

SYMMETRIES OF CAUCHY HORIZONS
AND
GLOBAL STABILITY OF COSMOLOGICAL MODELS

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DISSERTATION ABSTRACT

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This dissertation contains the results obtained from a study of two subjects in mathematical general relativity. The first part of this dissertation is about the existence of Killing symmetries in spacetimes containing a compact Cauchy horizon. We prove the existence of a nontrivial Killing symmetry in a large class of analytic cosmological spacetimes with a compact Cauchy horizon for any spacetime dimension. In doing so, we also remove the restrictive analyticity condition and obtain a generalization to the smooth case. The second part of the dissertation presents our results on the global stability problem for a class of cosmological models. We investigate the power law inflating cosmological models in the presence of electromagnetic fields. A stability result for such cosmological spacetimes is proved.

This dissertation includes unpublished co-authored material.

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CHAPTER I

INTRODUCTION

General relativity is a theory of space, time and gravity. A spacetime is a pair (M, g) such that M is a 4-dimensional manifold and g is a Lorentzian metric¹ on M . Einstein's equations for the gravitational field are the focus of general relativity; they relate a spacetime (M, g) to its matter and non-gravitational fields content

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}, \quad (\text{I.1})$$

where $T_{\mu\nu}$ is the energy momentum tensor of the matter and non-gravitational fields. To study the theory of general relativity, one can study the solutions of Einstein's equations coupled to various matter and non-gravitational fields. Spacetimes allowed by general relativity can exhibit a variety of exotic behaviors, such as singularities (causal geodesic incompleteness) and causality violation (closed world-lines). The question of spacetime singularities has led physicists to the singularity theorems of R. Penrose and S. Hawking [1, 2, 3, 4, 5] and to the conjecture of the BKL picture of cosmological singularities [6, 7]. Physicists now believe that a singularity can happen in a spacetime satisfying Einstein's equations under rather generic situations. The nature of singularities is still not clear to us. One possible picture is that given by Belinskii, Khalatnikov and Lifshitz (the BKL picture) [6, 7]. Determining the nature of singularities remains one of the more important research subjects in general relativity.

¹The definition is given in the glossary

The fact that there exist large numbers of spacetimes consistent with general relativity theory that violate causality surprises people. These spacetimes include the Taub-NUT spacetime [8] and large families of Taub-NUT-like spacetimes [9]. On the other hand, all of the known causality violating spacetimes are symmetric in some sense. Hence it is still possible to save predictability of *generic* spacetimes satisfying Einstein's equations at the classical level. This is the conjecture of strong cosmic censorship (SCC), which claims that the measure of the set of causality violating spacetimes is zero; i.e., generic spacetimes developed from non-singular initial data do not violate causality. SCC remains one of the main unsolved problems in general relativity. Chapter II and III are about our work in support of SCC.

Along with studying the properties of generic solutions of Einstein's equations, one can also use general relativity as a tool to study the specific spacetimes of interesting systems in nature. The important and interesting subjects of this kind include black hole formation, gravitational wave production and propagation in astrophysics and the whole universe. The universe as a cosmological system is among the most interesting subjects to which general relativity is applied. It is also one of the few arenas in physics in which general relativity plays a major role. The universe has been observed to be highly spatially isotropic on a very large scale. It has also been observed to be expanding. Cosmological models together with expansion-driving mechanisms are proposed to match the observation. Many forever expanding cosmological models have the property of future completeness. Future completeness means every object exists forever. One interesting question about future completeness of cosmological models is whether this property is stable

under the perturbation of its initial data. Progress has been made for inflationary cosmological models. Chapter IV is about such a stability theorem.

I.1. Cauchy Problem in General Relativity

In the first few decades after the birth of general relativity, very few results on the evolution of spacetimes satisfying Einstein's equations were obtained. Only the special cases were studied, such as the FRW model of the universe. Relativists were trying to find explicit model solutions to Einstein's equations and draw physical implications from those model solutions. These models play an important role in our understanding of general relativity and properties of spacetimes. In trying to understand the spherically symmetric Schwarzschild solution, people discovered important physical concepts such as event horizons and singularities. The singularity theorems of R. Penrose and S. Hawking [1, 2, 3, 4, 5] say that under some rather generic conditions, a singularity must occur in spacetimes satisfying Einstein's equations. However, the singularity theorems do not say what the singularities are like or how they develop from a Cauchy surface². Much has been discovered by studying explicit solutions to Einstein's equations and the causal structures of them, but not much insight regarding the dynamical pictures and general solutions has been gained. Some dynamical pictures have been obtained for the propagation of gravitational waves, but this is done by weak field approximation. To investigate the evolution of general solutions to Einstein's equations, we must have some initial value formulation for generic initial data.

It was not until 1952 that Y. Choquet-Bruhat [10] first showed that the initial value formulation of general relativity is well-posed. An initial value formulation

²Definition is given in the glossary

is well-posed if i) a small change in initial data only induces a small change in the solution; ii) a change in the initial data in a region S of the initial data surface should not produce any change in the solution outside the future set of S . In particular, i) implies that there exists a unique solution corresponding to an initial data set. The reason it took so long to develop a well-posed initial value formulation is because the gravitational field couples to itself causing the Einstein's equations to be nonlinear. The other reason is that there is a lot of redundancy in Einstein's equations. A solution to Einstein's equations is really a diffeomorphism³ class of Lorentzian manifolds. Only by choosing a coordinate system can Einstein's equations be cast into a hyperbolic system of equations. After proving the well-posedness of the initial value formulation of general relativity, Y. Choquet-Bruhat and R. Geroch [11] further showed that given initial data, there is a unique maximal Cauchy development (MCD)⁴ up to diffeomorphism. MCD is the largest development that can be obtained by evolving initial data.

An initial data set (Σ, h, k) of the vacuum Einstein's equations consists of an initial "spacetime slice" (Σ, h) , which is a Riemannian 3-manifold with a Riemannian metric h , and a symmetric 2-tensor k on Σ , which roughly speaking, corresponds to the first time derivative of the initial "spacetime slice". The initial data set (Σ, h, k) can not be chosen arbitrarily. Instead (Σ, h, k) has to satisfy the Einstein constraints equations for it to qualify as an initial data set of the vacuum Einstein's equations. If some non-gravitational fields are considered, an initial data set of the Einstein-Source field equations consists of (Σ, h, k) together with initial data of the non-gravitational fields on (Σ, h, k) . Again the initial data set

³See the definition in the glossary

⁴See the definition in the glossary

has to satisfy the Einstein-Source constraints equations. The details of the initial value formulation of the Einstein-Maxwell-scalar equations is presented in the introduction of Chapter IV. Much research involving initial data has been carried out since 1952. A. Rendall [12] proved an alternative initial value formulation with the initial data prescribed on null hypersurfaces⁵ instead of on spacelike Cauchy surfaces. Construction of initial data is one of the hot research areas today. Progress has been made in constructing new initial data from two given initial data sets. Various gluing techniques have been developed to construct new initial data. Recently, gluing of asymptotically hyperbolic initial data and the construction of N-body initial data have been obtained [13, 14]. Another important research topic in initial value problem is the global stability of solutions to Einstein's equations, which will be discussed in Section I.3..

One recent exciting result involving the initial value problem is the study of black hole formation by D. Christodoulou [15]. One version of the singularity theorem proved by R. Penrose says that a spacetime (M, g) must be null geodesically incomplete⁶ if i) $R_{\mu\nu}N^\mu N^\nu \geq 0$ for any null vectors N ; ii) (M, g) contains a non-compact Cauchy hypersurface; and iii) there is a closed trapped surface. The question of how such a trapped surface forms out of initial data not containing a trapped surface and whether a generic initial data set can produce a trapped surface is unsolved. D. Christodoulou [15] proved the existence of an open set of initial data that develop a trapped surface in the Cauchy development.

Perhaps the greatest unsolved problems in mathematical general relativity are the weak cosmic censorship (WCC) and strong cosmic censorship (SCC)

⁵The definition of a null hypersurface is given in the glossary

⁶Definition is given in the glossary

conjectures. SCC is not a stronger version of WCC as the names seem to imply. Both WCC and SCC can be rigorously formulated in terms of an initial value problem. The WCC applies to isolated gravitational systems, and says that generic singularities have no effects on distant observers because of the formation of an event horizon. In other words, the measure of the set of spacetimes satisfying Einstein's equations and containing a naked singularity (a singularity that is not hidden inside an event horizon) is zero. On the other hand, SCC says that causality violation does not happen in a generic spacetime developed from nonsingular initial data. Progress on SCC is presented in the next section.

I.2. Strong Cosmic Censorship

According to a theorem of Y. Choquet-Bruhat and R. Geroch [11], there exists a unique maximal Cauchy development (MCD) to a given initial data set of Einstein's equations coupled to well-behaved matter fields. An interesting question to ask is the following: what is the end of the MCD like? By the end of the MCD, we mean the boundary of the MCD to which we evolve initial data until we can not continue. The end of the MCD can exhibit roughly three different behaviors or a mixture of these. The first possibility is that there is no such end. Every time-like curve has infinite proper time; i.e., the MCD is future complete. The second possibility is geodesic incompleteness. That is, an object ceases to exist in some finite time as it approaches the end of the MCD. The usual notion of singularity is defined as geodesic incompleteness. The singularity theorems of R. Penrose and S. Hawking tell us that singularities are generic. However, there are two possibilities of geodesic incompleteness. The first one is that the curvature blows up. In this case, everything approaching the singularity is destroyed by the

infinite tidal force. The second possibility is that the curvature does not blow up, but there is a Cauchy horizon⁷ where causal curves become closed. A Cauchy horizon of the MCD is defined as the boundary of the region of spacetime that is uniquely determined by the initial data. The MCD can generally be smoothly extended as a solution to the Einstein's equations coupled to well-behaved matter fields by attaching another spacetime at the Cauchy horizon. As one approaches the Cauchy horizon, the light cones tip. At the Cauchy horizon, light cones become tangent to the Cauchy horizon and closed null curves form on the horizon. Hence the Cauchy horizon is the boundary across which closed time-like curves form. Causality violation happens as one crosses the Cauchy horizon. Since the usual idea of physical phenomena is to favor causality, the strong cosmic censorship conjecture is proposed, which roughly speaking, says that causality violation is non-generic.

SCC was first proposed by R. Penrose in [16]. It was rigorously formulated by V. Moncrief and D. Eardley [17]. The following statement is a mathematical formulation of SCC essentially due to V. Moncrief and D. Eardley:

Consider the collection of initial data of an Einstein-matter system with the initial data manifolds being compact or the initial data being asymptotically flat. There is an open dense subset of initial data whose MCDs are inextendible provided the matter fields are well behaved.

Attempts at proving SCC have been carried out in two directions. One direction is to find an open dense subset of initial data whose MCDs are inextendible. Due to the complexity of the conjecture, the easier thing to do is to first prove SCC for some classes of spacetimes. Most progress in this direction has been on Gowdy

⁷Definition is given in the glossary.

spacetimes [18, 19]. The other direction is to prove by “contradiction”. The proof is to show that the spacetimes containing a Cauchy horizon are non-generic. One way to do this is to find a symmetry in the spacetimes containing a Cauchy horizon. Progress in this direction has only been achieved on compact Cauchy horizons [20, 21]. The first part of this dissertation (Chapter II and Chapter III) is about some extensions of the results of [20].

I.3. Cosmological Models and Global Stability

Explicit solutions to Einstein’s equations are interesting not only because of their simplicity, but also because of the physical reality they may represent and approximate. Minkowski spacetime represents flat spacetime and approximates local weak gravitational field metrics, while the Schwarzschild and the Kerr spacetimes represent the gravitational field around an isolated body. The Schwarzschild metric is a better approximation of the gravitational field around the Earth than the Minkowski metric. Of equal interest to physicists are the metrics modeling the whole universe. These are the expanding spatially homogeneous and isotropic Robertson-Walker metrics

$$g = -dt^2 + a^2(t)\gamma, \tag{I.2}$$

where γ is a maximally symmetric 3-metric and $a(t)$ is the scale factor. The metric g satisfies Einstein’s equations coupled to non-gravitational fields. The form of the scale factor a is determined by astronomical observations to be expanding and the expansion is accelerated. The question of why the metric is expanding leads

cosmologists to construct many cosmological models that include matter fields that could drive such an expanding metric.

Stability is an essential property that should be tested for any explicit solution expected to have some relevance to the description of physical reality. There are many notions of stability - the meaning of stability differs in different theories. For a future complete spacetime, stability can be interpreted to mean that the perturbed spacetimes stay close to the original spacetime and are future complete.

The Minkowski metric and certain expanding cosmological metrics are future complete. One of the first global stability results is [22], in which H. Friedrich shows that the De Sitter spacetime is globally stable for Einstein's equations with a cosmological constant. The proof of global stability of the Minkowski metric [23] is one of the most important results in global stability. Alternative approaches to the proof and extensions of the stability result have also been obtained thereafter [24, 25]. More recently, L. Andersson and V. Moncrief [26] have proven that the Milne spacetimes⁸ are globally stable solutions of the vacuum Einstein equations. H. Ringström [27, 28] has very recently shown that certain solutions of the Einstein-scalar field equations with accelerating expansion are globally stable. He does this for both exponentially expanding background solutions with fairly general scalar field potential functions $V(\phi)$ [27], and for power law expanding background solutions with a certain set of exponentially-decaying scalar potential functions [28]. In Chapter IV, we show that the power law expanding solutions considered by Ringström in [28] are globally stable with respect to the Einstein-Maxwell-scalar field equations.

⁸These expanding spacetimes are constructed by spatially compactifying the mass hyperboloids in Minkowski spacetime

I.4. Organization

This dissertation consists of two main parts. The first part is presented in Chapter II and III, and the second part is contained in Chapter IV. Each main part of the dissertation contains the work accomplished under the supervision of James Isenberg. The results contained in the first part are in preparation for publication and co-authored by James Isenberg; and the results in the second part are going to be published and co-authored by James Isenberg. I made primary contributions in both main parts.

Chapter II and III are about the symmetries of Cauchy horizons. The goal is to prove the existence of a non-trivial Killing vector field⁹ in a spacetime neighborhood of the Cauchy horizon, that is, to show spacetimes with Cauchy horizons are not generic. In Chapter II, we consider spacetimes of any dimensions with the presence of a electromagnetic field, a scalar field and particles describe by the Boltzmann-Vlasov equation. This is a generalization of the work done by J. Isenberg and V. Moncrief [20], which only considers 3+1 dimensional electrovacuum spacetimes. Like [20], we also impose the analyticity condition.

In Chapter III we try to remove the restrictive analyticity condition. We notice the relationship between spacetimes containing a compact Cauchy horizon and the stationary black hole spacetimes [29]. We also notice the smoothness result [30] on stationary black hole spacetimes. By applying these results we remove the analyticity condition.

Chapter IV is about our work on global stability of a class of inflationary cosmological spacetimes. We generalize a result of H. Ringström [28] to the case with electromagnetic fields. In doing so, we need to consider extra field

⁹See the definition of a Killing vector field in the glossary

equations (Maxwell equations) and more general perturbations (perturbations with electromagnetic field). We obtain the stability result by following the same procedure carried out by H. Ringström: i) reformulations of field equations; ii) energy functionals; iii) bootstrap assumptions; iv) differential inequalities; v) global existence; vi) geodesic completeness and vi) asymptotic expansions.

CHAPTER II

SYMMETRIES OF CAUCHY HORIZONS

In [20] V. Moncrief and J. Isenberg prove that 3+1 dimensional analytic, vacuum or electrovacuum spacetimes containing compact null hypersurfaces with closed null geodesics necessarily have a Killing symmetry. Their result requires the assumption that the spacetime is analytic, that the embedded null hypersurface is compact with closed null geodesics and that it has the structure of local product bundle¹. J. Isenberg and V. Moncrief [21] later removed the assumption of a local product bundle structure. Some progress in removing analyticity is made in [29]. J. Isenberg and V. Moncrief are presently working on removing the condition of closedness of null geodesics.

We generalize the results of [20] to higher dimensional spacetimes and also obtain similar results of spacetimes with the presence of scalar fields and matter fields that can be described by the Boltzmann-Vlasov equations. For the higher dimensional versions of spacetimes studied in [20], our theorem says that there is a non-trivial Killing symmetry. Our result concerning the Einstein-Boltzmann-Vlasov system implies that for any analytic spacetime containing a compact null hypersurface with closed null geodesics, the distribution function $f(x^\mu, p^i)$ for particles must vanish. In other words, such spacetimes do not allow the existence of particles modeled by the Boltzmann-Vlasov theory. A more general result is proven in [31]², which says that if the future Cauchy horizon of a partial Cauchy surface is compact, and if the energy momentum tensor satisfies a slightly stronger energy

¹The definition of a product bundle is given in the glossary

²See p. 295 - p. 298.

condition than the dominant energy condition (DEC)³, then the energy momentum tensor must vanish.

This chapter includes the work completed by me under the supervision of James Isenberg. The result is in preparation for publication and co-authored by James Isenberg. I made primary contribution to the work.

II.1. Preliminaries

The spacetime we investigate has the structure $V^{(n+1)} = M^n \times \mathbb{R}$, where M^n is a closed (compact without boundary) n dimensional analytic manifold. Let g be an analytic Lorentzian metric on $V^{(n+1)}$. We assume $V^{(n+1)}$ has a compact embedded null hypersurface N which has closed null geodesics and is diffeomorphic to M^n .

Furthermore, we assume N has a local product bundle structure. That is, if γ is a closed geodesic of N , then there exists a neighborhood U_γ of γ in $V^{(n+1)}$ such that

1. $U_\gamma \cap N$ is diffeomorphic to $B_\gamma \times \mathbb{S}^1$ for some $(n - 1)$ -manifold B_γ and some diffeomorphism $\phi_\gamma : U_\gamma \cap N \rightarrow B_\gamma \times \mathbb{S}^1$.
2. There exists a smooth surjective map $\pi_\gamma : B_\gamma \times \mathbb{S}^1 \rightarrow B_\gamma$, such that for any $p \in B_\gamma$, $B_\gamma \times \mathbb{S}^1 \approx B_\gamma \times \pi_\gamma^{-1}(p)$ and the fiber $\pi_\gamma^{-1}(p)$ is diffeomorphic to a closed null geodesic in $U_\gamma \cap N$ via ϕ_γ^{-1} .

We call such submanifolds $U_\gamma \cap N$ *elementary regions* of N . By compactness, N can be covered by finitely many elementary regions. We introduce coordinates on a spacetime neighborhood of an elementary region $U_\gamma \cap N$ by the same method that is used in [20]. First introduce coordinates $\{x^a, x^n\}$, $a = 1, \dots, n - 1$, on $B_\gamma \times \mathbb{S}^1$ such that x^a are constant along the fibers and x^n is the coordinate on the

³Statement of the definition is given in the glossary

fibers with period 2π . Hence we can use $\{x^a, x^n\}$, $a = 1, \dots, n-1$ as coordinates on $U_\gamma \cap N$ via ϕ_γ . We now construct coordinates in a neighborhood $\tilde{U}_\gamma \approx \mathbf{R} \times (U_\gamma \cap N)$ of $U_\gamma \cap N$. To do this, we define a null vector field $\tilde{k}|_N$ throughout the elementary region by the algebraic conditions

$$\tilde{k}^\mu \tilde{k}^\nu g_{\mu\nu}|_{t=0} = 0, \quad \tilde{k}^\mu g_{\mu n}|_{t=0} = 1, \quad \tilde{k}^\mu g_{\mu a}|_{t=0} = 0. \quad (\text{II.1})$$

It follows that $\tilde{k}|_N$ is analytic, nowhere zero and everywhere transversal to the elementary region. Now for each point p in the elementary region, construct the unique, affine, null geodesic through p with the initial condition $(p, \tilde{k}(p))$. Define coordinates in a neighborhood of the elementary region by the requirements that the ‘‘spatial’’ coordinates $\{x^{n'}, x^{a'}\}$ are constant along the null geodesics so constructed and that the ‘‘time’’ coordinate t' vanishes on N and coincides with the affine parameter along each of the transversal null geodesics.

In such a coordinate system, which we call *Gaussian null coordinates*, the metric takes on the form

$$g = 2dt'dx^{n'} + \phi(dx^{n'})^2 + 2\beta_a dx^{a'} dx^{n'} + \mu_{ab} dx^{a'} dx^{b'}, \quad (\text{II.2})$$

where μ_{ab} is positive definite near $t' = 0$ and

$$\phi|_{t'=0} = \beta_a|_{t'=0} = 0. \quad (\text{II.3})$$

II.2. Field Equations

We present here the Ricci tensor, the Maxwell field equations, the scalar field equations and the Boltzmann-Vlasov equations in a Gaussian null coordinate

chart $\{t, x^n, x^a\}$ with primes removed. Choose the sign convention $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ and set $G = c = 1$. We have the components of the Ricci tensor in the form as follows:

$$R_{tt} = -\frac{1}{2}\mu^{ab}\mu_{ab,tt} + \frac{1}{4}\mu^{ac}\mu^{bd}\mu_{ab,t}\mu_{cd,t}, \quad (\text{II.4})$$

$$R_{tn} = \frac{1}{\sqrt{\mu}} \left[\sqrt{\mu} \left(\frac{1}{2}\phi_{,t} - \frac{1}{2}\beta^a\beta_{a,t} \right) \right]_{,t} + \frac{1}{\sqrt{\mu}} \left[\sqrt{\mu} \left(\frac{\mu^{ab}}{2}\beta_{a,t} \right) \right]_{,b} - \frac{1}{2}\mu^{ab}\mu_{ab,tn} + \frac{1}{4}\mu^{ac}\mu^{bd}\mu_{ab,t}\mu_{cd,n}, \quad (\text{II.5})$$

$$R_{tb} = \frac{1}{\sqrt{\mu}} \left[\sqrt{\mu} \left(\frac{1}{2}\beta_{b,t} - \frac{1}{2}\beta^a\mu_{ab,t} \right) \right]_{,t} + {}^{(n-1)}\nabla_c \left(\frac{1}{2}\mu^{ac}\mu_{ab,t} \right) - {}^{(n-1)}\nabla_b \left(\frac{1}{2}\mu^{cd}\mu_{cd,t} \right), \quad (\text{II.6})$$

$$R_{nn} = \frac{1}{\sqrt{\mu}} \left[\sqrt{\mu} \left(\frac{1}{2}\phi_{,n} + \frac{1}{2}\phi\phi_{,t} + \frac{1}{2}\beta^a\phi_{,a} - \beta^a\beta_{a,n} - \frac{1}{2}\beta^a\beta_a\phi_{,t} \right) \right]_{,t} + \frac{1}{\sqrt{\mu}} \left[\sqrt{\mu}\mu^{ac} \left(-\frac{\phi_{,a}}{2} + \beta_{a,n} + \frac{\beta_a}{2}\phi_{,t} \right) \right]_{,c} - \frac{1}{\sqrt{\mu}} \left[\sqrt{\mu}\frac{\phi_{,t}}{2} \right]_{,n} - \frac{1}{2}\mu^{ab}\mu_{ab,nn} + \frac{1}{4}\mu^{ac}\mu^{bd}\mu_{ab,n}\mu_{cd,n} - \frac{1}{2}(\phi_{,t} - \beta^a\beta_{a,t})^2 + \frac{1}{4}\mu^{ac}\mu^{bd}(\beta_{a,b} - \beta_{b,a})(\beta_{c,d} - \beta_{d,c}) - \frac{1}{2}\mu^{dc}\beta_{c,t}(2\phi_{,d} + \phi\beta_{d,t} - 2\beta^a(\beta_{a,d} - \beta_{d,a}) - 2\beta_{d,n} - \beta^a\beta_a\beta_{d,t}), \quad (\text{II.7})$$

$$\begin{aligned}
R_{nb} = & \frac{1}{\sqrt{\mu}} \left[\sqrt{\mu} \left(\frac{1}{2} \phi_{,b} + \frac{1}{2} \phi \beta_{b,t} - \frac{1}{2} \beta^a (\beta_{a,b} - \beta_{b,a}) - \frac{1}{2} \beta^a \beta_a \beta_{b,t} \right. \right. \\
& \left. \left. - \frac{1}{2} \beta^a \mu_{ab,n} \right) \right]_{,t} + {}^{(n-1)}\nabla_c \left[\frac{\mu^{ac}}{2} (\beta_{a,b} - \beta_{b,a}) + \frac{1}{2} \beta^c \beta_{b,t} \right. \\
& \left. + \frac{1}{2} \mu^{ac} \mu_{ab,n} \right] - \frac{1}{\sqrt{\mu}} \left[\sqrt{\mu} \frac{\beta_{b,t}}{2} \right]_{,n} - \frac{1}{2} \mu^{cd} {}^{(n-1)}\nabla_b \mu_{cd,n} \\
& - \frac{1}{2} (\beta_{b,t} - \beta^a \mu_{ab,t}) (\phi_{,t} - \beta^c \beta_{c,t}) \\
& - \frac{1}{2} \mu^{cd} \mu_{bc,t} (\phi_{,d} + \phi \beta_{d,t} - \beta^a (\beta_{a,d} - \beta_{d,a}) - \beta_{d,n} - \beta^a \beta_a \beta_{d,t}) \\
& - \frac{1}{2} \mu^{ac} \beta_{a,t} (-\mu_{bc,n} + {}^{(n-1)}\nabla_c \beta_b), \tag{II.8}
\end{aligned}$$

$$\begin{aligned}
R_{ab} = & \frac{1}{\sqrt{\mu}} \left[\sqrt{\mu} \left(-\frac{1}{2} \mu_{ab,n} + \frac{1}{2} ({}^{(n-1)}\nabla_b \beta_a + {}^{(n-1)}\nabla_a \beta_b) + \frac{\phi}{2} \mu_{ab,t} \right. \right. \\
& \left. \left. - \frac{1}{2} \beta^c \beta_c \mu_{ab,t} \right) \right]_{,t} + {}^{(n-1)}\nabla_c \left[\frac{\beta^c}{2} \mu_{ab,t} \right] - \frac{1}{\sqrt{\mu}} \left[\sqrt{\mu} \frac{\mu_{ab,t}}{2} \right]_{,n} \\
& + {}^{(n-1)}R_{ab} - \frac{1}{2} (\beta_{a,t} - \beta^c \mu_{ac,t}) (\beta_{b,t} - \beta^d \mu_{bd,t}) \\
& - \frac{1}{4} \mu^{df} \mu_{fb,t} (-2\mu_{ad,n} + 2 {}^{(n-1)}\nabla_d \beta_a + (\phi - \beta^g \beta_g) \mu_{ad,t}) \\
& - \frac{1}{4} \mu^{df} \mu_{fa,t} (-2\mu_{bd,n} + 2 {}^{(n-1)}\nabla_d \beta_b + (\phi - \beta^g \beta_g) \mu_{bd,t}). \tag{II.9}
\end{aligned}$$

where μ^{ab} is the inverse of μ_{ab} , $\beta^a = \mu^{ab} \beta_b$, ${}^{(n-1)}R_{ab}$ is the Ricci tensor of μ_{ab} and ${}^{(n-1)}\nabla_a$ represents the Riemannian covariant derivative with respect to μ_{ab} . In applying ${}^{(n-1)}\nabla_a$, one treats ϕ , $\phi_{,t}$, $\phi_{,n}$ as scalars, β_a , $\beta_{a,t}$, $\beta_{a,n}$ as covariant vectors and μ_{ab} , $\mu_{ab,t}$, $\mu_{ab,n}$ as second rank tensors respectively. Finally, $\mu = \det(\mu_{ab})$.

