Monetary Policy, Indeterminacy and Learning

George W. Evans — University of Oregon
Bruce McGough — Oregon State University

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Abstract

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1 Introduction

The development of tractable forward looking models of monetary policy, together with the influential work of (Taylor 1993), has lead to an explosion of research on the implications of adopting Taylor-type interest rate rules. These rules take the nominal interest rate as the policy instrument and direct the central bank to set this rate according to some simple (typically linear) dependence on current, lagged, and/or expected inflation and output gap, and possibly on an inertial term to encourage interest rate smoothing.

While these simple policy rules for many reasons are advantageous to both researchers and policy makers, it has been noted by some authors, e.g. (Bernanke and Woodford 1997), (Woodford 1999) and (Svensson and Woodford 1999), that the corresponding models exhibit indeterminate steady-states for large regions of the reasonable parameter space. This presence of indeterminacy is thought undesirable because associated with each indeterminate steady-state is a continuum of sunspot equilibria, and the particular equilibrium on which agents ultimately coordinate may not exhibit wanted properties.

Though having their informal origins in Keynes' notion of animal spirits, analysis of sunspots has, in the past, been couched principally in the theoretical literature. However, applied macroeconomists began to take notice when, in the mid nineties, (Farmer and Guo 1994) showed that calibrated real business cycle models, modified to include externalities or other non-convexities, exhibited sunspots; and furthermore, these sunspots could be used to explain fluctuations at business cycle frequencies. This applied interest has spread to the literature on monetary policy, and, in an empirical sense, has culminated with the argument of (Clarida, Gali, and Gertler 2000) that the volatile inflation and output of the seventies may have been due to sunspot phenomena. In particular, they combine a standard forward-looking "New Keynesian" IS-AS model[1] with a simple estimated forward-looking Taylor rule, using data from the 1960's and 1970's, and find that the corresponding steady-state is indeterminate; they conclude that the fluctuations in output gap and inflation may be well explained by agents coordinating on a volatile sunspot equilibrium.

[1] This model has also been called the "New Phillips Curve" or "optimizing IS-AS" model, and is obtained as a linearization of an optimimizing equilibrium model with "Calvo" pricing. For discussion, derivation and citations to the earlier literature, see (Clarida, Gali, and Gertler 1999), (Woodford 1999) and (Woodford 2003).

The existence of sunspot equilibria raises the question of whether it is plausible that agents will actually coordinate on them. One natural criterion for this is that the sunspot equilibria should be stable under adaptive learning.[2] Although it has been shown by (Woodford 1990) that stable sunspots can exist in simple overlapping generations models,[3] the sunspots in many calibrated applied models are lacking this necessary stability. For example (Evans and Honkapohja 2001) show that sunspots in the Farmer-Guo model are unstable, and (Evans and McGough 2002a) describe a stability puzzle surrounding the lack of stable indeterminacies in a host of non-convex RBC-type models.[4]

The existence of indeterminacies in monetary models, together with the instability of indeterminacies in RBC-type models, raises a natural question: Are sunspot equilibria in the New Keynesian models stable under learning? This specific question has been addressed by (Honkapohja and Mitra 2001), who consider a purely forward looking AS equation (" Phillips" curve) and analyze a variety of interest rules including those dependent on current, lagged, and expected inflation and output gap, and those also dependent on an interest rate smoothing term. They find that if the interest rate depends only on expected inflation and expected output gap then there can exist stable equilibria that depend on finite state sunspots; otherwise, the sunspot equilibria they consider are not learnable.[5]

Independent of the monetary policy literature, work on multiple equilibria and stability in macroeconomic models has continued, and recent research has emphasized that stability under learning of sunspots can depend upon the way in which a particular equilibrium is viewed, or represented. (Evans and Honkapohja 2003c) found that finite state sunspots in a simple forward looking model can be stable even though previous research had suggested

[2] Eductive approaches could also be considered. See, for example, (Guesnerie 1992), (Evans and Guesnerie 2003) and (Desgranges and Negroni 2001). Stability under eductive learning appears to be somewhat more stringent than stability under adaptive learning.

[3] For the local stability conditions see (Evans and Honkapohja 1994) and (Evans and Honkapohja 2003b).

[4] Other examples of stable sunspot equilibria include (Howitt and McAfee 1992), (Evans, Honkapohja, and Romer 1998), (Evans, Honkapohja, and Marimon 2001) and (Evans and McGough 2003).

[5] For these models stability under learning of "fundamental" (minimal state variable) solutions has been studied by (Bullard and Mitra 2003), (Evans and Honkapohja 2003d) and others. For a survey with references see (Evans and Honkapohja 2003a). An important early instability result was obtained by (Howitt 1992).

that no stable sunspots exist in these models. The apparent paradox is resolved as follows: all sunspot equilibria in these models can be represented as a linear dependence on lagged endogenous variables and a sunspot variable taking the form of a martingale difference sequence. These representations are always unstable under learning. However, when the sunspot is a finite state Markov process, the associated equilibrium is also finite state and thus has an alternate representation depending solely on the sunspot. When represented in this manner, the associated learning dynamics indicate stability for some (but not all) regions of the parameter space.

In (Evans and McGough 2003), we studied sunspot equilibria in a univariate stochastic linear forward looking model that incorporates a lag. We found that *any* given equilibrium may be viewed, or represented, in two fundamentally different ways: in the usual way, as a linear dependence on once and twice lagged endogenous variables and on a sunspot having zero conditional mean; and in a new way, on once lagged endogenous variables and on a sunspot exhibiting serial correlation. We referred to the usual way of viewing sunspots as the "general form" representation of the equilibrium, and to the new way of viewing sunspots as the "common factor" representation of the equilibrium.[6] We found that the stability of the equilibrium in question depended on the chosen representation. In particular, for the model we considered, stable common factor sunspots were found to exist in abundance, even though, as was already well known, there exist no stable general form sunspots.

This new line of research indicates the need for careful analysis of sunspot stability in applied models. *Every sunspot equilibrium has a common factor representation, and the stability properties of common factor representations are different from their general form counterparts.* Thus, stability analysis must incorporate both general form and common factor representations. In this paper, we generalize common factor analysis to apply to standard models of monetary policy, and carefully investigate the stability of the resulting representations. We follow (Bullard and Mitra 2002) and (Honkapohja and Mitra 2001) by specifying a simple New Keynesian IS-AS model, except that, for added generality, as in (Galí and Gertler 1999) and much applied work, we

[6] In (Evans and McGough 2002b) we show that common factor sunspot representations exist in some cases where finite state Markov sunspot solutions do not exist.

We remark that (Evans and McGough 2002b) and (Evans and McGough 2003) focus on models with real roots, but we show elsewhere that coomon factor sunspots representations exist more generally when there are complex roots.

allow for some dependence on lagged inflation in the Phillips curve. We close the model with a variety of interest rate rules: like Bullard and Mitra, we consider rules depending on current, lagged, and expected inflation; and like Honkapohja and Mitra, we also consider rules depending on lagged nominal interest rates. For each model we consider three calibrations of the IS-AS structure, as well as some alternative parameter values. Analytic results are, in general, unavailable, and so we test stability numerically by considering, for each calibration, a lattice over the space of policy parameters. At each point in the lattice, indeterminacy and stability of the corresponding equilibria are examined. Our main result supports the findings and advice of Honkapohja and Mitra, and indeed it makes their cautionary note more urgent: All models in which the policy rule depends on some form of expectations of future variables exhibit stable common factor sunspots for some parameter values. To be sure, these parameter values are not always reasonable, but, in some cases, they closely match calibrations. Furthermore, these stable sunspots exist even when the policy rule also depends on other aggregates, such as current inflation or output, and lagged interest rate. We also find that no general form sunspots are stable, thus emphasizing the importance of analyzing common factor representations.

This paper is organized as follows. Section two presents the various monetary models under consideration, as well as the associated learning theory and the extension of common factor analysis to monetary models. To conserve space and facilitate comprehension, we include explicit computations of equilibrium representations in the Appendix and for only one policy rule, and simply note that the remaining policy rules can be analyzed in a similar fashion. Section three contains the results of our investigations. The policy rules are classified into four types and discussed in separate subsections. In each case we consider numerous permutations of calibration, Phillips curve structure, indeterminacy nature, and representation type, and thus a careful catalog of all possible results would be tedious if not infeasible. Therefore, we provide a summary of the main features followed by a more careful discussion of the particularly interesting results. Section four concludes.

2 Theory

In this section we develop the theory necessary to analyze the stability of sunspot equilibria. We begin by specifying the models of interest. Then, for expedience, we choose a particular specification and develop the associated equilibrium representations and learning analysis. It is straightforward to modify this developed theory for application to other model specifications, and thus we omit the details concerning these other models. We initially develop the theory under the rational expectations assumption. Then, beginning in Section 2.4, we relax this assumption and study the stability of the solutions under adaptive learning.

2.1 Monetary Models and Policy Rules

We explore the possibility of existence of stable sunspots using several variants of the New Keynesian Monetary model. All specifications have in common the following forward looking IS-AS curves:

$$IS : x_t = -\phi(i_t - E_t\pi_{t+1}) + E_t x_{t+1} + g_t \quad (1)$$
$$AS : \pi_t = \beta(\gamma E_t\pi_{t+1} + (1-\gamma)\pi_{t-1}) + \lambda x_t + u_t \quad (2)$$

Here x_t is the proportional output gap, π_t is the inflation rate, and g_t and u_t are independent, exogenous, stationary, zero mean AR(1) shocks with damping parameters $0 \leq \rho_g < 1$ and $0 \leq \rho_u < 1$ respectively.

Equation (1) is the "IS" relationship obtained by log-linearizing the Euler equation for consumer optimization and using the GDP identity. This yields a unit coefficient on $E_t x_{t+1}$.[7] When $\gamma = 1$, equation (2) is the pure forward-looking New Keynesian "AS" relationship based on "Calvo pricing," and employed in (Clarida, Gali, and Gertler 1999) and Ch. 3 of (Woodford 2003).[8] Here $0 < \beta < 1$ is the discount factor. Again, this equation is obtained as the linearization around a steady state. The specification of the AS curve in the case $0 < \gamma < 1$ incorporates an inertial term and is similar in spirit to (Fuhrer and Moore 1995), the Section 4 model of (Galí and Gertler 1999), and the Ch. 3, Section 3.2 model of (Woodford 2003), each of which allows for some backward looking elements. Models with $0 < \gamma < 1$ are often called "hybrid" models, and we remark that in some versions, such as (Fuhrer and Moore 1995), $\beta = 1$, so that the sum of forward and backward looking components sum to one, while in other versions $\beta < 1$ is possible.

The region and nature of a model's indeterminacy depends critically on the specification of the policy rule. To better understand the role of this

[7] See, for example, Chapter 2 and Chapter 4, Section 1, of (Woodford 2003).

[8] For the version with mark-up shocks see (Woodford 2003) Chapter 6, Section 4.6.

specification, we analyze a number of policy rules, which we parameterize as follows:

$$
\begin{aligned}
PR_1 : i_t &= \alpha_\pi \pi_t + \alpha_x x_t && (3)\\
PR_1' : i_t &= \alpha_\pi E_t \pi_t + \alpha_x E_t x_t && (4)\\
PR_2 : i_t &= \alpha_\pi \pi_{t-1} + \alpha_x x_{t-1} && (5)\\
PR_3 : i_t &= \alpha_\pi E_t \pi_{t+1} + \alpha_x E_t x_{t+1} && (6)\\
PR_4 : i_t &= \theta i_{t-1} + (1-\theta)\alpha_\pi E_t \pi_{t+1} + (1-\theta)\alpha_x x_t && (7)
\end{aligned}
$$

PR_1, PR_1', PR_2, and PR_3 are the rules examined by (Bullard and Mitra 2002). We have omitted the intercepts for convenience, and in each policy rule π_t can be interpreted as the deviation of inflation from its target. These are all Taylor-type rules in the spirit of (Taylor 1993). We assume throughout that $\alpha_\pi, \alpha_x \geq 0$ and thus the $\alpha_\pi \pi_t$ term in PR_1 indicates the degree to which monetary policy authorities raise nominal interest rates in response to an upward deviation of π_t from its target. PR_1 assumes that current data on inflation and the output gap are available to policymakers when interest rates are set. Given the criticism that this assumption is not realistic, a point emphasized in (McCallum 1999), (Bullard and Mitra 2002) look at three natural alternatives: a slight modification yields PR_1' in which policy makers condition their instrument on expected values of current inflation and the output gap; in PR_2 policy makers respond to the most recent observed values of these variables; and in PR_3 they respond instead to forecasts of future inflation and the output gap.[9] Finally, PR_4 is the rule examined in the theoretical part of (Clarida, Gali, and Gertler 2000), and is the simplest form of the empirical interest rate rules that they estimate. A parameter value $0 < \theta < 1$ corresponds to inertia in interest rate setting, with policy-makers responding gradually to changes in information.

