GENERALIZED NEAR-GROUP CATEGORIES

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JOSIAH E. THORNTON

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Student: Josiah E. Thornton

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This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Victor Ostrik	Chair
Arkady Berenstein	Member
Boris Botvinnik	Member
Alexander Polishchuk	Member
Catherine Page	Outside Member

and

Kimberly Andrews Espy	Vice President for Research & Innovation/Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

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DISSERTATION ABSTRACT

Josiah E. Thornton

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We give an exposition of near-group categories and generalized near-group categories. We show that both have a $\varphi-$pseudounitary structure. We complete the classification of braided near-group categories and discuss the inherent structures on both symmetric and modular generalized near-group categories.

CURRICULUM VITAE

NAME OF AUTHOR: Josiah E. Thornton

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR

DEGREES AWARDED:

Doctor of Philosophy, Mathematics, 2012, University of Oregon
Master of Science, University of Oregon, 2008
Bachelor of Science, University of Oregon, 2006

AREAS OF SPECIAL INTEREST:

Representation Theory
Tensor Categories

PROFESSIONAL EXPERIENCE:

Graduate Teaching Fellow, Department of Mathematics, University of Oregon, Eugene, 2006 to present

GRANTS, AWARDS AND HONORS:

Graduate Teaching Fellowship, Mathematics, 2006 to present

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CHAPTER I

INTRODUCTION

If G is a finite group and k is a field, it is well known that two $k-$representations of G can be tensored over k to produce another $k-$representation. If we restrict ourselves to finite dimensional $\mathbb{C}-$representations of G, then Maschke's Theorem tells us that $\mathrm{Rep}(G)$, the category of of finite dimensional $\mathbb{C}-$representations of G, is semisimple with finitely many simple objects up to isomorphism. A special class of categories, called fusion categories, are a generalization of $\mathrm{Rep}(G)$; that is, a fusion category is essentially a semisimple category equipped with an associative, unital tensor product with the notion of duality, such that there are finitely many isomorphism classes of simple objects.

At the moment, there is little hope to classify all fusion categories; such a classification would yield a classification of finite groups, which is itself an unsolved problem. Many classification problems have dealt with added structure on fusion categories. Given a fusion category, the Grothendieck group has a ring structure (see, e.g. [BK]). Given a ring, one may ask what fusion categories categorify that ring; that is, have that ring as its Grothendieck ring. For a general ring, this classification is not yet done, but "Ocneanu rigidity" (see, e.g. [ENO]) states that there are finitely many fusion categories with a given Grothendieck ring. One classification problem is to study fusion categories which categorify a given ring. One also might ask what rings can arise as Grothendieck rings of a fusion category. In [HR], Hong and Rowell are able to classify Grothendieck rings of a class of fusion category with restrictive properties.

Another approach to partial classification of fusion categories is to restrict the number of simple objects in the category. In [O1] and [O2], Ostrik discusses fusion categories with two and three isomorphism classes of simple objects, respectively. In [O1], he is able to give a complete classification of fusion categories with two simple objects, but only able to give a partial classification of fusion categories with three simple objects in [O2]. One of the restrictions that Hong and Rowell make in [HR] is to limit the number of simple objects to at most five.

In this dissertation, we combine these two classification methods. In Section II.2.8, we discuss the well-known classification of so called pointed fusion categories. These are categories where every simple object is invertible; that is, the tensor product of the object and its dual is the unit object. Our approach is to study fusion categories where most of the simple objects are invertible (Chapter III) or where the non-invertible simple objects are very closely related (Chapter IV).

Chapter II is broken into two main sections. The first section is dedicated to the definitions and basic results associated to Tensor Category theory. This section defines the notion of duality and braiding, a natural isomorphism between $U \otimes V$ and $V \otimes U$ in a tensor category, as well as gives implications of combining these structures with others such as semisimplicity. It also covers the definitions of fusion categories, three notions of dimension in fusion categories and a pivotal structure, which is a tensor isomorphism between the identity functor and the functor of taking duals twice. It is still unknown (see [ENO, Conjecture 2.8]) if all fusion categories have a pivotal structure. This section also includes a classification of pointed fusion categories in Section II.2.8 and of pointed, braided fusion categories in Section II.2.10.

In Chapter III, we define near-group categories, fusion categories with one non-invertible simple object, and discuss inherent structures of their Grothendieck rings. We show in Theorem III.2.6 that every near-group category has a pivotal structure, and therefore no counterexamples to [ENO, Conjecture 2.8] are near-group categories. If the non-invertible simple object X does not appear as a summand of $X \otimes X^*$, then these categories are classified up to tensor equivalence by Tambara and Yamagami [TY]. If they are equipped with a braiding, then they are classified up to braided equivalence by Siehler [Si1]. Because of these classifications, we restrict our attention to the case when X is a summand of $X \otimes X^*$. In this case, Proposition III.3.5 states that any symmetric (Definition II.2.19) near-group category is braided equivalent to $\mathrm{Rep}(G)$ for some finite group G. Finally, Theorem III.4.6 gives the classification of the seven non-symmetric braided near-group categories up to braided equivalence.

We generalize our definition of near-group categories to the class of fusion categories where the invertible objects transitively act on the non-invertible simple objects by tensor product. These so called generalized near-group categories are the main focus of Chapter IV. As in Chapter III, we discuss some characteristics of their Grothendieck rings and in Theorem IV.3.6 show that all generalized near-group categories have a pivotal structure. The remaining part of Chapter IV is dedicated to the inherent structures of symmetric generalized near-group categories (Theorem IV.4.7 and Theorem IV.4.9) and of modular (Definition II.2.19) generalized near-group categories (Theorem IV.5.2).

CHAPTER II

PRELIMINARIES

In this chapter we will define the fundamental theory needed to talk about near-group categories and generalized near-group categories. The first section covers topics arising in tensor categories. This mainly covers structures and compatibility. The treatment of this section will be similar to [BK]. The second section briefly covers fusion categories. In this section we discuss some results of [ENO] and [DGNO].

II.1. Tensor Categories

II.1.1. Tensor Product

Let $\mathcal{C}$ be an abelian category. We wish to consider categories where we can tensor objects together. That is if $X, Y \in \mathrm{ob}(\mathcal{C})$, we wish to have an object $X \otimes Y \in \mathrm{ob}(\mathcal{C})$. More formally

Definition II.1.1. Let $\mathcal{C}$ be an abelian category. A tensor product on $\mathcal{C}$ is an abelian bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

The first thing we want is that a tensor product should be associative. This means that we should have $(U \otimes V) \otimes W \simeq U \otimes (V \otimes W)$, where the isomorphism are compatible with each other. The following Pentagon Axiom gives the compatibilities we desire.

