha,n}$ as follows: for each $\alpha \in I$, let

$$\tau^{\alpha,n} = \begin{cases} 0, & \text{if } \alpha_0 > K_n, \\ \min\left\{\frac{i+1}{n} : 0 \leq i \leq |\alpha|, N^{\alpha|_i,n} = 0\right\}, & \text{if this set is not } \varnothing \text{ and } \alpha_0 \leq M(n), \\ \frac{1+|\alpha|}{n}, & \text{otherwise} \end{cases}$$

The stopped tree of processes, with branching, is the family of processes $X^{\alpha,n}$

defined by

$$X_t^{\alpha,n} = \begin{cases} Y_t^{\alpha,n}, & t < \tau^{\alpha,n}, \\ \Delta, & t \geq \tau^{\alpha,n}. \end{cases}$$

The measure-valued process for the finite system of particles is

$$\mu_t^{(n)}(U) = \frac{\#\{\alpha \sim_n t : X_t^{\alpha,n} \in U\}}{n},$$

for $U \in \mathcal{B}(\mathbb{R}^d)$, where for a topological space E, $\mathcal{B}(E)$ denotes the $\sigma-$field of Borel measurable sets in E.

We define the corresponding filtration $\mathcal{F}^n$ by

$$\mathcal{F}_t^n \triangleq \sigma(B^{\alpha,n}, N^{\alpha,n} : |\alpha| < k) \bigvee \sigma(W_s^n : s \leq t) \bigvee \sigma(B_s^{\alpha,n} : s \leq t, |\alpha| = k),$$

for $t \in [k_n, k_n + a_n)$, $k = 0, 1, \ldots$.

Let $C^k(E)$ be the space of continuous functions on E having continuous partial derivatives up to order k, and for $\phi \in C^k(R^d)$ let

$$\partial^k_{i_1 i_2 \cdots i_k}\phi(x) = \left(\frac{\partial^k \phi}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_k}}\right)(x).$$

For $\phi \in C^2(\mathbb{R}^d)$ define the second-order operators L and Λ by

$$(L\phi)(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x,x)\partial^2_{ij}\phi(x) \tag{I.1}$$

and

$$(\Lambda\phi)(x,y) = \sum_{i,j=1}^{d} \sigma_{ij}(x,y)\partial_i\phi(x)\partial_j\phi(y),$$

where

$$a_{ij}(x,y) = \delta_{ij} b_i(x) b_j(y) + \sigma_{ij}(x,y)$$

and

$$\sigma_{ij}(x,y) = \sum_{\ell=1}^{m} c_{i\ell}(x) c_{j\ell}(y)$$

$x, y \in \mathbb{R}^d$, $i, j = 1, ..., d$,

Furthermore, for each $n \in \mathbb{N}$, $\phi \in C^2(\mathbb{R}^{n\times d})$ define the second-order operator L^n by

$$(L^n \phi)(x) = \frac{1}{2} \sum_{p,q=1}^{n} \sum_{i,j=1}^{d} a_{ij}^{pq}(x) \partial_{p_i} \partial_{q_j} \phi(x), \tag{I.2}$$

where

$$a_{ij}^{pq}(x) = \delta_{pq} \delta_{ij} b_i(x_p) b_j(x_q) + \sigma_{ij}(x_p, x_q),$$

$x = (x_1, ..., x_n)$, $x_p \in \mathbb{R}^d$, $p = 1, ..., n$, and $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$. For any operator A on a Banach space $\mathcal{B}$, such that $A\phi = \lim_{t\to 0} t^{-1}\{T_t \phi - \phi\}$ for some semigroup T_t, we will denote by $\mathcal{D}(A) \subset \mathcal{B}$ the domain of A. That is

$$\mathcal{D}(A) = \{\phi \in \mathcal{B} : \lim_{t\to 0} t^{-1}\{T_t\phi - \phi\} \text{ exists}\},$$

where the limit is in the strong sense.

Definition I.1.1. *The operator*

$$A = \sum_{i,j=1}^{d} \alpha_{ij} \partial_{ij}^2 + \sum_{i=1}^{d} \beta_i \partial_i \tag{I.3}$$

is said to be uniformly elliptic if for each $N \geq 1$, there exists $\eta_N > 0$ such that

$$\sum_{i,j=1}^{d} \sum_{p,q=1}^{N} \xi_i^p \alpha_{ij}(x_p, x_q) \xi_j^q \geq \eta_N \sum_{p=1}^{N} \sum_{i=1}^{d} \xi_i^p,$$

for all $x_1, x_2, ..., x_N \in \mathbb{R}^d$ and $(\xi_1^1, \xi_2^1, ..., \xi_d^1) \otimes (\xi_1^2, \xi_2^2, ..., \xi_d^2) \otimes \cdots \otimes (\xi_1^N, \xi_2^N, ..., \xi_d^N) \in \mathbb{R}^{d \times N}$.

Assumption 1. For the remainder of this paper, the assumption will be made that L is uniformly elliptic.

Definition I.1.2. *For $x \in \mathbb{R}^d$ and $\phi \in \mathcal{D} \subset \mathcal{D}(A)$, we say that a $\mathbb{R}^d-$valued process $X = \{X_t : t \geq 0\}$ solves the $(A, \mathcal{D}, x)$ martingale problem if $X_0 = x$ and*

$$\phi(X_t) - \phi(x) - \int_0^t (A\phi)(X_s) ds, \quad t \geq 0,$$

is a martingale for each $\phi \in \mathcal{D}$. When $\mathcal{D} = \mathcal{D}(A)$, we say that X solves the $(A, x)-$ martingale problem.

For each $k \in \mathbb{N}$ and metric space E, we will denote by $C_0^k(E)$ the subspace of functions in $C^k(E)$ which vanish at infinity.

Lemma I.1.3. *If L^n is defined as I.2, then L^n is the generator of the diffusion which describes the joint motion of n particles in the aforementioned branching particle system.*

Proof: For $p = 1, 2, ..., n$ let $Y_t^p = (Y_t^{p,1}, ..., Y_t^{p,d})$, where

$$dY_t^{p,i} = b_i(Y_t^p) dB_t^{p,i} + \sum_{k=1}^{m} c_{ik}(Y_t^p) dW_t^k, \quad i = 1, 2, ..., d.$$

If $Y_t = (Y_t^1, ..., Y_t^n)$ and $\phi \in C_0^2(\mathbb{R}^{nd})$, then Itô's Lemma implies

$$\begin{aligned}
\phi(Y_t) - \phi(Y_0) &= \sum_{p=1}^{n}\sum_{i=1}^{d}\int_0^t \partial_{p_i}\phi(Y_s)dY_s^{p,i} + \frac{1}{2}\sum_{p,q=1}^{n}\sum_{i,j=1}^{d}\int_0^t \partial_{p_i}\partial_{q_j}\phi(Y_s)d\langle Y^{p,i}, Y^{q,j}\rangle_s \\
&= \sum_{p=1}^{n}\sum_{i=1}^{d}\int_0^t \partial_{p_i}\phi(Y_s)b_i(Y_s^p)dB_s^{p,i} + \sum_{p=1}^{n}\sum_{i=1}^{d}\sum_{k=1}^{m}\int_0^t \partial_{p_i}\phi(Y_s)c_{ik}(Y_t^p)dW_t^k \\
&+ \frac{1}{2}\sum_{p,q=1}^{n}\sum_{i,j=1}^{d}\int_0^t \partial_{p_i}\partial_{q_j}\phi(Y_s)\delta_{pq}\delta_{ij}b_i(Y_s^p)b_j(Y_s^q)ds \\
&+ \frac{1}{2}\sum_{p,q=1}^{n}\sum_{i,j=1}^{d}\sum_{k=1}^{m}\int_0^t \partial_{p_i}\partial_{q_j}\phi(Y_s)c_{ik}(Y_s^p)c_{jk}(Y_s^q)ds
\end{aligned}$$

Since $\partial_{p_i}\phi$, b_i, and c_{ik} are bounded for $i = 1, ..., d$, $k = 1, ..., m$, and $p = 1, ..., n$, it follows that $\int_0^t \partial_{p_i}\phi(Y_s)b_i(Y_s^p)dB_s^{p,i}$ and $\int_0^t \partial_{p_i}\phi(Y_s)c_{ik}(Y_t^p)dW_t^k$ are martingales, and so

$$\mathbb{E}\phi(Y_t) - \phi(Y_0) = \frac{1}{2}\mathbb{E}\sum_{p,q=1}^{n}\sum_{i,j=1}^{d}\int_0^t a_{ij}^{pq}(Y_s)\partial_{p_i}\partial_{q_j}\phi(Y_s)ds.$$

Therefore,

$$\begin{aligned}
\lim_{t\to 0}\frac{\mathbb{E}\phi(Y_t) - \phi(Y_0)}{t} &= \frac{1}{2}\mathbb{E}\sum_{p,q=1}^{n}\sum_{i,j=1}^{d}\lim_{t\to 0} t^{-1}\int_0^t a_{ij}^{pq}(Y_s)\partial_{p_i}\partial_{q_j}\phi(Y_s)ds \\
&= \frac{1}{2}\sum_{p,q=1}^{n}\sum_{i,j=1}^{d} a_{ij}^{pq}(Y_0)\partial_{p_i}\partial_{q_j}\phi(Y_0) \\
&= (L^n\phi)(Y_0).
\end{aligned}$$

□

Lemma I.1.4. *For each $n \in \mathbb{N}$, there exists a transition function q_t^n for the Markov process $Y_t = (Y_t^1, ..., Y_t^n)$. Furthermore, $\{Q_t^n : t \geq 0\}$, defined by*

$$Q_t^n\phi(x) = \int_{\mathbb{R}^d}\phi(y)q_t^n(x, y)$$

is a strongly continuous contraction semigroup on $C_0(\mathbb{R}^d)$.

Proof: Since it is assumed that Assumption 1 holds for L, it follows that for each $n \in \mathbb{N}$, L^n Assumption 1 also holds. Theorem 5.11 in [8] then completes the proof. $\square$

Lemma I.1.5. *With* q_t^n *and* Q_t^n, *defined as above, the following are satisfied:*

1. *For any* $t > 0$, $x, y \in \mathbb{R}^{nd}$, $q_t^n(x, y) > 0$
2. *On the set* $\{t > 0, x, y \in \mathbb{R}^{nd}\}$, $q_t^n(x, y)$ *is jointly continuous in* t, x, y; $q_t^n(\cdot, y) \in C^2(\mathbb{R}^{nd})$; *and* $\partial_t q_t^n(x, y) = L^n q_t^n(x, y)$.
3. *For any* $\phi \in C_b(\mathbb{R}^{nd})$, $x \in \mathbb{R}^{nd}$,

$$\lim_{t \to 0} \int_{\mathbb{R}^d} dy\, q_t^n(x, y)\phi(y) = \phi(x) \tag{I.4}$$

4. *For any* $\delta > 0$, $q_t^n(x, y)$ *is bounded in the domain* $t + |x - y| \geq \delta$.
5. $q_t^n(x, y) \leq C p_{\iota t}^n(x, y)$.
6. $|\partial_{1_i} q_t^n(x, y)| \leq C t^{-1/2} p_{\iota t}^n(x, y)$, $i = 1, ..., d$.
7. $\left|\partial_{1_i} \partial_{1_j} q_t^n(x, y)\right| \leq C t^{-1} p_{\iota t}^n(x, y)$, $i, j = 1, 2, ..., d$.
8. $|\partial_t q_t^n(x, y)| \leq C t^{-1} p_{\iota t}^n(x, y)$.
9. $q_t^n(x, y) \geq C_1 p_{\iota_1 t}^n(x, y) - C_2 t^\nu p_{\iota_2 t}^n(x, y)$
10. $(t, y) \to q_t^n(x, y)$ *satisfies*

$$\partial_t q^n = \sum_{p,q=1}^{n} \sum_{i,j=1}^{d} \partial_i \partial_j (a_{ij}^{pq}(y) q^n). \tag{I.5}$$

11. *If* $\phi \in C_b^2(\mathbb{R}^{nd}) \cap L^1(\mathbb{R}^{nd})$, *then*

$$\lim_{t \to 0} \partial_i Q_t^n \phi(x) = \partial_i \phi(x)$$

and

$$\lim_{t\to 0} \partial_i\partial_j Q_t^n\phi(x) = \partial_i\partial_j\phi(x),$$

$i, j = 1, ..., nd.$

Where $\iota, \iota_1, \iota_2, \nu$ *and* C, C_1, C_2 *are positive constants and* $p_t^n(x,y)$ *is the joint transition density of* n *independent d-dimensional Brownian motions.*

Proof: This follows immediately from Assumption 1 upon L^n (see the appendix (p.228) of [8], or alternatively [12]). □

For any topological space E, let $M_F(E)$ denote the space of finite Borel measures on E, $C_E[0,\infty)$ the space of continuous paths in E, and for any $\ell \in \mathbb{N}$, $C_K^\ell(E)$ the subspace of $C^\ell(E)$ for which the elements have compact support. Endow $M_F(E)$ with the topology of weak convergence, that is, $\mu^{(n)} \in M_F(E)$ converges to $\mu \in M_F(E)$ provided $\lim_{n\to\infty} \langle \phi, \mu^{(n)} \rangle = \langle \phi, \mu \rangle$ for any $\phi \in C_b(E)$, and let $\Rightarrow$ denote weak convergence. In addition, for any $\mu \in M_F(E)$ and $\ell \in \mathbb{N}$, denote by μ^ℓ the product measure $\mu \times \mu \times \cdot \times \mu \in M_F(\mathbb{R}^{\ell\times d})$. Under these assumptions, and Assumption 1 upon L, we arrive at the following theorem.

Theorem I.1.6. *Let* $\mu^{(n)}$ *be defined as above with* $\mu_0^{(n)} \Rightarrow \mu_0$*, then* $\mu^{(n)} \Rightarrow \mu$*, where* $\mu \in C_{M_F(\mathbb{R}^d)}[0,\infty)$ *is the unique solution of the following martingale problem:*

For all $\phi \in C_K^2(\mathbb{R}^d)$,

$$Z_t(\phi) = \langle \phi, \mu_t \rangle - \langle \phi, \mu_0 \rangle - \int_0^t ds\ \langle L\phi, \mu_s \rangle \tag{I.6}$$

is a continuous square integrable $\{\mathcal{F}_t^\mu\}$*-martingale such that* $Z_0(\phi) = 0$ *and has quadratic variation process*

$$\langle Z(\phi) \rangle_t = \int_0^t ds\ \left(\langle \phi^2, \mu_s \rangle + \langle \Lambda\phi, \mu_s^2 \rangle \right). \tag{I.7}$$

Proof: See Theorem 2.2.1 in [26]. □

Assumption 2. For the remainder of this work, it will be assumed that $\mu_0 \in M_F(\mathbb{R}^d)$ has compact support and satisfies

$$\mu_0(dx) \leq m(x)dx,$$

for some bounded $m \in L^1(\mathbb{R}^d)$.

CHAPTER II

MOMENT FORMULAS

II.1 Preliminary Results

As in most existence proofs for self-intersection local time of a superprocess, higher moments of the superprocess are required (cf. [1], [9]). Through finding the first and second moments of the branching process, and passing to the limit as $n \to \infty$, Skoulakis & Adler [26] found the first and second moments for the SSF. A variation of this method will now be employed to find higher moments of the SSF.

We denote by $C^\infty(E)$ the space of infinitely differentiable functions on E and by $C_K^\infty(E)$, the subspace of $C^\infty(E)$ of which the elements have compact support.

By a test function, we are referring to any $\phi \in C_K^\infty(\mathbb{R}^d)$. We denote by $\mathcal{D}'(\mathbb{R}^d)$ the space of distributions on $C_K^\infty(\mathbb{R}^d)$. Suppose $u, v \in L^1_{\text{loc}}(\mathbb{R}^d)$, the space of locally L^1−integrable functions on $\mathbb{R}^d$, and $\alpha = (\alpha_1, \alpha_2, ... \alpha_d)$ is a multiindex of order $|\alpha|$. We say that v is the α^{th}−weak partial derivative of u, written

$$D^\alpha u = v,$$

provided

$$\int u(x)(D^\alpha \phi)(x)dx = (-1)^{|\alpha|} \int v(x)\phi(x)dx$$

for all test functions ϕ. Note that a differentiable function will have a weak derivative that agrees with the functions derivative, and thus we will at times use a slight abuse in notation and write the weak derivative as

$$D^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}.$$

We denote by S_d the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$. That is,

$$S_d = \left\{ \phi \in C^\infty(\mathbb{R}^d) : \sup_{|\alpha| \le N} \sup_{x \in \mathbb{R}^d} (1 + |x|^2)^N \left| (D_\alpha \phi)(x) \right| < \infty, \ N = 0, 1, 2, ... \right\}.$$

The sequence ϕ_n converges to ϕ in S_d if

$$\lim_{n \to \infty} \sup_{|\alpha| \le N} \sup_{x \in \mathbb{R}^d} (1 + |x|^2)^N \left| D^\alpha \phi_n(x) - D^\alpha \phi(x) \right| = 0,$$

for any $N \in \mathbb{N}$ (cf. [32]).

If $i : C_K^\infty(\mathbb{R}^d) \to S_d$ is the identity mapping, if L is a continuous linear functional on S_d, and if

$$u_L = L \circ i$$

then the continuity of i (Theorem 7.10 of [25]) shows that $u_L \in \mathcal{D}'(\mathbb{R}^d)$. Again from [25], $L \mapsto u_L$ describes a vector space isomorphism between the dual space S_d' of S_d and a subspace of $D'(\mathbb{R}^d)$. Distributions that arise as such are called tempered, and are precisely those $u \in \mathcal{D}'(\mathbb{R}^d)$ that extend continously to an element in S_d ([25]). Thus, if u_L is identified with L, the space of tempered distributions is precisely S_d'.

For any two functions $\phi : E_1 \to \mathbb{R}$, $\psi : E_2 \to \mathbb{R}$ denote by $\phi \otimes \psi$ the concatenation of ϕ and ψ. That is, $\phi \otimes \psi : E_1 \times E_2 \to \mathbb{R}$ is the map defined by $(x_1, x_2) \mapsto \phi(x_1)\psi(x_2)$.

Lemma II.1.1. *Let $\phi \in S_{\ell \times d}$, then there exists $\{\phi_n : n \in \mathbb{N}\}$ such that*

(i) $\phi_n = \sum_{k=1}^n \phi_k^1 \otimes \phi_k^2 \otimes \cdots \otimes \phi_k^\ell$, for some $\phi_k^1, ..., \phi_k^\ell \in C_K^\infty(\mathbb{R}^d)$,

and

(ii) ϕ_n converges to ϕ in $S_{\ell \times d}$ as $n \to \infty$.

Proof: Taylor's Theorem implies the above holds for any $\phi \in C_K^\infty(\mathbb{R}^d)$ (cf. [23], [24]) . From Theorem 7.10 of [25] there exist $\{\phi_n : n \in \mathbb{N}\} \subset C_K^\infty(\mathbb{R}^{\ell\times d})$ such that ϕ_n converges to ϕ in S_d, and the result thus follows. □

Given $\phi \in \mathcal{B}(\mathbb{R}^{(n+\ell)\times d})$, $n, \ell \in \mathbb{N}$, define the projection π_ℓ by

$$(\pi_\ell Q_t^n \phi)(x_1, ..., x_{n+\ell}) = Q_t^n \phi_{(x_1,...,x_\ell)}(x_{\ell+1}, ..., x_{n+\ell}),$$

where

$$\phi_{(x_1,...,x_\ell)}(y_1, ..., y_n) = \phi((x_1, ..., x_\ell) \otimes (y_1, ..., y_n)).$$

For any function ϕ having as its domain $\mathbb{R}^{m\times d}$, define Φ_{ij} by

$$(\Phi_{ij}\phi)(x_1, ..., x_{m-1}) = \phi(x_1, ..., x_i, ..., x_{j-1}, x_i, x_j, ..., x_{m-1}),$$

for $i, j = 1, 2, ...m-1$, $i \neq j$.

The next Lemma comes from Skoulakis & Adler [26].

Lemma II.1.2. *Let $\phi, \phi_1, \phi_2 \in C_K^2(\mathbb{R}^d)$ and $t > 0$, then*

$$\text{(i)}\ \ \mathbb{E}\mu_t(\phi) = \langle Q_t\phi, \mu_0\rangle\,,$$

and

$$\begin{aligned}\text{(ii)}\ \ \mathbb{E}\mu_{t_1}(\phi_1)\mu_{t_2}(\phi_2) &= \left\langle Q_{t_1}^2(\pi_1 Q_{t_2-t_1}(\phi_1\otimes\phi_2)), \mu_0^2\right\rangle \\ &\quad + \int_0^{t_1} ds\ \left\langle Q_s \Phi_{12} Q_{t_1-s}^2(\pi_1 Q_{t_2-t_1}(\phi_1\otimes\phi_2), \mu_0\right\rangle,\end{aligned}$$

with the convention that $Q_0^n\phi = \phi$, $n \in \mathbb{N}$.

Proof: See [26], Proposition 3.2.1. □

Before our moment calculations, some needed definitions and Lemmas will be presented. In what follows (S, d) will refer to a metric space, in which it is assumed S is separable.

Definition II.1.3. *The Prohorov metric ρ on $M_F(S)$ is defined by*

$$\rho(\mu, \nu) = \inf\{\varepsilon > 0 : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon, \ \ \forall A \in \mathcal{C}\},$$

for $\mu, \nu \in M_F(S)$ where $\mathcal{C}$ denotes the family of closed subsets of S, and $A^\varepsilon = \{x \in S : \inf_{a \in A} d(x, a) \leq \varepsilon\}$.

Definition II.1.4. *Given $\mu, \nu \in M_F(S)$, a marginal for μ and ν is a measure λ on $M_F(S \times S)$ such that for any $A \subset S$, $\lambda(A \times S) = \mu(A)\nu(S)$ and $\lambda(S \times A) = \mu(S)\nu(A)$. The collection of marginals corresponding to μ and ν will be denoted by $\mathcal{M}(\mu, \nu)$.*

Lemma II.1.5. *Let $\{\mu^{(n)} : n \in \mathbb{N}\} \subset M_F(S)$, then a necessary and sufficient condition for $\mu^{(n)} \Rightarrow \mu \in M_F(S)$ is $\lim_{n\to\infty} \rho(\mu^{(n)}, \mu) = 0$.*

Proof: See Theorem 3.1 in chapter 3 of [10]. □

Lemma II.1.6. *For any $\mu, \nu \in M_F(S)$, with $\mathcal{M}(\mu, \nu)$ defined as above,*

$$\rho(\mu, \nu) = \inf_{\lambda \in \mathcal{M}(\mu,\nu)} \inf\{\varepsilon > 0 : \lambda\{(x, y) : d(x, y) \geq \varepsilon\} \leq \varepsilon\}.$$

Proof: See Theorem 1.2 in chapter 3 of [10]. □

Lemma II.1.7. *For $\phi \in C_b(S)$ define $\|\phi\|_{bL}$ by*

$$\|\phi\|_{bL} = \sup_x |\phi(x)| \vee \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{d(x, y)},$$

and for μ, $\nu \in M_F(S)$ such that $\mu(S) = \nu(S) = 1$, let

$$\|\mu - \nu\|_U = \sup_{\|\phi\|_{bL}=1} |\langle\phi, \mu\rangle - \langle\phi, \nu\rangle|.$$

then

$$\rho(\mu,\nu)^2 \le \|\mu-\nu\|_U \le 3\rho(\mu,\nu).$$

Proof: See Ethier & Kurtz [10], p.150.

Lemma II.1.8. *If $\{\mu^{(n)} : n \ge 0\} \subset M_F(\mathbb{R}^d)$ satisfies $\mu^{(n)} \Rightarrow \mu \in M_F(\mathbb{R}^d)$ then*

$$\left(\mu^{(n)}\right)^\ell \Rightarrow \mu^\ell,$$

for all $\ell \in \mathbb{N}$.

Proof: Define

$$M = \left\{\phi = \bigotimes_{k=1}^{\ell} \phi_k : \ell \ge 1, \phi_k \in C_K(\mathbb{R}^d) \cup \{1\}, k = 1,2,...,\ell\right\}.$$

From Proposition 4.4 of chapter 3 in [10], for any $\nu, \nu^{(n)} \in M_F(\mathbb{R}^d)$, $n = 1,2,...$ such that

$$\lim_{n\to\infty} \left\langle \phi, \nu^{(n)} \right\rangle = \langle \phi, \nu\rangle$$

for all $\phi \in C_K(\mathbb{R}^d)$, it follows that $\nu^{(n)} \Rightarrow \nu$. For any $\ell \in \mathbb{N}$, since

$$\mu^{(n)} \Rightarrow \mu,$$

$\lim_{n\to\infty} \langle \phi, (\mu^{(n)})^\ell \rangle = \langle \phi, \mu^\ell \rangle$, for any $\phi = \bigotimes_{k=1}^{\ell} \phi_k$ with $\phi_k \in C_K(\mathbb{R}^d)$ or $\phi_k \in \{1\}$, $k = 1,...,\ell$. Thus, for any $\phi = \bigotimes_{k=1}^{\ell} \phi_k \in M$, $\lim_{n\to\infty} \langle \phi, (\mu^{(n)})^\ell \rangle = \langle \phi, \mu^\ell \rangle$, which implies, by Proposition 4.6 of chapter 3 in [10], $(\mu^{(n)})^\ell \Rightarrow \mu^\ell$. □

Definition II.1.9. *The Skorohod space $D_S[0,\infty)$ on (S,d) is defined by*

$$D_S[0,\infty) = \left\{x : [0,\infty) \to S : \lim_{s\to t^+} x(s) = x(t), \text{ and } \lim_{s\to t^-} x(s) \triangleq x(s-) \text{ exists}\right\}.$$

That is, $D_S[0,\infty)$ is the space of all càdlàg mappings from $[0,\infty)$ to S.

Note that under the assumption that S is separable, $D_S[0,\infty)$ with the metric defined by (5.2) in Chapter 3 of [10], is a separable metric space. Moreover, if (S,d) is complete, $D_S[0,\infty)$ is complete (cf. [10] Theorem 5.6, Chapter 3). The next two Lemmas are essential in the moment proofs for the superprocess.

Lemma II.1.10. *For $k,\ell \in \mathbb{N}$, let $\psi : \mathbb{R}_+^k \times \mathbb{R}^\ell \to \mathbb{R}$ in $C_b(\mathbb{R}^d)$ satisfy*

$$\sup_{s\in\mathbb{R}_+^k} \|\psi(s,\cdot)\|_{bL} < \infty$$

and let μ_0 be an a.s. finite measure having compact support with $\mu_0^{(n)} \Rightarrow \mu_0$. Then

$$\lim_{n\to\infty} \Bigg| n^{-k} \sum_{r_k=k}^{[nt]-1} \sum_{r_{k-1}=0}^{r_k} \cdots \sum_{r_1=0}^{r_2} \left\langle \psi(rn^{-1},\cdot), (\mu_0^{(n)})^\ell \right\rangle - \int_0^t ds_k \int_0^{s_k} ds_{k-1} \cdots \int_0^{s_2} ds_1 \left\langle \psi(s,\cdot), \mu_0^\ell \right\rangle \Bigg| = 0.$$

Proof: We have

$$\begin{aligned}
&\Bigg| n^{-k} \sum_{r_k=k}^{[nt]-1} \sum_{r_{k-1}=0}^{r_k} \cdots \sum_{r_1=0}^{r_2} \left\langle \psi(rn^{-1},\cdot), (\mu_0^{(n)})^\ell \right\rangle - \int_0^t ds_k \int_0^{s_k} ds_{k-1} \cdots \int_0^{s_2} ds_1 \left\langle \psi(s,\cdot), \mu_0^\ell \right\rangle \Bigg| \\
&\le n^{-k} \Bigg| \sum_{r_k=k}^{[nt]-1} \sum_{r_{k-1}=0}^{r_k} \cdots \sum_{r_1=0}^{r_2} \left\langle \psi(rn^{-1},\cdot), (\mu_0^{(n)})^\ell \right\rangle - \sum_{r_k=k}^{[nt]-1} \sum_{r_{k-1}=0}^{r_k} \cdots \sum_{r_1=0}^{r_2} \left\langle \psi(rn^{-1},\cdot), \mu_0^\ell \right\rangle \Bigg| \\
&+ \Bigg| n^{-k} \sum_{r_k=k}^{[nt]-1} \sum_{r_{k-1}=0}^{r_k} \cdots \sum_{r_1=0}^{r_2} \left\langle \psi(rn^{-1},\cdot), \mu_0^\ell \right\rangle - \int_0^t ds_k \int_0^{s_k} ds_{k-1} \cdots \int_0^{s_2} ds_1 \left\langle \psi(s,\cdot), \mu_0^\ell \right\rangle \Bigg| \\
&\le n^{-k} \sum_{r_k=k}^{[nt]-1} \sum_{r_{k-1}=0}^{r_k} \cdots \sum_{r_1=0}^{r_2} \left| \left\langle \psi(rn^{-1},\cdot), (\mu_0^{(n)})^\ell \right\rangle - \left\langle \psi(rn^{-1},\cdot), \mu_0^\ell \right\rangle \right| \\
&+ \left\langle \Bigg| n^{-k} \sum_{r_k=k}^{[nt]-1} \sum_{r_{k-1}=0}^{r_k} \cdots \sum_{r_1=0}^{r_2} \psi(rn^{-1},\cdot) - \int_0^t ds_k \int_0^{s_k} ds_{k-1} \cdots \int_0^{s_2} ds_1 \psi(s,\cdot) \Bigg|, \mu_0^\ell \right\rangle .
\end{aligned}$$

By assumption $\sup_s \|\psi(s,\cdot)\|_{bL} < \infty$, and thus Lemma II.1.7 implies the first of the above terms converges to zero. Since ψ is continuous and bounded, and μ_0^ℓ is finite with compact support, it follows that the second term is also convergent towards zero. $\square$

Lemma II.1.11. *For any $\phi_i \in C_K^\infty(\mathbb{R}^d)$, $i = 1, 2, ..., \ell$, $\ell \in \mathbb{N}$, $0 < t < \infty$,*

$$\lim_{n\to\infty} \mathbb{E}\left\langle \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_\ell, \left(\mu_t^{(n)}\right)^\ell \right\rangle = \mathbb{E}\left\langle \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_\ell, \mu_t^\ell \right\rangle.$$

Proof: Let $\mu^{(n)} = \{\mu_t^{(n)} : t \geq 0\}$ be a branching process as defined above, let μ be a weak limit point of $\mu^{(n)}$, and let $\{n_k\}$ be the subsequence along which $\mu^{(n_k)} \Rightarrow \mu$. From Theorem 3.1, Chapter 3, of [10], there is a Skorohod representation for $\mu, \mu^{(n_k)}$, $k \in \mathbb{N}$. That is, there exist random variables X, X_k, $k \in \mathbb{N}$, defined on the same probability space, such that $X \stackrel{d}{=} \mu$, $X_k \stackrel{d}{=} \mu^{(n_k)}$, $k \in \mathbb{N}$, and $X_k \to X$ *a.s.* as $k \to \infty$.

For $X \in D_{M_F(\mathbb{R}^d)}[0,\infty)$, define $\mathbb{P}X(\phi_i)^{-1}$ to be the distribution of $X(\phi_i) \in D_{\mathbb{R}}[0,\infty)$ then, by dominated convergence

$$\begin{aligned}
&\lim_{k\to\infty} \sup_{\|\psi\|_{bL}=1} \left| \left\langle \psi, \prod_{i=1}^{\ell} \mathbb{P}X_k(\phi_i)^{-1} \right\rangle - \left\langle \psi, \prod_{i=1}^{\ell} \mathbb{P}X(\phi_i)^{-1} \right\rangle \right| \\
&\qquad = \lim_{k\to\infty} \sup_{\|\psi\|_{bL}=1} \left| \mathbb{E}\psi(X_k(\phi_1), ..., X_k(\phi_\ell)) - \mathbb{E}\psi(X(\phi_1), ..., X(\phi_\ell)) \right| \\
&\qquad = 0.
\end{aligned}$$

It then follows from Lemma II.1.7 that

$$\lim_{k\to\infty} \rho\left(\prod_{i=1}^{\ell} \mathbb{P}X_k(\phi_i)^{-1}, \prod_{i=1}^{\ell} \mathbb{P}X(\phi_i)^{-1} \right) = 0,$$

or equivalently,

$$(X_k(\phi_1), ..., X_k(\phi_\ell)) \Rightarrow (X(\phi_1), ..., X(\phi_\ell))$$

in $D_{\mathbb{R}^\ell}[0,\infty)$. Therefore, from Theorem I.1.6,

$$(\mu^{(n_k)}(\phi_1), ..., \mu^{(n_k)}(\phi_\ell)) \Rightarrow (\mu(\phi_1), ..., \mu(\phi_\ell))$$

in $D_{\mathbb{R}^\ell}[0,\infty)$. Thus, from Lemma A.3.9 [26], for $i = 1, 2, ...$, $\mu(\phi_i)$ is continuous. Therefore, the open mapping theorem ([10], Corollary 1.9, Chapter 3) implies that

$$(\mu_t^{(n_k)}(\phi_1), ..., \mu_t^{(n_k)}(\phi_\ell)) \Rightarrow (\mu_t(\phi_1), ..., \mu_t(\phi_\ell))$$

in $\mathbb{R}^\ell$, which further implies that

$$\mu_t^{(n_k)}(\phi_1) \cdot \mu_t^{(n_k)}(\phi_2) \cdot \cdots \cdot \mu_t^{(n_k)}(\phi_\ell) \Rightarrow \mu_t(\phi_1) \cdot \mu_t(\phi_2) \cdot \cdots \cdot \mu_t(\phi_\ell)$$

in $\mathbb{R}$. Note that (cf. (3.1) in [26]) for any $t \geq 0$, $\mathbb{E}\mu_t^{(n)}(1) = \mu_0^{(n)}(1)$, and thus $\{\mu_t^{(n)}(1) : t \geq 0\}$ is an $\mathcal{F}_t^n$–martingale. It follows from Doob's maximal inequality ([16], Theorem 3.8) that for any $T \geq 0$,

$$\mathbb{E} \sup_{0 \leq t \leq T} \left[\mu_t^{(n)}(1)\right]^\ell \leq \left(\frac{\ell}{\ell - 1}\right)^\ell \mathbb{E}\left[\mu_T^{(n)}(1)\right]^\ell.$$

Since $\mu_0^{(n)} \Rightarrow \mu_0$,

$$\lim_{n\to\infty} \mu_0^{(n)}(1) = \mu_0(1),$$

and thus,

$$\sup_{n\geq 1} \mu_0^{(n)}(1) < \infty.$$

Since $\mu_t^{(n)}(1)$ is the total mass process of the branching particle system, and is absent of influence by the stochastic flow, $[\mu_T^{(n)}(1)]^\ell$ is equivalent in distribution to a total mass process with an initial $M(n)^\ell$ particles, which implies $\mathbb{E}[\mu_T^{(n)}(1)]^\ell = [\mu_0^{(n)}(1)]^\ell$. Thus,

$$\sup_{n\geq 1} \mathbb{E} \sup_{0\leq t\leq T} \left[\mu_t^{(n)}(1)\right]^\ell < \infty. \tag{II.1}$$

Theorem 25.12 of [3] implies

$$\lim_{k\to\infty} \mathbb{E} \prod_{i=1}^{\ell} \mu_t^{(n_k)}(\phi_i) = \mathbb{E} \prod_{i=1}^{\ell} \mu_t(\phi_i),$$

and thus,

$$\lim_{n\to\infty} \mathbb{E} \left\langle \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_\ell, (\mu_t^{(n)})^\ell \right\rangle = \mathbb{E} \left\langle \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_\ell, \mu_t^\ell \right\rangle.$$

□

II.2 Moment Calculations For The Branching Particle System

In Skoulakis & Adler the first and second moment calculations are done via first finding

$$\mathbb{E} \left\langle \phi, \mu_t^{(n)} \right\rangle$$

and

$$\mathbb{E} \left\langle \phi_1 \otimes \phi_2, \mu_{t_1}^{(n)} \mu_{t_2}^{(n)} \right\rangle$$

then passing to the limit as $n \to \infty$.

