GEOMETRY AND COMBINATORICS PERTAINING TO THE HOMOLOGY OF SPACES OF KNOTS

by

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DISSERTATION ABSTRACT

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We produce explicit geometric representatives of non-trivial homology classes in $\mathrm{Emb}(S^1, \mathbb{R}^d)$, the space of knots, when d is even. We generalize results of Cattaneo, Cotta-Ramusino and Longoni to define cycles which live off of the vanishing line of a homology spectral sequence due to Sinha. We use configuration space integrals to show our classes pair non-trivially with cohomology classes due to Longoni.

We then give an alternate formula for the first differential in the homology spectral sequence due to Sinha. This differential connects the geometry of the cycles we define to the combinatorics of the spectral sequence. The new formula for the differential also simplifies calculations in the spectral sequence.

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CHAPTER I

INTRODUCTION

The study of knot theory concerns the components of the space $\mathrm{Emb}(S^1, S^3)$, as well as its zeroth cohomology – that is knot invariants. Alternatively, one studies the closely related space $\mathcal{K}_d$ of smooth embeddings $\mathbb{R} \hookrightarrow \mathbb{R}^d$ which are fixed outside of the unit interval $\mathbb{I}$, for $d = 3$. Also of topological interest is the full homotopy type of $\mathrm{Emb}(S^1, S^3)$. The first results in this area are due to Gramain in [11], where he studies the fundamental groups of the path components of $\mathcal{K}_3$. Hatcher proves in [14] that the component of $\mathcal{K}_3$ containing the unknot is contractible, which is equivalent to the Smale conjecture. Hatcher also proves in [12] that each component of $\mathcal{K}_3$ is a $K(\pi, 1)$, and with McCullough [13] that each path component has the homotopy type of a finite CW-complex. We focus on the study of the topology of the embedding spaces $\mathrm{Emb}(S^1, \mathbb{R}^d)$ and $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d)$ for $d \geq 4$.

In [25], Vassiliev constructs a spectral sequence for each dimension d, and asserts that this spectral sequence converges to the homology of $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d)$ when $d \geq 4$. Vassiliev also analyzes the combinatorics of this spectral sequence when $d = 3$ without any knowledge of its convergence. This analysis initiated the study of finite type invariants, also called Vassiliev invariants.

The study of finite type invariants was continued by Birman and Lin [3], Bar-Natan [2], Bott and Taubes [4], and many others. Using ideas from the work of Bott and Taubes, the group of Cattaneo, Cotta-Ramusino and Longoni [7] combine the idea of finite type invariants with the de Rham cohomology of $\mathrm{Emb}(S^1, \mathbb{R}^d)$. We give a summary of these results in Section 2.2. Briefly, they produce explicit, non-trivial, $k(d-3)-$dimensional cycles and cocycles in $\mathrm{Emb}(S^1, \mathbb{R}^d)$. They define a chain map from a graph complex to the de Rham complex of $\mathrm{Emb}(S^1, \mathbb{R}^d)$, and produce cocycles as images of graph cocycles consisting of trivalent graphs. To produce cycles, they use families of resolutions of singular knots with k transverse double points. To establish non-triviality, they show the pairing between certain cycles and cocycles is nonzero. Their techniques produce many more cocycles, whose (non)-triviality is an open question.

In [16], Longoni defines a cocycle which is the image of a non-trivalent graph and proposes that this cocycle is non-trivial. These results of Longoni were very influential for our work. We extend the results of [7] and [16] by giving an example of a cycle in $H^{3(d-1)}(\mathrm{Emb}(S^1, \mathbb{R}^d))$, and showing that this cycle (and Longoni's cocycle) are non-trivial by evaluation. Furthermore, instead of focusing on the de Rham cohomology and its formal linear dual as in [16], we treat homology from a geometric viewpoint.

Another approach to the study of the topology of spaces of knots was initiated by Goodwillie starting with ideas from his thesis [8]. This approach, now known as the calculus of embeddings (or the calculus of isotopy functors) was fully developed by Weiss, Goodwillie and Klein [9, 10, 27]. We summarize their results specialized to knots below, following Volic [26] and Sinha [23].

The space of embeddings $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d)$ can be approximated by the space of "punctured knots." Specifically, we fix a sequence of disjoint closed subintervals of $\mathbb{I}$, namely $J_i = [\frac{1}{2^i}, \frac{1}{2^i} + \frac{1}{10^i}]$.

Definition 1.1. For every subset $S \subset \mathbf{n} = \{1, 2, \ldots, n\}$, let $E_S = \mathrm{Emb}(\mathbb{I} - \cup_{i \in S} J_i, \mathbb{I}^d)$. If $S_1 \subset S_2$ then there is an embedding $E_{S_1} \hookrightarrow E_{S_2}$. For a fixed n, the set of spaces $\{E_S : S \subseteq \mathbf{n}, S \neq \varnothing\}$ forms a cubical diagram, which we denote $\mathcal{E}_n$. Define $P_n\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d)$ to be the homotopy limit of $\mathcal{E}_n$.

For every non-empty subset $S \subseteq \mathbf{n}$, there is also an embedding $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d) \to E_S$, and thus a canonical map $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d) \to P_n\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d)$. Furthermore, $\mathcal{E}_{n-1}$ is a sub-cubical diagram of $\mathcal{E}_n$, so a choice of inclusion gives a map $P_n\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d) \to P_{n-1}\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d)$.

Definition 1.2. The Taylor tower for $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d)$ is

$$P_0\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d) \leftarrow P_1\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d) \leftarrow \cdots \leftarrow P_{n-1}\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d) \leftarrow P_n\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d) \leftarrow \cdots .$$

The Taylor tower together with the following result of Goodwillie and Klein, has led to many results in the study of the topology of spaces of knots.

Theorem 1.1. *[9] If $d \geq 4$ the canonical map $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d) \to P_n\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d)$ is $(n-1)(d-3)-$connected. Thus the map from $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d)$ to the homotopy inverse limit of the Taylor tower induces isomorphisms on homotopy and homology.*

In [21, 23], Sinha establishes conjectures of Bott and Goodwillie that the Taylor tower for knot spaces has a natural cosimplicial description involving compactified configuration spaces. As a result, he obtains spectral sequences converging to the homology, cohomology and homotopy of $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d)$ and closely related spaces [20]. We briefly summarize these results for the homology spectral sequence below and study the first differential of this spectral sequence further in Chapter III.

Let E_d be the subspace of $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d)$ consisting of pairs of an embedding with the endpoints and tangent vectors at the endpoints fixed on opposite faces of $\mathbb{I}^d$ and an isotopy through immersions to the unkot. Sinha shows in [21] that this space is homotopy equivalent to $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d) \times \Omega\mathrm{Imm}(\mathbb{I}, \mathbb{I}^d)$.

Let $\mathcal{O}_d$ be the d-th Kontsevich operad defined in [21], whose entries are compactified configuration spaces. In [17], McClure and Smith present a method for associating a cosimplicial object to an operad. Let $\mathcal{O}_d^\bullet$ be the cosimplicial object associated to the Kontsevich operad. The following is Theorem 6.9 of [21].

Theorem 1.2. *The k-th stage of the Taylor tower for E_d, or P_kE_d, is weakly equivalent to* Tot^k *of a fibrant replacement of $\mathcal{O}_d^\bullet$.*

Using the homology spectral sequence for cosimplicial spaces, developed by Bousfield in [6], Sinha [22] shows that there is a spectral sequence with $E^1_{-*,*} = H_*(\mathcal{O}_d^\bullet)$. Furthermore, $H_*(\mathcal{O}_d^\bullet) = \mathcal{P}ois^{d-1}$, so the E^1 page of this spectral sequence can be expressed in terms of the Poisson operad. For $d \geq 4$, this spectral sequence converges to homology of E_d. There are similar spectral sequences, also due to Sinha [23], which converge to the homotopy and cohomology of E_d. By work of Arone, Lambrechts, Turchin and Volic [1, 15], the rational cohomology and homotopy spectral sequences collapse at the E^2 page.

By work of Turchin in [24], the $E^1_{-*,*}$ term of Vassiliev's spectral sequence agrees with the $E^2_{-*,*}$ term of Sinha's spectral sequence. These approaches allow one to combinatorially understand the ranks of the homology groups of the closely related space $\mathrm{Emb}(S^1, \mathbb{R}^d)$, but do not immediately give geometric understanding or representing cycles and cocycles in knot spaces.

The cycles defined by Cattaneo et al. in [7] live along the $(-2q, q(d-1))-$diagonal in the first page of the homology spectral sequence, and thus in degree $q(d-3)$. This diagonal also serves as a vanishing line. For d odd, Sakai [18] produces a $(3d-8)-$dimensional cocycle in the space of long knots coming from a non-trivalent graph cocycle. To establish the non-triviality of this cocycle, he evaluates it on a cycle produced using the Browder bracket coming from the action of the little two-cubes operad on the space of framed knots.

1.1. Main Results

We produce a non-trivial cycle which lives off of the vanishing line of the homology spectral sequence for d even by generalizing the methods of Cattaneo, Cotta-Ramusino and Longoni to families of resolutions of singular knots with triple points. In particular, we first define a topological manifold M_β and an embedding of M_β into $\mathrm{Emb}(S^1, \mathbb{R}^d)$, extending and correcting the results in the preprint of Longoni [16]. Longoni also defines a cocycle which is the image of a non-trivalent graph when d is even. We show that the pairing between Longoni's cocycle and our cycle is nonzero, and thus both are non-trivial.

We also give a new formula for the first differential in Sinha's spectral sequence in Chapter III. Our techniques were motivated by the search for geometric representatives. Building on ideas from Chapter II, we immediately have conjectured recipes for representatives of all cycles in Sinha's spectral sequence. This is in contrast to Sakai's approach, which would require new input for any Browder-indecomposible classes off of the $(-2q, q(d-1))-$diagonal. We plan to develop these conjectured representatives in future work.

CHAPTER II

GEOMETRIC REPRESENTATIVES OF NON-TRIVIAL CYCLES

2.1. Definition of the Cycle

The idea at the heart of our method to produce homology classes in knot spaces goes back to Vassiliev's seminal work [25]. In finite type knot theory, one defines the derivative of a knot invariant by taking an immersion with transverse double-points and evaluating the knot invariant on the resolutions of that immersion. We require a generalization of such immersions.

Definition 2.1. An immersion $\gamma : S^1 \hookrightarrow \mathbb{R}^d$ has a transverse intersection r-singularity at $\bar{t} = (t_1, t_2, \ldots, t_r) \in \mathbb{I}^{\times r}$ with $0 < t_1 < t_2 < \cdots < t_r < 1$, if all of the $\gamma(t_i)$ coincide and the derivatives $\gamma'(t_i)$ are generic in the sense that any d or fewer of them are linearly independent.

To connect with the language naturally produced by the embedding calculus spectral sequence, we use bracket expressions to encode singularity data. Sinha calculates in [23] that the subgroup of $\mathcal{P}ois^d(p)$, the $p-$th entry of the Poisson operad (see [19]), generated by expressions with q brackets such that each x_i appears inside a bracket pair and the multiplication " $\cdot$ " does not appear inside a bracket pair, is also a subgroup of $E^1_{-p,q(d-1)}$ in the reduced homology spectral sequence. This is the full $E^1_{-p,q(d-1)}$ in the spectral sequence converging to the homology of the homotopy fiber of the inclusion map $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d) \hookrightarrow \mathrm{Imm}(\mathbb{I}, \mathbb{I}^d)$. On this subgroup, the differential $d_1 : E^1_{-p,q(d-1)} \to E^1_{-p-1,q(d-1)}$ is $d^1 = \sum_{i=0}^{p} (-1)^i (\delta^i)_*$, where $(\delta^0)_*$ is defined by adding x_1 in front of the expression and replacing each x_j by x_{j+1}, $(\delta^{p+1})_*$ is defined by adding x_{p+1} to the end, and for $1 \le i \le p$, the map $(\delta^i)_*$ is defined by replacing x_i by $x_i \cdot x_{i+1}$

and x_j by x_{j+1} for $j > i$. In [24], Tourtchine does further calculations in this spectral sequence.

Example 2.1. The bracket expression $\beta = \beta_1 + \beta_2$ where $\beta_1 = [[x_1, x_4], x_3] \cdot [x_2, x_5]$ and $\beta_2 = [x_1, x_4] \cdot [[x_2, x_5], x_3]$ is a cycle in $E^1_{-5,3(d-1)}$.

Definition 2.2. A pair $(\gamma, \bar{t})$ of an immersion and a sequence $\bar{t} = 0 < t_1 < t_2 < \cdots < t_p < 1$ respects a bracket expression $\beta \in \mathcal{P}ois^d(p)$ if γ has a transverse r-singularity at the sequence $0 < t_{i_1} < \ldots < t_{i_r} < 1$ whenever $x_{i_1}, \ldots, x_{i_r}$ appear inside of a bracket in β.

For example, the knots K_1 and K_2 in Figure 2.1 respect β_1 and β_2, respectively. A knot can respect a bracket expression but have higher singularities; for example K_1 also respects $[x_1, x_3] \cdot [x_2, x_4]$.

Definition 2.3. We will denote the subspace of all pairs $(\gamma, \bar{t}) \in \mathrm{Imm}(S^1, \mathbb{R}^d) \times \mathbb{I}^{\times r}$ respecting a bracket expression by $\mathrm{Imm}_{\geq\beta}(S^1, \mathbb{R}^d)$, with the convention $\mathrm{Imm}_{\phi}(S^1, \mathbb{R}^d) = \mathrm{Imm}(S^1, \mathbb{R}^d)$. The subspace of $\mathrm{Imm}_{\geq\beta}(S^1, \mathbb{R}^d)$ consisting of immersions which do not have higher singularities will be denoted by $\mathrm{Imm}_{=\beta}(S^1, \mathbb{R}^d)$.

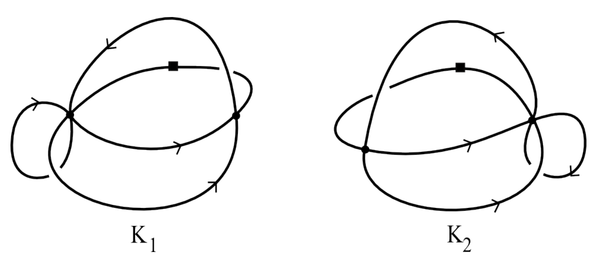


Figure 2.1. The singular knots K_1 and K_2.

In the spectral sequence, bracket expressions of the form $\prod_{m=1}^{k} [x_{i_m}, x_{j_m}]$ are E^1-cycles. Submanifolds representing these cycles are well known and described in

Section 2 of [7]. Informally, we start with a singular knot $K \subset \mathbb{R}^d$ with k double points which respects $\prod_{m=1}^{k}[x_{i_m}, x_{j_m}]$, and resolve each double point by moving one strand passing through the double point off of the other. For each vector in S^{d-3} we have a possible direction in which to move the strand, and therefore a possible way to resolve the double point. The subset of $\mathrm{Emb}(S^1, \mathbb{R}^d)$ consisting of all such resolutions of K is a submanifold parameterized by $\prod_m S^{d-3}$, and its fundamental class corresponds to the cycle $\prod_{m=1}^{k}[x_{i_m}, x_{j_m}]$ of the spectral sequence.

For higher singularities, we start with ideas of Longoni [16] and produce resolutions of transverse intersection singularities by moving one strand at a time off the intersection point. Assume the rank of the singularity r is less than d, so the (tangent vectors of the) strands in question span a proper subspace. There are two cases - resolving a double point and resolving a higher singularity. If $r \geq 3$, we are moving a strand off the intersection point. The complementary subspace to the (tangent vector of the) strand has a unit sphere S^{d-2} which parametrizes the directions to move one strand off the intersection point. If $r = 2$, we consider a unit sphere S^{d-3} in the complimentary subspace which parametrizes the directions to move one strand off another.

Resolutions of triple point singularities (and higher singularities) can produce further singularities (see Figure 2.4). By restricting away from neighborhoods of those "additional singularity" resolutions, we produce submanifolds with boundary which we show can be pieced together to build representatives of E^1-cycles in the spectral sequence. We formalize as follows.

Definition 2.4. If β is a bracket expression, let $\beta(\hat{i})$ denote the bracket expression obtained from β by removing x_i and the minimal set of other symbols as required to have a bracket expression, and replacing x_k by x_{k-1} for all $k > i$.

Remark 2.1. This will be called the restriction (and relabeling) of the bracket expression in Chapter III.

For example, with $\beta_1 = [[x_1, x_4], x_3] \cdot [x_2, x_5]$, we have $\beta_1(\hat{4}) = [x_1, x_3][x_2, x_4]$. For each strand through a transverse intersection r-singularity, we can define a resolution map which moves that strand off of the singularity. To accommodate the two cases, we let

$$d(r) = \begin{cases} d-2 & \text{if } r > 2 \\ d-3 & \text{if } r = 2 \end{cases}.$$

By the rank of x_i in a bracket expression β, we will mean the number of variables in β (counting x_i) which appear inside of common brackets with x_i. In β_1, x_3 has rank three and x_5 has rank two.

Definition 2.5. If β is a bracket expression in which x_i has rank r (with $r > 0$) define the resolution map

$$\rho_i : \mathrm{Imm}_{\geq \beta}(S^1, \mathbb{R}^d) \times S^{d(r)} \times \mathbb{I} \times \mathbb{I} \to \mathrm{Imm}_{\geq \beta(\hat{i})}(S^1, \mathbb{R}^d)$$

by

$$\rho_i(\gamma, \bar{t}, v, a, \varepsilon)(t) = \begin{cases} \gamma(t) + a \cdot v \exp\left(\frac{1}{(t-t_i)^2 - \varepsilon^2}\right) & \text{if } t \in (t_i - \varepsilon, t_i + \varepsilon) \\ \gamma(t) & \text{otherwise} \end{cases}.$$

We call the triple $(v, a, \varepsilon) \in S^{d(r)} \times \mathbb{I} \times \mathbb{I}$ the resolution data. We often fix a and ε so that the resolutions do not have unexpected singularities and by abuse denote the restriction by ρ_i as well. The resolution map produces immersions in which the strand (between times $t_i - \varepsilon$ and $t_i + \varepsilon$) is moved in the direction of v, as shown in Figure 2.2.

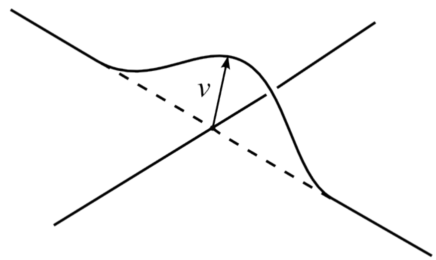


Figure 2.2. The resolution of a double point.

Definition 2.6. Let $S = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ be an ordered subset of the variables in β. Define $\rho_{\beta,S}$ to be the composite

$$\rho_{i_k} \circ (\rho_{i_{k-1}} \times id) \circ \cdots \circ (\rho_{i_1} \times id) : \mathrm{Imm}_{\geq \beta}(S^1, \mathbb{R}^d) \times \prod_m \left(S^{d(r_m)} \times \mathbb{I} \times \mathbb{I}\right) \to \mathrm{Imm}_{\geq \varnothing}(S^1, \mathbb{R}^d).$$

Let $\tilde{x}_{i_m}$ be x_{i_m} after the relabeling needed to define $\beta(\hat{i_1}, \ldots, \hat{i}_{m-1})$, and let r_m is the rank of $\tilde{x}_{i_m}$ in $\beta(\hat{i_1}, \ldots, \hat{i}_{m-1})$.

The set S encodes which strands get moved in the resolution defined by $\rho_{\beta,S}$.