The Lagrangian for a scalar field is given by

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \psi_{;\mu} \psi_{;\nu} - V(\psi), \tag{II.10}$$

from which we have the scalar field equation

$$g^{\mu\nu} \psi_{;\mu\nu} - \frac{dV}{d\psi} = 0, \tag{II.11}$$

where “;” represents the covariant derivative with respect to the spacetime metric, and the stress energy tensor related quantities

$$T_{\mu\nu}^S = \psi_{,;\mu}\psi_{,;\nu} - g_{\mu\nu} \left(\frac{1}{2} g^{\rho\sigma} \psi_{,;\rho}\psi_{,;\sigma} + V(\psi) \right), \quad (\text{II.12})$$

$$T^S = \frac{1-n}{2} g^{\mu\nu} \psi_{,;\mu}\psi_{,;\nu} - (n+1)V(\psi), \quad (\text{II.13})$$

$$T_{\mu\nu}^S - \frac{1}{n-1} T^S g_{\mu\nu} = \psi_{,;\mu}\psi_{,;\nu} + \frac{2}{n-1} g_{\mu\nu} V(\psi). \quad (\text{II.14})$$

The Boltzmann-Vlasov equation for charged particles is

$$p^\mu \frac{\partial f}{\partial x^\mu} + (q F^i{}_\nu \frac{p^\nu}{m} - \Gamma^i{}_{\alpha\beta} \frac{p^\alpha p^\beta}{m^2}) \frac{\partial f}{\partial p^i} = Q(f, f), \quad (\text{II.15})$$

where $F^\mu{}_\nu$ is the Faraday tensor, $f(x^\mu, p^i)$ is the distribution function⁴ of charged particles, and $Q(f, g)$ represents the collision operator due to some short range interaction, such as the strong interaction between nuclei in the center of Sun. $F^\mu{}_\nu$ satisfies the Maxwell equations with charged fluid source

$$F_{[\mu\nu, \sigma]} = 0, \quad F^{\mu\nu}{}_{; \nu} = 4\pi J^\mu, \quad J^\mu = \frac{q}{m} \int f p^\mu \frac{\sqrt{-g} dp^1 dp^2 \cdots dp^n}{dp^0}, \quad (\text{II.16})$$

⁴In kinetic theory, the distribution function $f(x^\mu, p^i)$ represents the number density of particles with momentum p^i at the spacetime point x^μ .

where

$$F^{t\beta}{}_{;\beta} = \frac{1}{\sqrt{\mu}}(\sqrt{\mu}\beta^a F_{ta})_{,n} + \frac{1}{\sqrt{\mu}}(\sqrt{\mu}F_{nt})_{,n} + \frac{1}{\sqrt{\mu}}[\sqrt{\mu}((\beta^a\beta_a - \phi)\mu^{bc}F_{tc} + \beta^b F_{tn} + \mu^{ab}F_{na} + \beta^a\beta^b F_{at} + \mu^{bc}\beta^a F_{ca})]_{,b}, \quad (\text{II.17})$$

$$F^{n\beta}{}_{;\beta} = \frac{1}{\sqrt{\mu}}(\sqrt{\mu}\beta^a F_{at})_{,t} + \frac{1}{\sqrt{\mu}}(\sqrt{\mu}F_{tn})_{,t} + \frac{1}{\sqrt{\mu}}(\sqrt{\mu}\mu^{ab}F_{ta})_{,b}, \quad (\text{II.18})$$

$$F^{a\beta}{}_{;\beta} = \frac{1}{\sqrt{\mu}}[\sqrt{\mu}((\mu^{ac}F_{cn} + \mu^{ac}\beta^d F_{dc} + \mu^{ac}(\beta^d\beta_d - \phi)F_{ct} + \beta^a F_{nt} + \beta^a\beta^c F_{tc})]_{,t} + \frac{1}{\sqrt{\mu}}[\sqrt{\mu}\mu^{ac}F_{ct}]_{,n} + \frac{1}{\sqrt{\mu}}[\sqrt{\mu}(\mu^{ac}\mu^{bd}F_{cd} + \mu^{ac}\beta^b F_{tc} + \beta^a\mu^{bc}F_{ct})]_{,b}. \quad (\text{II.19})$$

The stress energy tensor related quantities of the electromagnetic field are given by

$$T_{\mu\nu}^{EM} = \frac{1}{4\pi}(F_{\mu\sigma}F_{\nu}{}^{\sigma} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}), \quad (\text{II.20})$$

$$T^{EM} = \frac{1}{4\pi}\frac{3-n}{4}F_{\alpha\beta}F^{\alpha\beta}, \quad (\text{II.21})$$

$$T_{\mu\nu}^{EM} - \frac{1}{n-1}T^{EM}g_{\mu\nu} = \frac{1}{4\pi}F_{\mu\sigma}F_{\nu}{}^{\sigma} - \frac{1}{8\pi}\frac{1}{n-1}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \quad (\text{II.22})$$

and the stress energy tensor related quantities of the particles are

$$T_{\mu\nu}^P = \int f p_{\mu}p_{\nu} \frac{\sqrt{-g} dp^1 dp^2 \cdots dp^n}{dp^0}, \quad (\text{II.23})$$

$$T^P = -m^2 \int f \frac{\sqrt{-g} dp^1 dp^2 \cdots dp^n}{dp^0}, \quad (\text{II.24})$$

$$T_{\mu\nu}^P - \frac{1}{n-1}T^P g_{\mu\nu} = \int f \left(p_{\mu}p_{\nu} + \frac{m^2}{n-1}g_{\mu\nu} \right) \frac{\sqrt{-g} dp^1 dp^2 \cdots dp^n}{dp^0}. \quad (\text{II.25})$$

If a scalar field, electromagnetic fields and charged particles are all present, we have

$$T_{\mu\nu} = T_{\mu\nu}^S + T_{\mu\nu}^{EM} + T_{\mu\nu}^P, \quad T = T^S + T^{EM} + T^P, \quad (\text{II.26})$$

and

$$\begin{aligned}
T_{\mu\nu} - \frac{1}{n-1}g_{\mu\nu}T &= \psi_{,\mu}\psi_{,\nu} + \frac{1}{4\pi}F_{\mu\sigma}F_{\nu}{}^{\sigma} + \int f p_{\mu}p_{\nu} \frac{\sqrt{-g} dp^1 dp^2 \cdots dp^n}{dp^0} \\
&+ \frac{1}{n-1}g_{\mu\nu} \left(2V(\psi) - \frac{1}{8\pi}F_{\alpha\beta}F^{\alpha\beta} + m^2 \int f \frac{\sqrt{-g} dp^1 dp^2 \cdots dp^n}{dp^0} \right). \tag{II.27}
\end{aligned}$$

II.3. Construction of a Killing Symmetry

We prove the main theorem in this section, which says that any analytic vacuum, scalar-vacuum and electro-vacuum spacetime of dimension $n+1 \geq 4$ containing a compact null hypersurface with closed geodesics must have a nontrivial Killing symmetry and that such a spacetime does not allow any existence of particles.

Local Arguments

Before starting our local arguments, we present the following simple version of strong elliptic maximum principle, which will be used many times during the analysis.

Lemma II.1 (Maximum Principle). *If $u(\theta)$ satisfies the following differential inequality on \mathbb{S}^1*

$$\partial_{\theta}^2 u(\theta) + \lambda(\theta)\partial_{\theta} u(\theta) \geq 0, \tag{II.28}$$

where θ is the coordinate on \mathbb{S}^1 , then $u(\theta)$ is a constant function on \mathbb{S}^1 . □

Proof. Suppose $u(\theta)$ is not constant. Then there exists a point θ_0 such that $u(\theta_0)$ is a local maximum. It follows that $\partial_{\theta} u(\theta_0) = 0$ and $\partial_{\theta}^2 u(\theta_0) < 0$. This contradicts the assumption. Thus $u(\theta)$ is a constant function on \mathbb{S}^1 . □

The Einstein equation in $n + 1$ dimensions can be written as

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{n-1}g_{\mu\nu}T). \quad (\text{II.29})$$

Restricting $R_{nn} = 8\pi(T_{nn} - \frac{1}{n-1}g_{nn}T)$ to the null hypersurface N , $t = 0$, yields

$$\begin{aligned} & - \left[(\ln \sqrt{\mu})_{,nn} + \frac{1}{2}\phi_{,t}(\ln \sqrt{\mu})_{,n} + \frac{1}{4}\mu^{ac}\mu^{bd}\mu_{ab,n}\mu_{cd,n} \right] \Big|_{t=0} \\ = & 8\pi \left[\int p_n p_n f \frac{\sqrt{-g} dp^1 dp^2 \cdots dp^n}{dp^0} + \frac{1}{4\pi}\mu^{ab}F_{an}F_{bn} + \psi_{,n}\psi_{,n} \right] \Big|_{t=0}. \end{aligned} \quad (\text{II.30})$$

Considering the above equation as an elliptic equation for $-\ln \sqrt{\mu}$ on S^1 and applying the maximum principle, we obtain $\mu_{,n}|_{t=0} = 0$. Applying this result back to the above equation, we get

$$\begin{aligned} 8\pi \left[\int p_n p_n f \frac{\sqrt{-g} dp^1 dp^2 \cdots dp^n}{dp^0} + \frac{1}{4\pi}\mu^{ab}F_{an}F_{bn} + \psi_{,n}\psi_{,n} \right] \Big|_{t=0} \\ + \frac{1}{4} [\mu^{ac}\mu^{bd}\mu_{ab,n}\mu_{cd,n}] \Big|_{t=0} = 0. \end{aligned} \quad (\text{II.31})$$

Since each term is nonnegative, all the terms must be zero. It follows that

$$\psi_{,n}|_{t=0} = 0, \quad \mu_{ab,n}|_{t=0} = 0, \quad F_{na}|_{t=0} = 0, \quad f|_{t=0} = 0. \quad (\text{II.32})$$

Using (II.32), we get from $F^{t\beta};_{\beta}|_{t=0} = 4\pi J^t|_{t=0} = 0$ the result

$$F_{nt,n}|_{t=0} = 0. \quad (\text{II.33})$$

Combining (II.32), (II.33) with $F_{[ab,n]}|_{t=0} = 0$, one obtains

$$F_{ab,n}|_{t=0} = 0. \quad (\text{II.34})$$

We now restrict $R_{nb} = 8\pi(T_{nb} - \frac{1}{n-1}g_{nb}T)$ to N , $t = 0$, and we use the above results to obtain

$$(\phi_{,tb} - \beta_{b,tn})|_{t=0} = 0. \quad (\text{II.35})$$

If we integrate (II.35) along the fiber, by periodicity, we derive

$$\frac{\partial}{\partial x^b} \int \phi_{,t}|_{t=0} dx^n = 0. \quad (\text{II.36})$$

Thus, we have

$$\int \phi_{,t}|_{t=0} dx^n = 2\pi k, \quad (\text{II.37})$$

for some constant k .

Next we construct new Gaussian null coordinates $\{t', x^{n'}, x^{a'}\}$ in a neighborhood of the elementary region by the same argument in [20] (in our case, the dimension is higher, but this does not introduce any difference), such that $\phi'_{,t'}|_{t'=0} = k$ in the elementary region without changing the results obtained so far. To begin, assume there is a new coordinate system $(x^{n'}, x^{a'})$ of the elementary region related to our old coordinates through the following transformation

$$x^{n'} = h(x^n, x^a), \quad x^{a'} = x^a. \quad (\text{II.38})$$

Construct coordinates $(t', x^{n'}, x^{a'})$ on a neighborhood of the elementary region by the same method that our old coordinates (t, x^n, x^a) on a neighborhood of the elementary region is constructed. It follows that $t = t' = 0$ on N .

Write out the transformation equations and express $\phi'_{,t'}|_{t'=0}$ in terms of $\phi_{,t}|_{t=0}$ and $h(x^n, x^a)$; we therefore derive the following Riccati equation for $\frac{\partial h}{\partial x^n}$

$$2\frac{\partial}{\partial x^n}\frac{\partial h}{\partial x^n} + \frac{\partial h}{\partial x^n}\phi_{,t}|_{t=0} - \left(\frac{\partial h}{\partial x^n}\right)^2\phi'_{,t'}|_{t'=0} = 0. \quad (\text{II.39})$$

Set $\phi'_{,t'}|_{t'=0} = k$ and $u = \frac{\partial h}{\partial x^n}$ to obtain the following equation

$$\frac{\partial u}{\partial x^n} + \frac{1}{2}(\phi_{,t}|_{t=0})u - \frac{k}{2}u^2 = 0. \quad (\text{II.40})$$

To find a solution to the above equation, we need to consider the non-degenerate case ($k \neq 0$) and the degenerate case ($k = 0$) separately. First consider the non-degenerate case. For simplicity, assume $k > 0$ (if $k < 0$, we can reduce it to the case $k > 0$ by the change of Gaussian null coordinates: $t \rightarrow -t$, $x^n \rightarrow -x^n$, $x^a \rightarrow x^a$) and define the following analytic function

$$p(x^n, x^a) = \exp\left(-\frac{1}{2}\int_0^{x^n} dy(\phi_{,t}|_{t=0})(y, x^a)\right). \quad (\text{II.41})$$

From (II.37), $p(x^n, x^a)$ has the following property

$$p(x^n + 2\pi, x^a) = \exp(-\pi k)p(x^n, x^a), \quad (\text{II.42})$$

where we consider $\phi_{,t}|_{t=0}(x^n, x^a)$ as a periodic function of x^n .

We claim that the following function

$$u(x^n, x^a) = \frac{\partial h}{\partial x^n} = -\frac{2}{k}\frac{\partial}{\partial x^n} \ln\left(\frac{\int_0^{2\pi} dy p(y, x^a)}{1 - e^{-\pi k}} - \int_0^{x^n} dy p(y, x^a)\right) \quad (\text{II.43})$$

is an analytic, nowhere vanishing solution of (II.40). To show this, note that by (II.42), the quantity

$$D(x^n, x^a) := \frac{\int_0^{2\pi} dy p(y, x^a)}{1 - e^{-\pi k}} - \int_0^{x^n} dy p(y, x^a), \quad (\text{II.44})$$

satisfies

$$D(x^n + 2\pi, x^a) = e^{\pi k} D(x^n, x^a), \quad x^n \in \mathbb{R}. \quad (\text{II.45})$$

Hence $D(x^n, x^a)$ never vanishes. It is easy to verify that (II.43) satisfies (II.40) by straightforward computation. Thus, it follows that (II.43) is an analytic, nowhere vanishing solution of (II.40). Furthermore, (II.43) is periodic in x^n and have the integral property $h(x^n + 2\pi, x^a) - h(x^n, x^a) = 2\pi$. To verify, straightforward computation using (II.42) and (II.45) gives us the results.

With the above analysis, we find that the transformation of coordinates

$$x^{n'} = g(x^a) - \frac{2}{k} \ln D(x^n, x^a), \quad x^{a'} = x^a, \quad (\text{II.46})$$

where $g(x^a)$ is an analytic function, gives us the desired new coordinates on a neighborhood of the elementary region such that $\phi'_{,t'}|_{t'=0} = k$ in the elementary region.

For the degenerate case ($k = 0$), solve (II.40) directly to obtain

$$u(x^n, x^a) = \frac{2\pi p(x^n, x^a)}{\int_0^{2\pi} dy p(y, x^a)}, \quad (\text{II.47})$$

where $p(y, x^a)$ is given by (II.41). It is easy to show that the above function is an analytic, 2π -periodic, nowhere vanishing solution of (II.40) with $k = 0$ and has the integral property that $h(x^n + 2\pi, x^a) - h(x^n, x^a) = 2\pi$. Thus, coordinate

transformations of the form

$$x^{n'} = g(x^a) + \frac{2\pi \int_0^{x^n} dy p(y, x^a)}{\int_0^{2\pi} dy p(y, x^a)}, \quad x^{a'} = x^a, \quad (\text{II.48})$$

with $g(x^a)$ an analytic function give us a new coordinate system on a neighborhood of the elementary region such that $\phi'_{,t'}|_{t'=0} = 0$ in the elementary region.

To keep notation clean, we remove the primes. With the new Gaussian null coordinates, we have

$$\phi_{,t}|_{t=0} = k, \quad \beta_{b,tn}|_{t=0} = 0, \quad (\text{II.49})$$

where k is a constant. Next, restrict the scalar field equation (II.11) to $t = 0$ and make use of the results (II.32) and (II.49) to show that

$$\left[2\psi_{,tn} + \mu^{ab}\psi_{,ab} - \phi_{,t}\psi_{,t} - 2\mu^{ab}\beta_{b,t}\psi_{,a} - \mu^{ab(n-1)}\Gamma^c{}_{ab}\psi_{,c} - \frac{dV}{d\psi} \right] \Big|_{t=0} = 0. \quad (\text{II.50})$$

Taking the derivative with respect to x^n , we have

$$(2\psi_{,tnn} - k\psi_{,tn})|_{t=0} = 0. \quad (\text{II.51})$$

Applying the maximum principle yields

$$\psi_{,tn}|_{t=0} = 0. \quad (\text{II.52})$$

Now combining $F^{a\beta}{}_{;\beta}|_{t=0} = 4\pi J^a|_{t=0} = 0$ (since $f(x^\mu, p^i) = 0$), $F_{[an,t]}|_{t=0} = 0$ and taking the derivative with respect to x^n , we have

$$(-\phi_{,t}F_{ct,n} + 2F_{ct,nn})|_{t=0} = 0. \quad (\text{II.53})$$

By the maximum principle, we find

$$F_{ct,n}|_{t=0} = 0. \quad (\text{II.54})$$

Taking the derivative of $R_{ab}|_{t=0} = 8\pi(T_{ab} - \frac{1}{n-1}g_{ab}T)|_{t=0}$ with respect to x^n gives

$$(-\mu_{ab,tnn} + \frac{k}{2}\mu_{ab,tn})|_{t=0} = 0. \quad (\text{II.55})$$

Again it follows from the maximum principle that

$$\mu_{ab,tn}|_{t=0} = 0. \quad (\text{II.56})$$

Finally, restrict the Boltzmann-Vlasov equation (II.15) to $t = 0$ to derive

$$f_{,t}|_{t=0} = 0. \quad (\text{II.57})$$

So far, we have obtained (II.32), (II.33), (II.34), (II.49), (II.52), (II.54), (II.56), (II.57). To summarize, we list these results in the following

$$\begin{aligned} \mu_{ab,n}|_{t=0} = 0, \quad \phi_{,t}|_{t=0} = k, \quad \beta_{b,tn}|_{t=0} = 0, \quad \mu_{ab,tn}|_{t=0} = 0, \\ F_{na}|_{t=0} = 0, \quad F_{nt,n}|_{t=0} = 0, \quad F_{ab,n}|_{t=0} = 0, \quad F_{ct,n}|_{t=0} = 0, \\ \psi_{,n}|_{t=0} = 0, \quad \psi_{,tn}|_{t=0} = 0, \quad f|_{t=0} = 0, \quad f_{,t}|_{t=0} = 0. \end{aligned} \quad (\text{II.58})$$

We also have

$$\phi|_{t=0} = \beta_a|_{t=0} = 0. \quad (\text{II.59})$$

Note that (II.15) and $f|_{t=0} = f_{,t}|_{t=0} = 0$ imply that f vanishes identically. Indeed, by taking the time derivative of (II.15) and restricting to $t = 0$ repeatedly, one obtains $\frac{\partial^m}{\partial t^m} f = 0$, for $m = 0, 1, 2, \dots$. Thus the spacetime we consider does not allow the existence of particles that are described by the Boltzmann-Vlasov equation.

We show that $\frac{\partial}{\partial x^n}$ is a local Killing field by showing that any time derivative of the metric coefficients and the fields is independent of x^n at $t = 0$. To do this, we proceed inductively. Assume that

$$\begin{aligned} \left(\frac{\partial^k \phi}{\partial t^k} \Big|_{t=0} \right)_{,n} &= \left(\frac{\partial^k \beta_a}{\partial t^k} \Big|_{t=0} \right)_{,n} = \left(\frac{\partial^k \mu_{ab}}{\partial t^k} \Big|_{t=0} \right)_{,n} \\ &= \left(\frac{\partial^k \psi}{\partial t^k} \Big|_{t=0} \right)_{,n} = \left(\frac{\partial^{k-1} F_{\mu\nu}}{\partial t^{k-1}} \Big|_{t=0} \right)_{,n} = 0, \quad 0 \leq k \leq m. \end{aligned} \quad (\text{II.60})$$

Differentiating $R_{tn} = 8\pi(T_{tn} - \frac{1}{n-1}g_{tn}T)$, $m - 1$ times with respect to t and restrict to $t = 0$, we get

$$\left[\frac{1}{2} \frac{\partial^{m+1}}{\partial t^{m+1}} \beta_b - \frac{1}{2} \beta^a \frac{\partial^{m+1}}{\partial t^{m+1}} \mu_{ab} + \{\text{terms independent of } x^n\} \right]_{t=0} = 0. \quad (\text{II.61})$$

Then taking the derivative with respect to x^n , we obtain

$$\left(\frac{\partial^{m+1} \beta_a}{\partial t^{m+1}} \Big|_{t=0} \right)_{,n} = 0 \quad (\text{II.62})$$

Similarly, by differentiating $R_{ta} = 8\pi(T_{ta} - \frac{1}{n-1}g_{ta}T)$, $m - 1$ times with respect to t and setting $t = 0$, we get

$$\left[\frac{1}{2} \frac{\partial^{m+1}}{\partial t^{m+1}} \phi - \frac{1}{2} \beta^a \frac{\partial^{m+1}}{\partial t^{m+1}} \beta_a + \{\text{terms independent of } x^n\} \right]_{t=0} = 0. \quad (\text{II.63})$$

And taking the derivative with respect to x^n gives us

$$\left(\frac{\partial^{m+1}\phi}{\partial t^{m+1}} \Big|_{t=0} \right)_{,n} = 0 \quad (\text{II.64})$$

Now differentiate $F_{[ab,t]} = 0$, $F_{[an,t]} = 0$ and $F^{n\beta}{}_{;\beta} = 0$, $m - 1$ times with respect to t and set $t = 0$ to show

$$\left(\frac{\partial^m F_{ab}}{\partial t^m} \Big|_{t=0} \right)_{,n} = \left(\frac{\partial^m F_{an}}{\partial t^m} \Big|_{t=0} \right)_{,n} = \left(\frac{\partial^m F_{tn}}{\partial t^m} \Big|_{t=0} \right)_{,n} = 0. \quad (\text{II.65})$$

Next, differentiating $R_{ab} = 8\pi(T_{ab} - \frac{1}{n-1}g_{ab}T)$, $F^{a\beta}{}_{;\beta} = 0$ and the scalar field equation (II.11) m times with respect to t and restricting to $t = 0$, we get

$$-\left(\frac{\partial^{m+1}\mu_{ab}}{\partial t^{m+1}} \Big|_{t=0} \right)_{,nn} + \frac{k}{2} \left(\frac{\partial^{m+1}\mu_{ab}}{\partial t^{m+1}} \Big|_{t=0} \right)_{,n} = 0, \quad (\text{II.66})$$

$$\left(\frac{\partial^m F_{at}}{\partial t^m} \Big|_{t=0} \right)_{,n} + \frac{m+1}{2} k \frac{\partial^{m+1}\mu_{ab}}{\partial t^{m+1}} \Big|_{t=0} + \{\text{terms independent of } x^n\} = 0, \quad (\text{II.67})$$

$$2 \left(\frac{\partial^{m+1}\psi}{\partial t^{m+1}} \Big|_{t=0} \right)_{,n} + (mg^{tt}{}_{,t} - g^{\mu\nu}\Gamma^t{}_{\mu\nu})|_{t=0} \frac{\partial^{m+1}\psi}{\partial t^{m+1}} \Big|_{t=0} + \{\text{terms independent of } x^n\} = 0. \quad (\text{II.68})$$

If we differentiate (II.67) and (II.68) with respect to x^n and apply the maximum principle to the resultant equations and (II.66), we derive

$$\left(\frac{\partial^{m+1}\mu_{ab}}{\partial t^{m+1}} \Big|_{t=0} \right)_{,n} = \left(\frac{\partial^m F_{at}}{\partial t^m} \Big|_{t=0} \right)_{,n} = \left(\frac{\partial^{m+1}\psi}{\partial t^{m+1}} \Big|_{t=0} \right)_{,n} = 0. \quad (\text{II.69})$$

Thus (II.60) holds for k up to $m + 1$, which completes the inductive arguments. It follows from analyticity that $\frac{\partial}{\partial x^n}$ is a local Killing field in a neighborhood of the corresponding elementary region.

Global Arguments

We have constructed a Killing vector field in a neighborhood of an elementary region. Since the null hypersurface N is compact, it can be covered by finitely many such elementary regions. For each elementary region, we can construct a Killing field in a neighborhood of the corresponding elementary region. We want to construct a Killing vector field in a neighborhood of the whole null hypersurface N . To do this, we only need to show that if two elementary regions have an overlap, then the two corresponding Killing fields coincide in a neighborhood of the overlap.

Assume that $K = \frac{\partial}{\partial x^n}$ and $\tilde{K} = \frac{\partial}{\partial \tilde{x}^n}$ are two Killing fields constructed in the neighborhoods of two elementary regions which have an overlap N_\cap in N . Suppose that $\{t, x^n, x^a\}$ and $\{\tilde{t}, \tilde{x}^n, \tilde{x}^a\}$ are the Gaussian null coordinates defined in neighborhoods of the two elementary regions. By construction, $\phi_{,t}|_{t=0} = k$ and $\tilde{\phi}_{,\tilde{t}}|_{\tilde{t}=0} = \tilde{k}$ for some constants k and \tilde{k} . Without loss of generality, assume that $\frac{\partial \tilde{x}^n}{\partial x^n}|_{N_\cap} > 0$.

Let $\gamma(\lambda)$ be an affine null geodesic in N_\cap . Then $t \circ \gamma(\lambda) = \tilde{t} \circ \gamma(\lambda) = 0$, $x^a \circ \gamma(\lambda) = \text{constant}$, $\tilde{x}^a \circ \gamma(\lambda) = \text{constant}$, and we have

$$0 = \frac{d^2 x^n}{d\lambda^2} - \frac{k}{2} \left(\frac{dx^n}{d\lambda} \right)^2 = \frac{d^2 \tilde{x}^n}{d\lambda^2} - \frac{\tilde{k}}{2} \left(\frac{d\tilde{x}^n}{d\lambda} \right)^2. \quad (\text{II.70})$$

First consider the nondegenerate case for which $k \neq 0$, $\tilde{k} \neq 0$. The solutions are

$$\begin{aligned} x^n(\lambda) &= x^n(0) - \frac{2}{k} \ln \left[1 - \frac{k}{2} \lambda \frac{dx^n(0)}{d\lambda} \right], \\ \tilde{x}^n(\lambda) &= \tilde{x}^n(0) - \frac{2}{\tilde{k}} \ln \left[1 - \frac{\tilde{k}}{2} \lambda \frac{d\tilde{x}^n(0)}{d\lambda} \right]. \end{aligned} \quad (\text{II.71})$$

The initial values satisfy

$$\frac{d\tilde{x}^n(0)}{d\lambda} = \frac{\partial \tilde{x}^n}{\partial x^n}(x^\mu(0)) \frac{dx^n(0)}{d\lambda}. \quad (\text{II.72})$$

Assume $\frac{d\tilde{x}^n(0)}{d\lambda} > 0$ and let λ_m be the value of λ after m full cycles so that

$$\begin{aligned} x^n(\lambda_m) &= x^n(0) + 2\pi m, \\ \tilde{x}^n(\lambda_m) &= \tilde{x}^n(0) + 2\pi m. \end{aligned} \quad (\text{II.73})$$

Combining (II.71) and (II.73), we have

$$\begin{aligned} 1 - \frac{k}{2} \lambda_m \frac{dx^n(0)}{d\lambda} &= e^{-m\pi k}, \\ 1 - \frac{\tilde{k}}{2} \lambda_m \frac{d\tilde{x}^n(0)}{d\lambda} &= e^{-m\pi \tilde{k}}. \end{aligned} \quad (\text{II.74})$$

From (II.72), we obtain for $n \neq 0$,

$$\frac{\partial \tilde{x}^n}{\partial x^n}(x^\mu(0)) = \frac{k(1 - e^{-m\pi \tilde{k}})}{\tilde{k}(1 - e^{-m\pi k})}. \quad (\text{II.75})$$

Since the left hand side does not depend on m and the right hand side does not depend on $x^\mu(0)$, equality holds only if $k = \tilde{k}$ and $\frac{\partial \tilde{x}^n}{\partial x^n} \Big|_{N_\Gamma} = 1$. Thus we have

$$\frac{\partial}{\partial x^n} \Big|_{N_\Gamma} = \left(\frac{\partial \tilde{x}^\alpha}{\partial x^n} \frac{\partial}{\partial \tilde{x}^\alpha} \right) \Big|_{N_\Gamma} = \frac{\partial}{\partial \tilde{x}^n} \Big|_{N_\Gamma}. \quad (\text{II.76})$$

which establishes $K|_{N_\cap} = \tilde{K}|_{N_\cap}$. An analogous argument applies in the degenerate case $\tilde{k} = k = 0$.