This works well when the number of cases to consider are small, but due to the rapid growth in cases to consider as the moments increase, the following method will vary slightly. The method first calculates

$$\mathbb{E} \left\langle \phi, (\mu_t^{(n)})^3 \right\rangle$$

and

$$\mathbb{E}\left\langle \psi, (\mu_t^{(n)})^4 \right\rangle$$

for $\phi \in C_K^\infty(\mathbb{R}^{3\times d}), \psi \in C_K^\infty(\mathbb{R}^{4\times d})$, $t \geq 0$, then passes to the limit before utilizing the Markov property to find

$$\mathbb{E}\left\langle \phi_1 \otimes \phi_2 \otimes \phi_3, \mu_{t_1}\mu_{t_2}\mu_{t_3} \right\rangle$$

and

$$\mathbb{E}\left\langle \psi_1 \otimes \psi_2 \otimes \psi_3 \otimes \psi_4, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4} \right\rangle,$$

where $\phi_i, \psi_j \in C_K^\infty(\mathbb{R}^d)$, $i = 1, 2, 3$, $j = 1, 2, 3, 4$, and $0 < t_1 \leq t_2 \leq t_3 \leq t_4$.

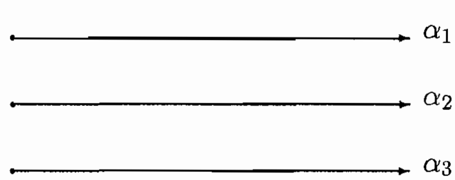
Note that

$$\mathbb{E}\left\langle \phi, (\mu_t^{(n)})^\ell \right\rangle = n^{-\ell} \sum_{\alpha_1 \sim_n t, \ldots, \alpha_\ell \sim_n t} \mathbb{E}\phi(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, \ldots, Y_t^{\alpha_\ell,n}) \mathbb{E} \prod_{i=1}^{\ell} 1_{\alpha_i,n}(t), \tag{II.2}$$

where $1_{\alpha_i,n}(t)$ is the indicator on the event that the particle α_i is alive at time t. Thus, for the third moment, if $\alpha_i \sim_n t$, $i = 1, 2, 3$, and $N = [tn]$, we will have the following cases to consider:

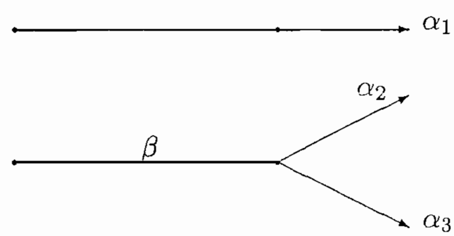
(I) Each particle resides on its own tree.

Figure 1: Third Moment, Case I.



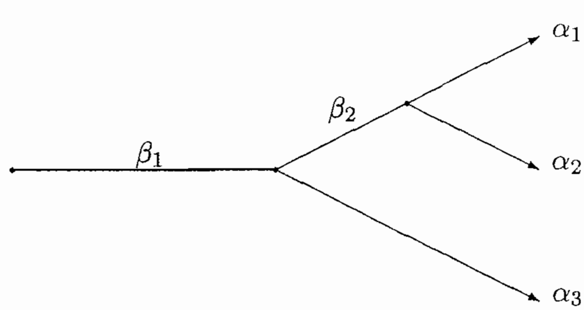
(II) Two particles reside on one tree, and the third particle on its own tree. Thus, two particles share a common ancestor β with $|\beta| = r$, $r \in \{0, 1, ..., N-1\}$.

Figure 2: Third Moment, Case II.



(III) All three particles are on one tree. Thus there exists a common ancestor β_1 for all three particles, and a common ancestor β_2 for two of the particles such that β_1 is an ancestor of β_2, and $|\beta_1| = r_1$, $|\beta_2| = r_2$, with $r_1 \in \{0, 1, ..., r_2 - 1\}, r_2 \in \{1, ..., N-1\}$.

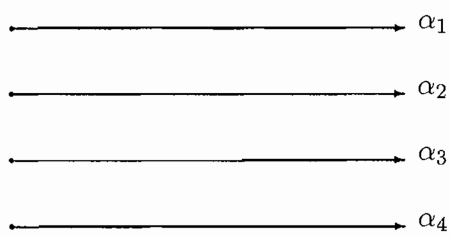
Figure 3: Third Moment, Case III.



The cases for the third moment are thus exhausted. For the fourth moment, we obtain:

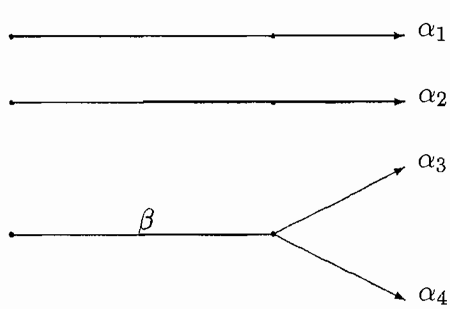
(I) Each particle resides on its own tree.

Figure 4: Fourth Moment, Case I.



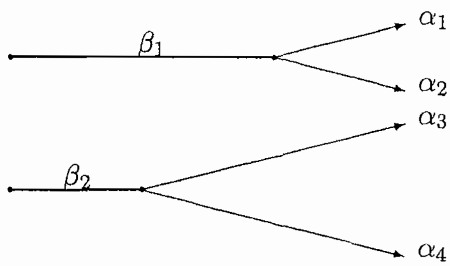
(II) Two particles reside on one tree, the other two reside on their own trees. Thus, the two particles on the common tree share a common ancestor β with $|\beta| = r$ and $r \in \{0, 1, ..., N-1\}$.

Figure 5: Fourth Moment, Case II.



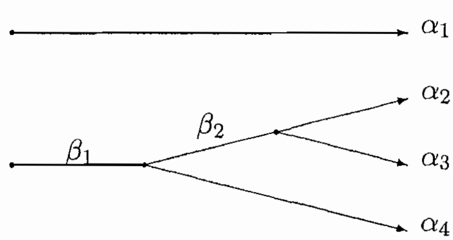
(III) Two particles reside on one tree, the other two on a second tree. Thus, the two particles on one tree share a common ancestor β_1 with $|\beta_1| = r_1$, the two particles on the second tree have a common ancestor β_2 with $|\beta_2| = r_2$, and $r_1, r_2 \in \{0, 1, ..., N-1\}$.

Figure 6: Fourth Moment, Case III.



(IV) Three particles reside on one tree, the fourth on its own tree. Thus, two of the three particles share a common ancestor β_2 with $|\beta_2| = r_2$, and all three share a common ancestor β_1 with $|\beta_1| = r_1$, such that $r_1 \in \{0, 1, ..., r_2 - 1\}$ and $r_2 \in \{1, ..., N-1\}$.

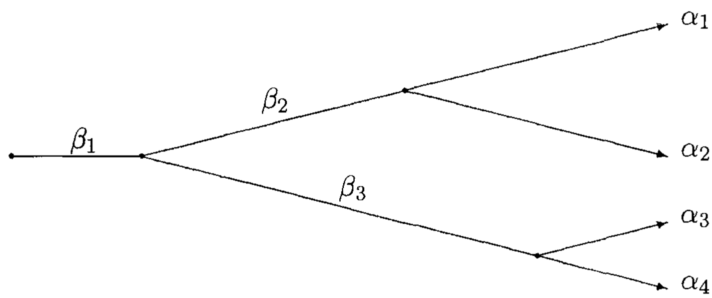
Figure 7: Fourth Moment, Case IV.



(V) All four particles reside on one tree. This gives the following two sub-cases:

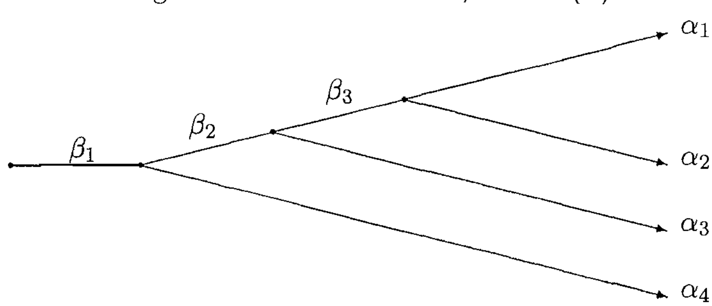
(A) Two of the particles share a common ancestor β_3 with $|\beta_3| = r_3$, the other two share a common ancestor β_2 also with $|\beta_2| = r_2$, all four share a common ancestor β_1 with $|\beta_1| = r_1$, β_2 and β_3 are both descendants of β_1, and $r_1 \in \{0, 1, ..., (r_2 - 1) \wedge (r_3 - 1)\}$ and $r_2, r_3 \in \{1, ..., N-1\}$.

Figure 8: Fourth Moment, Case V(A).



(B) Two of the particles share a common ancestor β_3, another particle shares a common ancestor β_2 with β_3, all four particles share a common ancestor β_1, and β_1 is an ancestor of β_2 which is an ancestor of β_3, with $|\beta_1| = r_1$, $|\beta_2| = r_2$, $|\beta_3| = r_3$ and $r_1 \in \{0, 1, ..., r_2 - 1\}$, $r_2 \in \{1, ..., r_3 - 1\}$, $r_3 \in \{2, ..., N - 1\}$.

Figure 9: Fourth Moment, Case V(B).



We now proceed with the third moment calculations. Much of what follows will be a consequence of the Markov property, and the reader is referred to Skoulakis & Adler [26] for a similar calculation for the first and second moments. Note that if $t \geq 0$ and $r \in \mathbb{N}$, we define $N \in \mathbb{N}$ and $r(n) \in [0, r]$ by

$$N = [nt]$$

and

$$r(n) = \frac{r}{n}.$$

Recall,

$$\mathbb{E}\left\langle \phi, (\mu_t^{(n)})^3 \right\rangle = n^{-3} \sum_{\alpha_1 \sim_n t, \alpha_2 \sim_n t, \alpha_3 \sim_n t} \mathbb{E}\phi(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, Y_t^{\alpha_3,n}) \mathbb{E} \prod_{i=1}^{3} 1_{\alpha_i,n}(t), \tag{II.3}$$

In case (I), given $\alpha_1(0), \alpha_2(0), \alpha_3(0)$, we have

$$\mathbb{E}\phi(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, Y_t^{\alpha_3,n}) = Q_t^3 \phi(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\alpha_3(0)}),$$

and

$$\begin{aligned} \mathbb{E} \prod_{i=1}^{3} 1_{\alpha_i,n}(t) &= \prod_{i=1}^{3} \mathbb{E} 1_{\alpha_i,n}(t) \\ &= \left(\frac{1}{2}\right)^{3N}. \end{aligned}$$

Since the number of possible triples $(\alpha_1, \alpha_2, \alpha_3)$ corresponding to the three initial ancestors is equal to 2^{3N}, case (I) gives the contribution:

$$\begin{aligned} n^{-3} &\sum_{\substack{\alpha_1(0), \alpha_2(0), \alpha_3(0) = 1 \\ \alpha_i(0) \neq \alpha_j(0), i \neq j}} Q_t^3 \phi(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\alpha_3(0)}) \\ &= n^{-3} \sum_{\alpha_1(0), \alpha_2(0), \alpha_3(0) = 1} Q_t^3 \phi(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\alpha_3(0)}) \\ &- n^{-3} \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \sum_{\substack{\alpha_1(0), \alpha_2(0), \alpha_3(0) = 1 \\ \alpha_i(0) = \alpha_j(0)}} Q_t^3 \phi(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\alpha_3(0)}) \\ &- n^{-3} \sum_{\substack{\alpha_1(0), \alpha_2(0), \alpha_3(0) = 1 \\ \alpha_1(0) = \alpha_2(0) = \alpha_3(0)}} Q_t^3 \phi(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\alpha_3(0)}) \end{aligned}$$

$$= \left\langle Q_t^3\phi, (\mu_0^{(n)})^3 \right\rangle - n^{-1} \sum_{\substack{i,j=1 \\ i\neq j}}^{3} \left\langle \Phi_{ij} Q_t^3 \phi, (\mu_0^{(n)})^2 \right\rangle - n^{-2} \left\langle \Phi_{12}\Phi_{12} Q_t^3\phi, \mu_0^{(n)} \right\rangle .$$

Note that from Lemma II.1.8, $\left\langle \Phi_{ij} Q_t^3 \phi, (\mu_0^{(n)})^2 \right\rangle$ and $\left\langle \Phi_{12}\Phi_{12} Q_t^3\phi, \mu_0^{(n)} \right\rangle$ converge to finite limits and thus

$$n^{-3} \sum_{\substack{\alpha_1(0),\alpha_2(0),\alpha_3(0)=1 \\ \alpha_i(0)\neq\alpha_j(0), i\neq j}} Q_t^3\phi(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\alpha_3(0)}) = \left\langle Q_t^3\phi, (\mu_0^{(n)})^3 \right\rangle + o(1), \tag{II.4}$$

where for any function ψ we write $\psi = o(1)$ exactly when $\lim_{n\to\infty} \psi(n) = 0$.

For case (II), given $\alpha(0), \beta(0)$, and r,

$$\begin{aligned}
\mathbb{E}\phi(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, Y_t^{\alpha_3,n}) &= \mathbb{E}\mathbb{E}\left[\phi(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, Y_t^{\alpha_3,n}) \middle| \mathcal{F}_{r(n)}^n\right] \\
&= \frac{1}{6}\sum_{\substack{i,j=1 \\ i\neq j}}^{3} \mathbb{E}\mathbb{E}_{(Y_{r(n)}^{\alpha,n}, Y_{r(n)}^{\beta,n})} \left[(\Phi_{ij}\phi)(Y_{t-r(n)}^{\alpha,n}, Y_{t-r(n)}^{\beta,n})\right] \\
&= \frac{1}{6}\sum_{\substack{i,j=1 \\ i\neq j}}^{3} \mathbb{E}(\Phi_{ij} Q_{t-r(n)}^3 \phi)(Y_{r(n)}^{\alpha,n}, Y_{r(n)}^{\beta,n}) \\
&= \frac{1}{6}\sum_{\substack{i,j=1 \\ i\neq j}}^{3} (Q_{r(n)}^2 \Phi_{ij} Q_{t-r(n)}^3 \phi)(x_{\alpha(0)}, x_{\beta(0)}),
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\prod_{i=1}^{3} 1_{\alpha_i,n}(t) &= \left(\frac{1}{2}\right)^N \left(\frac{1}{2}\right)^N \left(\frac{1}{2}\right)^{N-r-1} \\
&= \left(\frac{1}{2}\right)^{3N-r-1} .
\end{aligned}$$

For any $\alpha(0), \beta(0)$,and r, there are $2^N \cdot 2^N \cdot 2^{N-r-1}$ corresponding $(\alpha_1, \alpha_2, \alpha_3)$ which result from binary branching over N steps, $2 \cdot \binom{3}{2}$ possible arrangements for $(\alpha_1, \alpha_2, \alpha_3)$, and $r \in \{0, 1, ..., N-1\}$. We thus arrive at the following contribution from case (II):

$$n^{-3}\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\i\neq j}}^{3}\sum_{\substack{\alpha(0),\beta(0)=1\\\alpha(0)\neq\beta(0)}} Q^2_{r(n)}\Phi_{ij}Q^3_{t-r(n)}\phi(x_{\alpha(0)},x_{\beta(0)})$$
$$= n^{-3}\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\i\neq j}}^{3}\sum_{\alpha(0),\beta(0)=1} Q^2_{r(n)}\Phi_{ij}Q^3_{t-r(n)}\phi(x_{\alpha(0)},x_{\beta(0)})$$
$$- n^{-3}\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\i\neq j}}^{3}\sum_{\alpha(0)=1} \Phi_{12}Q^2_{r(n)}\Phi_{ij}Q^3_{t-r(n)}\phi(x_{\alpha(0)})$$
$$= n^{-1}\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\i\neq j}}^{3}\left\langle Q^2_{r(n)}\Phi_{ij}Q^3_{t-r(n)}\phi,(\mu_0^{(n)})^2\right\rangle$$
$$- n^{-2}\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\i\neq j}}^{3}\left\langle \Phi_{12}Q^2_{r(n)}\Phi_{ij}Q^3_{t-r(n)}\phi,\mu_0^{(n)}\right\rangle.$$

With regards to the above two terms, Lemma II.1.10 implies that the second term will vanish as $n\to\infty$. Therefore, case (II) gives

$$n^{-3}\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\i\neq j}}^{3}\sum_{\substack{\alpha(0),\beta(0)=1\\\alpha(0)\neq\beta(0)}} Q^2_{r(n)}\Phi_{ij}Q^3_{t-r(n)}\phi(x_{\alpha(0)},x_{\beta(0)})$$
$$= n^{-1}\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\i\neq j}}^{3}\left\langle Q^2_{r(n)}\Phi_{ij}Q^3_{t-r(n)}\phi,(\mu_0^{(n)})^2\right\rangle + o(1) \tag{II.5}$$

Finally for case (III), given r_1, r_2, and $\beta_1(0)$

$$E\phi(Y_t^{\alpha_1,n},Y_t^{\alpha_2,n},Y_t^{\alpha_3,n}) = \mathbb{E}\mathbb{E}\left[\mathbb{E}\left[\phi(Y_t^{\alpha_1,n},Y_t^{\alpha_2,n},Y_t^{\alpha_3,n})\middle|\mathcal{F}^n_{r_2(n)}\right]\mathcal{F}^n_{r_1(n)}\right]$$
$$= \frac{1}{6}\sum_{\substack{i,j=1\\i\neq j}}^{3}\mathbb{E}\mathbb{E}\left[(\Phi_{ij}Q^3_{t-r_2(n)})(Y^{\alpha,n}_{r_2(n)},Y^{\beta_2,n}_{r_2(n)})\middle|\mathcal{F}^n_{r_1(n)}\right]$$

$$= \frac{1}{6} \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \mathbb{E}(\Phi_{12} Q^2_{r_2(n)-r_1(n)} \Phi_{ij} Q^3_{t-r_2(n)})(Y^{\beta_1,n}_{r_1(n)})$$

$$= \frac{1}{6} \sum_{\substack{i,j=1 \\ i \neq j}}^{3} (Q_{r_1(n)} \Phi_{12} Q^2_{r_2(n)-r_1(n)} \Phi_{ij} Q^3_{t-r_2(n)})(x_{\beta_1(0)})$$

and

$$\begin{aligned}
\mathbb{E} \prod_{i=1}^{3} 1_{\alpha_i,n}(t) &= \mathbb{E}\mathbb{E}\left[1_{\alpha_{6-i-j},n}(t) \middle| \mathcal{F}^n_{r_1(n)} \right] \mathbb{E}\left[1_{\alpha_i,n}(t) 1_{\alpha_j,n}(t) \middle| \mathcal{F}^n_{r_1(n)} \right] \\
&= \left(\frac{1}{2}\right)^{N-r_1-1} \mathbb{E} 1_{\beta_1,n}(r_1(n)) \mathbb{E}\mathbb{E}\left[\left[1_{\alpha_i,n}(t) 1_{\alpha_j,n}(t) \middle| \mathcal{F}^n_{r_1(n)} \right] \middle| \mathcal{F}^n_{r_2(n)} \right] \\
&= \left(\frac{1}{2}\right)^{N-r_1-1} \left(\frac{1}{2}\right)^{2(N-r_2-1)} \mathbb{E} 1_{\beta_1,n}(r_1(n)) \mathbb{E}\left[1_{\beta_2,n}(r_2(n)) \middle| \mathcal{F}^n_{r_1(n)} \right] \\
&= \left(\frac{1}{2}\right)^{3N-r_1-2r_2-3} \left(\frac{1}{2}\right)^{r_2-r_1} \mathbb{E} 1_{\beta_1,n}(r_1(n)) \\
&= \left(\frac{1}{2}\right)^{3N-r_1-r_2-2}
\end{aligned}$$

For any r_1, r_2, and $\beta_1(0)$, there are $2^{N-r_1-1} \cdot 2 \cdot 2^{N-r_2-1} \cdot 2^{N-r_2-1} \cdot 2^{r_2-r_1} \cdot 2^{r_1+1}$ corresponding $(\alpha_1, \alpha_2, \alpha_3)$ which result from binary branching over N steps and $2 \cdot \binom{3}{2}$ possible arrangements for $(\alpha_1, \alpha_2, \alpha_3)$. We thus arrive at the following contribution from case (III):

$$n^{-3} \sum_{r_2=0}^{N-1} \sum_{r_1=0}^{r_2} \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \sum_{\beta(0)=1} (Q_{r_1(n)} \Phi_{12} Q^2_{r_2(n)-r_1(n)} \Phi_{ij} Q^3_{t-r_2(n)} \phi)(x_{\beta(0)})$$

$$= n^{-2} \sum_{r_2=0}^{N-1} \sum_{r_1=0}^{r_2} \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \left\langle (Q_{r_1(n)} \Phi_{12} Q^2_{r_2(n)-r_1(n)} \Phi_{ij} Q^3_{t-r_2(n)} \phi), \mu_0^{(n)} \right\rangle . \qquad \text{(II.6)}$$

Consequently, from II.4, II.5, and II.6,

$$\mathbb{E}\left\langle \phi, \mu_t^3 \right\rangle = \left\langle Q_t^3 \phi, (\mu_0^{(n)})^3 \right\rangle + n^{-1} \sum_{r=0}^{N-1} \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \left\langle Q_{r(n)}^2 \Phi_{ij} Q_{t-r(n)}^3 \phi, (\mu_0^{(n)})^2 \right\rangle$$
$$+ n^{-2} \sum_{r_2=0}^{N-1} \sum_{r_1=0}^{r_2} \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \left\langle (Q_{r_1(n)} \Phi_{12} Q_{r_2(n)-r_1(n)}^2 \Phi_{ij} Q_{t-r_2(n)}^3 \phi), \mu_0^{(n)} \right\rangle + o(1) \quad \text{(II.7)}$$

In regards the fourth moment, if $\alpha_1(0), \alpha_2(0), \alpha_3(0)$, and $\alpha_4(0)$ are given, case (I) gives

$$\mathbb{E}\phi(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, Y_t^{\alpha_3,n}, Y_t^{\alpha_4,n}) = Q_t^4 \phi(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\alpha_3(0)}, x_{\alpha_4(0)}),$$

and

$$\mathbb{E} \prod_{i=1}^{4} 1_{\alpha_i,n}(t) = \left(\frac{1}{2}\right)^{4N}.$$

For any $\alpha_1(0)$, $\alpha_2(0)$, $\alpha_3(0)$, $\alpha_4(0)$, there are 2^{4N} corresponding $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ which result from binary branching over N steps. We thus arrive at the following contribution from case (I):

$$n^{-4} \sum_{\substack{\alpha_1(0),\alpha_2(0),\alpha_3(0),\alpha_4(0)=1 \\ \alpha_i(0)\neq\alpha_j(0), i\neq j}} Q_t^4 \phi(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\alpha_3(0)}, x_{\alpha_4(0)})$$
$$= n^{-4} \sum_{\alpha_1(0),\alpha_2(0),\alpha_3(0),\alpha_4(0)=1} Q_t^4 \phi(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\alpha_3(0)}, x_{\alpha_4(0)})$$
$$- n^{-4} \sum_{\alpha_1(0),\alpha_2(0),\alpha_3(0)=1} (\Phi_{ij} Q_t^4 \phi)(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\alpha_3(0)})$$
$$- n^{-4} \sum_{\alpha_1(0),\alpha_2(0)=1} (\Phi_{i_2 j_2} \Phi_{i_1 j_1} Q_t^4 \phi)(x_{\alpha_1(0)}, x_{\alpha_2(0)}) - n^{-4} \sum_{\alpha(0)=1} (\Phi_{12} \Phi_{i_2 j_2} \Phi_{i_1 j_1} Q_t^4 \phi)(x_{\alpha_1(0)})$$

$$= \left\langle Q^4\phi, (\mu_0^{(n)})^4 \right\rangle - n^{-1}\left\langle \Phi_{ij} Q_t^4\phi, (\mu_0^{(n)})^3 \right\rangle$$
$$- n^{-2}\left\langle \Phi_{i_2j_2}\Phi_{i_1j_1} Q_t^4\phi, (\mu_0^{(n)})^2 \right\rangle - n^{-3}\left\langle \Phi_{12}\Phi_{i_2j_2}\Phi_{i_1j_1} Q_t^4\phi, \mu_0^{(n)} \right\rangle$$

Again, from Lemma II.1.8 all but the first term on the right hand side will vanish as $n \to \infty$, and thus the above implies

$$n^{-4} \sum_{\substack{\alpha_1(0),\alpha_2(0),\alpha_3(0),\alpha_4(0)=1 \\ \alpha_i(0)\neq\alpha_j(0), i\neq j}} Q_t^4\phi(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\alpha_3(0)}, x_{\alpha_4(0)}) = \left\langle Q^4\phi, (\mu_0^{(n)})^4 \right\rangle + o(1). \tag{II.8}$$

For case (II), given $\alpha_1(0), \alpha_2(0), \beta(0)$ and r,

$$\begin{aligned}
\mathbb{E}\phi(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, Y_t^{\alpha_3,n}, Y_t^{\alpha_4,n}) &= \mathbb{E}\mathbb{E}\left[\phi(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, Y_t^{\alpha_3,n}, Y_t^{\alpha_4,n}) \middle| \mathcal{F}_{r(n)}^n \right] \\
&= \frac{1}{12}\sum_{\substack{i,j=1 \\ i\neq j}}^{4} \mathbb{E}(\Phi_{ij} Q_{t-r(n)}^4 \phi)(Y_{r(n)}^{\alpha_1,n}, Y_{r(n)}^{\alpha_2,n}, Y_{r(n)}^{\beta,n}) \\
&= \frac{1}{12}\sum_{\substack{i,j=1 \\ i\neq j}}^{4} (Q_{r(n)}^3 \Phi_{ij} Q_{t-r(n)}^4 \phi)(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\beta(0)}),
\end{aligned}$$

and if for any distinct $i, j \in \{1,2,3,4\}$ we define i', j' to be the exhaustive elements of $\{1,2,3,4\} \setminus \{i,j\}$,

$$\begin{aligned}
\mathbb{E}\prod_{i=1}^{4} 1_{\alpha_i,n}(t) &= (\mathbb{E}1_{\alpha_{i'},n}(t)) \left(\mathbb{E}1_{\alpha_{j'},n}(t)\right) \mathbb{E}\mathbb{E}\left[1_{\alpha_i,n}(t) 1_{\alpha_j,n}(t) \middle| \mathcal{F}_{r(n)}^n \right] \\
&= 2^{-N} \cdot 2^{-N} \cdot 2 \cdot 2^{-(N-r-1)} \cdot 2^{-(r+1)} \\
&= 2^{-(4N-r-1)}.
\end{aligned}$$

For any $\alpha_1(0)$, $\alpha_2(0)$, $\beta_1(0)$, and r, there are 2^{4N-r-1} corresponding $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ which result from binary branching over N steps and $2 \cdot \binom{4}{2}$ possible arrangements for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. We thus arrive at the following contribution from case (II):

$$n^{-4}\cdot\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\ i\neq j}}^{4}\sum_{\substack{\alpha_1(0),\alpha_2(0),\beta(0)=1\\ \alpha_1(0)\neq\alpha_2(0),\alpha_\ell(0)\neq\beta(0)\\ \ell=1,2}}(Q^3_{r(n)}\Phi_{ij}Q^4_{t-r(n)}\phi)(x_{\alpha_1(0)},x_{\alpha_2(0)},x_{\beta(0)})$$

$$= n^{-4}\cdot\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\ i\neq j}}^{4}\sum_{\alpha_1(0),\alpha_2(0),\beta(0)=1}(Q^3_{r(n)}\Phi_{ij}Q^4_{t-r(n)}\phi)(x_{\alpha_1(0)},x_{\alpha_2(0)},x_{\beta(0)})$$

$$- n^{-4}\cdot\sum_{r=0}^{N-1}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\sum_{\alpha(0),\beta(0)=1}(\Phi_{i_2j_2}Q^3_{r(n)}\Phi_{i_1j_1}Q^4_{t-r(n)}\phi)(x_{\alpha(0)},x_{\beta(0)})$$

$$- n^{-4}\cdot\sum_{r=0}^{N-1}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\sum_{\alpha(0)=1}(\Phi_{12}\Phi_{i_2j_2}Q^3_{r(n)}\Phi_{i_1j_1}Q^4_{t-r(n)}\phi)(x_{\alpha(0)})$$

$$= n^{-1}\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\ i\neq j}}^{4}\left\langle Q^3_{r(n)}\Phi_{ij}Q^4_{t-r(n)}\phi,(\mu_0^{(n)})^3\right\rangle$$

$$- n^{-2}\sum_{r=0}^{N-1}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\left\langle \Phi_{i_2j_2}Q^3_{r(n)}\Phi_{i_1j_1}Q^4_{t-r(n)}\phi,(\mu_0^{(n)})^2\right\rangle$$

$$- n^{-3}\sum_{r=0}^{N-1}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\left\langle \Phi_{12}\Phi_{i_2j_2}Q^3_{r(n)}\Phi_{i_1j_1}Q^4_{t-r(n)}\phi,(\mu_0^{(n)})\right\rangle .$$

Again from Lemma II.1.10, all but the first term on the right hand side will vanish as $n\to\infty$ and thus,

$$n^{-4}\cdot\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\ i\neq j}}^{4}\sum_{\substack{\alpha_1(0),\alpha_2(0),\beta(0)=1\\ \alpha_1(0)\neq\alpha_2(0),\alpha_\ell(0)\neq\beta(0)\\ \ell=1,2}}(Q^3_{r(n)}\Phi_{ij}Q^4_{t-r(n)}\phi)(x_{\alpha_1(0)},x_{\alpha_2(0)},x_{\beta(0)})$$

$$= n^{-1}\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\ i\neq j}}^{4}\left\langle Q^3_{r(n)}\Phi_{ij}Q^4_{t-r(n)}\phi,(\mu_0^{(n)})^3\right\rangle + o(1). \quad \text{(II.9)}$$

Cases (III) and (IV) will now be considered together. For case (III), given $\beta_1(0)$,

$\beta_2(0)$, r_1, and r_2,

$$\begin{aligned}
\mathbb{E}\phi(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, Y_t^{\alpha_3,n}, Y_t^{\alpha_4,n}) &= 2\mathbb{E}\mathbb{E}\left[\phi(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, Y_t^{\alpha_3,n}, Y_t^{\alpha_4,n}); r_1 < r_2 \middle| \mathcal{F}^n_{r_2(n)}\right] \\
&= \frac{1}{6}\sum_{\substack{i,j=1 \\ i\neq j}}^{4} \mathbb{E}\mathbb{E}\left[(\Phi_{ij}Q^4_{t-r_2(n)}\phi)(Y^{\alpha_{i'},n}_{r_2(n)}, Y^{\alpha_{j'},n}_{r_2(n)}, Y^{\beta_1,n}_{r_2(n)}) \middle| \mathcal{F}^n_{r_1(n)}\right] \\
&= \frac{1}{12}\sum_{\substack{i_1,j_1=1 \\ i_1\neq j_1}}^{4} \sum_{\substack{i_2,j_2=1 \\ i_2\neq j_2 \\ i_2,j_2\neq i_1}}^{3} \mathbb{E}(\Phi_{i_2j_2}Q^3_{r_2(n)-r_1(n)}\Phi_{i_1j_1}Q^4_{t-r_2(n)}\phi)(Y^{\beta_1,n}_{r_1(n)}, Y^{\beta_2,n}_{r_1(n)}) \\
&= \frac{1}{12}\sum_{\substack{i_1,j_1=1 \\ i_1\neq j_1}}^{4} \sum_{\substack{i_2,j_2=1 \\ i_2\neq j_2 \\ i_2,j_2\neq i_1}}^{3} (Q^2_{r_1(n)}\Phi_{i_2j_2}Q^3_{r_2(n)-r_1(n)}\Phi_{i_1j_1}Q^4_{t-r_2(n)}\phi)(x_{\beta_1(0)}, x_{\beta_2(0)})
\end{aligned}$$

and,

$$\begin{aligned}
\mathbb{E}\prod_{i=1}^{4} 1_{\alpha_i,n}(t) &= \left(\mathbb{E}1_{\alpha_i,n}(t)1_{\alpha_j,n}(t)\right)\left(\mathbb{E}1_{\alpha_{i'},n}(t)1_{\alpha_{j'},n}(t)\right) \\
&= 2^{-(2N-r_1-1)}2^{-(2N-r_2-1)} \\
&= 2^{-(4N-r_1-r_2-2)}.
\end{aligned}$$