Definition II.1.2 (Pentagon Axiom)**.** Let $\mathcal{C}$ be an abelian category and $\otimes$ be a tensor product on $\mathcal{C}$. We say that $\otimes$ is associative if there exist natural transformations

$$\alpha_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$$

such that the following diagram commutes for all objects $U, V, W, X \in \mathrm{ob}(\mathcal{C})$

$$\begin{tikzcd}
 & ((U \otimes V) \otimes W) \otimes X \arrow[dl, "\alpha_{U,V,W} \otimes \mathrm{id}_X"'] \arrow[dr, "\alpha_{U \otimes V, W, X}"] & \\
(U \otimes (V \otimes W)) \otimes X \arrow[d, "\alpha_{U, V \otimes W, X}"'] & & (U \otimes V) \otimes (W \otimes X) \arrow[d, "\alpha_{U,V,W \otimes X}"] \\
U \otimes ((V \otimes W) \otimes X) \arrow[rr, "\mathrm{id}_U \otimes \alpha_{V,W,X}"'] & & U \otimes (V \otimes (W \otimes X))
\end{tikzcd}$$

We also want a tensor category to have a unit object. That is an object 1 such that $U \otimes 1 \simeq U \simeq 1 \otimes U$. The following Triangle Axiom gives the compatibility conditions we want the isomorphsims to have.

Definition II.1.3 (Triangle Axiom)**.** Let $\mathcal{C}$ be an abelian category and $\otimes$ be an associative tensor product on $\mathcal{C}$. We say that $\otimes$ is unital if there exists an object $1 \in \mathrm{ob}(\mathcal{C})$ and natural transformations $\lambda_U : 1 \otimes U \xrightarrow{\sim} U$ and $\rho_U : U \otimes 1 \xrightarrow{\sim} U$ such that the following diagram commutes for all objects $U, V \in \mathrm{ob}(\mathcal{C})$

$$\begin{tikzcd}
(U \otimes 1) \otimes V \arrow[rr, "\alpha_{U,1,V}"] \arrow[dr, "\rho_U \otimes \mathrm{id}_V"'] & & U \otimes (1 \otimes V) \arrow[dl, "\mathrm{id}_U \otimes \lambda_V"] \\
 & U \otimes V &
\end{tikzcd}$$

Definition II.1.4. An abelian category $\mathcal{C}$ over a field k is called a <u>tensor category</u> if it comes equipped with a unital, associative tensor product $\otimes$. That is $\mathcal{C}$ has the following data

- A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
- Natural transformations $\alpha_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$

- An object 1 and natural transformations $\lambda_U : 1 \otimes U \to U$ and $\rho_U : U \otimes 1 \to U$

such that

1. $(\mathcal{C}, \otimes, \alpha)$ satisfy the Pentagon Axiom,
2. $(\mathcal{C}, \otimes, \alpha, 1, \lambda, \rho)$ satisfy the Triangle Axiom.
3. $\mathrm{End}(1) = k$.

Example II.1.5. The following are examples of tensor categories.

1. Vect, the category of vector spaces over a field k, is a tensor category, where $\otimes$ is tensor product of vector spaces,
2. Vec, the category of finite dimensional vector spaces over a field k, where $\otimes$ is tensor product of vector spaces,
3. $\mathrm{Rep}(G)$, the category of finite dimensional k-representations of a finite group G, where $\otimes$ is tensor product of k-representations.

II.1.2. Semisimplicity

In the category Vec of finite dimensional vector spaces, we can decompose any vector space into a (finite) direct sum of one-dimensional vector spaces. The analogous structure on tensor categories is called semisimplicity.

Definition II.1.6. An object $U \in \mathrm{ob}(\mathcal{C})$ of a tensor category $\mathcal{C}$ is said to be simple if every injection $X \hookrightarrow U$ is an isomorphism. A tensor category $\mathcal{C}$ is said to be semisimple if every object $U \in \mathrm{ob}(\mathcal{C})$ can be written as a direct sum of finitely many simple objects. We let $\mathcal{O}(\mathcal{C})$ denote the set of simple objects of $\mathcal{C}$.

Definition II.1.7. Let $\mathcal{C}$ be a semisimple tensor category and $U, V, W \in \mathrm{ob}(\mathcal{C})$ be simple objects. Define

$$N_{U,V}^{W} = \dim \mathrm{Hom}(U \otimes V, W).$$

We call such a number a fusion coefficient of $\mathcal{C}$.

II.1.3. Rigidity

In the category Vec of finite dimensional $\mathbb{C}$-vector spaces, there is a notion of duality. More precisely, for every vector space V, there is a vector space $V^* := \mathrm{Hom}(V, \mathbb{C})$. Let $\{v_1, \ldots, v_n\}$ be a basis for V. It is clear that V^* is spanned by linear functions defined by $v_j^*(v_i) = \delta_{i,j}$. It is also clear that v_j^* are linearly independent and thus $\{v_1^*, \ldots, v_n^*\}$ is a basis for V^*. Now given a basis $\{v_1, \ldots, v_n\}$ of V and (dual) basis $\{v_1^*, \ldots, v_n^*\}$ of V^*, we can define maps

$$\begin{aligned} \mathrm{eval}_V : V^* \otimes V &\to \mathbb{C} \\ (f, v) &\mapsto f(v) \end{aligned}$$

$$\begin{aligned} \mathrm{coeval}_V : \mathbb{C} &\to V \otimes V^* \\ 1 &\mapsto \sum_{i=1}^{n} v_i \otimes v_i^* \end{aligned}$$

Moreover, if we look at the composition

$$V \xrightarrow{\mathrm{coeval}_V \otimes \mathrm{id}_V} V \otimes V^* \otimes V \xrightarrow{\mathrm{id}_V \otimes \mathrm{eval}_V} V \tag{II.1}$$

we get

$$(\mathrm{coeval}_V \otimes \mathrm{id}_V)\left(1 \otimes \sum_{j=1}^{n} \alpha_j v_j\right) = \left(\sum_{i=1}^{n} v_i \otimes v_i^*\right) \otimes \sum_{j=1}^{n} \alpha_j v_j = \sum_{i,j=1}^{n} \alpha_j v_i \otimes v_i^* \otimes v_j,$$

and

$$(\mathrm{id}_V \otimes \mathrm{eval}_V)\left(\sum_{i,j=1}^{n} \alpha_j v_i \otimes v_i^* \otimes v_j\right) = \sum_{i,j=1}^{n} \alpha_j v_i \otimes v_i^*(v_j) = \sum_{i=1}^{n} \alpha_i v_i.$$

Therefore the composition in Equation (II.1) is the identity on V. Smilarly, the composition

$$V^* \xrightarrow{\mathrm{id}_{V^*} \otimes \mathrm{coeval}_V} V^* \otimes V \otimes V^* \xrightarrow{\mathrm{eval}_V \otimes \mathrm{id}_{V^*}} V^*$$

is the identity on V^*. This motivates the following definition.

Definition II.1.8. Let $\mathcal{C}$ be a tensor category. An object $U \in \mathrm{ob}(\mathcal{C})$ is said to have a right dual in $\mathcal{C}$ if there exists an object $U^* \in \mathcal{C}$ and morphisms

$$\mathrm{eval}_U : U^* \otimes U \to 1$$

$$\mathrm{coeval}_U : 1 \to U \otimes U^*$$

such that the composition

$$U \xrightarrow{\mathrm{coeval}_U \otimes \mathrm{id}_U} U \otimes U^* \otimes U \xrightarrow{\mathrm{id}_U \otimes \mathrm{eval}_U} U$$

is the identity on U, and the composition

$$U^* \xrightarrow{\mathrm{id}_{U^*} \otimes \mathrm{coeval}_U} U^* \otimes U \otimes U^* \xrightarrow{\mathrm{eval}_U \otimes \mathrm{id}_{U^*}} U^*$$

is the identity on U^*.