K and \tilde{K} are both Killing fields in a neighborhood of N_\cap and coincide on N_\cap . It follows that $X = K - \tilde{K}$ is a Killing field in a neighborhood of N_\cap which vanishes on N_\cap . By the Killing equation

$$X_{\mu,t} + X_{t,\mu} - 2^{(n+1)}\Gamma^\nu{}_{\mu t}X_\nu = 0, \quad (\text{II.77})$$

X vanishes through its domain of definition. Thus $K = \tilde{K}$. Therefore, we have proved that K is a Killing field in a neighborhood of the entire compact null hypersurface N .

Theorem II.1. *Every analytic electro-scalar-vacuum spacetime $(^{(n+1)}V, g)$ of dimension $n + 1 \geq 4$ containing a compact null hypersurface N ruled by closed null geodesic geodesics with the structure of a local product bundle has an analytic Killing field K in a neighborhood of N in $^{(n+1)}V$, and K has closed integral curves in this neighborhood. $K|_N$ is null and tangent to the geodesics of N . Furthermore such a spacetime does not allow the existence of particles that can be described by Boltzmann-Vlasov equation in this neighborhood of the null hypersurface N . \square*

II.4. Conclusion

Now we have obtained the conclusion that a very large class of spacetimes containing a cosmological Cauchy horizon must have a nontrivial Killing symmetry in a neighborhood of the cosmological Cauchy horizon. To prove the Strong Cosmic Censorship, one still need to prove a Killing symmetry for any cosmological Cauchy horizon that is generated by non-closed null geodesics.

Theorem II.1 contains the restrictive analyticity condition. In the next chapter we discuss why the analyticity condition is not desired and how it can be removed.

CHAPTER III

SYMMETRY WITHOUT ANALYTICITY

Analyticity is not a desired condition to be imposed. In mathematics, analytic functions are the most regular functions. In physical applications, some functions used by physicists are much less regular, such as the Dirac delta function. Furthermore, physical measurements always introduce some errors. The more important reason is that the unique continuation property of analytic functions is not physical. For an analytic function, the value in an open subset determines not only the value of the function in the domain of dependence, but also the value of the function in the regions that are not causally connected. Hence analytic functions should not be used to describe a spacetime. Any theorems of general relativity expecting to have some physical applications should not contain the analyticity condition.

Hawking's local rigidity theorem of black hole spacetimes is proven in a small neighborhood of the bifurcate sphere without the analyticity condition in a paper [30] by S. Alexakis, A. Ionescu and S. Klainerman. We realize that together with the results of [20], [21] and that of [29], we can actually remove the analyticity condition in the theorem of symmetries of compact Cauchy horizons with closed orbits by a simple argument. Thus, we obtain a Killing symmetry in a neighborhood of a compact Cauchy horizon with closed null geodesics without the analyticity assumption.

This chapter includes the work completed by me under the supervision of James Isenberg. The result is probably going to be published and coauthored by James Isenberg. I made primary contribution to the work.

III.1. Preliminaries

We consider a smooth vacuum spacetime (M^4, g_{ab}) . We assume M has a compact null hypersurface N_0 which has closed null geodesics. For simplicity, we assume N_0 has a local product bundle structure¹. That is, for each closed null geodesic γ of N_0 , there exists a neighborhood N_γ of γ in N_0 such that

1. N_γ is diffeomorphic to $B_\gamma \times \mathbb{S}^1$ for some 2-manifold B_γ and some diffeomorphism ϕ_γ .
2. There exists a smooth surjective map $\pi_\gamma : B_\gamma \times \mathbb{S}^1 \rightarrow B_\gamma$, such that for any $p \in B_\gamma$, $B_\gamma \times \mathbb{S}^1 \approx B_\gamma \times \pi_\gamma^{-1}(p)$ and the fiber $\pi_\gamma^{-1}(p)$ is diffeomorphic to a closed null geodesic in N_γ via ϕ_γ^{-1} .

We shall call the class of spacetimes described above *Isenberg-Moncrief spacetimes*. Choose a tubular spacetime neighborhood \mathcal{U}_γ of N_γ with topology $B_\gamma \times \mathbb{R} \times \mathbb{S}^1$. Let \mathcal{O}_γ denote the universal covering space of \mathcal{U}_γ and let $\psi_\gamma : \mathcal{O}_\gamma \rightarrow \mathcal{U}_\gamma \subset M$ be the projection map. Let $\tilde{N}_\gamma = \psi_\gamma^{-1}(N_\gamma)$. We call $(\mathcal{O}_\gamma, \psi_\gamma^* g_{ab})$ an *elementary spacetime region*.

III.2. Theorems

We present here the theorems from [20], [21], [29] and [30] that are needed for our conclusion. The first theorem needed is proved in [20]. In [29] the same theorem is stated in a slightly different way. The following theorem corresponds to Theorem 3.1 in [29].

¹For N_0 having exceptional orbits, we can consider its covering space which is of the above type. See [21, 29].

Theorem III.1 (Moncrief and Isenberg). *Let (M, g_{ab}) be a smooth Isenberg-Moncrief spacetime and let $(\mathcal{O}_\gamma, \psi_\gamma^* g_{ab})$ be an elementary spacetime region. Then there exists an Eddington-Finkelstein-type coordinate system (also called Gaussian null coordinates in [20, 29]) (u, r, x^3, x^4) covering a neighborhood \mathcal{O}'_γ of \tilde{N}_γ in \mathcal{O}_γ such that*

1) *the range of u is $(-\infty, \infty)$ whereas the coordinate range of r is $(-\epsilon, \epsilon)$ for some $\epsilon > 0$, with the surface $r = 0$ being \tilde{N}_γ .*

2) *In \mathcal{O}'_γ , the projection map $\psi_\gamma : \mathcal{O}_\gamma \rightarrow \mathcal{U}_\gamma$ is obtained by periodically identifying the coordinate u with some period $P \in \mathbb{R}$. Thus the components of $\psi_\gamma^* g_{ab}$ in these coordinates are periodic functions of u with period P .*

3) *Writing g_{ab} instead of $\psi_\gamma^* g_{ab}$ for convenience, we have*

$$g = -r \cdot f du^2 + 2drdu + 2r \cdot h_A dx^A + g_{AB} dx^A dx^B, \quad (\text{III.1})$$

where g_{AB} are smooth functions of u, r, x^3, x^4 in \mathcal{O}'_γ such that $g_{AB} dx^A dx^B$ is a Riemannian metric, and where

$$f|_{\tilde{N}_\gamma} = -2\kappa_0, \quad \kappa_0 \in \mathbb{R}. \quad (\text{III.2})$$

4) *On \tilde{N}_γ , the r -derivatives of the metric components up to any order are u -independent; i.e.*

$$\left. \frac{\partial}{\partial u} \left[\frac{\partial^n}{\partial r^n} \{f, h_A, g_{AB}\} \right] \right|_{\tilde{N}_\gamma} = 0, \quad (\text{III.3})$$

for all $n \in \mathbb{N} \cup \{0\}$. □

Remark III.1. In [20], analyticity is assumed. Given that assumption, the above theorem immediately implies that $k = \partial/\partial u$ is a Killing field in a neighborhood

of \tilde{N}_γ . Since the projection map ψ_γ is obtained by periodically identifying the coordinate u , it follows that k projects to a Killing field in a neighborhood of N_γ . □

Remark III.2. If the constant κ_0 vanishes, then the corresponding null geodesics of the null hypersurface are complete in both direction. We call the Isenberg-Moncrief spacetimes with $\kappa_0 = 0$ *degenerate*. On the other hand, if $\kappa_0 \neq 0$, then the null geodesics are incomplete in one direction and complete in the other direction. We call the Isenberg-Moncrief spacetimes with $\kappa_0 \neq 0$ *non-degenerate*. For a non-degenerate Isenberg-Moncrief spacetime, we can always assume $\kappa_0 > 0$, since otherwise, the coordinate transformation $t \rightarrow -t, x^n \rightarrow -x^n, x^a \rightarrow x^a$ gives us a positive κ_0 . In the rest of this chapter, we shall only consider non-degenerate Isenberg-Moncrief spacetimes. □

The next theorem we need corresponds to Proposition 4.1 in [29]:

Theorem III.2 (Friedrich, Rácz and Wald). *Let $(\mathcal{O}_\gamma, g_{ab}|_{\mathcal{O}_\gamma})$ be an elementary spacetime region associated with a smooth Isenberg-Moncrief spacetime with $\kappa_0 > 0$. Then, there exists an open neighborhood \mathcal{O}_γ'' of \tilde{N}_γ in \mathcal{O}_γ such that $(\mathcal{O}_\gamma'', g_{ab}|_{\mathcal{O}_\gamma''})$ can be extended to a smooth vacuum spacetime $(\mathcal{O}^*, g_{ab}^*)$ that possesses a bifurcate null hypersurface N_γ^* – i.e., N_γ^* is the union of two null hypersurfaces, $N_{\gamma 1}^*$ and $N_{\gamma 2}^*$, which intersect on a 2-dimensional spacelike surface S – such that \tilde{N}_γ corresponds to the portion of N_γ^* that lies to the future of S and $I^+[S] = \mathcal{O}_\gamma'' \cap I^+[\tilde{N}_\gamma]$. Furthermore, the expansion and shear of both $N_{\gamma 1}^*$ and $N_{\gamma 2}^*$ vanish. □*

Before presenting the last theorem, we introduce some notation. Let S be a smooth spacelike 2-manifold embedded in a smooth vacuum spacetime. Let N_1, N_2 be the null boundaries of the causal set of S (the union of the future and past sets

of S). Fix \mathcal{O} to be a small neighborhood of S such that both N_1, N_2 are regular, achronal, null hypersurfaces in \mathcal{O} spanned by null geodesics orthogonal to S . We call (S, N_1, N_2) a *local, regular, bifurcate, non-expanding horizon* in \mathcal{O} if both N_1 and N_2 are expansion free and shear free.

Now we define Kruskal-type coordinates in \mathcal{O} . Fix a smooth future-directed null pair (L_1, L_2) along S , satisfying

$$g(L_1, L_1) = g(L_2, L_2) = 0, \quad g(L_1, L_2) = -1. \quad (\text{III.4})$$

such that L_1 is tangent to N_1 and L_2 is tangent to N_2 . In a small neighborhood of S , we extend L_1 (resp. L_2) along the null geodesics of N_1 (resp. N_2) by parallel transport; i.e. $\nabla_{L_1} L_1 = 0$ (resp. $\nabla_{L_2} L_2 = 0$). We define the function U (resp. V) along N_1 (resp. N_2) by setting $U = V = 0$ on S and solving $L_1(U) = 1$ (resp. $L_2(V) = 1$). Let S_U (resp. S_V) be the level surfaces of U (resp. V) along N_1 (resp. N_2). We define L_2 at every point of N_1 (resp. L_1 at every point of N_2) as the unique, future directed null vector-field orthogonal to the surface S_U (resp. S_V) passing through that point and such that $g(L_1, L_2) = -1$. We now define the null hypersurface N_U to be the congruence of null geodesics initiating on $S_U \subset N_1$ in the direction of L_2 . Similarly we define N_V to be the congruence of null geodesics initiating on $S_V \subset N_2$ in the direction of L_1 . Both congruences are well defined in a sufficiently small neighborhood of S in \mathcal{O} , which we continue to call \mathcal{O} . The null hypersurfaces N_U (resp. N_V) are the level sets of a function U (resp. V) vanishing on N_2 (resp. N_1). By construction

$$L_1 = -g^{ab} \partial_a V \partial_b, \quad L_2 = -g^{ab} \partial_a U \partial_b. \quad (\text{III.5})$$

The following theorem corresponds to Theorem 1.1 in [30].

Theorem III.3 (Alexakis, Ionescu and Klainerman). *Given a local, regular, bifurcate, non-expanding horizon (S, N_1, N_2) in a smooth vacuum spacetime (O, g) , where S is a 2-sphere, there exist an open neighborhood $O' \subset O$ of S and a non-trivial Killing vector field K in O' , which is tangent to the null geodesics of N_1 and N_2 . In other words, every local, regular, bifurcate, non-expanding horizon is a Killing bifurcate horizon.* \square

Remark III.3. In Theorem III.3, S is assumed to be a 2-sphere. The proof is carried out by first proving the conclusion for a neighborhood $S'(x)$ of x in S and then using the compactness of S to obtain the result. We shall only need the result on a neighborhood $S'(x)$ of a given point x in S in proving our conclusion. Hence we present below a weaker version of Theorem III.3 which will be applied to prove our main result of this chapter. \square

Theorem III.4. *Assume (S, N_1, N_2) is a local, regular, bifurcate, non-expanding horizon in a smooth, vacuum spacetime (O, g) , where S is a 2-manifold of any topology. Given a point x in S , there exist a neighborhood S' of x in S , a spacetime neighborhood $O' \subset O$ of S' and a non-trivial Killing vector field K in O' , such that K is tangent to the null geodesics of N_1 and N_2 .* \square

We present here the main ideas used in constructing the Killing field K . For details, See [30].

Step 1 Define a smooth vector field K in the domain of dependence of $N_1 \cup N_2$ by solving the following characteristic initial value problem,

$$\square_g K = 0, \quad K = UL_1 - VL_2 \text{ on } (N_1 \cup N_2) \cap \mathcal{O}. \quad (\text{III.6})$$

Due to the well-posedness of the characteristic initial value problem in the domain of dependence [12], K is well-defined and smooth in the domain of dependence of $N_1 \cup N_2$ in \mathcal{O} .

Step 2 Show that $\pi_{ab} = \nabla_a K_b + \nabla_b K_a$ vanishes in the domain of dependence of $N_1 \cup N_2$ in \mathcal{O} , by using the following characteristic initial value problem in the domain of dependence

$$\square_g \pi_{ab} = 2R^c{}_{ab}{}^d \pi_{cd}, \quad (\text{III.7})$$

and by showing that π_{ab} vanishes initially on $(N_1 \cup N_2) \cap \mathcal{O}$. The Killing field K obtained has the property that $[L_2, K] = -L_2$.

Step 3 Define a vector field K' in $\mathcal{O} \setminus D(S)$ by solving the ODE

$$[L_2, K'] = -L_2, \quad K' = UL_1 \text{ on } N_1 \cap \mathcal{O}. \quad (\text{III.8})$$

Let K denote the extended vector field.

Step 4 Let $g' = \Psi_t^*(g)$ for some small t , where Ψ_t is the flow generated by K . By construction, we have $\nabla'_{L_2} L_2 = 0$ in a small neighborhood of S . Now consider Proposition 4.3 of [30], which asserts the following:

Proposition III.1. *Assume that g' is a smooth Lorentzian metric on \mathcal{O} , such that (\mathcal{O}, g') is a smooth vacuum spacetime. Assume that*

$$g' = g \text{ in } \mathcal{O} \setminus D(S) \quad \text{and} \quad \nabla'_{L_2} L_2 = 0 \text{ in } \mathcal{O}.$$

Then $g' = g$ in a small neighborhood $\mathcal{O}' \subset \mathcal{O}$ of S . □

From this proposition, it follows that K is a Killing field in a small neighborhood $\mathcal{O}' \subset \mathcal{O}$ of S .

The proof of Proposition III.1 requires a Carleman estimate obtained by two of the authors of [30] in a previous paper [32]. Since it is much more complicated, we do not present it here.

III.3. Main Result

Corollary III.1. *Every smooth non-degenerate Isenberg-Moncrief spacetime (M^4, g) with a null hypersurface N_0 has a smooth Killing field K defined on some neighborhood of N_0 in M , and K has closed integral curves in this neighborhood. Furthermore, $K|_{N_0}$ is null and thus tangent to the null geodesics of N_0 . \square*

Proof. Let \mathcal{O}_γ denote the elementary spacetime region obtained by “unwrapping” a neighborhood of N_γ . It is shown in [20] that from the non-degeneracy the geodesic γ is incomplete in one direction and complete in the other. Without loss of generality, we assume that γ is past incomplete, so consequently we have $\kappa_0 > 0$. Then from Theorem III.2, (\mathcal{O}_γ, g) can be extended to a smooth vacuum spacetime $(\mathcal{O}_\gamma^*, g_{ab}^*)$ that possesses a bifurcate null hypersurface N_γ^* , such that \tilde{N}_γ corresponds to the portion of N_γ^* that lies to the future of S and $I^+[S] = \mathcal{O}_\gamma^* \cap I^+[\tilde{N}_\gamma]$. As well the expansion and shear of both $N_{\gamma_1}^*$ and $N_{\gamma_2}^*$ vanish.

Choose an arbitrary point $x \in S$. By Theorem III.4, there exist a neighborhood S' of x in S and a smooth Killing vector field K in a small spacetime neighborhood of S' . Instead of considering the extended spacetime $(\mathcal{O}_\gamma^*, g_{ab}^*)$ associated with S , we restrict to the extended spacetime associated with S' . To keep notation easy to read, we still denote the extended spacetime associated with S' by $(\mathcal{O}_\gamma^*, g_{ab}^*)$ and remove the prime of S' . Thus there is a smooth Killing vector

field K in a small neighborhood $\{p \in \mathcal{O}_\gamma^* : |U(p)| < \epsilon, |V(p)| < \epsilon\}$ of S . By restricting to \mathcal{O}_γ , we have a Killing field K in $\{p \in \mathcal{O}_\gamma : |U(p)| < \epsilon, |V(p)| < \epsilon\} = \{p \in \mathcal{O}_\gamma : 0 < U(p) < \epsilon, |V(p)| < \epsilon\}$. Both K and $k = \partial/\partial u$ are tangent to the null geodesics of \tilde{N}_γ , so we have $K = \phi k$ on \tilde{N}_γ for some function ϕ . Notice that on \tilde{N}_γ , we have $\mathcal{L}_K g = 0$ and $\mathcal{L}_k g = 0$. It follows that $\nabla_a \phi = 0$. We may rescale u so that $K = k$ on \tilde{N}_γ . Since the construction of $k = \partial/\partial u$ is the same as that of a Killing field, it follows that $K = k$ on their common domain of definition. Thus $K = k$ is a Killing field in $\{p \in \mathcal{O}_\gamma : 0 < U(p) < \epsilon, |V(p)| < \epsilon\}$.

Near \tilde{N}_γ , K has the form

$$K = U(1 + Vf) \frac{\partial}{\partial U} - V(1 + Ug) \frac{\partial}{\partial V} + UV h^i e_i, \quad (\text{III.9})$$

where f , g and h^i , $i = \{1, 2\}$, are functions. Along each integral curve of $K = k = \partial/\partial u$ on or near \tilde{N}_γ , we have

$$\frac{dU}{du} = U(1 + Vf), \quad \frac{dV}{du} = -V(1 + Ug). \quad (\text{III.10})$$

It follows that

$$\frac{dU}{U} = (1 + Vf) du, \quad \frac{d(UV)}{UV} + (g dU + f dV) = 0. \quad (\text{III.11})$$

Thus, we have

$$u = (1 + o(\epsilon)) \ln U + u_1, \quad UV = \delta(1 + o(\epsilon)), \quad (\text{III.12})$$

where δ is a small constant. As U varies from $(\delta/\epsilon)(1 + o(\epsilon))$ to ϵ , u varies from $[1 + o(\epsilon)] \ln(\delta/\epsilon) + u_1 + o(\epsilon)$ to $[1 + o(\epsilon)] \ln \epsilon + u_1$. Thus if we choose $|\delta| < \epsilon^2 \exp(-P + o(\epsilon))$, u varies through a full period P . Since (\mathcal{O}_γ, g) is periodic in u , $k = \partial/\partial u$ is

a Killing field in a full neighborhood of \tilde{N}_γ . The projection map ψ_γ then projects k to a well-defined Killing field in full neighborhood of N_γ .

Since N_0 is compact, it can be covered by finitely many elementary spacetime regions \mathcal{U}_{γ_i} . It then remains to show the Killing fields k_{γ_i} coincide on their overlap of domain of definition, which is easy to show. \square

III.4. Conclusion

Recently, P. Yu generalized [30] to the charged case [33]. We expect that analyticity can be removed for electrovacuum spacetimes also along with the results for electrovacuum spacetimes in [29, 20]. Our argument can also be applied to stationary black hole spacetimes of class (A) in [29], which can be compactified to a cosmological spacetime of this paper according to Proposition 3.1 of [29]. Thus Hawking's local rigidity theorem can be proved to hold in spacetimes of class (A) without analyticity.

CHAPTER IV

GLOBAL STABILITY

The well-posedness of the Einstein vacuum equations, the Einstein-Maxwell equations, and other related field equation systems has been established for many years [10, 34]. Specifically, it is known that for any given set of initial data satisfying the constraint equations, there exists a unique solution to the Einstein (or Einstein-Maxwell, etc.) equations for some amount of time to the future. Well-posedness also establishes that small perturbations to an initial data set only lead to small changes to the corresponding solution in finite time, and that if those changes to the data are confined to a subset of the initial hypersurface then the changes in the solution occur strictly in the domain of dependence of that subset. However, well-posedness gives no information about global-in-time behavior of the development.

One way to formulate this issue for Einstein's and other relativistic equations is in terms of *global future causal stability* (“GFC-stability”) which addresses the following question: Given a particular set of initial data for which the maximal Cauchy development (MCD) is future causally geodesically complete, if one makes small perturbations to that initial data set, is the resulting MCD also future causally geodesically complete? Note that while the property of well-posedness generally characterizes a PDE system together with all (or none) of its solutions, GFC-stability pertains to a particular solution or family of solutions of the system. Note also that GFC-stability concerns the full nonlinear PDE system (Einstein, Einstein-scalar, etc.), not a linearization of the system, and in referring to “perturbations to the initial data set”, we mean new data sets suitably near the

original given set of initial data (which, for convenience here, we will refer to as the “background” data set).

Global stability (GFC-stability or other related varieties) has been studied extensively for Einstein’s theory, but has been established for only a very small number of solutions: The epic work of Christodoulou and Klainerman [23] (see also the later generalizations and simplifications in [25, 24]) proves the global stability of Minkowski spacetime for the vacuum and Einstein-Maxwell equations, while that of Friedrich [22] shows that the DeSitter spacetime is globally stable for Einstein’s equations with a cosmological constant. More recently, Andersson and Moncrief [26] have proven that the Milne spacetimes are globally stable solutions of the vacuum equations.

One feature of the DeSitter and the Milne solutions which makes it a bit easier to establish global stability for them is the fact that they are expanding solutions. In a rough sense, this property acts to inhibit the concentration of curvature, so that perturbations do not tend to lead to singularity forming. Hence in searching for solutions expected to be globally stable, one is led to consider expanding solutions.

Motivated both by this consideration and by the recent astrophysical evidence [35] that our universe is likely expanding at an accelerated rate, Ringström has very recently shown that certain solutions of the Einstein-scalar field equations with accelerating expansion are GFC-stable. He does this for both exponentially expanding background solutions with fairly general scalar field potential functions $V(\phi)$ [27], and for power law expanding background solutions with a certain set of exponentially-decaying scalar potential functions [28].

In this work, we show that the power law expanding solutions considered by Ringström in [28] are globally stable with respect to the Einstein-*Maxwell*-scalar field equations. By choosing the electromagnetic fields to vanish, we may consider Ringström's solutions from [28] to be solutions of the Einstein-Maxwell-scalar system. To prove stability of these background solutions in the larger PDE system of course requires us to allow the perturbation solutions to include non vanishing electromagnetic fields. This paper shows that this can be done, and that GFC-stability holds.

A key feature of background solutions with accelerated expansion is that the analysis can be strongly localized. This is because the entire future of a small subset in the initial hypersurface is determined completely by the initial data on a small neighborhood of that subset. Effectively then, the topology of the Cauchy slices of the background solutions being tested for stability and of the perturbed solutions is irrelevant.

The general structure of our proof is very similar to that of [28]: i) localizing the analysis to the development of data sets on open sets in the initial Cauchy slice, with the formal extension of such local data sets to spatial tori; ii) establishing the well-posedness of the Cauchy problem for the field perturbations relative to the background solutions (with appropriate handling of the gauge choice); iii) defining energy-type functionals for the perturbation fields and their derivatives, and (with the help of bootstrap assumptions) proving monotonicity estimates for them; iv) using the energy estimates together with bootstrap arguments to prove long time existence, regularity, and global estimates for the MCDs of the perturbed initial data; v) using the global estimates to analyze the dynamics of causal paths in the perturbed spacetimes, and thereby verifying future causal completeness. The theme

of this paper is showing that all of these steps work for the Einstein-Maxwell-scalar field equations (with exponentially-decaying scalar field potentials).

This chapter includes the work completed by me under the supervision of James Isenberg. The result is going to be published and coauthored by James Isenberg. I made primary contribution to the work.

IV.1. Introduction

Field Equations and Background Solutions

Before stating our main results and then proceeding to prove them, we wish to set up the field equations for the parametrized set of Einstein-Maxwell-scalar field theories which we work with here, and we wish to also state what the background solutions are, explicitly.

The field variables for the Einstein-Maxwell-scalar field theories include the spacetime metric g , the electromagnetic vector potential A , and the scalar field ϕ . Letting $R_{\mu\nu}$ and R denote the Ricci tensor and the scalar curvature for g , letting F denote the electromagnetic tensor for A , and choosing the scalar field potential to take the form $V(\phi) = V_0 e^{-\lambda\phi}$ (for constants V_0 and λ), we can write the field equations for this theory (for $n + 1$ dimensional spacetimes) in the following (index) form:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}, \tag{IV.1}$$

$$\nabla^\mu \nabla_\mu \phi - V'(\phi) = 0, \tag{IV.2}$$

$$\nabla^\mu F_{\mu\nu} = 0. \tag{IV.3}$$

Here the stress-energy tensor for this system is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + V(\phi)\right) + (F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}), \quad (\text{IV.4})$$

and $V'(\phi) = -\lambda V(\phi)$. Note that (IV.1) can be rewritten as

$$R_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + \frac{2}{n-1}V(\phi)g_{\mu\nu} + F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{2(n-1)}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}. \quad (\text{IV.5})$$

Note also that three parameters characterize these systems of field equations: the spatial dimension $n \geq 3$, the scalar potential scale $V_0 > 0$ and the scalar potential decay $\lambda > 0$; hence for convenience, we shall denote a particular choice of these theories by ‘‘Einstein-Maxwell-scalar $_{\{n, V_0, \lambda\}}$ ’’. Finally, note that the form of the field equations (IV.1)-(IV.4) is consistent with the assumption here that the scalar fields are not charged, and so the interaction between the electromagnetic and the scalar fields is indirect (through the gravitational fields).

We now wish to specify the background fields $(\hat{g}, \hat{\phi}, \hat{A})$, which i) are solutions of the system (IV.1)-(IV.4), ii) have accelerating expansion, and iii) (as we shall show) are GFC-stable. The fields are defined on the manifold $M^{n+1} = \mathbf{T}^n \times \mathbf{R}_+$, on which we choose the time coordinate $t > 0$ and the global periodic spatial coordinates x^i . If we now choose the constant parameters $t_0 > 0, p > 1, c_0$, and κ , we write the following:

$$\hat{g} = -dt^2 + e^{2\kappa}(t/t_0)^{2p}\delta_{ij}dx^i dx^j, \quad (\text{IV.6})$$

$$\hat{\phi} = \frac{2}{\lambda} \ln t - \frac{c_0}{\lambda}, \quad (\text{IV.7})$$

$$\hat{A}_\mu = 0. \quad (\text{IV.8})$$

These fields do not generally satisfy the field equations (IV.1)-(IV.3). However, if we require the field equation parameters $\{n, V_0, \lambda\}$ and the field parameters $\{t_0, p, c_0, \kappa\}$ to satisfy the constraining relations

$$\lambda = \frac{2}{[(n-1)p]^{1/2}}, \quad (\text{IV.9})$$

$$c_0 = \ln \left[\frac{(n-1)(np-1)p}{2V_0} \right], \quad (\text{IV.10})$$

then indeed the fields $(\hat{g}, \hat{\phi}, \hat{A})$ do constitute a solution. Note that for a fixed spatial dimension n , (IV.9) expresses a one-to-one correspondence between the solution coefficient of expansion p , and the scalar potential exponent λ . So in effect, once one fixes the three field equation parameters $\{n, V_0, \lambda\}$, there remains a two parameter family of these background solutions. Note also that these solutions are identical to those appearing in [28], with the simple addition of the condition (IV.8). For convenience, we shall denote by “ $(\hat{g}, \hat{\phi}, \hat{A})_{\{t_0, p, c_0, \kappa\}}$ ” a particular choice of the background solution to the Einstein-Maxwell-scalar $_{\{n, V_0, \lambda\}}$ field theory; in using this notation, we presume that the conditions (IV.9)-(IV.10) hold.