2.2 Determinacy

As usual, the model is said to be determinate if there is a unique nonexplosive REE and indeterminate if there are multiple nonexplosive solutions (though

[9] Because at the moment we are assuming rational expectations and a common information set, we do not need to specify whose forecasts are represented in the interest rate rules (4), (6) and (7). We will return to this matter when we discuss the economy under learning.

we will see below that this definition can be refined in a helpful way).[10] The determinacy of a model can be analyzed by writing the reduced form equation as a discrete difference equation with the associated extraneous noise terms capturing the errors in the agents' forecasts of the free variables. If the nonexplosive requirement of a rational expectations equilibrium pins down the forecast errors, that is, if the dimension of the unstable manifold is equal to the number of free variables, then the model is determinate. On the other hand, if the errors are not pinned down, that is, if the dimension of the unstable manifold is less than the number of free variables, these forecast errors can capture extrinsic fluctuations in agents' expectations that are not inconsistent with rationality. In this case, multiple equilibria exist; these types of equilibria are sometimes called sunspots.

To illustrate our methodology, consider PR_1 or PR_3. Combining the policy rule (3) or (6) with (1) and (2) leads to the first-order reduced form

$$H\begin{pmatrix} x_t \\ \pi_t \\ \pi_{t-1} \\ g_t \\ u_t \end{pmatrix} = F\begin{pmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ \pi_t \\ g_{t+1} \\ u_{t+1} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ w_t^g \\ w_t^u \end{pmatrix}.$$

The specific form of F, H are given in the Appendix for PR_1. Let the free variables be written $y_t = (x_t, \pi_t)'$, so that $\varepsilon_t = y_t - E_{t-1}y_t$ is the forecast error, thus capturing potential sunspots. Writing also $\hat{y}_t = (x_t, \pi_t, \pi_{t-1}, g_t, u_t)'$ and $w_t = (w_t^g, w_t^u)'$ the model can be rewritten as

$$\hat{y}_t = F^{-1}H\hat{y}_{t-1} + F^{-1}Mw_t + N\varepsilon_t \tag{8}$$

for suitable M, N. Here we are using the fact that F is invertible. Note that by virtue of the rational expectations assumption ε_t is a martingale difference sequence, i.e. a stochastic process such that $E_t\varepsilon_{t+1} = 0$.

Recall that a *Rational Expectations Equilibrium* (REE) is any process y_t that satisfies the reduced form equations and is nonexplosive. The above analysis has shown that if y_t is an REE then there is a martingale difference sequence (mds) ε_t such that the associated process $\hat{y}_t$ solves (8). However,

[10] By "nonexplosive" we mean that the conditional expectation of the absolute value of future variables is uniformly bounded over the horizon. For a detailed discussion of this and related concepts see (Evans and McGough 2003).

there is no guarantee that a given mds ε_t yields a nonexplosive solution; it is precisely this issue that is addressed by the nature of the indeterminacy.

To understand for which mds ε_t the model is nonexplosive, we assume that $F^{-1}H$ is diagonalizable and factor it as $F^{-1}H = S(\Lambda \oplus \rho)S^{-1}$. Here we employ the direct sum notation

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

for matrices A and B. ρ is a diagonal 2×2 matrix with diagonal elements ρ_g and ρ_u and Λ contains the remaining three eigenvalues of $F^{-1}H$. We then change coordinates to $z_t = S^{-1}\hat{y}_t$, thus allowing us to rewrite (8) as

$$z_t = (\Lambda \oplus \rho)z_{t-1} + \tilde{w}_t + \tilde{\varepsilon}_t, \tag{9}$$

where $\tilde{w}_t = S^{-1}F^{-1}Mw_t$ and $\tilde{\varepsilon}_t = S^{-1}N\varepsilon_t$. If the eigenvalues of $F^{-1}H$ are all real then the columns of S are the corresponding linearly independent eigenvectors. If two eigenvalues are complex then we assume that the matrix of eigenvectors, S and the matrix of eigenvalues Λ are altered to allow for a matrix factorization with real entries. This can be achieved via the following observation: If A is a real 2×2 matrix with complex eigenvalues $\mu \pm i\nu$ and complex eigenvectors $u \pm iv$ then

$$A = S \begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix} S^{-1}$$

where the columns of S are v and u. If A is $n \times n$ then it can be decomposed similarly as SDS^{-1} with D a block diagonal matrix with the real eigenvalues and 2×2 blocks corresponding to the complex eigenvalues on its diagonal. Also, here and throughout the paper, the eigenvalues in Λ are assumed ordered in decreasing magnitude.

Now the conditions for determinacy are clear. If λ_1 and λ_2 lie outside the unit circle, then nonexplosiveness requires $z_{it} = 0$ for $i = 1, 2$. Thus the forecast errors are pinned down by the requirement that $\tilde{w}_{it} + \tilde{\varepsilon}_{it} = 0$ for $i = 1, 2$; that is, the dimension of the unstable manifold, which in this simple linear framework is the direct sum of the eigenspaces corresponding to the explosive eigenvalues, is two, and there is a unique mds ε_t such that the associated process y_t is nonexplosive.

If $|\lambda_1| > 1$ and $|\lambda_2| < 1$ then the only implied restriction is that $z_{1t} = 0$. The forecast errors must satisfy $\tilde{w}_{1t} + \tilde{\varepsilon}_{1t} = 0$ but are otherwise unrestricted.

Thus there is a one dimensional continuum of equilibria, and, consequently, we say the model exhibits *order one indeterminacy*. Finally, if λ_i is in the unit circle for all i, the process y_t is nonexplosive regardless of the mds ε_t. There is a two dimensional continuum of equilibria, and we say the model exhibits *order two indeterminacy*.

We have focused on cases of determinacy and indeterminacy, but one other possibility should be noted. If $|\lambda_3| > 1$, so that there are three roots outside the unit circle, then the model is explosive: there exist no nonexplosive solutions and with probability one at least one of the components of y_t tends to infinity in absolute value as $t \to \infty$.

2.3 Representations

A *rational expectations equilibrium representation* (REER) is a discrete difference equation, any solution to which is an REE. As is now well known, see e.g. Chapters 8 and 9 of (Evans and Honkapohja 2001), (Evans and Honkapohja 2003c) and (Evans and McGough 2003), a given REE may have many representations, and the stability of the REE under learning may be representation dependent. In this subsection we construct the representations of interest, noting that the particular form of the representation depends on the nature of the indeterminacy.

Assume first that for $i = 1, 2$ we have $|\lambda_i| > 1$ and also assume that $|\lambda_3| < 1$. For the solution to be nonexplosive the mds sunspots ε_{1t} and ε_{2t} must be chosen so that $\tilde{\varepsilon}_{it} + \tilde{w}_{it} = 0$ for $i = 1$ and 2. The associated representation is given by

$$y_t = -(S_2^{11})^{-1} \begin{pmatrix} 0 & S^{13} \\ 0 & S^{23} \end{pmatrix} y_{t-1} - (S_2^{11})^{-1} S_2^{14} \hat{g}_t, \qquad (10)$$

where for convenience, we write $\hat{g}_t = (g_t, u_t)'$. Here and in the sequel, $S^{ij} = (S^{-1})_{ij}$ and

$$S_k^{ij} = \begin{pmatrix} S^{ij} & S^{ij+1} \\ S^{kj} & S^{kj+1} \end{pmatrix}.$$

Thus, in the determinate case, the unique nonexplosive solution takes the form

$$y_t = a + by_{t-1} + c\hat{g}_t, \qquad (11)$$

where $a = 0$ because in the structural equations we have omitted intercepts. We include the intercept term a here and below because under learning agents will be assumed to estimate its value.

2.3.1 Order One Indeterminacy.

Order one indeterminacy occurs when $|\lambda_1| > 1$ and the remaining eigenvalues have norm less than one. Notice this implies λ_1 is real; however λ_i for $i > 1$ may be complex. For reasons discussed below, in the indeterminate case, we only consider real eigenvalues. To obtain a nonexplosive solution, we require that $z_{1t} = 0$, and that the mds ε_t satisfy $\tilde{\varepsilon}_{1t} + \tilde{w}_{1t} = 0$. We now proceed to develop the general form and common factor representations.

General Form Representations: As is shown in the Appendix, imposing the restriction $z_{1t} = 0$ for all t leads to General Form (GF) representations

$$y_t = a + by_{t-1} + hy_{t-2} + c\hat{g}_t + f\hat{g}_{t-1} + e\xi_t. \tag{12}$$

where ξ_t is an arbitrary one-dimensional mds and $a = 0$. There are actually two representations of this form, i.e. two distinct nontrivial sets of parameter coefficients (b, h, c, d) which yield solutions of this form. These are obtained by combining $z_{1t} = 0$ with either the $i = 2$ or $i = 3$ equation from (9), i.e. with

$$z_{it} = \lambda_i z_{it-1} + \tilde{w}_{it} + \tilde{\varepsilon}_{it} \tag{13}$$

and then using the definition of z_t to rewrite the equation in term of y_t. The Appendix gives details. Note that the General Form representations express the REE as a dependence of the endogenous variables on two lags, current and lagged intrinsic noise, and a sunspot exhibiting no serial correlation.[11]

Common Factor Representations: As with the general form representation we impose $z_{1t} = 0$ and that the mds ε_t satisfy $\tilde{\varepsilon}_{1t} + \tilde{w}_{1t} = 0$. There will be two Common Factor (CF) representations, again obtained by combining $z_{1t} = 0$ with (13) for either $i = 2$ or $i = 3$. However, we now rewrite (13) as

$$z_{it} = (1 - \lambda_i L)^{-1}(\tilde{w}_{it} + \tilde{\varepsilon}_{it}).$$

We interpret the noise term on the right to be a sunspot ζ_t and thus write $z_{it} = \zeta_t$ with

$$\zeta_t = \lambda_i \zeta_{t-1} + \breve{\varepsilon}_t,$$

where $\breve{\varepsilon}_t = \tilde{w}_{it} + \tilde{\varepsilon}_{it}$. Note that since only one dimension of ε_t is restricted, we can take $\breve{\varepsilon}_t$ to be an arbitrary univariate mds. Combining this with the

[11] Our method for computing the GF representations (12) is closely related to the solution method for the "irregular case" described in Chapter 10, Appendix 2, of (Evans and Honkapohja 2001) and which was used to obtain the VARMA solutions given in (Honkapohja and Mitra 2001).

restriction $z_{1t} = 0$ and using the definition of z_t yields a CF representation of the form

$$y_t = a + by_{t-1} + c\hat{g}_t + d\zeta_t, \tag{14}$$

and $a = 0$. Again there are two CF representations of this form. Note that in these representations, the endogenous variables depend on one lag, current intrinsic noise, and a serially correlated sunspot.

For reasons that are now apparent, complex eigenvalues pose difficulties for common factor representations; if λ_i is complex, it is not possible to write the sunspot ζ_t as a serially correlated process with real damping parameter. This problem is not insurmountable - in fact we consider it in another paper - however, we feel it is best avoided for now, as our story is well told by focusing on the real case.[12] Therefore, throughout the paper, our analysis in the indeterminate region pertains to representations obtained via real eigenvalues.

2.3.2 Order Two Indeterminacy

General Form Representations: Now all eigenvalues are in the unit circle and thus there is no concern over the nonexplosiveness restriction. Pick real eigenvalues λ_i, and λ_j, where $i, j = 1, 2, 3$ and $i \neq j$. Combining the two equations (13) for i and j with the definition of z_t we obtain three representations of the form (12) except that now ξ_t is an arbitrary two-dimensional mds. See the Appendix for details on the required (b, h, c, d).

Common Factor Representations: Combining the two $i \neq j$ equations from (13) and defining the VAR sunspot

$$\zeta_t = (\lambda_i \oplus \lambda_j)\zeta_{t-1} + \begin{pmatrix} \tilde{w}_{it} \\ \tilde{w}_{jt} \end{pmatrix} + \begin{pmatrix} \tilde{\varepsilon}_{it} \\ \tilde{\varepsilon}_{jt} \end{pmatrix},$$

[12] In the case of complex eigenvalues, one must simultaneously incorporate the nonexplosiveness condition $z_{1t} = 0$ and *both* complex eigenvalues, thus representing the sunspot as a VAR. This does not pose a problem; however, once learning is incorporated, difficulties arise. Specifically, the PLM associated with a complex common factor representation is underspecified out of equilibrium, as there is no explicit dependence on lagged endogenous variables. This implies that the T-map must be formed using orthogonal projections. Again, this is straightforward, in theory; however, to compute the eigenvalues of the Jacobian of the T-map, one must differentiate endogenously determined second moments. We have worked out the details of this analysis for the models considered here, and initial investigations reveal no stable sunspots. Therefore, due to the technical nature of this exercise, as well as the fact that it does not appear to add important details to our current story, we present these results in a different work.

we obtain CF representations of the form (14) except that now of course ζ_t is two-dimensional. Again, further details on (b, c) are given in the Appendix.

2.3.3 Discussion

For PR_1 and PR_3 with $\gamma = 1$ the procedure to provide solution representations could be simplified since π_{t-1} no longer appears in the structural equations. π_{t-1} could thus be dropped from the first-order form of the model. However, the above analysis does also cover this case. If $\gamma = 1$ then in the determinate case $b = 0$. Similarly, when $\gamma = 1$ and the model is indeterminate, the general form representations satisfy $h = 0$. In this case one of the CF representations satisfies $b = 0$, with a serially correlated sunspot ζ_t, and the other CF representation has a serially uncorrelated sunspot with $b \neq 0$.