For a finite dimensional vector space V, the dual object $V^* = \mathrm{Hom}(V, \mathbb{C})$ is canonically defined. The following Lemma gives a similar result for tensor categories.

Lemma II.1.9. [BK, Lemma 2.1.5] Let $\mathcal{C}$ be a tensor category and $U \in \mathrm{ob}(\mathcal{C})$ be an object with a right dual. Then U^* is unique up to a unique isomorphism compatible with eval_U and coeval_U.

The following Lemma gives a canonical isomorphism of some spaces of morphisms.

Lemma II.1.10. [BK, Lemma 2.1.6] Let $\mathcal{C}$ be a tensor category and $V \in \mathrm{ob}(\mathcal{C})$ be an object with a right dual. Then there are canonical isomorphisms

$$\mathrm{Hom}(U \otimes V, W) = \mathrm{Hom}(U, W \otimes V^*)$$
$$\mathrm{Hom}(U, V \otimes W) = \mathrm{Hom}(V^* \otimes U, W)$$

One immediate result of Lemma II.1.10 is the ability to dualize morphisms. Let U, V be objects of a tensor category with right duals, and let $f : U \to V$ be any morphism. Then define $f^* : V^* \to U^*$ to be the composition

$$V^* \xrightarrow{\mathrm{id}_{V^*} \otimes \mathrm{coeval}_U} V^* \otimes U \otimes U^* \xrightarrow{\mathrm{id}_{V^*} \otimes f \otimes \mathrm{id}_{U^*}} V^* \otimes V \otimes U^* \xrightarrow{\mathrm{eval}_V \otimes \mathrm{id}_{U^*}} U^*.$$

Analogous to above, we may define left duals in the following way.

Definition II.1.11. Let $\mathcal{C}$ be a tensor category. An object $U \in \mathrm{ob}(\mathcal{C})$ is said to have a left dual in $\mathcal{C}$ if there exists an object ${}^*U \in \mathcal{C}$ and morphisms

$$\mathrm{eval}'_U : U \otimes {}^*U \to 1$$
$$\mathrm{coeval}'_U : 1 \to {}^*U \otimes U$$

such that the composition

$$U \xrightarrow{\mathrm{id}_U \otimes \mathrm{coeval}'_U} U \otimes {}^*U \otimes U \xrightarrow{\mathrm{eval}'_U \otimes \mathrm{id}_U} U$$

is the identity on U, and the composition

$${}^*U \xrightarrow{\mathrm{coeval}'_U \otimes \mathrm{id}_{{}^*U}} {}^*U \otimes U \otimes {}^*U \xrightarrow{\mathrm{id}_{{}^*U} \otimes \mathrm{eval}'_U} U^*$$

is the identity on ${}^*U$.

Definition II.1.12. A tensor category $\mathcal{C}$ is called rigid if every object of $\mathcal{C}$ has both a right and a left dual.

By Lemma II.1.9, if U is an object of a rigid tensor category, then there are canonical isomorphisms

$$U = {}^*(U^*) = ({}^*U)^*.$$

Lemma II.1.13. Let $\mathcal{C}$ be a semisimple, rigid, tensor category. Let $U \in \mathrm{ob}(\mathcal{C})$. Then

1. U^* is simple if and only if U is simple (the same result holds for ${}^*U$),
2. $U^* \simeq {}^*U$.
3. If U is simple, then $U^{**} \simeq U$.

Proof. Let $U \in \mathrm{ob}(\mathcal{C})$.

1. Assume $U \simeq V \oplus W$ for some objects of $\mathcal{C}$. Then $U^* \simeq V^* \oplus W^*$.

   Now assume $U^* \simeq V \oplus W$ for some objects of $\mathcal{C}$. Then $U = {}^*(U^*) \simeq {}^*V \oplus {}^*W$

2. Since * commutes with direct sum, we may assume U is simple.

   $$\mathrm{Hom}(1, {}^*U \otimes U) \neq 0.$$

   Since $\mathcal{C}$ is semisimple, ${}^*U \otimes U = 1 \oplus X$ for some object X of $\mathcal{C}$. Therefore

   $$\mathrm{Hom}({}^*U, U^*) = \mathrm{Hom}({}^*U \otimes U, 1) \neq 0.$$

   Since ${}^*U$ and U^* are simple, they are isomorphic.

3. By Lemma II.1.10, $\mathrm{Hom}(U^{**}, U) = \mathrm{Hom}(1, U^* \otimes U) \neq 0$, where the second inequality follows since $\mathcal{C}$ is semisimple. Therefore $U^{**} \simeq U$.

$\square$

Note that the isomorphisms above are not canonical.

The following Lemma displays some of the compatibility between rigidity and the tensor category structure.

Lemma II.1.14. [BK, Lemma 2.1.11] Let $\mathcal{C}$ be a rigid tensor category. Then

1. $1^* = 1 = {}^*1$,
2. $(U \otimes V)^* = V^* \otimes U^*$,
3. $(\alpha_{U,V,W})^* = \alpha_{W^*,V^*,U^*}$,

where equality in (1) and (2) mean the existence of a unique isomorphism.

II.1.4. Braiding

In the category Vec of finte dimensional vector spaces, there are isomorphisms $V \otimes W \simeq W \otimes V$. The analogous isomorphisms for tensor categories are called braidings.

Definition II.1.15. Let $\mathcal{C}$ be a tensor category. A braiding of $\mathcal{C}$ is a family of functorial isomorphisms

$$\sigma_{U,V} : U \otimes V \xrightarrow{\simeq} V \otimes U$$

for every pair of objects $U, V \in \mathrm{ob}(\mathcal{C})$.

Similar to the Pentagon and Triangle Axioms before, we want a braiding on a tensor category to satisfy commutative diagrams. In this case we want them to follow the proceeding Hexagon Axioms.

Definition II.1.16 (Hexagon Axioms)**.** Let $\mathcal{C}$ be a tensor category with associativity given by α. A braiding σ on $\mathcal{C}$ is said to fulfill the Hexagon axioms if for every $U, V, W \in \mathrm{ob}(\mathcal{C})$

the following diagrams commute:

(a)

$$\begin{tikzcd}[column sep=small]
& U \otimes (V \otimes W) \arrow[r, "\sigma_{U,V\otimes W}"] & (V \otimes W) \otimes U \arrow[dr, "\alpha_{V,W,U}"] & \\
(U \otimes V) \otimes W \arrow[ur, "\alpha_{U,V,W}"] \arrow[dr, "\sigma_{U,V} \otimes \mathrm{id}_W"'] & & & V \otimes (W \otimes U) \\
& (V \otimes U) \otimes W \arrow[r, "\alpha_{V,U,W}"'] & V \otimes (U \otimes W) \arrow[ur, "\mathrm{id}_V \otimes \sigma_{U,W}"'] &
\end{tikzcd}$$

(b)

$$\begin{tikzcd}[column sep=small]
& U \otimes (V \otimes W) \arrow[r, "\sigma^{-1}_{V\otimes W,U}"] & (V \otimes W) \otimes U \arrow[dr, "\alpha_{V,W,U}"] & \\
(U \otimes V) \otimes W \arrow[ur, "\alpha_{U,V,W}"] \arrow[dr, "\sigma^{-1}_{V,U} \otimes \mathrm{id}_W"'] & & & V \otimes (W \otimes U) \\
& (V \otimes U) \otimes W \arrow[r, "\alpha_{V,U,W}"'] & V \otimes (U \otimes W) \arrow[ur, "\mathrm{id}_V \otimes \sigma^{-1}_{W,U}"'] &
\end{tikzcd}$$

Definition II.1.17. A braided tensor category is a tensor category $\mathcal{C}$ equipped with a braiding which satisfies the Hexagon Axioms.