One of the key properties of any of the background spacetimes $(\mathbf{T}^n \times \mathbf{R}_+, \hat{g})_{\{t_0, p, c_0, \kappa\}}$ corresponding to the solutions (IV.6)-(IV.8) is the accelerated expansion they exhibit, and the somewhat peculiar causal structure which

consequently characterizes them. In particular, one finds that if one fixes a time t_0 and the corresponding Cauchy slice $\mathbf{T}_{t_0}^n$ in a background spacetime with expansion parameter p , and if for any point $q \in \mathbf{T}_{t_0}^n$ one considers a pair of coordinate balls $B_{\ell_0}(q)$ and $B_{3\ell_0}(q)$ in $\mathbf{T}_{t_0}^n$ for the characteristic length $\ell_0 := \frac{t_0}{p-1}$, then the causal future of $B_{\ell_0}(q)$ is contained in the future domain of dependence of $B_{3\ell_0}(q)$; in terms of standard notation (see, e.g., Wald [36]), one has

$$J^+[B_{\ell(t_0)}(q) \times \{t_0\}] \subseteq D^+[B_{3\ell(t_0)}(q) \times \{t_0\}]. \quad (\text{IV.11})$$

The basis for this result is the fact that, if one considers any future causal path with starting point (q, t_0) on the t_0 Cauchy surface $\mathbf{T}_{t_0}^n$, and if one calculates the projected spatial distance (relative to the induced metric) that the path can stray from q on $\mathbf{T}_{t_0}^n$, a straightforward calculation¹ (see also [28]) shows that it is bounded from above by ℓ_0 . Hence no causal path starting inside $B_{\ell_0}(q)$ can reach a spacetime point for which there are inextendible past directed paths which avoid $B_{3\ell_0}(q)$; the result follows.

Relying on this result, we can spatially localize the study of the GFC-stability of our background spacetimes, since in analyzing the future causal behavior of the spacetime evolved from perturbed data in $B_{\ell_0}(q)$, we need not consider the influence of the development of any data outside of $B_{3\ell_0}(q)$. Note that there is a small simplification of the proof of our results below if we work with an exterior ball of radius $4\ell_0$ rather than $3\ell_0$. Also, it is convenient to rescale the spatial metric in our background solutions by choosing the constant $\kappa = \ln[4\ell(t_0)]$. Doing this, we

¹It follows from the expression (IV.6) for the metric that any causal path $\gamma(s)$ satisfies the condition $-(\dot{\gamma}^t)^2 + e^{2\kappa}(\frac{t}{t_0})^{2p}\delta_{ij}\dot{\gamma}^i\dot{\gamma}^j \leq 0$, which can be rewritten as $e^{2\kappa}(\frac{t}{t_0})^{2p}\dot{\gamma}^i\dot{\gamma}^j \leq (\frac{t_0}{t})^{2p}(\dot{\gamma}^t)^2$. One then calculates the projected displacement as $\int_{s_0}^{s_1} [e^{2\kappa}(\frac{t}{t_0})^{2p}\dot{\gamma}^i\dot{\gamma}^j]^{1/2} ds \leq \int_{t_0}^{t_1} (\frac{t_0}{t})^p dt = \frac{t_0^p}{1-p} t^{1-p} \Big|_{t_0}^{t_1} \leq \ell_0$.

have the slightly simpler causal condition $J^+[B_{\frac{1}{4}}(q) \times \{t_0\}] \subseteq D^+[B_1(q)] \times \{t_0\}$, which we can exploit in stating and proving our results here.

Initial Value Formulation of the Field Equations

The statement of our results, as well as the proof, rely strongly on a formulation of the field equations (IV.1)-(IV.4) as an initial value problem. The standard $n + 1$ ADM-type initial value formulation is as follows: The initial data consist of a choice of (i) a spatial manifold Σ^n , (ii) a Riemannian metric h_{ab} and a symmetric tensor K_{cd} on Σ^n which together comprise the gravitational initial data, (iii) a pair of scalar fields φ and π on Σ^n which provide initial data for the scalar field, and (iv) a two-form B and one-form E on Σ^n which make up the electromagnetic initial data. The initial data set $(\Sigma^n, h, K, \varphi, \pi, B, E)$ satisfies the *Einstein-Maxwell-scalar constraint equations* (consisting of certain components of the field equations (IV.1)-(IV.4)) if the following hold ²

$$R - K_{ij}K^{ij} + (\text{tr}K)^2 = \pi^2 + \nabla^i \varphi \nabla_i \varphi + 2V(\varphi) + (E_j E^j + \frac{1}{2} B_{ij} B^{ij}), \quad (\text{IV.12})$$

$$\nabla^j K_{ji} - \nabla_i (\text{tr}K) = \varphi \nabla_i \varphi + E_j B^j{}_i, \quad (\text{IV.13})$$

$$\nabla_i E^i = 0, \quad (\text{IV.14})$$

Here ∇ is the Levi-Civita connection of h , R is its scalar curvature, and the indices are raised and lowered using h .

Note that if we choose one of the natural $t = \text{const.}$ Cauchy surfaces (say, $t = t_1$) of the background solution $(\hat{g}, \hat{\phi}, \hat{A})_{\{t_0, p, c_0, \kappa\}}$, then the initial data on this

²Here and throughout the paper, Latin indices run from 1 to n (space only) while Greek indices run from 0 to n (space plus time)

Cauchy surface is $\hat{h} = e^{2\kappa}(t_1/t_0)^{2p}\delta_{ij}dx^i dx^j$, $\hat{K} = -pe^{2\kappa}(t_1/t_0)^{2p-1}\delta_{ij}dx^i dx^j$, $\hat{\varphi} = \frac{2}{\lambda} \ln t_1 - \frac{c_0}{\lambda}$, $\pi = \frac{2}{\lambda} \frac{1}{t_1}$, $\hat{B} = 0$, and $\hat{E} = 0$.

Given a choice of initial data satisfying the constraint equations, one seeks a *Cauchy development* of the data, which is a set (M^{n+1}, g, ϕ, A) such that the following hold true: (a) (M^{n+1}, g) is a globally hyperbolic spacetime, with M^{n+1} diffeomorphic to $\Sigma^n \times \mathbf{R}_+$; (b) (M^{n+1}, g, ϕ, A) satisfies the Einstein-Maxwell-scalar field equations (IV.1)-(IV.4); (c) there exists an embedding $i : \Sigma^n \rightarrow M^{n+1}$ such that $i(\Sigma^n)$ is a Cauchy hypersurface for (M^{n+1}, g) , with first and second fundamental forms h and K , with $\phi \circ i = \varphi$ and $\nabla_{e_\perp} \phi \circ i = \pi$ (for e_\perp the future-directed unit normal vector field on $i(\Sigma^n)$), and with $B = i^*F$ and $E = i^*F(e_\perp, \cdot)$ for $F = dA$.

With small modifications (to generalize from the vacuum Einstein equations to the Einstein-Maxwell-scalar field equations), the well-known results of Choquet-Bruhat [10] (see also [34]) and of Choquet-Bruhat and Geroch [11] guarantee that for any smooth set of initial data satisfying the constraint equations (IV.12)-(IV.14) there exists a Cauchy development; moreover, for such data there exists a *maximal Cauchy development* (MCD) unique up to isometry, which is maximal in the sense of containment (with appropriate isometry map). The existence and uniqueness of MCDs plays a crucial role in the statement of our results, and in the proof of GFC-stability.

While it is useful to state our main theorem (below) in terms of initial data sets of the form $(\Sigma^n, h, K, \varphi, \pi, B, E)$, in carrying out the proof of our results we are led to work with modified specifications of initial data sets, which include quantities such as g_{0i} and $1 + g_{00}$. Inclusion of these quantities is closely tied with the need to control gauges in the analysis, as we see below.

Main Results

The standard idea of a stability theorem is that one fixes a solution of the field equations, noting certain important properties of the solution, one considers certain classes of perturbations of the solution, and one shows that the properties of interest remain true for the perturbed solutions. As a consequence of the localized character of the causal structure of the expanding solutions under study here, following [28] we state our main theorems here in a slightly different way (which effectively leads to slightly stronger results). We consider sets of initial data $(\Sigma^n, h, K, \varphi, \pi, B, E)$ for an Einstein-Maxwell-scalar $_{\{n, V_0, \lambda\}}$ field theory which *in local regions* are small perturbations of local initial data for one of our background solutions, and proceed to prove that the future development of the data restricted to a somewhat smaller region has the desired properties (causal geodesic completeness, etc.). In stating our results this way, they apply to solutions which may only locally be a small perturbation of one of the $(\hat{g}, \hat{\phi}, \hat{A})_{\{t_0, p, c_0, \kappa\}}$ solutions, or may be a perturbation of one of them in one region, and a different one in another region. This allows the results to hold for solutions with unrestricted topologies (unlike the background solutions, which are assumed to have the topology $\mathbf{T}^n \times \mathbf{R}_+$).

To measure the degree to which initial data for the perturbed solutions locally deviate from that of the background solutions, we need to work with a set of norms. Since the proof here depends on control of these norms via energy functionals, we are led to work with Sobolev norms; since the analysis is essentially local, we work with local Sobolev norms. In particular, for an open set $U \in \Sigma^n$ diffeomorphic to a ball in \mathbf{R}^n and therefore covered by Euclidean coordinates (x^1, \dots, x^n) , for a tensor field Ψ on Σ^n with x^j coordinate-basis components $\Psi_{j_1 \dots j_r}^{i_1 \dots i_q}$,

and for a non-negative integer m , we work with Sobolev norms defined as follows

$$\|\Psi\|_{H^m(U)} = \left(\sum_{i_1, \dots, i_q=1}^n \sum_{j_1, \dots, j_r=1}^n \sum_{|\alpha| \leq m} \int_{x(U)} |\partial^\alpha \Psi_{j_1 \dots j_r}^{i_1 \dots i_q} \circ x^{-1}|^2 dx^1 \dots dx^n \right)^{1/2}.$$

Here the collective multi-index notation “ ∂^α ” is used for the partial derivatives, all of which are calculated using the x^j coordinate basis.

In comparing (locally) a given set of initial data $(\Sigma^n, h, K, \varphi, \pi, B, E)$ for a perturbed solution with the data of a background solution, it is useful to find the “closest” background solution for the comparison. Presuming that the parameters n, V_0 , and λ have been chosen—thereby fixing the field theory and also thereby fixing (via (IV.9) and (IV.10)) p and c_0 —it remains to determine t_0 and κ . As discussed above, it is convenient to choose (as a scaling) $\kappa := \ln[4\ell(t_0)]$. Hence, one needs only to determine t_0 .

The idea for determining the appropriate choice of t_0 is based on equation (IV.7), which (for a given λ and c_0) gives the time dependence of the background scalar field $\hat{\phi}$. Roughly speaking, to determine t_0 one calculates from the given (perturbed) data a local average of the scalar field, and then setting $\phi(t_0)$ equal to this average and inverting (IV.7), one obtains t_0 . More precisely, one chooses an open set $U \in \Sigma^n$, together with a diffeomorphism $\zeta : U \rightarrow B_1(0) \in \mathbf{R}^n$. Then, one calculates $\langle \varphi \rangle := \frac{1}{\omega_n} \int_{B_1(0)} \varphi \circ \zeta^{-1} dx$, where ω_n is the volume of the unit ball in \mathbf{R}^n with respect to the Euclidean metric. Finally, one sets

$$t_0 := \exp \left[\frac{1}{2} (\lambda \langle \varphi \rangle + c_0) \right]. \quad (\text{IV.15})$$

We note that for a given set of initial data, this procedure for mapping to a comparison background solution depends only on the choice of the open set U and on the choice of the map $\zeta : U \rightarrow B_1(0)$; once that choice is made, t_0 (and therefore the comparison solution) is uniquely determined, regardless of whether the initial data is indeed close to a background solution. To simplify the discussion below, we use the notation $\Theta_{\{U,\zeta\}}(\Sigma^n, h, K, \varphi, \pi, B, E)$ to denote the map taking the indicated data set to t_0 , as defined above.

We are now ready to state our main theorem:

Theorem IV.1. *Let $(\Sigma^n, h, K, \varphi, \pi, B, E)$ be a set of initial data satisfying the constraint equations (IV.12)-(IV.14) for a fixed choice of the Einstein-Maxwell-scalar $_{\{n,V_0,\lambda\}}$ field theory. There exists an $\epsilon > 0$ (depending only on n and p) such that if for some open set $U \in \Sigma^n$ and for some diffeomorphism $\zeta : U \rightarrow B_1(0) \in \mathbf{R}^n$ the data satisfy the smallness condition*

$$\begin{aligned} & \|e^{-2\kappa}h - \delta\|_{H^{m_0+1}(U)} + \|e^{-2\kappa}t_0K - p\delta\|_{H^{m_0}(U)} \\ & + \|\varphi - \langle\varphi\rangle\|_{H^{m_0+1}(U)} + \|t_0\pi - t_0\frac{2}{\lambda}\|_{H^{m_0}(U)} \\ & + \sum_i \|t_0E_i\|_{H^{m_0}(U)} + \sum_i \|B_{ik}x^k\|_{H^{m_0+1}(U)} + \|t_0\delta^{ij}\partial_iB_{jk}x^k\|_{H^{m_0}(U)} \leq \epsilon, \end{aligned} \quad (\text{IV.16})$$

with $t_0 = \Theta_{\{U,\zeta\}}(\Sigma^n, h, K, \varphi, \pi, B, E)$, with κ chosen as above, and with m_0 the smallest integer satisfying $m_0 > n/2 + 1$; then the MCD $(\Sigma^n \times \mathbf{R}_+, g, \phi, A)$ of $(\Sigma^n, h, K, \varphi, \pi, B, E)$ has the property that if $i : \Sigma^n \rightarrow M$ labels the embedding corresponding to the initial data, then all causal geodesics starting in $i\{\zeta^{-1}[B_{1/4}(0)]\}$ are future complete. □

This theorem shows that for a data set which in a local region is sufficiently close to data for one of the background solutions, geodesic completeness holds.

If this holds for data in a neighborhood of every point in Σ^n , then clearly the entire MCD of the data set is future geodesically complete. It follows as a special case that if one chooses a Cauchy surface $\mathbf{T}_{t_0}^n$ for one of the background solutions $(\hat{g}, \hat{\phi}, \hat{A})_{\{t_0, p, c_0, \kappa\}}$, and if one considers sufficiently small Einstein-Maxwell-scalar $_{\{n, V_0, \lambda\}}$ field perturbations of the data on $\mathbf{T}_{t_0}^n$, then the MCD of that data is future geodesically complete; hence the solutions $(\hat{g}, \hat{\phi}, \hat{A})_{\{t_0, p, c_0, \kappa\}}$ are all GFC-stable.

One might ask if, in addition to the property of future geodesic completeness, the MCD of a set of perturbed data has the property that its fields in some sense approach those of the corresponding background solution. This is in fact the case, in a certain weak sense:

Theorem IV.2. *Let $(\Sigma^n, h, K, \varphi, \pi, B, E)$ be a set of initial data satisfying the constraint equations (IV.12)-(IV.14) for a fixed choice of the Einstein-Maxwell-scalar $_{\{n, V_0, \lambda\}}$ field theory, and also satisfying the ϵ -smallness condition (IV.16) from Theorem IV.1. Let (M^{n+1}, g, ϕ, A) denote the MCD of this data. There are constants $t_- \in (0, t_0)$, $a > 0$, and $\alpha_m > 0$ for all non-negative integers m , there is a smooth map $\Psi : (t_-, \infty) \times B_{5/8}(0) \rightarrow M^{n+1}$ which is a diffeomorphism onto its image and satisfies $\Psi(0, q) = i \circ \zeta^{-1}(q)$ for $q \in B_{5/8}(0)$, and there is a Riemannian metric H_{ab} on $B_{5/8}(0)$ such that the following are true:*

All causal paths that start in $i\{\zeta^{-1}[B_{1/4}(0)]\}$ remain in $\text{Imag}\{\Psi\}$ for all of the future.

Letting $\|\cdot\|_{C^m}$ denote the C^m norm on $B_{5/8}(0)$, letting (g, ϕ, A) denote the pullback of the MCD fields via Ψ , and letting $(\hat{g}, \hat{\phi}, \hat{A})$ denote the corresponding

background fields, we have, for $t \geq t_0$, the following decay estimates:

$$\|\phi(t, \cdot) - \hat{\phi}(t)\|_{C^m} + \|(t\partial_t\phi)(t, \cdot) - (t\partial_t\hat{\phi})(t)\|_{C^m} \leq \alpha_m(t/t_0)^{-a}, \quad (\text{IV.17})$$

$$\|E_i\|_{C^m} = \|\partial_i A_0 - \partial_0 A_i\|_{C^m} \leq \alpha_m e^\kappa (t/t_0)^p (t/t_0)^{-1-a}, \quad (\text{IV.18})$$

$$\|B_{ij}\|_{C^m} = \|\partial_i A_j - \partial_j A_i\|_{C^m} \leq \alpha_m e^{2\kappa} (t/t_0)^{2p} (t/t_0)^{-1-a}, \quad (\text{IV.19})$$

$$\|(1 + g_{00})(t, \cdot)\|_{C^m} + \|(t\partial_t g_{00})(t, \cdot)\|_{C^m} \leq \alpha_m (t/t_0)^{-a}, \quad (\text{IV.20})$$

$$\begin{aligned} \left\| \frac{1}{t} g_{0i}(t, \cdot) - \frac{1}{(n-2)p+1} H^{jl} \gamma_{jil} \right\|_{C^m} + \left\| t\partial_t \left(\frac{1}{t} g_{0i} \right)(t, \cdot) \right\|_{C^m} \\ \leq \alpha_m (t/t_0)^{-a}, \end{aligned} \quad (\text{IV.21})$$

$$\begin{aligned} \left\| (t/t_0)^{-2p} e^{-2\kappa} g_{ij}(t, \cdot) - H_{ij} \right\|_{C^m} + \left\| (t/t_0)^{-2p} e^{-2\kappa} (t\partial_t g_{ij})(t, \cdot) - 2pH_{ij} \right\|_{C^m} \\ \leq \alpha_m (t/t_0)^{-a}, \end{aligned} \quad (\text{IV.22})$$

$$\left\| (t/t_0)^{2p} e^{2\kappa} g^{ij}(t, \cdot) - H^{ij} \right\|_{C^m} \leq \alpha_m (t/t_0)^{-a}, \quad (\text{IV.23})$$

$$\left\| (t/t_0)^{-2p} e^{-2\kappa} t K_{ij}(t, \cdot) - pH_{ij} \right\|_{C^m} \leq \alpha_m (t/t_0)^{-a}. \quad (\text{IV.24})$$

Here γ_{jil} are the (lowered index) Christoffel symbols for the metric H on $B_{5/8}(0)$, and K_{ij} is the (evolving) second fundamental form for the hypersurface $B_{5/8}(0) \times t$.

□

We remark that while most of these inequalities clearly indicate decay, two of them appear not to do so: the second and the third, involving electromagnetic fields. We note, however, that the electromagnetic fields are vector components relative to coordinates in which the metric is expanding. If one considers locally measured fields (factoring out the expansion), then these fields do decay.

We also remark that these decay results do *not* show that the developments of the perturbed data sets decay to the original background metrics directly. Rather, one obtains decay *only* if one adds a diffeomorphism, and also adds the fiducial

metric H on $B_{5/8}$. It may be that sharper and more direct decay rates can be proven. We do not pursue this question here.

Finally, we note that while the results we prove here are generalizations of Theorem 2 in [28], we have not gone on to prove a generalization of Ringström's Theorem 3, which applies his Theorem 2 to prove global stability for a class of locally spatially homogeneous spacetimes. Such results likely could be obtained for spatially homogeneous spacetimes containing electromagnetic fields; we do not, however, consider that issue here.

Outline of the Proof

A key feature of the proofs of Theorems IV.1 and IV.2 is the spatial localizability of the analysis, noted above. This allows the analysis to be carried out independently on each open set U satisfying the hypotheses of Theorem IV.1. However in order to avoid working on regions with free boundaries, it is useful to patch the data set on U into a set of background solution data on $\mathbf{T}^n \setminus U$, and then study the development of the patched-together data on \mathbf{T}^n . The expansion behavior of the background solutions as well as their perturbations guarantee that the development of the data on sufficiently small subsets of U is independent of the externally patched-in data; hence the proof can be done via analysis on these patched data sets.

In general, the patched initial data sets violate the constraints in an annular region around U . It is thus necessary to formulate a global stability analysis that works for initial data sets which violate the constraint equations. We begin to set up such an analysis in Section IV.2.. To start, we modify the field equations by introducing gauge source functions \mathcal{D}^μ and \mathcal{G} , and using them to hyperbolize the

field equations. The gauge source function \mathcal{D}^μ , if it is zero, replaces the contracted Christoffel symbols of the perturbed unknown metric by that of the background metric we perturb around and therefore is related to the “wave coordinate gauge”, while $\mathcal{G} = 0$ corresponds to the Lorentz gauge of electromagnetism. For handling a global stability problem, hyperbolization of the field equations is not enough. We also need to add to the field equations some correction quantities that are expressed in terms of the gauge source functions. The purpose of these correction terms is to partially decouple the field equations to linear order in the field perturbed, and to insert damping terms in these equations. These features are helpful in proving stability for the modified system. Since we are interested in global stability of the original field equations, we reformulate the equations in such a way that the gauge source functions satisfy a system of homogeneous hyperbolic equations, and so that it is possible to prepare initial data for the modified equations so that the gauge source functions and their first order time derivatives vanish initially. Thus, the gauge source functions vanish identically and by proving global stability for the modified equations, we obtain global stability for our original field equations.

We express our reformulated equations as PDEs for variables which are essentially the differences between the perturbed fields and the corresponding background fields. More specifically, we work with $u := 1 + g_{00}$, $u_i := g_{0i}$, $\psi := \phi - \hat{\phi}$, A_0 , and A_i , all of which vanish for the background solution, and also $h_{ij} := (t/t_0)^{-2p} g_{ij}$ which is time independent for $g_{ij} = \hat{g}_{ij}$. We further make a change of time coordinate from t to τ , such that $t\partial_t = \partial_\tau$, by defining $\tau = \ln(t/t_0)$. The reason for doing this is to eliminate the time dependences of the linear terms. Thus we obtain a system of equations (IV.63) - (IV.68) such that each equation is in the form of a hyperbolic equation with dissipation and dispersion, plus some

extra terms. Note that in the modified system, the equations for u , ψ and A_0 are decoupled to first order from the equations for the other field variables.

In Section IV.3., we define a sequence of energy functionals for the field variables, and in Section IV.4. we specify the bootstrap assumptions which we use to prove global existence in τ . The bootstrap assumptions state that for all $\tau \in I = [0, s)$ for some unspecified s , we have solutions to the field equations, and the energy functionals are controlled by some small number ϵ . Note that the specific bootstrap assumptions we use here are not optimal; they are chosen because they are sufficient to carry out the global existence argument, and because they make the estimation of the nonlinear terms (following the algorithm for estimates introduced in [27]) applicable in our case.

One of the key tools for proving global existence is the set of differential inequalities (IV.122) - (IV.126) for the energy functionals. We derive these in Section IV.5., and we also show that as a consequence of the hierachical structure of the equations, the differential inequalities exhibit an hierachical structure as well. Relying on these inequalities, we prove global existence in Section IV.3 using a bootstrap argument. The main work needed to carry through this proof is that of improving the bootstrap assumption; this can be done as a consequence of the hierachical structure of the differential inequalities. Once the improvement has been demonstrated, one proceeds to show via an “open-closed” type argument that indeed the interval I is non empty and extends to $s = \infty$.

The remaining work in proving our two theorems is first to show that the geodesics which start from a subset of the domain $U \times t_0$ are complete (we do this in Section IV.7.), and then to verify the asymptotic expansions (i.e., the decay results) (IV.155) - (IV.162) for the fields in MCD of our initial data, as stated in Theorem

IV.2; we do this in Section IV.8.. We conclude our proof with remarks in Section IV.9..

IV.2. Field Equation Reformulation

As noted in Section IV.1., as a consequence of the accelerated expansion of the background solutions, we can carry out the analysis in spatially local regions, which for convenience are patched into data for a background solution on \mathbf{T}^n . We defer discussion of the details of the patching to Section IV.9.. For now, we presume that the patching has been done, and that we are consequently working with data on \mathbf{T}^n which satisfies the hypotheses of Theorem IV.1, but does not necessarily satisfy the constraints everywhere. The MCD of the data consists of fields defined on the spacetime manifold $\mathbf{T}^n \times I$ for some interval I . Working on $\mathbf{T}^n \times I$, and working with fields which are small perturbations of the background fields, we can always choose coordinates (x^i, t) with x^i global periodic spatial coordinates.

The aim of the reformulation of the field equations (IV.1)-(IV.4) we carry out here is to replace them by alternative equations which are manifestly hyperbolic and lead to well-posedness even if the constraints are not satisfied, and also to make sure that the reformulated equations can be used to develop a set of energy functionals which are controlled in time and lead to global existence. The replacement equations are obtained by adding terms to equations (IV.1)-(IV.4) which can be made to vanish via gauge choice. We carry out the reformulation in two steps. In the first step, we add gauge terms which result in equations which are manifestly hyperbolic for the fields $g_{\mu\nu}, \phi$ and A_μ (with components defined using the coordinate basis (x^i, t) defined above). In the second step, we add further gauge

terms and we rewrite the field equations in terms of the variables $u, u_i, \gamma_{ij}, \psi, A_0$, and A_i in a certain semi-decoupled form which is very useful for the analysis.

Reformulation I: Hyperbolization

The gauge functions we use to carry out the first stage of reformulation of the field equations (IV.1)-(IV.4) are

$$\mathcal{D}^\mu := \hat{\Gamma}^\mu - \Gamma^\mu, \quad (\text{IV.25})$$

$$\mathcal{G} := \nabla^\mu A_\mu, \quad (\text{IV.26})$$

where $\Gamma^\mu := \frac{1}{2}g^{\alpha\beta}g^{\mu\nu}(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta})$ is the contracted Christoffel symbol for the metric $g_{\alpha\beta}$, where $\hat{\Gamma}^\mu := \Gamma^\mu(\hat{g}) = \frac{np}{t}\delta_0^\mu$ is the same for the background metric \hat{g} , and where ∇ is the covariant derivative compatible with $g_{\alpha\beta}$. Note that \mathcal{D}^μ is not covariant. However, we define $\mathcal{D}_\nu := g_{\nu\mu}\mathcal{D}^\mu$, and also $\nabla_\mu \mathcal{D}_\nu := \partial_\mu \mathcal{D}_\nu - \Gamma_\mu{}^\gamma{}_\nu \mathcal{D}_\gamma$.

Using these gauge quantities, we define modified versions of the Ricci and Faraday tensors

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + \nabla_{(\mu} \mathcal{D}_{\nu)}, \quad (\text{IV.27})$$

$$\tilde{F}_{\mu\nu} = F_{\mu\nu} + g_{\mu\nu}(\mathcal{G} - \mathcal{D}_\gamma A^\gamma), \quad (\text{IV.28})$$

and we also define certain arrays of correction terms

$$M_{\mu\nu}^{[g]} := \frac{2p}{t} \begin{bmatrix} -\mathcal{D}^0 & \mathcal{D}_i \\ \mathcal{D}_i & 0 \end{bmatrix}, \quad (\text{IV.29})$$

$$M^{[\phi]} := -g^{\mu\nu} \mathcal{D}_\mu \partial_\nu \phi, \quad (\text{IV.30})$$

$$M_\nu^{[A]} := -g^{\alpha\beta} \mathcal{D}_\alpha F_{\beta\nu} + \frac{2p}{t} g_{0\nu} (\mathcal{D}_\gamma A^\gamma - \mathcal{G}). \quad (\text{IV.31})$$

We then construct the following set of “gauge-modified” Einstein-Maxwell-scalar $_{\{n, V_0, \lambda\}}$ field equations

$$\partial_\mu \phi \partial_\nu \phi + \frac{2}{n-1} V(\phi) g_{\mu\nu} + F_{\mu\sigma} F_\nu{}^\sigma - \frac{1}{2(n-1)} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} = \tilde{R}_{\mu\nu} + M_{\mu\nu}^{[g]}, \quad (\text{IV.32})$$

$$g^{\alpha\beta} \partial_\alpha \partial_\beta \phi - \Gamma^\mu \partial_\mu \phi - V'(\phi) + M^{[\phi]} = 0, \quad (\text{IV.33})$$

$$\nabla^\mu \tilde{F}_{\mu\nu} + M_\nu^{[A]} = 0, \quad (\text{IV.34})$$

noting the following properties: (i) If the gauge quantities \mathcal{D}^μ and \mathcal{G} vanish, then this system (IV.32)-(IV.34) is satisfied if and only if the Einstein-Maxwell-scalar $_{\{n, V_0, \lambda\}}$ system (IV.1)-(IV.4) is satisfied. (ii) The gauge modified system (IV.32)-(IV.34) is manifestly (second order) hyperbolic for the fields (g, ϕ, A) , so long as the gauge quantities vanish.