We also provide a brief discussion of REERs under policy rules PR_2 and PR_4. In the case of PR_2, given by (5), the state variable in the first-order form must be enlarged to include x_{t-1}. Then $\hat{y}_t = (x_t, \pi_t, \pi_{t-1}, x_{t-1}, g_t, u_t)'$ and z_t becomes 6×1. The methodology for obtaining solutions is analogous and in fact the form of the solution representations for $y_t = (x_t, \pi_t)$ is as above. For PR_4, given by (7), the state vector is written as $\hat{y}_t = (x_t, \pi_t, i_t, \pi_{t-1}, i_{t-1}, g_t, u_t)'$, z_t is 7×1 and $y_t = (x_t, \pi_t, i_t)'$. However, the procedure for determining REERs remains analogous.

We close this section with a brief remark on an aspect of the time series properties of sunspot equilibria. Consider the CF representations (14). If the exogenous sunspot variable ζ_t is independent of the intrinsic shocks $\hat{g}_t$, then it is easily verified that the endogenous variables y_t have larger variances than are present in the corresponding minimal state variable solution $y_t = a + by_{t-1} + c\hat{g}_t$. Policy makers that aim to minimize output and inflation volatility would thus want to avoid interest rate rules consistent with the existence of CF sunspot solutions, at least if these solutions are stable under learning. We now turn to this issue, i.e. to the question of the stability of the various solutions under least squares learning.

2.4 Learning

We use expectational stability as our criterion for judging whether agents may be able to coordinate on specific solutions, including in particular sunspot equilibria. This is because, for a wide range of models and solutions, E-stability has been shown to govern the local stability of rational expectations equilibria under least squares learning. In many cases this correspondence can be proved, and in cases where this cannot be formally demonstrated the "E-stability principle" has been validated through simulations. Before giving details, we provide an overview of E-stability; for further reading see (Evans and Honkapohja 2001).

The models analyzed in this paper can be written in reduced form as follows:[13]

$$y_t = AE_t^* y_{t+1} + By_{t-1} + C\hat{g}_t. \tag{15}$$

We now write $E_t^* y_{t+1}$ to indicate that we no longer impose rational expectations, and at issue is how agents form their time t expectations E_t^*. Backing away from the benchmark that agents are fully rational, we assume that agents believe the endogenous variable y_t depends linearly on lagged endogenous variables, current (and possibly lagged) exogenous shocks $\hat{g}_t$, and exogenous sunspots. The latter will either be serially uncorrelated or have an AR(1) structure. Combining these regressors into the vector X_t, we postulate a perceived law of motion (PLM) $y_t = \Theta' X_t$. Agents then use this perceived law of motion to form their expectations of y_{t+1}. A rational expectations solution will correspond to one or more values for the parameter vector Θ.

Under real-time learning agents will estimate Θ using an algorithm such as recursive least squares and these estimates will be updated over time. Given a particular value for Θ the corresponding expectations $E_t^* y_{t+1}$ can be computed, the expectations can be substituted in the reduced form equation above, and the true data generating process, or actual law of motion (ALM), thus determined. If the perceived law of motion is well specified then the actual law of motion will have the same form: $y_t = T(\Theta)' X_t$. In particular, the ALM will depend linearly on the same variables as did the PLM. Thus a map, known as the T-map, is constructed, taking the perceived parameters to the implied parameters. A fixed point of this map constitutes a representation of a rational expectations equilibrium.

We note that associated with a given reduced form model there may be multiple well-specified PLMs, and the specification of the PLM determines the representation of the REE that agents are trying to learn. For example, it is reasonable for X_t to include a constant, once lagged y, current $\hat{g}$, and the serially correlated sunspot ζ; in this case agents would be trying to learn a common factor representation. It is also reasonable for X_t to include a

[13] When PR_1' is used the reduced form also depends on expectations of contemporaneous endogenous variables $E_t^* y_t$. This extended reduced form is considered in the Appendix.

constant, once and twice lagged y, current and once lagged $\hat{g}$, and a mds noise term ξ; in this case agents would be trying to learn a general form representation. Finally, we note that a fixed point of the T-map defines not just an equilibrium, but also a representation of that equilibrium.

Once the T-map is obtained, the stability under learning of a particular representation can be addressed as follows. Let the equilibrium representation be characterized by the fixed point Θ^*, and consider the differential equation

$$\frac{d\Theta}{d\tau} = T(\Theta) - \Theta. \tag{16}$$

Notice that Θ^* is a rest point of this ordinary differential equation. The representation corresponding to the fixed point is said to be *E-stable* if it is a locally asymptotically stable equilibrium of (16). The E-stability principle tells us that E-stable representations are locally learnable for Least Squares and closely related algorithms. That is, if Θ_t is the time t estimate of the coefficient vector Θ, and if Θ_t is updated over time using recursive least squares, then Θ^* is a possible convergence point, i.e. locally $\Theta_t \to \Theta^*$ if and only if Θ^* is E-stable. The intuition behind this principle is that a reasonable learning algorithm, such as least squares, would gradually adjust estimates Θ_t in the direction of the actual parameters $T(\Theta_t)$ that are generating the data. For an E-stable fixed point Θ^* such a procedure would then be expected to converge locally.

The above discussion has implicitly assumed a rest point Θ^* that is locally isolated. In this case it is locally asymptotically stable under (16) provided all eigenvalues of the Jacobian of T at Θ^* have real parts less than one, and it is unstable if the Jacobian has at least one eigenvalue with real part greater than one. Because we are studying sunspot equilibria, the set of rest points of (16) may have unbounded continua as connected components. Along these components the T map will always be neutrally stable, and thus will have at least one eigenvalue equal to unity.[14] In this case we say a sunspot equilibrium representation is E-stable if the Jacobian of the T-map has eigenvalues with real part less than one, apart from unit eigenvalues arising from the equilibrium connected components.

We consider separately the determinate and two indeterminate cases.

[14]The number of unit eigenvalues will be equal to the dimension of these components.

2.4.1 Determinate case

The real and complex cases can be handled simultaneously. Agents are assumed to have the PLM (11). As indicated above, we make the (fairly standard) assumption that, for agents forming expectations at time t, the current value of y_t is not in the information set, but all time t exogenous variables, as well as lagged values of y, are known at t. From the PLM $y_t = a + by_{t-1} + c\hat{g}_t$ we compute $E_t^* y_{t+1} = a + bE_t^* y_t + cE_t^* \hat{g}_{t+1}$. Using $E_t^* y_t = a + by_{t-1} + c\hat{g}_t$, and assuming for convenience that ρ is known so that $E_t^* \hat{g}_{t+1} = \rho \hat{g}_t$, yields

$$E_t^* y_{t+1} = (I_2 + b)a + b^2 y_{t-1} + (bc + c\rho)\hat{g}_t.$$

Inserting this expression into (15) and solving for y_t as a linear function of an intercept, y_{t-1} and $\hat{g}_t$ yields the T-map given by

$$a \rightarrow A(I_2 + b)a \quad (17)$$

$$b \rightarrow Ab^2 + B \quad (18)$$

$$c \rightarrow A(bc + c\rho) + C. \quad (19)$$

The relevant Jacobians are given by

$$DT_a = A(I_2 + b) \quad (20)$$

$$DT_b = b' \otimes A + I_2 \otimes Ab \quad (21)$$

$$DT_c = I_2 \otimes Ab + \rho' \otimes A, \quad (22)$$

where $\otimes$ denotes the Kronecker product of two matrices.

2.4.2 Order One Indeterminacy

We employ the same notation as above and consider common factor and general form representations separately. We consider common factor representations first because their form is quite similar to the representation of determinate equilibria.[15] In each case we compute $E_t^* y_{t+1}$ for the assumed PLM, insert into (15) and solve for y_t as a linear function of the explanatory variables contained in the PLM. We omit the details, which are straightforward, and simply write down the T-map and corresponding Jacobians.

[15] This is not simple coincidence and helps explain why common factor representations may be stable when general form representations are not.

Common Factor Representations: If the roots are real then ζ_t is AR(1). Agents are assumed to have the PLM (14). The T-map is given by equations (17)-(19) and

$$d \to A(bd + d\lambda_i). \tag{23}$$

The relevant Jacobians are given by (20)-(22) and

$$DT_d = A(b + I_2(\lambda_i)),$$

where $I_2(x) = x \oplus x$.

General Form Representations: Agents are assumed to have the PLM (12). The corresponding T-map is given by equation (17) together with

$$\begin{aligned} b &\to A(b^2 + h) + B && (24)\\ h &\to Abh && (25)\\ c &\to A(bc + c\rho + d) + C && (26)\\ f &\to Abf && (27)\\ e &\to Abe. && (28) \end{aligned}$$

The relevant Jacobians are given by (20), (22), $DT_f = I_2 \otimes Ab$, $DT_e = Ab$, and

$$DT_{bh} = \begin{pmatrix} b' \otimes A + I_2 \otimes Ab & I_2 \otimes A \\ h' \otimes A & I_2 \otimes Ab \end{pmatrix}$$

2.4.3 Order Two Indeterminacy

The analysis is almost the same as for order one indeterminacy. Learning the CF-representation in the case of order two indeterminacy is affected only in that the sunspot is now a VAR so that the T-map in the d variable is amended to have the form

$$d \to Abd + Ad(\lambda_i \oplus \lambda_j).$$

The associated Jacobian is

$$DT_d = I_2 \otimes Ab + (\lambda_i \oplus \lambda_j) \otimes A.$$

Analysis of learning the GF-representation in case of order two indeterminacy is the same as for order one indeterminacy.

3 Results

We studied stability of general form and common factor sunspots in five models, which differed only in the specification of the monetary policy rule, and the models are identified by the number of the corresponding policy rule as given by equations (3)-(7). The models were analyzed using three different calibrations of the parameters in the IS-AS curves, as due to (Woodford 1999), (Clarida, Gali, and Gertler 2000) and (McCallum and Nelson 1999); the relevant parameter values are given in Table 1 below.

Table 1: Calibrations

Author(s)	ϕ	λ
W	$1/.157$	.024
CGG	1	.3
MN	.164	.3

For convenience in interpreting the numerical results below we note that $1/.157 \simeq 6.3694$.

Also, each policy rule was analyzed both with and without lagged inflation in the AS equation (or Phillips curve). With pure Calvo pricing, and thus no inflation inertia in the AS equation, the discount rate β was set equal to .99 following W, CGG, and MN[16]. When lagged inflation was included γ was set equal to one half, and we set $\beta = 1$, thereby imposing that the sum of the coefficients on inflation equals one. Finally, for all policy rules, the exogenous noise terms were taken to have damping parameter equal to .9. For each calibration (and for $\gamma = 1$ and .5), a lattice over the square $(0, 10) \times (0, 10)$ in policy space (α_π, α_x) was analyzed. For PR_4 we also computed results for several values of θ.

Some general results were found across all or most of the policy rules and calibrations investigated, and are therefore worth summarizing before presenting more specific results in detail. Throughout this section we will use "stable" to mean "stable under learning" as determined by E-stability.

1. In no case were General Form sunspot solutions stable.

[16] With $\gamma = 1$, the coefficient β modifying the expectations term in the AS equation results from the optimal price setting behavior of the agents. However, more traditional specifications of the Phillips curve often specify this coefficient equal to one. For the case $\gamma = 1$ and $\beta = 0.99$ we checked for robustness against $\gamma = 1$ and $\beta = 1$ and, except where indicated, found little impact on our results.

2. In the determinate case the unique nonexplosive solution is usually stable under learning: the exceptions are PR_2, and with low γ, PR_3, in which unstable cases exist.

3. The explosive case arises for PR_2, and with low values of γ, for all policy rules.

4. Order two indeterminacy exists only for PR_3 and is never stable.

5. Common Factor sunspot solutions exist[17] for all forms of the policy rule, but they are only stable for PR_3 and PR_4.

6. Stable CF sunspots arise with $\gamma = 0.5$ as well as in the purely forward looking case $\gamma = 1$.

3.1 Policy Rule 1

As noted in Section 2, policy rule 1 has a natural variant, which we label PR_1'. We summarize the results for PR_1 and PR_1' in separate subsections below.

3.1.1 PR_1

For ease of exposition, we restate, at the beginning of each subsection, the specification of the relevant policy rule. PR_1 is given by

$$PR_1 : i_t = \alpha_\pi \pi_t + \alpha_x x_t.$$

PR_1 with $\gamma = 1$ (i.e. no lagged inflation in the AS) has been analyzed by a number of authors, including (Bullard and Mitra 2002) and (Honkapohja and Mitra 2001). Bullard and Mitra found that the region in policy space corresponding to E-stability is precisely the region corresponding to determinacy; this result was obtained analytically and is independent of calibration.[18] Because of this result, Bullard and Mitra would recommend this policy rule if it

[17] In line with our earlier discussion, the statement "Common Factor sunspot solutions exist" is now to be interpreted as asserting the existence of CF sunspot representations based on real roots.

[18] (Honkapohja and Mitra 2001) obtain the additional result, for $\gamma = 1$, that what we call GF sunspot solutions are not stable. See their Proposition 4, which also covers PR_3 for the case $\gamma = 1$.

were feasible. However, as discussed above, it is widely agreed that current values of inflation and GDP are not available to policymakers. We include results for this policy rule primarily because it serves as a useful benchmark. We find that the result of Bullard and Mitra is robust to the inclusion of inflation inertia in the Phillips curve: in all cases investigated, determinacy implies stability under learning. In addition, when the model is indeterminate no solutions are stable, including CF sunspot solutions.