The Hexagon Axioms identify morphisms that should be equivalent. For instance (omitting associativity isomorphisms), the Hexagon Axioms state that

$$\sigma_{U,V\otimes W} = (\sigma_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes \sigma_{V,W})$$

and

$$\sigma^{-1}_{V\otimes W,U} = (\sigma^{-1}_{W,U} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes \sigma^{-1}_{W,V})$$
$$[\text{equivalently } \sigma_{V\otimes W,U} = (\mathrm{id}_U \otimes \sigma_{W,V}) \circ (\sigma_{W,U} \otimes \mathrm{id}_V)]. \qquad \text{(II.2)}$$

The following Lemma is another example of maps identified by the Hexagon Axioms.

Lemma II.1.18. Let $\mathcal{C}$ be a braided tensor category and $U, V, W, X \in \mathrm{ob}(\mathcal{C})$. Then (omitting associativity isomorphisms)

$$\sigma_{U\otimes V,W\otimes X} = (\mathrm{id}_W \otimes \sigma_{U,X} \otimes \mathrm{id}_V) \circ (\sigma_{U,W} \otimes \sigma_{V,X}) \circ (\mathrm{id}_U \otimes \sigma_{V,W} \otimes \mathrm{id}_X) \tag{II.3}$$

Proof. Consider the following diagram.

$$\begin{tikzcd}
& U\otimes V\otimes W\otimes X \arrow[dl, "\mathrm{id}_U \otimes \sigma_{V,W} \otimes \mathrm{id}_X"'] \arrow[d, "\sigma_{U\otimes V,W} \otimes \mathrm{id}_X"] \arrow[dr, "\sigma_{U\otimes V,W\otimes X}"] & \\
U\otimes W\otimes V\otimes X \arrow[r, "\sigma_{U,W} \otimes \mathrm{id}_{V\otimes X}"] \arrow[dr, "\sigma_{U,W} \otimes \sigma_{V,X}"'] & W\otimes U\otimes V\otimes X \arrow[r, "\mathrm{id}_W \otimes \sigma_{U\otimes V,X}"] \arrow[d, "\mathrm{id}_{W\otimes U} \otimes \sigma_{V,X}"] & W\otimes X\otimes U\otimes V \\
& W\otimes U\otimes X\otimes V \arrow[ur, "\mathrm{id}_W \otimes \sigma_{U,X} \otimes \mathrm{id}_V"'] &
\end{tikzcd}$$

The top left triangle commutes by Hexagon Axiom (a), while the top right and bottom right triangles commute by Hexagon Axiom (b) (more precisely by Equation (II.2)). The bottom left triangle commutes because it is the composition of two functions. Therefore the diagram commutes. Since the left-hand side of Equation (II.3) is the clockwise perimeter of the diagram and the right-hand side of Equation (II.3) is the counter-clockwise perimeter of the equation, the Lemma is proved. □

Rigidity and braiding are compatible structures on a tensor category, and given both structures we get the following relation.

Lemma II.1.19. [BK, Lemma 2.1.11] If $\mathcal{C}$ is a rigid, braided tensor category, then

$$(\sigma_{U,V})^* = \sigma_{U^*,V^*}.$$

II.2. Fusion Categories

Fusion categories are a natural generalization of the categories of representations of finite groups or semisimple Hopf algebras. We will restrict ourselves to the case of fusion categories over $\mathbb{C}$. Many of the results from this section are from [ENO] and [DGNO].

II.2.1. Definition and Examples

Definition II.2.1. A fusion category is a semisimple, rigid tensor category over $\mathbb{C}$ with finitely many simple objects such that $\mathrm{End}(1) = \mathbb{C}$. we say a fusion category $\mathcal{C}$ is a braided fusion category if it also comes equipped with a braiding.

Example II.2.2. Let G be a finite group and let $\mathrm{Rep}(G)$ be the category of finite dimensional $\mathbb{C}$-representations of G. Then $\mathrm{Rep}(G)$ is a braided fusion category with the following structure.

1. The tensor product on $\mathrm{Rep}(G)$ is given by $U \otimes V = U \otimes_{\mathbb{C}} V$, where $\otimes_{\mathbb{C}}$ is the tensor production of two $\mathbb{C}[G]-$modules. The unit object, 1, is the trivial representation of G.

2. The number of irreducible representations (up to isomorphism) is the number of conjugacy classes of G, which is finite. By Maschke's Theorem, $\mathrm{Rep}(G)$ is semisimple.

3. Let V be a representation of G. Define V^* to be the $\mathbb{C}[G]-$module with action $(g \cdot \omega)(v) = \omega(g^{-1} \cdot v)$, where $g \in G$, $v \in V$ and $\omega \in V^*$. Let $\mathrm{eval}_V : V^* \otimes V \to 1$ be defined by $\mathrm{eval}_V(\omega \otimes v) = \omega(v)$. This is a morphism of $\mathbb{C}[G]-$modules since

$$\begin{aligned}\mathrm{eval}_V(g \cdot (\omega \otimes v)) = \mathrm{eval}_V(g \cdot \omega \otimes g \cdot v) = (g \cdot \omega)(g \cdot v) &= \omega(g^{-1} \cdot g \cdot v) \\ &= \omega(v) \\ &= \mathrm{eval}_V(\omega \otimes v) \\ &= g \cdot \mathrm{eval}_V(\omega \otimes v).\end{aligned}$$

Furthermore, let $v_1, \ldots, v_n$ be a basis for V and $\omega_1, \ldots, \omega_n$ be the dual basis for V^*. Define $\mathrm{coeval}_V : 1 \to V \otimes V^*$ by $\mathrm{coeval}_V(1) = \sum_{i=1}^{n} v_i \otimes \omega_i$. To see that coeval_V is a morpshism of $\mathbb{C}[G]-$modules, we need to check that $g \cdot \mathrm{coeval}_V(1) = \mathrm{coeval}_V(1)$ for all $g \in G$. Let $g \in G$ and

$$g \cdot v_k = \sum_{i=1}^{n} c_{ki} v_i \qquad \text{and} \qquad g^{-1} \cdot v_k = \sum_{i=1}^{n} d_{ki} v_i.$$

By computing $g^{-1} \cdot g \cdot v_k = v_k$, we see $\sum_{k=1}^{n} c_{ki} d_{jk} = \delta_{i,j}$. Moreover,

$$g \cdot \omega_k(v_j) = \omega_k(g^{-1} \cdot v_j) = \omega_k\left(\sum_{i=1}^{n} d_{ji} v_i\right) = d_{jk}.$$

Therefore $g \cdot \omega_k = \sum_{j=1}^{n} d_{jk} \omega_j$. This gives

$$g \cdot v_k \otimes g \cdot \omega_k = \left(\sum_{i=1}^{n} c_{ki} v_i\right) \otimes \left(\sum_{j=1}^{n} d_{jk} \omega_j\right) = \sum_{i,j=1}^{n} c_{ki} d_{jk} (v_i \otimes \omega_j),$$

and

$$\begin{aligned} g \cdot \mathrm{eval}_V(1) = \sum_{k=1}^{n} g \cdot v_k \otimes g \cdot \omega_k &= \sum_{i,j,k=1}^{n} c_{ki} d_{jk} (v_i \otimes \omega_j) \\ &= \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} c_{ki} d_{jk}\right) v_i \otimes \omega_j \\ &= \sum_{i,j=1}^{n} \delta_{i,j} v_i \otimes \omega_j \\ &= \sum_{i=1}^{n} v_i \otimes \omega_i = \mathrm{eval}_V(1). \end{aligned}$$

Therefore eval_V is a morphism of $\mathbb{C}[G]-$modules. That these morphisms satisfy the rigidity axioms is the same computation done for vector spaces in Section II.1.3.