In view of these two properties of the gauge-modified system, our goal now is to show that for any given set of initial data $(\mathbf{T}^n, \tilde{h}, \tilde{K}, \tilde{\varphi}, \tilde{\pi}, \tilde{B}, \tilde{E})$ for the Einstein-Maxwell-scalar $_{\{n, V_0, \lambda\}}$ system (IV.1)-(IV.4), there is a corresponding set of initial data $(g, \partial_t g, \varphi, \partial_t \varphi, A, \partial_t A)$ for (IV.32)-(IV.34) which enforces the condition that the gauge quantities \mathcal{D}^μ and \mathcal{G} vanish for as long as the solution exists. We show in the next subsection that this can be done.

Initial Data and Gauge Choice

There are many choices of the data $(g, \partial_t g, \varphi, \partial_t \varphi, A, \partial_t A)$ which are consistent with a given set of initial data $(\tilde{h}, \tilde{K}, \tilde{\varphi}, \tilde{\pi}, \tilde{B}, \tilde{E})$ specified on \mathbf{T}^n ; this is a manifestation of the gauge freedom in the Einstein-Maxwell-scalar field system. The following choice restricts this freedom, and in so doing, it enforces the condition that the gauge functions \mathcal{D}^μ and \mathcal{G} vanish at the initial time³ t_0 :

$$g_{ij}(t_0, \cdot) = \tilde{h}(\partial_i, \partial_j), \quad g_{00}(t_0, \cdot) = -1, \quad g_{0j}(t_0, \cdot) = 0, \quad (\text{IV.35})$$

$$\partial_t g_{ij}(t_0, \cdot) = 2\tilde{K}(\partial_i, \partial_j), \quad (\text{IV.36})$$

$$\partial_t g_{00}(t_0, \cdot) = 2\hat{\Gamma}^0(t_0, \cdot) - 2 \operatorname{tr} \tilde{K}, \quad (\text{IV.37})$$

$$\partial_t g_{0l}(t_0, \cdot) = \frac{1}{2} \tilde{h}^{ij} (2\partial_i \tilde{h}_{jl} - \partial_l \tilde{h}_{ij})(t_0, \cdot), \quad (\text{IV.38})$$

$$\phi(t_0, \cdot) = \tilde{\varphi}, \quad \partial_t \phi(t_0, \cdot) = \tilde{\pi}, \quad (\text{IV.39})$$

$$A_0(t_0, \cdot) = 0, \quad A_i(t_0, \cdot) = -\frac{1}{2} \tilde{B}_{ik} x^k, \quad (\text{IV.40})$$

$$\partial_t A_0(t_0, \cdot) = \tilde{h}^{ij} \partial_i A_j(t_0, \cdot), \quad \partial_t A_i(t_0, \cdot) = -\tilde{E}_i. \quad (\text{IV.41})$$

The vanishing of the first time derivatives of \mathcal{D}^μ and \mathcal{G} on (portions of) the initial surface follows not from further restrictions on the gauge choice, but rather from the constraint equations (IV.12)-(IV.14). The constraints are not satisfied everywhere (recall the consequences of the patching of the initial data) but they do hold on a subset S of the initial hypersurface. To see that $\partial_t \mathcal{D}^\mu$ and $\partial_t \mathcal{G}$ vanish on S , we first calculate the quantity $G_{\mu\nu} - T_{\mu\nu}$, *assuming that the gauge-modified field*

³Note that $\mathcal{D}_0 = 0$ corresponds to (IV.37), $\mathcal{D}_i = 0$ corresponds to (IV.38), and $\mathcal{G} = 0$ corresponds to the first part of (IV.41).

equations hold, rather than the original system (IV.1)-(IV.4); we obtain

$$G_{\mu\nu} - T_{\mu\nu} = -\nabla_{(\mu}\mathcal{D}_{\nu)} + \frac{1}{2}(\nabla^\rho\mathcal{D}_\rho)g_{\mu\nu} - M_{\mu\nu}^{[g]} + \frac{1}{2}(g^{\alpha\beta}M_{\alpha\beta}^{[g]})g_{\mu\nu}. \quad (\text{IV.42})$$

If we now let e_\perp be the unit normal to the initial surface and let X be any vector tangent to the surface, and if we contract (IV.42) with e_\perp and X while assuming that the constraints (IV.12)-(IV.14) hold on S , then we obtain

$$-\frac{1}{2}(e_\perp)^\mu X^\nu (\partial_\mu\mathcal{D}_\nu + \partial_\nu\mathcal{D}_\mu) = 0. \quad (\text{IV.43})$$

Setting $t = t_0$, and noting that since $\mathcal{D}_i(t_0, \cdot) = 0$ we must have $X^\nu\partial_\nu\mathcal{D}_\mu(t_0, \cdot) = 0$, it follows that

$$\partial_t\mathcal{D}_i(t_0, \cdot) = 0 \text{ on } S \subseteq \mathbf{T}^n. \quad (\text{IV.44})$$

Arguing similarly, but now contracting (IV.42) twice with e_\perp while assuming that the constraints hold at t_0 , we obtain

$$\partial_t\mathcal{D}_0(t_0, \cdot) = 0 \text{ on } S \subseteq \mathbf{T}^n. \quad (\text{IV.45})$$

Finally, if we contract (IV.34) with e_\perp , then (setting $t = t_0$) we may use the constraint (IV.14) to argue that $e_\perp^\nu\nabla^\mu F_{\mu\nu}(t_0, \cdot) = 0$; combining this with the vanishing of $\mathcal{D}_\mu(t_0, \cdot)$, $\mathcal{G}(t_0, \cdot)$ and $\partial_t\mathcal{D}_\mu(t_0, \cdot)$ (as shown above), we are left with

$$\partial_t\mathcal{G}(t_0, \cdot) = 0 \text{ on } S \subseteq \mathbf{T}^n. \quad (\text{IV.46})$$

Now that we have determined that for any choice of initial data we may choose a gauge so that \mathcal{G} , \mathcal{D}_μ , and their first time derivatives vanish on the subset

S of the initial slice, if we wish to show that \mathcal{G} and \mathcal{D}_μ vanish in a spacetime neighborhood of S (within its domain of dependence) it is sufficient to show that these quantities satisfy homogeneous wave equations. To do this, we first take the divergence of equation (IV.42), noting that

$$\nabla^\mu T_{\mu\nu} = -M^{[\phi]}\partial_\nu\phi - F_\nu{}^\sigma(M_\sigma^{[A]} + \partial_\sigma(\mathcal{G} - \mathcal{D}_\gamma A^\gamma)). \quad (\text{IV.47})$$

It follows that, presuming the gauge-modified equations have smooth solutions, we obtain

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\mathcal{D}_\mu + Q_\mu{}^{\alpha\beta}\partial_\alpha\mathcal{D}_\beta + S_\mu{}^\nu\partial_\nu\mathcal{G} + V_\mu{}^\nu\mathcal{D}_\nu + W_\mu\mathcal{G} = 0 \quad (\text{IV.48})$$

for smooth functions $Q_\mu{}^{\alpha\beta}$, $S_\mu{}^\nu$, $V_\mu{}^\nu$ and W_μ . Similarly, taking the divergence of (IV.34), we obtain

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\mathcal{G} + H^\mu\partial_\mu\mathcal{G} + I^{\alpha\beta}\partial_\alpha\mathcal{D}_\beta + J\mathcal{G} + L^\mu\mathcal{D}_\mu = 0. \quad (\text{IV.49})$$

for smooth functions H^μ , $I^{\alpha\beta}$, J and L^μ . This pair of equations together constitute the desired homogeneous hyperbolic system for \mathcal{G} and \mathcal{D}_μ .

Reformulation II: Perturbation Variables and First Order Semi-Decoupling

The system of PDEs (IV.32)-(IV.34) for the field variables $(g_{\mu\nu}, A_\mu, \phi)$ together with the system (IV.48)-(IV.49) for the gauge quantities $(\mathcal{D}_\mu, \mathcal{G})$ constitute a coupled PDE system which could be used to argue local existence of solutions. To be able to show long time existence, however, it is advantageous to replace the variables $(g_{\mu\nu}, A_\mu, \phi)$ by others which are closely related to perturbations of the

background solution; following [28], we choose to work with $u := 1 + g_{00}$, $u_i := g_{0i}$, $h_{ij} := (t/t_0)^{-2p}g_{ij}$, $\psi := \phi - \hat{\phi}$, A_0 , and A_i .

In deriving (from (IV.32)-(IV.34)) the evolution PDEs for these new variables, we wish to segregate those terms which are linear in perturbations of the background solution (i.e., linear in $u := 1 + g_{00}$, $u_i := g_{0i}$, $h_{ij} := (t/t_0)^{-2p}g_{ij}$, $\psi := \phi - \hat{\phi}$, A_0 , and A_i) from those which are higher order. Doing this, we obtain the following

$$\begin{aligned} -g^{\mu\nu}\partial_\mu\partial_\nu u + \frac{(n+2)p}{t}\partial_t u + \frac{2p[n(p-1)+1]}{t^2}u \\ - \frac{8}{\lambda t}\partial_t\psi - \frac{2\lambda p(np-1)}{t^2}\psi + \Delta'_{00} = 0, \end{aligned} \quad (\text{IV.50})$$

$$\begin{aligned} -g^{\mu\nu}\partial_\mu\partial_\nu u_i + \frac{np}{t}\partial_t u_i + \frac{p(n-2)(2p-1)}{t^2}u_i \\ - \frac{4}{\lambda t}\partial_i\psi - \frac{2p}{t}g^{lm}\Gamma_{lim} + \Delta'_{0i} = 0, \end{aligned} \quad (\text{IV.51})$$

$$-g^{\mu\nu}\partial_\mu\partial_\nu h_{ij} + \frac{np}{t}\partial_t h_{ij} + \frac{2p}{t^2}(\lambda(np-1)\psi - u)h_{ij} + \Delta'_{ij} = 0, \quad (\text{IV.52})$$

$$-g^{\mu\nu}\partial_\mu\partial_\nu\psi + \frac{np}{t}\partial_t\psi + \frac{2(np-1)}{t^2}\psi - \frac{2}{\lambda t^2}u + \Delta'_\psi = 0, \quad (\text{IV.53})$$

$$-g^{\mu\nu}\partial_\mu\partial_\nu A_0 + \frac{(n+2)p}{t}\partial_t A_0 + \frac{np(2p-1)}{t^2}A_0 + \Delta'_e = 0, \quad (\text{IV.54})$$

$$-g^{\mu\nu}\partial_\mu\partial_\nu A_i + \frac{(n-2)p}{t}\partial_t A_i + \frac{2p}{t}\partial_i A_0 + \Delta'_{b,i} = 0. \quad (\text{IV.55})$$

Here, the quantities $\Delta'_{\mu\nu}$, Δ'_ψ , Δ'_e and $\Delta'_{b,i}$ are all of quadratic order or higher in the perturbations variables $(u, u_i, h_{ij}, \psi, A_0, A_i)$, and their first derivatives⁴. More

⁴We include the primes on these Δ' quantities here because below, we replace them by slightly changed Δ quantities without primes.

specifically, one can write

$$\Delta'_{\mu\nu} = \tilde{\Delta}_{\mu\nu} - 2 \left[F_{\mu\sigma} F_{\nu}{}^{\sigma} - \frac{1}{2(n-1)} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right], \quad (\text{IV.56})$$

$$\Delta'_{\psi} = \tilde{\Delta}_{\psi}, \quad (\text{IV.57})$$

$$\Delta'_e = 2 \left[(g^{\mu\rho} \Gamma_{0\ \mu}{}^{\sigma} - \frac{p}{t} g^{\rho\sigma}) \partial_{\rho} A_{\sigma} - \frac{p}{t} \partial_t A_0 - \frac{p}{t} u (\hat{\Gamma}^{\mu} A_{\mu} - g^{\rho\sigma} \partial_{\rho} A_{\sigma}) \right], \quad (\text{IV.58})$$

$$\Delta'_{b,i} = 2 \left[\frac{p}{t} F_{0i} + g^{\rho\sigma} \Gamma_i{}^{\mu}{}_{\rho} \partial_{\sigma} A_{\mu} - \frac{p}{t} g_{0i} (\hat{\Gamma}^{\mu} A_{\mu} - g^{\rho\sigma} \partial_{\rho} A_{\sigma}) \right], \quad (\text{IV.59})$$

where $\tilde{\Delta}_{\mu\nu}$ and $\tilde{\Delta}_{\psi}$ are written out explicitly in the work of Ringström; see equations (51), (52), (55) and (56) in [28], together with equations (82)-(87) and (92)-(93) in [27].

We note that the parameters n, p , and λ appear in (IV.50)-(IV.55) because the new variables $(u, u_i, h_{ij}, \psi, A_0, A_i)$ are defined in terms of the background solution $(\hat{g}, \hat{\phi}, \hat{A})_{\{t_0, p, c_0, \kappa\}}$. We also note that the quantity $g^{\mu\nu}$ as well as other metric and Christoffel quantities appearing in (IV.50)-(IV.55) may be viewed as functions of u, u_i , and h_{ij} .

The calculations leading from the PDEs (IV.32)-(IV.34) to (IV.50)-(IV.53) are essentially the same as those done in proving Lemma 3 in [28]. To derive (IV.54)

and (IV.55), we calculate as follows:

$$\begin{aligned}
0 &= \nabla^\mu \tilde{F}_{\mu\alpha} + M_\alpha^{[A]} \\
&= g^{\rho\sigma} (\partial_\rho F_{\sigma\alpha} - \Gamma_\sigma{}^\gamma{}_\rho F_{\gamma\alpha} - \Gamma_\alpha{}^\gamma{}_\rho F_{\sigma\gamma}) + \partial_\alpha (\nabla^\mu A_\mu) \\
&\quad - \mathcal{D}^\mu \partial_\mu A_\alpha - A_\mu \partial_\alpha \mathcal{D}^\mu + \frac{2p}{t} g_{0\alpha} (\hat{\Gamma}^\mu A_\mu - g^{\rho\sigma} \partial_\rho A_\sigma) \\
&= g^{\rho\sigma} \partial_\rho \partial_\sigma A_\alpha - g^{\rho\sigma} \partial_\alpha \partial_\rho A_\sigma - \Gamma^\mu (\partial_\mu A_\alpha - \partial_\alpha A_\mu) \\
&\quad - g^{\rho\sigma} \Gamma_\alpha{}^\mu{}_\rho \partial_\sigma A_\mu + g^{\rho\sigma} \Gamma_\alpha{}^\mu{}_\rho \partial_\mu A_\sigma + \partial_\alpha (g^{\rho\sigma} \partial_\rho A_\sigma) \\
&\quad - \partial_\alpha (\Gamma^\mu A_\mu) - \mathcal{D}^\mu \partial_\mu A_\alpha - A_\mu \partial_\alpha \mathcal{D}^\mu + \frac{2p}{t} g_{0\alpha} (\hat{\Gamma}^\mu A_\mu - g^{\rho\sigma} \partial_\rho A_\sigma) \\
&= g^{\rho\sigma} \partial_\rho \partial_\sigma A_\alpha - \hat{\Gamma}^\mu \partial_\mu A_\alpha - A_\mu \partial_\alpha \hat{\Gamma}^\mu - g^{\rho\sigma} \Gamma_\alpha{}^\mu{}_\rho \partial_\sigma A_\mu \\
&\quad + g^{\rho\sigma} \Gamma_\alpha{}^\mu{}_\rho \partial_\mu A_\sigma + (\partial_\alpha g^{\rho\sigma}) \partial_\rho A_\sigma + \frac{2p}{t} g_{0\alpha} (\hat{\Gamma}^\mu A_\mu - g^{\rho\sigma} \partial_\rho A_\sigma) \\
&= g^{\rho\sigma} \partial_\rho \partial_\sigma A_\alpha - \hat{\Gamma}^\mu \partial_\mu A_\alpha - A_\mu \partial_\alpha \hat{\Gamma}^\mu - 2g^{\rho\sigma} \Gamma_\alpha{}^\mu{}_\rho \partial_\sigma A_\mu \\
&\quad + \frac{2p}{t} g_{0\alpha} (\hat{\Gamma}^\mu A_\mu - g^{\rho\sigma} \partial_\rho A_\sigma) \tag{IV.60}
\end{aligned}$$

For $\alpha = 0$, we obtain

$$\begin{aligned}
0 &= \nabla^\mu \tilde{F}_{\mu 0} + M_0^{[A]} \\
&= g^{\rho\sigma} \partial_\rho \partial_\sigma A_0 - \hat{\Gamma}^\mu \partial_\mu A_0 - A_\mu \partial_0 \hat{\Gamma}^\mu - 2g^{\rho\sigma} \Gamma_0{}^\mu{}_\rho \partial_\sigma A_\mu \\
&\quad + \frac{2p}{t} g_{00} (\hat{\Gamma}^\mu A_\mu - g^{\rho\sigma} \partial_\rho A_\sigma) \\
&= g^{\rho\sigma} \partial_\rho \partial_\sigma A_0 - \frac{np}{t} \partial_t A_0 + \frac{np}{t^2} A_0 - 2g^{\rho\sigma} \Gamma_0{}^\mu{}_\rho \partial_\sigma A_\mu \\
&\quad - \frac{2p}{t} \hat{\Gamma}^\mu A_\mu + \frac{2p}{t} g^{\rho\sigma} \partial_\rho A_\sigma + \frac{2p}{t} (1 + g_{00}) (\hat{\Gamma}^\mu A_\mu - g^{\rho\sigma} \partial_\rho A_\sigma) \\
&= g^{\rho\sigma} \partial_\rho \partial_\sigma A_0 - \frac{np}{t} \partial_t A_0 - \frac{np(2p-1)}{t^2} A_0 \\
&\quad + \left[-2g^{\rho\sigma} \Gamma_0{}^\mu{}_\rho \partial_\sigma A_\mu + \frac{2p}{t} g^{\rho\sigma} \partial_\rho A_\sigma + \frac{2p}{t} u (\hat{\Gamma}^\mu A_\mu - g^{\rho\sigma} \partial_\rho A_\sigma) \right] \\
&= g^{\rho\sigma} \partial_\rho \partial_\sigma A_0 - \frac{(n+2)p}{t} \partial_t A_0 - \frac{np(2p-1)}{t^2} A_0 \\
&\quad + 2 \left[\left(\frac{p}{t} g^{\rho\sigma} - g^{\mu\rho} \Gamma_0{}^\sigma{}_\mu \right) \partial_\rho A_\sigma + \frac{p}{t} u (\hat{\Gamma}^\mu A_\mu - g^{\rho\sigma} \partial_\rho A_\sigma) + \frac{p}{t} \partial_t A_0 \right] \tag{IV.61}
\end{aligned}$$

For $\alpha = i$, we obtain

$$\begin{aligned}
0 &= \nabla^\mu \tilde{F}_{\mu i} + M_i^{[A]} \\
&= g^{\rho\sigma} \partial_\rho \partial_\sigma A_i - \hat{\Gamma}^\mu \partial_\mu A_i - A_\mu \partial_i \hat{\Gamma}^\mu - 2g^{\rho\sigma} \Gamma_i{}^\mu{}_\rho \partial_\sigma A_\mu \\
&\quad + \frac{2p}{t} g_{0i} (\hat{\Gamma}^\mu A_\mu - g^{\rho\sigma} \partial_\rho A_\sigma) \\
&= g^{\rho\sigma} \partial_\rho \partial_\sigma A_i - \frac{(n-2)p}{t} \partial_t A_i - \frac{2p}{t} \partial_i A_0 \\
&\quad + 2 \left[-\frac{p}{t} F_{0i} + \frac{p}{t} g_{0i} (\hat{\Gamma}^\mu A_\mu - g^{\rho\sigma} \partial_\rho A_\sigma) - g^{\rho\sigma} \Gamma_i{}^\mu{}_\rho \partial_\sigma A_\mu \right]. \tag{IV.62}
\end{aligned}$$

The evolution PDEs (IV.50)-(IV.55) involve a number of factors of t^{-1} . These can be conveniently removed by multiplying all of the equations (IV.50)-(IV.55) by t^2 , and by replacing t by $\tau := \ln(t/t_0)$ (so that consequently $\partial_\tau = t\partial_t$); we then obtain the following system of PDEs, to be solved for $(u(x, \tau) = 1 + g_{00}(x, t_0 e^\tau))$,

$u_i(x, \tau) = \text{etc...}$):

$$\begin{aligned} \tilde{\square}_g u + [(n+2)p-1]\partial_\tau u + 2p[n(p-1)+1]u \\ - \frac{8}{\lambda}\partial_\tau \psi - 2\lambda p(np-1)\psi + \Delta_{00} = 0, \end{aligned} \quad (\text{IV.63})$$

$$\begin{aligned} \tilde{\square}_g u_i + (np-1)\partial_\tau u_i + p(n-2)(2p-1)u_i \\ - e^{\tau+\tau_0}\left[\frac{4}{\lambda}\partial_i \psi + 2pg^{lm}\Gamma_{lim}\right] + \Delta_{0i} = 0, \end{aligned} \quad (\text{IV.64})$$

$$\tilde{\square}_g h_{ij} + (np-1)\partial_\tau h_{ij} + 2p(\lambda(np-1)\psi - u)h_{ij} + \Delta_{ij} = 0, \quad (\text{IV.65})$$

$$\tilde{\square}_g \psi + (np-1)\partial_\tau \psi + 2(np-1)\psi - \frac{2}{\lambda}u + \Delta_\psi = 0, \quad (\text{IV.66})$$

$$\tilde{\square}_g A_0 + [(n+2)p-1]\partial_\tau A_0 + np(2p-1)A_0 + \Delta_e = 0, \quad (\text{IV.67})$$

$$\tilde{\square}_g A_i + [(n-2)p-1]\partial_\tau A_i + 2pe^{\tau+\tau_0}\partial_i A_0 + \Delta_{b,i} = 0. \quad (\text{IV.68})$$

Here we define the hyperbolic operator $\tilde{\square}_g$ via

$$\tilde{\square}_g := -g^{00}\partial_\tau^2 - 2e^{\tau+\tau_0}g^{0i}\partial_\tau\partial_i - e^{2(\tau+\tau_0)}g^{ij}\partial_i\partial_j,$$

and we define the nonlinear remainder terms $\Delta_{\mu\nu}$, Δ_ψ , Δ_e and $\Delta_{b,i}$ via

$$\Delta_{00} := (1+g^{00})\partial_\tau u + e^{2(\tau+\tau_0)}\Delta'_{00}, \quad (\text{IV.69})$$

$$\Delta_{0i} := (1+g^{00})\partial_\tau u_i + e^{2(\tau+\tau_0)}\Delta'_{0i}, \quad (\text{IV.70})$$

$$\Delta_{ij} := (1+g^{00})\partial_\tau h_{ij} + e^{2(\tau+\tau_0)}\Delta'_{ij}, \quad (\text{IV.71})$$

$$\Delta_\psi := (1+g^{00})\partial_\tau \psi + e^{2(\tau+\tau_0)}\Delta'_\psi, \quad (\text{IV.72})$$

$$\Delta_e := (1+g^{00})\partial_\tau A_0 + e^{2(\tau+\tau_0)}\Delta'_e, \quad (\text{IV.73})$$

$$\Delta_{b,i} := (1+g^{00})\partial_\tau A_i + e^{2(\tau+\tau_0)}\Delta'_{b,i}. \quad (\text{IV.74})$$

Our stability analysis in the rest of the paper focuses on the evolution PDEs (IV.63)-(IV.68). We note that, *if we ignore the Δ terms in these equations and also ignore (for the moment) the dependence of the wave operator $\tilde{\square}_g$ on the metric*, then we have the following semi-decoupled setup: (i) Equation (IV.67) involves A_0 alone. (ii) Equation (IV.68) involves only A_i and A_0 . (iii) Equations (IV.63) and (IV.66) together form a coupled system for u and ψ , independent of the other variables. (iv) Equation (IV.65) involves only h_{ij} and u and ψ . (v) Equation (IV.64) involves u_i and u and ψ and h_{ij} , but not the electromagnetic variables. This semi-decoupled structure plays an important role in the analysis we carry out below.

IV.3. Energy Functionals

The key tool for proving global existence for solutions to a Cauchy problem for a hyperbolic PDE system is the set of energy functionals for the system. For a general (nonlinear) system, these functionals are neither canonically determined nor unique. However they may be obtained, the necessary properties are (i) that (perhaps assuming certain a priori conditions on the field) their future evolution is bounded, and (ii) that they control appropriate norms of the field variables. In this section, we obtain energy functionals for the PDE system (IV.63)-(IV.68).

The field equations (IV.63)-(IV.68) all involve the differential operator $\tilde{\square}_g$, which of course involves the metric g . By definition of the field variables u , u_i , and h_{ij} , the metric g is closely tied to them, and consequently g evolves with them. This fact (a key feature of Einstein's gravitational field equations) must be taken into account in setting up the energy functionals and verifying their evolution properties.

However, we start our discussion of the energy functionals by artificially decoupling the metric as it appears in $\tilde{\square}_g$ and elsewhere in coefficients of (IV.63)-(IV.68) from the evolving fields u, u_i , and h_{ij} . We do this, in this section, by fixing a (generally time dependent) spacetime metric g , basing $\tilde{\square}_g$ and the other coefficients on this fixed g , and treating u, u_i , and h_{ij} as independent. We recouple g and the field variables in the next section, with the help of bootstrap assumptions.