We now specialize. Let $\beta_1 = [[x_1, x_4], x_3] \cdot [x_2, x_5]$, $\beta_2 = [x_1, x_4] \cdot [[x_2, x_5], x_3]$ and choose the ordered subset of variables for each to be $S = \{x_3, x_4, x_5\}$. We choose embeddings K_1 and K_2 of S^1 in $\mathbb{R}^3 \hookrightarrow \mathbb{R}^d$ as shown in Figure 2.1, as well as a sequence $0 < t_1 < t_2 < \cdots < t_5 < 1$ so that $(K_1, \bar{t})$ respects β_1 and $(K_2, \bar{t})$ respects β_2.

We restrict the directions in which the singularities are resolved to ensure we produce not just immersions but embeddings. We assume that in the disk of radius $1/10$ centered at each singularity, both K_1 and K_2 consist of linear segments intersecting transversely, as shown in Figure 2.3. Fix $\varepsilon > 0$ so that the intervals $[t_i - \varepsilon, t_i + \varepsilon]$, $i = 1, 2, \ldots, 5$, are disjoint and $K_1([t_i - \varepsilon, t_i + \varepsilon])$ is contained in

$B_{\frac{1}{10}}(K_1(s_i))$ for $i = 1, 2, \ldots, 5$. These intervals are the strands we will move to resolve the singularities.

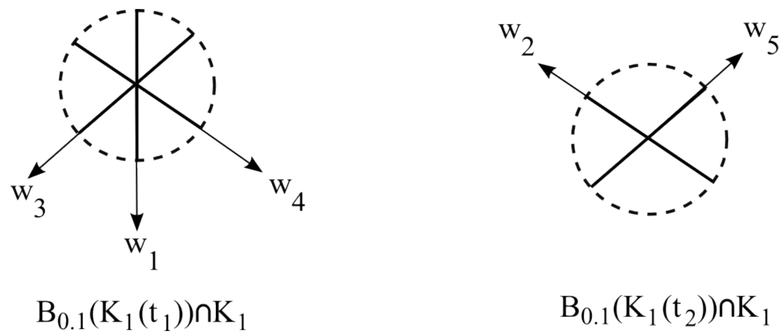


Figure 2.3. $B_{\frac{1}{10}}(K_1(t_1)) \cap K_1$ and $B_{\frac{1}{10}}(K_1(t_2)) \cap K_1$.

Let $w_1, \ldots, w_5$ be the unit tangent vectors to each line segment at the singular points of K_1. Fix $\delta > 0$ so that $\{v \in S^{d-2} : \| v - w_1 \| < \delta\}$ and $\{v \in S^{d-2} : \| v - w_4 \| < \delta\}$ are disjoint. As mentioned above, we avoid moving the third strand off of the triple point in these directions to prevent the introduction of a double point.

We produce a manifold $\mathcal{M}_\beta$ as the image of a topological manifold M_β embedded in $\mathrm{Emb}(S^1, \mathbb{R}^d)$ by resolving singular knots with triple and double points. The manifold $\mathcal{M}_\beta$ decomposes as the union $\bigcup_{i=1}^6 \mathcal{M}_i$, where each $\mathcal{M}_i$ is the image in $\mathrm{Emb}(S^1, \mathbb{R}^d)$ of a resolution map defined below. The domains of the resolution maps for the main pieces, $\mathcal{M}_1$ and $\mathcal{M}_2$, are denoted M_1 and M_2 and are homeomorphic to $\left(S^{d-2} \setminus \cup_4 B_\delta\right) \times S^{d-3} \times S^{d-3}$. The domains of resolution maps defining the remaining four families are denoted $M_i \times \mathbb{I}$, where M_i is homeomorphic to $S^{d-3} \times S^{d-3} \times S^{d-3}$ for $i = 3, 4, 5, 6$.

Definition 2.7. For any triple $(\varepsilon_3, \varepsilon_4, \varepsilon_5)$ with each $\varepsilon_i \leq \varepsilon$ for ε as above, define

$$M_1(\varepsilon_3, \varepsilon_4, \varepsilon_5) \subset \mathrm{Imm}_{\geq \beta_1}(S^1, \mathbb{R}^d) \times \prod_{k=3}^{5} (S^{d(r_k)} \times \mathbb{I} \times \mathbb{I})$$

as the subspace of all $K_1 \times \prod(v_i, a_i, \varepsilon_i)$, where $a_3 = \frac{1}{10}$, $a_4 = a_5 = \frac{\delta}{10}$, and v_3 is such that the distances between v_3 and the vectors $\pm w_1$ and $\pm w_4$ are all greater than or equal to δ. There are no restrictions on $v_4, v_5 \in S^{d-3}$.

We will suppress the dependence of M_1 on the values of $\varepsilon_3, \varepsilon_4, \varepsilon_5 \leq \varepsilon$ as well as δ except when needed.

Lemma 2.1. *The restriction of* $\rho_{\beta_1, S}$ *to* M_1 *maps to* $\mathrm{Emb}(S^1, \mathbb{R}^d) \subset \mathrm{Imm}_{\geq \phi}(S^1, \mathbb{R}^d)$.

Choose the immersion K_2 as shown in Figure 2.1, and assume that the constants $\delta > 0$ and $\varepsilon > 0$ chosen above satisfy similar conditions for K_2, to define M_2 analogously. The restriction of $\rho_{\beta_2, S}$ maps M_2 to $\mathrm{Emb}(S^1, \mathbb{R}^d) \subset \mathrm{Imm}_{\geq \phi}(S^1, \mathbb{R}^d)$. We denote the families of embeddings $\rho_{\beta_1, S}(M_1)$ and $\rho_{\beta_2, S}(M_2)$ by $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively, and connect the boundary components of $\mathcal{M}_1$ to those of $\mathcal{M}_2$ to build a family without boundary.

Each boundary component can also be described as the family of knots obtained by resolving a singular knot with three double points. In fact, resolving the triple point in K_1 by moving the strand $K_1\left([t_3 - \varepsilon_3, t_3 + \varepsilon_3]\right)$ in the direction of $\pm w_1$ or $\pm w_4$ yields an immersion with three double points. The four boundary components of $\mathcal{M}_1$ are families of resolutions of these four "newly singular" knots.

Definition 2.8. Let K_3, K_4, K_5 and K_6 be the singular knots, each with three double points, defined below and shown in Figure 2.4.

$$K_3 = \rho_3\left(K_1, w_4, \tfrac{1}{10}, \varepsilon_3\right)$$
$$K_4 = \rho_3\left(K_1, -w_4, \tfrac{1}{10}, \varepsilon_3\right)$$
$$K_5 = \rho_3\left(K_1, w_1, \tfrac{1}{10}, \varepsilon_3\right)$$
$$K_6 = \rho_3\left(K_1, -w_1, \tfrac{1}{10}, \varepsilon_3\right)$$

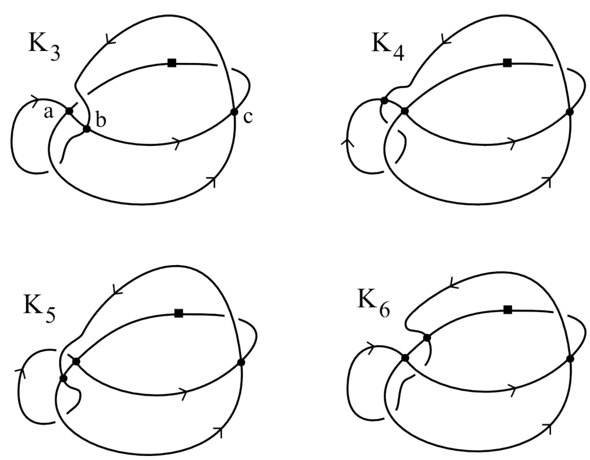


Figure 2.4. Singular knots K_3, K_4, K_5, and K_6.

We resolve these knots, restricting the directions so the resulting embeddings are those in the boundary components of $\mathcal{M}_1$. Initially, we focus on K_3. The double points corresponding to $[x_1, x_4]$ and $[x_2, x_6]$, labeled a and c, are resolved in the same way as the double points in K_1. The double point corresponding to $[x_3, x_5]$, labeled b, is resolved using only vectors in the direction $v - w_4$ for some v such that $\| v - w_4 \| = \delta$. This guarantees that resolving this double point in K_3 yields the $\| v_3 - w_4 \| = \delta$ boundary component of $\mathcal{M}_1$.

Definition 2.9. Define $M_3(\varepsilon_3, \varepsilon_4, \varepsilon_5) \subset \mathrm{Imm}_{\geq\beta_3}(S^1, \mathbb{R}^d) \times \prod_{i=3,4,6} \left(S^{d-3} \times \mathbb{I} \times \mathbb{I}\right)$ where $\beta_3 = [x_1, x_4] \cdot [x_2, x_6] \cdot [x_3, x_5]$ as the subset of all $K_3 \times \prod_{i=3,4,6} \left(u_i, \frac{\delta}{10}, \varepsilon_i\right)$ where u_4 and u_6 are unrestricted and u_3 satisfies $\| w_4 + \delta u_3 \| = 1$.

Proposition 2.1. *Let $S_3 = \{x_3, x_4, x_6\}$. The restriction of ρ_{β_3,S_3} maps M_3 to $\mathrm{Emb}(S^1, \mathbb{R}^d) \subset \mathrm{Imm}_{\geq\phi}(S^1, \mathbb{R}^d)$, and $\rho_{\beta_3,S_3}(M_3)$ is the $\| v_3 - w_4 \| = \delta$ boundary component of $\mathcal{M}_1$.*

Proof. The resolution $\rho_{\beta_3,S_3}(K_3) = \rho_{\beta_3}\left(\rho_3\left(K_1, w_4, \frac{1}{10}, \varepsilon_3\right)\right)$ using u_3 as in the definition of $M_3(\varepsilon_3, \varepsilon_4, \varepsilon_5)$ is the same embedding as the resolution $\rho_{\beta_1}(K_1)$ using $v_3 = w_4 + \delta u_3$, since

$$\frac{1}{10} w_4 \exp\left(\frac{1}{(t-t_3)^2 + \varepsilon_3^2}\right) + \frac{\delta}{10} u_3 \exp\left(\frac{1}{(t-t_3)^2 + \varepsilon_3^2}\right) = \frac{1}{10} v_3 \exp\left(\frac{1}{(t-t_3)^2 + \varepsilon_3^2}\right).$$

□

Similarly, resolving the knots K_4, K_5, and K_6 yields the boundary components of $\mathcal{M}_1$ corresponding to $\| v_3 + w_4 \| = \delta$, $\| v_3 - w_1 \| = \delta$, and $\| v_3 + w_1 \| = \delta$ respectively. This process can also be applied to the boundary components of $\mathcal{M}_2$. Let K_7, K_8, K_9, and K_{10} be the four singular knots obtained from K_2 by moving $K_2\left([t_3 - \varepsilon_3, t_3 + \varepsilon_3]\right)$ in the direction of the tangent vectors to the other two strands intersecting at the triple point, as shown in Figure 2.5. As with K_1, resolving these singular knots gives the four boundary components of $\mathcal{M}_2$.

Since each of the four knots K_3, K_4, K_5, K_6 has the same singularity data as one of K_7, K_8, K_9, K_{10}, we have four pairs of knots which are isotopic in $\mathrm{Imm}_{=\beta_i}(S^1, \mathbb{R}^4)$, and thus in $\mathrm{Imm}_{=\beta_i}(S^1, \mathbb{R}^d)$ with $d \geq 4$, where $\beta_3, \beta_4, \beta_5, \beta_6$ each encodes singularity data for a knot with exactly three double points. If $d > 4$, we require that the isotopy

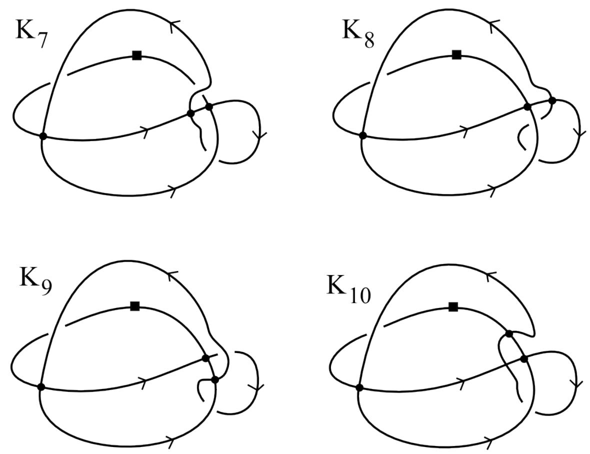


Figure 2.5. Singular knots K_7, K_8, K_9, and K_{10}.

be through knots in $\mathbb{R}^4 \subset \mathbb{R}^d$ (with the standard embedding). If $d = 4$, we restrict the steps of the isotopy, as described below, to simplify evaluation of Longoni cocycle on the cycle. Resolving each singular knot in these four isotopies yields four families, denoted $\mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_5$, and $\mathcal{M}_6$, parametrized by $S^{d-3} \times S^{d-3} \times S^{d-3} \times \mathbb{I}$. Specifically, if $h_i : \mathbb{I} \to \mathrm{Imm}_{=\beta_i}(S^1, \mathbb{R}^d)$ is an isotopy, then these $\mathcal{M}_i$ are be the images of the composites

$$\begin{tikzcd}
M_i \times \mathbb{I} = S^{d-3} \times S^{d-3} \times S^{d-3} \times \mathbb{I} \arrow[dr, bend right, "\mathrm{Id} \times h_i"'] & & \\
& S^{d-3} \times S^{d-3} \times S^{d-3} \times \mathrm{Imm}_{=\beta_i}(S^1, \mathbb{R}^d) \arrow[r, "\rho_{\beta_i, S_i}"] & \mathrm{Emb}(S^1, \mathbb{R}^d)
\end{tikzcd}$$

For $i = 3, 4, 5, 6$, the boundary of $\mathcal{M}_i$ is the disjoint union of a boundary component of $\mathcal{M}_1$ and a boundary component of $\mathcal{M}_2$, providing a way to glue the boundary of $\mathcal{M}_1$ to the boundary of $\mathcal{M}_2$.

The union of these six $(3d-8)$-dimensional families in $\mathrm{Emb}(S^1, \mathbb{R}^d)$ gives a single family without boundary. Let

$$M_\beta = \left(M_1 \sqcup M_2 \sqcup \left(\sqcup_{i=3}^{6} M_i \times \mathbb{I}\right)\right) / \sim$$

where each boundary component of M_3, M_4, M_5, M_6 is identified with a boundary component of M_1 or M_2 so as to be compatible with Proposition 2.1. Let $\mathcal{M}_\beta$ be the image of the orientable topological manifold M_β under the resolution map defined above. We can now precisely state the first version of our main result.

Theorem 2.1. *If $d > 4$ is even then the fundamental class of $\mathcal{M}_\beta$ is a non-trivial homology class in $\mathrm{Emb}(S^1, \mathbb{R}^d)$ for any choice of isotopies h_i through $\mathrm{Imm}_{=\beta_i}(S^1, \mathbb{R}^d)$. For $d = 4$ the fundamental class of $\mathcal{M}_\beta$ is a non-trivial homology class in $\mathrm{Emb}(S^1, \mathbb{R}^d)$ if the isotopies h_i satisfy a sequence of specified steps, given in Section 2.1.1.*

2.1.1. Isotopies for the Case $d = 4$.

We can construct isotopies whose images are in $\mathbb{R}^3$ except for near crossing changes. This forces the calculation we will do in Section 2.3 to be the same for $d = 4$ as in the higher dimensional cases.

By a slide isotopy we will mean an isotopy through singular knots in which a singular point is moved along one of the strands through the singularity while the other strand moves along with the singular point. By a planar isotopy we will mean an isotopy which can be represented by an isotopy of knot diagrams. Isotopies corresponding to the Reidemeister moves in classical knot theory generalize to singular knots in $\mathbb{R}^d$. In addition to the usual Reidemeister I and II moves, we use Reidemeister

III moves to move a strand past a crossing (as in classical theory) or past a singularity, as shown in Figure 2.6.

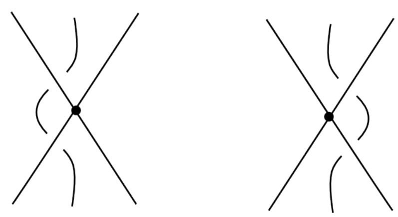


Figure 2.6. Reidemeister III move for singular knots.

By a "rotate the disk isotopy," we mean an isotopy in which the disk centered at a singularity is rotated by 180° about the axis perpendicular to a particular great circle. Specifically, we take two distinct nested disks centered at the singular point with radii small enough that the intersection of the knot with the disks is the two strands intersecting at the singular point. The smaller of the two disks is rotated by 180° without changing anything inside of this disk. The strands inside of the larger disk but outside of the smaller disk are stretched through a planar isotopy. This isotopy is shown in Figure 2.7 from the perspective of the north pole of the larger disk. The knot remains unchanged outside of the larger disk.

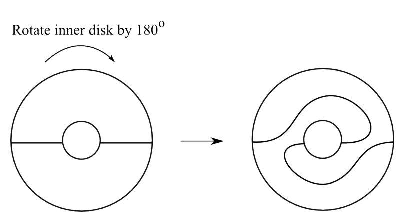


Figure 2.7. View of a rotate the disk isotopy from the north pole.

A suitable type of isotopy from K_3 to K_9 is shown in Figure 2.8, and the steps are given below. Each step occurs in $\mathbb{R}^3 \subset \mathbb{R}^4$ except (4), (6) and (10), in which one strand of the knot briefly moves into $\mathbb{R}^4$.

1. Simplify the shape of the strand from b_1 to c_1 and perform a Reidemeister II move on the strand from a_1 to b_1 to eliminate crossings.

2. Move the points a_1, b_1 and c_1 to a_2, b_2 and c_2 through a planar isotopy.

3. Rotate the disk centered at c_2 by 180° about the axis perpendicular to the great circle shown.

4. The crossing is changed, briefly moving the strand from b_2 to c_2 in the direction of the fourth standard basis vector.

5. Perform a sequence of Redemeister I,II and III moves on the strand from b_2 to c_2.

6. The crossing is changed, briefly moving the strand from c_2 to a_2 in the direction of the fourth standard basis vector.

7. Perform a sequence of Reidemeister I, II and III moves on the strand from c_2 to a_2.

8. Rotate the disk centered at a_2 by 180° about the axis perpendicular to the great circle shown.

9. Perform a sequence of Reidemeister I, II and III moves on the strand from c_2 to a_2 and the strand from a_2 to b_2.

10. The crossing is changed, briefly moving the strand from a_2 to b_2 in the direction of the fourth standard basis vector.

11. Perform a sequence of Reidemeister I, II and III moves on the strand from a_2 to b_2.

12. Through a planar isotopy, the points a_2, b_2 and c_2 are moved to the positions of the double points of K_9, denoted a_3, b_3 and c_3 and the strands are moved to give the knot the same shape as K_9.

A suitable type of isotopy from K_6 to K_{10} is shown in Figure 2.9, and the steps are given below. Each step occurs in $\mathbb{R}^3 \subset \mathbb{R}^4$.

1. Through a slide isotopy, the point b_1 is moved to b_2, and the shape of the strand from c_1 to a_1 is simplified.

2. Rotate the disk centered at a_2 by 180° about the axis perpendicular to the great circle shown.

3. Perform a sequence of Reidemeister I, II and III moves on the strand from a_2 to b_2 to remove crossings.

4. Rotate the disk centered at c_2 by 180° about the axis perpendicular to the great circle shown.

5. Perform a sequence of Reidemeister I, II and III moves on the strand from c_2 to the base point to eliminate crossings.

6. Through a planar isotopy, move the points a_2, b_2 and c_2 to the positions of the double points of K_{10}, denoted a_3, b_3 and c_3.