For case (IV), given $\alpha(0)$, $\beta_1(0)$, r_1, and r_2,

$$\begin{aligned}
\mathbb{E}\phi(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, Y_t^{\alpha_3,n}, Y_t^{\alpha_4,n}) &= \mathbb{E}\mathbb{E}\left[\phi(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, Y_t^{\alpha_3,n}, Y_t^{\alpha_4,n}) \middle| \mathcal{F}^n_{r_2(n)}\right] \\
&= \frac{1}{12}\sum_{\substack{i,j=1 \\ i\neq j}}^{4} \mathbb{E}\mathbb{E}\left[(\Phi_{ij}Q^4_{t-r_2(n)}\phi)(Y^{\alpha_1,n}_{r_2(n)}, Y^{\alpha_2,n}_{r_2(n)}, Y^{\beta_2,n}_{r_2(n)}) \middle| \mathcal{F}^n_{r_1(n)}\right] \\
&= \frac{1}{48}\sum_{\substack{i_1,j_1=1 \\ i_1\neq j_1}}^{4} \sum_{\substack{i_2,j_2=1 \\ i_2\neq j_2 \\ i_2=i_1 \text{ or } j_2=i_1}}^{3} \mathbb{E}(\Phi_{i_2j_2}Q^3_{r_2(n)-r_1(n)}\Phi_{i_1j_1}Q^4_{t-r_2(n)}\phi)(Y^{\alpha,n}_{r_1(n)}, Y^{\beta_1,n}_{r_1(n)})
\end{aligned}$$

$$= \frac{1}{48} \sum_{\substack{i_1,j_1=1 \\ i_1 \neq j_1}}^{4} \sum_{\substack{i_2,j_2=1 \\ i_2 \neq j_2 \\ i_2=i_1 \text{ or } j_2=i_1}}^{3} (Q^2_{r_1(n)} \Phi_{i_2j_2} Q^3_{r_2(n)-r_1(n)} \Phi_{i_1j_1} Q^4_{t-r_2(n)} \phi)(x_{\alpha(0)}, x_{\beta_1(0)}).$$

and,

$$\begin{aligned}
\mathbb{E} \prod_{i=1}^{4} 1_{\alpha_i,n}(t) &= 2^{-N} \mathbb{E} \prod_{i=1}^{3} 1_{\alpha_i,n}(t) \\
&= 2^{-N} 2^{-(3N-r_1-r_2-2)} \\
&= 2^{-(4N-r_1-r_2-2)}.
\end{aligned}$$

Given two initial ancestors, there are $2^{4N-r_1-r_2-2}$ possible trees, and a possible $2 \cdot \binom{4}{2}$ arrangements for α_1, α_2, α_3, α_4 upon each tree (requiring $r_1 < r_2$) that result in case(III). Furthermore, there are $2^{4N-r_1-r_2-2}$ possible trees, and a possible $2 \cdot \binom{2}{1} \cdot \binom{3}{2} \cdot \binom{4}{3}$ arrangements for α_1, α_2, α_3, α_4 upon each tree that result in case(IV). It follows that the contribution coming from the sum of case(III) and case(IV) is given by

$$\begin{aligned}
&n^{-4} \sum_{r_2=0}^{N-1} \sum_{r_1=0}^{r_2} \sum_{\substack{i_1,j_1=1 \\ i_1 \neq j_1}}^{4} \sum_{\substack{i_2,j_2=1 \\ i_2 \neq j_2}}^{3} \sum_{\substack{\alpha(0),\beta(0)=1 \\ \alpha(0) \neq \beta(0)}} (Q^2_{r_1(n)} \Phi_{i_2j_2} Q^3_{r_2(n)-r_1(n)} \Phi_{i_1j_1} Q^4_{t-r_2(n)} \phi)(x_{\alpha(0)}, x_{\beta(0)}) \\
&= n^{-4} \sum_{r_2=0}^{N-1} \sum_{r_1=0}^{r_2} \sum_{\substack{i_1,j_1=1 \\ i_1 \neq j_1}}^{4} \sum_{\substack{i_2,j_2=1 \\ i_2 \neq j_2}}^{3} \sum_{\alpha(0),\beta(0)=1} (Q^2_{r_1(n)} \Phi_{i_2j_2} Q^3_{r_2(n)-r_1(n)} \Phi_{i_1j_1} Q^4_{t-r_2(n)} \phi)(x_{\alpha(0)}, x_{\beta(0)}) \\
&- n^{-4} \sum_{r_2=0}^{N-1} \sum_{r_1=0}^{r_2} \sum_{\substack{i_1,j_1=1 \\ i_1 \neq j_1}}^{4} \sum_{\substack{i_2,j_2=1 \\ i_2 \neq j_2}}^{3} \sum_{\alpha(0)=1} (\Phi_{12} Q^2_{r_1(n)} \Phi_{i_2j_2} Q^3_{r_2(n)-r_1(n)} \Phi_{i_1j_1} Q^4_{t-r_2(n)} \phi)(x_{\alpha(0)}) \\
&= n^{-2} \sum_{r_2=0}^{N-1} \sum_{r_1=0}^{r_2} \sum_{\substack{i_1,j_1=1 \\ i_1 \neq j_1}}^{4} \sum_{\substack{i_2,j_2=1 \\ i_2 \neq j_2}}^{3} \left\langle Q^2_{r_1(n)} \Phi_{i_2j_2} Q^3_{r_2(n)-r_1(n)} \Phi_{i_1j_1} Q^4_{t-r_2(n)} \phi, (\mu_0^{(n)})^2 \right\rangle \\
&- n^{-3} \sum_{r_2=0}^{N-1} \sum_{r_1=0}^{r_2} \sum_{\substack{i_1,j_1=1 \\ i_1 \neq j_1}}^{4} \sum_{\substack{i_2,j_2=1 \\ i_2 \neq j_2}}^{3} \left\langle \Phi_{12} \Phi_{12} Q^2_{r_1(n)} \Phi_{i_2j_2} Q^3_{r_2(n)-r_1(n)} \Phi_{i_1j_1} Q^4_{t-r_2(n)} \phi, \mu_0^{(n)} \right\rangle.
\end{aligned}$$

Thus, again from Lemma II.1.10, the second term vanishes as $n \to \infty$, and we have the contribution

$$n^{-4}\sum_{r_2=0}^{N-1}\sum_{r_1=0}^{r_2}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\sum_{\substack{\alpha(0),\beta(0)=1\\ \alpha(0)\neq\beta(0)}}(Q^2_{r_1(n)}\Phi_{i_2j_2}Q^3_{r_2(n)-r_1(n)}\Phi_{i_1j_1}Q^4_{t-r_2(n)}\phi)(x_{\alpha(0)},x_{\beta(0)})$$

$$= n^{-2}\sum_{r_2=0}^{N-1}\sum_{r_1=0}^{r_2}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\left\langle Q^2_{r_1(n)}\Phi_{i_2j_2}Q^3_{r_2(n)-r_1(n)}\Phi_{i_1j_1}Q^4_{t-r_2(n)}\phi,(\mu_0^{(n)})^2\right\rangle + o(1). \tag{II.10}$$

Considering subcase(V)(A), given r_1, r_2, r_3, and $\beta_1(0)$,

$$\mathbb{E}\phi(Y_t^{\alpha_1,n},Y_t^{\alpha_2,n},Y_t^{\alpha_3,n},Y_t^{\alpha_4,n}) = 2\mathbb{E}\mathbb{E}\left[\phi(Y_t^{\alpha_1,n},Y_t^{\alpha_2,n},Y_t^{\alpha_3,n},Y_t^{\alpha_4,n});r_2<r_3\middle|\mathcal{F}^n_{r_3(n)}\right]$$

$$= \frac{1}{6}\sum_{\substack{i,j=1\\ i\neq j}}^{4}\mathbb{E}(\Phi_{ij}Q^4_{t-r_3(n)}\phi)(Y^{\alpha_1,n}_{r_3(n)},Y^{\alpha_2,n}_{r_3(n)},Y^{\beta_3,n}_{r_3(n)})$$

$$= \frac{1}{6}\sum_{\substack{i,j=1\\ i\neq j}}^{4}\mathbb{E}\mathbb{E}\left[(\Phi_{ij}Q^4_{t-r_3(n)}\phi)(Y^{\alpha_1,n}_{r_3(n)},Y^{\alpha_2,n}_{r_3(n)},Y^{\beta_3,n}_{r_3(n)})\middle|\mathcal{F}^n_{r_2(n)}\right]$$

$$= \frac{1}{12}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2\\ i_2,j_2\neq i_1}}^{3}\mathbb{E}(\Phi_{i_2j_2}Q^3_{r_3(n)-r_2(n)}\Phi_{i_1j_1}Q^4_{t-r_3(n)}\phi)(Y^{\beta_2,n}_{r_2(n)},Y^{\beta_3,n}_{r_2(n)})$$

$$= \frac{1}{12}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2\\ i_2,j_2\neq i_1}}^{3}\mathbb{E}\mathbb{E}\left[(\Phi_{i_2j_2}Q^3_{r_3(n)-r_2(n)}\Phi_{i_1j_1}Q^4_{t-r_3(n)}\phi)(Y^{\beta_2,n}_{r_2(n)},Y^{\beta_3,n}_{r_2(n)})\middle|\mathcal{F}^n_{r_1(n)}\right]$$

$$= \frac{1}{12}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2\\ i_2,j_2\neq i_1}}^{3}\mathbb{E}(\Phi_{12}Q^2_{r_2(n)-r_1(n)}\Phi_{i_2j_2}Q^3_{r_3(n)-r_2(n)}\Phi_{i_1j_1}Q^4_{t-r_3(n)}\phi)(Y^{\beta_1,n}_{r_1(n)})$$

$$= \frac{1}{12}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2\\ i_2,j_2\neq i_1}}^{3}(Q_{r_1(n)}\Phi_{12}Q^2_{r_2(n)-r_1(n)}\Phi_{i_2j_2}Q^3_{r_3(n)-r_2(n)}\Phi_{i_1j_1}Q^4_{t-r_3(n)}\phi)(x_{\beta_1(0)}),$$

and

$$\begin{aligned}
\mathbb{E}\prod_{i=1}^{4} 1_{\alpha_i,n}(t) &= \mathbb{E}\mathbb{E}\left[\prod_{i=1}^{4} 1_{\alpha_i,n}(t)\middle|\mathcal{F}^n_{r_1(n)}\right] \\
&= \mathbb{E}\left\{\mathbb{E}\left[1_{\alpha_i,n}(t)1_{\alpha_j,n}(t)\middle|\mathcal{F}^n_{r_1(n)}\right]\cdot\mathbb{E}\left[1_{\alpha_{i'},n}(t)1_{\alpha_{j'},n}(t)\middle|\mathcal{F}^n_{r_1(n)}\right]\right\} \\
&= \mathbb{E}\left\{\mathbb{E}\left[\mathbb{E}\left[1_{\alpha_i,n}(t)1_{\alpha_j,n}(t)\middle|\mathcal{F}^n_{r_3(n)}\right]\middle|\mathcal{F}^n_{r_1(n)}\right]\cdot\mathbb{E}\left[\mathbb{E}\left[1_{\alpha_{i'},n}(t)1_{\alpha_{j'},n}(t)\middle|\mathcal{F}^n_{r_2(n)}\right]\middle|\mathcal{F}^n_{r_1(n)}\right]\right\} \\
&= 2^{-(2N-2r_3-2)}\cdot 2^{-(2N-2r_2-2)}\mathbb{E}\left\{\mathbb{E}\left[1_{\beta_3,n}(r_3(n))\middle|\mathcal{F}^n_{r_1(n)}\right]\cdot\mathbb{E}\left[1_{\beta_2,n}(r_2(n))\middle|\mathcal{F}^n_{r_1(n)}\right]\right\} \\
&= 2^{-(4N-2r_3-2r_2-4)}\cdot 2^{-(r_3-r_1)}\cdot 2^{-(r_2-r_1)}\mathbb{E}\left\{1_{\beta_1,n}(r_1(n))\right\} \\
&= 2^{-(4N-r_3-r_2-2r_1-4)}\cdot 2^{-(r_1+1)} \\
&= 2^{-(4N-r_3-r_2-r_1-3)}.
\end{aligned}$$

Then for subcase(V)(B), given r_1, r_2, r_3, and $\beta_1(0)$,

$$\begin{aligned}
\mathbb{E}\phi(Y_t^{\alpha_1,n},Y_t^{\alpha_2,n},Y_t^{\alpha_3,n},Y_t^{\alpha_4,n}) &= \mathbb{E}\mathbb{E}\left[\phi(Y_t^{\alpha_1,n},Y_t^{\alpha_2,n},Y_t^{\alpha_3,n},Y_t^{\alpha_4,n})\middle|\mathcal{F}^n_{r_3(n)}\right] \\
&= \frac{1}{12}\sum_{\substack{i,j=1\\ i\neq j}}^{4}\mathbb{E}(\Phi_{ij}Q^4_{t-r_3(n)}\phi)(Y^{\alpha_1,n}_{r_3(n)},Y^{\alpha_2,n}_{r_3(n)},Y^{\beta_3,n}_{r_3(n)}) \\
&= \frac{1}{12}\sum_{\substack{i,j=1\\ i\neq j}}^{4}\mathbb{E}\mathbb{E}\left[(\Phi_{ij}Q^4_{t-r_3(n)}\phi)(Y^{\alpha_1,n}_{r_3(n)},Y^{\alpha_2,n}_{r_3(n)},Y^{\beta_3,n}_{r_3(n)})\middle|\mathcal{F}^n_{r_2(n)}\right] \\
&= \frac{1}{48}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2\\ i_2=i_1\text{ or }j_2=i_1}}^{3}\mathbb{E}(\Phi_{i_2j_2}Q^3_{r_3(n)-r_2(n)}\Phi_{i_1j_1}Q^4_{t-r_3(n)}\phi)(Y^{\alpha_1,n}_{r_2(n)},Y^{\beta_2,n}_{r_2(n)}) \\
&= \frac{1}{48}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2\\ i_2=i_1\text{ or }j_2=i_1}}^{3}\mathbb{E}\mathbb{E}\left[(\Phi_{i_2j_2}Q^3_{r_3(n)-r_2(n)}\Phi_{i_1j_1}Q^4_{t-r_3(n)}\phi)(Y^{\alpha_1,n}_{r_2(n)},Y^{\beta_2,n}_{r_2(n)})\middle|\mathcal{F}^n_{r_1(n)}\right] \\
&= \frac{1}{48}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2\\ i_2=i_1\text{ or }j_2=i_1}}^{3}\mathbb{E}(\Phi_{12}Q^2_{r_2(n)-r_1(n)}\Phi_{i_2j_2}Q^3_{r_3(n)-r_2(n)}\Phi_{i_1j_1}Q^4_{t-r_3(n)}\phi)(Y^{\beta_1,n}_{r_1(n)})
\end{aligned}$$

$$= \frac{1}{48} \sum_{\substack{i_1, j_1 = 1 \\ i_1 \neq j_1}}^{4} \sum_{\substack{i_2, j_2 = 1 \\ i_2 \neq j_2 \\ i_2 = i_1 \text{ or } j_2 = i_1}}^{3} (Q_{r_1(n)} \Phi_{12} Q^2_{r_2(n) - r_1(n)} \Phi_{i_2 j_2} Q^3_{r_3(n) - r_2(n)} \Phi_{i_1 j_1} Q^4_{t - r_3(n)} \phi)(x_{\beta_1(0)}),$$

and

$$\begin{aligned}
\mathbb{E} \prod_{i=1}^{4} 1_{\alpha_i, n}(t) &= \mathbb{E}\mathbb{E}\left[\prod_{i=1}^{4} 1_{\alpha_i, n}(t) \middle| \mathcal{F}^n_{r_1(n)} \right] \\
&= 2^{-(N - r_1 - 1)} \mathbb{E} \left\{ 1_{\beta_1, n}(r_1(n)) \cdot \mathbb{E} \left[1_{\alpha_i, n}(t) 1_{\alpha_j, n}(t) 1_{\alpha_{i'}, n}(t) \middle| \mathcal{F}^n_{r_1(n)} \right] \right\} \\
&= 2^{-(N - r_1 - 1)} \mathbb{E} \left\{ 1_{\beta_1, n}(r_1(n)) \cdot \mathbb{E} \left[\mathbb{E} \left[1_{\alpha_i, n}(t) 1_{\alpha_j, n}(t) 1_{\alpha_{i'}, n}(t) \middle| \mathcal{F}^n_{r_2(n)} \right] \middle| \mathcal{F}^n_{r_1(n)} \right] \right\} \\
&= 2^{-(2N - r_1 - r_2 - 2)} \\
&\quad \times \mathbb{E} \left\{ 1_{\beta_1, n}(r_1(n)) \cdot \mathbb{E} \left[1_{\beta_2, n}(r_2(n)) \cdot \mathbb{E} \left[\mathbb{E} \left[1_{\alpha_i, n}(t) 1_{\alpha_j, n}(t) \middle| \mathcal{F}^n_{r_3(n)} \right] \middle| \mathcal{F}^n_{r_2(n)} \right] \middle| \mathcal{F}^n_{r_1(n)} \right] \right\} \\
&= 2^{-(2N - r_1 - r_2 - 2)} \cdot 2^{-2(N - r_3 - 1)} \cdot 2^{-(r_3 - r_2)} \mathbb{E} \left\{ \mathbb{E} \left[1_{\beta_2, n}(r_2(n)) \middle| \mathcal{F}^n_{r_1(n)} \right] \right\} \\
&= 2^{-(4N - r_1 - 2r_2 - r_3 - 4)} \cdot 2^{-(r_2 - r_1)} \cdot 2^{-(r_1 + 1)} \\
&= 2^{-(4N - r_1 - r_2 - r_3 - 3)}
\end{aligned}$$

Given one initial ancestor, there are $2^{4N - r_1 - r_2 - r_3 - 3}$ possible trees, and a possible $\binom{1}{1} \cdot \binom{2}{1} \cdot \binom{4}{2}$ arrangements for α_1, α_2, α_3, α_4 upon each tree (requiring $r_2 < r_3$) that result in case(V)(A). Furthermore, there are $2^{4N - r_1 - r_2 - r_3 - 3}$ possible trees, and a possible $\binom{1}{1} \cdot \binom{2}{1} \cdot \binom{3}{2} \cdot \binom{4}{3}$ arrangements for α_1, α_2, α_3, α_4 upon each tree that result in case(V)(B). It follows that the contribution coming from the sum of subcase(V)(A) and subcase(V)(B) is given by

$$n^{-4}\sum_{r_3=0}^{N-1}\sum_{r_2=0}^{r_3}\sum_{r_1=0}^{r_2}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\sum_{\alpha(0)=1}$$

$$(Q_{r_1(n)}\Phi_{12}Q^2_{r_2(n)-r_1(n)}\Phi_{i_2j_2}Q^3_{r_3(n)-r_2(n)}\Phi_{i_1j_1}Q^4_{t-r_3(n)}\phi)(x_{\alpha(0)})$$

$$= n^{-3}\sum_{r_3=0}^{N-1}\sum_{r_2=0}^{r_3}\sum_{r_1=0}^{r_2}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}$$

$$\left\langle Q_{r_1(n)}\Phi_{12}Q^2_{r_2(n)-r_1(n)}\Phi_{i_2j_2}Q^3_{r_3(n)-r_2(n)}\Phi_{i_1j_1}Q^4_{t-r_3(n)}\phi, \mu_0^{(n)}\right\rangle. \tag{II.11}$$

Therefore, from II.7, II.8, II.9, II.10, and II.11, we conclude this section with the following Lemma.

Lemma II.2.1. *Given $\phi \in C_K^2(\mathbb{R}^{3\times d})$ and $\psi \in C_K^2(\mathbb{R}^{4\times d})$, for all $n \in \mathbb{N}$, $t > 0$, it follows that*

$$(i)\ \mathbb{E}\left\langle \phi, (\mu_t^{(n)})^3\right\rangle = \left\langle Q_t^3\phi, (\mu_0^{(n)})^3\right\rangle + n^{-1}\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\ i\neq j}}^{3}\left\langle Q^2_{r(n)}\Phi_{ij}Q^3_{t-r(n)}\phi, (\mu_0^{(n)})^2\right\rangle$$

$$+ n^{-2}\sum_{r_2=0}^{N-1}\sum_{r_1=0}^{r_2}\sum_{\substack{i,j=1\\ i\neq j}}^{3}\left\langle (Q_{r_1(n)}\Phi_{12}Q^2_{r_2(n)-r_1(n)}\Phi_{ij}Q^3_{t-r_2(n)}\phi), \mu_0^{(n)}\right\rangle + o(1), \tag{II.12}$$

and

$$(ii)\ \mathbb{E}\left\langle \psi, (\mu_t^{(n)})^4 \right\rangle = \left\langle Q_t^4 \psi, (\mu_0^{(n)})^4 \right\rangle + n^{-1} \sum_{r=0}^{N-1} \sum_{\substack{i,j=1 \\ i \neq j}}^{4} \left\langle Q_{r(n)}^3 \Phi_{ij} Q_{t-r(n)}^4 \psi, (\mu_0^{(n)})^3 \right\rangle$$

$$+ n^{-2} \sum_{r_2=0}^{N-1} \sum_{r_1=0}^{r_2} \sum_{\substack{i_1,j_1=1 \\ i_1 \neq j_1}}^{4} \sum_{\substack{i_2,j_2=1 \\ i_2 \neq j_2}}^{3} \left\langle Q_{r_1(n)}^2 \Phi_{i_2 j_2} Q_{r_2(n)-r_1(n)}^3 \Phi_{i_1 j_1} Q_{t-r_2(n)}^4 \psi, (\mu_0^{(n)})^2 \right\rangle$$

$$+ n^{-3} \sum_{r_3=0}^{N-1} \sum_{r_2=0}^{r_3} \sum_{r_1=0}^{r_2} \sum_{\substack{i_1,j_1=1 \\ i_1 \neq j_1}}^{4} \sum_{\substack{i_2,j_2=1 \\ i_2 \neq j_2}}^{3}$$

$$\times \left\langle Q_{r_1(n)} \Phi_{12} Q_{r_2(n)-r_1(n)}^2 \Phi_{i_2 j_2} Q_{r_3(n)-r_2(n)}^3 \Phi_{i_1 j_1} Q_{t-r_3(n)}^4 \psi, \mu_0^{(n)} \right\rangle$$

$$+ o(1). \tag{II.13}$$

Thus a formula for both the third and fourth moment of the branching process has been found. With the exception of a some small technicalities to mention, the moment formulae for the superprocess will follow almost immediately from Lemmas II.1.10, II.1.11, and II.2.1.

II.3 Moment Calculations For The Superprocess

Since the preliminary calculations necessary for this section have already been worked out, we go straight to the moment formulae.

Lemma II.3.1. *Given $\phi_k, \psi_j \in C_K^\infty(\mathbb{R}^d)$, $k = 1,2,3$, $j = 1,2,3,4$, for any $t \geq 0$, the following hold:*

(i) $\mathbb{E}\left\langle \phi_1 \otimes \phi_2 \otimes \phi_3, \mu_t^3 \right\rangle$

$$= \left\langle Q_t^3(\phi_1 \otimes \phi_2 \otimes \phi_3), \mu_0^3 \right\rangle + \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \int_0^t ds \, \left\langle Q_s^2 \Phi_{ij} Q_{t-s}^3(\phi_1 \otimes \phi_2 \otimes \phi_3), \mu_0^2 \right\rangle$$

$$+ \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \int_0^t ds_2 \int_0^{s_2} ds_1 \, \left\langle Q_{s_1} \Phi_{12} Q_{s_2-s_1}^2 \Phi_{ij} Q_{t-s_2}^3(\phi_1 \otimes \phi_2 \otimes \phi_3), \mu_0 \right\rangle \tag{II.14}$$

and

(ii) $\mathbb{E}\left\langle (\psi_1 \otimes \psi_2 \otimes \psi_3 \otimes \psi_4), \mu_t^4 \right\rangle$

$$= \left\langle Q_t^4(\psi_1 \otimes \psi_2 \otimes \psi_3 \otimes \psi_4), \mu_0^4 \right\rangle + \sum_{\substack{i,j=1 \\ i \neq j}}^{4} \int_0^t ds \, \left\langle Q_s^3 \Phi_{ij} Q_{t-s}^4(\psi_1 \otimes \psi_2 \otimes \psi_3 \otimes \psi_4), \mu_0^3 \right\rangle$$

$$+ \sum_{\substack{i_1,j_1=1 \\ i_1 \neq j_1}}^{4} \sum_{\substack{i_2,j_2=1 \\ i_2 \neq j_2}}^{3} \int_0^t ds_2 \int_0^{s_2} ds_1 \, \left\langle Q_{s_1}^2 \Phi_{i_2 j_2} Q_{s_2-s_1}^3 \Phi_{i_1 j_1} Q_{t-s_2}^4(\psi_1 \otimes \psi_2 \otimes \psi_3 \otimes \psi_4), \mu_0^2 \right\rangle$$

$$+ \sum_{\substack{i_1,j_1=1 \\ i_1 \neq j_1}}^{4} \sum_{\substack{i_2,j_2=1 \\ i_2 \neq j_2}}^{3} \int_0^t ds_3 \int_0^{s_3} ds_2 \int_0^{s_2} ds_1$$

$$\times \left\langle Q_{s_1} \Phi_{12} Q_{s_2-s_1}^2 \Phi_{i_2 j_2} Q_{s_3-s_2}^3 \Phi_{i_1 j_1} Q_{t-s_3}^4(\psi_1 \otimes \psi_2 \otimes \psi_3 \otimes \psi_4), \mu_0 \right\rangle \tag{II.15}$$

Proof: To begin, note that Lemma II.1.8 implies $(\mu_0^{(n)})^\ell \Rightarrow \mu_0^\ell$ for any $\ell \in \mathbb{N}$, and thus the first term of the right-hand sides of II.12 and II.13 converge respectively to the first term of the right-hand sides of II.14 and II.15 as $n \to \infty$. Since Q_t^k is a strongly continuous contraction semigroup for $k \in \mathbb{N}$ (Lemma I.1.4), for any $\phi \in C_b(\mathbb{R}^d)$ which satisfies

$\|\phi\|_{bL} = 1,$

$$\left\|Q_t^k \phi\right\|_\infty \leq \|\phi\|_\infty$$
$$\leq 1,$$

and

$$\sup_{x \neq y} \frac{\left|Q_t^k \phi(x) - Q_t^k \phi(y)\right|}{|x-y|} \leq \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x-y|}$$
$$\leq 1.$$

Thus, for any $k \in N$, $\|\phi\|_{bL} = 1$ implies $\left\|Q_t^k \phi\right\|_{bL} \leq 1$. From Lemma II.1.10, the remaining terms on the right-hand sides of II.12 and II.13 converge respectively to the remaining terms of the right-hand sides of II.14 and II.15 as $n \to \infty$. It remains to show that the left hand sides of II.12 and II.13 converge respectively to the left hand sides of II.14 and II.15, but this follows immediately from Lemma II.1.11 □

To conclude this section, the above will be used to find the needed extensions to moment formulae.

Theorem II.3.2. *Let $\phi \in C_K^\infty(\mathbb{R}^{3\times d})$ and $\psi \in C_K^\infty(\mathbb{R}^{4\times d})$ be respectfully defined by*

$$\phi = \phi_1 \otimes \phi_2 \otimes \phi_3$$

and

$$\psi = \psi_1 \otimes \psi_2 \otimes \psi_3 \otimes \psi_4,$$

for $\phi_i,\ \psi_j \in C_K^\infty(\mathbb{R}^d)$, $i=1,2,3,\ j=1,2,3,4$. *Then for all* $0 \le t_1 \le t_2 \le t_3 \le t_4 < \infty$,

$$
\begin{aligned}
&(i)\ \mathbb{E}\langle \phi, \mu_{t_1}\mu_{t_2}\mu_{t_3}\rangle \\
&= \langle Q^3_{t_1}\pi_1 Q^2_{t_2-t_1}\pi_1 Q_{t_3-t_2}\phi, \mu_0^3\rangle + \sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_0^{t_1} ds\ \langle Q^2_s \Phi_{ij} Q^3_{t_1-s}\pi_1 Q^2_{t_2-t_1}\pi_1 Q_{t_3-t_2}\phi, \mu_0^2\rangle \\
&+ \sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_0^{t_1} ds_2 \int_0^{s_2} ds_1\ \langle Q_{s_1}\Phi_{12}Q^2_{s_2-s_1}\Phi_{ij}Q^3_{t_1-s_2}\pi_1 Q^2_{t_2-t_1}\pi_1 Q_{t_3-t_2}\phi, \mu_0\rangle \\
&+ \int_{t_1}^{t_2} ds\ \langle Q^2_{t_1}\pi_1 Q_{s-t_1}\Phi_{12}Q^2_{t_2-s}\pi_1 Q_{t_3-t_2}\phi, \mu_0^2\rangle \\
&+ \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1\ \langle Q_{s_1}\Phi_{12}Q^2_{t_1-s_1}\pi_1 Q_{s_2-t_1}\Phi_{12}Q^2_{t_2-s_2}\pi_1 Q_{t_3-t_2}\phi, \mu_0\rangle
\end{aligned}
\tag{II.16}
$$

and

$$
\begin{aligned}
&(ii)\ \mathbb{E}\langle \psi, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4}\rangle \\
&= \langle Q^4_{t_1}\pi_1 Q^3_{t_2-t_1}\pi_1 Q^2_{t_3-t_2}\pi_1 Q_{t_4-t_3}\psi, \mu_0^4\rangle \\
&+ \sum_{\substack{i,j=1\\ i\neq j}}^{4}\int_0^{t_1} ds\ \langle Q^3_s\Phi_{ij}Q^4_{t_1-s}\pi_1 Q^3_{t_2-t_1}\pi_1 Q^2_{t_3-t_2}\pi_1 Q_{t_4-t_3}\psi, \mu_0^3\rangle \\
&+ \sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_{t_1}^{t_2} ds\ \langle Q^3_{t_1}\pi_1 Q^2_{s-t_1}\Phi_{ij}Q^3_{t_2-s}\pi_1 Q^2_{t_3-t_2}\pi_1 Q_{t_4-t_3}\psi, \mu_0^3\rangle \\
&+ \int_{t_2}^{t_3} ds\ \langle Q^3_{t_1}\pi_1 Q^2_{t_2-t_1}\pi_1 Q_{s-t_2}\Phi_{12}Q^2_{t_3-s}\pi_1 Q_{t_4-t_3}\psi, \mu_0^3\rangle \\
&+ \sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\ \sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\int_0^{t_1} ds_2\int_0^{s_2} ds_1 \\
&\qquad \times \langle Q^2_{s_1}\Phi_{i_2j_2}Q^3_{s_2-s_1}\Phi_{i_1j_1}Q^4_{t_1-s_2}\pi_1 Q^3_{t_2-t_1}\pi_1 Q^2_{t_3-t_2}\pi_1 Q_{t_4-t_3}\psi, \mu_0^2\rangle \\
&+ \sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{3}\ \sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\int_{t_1}^{t_2} ds_2\int_0^{t_1} ds_1 \\
&\qquad \times \langle Q^2_{s_1}\Phi_{i_2j_2}Q^3_{t_1-s_1}\pi_1 Q^2_{s_2-t_1}\Phi_{i_1j_1}Q^3_{t_2-s_2}\pi Q^2_{t_3-t_2}\pi Q_{t_4-t_3}\psi, \mu_0^2\rangle
\end{aligned}
$$

$$+\sum_{\substack{i,j=1\\i\neq j}}^{3}\int_{t_1}^{t_2}ds_2\int_{t_1}^{s_2}ds_1\ \left\langle Q_{t_1}^2\pi_1 Q_{s_1-t_1}\Phi_{12}Q_{s_2-s_1}^2\Phi_{ij}Q_{t_2-s_2}^3\pi_1 Q_{t_3-t_2}^2\pi_1 Q_{t_4-t_3}\psi,\mu_0^2\right\rangle$$

$$+\sum_{\substack{i,j=1\\i\neq j}}^{3}\int_{t_2}^{t_3}ds_2\int_{0}^{t_1}ds_1\ \left\langle Q_{s_1}^2\Phi_{ij}Q_{t_1-s_1}^3\pi_1 Q_{t_2-t_1}^2\pi_1 Q_{s_2-t_2}\Phi_{12}Q_{t_3-s_2}^2\pi_1 Q_{t_4-t_3}\psi,\mu_0^2\right\rangle$$

$$+\int_{t_2}^{t_3}ds_2\int_{t_1}^{t_2}ds_1\ \left\langle Q_{t_1}^2\pi_1 Q_{s_1-t_1}\Phi_{12}Q_{t_2-s_1}^2\pi_1 Q_{s_2-t_2}\Phi_{12}Q_{t_3-s_2}^2\pi_1 Q_{t_4-t_3}\psi,\mu_0^2\right\rangle$$

$$+\sum_{\substack{i_1,j_1=1\\i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\i_2\neq j_2}}^{3}\int_0^{t_1}ds_3\int_0^{s_3}ds_2\int_0^{s_2}ds_1$$

$$\times\left\langle Q_{s_1}\Phi_{12}Q_{s_2-s_1}^2\Phi_{i_2j_2}Q_{s_3-s_2}^3\Phi_{i_1j_1}Q_{t_1-s_3}^4\pi_1 Q_{t_2-t_1}^3\pi_1 Q_{t_3-t_2}^2\pi_1 Q_{t_4-t_3}\psi,\mu_0\right\rangle$$

$$+\sum_{\substack{i_1,j_1=1\\i_1\neq j_1}}^{3}\sum_{\substack{i_2,j_2=1\\i_2\neq j_2}}^{3}\int_{t_1}^{t_2}ds_3\int_0^{t_1}ds_2\int_0^{s_2}ds_1$$

$$\times\left\langle Q_{s_1}\Phi_{12}Q_{s_2-s_1}^2\Phi_{i_2j_2}Q_{t_1-s_2}^3\pi_1 Q_{s_3-t_1}^2\Phi_{i_1j_1}Q_{t_2-s_3}^3\pi_1 Q_{t_3-t_2}^2\pi_1 Q_{t_4-t_3}\psi,\mu_0\right\rangle$$

$$+\sum_{\substack{i,j=1\\i\neq j}}^{3}\int_{t_1}^{t_2}ds_3\int_{t_1}^{s_3}ds_2\int_0^{t_1}ds_1$$

$$\times\left\langle Q_{s_1}\Phi_{12}Q_{t_1-s_1}^2\pi_1 Q_{s_2-t_1}\Phi_{12}Q_{s_3-s_2}^2\Phi_{ij}Q_{t_2-s_3}^3\pi_1 Q_{t_3-t_2}^2\pi_1 Q_{t_4-t_3}\psi,\mu_0\right\rangle$$

$$+\sum_{\substack{i,j=1\\i\neq j}}^{3}\int_{t_2}^{t_3}ds_3\int_0^{t_1}ds_2\int_0^{s_2}ds_1$$

$$\times\left\langle Q_{s_1}\Phi_{12}Q_{s_2-s_1}^2\Phi_{i_2j_2}Q_{t_1-s_2}^3\pi_1 Q_{t_2-t_1}^2\pi_1 Q_{s_3-t_2}\Phi_{12}Q_{t_3-s_3}^2\pi_1 Q_{t_4-t_3}\psi,\mu_0\right\rangle$$

$$+\int_{t_2}^{t_3}ds_3\int_{t_1}^{t_2}ds_2\int_0^{t_1}ds_1$$

$$\times\left\langle Q_{s_1}\Phi_{12}Q_{t_1-s_1}^2\pi_1 Q_{s_2-t_1}\Phi_{12}Q_{t_2-s_2}^2\pi_1 Q_{s_3-t_2}\Phi_{12}Q_{t_3-s_3}^2\pi_1 Q_{t_4-t_3}\psi,\mu_0\right\rangle \quad \text{(II.17)}$$

Proof: Using the Markov property, and Lemma II.1.2, it follows that

$$\mathbb{E}\left\langle\phi,\mu_{t_1}\mu_{t_2}\mu_{t_3}\right\rangle$$

$$=\mathbb{E}\,\mu_{t_1}(\phi_1)\mu_{t_2}(\phi_2)\mathbb{E}_{\mu_{t_2}}\mu_{t_3-t_2}(\phi_3)$$

$$= \mathbb{E}\,\mu_{t_1}(\phi_1)\mu_{t_2}^2(\phi_2 \otimes Q_{t_3-t_2}\phi_3)$$

$$= \mathbb{E}\,\mu_{t_1}(\phi_1)\mathbb{E}_{\mu_{t_1}}\mu_{t_2-t_1}^2(\phi_2 \otimes Q_{t_3-t_2}\phi_3)$$

$$= \mathbb{E}\,\mu_{t_1}^3(\phi_1 \otimes Q_{t_2-t_1}^2(\phi_2 \otimes Q_{t_3-t_2}\phi_3)) + \int_0^{t_2-t_1} ds\,\mathbb{E}\,\mu_{t_1}^2(\phi_1 \otimes Q_s\Phi_{12}Q_{t_2-t_1-s}^2(\phi_2 \otimes Q_{t_3-t_2}\phi_3))$$

$$= \mathbb{E}\,\mu_{t_1}^3(\pi_1 Q_{t_2-t_1}^2\pi_1 Q_{t_3-t_2}\phi) + \int_{t_1}^{t_2} ds\,\mathbb{E}\,\mu_{t_1}^2(\pi_1 Q_{s-t_1}\Phi_{12}Q_{t_2-s}^2\pi_1 Q_{t_3-t_2}\phi).$$

From II.14

$$\mathbb{E}\,\mu_{t_1}^3(\pi_1 Q_{t_2-t_1}^2\pi_1 Q_{t_3-t_2}\phi)$$

$$= \mu_0^3(Q_{t_1}^3\pi_1 Q_{t_2-t_1}^2\pi_1 Q_{t_3-t_2}\phi)$$

$$+ \sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_0^{t_1} ds\,\mu_0^2(Q_s^2\Phi_{12}Q_{t_1-s}^3\pi_1 Q_{t_2-t_1}^2\pi_1 Q_{t_3-t_2}\phi)$$

$$+ \sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_0^{t_1} ds_2 \int_0^{s_2} ds_1\,\mu_0(Q_{s_1}\Phi_{12}Q_{s_2-s_1}^2\Phi_{12}Q_{t_1-s_2}^3\pi_1 Q_{t_2-t_1}^2\pi_1 Q_{t_3-t_2}\phi)$$

and from Lemma II.1.2,

$$\int_{t_1}^{t_2} ds\,\mathbb{E}\,\mu_{t_1}^2(\pi_1 Q_{s-t_1}\Phi_{12}Q_{t_2-s}^2\pi_1 Q_{t_3-t_2}\phi)$$

$$= \int_{t_1}^{t_2} ds\,\mu_0^2(Q_{t_1}^2\pi_1 Q_{s-t_1}\Phi_{12}Q_{t_2-s}^2\pi_1 Q_{t_3-t_2}\phi)$$

$$+ \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1\,\mathbb{E}\,\mu_0(Q_{s_1}\Phi_{12}Q_{t_1-s_1}^2\pi_1 Q_{s_2-t_1}\Phi_{12}Q_{t_2-s_2}^2\pi_1 Q_{t_3-t_2}\phi),$$

thus showing II.16.