In defining the energy functionals, it is useful to treat the field variables in blocks, according to the semi-decoupling of the (linearized) evolution equation, noted above. So we start by working with just u and ψ , which we write collectively as the 2-vector

$$\mathbf{u} = \begin{bmatrix} u \\ \psi \end{bmatrix}. \quad (\text{IV.75})$$

The evolution PDEs (IV.63) and (IV.66) then take the form

$$\tilde{\square}_g \mathbf{u} + C \partial_\tau \mathbf{u} + J \mathbf{u} + \mathbf{\Delta} = 0, \quad (\text{IV.76})$$

where J and C are the constant matrices

$$J = \begin{bmatrix} 2p[n(p-1) + 1] & -2\lambda p(np-1) \\ -2/\lambda & 2(np-1) \end{bmatrix},$$

$$C = \begin{bmatrix} (n+2)p-1 & -8/\lambda \\ 0 & np-1 \end{bmatrix},$$

and Δ is the vector of nonlinear terms

$$\Delta = \begin{bmatrix} \Delta_{00} \\ \Delta_\psi \end{bmatrix}. \quad (\text{IV.77})$$

Since the only effect of the electromagnetic field on the evolution of \mathbf{u} is via the Δ term, the form of the energy functionals we use for \mathbf{u} and the calculation of their evolutions are formally very similar to that of Ringström in [28] (Section 4). Following that narrative, we first obtain a matrix T (see equation (76) in [28]) which diagonalizes the matrix J , and then setting $\tilde{\mathbf{u}} = T^{-1}\mathbf{u}$, $\tilde{\Delta} = T^{-1}\Delta$, $\tilde{J} = T^{-1}JT = \text{diag}\{\lambda_-, \lambda_+\}$ and $\tilde{C} = T^{-1}CT$, we have

$$\tilde{\square}_g \tilde{\mathbf{u}} + \tilde{C} \partial_\tau \tilde{\mathbf{u}} + \tilde{J} \tilde{\mathbf{u}} + \tilde{\Delta} = 0. \quad (\text{IV.78})$$

We next define the base energy functional we shall use for $\tilde{\mathbf{u}}$. Letting $c_{\mathcal{L}S}$, b_1 , and b_2 be any set of positive definite constants, and using the notation $\tilde{g}^{ij} = e^{2(\tau+\tau_0)}g^{ij}$, we define

$$\mathcal{E}[\tilde{\mathbf{u}}] := \frac{1}{2} \int_{\mathbf{T}^n} (-g^{00} \partial_\tau \tilde{\mathbf{u}}^t \partial_\tau \tilde{\mathbf{u}} + \tilde{g}^{ij} \partial_i \tilde{\mathbf{u}}^t \partial_j \tilde{\mathbf{u}} - 2c_{\mathcal{L}S} g^{00} \tilde{\mathbf{u}}^t \partial_\tau \tilde{\mathbf{u}} + b_1 \tilde{u}^2 + b_2 \tilde{\psi}^2) dx, \quad (\text{IV.79})$$

where the superscript t on $\tilde{\mathbf{u}}$ indicates the transpose. We then verify the following:

Lemma IV.1. *Let g be a fixed Lorentz metric on the spacetime $I \times \mathbf{T}^n$ for some interval I , and let \mathbf{u} be a solution to equation (IV.78) on $I \times \mathbf{T}^n$ for some choice of the constants $p > 1$ and $n \geq 3$ characterizing an Einstein-Maxwell-scalar field theory. There exist positive constants $\eta_{\mathcal{L}S}$, $\zeta_{\mathcal{L}S}$, b_1 , b_2 and $c_{\mathcal{L}S}$ (depending on n and*

p) such that if we define \mathcal{E} with these constants and so long as

$$|g^{00} + 1| \leq \eta_{\mathcal{L}\mathcal{S}}, \quad (\text{IV.80})$$

then

$$\mathcal{E}[\tilde{\mathbf{u}}] \geq \zeta_{\mathcal{L}\mathcal{S}} \int_{\mathbf{T}^n} (\partial_\tau \tilde{\mathbf{u}}^t \partial_\tau \tilde{\mathbf{u}} + \tilde{g}^{ij} \partial_i \tilde{\mathbf{u}}^t \partial_j \tilde{\mathbf{u}} + \tilde{\mathbf{u}}^t \tilde{\mathbf{u}}) dx \quad (\text{IV.81})$$

and

$$\frac{d\mathcal{E}[\tilde{\mathbf{u}}]}{d\tau} \leq -2\eta_{\mathcal{L}\mathcal{S}}\mathcal{E} + \int_{\mathbf{T}^n} [-(\partial_\tau \tilde{\mathbf{u}}^t + c_{\mathcal{L}\mathcal{S}}\tilde{\mathbf{u}}^t)\tilde{\Delta} + \Delta_E[\tilde{\mathbf{u}}]] dx, \quad (\text{IV.82})$$

where $\Delta_E[\tilde{\mathbf{u}}]$ is a function quadratic in \mathbf{u} and its derivatives, defined in equation (IV.83). □

Proof. First, we want to choose constants $c_{\mathcal{L}\mathcal{S}}$, b_1 and b_2 such that (IV.81) holds for some constant $\zeta_{\mathcal{L}\mathcal{S}}$. This implies $c_{\mathcal{L}\mathcal{S}}^2 < b_i$ for $i = 1, 2$. Let

$$b_1 = \lambda_- + c_{\mathcal{L}\mathcal{S}}\tilde{C}_{11}, \quad b_2 = \lambda_+ + c_{\mathcal{L}\mathcal{S}}\tilde{C}_{22}.$$

Then $c_{\mathcal{L}\mathcal{S}}^2 < b_i$ is satisfied if we choose $c_{\mathcal{L}\mathcal{S}}$ to be small enough. Thus (IV.81) holds for some constant $\zeta_{\mathcal{L}\mathcal{S}}$, provided g^{00} is close enough to -1 . Differentiating $\mathcal{E}[\tilde{\mathbf{u}}]$, we

get

$$\begin{aligned}
\frac{d\mathcal{E}}{d\tau} &= \int_{\mathbf{T}^n} \left\{ -\frac{1}{2} \partial_\tau \tilde{\mathbf{u}}^t (\tilde{C} + \tilde{C}^t) \partial_\tau \tilde{\mathbf{u}} - \partial_\tau \tilde{\mathbf{u}}^t \tilde{J} \tilde{\mathbf{u}} - \partial_\tau \tilde{\mathbf{u}}^t \tilde{\Delta} \right. \\
&\quad - (p-1 + c_{\mathcal{L}\mathcal{S}}) \tilde{g}^{ij} \partial_i \tilde{\mathbf{u}}^t \partial_j \tilde{\mathbf{u}} + c_{\mathcal{L}\mathcal{S}} |\partial_\tau \tilde{\mathbf{u}}|^2 - c_{\mathcal{L}\mathcal{S}} \tilde{\mathbf{u}}^t \tilde{C} \partial_\tau \tilde{\mathbf{u}} - c_{\mathcal{L}\mathcal{S}} \tilde{\mathbf{u}}^t \tilde{J} \tilde{\mathbf{u}} \\
&\quad \left. - c_{\mathcal{L}\mathcal{S}} \tilde{\mathbf{u}}^t \tilde{\Delta} + b_1 \tilde{u} \partial_\tau \tilde{u} + b_2 \tilde{\psi} \partial_\tau \tilde{\psi} + \Delta_E[\tilde{\mathbf{u}}] \right\} dx, \\
\Delta_E[\tilde{\mathbf{u}}] &= -c_{\mathcal{L}\mathcal{S}} (g^{00} + 1) \partial_\tau \tilde{\mathbf{u}}^t \partial_\tau \tilde{\mathbf{u}} - 2c_{\mathcal{L}\mathcal{S}} (\tilde{g}^{0i} \partial_i \tilde{\mathbf{u}}^t \partial_\tau \tilde{\mathbf{u}} + (\partial_i \tilde{g}^{0i}) \tilde{\mathbf{u}}^t \partial_\tau \tilde{\mathbf{u}}) \\
&\quad - c_{\mathcal{L}\mathcal{S}} (\partial_j \tilde{g}^{ij}) \partial_i \tilde{\mathbf{u}}^t \tilde{\mathbf{u}} - \frac{1}{2} \partial_\tau g^{00} \partial_\tau \tilde{\mathbf{u}}^t \partial_\tau \tilde{\mathbf{u}} + \left[\frac{1}{2} \partial_\tau \tilde{g}^{ij} + (p-1) \tilde{g}^{ij} \right] \partial_i \tilde{\mathbf{u}}^t \partial_j \tilde{\mathbf{u}} \\
&\quad - \partial_i \tilde{g}^{0i} \partial_\tau \tilde{\mathbf{u}}^t \partial_\tau \tilde{\mathbf{u}} - \partial_j \tilde{g}^{ij} \partial_\tau \tilde{\mathbf{u}}^t \partial_i \tilde{\mathbf{u}} - c_{\mathcal{L}\mathcal{S}} \partial_\tau g^{00} \tilde{\mathbf{u}}^t \partial_\tau \tilde{\mathbf{u}}. \tag{IV.83}
\end{aligned}$$

With our choice of $c_{\mathcal{L}\mathcal{S}}$, b_1 and b_2 , we obtain

$$\begin{aligned}
\frac{d\mathcal{E}}{d\tau} &= \int_{\mathbf{T}^n} \left\{ -\frac{1}{2} \partial_\tau \tilde{\mathbf{u}}^t (\tilde{C} + \tilde{C}^t) \partial_\tau \tilde{\mathbf{u}} + c_{\mathcal{L}\mathcal{S}} |\partial_\tau \tilde{\mathbf{u}}|^2 - (p-1 + c_{\mathcal{L}\mathcal{S}}) \tilde{g}^{ij} \partial_i \tilde{\mathbf{u}}^t \partial_j \tilde{\mathbf{u}} \right. \\
&\quad \left. - c_{\mathcal{L}\mathcal{S}} \left[\lambda_- \tilde{u}^2 + \lambda_+ \tilde{\psi}^2 \right] - c_{\mathcal{L}\mathcal{S}} \left[\tilde{C}_{21} \tilde{\psi} \partial_\tau \tilde{u} + \tilde{C}_{12} \tilde{u} \partial_\tau \tilde{\psi} \right] \right\} dx, \\
&\quad + \int_{\mathbf{T}^n} \left[-(\partial_\tau \tilde{\mathbf{u}}^t + c_{\mathcal{L}\mathcal{S}} \tilde{\mathbf{u}}^t) \tilde{\Delta} + \Delta_E[\tilde{\mathbf{u}}] \right] dx, \tag{IV.84}
\end{aligned}$$

By Lemma 5 of [28], $\tilde{C} + \tilde{C}^t$ is positive definite. Thus provided that g^{00} is close enough to -1 and that we choose $c_{\mathcal{L}\mathcal{S}}$ to be small enough, we get (IV.82) for some constant $\eta_{\mathcal{L}\mathcal{S}}$. \square

It is important to note that the differential inequality (IV.82) is useful for controlling the evolution of \mathcal{E} only if we can establish estimates for $\Delta(\tilde{\Delta})$ and Δ_E . We do this below, using the bootstrap assumptions.

Since we need to control derivatives of \mathbf{u} as well as \mathbf{u} itself, it is very useful to work with energy functionals which involve higher spatial derivatives. For that

purpose, we define the following sequence of energy functionals

$$E_{\mathcal{L}\mathcal{S},k} = \sum_{|\alpha| \leq k} \mathcal{E}[\partial^\alpha \tilde{\mathbf{u}}], \quad (\text{IV.85})$$

where ∂^α indicates the usual multi-index spatial derivate, of order $|\alpha|$. It follows readily from this definition of $E_{\mathcal{L}\mathcal{S},k}$ and from Lemma IV.1 that the following differential inequality holds:

Corollary IV.1. *Presuming the hypotheses of Lemma IV.1, $E_{\mathcal{L}\mathcal{S},k}$ satisfies*

$$\begin{aligned} \frac{dE_{\mathcal{L}\mathcal{S},k}}{d\tau} &\leq -2\eta_{\mathcal{L}\mathcal{S}}E_{\mathcal{L}\mathcal{S},k} \\ &\quad + \sum_{|\alpha| \leq k} \int_{\mathbf{T}^n} [(\partial^\alpha \partial_\tau \tilde{\mathbf{u}}^t + c_{\mathcal{L}\mathcal{S}} \partial^\alpha \tilde{\mathbf{u}}^t)(-\partial^\alpha \tilde{\Delta} + [\tilde{\square}_g, \partial^\alpha] \tilde{\mathbf{u}}) + \Delta_E[\partial^\alpha \hat{\mathbf{u}}]] dx. \end{aligned} \quad (\text{IV.86})$$

□

Thus far, we have developed a sequence of energy functionals only for the pair of field variables u and ψ (with artificially fixed g). To obtain a similar sequence of energy functionals for u_i, h_{ij}, A_0 , and A_j , again with the metric g fixed, it is useful to work with solutions of a model PDE

$$\tilde{\square}_g v + \alpha \partial_\tau v + \beta v = F \quad (\text{IV.87})$$

for the scalar function v , where $\alpha > 0$ and $\beta \geq 0$ are constants, g is a fixed Lorentz metric on the spacetime manifold $\mathbf{T}^n \times I$ (for $n \geq 3$), and F is a fixed function on the spacetime. We have the following:

Lemma IV.2. *Let v be a solution of equation (IV.87) on $\mathbf{T}^n \times I$, with g, α, β and F be as stated above. There are constants $\eta_c, \zeta > 0$ and $\gamma, \delta \geq 0$, depending on α and β , such that if the given metric satisfies*

$$|g^{00} + 1| \leq \eta_c, \quad (\text{IV.88})$$

and if we define the energy functional via

$$\mathcal{E}_{\gamma, \delta}[v] := \frac{1}{2} \int_{\mathbf{T}^n} (-g^{00}(\partial_\tau v)^2 + \tilde{g}^{ij} \partial_i v \partial_j v - 2\gamma g^{00} v \partial_\tau v + \delta v^2) dx, \quad (\text{IV.89})$$

then $\mathcal{E}_{\gamma, \delta}[v]$ bounds the following quadratic integral

$$\mathcal{E}_{\gamma, \delta}[v] \geq \zeta \int_{\mathbf{T}^n} [(\partial_\tau v)^2 + \tilde{g}^{ij} \partial_i v \partial_j v + \iota_\beta v^2] dx, \quad (\text{IV.90})$$

where $\iota_\beta = 0$ if $\beta = 0$ and $\iota_\beta = 1$ otherwise, and $\mathcal{E}_{\gamma, \delta}[v]$ satisfies the differential inequality

$$\frac{d\mathcal{E}_{\gamma, \delta}}{d\tau} \leq -2\eta_c \mathcal{E}_{\gamma, \delta} + \int_{\mathbf{T}^n} [(\partial_\tau v + \gamma v)F + \Delta_{E, \gamma, \delta}[v]] dx, \quad (\text{IV.91})$$

where $\Delta_{E, \gamma, \delta}[v]$ is given by (IV.92). If $\beta = 0$, then $\delta = \gamma = 0$. □

Proof. If $\beta > 0$, choose $\gamma = \alpha/2$, $\delta = \beta + \alpha^2/2$ and if $\beta = 0$, simply let $\gamma = \delta = 0$. It is easy to check that there exists a constant ζ such that (IV.90) holds provided g^{00}

is close enough to -1 . Differentiate $\mathcal{E}_{\gamma,\delta}[v]$, we get

$$\begin{aligned}
\frac{d\mathcal{E}_{\gamma,\delta}[v]}{d\tau} &= \int_{\mathbf{T}^n} \{ -(\alpha - \gamma)(\partial_\tau v)^2 + (\delta - \beta - \gamma\alpha)v\partial_\tau v - \beta\gamma v^2 \\
&\quad - (p - 1 + \gamma)\tilde{g}^{ij}\partial_i v\partial_j v + (\partial_\tau v + \gamma v)F + \Delta_{E,\gamma,\delta}[v] \} dx, \\
\Delta_{E,\gamma,\delta}[v] &= -\gamma(\partial_i \tilde{g}^{ij})v\partial_j v - 2\gamma(\partial_i \tilde{g}^{0i})v\partial_\tau v - 2\gamma\tilde{g}^{0i}\partial_i v\partial_\tau v - (\partial_i \tilde{g}^{0i})(\partial_\tau v)^2 \\
&\quad - (\partial_j \tilde{g}^{ij})\partial_i v\partial_\tau v - \frac{1}{2}(\partial_\tau g^{00})(\partial_\tau v)^2 + \left(\frac{1}{2}\partial_\tau \tilde{g}^{ij} + (p-1)\tilde{g}^{ij} \right) \partial_i v\partial_j v \\
&\quad - \gamma\partial_\tau g^{00}v\partial_\tau v - \gamma(g^{00} + 1)(\partial_\tau v)^2. \tag{IV.92}
\end{aligned}$$

In the case $\beta > 0$, we have

$$\begin{aligned}
\frac{d\mathcal{E}_{\gamma,\delta}[v]}{d\tau} &= -\frac{1}{2} \int_{\mathbf{T}^n} \{ \alpha(\partial_\tau v)^2 + \alpha\beta v^2 + (2(p-1) + \alpha)\tilde{g}^{ij}\partial_i v\partial_j v \} dx \\
&\quad + \int_{\mathbf{T}^n} \{ (\partial_\tau v + \gamma v)F + \Delta_{E,\gamma,\delta}[v] \} dx.
\end{aligned}$$

Use the opposite of (IV.90), we obtain the differential inequality for some constant η_c . Similar for the case $\beta = 0$. \square

As with the energy functionals for \mathbf{u} , discussed above, it is useful to proceed from $\mathcal{E}_{\gamma,\delta}$ to a sequence of energy functionals involving higher derivatives of v :

$$E_{\mathcal{V},k}[v] := \sum_{|\alpha| \leq k} \mathcal{E}_{\gamma,\delta}[\partial^\alpha v]. \tag{IV.93}$$

One then proves the following differential inequality result:

Corollary IV.2. *Presuming that the hypotheses of Lemma IV.2 hold, the sequence of higher order energy functionals $E_{\mathcal{V},k}$ satisfy the following inequalities:*

$$\frac{dE_{\mathcal{V},k}}{d\tau} \leq -2\eta_c E_{\mathcal{V},k} + \sum_{|\alpha| \leq k} \int_{\mathbf{T}^n} \{(\partial_\tau \partial^\alpha v + \gamma \partial^\alpha v)(\partial^\alpha F + [\tilde{\square}_g, \partial^\alpha]v) + \Delta_{E,\gamma,\delta}[\partial^\alpha v]\} dx. \quad (\text{IV.94})$$

□

Proof. If one differentiates equation (IV.87) and then applies Lemma IV.2, the corollary immediately follows. □

To obtain the energy functionals for the fields u_i, h_{ij}, A_0 , and A_j , we now manipulate the evolution equations for these fields so that, with varying specifications of the function F and of the constants α and β , these evolution equations (for *each of the components* of u_i, h_{ij} , etc.) match with the model equation (IV.87). For present purposes, we presume that g is a fixed Lorentz metric.

For u_i , we work with equation (IV.64). If we set $\alpha = np - 1 > 0$ and $\beta = p(n - 2)(2p - 1) > 0$, and if we set F equal to the negative of all except the first three terms ⁵ in (IV.64), then we have an equation of the form (IV.87) for each of the components of u_i . It then follows from Lemma IV.2 and Corollary IV.2 that there exists a set of positive constants $\gamma_{S\mathcal{H}}, \delta_{S\mathcal{H}}, \eta_{S\mathcal{H}}$ and $\zeta_{S\mathcal{H}}$ such that the conclusions of Lemma IV.2, including the bounding condition (IV.90) and the differential inequality (IV.91), hold for u_i . Hence we define the following energy

⁵We note that, according to this construction, the function F includes information—terms depending on h_{ij} , on A_i , etc.—which is not known. The formal derivation of the differential inequalities of the form (IV.82) or (IV.86) still works, however.

functional

$$E_{S\mathcal{H},k} := \sum_i E_{u_i,k} = \sum_i \sum_{|\lambda| \leq k} \mathcal{E}_{\gamma_{S\mathcal{H}}, \delta_{S\mathcal{H}}}[\partial^\lambda u_i]. \quad (\text{IV.95})$$

For h_{ij} , we work with equation (IV.65). In this case, we set $\alpha = np - 1, \beta = 0$, and $F = -\Delta_{ij} - 2p(\lambda(np - 1)\psi - u)h_{ij}$; we then have an equation of the form (IV.87) for the components of h_{ij} . Lemma IV.2 and Corollary IV.2 now imply that there exist constants $\eta_{\mathcal{M}} > 0$ and $\zeta_{\mathcal{M}} > 0$ (with $\gamma_{\mathcal{M}} = 0$ and $\delta_{\mathcal{M}} = 0$) such that a bounding condition of the form (IV.90) and a differential inequality of the form (IV.91) hold for h_{ij} . We define the energy functional

$$E_{\mathcal{M},k} := \frac{1}{2} \sum_{i,j} \sum_{|\lambda| \leq k} \left(\mathcal{E}_{\gamma_{\mathcal{M}}, \delta_{\mathcal{M}}}[\partial^\lambda h_{ij}] + \int_{\mathbf{T}^n} e^{-2a\tau} a_\lambda (\partial^\lambda h_{ij})^2 dx \right), \quad (\text{IV.96})$$

with $a_\lambda = 0$ for $|\lambda| = 0$ and $a_\lambda = 1$ for $|\lambda| > 0$ ($a > 0$ is a constant to be determined below, by (IV.107)). We note the inclusion in this expression (IV.96) of the unfamiliar second term (with coefficient a_λ). The reason we include this extra term is that some terms are missing in $\mathcal{E}_{\gamma_{\mathcal{M}}, \delta_{\mathcal{M}}}[\partial^\lambda h_{ij}]$ because of the vanishing of $\gamma_{\mathcal{M}}$ and $\delta_{\mathcal{M}}$. We need to include this extra term in order for $E_{\mathcal{M},k}$ to control some norm of h_{ij} . As discussed in Section 7 of [27], this term is consistent with the conclusions we derive from Lemma IV.2 and Corollary IV.2, and also enforces the condition that the sequence of energies $E_{\mathcal{M},k}$ all vanish for fields which are the same as the background fields $(\hat{g}, \hat{\phi}, \hat{A})_{\{t_0, p, c_0, \kappa\}}$.

We proceed to the energy functionals for the electromagnetic fields. For A_0 , we work with the evolution equation (IV.67). It takes the desired form if we set $\alpha = (n + 2)p - 1, \beta = np(2p - 1)$, and $F = -\Delta_e$. Applying Lemma IV.2 and Corollary IV.2, we determine that there are positive constants $\gamma_{S\mathcal{P}}, \delta_{S\mathcal{P}}, \eta_{S\mathcal{P}}$ and $\zeta_{S\mathcal{P}}$ such that results of the form (IV.90) and of the form (IV.91) hold. Hence we

define

$$E_{\mathcal{SP},k} = \sum_{|\lambda| \leq k} \mathcal{E}_{\gamma_{\mathcal{SP}}, \delta_{\mathcal{SP}}}[\partial^\lambda A_0], \quad (\text{IV.97})$$

as the energy functional for A_0 .

Finally for A_j , we work with equation (IV.68). To match the form of equation (IV.87), we set $\alpha = (n-2)p-1$, $\beta = 0$, and $F = -\Delta_{b,i} - 2pe^{\tau+\tau_0}\partial_i A_0$. Then from Lemma IV.2 and Corollary IV.2, we determine that there are constants $\eta_{\mathcal{VP}} > 0$ and $\zeta_{\mathcal{VP}} > 0$ (with $\gamma_{\mathcal{VP}} = 0$ and $\delta_{\mathcal{VP}} = 0$) such that a bounding condition of the form (IV.90) and a differential inequality of the form (IV.91) hold. We define the following energy functional for A_i

$$E_{\mathcal{VP},k} = \sum_i \sum_{|\lambda| \leq k} \left(\mathcal{E}_{\gamma_{\mathcal{VP}}, \delta_{\mathcal{VP}}}[\partial^\lambda A_i] + \int_{\mathbf{T}^n} e^{-2a\tau} (\partial^\lambda A_i)^2 dx \right), \quad (\text{IV.98})$$

with $a > 0$ a constant to be determined below, by (IV.107). The role of the second term in this expression for $E_{\mathcal{VP},k}$ is much the same as that of the equivalent term in the expression (IV.96) for $E_{\mathcal{M},k}$.

For each of the energy functionals $E_{\mathcal{LS},k}$, $E_{\mathcal{SH},k}$, $E_{\mathcal{M},k}$, $E_{\mathcal{SP},k}$, and $E_{\mathcal{VP},k}$, there is a corresponding differential inequality of the form (IV.91) which could be used to control the evolution of that energy functional *if* one knew the enough about the corresponding function F . In the next section, we use bootstrap assumptions to establish this knowledge. Bootstrap assumptions also play a role in controlling the nature of the evolving metric g which appears in the differential operator $\tilde{\square}_g$ as well as elsewhere in the evolution equations (IV.63)-(IV.68), once we restore the relation between g and the evolving field variables u , u_i , and h_{ij} . We also discuss this in the next section

IV.4. Bootstrap Assumptions

In this section we state our bootstrap assumptions. The idea of bootstrap assumptions is that one assumes that the evolving fields satisfy certain conditions for $t \in I \subset \mathbf{R}^1$, one uses those conditions to prove that certain estimates consequently hold (on I), and then one uses these estimates together with the evolution equations to argue that the solutions (satisfying the bootstrap assumptions) exist for all time $t \in \mathbf{R}^1$.

We use two bootstraps assumptions here. The first (which, following Ringström. we call the “primary bootstrap assumption”) essentially says that the evolving metric stays Lorentzian for $t \in I$. The second (called the “main bootstrap assumption”) says that the energy functionals for the fields evolving from the perturbed initial data set remain small for $t \in I$. We now state these more precisely, and discuss their immediate consequences.

To state the primary bootstrap assumption, it is useful to work with the following notation: Let g be a real valued $(n + 1) \times (n + 1)$ matrix, with components $g_{\mu\nu}$ for $\mu \in \{i, 0\}$ and $i \in \{1, \dots, n\}$. We use g_b to denote the $n \times n$ submatirix with components g_{ij} , and we write $g_b > 0$ if this matrix is positive definite. We use g^{-1} to denote the inverse of g and we use g^\sharp to denote the $n \times n$ submatirix with components g^{ij} , presuming that these inverses exist. As before, we let $u[g] := 1 + g_{00}$, and we write $v[g] := (g_{01}, g_{02}, \dots, g_{0n})$. We recall that g is defined to be a Lorentz metric if it is symmetric, has n positive eigenvalues and 1 negative one. Noting that if $u[g] < 1$ and if $g_b > 0$, then g is Lorentzian we call g a canonical Lorentz metric if these two inequalities hold. We use \mathcal{L}_n to denote the set of canonical $n \times n$ Lorentz metrics.

Now, given a symmetric positive definite $n \times n$ matrix M with components M_{ij} , and given a vector $w \in \mathbf{R}^n$, we write (presuming the Einstein summation convention)

$$|w|_M := (M_{ij}w^iw^j)^{1/2}.$$

If M is the identity matrix, we simply write $|w| := |w|_{Id}$. We can now state the following:

Definition IV.1. Let $a > 0$, $c_1 > 1$, $\eta \in (0, 1)$, κ_0 and τ_0 be real numbers. We say that a function $g : I \times \mathbf{T}^n \rightarrow \mathcal{L}_n$, satisfies the **primary bootstrap assumption** $\mathcal{PBA}\{a, c_1, \eta, \kappa_0, \tau_0\}$ on an interval I if

$$\frac{1}{c_1}|w|^2 \leq e^{-2p\tau-2\kappa}|w|_{g^\#}^2 \leq c_1|w|^2, \quad (\text{IV.99})$$

$$|u[g]| \leq \eta, \quad (\text{IV.100})$$

$$|v[g]|^2 \leq \eta c_1^{-1} e^{2p\tau-2a\tau+2\kappa}, \quad (\text{IV.101})$$

for all $w \in \mathbf{R}^n$ and all $(\tau, x) \in I \times \mathbf{T}^n$, where $\kappa = \tau_0 + \kappa_0$. □

The following lemma follows from Lemma 7 of [27].