7. Perform a sequence of Reidemeister I, II and III moves on the strand from b_3 to c_3.

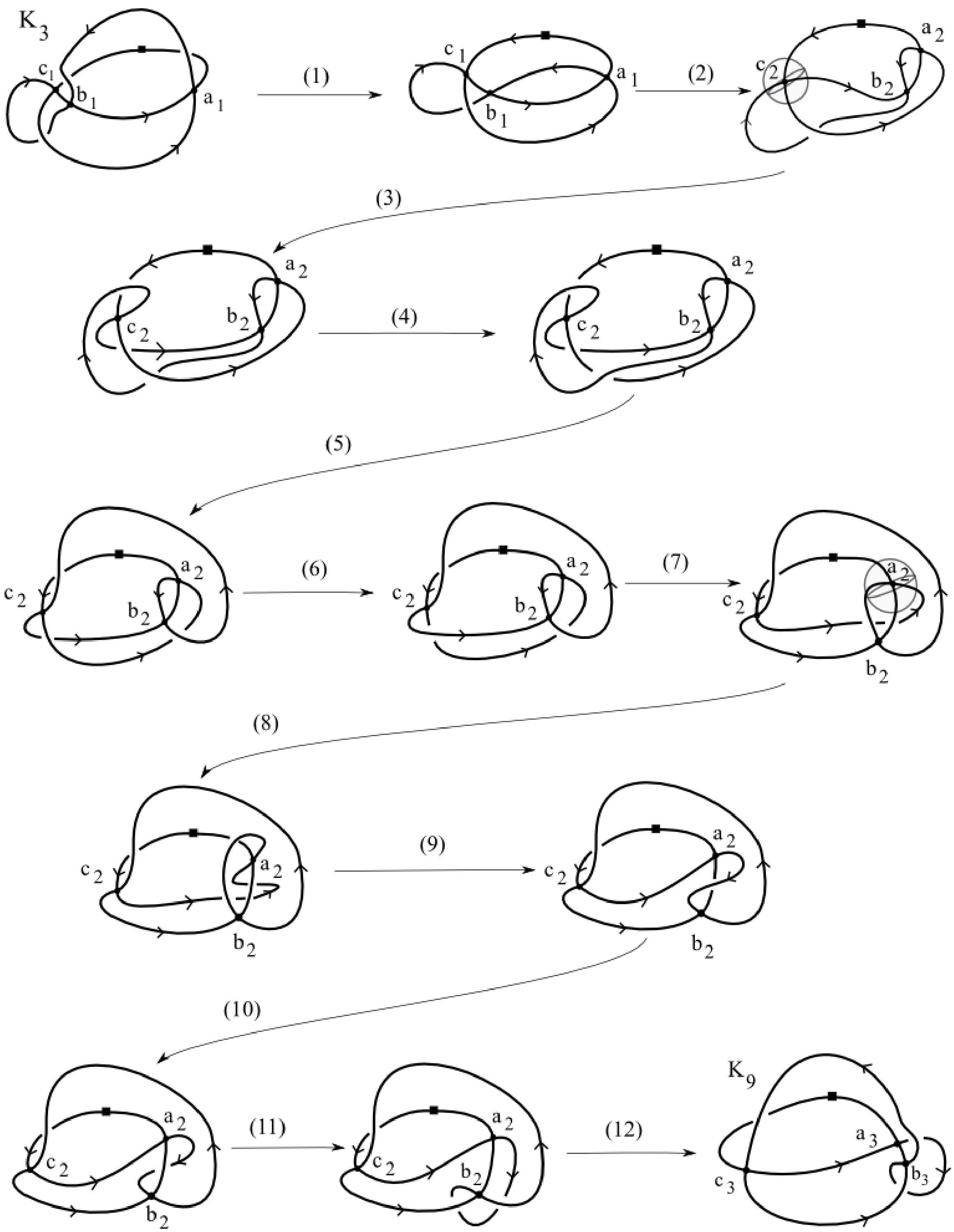


Figure 2.8. Isotopy from K_3 to K_9.

8. Through a planar isotopy, move the strands to give the knot the same shape and K_{10}.

A suitable type of isotopy from K_5 to K_7 is shown in Figure 2.10, and the steps are given below. Each step occurs in $\mathbb{R}^3 \subset \mathbb{R}^4$.

1. Through a slide isotopy, move point b_1 to b_2 and simplify the shape of the strands with a planar isotopy.

2. Rotate the disk centered at b_2 by 180° about the axis perpendicular to the great circle shown.

3. Perform a sequence of Reidemeister I, II and III moves on the strand from b_2 to a_2 to eliminate crossings.

4. Perform a sequence of Reidemeister I, II and III moves on the strand from c_2 to b_2.

5. Rotate the disk centered at b_2 by 180° about the axis perpendicular to the great circle shown.

6. Rotate the disk centered at a_2 by 180° about the axis perpendicular to the great circle shown.

7. Perform a sequence of Reidemeister I, II and III moves on the strand from b_2 to a_2 to eliminate crossings.

8. Rotate the disk centered at c_2 by 180° about the axis perpendicular to the great circle shown.

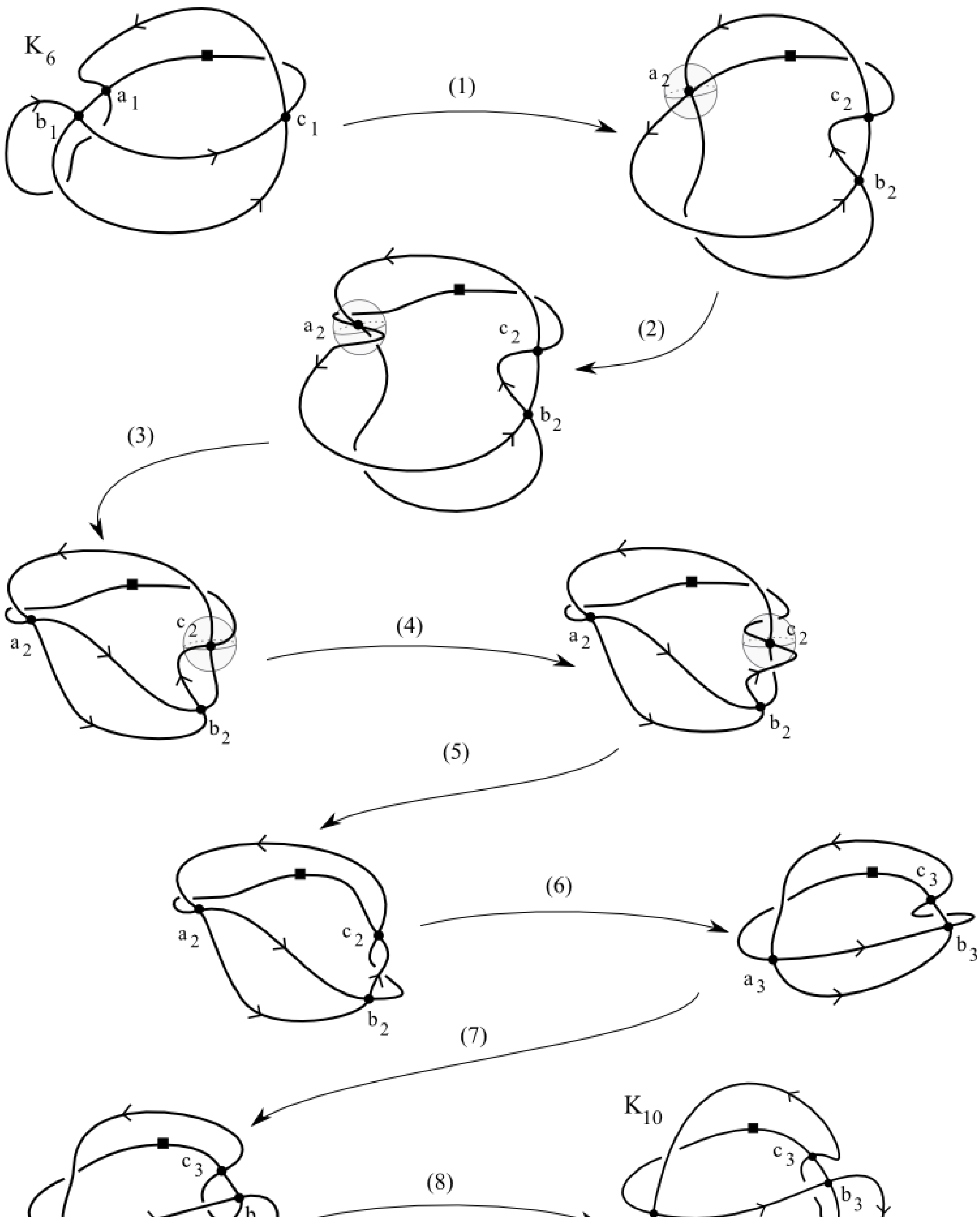


Figure 2.9. Isotopy from K_6 to K_{10}.

9. Perform a sequence of Reidemeister I, II and III moves on the strand from c_2 to b_2 to eliminate crossings.

10. Perform a sequence of Reidemeister I, II and III moves on the strand from b_2 to c_2.

11. Perform a sequence of Reidemeister I, II and III moves on the strand from a_2 to c_2.

12. Through a slide isotopy, move the point b_2 to b_3.

13. Through a planar isotopy, move the points a_3, b_3 and c_3 to the positions of the double points of K_7, denoted a_4, b_4 and c_4.

14. Perform a sequence of Reidemeister I, II and III moves on the strand from b_4 to a_4. Through a planar isotopy, move the strands to give the knot the same shape and K_7.

A suitable type of isotopy from K_4 to K_8 is shown in Figure 2.11, and the steps are given below. Each step occurs in $\mathbb{R}^3 \subset \mathbb{R}^4$, except (6), in which one strand of the knot briefly moves into $\mathbb{R}^4$.

1. Simplify the shape of the strand from c_1 to a_1 and perform a sequence of Reidemeister I, II and III moves on the loop based at a_1 to eliminate crossings.

2. Through a planar isotopy, move point a_1 to the point a_2.

3. Rotate the disk centered at b_2 by 180° about the axis perpendicular to the great circle shown.

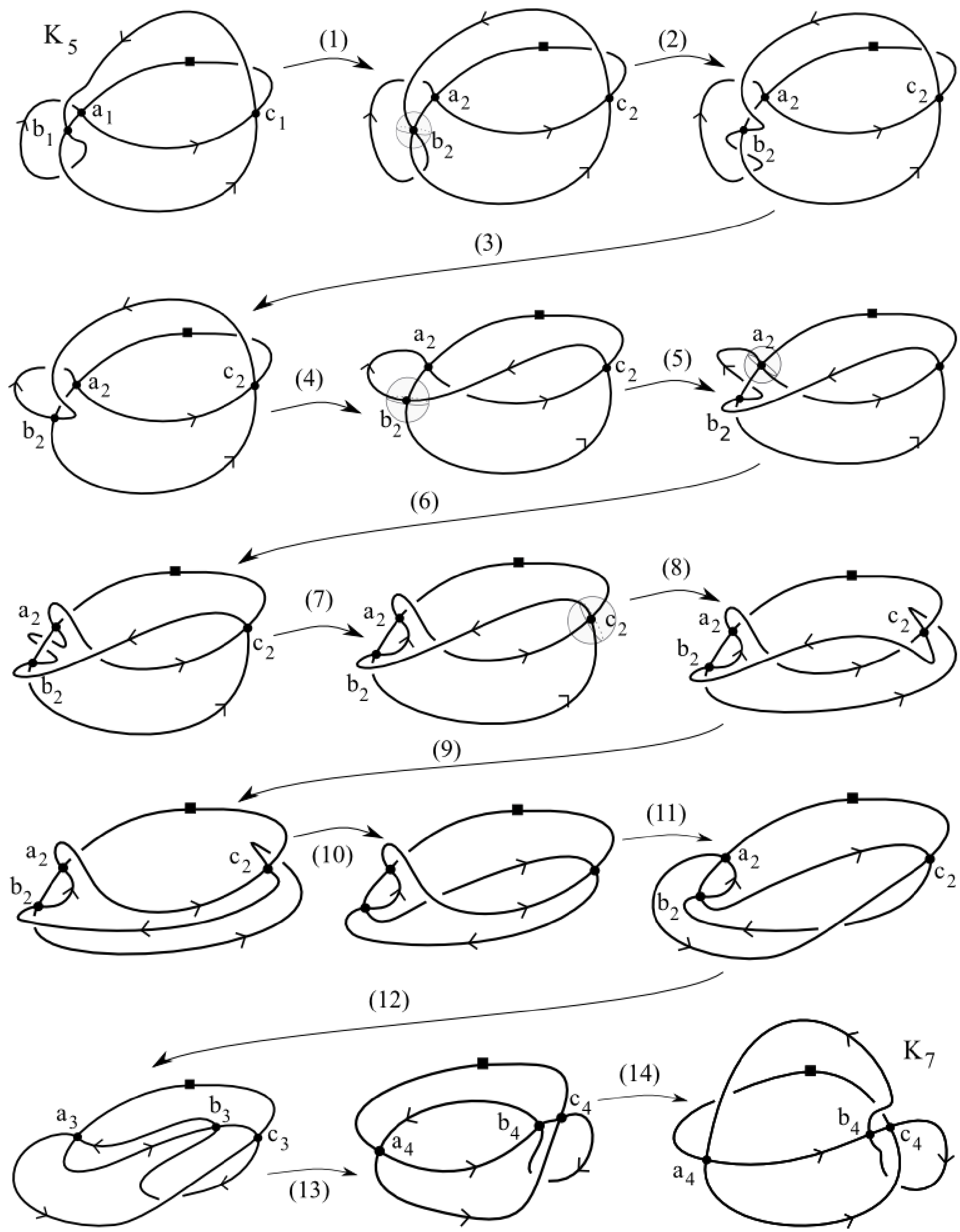


Figure 2.10. Isotopy from K_5 to K_7.

4. Rotate the disk centered at c_2 by $180°$ about the axis perpendicular to the great circle shown.

5. Perform a sequence of Reidemeister I, II and III moves on one of the strands from b_2 to c_2, and simplify the shape of the other strand from b_2 to c_2. Simplify the shape of the strand from a_2 to b_2.

6. The crossing is changed, briefly moving the strand from c_2 to a_2 in the direction of the fourth standard basis vector.

7. Perform a Reidemeister II move on the strand from c_2 to a_2. Through a planar isotopy, move the points a_2, b_2 and c_2 to positions of the double points of K_8, denoted a_3, b_3 and c_3.

8. Through a planar isotopy, the strands are moved to give the knot the same shape as K_8.

2.2. The Longoni Cocycle

In [7], Cattaneo, Cotta-Ramusino, and Longoni use configuration space integrals to define a chain map I from a complex of decorated graphs to the de Rham complex of $\mathrm{Emb}(S^1, \mathbb{R}^d)$. The starting point is the evaluation map $ev : C_q[S^1] \times \mathrm{Emb}(S^1, \mathbb{R}^d) \to C_q[\mathbb{R}^d]$, where $C_q[M]$ is the Fulton-MacPherson compactified configuration space. See [20] for more details. For some graphs G (namely those with no internal vertices), the image of the chain map I is defined by pulling back a form determined by G from $C_q[\mathbb{R}^d]$ to $C_q[S^1] \times \mathrm{Emb}(S^1, \mathbb{R}^d)$ and then pushing forward to $\mathrm{Emb}(S^1, \mathbb{R}^d)$.

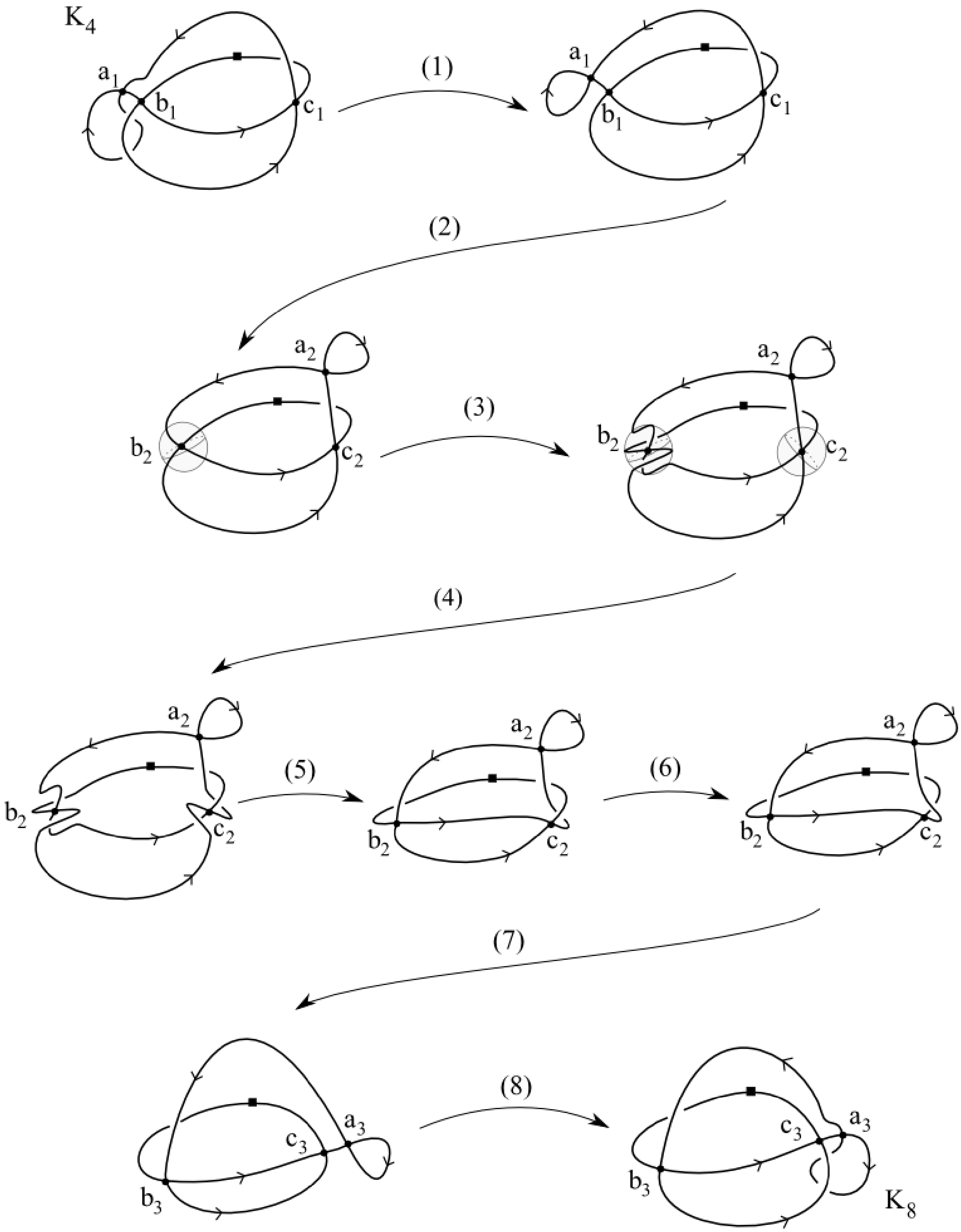


Figure 2.11. Isotopy from K_4 to K_8.

To understand the general case, let $ev^*C_{q,r}[\mathbb{R}^d]$ be the total space of the pull-back bundle shown below:

$$\begin{tikzcd}
ev^*C_{q,r}[\mathbb{R}^d] \arrow[r, "\hat{ev}"] \arrow[d] & C_{q+r}[\mathbb{R}^d] \arrow[d] \\
C_q^{ord}[S^1] \times \mathrm{Emb}(S^1, \mathbb{R}^d) \arrow[r, "ev"] & C_q[\mathbb{R}^d] \ ,
\end{tikzcd}$$

where $C_q^{ord}[S^1]$ is the connected component of $C_q[S^1]$ in which the ordering on the points in the configuration agrees with the ordering induced by the orientation of S^1. Fix an antipodally symmetric volume form on S^{d-1}, denoted α. A choice of α determines tautological $(d-1)-$forms on $ev^*C_{q,r}[\mathbb{R}^d]$, defined by

$$\theta_{ij} = \hat{ev}^*\phi_{ij}^*(\alpha)$$

where $\phi_{ij} : C_q(\mathbb{R}^d) \to S^{d-1}$ sends a configuration to the unit vector from the $i-$th point to the $j-$th point in the configuration. We use integration over the fiber of the bundle $ev^*C_{q,r}[\mathbb{R}^d] \to \mathrm{Emb}(S^1, \mathbb{R}^d)$, which is the composite of the projections

$$ev^*C_{q,r}[\mathbb{R}^d] \to C_q^{ord}[S^1] \times \mathrm{Emb}(S^1, \mathbb{R}^d) \to \mathrm{Emb}(S^1, \mathbb{R}^d),$$

to push forward products of the tautological forms to forms on $\mathrm{Emb}(S^1, \mathbb{R}^d)$.