4. The braiding on $\mathrm{Rep}(G)$ is the isomorphisms $\sigma_{U,V} : U \otimes V \to V \otimes U$ of $\mathbb{C}-$vector spaces. It is clear that this is a morphism of $\mathbb{C}[G]-$modules.

Another example comes from a small generalization of the category of finite dimensional vector spaces.

Example II.2.3. Let G be a finite group and $\mathcal{C}$ be the category of $G-$graded vector spaces. That is the semisimple abelian category generated by (non-isomorphic) one dimensional vector spaces, V_g, for each element in $g \in G$. For $g, h \in G$ we define a tensor product on $\mathcal{C}$ by $V_g \otimes V_h \simeq V_{gh}$, where $1_g \otimes 1_h \mapsto 1_{gh}$. $V_g^* \simeq V_{g^{-1}}$ with $1_g \otimes 1_{g^{-1}} \mapsto 1_e$ and $1_e \mapsto 1_{g^{-1}} \otimes 1_g$. If G is abelian, then $\mathcal{C}$ has a braiding given by $1_g \otimes 1_h \mapsto 1_h \otimes 1_g$.

II.2.2. Traces and Squared Norm Dimension

Definition II.2.4. Let $\mathcal{C}$ be a fusion category and $U \in \mathrm{ob}(\mathcal{C})$. For a morphism $f : U \to U^{**}$, we define the <u>trace</u> of f to be the composition

$$1 \xrightarrow{\mathrm{coeval}_U} U \otimes U^* \xrightarrow{f \otimes \mathrm{id}_U} U^{**} \otimes U^* \xrightarrow{\mathrm{eval}_{U^*}} 1.$$

We write the trace of f as $\mathrm{Tr}_U(f)$, and since $\mathrm{End}(1) = \mathbb{C}$, we may simply think $\mathrm{Tr}_U(f) \in \mathbb{C}$.

Let U be a simple object of a fusion category $\mathcal{C}$. By Lemma II.1.13 (iii), there exists an isormorphism $\gamma : U \to U^{**}$. Since U is simple and γ is non-zero, $\mathrm{Tr}(\gamma) \neq 0$. The following Lemma will allow us to make a canonical definition of a dimension.

Lemma II.2.5. Let U be a simple object of a fusion category $\mathcal{C}$. Let $\gamma : U \to U^{**}$ be an isomorphism and $k \in \mathbb{C}^{\times}$. Then

1. $\mathrm{Tr}_U(k\gamma) = k\,\mathrm{Tr}_U(\gamma)$,
2. $\mathrm{Tr}_{U^*}(((k\gamma)^{-1})^*) = \frac{1}{k}\,\mathrm{Tr}_{U^*}((\gamma)^{-1})^*)$.

Proof. 1. Since morphisms are $\mathbb{C}-$linear, we have

$$\mathrm{Tr}_U(k\gamma) = \mathrm{eval}_{V^*} \circ (k\gamma \otimes \mathrm{id}_{V^*}) \circ \mathrm{coeval}_V = k(\mathrm{eval}_{V^*} \circ (\gamma \otimes \mathrm{id}_{V^*}) \circ \mathrm{coeval}_V = k\,\mathrm{Tr}_U(\gamma).$$

2. Furthermore $((k\gamma)^{-1})^* = \frac{1}{k}(\gamma^{-1})^*$, so applying 1. to $\frac{1}{k}(\gamma^{-1})^*$ gives 2.

□

Using Lemma II.2.5, the product $\mathrm{Tr}_U(\gamma)\,\mathrm{Tr}_{U^*}((\gamma^{-1})^*)$ is well-defined for any isomorphism $\gamma : U \to U^{**}$.

Definition II.2.6. [Mu1, ENO] Let U be a simple object of a fusion category $\mathcal{C}$. Define the squared norm of U to be the product $|U|^2 = \mathrm{Tr}_U(\gamma)\,\mathrm{Tr}_{U^*}((\gamma^{-1})^*)$, where $\gamma : U \to U^{**}$ is any isomorphism.

II.2.3. Pivotal Structures and Quantum Dimension

Definition II.2.7. [ENO, Definition 2.7] A fusion category $\mathcal{C}$ is said to have a pivotal structure if there is an isomorphism of tensor functors $\gamma : \mathrm{id} \to **$. Such a fusion category is called a pivotal fusion category.

It is conjectured (see [ENO, Conjecture 2.8]) that all fusion categories are pivotal.

Given a pivotal structure on a fusion category, we can define the dimension of a simple object in another way.

Definition II.2.8. [BK, Definition 2.3.2] Let $\mathcal{C}$ be a pivotal fusion category with pivotal structure $\gamma : \mathrm{id} \to **$. Let U be a simple object of $\mathcal{C}$. then we define the quantum dimension of U to be $\dim(U) = \mathrm{Tr}_U(\gamma)$.

The following proposition relates the squared norm dimension and the quantum dimension. Recall that we assume our fusion categories are over $\mathbb{C}$.

Proposition II.2.9. [ENO, Proposition 2.9] Let $\mathcal{C}$ be a pivotal fusion category and $U \in \mathrm{ob}(\mathcal{C})$. Then $|U|^2 = |\dim(U)|^2$. Moreover if $\dim(U)$ is real, then $|U|^2 = (\dim(U))^2$.

We define the global dimension of a fusion category to be the sum of the squared norms of the simple objects. We write $\dim(\mathcal{C})$ for the global dimension of the category.

II.2.4. Spherical and φ-Pseudounitary

Definition II.2.10. A pivotal fusion category $\mathcal{C}$ is called spherical if $\dim(U)$ is real for all objects of $\mathcal{C}$.

For any fusion category, $\mathcal{C}$, we may define the sphericalization of $\mathcal{C}$, which is a spherical fusion category containing $\mathcal{C}$ as a fusion subcategory. By [ENO, Theorem 2.6], we are given $\gamma : Id \to ****$ an isomorphism of tensor functors.

Definition II.2.11. [ENO, Remark 3.1] The sphericalization, $\tilde{\mathcal{C}}$, of a fusion category $\mathcal{C}$ is the fusion category whose simple objects are pairs (V, α) where $V \in \mathcal{O}(\mathcal{C})$ and $\alpha : V \xrightarrow{\sim} V^{**}$ satisfies $\alpha^{**}\alpha = \gamma$. This category has a canonical spherical structure $i : \mathrm{Id} \to **$.