Lemma IV.3. *Assume $g : I \times \mathbf{T}^n \rightarrow \mathcal{L}_n$ satisfies $\mathcal{PBA}\{a, c_1, \eta, \kappa_0, \tau_0\}$ on I . There is a constant $\eta_0 \in (0, 1/4)$ such that if $\eta \leq \eta_0$, then*

$$|v[g^{-1}]| \leq 2c_1 e^{-2p\tau-2\kappa} |v[g]|, \quad (\text{IV.102})$$

$$|(v[g], v[g^{-1}])| \leq 2c_1 e^{-2p\tau-2\kappa} |v[g]|^2, \quad (\text{IV.103})$$

$$|u[g^{-1}]| \leq 4\eta, \quad (\text{IV.104})$$

$$\frac{2}{3c_1}|w|^2 \leq e^{2p\tau+2\kappa}|w|_{g^\#}^2 \leq \frac{3c_1}{2}|w|^2, \quad (\text{IV.105})$$

hold for all $w \in \mathbf{R}^n$ and all $(\tau, x) \in I \times \mathbf{T}^n$, where $(v[g], v[g^{-1}]) = g_{0i}g^{0i}$. \square

From now on, we assume that g satisfies $\mathcal{PBA}\{a, c_1, \eta, \kappa_0, \tau_0\}$ on an interval I , where we define η and a by

$$\eta := \min\{\eta_0, \eta_{\mathcal{LS}}/4, \eta_{\mathcal{SH}}/4, \eta_{\mathcal{M}}/4, \eta_{\mathcal{SP}}/4, \eta_{\mathcal{VP}}/4\}, \quad (\text{IV.106})$$

$$a := \frac{1}{4} \min\{p-1, \eta_{\mathcal{LS}}, \eta_{\mathcal{SH}}, \eta_{\mathcal{M}}, \eta_{\mathcal{SP}}, \eta_{\mathcal{VP}}\}. \quad (\text{IV.107})$$

As a consequence, the conclusions of Lemma IV.1 and Lemma IV.2 hold for the energies of interest. Note that a and η only depend on n and p . Furthermore, we define the following rescaled energies

$$\begin{aligned} \tilde{E}_{\mathcal{LS},k} &= e^{2a\tau} E_{\mathcal{LS},k}, & \tilde{E}_{\mathcal{SH},k} &= e^{-2p\tau-2\kappa+2a\tau} E_{\mathcal{SH},k}, \\ \tilde{E}_{\mathcal{M},k} &= e^{-4\kappa+2a\tau} E_{\mathcal{M},k}, & \tilde{E}_{\mathcal{SP},k} &= e^{2a\tau} E_{\mathcal{SP},k}, & \tilde{E}_{\mathcal{VP},k} &= e^{-2p\tau-2\kappa+2a\tau} E_{\mathcal{VP},k}, \end{aligned}$$

and

$$\tilde{E}_k = \tilde{E}_{\mathcal{LS},k} + \tilde{E}_{\mathcal{SH},k} + \tilde{E}_{\mathcal{M},k} + \tilde{E}_{\mathcal{SP},k} + \tilde{E}_{\mathcal{VP},k}.$$

Define the notation

$$\|f\|_{H^k} = \left(\sum_{|\alpha| \leq k} \int_{\mathbf{T}^n} (\partial^\alpha f)^2 dx \right)^{1/2}$$

for the Sobolev norms. By definition, we have the following lemma

Lemma IV.4. *Let $c_1 > 1$, κ_0 and τ_0 be real numbers. Let η and a be defined by (IV.106) and (IV.107) respectively and assume that $g : I \times \mathbf{T}^n \rightarrow \mathcal{L}_n$ satisfies*

$\mathcal{PBA}\{a, c_1, \eta, \kappa_0, \tau_0\}$ on a time interval I . Then

$$e^{a\tau} [\|\psi\|_{H^k} + \|\partial_\tau \psi\|_{H^k} + e^{-(p-1)\tau - \kappa_0} \|\partial_i \psi\|_{H^k}] \leq C \tilde{E}_{\mathcal{LS}, k}^{1/2}, \quad (\text{IV.108})$$

$$e^{a\tau} [\|u\|_{H^k} + \|\partial_\tau u\|_{H^k} + e^{-(p-1)\tau - \kappa_0} \|\partial_i u\|_{H^k}] \leq C \tilde{E}_{\mathcal{LS}, k}^{1/2}, \quad (\text{IV.109})$$

$$e^{-p\tau - \kappa + a\tau} [\|u_m\|_{H^k} + \|\partial_\tau u_m\|_{H^k} + e^{-(p-1)\tau - \kappa_0} \|\partial_i u_m\|_{H^k}] \leq C \tilde{E}_{\mathcal{SH}, k}^{1/2}, \quad (\text{IV.110})$$

$$e^{-2p\tau - 2\kappa + a\tau} [\|\partial_\tau g_{ij} - 2pg_{ij}\|_{H^k} + e^{-(p-1)\tau - \kappa_0} \|\partial_l g_{ij}\|_{H^k}] \leq C \tilde{E}_{\mathcal{M}, k}^{1/2}, \quad (\text{IV.111})$$

$$e^{-2p\tau - 2\kappa} \|\partial^\alpha g_{ij}\|_2 \leq C \tilde{E}_{\mathcal{M}, k}^{1/2}, \quad 0 < |\alpha| \leq k, \quad (\text{IV.112})$$

$$e^{a\tau} [\|A_0\|_{H^k} + \|\partial_\tau A_0\|_{H^k} + e^{-(p-1)\tau - \kappa_0} \|\partial_i A_0\|_{H^k}] \leq C \tilde{E}_{\mathcal{SP}, k}^{1/2}, \quad (\text{IV.113})$$

$$e^{-p\tau - \kappa + a\tau} [\|\partial_\tau A_i\|_{H^k} + e^{-a\tau} \|A_i\|_{H^k} + e^{-(p-1)\tau - \kappa_0} \|\partial_l A_i\|_{H^k}] \leq C \tilde{E}_{\mathcal{VP}, k}^{1/2}, \quad (\text{IV.114})$$

hold on I , where $\kappa = \tau_0 + \kappa_0$ and the constants depend on c_1 , n and p . \square

Now we define the main bootstrap assumption that we use in our proof of global existence.

Definition IV.2. Let $c_1 > 1$, κ_0 and τ_0 be real numbers and $k_0 > n/2 + 1$ be an integer. Let η and a be defined by (IV.106) and (IV.107) respectively. We say that (g, ψ, A) satisfy the **main bootstrap assumption** $\mathcal{MBA}\{a, c_1, \eta, \kappa_0, \tau_0, \epsilon\}$ on I , if

1. $g : I \times \mathbf{T}^n \rightarrow \mathcal{L}_n$, $\psi : I \times \mathbf{T}^n \rightarrow \mathbf{R}$ and $A_\mu : I \times \mathbf{T}^n \rightarrow \mathbf{R}$ are C^∞ ,
2. g satisfies $\mathcal{PBA}\{a, c_1, \eta, \kappa_0, \tau_0\}$ on I ,
3. g , ψ and A satisfy

$$\tilde{E}_{k_0}^{1/2}(\tau) \leq \epsilon, \quad (\text{IV.115})$$

for $\tau \in I$, where $\kappa = \tau_0 + \kappa_0$. \square

IV.5. Estimates and Differential Inequalities

In this section we derive estimates to control nonlinear terms (such as those contained in the Δ terms) and prove the differential inequalities which, so long as the bootstrap assumptions hold, control the evolution of the energy functionals. The differential inequalities (IV.122) - (IV.126) are crucial in our proof of global existence.

First we show some estimates that we need in order to derive the differential inequalities.

Lemma IV.5. *Assume that (g, ψ, A) satisfies $\mathcal{MBA}\{a, c_1, \eta, \kappa_0, \tau_0, \epsilon\}$ on I . Then*

$$\|\Delta_e\|_{H^k} \leq C\epsilon e^{-2a\tau} \tilde{E}_k^{1/2}, \quad (\text{IV.116})$$

$$\|\Delta_{b,i}\|_{H^k} \leq C\epsilon e^{p\tau-2a\tau+\kappa} \tilde{E}_k^{1/2}, \quad (\text{IV.117})$$

hold on I , where the constant coefficients depend on n, p, k and c_1 , and

$$\|\Delta_{E,\gamma_{\mathcal{SP}},\delta_{\mathcal{SP}}}[\partial^\alpha A_0]\|_{L^1} \leq C\epsilon e^{-a\tau} E_{\mathcal{SP},k}, \quad (\text{IV.118})$$

$$\|\Delta_{E,\gamma_{\mathcal{VP}},\delta_{\mathcal{VP}}}[\partial^\alpha A_i]\|_{L^1} \leq C\epsilon e^{-a\tau} E_{\mathcal{VP},k}, \quad (\text{IV.119})$$

hold on I for $|\alpha| \leq k$, where the constant coefficients depend on n, p, k, c_1 and an upper bound on $e^{-\kappa_0}$. If we further assume that (IV.67) and (IV.68) are satisfied, then for $0 < |\alpha| \leq k$,

$$\|[\tilde{\square}_g, \partial^\alpha]A_0\|_{L^2} \leq C\epsilon e^{-2a\tau} \tilde{E}_k^{1/2}, \quad (\text{IV.120})$$

$$\|[\tilde{\square}_g, \partial^\alpha]A_i\|_{L^2} \leq C\epsilon e^{p\tau-2a\tau+\kappa} \tilde{E}_k^{1/2}, \quad (\text{IV.121})$$

hold on I , where the constant coefficients depend on n, p, k, c_1 and an upper bound on $e^{-\kappa_0}$. \square

Proof. Note that the algorithm for estimating nonlinear terms used in [28] applies in our case. (IV.116) - (IV.121) follow by the same arguments as that in Lemma 11, Lemma 15 and Lemma 13 of [27]. \square

Lemma IV.6. *Assume that (g, ψ, A) satisfies $\mathcal{MBA}\{a, c_1, \eta, \kappa_0, \tau_0, \epsilon\}$ on I . If (IV.63) - (IV.68) are satisfied, then*

$$\frac{d\tilde{E}_{\mathcal{LS},k}}{d\tau} \leq -2a\tilde{E}_{\mathcal{LS},k} + C\epsilon e^{-a\tau} \tilde{E}_{\mathcal{LS},k}^{1/2} \tilde{E}_k^{1/2}, \quad (\text{IV.122})$$

$$\frac{d\tilde{E}_{\mathcal{SH},k}}{d\tau} \leq -2a\tilde{E}_{\mathcal{SH},k} + C\tilde{E}_{\mathcal{SH},k}^{1/2} (\tilde{E}_{\mathcal{LS},k}^{1/2} + \tilde{E}_{\mathcal{M},k}^{1/2}) + C\epsilon e^{-a\tau} \tilde{E}_{\mathcal{SH},k}^{1/2} \tilde{E}_k^{1/2}, \quad (\text{IV.123})$$

$$\frac{d\tilde{E}_{\mathcal{M},k}}{d\tau} \leq C e^{-a\tau} \tilde{E}_{\mathcal{M},k} + C\tilde{E}_{\mathcal{LS},k_0}^{1/2} \tilde{E}_{\mathcal{M},k} + C\tilde{E}_{\mathcal{LS},k}^{1/2} \tilde{E}_{\mathcal{M},k}^{1/2} + C\epsilon e^{-a\tau} \tilde{E}_{\mathcal{M},k}^{1/2} \tilde{E}_k^{1/2}, \quad (\text{IV.124})$$

$$\frac{d\tilde{E}_{\mathcal{SP},k}}{d\tau} \leq -2a\tilde{E}_{\mathcal{SP},k} + C\epsilon e^{-a\tau} \tilde{E}_{\mathcal{SP},k}^{1/2} \tilde{E}_k^{1/2}, \quad (\text{IV.125})$$

$$\frac{d\tilde{E}_{\mathcal{VP},k}}{d\tau} \leq -2p\tilde{E}_{\mathcal{VP},k} + C e^{-a\tau} \tilde{E}_{\mathcal{VP},k} + C\epsilon e^{-a\tau} \tilde{E}_{\mathcal{VP},k}^{1/2} \tilde{E}_k^{1/2} + C\tilde{E}_{\mathcal{VP},k}^{1/2} \tilde{E}_{\mathcal{SP},k}^{1/2}, \quad (\text{IV.126})$$

hold on I , where the constants depend on n, p, k, c_1 and an upper bound on $e^{-\kappa_0}$. \square

Proof. Notice that the equations (IV.63), (IV.64) and (IV.65) for u, u_i and h_{ij} differ from that in [28] only in the quadratic terms, which can easily be shown to have the same estimates as those in [28]. The equation (IV.66) is exactly the same as that in [28]. Thus the differential inequalities (139), (140) and (141) (with \hat{H} replaced by \tilde{E} and the subscripts ls, s and m replaced by $\mathcal{LS}, \mathcal{SH}$ and \mathcal{M} respectively) of [28] hold without any change for our case.

Thus we only need to show (IV.125) and (IV.126). Applying Corollary IV.2 to equation (IV.67), we obtain

$$\begin{aligned}
\frac{E_{\mathcal{SP},k}}{d\tau} &\leq -2\eta_{\mathcal{SP}}E_{\mathcal{SP},k} + \sum_{|\alpha|\leq k} \int_{\mathbf{T}^n} (\partial_\tau \partial^\alpha A_0 + \gamma_{\mathcal{SP}} \partial^\alpha A_0)(-\partial^\alpha \Delta_e + [\tilde{\square}_g, \partial^\alpha] A_0) dx \\
&\quad + \sum_{|\alpha|\leq k} \int_{\mathbf{T}^n} \Delta_{E,\gamma_{\mathcal{SP}},\delta_{\mathcal{SP}}} [\partial^\alpha A_0] dx \\
&\leq -2\eta_{\mathcal{SP}}E_{\mathcal{SP},k} + C \sum_{|\alpha|\leq k} (\|\partial_\tau \partial^\alpha A_0\|_2 + \|\partial^\alpha A_0\|_2)(\|\partial^\alpha \Delta_e\|_2 + \|[\tilde{\square}_g, \partial^\alpha] A_0\|_2) \\
&\quad + C \sum_{|\alpha|\leq k} \|\Delta_{E,\gamma_{\mathcal{SP}},\delta_{\mathcal{SP}}} [\partial^\alpha A_0]\|_1 \\
&\leq -2\eta_{\mathcal{SP}}E_{\mathcal{SP},k} + C(\|\partial_\tau A_0\|_{H^k} + \|A_0\|_{H^k})(\|\Delta_e\|_{H^k} + C\epsilon e^{-2a\tau} \tilde{E}_k^{1/2}) \\
&\quad + C\epsilon e^{-a\tau} E_{\mathcal{SP},k} \\
&\leq -2\eta_{\mathcal{SP}}E_{\mathcal{SP},k} + C e^{-a\tau} \tilde{E}_{\mathcal{SP},k}^{1/2} \epsilon e^{-2a\tau} \tilde{E}_k^{1/2} + C\epsilon e^{-a\tau} E_{\mathcal{SP},k} \tag{IV.127}
\end{aligned}$$

thus

$$-2ae^{-2a\tau} \tilde{E}_{\mathcal{SP},k} + e^{-2a\tau} \frac{d\tilde{E}_{\mathcal{SP},k}}{d\tau} \leq -2\eta_{\mathcal{SP}}E_{\mathcal{SP},k} + C\epsilon e^{-3a\tau} \tilde{E}_{\mathcal{SP},k}^{1/2} \tilde{E}_k^{1/2} + C\epsilon e^{-a\tau} E_{\mathcal{SP},k}.$$

and equation (IV.125) follows.

To show (IV.125), note that $\gamma_{\mathcal{VP}} = \delta_{\mathcal{VP}} = 0$. Applying Corollary IV.2 to equation (IV.68), we obtain

$$\begin{aligned}
\frac{E_{\mathcal{VP},k}}{d\tau} &\leq -2\eta_{\mathcal{VP}} \sum_i \sum_{|\alpha| \leq k} \mathcal{E}_{\gamma_{\mathcal{VP}}, \delta_{\mathcal{VP}}}[\partial^\alpha A_i] + \sum_i \sum_{|\alpha| \leq k} \int_{\mathbf{T}^n} \Delta_{E, \gamma_{\mathcal{VP}}, \delta_{\mathcal{VP}}}[\partial^\alpha A_i] dx \\
&\quad + \sum_i \sum_{|\alpha| \leq k} \int_{\mathbf{T}^n} \partial_\tau \partial^\alpha A_i [-\partial^\alpha (2pe^{\tau+\tau_0} \partial_i A_0 + \Delta_{b,i}) + [\tilde{\square}_g, \partial^\alpha] A_i] dx \\
&\quad + 2 \sum_i \sum_{|\alpha| \leq k} e^{-2a\tau} \int_{\mathbf{T}^n} [(\partial^\alpha A_i)(\partial_\tau \partial^\alpha A_i) - a(\partial^\alpha A_i)^2] dx \\
&\leq -2aE_{\mathcal{VP},k} + C\epsilon e^{-a\tau} E_{\mathcal{VP},k} \\
&\quad + \sum_i \sum_{|\alpha| \leq k} \int_{\mathbf{T}^n} \partial_\tau \partial^\alpha A_i [-\partial^\alpha (2pe^{\tau+\tau_0} \partial_i A_0 + \Delta_{b,i}) + [\tilde{\square}_g, \partial^\alpha] A_i] dx \\
&\quad + 2 \sum_i \sum_{|\alpha| \leq k} e^{-2a\tau} \int_{\mathbf{T}^n} (\partial^\alpha A_i)(\partial_\tau \partial^\alpha A_i) dx \\
&\leq -2aE_{\mathcal{VP},k} + C\epsilon e^{-a\tau} E_{\mathcal{VP},k} + 2e^{-2a\tau} \|A_i\|_{H^k} \|\partial_\tau A_i\|_{H^k} \\
&\quad + \|\partial_\tau A_i\|_{H^k} (C\epsilon e^{p\tau+\kappa-2a\tau} \tilde{E}_k^{1/2} + Ce^{\tau+\tau_0} \|\partial_i A_0\|_{H^k} + \|\Delta_{b,i}\|_{H^k}), \quad (\text{IV.128})
\end{aligned}$$

thus

$$\begin{aligned}
2(p-a)e^{2p\tau+2\kappa-2a\tau} \tilde{E}_{\mathcal{VP},k} + e^{2p\tau+2\kappa-2a\tau} \frac{d\tilde{E}_{\mathcal{VP},k}}{d\tau} &\leq -2aE_{\mathcal{VP},k} + C\epsilon e^{-a\tau} E_{\mathcal{VP},k} \\
+Ce^{2p\tau+2\kappa-3a\tau} \tilde{E}_{\mathcal{VP},k} + C\epsilon e^{2p\tau+2\kappa-3a\tau} \tilde{E}_{\mathcal{VP},k}^{1/2} \tilde{E}_k^{1/2} + Ce^{2p\tau+2\kappa-2a\tau} \tilde{E}_{\mathcal{VP},k}^{1/2} \tilde{E}_{SP,k}^{1/2},
\end{aligned}$$

and (IV.126) follows. \square

IV.6. Global Existence

We are now ready to prove that the solutions of (IV.63) - (IV.68) with small data exist for all $\tau \geq 0$. First we need to relate initial data for (IV.32) - (IV.34)

to that for (IV.63) - (IV.68). Before we do so, we need to determine initial time t_0 from the given initial data $(\tilde{h}, \tilde{K}, \tilde{\varphi}, \tilde{\pi}, \tilde{E}, \tilde{B})$ on \mathbf{T}^n . Since we assume $\tilde{\varphi}$ to be close to the background solution, it is natural to determine the initial time by

$$\langle \tilde{\varphi} \rangle = \frac{2}{\lambda} \ln t_0 - \frac{1}{\lambda} c_0,$$

where $\langle \tilde{\varphi} \rangle$ denotes the mean value of $\tilde{\varphi}$ over \mathbf{T}^n ,

$$\langle \tilde{\varphi} \rangle = \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} \tilde{\varphi} dx.$$

Thus we define the initial data for (IV.63) - (IV.68) as follows

Definition IV.3. Let $V_0 > 0$, $p > 1$ and let $n \geq 3$ be an integer. Define λ by (IV.9), $V = V_0 e^{-\lambda\phi}$ and let c_0 be given by (IV.10). Let $(\tilde{h}, \tilde{K}, \tilde{\varphi}, \tilde{\pi}, \tilde{E}, \tilde{B})$ be given on \mathbf{T}^n , where \tilde{h} is a Riemannian metric, \tilde{K} is a symmetric covariant 2-tensor, $\tilde{\varphi}$ and $\tilde{\pi}$ are smooth functions, \tilde{E} is a 1-form and \tilde{B} is a 2-form on \mathbf{T}^n . Define the **initial time associated with** $(\tilde{h}, \tilde{K}, \tilde{\varphi}, \tilde{\pi}, \tilde{E}, \tilde{B})$ as

$$t_0 = \exp \left[\frac{1}{2} (\lambda \langle \tilde{\varphi} \rangle + c_0) \right], \quad (\text{IV.129})$$

and define the **initial data for (IV.63) - (IV.68) associated with**

$(\tilde{h}, \tilde{K}, \tilde{\varphi}, \tilde{\pi}, \tilde{E}, \tilde{B})$ to be

$$u(0, \cdot) = 0, \quad (\partial_\tau u)(0, \cdot) = 2np - 2t_0(\text{tr}\tilde{K}), \quad (\text{IV.130})$$

$$u_i(0, \cdot) = 0, \quad (\partial_\tau u_i)(0, \cdot) = \frac{1}{2}t_0\tilde{h}^{kl}(2\partial_k\tilde{h}_{li} - \partial_i\tilde{h}_{kl}), \quad (\text{IV.131})$$

$$h_{ij}(0, \cdot) = \tilde{h}_{ij}, \quad (\partial_\tau h_{ij})(0, \cdot) = 2t_0\tilde{K}_{ij} - 2p\tilde{h}_{ij}, \quad (\text{IV.132})$$

$$\psi(0, \cdot) = \tilde{\varphi} - \langle \tilde{\varphi} \rangle, \quad (\partial_\tau \psi)(0, \cdot) = t_0\tilde{\pi} - \frac{2}{\lambda}, \quad (\text{IV.133})$$

$$A_0(0, \cdot) = 0, \quad \partial_\tau A_0(0, \cdot) = -\frac{1}{2}t_0\tilde{h}^{ij}(\tilde{B}_{ji} + \partial_i\tilde{B}_{jk}x^k), \quad (\text{IV.134})$$

$$A_i(0, \cdot) = -\frac{1}{2}\tilde{B}_{ik}x^k, \quad \partial_\tau A_i(0, \cdot) = -t_0\tilde{E}_i. \quad (\text{IV.135})$$

where all the indices are with respect to the standard coordinates on \mathbf{T}^n . \square

Before we prove global existence, we define

$$\kappa = \ln[4\ell(t_0)]. \quad (\text{IV.136})$$

Since we defined $\kappa = \tau_0 + \kappa_0$, we have

$$\kappa_0 = \ln \frac{4}{p-1}. \quad (\text{IV.137})$$

Theorem IV.3. *Let η and a be given by (IV.106) and (IV.107) respectively and let $k_0 > n/2 + 1$ be an integer. Let $(\tilde{h}, \tilde{K}, \tilde{\varphi}, \tilde{\pi}, \tilde{E}, \tilde{B})$ be given on \mathbf{T}^n , where \tilde{h} is a Riemannian metric, \tilde{K} is a symmetric covariant 2-tensor, $\tilde{\varphi}$ and $\tilde{\pi}$ are smooth functions, \tilde{E} is a 1-form and \tilde{B} is a 2-form on \mathbf{T}^n . Define the initial data for (IV.63) - (IV.68) by (IV.130) - (IV.135), where $\tau_0 = \ln t_0$ and t_0 is given by*

(IV.129). Let κ be given by (IV.136). Assume there is a constant $c_1 > 2$ such that

$$\frac{2}{c_1}|v|^2 \leq e^{-2\kappa} h_{ij}(0, x) v^i v^j \leq \frac{c_1}{2}|v|^2 \quad (\text{IV.138})$$

holds for any $v \in \mathbf{R}^n$ and $x \in \mathbf{T}^n$. There exist constants $\epsilon_0 > 0$ and $c_b \in (0, 1)$, which should be small enough and depend on n, k_0, p and c_1 , such that if

$$\tilde{E}_{k_0}^{1/2}(0) \leq c_b \epsilon, \quad (\text{IV.139})$$

holds for some $\epsilon \leq \epsilon_0$, then there is a global solution to (IV.63) - (IV.68).

Furthermore, (IV.99) - (IV.101) together with

$$\tilde{E}_{k_0}^{1/2}(\tau) \leq \epsilon, \quad (\text{IV.140})$$

hold for all $\tau \geq 0$. □

Proof. Let $0 < \epsilon \leq 1$ and let \mathcal{A} denote the set of $s \in [0, \infty)$ such that i) (g, ψ, A) satisfy the main bootstrap assumption $\mathcal{MBA}\{a, c_1, \eta, \kappa_0, \tau_0, \epsilon\}$ on $I = [0, s)$; ii) (g, ψ, A) is a solution to (IV.63) - (IV.68) on $I \times \mathbf{T}^n$ with initial data specified by (IV.130) - (IV.135).

Note that the initial value problem (IV.63) - (IV.68) with initial data (IV.130) - (IV.135) is equivalent to that of (IV.32) - (IV.34) with initial data defined by (IV.35) - (IV.41). Applying Proposition 1 of [27] to (IV.32) - (IV.34) with initial data (IV.35) - (IV.41), we obtain a unique smooth solution to (IV.32) - (IV.34) on some time interval (T_{min}, T_{max}) . Hence we also obtain the same unique smooth solution to (IV.63) - (IV.68) on the same interval. Assuming $c_b \leq 1/2$, then (IV.140) is satisfied with a margin for $\tau = 0$, so it is satisfied on an open time

interval containing 0. It follows from (IV.138), $u(0, \cdot) = 0$ and $u_i(0, \cdot) = 0$ that (IV.99) - (IV.101) are satisfied on an open time interval containing 0. Thus a solution of (IV.63) - (IV.68) exists and $\mathcal{MBA}\{a, c_1, \eta, \kappa_0, \tau_0, \epsilon\}$ is satisfied on an open time interval containing 0.

So far, we have shown that \mathcal{A} is nonempty. In order to show $\mathcal{A} = [0, \infty)$, we need to show that it is closed and open. Assume $0 < T < \infty$ and $T \in \mathcal{A}$. From the bootstrap assumptions and the equations, u , u_i , h_{ij} , ψ and A_μ do not blow up in C^2 . Due to (IV.99) and (IV.100), g_{00} and the eigenvalues of $\{h_{ij}\}$ are bounded away from zero. It follows from Proposition 1 of [27] that $T < T_{max}$. Thus there is a smooth solution beyond T , and the bootstrap assumption $\mathcal{MBA}\{a, c_1, \eta, \kappa_0, \tau_0, \epsilon\}$ holds on $[0, T]$. This proves that \mathcal{A} is closed.

We still need to prove that \mathcal{A} is open. Let $T \in \mathcal{A}$. There is a smooth solution beyond T from the above arguments. To prove that \mathcal{A} is open, we need to show that we can improve the bootstrap assumptions in $[0, T)$. Note that Lemma IV.6 applies on the interval $[0, T)$. Thus (IV.122) - (IV.126) hold on $[0, T)$. It has been shown in the proof of Theorem 4 of [28] that (replace “ \hat{H} ” by “ \tilde{E} ” and “lp” by “ \mathcal{LS} ”)

$$\tilde{E}_{\mathcal{LS}, k_0}^{1/2}(\tau) \leq C_{\mathcal{LS}}(c_b \epsilon + \epsilon^2)e^{-a\tau/2}, \quad (\text{IV.141})$$

$$\tilde{E}_{\mathcal{M}, k_0}(\tau) \leq C_{\mathcal{M}}(c_b \epsilon + \epsilon^2)\epsilon, \quad (\text{IV.142})$$

$$\tilde{E}_{\mathcal{SH}, k_0}^{1/2} \leq \frac{C_{\mathcal{SH}}}{2a}(c_b^{1/2} \epsilon + \epsilon^{3/2}), \quad (\text{IV.143})$$

$$\tilde{E}_{\mathcal{SP}, k_0}^{1/2}(\tau) \leq C_{\mathcal{SP}}(c_b \epsilon + \epsilon^2)e^{-a\tau/2}, \quad (\text{IV.144})$$

assuming $c_b \leq 1$. We only need to deal with $\tilde{E}_{\mathcal{VP},k_0}(\tau)$. Note that with $k = k_0$, (IV.126) differs from (IV.124) only by the decaying term $-2p\tilde{E}_{\mathcal{VP},k_0}$, thus

$$\tilde{E}_{\mathcal{VP},k_0}(\tau) \leq C_{\mathcal{VP}}(c_b\epsilon + \epsilon^2)\epsilon \quad (\text{IV.145})$$

assuming $c_b \leq 1$. By assuming c_b and ϵ to be small enough, depending on $C_{\mathcal{LS}}$, $C_{\mathcal{M}}$, $C_{\mathcal{SH}}$, $C_{\mathcal{SP}}$ and $C_{\mathcal{VP}}$, we conclude that

$$\tilde{E}_{k_0}^{1/2}(\tau) \leq \frac{1}{3}\epsilon$$

holds in $[0, T)$. Thus \mathcal{A} is open and the theorem follows. \square

Theorem IV.4. *Consider a solution to (IV.63) - (IV.68) constructed in Theorem IV.3. For every integer $k \geq 0$, the inequality*

$$\tilde{E}_k^{1/2}(\tau) \leq C_k. \quad (\text{IV.146})$$

holds for all $\tau \geq 0$ and for some constant C_k . \square

Proof. As a result of Theorem IV.3, we have (IV.122) - (IV.126) for all $\tau \geq 0$.