Which forms to push forward will be determined by graphs. Consider connected graphs which satisfy the following conditions. A *decorated graph (of even type)* is a connected graph consisting of an oriented circle, vertices on the circle (called *external vertices*), vertices which are not on the circle (called *internal vertices*), and edges. We require that all vertices are at least trivalent. The decoration consists of an enumeration of the edges and an enumeration of the external vertices that is cyclic with respect to the orientation of the circle. We will call the portion of the oriented circle between two external vertices an *arc*.

Definition 2.10. Let $\mathcal{D}_e$ be the vector space generated by decorated graphs of even type with the following relations. We set $G = 0$ if there are two edges in G with the same endpoints, or if there is an edge in G whose endpoints are the same internal vertex. The graphs G and G' are equal if they are isomorphic as graphs and the enumerations of their edges differ by an even permutation.

The vector space $\mathcal{D}_e$ admits a bigrading as follows. Let v_e and v_i be the number of external and internal vertices, respectively, and let e be the number edges. The order of a graph is given by

$$\operatorname{ord} G = e - v_i$$

and the degree of a graph is defined by

$$\deg G = 2e - 3v_i - v_e.$$

Let $\mathcal{D}_e^{k,m}$ be the vector space of equivalence classes with order k and degree m. In [7], Cattaneo, Cotta-Ramusino and Longoni define a map from this vector space to the space of $(m + (d-3)k)-$ forms on $\mathrm{Emb}(S^1, \mathbb{R}^d)$.

Definition 2.11. Define $I(\alpha) : \mathcal{D}_e^{k,m} \to \Omega^{m+(d-3)k}\left(\mathrm{Emb}(S^1, \mathbb{R}^d)\right)$ as follows.

1. Choose an ordering on the internal vertices.

2. Associate each edge in G joining vertex i and vertex j to the tautological form θ_{ij}.

3. Take the product of these tautological forms with the order of multiplication determined by the enumeration of the edges, to define a form on $ev^* C_{q,r}[\mathbb{R}^d]$.

4. Integrate this form over the fiber to obtain a form on $\mathrm{Emb}(S^1, \mathbb{R}^d)$.

This integration over the fiber defines the pushforward and in this case is often called a configuration space integral. There is a coboundary map on $\mathcal{D}_e$ which makes $I(\alpha)$ a cochain map.

Definition 2.12. Define a coboundary operator on $\mathcal{D}_e$ by taking δG to be the signed sum of the decorated graphs obtained from G by contracting, one at a time, the arcs of G and the edges of G which have at least one endpoint at an external vertex. After contracting, the edges and vertices are relabeled in the obvious way - if the edge (respectively vertex) labeled i is removed, we replace the label j by $j-1$ for all $j > i$. When contracting an arc joining vertex i to $i+1$, the sign is given by $\sigma(i, i+1) = (-1)^{i+1}$, and when contracting the arc joining vertex j to vertex 1, the sign is given by $\sigma(j, 1) = (-1)^{j+1}$. When contracting the edge l, the sign is given by $\sigma(l) = l + 1 + v_e$, where v_e is the number of external vertices.

Theorem 2.2. *[7] The map $I(\alpha)$ determines a cochain map and therefore induces a map on cohomology, which we denote $I(\alpha) : H^{k,m}(\mathcal{D}_e) \to H^{m+(d-3)k}(\mathrm{Emb}(S^1, \mathbb{R}^d))$.*

At the level of forms, $I(\alpha)$ depends on the choice of antipodally symmetric volume form α. On cohomology, when $d > 4$ this is independent of α.

Example 2.2. From [7], we have the graph cocycle shown in Figure 2.12, originally investigated by Bott and Taubes [4] for $d = 3$.

This induces the cocycle

$$\frac{1}{4}\int_{ev^* C_{4,0}[\mathbb{R}^d]} \theta_{13}\theta_{24} - \frac{1}{3}\int_{ev^* C_{3,1}[\mathbb{R}^d]} \theta_{14}\theta_{24}\theta_{34} \in H^{2d-6}\left(\mathrm{Emb}(S^1, \mathbb{R}^d)\right).$$

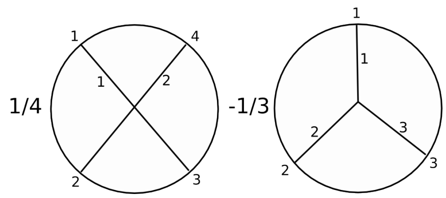


Figure 2.12. Graph cocycle given by Cattaneo et al. in [7].

In [7], Cattaneo et al. show that this cocycle evaluates non-trivially on $\rho_{[x_1,x_3]\cdot[x_2,x_4]}\left(K \times S^{d-3} \times S^{d-3}\right)$, where K is a singular knot with two double points respecting $[x_1, x_3] \cdot [x_2, x_4]$ (in this case, the cycle does not depend on the ordered subset $S \subseteq \{x_1, x_2, x_3, x_4\}$).

Example 2.3. In [16], Longoni gives the example shown in Figure 2.13 of a graph cocycle G_L in $H^{3,1}(\mathcal{D}_e)$ which uses nontrivalent graphs. There $I(\alpha)(G_L) \in H^{3(d-3)+1}(\mathrm{Emb}(S^1, \mathbb{R}^d))$ is the form

$$\omega = \int_{ev^* C_{4,1}[\mathbb{R}^d]} \theta_{15}\theta_{45}\theta_{35}\theta_{25} + 2\int_{ev^* C_{5,0}[\mathbb{R}^d]} \theta_{13}\theta_{14}\theta_{25}.$$

We pair this cocycle with the cycle $[M_\beta]$ defined in Section 2.1 to see that both are non-trivial.

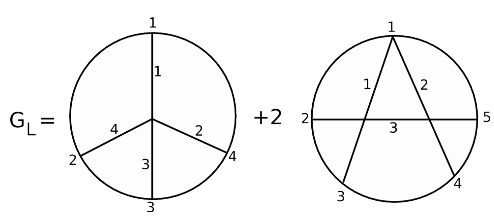


Figure 2.13. Graph cocycle given by Longoni in [16].

2.3. Non-triviality

Theorem 2.3. *Assume $d \geq 4$ is even. Let $[\mathcal{M}_\beta] \in H_{3(d-3)+1}(\mathrm{Emb}(S^1, \mathbb{R}^d))$ be the cycle defined in Section 2.1, and let $\omega \in H^{3(d-3)+1}(\mathrm{Emb}(S^1, \mathbb{R}^d))$ be the Longoni cocycle defined in the last section. Then $\omega([\mathcal{M}_\beta]) = \pm 2$. In particular, $\omega([\mathcal{M}_\beta])$ is nonzero, and therefore both ω and $[\mathcal{M}_\beta]$ are non-trivial.*

In [24], Turchin calculates that $E^2_{-5,3(d-1)}$ in the spectral sequence for $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d)$ has rank one, so $[\mathcal{M}_\beta]$ is a generator of this vector space.

Proof. Write $\omega = \omega_1 + 2\omega_2$ where $\omega_1 = \int_{ev^* C_{4,1}[\mathbb{R}^d]} \theta_{15}\theta_{45}\theta_{35}\theta_{25}$ and $\omega_2 = \int_{ev^* C_{5,0}[\mathbb{R}^d]} \theta_{13}\theta_{14}\theta_{25}$.

First we show that $\omega_2([\mathcal{M}_\beta]) = \pm 1$. Let $g : ev^* C_{5,0}[\mathbb{R}^d] \to S^{d-1} \times S^{d-1} \times S^{d-1}$ be the map shown in the diagram below, where $\bar{\psi} = \phi_{13} \times \phi_{14} \times \phi_{25}$. Then ω_2 is the pushforward along $\pi : ev^* C_{5,0}[\mathbb{R}^d] \to \mathrm{Emb}(S^1, \mathbb{R}^d)$ of $g^*(\alpha \otimes \alpha \otimes \alpha)$.

$$\begin{tikzcd}
 & ev^* C_{5,0}[\mathbb{R}^d] \arrow[r] \arrow[d] \arrow[rr, bend left, "g"] \arrow[dd, bend right, "\pi"'] & C_5[\mathbb{R}^d] \arrow[r, "\bar{\psi}"] \arrow[d, "id"] & S^{d-1} \times S^{d-1} \times S^{d-1} \\
 & C_5^{ord}[S^1] \times \mathrm{Emb}(S^1, \mathbb{R}^d) \arrow[r] \arrow[d] & C_5[\mathbb{R}^d] & \\
\mathcal{M}_\beta \arrow[r, hookrightarrow] & \mathrm{Emb}(S^1, \mathbb{R}^d) & &
\end{tikzcd}$$

By naturality of pushforwards, $\omega_2([\mathcal{M}_\beta]) = g^*(\alpha \otimes \alpha \otimes \alpha)([\pi^{-1}(\mathcal{M}_\beta)])$. The bundle $\pi : ev^* C_{5,0}[\mathbb{R}^d] \to \mathrm{Emb}(S^1, \mathbb{R}^d)$ is trivial, so

$$g^*(\alpha \otimes \alpha \otimes \alpha)([\pi^{-1}(\mathcal{M}_\beta)]) = \int_{C_5^{ord}[S^1] \times \mathcal{M}_\beta} g^*(\alpha \otimes \alpha \otimes \alpha).$$

To calculate $\int_{C_5^{ord}[S^1]\times\mathcal{M}_\beta} g^*(\alpha\otimes\alpha\otimes\alpha)$, we first partition $C_5^{ord}[S^1]$. For $i = 1, \ldots, 5$ let $N_i = (t_i - \varepsilon, t_i + \varepsilon)$, where the t_i are the times of singularity in K_1 and K_2, and ε is as in Section 2.1. Define

$$C_5^{(i)} = \left\{ \bar{s} \in C_5^{ord} : s_j \notin N_i \text{ for } j = 1, \ldots, 5 \text{ and } \bar{s} \notin C_5^{(m)} \text{ for } m < i \right\},$$

and $C_5^c = C_5^{ord}[S^1] \backslash \left(\cup_{i=1}^5 C_5^{(i)} \right)$, so C_5^c is the set of all $\bar{s} \in C_5^{ord}[S^1]$ such that $t_i - \varepsilon < s_i < t_i + \varepsilon$ for $i = 1, \ldots, 5$. Then $C_5^{ord}[S^1]$ decomposes as

$$C_5^{ord}[S^1] = C_5^c \sqcup C_5^{(1)} \sqcup \cdots \sqcup C_5^{(5)},$$

and we obtain a corresponding decomposition of $\int_{C_5^{ord}[S^1]\times\mathcal{M}_\beta} g^*(\alpha \otimes \alpha \otimes \alpha)$. We will show that $\int_{C_5^{(m)}\times\mathcal{M}_\beta} g^*(\alpha \otimes \alpha \otimes \alpha) = 0$ for $m = 1, \ldots, 5$, so calculating $\omega_2([\mathcal{M}_\beta])$ reduces to evaluating the integrals $\int_{C_5^c\times\mathcal{M}_i} g^*(\alpha \otimes \alpha \otimes \alpha)$.

For $m = 3, 4, 5$, we show $\int_{C_5^{(m)}\times\mathcal{M}_\beta} g^*(\alpha\otimes\alpha\otimes\alpha) = 0$ by showing $\int_{C_5^{(m)}\times\mathcal{M}_i} g^*(\alpha \otimes \alpha \otimes \alpha) = 0$ for $i = 1, \ldots, 6$. Recall that manifolds have only trivial forms in degrees above their dimension, so a form pulled back through a smaller dimensional manifold is always zero. To prove that the integrals $\int_{C_5^{(m)}\times\mathcal{M}_i} g^*(\alpha \otimes \alpha \otimes \alpha)$ are zero, we show that the map g factors through spaces of smaller dimension when restricted to each of the subspaces $C_5^{(m)} \times \mathcal{M}_i$.

First, consider the case $\int_{C_5^{(3)}\times\mathcal{M}_1} g^*(\alpha \otimes \alpha \otimes \alpha)$. Recall that $\mathcal{M}_1$ is $\rho_{\beta_1,S}\left(K_1 \times \prod_{k=3}^5 (v_k, a_k, \varepsilon)\right)$. If $t \notin N_3$ and $\gamma \in \mathcal{M}_1$, the point $\gamma(t)$ does not depend on the value of v_3 in the preimage of γ. This gives us the following factorization of

$g\big|_{C_5^{(3)}\times\mathcal{M}_1}$:

$$\begin{tikzcd}[column sep=small]
C_5^{(3)} \times \mathcal{M}_1 \arrow[rr, "g"] \arrow[dr] & & S^{d-1}\times S^{d-1}\times S^{d-1} \\
& C_5^{(3)} \times S^{d-3} \times S^{d-3} \arrow[ur] &
\end{tikzcd}$$

Since $\dim(C_5^{(3)} \times S^{d-3} \times S^{d-3}) = 2d-1$ is less than $\dim(S^{d-1}\times S^{d-1}\times S^{d-1}) = 3d-3$, we have $\int_{C_5^{(3)}\times\mathcal{M}_1} g^*(\alpha\otimes\alpha\otimes\alpha) = 0$.

Similarly, for $m = 4$ or $m = 5$, the restriction $g\big|_{C_5^{(m)}\times\mathcal{M}_1}$ factors through

$$C_5^{(m)} \times \{v_3 \in S^{d-2} : \parallel v_3 \pm w_1 \parallel > \delta \text{ and } \parallel v_3 \pm w_4 \parallel > \delta\} \times S^{d-3},$$

so the corresponding integrals are zero. This argument also shows that $\int_{C_5^{(m)}\times\mathcal{M}_2} g^*(\alpha\otimes\alpha\otimes\alpha) = 0$ for $m = 3,4,5$. For $i = 3,4,5,6$ and $m = 3,4,5$, the restriction $g\big|_{C_5^{(m)}\times\mathcal{M}_i}$ factors through $S^{d-3}\times S^{d-3}\times\mathbb{I}$ and therefore $\int_{C_5^{(m)}\times\mathcal{M}_i} g^*(\alpha\otimes\alpha\otimes\alpha)$ is zero.

We show $\int_{C_5^{(1)}\times\mathcal{M}_\beta} g^*(\alpha\otimes\alpha\otimes\alpha) = 0$ by replacing $\mathcal{M}_\beta$ with the family of embeddings obtained by moving the first strand (instead of the fourth) off of the double point $K_i(t_1) = K_i(t_4)$, over which g^* factors through a space of lower dimension. We replace $\mathcal{M}_\beta$ in two steps - first with the family of embeddings in which both strands are moved off the double point, and then by the family in which only the first strand is moved.

Let $\mathcal{M}'_\beta$ be the piecewise smooth subspace of $\mathrm{Emb}(S^1,\mathbb{R}^d)$ defined similarly to $\mathcal{M}_\beta$, but by choosing the ordered subset of variables in β_1 and β_2 to be $S = \{x_1,x_3,x_4,x_5\}$, and fixing $a_1 = a_4$ and $v_1 = -v_4$. In other words, $\mathcal{M}'_\beta$ is obtained from $K_1, \ldots, K_6$ by moving both strands off the double point $K_i(t_1) = K_i(t_4)$ in antipodal directions.

We define a cobordism W_1 between $\mathcal{M}_\beta$ and $\mathcal{M}'_\beta$ as the subspace of $\mathrm{Emb}(S^1, \mathbb{R}^d)$ parametrized by $(\sqcup_i M_i) \times \mathbb{I}$, with the embedding corresponding to the parameter $u \in \mathbb{I}$ determined by $a_1 = ua_4$ (so the $\mathbb{I}$ parametrizes how far the strand with $K_i(t_1)$ is moved off the double point).

By Stokes' theorem,

$$\int_{C_5^{(1)} \times W_1} dg^*(\alpha \otimes \alpha \otimes \alpha) = \int_{\partial(C_5^{(1)} \times W_1)} g^*(\alpha \otimes \alpha \otimes \alpha).$$

Since $dg^*(\alpha \otimes \alpha \otimes \alpha) = g^* d(\alpha \otimes \alpha \otimes \alpha) = 0$, we have

$$0 = \int_{\partial C_5^{(1)} \times W_1} g^*(\alpha \otimes \alpha \otimes \alpha) + \int_{C_5^{(1)} \times \mathcal{M}_\beta} g^*(\alpha \otimes \alpha \otimes \alpha) - \int_{C_5^{(1)} \times \mathcal{M}'_\beta} g^*(\alpha \otimes \alpha \otimes \alpha). \quad (2.1)$$

The restriction $g^*\big|_{\partial C_5^{(1)} \times W_1}$ factors through $\partial C_5^{(1)} \times (\sqcup_i \mathcal{M}_i)$. If $\bar{s} \in \partial C_5^{(1)}$ then the parameter, $u \in \mathbb{I}$ determining how far the first strand is moved does not affect $g(\bar{s}, \gamma)$ for $\gamma \in W_1$. Thus, $\int_{\partial C_5^{(1)} \times W_1} g^*(\alpha \otimes \alpha \otimes \alpha) = 0$ and

$$\int_{C_5^{(1)} \times \mathcal{M}_\beta} g^*(\alpha \otimes \alpha \otimes \alpha) = \int_{C_5^{(1)} \times \mathcal{M}'_\beta} g^*(\alpha \otimes \alpha \otimes \alpha).$$

Let $\mathcal{M}''_\beta$ be the piecewise smooth subspace of $\mathrm{Emb}(S^1, \mathbb{R}^d)$ obtained by choosing the ordered subset of variables in β_1 and β_2 to be $S = \{x_1, x_3, x_5\}$. In other words, $\mathcal{M}''_\beta$ is obtained from $K_1, \ldots, K_6$ by moving only the first strand off the double point $K_i(t_1) = K_i(t_4)$. Let $W_2 \subset \mathrm{Emb}(S^1, \mathbb{R}^d)$ be parametrized by $(\sqcup_i M_i) \times \mathbb{I}$, with the embedding corresponding to the parameter $u \in \mathbb{I}$ given by choosing $a''_4 = ua_4$ (so the interval parametrizes how far the strand with $K_i(t_4)$ is moved off the double point).

Then W_2 gives a cobordism between $\mathcal{M}'_\beta$ and $\mathcal{M}''_\beta$, as $\partial W_2 = \mathcal{M}_\beta \sqcup (-\mathcal{M}'_\beta)$. Using Stokes' Theorem and naturality again, we have

$$0 = \int_{\partial C_5^{(1)} \times W_2} g^*(\alpha \otimes \alpha \otimes \alpha) + \int_{C_5^{(1)} \times \mathcal{M}'_\beta} g^*(\alpha \otimes \alpha \otimes \alpha) - \int_{C_5^{(1)} \times \mathcal{M}''_\beta} g^*(\alpha \otimes \alpha \otimes \alpha). \quad (2.2)$$

The restriction $g^*\big|_{\partial C_5^{(1)} \times W_2}$ does not factor through $\partial C_5^{(1)} \times (\sqcup_i M_i)$. To show the first integral in (2.2) is zero, we consider W_2 as a subspace of $\mathrm{Imm}_{\leq [x_1,x_2],t_1,t_4}(S^1, \mathbb{R}^d)$, the subset of $\mathrm{Imm}(S^1, \mathbb{R}^d)$ consisting of all immersions γ with at most one singularity – a double point $\gamma(t_1) = \gamma(t_4)$. Since a configuration in $\partial C_5^{(1)}$ does not contain the point t_1, the map g is well-defined on $\partial C_5^{(1)} \times \mathrm{Imm}_{\leq [x_1,x_2],t_1,t_4}(S^1, \mathbb{R}^d)$. Letting the dependance on the lengths of the strands be apparent, we now work with $W_2 = W_2(\varepsilon_3, \varepsilon_4, \varepsilon_4)$ as a subspace of $\mathrm{Imm}_{\leq [x_1,x_2],t_1,t_2}(S^1, \mathbb{R}^d)$. In this larger space, $W_2(\varepsilon_3, \varepsilon_4, \varepsilon_5)$ is cobordant to $W_2(\varepsilon_3, 0, \varepsilon_5)$. The cobordism is given by $W_3 \subset \mathrm{Imm}_{[x_1,x_2],t_1,t_4}(S^1, \mathbb{R}^d)$ parametrized by $(\sqcup_i M_i) \times \mathbb{I} \times \mathbb{I}$ where the second unit interval parametrizes the length of the strand centered at t_4 moved by the resolution map.