The proof of II.17 while similar to the above, requires many more calculations. We begin

$$\mathbb{E}\,\langle \psi_1 \otimes \psi_2 \otimes \psi_3 \otimes \psi_4, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4}\rangle$$

$$= \mathbb{E}\,\mu_{t_1}(\psi_1)\mu_{t_2}(\psi_2)\mu_{t_3}^2(\psi_3 \otimes Q_{t_4-t_3}\psi_4)$$

$$= \mathbb{E}\,\mu_{t_1}(\psi_1)\mu^3_{t_2}(\psi_2 \otimes Q^2_{t_3-t_2}(\psi_3 \otimes Q_{t_4-t_3}\psi_4))$$

$$+ \int_0^{t_3-t_2} ds\, \mu_{t_1}(\psi_1)\mu^2_{t_2}(\psi_2 \otimes Q_s\Phi_{12}Q^2_{t_3-t_2-s}(\psi_3 \otimes Q_{t_4-t_3}\psi_4))$$

$$= \mathbb{E}\,\mu_{t_1}(\psi_1)\mu^3_{t_2}(\pi_1 Q^2_{t_3-t_2}\pi_1 Q_{t_4-t_3}(\psi_2 \otimes \psi_3 \otimes \psi_4))$$

$$+ \int_{t_2}^{t_3} ds\, \mathbb{E}\,\mu_{t_1}(\psi_1)\mu^2_{t_2}(\pi_1 Q_{s-t_2}\Phi_{12}Q^2_{t_3-s}\pi_1 Q_{t_4-t_3}(\psi_2 \otimes \psi_3 \otimes \psi_4))$$

$$= \mathbb{E}\,\mu^4_{t_1}(\pi_1 Q^3_{t_2-t_1}\pi_1 Q^2_{t_3-t_2}\pi_1 Q_{t_4-t_3}\psi)$$

$$+ \sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_{t_1}^{t_2} ds\, \mathbb{E}\,\mu^3_{t_1}(\pi_1 Q^2_{s-t_1}\Phi_{ij}Q^3_{t_2-s}\pi_1 Q^2_{t_3-t_2}\pi_1 Q_{t_4-t_3}\psi)$$

$$+ \int_{t_2}^{t_3} ds\, \mathbb{E}\,\mu^3_{t_1}(\pi_1 Q^2_{t_2-t_1}\pi_1 Q_{s-t_2}\Phi_{12}Q^2_{t_3-s}\pi_1 Q_{t_4-t_3}\psi)$$

$$+ \sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_{t_1}^{t_2} ds_2 \int_{t_1}^{s_2} ds_1\, \mathbb{E}\,\mu^2_{t_1}(\pi_1 Q_{s_1-t_1}\Phi_{12}Q^2_{s_2-s_1}\Phi_{ij}Q^3_{t_2-s_2}\pi_1 Q^2_{t_3-t_2}\pi_1 Q_{t_4-t_3}\psi)$$

$$+ \int_{t_2}^{t_3} ds_2 \int_{t_1}^{t_2} ds_1\, \mathbb{E}\,\mu^2_{t_1}(\pi_1 Q_{s_1-t_1}\Phi_{12}Q^2_{t_2-s_1}\pi_1 Q_{s_2-t_2}\Phi_{12}Q^2_{t_3-s_2}\pi_1 Q_{t_4-t_3}\psi).$$

To make sense of the remained of the proof, each of the above five terms will now be considered separately:

Let $F_1 = \pi_1 Q^3_{t_2-t_1}\pi_1 Q^2_{t_3-t_2}\pi_1 Q_{t_4-t_3}\psi$, then from II.15

$$\mathbb{E}\,\mu^4_{t_1}(F_1)$$

$$= \mu^4_0(Q^4_t F_1) + \sum_{\substack{i,j=1\\ i\neq j}}^{4} \int_0^{t_1} ds\, \mu^3_0(Q^3_s\Phi_{ij}Q^4_{t_1-s}F_1)$$

$$+ \sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4} \sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3} \int_0^{t_1} ds_2 \int_0^{s_2} ds_1\, \mu^2_0(Q^2_{s_1}\Phi_{i_2j_2}Q^3_{s_2-s_1}\Phi_{i_1j_1}Q^4_{t-s_2}F_1)$$

$$+ \sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4} \sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3} \int_0^{t} ds_3 \int_0^{s_3} ds_2 \int_0^{s_2} ds_1\, \mu_0(Q_{s_1}\Phi_{12}Q^2_{s_2-s_1}\Phi_{i_2j_2}Q^3_{s_3-s_2}\Phi_{i_1j_1}Q^4_{t-s_3}F_1). \quad \text{(II.18)}$$

Let $F_2^{ij}(s) = \pi_1 Q^2_{s-t_1}\Phi_{ij}Q^3_{t_2-s}\pi_1 Q^2_{t_3-t_2}\pi_1 Q_{t_4-t_3}\psi$, then from II.14

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_{t_1}^{t_2} ds\,\mathbb{E}\,\mu^3_{t_1}(F_2^{ij}(s))$$
$$= \sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_{t_1}^{t_2} ds\,\mu_0^3(Q^3_{t_1}F_2^{ij}(s)) + \sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{3}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\int_{t_1}^{t_2} ds_2\int_0^{t_1} ds_1\,\mu_0^2(Q^2_{s_1}\Phi_{i_2j_2}Q^3_{t_1-s_1}F_2^{i_1j_1}(s_2))$$
$$+ \sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{3}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\int_{t_1}^{t_2} ds_3\int_0^{t_1} ds_2\int_0^{s_2} ds_1\,\mu_0(Q_{s_1}\Phi_{12}Q^2_{s_2-s_1}\Phi_{i_2j_2}Q^3_{t_1-s_2}F_2^{i_1j_1}(s_3)). \qquad \text{(II.19)}$$

Let $F_3(s) = \pi_1 Q^2_{t_2-t_1}\pi_1 Q_{s-t_2}\Phi_{12}Q^2_{t_3-s}\pi_1 Q_{t_4-t_3}\psi$, then from II.14

$$\int_{t_2}^{t_3} ds\,\mathbb{E}\,\mu^3_{t_1}(F_3(s))$$
$$= \int_{t_2}^{t_3} ds\,\mu_0^3(Q^3_{t_1}F_3(s)) + \sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_{t_2}^{t_3} ds_2\int_0^{t_1} ds_1\,\mu_0^2(Q^2_{s_1}\Phi_{ij}Q^3_{t_1-s_1}F_3(s_2))$$
$$+ \sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_{t_2}^{t_3} ds_3\int_0^{t_1} ds_2\int_0^{s_2} ds_1\,\mu_0(Q_{s_1}\Phi_{12}Q^2_{s_2-s_1}\Phi_{ij}Q^3_{t_1-s_2}F_3(s_3)). \qquad \text{(II.20)}$$

From Lemma II.1.2, if

$$F_4^{ij}(s_1,s_2) = \pi_1 Q_{s_1-t_1}\Phi_{12}Q^2_{s_2-s_1}\Phi_{ij}Q^3_{t_2-s_2}\pi_1 Q^2_{t_3-t_2}\pi_1 Q_{t_4-t_3}\psi$$

and

$$F_5(s_1,s_2) = \pi_1 Q_{s_1-t_1}\Phi_{12}Q^2_{t_2-s_1}\pi_1 Q_{s_2-t_2}\Phi_{12}Q^2_{t_3-s_2}\pi_1 Q_{t_4-t_3}\psi,$$

then

$$\sum_{\substack{i,j=1\\i\neq j}}^{3}\int_{t_1}^{t_2}ds_2\int_{t_1}^{s_2}ds_1\,\mathbb{E}\,\mu_{t_1}^2(F_4^{ij}(s_1,s_2))=\sum_{\substack{i,j=1\\i\neq j}}^{3}\int_{t_1}^{t_2}ds_2\int_{t_1}^{s_2}ds_1\,\mu_0^2(Q_{t_1}^2F_4^{ij}(s_1,s_2))$$
$$+\sum_{\substack{i,j=1\\i\neq j}}^{3}\int_{t_1}^{t_2}ds_3\int_{t_1}^{s_3}ds_2\int_0^{t_1}ds_1\,\mu_0(Q_{s_1}\Phi_{12}Q_{t_1-s_1}^2F_4^{ij}(s_2,s_3))\quad\text{(II.21)}$$

and

$$\int_{t_2}^{t_3}ds_2\int_{t_1}^{t_2}ds_1\,\mathbb{E}\,\mu_{t_1}^2(F_5(s_1,s_2))=\int_{t_2}^{t_3}ds_2\int_{t_1}^{t_2}ds_1\,\mu_0^2(Q_{t_1}^2F_5(s_1,s_2))$$
$$+\int_{t_2}^{t_3}ds_3\int_{t_1}^{t_2}ds_2\int_0^{t_1}ds_1\,\mu_0(Q_{s_1}\Phi_{12}Q_{t_1-s_1}^2F_5(s_2,s_3))\qquad\text{(II.22)}$$

Combining II.18,II.19,II.20,II.21, and II.22, the desired formula follows. □

The following Corollary will be helpful in the construction of a needed Itô formula (cf. chapter III). For the remainder of this paper, any arbitrary constant value, dependent only upon $0 \le T$, will be denoted by $C = C(T)$.

Corollary II.3.3. *For $i, j = 1,2,3,4$, let $\phi_j^i \in C_K^\infty(\mathbb{R}^d)$ and define $\phi_i \in C_K^\infty(\mathbb{R}^{i\times d})$ by*

$$\phi_i = \phi_1^i \otimes \cdots \otimes \phi_i^i.$$

Then if $0 \le t_1 \le t_2 \le t_3 \le t_4 \le T < \infty$,

$$\mathbb{E}\,\langle \phi_1, \mu_{t_1}\rangle \le C(T)\,\|\phi_1\|_\infty\,,$$

$$\mathbb{E}\,\langle \phi_2, \mu_{t_1}\mu_{t_2}\rangle \le C(T)\,\|\phi_2\|_\infty\,,$$

$$\mathbb{E}\,\langle \phi_3, \mu_{t_1}\mu_{t_2}\mu_{t_3}\rangle \le C(T)\,\|\phi_3\|_\infty\,,$$

and

$$\mathbb{E}\left\langle \phi_4, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4}\right\rangle \leq C(T)\left\|\phi_4\right\|_\infty .$$

Proof: Since $\int dy\, q_t^k(x,y) = 1$ for any $k \in \mathbb{N}$ and all $x \in \mathbb{R}^{k\times d}$, and since μ_0 is a finite measure having compact support, this follows immediately from the preceding lemma. □

Corollary II.3.4. *The equations II.16 and II.17 continue to hold for $\phi \in S_{3\times d}$ and $\psi \in S_{4\times d}$.*

Proof: From Lemma II.1.1 there exist $\{\phi_n \triangleq \sum_{k=1}^n \phi_k^1 \otimes \phi_k^2 \otimes \phi_k^3 : k \in \mathbb{N}\}$ and $\{\psi_n \triangleq \sum_{k=1}^n \psi_k^1 \otimes \psi_k^2 \otimes \psi_k^3 \otimes \psi_k^4 : k \in \mathbb{N}\}$ such that $\phi_k^j, \psi_m^i \in C_K^\infty(\mathbb{R}^d)$, $i = 1,2,3,4$, $j = 1,2,3$, $k, m \in \mathbb{N}$, and $\lim_{n\to\infty} \phi_n = \phi$, $\lim_{n\to\infty}\psi_n = \psi$, where the convergence is uniform. For any $n, m \in N$, from equations II.16 and II.17 it follows, respectively, that

$$\begin{aligned}
&\mathbb{E}\left\langle |\phi_n - \phi_m|, \mu_{t_1}\mu_{t_2}\mu_{t_3}\right\rangle \\
&= \left\langle Q_{t_1}^3 \pi_1 Q_{t_2-t_1}^2 \pi_1 Q_{t_3-t_2} |\phi_n - \phi_m|, \mu_0^3\right\rangle \\
&+ \sum_{\substack{i,j=1\\ i\neq j}}^3 \int_0^{t_1} ds\, \left\langle Q_s^2 \Phi_{ij} Q_{t_1-s}^3 \pi_1 Q_{t_2-t_1}^2 \pi_1 Q_{t_3-t_2} |\phi_n - \phi_m|, \mu_0^2\right\rangle \\
&+ \sum_{\substack{i,j=1\\ i\neq j}}^3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1\, \left\langle Q_{s_1}\Phi_{12} Q_{s_2-s_1}^2 \Phi_{ij} Q_{t_1-s_2}^3 \pi_1 Q_{t_2-t_1}^2 \pi_1 Q_{t_3-t_2} |\phi_n - \phi_m|, \mu_0\right\rangle \\
&+ \int_{t_1}^{t_2} ds\, \left\langle Q_{t_1}^2 \pi_1 Q_{s-t_1} \Phi_{12} Q_{t_2-s}^2 \pi_1 Q_{t_3-t_2} |\phi_n - \phi_m|, \mu_0^2\right\rangle \\
&+ \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1\, \left\langle Q_{s_1}\Phi_{12} Q_{t_1-s_1}^2 \pi_1 Q_{s_2-t_1} \Phi_{12} Q_{t_2-s_2}^2 \pi_1 Q_{t_3-t_2} |\phi_n - \phi_m|, \mu_0\right\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left\langle |\psi_n - \psi_m|, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4}\right\rangle \\
&= \left\langle Q_{t_1}^4 \pi_1 Q_{t_2-t_1}^3 \pi_1 Q_{t_3-t_2}^2 \pi_1 Q_{t_4-t_3} |\psi_n - \psi_m|, \mu_0^4\right\rangle
\end{aligned}$$

$$+\sum_{\substack{i,j=1\\i\neq j}}^{4}\int_0^{t_1} ds\ \left\langle Q_s^3\Phi_{ij}Q_{t_1-s}^4\pi_1Q_{t_2-t_1}^3\pi_1Q_{t_3-t_2}^2\pi_1Q_{t_4-t_3}\left|\psi_n-\psi_m\right|,\mu_0^3\right\rangle$$

$$+\sum_{\substack{i,j=1\\i\neq j}}^{3}\int_{t_1}^{t_2} ds\ \left\langle Q_{t_1}^3\pi_1Q_{s-t_1}^2\Phi_{ij}Q_{t_2-s}^3\pi_1Q_{t_3-t_2}^2\pi_1Q_{t_4-t_3}\left|\psi_n-\psi_m\right|,\mu_0^3\right\rangle$$

$$+\int_{t_2}^{t_3} ds\ \left\langle Q_{t_1}^3\pi_1Q_{t_2-t_1}^2\pi_1Q_{s-t_2}\Phi_{12}Q_{t_3-s}^2\pi_1Q_{t_4-t_3}\left|\psi_n-\psi_m\right|,\mu_0^3\right\rangle$$

$$+\sum_{\substack{i_1,j_1=1\\i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\i_2\neq j_2}}^{3}\int_0^{t_1} ds_2\int_0^{s_2} ds_1$$

$$\times\left\langle Q_{s_1}^2\Phi_{i_2j_2}Q_{s_2-s_1}^3\Phi_{i_1j_1}Q_{t_1-s_2}^4\pi_1Q_{t_2-t_1}^3\pi_1Q_{t_3-t_2}^2\pi_1Q_{t_4-t_3}\left|\psi_n-\psi_m\right|,\mu_0^2\right\rangle$$

$$+\sum_{\substack{i_1,j_1=1\\i_1\neq j_1}}^{3}\sum_{\substack{i_2,j_2=1\\i_2\neq j_2}}^{3}\int_{t_1}^{t_2} ds_2\int_0^{t_1} ds_1$$

$$\times\left\langle Q_{s_1}^2\Phi_{i_2j_2}Q_{t_1-s_1}^3\pi_1Q_{s_2-t_1}^2\Phi_{i_1j_1}Q_{t_2-s_2}^3\pi_1Q_{t_3-t_2}^2\pi_1Q_{t_4-t_3}\left|\psi_n-\psi_m\right|,\mu_0^2\right\rangle$$

$$+\sum_{\substack{i,j=1\\i\neq j}}^{3}\int_{t_1}^{t_2} ds_2\int_{t_1}^{s_2} ds_1\ \left\langle Q_{t_1}^2\pi_1Q_{s_1-t_1}\Phi_{12}Q_{s_2-s_1}^2\Phi_{ij}Q_{t_2-s_2}^3\pi_1Q_{t_3-t_2}^2\pi_1Q_{t_4-t_3}\left|\psi_n-\psi_m\right|,\mu_0^2\right\rangle$$

$$+\sum_{\substack{i,j=1\\i\neq j}}^{3}\int_{t_2}^{t_3} ds_2\int_0^{t_1} ds_1\ \left\langle Q_{s_1}^2\Phi_{ij}Q_{t_1-s_1}^3\pi_1Q_{t_2-t_1}^2\pi_1Q_{s_2-t_2}\Phi_{12}Q_{t_3-s_2}^2\pi_1Q_{t_4-t_3}\left|\psi_n-\psi_m\right|,\mu_0^2\right\rangle$$

$$+\int_{t_2}^{t_3} ds_2\int_{t_1}^{t_2} ds_1\ \left\langle Q_{t_1}^2\pi_1Q_{s_1-t_1}\Phi_{12}Q_{t_2-s_1}^2\pi_1Q_{t_3-s_2}\Phi_{12}Q_{s_2-t_2}^2\pi_1Q_{t_4-t_3}\left|\psi_n-\psi_m\right|,\mu_0^2\right\rangle$$

$$+\sum_{\substack{i_1,j_1=1\\i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\i_2\neq j_2}}^{3}\int_0^{t_1} ds_3\int_0^{s_3} ds_2\int_0^{s_2} ds_1$$

$$\times\left\langle Q_{s_1}\Phi_{12}Q_{s_2-s_1}^2\Phi_{i_2j_2}Q_{s_3-s_2}^3\Phi_{i_1j_1}Q_{t_1-s_3}^4\pi_1Q_{t_2-t_1}^3\pi_1Q_{t_3-t_2}^2\pi_1Q_{t_4-t_3}\left|\psi_n-\psi_m\right|,\mu_0\right\rangle$$

$$+\sum_{\substack{i_1,j_1=1\\i_1\neq j_1}}^{3}\sum_{\substack{i_2,j_2=1\\i_2\neq j_2}}^{3}\int_{t_1}^{t_2} ds_3\int_0^{t_1} ds_2\int_0^{s_2} ds_1$$

$$\times\left\langle Q_{s_1}\Phi_{12}Q_{s_2-s_1}^2\Phi_{i_2j_2}Q_{t_1-s_2}^3\pi_1Q_{s_3-t_1}^2\Phi_{i_1j_1}Q_{t_2-s_3}^3\pi_1Q_{t_3-t_2}^2\pi_1Q_{t_4-t_3}\left|\psi_n-\psi_m\right|,\mu_0\right\rangle$$

$$+\sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_{t_1}^{t_2}ds_3\int_{t_1}^{s_3}ds_2\int_0^{t_1}ds_1$$

$$\times\left\langle Q_{s_1}\Phi_{12}Q^2_{t_1-s_1}\pi_1 Q_{s_2-t_1}\Phi_{12}Q^2_{s_3-s_2}\Phi_{ij}Q^3_{t_2-s_3}\pi_1 Q^2_{t_3-t_2}\pi_1 Q_{t_4-t_3}\left|\psi_n-\psi_m\right|,\mu_0\right\rangle$$

$$+\sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_{t_2}^{t_3}ds_3\int_0^{t_1}ds_2\int_0^{s_2}ds_1$$

$$\times\left\langle Q_{s_1}\Phi_{12}Q^2_{s_2-s_1}\Phi_{i_2j_2}Q^3_{t_1-s_2}\pi_1 Q^2_{t_2-t_1}\pi_1 Q_{s_3-t_2}\Phi_{12}Q^2_{t_3-s_3}\pi_1 Q_{t_4-t_3}\left|\psi_n-\psi_m\right|,\mu_0\right\rangle$$

$$+\int_{t_2}^{t_3}ds_3\int_{t_1}^{t_2}ds_2\int_0^{t_1}ds_1$$

$$\times\left\langle Q_{s_1}\Phi_{12}Q^2_{t_1-s_1}\pi_1 Q_{s_2-t_1}\Phi_{12}Q^2_{t_2-s_2}\pi_1 Q_{s_3-t_2}\Phi_{12}Q^2_{t_3-s_3}\pi_1 Q_{t_4-t_3}\psi,\mu_0\right\rangle$$

Using Corollary II.3.3, it follows that

$$\mathbb{E}\left\langle\left|\phi_n-\phi_m\right|,\mu_{t_1}\mu_{t_2}\mu_{t_3}\right\rangle\leq C(T)\left\|\phi_n-\phi_m\right\|_\infty$$

and

$$\mathbb{E}\left\langle\left|\psi_n-\psi_m\right|,\mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4}\right\rangle\leq C(T)\left\|\psi_n-\psi_m\right\|_\infty.$$

Thus $\langle\phi_n,\mu_{t_1}\mu_{t_2}\mu_{t_3}\rangle$ and $\langle\psi_n,\mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_1}\rangle$ are Cauchy in the complete space $L^1(\mathbb{P})$, and hence convergent. Uniform convergence of ϕ_n and ψ_n implies

$$\lim_{n\to\infty}\langle\phi_n,\mu_{t_1}\mu_{t_2}\mu_{t_3}\rangle=\langle\phi,\mu_{t_1}\mu_{t_2}\mu_{t_3}\rangle,\quad a.s.$$

and

$$\lim_{n\to\infty}\langle\psi_n,\mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4}\rangle=\langle\psi,\mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4}\rangle,\quad a.s.$$

Since the L^1 and $a.s.$ limits must agree when they both exist,

$$\lim_{n\to\infty}\mathbb{E}\langle\phi_n,\mu_{t_1}\mu_{t_2}\mu_{t_3}\rangle=\langle\phi,\mu_{t_1}\mu_{t_2}\mu_{t_3}\rangle$$

and

$$\lim_{n\to\infty} \mathbb{E}\left\langle \psi_n, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4}\right\rangle = \left\langle \psi, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4}\right\rangle .$$

Considering now the right hand sides of the above equations, by uniform convergence, and since μ_0 is finite with compact support, the desired convergence is shown. □

Lemma II.3.5. *Let $\phi \in S_d$, $d \leq 3$, and define for $x = (x_1, x_2, x_3) \in \mathbb{R}^{3\times d}$, $y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^{4\times d}$,*

$$\psi(x) \triangleq \phi(x_1 - x_3)\phi(x_2 - x_3),$$

and

$$\varphi(y) \triangleq \phi(y_1 - y_3)\phi(y_2 - y_4).$$

Suppose that $\mu = \{\mu_t : t \geq 0\}$ is a superprocess over a stochastic flow such that $\mu_0 \in M_F(\mathbb{R}^d)$ satisfies Assumption 2. Then, for any $0 \leq t_1 \leq t_2 \leq t_3 \leq T < \infty$,

(i)

$$\int_0^T dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1\, \mathbb{E}\left\langle \psi, \mu_{t_1}\mu_{t_2}\mu_{t_3}\right\rangle \leq C \left\|\psi\right\|_{L^1}^2 ,$$

and

(ii)

$$\int_0^{t_3} dt_2 \int_0^{t_2} dt_1\, \mathbb{E}\left\langle \varphi, \mu_{t_1}\mu_{t_2}\mu_{t_3}^2\right\rangle \leq C \left\|\phi\right\|_{L^1}^2 ,$$

where $C = C(T)$.

Proof: Throughout this proof, the norm on L^p will be denoted by $\|\cdot\|_p$. From the moment equation II.16, it follows that

$$
\begin{aligned}
&\mathbb{E}\left\langle \psi, \mu_{t_1}\mu_{t_2}\mu_{t_3}\right\rangle \\
&= \left\langle Q^3_{t_1}\pi_1 Q^2_{t_2-t_1}\pi_1 Q_{t_3-t_2}\psi, \mu_0^3\right\rangle \\
&+ \sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_0^{t_1} ds\ \left\langle Q^2_s \Phi_{ij} Q^3_{t_1-s}\pi_1 Q^2_{t_2-t_1}\pi_1 Q_{t_3-t_2}\psi, \mu_0^2\right\rangle \\
&+ \int_{t_1}^{t_2} ds\ \left\langle Q^2_{t_1}\pi_1 Q_{s-t_1}\Phi_{12} Q^2_{t_2-s}\pi_1 Q_{t_3-t_2}\psi, \mu_0^2\right\rangle \\
&+ \sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_0^{t_1} ds_2 \int_0^{s_2} ds_1\ \left\langle Q_{s_1}\Phi_{12} Q^2_{s_2-s_1}\Phi_{ij} Q^3_{t_1-s_2}\pi_1 Q^2_{t_2-t_1}\pi_1 Q_{t_3-t_2}\psi, \mu_0\right\rangle \\
&+ \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1\ \left\langle Q_{s_1}\Phi_{12} Q^2_{t_1-s_1}\pi_1 Q_{s_2-t_1}\Phi_{12} Q^2_{t_2-s_2}\pi_1 Q_{t_3-t_2}\psi, \mu_0\right\rangle .
\end{aligned}
$$

Thus, if we write

$$
\mathbb{E}\left\langle \psi, \mu_{t_1}\mu_{t_2}\mu_{t_3}\right\rangle = \sum_{k=1}^{5} I_k(t_1,t_2,t_3)
$$

as defined in equation II.16, it suffices to show that

$$
\int_0^T dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1\, I_k(t_1,t_2,t_3) \le C(T)\left\|\psi\right\|_1^2,
$$

for $k = 1,2,3,4,5$.

The following arguments will rely heavily upon the Markov property, the Kolmogorov Chapmann equation, and Lemma I.1.5.

To begin, note that for any $s \in [0,t_1]$, $x \in \mathbb{R}^{3\times d}$

$$
\begin{aligned}
&Q^3_{t_1-s}\pi_1 Q^2_{t_2-t_1}\pi_1 Q_{t_3-t_2}\psi \\
&\le C\int dy\, p^3_{\iota(t_1-s)}(x,y)\int dz\, p^2_{\iota(t_2-t_1)}((y_2,y_3),z)\int dw\, p_{\iota(t_3-t_2)}(z_2,w)\psi(y_1,z_1,w) \\
&\le C\int da_1 da_2 da_3 \prod_{q=1}^{3} p_{\iota(t_q-s)}(x_q,a_q)\psi(a_1,a_2,a_3),
\end{aligned}
$$

and for any s_1, s_2 with $0 \le s_1 \le t_1$, $t_1 \le s_2 \le t_2$, and $x \in \mathbb{R}^{2\times d}$,

$$\begin{aligned}
&Q^2_{t_1-s_1}\pi_1 Q_{t_2-s_2}\Phi_{12}Q^2_{s_2-t_1}\pi_1 Q_{t_3-t_2}\phi \\
&\le C\int dy\, p_{\iota(t_1-s_1)}(x_1,y_1)p_{\iota(t_1-s_1)}(x_2,y_2)\int dz\, p_{\iota(t_2-s_2)}(y_2,z) \\
&\qquad \times \int dw\, p_{\iota(s_2-t_1)}(z,w_1)p_{\iota(s_2-t_1)}(z,w_2)\int dv\, p_{\iota(t_3-t_2)}(w_2,v)\psi(y_1,w_1,v) \\
&\le C\int db\, p_{\iota(t_2+t_1-s_2-s_1)}(x_2,b)\int da_1 da_2 da_3\, p_{\iota(t_1-s_1)}(x_1,a_1) \\
&\qquad \times p_{\iota(s_2-t_1)}(b,a_2)p_{\iota(t_3-t_2+s_2-t_1)}(b,a_3)\psi(a_1,a_2,a_3).
\end{aligned}$$

It thus follows that,

$$\begin{aligned}
&\langle Q^3_{t_1}\pi_1 Q^2_{t_2-t_1}\pi_1 Q_{t_3-t_2}\psi, \mu_0^3\rangle \\
&\le C\int \mu_0(dx_3)\int dy_3\, p_{\iota t_3}(x_3,y_3)\int dy_2 \phi(y_2-y_3)\int dy_1\, \phi(y_1-y_3) \\
&\qquad \times \int \mu_0(dx_2)\, p_{\iota t_2}(x_2,y_2)\int \mu_0(dx_1)\, p_{\iota t_1}(x_1,y_1) \\
&\le C\,\|m\|^2_\infty\,\|\phi\|^2_1\int \mu_0(dx)\int dy\, p_{\iota t_3}(x,y) \\
&\le C\,\|m\|^2_\infty\,\|\phi\|^2_1\,\mu_0(1) \\
&\le C\,\|\phi\|^2_1,
\end{aligned}$$

which implies that

$$\int_0^T dt_3\int_0^{t_3} dt_2\int_0^{t_2} dt_1\, I_1(t_1,t_2,t_3) \le C(T)\,\|\phi\|^2_1. \tag{II.23}$$

Next,

$$\sum_{\substack{i,j=1\\ i\ne j}}^3 \int_0^{t_1} ds\, \langle Q^2_s\Phi_{ij}Q^3_{t_1-s}\pi_1 Q^2_{t_2-t_1}\pi_1 Q_{t_3-t_2}\psi, \mu_0^2\rangle$$

$$\leq C\sum_{\substack{i,j=1\\i<j}}^{3}\int_0^{t_1}ds\ \int\mu_0(dx_1)\mu_0(dx_2)\int dy\,p_{\iota s}(x_1,y)\int dz_1dz_2dz_3\,p_{\iota t_{6-i-j}}(x_2,z_{6-i-j})$$

$$\times\,p_{\iota(t_i-s)}(y,z_i)p_{\iota(t_j-s)}(y,z_j)\psi(z_1,z_2,z_3)$$

$$\leq C\sum_{i=1}^{2}\int_0^{t_1}ds\ \int\mu_0(dx_1)\int dy\,p_{\iota s}(x_1,y)\int dz_3\,p_{\iota(t_3-s)}(y,z_3)\int dz_i\,p_{\iota(t_i-s)}(y,z_i)\phi(z_i-z_3)$$

$$\times\int dz_{3-i}\,\phi(z_{3-i}-z_3)\int\mu_0(dx_2)\,p_{\iota t_{3-i}}(x_2,z_{3-i})$$

$$+\,C\int_0^{t_1}ds\int\mu_0(dx_2)\int dz_1dz_2dz_3\,p_{\iota t_3}(x_2,z_3)\phi(z_1-z_3)\phi(z_2-z_3)$$

$$\times\int dy\,p_{\iota(t_1-s)}(y,z_1)p_{\iota(t_2-s)}(y,z_2)\int\mu_0(dx_1)\,p_{\iota s}(x_1,y)$$

$$\leq C\|m\|_\infty\|\phi\|_1\int_0^{t_1}ds\int\mu_0(dx)\int dy\,p_{\iota s}(x,y)\int dz_3\,p_{\iota(t_3-s)}(y,z_3)\int dz_i\,p_{\iota(t_i-s)}(y,z_i)\phi(z_i-z_3)$$

$$+\,C\,\|m\|_\infty\int_0^{t_1}ds\int\mu_0(dx)\int dz_1dz_2dz_3\,p_{\iota t_3}(x,z_3)p_{\iota(t_2+t_1-2s)}(z_1,z_2)\phi(z_1-z_3)\phi(z_2-z_3)$$

$$\leq C\,\|\phi\|_1^2\int_0^{t_1}ds\Big[(t_3+t_1-2s)^{-d/2}+(t_2+t_1-2s)^{-d/2}\Big]$$

$$\leq C\,\|\phi\|_1^2\int_0^{t_1}ds\,(t_2+t_1-2s)^{-d/2},$$

and since $d\leq 3$,

$$\int_0^T dt_3\int_0^{t_3}dt_2\int_0^{t_2}dt_1\,I_2(t_1,t_2,t_3)$$
$$\leq C\,\|\phi\|_1^2\int_0^T dt_3\int_0^{t_3}dt_2\int_0^{t_2}dt_1\ \int_0^{t_1}ds\,(t_2+t_1-2s)^{-d/2}$$
$$\leq C(T)\,\|\phi\|_1^2. \tag{II.24}$$