Define

$$\tilde{E}_{s,k} = e^{-a\tau/2}\tilde{E}_{\mathcal{SH},k}, \quad \tilde{E}_{ls,k} = e^{a\tau/2}\tilde{E}_{\mathcal{LS},k}.$$

From (IV.122) and (IV.123), we obtain

$$\begin{aligned} \frac{d\tilde{E}_{ls,k}}{d\tau} &\leq -a\tilde{E}_{ls,k} + C\epsilon e^{-3a\tau/4}\tilde{E}_{ls,k}^{1/2}\tilde{E}_k^{1/2}, \\ \frac{d\tilde{E}_{s,k}}{d\tau} &\leq -2a\tilde{E}_{s,k} + C\epsilon e^{-a\tau/4}\tilde{E}_{s,k}^{1/2}(\tilde{E}_{\mathcal{LS},k}^{1/2} + \tilde{E}_{\mathcal{M},k}^{1/2}) + C\epsilon e^{-5a\tau/4}\tilde{E}_{s,k}^{1/2}\tilde{E}_k^{1/2}. \end{aligned}$$

Due to (IV.124), (IV.125) and the above inequalities, we obtain

$$\frac{d\mathcal{E}_k}{d\tau} \leq Ce^{-a\tau/4}\mathcal{E}_k + CE_{\mathcal{LS},k_0}^{1/2}\tilde{E}_{\mathcal{M},k}, \quad (\text{IV.147})$$

where

$$\mathcal{E}_k = \tilde{E}_{ls,k} + \tilde{E}_{s,k} + \tilde{E}_{\mathcal{M},k} + \tilde{E}_{\mathcal{SP},k} + \tilde{E}_{\mathcal{VP},k}.$$

Since $\tilde{E}_{\mathcal{M},k_0}$ is bounded for all $\tau \geq 0$ and due to (IV.147), we have

$$\frac{d\mathcal{E}_{k_0}}{d\tau} \leq Ce^{-a\tau/4}\mathcal{E}_{k_0}.$$

Thus \mathcal{E}_{k_0} is bounded and consequently, $\tilde{E}_{\mathcal{LS},k_0}^{1/2} \leq Ce^{-a\tau/4}$. Together with (IV.147), this inequality yields

$$\frac{d\mathcal{E}_k}{d\tau} \leq Ce^{-a\tau/4}\mathcal{E}_k.$$

Thus \mathcal{E}_k is bounded for all k and this leads to $\tilde{E}_{\mathcal{LS},k}$, $\tilde{E}_{\mathcal{M},k}$, $\tilde{E}_{\mathcal{SP},k}$ and $\tilde{E}_{\mathcal{VP},k}$ being bounded. Applying this fact to (IV.123), we conclude that

$$\frac{d\tilde{E}_{\mathcal{SH},k}}{d\tau} \leq -2a\tilde{E}_{\mathcal{SH},k} + Ce^{-a\tau}\tilde{E}_{\mathcal{SH},k} + CE_{\mathcal{SH},k}^{1/2}.$$

Assuming τ to be large enough, the second term can be absorbed into the first term. Thus, $\tilde{E}_{\mathcal{SH},k}$ is bounded since it decays as soon as it exceeds certain value.

This proves the theorem. \square

IV.7. Causal Geodesic Completeness

Proposition IV.1. *Consider a solution to (IV.63) - (IV.68) constructed in Theorem IV.3. Let γ be a future directed causal curve on $[s_0, s_{max})$ with $\gamma^0(s_0) = t_0$*

(t_0 is given by (IV.129)). If the ϵ in Theorem IV.3 is small enough (depending only on n , p and c_1), then $\dot{\gamma}^0 > 0$ and

$$\int_{s_0}^{s_{max}} [g_{ij}(t_0, \gamma_b) \dot{\gamma}^i \dot{\gamma}^j]^{1/2} ds \leq d(\epsilon) \ell(t_0), \quad (\text{IV.148})$$

holds, where $d(\epsilon)$ is independent of γ , $d(\epsilon) \rightarrow 1$ as $\epsilon \rightarrow 0$, and where $\gamma_b(s) = (\gamma^1(s), \gamma^2(s), \dots, \gamma^n(s))$. Furthermore, assuming that γ is future inextendible, we have $\gamma^0(s) \rightarrow \infty$ as $s \rightarrow s_{max}$. \square

Proof. Since γ is a future directed causal curve, we have the following inequalities

$$g_{\mu\nu} \dot{\gamma}^\mu \dot{\gamma}^\nu \leq 0, \quad (\text{IV.149})$$

$$g_{00} \dot{\gamma}^0 + g_{0i} \dot{\gamma}^i < 0. \quad (\text{IV.150})$$

Due to (IV.101), we obtain

$$|2g_{0i} \dot{\gamma}^0 \dot{\gamma}^i| \leq \eta^{1/2} |\dot{\gamma}^0|^2 + \eta^{-1/2} |g_{0i} \dot{\gamma}^i|^2 \leq \eta^{1/2} |\dot{\gamma}^0|^2 + \eta^{1/2} c_1^{-1} e^{2p\tau + 2\kappa - 2a\tau} \delta_{ij} \dot{\gamma}^i \dot{\gamma}^j.$$

From (IV.99), the last term is bounded by $\eta^{1/2} g_{ij} \dot{\gamma}^i \dot{\gamma}^j$. Applying (IV.100) and (IV.149), we find that

$$g_{ij} \dot{\gamma}^i \dot{\gamma}^j \leq c(\eta) \dot{\gamma}^0 \dot{\gamma}^0, \quad (\text{IV.151})$$

where $c(\eta) \rightarrow 1$ as $\eta \rightarrow 0+$. (IV.99) yields

$$\delta_{ij} \dot{\gamma}^i \dot{\gamma}^j \leq c_1 c(\eta) e^{-2p\tau - 2\kappa} \dot{\gamma}^0 \dot{\gamma}^0 = c_1 c(\eta) (t/t_0)^{-2p} e^{-2\kappa} \dot{\gamma}^0 \dot{\gamma}^0. \quad (\text{IV.152})$$

Due to (IV.140) and Sobolev embedding, we obtain, cf. (IV.111),

$$e^{a\tau-2\kappa}\|\partial_\tau h_{ij}\|_\infty \leq C\epsilon.$$

Consequently,

$$\|(t/t_0)^{-2p}e^{-2\kappa}g_{ij}(t, \cdot) - e^{-2\kappa}g_{ij}(t_0, \cdot)\|_\infty \leq Ca^{-1}\epsilon,$$

where C only depends on n , p and c_1 . Combining this with (IV.152), we obtain

$$|e^{-2\kappa}g_{ij}(t_0, \gamma_b)\dot{\gamma}^i\dot{\gamma}^j - (t/t_0)^{-2p}e^{-2\kappa}g_{ij}\dot{\gamma}^i\dot{\gamma}^j| \leq Ca^{-1}\epsilon c_1 c(\eta)(t/t_0)^{-2p}e^{-2\kappa}\dot{\gamma}^0\dot{\gamma}^0.$$

Note that η in (IV.100) and (IV.101) can be replaced by $C\epsilon$, where C only depends on n , p and c_1 , due to (IV.109), (IV.110) and (IV.140). The above inequality together with (IV.151) yields

$$e^{-2\kappa}g_{ij}(t_0, \gamma_b)\dot{\gamma}^i\dot{\gamma}^j \leq d^2(\epsilon)(t/t_0)^{-2p}e^{-2\kappa}\dot{\gamma}^0\dot{\gamma}^0, \quad (\text{IV.153})$$

where $d(\epsilon) \rightarrow 1$ as $\epsilon \rightarrow 0+$. Consider (IV.150). Due to (IV.101) and (IV.152), we find that

$$|g_{0i}\dot{\gamma}^i| \leq [e^{-2p\tau-2\kappa}\delta^{ij}g_{0i}g_{0j}]^{1/2}[e^{2p\tau+2\kappa}\delta_{ij}\dot{\gamma}^i\dot{\gamma}^j]^{1/2} \leq \xi(\epsilon)|\dot{\gamma}^0|,$$

where $\xi(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0+$. By making ϵ small enough (depending only on n , p and c_1), we find that $\dot{\gamma}^0 > 0$. Combine this result with (IV.153), we obtain (IV.148).

Finally, assume that γ is future inextendible and suppose that γ^0 does not tend to ∞ . Since $\dot{\gamma}^0 > 0$, γ^0 has to converge to a finite number. Due to (IV.152), γ_b must converge to a point on \mathbf{T}^n . We have a contradiction. \square

Proposition IV.2. *Consider a solution to (IV.63) - (IV.68) constructed in Theorem IV.3. If we assume that the ϵ in Theorem IV.3 is small enough (depending on n , p and c_1), then the spacetime is future causally geodesically complete. \square*

Proof. Let γ be a future directed causal geodesic and assume that (s_{min}, s_{max}) is the maximum range of the proper parameter. Let $t = \gamma^0(s)$. The geodesic equation for $\gamma'' = 0$ yields

$$\ddot{\gamma}^0 + \Gamma_{\mu}{}^0{}_{\nu} \dot{\gamma}^{\mu} \dot{\gamma}^{\nu} = 0. \quad (\text{IV.154})$$

Due to (IV.140) and the algorithm of Subsection 9.1 of [27], we have the estimates

$$\begin{aligned} |\Gamma_0{}^0{}_0| &\leq C\epsilon(p/t)e^{-a\tau}, \\ |\Gamma_0{}^0{}_i| &\leq C\epsilon(p/t)e^{p\tau+\kappa-a\tau}, \\ |\Gamma_i{}^0{}_j - (p/t)g_{ij}| &\leq C\epsilon(p/t)e^{2p\tau+2\kappa-a\tau}. \end{aligned}$$

Consequently, for t large enough or ϵ small enough, we have $\Gamma_i{}^0{}_j \dot{\gamma}^i \dot{\gamma}^j \geq 0$.

Combining these estimates with (IV.152), we conclude that

$$|\Gamma_0{}^0{}_0 \dot{\gamma}^0 \dot{\gamma}^0| + 2|\Gamma_0{}^0{}_i \dot{\gamma}^0 \dot{\gamma}^i| \leq C\epsilon(p/t)e^{-a\tau} |\dot{\gamma}^0|^2,$$

where C only depends on n , p and c_1 . From these conclusions and (IV.154), we find that

$$\ddot{\gamma}^0 \leq C\epsilon(p/t)e^{-a\tau} \dot{\gamma}^0 \dot{\gamma}^0 = C\epsilon(p/t)(t/t_0)^{-a} \dot{\gamma}^0 \dot{\gamma}^0,$$

for $s \geq s_1$. Note that $\dot{\gamma}^0 > 0$ and assume that ϵ is small enough (depending only on n, p and c_1). If we divide this equation by $\dot{\gamma}^0$ and integrate, we find that

$$\ln \frac{\dot{\gamma}^0(s)}{\dot{\gamma}^0(s_1)} \leq C\epsilon p \int_{s_1}^s t^{-1}(t/t_0)^{-a} \dot{\gamma}^0 ds = C\epsilon p \int_{\gamma^0(s_1)}^{\gamma^0(s)} t^{-1}(t/t_0)^{-a} dt \leq C\epsilon p/a,$$

where we let s_1 be large enough such that $\gamma^0(s_1) \geq t_0$ if necessary. It follows that $\dot{\gamma}^0$ is bounded away from 0. Hence we have

$$\gamma^0(s) - \gamma^0(s_1) = \int_{s_1}^s \dot{\gamma}^0(s) ds \leq C|s - s_1|,$$

for some constant C . Since $\gamma^0(s) \rightarrow \infty$ as $s \rightarrow s_{max}$, we conclude that $s_{max} = \infty$.

Thus γ is future complete. □

IV.8. Asymptotic Expansions

We obtain some detailed information about the asymptotic behavior in the following proposition.

Proposition IV.3. *Consider a spacetime constructed in Theorem IV.3. If the ϵ in Theorem IV.3 is small enough (depending only on n, p, c_1 and k_0), then there is a smooth Riemannian metric H on \mathbf{T}^n and for every integer $l \geq 0$, a constant α_l (depending only on n, l, p and c_1) such that for all $t \geq t_0$, we have the following*

asymptotic expansions

$$\|\phi(t, \cdot) - \frac{2}{\lambda} \ln t + \frac{c_0}{\lambda}\|_{C^l} + \|(t\partial_t\phi)(t, \cdot) - \frac{2}{\lambda}\|_{C^l} \leq \alpha_l(t/t_0)^{-a}, \quad (\text{IV.155})$$

$$\|E_i\|_{C^l} = \|\partial_i A_0 - \partial_0 A_i\|_{C^l} \leq \alpha_l e^\kappa (t/t_0)^p (t/t_0)^{-1-a}, \quad (\text{IV.156})$$

$$\|B_{ij}\|_{C^l} = \|\partial_i A_j - \partial_j A_i\|_{C^l} \leq \alpha_l e^{2\kappa} (t/t_0)^{2p} (t/t_0)^{-1-a}, \quad (\text{IV.157})$$

$$\|(1 + g_{00})(t, \cdot)\|_{C^l} + \|(t\partial_t g_{00})(t, \cdot)\|_{C^l} \leq \alpha_l (t/t_0)^{-a}, \quad (\text{IV.158})$$

$$\begin{aligned} \left\| \frac{1}{t} g_{0i}(t, \cdot) - \frac{1}{(n-2)p+1} H^{jm} \gamma_{jim} \right\|_{C^l} + \left\| t \partial_t \left(\frac{1}{t} g_{0i} \right)(t, \cdot) \right\|_{C^l} \\ \leq \alpha_l (t/t_0)^{-a}, \end{aligned} \quad (\text{IV.159})$$

$$\begin{aligned} \|e^{-2\kappa} (t/t_0)^{-2p} g_{ij}(t, \cdot) - H_{ij}\|_{C^l} \\ + \|e^{-2\kappa} (t/t_0)^{-2p} (t\partial_t g_{ij})(t, \cdot) - 2p H_{ij}\|_{C^l} \leq \alpha_l (t/t_0)^{-a}, \end{aligned} \quad (\text{IV.160})$$

$$\|e^{2\kappa} (t/t_0)^{2p} g^{ij}(t, \cdot) - H^{ij}\|_{C^l} \leq \alpha_l (t/t_0)^{-a}, \quad (\text{IV.161})$$

$$\|e^{-2\kappa} (t/t_0)^{-2p} t K_{ij}(t, \cdot) - p H_{ij}\|_{C^l} \leq \alpha_l (t/t_0)^{-a}, \quad (\text{IV.162})$$

where γ_{jim} are the Christoffel symbols associated with the metric H and $K_{ij}(t, \cdot)$ are the components of the second fundamental form induced on the hyperspace $\{t\} \times \mathbf{T}^n$ by $g_{\mu\nu}$ with respect to the standard coordinates on \mathbf{T}^n . In the above inequalities $\|\cdot\|_{C^l}$ denotes the C^l norm on \mathbf{T}^n . \square

Remark IV.1. Although it seems that the electromagnetic field grows exponentially from (IV.156) and (IV.157), the electromagnetic field really decays. From our asymptotic expansions, the observed electric field and magnetic field decay as $(t/t_0)^{-(1+a)}$. Much stronger decay rate can be expected with further analysis. \square

Proof. (IV.146) combined with (IV.108), (IV.109), (IV.113) and (IV.114) gives us (IV.155) - (IV.158). From (IV.111), the C_l norm of $e^{-2\kappa} \partial_\tau h_{ij}$ decays as $e^{-a\tau}$. Thus

there exist some smooth functions H_{ij} such that for every integer $l \geq 0$,

$$\|e^{-2\kappa}h_{ij}(\tau, \cdot) - H_{ij}\|_{C_l} \leq \alpha_l e^{-a\tau}$$

holds for some α_l and for all $\tau > 0$. As a consequence, we obtain (IV.160).

Consider

$$e^{2\kappa}\partial_\tau(e^{2p\tau}g^{ij}) = 2pe^{2p\tau+2\kappa}g^{ij} - e^{2p\tau+2\kappa}g^{i\mu}g^{j\nu}\partial_\tau g_{\mu\nu}.$$

The C_l norm of the right hand side of the above decays as $e^{-a\tau}$. It follows that there exist some smooth functions H^{ij} on \mathbf{T}^n such that (IV.161) holds for all $\tau > 0$. We also conclude from above that $H^{ij}H_{jk} = \delta_k^i$, which implies H_{ij} is a Riemannian metric on \mathbf{T}^n and H^{ij} is the inverse.

To obtain (IV.159) and (IV.162), we need to the apply the argument of [28].

The same argument applies here with one more term to be estimated, i.e. $F_{0\sigma}F_i^\sigma - \frac{1}{2(n-1)}g_{0i}F_{\alpha\beta}F^{\alpha\beta}$, which has good decay property. We refer reader to pages 119 – 203 of [28] for details. \square

IV.9. Proof of the Main Theorem

Now we prove our main results Theorem IV.1 and Theorem IV.2.

Proof. Since most of the proof is the same as that of the corresponding proof in [28, 27], we briefly sketch the ideas in the following steps.

Step 1: construction of a global in time patch. We construct a patch of development of the data on U , of which we have control. To apply Theorem IV.3 to the piece of initial data on U , we need to construct initial data on \mathbf{T}^n . Define a cutoff function $f_c(x) \in C_0^\infty(B_1(0))$ such that $f_c(x) = 1$ for $|x| \leq 15/16$ and

$0 \leq f_c(x) \leq 1$. Define initial data $(\tilde{h}, \tilde{K}, \tilde{\varphi}, \tilde{\pi}, \tilde{E}, \tilde{B})$ on \mathbf{T}^n as follows

$$\tilde{h}_{ij} = f_c(h_{ij} \circ x^{-1}) + (1 - f_c)e^{2\kappa}\delta_{ij}, \quad (\text{IV.163})$$

$$\tilde{K}_{ij} = f_c(K_{ij} \circ x^{-1}) + (1 - f_c)\frac{p}{t_0}e^{2\kappa}\delta_{ij}, \quad (\text{IV.164})$$

$$\begin{aligned} \tilde{\varphi} &= f_c\varphi \circ x^{-1} + (1 - f_c)\langle\varphi\rangle \\ &\quad - \frac{1 - f_c}{1 - \langle f_c \rangle} [\langle f_c(\varphi \circ x^{-1}) \rangle - \langle f_c \rangle \langle \varphi \rangle], \end{aligned} \quad (\text{IV.165})$$

$$\tilde{\pi} = f_c(\pi \circ x^{-1}) + (1 - f_c)\frac{2}{\lambda t_0}, \quad (\text{IV.166})$$

$$\tilde{E} = f_c(E \circ x^{-1}), \quad (\text{IV.167})$$

$$\tilde{B} = f_c(B \circ x^{-1}), \quad (\text{IV.168})$$

where t_0 and κ are as in the statement of Theorem IV.1. The indices on the right hand side are with respect to the coordinates on U defined by $x : U \rightarrow B_1(0)$ and that on the left hand side are with respect to the standard coordinates on \mathbf{T}^n . Note that the last term in (IV.165) is to ensure that t_0 defined in Theorem IV.3 equals that defined in Theorem IV.1. From the initial data set $(\tilde{h}, \tilde{K}, \tilde{\varphi}, \tilde{\pi}, \tilde{E}, \tilde{B})$ constructed above, we further define the initial data for (IV.63) - (IV.68) according to (IV.130) - (IV.135).

From (IV.16), we conclude, cf. Section 11 of [28], that

$$\tilde{E}_{k_0}^{1/2}(0) \leq C\epsilon,$$

where the constant depends on n and p . We also have (IV.138) for some $c_1 > 2$, κ_0 only depending on p and k_0 only depending on n . As a consequence, by assuming ϵ to be small enough depending on n , p and applying Theorem IV.3, we obtain a

solution (g', ψ', A') on $(t_-, \infty) \times \mathbf{T}^n$ for some $0 < t_- < t_0$. Due to Proposition IV.3, we also have the asymptotic expansions (IV.155) - (IV.162).

Thus, we obtain a solution (g', ϕ', A') to the modified field equations (IV.32), (IV.33) and (IV.34). Furthermore, since the constraints (IV.12) - (IV.14) are satisfied on $B_{15/16}(0)$ and the initial data are constructed such that $\mathcal{D}_\mu(t_0, \cdot) = \partial_t \mathcal{D}_\mu(t_0, \cdot) = 0$ and $\mathcal{G}(t_0, \cdot) = \partial_t \mathcal{G}(t_0, \cdot) = 0$, by local existence and uniqueness of solutions to hyperbolic equations, cf. Proposition 1 of [27], the solution (g', ϕ', A') satisfies the original Einstein-Scalar-Maxwell system of equations (IV.1) - (IV.3). By assuming ϵ to be small enough, we get from Proposition IV.1, cf. Section 11 of [28], that

$$(t_-, \infty) \times B_{5/8}(0) \subseteq D[\{t_0\} \times B_{29/32}(0)], \quad (\text{IV.169})$$

where we increase t_- if necessary. Define the sets

$$U_{0,exc} = D[\{t_0\} \times B_{15/16}(0)], \quad U_{1,exc} = D[\{t_0\} \times B_{29/32}(0)],$$

$$U_{2,exc} = D[\{t_0\} \times \bar{B}_{29/32}(0)],$$

and $W_{i,exc} = (Id \times x^{-1}) U_{i,exc}$ for $i = 0, 1, 2$.

Step 2: construction of a reference metric. To show that the patches of spacetime development fit together to give a Cauchy development, we construct a reference metric

$$\tilde{g} = (1 - f_c \circ x)(-dt^2 + h) + (f_c \circ x)(Id \times x)^* g',$$

which is Lorentzian on $(t_-, \infty) \times \Sigma$, cf. Section 11 of [28].

Step 3: construction of local in time patches. To obtain a Cauchy development of the initial data on Σ , we also need patches that are developments of initial data on subsets of Σ that are not contained in U . Let $p \in \Sigma$ be a point. Consider an open subset S of Σ such that we have coordinates $\{y^1, \dots, y^n\}$ on S and $p \in S$. Define coordinates $\{y^0, y^1, \dots, y^n\}$ on $\mathbf{R} \times S$ with $y^0 = t$. Consider the equations

$$\begin{aligned} \nabla^\alpha \nabla_\alpha \phi - V'(\phi) &= 0, \quad \nabla^\mu \tilde{F}_{\mu\nu} = \nabla^\mu F_{\mu\nu} + \partial_\nu (\mathcal{G} - \mathcal{D}_\gamma A^\gamma) = 0, \\ \tilde{R}_{\mu\nu} &= \partial_\mu \phi \partial_\nu \phi + \frac{2}{n-1} V(\phi) g_{\mu\nu} + F_{\mu\sigma} F_\nu{}^\sigma - \frac{1}{2(n-1)} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}, \end{aligned}$$

where

$$\begin{aligned} \tilde{R}_{\mu\nu} &= R_{\mu\nu} + \nabla_{(\mu} \mathcal{D}_{\nu)}, \quad \mathcal{D}_\mu = \tilde{\Gamma}_\mu - \Gamma_\mu, \quad \tilde{\Gamma}_\mu = g_{\mu\nu} g^{\alpha\beta} \tilde{\Gamma}_\alpha{}^\nu{}_\beta, \\ \mathcal{G} &= \nabla^\mu A_\mu, \quad \tilde{F}_{\mu\nu} = F_{\mu\nu} + g_{\mu\nu} (\mathcal{G} - \mathcal{D}_\gamma A^\gamma). \end{aligned}$$

In the above equations, $\tilde{\Gamma}_\alpha{}^\nu{}_\beta$ is the Christoffel symbol of the reference metric \tilde{g} . Let $\mathcal{D}_\mu = \mathcal{G} = 0$ on S . Since the constraints (IV.12) - (IV.14) are satisfied on S , we have $\partial_t \mathcal{D}_\mu = \partial_t \mathcal{G} = 0$ on S . By local existence and uniqueness results and arguments similar to that of [28, 27], we obtain a piece of development (W_p, g_p, ϕ_p, A_p) satisfying (IV.1) - (IV.3), where W_p is a spacetime neighborhood containing p .

Step 4: gluing together of patches and embedding into a maximal Cauchy development. Finally we glue the global in time patch $W_{1,exc}$ and the local in time patches W_p together to get a Cauchy development (M, g, ϕ, A) of the initial data on Σ . For details, see corresponding arguments of Section 16 of [27]. \square

IV.10. Conclusion

In this chapter we proved the asymptotic stability of the following inflationary cosmological model

$$\begin{aligned}\hat{g} &= -dt^2 + (t/t_0)^{2p}\delta_{ij}dx^i dx^j, \\ \hat{\phi} &= \frac{2}{\lambda} \ln t - \frac{c_0}{\lambda}, \\ \hat{A}_\mu &= 0,\end{aligned}$$

as a solution to the Einstein-Maxwell-Scalar field equations

$$\begin{aligned}R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= T_{\mu\nu}, \\ \nabla^\mu \nabla_\mu \phi - V'(\phi) &= 0, \\ \nabla^\mu F_{\mu\nu} &= 0,\end{aligned}$$

where

$$\begin{aligned}\lambda &= \frac{2}{[(n-1)p]^{1/2}}, \quad c_0 = \ln \left[\frac{(n-1)(np-1)p}{2V_0} \right], \quad V(\phi) = V_0 e^{-\lambda\phi}, \\ T_{\mu\nu} &= \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi + V(\phi) \right) + (F_{\mu\sigma} F_\nu{}^\sigma - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}).\end{aligned}$$

There are other models where one might be able to obtain similar conclusions. These other models include charged scalar fields and multiple scalar fields. Along with considering various fields coupled to the Einstein equations, one can also consider the models with non-flat spatial slices.

CHAPTER V

CONCLUSION

This dissertation has presented the results of a study in two subjects of mathematical general relativity. The first subject is on the properties of cosmological Cauchy horizons. The existence of a cosmological Cauchy horizon signals causality violation. The Strong Cosmic Censorship (SCC) conjecture is that causality violation is non-generic. Hence the study of cosmological Cauchy horizons is important to the understanding of SCC. Our results on cosmological Cauchy horizons support SCC. We proved the existence of a Killing vector field in the spacetime neighborhood of a Cauchy horizon for a large class of spacetimes (Theorem II.1). This shows that the class of spacetimes considered, which contain a compact Cauchy horizon with closed null geodesics, are non-generic. Theorem II.1 is a generalization of a theorem obtained by V. Moncrief and J. Isenberg [20] to higher dimensions. In their theorem, only 4 dimensional spacetimes with electromagnetic fields have been considered. Along with extending the theorem of V. Moncrief and J. Isenberg, we also remove the restrictive analyticity condition (Corollary III.1) for vacuum spacetimes.

The second subject is on the stability of inflationary cosmological models. We proved asymptotic stability for the model

$$\begin{aligned}\hat{g} &= -dt^2 + (t/t_0)^{2p}\delta_{ij}dx^i dx^j, \\ \hat{\phi} &= \frac{2}{\lambda} \ln t - \frac{c_0}{\lambda}, \\ \hat{A}_\mu &= 0,\end{aligned}$$

as a solution to the Einstein-Maxwell-Scalar field equations. The precise statement of the result is contained in Theorem IV.1 and IV.2. Theorem IV.1 says that the perturbed spacetimes are also future causally geodesically complete if the perturbation is small and Theorem IV.2 says that the perturbed spacetimes of Theorem IV.1 asymptotically decay to the model spacetime. Our result generalizes a theorem obtained by H. Ringström [28] to include electromagnetic fields.

APPENDIX

GLOSSARY

Cauchy horizon A Cauchy horizon is the boundary of the domain of dependence of a given spacetime subset.

Cauchy surface A Cauchy surface is a spacelike hypersurface of a spacetime such that the spacetime is an evolution of the data given on that hypersurface.

Diffeomorphism A diffeomorphism $\phi : M \rightarrow N$ between two smooth manifolds M and N is a smooth bijective map such that there exists a smooth inverse.

Dominant energy condition (DEC) The energy momentum tensor $T_{\mu\nu}$ satisfies the dominant energy condition (DEC) if for all future directed timelike vector V , $-T^\mu{}_\nu V^\nu$ is a future directed timelike or null vector. It can be interpreted as the condition that the speed of energy flow is always less than the speed of light.

Fiber bundle (locally trivial) A (locally trivial) fiber bundle is a four-tuple (E, B, F, p) , where E , B and F are topological spaces and are called the total space, the base space and the fiber respectively, and that $p : E \rightarrow B$ is a map, such that for each point $x \in B$, there is a neighborhood $U \subset B$ of x , such that $p^{-1}(U)$ is homeomorphic to $U \times F$. Moreover, the homeomorphism $\varphi_U : p^{-1}(U) \rightarrow U \times F$ satisfies $P \circ \varphi_U = p : p^{-1}(U) \rightarrow U$ (local triviality), where $P : U \times F \rightarrow U$ is the projection map.

Killing vector field A Killing vector field is a vector field on a smooth manifold with a metric such that the flow generated by the vector field leaves the metric invariant.

Lorentzian metric A Lorentzian metric is a metric such that if expressed with respect to any given basis, the matrix of metric coefficients has one negative eigenvalue and all the other eigenvalues are positive.

Maximal Cauchy development (MCD) A maximal Cauchy development (M, g) of initial data (Σ, h, k) is a spacetime development of the initial data (Σ, h, k) such that any other spacetime development of the same initial data can be isometrically mapped into a subset of (M, g) .

Null geodesically incomplete A spacetime is null geodesically incomplete if there exists an inextendible null geodesic with a finite proper parameter range.

Null hypersurface A hypersurface of a spacetime is null if the normal vector of the hypersurface is null at every point on the hypersurface.

Product bundle A product bundle (also called a trivial bundle) (E, B, F, p) is a fiber bundle such that E is homeomorphic to $B \times F$.

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