By Stokes' Theorem and naturality,

$$0 = \int_{\partial C_5^{(1)} \times W_3} dg^*(\alpha \otimes \alpha \otimes \alpha) = \int_{\partial\left(\partial C_5^{(1)} \times W_3\right)} g^*(\alpha \otimes \alpha \otimes \alpha),$$

and thus,

$$\begin{aligned} 0 = \int_{\partial(\partial C_5^{(1)}) \times W_3} g^*(\alpha \otimes \alpha \otimes \alpha) + \int_{\partial C_5^{(1)} \times W_2(\varepsilon_3,\varepsilon_4,\varepsilon_5)} g^*(\alpha \otimes \alpha \otimes \alpha) - \int_{\partial C_5^{(1)} \times W_2(\varepsilon_3,0,\varepsilon_5)} g^*(\alpha \otimes \alpha \otimes \alpha) \\ = \int_{\partial C_5^{(1)} \times W_2(\varepsilon_3,\varepsilon_4,\varepsilon_5)} g^*(\alpha \otimes \alpha \otimes \alpha). \quad (2.3) \end{aligned}$$

The second equality holds because $\partial(\partial C_5^{(1)}) = \varnothing$ and the dimension of $W_2(\varepsilon_3, 0, \varepsilon_5)$ is $2d-3$.

By the same argument, $\int_{C_5^{(5)} \times \mathcal{M}_\beta} g^*(\alpha \otimes \alpha \otimes \alpha) = 0$. Calculating $\int_{C_5 \times \mathcal{M}_\beta} g^*(\alpha \otimes \alpha \otimes \alpha)$ thus reduces to calculating $\int_{C_5^c \times \mathcal{M}_i} g^*(\alpha \otimes \alpha \otimes \alpha)$ for $i = 1, \ldots, 6$.

We chose the antipodally symmetric volume form, α, to be concentrated near the points $\bar{x}_1 = (0, \ldots, 0, 1) \in S^{d-1}$ and $\bar{x}_2 = (0, \ldots, 0, -1) \in S^{d-1}$. Let $\tau_{\bar{x}_1}$ and $\tau_{\bar{x}_2}$ be the Thom classes of these points, as defined in Section 6 of [5], so $\alpha = \frac{1}{2}\left(\tau_{\bar{x}_1} + \tau_{\bar{x}_2}\right)$. Let y be the arc in S^{d-1} connecting $(0, \ldots, 0, 1)$ and $(0, \ldots, 0, -1)$, defined as

$$y = \left\{ \left(0, \ldots, 0, \sqrt{1-s^2}, s\right) \in S^{d-1} \;:\; -1 \leq s \leq 1 \right\}.$$

The Thom class τ_y of y can be chosen so that $d\tau_y = \tau_{\bar{x}_1} - \tau_{\bar{x}_2} = 2(\tau_{\bar{x}_1} - \alpha)$.

We have

$$\begin{aligned}
\int_{C_5^c \times \mathcal{M}_i} g^*(\alpha \otimes \alpha \otimes \alpha - \tau_{\bar{x}_1} \otimes \alpha \otimes \alpha) &= \int_{C_5^c \times \mathcal{M}_i} g^*\left(-\tfrac{1}{2} d\tau_y \otimes \alpha \otimes \alpha\right) \\
&= -\tfrac{1}{2} \int_{C_5^c \times \mathcal{M}_i} dg^*(\tau_y \otimes \alpha \otimes \alpha) \\
&= -\tfrac{1}{2} \int_{\partial(C_5^c \times \mathcal{M}_i)} g^*(\tau_y \otimes \alpha \otimes \alpha).
\end{aligned}$$

If $(v_1, v_2, v_3) \in g\left(\partial(C_5^c \times \mathcal{M}_i)\right)$ at least one of the first two coordinates of v_1 is non-zero, but every $\bar{x} \in y \subset S^{d-1}$ has $x_1, x_2 = 0$. Thus, the sets y and $g\left(\partial(C_5^c \times \mathcal{M}_i)\right)$ are disjoint and $\int_{C_5^c \times \mathcal{M}_i} g^*(\tau_y \otimes \alpha \otimes \alpha) = 0$, which means $\int_{C_5^c \times \mathcal{M}_i} g^*(\alpha \otimes \alpha \otimes \alpha) = \int_{C_5^c \times \mathcal{M}_i} g^*(\tau_{\bar{x}_1} \otimes \alpha \otimes \alpha)$. By a similar argument,

$$\int_{C_5^c \times \mathcal{M}_i} g^*(\alpha \otimes \alpha \otimes \alpha) = \int_{C_5^c \times \mathcal{M}_i} g^*(\tau_{\bar{x}_1} \otimes \tau_{\bar{x}_1} \otimes \tau_{\bar{x}_1}).$$

This integral can be calculated by counting the transverse intersections of $g(C_5^c \times \mathcal{M}_i)$ and $(\bar{x}_1, \bar{x}_1, \bar{x}_1)$ in $S^{d-1} \times S^{d-1} \times S^{d-1}$.

Recall that

$$g(\bar{s}, \gamma) = \left(\frac{\gamma(s_3) - \gamma(s_1)}{\|\gamma(s_3) - \gamma(s_1)\|}, \frac{\gamma(s_4) - \gamma(s_1)}{\|\gamma(s_4) - \gamma(s_1)\|}, \frac{\gamma(s_5) - \gamma(s_2)}{\|\gamma(s_5) - \gamma(s_2)\|} \right).$$

Thus, we are counting the number of pairs $(\bar{s}, \gamma) \in C_5^c \times \mathcal{M}_i$ for which

$$\frac{\gamma(s_3) - \gamma(s_1)}{\|\gamma(s_3) - \gamma(s_1)\|} = \frac{\gamma(s_4) - \gamma(s_1)}{\|\gamma(s_4) - \gamma(s_1)\|} = \frac{\gamma(s_5) - \gamma(s_2)}{\|\gamma(s_5) - \gamma(s_2)\|} = (0, \ldots, 0, 1). \qquad (2.4)$$

If $\gamma \in \mathcal{M}_\beta$ the image of γ is in $\mathbb{R}^3 \subset \mathbb{R}^d$ except for along the intervals which are moved to resolve the singularities (and at the crossing changes in the isotopies for $d = 4$). Thus, the equality in Equation 2.4 is only possible if $s_i = t_i$ for $i = 1, \ldots, 5$.

If $\gamma \in \mathcal{M}_1$, then

$$\frac{\gamma(t_3) - \gamma(t_1)}{\|\gamma(t_3) - \gamma(t_1)\|} = \frac{\gamma(t_4) - \gamma(t_1)}{\|\gamma(t_4) - \gamma(t_1)\|} = \frac{\gamma(t_5) - \gamma(t_2)}{\|\gamma(t_5) - \gamma(t_2)\|} = (0, \ldots, 0, 1)$$

exactly when $v_3 = v_4 = v_5 = (0, \ldots, 0, 1)$ and so $\int_{C_5^c \times \mathcal{M}_1} g^*(\alpha \otimes \alpha \otimes \alpha) = \pm 1$. If $\gamma \in \mathcal{M}_i$ for $i = 2, \ldots, 6$, then

$$\frac{\gamma(t_3) - \gamma(t_1)}{\|\gamma(t_3) - \gamma(t_1)\|} \neq (0, \ldots, 0, 1),$$

and $\int_{C_5^c \times \mathcal{M}_i} g^*(\alpha \otimes \alpha \otimes \alpha) = 0$. Thus, $\omega_2([\mathcal{M}_\beta]) = \pm 1$.

Next, we show that $\omega_1([\mathcal{M}_\beta]) = 0$. Let

$$f : ev^* C_{4,1}(\mathbb{R}^d) \to S^{d-1} \times S^{d-1} \times S^{d-1} \times S^{d-1}$$

be the map shown in the diagram below, where $\bar{\varphi} = \phi_{15} \times \phi_{45} \times \phi_{35} \times \phi_{25}$. Then ω_1 is the pushforward of $f^*(\alpha \otimes \alpha \otimes \alpha \otimes \alpha)$ along $p : ev^*C_{4,1}(\mathbb{R}^d) \to \mathrm{Emb}(S^1, \mathbb{R}^d)$.

$$\begin{tikzcd}
 & ev^*C_{4,1}(\mathbb{R}^d) \arrow[r] \arrow[d, "p_1"] \arrow[dd, bend right=60, "p"'] \arrow[rr, bend left, "f"] & C_5[\mathbb{R}^d] \arrow[r, "\bar{\varphi}"] \arrow[d] & S^{d-1} \times S^{d-1} \times S^{d-1} \times S^{d-1} \\
 & C_4^{ord}[S^1] \times \mathrm{Emb}(S^1, \mathbb{R}^d) \arrow[r] \arrow[d, "p_2"] & C_4[\mathbb{R}^d] & \\
M_\beta \arrow[r, hook] & \mathrm{Emb}(S^1, \mathbb{R}^d) & &
\end{tikzcd}$$

Since $p^{-1}(\mathcal{M}_\beta) = p_1^{-1}(C_4^{ord}[S^1] \times \mathcal{M}_\beta)$, we have

$$\omega_1([\mathcal{M}_\beta]) = \int_{p_1^{-1}(C_4^{ord}[S^1] \times \mathcal{M}_\beta)} f^*(\alpha \otimes \alpha \otimes \alpha \otimes \alpha).$$

Following the calculation of $\omega_2([\mathcal{M}_\beta])$, define

$$C_4^{(i)} = \left\{ \bar{s} \in C_4^{ord}[S^1] : s_j \notin N_i \text{ for } j = 1, \ldots, 4 \text{ and } \bar{s} \notin C_4^{(m)} \text{ for } m < i \right\}.$$

Each configuration in $C_4^{ord}[S^1]$ has four points, so $C_4^{ord}[S^1] = C_4^{(1)} \sqcup \cdots \sqcup C_4^{(5)}$. The arguments used to prove that $\int_{C_5^{(m)} \times \mathcal{M}_\beta} g^*(\alpha \otimes \alpha \otimes \alpha) = 0$ also show $\int_{p_1^{-1}(C_4^{(m)} \times \mathcal{M}_\beta)} f^*(\alpha \otimes \alpha \otimes \alpha \otimes \alpha) = 0$ for $m = 1, \ldots, 5$.

$\square$

CHAPTER III

FIRST DIFFERENTIAL IN SINHA'S SPECTRAL SEQUENCE

3.1. Graphs and Trees

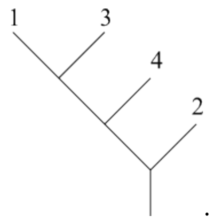
In [22], Sinha gives a characterization of the Poisson operad $\mathcal{P}ois^d$ and its dual co-operad $\mathcal{S}iop^d$ in terms of trees and graphs and gives a perfect pairing between the two. We summarize these results below, introducing the language of graphs and trees which will be used throughout the rest of this chapter.

Definition 3.1. A *tree* is an isotopy class of a planar graph consisting of only trivalent and univalent vertices, with a distinguished univalent vertex called the *root*. The remaining univalent vertices are called *leaves* and are labeled by elements of $L(T) \subseteq \mathbf{n} = \{1, 2, \ldots, n\}$. A *forest* is an ordered collection of trees embedded in the upper half plane with their roots on the x-axis. If the forest has n leaves, the leaves are labeled by the set $\mathbf{n}$. We call the trivalent vertices *internal vertices*.

The *root path* of an interior vertex or a leaf is the path in the tree from the vertex to the root. The *height* of an interior vertex or a leaf is the number of edges in the root path of the vertex. If v is an interior vertex, then another vertex u (either interior or a leaf) is *above* v if the root path of u passes through the vertex v.

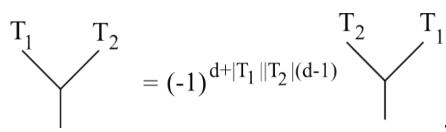
Definition 3.2. A *right* (respectively, *left*) *tall tree* is a tree for which the left (respectively, right) branch of every internal vertex is a leaf, and the leaf with the maximal height has the minimal label.

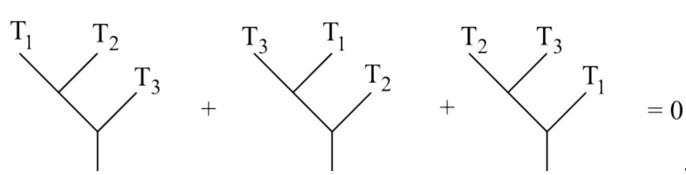
Example 3.1. The tree shown below is a tall tree



3.1.1. Sinha's Spectral Sequence

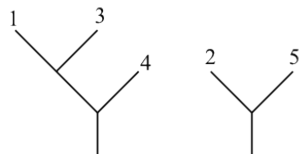
Recall that the group $E^1_{-p,q(d-1)}$ in the reduced homology spectral sequence for $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d)$ has a subgroup isomorphic to the subgroup of $\mathcal{P}ois^d(p)$ generated by expressions with q brackets such that each x_i appears inside a bracket and the multiplication "$\cdot$" does not appear inside of a bracket. This subgroup can also be described as the vector space generated by forests with q internal vertices and leaves labeled by the set $\mathbf{p}$, modulo the antisymmetry, commutativity and Jacobi identities. The anti-symmetry and Jacobi Identities are shown below, where $|T_i|$ is the number of internal vertices in the tree T_i:

$Anti - symmetry :$ 

$Jacobi :$ 

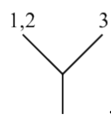
For commutativity, if two forests F_1 and F_2 contain the same trees, then $F_1 = \sigma^{d-1} F_2$ where σ is the sign of the permutation taking the ordered set of the internal vertices of F_1 to the ordered set of the internal vertices of F_2.

Example 3.2. The following forest represents the element $[[x_1, x_3], x_4] \cdot [x_2, x_5]$ of $E^1_{-5,3(d-1)}$:



Definition 3.3. We will call trees with leaves labeled by ordered subsets of $\mathbf{p}$ (instead of just elements of $\mathbf{p}$) *multi-labeled trees.*

Using multi-labeled trees, we can encode the multiplication "$\cdot$" inside of the tree and represent the standard spanning set for $\mathcal{P}ois^d$ using multi-labeled trees. For example, the element $[x_1 \cdot x_2, x_3] \in \mathcal{P}ois^d(3)$ is represented by the following multi-labeled tree:



.

Multi-labeled trees can be written as sums of trees through the Leibniz identity, just as bracket expressions with "$\cdot$" inside of a bracket pair can be rewritten using the Leibniz rule so the multiplication only appears outside of all brackets. Recall that for bracket expressions, the Leibniz rules states

$$[X \cdot Y, Z] = X \cdot [Y, Z] + [X, Z] \cdot Y,$$

where X, Y, and Z are sub-expressions. For multi-labeled trees in which the leaves labeled by sets have height greater than two, the intermediate steps cannot be written using multi-labeled trees. Bracket expressions are much better suited to calculation with the Leibniz rule.

For any two leaves x and y in a (multi-labeled) tree T, there is a unique embedded path in T from x to y, which we will denote $p_T(x, y)$. Similarly, for any two internal vertices v and w in T, there is a unique embedded path in T from v to w, which we will denote $p_T(v, w)$.

Definition 3.4. We define the differential d_1 using multi-labeled trees instead of brackets:

$$d_1 T = F_0 + \sum_{a=1}^{p} (-1)^a T_a + (-1)^{p+1} F_{p+1},$$

where F_0 (respectively, F_{p+1}) is the forest obtained by adding the tree with exactly one leaf, labeled 1 (respectively, p) before (respectively, after) the tree T, and in F_0 the label on leaf i is replaced by $i + 1$ for each leaf in T. The tree T_a is the multi-labeled tree obtained from T by replacing the label on leaf a by $\{a, a + 1\}$ and the label on leaf i by $i + 1$ for all $i > a$.

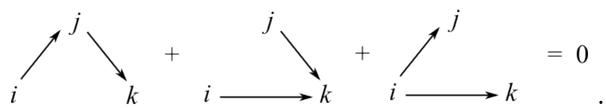
3.1.2. The Perfect Pairing Between Graphs and Trees.

In [22], Sinha shows that the following defines a perfect pairing between $\mathcal{P}ois^d(p)$ and $\mathcal{S}iop^d(p)$.

Following Sinha in [19], let $\Gamma(p)$ be the free module generated by graphs whose vertices are labeled by $\mathbf{p}$ and whose edges are oriented and ordered. If two graphs, G and G', differ by the orientations on k edges and a reordering of the edges given by a permutation σ, then the arrow reversing identity is given by

$$G - (-1)^{k(d-1)} (\text{sign } \sigma)^d G' = 0.$$

The Arnold identity for three graphs which differ only in the edges between the three vertices i, j and k, is shown below:



Definition 3.5. A *long graph* is graph for which each component is a linear graph such that one endpoint vertex has the minimal label. Also, the edges are ordered consecutively and oriented away from the vertex with the minimal label.

The following is an example of a long graph:

$$1 \to 3 \to 4 \qquad 2 \to 5.$$

Sinha shows in Lemma 4.4 of [19] that $\mathcal{P}ois^d(p)$ is spanned by forests with leaf set $\mathbf{p}$ whose trees are all tall, and $\mathcal{S}iop^d(p)$ is spanned by long graphs with p vertices.

Definition 3.6. For any graph $G \in \mathcal{S}iop^d(p)$ and tree $T \in \mathcal{P}ois^d(p)$, define a function

$$\beta(G,T) : \{\text{edges of } G\} \to \{\text{internal vertices of } T\}$$

which takes the edge $i \to j$ in G to the nadir of the path in T from the leaf labeled i to the leaf labeled j.

A graph $G \in \mathcal{S}iop^d(p)$ defines a partition of $\mathbf{p}$ through its components, and a forest $F \in \mathcal{P}ois^d(p)$ defines a partition of $\mathbf{p}$ through the labels of its trees. If these partitions are the same, then the map defined above can be extended to a map

$$\beta(G,F) : \{\text{edges of } G\} \to \{\text{internal vertices of } F\}.$$

Definition 3.7. If d is even, let $\tau^d(G,F) = (-1)^N$, where N is the number of paths in F corresponding to edges $i \to j$ in G which travel from left to right. If d is odd, let $\tau^d(G,F)$ be the sign of the permutation taking the ordered set of the edges of G to the internal vertices of F given by the map $\beta(G,F)$. The internal vertices of F are ordered left to right according to the planar embedding of F. For any graph $G \in \mathcal{S}iop^d(p)$ and forest $F \in \mathcal{P}ois^d(p)$, define the pairing $\langle G,F\rangle$ as

$$\langle G,F\rangle = \begin{cases} \tau^d(G,F) & \text{if } \beta(G,F) \text{ exists and is a bijection} \\ 0 & \text{otherwise} \end{cases}.$$

We extend linearly as is standard, and the remarkable fact is that this well-defined – that is the pairing vanishes on the Jacobi, Arnold, and anti-symmetry identities.