And,

$$C\sum_{\substack{i,j=1\\i\neq j}}^{3}\int_0^{t_1}ds_2\int_0^{s_2}ds_1\ \langle Q_{s_1}\Phi_{12}Q^2_{s_2-s_1}\Phi_{ij}Q^3_{t_1-s_2}\pi_1Q^2_{t_2-t_1}\pi_1Q_{t_3-t_2}\psi,\mu_0\rangle$$

$$\leq C \sum_{\substack{i,j=1 \\ i<j}}^{3} \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y) \int dz\, p_{\iota(s_2-s_1)}(y,z)$$

$$\times \int dw_1 dw_2 dw_3\, p_{\iota(t_i-s_2)}(z,w_i) p_{\iota(t_j-s_2)}(z,w_j) p_{\iota(t_{6-i-j}-s_1)}(y,w_{6-i-j}) \psi(w_1,w_2,w_3)$$

$$\leq C \sum_{\substack{i,j=1 \\ i<j}}^{3} \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 (t_{6-i-j}-s_1)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y) \int dz\, p_{\iota(s_2-s_1)}(y,z)$$

$$\times \int dw_i dw_j\, p_{\iota(t_i-s_2)}(z,w_i) p_{\iota(t_j-s_2)}(z,w_j) \int dw_{6-i-j} \phi(w_1-w_3)\phi(w_2-w_3)$$

$$\leq C \sum_{\substack{i,j=1 \\ i<j}}^{3} \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 (t_{6-i-j}-s_1)^{-d/2} (t_j-s_2)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y)$$

$$\times \int dz\, p_{\iota(s_2-s_1)}(y,z) \int dw_i\, p_{\iota(t_i-s_2)}(z,w_i) \int dw_j dw_{6-i-j} \phi(w_1-w_3)\phi(w_2-w_3)$$

$$\leq C\, \|\phi\|_1^2 \sum_{\substack{i,j=1 \\ i<j}}^{3} \int_0^{t_1} ds_2 \int_0^{s_2} ds_1\, (t_{6-i-j}-s_1)^{-d/2} (t_j-s_2)^{-d/2}$$

$$\leq C\, \|\phi\|_1^2 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1\, \left[(t_3-s_1)^{-d/2}(t_2-s_2)^{-d/2} + (t_1-s_1)^{-d/2}(t_3-s_2)^{-d/2} \right]$$

Since $d \leq 3$, it follows that

$$\begin{aligned} &\int_0^T dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1\, I_3(t_1,t_2,t_3) \\ &\leq C\, \|\phi\|_1^2 \int_0^T dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_0^{t_1} ds_2 \\ &\quad \times \int_0^{s_2} ds_1\, \left[(t_3-s_1)^{-d/2}(t_2-s_2)^{-d/2} + (t_1-s_1)^{-d/2}(t_3-s_2)^{-d/2} \right] \\ &\leq C(T)\, \|\phi\|_1^2 . \end{aligned} \tag{II.25}$$

Now,

$$\begin{aligned} &C \int_{t_1}^{t_2} ds\, \langle Q_{t_1}^2 \pi_1 Q_{s-t_1} \Phi_{12} Q_{t_2-s}^2 \pi_1 Q_{t_3-t_2} \psi, \mu_0^2 \rangle \\ &\leq C \int_{t_1}^{t_2} ds \int \mu_0(dx_1)\mu_0(dx_2) \int dy\, p_{\iota t_1}(x_1,y) \int dz\, p_{\iota s}(x_2,z) \\ &\qquad \times \int dw_1 dw_2\, p_{\iota(t_2-s)}(z,w_1) p_{\iota(t_3-s)}(z,w_2) \psi(y,w_1,w_2) \end{aligned}$$

$$
\begin{aligned}
&\le C \int_{t_1}^{t_2} ds \int \mu_0(dx_2) \int dz\, p_{\iota s}(x_2, z) \int dw_1 dw_2\, p_{\iota(t_2-s)}(z, w_1) \\
&\quad \times p_{\iota(t_3-s)}(z, w_2)\phi(w_1 - w_2) \int dy\, \phi(y - w_2) \int \mu_0(dx_1)\, p_{\iota t_1}(x_1, y) \\
&\le C \, \|m\|_\infty \, \|\phi\|_1 \int_{t_1}^{t_2} ds \int \mu_0(dx_2) \int dz\, p_{\iota s}(x_2, z) \\
&\quad \times \int dw_1 dw_2\, p_{\iota(t_2-s)}(z, w_1) p_{\iota(t_3-s)}(z, w_2) \phi(w_1 - w_2) \\
&\le C \, \|m\|_\infty \, \|\phi\|_1 \int_{t_1}^{t_2} ds\, (t_3 - s)^{-d/2} \int \mu_0(dx_2) \int dz\, p_{\iota s}(x_2, z) \\
&\quad \times \int dw_1\, p_{\iota(t_2-s)}(z, w_1) \int dw_2\, \phi(w_1 - w_2) \\
&\le C \, \|\phi\|_1^2 \int_{t_1}^{t_2} ds\, (t_3 - s)^{-d/2}
\end{aligned}
$$

And since $d \le 3$, it follows that

$$
\begin{aligned}
&\int_0^T dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1\, I_4(t_1, t_2, t_3) \\
&\le C \, \|\phi\|_1^2 \int_0^T dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_1}^{t_2} ds\, (t_3 - s)^{-d/2} \\
&\le C(T) \, \|\phi\|_1^2 . \qquad \text{(II.26)}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&C \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1 \, \langle Q_{s_1} \Phi_{12} Q^2_{t_1 - s_1} \pi_1 Q_{s_2 - t_1} \Phi_{12} Q^2_{t_2 - s_2} \pi_1 Q_{t_3 - t_2} \psi, \mu_0 \rangle \\
&\le C \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1 \int \mu_0(dx) \int dy\, p_{\iota s_1}(x, y) \int dz dw dv_1 dv_2\, p_{\iota(t_1 - s_1)}(y, z) \\
&\quad \times p_{\iota(s_2 - s_1)}(y, w) p_{\iota(t_2 - s_2)}(w, v_1) p_{\iota(t_3 - s_2)}(w, v_2) \psi(z, v_1, v_2) \\
&\le C \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1\, (t_3 - s_2)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_1}(x, y) \\
&\quad \times \int dz dv_1\, p_{\iota(t_1 - s_1)}(y, z) p_{\iota(t_2 - s_1)}(y, v_1) \int dv_2\, \phi(z - v_2) \phi(v_1 - v_2) \\
&\le C \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1\, (t_3 - s_2)^{-d/2} (t_2 - s_1)^{-d/2} \int \mu_0(dx) \\
&\quad \times \int dy\, p_{\iota s_1}(x, y) \int dz\, p_{\iota(t_1 - s_1)}(y, z) \int dv_2\, \phi(z - v_2) \int dv_1\, \phi(v_1 - v_2)
\end{aligned}
$$

$$\leq C\,\|\phi\|_1^2 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1\,(t_3-s_2)^{-d/2}(t_2-s_1)^{-d/2}.$$

Since $d \leq 3$, it follows that

$$\begin{aligned}
&\int_0^T dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1\, I_5(t_1,t_2,t_3) \\
&\leq C\,\|\phi\|_1^2 \int_0^T dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1\,(t_3-s_2)^{-d/2}(t_2-s_1)^{-d/2} \\
&\leq C(T)\,\|\phi\|_1^2. \qquad \text{(II.27)}
\end{aligned}$$

Therefore, by II.23,II.24, II.25, II.26, and II.27, it follows that

$$\int_0^T dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1\, \mathbb{E}\,\langle \psi, \mu_{t_1}\mu_{t_2}\mu_{t_3}\rangle \leq C(T)\,\|\phi\|_1^2,$$

and (i) is thus shown.

The proof of (ii) is very similar to that of (i), but clearly involves more calculations. From the moment equation II.17 and the preceding corollary, it follows that

$$\int_0^{t_3} dt_2 \int_0^{t_2} dt_1\, \mathbb{E}\,\langle \varphi, \mu_{t_1}\mu_{t_2}\mu_{t_3}^2\rangle \leq C \sum_{k=1}^{14} J_k(t_1,t_2,t_3),$$

where the definition of each J_k is implicit in equation II.17.

To begin, note that

$$\begin{aligned}
&Q^4_{t_1-s}\pi_1 Q^3_{t_2-t_1}\pi_1 Q^2_{t_3-t_2}\varphi(x) \\
&\leq C \int dy\, p_{\iota(t_1-s)}(x_1,y_1)p_{\iota(t_1-s)}(x_2,y_2)p_{\iota(t_1-s)}(x_3,y_3)p_{\iota(t_1-s)}(x_4,y_4) \\
&\quad \times \int dz\, p_{\iota(t_2-t_1)}(y_2,z_1)p_{\iota(t_2-t_1)}(y_3,z_2)p_{\iota(t_2-t_1)}(y_4,z_3) \\
&\quad \times \int dw\, p_{\iota(t_3-t_2)}(z_2,w_1)p_{\iota(t_3-t_2)}(z_3,w_2)\varphi(y_1,z_1,w_1,w_2) \\
&\leq C \int da\, p_{\iota(t_1-s)}(x_1,a_1)p_{\iota(t_2-s)}(x_2,a_2)p_{\iota(t_3-s)}(x_3,a_3)p_{\iota(t_3-s)}(x_4,a_4)\varphi(a_1,a_2,a_3,a_4),
\end{aligned}$$

(II.28)

for all $x \in \mathbb{R}^{4\times d}$, $s \in [0, t_1]$.

Using the inequality II.28, it follows that

$$
\begin{aligned}
&\langle Q^4_{t_1-s}\pi_1 Q^3_{t_2-t_1}\pi_1 Q^2_{t_3-t_2}\varphi, \mu_0^4\rangle \\
&\le C \int \mu_0^4(dx) \int da\, p_{\iota t_3}(x_4, a_4) \prod_{i=1}^{3} p_{\iota t_i}(x_i, a_i)\varphi(a_1, a_2, a_3, a_4) \\
&\le C \int \mu_0(dx_3)\mu_0(dx_4) \int da_3 da_4\, p_{\iota t_3}(x_3, a_3) p_{\iota t_3}(x_4, a_4) \int da_1\, \phi(a_1 - a_3) \\
&\quad \times \int da_2 \phi(a_2 - a_4) \int \mu_0(dx_1) p_{\iota t_1}(x_1, a_1) \int \mu_0(dx_2)\, p_{\iota t_2}(x_2, a_2) \\
&\le C \, \|m\|^2_\infty \, \mu_0(1)^2 \, \|\phi\|_1^2 \\
&= C \, \|\phi\|_1^2 ,
\end{aligned}
$$

and thus, since $d \le 3$,

$$
\int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \, \langle Q^4_{t_1-s}\pi_1 Q^3_{t_2-t_1}\pi_1 Q^2_{t_3-t_2}\varphi, \mu_0^4\rangle \le C(T)\, \|\phi\|_1^2 .
$$

Let $\{i', j'\} = \{1,2,3,4\} \setminus \{i, j\}$, $i' < j'$, then again from II.28,

$$
\begin{aligned}
&\sum_{\substack{i,j=1 \\ i\neq j}}^{4} \int_0^{t_1} ds \, \langle Q^3_s \Phi_{ij} Q^4_{t_1-s}\pi_1 Q^3_{t_2-t_1}\pi_1 Q^2_{t_3-t_2}\varphi, \mu_0^3\rangle \\
&\le C \sum_{\substack{i,j=1 \\ i<j}}^{4} \int_0^{t_1} ds \, \int \mu_0^3(dx) \int dy\, p_{\iota s}(x_1, y) \int da_1 da_2 da_3 da_4\, p_{\iota(t_i - s)}(y, a_i) \\
&\quad \times p_{\iota(t_{j\wedge 3} - s)}(y, a_j) p_{\iota t_{i'}}(x_2, a_{i'}) p_{\iota t_{j'\wedge 3}}(x_3, a_{j'}) \phi(a_1 - a_3)\phi(a_2 - a_4) \\
&\le C \sum_{\substack{i,j=1 \\ i<j, |i-j|\neq 2}}^{4} \int_0^{t_1} ds \int \mu_0(dx_1) \int dy\, p_{\iota s}(x_1, y) \int da_i da_j\, p_{\iota(t_i - s)}(y, a_i) p_{\iota(t_{j\wedge 3} - s)}(y, a_j) \\
&\quad \times \int da_{i'} da_{j'} \phi(a_1 - a_3)\phi(a_2 - a_4) \int \mu_0(dx_2)\, p_{\iota(t_{i'})}(x_2, a_{i'}) \int \mu_0(dx_3)\, p_{\iota(t_{j'\wedge 3})}(x_3, a_{j'})
\end{aligned}
$$

$$
\begin{aligned}
&+C\sum_{i=1}^{2}\int_0^{t_1} ds \int \mu_0(dx_1)\mu_0(dx_2) \int dy\, p_{\iota s}(x_1,y) \int da_i da_{i+2}\, p_{\iota(t_i-s)}(y,a_i)p_{\iota(t_3-s)}(y,a_{i+2}) \\
&\quad\times \phi(a_i - a_{i+2}) \int da_{i'}\, p_{\iota t_{i'}}(x_2,a_{i'}) \int da_{i'+2}\, \phi(a_{i'}-a_{i'+2}) \int \mu_0(dx_3)\, p_{\iota t_3}(x_3,a_{i'+2}) \\
&\le C(T)\, \|m\|_\infty^2\, \mu_0(1)\, \|\phi\|_1^2 \\
&+C\, \|m\|_\infty\, \mu_0(1)\, \|\phi\|_1 \int_0^{t_1} ds \int \mu_0(dx_1) \int dy p_{\iota s}(x_1,y) \\
&\quad\times \int da_i da_{i+2}\, p_{\iota(t_i-s)}(y,a_i)p_{\iota(t_3-s)}(y,a_{i+2})\phi(a_i-a_{i+2}) \\
&\le C(T)\, \|\phi\|_1^2 + C\int_0^{t_1} ds\, (t_3-s)^{-d/2},
\end{aligned}
$$

and since $d \le 3$, it follows that

$$
\begin{aligned}
&\sum_{\substack{i,j=1\\ i\neq j}}^{4} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_0^{t_1} ds\, \left\langle Q_s^3 \Phi_{ij} Q_{t_1-s}^4 \pi_1 Q_{t_2-t_1}^3 \pi_1 Q_{t_3-t_2}^2 \varphi, \mu_0^3 \right\rangle \\
&\le \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_0^{t_1} ds\, \left[C(T)\, \|\phi\|_1^2 + C\int_0^{t_1} ds\, (t_3-s)^{-d/2} \right] \\
&\le C(T)\, \|\phi\|_1^2 .
\end{aligned}
$$

This next case becomes quite a bit more complicated, so we explain with more detail. Consider

$$
\sum_{\substack{i,j=1\\ i\neq j}}^{4} \sum_{\substack{n,m=1\\ n\neq m}}^{3} \int_0^{t_1} ds_2 \int_0^{s_2} ds_1\, \left\langle Q_{s_1}^2 \Phi_{nm} Q_{s_2-s_1}^3 \Phi_{ij} Q_{t_1-s_2}^4 \pi_1 Q_{t_2-t_1}^3 \pi_1 Q_{t_3-t_2}^2 \varphi, \mu_0^2 \right\rangle ,
$$

wherein the presence of both Φ_{nm} and Φ_{ij} greatly increase the number of cases. In bounding, we again may assume, with the addition of a multiplicative constant to the bound, that $i < j$. Note first that when $m+n \neq 6-i$ it will follow that either

$$\begin{aligned}
&Q^2_{s_1}\Phi_{nm}Q^3_{s_2-s_1}\Phi_{ij}Q^4_{t_1-s_2}\pi_1Q^3_{t_2-t_1}\pi_1Q^2_{t_3-t_2}\varphi(x_1,x_2)\\
&\leq C\int dy\, p_{\iota s_1}(x_1,y)\int dz\, p_{\iota(s_2-s_1)}(y,z)\\
&\quad\times\int dw_1dw_2dw_3dw_4\, p_{\iota(t_i-s_2)}(z,w_i)p_{\iota(t_{j\wedge 3}-s_2)}(z,w_j)p_{\iota(t_{i'}-s_1)}(y,w_{i'})p_{\iota t_{j'\wedge 3}}(x_2,w_{j'})\varphi(w),
\end{aligned}$$

or

$$\begin{aligned}
&Q^2_{s_1}\Phi_{nm}Q^3_{s_2-s_1}\Phi_{ij}Q^4_{t_1-s_2}\pi_1Q^3_{t_2-t_1}\pi_1Q^2_{t_3-t_2}\varphi(x_1,x_2)\\
&\leq C\int dy\, p_{\iota s_1}(x_1,y)\int dz\, p_{\iota(s_2-s_1)}(y,z)\\
&\quad\times\int dw_1dw_2dw_3dw_4\, p_{\iota(t_i-s_2)}(z,w_i)p_{\iota(t_{j\wedge 3}-s_2)}(z,w_j)p_{\iota(t_{j'\wedge 3}-s_1)}(y,w_{j'})p_{\iota t_{i'}}(x_2,w_{i'})\varphi(w),
\end{aligned}$$

where again $\{i',j'\} = \{1,2,3,4\}\setminus\{i,j\}$, with $i' < j'$.

In the case that $m+n = 6-i$, we have the bound

$$\begin{aligned}
&Q^2_{s_1}\Phi_{nm}Q^3_{s_2-s_1}\Phi_{ij}Q^4_{t_1-s_2}\pi_1Q^3_{t_2-t_1}\pi_1Q^2_{t_3-t_2}\varphi(x_1,x_2)\\
&\leq C\int dy\, p_{\iota s_1}(x_1,y)\int dz\, p_{\iota s_2}(x_2,z)\\
&\times\int dw_1dw_2dw_3dw_4\, p_{\iota(t_i-s_2)}(z,w_i)p_{\iota(t_{j\wedge 3}-s_2)}(z,w_j)p_{\iota(t_{i'}-s_1)}(y,w_{i'})p_{\iota(t_{j'\wedge 3}-s_1)}(y,w_{j'})\varphi(w).
\end{aligned}$$

It thus follows that

$$\begin{aligned}
&\sum_{\substack{i,j=1\\ i\neq j}}^{4}\sum_{\substack{n,m=1\\ n\neq m}}^{3}\int_0^{t_1}ds_2\int_0^{s_2}ds_1\,\langle Q^2_{s_1}\Phi_{nm}Q^3_{s_2-s_1}\Phi_{ij}Q^4_{t_1-s_2}\pi_1Q^3_{t_2-t_1}\pi_1Q^2_{t_3-t_2}\varphi,\mu_0^2\rangle\\
&\leq C\sum_{\substack{i,j,i',j'=1\\ i'\neq i,j'\neq j\\ i<j,i'<j'}}^{4}\sum_{\substack{n,m=1\\ n<m\\ 6-n-m\neq i}}^{3}\int_0^{t_1}ds_2\int_0^{s_2}ds_1\int\mu_0(dx_1)\mu_0(dx_2)\int dy\, p_{\iota s_1}(x_1,y)\int dz\, p_{\iota(s_2-s_1)}(y,z)\\
&\quad\times\int dw_1dw_2dw_3dw_4\, p_{\iota t_{j'\wedge 3}}(x_2,w_{j'})p_{\iota(t_{i'}-s_1)}(y,w_{i'})p_{\iota(t_i-s_2)}(z,w_i)\\
&\quad\times p_{\iota(t_{j\wedge 3}-s_2)}(z,w_j)\varphi(w_1,w_2,w_3,w_4)
\end{aligned}$$

$$+C\sum_{\substack{i,j,i',j'=1\\ i'\neq i,j'\neq j\\ i<j,i'<j'}}^{4}\sum_{\substack{n,m=1\\ n<m\\ 6-n-m\neq i}}^{3}\int_0^{t_1}ds_2\int_0^{s_2}ds_1\int\mu_0(dx_1)\mu_0(dx_2)\int dy\,p_{\iota s_1}(x_1,y)\int dz\,p_{\iota(s_2-s_1)}(y,z)$$

$$\times\int dw_1dw_2dw_3dw_4\,p_{\iota t_{i'}}(x_2,w_{i'})p_{\iota(t_{j'\wedge 3}-s_1)}(y,w_{j'})p_{\iota(t_i-s_2)}(z,w_i)$$

$$\times\,p_{\iota(t_{j\wedge 3}-s_2)}(z,w_j)\varphi(w_1,w_2,w_3,w_4)$$

$$+C\sum_{\substack{i,j,i',j'=1\\ i'\neq i,j'\neq j\\ i<j,i'<j'}}^{4}\sum_{\substack{n,m=1\\ n<m\\ 6-n-m=i}}^{3}\int_0^{t_1}ds_2\int_0^{s_2}ds_1\int\mu_0(dx_1)\mu_0(dx_2)\int dy\,p_{\iota s_1}(x_1,y)\int dz\,p_{\iota s_2}(x_2,z)$$

$$\times\int dw_1dw_2dw_3dw_4\,p_{\iota(t_{i'}-s_1)}(y,w_{i'})p_{\iota(t_{j'\wedge 3}-s_1)}(y,w_{j'})p_{\iota(t_i-s_2)}(z,w_i)$$

$$\times\,p_{\iota(t_{j\wedge 3}-s_2)}(z,w_j)\varphi(w_1,w_2,w_3,w_4)$$

$$\leq C\sum_{\substack{i,j,i',j'=1\\ i'\neq i,j'\neq j\\ i<j,i'<j'}}^{4}\sum_{\substack{n,m=1\\ n<m\\ 6-n-m\neq i}}^{3}\int_0^{t_1}ds_2\int_0^{s_2}ds_1\int\mu_0(dx_1)\int dy\,p_{\iota s_1}(x_1,y)\int dz\,p_{\iota(s_2-s_1)}(y,z)$$

$$\times\int dw_idw_jdw_{i'}\,p_{\iota(t_{i'}-s_1)}(y,w_{i'})p_{\iota(t_i-s_2)}(z,w_i)p_{\iota(t_{j\wedge 3}-s_2)}(z,w_j)$$

$$\times\int dw_{j'}\,\phi(w_1-w_3)\phi(w_2-w_4)\int\mu_0(dx_2)\,p_{\iota t_{j'\wedge 3}}(x_2,w_{j'})$$

$$+C\sum_{\substack{i,j,i',j'=1\\ i'\neq i,j'\neq j\\ i<j,i'<j'}}^{4}\sum_{\substack{n,m=1\\ n<m\\ 6-n-m\neq i}}^{3}\int_0^{t_1}ds_2\int_0^{s_2}ds_1\int\mu_0(dx_1)\int dy\,p_{\iota s_1}(x_1,y)\int dz\,p_{\iota(s_2-s_1)}(y,z)$$

$$\times\int dw_idw_jdw_{j'}\,p_{\iota(t_{j'\wedge 3}-s_1)}(y,w_{j'})p_{\iota(t_i-s_2)}(z,w_i)p_{\iota(t_{j\wedge 3}-s_2)}(z,w_j)$$

$$\times\int dw_{i'}\,\phi(w_1-w_3)\phi(w_2-w_4)\int\mu_0(dx_2)\,p_{\iota t_{i'}}(x_2,w_{i'})$$

$$+C\sum_{\substack{i,j,i',j'=1\\ i'\neq i,j'\neq j\\ i<j,i'<j'}}^{4}\sum_{\substack{n,m=1\\ n<m\\ 6-n-m=i}}^{3}\int_0^{t_1}ds_2\int_0^{s_2}ds_1\int\mu_0(dx_1)\mu_0(dx_2)\int dy\,p_{\iota s_1}(x_1,y)\int dz\,p_{\iota s_2}(x_2,z)$$

$$\times\int dw_1dw_2dw_3dw_4\,p_{\iota(t_{i'}-s_1)}(y,w_{i'})p_{\iota(t_{j'\wedge 3}-s_1)}(y,w_{j'})p_{\iota(t_i-s_2)}(z,w_i)$$

$$\times\,p_{\iota(t_{j\wedge 3}-s_2)}(z,w_j)\varphi(w_1,w_2,w_3,w_4).$$

For the first of the above three terms, the process is as follows. Bound $\mu_0(dx_2)$

by $\|m\|_\infty \, dx_2$, integrate $p_{\iota t_{j'\wedge 3}}(x_2, w_j')$ with respect to dx_2, then ϕ with respect to $dw_{j'}$. In doing so, we may then integrate out one of the remianing w_i, w_j, or w_j'. In what remains, if $(i,j) \neq (1,2)$ there will be the term $p_{\iota(t_3-s_2)}(z, w_.)$, or if $(i,j) = (1,2)$, the term $p_{\iota(t_3-s_1)}(y, w_3)$. In either case, bound the respective term by $C(t_3 - s_.)^{-d/2}$. This allows for the integration of the second ϕ.

For the second of the two above terms, Bound $\mu_0(dx_2)$ by $\|m\|_\infty \, dx_2$, integrate $p_{\iota t_{i'\wedge 3}}(x_2, w_i')$ with respect to dx_2, then ϕ with respect to $dw_{i'}$. In doing so, we may then integrate out one of the remianing w_i, w_j, or w_j'. In what remains, if $(i,j) \in \{(1,2),(1,4),(2,3)\}$ there will be the term $p_{\iota(t_3-s_1)}(y, w_{j'})$, otherwise there will exist the term $p_{\iota(t_3-s_2)}(z, w_.)$. In either case, bound the respective term by $C(t_3 - s_.)^{-d/2}$. This allows for the integration of the second ϕ.

For the third and final term, if $(i,j) \notin \{(1,2),(3,4)\}$, there will exist the terms $p_{\iota(t_3-s_2)}(z, w_j)$ and $p_{\iota(t_3-s_1)}$, which are bounded respectively by $C(t_3-s_2)^{-d/2}$ and $C(t_3-s_1)^{-d/2}$. When $(i,j) = (1,2)$ we bound $p_{\iota(t_3-s_1)}(y, w_3)$ and $p_{\iota(t_2-s_2)}(z, w_2)$ respectively by $C(t_3 - s_1)^{-d/2}$ and $C(t_2 - s_2)^{-d/2}$. Finally, when $(i,j) = (3,4)$, bound the terms $p_{\iota(t_3-s_2)}(z, w_3)$ and $p_{\iota(t_2-s_1)}(y, w_2)$ respectively by $C(t_3-s_2)^{-d/2}$ and $C(t_2-s_1)^{-d/2}$. This allows for the desired integration of $\phi(w_1 - w_3)\phi(w_2 - w_4)$.

Combining the above, and since $d \leq 3$, we arrive at the bound

$$
\begin{aligned}
&\sum_{\substack{i,j=1 \\ i\neq j}}^{4} \sum_{\substack{n,m=1 \\ n\neq m}}^{3} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_0^{t_1} ds_1 \int_0^{s_1} ds_2 \\
&\qquad \times \left\langle Q^2_{s_1} \Phi_{nm} Q^3_{s_2-s_1} \Phi_{ij} Q^4_{t_1-s_2} \pi_1 Q^3_{t_2-t_1} \pi_1 Q^2_{t_3-t_2} \pi_1 Q_{t_4-t_3} \varphi, \mu_0^2 \right\rangle \\
&\leq C \|\phi\|_1^2 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_0^{t_1} ds_1 \int_0^{s_1} ds_2 \Big[(t_3 - s_1)^{-d/2} + (t_3 - s_2)^{-d/2} \\
&\quad + (t_3 - s_2)^{-d/2}(t_3 - s_1)^{-d/2} + (t_3 - s_1)^{-d/2}(t_2 - s_2)^{-d/2} + (t_3 - s_2)^{-d/2}(t_2 - s_1)^{-d/2} \Big] \\
&\leq C(T) \|\phi\|_1^2.
\end{aligned}
$$

Considering the next case, note first the similarities in the respective corresponding particle pictures of this and the previous case. This case can be seen as a modification of the previous case in which the two original paricles were both born from a common ancestor. Thus, arguing as before, we arrive at the bound

$$\sum_{\substack{i,j=1\\ i\neq j}}^{4}\sum_{\substack{n,m=1\\ n\neq m}}^{3}\int_0^{t_1}ds_3\int_0^{s_3}ds_2\int_0^{s_2}ds_1$$

$$\times\left\langle Q_{s_1}\Phi_{12}Q^2_{s_2-s_1}\Phi_{nm}Q^3_{s_3-s_2}\Phi_{ij}Q^4_{t_1-s_3}\pi_1Q^3_{t_2-t_1}\pi_1Q^2_{t_3-t_2}\varphi,\mu_0\right\rangle$$

$$\leq C\sum_{\substack{i,j,i',j'=1\\ i'\neq i,j'\neq j\\ i<j,i'<j'}}^{4}\sum_{\substack{n,m=1\\ n<m\\ 6-n-m\neq i}}^{3}\int_0^{t_1}ds_3\int_0^{s_3}ds_2\int_0^{s_2}ds_1\int\mu_0(dx)\int dy\,p_{\iota s_1}(x,y)\int dz\,p_{\iota(s_2-s_1)}(y,z)$$

$$\times\int dw\,p_{\iota(s_3-s_2)}(z,w)\int dv_1dv_2dv_3dv_4\,p_{\iota(t_{j'\wedge 3}-s_1)}(y,v_{j'})p_{\iota(t_{i'}-s_2)}(z,v_{i'})$$

$$\times p_{\iota(t_i-s_3)}(w,v_i)p_{\iota(t_{j\wedge 3}-s_3)}(w,v_j)\phi(v_1-v_3)\phi(v_2-v_4)$$

$$+C\sum_{\substack{i,j,i',j'=1\\ i'\neq i,j'\neq j\\ i<j,i'<j'}}^{4}\sum_{\substack{n,m=1\\ n<m\\ 6-n-m\neq i}}^{3}\int_0^{t_1}ds_3\int_0^{s_3}ds_2\int_0^{s_2}ds_1\int\mu_0(dx)\int dy\,p_{\iota s_1}(x,y)\int dz\,p_{\iota(s_2-s_1)}(y,z)$$

$$\times\int dw\,p_{\iota(s_3-s_2)}(z,w)\int dv_1dv_2dv_3dv_4\,p_{\iota(t_{i'}-s_1)}(y,v_{i'})p_{\iota(t_{j'\wedge 3}-s_2)}(z,v_{j'})$$

$$\times p_{\iota(t_i-s_3)}(w,v_i)p_{\iota(t_{j\wedge 3}-s_3)}(w,v_j)\phi(v_1-v_3)\phi(v_2-v_4)$$

$$+C\sum_{\substack{i,j,i',j'=1\\ i'\neq i,j'\neq j\\ i<j,i'<j'}}^{4}\sum_{\substack{n,m=1\\ n<m\\ 6-n-m=i}}^{3}\int_0^{t_1}ds_3\int_0^{s_3}ds_2\int_0^{s_2}ds_1\int\mu_0(dx)\int dy\,p_{\iota s_1}(x,y)$$

$$\times\int dz_1dz_2\,p_{\iota(s_2-s_1)}(y,z_1)p_{\iota(s_3-s_1)}(y,z_2)\int dv_1dv_2dv_3dv_4\,p_{\iota(t_i-s_3)}(z_2,v_i)$$

$$\times p_{\iota(t_{j\wedge 3}-s_3)}(z_2,v_j)p_{\iota(t_{i'}-s_2)}(z_1,v_{i'})p_{\iota(t_{j'\wedge 3}-s_2)}(z_1,v_{j'})\phi(v_1-v_3)\phi(v_2-v_4)$$

$$\leq C\,\|\phi\|_1^2\int_0^{t_1}ds_3\int_0^{s_3}ds_2\int_0^{s_2}ds_1\Big[(t_3-s_2)^{-d/2}(t_2-s_2)^{-d/2}+(t_3-s_2)^{-d/2}(t_3-s_3)^{-d/2}$$

$$+(t_2-s_2)^{-d/2}(t_3-s_3)^{-d/2}+(t_3-s_2)^{-d/2}(t_2-s_3)^{-d/2}\Big],$$

where the above bounds are obtained similar to the previous case.

And so, since $d \leq 3$,

$$\sum_{\substack{i,j=1\\ i\neq j}}^{4} \sum_{\substack{n,m=1\\ n\neq m}}^{3} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_0^{t_1} ds_3 \int_0^{s_3} ds_2 \int_0^{s_2} ds_1$$

$$\times \left\langle Q_{s_3} \Phi_{12} Q^2_{s_2-s_3} \Phi_{nm} Q^3_{s_1-s_2} \Phi_{ij} Q^4_{t_1-s_3} \pi_1 Q^3_{t_2-t_1} \pi_1 Q^2_{t_3-t_2} \varphi, \mu_0 \right\rangle$$

$$\leq C(T) \|\phi\|_1^2 .$$

This takes care of four of the fourteen J_k, we consider now the next three integrals which are dependent upon the expression

$$\begin{aligned}
& Q^3_{t_1-s_1} \pi_1 Q^2_{s_2-t_1} \Phi_{ij} Q^3_{t_2-s_2} \pi_1 Q^2_{t_3-t_2} \varphi(x) \\
& \leq C \int dy\, p_{\iota(t_1-s_1)}(x_1, y_1) p_{\iota(t_1-s_1)}(x_2, y_2) p_{\iota(t_1-s_1)}(x_3, y_3) \\
& \quad \times \int dz\, p_{\iota(s_2-t_1)}(y_2, z_1) p_{\iota(s_2-t_1)}(y_3, z_2) \\
& \quad \times \int dw\, p_{\iota(t_2-s_2)}(z_1^{ij}, w_1) p_{\iota(t_2-s_2)}(z_2^{ij}, w_2) p_{\iota(t_2-s_2)}(z_3^{ij}, w_3) \\
& \quad \times \int dv\, p_{\iota(t_3-t_2)}(w_2, v_1) p_{\iota(t_3-t_2)}(w_3, v_2) \varphi(y_1, w_1, v_1, v_2) \\
& \leq C \int db\, p_{\iota(s_2-s_1)}(x_3, b) \int da_1 da_2 da_3 da_4\, p_{\iota(t_1-s_1)}(x_1, a_1) p_{\iota(t_{4-i}-s_1)}(x_2, a_{7-i-j}) \\
& \quad \times p_{\iota(t_{i+1}-s_2)}(b, a_{i+1}) p_{\iota(t_3-s_2)}(b, a_{j+1}) \phi(a_1 - a_3) \phi(a_2 - a_4),
\end{aligned} \tag{II.29}$$

for all $x \in \mathbb{R}^{3\times d}$, $0 \leq s_1 \leq t_1$, $t_1 \leq s_2 \leq t_2$, and $i, j = 1, 2, 3$, $i \neq j$. In the above z^{ij} refers to the particular arrangement of z_1, z_2 given the pair (i, j).