An example lattice is plotted in Figure 1 for the Woodford calibration with $\gamma = 1$. Here and in all figures containing lattice plots, each lattice point is marked with a symbol indicating properties of the associated steady state: lattice points associated with determinate steady states are marked with an '$\times$'; lattice points associated with indeterminate steady states and for which common factor representations exist (i.e. there exist at least two real eigenvalues) are marked with a '$*$', and lattice points associated with indeterminate steady states for which no common factor representations exist (i.e. there exists at most one real eigenvalue) are marked with a '$\cdot$'. Also, if there exists a stable representation associated with the steady state, the symbol marking the lattice point is circled. CF sunspots exist for all indeterminate cases, a finding that extends to the other calibrations with $\gamma = 1$.

Figure 1 Here

Figure 2 shows the regions of determinacy and indeterminacy pertaining to the CGG calibration with $\gamma = .5$. The main result that all CF and GF sunspots are unstable still holds, but the region of determinacy is altered and here it is no longer the case that CF sunspots always exist. The failure of CF sunspots to exist seems to depend principally on the inertial term in the Phillips curve. When γ is set equal to one, all indeterminate steady states support common factor sunspots. The MN calibration yields much the same picture as the CGG calibration. Note that the range of policy parameters displayed does not necessarily coincide with the 10×10 grid, and also varies across figures; this was done to emphasize features particular to given figures.

Figure 2 Here

One further result of interest (not shown) is that sufficiently low values of $\gamma > 0$ together with passive response to inflation (low α_π) can yield the explosive case.

Under our information assumptions, when policymakers use PR_1 they effectively have an information advantage relative to private agents. This is because policymakers are conditioning policy on contemporaneous endogenous variables, which are assumed not available to private agents when their forming expectations. This information asymmetry does not arise under PR_1' to which we now turn.

3.1.2 $\mathbf{PR}_1'$

PR_1' is given by

$$PR_1' : i_t = \alpha_\pi E_t^* \pi_t + \alpha_x E_t^* x_t.$$

This policy can be thought of as a contemporaneous rule that is feasible even if current values of the endogenous variables π_t and x_t are not known at time t. We continue to make the assumption that all t-dated exogenous variables and all lagged variables are observed prior to expectations formation. As discussed further in Section 3.3, below, we are also making the homogeneous expectations assumption that policymakers and private agents form expectations in the same way. In contrast to our PR_1, under PR_1' policy makers and private agents are treated as having the same information set.

In a rational expectations equilibrium $E_t \pi_t = \pi_t$ and $E_t x_t = x_t$. This implies that the REE, their representations, and the regions of determinacy, indeterminacy, and explosiveness will be precisely as they were under PR_1. However, out of equilibrium, agents may make errors when forecasting current values of the endogenous variables. In particular, because learning dynamics are in part determined by out of equilibrium behavior, the stability properties of the model under PR_1' may be different than under PR_1: see the Appendix for details on how the learning analysis is altered.

(Bullard and Mitra 2002) analyzed PR_1', though with a slightly different interpretation of the timing structure: they assume that expectations formed at t use only information available at time $t-1$. Although this differs from our assumption that exogenous variables at time t are part of the information set, it can be verified that the E-stability conditions are identical for the two information assumptions. Bullard and Mitra show, analytically, that their results for PR_1 hold also for PR_1'. In particular, the regions of determinacy are the same for both policy rules, and, for each policy rule, a steady state is stable under learning if and only if it is determinate.

We obtain the same correspondence and find that it also extends to the stability of sunspot solutions. We have already noted that the regions of determinacy, indeterminacy, and explosiveness are the same for PR_1 and PR_1'. In addition, our numerical analysis indicates that for all parameter combinations considered, all determinate steady-states are stable under learning, and no stable indeterminacies exist. Our results thus tend to reinforce those of (Bullard and Mitra 2002), who in consequence recommend Taylor rules of the form PR_1' with $\alpha_\pi > 1$.

3.2 Policy Rule 2

PR_2 is given by

$$PR_2 : i_t = \alpha_\pi \pi_{t-1} + \alpha_x x_{t-1}.$$

PR_2, with $\gamma = 1$, was also studied by Bullard and Mitra. Numerically, and using the Woodford calibration, they found that, unlike PR_1, there were determinate cases for which the REE was not stable under learning. They concluded that policy rules dependent on lagged output gap and inflation may not be advisable because agents may fail to coordinate on the equilibrium even though it is unique. We find that this result is robust to the calibrations considered here and extends to the specification with inertial inflation in the Phillips curve. Figure 3 gives an example plot using the CGG calibration and with $\gamma = .5$. In this Figure, lattice points left unmarked correspond to explosive steady states.

Figure 3 Here

Aggressive response to output gap and inflation may yield explosive steady states. In fact, using the Woodford calibration one obtains that even passive response to output gap ($\alpha_x > .35$), together with a Taylor rule $\alpha_\pi > 1$, can yield an explosive steady state. In the indeterminate region CF representations exist but they are not stable.

3.3 Policy Rule 3

PR_3 is given by

$$PR_3 : i_t = \alpha_\pi E_t^* \pi_{t+1} + \alpha_x E_t^* x_{t+1}.$$

Before giving the results we discuss the interpretation of this rule under learning. Under least squares learning private agents are assumed to recursively estimate the parameters of their PLM and use the estimated forecasting model to form the expectations $E_t^* \pi_{t+1}$ and $E_t^* x_{t+1}$ that enter into their decisions as captured by the IS and AS curves. Under PR_3 and PR_4 forecasts also enter into the policy rule. Because we are now relaxing the rational expectations assumption, one can in principle distinguish between the forecasts of the private sector, which enter the IS and AS curves, and the forecasts of the Central Bank, which enter policy rule PR_3 or PR_4. We will instead adopt the simplest assumption for studying stability under learning, which is that the forecasts for the private sector and the Central Bank are identical. This can either be because private agents and the Central Bank use the same least squares learning scheme, or it could be because one group relies on the others' forecasts. In the latter case, for example, the Central Bank might be setting interest rates as a reaction to private sector forecasts, as in (Bernanke and Woodford 1997) or (Evans and Honkapohja 2003a). The homogeneous expectations assumption was also adopted in (Bullard and Mitra 2002).[19] Since we are searching for stable sunspot equilibria, the homogeneous expectations assumption appears to give the greatest likelihood for finding them.

We now turn to the results. For policy rule PR_3, under the calibrations studied, all determinate steady states are stable and no explosive steady states are observed.[20] However, in contrast to PR_1 and PR_2, policy rule PR_3 can exhibit stable sunspots, and the region of stability may include economically reasonable parameter values. This result corroborates and extends those of (Honkapohja and Mitra 2001), who showed the existence of stable noisy K-state Markov sunspots for this policy rule. We discuss the relationship of our results to theirs below.

For the W and CGG calibrations, and for both values of γ, stable common factor representations exist. For the CGG calibration the region of stability requires aggressive ("active") policy response to both the output gap and to inflation, i.e. $\alpha_\pi > 1$ and $\alpha_x > 1$. Although the "Taylor principle" specifically recommends $\alpha_\pi > 1$, output gap responses of $\alpha_x > 1$ might be considered too aggressive. However, using the W calibration, stable CF sunspots arise with very plausible policy settings, including in particular the benchmark choice $\alpha_\pi = 1.5$ and $\alpha_x = 0.5$ discussed by (Taylor 1993). (For the MN calibrations, stable CF representations exist, but not in the 10×10 benchmark policy

[19] The implications of heterogeneous expectations in the context of the New Keyensian monetary model is examined in (Honkapohja and Mitra 2002). This issue is further discussed in (Evans and Honkapohja 2003a)

[20] As noted earlier, in case of very low γ, unstable determinacy and explosive steady states are present.

space: $\alpha_x > 12$ yields stable sunspots).

Consider Figure 4, showing the regions of stability of CF representations for the Woodford calibration with $\gamma = 1$.[21]

Figures 4, 5 Here

For this calibration stable CF sunspots appear for large regions of plausible policy parameters. For the Woodford calibration, the results are almost as dramatic for the case $\gamma = 0.5$, as can be seen in Figure 5. Recall that when we set $\gamma = 1$ we also set $\beta = 0.99$ in line with W, CGG and MN, whereas for the case $\gamma = 0.5$ we set $\beta = 1$, as in (Fuhrer and Moore 1995). The existence of stable CF sunspots does not depend on the choice of β, but the precise region is sensitive to this choice for PR_3: for $\beta < 1$ (and either value of γ) the region of stable CF sunspots includes regions of part of the passive policy region $\alpha_\pi < 1$.

We now relate these results to those found elsewhere in the literature. (Bullard and Mitra 2002) studied this model with $\gamma = 1$, and showed that all determinate equilibria were stable under learning. Our findings indicate that this result extends to models that include lagged inflation in the AS curve. Bullard and Mitra also found that for indeterminate steady-states the MSV solution[22] may be stable, and pointed out that whether agents could learn sunspots in this case was an open question. Clearly the answer to this question is a resounding yes. In particular, common factor representations can be thought of as MSV representations together with serially correlated sunspots, and we have found that these sunspots may be stable.

(Honkapohja and Mitra 2001) studied this model, with $\gamma = 1$, and demonstrated that finite state "resonance frequency" sunspots exist and are stable for a region of the parameter space. These solutions take the form

$$y_t = c\hat{g}_t + ds_t$$

where s_t is a $K \times 1$ vector representing a K-state Markov process with transition probabilities that satisfy particular conditions sometimes called "resonant frequency conditions." This result is consistent with and, in fact, suggestive of ours.

[21] In Figures 4 and 5, unmarked lattice points indicate order two indeterminacy.

[22] By MSV solution is meant a solution that depends on a minimal number of state variables. For a discussion see (McCallum 1983) and (Evans and Honkapohja 2001).

In (Evans and McGough 2002b) we show that there is an intimate link between CF representations and finite-state sunspots in univariate models. For the current model with $\gamma = 1$ our solutions (14) satisfy $b = 0$ and thus take the form

$$y_t = c\hat{g}_t + d\zeta_t,$$

where, in the case of order-one indeterminacy, ζ_t is an AR(1) process $\zeta_t = \lambda_i \zeta_{t-1} + \breve{\varepsilon}_t$.[23] Particular choices of the mds $\breve{\varepsilon}_t$ yield solutions of the form $y_t = c\hat{g}_t + ds_t$ with the required transition probabilities. When $0 < \gamma < 1$ our CF solutions take the form

$$y_t = by_{t-1} + c\hat{g}_t + d\zeta_t,$$

where now $b \neq 0$. The condition that the AR(1) coefficient is λ_i, for $i = 2, 3$, i.e. equal to a critical eigenvalue, is the *resonant frequency condition* for CF solutions. Our PR_3 result can thus be thought of as a generalization of the Honkapohja-Mitra finding: we extend the economic model to the case $\gamma < 1$, in which the model has backward looking components, and we exhibit and study the more general representations taking the form of CF sunspot solutions.

(Evans and Honkapohja 2003d) studied optimal discretionary policy in the model (1)-(2), with $\gamma = 1$, and advise an interest rate designed specifically to offset any destabilizing forward looking behavior of agents. Their recommended interest rate rule takes the form

$$i_t = \delta_\pi E_t^* \pi_{t+1} + \delta_x E_t^* x_{t+1} + \delta_{\hat{g}} \hat{g}_t,$$

where $\delta_\pi = 1 + \lambda\beta\phi^{-1}(\alpha + \lambda^2)^{-1}$, $\delta_x = \phi^{-1}$, $\delta_{\hat{g}} = \left(\phi^{-1}, \lambda\phi^{-1}(\alpha + \lambda^2)^{-1}\right)$ and $\alpha \geq 0$ parameterizes the weight placed by the policy maker on output relative to inflation volatility. Note that optimal policy requires a dependence on $\hat{g}_t$ as well as on inflation and output forecasts, but the presence of this term does not affect determinacy or stability.[24] (Evans and Honkapohja 2003d) show that this rule is invariably determinate and that the REE is always E-stable and hence stable under least squares learning. This rule is recommended in

[23] If $\breve{\varepsilon}_t$ is white noise then ζ_t is a stationary AR(1) process since $|\lambda_i| < 1$. For general mds $\breve{\varepsilon}_t$ the process ζ_t need not be stationary, but ζ_t will be nonexplosive in conditional mean (and ζ_t can be expressed as an absolutely summable infinite moving average process in $\breve{\varepsilon}_{t-s}$).

[24] In the reduced form, only the coefficient C of $\hat{g}_t$ is affected. This does not affect the E-stability conditions or the conditions for determinacy, as can be seen from the Appendix.

preference to other interest rate policies, such as fundamentals based rules depending only on $\hat{g}_t$, which they show to be unstable under learning even though they are consistent with the REE corresponding to optimal discretionary policy.