Fix an isomorphism $f : V \to V^{**}$. Since $\mathrm{Hom}(V, V^{**})$ is one dimensional, we may write $\alpha = a \cdot f$ for some $a \in \mathbb{C}^{\times}$. We also have $\alpha^{**} = a \cdot f^{**}$. Similarly, we may write $\gamma = z \cdot f^{**}f$ for some $z \in \mathbb{C}^{\times}$. Then the condition $\alpha^{**}\alpha = \gamma$ is equivalent to $a^2 = z$. Therefore for each $V \in \mathcal{O}(\mathcal{C})$, we have two such α. Fixing one, we write $(V, \alpha) = V_+$ and $(V, -\alpha) = V_-$.

Definition II.2.12. Given a automorphism $\varphi \in \mathrm{Gal}(\mathbb{A}/\mathbb{Q})$, a fusion category $\mathcal{C}$ is called $\varphi-$pseudo unitary if $\varphi(\dim(U)) = \mathrm{FPdim}(U)$ for all objects U of $\mathcal{C}$.

By [DGNO, Proposition 2.16] a $\varphi-$pseudo unitary fusion category admits a spherical (and thus pivotal) structure.

II.2.5. Frobenius-Perron Dimension

The notion of dimension of objects in a fusion category can be defined in yet another way. This way uses the semisimple structure of the fusion category.

Let X be an object of a fusion category $\mathcal{C}$. Since $\mathcal{C}$ is semisimple, the Grothendieck ring of $\mathcal{C}$, $\mathcal{K}(\mathcal{C})$, is a free $\mathbb{Z}-$algebra with generators the simple objects of $\mathcal{C}$. X acts on $\mathcal{K}(\mathcal{C})$ by tensoring on the left. Let $[X]$ represent the matrix given by the action of tensoring on the left by X. That is $[X]_{ij} = \dim \mathrm{Hom}(X \otimes X_j, X_i)$.

Definition II.2.13. [ENO, Section 8.1] Let $\mathcal{C}$ be a fusion category and X be an object of $\mathcal{C}$. Define $\mathrm{FPdim}(X)$ to be the largest eigenvalue of the matrix $[X]$. We define $\mathrm{FPdim}(\mathcal{C})$ to be the sum of $(\mathrm{FPdim}(X))^2$ for each simple object X.

Two important results come from this definition. By the Frobenius-Perron Theorem (see e.g. [ENO, Theorem 8.1]) $\mathrm{FPdim}(X) > 0$ for all objects $X \in \mathcal{C}$. Furthermore, by [ENO, Theorem 8.6] the map $\mathcal{K}(\mathcal{C}) \to \mathbb{R}$ given by assigning $\mathrm{FPdim}(X)$ to each object of $\mathcal{C}$ extends to a homomorphism of $\mathbb{Z}-$algebras.

The following two results relate the quantum dimension and Frobenius-Perron dimension.

Proposition II.2.14. [ENO, Proposition 8.21] Let $\mathcal{C}$ be a fusion category. For any simple object U of $\mathcal{C}$, one has $|\dim(U)| \leq \mathrm{FPdim}(U)$. Therefore $\dim(\mathcal{C}) \leq \mathrm{FPdim}(\mathcal{C})$, with equality holding if and only if $\dim(U) = \mathrm{FPdim}(U)$ for all simple objects of $\mathcal{C}$.

Proposition II.2.15. [ENO, Proposition 8.22] Let $\mathcal{C}$ be a fusion category. Then $\frac{\dim(\mathcal{C})}{\mathrm{FPdim}(\mathcal{C})}$ is an algebraic integer.

II.2.6. Adjoint Categories and Gradings

In some situations, we may wish to study a fusion subcategory of a given fusion category. The following section discusses such a category and how it gives rise to some structure of the original category.

Definition II.2.16. [ENO, Section 2.2] Let $\mathcal{K} \subset \mathcal{C}$ be a fusion subcategory of a fusion category $\mathcal{C}$. The <u>adjoint category</u> to $\mathcal{K}$ is the fusion subcategory $\mathcal{K}_{\mathrm{ad}} \subset \mathcal{C}$ generated by $X \otimes X^*$ where $X \in \mathrm{ob}(\mathcal{K})$.

Definition II.2.17. Let $\mathcal{C}$ be a fusion category and G be a finite group. A grading of $\mathcal{C}$ by G is a map $o : \mathcal{O}(\mathcal{C}) \to G$ such that for any simple objects $U, V, W \in \mathcal{O}(\mathcal{C})$, o satisfies $o(U) = o(V)o(W)$ if $\mathrm{Hom}(U, V \otimes W) \neq 0$. We say a grading is faithful if the map $o : \mathcal{O}(\mathcal{C}) \to G$ is surjective.

Proposition II.2.18. [GN, Theorem 3.5,Proposition 3.9] Let $\mathcal{C}$ be a fusion category. Then there exists a finite group $U_{\mathcal{C}}$ and a faithful grading $o : \mathcal{O}(\mathcal{C}) \to U_{\mathcal{C}}$ of $\mathcal{C}$ with the following properties:

1. $\mathcal{C}_{\mathrm{ad}} = o^{-1}(e)$.
2. Let A be the maximal abelian quotient of $U_{\mathcal{C}}$. Then there is an isomorphism $\mathrm{Hom}(A, \mathbb{C}^\times) \cong \mathrm{Aut}_\otimes(\mathrm{id}_{\mathcal{C}})$, where $\mathrm{Aut}_\otimes(\mathrm{id}_{\mathcal{C}})$ is the group of tensor automorphisms of the identity functor.

II.2.7. Müger Center

Let $\mathcal{C}$ be a braided fusion category. From [Mu3], we define the Müger center of $\mathcal{C}$ to be the full tensor subcategory of $\mathcal{C}$ with objects

$$\{X \in \mathcal{C} | \sigma_{Y,X}\sigma_{X,Y} = \mathrm{id}_{X\otimes Y} \ \ \forall Y \in \mathcal{C}\}.$$

We denote the Müger center of $\mathcal{C}$ by $\mathcal{C}'$. The following definitions give another way to view symmetric and modular tensor categories. They are seen to be equivalent to the usual definitions in [Mu3].

Definition II.2.19. A braided fusion category is called symmetric if $\mathcal{C}' = \mathcal{C}$. A braided, spherical fusion category is called modular if $\mathcal{C}' = \mathrm{Vec}$.

II.2.8. Pointed Fusion Categories

Definition II.2.20. A fusion category $\mathcal{C}$ is called pointed if $U \otimes U^* \simeq 1$ for every simple object of $U \in \mathcal{O}(\mathcal{C})$.