Applying II.29 now gives,

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_{t_1}^{t_2} ds\ \left\langle Q^3_{t_1} \pi_1 Q^2_{s-t_1} \Phi_{ij} Q^3_{t_2-s} \pi_1 Q^2_{t_3-t_2} \varphi, \mu_0^3 \right\rangle$$

$$
\begin{aligned}
&\leq C \sum_{\substack{i,j=1 \\ i<j}}^{3} \int_{t_1}^{t_2} ds \int \mu_0^3(dx) \int dy\, p_{\iota s}(x_3, y) \\
&\times \int dz\, p_{\iota t_1}(x_1, z_1) p_{\iota t_{4-i}}(x_2, z_{7-i-j}) p_{\iota(t_{i+1}-s)}(y, z_{i+1}) \\
&\times p_{\iota(t_3-s)}(y, z_{j+1}) \phi(z_1 - z_3) \phi(z_2 - z_4) \\
&\leq C \sum_{\substack{i,j=1 \\ i<j}}^{3} \int_{t_1}^{t_2} ds \int \mu_0(dx_2)\mu_0(dx_3) \int dy\, p_{\iota s}(x_3, y) \int dz_{7-i-j} dz_{i+1}\, p_{\iota t_{4-i}}(x_2, z_{7-i-j}) \\
&\times p_{\iota(t_{i+1}-s)}(y, z_{i+1}) \int dz_{j+1}\, p_{\iota(t_3-s)}(y, z_{j+1}) \\
&\times \int dz_1\, \phi(z_1 - z_3)\phi(z_2 - z_4) \int \mu_0(dx_1)\, p_{\iota t_1}(x_1, z_1) \\
&\leq C \left\| m \right\|_\infty \left\| \phi \right\|_1 \sum_{\substack{i,j=1 \\ i<j}}^{3} \int_{t_1}^{t_2} ds \int \mu_0(dx_2)\mu_0(dx_3) \int dy\, p_{\iota s}(x_3, y) \\
&\times \int dz_2 dz_3 dz_4\, p_{\iota t_{4-i}}(x_2, z_{7-i-j}) p_{\iota(t_{i+1}-s)}(y, z_{i+1}) p_{\iota(t_3-s)}(y, z_{j+1}) \phi(z_2 - z_4) \\
&\leq C \left\| \phi \right\|_1 \sum_{i=1}^{2} \int_{t_1}^{t_2} ds \int \mu_0(dx_3) \int dy\, p_{\iota s}(x_3, y) \int dz_{i+1} dz_{i+2}\, p_{\iota(t_{i+1}-s)}(y, z_{i+1}) \\
&\times p_{\iota(t_3-s)}(y, z_{i+2}) \int dz_{6-2i}\, \phi(z_2 - z_4) \int \mu_0(dx_2) p_{\iota t_{4-i}}(x_2, z_{6-2i}) \\
&+ C \left\| \phi \right\|_1 \int_{t_1}^{t_2} ds\, (t_3 - s)^{-d/2} \int \mu_0(dx_2)\mu_0(dx_3) \int dy\, p_{\iota s}(x, y) \\
&\times \int dz_2\, p_{\iota(t_3-s)}(y, z_2) \int dz_3\, p_{\iota t_3}(x_2, z_3) \int dz_4\, \phi(z_2 - z_4) \\
&\leq C \left\| \phi \right\|_1^2 \int_{t_1}^{t_2} ds \left[1 + (t_3 - s)^{-d/2} \right].
\end{aligned}
$$

And so, since $d \leq 3$,

$$
\sum_{\substack{i,j=1 \\ i\neq j}}^{3} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_1}^{t_2} ds\, \langle Q^3_{t_1} \pi_1 Q^2_{t_2-s} \Phi_{ij} Q^3_{s-t_1} \pi_1 Q^2_{t_3-t_2} \varphi, \mu_0^3 \rangle \leq C(T) \left\| \phi \right\|_1^2.
$$

Again from II.29,

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3}\sum_{\substack{n,m=1\\ n\neq m}}^{3}\int_{t_1}^{t_2} ds_2\int_0^{t_1} ds_1\ \langle Q^2_{s_1}\Phi_{nm}Q^3_{t_1-s_1}\pi_1 Q^2_{s_2-t_1}\Phi_{ij}Q^3_{t_2-s_2}\pi_1 Q^2_{t_3-t_2}\varphi,\mu_0^2\rangle$$

$$\leq C\sum_{\substack{i,j=1\\ i<j}}^{3}\int_{t_1}^{t_2} ds_2\int_0^{t_1} ds_1\int \mu_0(dx_1)\mu_0(dx_2)\int dy\, p_{\iota s_1}(x_1,y)\int dz\, p_{\iota(s_2-s_1)}(y,z)$$

$$\times\int dw_1dw_2dw_3dw_4\, p_{\iota(t_1-s_1)}(y,w_1)p_{\iota(t_{i+1}-s_2)}(z,w_{i+1})p_{\iota(t_3-s_2)}(z,w_{j+1})$$

$$\times\, p_{\iota(t_{(7-i-j)\wedge 3})}(x_2,w_{7-i-j})\phi(w_1-w_3)\phi(w_2-w_4)$$

$$+\,C\sum_{\substack{i,j=1\\ i<j}}^{3}\int_{t_1}^{t_2} ds_2\int_0^{t_1} ds_1\int \mu_0(dx_1)\mu_0(dx_2)\int dy\, p_{\iota s_1}(x_1,y)\int dz\, p_{\iota(s_2-s_1)}(y,z)$$

$$\times\int dw_1dw_2dw_3dw_4\, p_{\iota(t_{(7-i-j)\wedge 3}-s_1)}(y,w_{7-i-j})p_{\iota(t_{i+1}-s_2)}(z,w_{i+1})$$

$$\times\, p_{\iota(t_3-s_2)}(z,w_{j+1})p_{\iota t_1}(x_2,w_1)\phi(w_1-w_3)\phi(w_2-w_4)$$

$$+\,C\sum_{\substack{i,j=1\\ i<j}}^{3}\int_{t_1}^{t_2} ds_2\int_0^{t_1} ds_1\int \mu_0(dx_1)\mu_0(dx_2)\int dy\ p_{\iota s_1}(x_1,y)\int dz\, p_{\iota s_2}(x_2,z)$$

$$\times\int dw_1dw_2dw_3dw_4\, p_{\iota(t_1-s_1)}(y,w_1)p_{\iota(t_{(7-i-j)\wedge 3}-s_1)}(y,w_{7-i-j})p_{\iota(t_{i+1}-s_2)}(z,w_{i+1})$$

$$\times\, p_{\iota(t_3-s_2)}(z,w_{j+1})\phi(w_1-w_3)\phi(w_2-w_4)$$

$$\leq C\sum_{\substack{i,j=1\\ i<j}}^{3}\int_{t_1}^{t_2} ds_2\int_0^{t_1} ds_1\int \mu_0(dx_1)\int dy\, p_{\iota s_1}(x_1,y)\int dz\, p_{\iota(s_2-s_1)}(y,z)$$

$$\times\int dw_1dw_{i+1}dw_{j+1}\, p_{\iota(t_1-s_1)}(y,w_1)p_{\iota(t_{i+1}-s_2)}(z,w_{i+1})p_{\iota(t_3-s_2)}(z,w_{j+1})$$

$$\times\int dw_{7-i-j}\,\phi(w_1-w_3)\phi(w_2-w_4)\int \mu_0(dx_2)\, p_{\iota(t_{(7-i-j)\wedge 3})}(x_2,w_{7-i-j})$$

$$+\,C\sum_{\substack{i,j=1\\ i<j}}^{3}\int_{t_1}^{t_2} ds_2\int_0^{t_1} ds_1\int \mu_0(dx_1)\int dy\, p_{\iota s_1}(x_1,y)\int dz\, p_{\iota(s_2-s_1)}(y,z)$$

$$\times\int dw_2dw_3dw_4\, p_{\iota(t_{(7-i-j)\wedge 3}-s_1)}(y,w_{7-i-j})p_{\iota(t_{i+1}-s_2)}(z,w_{i+1})$$

$$\times\, p_{\iota(t_3-s_2)}(z,w_{j+1})\phi(w_2-w_4)\int dw_1\,\phi(w_1-w_3)\int \mu_0(dx_2)\, p_{\iota t_1}(x_2,w_1)$$

$$
\begin{aligned}
&+ C \sum_{\substack{i,j=1\\ i<j}}^{3} \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1 \int \mu_0(dx_1)\mu_0(dx_2) \int dy\; p_{\iota s_1}(x_1, y) \int dz\, p_{\iota s_2}(x_2, z) \\
&\times \int dw_1 dw_2 dw_3 dw_4\, p_{\iota(t_1-s_1)}(y, w_1) p_{\iota(t_{(7-i-j)\wedge 3}-s_1)}(y, w_{7-i-j}) p_{\iota(t_{i+1}-s_2)}(z, w_{i+1}) \\
&\times p_{\iota(t_3-s_2)}(z, w_{j+1}) \phi(w_1 - w_3)\phi(w_2 - w_4) \\
&\le C \left\|\phi\right\|_1 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1\, (t_3 - s_2)^{-d/2} \\
&+ C \left\|\phi\right\|_1 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1\, \left[(t_3 - s_1)^{-d/2} + (t_3 - s_2)^{-d/2}\right] \\
&+ C \left\|\phi\right\|_1 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1\, \left[(t_3 - s_1)^{-d/2}(t_3 - s_2)^{-d/2} + (t_3 - s_2)^{-d/2}\right] \\
&\le C \left\|\phi\right\|_1^2 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1\, \left[(t_3 - s_1)^{-d/2} + (t_3 - s_1)^{-d/2}(t_3 - s_2)^{-d/2} + (t_3 - s_2)^{-d/2}\right],
\end{aligned}
$$

Thus, since $d \le 3$,

$$
\begin{aligned}
&\sum_{\substack{i,j=1\\ i\neq j}}^{3} \sum_{\substack{n,m=1\\ n\neq m}}^{3} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1 \langle Q^2_{s_1} \Phi_{nm} Q^3_{t_1-s_1} \pi_1 Q^2_{s_2-t_1} \Phi_{ij} Q^3_{t_2-s_2} \pi_1 Q^2_{t_3-t_2} \varphi, \mu_0^2 \rangle \\
&\le C(T) \left\|\phi\right\|_1^2 .
\end{aligned}
$$

After one final application of II.29,

$$\sum_{\substack{i,j=1\\i\neq j}}^{3}\sum_{\substack{n,m=1\\n\neq m}}^{3}\int_{t_1}^{t_2}ds_3\int_0^{t_1}ds_2\int_0^{s_2}ds_1\left\langle Q_{s_1}\Phi_{12}Q^2_{s_2-s_1}\Phi_{nm}Q^3_{t_1-s_2}\pi_1Q^2_{s_3-t_1}\Phi_{ij}Q^3_{t_2-s_3}\pi_1Q^2_{t_3-t_2}\varphi,\mu_0\right\rangle$$

$$\leq C\sum_{\substack{i,j=1\\i\neq j}}^{3}\int_{t_1}^{t_2}ds_3\int_0^{t_1}ds_2\int_0^{s_2}ds_1\int\mu_0(dx)\int dy\,p_{\iota s_1}(x,y)\int dz\,p_{\iota(s_2-s_1)}(y,z)$$

$$\times\int dw\,p_{\iota(s_3-s_2)}(z,w)\int dv_1dv_2dv_3dv_4\,p_{\iota(t_1-s_2)}(z,v_1)p_{\iota(t_{(7-i-j)\wedge 3}-s_1)}(y,v_{7-i-j})$$

$$\times\,p_{\iota(t_{i+1}-s_3)}(w,v_{i+1})p_{\iota(t_3-s_3)}(w,v_{j+1})\phi(v_1-v_3)\phi(v_2-v_4)$$

$$+\,C\sum_{\substack{i,j=1\\i\neq j}}^{3}\int_{t_1}^{t_2}ds_3\int_0^{t_1}ds_2\int_0^{s_2}ds_1\int\mu_0(dx)\int dy\,p_{\iota s_1}(x,y)\int dz\,p_{\iota(s_2-s_1)}(y,z)$$

$$\times\int dw\,p_{\iota(s_3-s_2)}(z,w)\int dv_1dv_2dv_3dv_4\,p_{\iota(t_{(7-i-j)\wedge 3}-s_2)}(z,v_{7-i-j})p_{\iota(t_1-s_1)}(y,v_1)$$

$$\times\,p_{\iota(t_{i+1}-s_3)}(w,v_{i+1})p_{\iota(t_3-s_3)}(w,v_{j+1})\phi(v_1-v_3)\phi(v_2-v_4)$$

$$+\,C\sum_{\substack{i,j=1\\i\neq j}}^{3}\int_{t_1}^{t_2}ds_3\int_0^{t_1}ds_2\int_0^{s_2}ds_1\int\mu_0(dx)\int dy\,p_{\iota s_1}(x,y)\int dz\,p_{\iota(s_2-s_1)}(y,z)$$

$$\times\int dw\,p_{\iota(s_3-s_1)}(y,w)\int dw_1dw_2dw_3dw_4\,p_{\iota(t_1-s_2)}(z,v_1)p_{\iota(t_{(7-i-j)\wedge 3}-s_2)}(z,v_{7-i-j})$$

$$\times\,p_{\iota(t_{i+1}-s_3)}(w,v_{i+1})p_{\iota(t_3-s_3)}(w,v_{j+1})\phi(v_1-v_3)\phi(v_2-v_4)$$

$$\leq C\,\|\phi\|_1^2\int_{t_1}^{t_2}ds_3\int_0^{t_1}ds_2\int_0^{s_2}ds_1\left[(t_3-s_1)^{-d/2}(t_3-s_3)^{-d/2}+(t_2-s_1)^{-d/2}(t_3-s_3)^{-d/2}\right]$$

$$+\,C\,\|\phi\|_1^2\int_{t_1}^{t_2}ds_3\int_0^{t_1}ds_2\int_0^{s_2}ds_1\left[(t_3-s_2)^{-d/2}(t_3-s_3)^{-d/2}+(t_2-s_2)^{-d/2}(t_3-s_3)^{-d/2}\right]$$

$$+\,C\,\|\phi\|_1^2\int_{t_1}^{t_2}ds_3\int_0^{t_1}ds_2\int_0^{s_2}ds_1\left[(t_3-s_2)^{-d/2}(t_3-s_3)^{-d/2}+(t_2-s_2)^{-d/2}(t_3-s_3)^{-d/2}\right]$$

$$\leq C\,\|\phi\|_1^2\sum_{k=1}^{2}\int_{t_1}^{t_2}ds_3\int_0^{t_1}ds_2\int_0^{s_2}ds_1\left[(t_3-s_3)^{-d/2}\left((t_3-s_k)^{-d/2}+(t_2-s_k)^{-d/2}\right)\right].$$

Therefore, since $d \leq 3$,

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3} \sum_{\substack{n,m=1\\ n\neq m}}^{3} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_1}^{t_2} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1$$
$$\times \left\langle Q_{s_1} \Phi_{12} Q^2_{s_2-s_1} \Phi_{nm} Q^3_{t_1-s_2} \pi_1 Q^2_{s_3-t_1} \Phi_{ij} Q^3_{t_2-s_3} \pi_1 Q^2_{t_3-t_2} \varphi, \mu_0 \right\rangle$$
$$\leq C(T) \, \|\phi\|_1^2 .$$

Thus seven of the fourteen J_k are now shown to have the desired bound, we continue with three more of the J_k.

$$\begin{aligned} &Q^3_{t_1-s_1} \pi_1 Q^2_{t_2-t_1} \pi_1 Q_{s_2-t_2} \Phi_{12} Q^2_{t_3-s_2} \varphi(x) \\ &\leq C \int db \, p_{\iota(s_2-s_1)}(x_3, b) \int da_1 da_2 da_3 da_4 \, p_{\iota(t_1-s_1)}(x_1, a_1) p_{\iota(t_2-s_1)}(x_2, a_2) \\ &\qquad \times p_{\iota(t_3-s_2)}(b, a_3) p_{\iota(t_3-s_2)}(b, a_4) \varphi(a_1, a_2, a_3, a_4), \end{aligned} \tag{II.30}$$

for all $x \in \mathbb{R}^{3\times d}$, $0 \leq s_1 \leq t_1, t_2 \leq s_2 \leq t_3$.

Now, from the inequality II.30

$$\begin{aligned} &\int_{t_2}^{t_3} ds \, \left\langle Q^3_{t_1} \pi_1 Q^2_{t_2-t_1} \pi_1 Q_{s-t_2} \Phi_{12} Q^2_{t_3-s} \varphi, \mu_0^3 \right\rangle \\ &\leq C \int_{t_2}^{t_3} ds \int \mu_0(dx_1) \mu_0(dx_2) \mu_0(dx_3) \int dy \, p_{\iota s}(x_3, y) \\ &\quad \times \int dz_1 dz_2 dz_3 dz_4 \, p_{\iota t_1}(x_1, z_1) p_{\iota t_2}(x_2, z_2) p_{\iota(t_3-s)}(y, z_3) p_{\iota(t_3-s)}(y, z_4) \phi(z_1 - z_3) \phi(z_2 - z_4) \\ &\leq C \int_{t_2}^{t_3} ds \int \mu_0(dx_3) \int dy \, p_{\iota s}(x_3, y) \int dz_3 dz_4 \, p_{\iota(t_3-s)}(y, z_3) p_{\iota(t_3-s)}(y, z_4) \\ &\quad \times \int dz_1 \, \phi(z_1 - z_3) \int dz_2 \, \phi(z_2 - z_4) \int \mu_0(dx_1) \, p_{\iota t_1}(x_1, z_1) \int \mu_0(dx_2) \, p_{\iota t_2}(x_2, z_2) \\ &\leq C(T) \, \|\phi\|_1^2 . \end{aligned}$$

It thus follows that

$$\int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_2}^{t_3} ds \, \langle Q^3_{t_1} \pi_1 Q^2_{t_2-t_1} \pi_1 Q_{s-t_2} \Phi_{12} Q^2_{t_3-s} \varphi, \mu_0^3 \rangle \le C(T) \, \|\phi\|_1^2 \, .$$

Again from II.30, we have that

$$\begin{aligned}
&\sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_{t_2}^{t_3} ds_2 \int_0^{t_1} ds_1 \, \langle Q^2_{s_1} \Phi_{ij} Q^3_{t_1-s_1} \pi_1 Q^2_{t_2-t_1} \pi_1 Q_{s_2-t_2} \Phi_{12} Q^2_{t_3-s_2} \varphi, \mu_0^2 \rangle \\
&\le C \sum_{k=1}^{2} \int_{t_2}^{t_3} ds_2 \int_0^{t_1} ds_1 \int \mu_0(dx_1)\mu_0(dx_2) \int dy \, p_{\iota s_1}(x_1, y) \int dz \, p_{\iota(s_2-s_1)}(y, z) \\
&\quad \times \int dw_1 dw_2 dw_3 dw_4 \, p_{\iota t_k}(x_2, w_k) p_{\iota(t_{3-k}-s_1)}(y, w_{3-k}) p_{\iota(t_3-s_2)}(z, w_3) \\
&\quad \times p_{\iota(t_3-s_2)}(z, w_4) \phi(w_1 - w_3) \phi(w_2 - w_4) \\
&+ C \int_{t_2}^{t_3} ds_2 \int_0^{t_1} ds_1 \int \mu_0(dx_1)\mu_0(dx_2) \int dy \, p_{\iota s_1}(x_1, y) \int dz \, p_{\iota s_2}(x_2, z) \\
&\quad \times \int dw_1 dw_2 dw_3 dw_4 \, p_{\iota(t_1-s_1)}(y, w_1) p_{\iota(t_2-s_1)}(y, w_2) p_{\iota(t_3-s_2)}(z, w_3) \\
&\quad \times p_{\iota(t_3-s_2)}(z, w_4) \phi(w_1 - w_3) \phi(w_2 - w_4) \\
&\le C \sum_{k=1}^{2} \int_{t_2}^{t_3} ds_2 \int_0^{t_1} ds_1 \int \mu_0(dx_1) \int dy \, p_{\iota s_1}(x_1, y) \int dz \, p_{\iota(s_2-s_1)}(y, z) \\
&\quad \times \int dw_{3-k} dw_{5-k} \, p_{\iota(t_{3-k}-s_1)}(y, w_{3-k}) p_{\iota(t_3-s_2)}(z, w_{5-k}) \\
&\quad \times \int dw_{k+2} p_{\iota(t_3-s_2)}(z, w_{k+2}) \int dw_k \, \phi(w_1 - w_3) \phi(w_2 - w_4) \int \mu_0(dx_2) p_{\iota t_k}(x_2, w_k) \\
&+ C \int_{t_2}^{t_3} ds_2 \int_0^{t_1} ds_1 \, (t_3 - s_2)^{-d/2} \int \mu_0(dx_1)\mu_0(dx_2) \int dw_1 dw_3 \, \phi(w_1 - w_3) \\
&\quad \times \int dy \, p_{\iota s_1}(x_1, y) p_{\iota(t_1-s_1)}(y, w_1) \int dz \, p_{\iota s_2}(x_2, z) p_{\iota(t_3-s_2)}(z, w_3) \\
&\quad \times \int dw_4 \, p_{\iota(t_3-s_2)}(z, w_4) \int dw_2 \, \phi(w_2 - w_4)
\end{aligned}$$

$$\le C\,\|\phi\|_1 \sum_{k=1}^{2} \int_{t_2}^{t_3} ds_2 \int_0^{t_1} ds_1 \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y)$$

$$\times \int dw_{3-k} dw_{5-k}\, p_{\iota(t_{3-k}-s_1)}(y, w_{3-k}) p_{\iota(t_3-s_1)}(y, w_{5-k}) \phi(w_{3-k} - w_{5-k})$$

$$+ C\,\|\phi\|_1 \int_{t_2}^{t_3} ds_2 \int_0^{t_1} ds_1\, (t_3 - s_2)^{-d/2} \int \mu_0(dx_1) \int dw_1\, p_{\iota t_1}(x_1, w_1) \int dw_3\, \phi(w_1 - w_3)$$

$$\times \int \mu_0(dx_2)\, p_{\iota t_3}(x_2, w_3)$$

$$\le C\,\|\phi\|_1^2 \int_{t_2}^{t_3} ds_2 \int_0^{t_1} ds_1\, (t_3 - s_1)^{-d/2} + C\,\|\phi\|_1^2 \int_{t_2}^{t_3} ds_2 \int_0^{t_1} ds_1\, (t_3 - s_2)^{-d/2}$$

$$\le C\,\|\phi\|_1^2 \int_{t_2}^{t_3} ds_2 \int_0^{t_1} ds_1\, \left[(t_3 - s_1)^{-d/2} + (t_3 - s_2)^{-d/2} \right].$$

Since $d \le 3$, it follows that

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_2}^{t_3} ds_2 \int_0^{t_1} ds_1\, \langle Q_{s_1}^2 \Phi_{ij} Q_{t_1-s_1}^3 \pi_1 Q_{t_2-t_1}^2 \pi_1 Q_{s_2-t_2} \Phi_{12} Q_{t_3-s_2}^2 \varphi, \mu_0^2 \rangle$$

$$\le C(T)\, \|\phi\|_1^2.$$

With one final application of II.30, we have

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_{t_2}^{t_3} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1\, \langle Q_{s_1} \Phi_{12} Q_{s_2-s_1}^2 \Phi_{ij} Q_{t_1-s_2}^3 \pi_1 Q_{t_2-t_1}^2 \pi_1 Q_{s_3-t_2} \Phi_{12} Q_{t_3-s_3}^2 \varphi, \mu_0 \rangle$$

$$\le C \sum_{i=1}^{2} \int_{t_2}^{t_3} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y) \int dz\, p_{\iota(s_2-s_1)}(y,z)$$

$$\times \int dw\, p_{\iota(s_3-s_2)}(z,w) \int dv_1 dv_2 dv_3 dv_4\, p_{\iota(t_i-s_2)}(z, v_i) p_{\iota(t_{3-i}-s_1)}(y, v_{3-i})$$

$$\times p_{\iota(t_3-s_3)}(w, v_3) p_{\iota(t_3-s_3)}(w, v_4) \phi(v_1 - v_3) \phi(v_2 - v_4)$$

$$+ C \int_{t_2}^{t_3} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y) \int dz_1 dz_2\, p_{\iota(s_2-s_1)}(y, z_1) p_{\iota(s_3-s_1)}(y, z_2)$$

$$\times \int dv_1 dv_2 dv_3 dv_4\, p_{\iota(t_1-s_2)}(z_1, v_1) p_{\iota(t_2-s_2)}(z_2, v_2) p_{\iota(t_3-s_3)}(z_3, v_3)$$

$$\times p_{\iota(t_3-s_3)}(z_4, v_4) \phi(v_1 - v_3) \phi(v_2 - v_4)$$

$$\le C \int_{t_2}^{t_3} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 (t_2 - s_1)^{-d/2} \int \mu_0(dx) \int dz\, p_{\iota s_2}(x,z) \int dw\, p_{\iota(s_3-s_2)}(z,w)$$
$$\times \int dv_1 dv_3\, p_{\iota(t_1-s_2)}(z,v_1) p_{\iota(t_3-s_3)}(w,v_3) \phi(v_1 - v_3) \int dv_4\, p_{\iota(t_3-s_3)}(w,v_4) \int dv_2\, \phi(v_2 - v_4)$$
$$+ C \int_{t_2}^{t_3} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 (t_2 - s_2)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y) \int dw\, p_{\iota(s_3-s_1)}(y,w)$$
$$\times \int dv_1 dv_3\, p_{\iota(t_1-s_1)}(y,v_1) p_{\iota(t_3-s_3)}(w,v_3) \phi(v_1 - v_3) \int dv_4\, p_{\iota(t_3-s_3)}(w,v_4) \int dv_2\, \phi(v_2 - v_4)$$
$$+ C \int_{t_2}^{t_3} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 (t_2 - s_2)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y) \int dw\, p_{\iota(s_3-s_1)}(y,w)$$
$$\times \int dv_1 dv_3\, p_{\iota(t_1-s_1)}(y,v_1) p_{\iota(t_3-s_3)}(w,v_3) \phi(v_1 - v_3) \int dv_4\, p_{\iota(t_3-s_3)}(w,v_4) \int dv_2\, \phi(v_2 - v_4)$$
$$\le C \|\phi\|_1 \int_{t_2}^{t_3} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 (t_2 - s_1)^{-d/2} \int \mu_0(dx) \int dz\, p_{\iota s_2}(x,z)$$
$$\times \int dv_1 dv_3\, p_{\iota(t_1-s_2)}(z,v_1) p_{\iota(t_3-s_2)}(z,v_3) \phi(v_1 - v_3)$$
$$+ C \|\phi\|_1 \int_{t_2}^{t_3} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 (t_2 - s_2)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_2}(x,y)$$
$$\times \int dv_1 dv_3\, p_{\iota(t_1-s_2)}(y,v_1) p_{\iota(t_3-s_1)}(y,v_3) \phi(v_1 - v_3)$$
$$\le C \|\phi\|_1^2 \int_{t_2}^{t_3} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 \left[(t_2 - s_1)^{-d/2} (t_3 - s_2)^{-d/2} + (t_2 - s_2)^{-d/2} (t_3 - s_1)^{-d/2} \right].$$

Therefore, since $d \le 3$,

$$\sum_{\substack{i,j=1 \\ i \ne j}}^{3} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_2}^{t_3} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1$$
$$\times \left\langle Q_{s_1} \Phi_{12} Q^2_{s_2-s_1} \Phi_{ij} Q^3_{t_1-s_2} \pi_1 Q^2_{t_2-t_1} \pi_1 Q_{s_3-t_2} \Phi_{12} Q^2_{t_3-s_3} \varphi, \mu_0 \right\rangle$$
$$\le C(T) \|\phi\|_1^2.$$

As a total count of the original fourteen J_k, the desired bound has now been shown for ten. We continue now with

$$
\begin{aligned}
&Q^2_{t_1-s_1}\pi_1 Q_{s_2-t_1}\Phi_{12}Q^2_{s_3-s_2}\Phi_{ij}Q^3_{t_2-s_3}\pi_1 Q^2_{t_3-t_2}\varphi(x)\\
&\le C\int db_1 db_2 p_{\iota(s_2-s_1)}(x_2,b_1)p_{\iota(s_3-s_2)}(b_1,b_2)\int da_1 da_2 da_3 da_4\, p_{\iota(t_1-s_1)}(x_1,a_1)\\
&\quad\times p_{\iota(t_{(7-i-j)\wedge 3}-s_2)}(b_1,a_{7-i-j})p_{\iota(t_{i+1}-s_3)}(b_2,a_{i+1})p_{\iota(t_3-s_3)}(b_2,a_{j+1})\varphi(a_1,a_2,a_3,a_4),
\end{aligned}
\tag{II.31}
$$

for any $x\in\mathbb{R}^{2\times d}$, $0\le s_1\le t_1\le s_2\le s_3\le t_2$, and $i,j=1,2,3,\ i<j$.

We now apply the inequality II.31 to show

$$
\begin{aligned}
&\sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_{t_1}^{t_2} ds_2\int_{t_1}^{s_2} ds_1\ \langle Q^2_{t_1}\pi_1 Q_{s_1-t_1}\Phi_{12}Q^2_{s_2-s_1}\Phi_{ij}Q^3_{t_2-s_2}\pi_1 Q^2_{t_3-t_2}\varphi,\mu_0^2\rangle\\
&\le C\sum_{\substack{i,j=1\\ i<j}}^{3}\int_{t_1}^{t_2} ds_2\int_{t_1}^{s_2} ds_1\int\mu_0(dx_2)\int dy\, p_{\iota s_1}(x_2,y)\int dz\, p_{\iota(s_2-s_1)}(y,z)\\
&\quad\times\int dw_2dw_4\,\phi(w_2-w_4)\int dw_3\, p_{\iota(t_{i+1}-s_2)}(z,w_{i+1})p_{\iota(t_3-s_2)}(z,w_{j+1})\\
&\quad\times p_{\iota(t_{(7-i-j)\wedge 3}-s_1)}(y,w_{7-i-j})\int dw_1\,\phi(w_1-w_3)\int\mu_0(dx_1)\,p_{\iota t_1}(x_1,w_1)\\
&\le C\,\|m\|_\infty\,\|\phi\|_1\sum_{\substack{i,j=1\\ i<j}}^{3}\int_{t_1}^{t_2} ds_2\int_{t_1}^{s_2} ds_1\int\mu_0(dx_2)\int dy\, p_{\iota s_1}(x_2,y)\int dz\, p_{\iota(s_2-s_1)}(y,z)\\
&\quad\times\int dw_2dw_4\,\phi(w_2-w_4)\int dw_3\, p_{\iota(t_{i+1}-s_2)}(z,w_{i+1})p_{\iota(t_3-s_2)}(z,w_{j+1})\\
&\quad\times p_{\iota(t_{(7-i-j)\wedge 3}-s_1)}(y,w_{7-i-j})\\
&\le C\,\|\phi\|_1^2\sum_{k=1}^{2}\int_{t_1}^{t_2} ds_2\int_{t_1}^{s_2} ds_1\int\mu_0(dx_2)\int dy\, p_{\iota s_k}(x_2,y)\\
&\quad\times\int dw_2dw_4\, p_{\iota(t_2-s_k)}(y,w_2)p_{\iota(t_3-s_k)}(y,w_4)\phi(w_2-w_4)\\
&\le C\,\|\phi\|_1^2\int_{t_1}^{t_2} ds_2\int_{t_1}^{s_2} ds_1\ \left[(t_3-s_1)^{-d/2}+(t_3-s_2)^{-d/2}\right].
\end{aligned}
$$

Therefore, since $d \le 3$, we have

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_1}^{t_2} ds_2 \int_{t_1}^{s_2} ds_1 \, \langle Q^2_{t_1}\pi_1 Q_{s_1-t_1}\Phi_{12} Q^2_{s_2-s_1}\Phi_{ij} Q^3_{t_2-s_2}\pi_1 Q^2_{t_3-t_2}\varphi, \mu_0^2\rangle$$
$$\le C(T)\, \|\phi\|_1^2 .$$

With a second, and final application of II.31, it follows that

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \int_0^{t_1} ds_1 \, \langle Q_{s_1}\Phi_{12} Q^2_{t_1-s_1}\pi_1 Q_{s_2-t_1}\Phi_{12} Q^2_{s_3-s_2}\Phi_{ij} Q^3_{t_2-s_3}\pi_1 Q^2_{t_3-t_2}\varphi, \mu_0\rangle$$
$$\le C \sum_{\substack{i,j=1\\ i< j}}^{3} \int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \int_0^{t_1} ds_1 \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y) \int dz\, p_{\iota(s_2-s_1)}(y,z)$$
$$\times \int dw\, p_{\iota(s_3-s_2)}(z,w) \int dv_1 dv_2 dv_3 dv_4\, p_{\iota(t_1-s_1)}(y,v_1) p_{\iota(t_{(7-i-j)\wedge 3}-s_2)}(z, v_{7-i-j})$$
$$\times p_{\iota(t_{i+1}-s_3)}(w, v_{i+1}) p_{\iota(t_3-s_3)}(w, v_{j+1}) \phi(v_1-v_3)\phi(v_2-v_4)$$
$$\le C \int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \int_0^{t_1} ds_1\, (t_3-s_2)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y)$$
$$\times \int dv_3 \int dz\, p_{\iota(s_2-s_1)}(y,z) \int dw\, p_{\iota(s_3-s_2)}(z,w) p_{\iota(t_3-s_3)}(w,v_3)$$
$$\times \int dv_1\, p_{\iota(t_1-s_1)}(y,v_1)\phi(v_1-v_3) \int dv_2\, p_{\iota(s_3-t_1)}(w,v_2) \int dv_4\, \phi(v_2-v_4)$$
$$+ C \int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \int_0^{t_1} ds_1\, (t_3-s_3)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y)$$
$$\times \int dv_1\, p_{\iota(t_1-s_1)}(y,v_1) \int dv_3\, \phi(v_1-v_3) \int dz\, p_{\iota(s_2-s_1)}(y,z) p_{\iota(t_3-s_2)}(z,v_3)$$
$$\times \int dw\, p_{\iota(s_3-s_2)}(z,w) \int dv_2\, p_{\iota(t_2-s_3)}(w,v_2) \int dv_4\, \phi(v_2-v_4)$$
$$+ C \int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \int_0^{t_1} ds_1\, (t_3-s_3)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y)$$
$$\times \int dv_3 \int dz\, p_{\iota(s_2-s_1)}(y,z) \int dw\, p_{\iota(s_3-s_2)}(z,w) p_{\iota(t_3-s_3)}(w,v_3)$$
$$\times \int dv_1\, p_{\iota(t_1-s_1)}(y,v_1)\phi(v_1-v_3) \int dv_2\, p_{\iota(t_3-s_2)}(z,v_2) \int dv_4\, \phi(v_2-v_4)$$

$$\leq C\,\|\phi\|_1 \int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \int_0^{t_1} ds_1\,(t_3-s_2)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y)$$
$$\times \int dv_1\, p_{\iota(t_1-s_1)}(y,v_1) \int dv_3\, p_{\iota(t_3-s_1)}(y,v_3)\phi(v_1-v_3)$$
$$+ C\,\|\phi\|_1 \int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \int_0^{t_1} ds_1\,(t_3-s_3)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y)$$
$$\times \int dv_1\, p_{\iota(t_1-s_1)}(y,v_1) \int dv_3\, p_{\iota(t_3-s_1)}(y,v_3)\phi(v_1-v_3)$$
$$\leq C\,\|\phi\|_1^2 \int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \int_0^{t_1} ds_1\,(t_3-s_1)^{-d/2}\left((t_3-s_2)^{-d/2}+(t_3-s_3)^{-d/2}\right).$$

And so, since $d \leq 3$,

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \int_0^{t_1} ds_3$$
$$\times \left\langle Q_{s_1}\Phi_{12}Q^2_{t_1-s_1}\pi_1 Q_{s_2-t_1}\Phi_{12}Q^2_{s_3-s_2}\Phi_{ij}Q^3_{t_2-s_3}\pi_1 Q^2_{t_3-t_2}\varphi, \mu_0\right\rangle$$
$$\leq C(T)\,\|\phi\|_1^2.$$

It thus remains to show the desired bound on two of the fourteen original J_k. As in the previous steps, the bounds will result from the following simpler bound.

$$Q^2_{t_1-s_1}\pi_1 Q_{s_2-t_1}\Phi_{12}Q^2_{t_2-s_2}\pi_1 Q_{s_3-t_2}\Phi_{12}Q^2_{t_3-s_3}\varphi(x)$$
$$\leq C \int db_1\, p_{\iota(s_2-s_1)}(x_2,b_1) \int db_2\, p_{\iota(s_3-s_2)}(b_1,b_2) \int da_1 da_2 da_3 da_4$$
$$\times\, p_{\iota(t_1-s_1)}(x_1,a_1)p_{\iota(t_2-s_2)}(b_1,a_2)p_{\iota(t_3-s_3)}(b_2,a_3)p_{\iota(t_3-s_3)}(b_2,a_4)\varphi(a_1,a_2,a_3,a_4), \quad \text{(II.32)}$$

for any $x \in \mathbb{R}^{2\times d}$, $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2 \leq s_3 \leq t_3$.