The size of the stable determinacy region surrounding this policy depends on the structural parameters. We investigated this point by analyzing nine lattices over the square $(0, 10) \times (0, 10)$ in policy space (α_π, α_x) corresponding to all permutations of $\lambda \in \{.024, .3, 1\}$ and $\phi \in \{.164, 1, 1/.157\}$, with $\gamma = 1$. In each case there are qualitatively similar regions: stable determinacy (Figure 4, region A) lies at least part way along the horizontal axis for $\alpha_\pi > 1$ and is bounded above by a downward sloping line, unstable indeterminacy (Figure 4, region B) lies at least part way along the vertical axis for $\alpha_\pi < 1$ and is bounded on the right by a downward sloping line. The area of stable CF sunspots (Figure 4, region C) is the region that remains. However, quantitatively we find the following: first, for fixed λ, as ϕ gets smaller, the region of stable indeterminacy shifts up, replaced by stable determinacy, and the region of unstable indeterminacy appears unaffected; second, for fixed ϕ, as λ gets smaller, the region of stable indeterminacy shifts up slightly, replaced by stable determinacy, and again, the region of unstable indeterminacy appears unaffected.[25]

The existence of stable sunspots in part of the parameter space provides an important *caveat* to following the advice of (Evans and Honkapohja 2003d). Policy makers may think the economy is in a determinate and stable region of its parameter space and thus that agents will learn the intended equilibrium; however, if policy makers are wrong about the values of the key parameters λ and ϕ, agents may instead coordinate on an inferior sunspot equilibrium.

Stability and determinacy respect small continuous movements in parameter values, and thus for any particular calibration, the Evans-Honkapohja rule will work well locally. (Evans and Honkapohja 2003d) also show that

[25] In the case $\gamma = 1$ analytic stability results are possible. We find that provided $\alpha_x \neq 1/\phi$ then necessary and sufficient conditions for the existence of stable sunspots are given by

$$\begin{aligned} \lambda(\alpha_\pi - 1) + \alpha_x(1-\beta) &> 0 \\ \lambda(\alpha_\pi - 1) + \alpha_x(1+\beta) &> 2(1+\beta)/\phi. \end{aligned}$$

These are precisely the same restrictions obtained by (Honkapohja and Mitra 2001) for noisy finite-state Markov sunspot solutions.

the system under learning remains locally stable even when policy makers are simultaneously updating estimates λ and ϕ.[26] However, numerical results suggest that the margin for error available to policy makers when attempting to follow the Evans-Honkapohja rule, can depend critically on the structural parameters. As an extreme, but perhaps not implausible example, we obtained results for $\lambda = 1$, $\phi = 6.3694$, and $\gamma = 1$. The value ϕ is the one used in the W calibration, and $\lambda = 1$ is within the range of estimates from the literature mentioned on p. 170, footnote 32, in (Clarida, Gali, and Gertler 2000).

Figure 6 Here

In Figure 6 we see that a triangle of stable determinacy exists, but it is bordered by unstable indeterminacy on the left, and, even more ominously, by stable indeterminacy on the right. We conclude that in some cases learnable sunspots abound in regions not far from those corresponding to optimal policy. Our findings also import a more general warning: simply following a Taylor rule with aggressive response to expected inflation is not necessarily stabilizing for the economy. This warning is emphasized for the parameter values used in Figure 6. Note that even for $\alpha_x = 0$, stable sunspots exist if $\alpha_\pi > 1.7$. In contrast, if the CGG values are correct then the Evans-Honkapohja rule is quite robust.

3.4 Policy Rule 4

PR_4 is given by

$$PR_4 : i_t = \theta i_{t-1} + (1-\theta)\alpha_\pi E_t^* \pi_{t+1} + (1-\theta)\alpha_x x_t,$$

where $\theta > 0$. This policy rule is of particular interest in part because it is the form of the rule specifically considered in part IV of (Clarida, Gali, and Gertler 2000). Furthermore, the issue of inertia in policy rules, captured by $\theta > 0$, has been discussed extensively in the literature. (Clarida, Gali, and Gertler 2000) use the value $\theta = 0.68$ based on (quarterly) estimates from the pre-Volker period, but there is no agreement that this is an appropriate value.

[26] That is, locally there is asymptotic convergence both of structural parameter estimates to their true values λ, ϕ and forecasts $E_t^* x_{t+1}, E_t^* \pi_{t+1}$ to their RE values. Hence locally the economy converges to the REE corresponding to optimal discretionary policy.

On the one hand (Rudebusch 2002) argues that the usual empirical evidence for monetary policy inertia may well be illusory and that the true value of θ may be zero or small. On the other hand (Rotemberg and Woodford 1999) have argued that $\theta > 1$ with $\alpha_\pi > 0$ may be close to optimal. Interest rate rules with $\theta > 1$ are often called "superinertial."

(Honkapohja and Mitra 2001) analyzed PR_4 numerically, for $\gamma = 1$, and found that for the CGG calibration of λ and ϕ and with their estimated values of α_x, α_π and θ for the pre-Volker era, the sunspot solutions that we call general form representations were unstable under learning. We confirm this result and we find also that although CF-sunspot solutions do exist, they are not stable under learning. Figure 7 shows the results for the CGG calibration of λ and ϕ with $\theta = 0.68$: in the 10×10 policy grid for (α_x, α_π), there are no stable CF sunspots, while all determinate steady states are learnable. Furthermore, these results are robust both to the inclusion of lagged inflation in the AS curve, $0 < \gamma < 1$, and to the magnitude of $0 < \theta < 1$.

Figure 7 Here

However, as with PR_3, we find that there are many cases in which stable CF sunspots exist, and the location of this region in policy space is very sensitive to the values of structural parameters assumed. Even with the CGG calibration for λ and ϕ and with $\theta = 0.68$, there exist stable CF sunspots for sufficiently large values of α_π. Furthermore, for other values of λ and ϕ we find that the possibility of stable CF sunspots needs to be taken seriously. For example, for the values $\phi = 1/.157$ and $\lambda = 1$ examined earlier, stable CF sunspots exist for passive responses to output gap, and aggressive responses to inflation; further, as θ gets small, the response to inflation required for stability becomes reasonably valued: see Figure 8 in which we set $\theta = 0.1$. A very similar figure is obtained in case $\gamma = .5$; in particular, the presence of inflation inertia in the AS curve does not preclude stable CF sunspots at reasonable parameter values.

Figure 8 Here

Because the region of stable indeterminacy depends on the structural parameters ϕ and λ, as well as the interest rate smoothing term θ, we again test the robustness of our results to alternative calibrations. We analyzed 27 lattices over the square $(0, 10) \times (0, 10)$ in policy space (α_π, α_x) corresponding to all permutations of $\lambda \in \{.024, .3, 1\}$, $\phi \in \{.164, 1, 1/.157\}$, and

$\theta \in \{.05, .5, .9\}$. For this exercise we set $\gamma = 1$. In general, there are two regions of indeterminacy: an unstable region along the vertical axis for $\alpha_\pi < 1$: see Figure 8, Region B; and a triangular region of stable CF sunspots in the southeast corner: see Figure 8, Region C; the remaining region corresponds to stable determinacy: see Figure 8, Region A. We find that as ϕ and λ get smaller, and as θ gets larger, the region of stable indeterminacy shifts to the right, replaced by stable determinacy. The unstable region appears unaffected.

One conclusion that emerges from this analysis is that while stable sunspots do exist under PR_4 for sufficiently aggressive responses to expected inflation, the policy maker may hedge against the danger, which depends on the true values of λ and ϕ, by setting the smoothing term $0 < \theta < 1$ to be fairly high.[27]

Finally, it is of interest to examine the case of superinertial rules with $\theta > 1$. To remain consistent with the superinertial rules already in the literature, we modify PR_4 so as to ensure that the coefficients on expected inflation and current output gap are positive:

$$PR_4' : i_t = \theta i_{t-1} + \chi_\pi E_t^* \pi_{t+1} + \chi_x x_t.$$

We examined equilibria corresponding to a 10×10 lattice over (χ_π, χ_x) policy space, and for $\theta = 1.1$ and 2, and $\gamma = 1$ and .5. For the W, CGG, and MN calibrations, this rule performed well; for all permutations of θ and γ and over the entire benchmark lattice the corresponding steady states were stable and determinate. However, stable indeterminacy was found for the alternative calibration $\phi = 1/.157$, and $\lambda = 1$: see Figure 9. The existence of these stable sunspots is robust to the permutations of θ and γ.

Figure 9 Here

3.5 Discussion

Sunspot equilibria are stable under learning only for Taylor-type policy rules that depend on forecasts of future inflation, and only for certain solution representations that we call "common factor" representations. However, for

[27](Bullard and Mitra 2003) consider in detail the impact of interest rate inertia on determinacy and stability of MSV solutions, and reach the same conclusion.

such forward looking rules stable CF sunspots are abundant. The location of the region of stable sunspot solutions depends on the structural parameters ϕ and λ in the IS and AS curves. When these parameters are large, the region of stable CF sunspots, in the policy parameter space, includes realistic values for feedback coefficients in the interest rate rule. In particular, following the Taylor principle $\alpha_\pi > 1$ is not sufficient to avoid stable CF sunspots, even if the output feedback α_x is small. Given uncertainty about the true values for ϕ and λ, the possibility of stable sunspots appears to be of genuine concern, and the possibility, in the indeterminate case, of all solutions being unstable is equally troubling.

Stable CF sunspot solutions can arise even if there are backward looking components to inflation and even if there is inertia (interest rate smoothing) in the monetary policy rule. This possibility had not been previously recognized in the literature. Interest rate inertia does, however, increase the region of stable determinacy relative to the benchmark policy square.

In general, inertia, or backward looking behavior, might be expected to reduce the risk of indeterminacy (i.e. reduce the proportion of the benchmark parameter space corresponding to indeterminate steady states). While we have demonstrated that inertial components in the AS curve and policy rule do not overturn our main results, it is possible that inertia in the IS curve – specifically, a dependence on lagged output gap, justified, say, by habit formation – may have an important impact. Investigation of this issue will receive a high priority in our future research.[28]

Our study of interest rate inertia focussed on interest rate rule PR_4. It would also be useful to investigate the influence on the regions of determinacy and stability of an interest rate smoothing term in rules PR_1 – PR_3. We restricted attention to the specification of these rules without the inertial term for expedience. However our preliminary investigations indicate that while the location and relative size of the regions of determinacy and stability are altered somewhat, the presence of interest rate inertia in policy rules PR_1 – PR_3 does not, in general, change our central results. This is also an issue that we intend to investigate more thoroughly in future work.

[28] In principle, stability under learning can be investigated using large scale macroeconometric models; see, for example, (Garratt and Hall 1997). Our techniques could be extended to study the stability of CF sunspot solutions in such models as well as to a larger class of policy rules.

4 Conclusion

This paper has examined the question of whether macroeconomic fluctuations, taking the form of coordination on extraneous exogenous variables, are likely to emerge under adaptive learning when the economy is characterized by New Keynesian IS-AS equations and monetary policy follows a form of Taylor rule. Both purely forward-looking and hybrid, partly backward-looking inflation equations were examined. We have emphasized that the possibility of "sunspot equilibria" that are stable under adaptive learning depends critically on the representation of the solution, i.e. on the econometric specification used by agents when they estimate and update their forecasting model.

In many cases stationary sunspot equilibria can be represented either as "general form" VARs, driven by serially uncorrelated sunspots, or as "common factor" sunspot solutions, in which the extraneous sunspot variables are autoregressive processes with resonant frequency coefficients. Common factor sunspots generalize finite state Markov sunspots, which were an early focus in the sunspot literature and which have recently been shown to yield the possibility of stable sunspots in purely forward looking linear models. In the New Keynesian model, we find that common factor sunspots can indeed be stable under learning, in many cases, even though the general form solutions with serially uncorrelated sunspots are not.

In particular, Taylor-type interest rate rules that depend on forecasts of future inflation can generate stable common factor sunspot solutions, and this risk is particularly high when there are strong IS and AS effects. This possibility arises even if the AS equation includes backward looking components and the interest rate rule includes inertia. This result is deeply troubling since monetary policy is often viewed as forward looking. If the structural model and its key parameters are known, or can be estimated fairly precisely, then an appropriately designed forward looking policy can deliver a stable determinate equilibrium (indeed an optimal stable equilibrium) and the sunspot problem will not arise. However, for some structural parameters the margin of error is small and the impact of an error is great. In contrast, policy rules depending on forecasts of current output and inflation do not appear to be subject to these difficulties.

Appendix

To illustrate the details of the technique we focus on the policy rule PR_1 given by (3). In this case

$$H = \begin{pmatrix} 1+\phi\alpha_x & \phi\alpha_\pi & 0 & -1 & 0 \\ -\lambda & 1 & \beta(\gamma-1) & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_g & 0 \\ 0 & 0 & 0 & 0 & \rho_u \end{pmatrix} \text{ and } F = \begin{pmatrix} 1 & \phi & 0 & 0 & 0 \\ 0 & \beta\gamma & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Order one indeterminacy. This occurs when $|\lambda_1| > 1$ and the remaining eigenvalues have norm less than one. Notice this implies λ_1 is real; however λ_i for $i > 1$ may be complex. For reasons discussed above, in the indeterminate case, we only consider real eigenvalues. To obtain a nonexplosive solution, we require $z_{1t} = 0$, and $\tilde{\varepsilon}_{1t} + \tilde{w}_{1t} = 0$.