This definition of the pairing can be extended to forests made up of multi-labeled trees, by extending the definition of $\beta(G,T)$. If T is a multi-labeled tree and for every edge $i \to j$ in the graph G, the labels i and j appear on different leaves of T, then the map β defined in Definition 3.6 can be extended to a map

$$\beta(G,T) : \{\text{edges of } G\} \to \{\text{internal vertices of } T\}.$$

The following is essentially Proposition 2.6 in [22].

Proposition 3.1. *If F is a forest made up of multi-labled trees with leaf set $L(F) = \mathbf{p}$ and $G \in \mathcal{S}iop^d(p)$, the extended pairing $\langle G,F\rangle$ is equal to the pairing $\langle G, \Sigma\tilde{F}_i\rangle$, where $\Sigma\tilde{F}_i$ is the sum of the forests obtained by expanding the multi-labeled trees in F using the Leibniz rule.*

We have already seen trees and bracket expressions as closely related combinatorial objects. We now introduce a third formulation.

Definition 3.8. An *exclusion relation* R on a set S is a subset of $S^{\times 3}$ such that all coordinates are distinct and

1. If $((x, y), z) \in R$ then $((y, x), z) \in R$ and $((x, z), y) \notin R$.

2. If $((x, y), z) \in R$ and $((w, x), y) \in R$ then $((w, x), z) \in R$.

If $((x, y), z) \in R$ we say that x and y exclude z. An exclusion relation is called *full* if for any three elements $x, y, z \in S$ one of $((x, y), z), ((z, x), y)$ or $((y, z), x)$ is an element of R.

In [20], Sinha defines a bijective correspondence between exclusion relations and possibly non-binary trees. The exclusion relation corresponding to a tree T is the subset $R_T \subset L(T)^{\times 3}$ defined by $((x, y), z) \in R_T$ if there is an internal vertex v in T such that the root paths of the leaves x and y pass through v but the root path of z does not. Under this correspondence, full exclusion relations correspond to binary trees.

An internal vertex v of a tree T corresponds to a subset U_v of the leaf $L(T)$ set that is closed under the exclusion relation R_T in the sense that if $x, y \in U_v$ then x and y exclude z for every $z \notin U_v$. In particular, U_v is the set of all leaves which are above v. Furthermore, if a vertex v is the nadir of the path $p_T(x, y)$, then U_v is the smallest subset containing x and y that is closed under exclusion.

Definition 3.9. Let T be a tree with leaf set $L(T)$, and let R_T be the corresponding exclusion relation.

1. A *restriction* S of T is defined by restricting the exclusion relation R_T to a subset $L(S)^{\times 3} \subset L(T)^{\times 3}$.

2. A *branch* S of T is a restriction where $L(S)$ is defined as the set of all $l \in L(T)$ above a chosen internal vertex of T.

3. A *tall branch* S of T is a branch which is a tall tree (except the leaf with the maximal height is not required to have the minimal label) and is maximal in the sense that if any leaves are added to S, then S is no longer tall.

A multi-labeled tree T corresponds to an exclusion relation R_T on a set $L(T) \subset \mathcal{P}(\mathbf{n})$ where $\mathcal{P}(\mathbf{n})$ is the power set of $\mathbf{n}$, such that if $X, Y \in L(T)$ then $X \cap Y = \varnothing$.

Definition 3.10. By a restriction S of a multi-labeled tree T, we will mean an exclusion relation on a subset $L(S) \subset \mathcal{P}(n)$ such that for each $X \in L(S)$, there is a set $\tilde{X} \in L(T)$ with $X \subseteq \tilde{X}$, and each set in $L(T)$ has at most one subset appearing in $L(S)$. The exclusion relation on $L(S)$ is defined as follows. If $X, Y, Z \in L(S)$ are subsets of $\tilde{X}, \tilde{Y}, \tilde{Z} \in L(T)$, respectively and $((\tilde{X}, \tilde{Y}), \tilde{Z}) \in R_T$, then $((X, Y), Z) \in R_S$.

Example 3.3. The tree, T, shown in Figure 3.1 has tall branches S_1, S_2 and S_3, shown in Figure 3.2.

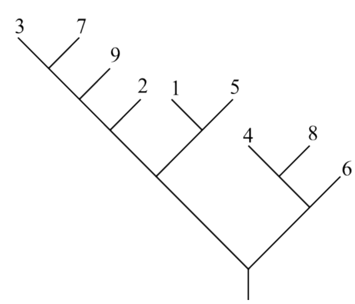


Figure 3.1. The tree T.

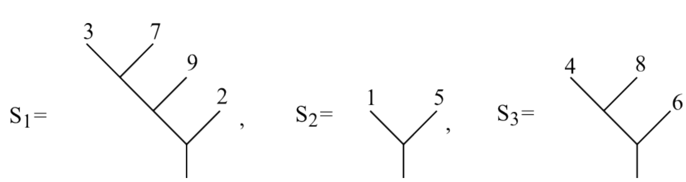
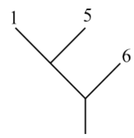


Figure 3.2. The tall branches of T.

The restriction

S = 

is not a tall branch of T (even though it is a tall tree) because it is not a branch of T.

Definition 3.11. An *almost tall branch* of a tree T is a restriction S of a tall branch $\tilde{S}$ of T such that S does not contain either of the two leaves of $\tilde{S}$ with maximal height.

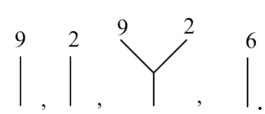
Definition 3.12. Let T be a tree and let S be a restriction of T. We say S is *admissible* if the leaf set $L(S)$ can be written as

$$L(S) = \bigcup_{i=1}^{n} L(S_i),$$

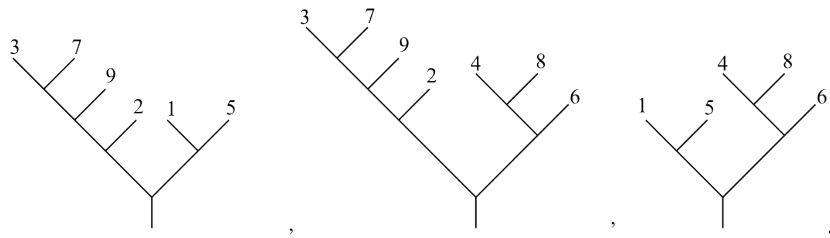
where at most one of the S_i is an almost tall branch, and the rest are tall branches.

If S is an admissible restriction of T with all of the S_i tall, we say that S is purely admissible. Otherwise, we say that S is impurely admissible. We will call the sets S_i the components of S.

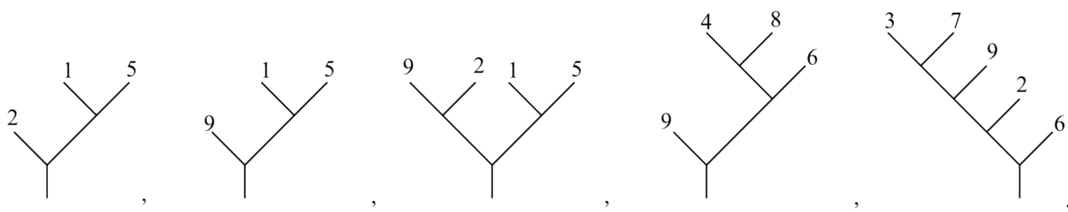
Example 3.4. The almost tall branches of the tree T from Figure 3.1 are shown below, where the first three are restrictions of S_1 and the last is a restriction of S_3:



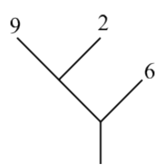
The restrictions of T shown below are admissible, as the leaf set of each is the union of the leaf sets of tall branches of T:



The examples below of admissible restrictions have leaf sets that are the union of the leaf sets of one almost tall branch and those of some tall branches:



The restriction



is not admissible because its leaf set is the union of the leaf sets of two almost tall branches.

We will make frequent us of the reindexing function,

$$r_{l,\delta}(i) = \begin{cases} i & \text{if } i < l+\delta \\ i+1 & \text{if } i \geq l+\delta \end{cases}$$

where $\delta = 0$ or $\delta = 1$. Recall that T_l is the multi-labeled tree obtained from T by replacing the label on leaf l by $\{l, l+1\}$ and the label on leaf i by $r_{l,1}(i)$.

Definition 3.13. Let S be an admissible restriction of T, and let $l \in L(T) \setminus L(S)$. The compliment of S with respect to l is the restriction $S^{c,l}$ with leaf set $L(S^{c,l}) = L(T) \setminus (L(S) \cup \{l\})$. We say the leaf l *distinguishes* S from $S^{c,l}$ if for any two leaves $l_1 \in L(S^{c,l})$ and $l_2 \in L(S)$, either $((l, r_{l,1}(l_1)), r_{l,1}(l_2)) \in R_{T_l}$ or $((l, r_{l,1}(l_2)), r_{l,1}(l_1)) \in R_{T_l}$.

Equivalently, l distinguishes S from $S^{c,l}$ if for any such pair of leaves l_1 and l_2, the path $p_{T_l}(l, r_{l,1}(l_1))$ has a different nadir than the path $p_{T_l}(l, r_{l,1}(l_2))$. By abuse, we simply say that l distinguishes S.

Example 3.5. Let T be the tree shown in Figure 3.3.

T =

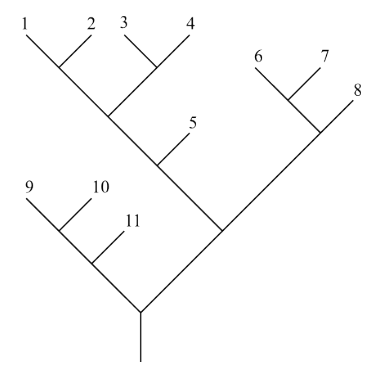
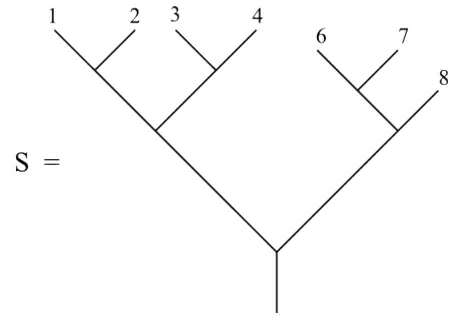


Figure 3.3. The tree T.

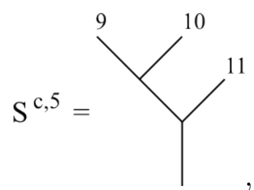
For the admissible restriction S with $L(S) = \{1, 2, 6, 7, 8\}$, the leaves labeled by 3 and 4 distinguish S while the leaves labeled $5, 9, 10$ and 11 do not distinguish S.

For the admissible restriction S with $L(S) = \{3, 4, 9, 10, 11\}$, the leaves labeled by 1 and 2 distinguish S while the leaves labeled $5, 6, 7$ and 8 do not.

Example 3.6. Let T be the tree shown in Figure 3.3. If S is the admissible restriction



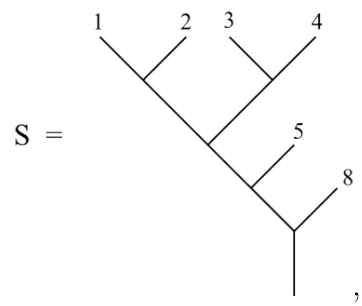
then only the leaf labeled by 5 distinguishes S. In this case, the compliment of S with respect to 5 is



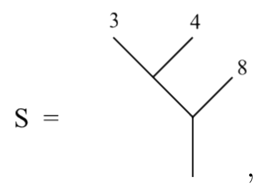
which is itself admissible. The leaves $1, 2, \ldots, 8$ all distinguish $S^{c,5}$. In particular, the leaf labeled 5 distinguishes S from $S^{c,5}$ and distinguishes $S^{c,5}$ from S. This symmetry can only occur if the leaf is itself a tall branch of the original tree T.

Remark 3.1. Suppose S is an impurely admissible restriction of T with almost tall component S_j, which is a restriction of the tall branch $\tilde{S}_j$ of T. If $l \in L(T) \setminus L(S)$ distinguishes S, then $l \in \tilde{S}_j$ and the height of l is greater than the height of every leaf in S_j.

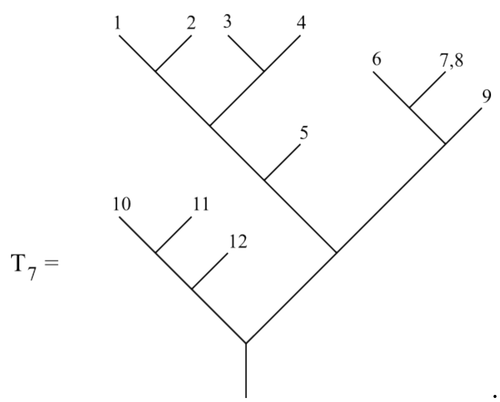
Example 3.7. If T is the tree shown in Figure 3.3, and S is the admissible restriction



then only the leaves 6 and 7 distinguish S. However, if S is the admissible restriction



then none of the leaves in $L(T) \setminus L(S)$ distinguish S. For example, if $l = 7$ then the nadir of $p_{T_7}(7, 4)$ will be the same vertex as the nadir of $p_{T_7}(7, 1)$.



3.2. A New Formula for the First Differential

To define a new formula for the differential, we will use admissible restrictions to define new forests.

Definition 3.14. Let T be a tree with leaves labeled by $\mathbf{p}$, let S be a restriction of T and let $l \in L(T) \setminus L(S)$. Define $F_{(S,l,\delta)}$ (here δ is either 0 or 1) to be a forest with leaves labeled by $\mathbf{p}+\mathbf{1}$ and two trees, T_S and $T_{L\setminus S}$, which are restrictions of T_l. The leaf sets of T_S and $T_{L\setminus S}$ are defined below.

1. $L(T_S) = r_{l,\delta}(S) \cup \{l+\delta\}$.
2. $L(T_{L\setminus S}) = r_{l,\delta}(L(T) \setminus L(S))$.

Remark 3.2. If $\delta = 0$, the leaf l is in T_S and if $\delta = 1$ the leaf l is in $T_{L\setminus S}$.

Example 3.8. Let T be the tree given in Figure 3.1. In Figures 3.4 and 3.5, we have examples of $F_{(S,l,\delta)}$ for different admissible restrictions, leaves, and values of δ.

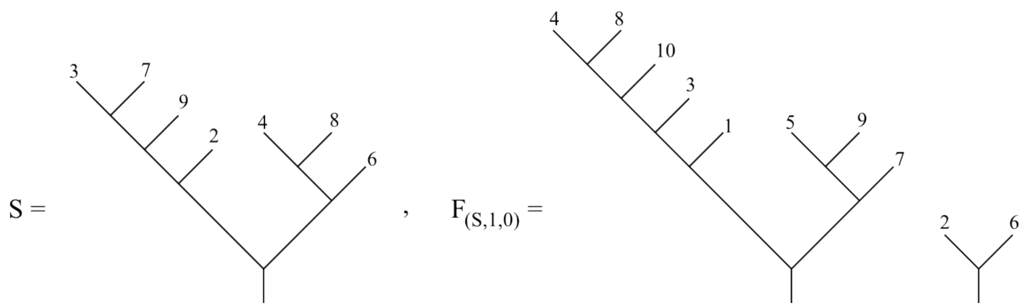


Figure 3.4. An example of $F_{(S,l,\delta)}$ with $l = 1$ and $\delta = 0$.

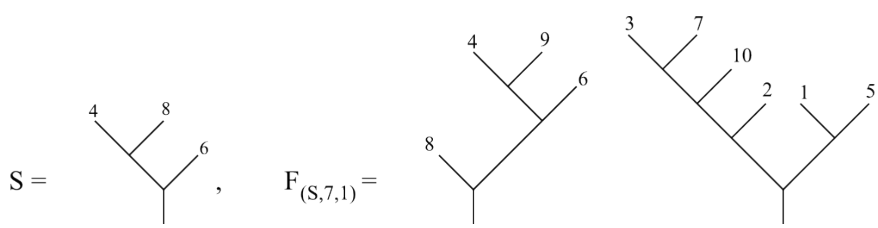


Figure 3.5. An example of $F_{(S,l,\delta)}$ with $l = 7$ and $\delta = 1$.

Remark 3.3. For the tree T and admissible restriction S in Example 3.6, the forests $F_{(S,5,0)}$ and $F_{(S^{c,5},5,1)}$ are equal up to a sign, as shown in Figure 3.6. Similarly, the forests $F_{(S,5,1)}$ and $F_{(S^{c,5},5,0)}$ are equal up to a sign.

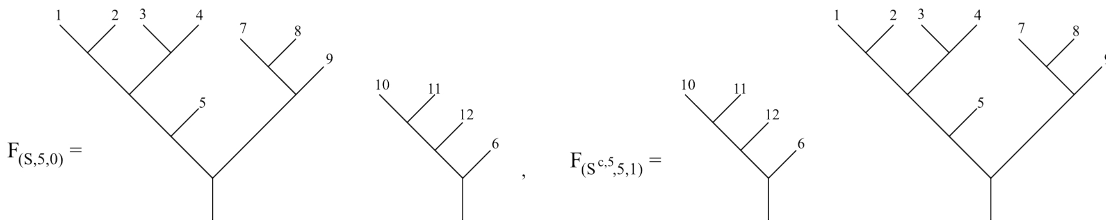


Figure 3.6. The forests $F_{(S,5,0)}$ and $F_{(S^{c,5},5,1)}$.

Definition 3.15. Let

$$B_T = \{(S, S^{c,l}, l); S \text{ or } S^{c,l} \text{ is an admissible restriction of } T, \text{ and } l \text{ distinguishes } S \text{ or } S^{c,l}\}.$$

Define a map $\tilde{d}_1 : E_{-p,q(d-1)} \to E_{-(p+1),q(d-1)}$ by

$$\tilde{d}_1 T = \sum_{(S,S^{c,l},l)\in B_T} (-1)^l \sigma^d_{S,l} \left(F_{(S,l,0)} + F_{(S,l,1)}\right),$$

where S is admissible. If d is even, $\sigma^d_{S,l} = 1$ and if d is odd, $\sigma^d_{S,l}$ is the sign of the permutation defined by $f_{S,l}$ from Definition 3.16.

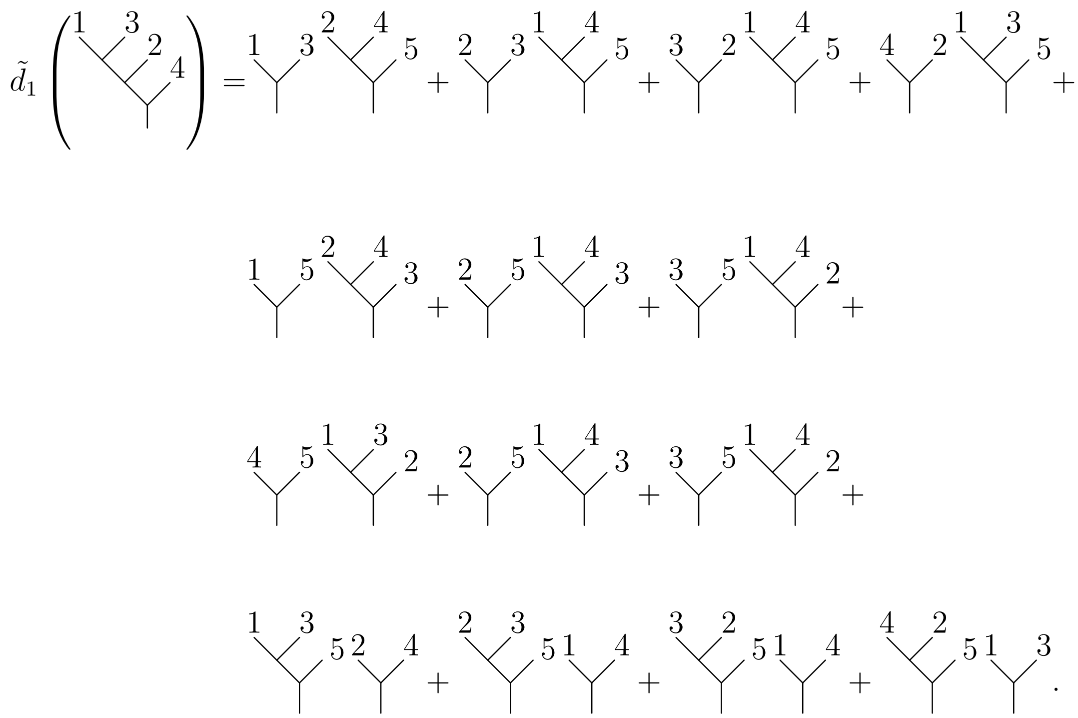
If S and $S^{c,l}$ are both admissible, then the term $\sigma^d_{S,l} \left(F_{(S,l,0)} + F_{(S,l,1)}\right)$ appears in the sum only once, and the term does not depend on which choice of S or $S^{c,l}$ is made for the admissible restriction.