Using the inequality II.32, it follows that

$$\int_{t_2}^{t_3} ds_2 \int_{t_1}^{t_2} ds_1\ \left\langle Q^2_{t_1}\pi_1 Q_{s_1-t_1}\Phi_{12}Q^2_{t_2-s_1}\pi_1 Q_{s_2-t_2}\Phi_{12}Q^2_{t_3-s_2}\varphi, \mu_0^2\right\rangle$$

$$
\begin{aligned}
&\leq C \int_{t_2}^{t_3} ds_2 \int_{t_1}^{t_2} ds_1 \int \mu_0(dx_1)\mu_0(dx_2) \int dy\, p_{\iota s_1}(x_2, y) \\
&\quad \times \int dz\, p_{\iota(s_2-s_1)}(y,z) \int dw_1 dw_2 dw_3 dw_4\, p_{\iota t_1}(x_1, w_1) p_{\iota(t_2-s_1)}(y, w_2) \\
&\quad \times p_{\iota(t_3-s_2)}(z, w_3) p_{\iota(t_3-s_2)}(z, w_4) \phi(w_1 - w_3)\phi(w_2 - w_4) \\
&\leq C \int_{t_2}^{t_3} ds_2 \int_{t_1}^{t_2} ds_1 \int \mu_0(dx_2) \int dy\, p_{\iota s_1}(x_2, y) \int dw_2\, p_{\iota(t_2-s_1)}(y, w_2) \\
&\quad \times \int dw_4\, \phi(w_2 - w_4) \int dz\, p_{\iota(s_2-s_1)}(y,z) p_{\iota(t_3-s_2)}(z, w_4) \\
&\quad \times \int dw_3\, p_{\iota(t_3-s_2)}(z, w_3) \int dw_1\, \phi(w_1 - w_3) \int \mu_0(dx_1)\, p_{\iota t_1}(x_1, w_1) \\
&\leq C \|\phi\|_1 \int_{t_2}^{t_3} ds_2 \int_{t_1}^{t_2} ds_1 \int \mu_0(dx_2) \int dy\, p_{\iota s_1}(x_2, y) \int dw_2\, p_{\iota(t_2-s_1)}(y, w_2) \\
&\quad \times \int dw_4\, p_{\iota(t_3-s_1)}(y, w_4)\phi(w_2 - w_4) \\
&\leq C \|\phi\|_1^2 \int_{t_2}^{t_3} ds_2 \int_{t_1}^{t_2} ds_1\, (t_3 - s_1)^{-d/2}.
\end{aligned}
$$

And so, since $d \leq 3$,

$$
\begin{aligned}
&\int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_2}^{t_3} ds_1 \int_{t_1}^{t_2} ds_2\, \langle Q^2_{t_1} \pi_1 Q_{s_1-t_1} \Phi_{12} Q^2_{t_2-s_1} \pi_1 Q_{s_2-t_2} \Phi_{12} Q^2_{t_3-s_2} \varphi, \mu_0^2 \rangle \\
&\leq C(T)\, \|\phi\|_1^2 .
\end{aligned}
$$

Finally, once again by II.32,

$$
\begin{aligned}
&\int_{t_2}^{t_3} ds_3 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1\, \langle Q_{s_1} \Phi_{12} Q^2_{t_1-s_1} \pi_1 Q_{s_2-t_1} \Phi_{12} Q^2_{t_2-s_2} \pi_1 Q_{s_3-t_2} \Phi_{12} Q^2_{t_3-s_3} \varphi, \mu_0 \rangle \\
&\leq C \int_{t_2}^{t_3} ds_3 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1 \int \mu_0(dx) \int dy\, p_{\iota s_1}(x, y) \int dz\, p_{\iota(s_2-s_1)}(y, z) \\
&\quad \times \int dw\, p_{\iota(s_3-s_2)}(z, w) \int dv_1 dv_3\, p_{\iota(t_1-s_1)}(y, v_1) p_{\iota(t_3-s_3)}(w, v_3) \phi(v_1 - v_3) \\
&\quad \times \int dv_2 dv_4\, p_{\iota(t_2-s_2)}(z, v_2) p_{\iota(t_3-s_3)}(w, v_4) \phi(v_2 - v_4)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C \int_{t_2}^{t_3} ds_3 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1\, (s_3 - s_2)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_1}(x, y) \\
&\quad \times \int dv_1\, p_{\iota(t_1 - s_1)}(y, v_1) \int dv_3\, \phi(v_1 - v_3) \int dv_4 \int dw\, p_{\iota(t_3 - s_3)}(w, v_3) p_{\iota(t_3 - s_3)}(w, v_4) \\
&\quad \times \int dv_2\, \phi(v_2 - v_4) \int dz\, p_{\iota(s_2 - s_1)}(y, z) p_{\iota(t_2 - s_2)}(z, v_2) \\
&\leq C\, \|\phi\|_1^2 \int_{t_2}^{t_3} ds_3 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1\, (t_2 - s_1)^{-d/2} (s_3 - s_2)^{-d/2}.
\end{aligned}
$$

It thus follows,

$$
\begin{aligned}
&\int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_2}^{t_3} ds_3 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1 \\
&\qquad\qquad \times \left\langle Q_{s_1} \Phi_{12} Q^2_{t_1 - s_1} \pi_1 Q_{s_2 - t_1} \Phi_{12} Q^2_{t_2 - s_2} \pi_1 Q_{s_3 - t_2} \Phi_{12} Q^2_{t_3 - s_3} \varphi, \mu_0 \right\rangle \\
&\leq C(T)\, \|\phi\|_1^2 .
\end{aligned}
$$

Therefore, from the bounds established above for each J_k, $k = 1, ..., 14$, it follows that

$$
\int_0^{t_3} dt_2 \int_0^{t_2} dt_1\, \mathbb{E} \left\langle \varphi, \mu_{t_1} \mu_{t_2} \mu^2_{t_3} \right\rangle \leq C(T)\, \|\phi\|^2_{L^1} .
$$

□

Using the above moment formulas, and knowledge of the transition function q_t^k, existence of GSILT can now be shown in a constructive manner.

CHAPTER III

EXISTENCE OF GENERALIZED SELF-INTERSECTION LOCAL TIME

The proof of existence is now completed as follows:

III.1 Preliminary Results

Generalized self-intersection local time (GSILT) at $u \in \mathbb{R}^d$, over $B \subset \mathcal{B}(\mathbb{R}^2)$, is defined formally as

$$\mathcal{L}(u; B) \triangleq \int_B dtds \, \langle \delta_u, \mu_s \mu_t \rangle \,,$$

where

$$\delta_u(x) \triangleq \begin{cases} \infty, & x = u \\ 0, & x \neq u \end{cases}$$

is the Dirac point-mass measure at u.

Note that in the above, and throughout the remainder of this paper, if $\varphi : \mathbb{R}^d \to \mathbb{R}$, the convention

$$\langle \varphi, \mu_s \mu_t \rangle = \int \mu_s(dx) \mu_t(dy) \varphi(x - y)$$

is made.

Since $\mu_s \mu_t = \mu_t \mu_s$, it makes sense to restrict GSILT either above or below the diagonal, and so we set

$$\mathcal{L}(u, T) = \mathcal{L}(u; \{(s,t) : 0 \leq s \leq t \leq T\}),$$

for fixed $T \in [0, \infty)$.

The above definition is clearly formal, and thus to make sense of this a limiting process will be constructed. For fixed $\lambda > 0$, define

$$G^{\lambda,u}(x) = \int_0^\infty dt\, e^{-\lambda t} q_t(u,x),$$

then in the sense of distributions,

$$\begin{aligned} LG^{\lambda,u}(x) &= \int_0^\infty dt\, e^{-\lambda t} Lq_t(u,x) \\ &= \int_0^\infty dt\, e^{-\lambda t} \partial_t q_t(u,x) \\ &= e^{-\lambda t} q_t(u,x)\Big|_{t=0}^{\infty} + \lambda \int_0^t dt\, e^{-\lambda t} q_t(u,x) \\ &= -\delta_u(x) + \lambda G^{\lambda,u}. \end{aligned}$$

Or equivalently, in the sense of distributions,

$$(\lambda - L)G^{\lambda,u} = \delta_u.$$

The above implies that $G^{\lambda,u}$ is the resolvent to L at λ, and thus

$$\left\| G^{\lambda,u} \right\|_{L^1} \leq \lambda^{-1}.$$

Since $G^{\lambda,u} \in L^1$, for any $\phi \in C_K^\infty(\mathbb{R}^d)$

$$\left\langle \phi, G^{\lambda,u} \right\rangle \triangleq \int dx\, G^{\lambda,u}(x)\phi(x) < \infty,$$

which implies $G^{\lambda,u}$ can be regarded as the element of S'_d which sends $\phi \in S_d$ to $\langle \phi, G^{\lambda,u} \rangle$. Thus, Theorem 7.10 of [18] implies the existence of a family $\{G_\varepsilon^{\lambda,u} : \varepsilon > 0\} \subset C_K^\infty$ such that $G_\varepsilon^{\lambda,u} \to G^{\lambda,u}$ as $\varepsilon \to 0$, in S'_d.

From [13], L is a continuous operator on S_d', and it is concluded that

$$\lim_{\varepsilon \to 0} (\lambda - L) G_\varepsilon^{\lambda,u} = \delta_u,$$

where convergence is in the sense of distributions, and so a limiting process is defined by

$$\gamma_\varepsilon^\lambda(u,T) \triangleq \int_0^T dt \int_0^t ds \left\langle (\lambda - L) G_\varepsilon^{\lambda,u}, \mu_s \mu_t \right\rangle,$$

$\lambda > 0$, $\varepsilon > 0$, $0 \leq T < \infty$.

The goal now is to make sense of the operator L appearing in the integrand.

III.2 An Itô Formula

As in the independent case, the derivation of the evolution equation is accomplished through the construction, and careful application, of an appropriate Itô formula. This construction will mimic that of Adler & Lewin (1991), which begins with application of Itô's Lemma to the non-anticipative functional f, given by

$$f(t,x) = x \int_0^t ds \; \mu_s(\psi),$$

where $\psi \in C_K^2(\mathbb{R}^d)$, and x is a $\mathbb{R}$-valued random variable. Note that from the SPDE I.6, if $\phi \in C_K^\infty(\mathbb{R}^d)$, $\mu_t(\phi)$ is a continuous semi-martingale with decomposition

$$\mu_t(\phi) = \mu_0(\phi) + Z_t(\phi) + V_t(\phi),$$

where

$$V_t(\phi) \triangleq \int_0^t ds \, \mu_s(L\phi).$$

Theorem III.2.1. *If $\phi \in S_d$ then $\mu_t(\phi)$ is an a.s. continuous semimartingale.*

Proof: From Doob's maximal inequality for martingales and Theorem I.1.6 we have that for $\phi \in C_K^\infty(\mathbb{R}^d)$, $0 \le T < \infty$,

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le T} \mu_t(\phi)\right)^2 &\le 2\mu_0(\phi)^2 + 2\mathbb{E}\left(\sup_{0\le t\le T} Z_t(\phi)\right)^2 + 2\mathbb{E}\left(\int_0^T ds\, \mu_s(L\phi)\right)^2 \\ &\le 2\mu_0(\phi)^2 + 8\mathbb{E}Z_T(\phi)^2 + 2\mathbb{E}\left(\int_0^T ds\, \mu_s(L\phi)\right)^2\end{aligned}$$

For the second term, from equation II.1.2

$$\begin{aligned}\mathbb{E}Z_T(\phi)^2 &= \mathbb{E}\left\langle Z(\phi)\right\rangle_T \\ &= \int_0^T ds\, \mathbb{E}\mu_s(\phi^2) + \int_0^T ds\, \mathbb{E}\mu_s^2(\Lambda\phi) \\ &= \int_0^T ds\, \mu_0(Q_s\phi^2) + \int_0^T ds\, \mu_0^2(Q_s^2\Lambda\phi) + \int_0^T ds_1 \int_0^{s_1} ds_2\, \mu_0(Q_{s_2}\Phi_{12}Q_{s_1-s_2}^2\Lambda\phi) \\ &\le \|m\|_\infty \int_0^T ds\, \left\|Q_s\phi^2\right\|_{L^1} + \|m\|_\infty^2 \int_0^T ds\, \left\|Q_s^2\Lambda\phi\right\|_{L^1} \\ &\quad + \|m\|_\infty \int_0^T ds_1 \int_0^{s_1} ds_2\, \left\|\Phi_{12}Q_{s_1-s_2}^2\Lambda\phi\right\|_{L^1} \\ &\le C(T)\,\|\phi\|_{L^2}^2 + C(T)\,\|\Lambda\phi\|_{L^1} + C(T)\sum_{i,j=1}^d \left\|(S_{\iota(s_1-s_2)}\partial_i\phi)(S_{\iota(s_1-s_2)}\partial_j\phi)\right\|_{L^1}\end{aligned}$$

where in the above $\{S_t : t \ge 0\}$ is the Brownian transition semigroup. Thus, from Hölders inequality

$$\mathbb{E}Z_T(\phi)^2 \le C(T)\,\|\phi\|_{L^2}^2 + C(T)\sum_{i,j=1}^d \|\partial_i\phi\|_{L^1}\|\partial_j\phi\|_{L^1} + C(T)\sum_{i,j=1}^d \|\partial_i\phi\|_{L^2}\|\partial_j\phi\|_{L^2}\,.$$

With regards to the third term above,

$$\begin{aligned}
\mathbb{E}\left(\int_0^T ds\,\mu_s(L\phi)\right)^2 &= \int_0^T ds_1 \int_0^T ds_2\, \mathbb{E}\mu_{s_1}(L\phi)\mu_{s_2}(L\phi) \\
&\leq \int_0^T ds_1 \int_0^T ds_2 \left(\mathbb{E}\mu_{s_1}(L\phi)^2 \mathbb{E}\mu_{s_2}(L\phi)^2\right)^{1/2} \\
&\leq T^2 \sup_{0\leq s\leq T} \mathbb{E}\mu_s(L\phi)^2 \\
&\leq T^2 \sup_{0\leq s\leq T} \left(\mu_0^2(Q_s^2(L\phi\otimes L\phi)) + \int_0^s dr\,\mu_0(Q_r\Phi_{12}Q_{s-r}^2(L\phi\otimes L\phi))\right) \\
&\leq T^2 \left(C\left\|L\phi\otimes L\phi\right\|_{L^1} + \sup_{0\leq s\leq T}\int_0^s dr \int dy\,(S_{\iota(s-r)}L\phi)(y)^2\right) \\
&\leq C(T)\left\|L\phi\right\|_{L^1}^2 + C(T)\left\|L\phi\right\|_{L^2}^2 \\
&\leq C(T)\sum_{i,j,p,q=1}^d \left(\left\|\partial_i\partial_j\phi\right\|_{L^1}\left\|\partial_p\partial_q\phi\right\|_{L^1} + \left\|\partial_i\partial_j\phi\right\|_{L^2}\left\|\partial_p\partial_q\phi\right\|_{L^2}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\leq t\leq T}\mu_t(\phi)\right)^2 \leq C(T)\left\|\phi\right\|_{L^2}^2 &+ \sum_{i,j=1}^d \left(\left\|\partial_i\phi\right\|_{L^1}\left\|\partial_j\phi\right\|_{L^1} + \left\|\partial_i\phi\right\|_{L^2}\left\|\partial_j\phi\right\|_{L^2}\right) \\
&+ \sum_{i,j,p,q=1}^d \left(\left\|\partial_i\partial_j\phi\right\|_{L^1}\left\|\partial_p\partial_q\phi\right\|_{L^1} + \left\|\partial_i\partial_j\phi\right\|_{L^2}\left\|\partial_p\partial_q\phi\right\|_{L^2}\right).
\end{aligned} \tag{III.1}$$

If $\phi \in S_d$, from Theorem 7.10 of [25], there exists a Cauchy sequence $\{\phi_n\} \subset C_K^\infty(\mathbb{R}^d)$ converging to ϕ in S_d. Thus, from III.1, Chebychev's inequality, and a subsequence argument from the Borel-Cantelli Lemma (cf. Theorem 4.2.3 of [4]), there is a subsequence $\{\phi_{n_k}\}$ such that $\mu_t(\phi_{n_k})$ converges uniformly in $t \in [0,T]$ to $\mu_t(\phi)$ with probability one. Therefore, $\mu_t(\phi)$ is an a.s. continuous semimartingale for S_d. $\square$

Fix $T \geq 0$, and set $\phi \in C_K^\infty(\mathbb{R}^d)$, then from Itô's Lemma (cf. [15]),

$$f(T, \mu_T(\phi)) = f(0, \mu_0(\phi)) + \int_0^T dt\, \partial_t f(t, \mu_t(\phi)) + \int_0^T dV_t(\phi)\, \partial_x f(t, \mu_t(\phi)) \\ + \int_0^T dZ_t(\phi)\, \partial_x f(t, \mu_t(\phi)) + \frac{1}{2} \int_0^T d\left<Z(\phi)\right>_t \partial_{xx}^2 f(t, \mu_t(\phi)).$$

Equivalently,

$$\int_0^T dt\ \left<\psi \otimes \phi, \mu_t \mu_T\right> = \int_0^T dt\, \left<\psi \otimes \phi, \mu_t \mu_t\right> + \int_0^T dt \int_0^t ds\ \left<\psi \otimes L\phi, \mu_s \mu_t\right> \\ + \int_0^T dZ_t(\phi) \int_0^t ds\, \left<\psi, \mu_s\right>.$$

Lemma III.2.2. *For any $\phi, \psi \in C_K^\infty(\mathbb{R}^d)$,*

$$\int_0^T dZ_t(\phi) \int_0^t ds \mu_s(\psi) = \int_0^T \int_{\mathbb{R}^d} Z(dt, dx) \int_0^t ds\, \phi(x) \mu_s(\psi).$$

Proof: Let $0 \leq t \leq T$. It follows from I.6 and Corollary II.3.3 that

$$\begin{aligned} &\mathbb{E}\left(\int_0^t dZ_s(\phi) \int_0^s dv \mu_v(\psi)\right)^2 \\ &= \mathbb{E} \int_0^t d\left<Z(\phi).\right>_s \left(\int_0^s dv\, \mu_v(\psi)\right)^2 \\ &= \mathbb{E} \int_0^t ds\, (\mu_s(\phi^2) + \mu_s(\Lambda\phi)^2) \left(\int_0^s dv\, \mu_v(\psi)\right)^2 \\ &= \int_0^t ds \int_0^s dv_1 \int_0^s dv_2\, \mathbb{E}\mu_s(\phi^2) \mu_{v_1}(\psi) \mu_{v_2}(\psi) + \int_0^t ds \int_0^s dv_1 \int_0^s dv_2\, \mathbb{E}\mu_s(\Lambda\phi)^2 \mu_{v_1}(\psi) \mu_{v_2}(\psi) \\ &\leq C(T) \left(\|\phi\|_\infty^2 \|\psi\|_\infty^2 + \|\Lambda\phi\|_\infty^2 \|\psi\|_\infty^2\right). \end{aligned}$$

By assumption on Λ and since $\phi, \psi \in C_K^\infty(\mathbb{R}^d)$,

$$\|\Lambda\phi\|_\infty < \infty, \quad \|\Lambda\psi\|_\infty < \infty,$$

which, since $\|\psi\|_\infty, \|\phi\|_\infty < \infty$, implies by the definition of the stochastic integral (cf. [16], chapter 3) that

$$\int_0^t dZ_s(\phi) \int_0^s dv\, \mu_v(\psi) \in L^2(\mathbb{P}).$$

In addition, it is clear from Lemma II.3.3 that

$$\int_0^s dv\, \mu_v(\psi) \in L^2(\mathbb{P}),$$

and thus, again from the definition of the stochastic integral, $\int_0^s dv\, \mu_v(\psi)$ can be approximated in $L^2(\mathbb{P})$ by simple functions of the form

$$\sum_i^n \sum_{A_i} c_{A_i} 1_{A_i^{(n)}}(\omega) 1_{(t_i^{(n)}, t_{i+1}^{(n)}]}(s),$$

where $\bigcup_i A_i^{(n)} = \Omega$, $A_i^{(n)} \cap A_j^{(n)} = \varnothing$ if $i \neq j$, $\bigcup_i (t_i^{(n)}, t_{i+1}^{(n)}] = [0, \infty)$, and $(t_i^{(n)}, t_{i+1}^{(n)}] \cap (t_k^{(n)}, t_{k+1}^{(n)}] = \varnothing$ if $i \neq k$.

It follows that an L^2 approximation to $\int_0^t dZ_s(\phi) \int_0^s dv\, \mu_v(\psi)$ is given by

$$\int_0^t dZ_s(\phi) \sum_i^n \sum_{A_i} c_{A_i} 1_{A_i^{(n)}}(\omega) 1_{(t_i^{(n)}, t_{i+1}^{(n)}]}(s)$$
$$= \sum_i^n \sum_{A_i} c_{A_i} 1_{A_i^{(n)}}(\omega) 1_{(0,t]}(t_i) \left(Z_{t_{i+1}}(\phi) - Z_{t_i}(\phi) \right).$$

Clearly $f(s, \phi(x)) = \phi(x) \int_0^s dv\, \mu_v(\psi)$ is also in $L^2(\mathbb{P})$, and thus we have the simple functions of the form

$$\sum_i^n \sum_{A_i} c_{A_i} 1_{A_i^{(n)}}(\omega) 1_{(t_i^{(n)}, t_{i+1}^{(n)}]}(s) \phi(x),$$

converging to $f(s, \phi(x))$ in $L^2(\mathbb{P})$.

From Walsh's construction of the stochastic integral with respect to a martingale measure ([27]), an L^2 approximation to

$$\int_0^t \int Z(ds, dx)\phi(x) \int_0^s dv\, \mu_v(\psi)$$

is then given by

$$\sum_i^n \sum_{A_i} \int c_{A_i} 1_{A_i^{(n)}}(\omega) 1_{(0,t]}(t_i)\phi(x)(Z_{t_{i+1}} - Z_{t_i})(dx)$$
$$= \sum_i^n \sum_{A_i} c_{A_i} 1_{A_j^{(n)}}(\omega) 1_{(0,t]}(t_i) \left(Z_{t_{i+1}}(\phi) - Z_{t_i}(\phi)\right).$$

Since any two L^2 limits of a sequence must agree, it follows that

$$\int_0^T dZ_t(\phi) \int_0^t ds \mu_s(\psi) = \int_0^T \int_{\mathbb{R}^d} Z(dt, dx) \int_0^t ds\, \phi(x)\mu_s(\psi).$$

□

Immediately, we arrive at the Corollary:

Corollary III.2.3.

$$\int_0^T dt \int_0^t ds\, \mu_s(\psi)\mu_t(L\phi) = \int_0^T dt\, \mu_t(\psi)\mu_T(\phi) - \int_0^T dt\, \mu_t(\psi)\mu_t(\phi)$$
$$- \int_0^T \int_{\mathbb{R}^d} Z(dt, dx) \int_0^t ds\, \phi(x)\mu_s(\psi), \tag{III.2}$$

for any $\psi, \phi \in C_K^\infty(\mathbb{R}^d)$.

This will now lead to the following Lemma.

Lemma III.2.4. *Given* $\Psi \in S_{2d}$,

$$\int_0^T dt \int_0^t ds\, \langle L_2\Psi, \mu_s\mu_t\rangle = \int_0^T dt\, \langle \Psi, \mu_t\mu_T\rangle - \int_0^T dt\, \langle \Psi, \mu_t\mu_t\rangle$$
$$- \int_0^T \int_{\mathbb{R}^d} Z(dt, dy) \int_0^t ds\, \langle \Psi(\cdot, y), \mu_s\rangle, \tag{III.3}$$

where

$$(L_2\Psi)(x,y) \triangleq \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(y)\partial_{2_i}\partial_{2_j}\Psi(x,y).$$

Proof: Assume that $\Psi \in S_{2d}$, then from Lemma II.1.1 we can choose $\{\Psi_n; n \in \mathbb{N}\}$ such that

$$\Psi_n(x,y) = \sum_{k=1}^{n}(\psi_k \otimes \phi_k)(x,y),$$

for some $\{\psi_k : k \in \mathbb{N}\}, \{\phi_k : k \in \mathbb{N}\} \subset C_K^\infty(\mathbb{R}^d)$, and Ψ_n converges to Ψ in S_{2d} as $n \to \infty$.

It is clear from III.2 that

$$\begin{aligned}\int_0^T dt \int_0^t ds\ \langle L_2\Psi_n, \mu_s\mu_t\rangle = \int_0^T dt\ \langle \Psi_n, \mu_t\mu_T\rangle - \int_0^T dt\ \langle \Psi_n, \mu_t\mu_t\rangle \\ - \int_0^T \int_{\mathbb{R}^d} Z(dt,dy)\int_0^t ds\ \langle \Psi_n(\cdot,y), \mu_s\rangle.\end{aligned} \tag{III.4}$$

From Corollary II.3.3,

$$\begin{aligned}\mathbb{E}\left\{\int_0^T dt\ \langle \Psi_n - \Psi_m, \mu_t\mu_T\rangle\right\}^2 &= \int_0^T dt \int_0^T ds\, \mathbb{E}\,\langle (\Psi_n - \Psi_m)\otimes(\Psi_n - \Psi_m), \mu_t\mu_T\mu_s\mu_T\rangle \\ &\leq C(T)\,\|\Psi_n - \Psi_m\|_\infty^2,\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left\{\int_0^T dt\ \langle \Psi_n - \Psi_m, \mu_t\mu_t\rangle\right\}^2 &= \int_0^T dt \int_0^T ds\, \mathbb{E}\,\langle (\Psi_n - \Psi_m)\otimes(\Psi_n - \Psi_m), \mu_t\mu_t\mu_s\mu_s\rangle \\ &\leq C(T)\,\|\Psi_n - \Psi_m\|_\infty^2,\end{aligned}$$

Since Ψ_n converges in S_d to Ψ,

$$\lim_{n\to\infty} \|\Psi_n - \Psi\|_\infty$$

and both of the above two terms are L^2 convergent. It remains to show that the convergence it to the desired limit.

For any $t, s \geq 0$, since $\mu_{\cdot} \in C_{M_F(\mathbb{R}^d)}[0,\infty)$, and $\Psi_n \to \Psi$ uniformly,

$$\langle \Psi_n, \mu_s \mu_t \rangle \to \langle \Psi, \mu_s \mu_t \rangle \qquad \text{a.s.}$$

Since the L^2 limit must agree with the a.s. limit,

$$L^2 - \lim_{n\to\infty} \langle \Psi_n, \mu_s \mu_t \rangle = \langle \Psi, \mu_s \mu_t \rangle .$$

Thus,

$$L^2 - \lim_{n\to\infty} \int_0^T dt \ \langle \Psi_n, \mu_t \mu_T \rangle = \int_0^T dt \ \langle \Psi, \mu_t \mu_T \rangle ,$$

and

$$L^2 - \lim_{n\to\infty} \int_0^T dt \ \langle \Psi_n, \mu_t \mu_t \rangle = \int_0^T dt \ \langle \Psi, \mu_t \mu_t \rangle .$$

Consider next the stochastic integral term and term involving the generator L. From Lemma II.3.3 it follows that

$$\begin{aligned}
&\mathbb{E}\left\{ \int_0^T dt \int_0^t ds \ \langle L_2 \Psi_n - L_2 \Psi_m, \mu_s \mu_t \rangle \right\}^2 \\
&\leq C(T) \left\| L_2(\Psi_n - \Psi_m) \right\|_\infty^2 \\
&\leq C(T) \sum_{i,j,p,q=1}^{d} \left\| \partial_{2_i} \partial_{2_j} (\Psi_n - \Psi_m) \right\|_\infty \left\| \partial_{2_p} \partial_{2_q} (\Psi_n - \Psi_m) \right\|_\infty
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left\{ \int_0^T \int_{\mathbb{R}^d} Z(dt, dy) \int_0^t ds \ \langle \Psi_n(\cdot, y) - \Psi_m(\cdot, y), \mu_s \rangle \right\}^2 \\
&= \int_0^T dt \, \mathbb{E} \left\langle \left\{ \int_0^t ds \, \langle \Psi_n(\cdot, \cdot\cdot) - \Psi_m(\cdot, \cdot\cdot), \mu_s(\cdot) \rangle \right\}^2 , \mu_t(\cdot\cdot) \right\rangle \\
&\qquad + \int_0^T dt \, \mathbb{E} \left\langle \Lambda \int_0^t ds \ \langle \Psi_n(\cdot, \cdot\cdot) - \Psi_m(\cdot, \cdot\cdot), \mu_s(\cdot) \rangle , \mu_t \mu_t \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T dt \int_0^t ds_1 \int_0^t ds_2 \, \mathbb{E} \left\langle (\Psi_n - \Psi_m)(x-z) \cdot (\Psi_n - \Psi_m)(y-z), \mu_{s_1}(dx)\mu_{s_2}(dy)\mu_t(dz) \right\rangle \\
&\quad + \sum_{i,j=1}^d \int_0^T dt \int_0^t ds_2 \int_0^t ds_1 \\
&\qquad \times \mathbb{E} \left\langle \partial_i(\Psi_n - \Psi_m)(x_1,y_1) \cdot \partial_j(\Psi_n - \Psi_m)(x_2,y_2), \mu_{s_1}(dx_1)\mu_{s_2}(dx_2)\mu_t(dy_1)\mu_t(dy_2) \right\rangle \\
&\le C \int_0^T dt \int_0^t ds_2 \int_0^{s_2} ds_1 \, \mathbb{E} \left\langle (\Psi_n - \Psi_m)(x-z) \cdot (\Psi_n - \Psi_m)(y-z), \mu_{s_1}(dx)\mu_{s_2}(dy)\mu_t(dz) \right\rangle \\
&\quad + C \sum_{i,j=1}^d \int_0^T dt \int_0^t ds_2 \int_0^{s_2} ds_1 \\
&\qquad \times \mathbb{E} \left\langle \partial_i(\Psi_n - \Psi_m)(x_1,y_1) \cdot \partial_j(\Psi_n - \Psi_m)(x_2,y_2), \mu_{s_1}(dx_1)\mu_{s_2}(dx_2)\mu_t(dy_1)\mu_t(dy_2) \right\rangle \\
&\le C(T) \left\| \Psi_n - \Psi_m \right\|_\infty^2 + C(T) \sum_{i,j=1}^d \left\| \partial_{2_i}(\Psi_n - \Psi_m) \right\|_\infty \left\| \partial_{2_j}(\Psi_n - \Psi_m) \right\|_\infty .
\end{aligned}
$$

Lemma II.1.1 implies Ψ_n converges in the Schwartz space $S_{2\times d}$, and thus

$$\{\partial_{2_i}\Psi_n : n \in \mathbb{N}\}$$

and

$$\{\partial_{2_i}\partial_{2_j}\Psi_n : n \in \mathbb{N}\},$$

for all $i, j = 1, 2, ..., d$, are uniformly Cauchy sequences. For any $t, s \ge 0$, since $\mu_\cdot \in C_{M_F(\mathbb{R}^d)}[0,\infty)$, and $D^\alpha \Psi_n \to D^\alpha \Psi$ uniformly for any multiindex α,

$$\langle D^\alpha \Psi_n, \mu_s \mu_t \rangle \to \langle D^\alpha \Psi, \mu_s \mu_t \rangle \qquad \text{a.s.}$$

Since the L^2 limit must agree with the a.s. limit,

$$L^2 - \lim_{n\to\infty} \langle D^\alpha \Psi_n, \mu_s \mu_t \rangle = \langle D^\alpha \Psi, \mu_s \mu_t \rangle ,$$

and so

$$L^2 - \lim_{n\to\infty} \int_0^T dt \int_0^t ds \, \langle L_2 \Psi_n, \mu_s \mu_t \rangle = \int_0^T dt \int_0^t ds \, \langle L_2 \Psi, \mu_s \mu_t \rangle .$$

Finally,

$$\left\{\int_0^T \int Z(dt,dy) \int_0^t ds \ \langle (\Psi_n - \Psi_m)(x,y), \mu_s(dx)\rangle \right\}$$

is a Cauchy sequence in L^2. Now, for each $y \in \mathbb{R}^d$, and $t \in [0,T]$, $\langle \Psi_n(\cdot,y), \mu_t\rangle$ is Cauchy in L^2, and so there exists an a.s convergent subsequence $\langle \Psi_{n_k}(\cdot,y), \mu_t\rangle$. Since μ_t is almost surely finite and $\Psi_n \to \Psi$ uniformly,

$$\langle \Psi_{n_k}(\cdot,y), \mu_t\rangle \to \langle \Psi(\cdot,y), \mu_t\rangle \qquad \text{a.s.,} \quad \text{as } k \to \infty.$$

Furthermore, both $\langle \Psi_n(\cdot,y), \mu_t\rangle$ and $\langle \Psi(\cdot,y), \mu_t\rangle$ are uniformly continuous in $y \in \mathbb{R}^d$ and $t \in [0,T]$, and so,

$$\lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^d} Z(dt,dy) \int_0^t ds \ \langle \Psi_{n_k}(\cdot,y), \mu_s\rangle = \int_0^T \int_{\mathbb{R}^d} Z(dt,dy) \int_0^t ds \ \langle \Psi(\cdot,y), \mu_s\rangle \qquad \text{a.s.}$$

Since the L^2 limit must agree with the a.s. limit,

$$L^2-\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^d} Z(dt,dy) \int_0^t ds \ \langle \Psi_n(\cdot,y), \mu_s\rangle = \int_0^T \int_{\mathbb{R}^d} Z(dt,dy) \int_0^t ds \ \langle \Psi(\cdot,y), \mu_s\rangle .$$

□

Our much needed Itô formula has thus been developed, and existence of GSILT will follow shortly.