General Form Representations: A general form representation is the usual recursive system describing the equilibrium and is characterized by a sunspot that forms a martingale difference sequence. Fix $i = 2$ or 3. We may then use the nonexplosiveness condition $z_{1t} = 0$ together with the equation

$$z_{it} = \lambda_i z_{it-1} + \tilde{w}_{it} + \tilde{\varepsilon}_{it}$$

to obtain the following representation.

$$\begin{aligned} y_t \;=\; & (S_i^{11})^{-1} \begin{pmatrix} 0 & -S^{13} \\ \lambda_i S^{i1} & \lambda_i S^{i2} - S^{i3} \end{pmatrix} y_{t-1} + (S_i^{11})^{-1} \begin{pmatrix} 0 & 0 \\ 0 & \lambda_i S^{i3} \end{pmatrix} y_{t-2} \\ & -(S_i^{11})^{-1} S_i^{14} \hat{g}_t + (S_i^{11})^{-1} \begin{pmatrix} 0 & 0 \\ \lambda_i S^{i4} & \lambda_i S^{i5} \end{pmatrix} \hat{g}_{t-1} \\ & +(S_i^{11})^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tilde{w}_{it} + (S_i^{11})^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tilde{\varepsilon}_{it}. \end{aligned}$$

The identities

$$\begin{aligned} \tilde{w}_{it} &= \left((FS)^{i4}, (FS)^{i5}\right) w_t \\ \tilde{\xi}_{it} &= \left(S^{i1}, S^{i2}\right) \varepsilon_t \\ w_t &= \hat{g}_t - \rho \hat{g}_{t-1} \end{aligned}$$

may be used to place this representation in the form given in the text.

Common Factor Representations: Again assume $|\lambda_1| > 1$ and the remaining eigenvalues have norm less than one. Let ε_t be a mds with $\tilde{\varepsilon}_{1t} + \tilde{w}_{1t} = 0$. Pick $i = 2$ or 3. The REE associated to ε_t must satisfy the equation

$$z_{it} = \lambda_i z_{it-1} + \tilde{w}_{it} + \tilde{\varepsilon}_{it},$$

or

$$z_{it} = (1 - \lambda_i L)^{-1}(\tilde{w}_{it} + \tilde{\varepsilon}_{it}).$$

We interpret the noise term on the right to be a sunspot ζ_t and thus write $z_{it} = \zeta_t$ with

$$\zeta_t = \lambda_i \zeta_{t-1} + \tilde{w}_{it} + \tilde{\varepsilon}_{it}.$$

Combining this with the restriction $z_{1t} = 0$ yields two common factor representations of the form

$$y_t = -(S_i^{11})^{-1} \begin{pmatrix} 0 & S^{13} \\ 0 & S^{i3} \end{pmatrix} y_{t-1} - (S_i^{11})^{-1} S_i^{14} \hat{g}_t + (S_i^{11})^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \zeta_t.$$

Order two indeterminacy.

General Form Representations: Now all eigenvalues are in the unit circle and thus there is no concern over the nonexplosiveness restriction. Pick real eigenvalues λ_i, and λ_j. We can write

$$\begin{pmatrix} z_{it} \\ z_{jt} \end{pmatrix} = (\lambda_i \oplus \lambda_j) \begin{pmatrix} z_{it-1} \\ z_{jt-1} \end{pmatrix} + \begin{pmatrix} \tilde{w}_{it} \\ \tilde{w}_{jt} \end{pmatrix} + \begin{pmatrix} \tilde{\varepsilon}_{it} \\ \tilde{\varepsilon}_{jt} \end{pmatrix}.$$

This can be rearranged to yield the following representation:

$$\begin{aligned} y_t \;=\; & (S_j^{i1})^{-1} \begin{pmatrix} \lambda_i S^{i1} & \lambda_i S^{i2} - S^{i3} \\ \lambda_j S^{j1} & \lambda_j S^{j2} - S^{j3} \end{pmatrix} y_{t-1} + (S_j^{i1})^{-1} \begin{pmatrix} 0 & \lambda_i S^{i3} \\ 0 & \lambda_j S^{j3} \end{pmatrix} y_{t-2} \\ & -(S_j^{i1})^{-1} S_j^{i4} \hat{g}_t + (S_j^{i1})^{-1} \begin{pmatrix} \lambda_i S^{i4} & \lambda_i S^{i5} \\ \lambda_j S^{j4} & \lambda_j S^{j5} \end{pmatrix} \hat{g}_{t-1} \\ & +(S_j^{i1})^{-1} \begin{pmatrix} \tilde{w}_{it} \\ \tilde{w}_{jt} \end{pmatrix} + (S_j^{i1})^{-1} \begin{pmatrix} \tilde{\varepsilon}_{it} \\ \tilde{\varepsilon}_{jt} \end{pmatrix}. \end{aligned}$$

Common Factor Representations: Again, pick real eigenvalues λ_i, and λ_j. Then we may define the VAR sunspot

$$\zeta_t = (\lambda_i \oplus \lambda_j)\zeta_{t-1} + \begin{pmatrix} \tilde{w}_{it} \\ \tilde{w}_{jt} \end{pmatrix} + \begin{pmatrix} \tilde{\varepsilon}_{it} \\ \tilde{\varepsilon}_{jt} \end{pmatrix}.$$

The resulting CF representation has the form

$$y_t = -(S_j^{i1})^{-1} \begin{pmatrix} 0 & S^{i3} \\ 0 & S^{j3} \end{pmatrix} y_{t-1} - (S_j^{i1})^{-1} S_j^{i4} \hat{g}_t + (S_j^{i1})^{-1} \zeta_t.$$

Learning. For PR_1 the reduced form is

$$\begin{pmatrix} x_t \\ \pi_t \end{pmatrix} = AE_t \begin{pmatrix} x_{t+1} \\ \pi_{t+1} \end{pmatrix} + B \begin{pmatrix} x_{t-1} \\ \pi_{t-1} \end{pmatrix} + C \begin{pmatrix} g_t \\ u_t \end{pmatrix},$$

where

$$\begin{aligned} A &= \begin{pmatrix} \delta & \phi(1-\alpha_\pi \beta\gamma)\delta \\ \lambda\delta & \beta\gamma + \lambda\phi(1-\alpha_\pi\beta\gamma)\delta \end{pmatrix} \\ B &= \begin{pmatrix} 0 & -\phi\alpha_\pi\beta(1-\gamma)\delta \\ 0 & \beta(1-\gamma) - \lambda\phi\alpha_\pi\beta(1-\gamma)\delta \end{pmatrix} \\ C &= \begin{pmatrix} \delta & -\phi\alpha_\pi\delta \\ \lambda\delta & 1-\lambda\phi\alpha_\pi\delta \end{pmatrix}, \end{aligned}$$

and $\delta = (1 + \phi(\alpha_x + \lambda\alpha_\pi))^{-1}$.

For PR_1' the reduced form is

$$\begin{pmatrix} x_t \\ \pi_t \end{pmatrix} = AE_t^* \begin{pmatrix} x_{t+1} \\ \pi_{t+1} \end{pmatrix} + B \begin{pmatrix} x_{t-1} \\ \pi_{t-1} \end{pmatrix} + C \begin{pmatrix} g_t \\ u_t \end{pmatrix} + DE_t^* \begin{pmatrix} x_t \\ \pi_t \end{pmatrix}$$

where

$$\begin{aligned} A &= \begin{pmatrix} 1 & \phi \\ \lambda & \beta\gamma + \lambda\phi \end{pmatrix} \\ B &= \begin{pmatrix} 0 & 0 \\ 0 & \beta(1-\gamma) \end{pmatrix} \\ C &= \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \\ D &= \begin{pmatrix} -\phi\alpha_x & -\phi\alpha_\pi \\ -\lambda\phi\alpha_x & -\lambda\phi\alpha_\pi \end{pmatrix} \end{aligned}$$

For all of our policy rules the CF-PLM is given by

$$\begin{aligned} y_t &= a + by_{t-1} + c\hat{g}_t + d\zeta_t \\ \zeta_t &= \lambda_i \zeta_{t-1} + \breve{\varepsilon}_t. \end{aligned}$$

The associated T-map is

$$\begin{aligned}
a &\rightarrow (A(I_2 + b) + D)a \\
b &\rightarrow Ab^2 + Db + B \\
c &\rightarrow A(bc + c\rho) + Dc + C \\
d &\rightarrow A(bd + d\lambda_i) + Dd
\end{aligned}$$

The relevant Jacobians are given by

$$\begin{aligned}
DT_a &= A(I_2 + b) + D \\
DT_b &= b' \otimes A + I_2 \otimes Ab + I_2 \otimes D \\
DT_c &= I_2 \otimes Ab + \rho' \otimes A + I_2 \otimes D \\
DT_d &= Ab + \lambda_i \otimes A + D.
\end{aligned}$$

The E-stability conditions are that the real part is less than 1 for every eigenvalue of DT_i, $i = a, b, c, d$. The general form case can be handled similarly.

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Figure 1: Policy Rule 1, Woodford Calibration, γ=1