Let $\mathcal{C}$ be a pointed fusion category. Let $U, V \in \mathcal{O}(\mathcal{C})$. Then $U \otimes V \in \mathcal{O}(\mathcal{C})$, since otherwise

$$U \simeq U \otimes V \otimes V^* \simeq (X \oplus Y) \otimes V^* \simeq (X \otimes V^*) \oplus (Y \otimes V^*),$$

which is a contradiction. By Lemma II.1.13, $U^* \in \mathcal{O}(\mathcal{C})$. Therefore $G := \mathcal{O}(\mathcal{C})$ is a group where multiplication is tensor product and inverses are duals. Let $a, b, c, d \in G$. Then associativity of the tensor product is a map $\alpha : G \times G \times G \to \mathrm{End}(- \otimes - \otimes -) = \mathrm{End}(1) = \mathbb{C}^\times$ such that (by the Pentagon Axiom)

$$\alpha(a, b, c)\alpha(a, bc, d)\alpha(b, c, d) = \alpha(ab, c, d)\alpha(a, b, cd).$$

Therefore $\alpha \in Z^3(G, \mathbb{C}^\times)$ is a 3−cocylce of G. Cohomologous cocycles give equivalent fusion categories, so a pointed fusion category is classified by its group $G = \mathcal{O}(\mathcal{C})$ of simple objects and an element $\alpha \in H^3(G, \mathbb{C}^\times)$ of the 3−cohomology of G.

II.2.9. Equivariantization

Let $\mathcal{C}$ be a braided tensor category.

Definition II.2.21. [DGNO, Section 4.2] (i) Let $\underline{\mathrm{Aut}}^{\mathrm{br}}(\mathcal{C})$ be the category whose objects are braided tensor equivalences and whose morphisms are isomorphism of braided tensor functors.

(ii) For a group G, let $\underline{G}$ be the tensor category whose objects are elements of G, whose morphisms are the identity morphisms and whose tensor product is given by group multiplication.

(iii) We say that G *acts on* $\mathcal{C}$ *viewed as a braided tensor category* if there is a tensor functor $\underline{G} \to \underline{\mathrm{Aut}}^{\mathrm{br}}(\mathcal{C})$.

(iv) We say $\mathcal{C}$ is a *braided tensor category* $\mathcal{C}$ *over* $\mathcal{E}$ if it is equipped with a braided functor $\mathcal{E} \to \mathcal{C}'$.

Let G be a group, and G act on $\mathcal{C}$ viewed as a braided tensor category. Then define the equivariantization of $\mathcal{C}$ by G.

Definition II.2.22. [DGNO, Definition 4.2.2] Let $\mathcal{C}^G$ be the category with objects G-equivariant objects. That is an object $X \in \mathcal{C}$ along with an isomorphism $\mu_g : F_g(X) \to X$ such that the following diagram commutes for all $g, h \in G$.

$$\begin{tikzcd} F_g(F_h(X)) \arrow[r, "F_g(u_h)"] \arrow[d, "\gamma_{g,h}(X)"'] & F_g(X) \arrow[d, "u_g"] \\ F_{gh}(X) \arrow[r, "u_{gh}"] & X \end{tikzcd}$$

The morphisms in $\mathcal{C}^G$ are morphisms in $\mathcal{C}$ which commute with u_g. The tensor product on $\mathcal{C}^G$ is the obvious one induced by the tensor product on $\mathcal{C}$. Since the action of G on $\mathcal{C}$ respects the braiding, there is an induced braiding on $\mathcal{C}^G$.

The following propostition relates actions of G on $\mathcal{C}$ and equivariantization.

Proposition II.2.23. [DGNO, Theorem 4.18(ii)] Let G be a finite group and $\mathcal{C}$ be a braided tensor category over $\mathrm{Rep}(G)$. Then there is a braided tensor category $\mathcal{D}$ equipped with an action of G, such that $\mathcal{D}^G \cong \mathcal{C}$.

II.2.10. Pointed Braided Fusion Categories and Premetric Groups

Recall that a fusion category $\mathcal{C}$ is called pointed if $U \otimes U^* \simeq 1$ for every simple object $U \in \mathcal{O}(\mathcal{C})$. Let $\mathcal{C}$ be a pointed, braided fusion category and $A := \mathcal{O}(\mathcal{C})$ be the group of simple objects of $\mathcal{C}$. Since $\mathcal{C}$ is braided, for each $U, V \in A$ there is a braiding $\sigma_{U,V} : U \otimes V \simeq V \otimes U$. Therefore A is an abelian group. Since $\sigma_{U,U} \in \mathrm{Hom}(U \otimes U, U \otimes U) \simeq \mathrm{Hom}(1, 1) = \mathbb{C}$, we have a map $q : A \to \mathbb{C}^\times$ defined by $q(U) = \sigma_{U,U}$. $q(U)$ is non-zero since $\sigma_{U,U}$ is an isomorphism.

Lemma II.2.24. Let $\mathcal{C}$ be a pointed, braided fusion category and $A := \mathcal{O}(\mathcal{C})$ be the abelian group of simple objects of $\mathcal{C}$. Define $q : A \to \mathbb{C}^\times$ by $q(U) = \sigma_{U,U}$. and $b : A \times A \to \mathbb{C}^\times$ by $b(U, V) = \frac{q(U \otimes V)}{q(U)q(V)}$. Then

1. $q(U^*) = q(U)$,

2. $b(U,V) = \sigma^2_{U,V}$, where $\sigma_{U,V} \in \mathrm{Hom}(U \otimes V, V \otimes U) \simeq \mathrm{Hom}(1,1) = \mathbb{C}$,

3. b is bimultiplicative. That is $b(U_1 \otimes U_2, V) = b(U_1, V)b(U_2, V)$.

Proof. 1. Let $U \in \mathcal{O}(\mathcal{C})$. By Lemma II.1.19 (and since all spaces of morphsims are one-dimensional)

$$q(U^*) = \sigma_{U^*,U^*} = (\sigma_{U,U})^* = \sigma_{U,U} = q(U).$$

2. Let $U, V \in \mathcal{O}(\mathcal{C})$. By Lemma II.1.18

$$b(U,V) = \frac{\sigma_{U\otimes V,U\otimes V}}{\sigma_{U,U}\sigma_{V,V}} = \frac{\sigma_{U,U}\sigma_{V,V}\sigma^2_{U,V}}{\sigma_{U,U}\sigma V, V} = \sigma^2_{U,V}.$$

3. Let $U_1, U_2, V \in \mathcal{O}(\mathcal{C})$. Then, by Equation (II.2),

$$b(U_1 \otimes U_2, V) = \sigma^2_{U_1\otimes U_2,V} = (\sigma_{U_1,V}\sigma_{U_2,V})^2 = \sigma^2_{U_1,V}\sigma^2_{U_2,V} = b(U_1,V)b(U_2,V).$$

□

The structure arising from a pointed, braided fusion category motivates the following definition.

Definition II.2.25. A pre-metric group is an a finite abelian group A along with a quadratic form $q : A \to \mathbb{C}^\times$, that is $q(a) = q(a^{-1})$ and the map $b : A \times A \to \mathbb{C}^\times$ defined by $b(a,b) = \frac{q(ab)}{q(a)q(b)}$ is bimultiplicative. A morphism of pre-metric groups $(A_1, q_1) \to (A_2, q_2)$ is a homomorphism $\varphi : A_1 \to A_2$ such that $q_2(\varphi(a)) = q_1(a)$ for all $a \in A$.
A pre-metric group (A, q) is called a metric group if the map $b : A \times A \to \mathbb{C}^\times$ is non-degenerate. In this case, we say q is non-degenerate.