III.3 Existence

Using lemma III.3 with $G_\varepsilon^{\lambda,u}$ in place of Ψ, we now have

$$\begin{aligned}\gamma_\varepsilon^\lambda(u,T) = \lambda \int_0^T dt \int_0^s ds \left\langle G_\varepsilon^{\lambda,u}, \mu_s\mu_t \right\rangle - \int_0^T dt \left\langle G_\varepsilon^{\lambda,u}, \mu_t\mu_T \right\rangle \\ + \int_0^T dt \left\langle G_\varepsilon^{\lambda,u}, \mu_t\mu_t \right\rangle + \int_0^T \int_{\mathbb{R}^d} Z(dt,dy) \int_0^t ds \left\langle G_\varepsilon^{\lambda,u}(\cdot - y), \mu_s \right\rangle .\end{aligned}$$

As in Rosen [22] and Adler & Lewin [1] & [2] the issue of "local double points" must be addressed, that is, the set of points lying on the diagonal in $\mathbb{R}^2$, which will be (falsely) counted as points of self-intersection when $u = 0$, and will lead to singularities in dimensions greater than one. Due to this we follow the idea first proposed by Adler & Lewin, and renormalize our GSILT via subtraction of the term involving "local double points". It is easy enough to see that the term involving the "local double points" is given by

$$\int_0^T dt \left\langle G_\varepsilon^{\lambda,u}, \mu_t\mu_t \right\rangle,$$

and thus we define our renormalized limiting process to generalized self-intersection local time at $u \in \mathbb{R}^d$, over the set $\{(s,t) : 0 \le s < t \le T\}$ by

$$\begin{aligned}
&\mathcal{L}_\varepsilon^\lambda(u,T) \\
&= \gamma_\varepsilon^\lambda(u,T) - \int_0^T dt \left\langle G_\varepsilon^{\lambda,u}, \mu_t\mu_t \right\rangle \\
&= \lambda \int_0^T dt \int_0^t ds \left\langle G_\varepsilon^{\lambda,u}, \mu_s\mu_t \right\rangle - \int_0^T dt \left\langle G_\varepsilon^{\lambda,u}, \mu_t\mu_T \right\rangle \\
&\quad + \int_0^T \int_{\mathbb{R}^d} Z(dt,dy) \int_0^t ds \left\langle G_\varepsilon^{\lambda,u}(\cdot - y), \mu_s \right\rangle.
\end{aligned}$$

Using Lemma II.3.5 existence follows almost immediately.

Theorem III.3.1. *Suppose that $\mu = \{\mu_t : t \ge 0\}$ is a $d-$dimensional superprocess over a stochastic flow such that $\mu_0 \in M_F(\mathbb{R}^d)$, $d \le 3$, satisfies Assumption 2. Fix $T \in [0,\infty)$ and define $\mathcal{L}_\varepsilon^\lambda(u,T)$ as above, then for $0 \le s < t \le T$,*

$$L^2 - \lim_{\varepsilon \to 0} \mathcal{L}_\varepsilon^\lambda(u,T) = \mathcal{L}^\lambda(u,T),$$

uniformly in $u \in \mathbb{R}^d$, *where* $\mathcal{L}^\lambda(u,T)$ *is defined by*

$$\begin{aligned}\mathcal{L}^\lambda(u,T) = \lambda \int_0^T dt \int_0^t ds \left\langle G^{\lambda,u}, \mu_s\mu_t \right\rangle - \int_0^T dt \left\langle G^{\lambda,u}, \mu_t\mu_T \right\rangle \\ + \int_0^T \int_{\mathbb{R}^d} Z(dt,dy) \int_0^t ds \left\langle G^{\lambda,u}(\cdot - y), \mu_s \right\rangle .\end{aligned}$$

Proof: Let $\{G_\varepsilon\} \subset C^\infty(\mathbb{R}^d)$ be any sequence such that G_ε and $\partial_i G_\varepsilon$ converge respectively in L^1 to $G^{\lambda,u}$ and $\partial_i G^{\lambda,u}$, and for $\varepsilon_1, \varepsilon_2 > 0$, $x \in \mathbb{R}^d$, define

$$\phi_{\varepsilon_1,\varepsilon_2}(x) = G_{\varepsilon_1}(x) - G_{\varepsilon_2}(x).$$

Then for the two non-stochastic integral terms, it is clear that

$$\begin{aligned}&\mathbb{E}\left[\int_0^T dt \int_0^t ds \, \langle \phi_{\varepsilon_1,\varepsilon_2}, \mu_s\mu_t\rangle\right]^2 \\ &\le \int_0^T dt \, \mathbb{E}\left[\int_0^s ds \, \langle \phi_{\varepsilon_1,\varepsilon_2}, \mu_s\mu_t\rangle\right]^2 \\ &\le \int_0^T dt \int_0^t ds_1 \int_0^t ds_2 \, \mathbb{E}\, \langle \phi_{\varepsilon_1,\varepsilon_2} \otimes \phi_{\varepsilon_1,\varepsilon_2}, \mu_{s_1}\mu_t\mu_{s_1}\mu_t\rangle \\ &\le C \int_0^T dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \mathbb{E}\, \langle \varphi_{\varepsilon_1,\varepsilon_2}, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_3}\rangle \, ,\end{aligned} \tag{III.5}$$

and

$$\begin{aligned}&\mathbb{E}\left[\int_0^T dt \, \langle \phi_{\varepsilon_1,\varepsilon_2}, \mu_t\mu_T\rangle\right]^2 \\ &= \int_0^T dt_1 \int_0^T dt_2 \, \mathbb{E}\, \langle \phi_{\varepsilon_1,\varepsilon_2}, \mu_{t_1}\mu_T\mu_{t_2}\mu_T\rangle \\ &\le C \int_0^T dt_2 \int_0^{t_2} dt_1 \, \mathbb{E}\, \langle \varphi_{\varepsilon_1,\varepsilon_2}, \mu_{t_1}\mu_{t_2}\mu_T\mu_T\rangle \, ,\end{aligned} \tag{III.6}$$

where

$$\varphi_{\varepsilon_1,\varepsilon_2}(x_1,x_2,x_3,x_4) \triangleq \phi_{\varepsilon_1,\varepsilon_2}(x_1 - x_3)\phi_{\varepsilon_1,\varepsilon_2}(x_2 - x_4).$$

For the stochastic integral term, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^d} Z(dt,dy)\int_0^t ds\ \langle\phi_{\varepsilon_1,\varepsilon_2}(\cdot-y),\mu_s\rangle\right]^2\\
&=\int_0^T dt\,\mathbb{E}\left\langle\left[\int_0^t ds\ \langle\phi_{\varepsilon_1,\varepsilon_2}(\cdot-\cdot\cdot),\mu_s(\cdot)\rangle\right]^2,\mu_t\right\rangle\\
&+\int_0^T dt\ \left\langle\Lambda\left[\int_0^t ds\ \langle\phi_{\varepsilon_1,\varepsilon_2}(\cdot-\cdot\cdot),\mu_s(\cdot)\rangle\right],\mu_t\mu_t\right\rangle\\
&\le C\int_0^T dt_3\int_0^{t_3} dt_2\int_0^{t_2} dt_1\ \left[\mathbb{E}\langle\varphi_{\varepsilon_1,\varepsilon_2},\mu_{s_1}\mu_{s_2}\mu_t\rangle+\sum_{p,q=1}^d\mathbb{E}\left\langle\varphi^{pq}_{\varepsilon_1,\varepsilon_2},\mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_3}\right\rangle\right],\quad\text{(III.7)}
\end{aligned}$$

where

$$\varphi^{pq}_{\varepsilon_1,\varepsilon_2}(x_1,x_2,x_3,x_4)\triangleq\partial_p\phi_{\varepsilon_1,\varepsilon_2}(x_1-x_3)\partial_q\phi_{\varepsilon_1,\varepsilon_2}(x_2-x_4),$$

for each $p,q=1,2,...,d$, and $(x_1,x_2,x_3,x_4)\in\mathbb{R}^{4\times d}$.

Thus, from III.5, III.6, III.7, and lemma II.3.5, it follows that

$$\mathbb{E}\left[\int_0^T dt\int_0^t ds\ \langle\phi_{\varepsilon_1,\varepsilon_2},\mu_s\mu_t\rangle\right]^2\le C(T)\left\|\phi_{\varepsilon_1,\varepsilon_2}\right\|_1^2,$$

$$\mathbb{E}\left[\int_0^T dt\ \langle\phi_{\varepsilon_1,\varepsilon_2},\mu_t\mu_T\rangle\right]^2\le C(T)\left\|\phi_{\varepsilon_1,\varepsilon_2}\right\|_1^2,$$

and

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^d} Z(dt,dy)\int_0^t ds\ \langle\phi_{\varepsilon_1,\varepsilon_2}(\cdot-y),\mu_s\rangle\right]^2\\
&\le C(T)\left\|\phi_{\varepsilon_1,\varepsilon_2}\right\|_1^2+C(T)\sum_{p,q=1}^d\left\|\partial_p\phi_{\varepsilon_1,\varepsilon_2}\right\|_1\left\|\partial_q\phi_{\varepsilon_1,\varepsilon_2}\right\|_1.
\end{aligned}$$

Since G_ε and $\partial_i G_\varepsilon$ converge respectively to $G^{\lambda,u}$ and $\partial_i G^{\lambda,u}$ in L^1, $i=1,...,d$, we have that

$$\lim_{\varepsilon_1,\varepsilon_2\to 0}\left\|\phi_{\varepsilon_1,\varepsilon_2}\right\|_1=0$$

and

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \|\partial_i \phi_{\varepsilon_1,\varepsilon_2}\|_1 = 0.$$

Since the choice of the $\{G_\varepsilon\}$ is arbitrary, it may be assumed that $G_\varepsilon = G_\varepsilon^{\lambda,u}$ for each $\varepsilon > 0$, and the result follows. $\square$

Existence of GSILT for the SSF is thus shown to exist, in a Tanaka-like form, for dimensions $d \leq 3$. A desired, yet still open result is to show that the mapping

$$(u,T) \mapsto \mathcal{L}^\lambda(u,T)$$

is continuous for each fixed $\lambda > 0$. The progress made so far follows in some Lemmas and Theorems with extend Kolmogorov's continuity criterion (cf. [16]).

CHAPTER IV

CONTINUITY LEMMA

IV.1 Stochastic Fields and Continuity

In order to show joint continuity of the map $(u,t) \to L^{\lambda}(u,t)$, we will now prove an extension of Kolmogorov's continuity criterion from a stochastic process to a stochastic field. We begin with some definitions.

Definition IV.1.1. *A stochastic field is a collection of random variables $\{X_a; a \in A\}$, where A is a partially ordered set.*

Definition IV.1.2. *Given two stochastic fields $X = \{X_a; a \in A\}$ and $Y = \{Y_a; a \in A\}$ defined on the same probability space. If for each $a \in A$,*

$$\mathbb{P}(X_a = Y_a) = 1,$$

then Y is called a modification of X.

Definition IV.1.3. *A stochastic field $X = \{X_a; a \in A\}$ is said to be locally Hölder continuous on $B \subset A$ with exponent δ if*

$$\mathbb{P}\left(\omega; \sup_{\substack{0<|a_1-a_2|<2^{-N(\omega)} \\ a_1,a_2 \in A}} \frac{|X_{a_1}(\omega) - X_{a_2}(\omega)|}{|a_1 - a_2|^{\delta}} \leq \varepsilon\right) = 1,$$

where N is an a.s. positive random variable and $\varepsilon > 0$ is an appropriate constant.

The following lemma extends Kolmogorov's continuity criterion to a stochastic field indexed on $\mathbb{R}^d \times [0,\infty) \times (0,\infty)$. A similar result was shown first by Meyer [19], which this extends through an independent proof.

Lemma IV.1.4. *Fix $T \in (0,\infty)$ and set $U_T \triangleq [-T,T] \times [0,T] \times (0,T)$. Let $X = \{X_u; u \in U_T\}$ be a stochastic field defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that*

$$\mathbb{E}\left|X_{u_1} - X_{u_2}\right|^{\gamma} \leq C\left|u_1 - u_2\right|^{d+\beta+1}, \qquad u_1, u_2 \in U_T$$

for some positive constants γ, β, and C. Then, their exists a continuous modification $\widetilde{X} = \left\{\widetilde{X}_u; u \in U_T\right\}$ of X, which is locally Hölder continuous with exponent δ for every $\delta \in (0, \beta/\gamma)$.

Proof: For simplicity, assume $T \equiv 1$, and write $U \triangleq U_1$. For any $n \in \mathbb{N}$, by Čebyčev's inequality,

$$\begin{aligned}\mathbb{P}(\left|X_{u_1} - X_{u_2}\right| \geq 2^{-\delta n}) &\leq 2^{\gamma\delta n}\mathbb{E}\left|X_{u_1} - X_{u_2}\right|^{\gamma} \\ &\leq C2^{\gamma\delta n}\left|u_1 - u_2\right|^{d+2+\beta}.\end{aligned} \qquad \text{(IV.1)}$$

Thus,

$$\mathbb{P}-\lim_{u_1 \to u_2} X_{u_1} = X_{u_2}, \qquad \forall u_1, u_2 \in U. \qquad \text{(IV.2)}$$

Define

$$D_n \triangleq \{u = (u^1, ..., u^{d+2}) | u^i = k^i 2^{-n}, \left|k^i\right| \leq 2^n, i = 1, ..., d, 0 \leq k^{d+1} \leq 2^n, 0 < k^{d+2} < 2^n\}$$

for each $n \in \mathbb{N}$,

$$D \triangleq \bigcup_{n=1}^{\infty} D_n,$$

and write $u_1 \overset{n}{\sim} u_2$ exactly when $u_1, u_2 \in D_n$ and $|x_1 - x_2| = 2^{-n}$.

If $u_1 \overset{n}{\sim} u_2$, then (IV.1) implies

$$\mathbb{P}(|X_{u_1} - X_{u_2}| \geq 2^{-\delta n}) \leq C 2^{-n(d+2+\beta-\gamma\delta)}.$$

Since

$$\begin{aligned} |D_n| &= (2^{n+1}+1)^d (2^n+1)(2^n-2) \\ &\leq (2 \cdot 3^d) 2^{n(d+2)} \\ &= C\, 2^{n(d+2)} \end{aligned}$$

and each $u \in D_n$ has at most $2(d+2)$ neighbors,

$$\begin{aligned} \mathbb{P}\left(\max_{u_1 \overset{n}{\sim} u_2} |X_{u_1} - X_{u_2}| \geq 2^{-\delta n} \right) &= \mathbb{P}\left(\bigcup_{u_1 \in D_n} \bigcup_{\substack{u_2 \in D_n \\ u_2 \overset{n}{\sim} u_1}} |X_{u_1} - X_{u_2}| \geq 2^{-\delta n} \right) \\ &\leq \sum_{u_1 \in D_n} \sum_{\substack{u_2 \in D_n \\ u_2 \overset{n}{\sim} u_1}} \mathbb{P}\left(|X_{u_1} - X_{u_2}| \geq 2^{-\delta n}; u_1 \overset{n}{\sim} u_2 \right) \\ &\leq C\, 2^{-n(\beta-\gamma\delta)}. \end{aligned}$$

Since $\beta - \gamma\delta > 0$, the Borel-Cantelli Lemma implies the existence of $\Omega_0 \in \mathcal{F}$ such that $\mathbb{P}(\Omega \setminus \Omega_0) = 1$ and for any $\omega \notin \Omega_0$, $u_1 \overset{n}{\sim} u_2$,

$$|X_{u_1} - X_{u_2}|(\omega) < 2^{-\delta n}, \qquad \forall\, n \geq n_0(\omega), \tag{IV.3}$$

where n_0 is a positive, integer-valued random variable.

Consider the claim.

Claim IV.1.5. *Fix $\omega \notin \Omega_0$ and $n \geq n_0(\omega)$. For any $m > n$, and any $u_1, u_2 \in D_m$ with*

$|u_1 - u_2| < 2^{-n}$,

$$|X_{u_1}(\omega) - X_{u_2}(\omega)| \leq 2(d+2) \sum_{j=n+1}^{m} 2^{-\delta j}. \tag{IV.4}$$

Proof of Claim IV.1.5: The proof will follow from an induction on $m > n$. If it is first assumed that $m = n+1$, then $u_1, u_2 \in D_{n+1}$ and $|u_1 - u_2| < 2^{-n}$ imply $u_1 \overset{n+1}{\sim} u_2$, and so by (IV.3),

$$\begin{aligned} |X_{u_1}(\omega) - X_{u_2}(\omega)| &< 2^{-\delta(n+1)} \\ &< 2(d+2)2^{-\delta(n+1)}. \end{aligned}$$

Assume now that for some $m > n$,

$$|X_{u_1}(\omega) - X_{u_2}(\omega)| \leq 2(d+2) \sum_{j=n+1}^{m-1} 2^{-\delta j}, \tag{IV.5}$$

for any $u_1, u_2 \in D_{m-1}$ with $|u_1 - u_2| < 2^{-n}$. For $p = 1, 2$, define $v_p = (v_p^1, ..., v_p^{d+2})$ as follows. Given each pair (u_1^i, u_2^i), $i = 1, 2, ..., d+2$, if $u_p^i \geq u_q^i$, set

$$v_p^i = \max\left\{k2^{-(m-1)}; k2^{-(m-1)} \leq u_p^i\right\},$$

and

$$v_q^i = \min\left\{k2^{-(m-1)}; k2^{-(m-1)} \geq u_q^i\right\}.$$

For each $i = 1, 2, ..., d+2$, define

$$u(i) = (v_1^1, ..., v_1^i) \otimes (u_1^{i+1}, ..., u_1^{d+2}),$$

$$u(0) = u_1,$$

and

$$v(i) = (u_2^1, ..., u_2^i) \otimes (v_2^{i+1}, ..., v_2^{d+2}),$$

$$v(0) = v_2,$$

and note that

$$u(i) \overset{m}{\sim} u(i-1)$$

$$v(i) \overset{m}{\sim} v(i-1).$$

It follows from (IV.3) that

$$\left|X_{u(i)}(\omega) - X_{u(i-1)}(\omega)\right| \leq 2^{-\delta m},$$

and

$$\left|X_{v(i)}(\omega) - X_{v(i-1)}(\omega)\right| \leq 2^{-\delta m}.$$

Since $|v_1 - v_2| < 2^{-n}$, (IV.5) implies

$$|X_{v_1}(\omega) - X_{v_2}(\omega)| \leq 2(d+2) \sum_{j=n+1}^{m-1} 2^{-\delta j}.$$

Finally,

$$\begin{aligned}
&|X_{u_1}(\omega) - X_{u_2}(\omega)| \\
&\le \sum_{i=1}^{d+2} \left|X_{u(i)}(\omega) - X_{u(i-1)}(\omega)\right| + \sum_{i=1}^{d+2} \left|X_{v(i)}(\omega) - X_{v(i-1)}(\omega)\right| + |X_{v_1}(\omega) - X_{v_2}(\omega)| \\
&\le 2(d+2)2^{-\delta m} + 2(d+2) \sum_{j=n+1}^{m-1} 2^{-\delta j} \\
&= 2(d+2) \sum_{j=n+1}^{m} 2^{-\delta j},
\end{aligned}$$

thus proving the claim. □

(Continuation of proof of lemma.) Now, if $u_1, u_2 \in D$ with $0 < |u_1 - u_2| < 2^{-n_0(\omega)}$, there exists a unique $n \ge n_0(\omega)$ such that $2^{-(n+1)} \le |u_1 - u_2| < 2^{-n}$. If $\eta > 0$ is defined by $\eta \triangleq \dfrac{2(d+2)}{1-2^{-\delta}}$, it follows from the claim that

$$\begin{aligned}
|X_{u_1}(\omega) - X_{u_2}(\omega)| &\le 2(d+2) \sum_{j=n+1}^{\infty} 2^{-\delta j} \\
&= \frac{2(d+2)}{1-2^{-\delta}} \left(2^{-(n+1)}\right)^{\delta} \\
&\le \eta \, |u_1 - u_2|^{\delta} . \qquad \text{(IV.6)}
\end{aligned}$$

Thus, $u \mapsto X_u(\omega)$ is uniformly continuous for any $\omega \notin \Omega_0$.

Define $\widetilde{X}$ as follows. For $\omega \in \Omega_0, u \in U$, set $\widetilde{X}_u(\omega) = 0$. If $\omega \notin \Omega_0$, $u \in D$, set $\widetilde{X}_u(\omega) = X_u(\omega)$, and for $\omega \notin \Omega_0$, $u \in U \cap D^c$, choose a sequence $\{u_n; n \in \mathbb{N}\} \subset D$ such that $\lim_{n\to\infty} u_n = u$. For m, n large enough,

$$|X_{u_n}(\omega) - X_{u_m}(\omega)| \le \eta \, |u_n - u_m|^{\delta},$$

which implies $\lim_{n\to\infty} X_{u_n}(\omega)$ exists, and is dependent only upon u (independent of the choice of sequence). Thus set

$$\widetilde{X}_u(\omega) = \lim_{u_n \to u} X_{u_n}(\omega).$$

By construction it thus follows that $\widetilde{X}_u = X_u$ *a.s.* for all $u \in D$. If $u \in U \cap D^c$, and $\{u_n\} \subset D$ with $u_n \to u$, it has been shown that

$$\lim_{u_n \to u} X_{u_n} = \widetilde{X}_u \quad a.s.$$

From (IV.2)

$$\mathbb{P}-\lim_{u_n \to u} X_{u_n} = X_u,$$

and since the *a.s.* and probabilistic limits must agree, $\widetilde{X_u} = X_u$ *a.s.*.

Thus, it is shown that $\widetilde{X_u} = X_u$ *a.s.* for all $u \in U$, and so, $\widetilde{X}$ is the desired modification of X.

Finally, by (IV.6),

$$\mathbb{P}\left(\omega;\ \sup_{\substack{0<|u_1-u_2|<2^{-n_0(\omega)} \\ u_1,u_2 \in U}} \frac{|X_{u_1}(\omega) - X_{u_2}(\omega)|}{|u_1 - u_2|^{\delta}} \leq \eta\right) = 1,$$

showing the desired Hölder continuity.

CHAPTER V

FINAL REMARKS

V.1 Open Questions

We conclude this work with some open questions.

Assume for fixed $\lambda > 0$ that $\mathcal{L}^\lambda(z,T)$ is jointly continuous in (z,T), and write $\alpha_T(dz) = \mathcal{L}^\lambda(z,T)\,dz$. Then, for any bounded Borel measurable ϕ,

$$\begin{aligned}
\int_{\mathbb{R}^d} \alpha_T(dz)\phi(z) &= \int_{\mathbb{R}^d} dz\, \mathcal{L}^\lambda(z,T)\phi(z) \\
&= \int_{\mathbb{R}^d} dz \int_0^T dt \int_0^t ds\, \langle \delta_z, \mu_s\mu_t\rangle\, \phi(z) \\
&= \int_{\mathbb{R}^d} dz \int_0^T dt \int_0^t ds \int_{\mathbb{R}^{2d}} \mu_s(dx)\mu_t(dy)\, \delta_z(x-y)\phi(z) \\
&= \int_0^T dt \int_0^t ds \int_{\mathbb{R}^{2d}} \mu_s(dx)\mu_t(dy) \int_{\mathbb{R}^d} dz\, \delta_z(x-y)\phi(z) \\
&= \int_0^T dt \int_0^t ds \int_{\mathbb{R}^{2d}} \mu_s(dx)\mu_t(dy)\, \phi(x-y) \\
&= \int_0^T dt \int_0^t ds\, \langle \phi, \mu_s\mu_t\rangle\,.
\end{aligned}$$

Thus,

$$\mathcal{L}^\lambda(z,T) = \frac{d\alpha_T}{dz}$$

gives the local time for $\nu_{s,t} = \mu_t - \mu_s$ over $\{(s,t) : 0 \le s \le t \le T\}$ (cf. [22]). The first question follows naturally.

Question 1: For fixed $\lambda > 0$, is the process $L^\lambda(u,T)$ jointly continuous in (u,T)?

It is likely that some difficulties will arise when attempting to use Lemma IV.1.4

to prove continuity. Namely, it is clear that moments at least as great as the sixth must be calculated, which will be at the very least quite tedious. From the work on the third and fourth moments, it is conjectured that a general moment formula could be employed to bound

$$\mathbb{E}\left\{\int_0^T dt \int_0^t ds \ \langle \phi, \mu_s \mu_t \rangle \right\}^n,$$

$$\mathbb{E}\left\{\int_0^T dt \ \langle \phi, \mu_t \mu_T \rangle \right\}^n,$$

and

$$\mathbb{E}\left\{\int_0^T \int_{\mathbb{R}^d} Z(dt, dy) \int_0^t ds \, \langle \phi(\cdot - y), \mu_s \rangle \right\}^n,$$

in a manner that allows Lemma IV.1.4 to show the desired continuity. Thus, a new method for calculating moments is desired. It was conjectured in [1], and shown in [30] and [31] that a conditional log-Laplace functional can be defined for the superprocess over a stochastic flow. Let ξ_t be the diffusion process described by

$$d\xi_t = b(\xi_t) dB_t^\alpha + c(\xi_t) dW_t$$

for independent Brownian motions B^α and W, let $\mathbb{E}^W$ denote the conditional expectation given the Brownian motion W and let $\mathcal{F}_t^\xi = \sigma(\xi_s : s \leq t)$, then we have the following lemma.

Lemma V.1.1. $\xi = \{\xi_t : t \geq 0\}$ *satisfies:*

$$\mathbb{E}^W(\phi(\xi_t)|\mathcal{F}_s^\xi) = \mathbb{E}^W(\phi(\xi_t)|\xi_s), \qquad \text{a.s.},$$

for all $s < t$, $\phi \in C_b(\mathbb{R}^d)$. *That is,* ξ *is a conditional Markov process given* W.

Proof: See Lemma 1 of Xiong [31]. □

Given W, denote the conditional transition function by

$$q_{s,t}^W(x, \cdot) \triangleq \mathbb{P}^W(\xi_t \in \cdot | \xi_s = x),$$

and the associated transition semigroup by

$$Q_{s,t}^{W}\phi(x) = \int_{\mathbb{R}^d} q_{s,t}^{W}(x,dy)\phi(y).$$

Let $\hat{d}W_r$ represent the the backward Itô integral:

$$\int_s^t \hat{d}W_r\,\psi(r) = \lim_{|\Delta|\to 0}\sum_{i=1}^{n}\psi(r_i)(W_{r_i} - W_{r_{i-1}})$$

where $\Delta = \{r_0, r_1, ..., r_n\}$ is a partition of $[s,t]$ and $|\Delta|$ is the maximum length of its subintervals. Furthermore, for $\phi \in C_K^2(\mathbb{R}^d)$, let $y_{s,t}$ be defined as the solution to the following SPDE:

$$y_{s,t}(x) = \phi(x) + \int_s^T dr\,(Ly_{r,t}(x) - y_{r,t}(x)^2) + \int_s^t \nabla^T y_{r,t}(x)c(x)\hat{d}W_r. \qquad \text{(V.1)}$$

It was shown in [30] and [31] that the SPDE V.1 has a unique $L^2(\mathbb{R}^d)^+$-valued solution in the following sense: for any $\psi \in C_K^\infty(\mathbb{R}^d)$, for any $s \le t$,

$$\langle y_{s,t}, \psi\rangle = \langle \phi, \psi\rangle + \int_s^t dr\;\langle y_{r,t}, L^*\phi - y_{r,t}\phi\rangle + \int_s^t \langle y_{r,t}, \nabla^T(c\phi)\rangle\,\hat{d}W_r$$

where for any space of real valued functions $\mathcal{A}$, $\mathcal{A}^+ \subset \mathcal{A}$ is the subset of positive valued functions and L^* is the dual operator to L given by

$$L^*\phi = \frac{1}{2}\sum_{i,j=1}^{d}\partial_{ij}^2(a_{ij}\phi).$$

Furthermore, it was shown that

$$\mathbb{E}\sup_{0\le r\le t}\|\partial_x y_{r,t}\|_{L^2(\mathbb{R}^d)}^2 \le \infty,$$

where $\partial_x y_{r,t}$ is the derivative in the weak sense, and that

$$0 \leq y_{r,t}(x) \leq \|\phi\|_{\infty}.$$

The following log-Laplace functional was conjectured first in [26], then shown to hold (under certain nice conditions) by Xiong in [30] and [31].

Theorem V.1.2. *For any $\phi \in C_K^2(\mathbb{R}^d)^+$,*

$$\mathbb{E}_{\mu_0}^W e^{-\langle \phi, \mu_t \rangle} = e^{-\langle y_{0,t}, \mu_0 \rangle}.$$

If y_t is defined by

$$y_t(x) = \phi(x) + \int_0^t ds\,(Ly_s(x) - y_s(x)^2) + \int_0^t \nabla^T y_s(x) c(x) dW_s,$$

Xiong [31] has shown that

$$\mathbb{E}_{\mu_0}^W e^{-\langle \phi, \mu_t \rangle} \stackrel{d}{=} e^{-\langle y_t, \mu_0 \rangle},$$

and that

$$y_t = Q_{s,t}^W \phi - \int_0^t ds\, Q_{s,t}^W (y_s)^2.$$

Therefore, by taking expectations, it follows that

$$\mathbb{E}_{\mu_0} e^{-\langle \phi, \mu_t \rangle} = \mathbb{E} e^{-\langle y_t, \mu_0 \rangle}.$$

If y_t^α is defined by

$$y_t^\alpha(x) = \alpha\phi(x) + \int_0^t ds\,(Ly_s^\alpha(x) - y_s^\alpha(x)^2) + \int_0^t \nabla^T y_s^\alpha(x) c(x) dW_s,$$

it thus follows that

$$\mathbb{E}_{\mu_0} e^{-\alpha\langle\phi,\mu_t\rangle} = \mathbb{E} e^{-\langle y_t^\alpha,\mu_0\rangle},$$

or equivalently, we have the log-Laplace transform given by

$$\mathcal{L}_\alpha(\mu_t) = \mathbb{E} e^{-\langle y_t^\alpha,\mu_0\rangle}.$$

And so it is now theoretically possible to use this method to calculate any higher moments of the superprocess. This leads to the next question.

Question 2. Using the above log-Laplace transform, for any even m and some $\gamma > 0$, can the following bounds be established?

(i) $\mathbb{E}\left[\mathcal{L}_\varepsilon^\lambda(u,T) - L_\varepsilon^\lambda(u,T')\right]^m \leq C\,|T-T'|^{\gamma m}$,

(ii) $\mathbb{E}\left[\mathcal{L}_\varepsilon^\lambda(u,T) - L_\varepsilon^\lambda(u',T)\right]^m \leq C\,|u-u'|^{\gamma m}$,

(iii) $\mathbb{E}\left[\mathcal{L}_\varepsilon^\lambda(u,T) - L_{\varepsilon'}^\lambda(u,T)\right]^m \leq C\,|\varepsilon-\varepsilon'|^{\gamma m}$

where the constant C depends only upon m.

Under the assumption that the above can be shown, Lemma IV.1.4 would further imply the property of Hölder continuity, with Hölder exponent δ for every $\delta \in (0, \gamma - \frac{d-1}{m})$. Since the assumption of Question 2 holding implies the bounds hold for every even m, the Hölder continuity follows for any exponent $\delta \in (0,\gamma)$.

Question 3. Given the above, can a maximal γ be found. Failing this, is it possible to find $0 < a < b < \infty$ such that $\gamma \in (a,b)$.

Xiong examined the long-term behavior for superprocesses over a stochastic flow in [31], and produced the following two theorems.

Theorem V.1.3. *Suppose that $d \leq 2$,*

$$\int \mu_0(dx)\, q^W_{s,t}(x,\cdot) = \mu_0(\cdot), \quad \mu_0 << \lambda, \quad \text{and } 0 < c_1 \leq \frac{d\mu}{d\lambda} \leq c_2 < \infty$$

for constants c_1, c_2. For any bounded Borel set $B \subset \mathbb{R}^d$, we have

$$\mathbb{P}\!-\!\lim_{t\to\infty} \mu_t(B) = 0.$$

Theorem V.1.4. *Suppose that $d \geq 3$,*

$$\int \mu_0(dx)\, q^W_{s,t}(x,\cdot) = \mu_0(\cdot),$$

and that μ_0 has a density that is bounded by $c_1 e^{c_2|x|}$ where c_1 and c_2 are constants. Then $\mu_t \Rightarrow \mu_\infty$ for some μ_∞ as $t \to \infty$. Furthermore, $\mathbb{E}\mu_\infty = \mu_0$.

Since "local double points" are excluded from the SILTSSF, it thus seems reasonable that when $d \leq 2$,

$$P\!-\!\lim_{t\to\infty} L^\lambda(u,t) = 0.$$

This gives the next question.

Question 4. Using the above Theorem, does it follow that for $d \leq 2$,

$$P\!-\!\lim_{t\to\infty} L^\lambda(u,t) = 0?$$

Will this also hold if the limit is taken in L^2?

The more interesting case occurs when $d = 3$, and leads to the final question.

Question 5. Since when $d = 3$ the SSF converges in distribution to a random measure, having expectation μ_0, what is the behavior of $L^\lambda(u,t)$ as $t \to \infty$?

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