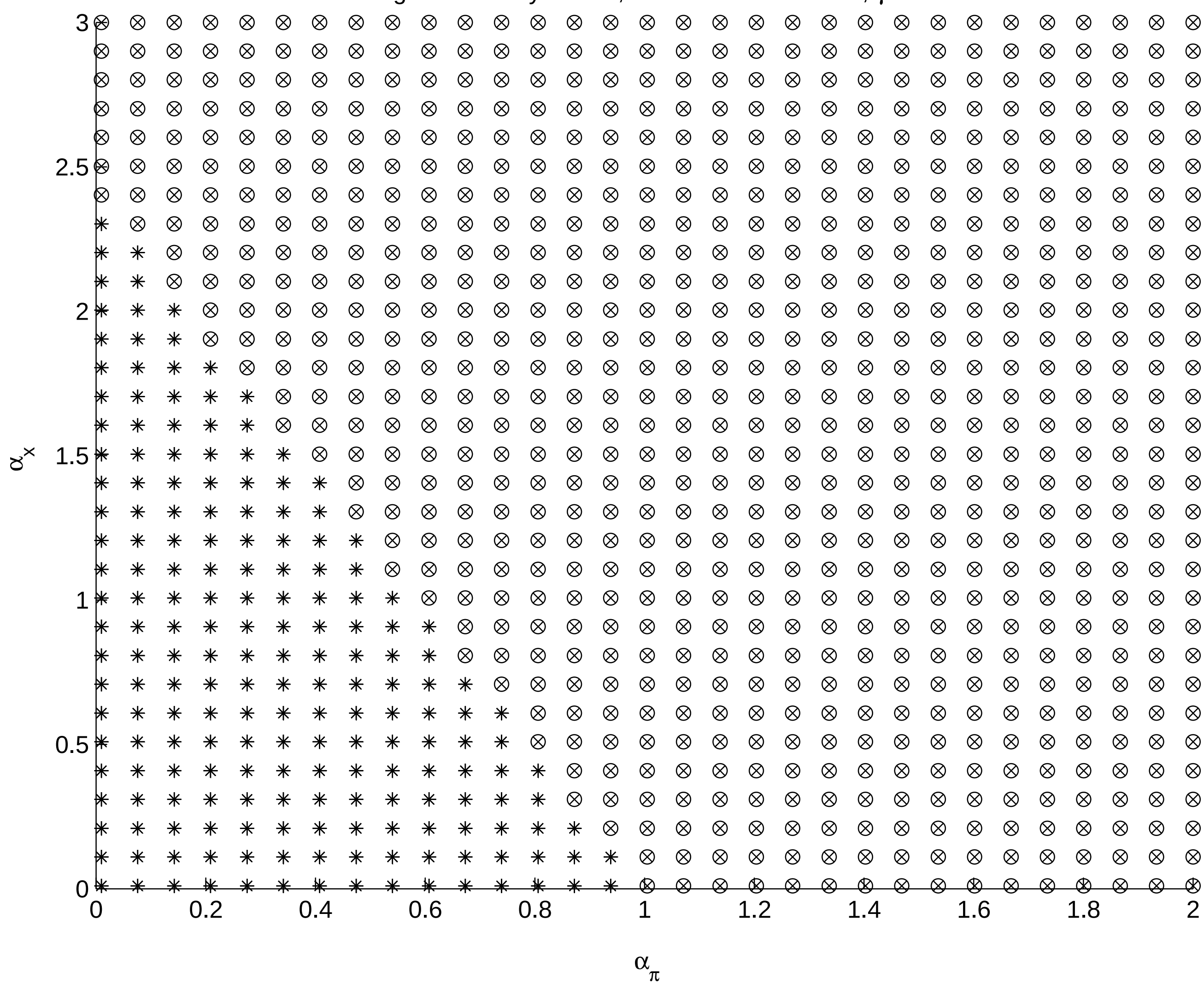


Figure 2: Policy Rule 1, CGG Calibration, $\gamma = 1/2$, $\beta = 1$

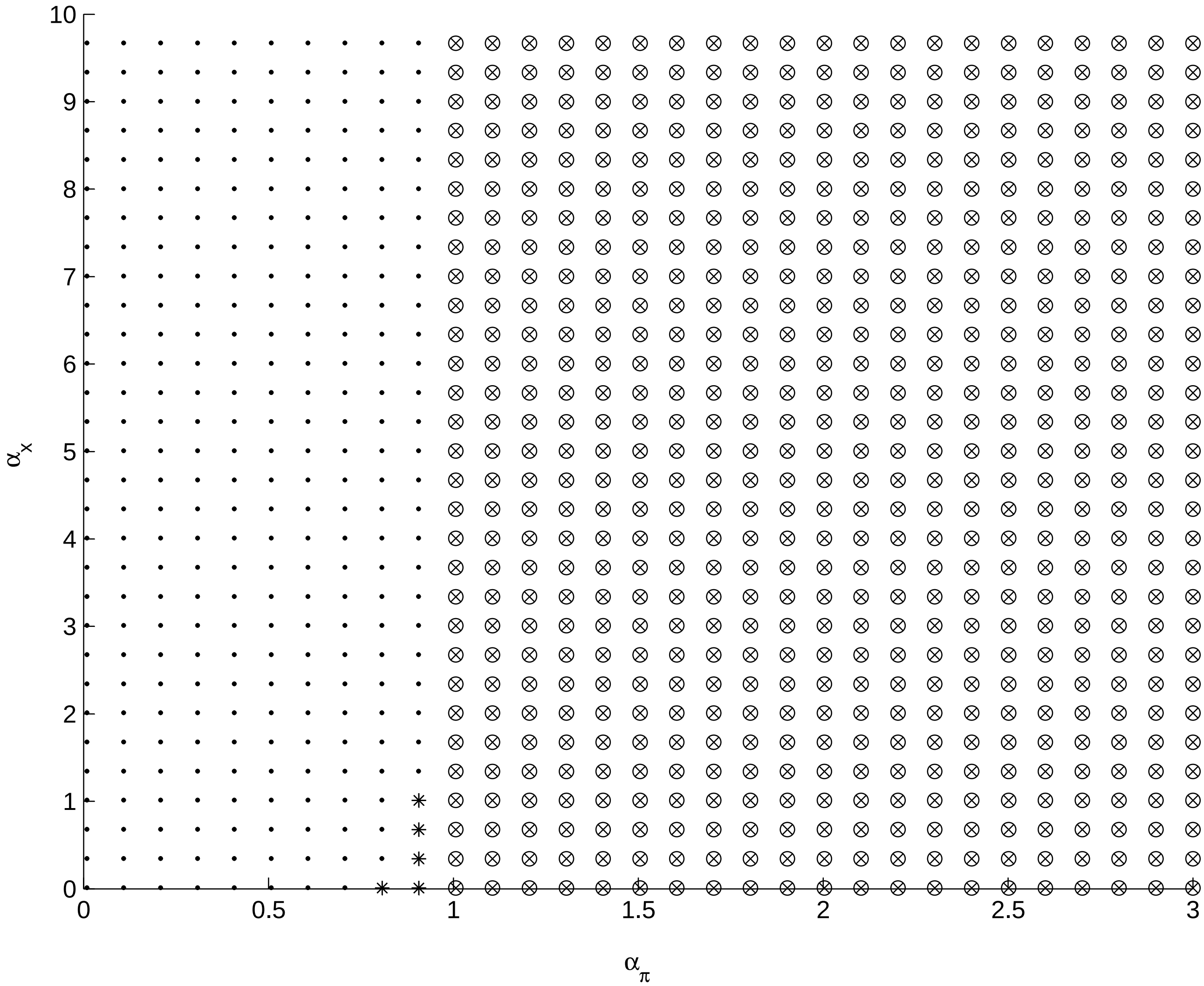


Figure 3: Policy Rule 2, CGG Calibration, $\gamma = 1/2$, $\beta = 1$

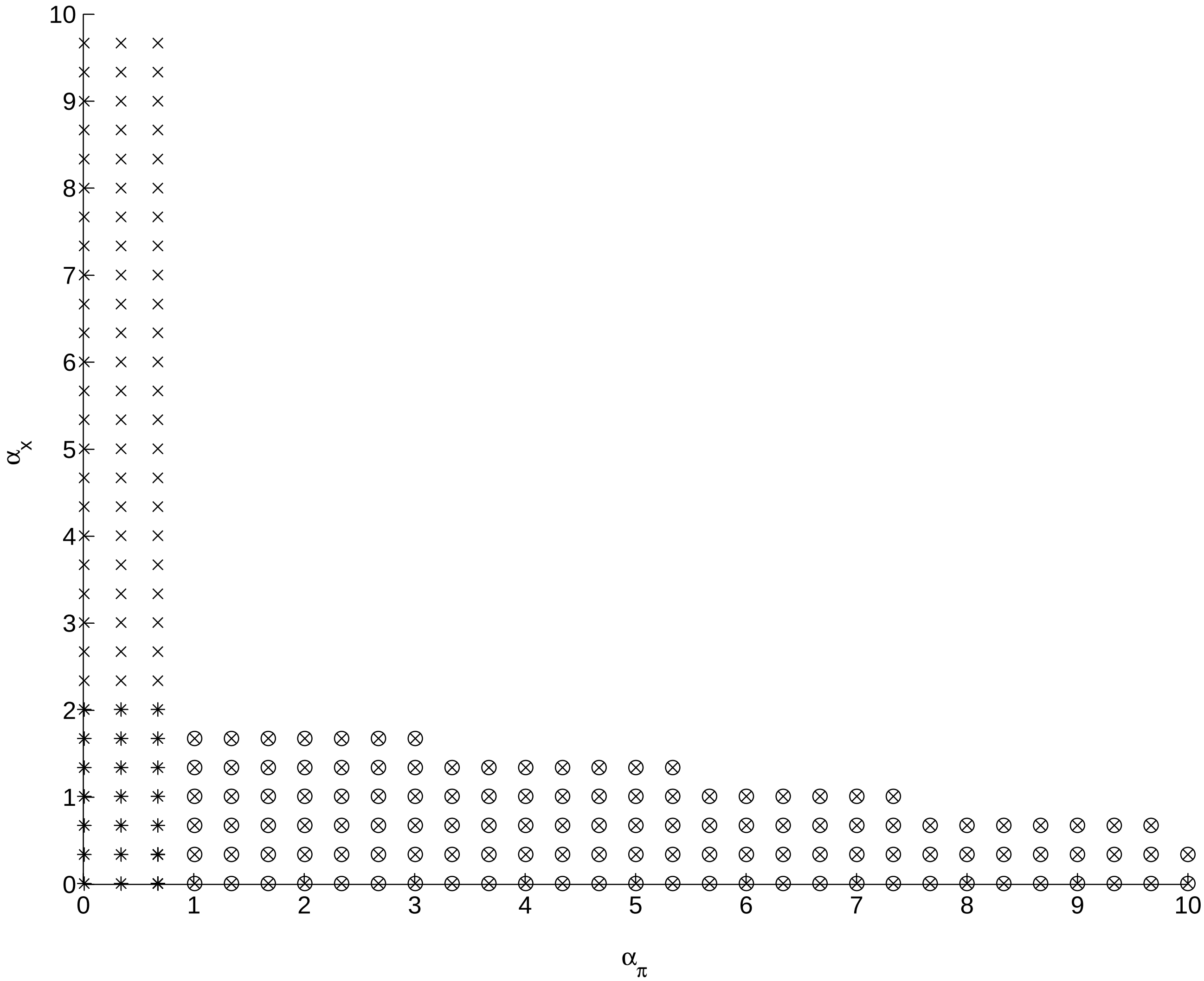


Figure 4: Policy Rule 3, Woodford Calibration, $\gamma = 1$

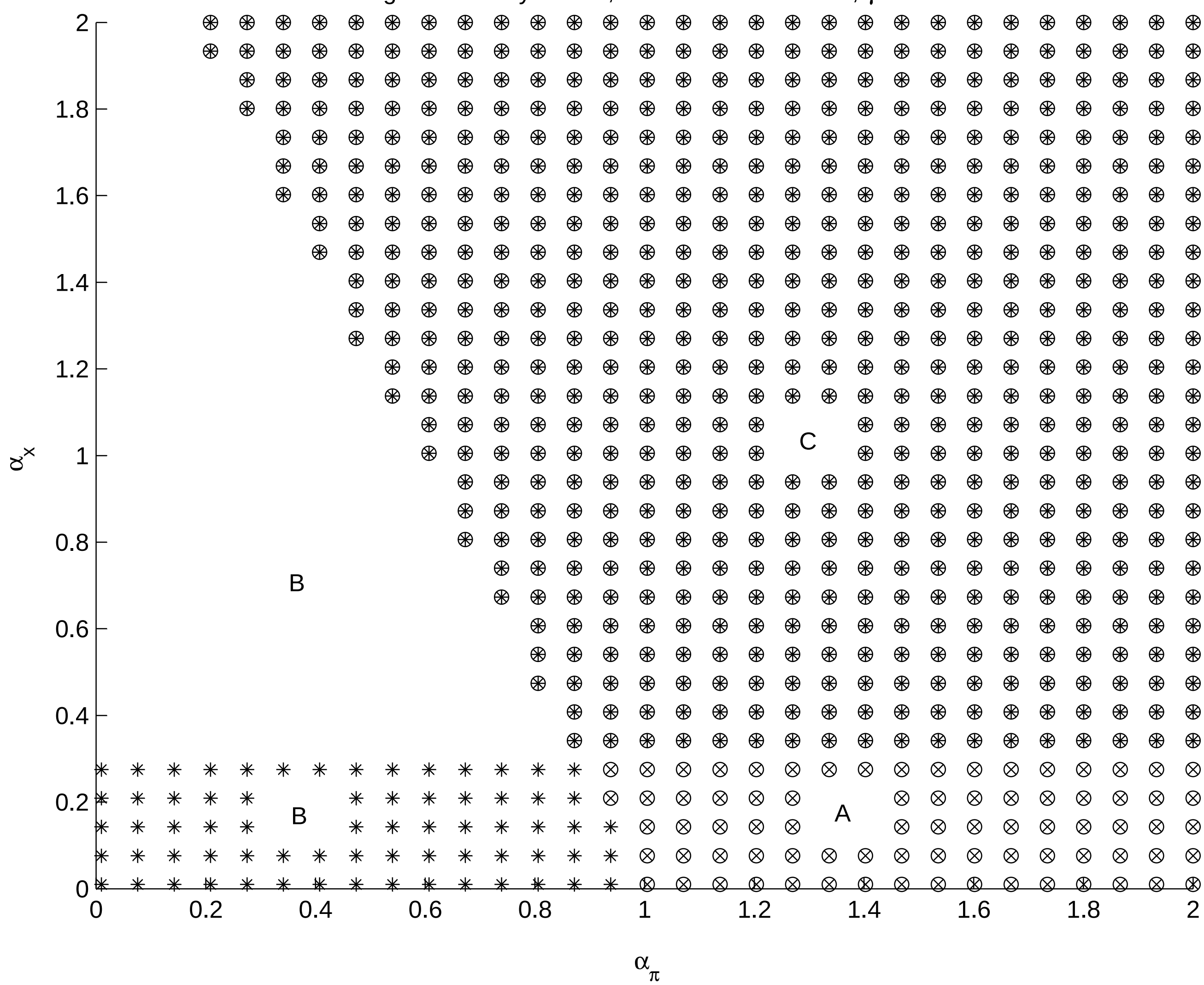
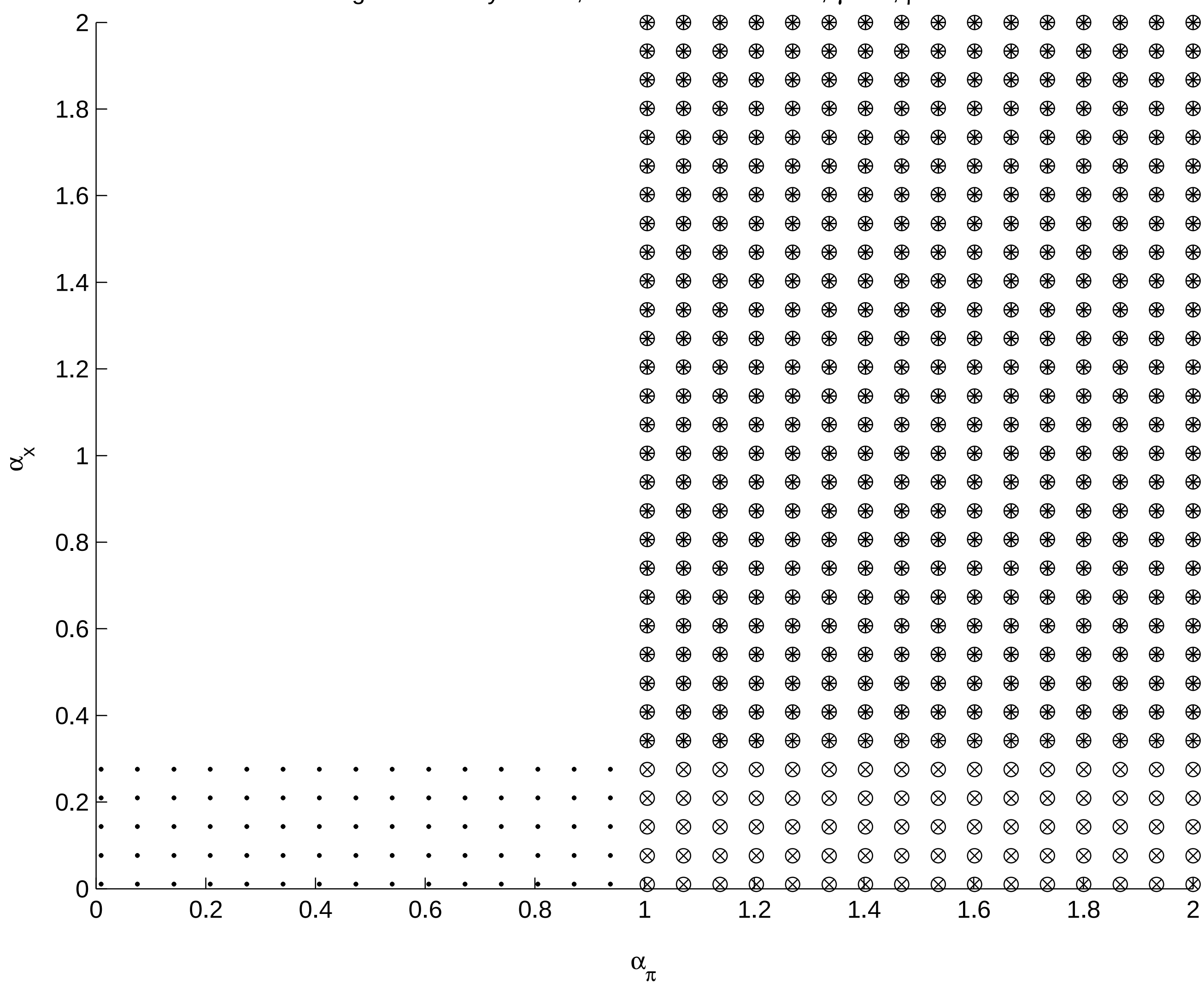