Remark 3.4. If the tree T is tall with leaf set $L(T) = \{l_1, l_2, \ldots, l_n\}$, where l_1 and l_2 have the greatest height, then this formula reduces to

$$\tilde{d}_1 T = \sum_{S\subseteq L(T)\setminus\{l_1,l_2\}} \left(\sum_{l \text{ distinguishes } S} (-1)^l \sigma^d_{S,l} \left(F_{(S,l,0)} + F_{(S,l,1)}\right) \right).$$

In this case, l distinguishes S exactly when $l \in L(T) \setminus L(S)$ such that the height of l is greater than the height of every leaf in $L(S)$.

Example 3.9. To calculate $d_1([[[x_1, x_3], x_2], x_4])$ using the definition of d_1 from the spectral sequence, we have 24 terms after expanding using Leibniz' Rule, many of which cancel. Using the definition of $\tilde{d}_1$ above, we immediately have the mod 2 equality



Theorem 3.1. *The maps d_1 and $\tilde{d}_1$ are equal. In other words, $d_1 T = \tilde{d}_1 T$ for any $T \in E^1_{-p,q(d-1)}$.*

We immediately have the following, which is not obvious from the definition of d_1.

Corollary 3.2. *For a forest, F, the differential $d_1 F$ is equal to a linear combination of forests with coefficients ± 1.*

We spend the rest of this section proving Theorem 3.1. Recall from Definition 3.4 that T_l is the multi-labeled tree obtained from T by replacing the label on leaf l by $\{l, l+1\}$, and replacing $i \neq l$ by $r_{l,1}(i)$.

Definition 3.16. If S is an admissible restriction of T and l distinguishes S, we will define a map

$$f_{S,l} : \{\text{internal vertices of } F_{(S,l,\delta)}\} \to \{\text{internal vertices of } T_l\}$$

as follows. For an internal vertex v in $F_{(S,l,\delta)}$, choose leaves x and y in $F_{(S,l,\delta)}$ such that v is the nadir of the path $p_{F_{(S,l,\delta)}}(x, y)$. Define $f_{S,l}(v)$ to be the nadir of the path $p_{T_l}(x, y)$.

Lemma 3.1. *Let T be a (multi-labeled) tree with $x, y, r, s \in L(T)$. The paths $p_T(x, y)$ and $p_T(r, s)$ have the same nadir if and only if none of $((x, y), r), ((x, y), s), ((r, s), x)$ and $((r, s), y)$ are in R_T.*

Proof. If the paths $p_T(x, y)$ and $p_T(r, s)$ have the same nadir, then clearly none of $((x, y), r), ((x, y), s), ((r, s), x)$ or $((r, s), y)$ are in R_T. For example, if $((x, y), r) \in R_T$ then the root path of r does not pass through the nadir of $p_T(x, y)$, and so the nadir of $p_T(x, y)$ cannot be equal to the nadir of $p_T(r, s)$.

Suppose the paths $p_T(x, y)$ and $p_T(r, s)$ have different nadirs. If the nadir of $p_T(x, y)$ is above the nadir of $p_T(r, s)$, then either $((x, y), r)$ or $((x, y), s)$ is in R_T. Similarly, if the nadir of $p_T(r, s)$ is above the nadir of $p_T(x, y)$, then either $((r, s), x)$ or $((r, s), y)$ is in R_T. If neither nadir is above the other, then all four of $((x, y), r), ((x, y), s), ((r, s), x)$ and $((r, s), y)$ are in R_T. Thus, if the paths $p_T(x, y)$ and $p_T(r, s)$ have different nadirs, then at least one of $((x, y), r), ((x, y), s), ((r, s), x)$ and $((r, s), y)$ is in R_T. Equivalently, if none of $((x, y), r), ((x, y), s), ((r, s), x)$ and $((r, s), y)$ are in R_T, then the paths $p_T(x, y)$ and $p_T(r, s)$ have the same nadir. $\square$

Lemma 3.2. *If S is an admissible restriction of T and the leaf l distinguishes S, then for $\delta = 0$ or $\delta = 1$ the map $f_{S,l}$ is well-defined.*

Proof. Suppose the vertex v in $F_{(S,l,\delta)}$ is the nadir of both the path $p_{F_{(S,l,\delta)}}(x,y)$ and the path $p_{F_{(S,l,\delta)}}(r,s)$. Then the leaves x,y,r,s are all in the same tree in $F_{(S,l,\delta)}$ – call this tree T_v. By Lemma 3.1, none of $((x,y),r),((x,y),s),((r,s),x)$ or $((r,s),y)$ are in R_{T_v}. Since T_v is a restriction of T_l, this implies that none of $((x,y),r),((x,y),s),((r,s),x)$ or $((r,s),y)$ are in R_{T_l}. By Lemma 3.1, the paths $p_{T_l}(x,y)$ and $p_{T_l}(r,s)$ have the same nadir. Thus, $f_{S,l}$ is well-defined. $\square$

Definition 3.17. Let T be a tree and S a restriction of T. A set $A \subset L(T)$ of leaves is *not separated* by S if either $A \subset L(S)$ or $A \subset L(T_{L\setminus S})$.

To construct an inverse to $f_{S,l}$, we will need the following.

Lemma 3.3. *Let S be an admissible restriction of T and assume the leaf l distinguishes S. Let the leaves $x,y,r,s \in L(T_l)$ be such that the leaf sets $\{x,y\}$ and $\{r,s\}$ are not separated by S. If the paths $p_{T_l}(x,y)$ and $p_{T_l}(r,s)$ have the same nadir, then $\{x,y,r,s\}$ is not separated by S.*

Proof. Since the exclusion relation is full, one of $((r,s),l),((r,l),s)$ or $((s,l),r)$ is in R_{T_l}.

If $((r,s),l) \in R_{T_l}$, then the root path of l does not pass through the nadir of $p_{T_l}(r,s)$. This implies that x and y also exclude l. Suppose, by way of a contradiction, that $x,y \in L(T_S)$ and $r,s \in L(T_{L\setminus S})$. Since l distinguishes S, either $((r,l),x) \in R_{T_l}$ or $((x,l),r) \in R_{T_l}$. If $((r,l),x) \in R_{T_l}$ then $((r,s),l) \in R_{T_l}$ implies that $((r,s),x) \in R_{T_l}$. By Lemma 3.1 this contradicts that the paths $p_{T_l}(x,y)$ and $p_{T_l}(r,s)$ have the same nadir. If $((x,l),r) \in R_{T_l}$ then $((x,y),l) \in R_{T_l}$ implies that $((x,y),r) \in R_{T_l}$, contradicting that the paths $p_{T_l}(x,y)$ and $p_{T_l}(r,s)$ have the same nadir. Thus, $\{x,y,r,s\}$ is not separated by S.

Suppose $((r,l),s) \in R_{T_l}$. Since the paths $p_{T_l}(x,y)$ and $p_{T_l}(r,s)$ have the same nadir (call the nadir v), we may assume that $((s,x),r) \in R_{T_l}$. This means that the root paths of s and x pass through the vertex v from one side, while the root paths of r, y and l pass through v from the other side, as shown in Figure 3.7.

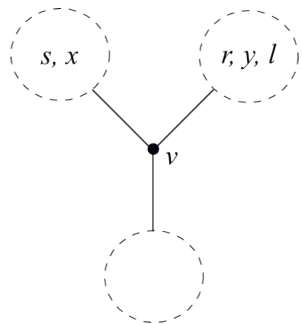


Figure 3.7. The arrangement of the leaves with respect to the vertex v.

From Figure 3.7, we see that $((s, x), l) \in R_{T_l}$. Thus, $((l, x), s) \notin R_{T_l}$ and $((l, s), x) \notin R_{T_l}$. Since l distinguishes S, the set $\{s, x\}$ is not separated by S. Thus, $\{x, y, r, s\}$ is also not separated by S.

Similarly, if $((s, l), r) \in R_{T_l}$ then $\{x, y, r, s\}$ is not separated by S.

□

Proposition 3.2. *If S is an admissible restriction of T and the leaf l distinguishes S, then for $\delta = 0$ or $\delta = 1$ the map $f_{S,l}$ is a bijection.*

Proof. Let w_1 and w_2 be two vertices in $F_{(S,l,\delta)}$ such that $f_{S,l}(w_1) = f_{S,l}(w_2) = v$. Choose leaves x and y in $F_{(S,l,\delta)}$ so that w_1 is the nadir of the path $p_{F_{(S,l,\delta)}}(x, y)$. Similarly, choose leaves r and s in $F_{(S,l,\delta)}$ so that w_2 is the nadir of the path $p_{F_{(S,l,\delta)}}(r, s)$. Then the vertex v is the nadir of both the path $p_{T_l}(x, y)$ and the path $p_{T_l}(r, s)$. By Lemma 3.3, the leaves x, y, r, s are all in the same tree in $F_{(S,l,\delta)}$. Call this tree T_v.

By Lemma 3.1 none of $((x, y), r), ((x, y), s), ((r, s), x)$ or $((r, s), y)$ are in R_{T_l}. Since T_v is a restriction of T_l, this means none of $((x, y), r), ((x, y), s), ((r, s), x)$ or

$((r,s),y)$ are in R_{T_v}. By Lemma 3.1, the paths $p_{T_v}(x,y)$ and $p_{T_v}(r,s)$ have the same nadir. The vertices w_1 and w_2 in $F_{(S,l,\delta)}$ must then be the same, and $f_{S,l}$ is injective. Since both $F_{(S,l,\delta)}$ and T_l have $p-1$ vertices, f is a bijection. $\square$

Let T be any tree in $E^1_{-p,q(d-1)}$ and let $G \in \mathcal{S}iop^d(p+1)$. Recall from the definition of d_1 that the pairing $\langle G, d_1T\rangle$ is given by

$$\langle G, d_1T\rangle = \langle G, F_0\rangle + \sum_{a=1}^{p}\langle G, (-1)^a T_a\rangle + \langle G, (-1)^{p+1}F_{p+1}\rangle.$$

Recall as well that the pairing $\langle G, \tilde{d}_1T\rangle$ is given by

$$\langle G, \tilde{d}_1T\rangle = \sum_{(S,S^{c,l},l)\in B_T}(-1)^l\sigma^d_{S,l}\langle G, F_{(S,l,0)} + F_{(S,l,1)}\rangle.$$

We will show $d_1T = \tilde{d}_1T$ by showing that $\langle G, d_1T\rangle = \langle G, \tilde{d}_1T\rangle$ for every long graph in $\mathcal{S}iop^d(p+1)$. The following is immediate.

Lemma 3.4. *Let $G \in \mathcal{S}iop^d(p+1)$ be a long graph and $F \in \mathcal{P}ois^d(p+1)$. The pairing $\langle G, F\rangle$ is non-zero if and only if the partition of $\mathbf{p+1}$ defined by the components of G is equal to the partition defined by the trees in F and $\langle G', T\rangle \neq 0$ for each pair of a component G' of G and a tree T of F such that*

$$\{\text{vertices of } G'\} = \{\text{leaves of } T\}.$$

Fix a long graph $G \in \mathcal{S}iop^d(p+1)$. Both T_a and $F_{(S,l,\delta)}$ have $p-1$ vertices, so we may assume that G has $p-1$ edges and therefore is a long graph with exactly two components.

Lemma 3.5. *If one of the components of G has one vertex and no edges, then* $\langle G, d_1 T\rangle = 0$ *and* $\langle G, \tilde{d}_1 T\rangle = 0$.

Proof. Suppose one of the two components of G has zero edges and one vertex. If $\langle G, F_0\rangle \neq 0$ then the vertex in the component of G with one vertex is 1, and $\langle G, T_1\rangle = \langle G, F_0\rangle$. Similarly, if $\langle G, F_{p+1}\rangle \neq 0$ then the vertex in the component of G with one vertex is $p+1$ and $\langle G, T_p\rangle = \langle G, F_p\rangle$. If $\langle G, T_a\rangle \neq 0$ then the vertex of the component of G with one vertex is either a or $a+1$. If this vertex is a, then $\langle G, T_{a-1}\rangle = \langle G, T_a\rangle$, otherwise $\langle G, T_{a+1}\rangle = \langle G, T_a\rangle$. Thus, in the sum $\langle G, d_1 T\rangle$ any non-zero terms occur in pairs with opposite signs, so $\langle G, d_1 T\rangle = 0$. Furthermore, $\langle G, \tilde{d}_1 T\rangle = 0$ because every forest in $\tilde{d}_1 T$ consists of two trees, each with two or more leaves. □

Thus, we may assume G is a long graph with two components and each component has at least one edge. In particular, $\langle G, F_0\rangle = 0$ and $\langle G, F_{p+1}\rangle = 0$.

Lemma 3.6. *Let G be a long graph with two components. If $\langle G, T_a\rangle \neq 0$, then each component of G contains one, but not both, of the vertices with labels a and $a+1$.*

Proof. If either component of G contains both the vertex labeled a and the vertex labeled $a+1$, then this component defines a path on T_a starting and ending at the leaf labeled $\{a, a+1\}$. Every internal vertex in this path appears at least twice. The vertex with minimal height must then be the nadir of the paths determined by two edges in this component of G. If $\langle G, T_a\rangle \neq 0$ this is impossible, so each component of G has a vertex labeled by one of a or $a+1$, but not both. □

Definition 3.18. Assume $\langle G, T_a\rangle \neq 0$, and let G' be the component of G which does not have the edge corresponding to the vertex in T_a directly below $\{a, a+1\}$. If G' has a vertex labeled by a, let $\delta_a = 0$; otherwise, let $\delta_a = 1$. Let S_a be the restriction of T whose leaf set is $r^{-1}_{a,\delta_a}(\{\text{vertices in } G'\} \setminus \{a, a+1\})$.

Definition 3.19. The *root vertex* of a restriction S of T is the second highest vertex of T through which all the root paths of $l \in L(S)$ pass.

Lemma 3.7. *If $\langle G, T_a \rangle \neq 0$, then at least one of S_a or $S_a^{c,a}$ is admissible and a distinguishes that set. In other words, $(S_a, S_a^{c,a}, a) \in B_T$.*

Proof. First we show that S_a is admissible. The leaf set $L(S_a)$ can be written as $L(S_a) = \bigcup_{i=1}^{n} L(S_i)$, where each S_i is either a tall branch or an almost tall branch. We can choose this union to be maximal in the sense that no two almost tall branches in the union are restrictions of the same tall branch. For each S_i, let $\tilde{S}_i$ be the restriction of T_a with leaf set $r_{a,1}(L(S_i))$.

Recall that S_a is defined as the restriction of T whose leaf set is $r_{a,\delta_a}^{-1}(\{\text{vertices in } G'\} \setminus \{a, a+1\})$. Thus, G' contains the vertices with labels corresponding to the leaves in $L(S_i)$ for each i. For each pair of indices $i \neq j$, the path on T_a determined by this component passes through the nadir of the path $p_{T_a}(\text{root of } \tilde{S}_i, \text{root of } \tilde{S}_j)$.

If both S_i and S_j are almost tall branches of T, then $\tilde{S}_i$ and $\tilde{S}_j$ are restriction of tall branches, $\tilde{S}_i'$ and $\tilde{S}_j'$, of T_a. The second component of G contains vertices with the same labels as the leaves in $L(\tilde{S}_i') \setminus L(\tilde{S}_i)$ and $L(\tilde{S}_j') \setminus (\tilde{S}_j)$. Thus, $\beta(G, T_a)$ will map one of the edges from the second component of G to the nadir of the path $p_{T_a}(\text{root of } \tilde{S}_i, \text{root of } \tilde{S}_j)$, which is equal to the nadir of the path $p_{T_a}(\text{root of } \tilde{S}_i', \text{root of } \tilde{S}_j')$. Since $\langle G, T_a \rangle \neq 0$, this is impossible. Thus, at most one of the S_i can be an almost tall branch of T, and S_a is admissible.

Now we show that a distinguishes S_a. Let $l_1 \in L(S_a^{c,l})$ and $l_2 \in L(S_a)$. Suppose, by way of a contradiction, that $((r_{a,1}(l_1), r_{a,1}(l_2)), a) \in R_{T_a}$. This implies that the paths $p_{T_a}(r_{a,1}(l_1), \{a, a+1\})$ and $p_{T_a}(r_{a,1}(l_2), \{a, a+1\})$ have the same nadir – call this vertex v. Since $r_{a,1}(l_2)$ is the label of a vertex in G', this component determines a path on T_a between the vertex v and the leaf $\{a, a+1\}$. However, $r_{a,1}(l_1)$ is the label of a vertex in the other component of G, so this component determines another path on T_a between v and $\{a, a+1\}$. Since $\langle G, T_a \rangle \neq 0$, the vertex in each of these paths with minimal height must be v, so v the image under $\beta(G, T_a)$ of one edge from each component of G. This is impossible, because $\beta(G, T_a)$ is a bijection, and so $((r_{a,1}(l_1), r_{a,1}(l_2)), a) \notin R_{T_a}$. The equivalence relation is full, so either $((a, r_{a,1}(l_1)), r_{a,1}(l_2)) \in R_{T_a}$ or $((a, r_{a,1}(l_2)), r_{a,1}(l_1)) \in R_{T_a}$. Thus, a distinguishes S_a.

□

In Example 3.11, the leaf set of S_7 is the union of the leaf sets of one almost tall branch of T and one tall branch of T. In Example 3.12, the leaf set of S_4 is the union the the leaf sets of two tall branches of T.

Define sets $A_{G,T}$ and $\tilde{A}_{G,T}$ to keep track of which terms in $\langle G, d_1 T \rangle$ or $\langle G, \tilde{d}_1 T \rangle$ are nonzero. In other words,

$$A_{G,T} = \{a \in \mathbf{p} \;:\; \langle G, T_a \rangle = \pm 1\},$$

and

$$\tilde{A}_{G,T} = \{(S, l, \delta) \;:\; (S, S^{c,l}, l) \in B_T, \text{ and } \langle G, F_{(S,l,\delta)} \rangle = \pm 1\}.$$

We will show $A_{G,T}$ and $\tilde{A}_{G,T}$ are isomorphic as sets, and thus $\langle G, d_1 T \rangle \equiv \langle G, \tilde{d}_1 T \rangle$ mod 2.

Provisionally, we define a map $\varphi : \tilde{A}_{G,T} \to A_{G,T}$ by $\varphi(S, l, \delta) = l$.