Figure 5: Policy Rule 3, Woodford Calibration, γ=1/2, β=1



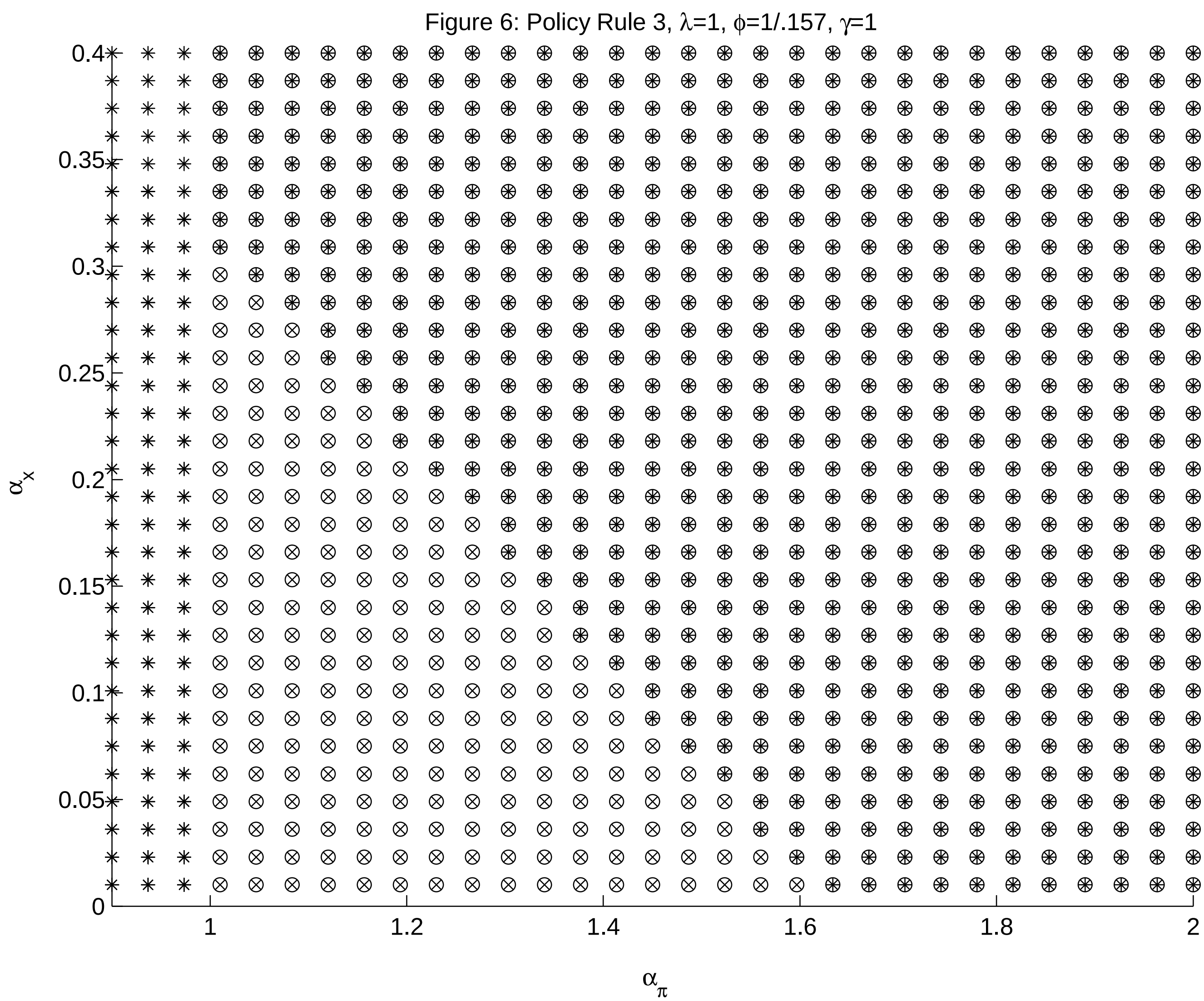


Figure 6: Policy Rule 3, λ=1, ϕ=1/.157, γ=1
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
α_x
1
1.2
1.4
1.6
1.8
2
α_π

Figure 7: Policy Rule 4, CGG Calibration, θ=.68, γ = 1

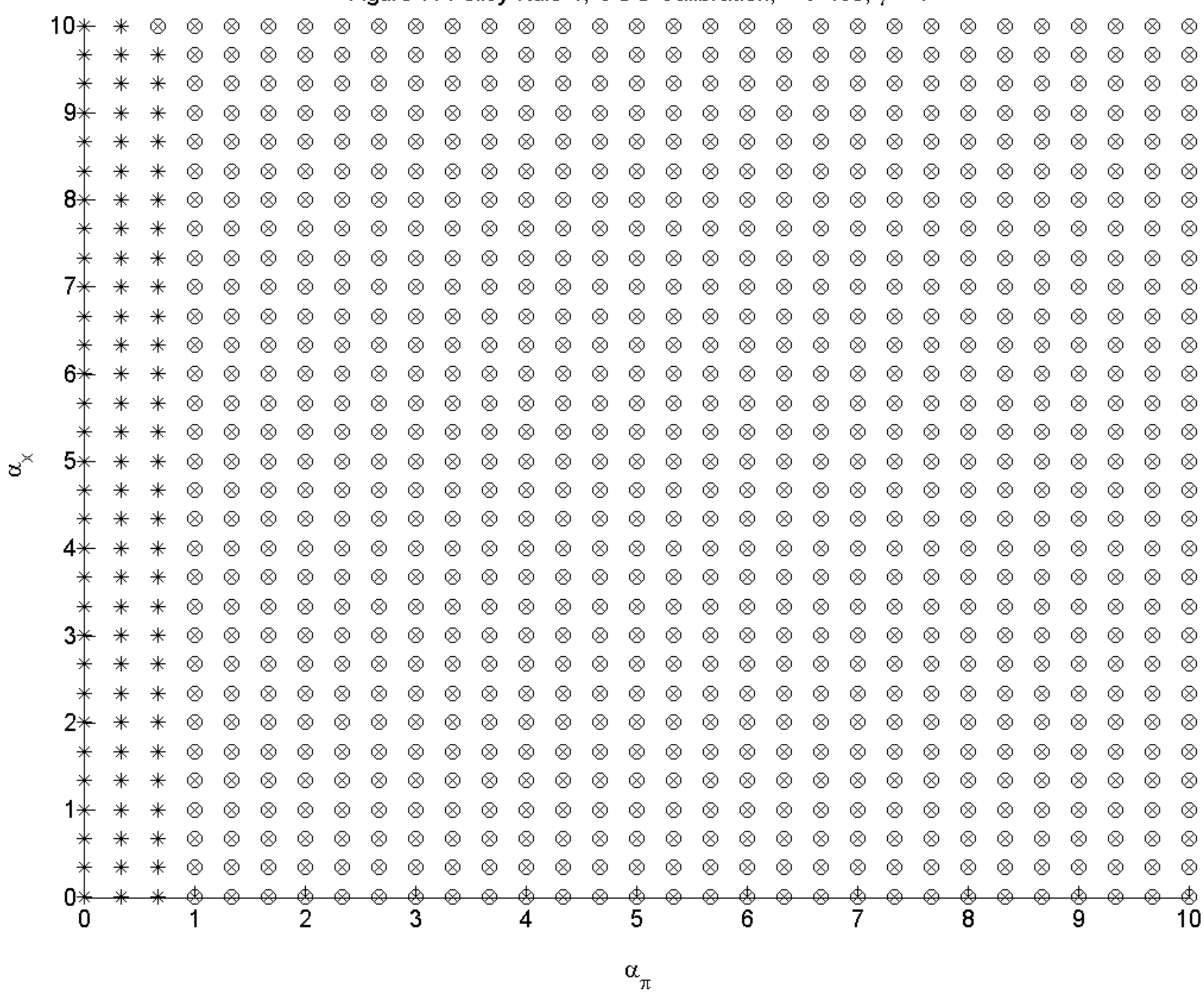


Figure 8: Policy Rule 4, ϕ=1/.157, λ=1, θ =.1, γ=1

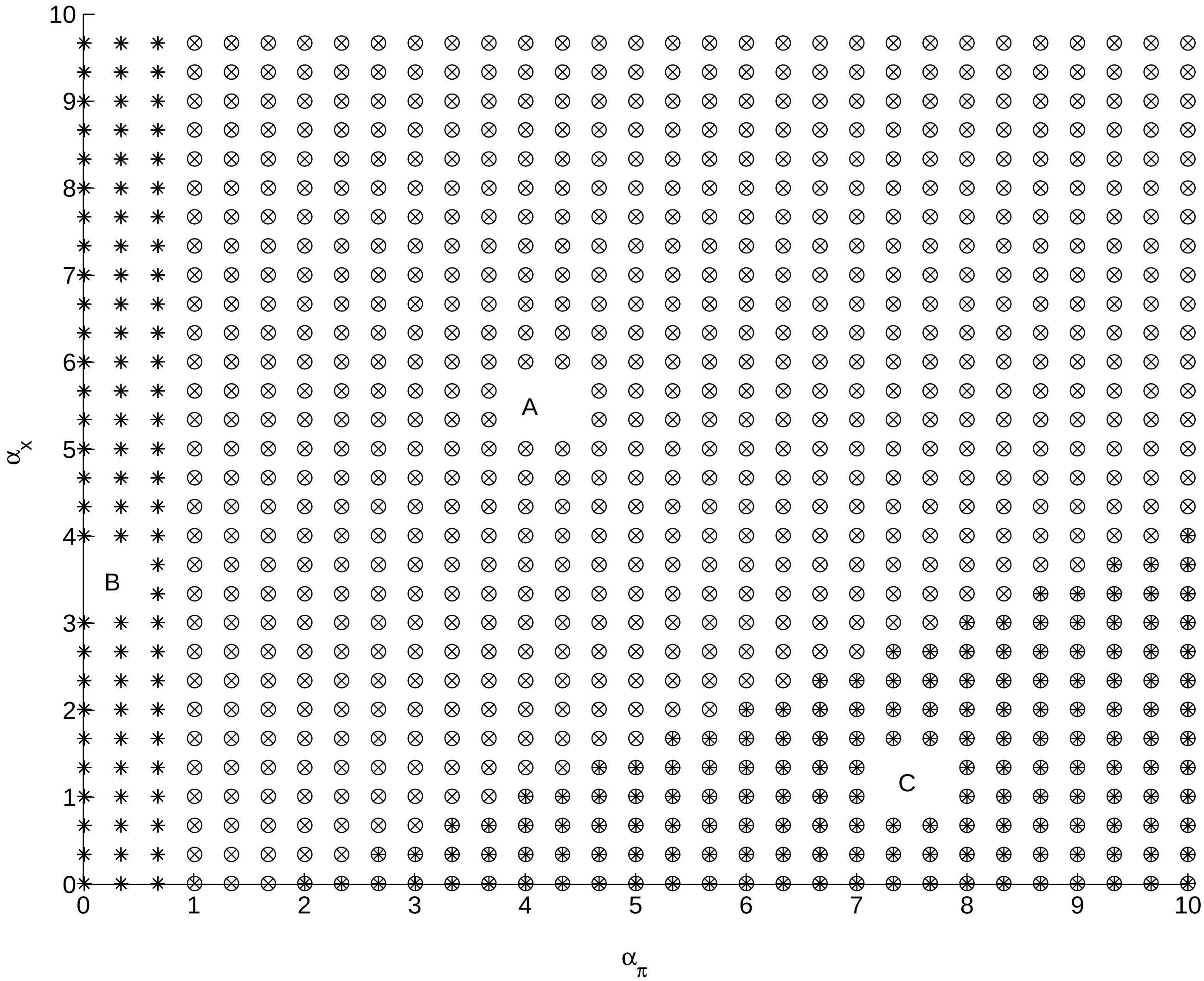


Figure 9: Policy Rule 4', $\phi = 6.3694$, $\lambda = 1$, $\theta = 2$, $\gamma = 1$