Lemma 3.8. *The map $\varphi : \tilde{A}_{G,T} \to A_{G,T}$ is well-defined.*

Proof. Assume that $(S, l, \delta) \in \tilde{A}_{G,T}$, so $\langle G, F_{(S,l,\delta)} \rangle \neq 0$. We have the following sequence of bijections:

$$\{\text{Edges of } G\} \xrightarrow{\beta(G,F_{(S,l,\delta)})} \{\text{Vertices of } F_{(S,l,\delta)}\} \xrightarrow{f_{S,l}} \{\text{Vertices of } T_l\} .$$

Recall from Definition 3.6 that if $\beta(G, F_{(S,l,\delta)})$ is defined, then it sends the edge $i \to j$ in G to the nadir of the path $p_{F_{(S,l,\delta)}}(i, j)$. The map $f_{S,l}$ then sends this internal vertex to the nadir of the path $p_{T_l}(i, j)$. Thus, the composition of these bijections is $\beta(G, T_l)$, the set map used in the pairing $\langle G, T_l \rangle$. The map $\beta(G, T_l)$ is therefore a bijection, and $\langle G, T_l \rangle \neq 0$, as needed for φ to be well-defined. $\square$

We work out an example of the composition $f_{S,l} \circ \beta(G, F_{(S,l,\delta)})$ in Example 3.10.

Provisionally, we define another map $\psi : A_{G,T} \to \tilde{A}_{G,T}$ by $\psi(a) = (S_a, a, \delta_a)$. The image $\psi(a)$ is calculated in Examples 3.10, 3.11 and 3.12 for various situations.

Lemma 3.9. *The map $\psi : A_{G,T} \to \tilde{A}_{G,T}$ is well-defined.*

Proof. Fix $a \in A_{G,T}$. By Lemma 3.7, $(S_a, S_a^{c,a}, \delta_a) \in B_T$. Since $\langle G, T_a \rangle \neq 0$, we have the following sequence of bijections:

$$\{\text{Edges of } G\} \xrightarrow{\beta(G,T_a)} \{\text{Vertices of } T_a\} \xrightarrow{f_{S_a,a}^{-1}} \{\text{Vertices of } F_{(S_a,a,\delta_a)}\} .$$

The map $\beta(G, T_a)$ sends the edge $i \to j$ to the nadir of the path $p_{T_a}(i, j)$. If the edge $i \to j$ is in the component G' of G, then the leaves i and j in $F_{(S_a,a,\delta_a)}$ appear on the tree T_{S_a}. Otherwise, the leaves i and j appear on the tree $T_{L \setminus S_a}$. Thus, the map $f_{S_a,a}^{-1}$ maps the nadir of the path $p_{T_a}(i, j)$ to the nadir of the path $p_{F_{(S_a,a,\delta_a)}}(i, j)$. An example calculation of this composition is done in Example 3.10. The composition of these bijections is $\beta(G, F_{(S_a,a,\delta_a)})$, the set map used in the pairing $\langle G, F_{(S_a,a,\delta_a)}\rangle$. Thus, this map is a bijection and $\langle G, F_{(S_a,a,\delta_a)}\rangle \neq 0$. $\square$

Lemma 3.10. *The maps ψ and φ are inverse set isomorphisms.*

Proof. Clearly, $\varphi \circ \psi$ is the identity, so it remains to show that the composition $\psi \circ \varphi : \tilde{A}_{G,T} \to \tilde{A}_{G,T}$ is the identity. An explicit calculation of these as inverse isomorphisms is shown in Example 3.10. If $\langle G, F_{(S,l,\delta)}\rangle \neq 0$ then $\psi \circ \varphi(S, l, \delta) = \psi(l) = (S_l, l, \delta_l)$. Since $\langle G, F_{(S,l,\delta)}\rangle \neq 0$ and $\langle G, F_{(S_l,l,\delta_l)}\rangle \neq 0$, the partition of $\mathbf{p+1}$ defined by the trees in $F_{(S,l,\delta)}$ is equal to the partition defined by the trees in $F_{(S_l,l,\delta_l)}$. Since the same leaf, l, is used in the definition of both forests, the admissible sets S and S_l must be the same and $\delta = \delta_l$.

$\square$

Proof of Theorem 3.1. We have now shown that $\langle G, d_1T\rangle \equiv \langle G, \tilde{d}_1T\rangle \mod 2$ for every tree $T \in \mathcal{P}ois^d(p)$ and every long graph $G \in \mathcal{S}iop^d(p+1)$ by showing that the sets $A_{G,T}$ and $\tilde{A}_{G,T}$ are isomorphic. To show that $d_1T = \tilde{d}_1T$, we will show that the isomorphisms φ and ψ also preserve the signs.

That is, we will show that for any long graph $G \in \mathcal{S}iop^d(p+1)$ and tree $T \in \mathcal{P}ois^d(p)$, we have

$$\langle G, (-1)^a T_a\rangle = \langle G, (-1)^a \sigma_{S_a,a} F_{(S_a,a,\delta_a)}\rangle. \tag{3.1}$$

This pairing is calculated in Examples 3.12 and 3.11. Recall that if d is even $\sigma^d_{S,l} = 1$ and if d is odd $\sigma^d_{S,l}$ is the sign of the permutation defined by the map $f_{S,l}$ from Lemma 3.2.

When d is even, the equality in Equation 3.1 reduces to $\langle G, T_a\rangle = \langle G, F_{(S_a,a,\delta_a)}\rangle$. Since the trees in $F_{(S_a,a,\delta_a)}$ are restrictions of T_a, the paths used to define the maps

$\beta(G, T_a)$ and $\beta(G, F_{(S_a,a,\delta_a)})$ have the same number of paths traveling from left to right, and $\langle G, T_a\rangle = \langle G, F_{(S_a,a,\delta_s)}\rangle$. Thus, $\langle G, T_a\rangle = \langle G, F_{(S_a,a,\delta_a)}\rangle$.

When d is odd, $\langle G, (-1)^a T_a\rangle = (-1)^a \sigma^d(G, T_a)$, where $\sigma^d(G, T_a)$ is the sign of the permutation given by $\beta(G, T_a)$. Also

$$\langle G, (-1)^a \sigma_{S_a,a} F_{(S_a,a,\delta_a)}\rangle = (-1)^a \sigma_{S_a,a} \cdot \sigma^d(G, F_{(S_a,a,\delta_a)}),$$

where $\sigma_{S_a,a} \cdot \sigma^d(G, F_{(S_a,a,\delta_a)})$ is the sign of the permutation given by the composition

$$\{\text{edges of } G\} \xrightarrow{\beta(G, F_{(S_a,a,\delta_a)})} \{\text{internal vertices of } F_{(S_a,a,\delta_a)}\} \xrightarrow{f_{S_a,a}} \{\text{vertices of } T_a\}\ .$$

Since $f \circ \beta(G, F_{(S_a,a,\delta_a)}) = \beta(G, T_a)$, we have $\sigma_{S_a,a} \cdot \sigma^d(G, F_{(S_a,a,\delta_a)}) = \sigma(G, T_a)$. Thus, $\langle G, (-1)^a T_a\rangle = \langle G, (-1)^a \sigma_{S_a,a} F_{(S_a,a,\delta_a)}\rangle$.

We now have that $\langle G, d_1 T\rangle = \langle G, \tilde{d}_1 T\rangle$ for every tree $T \in \mathcal{P}ois^d(p)$ and every long graph $G \in \mathcal{S}iop^d(p)$, so the maps d_1 and $\tilde{d}_1$ are equal. $\square$

Example 3.10. Let T be the tree shown in Figure 3.1, and let S be the admissible subset of T shown in Figure 3.4. The multi-labeled tree T_1 and the forest $F_{(S,1,0)}$ are shown in Figure 3.8, with their internal vertices ordered from left to right according to the planar embedding. We will check that $\varphi(S, 1, 0)$ and $\psi(1)$ are well-defined, that $\psi \circ \varphi(S, 1, 0) = (S, 1, 0)$ and that $\varphi \circ \psi(1) = 1$.

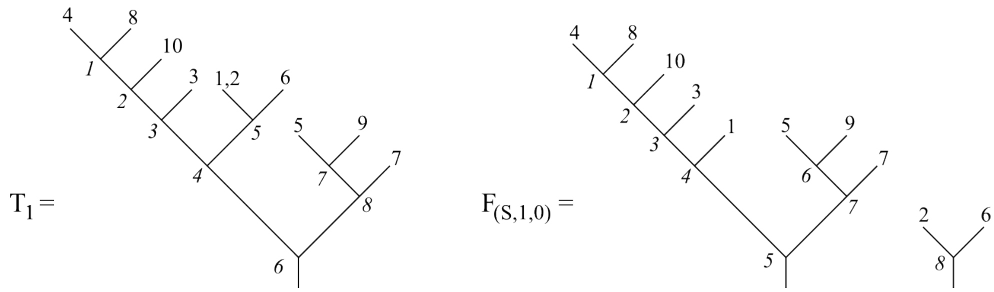


Figure 3.8. The tree T_1 and the forest $F_{(S,1,0)}$ with internal vertices labeled.

The map $f_{S,1}$: {internal vertices of $F_{(S,1,0)}$} $\to$ {internal vertices of T_1} is given by the permutation (5678). Let G be the graph shown below, with edges labeled from left to right:

$$G = \; 1 \xrightarrow{1} 3 \xrightarrow{2} 10 \xrightarrow{3} 8 \xrightarrow{4} 4 \xrightarrow{5} 5 \xrightarrow{6} 9 \xrightarrow{7} 7 \qquad 2 \xrightarrow{8} 6\,.$$

Then $\langle G, F_{(S,1,0)}\rangle = \pm 1$ and $\langle G, T_1\rangle = \pm 1$, with the bijections given by $\beta(G, F_{(S,1,0)}) =$ (14)(23) and $\beta(G, T_1) = (14)(23)(5678)$.

First we will check that $\varphi(S,1,0) = 1$ is an element of $A_{G,T}$. We find that the permutation defined by $\beta(G,T_1)$ is (14)(23)(5678), which is clearly a bijection. Alternatively, the composition below

$$\{\text{Edges of } G\} \xrightarrow{\beta(G,F_{(S,1,0)})} \{\text{Vertices of } F_{(S,1,0)}\} \xrightarrow{f_{S,1}} \{\text{Vertices of } T_1\}$$

is given by (14)(23)(5678). This illustrates that $f \circ \beta(G, F_{(S,1,0)}) = \beta(G, T_1)$, as in the proof of Lemma 3.8.

Next we will calculate $\psi(1) = (S_1, 1, \delta_1)$ and see that it is an element of $\tilde{A}_{G,T}$. The map $\beta(G,T_1)$ sends edge 8 to vertex 5, the vertex directly below $\{1,2\}$ in T_1.

The component of G which does not contain edge 8 is

$$G' = \; 1 \xrightarrow{1} 3 \xrightarrow{2} 10 \xrightarrow{3} 8 \xrightarrow{4} 4 \xrightarrow{5} 5 \xrightarrow{6} 9 \xrightarrow{7} 7\,.$$

From here, we can see that the leaf set of S_1 is $L(S_1) = \{2, 9, 7, 3, 4, 8, 6\}$. Since their leaf sets are the same, S_1 and S are the same restriction of T. Since G' contains the vertex 1, we have $\delta_1 = 0$. This gives us that $\psi(1) = (S, 1, 0)$, which we have already checked is an element of $\tilde{A}_{G,T}$. We can also check explicitly that the composition

$$\{\text{Edges of } G\} \xrightarrow{\beta(G,T_1)} \{\text{Vertices of } T_1\} \xrightarrow{f_{S,1}^{-1}} \{\text{Vertices of } F_{(S_1,1,\delta_1)}\}$$

is $\beta(G, F_{(S_1,1,\delta_1)})$:

$$f_{S,1}^{-1} \circ \beta(G, T_1) = (5876)(14)(23)(5678) = (14)(23) = \beta(G, F_{(S_1,1,\delta_1)}).$$

We now have that

$$\psi \circ \varphi(S, 1, 0) = \psi(1) = (S, 1, 0),$$

and

$$\varphi \circ \psi(1) = \varphi(S, 1, 0) = 1.$$

Example 3.11. If T is the tree in Figure 3.1, then T_7 is the multi-labled tree shown in Figure 3.9. We will check that $\psi(7)$ is well defined and that ψ respects the signs.

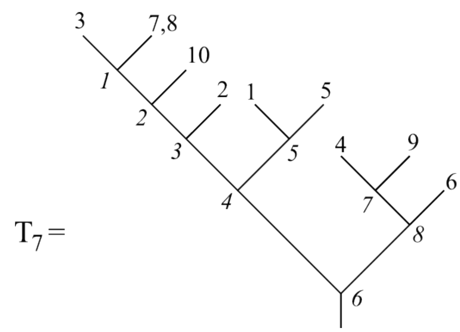


Figure 3.9. The mulit-labeled tree T_7 with internal vertices labeled.

The graph G, shown below, pairs non-trivially with T_7 since $\beta(G, T_7) = (1524)$ is a bijection.

$$G = \ 1 \xrightarrow{1} 5 \xrightarrow{2} 8 \xrightarrow{3} 2 \qquad 3 \xrightarrow{4} 7 \xrightarrow{5} 10 \xrightarrow{6} 4 \xrightarrow{7} 9 \xrightarrow{8} 6 \ .$$

The map $\beta(G, T_7)$ sends edge 4 of G to vertex 1, and the component of G which does not contain edge 4 is

$$G' = \ 1 \xrightarrow{1} 5 \xrightarrow{2} 8 \xrightarrow{3} 2 \ .$$

This gives us that the leaf set of S_7 is $L(S_7) = \{1, 5, 2\}$. Thus, S_7 is the restriction shown in Figure 3.10.

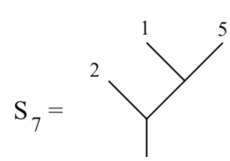


Figure 3.10. The restriction S_7 of T.

The restriction S_7 is admissible because the leaf set $L(S_7) = \{2\} \cup \{1, 5\}$ is the union of the leaf sets of one tall branch and one almost tall branch of T. The leaf 7 distinguishes the restriction S_7, as the nadir of the path $p_{T_7}(7, i)$ is not equal to the nadir of the path $p_{T_7}(7, j)$ for $i = 3, 8, 10, 4, 9, 6$ and $j = 2, 1, 5$. We have that $\delta_7 = 1$ because G' contains the vertex 8. The forest $\psi(7) = F_{(S_7,7,1)}$ is shown in Figure 3.11.

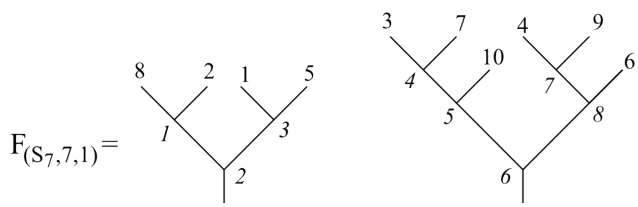


Figure 3.11. The forest $F_{(S_7,7,1)}$ with internal vertices labeled.

The pairing $\langle G, F_{(S_7,7,1)}\rangle = \pm 1$ as $\beta(G, F_{(S_7,7,1)}) = (13)$ is a bijection.

To check that ψ respects the signs, we need to check that $\langle G, (-1)^7 T_7\rangle = \langle G, (-1)^7 \sigma^d_{S_7,7} F_{(S_7,7,1)}\rangle$.

If d is even, we can see that the paths determined by G on T_7 travel in the same direction as the paths determined by G on $F_{(S_7,7,1)}$, and each has seven traveling from left to right, so

$$\langle G, (-1)^7 T_7\rangle = (-1)^7(-1)^7 = \langle G, (-1)^7 F_{(S_7,7,1)}\rangle.$$

If d is odd, we find that $f_{S,7} = (13524)$ has sign 1, so $\sigma^d_{S_7,7} = 1$. Furthermore, $\beta(G, T_7) = (1524)$ and $\beta(G, F_{(S_7,7,1)}) = (13)$ both have sign -1. Then $\langle G, (-1)^7 T_7\rangle = (-1)^7(-1) = 1$ and $\langle G, (-1)^7 \sigma^d_{S_7,7} F_{(S_7,7,1)}\rangle = (-1)^7(1)(-1) = 1$.

Example 3.12. If T is the tree in Figure 3.1, then T_4 is the multi-labled tree shown in Figure 3.12. We will check that $\psi(4)$ is well defined and that ψ respects the signs.

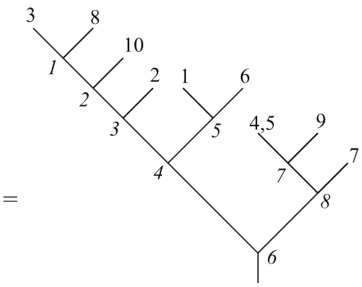
$T_4 =$ 

Figure 3.12. The mulit-labeled tree T_4 with internal vertices labeled.

The graph G, shown below, pairs non-trivially with T_4 since $\beta(G, T_4) = (153)(24)$ is a bijection.

$$G = 1 \xrightarrow{1} 6 \xrightarrow{2} 3 \xrightarrow{3} 8 \xrightarrow{4} 10 \xrightarrow{5} 2 \xrightarrow{6} 5 \qquad 4 \xrightarrow{7} 9 \xrightarrow{8} 7 .$$

The map $\beta(G, T_4)$ sends edge 7 of G to vertex 7 of T_4, and the component of G which does not contain edge 7 is

$$G' = 1 \xrightarrow{1} 6 \xrightarrow{2} 3 \xrightarrow{3} 8 \xrightarrow{4} 10 \xrightarrow{5} 2 \xrightarrow{6} 5 .$$

This gives us that the leaf set of S_4 is $L(S_4) = \{1, 5, 3, 7, 9, 2\}$, and S_4 is the restriction shown in Figure 3.13

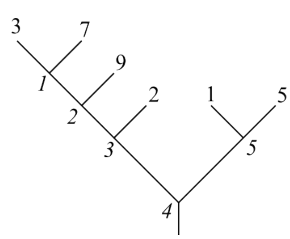
$S_4 =$ 

Figure 3.13. The restriction S_4 of T with internal vertices labeled.

The restriction S_4 is admissible because the leaf set $L(S_4) = \{3, 7, 9, 2\} \cup \{1, 5\}$ is the union of the leaf sets of two tall branches of T. The leaf 4 distinguishes the restriction S_4. We have that $\delta_4 = 1$ because G' contains the vertex 5. The forest $\psi(4) = F_{(S_4,4,1)}$ is shown in Figure 3.14.

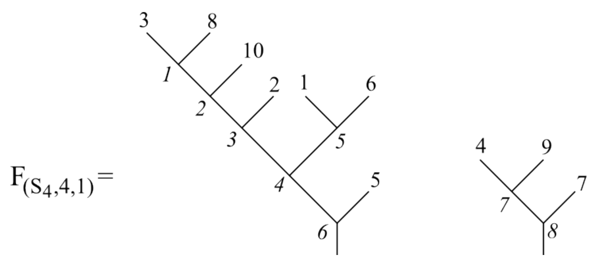


Figure 3.14. The forest $F_{(S_4,4,1)}$ with internal vertices labeled.

The pairing $\langle G, F_{(S_4,4,1)}\rangle = \pm 1$ as $\beta(G, F_{(S_4,4,1)}) = (153)(24)$. In this example, $f_{S,4}$ is the identity, so $\sigma^d_{S_a,a} = 1$ for all d. To check that ψ preserves the signs, we need to check that $\langle G, (-1)^a T_a\rangle = \langle G, (-1)^a F_{(S_a,a,\delta_a)}\rangle$.

If d is even, we can see in Figures 3.12 and 3.14 that the paths used to determine the sign of $\langle G, F_{(S_4,4,1)}\rangle$ travel in the same direction as the paths used to determine the sign of $\langle G, T_4\rangle$, and $\langle G, F_{(S_4,4,1)}\rangle = -1 = \langle G, T_4\rangle$.

If d is odd, then $\langle G, F_{(S_4,4,1)}\rangle = -1$, the sign of the permutation $(153)(24)$. But this permutation is the same as the permutation

$$f_{S_4,4} \circ \beta(G, F_{(S_4,4,1)}) = (153)(24) = \beta(G, T_4).$$

Thus, $\langle G, (-1)^4 T_4\rangle = \langle G, (-1)^4 F_{(S_4,4,\delta_4)}\rangle$ for all d.

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