The following proposition allows us to study pre-metric groups (resp. metric groups) when we wish to study pointed, braided fusion categories (resp. pointed, modular fusion categories).

Proposition II.2.26. [DGNO, Proposition 2.41] Let $\mathcal{P}B$ be the category whose objects are pointed, braided fusion categories and whose morphisms are braided functors up to braided isomorphism. Let $\mathcal{P}M$ be the category whose objects are pre-metric groups and whose morphisms are morphisms of pre-metric groups. Then the assinment $(\mathcal{C}, \sigma) \to (\mathcal{O}(\mathcal{C}), \sigma_{U,U})$ discussed above gives rise to an equivalence of categories $\mathcal{F} : \mathcal{P}B \to \mathcal{P}M$.

Note that if we restict the functor to acting on pointed, modular fusion categories we get exactly the metric groups.

The following in an example of a non-symmetric fusion category which was claimed to not exist in [**?**].

Example II.2.27. Let $A = \mathbb{Z}/3\mathbb{Z}$, and define a map $q : A \to \mathbb{C}^\times$ by $q(0) = 1$ and $q(\pm 1) = \omega = e^{2\pi i/3}$. Then $q(a) = q(q^{-1})$ for all $a \in A$ and $b(a, b) = \frac{q(ab)}{q(a)q(b)}$ is given by the following table

$q(a,b)$	0	1	-1
0	1	1	1
1	1	ω^2	ω
-1	1	ω	ω^2

Table 1: Values of $q(a, b)$.

It is clear from Table 1 that b is bimultiplicative and non-degenerate. Therefore by Proposition II.2.26 there is a pointed, modular fusion category $\mathcal{C}(\mathbb{Z}/3\mathbb{Z}, q)$ with three simple objects. We may let $\mathbb{Z}/2\mathbb{Z}$ act on $\mathcal{C}(\mathbb{Z}/3\mathbb{Z}, q)$ by permuting the non-unit simple objects.

Let $\mathcal{C} = \mathcal{C}(\mathbb{Z}/3\mathbb{Z}, q)^{\mathbb{Z}/2\mathbb{Z}}$. By [DGNO, Proposition 4.26],

$$\dim(\mathcal{C}) = |\mathbb{Z}/2\mathbb{Z}| \cdot \dim(\mathcal{C}(\mathbb{Z}/3\mathbb{Z}, q)) = 6.$$

$\mathcal{C}$ has two invertible objects given by $\mathrm{Vec}^{\mathbb{Z}/2\mathbb{Z}} \cong \mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$ and at least one simple object of dimension two given by the direct sum of the non-unit simple objects in $\mathcal{C}(\mathbb{Z}/3\mathbb{Z}, q)$. Since $\dim(\mathcal{C}) = 6$, these must be the only simple objects.

By [DGNO, Proposition 4.30], $\mathcal{C}' = (\mathcal{C}(\mathbb{Z}/3\mathbb{Z}, q)')^{\mathbb{Z}/2\mathbb{Z}} = \mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$, therefore $\mathcal{C}$ is not symmetric and $\mathcal{C} \ncong \mathrm{Rep}(S_3)$. By Proposition III.2.6, $\mathcal{C}$ is spherical and thus has a ribbon structure. Note that this is a contradiction to [O2, Section 4], where it was claimed that no such category exists.

CHAPTER III

NEAR-GROUP CATEGORIES

This chapter is dedicated to a class of fusion categories called near-group categories. The first main theorem of this chapter is to prove that any near-group category admits a $\varphi-$pseudounitary structure, in particular they are pivotal. Therefore no near-group categories are counterexampls to [ENO, Conjecture 2.8]. The second part of the chapter is the study of near-group categories that admit a braiding. In Theorem III.4.6, we classify the braided near-group categories not discussed in [Si1].

III.1. Definition, Examples and Known Results

In Section II.2.8 we classify fusion categories whose simple objects are invertible. In Section II.2.10, we classify braided fusion categories whose simple objects are invertible (up to braided equivalence). A (braided) near-group category generalizes these concepts slightly by asking all but one simple object to be invertible.

Definition III.1.1. [Si1] A near-group category is a semisimple, rigid tensor category with finitely many simple objects (up to isomorphism) such that all but one of the simple objects is invertible. In the language of fusion categories, a near-group category is a fusion category with one non-invertible simple object. If such a category comes equipped with a braiding, then we call it a braided near-group category.

Example III.1.2. The following are examples of braided near-group fusion categories.

1. Let $\mathcal{C} = \mathrm{Rep}(S_3)$, where S_3 is the symmetric group on three letters. S_3 has three

conjugacy classes, each given by the cycle decomposition of permutations. Therefore $\mathcal{C}$ has three simple objects (up to isomorphism). Since the sum of the squares of the dimensions of the simple objects is equal $|S_3| = 6$, $\mathcal{C}$ must have simple objects of dimensions one, one and two. Since $\mathcal{C}$ has one simple object of dimension greater than one (ans since it is fusion by Example II.2.2), it is a near-group category. Note that if $1, T$ and X are the simple objects of $\mathcal{C}$, with X the non-invertible simple, then $X \otimes X \simeq 1 \oplus T \oplus X$.

2. Let $\mathcal{C} = \mathrm{Rep}(A_4)$, where A_4 is the alternating group on four letters. A_4 has four conjugacy classes: the two conjugacy classes of S_4 represented by elements 1 and (12)(34) are transitively permuted by conjugation by A_4. The conjugacy class of S_4 represented by (123) breaks into two orbits under the action of A_4, the first represented by (123) and the second represented by (132). Since the only way to partition $|A_4| = 12$ into a sum of four square is $1+1+1+9$, $\mathcal{C}$ must have simple objects of dimensions one, one, one and three. Therefore $\mathcal{C}$ is a near group category. Again note that if $1, T_1, T_2$ and X are the simple objects of $\mathcal{C}$, then $X \otimes X = 1 \oplus T_1 \oplus T_2 \oplus 2X$.

III.1.1. Near-Group Fusion Rule

Let $\mathcal{C}$ be a near-group category with non-invertible object X. The set of invertible objects of $\mathcal{C}$, denoted $\mathcal{O}(\mathcal{C})$, forms a group where multiplication is given by the tensor product structure on the category. Therefore we can associate a finite group G to any near-group category. Let $g \in G$ represent an invertible object of $\mathcal{C}$. Since g is invertible and X is not invertible, $g \otimes X$ is a non-invertible simple object of $\mathcal{C}$, therefore $g \otimes X \simeq X$ for all $g \in G$. Similarly, X^* is a non-invertible simple object, and therefore $X^* \simeq X$. Therefore

$$\mathrm{Hom}(g, X \otimes X) \cong \mathrm{Hom}(1, g^* \otimes X \otimes X) \cong \mathrm{Hom}(1, X \otimes X) \neq 0.$$

Thus g appears as a summand of $X \otimes X$ for each $g \in G$. Since $\dim \operatorname{Hom}(1, X \otimes X) = 1$, g appears as a summand of $X \otimes X$ exactly once. Therefore we may decompose

$$X \otimes X \simeq \bigoplus_{g \in G} g \oplus kX$$

for some $k \in \mathbb{Z}_{\geq 0}$. We call the data (G, k) the <u>near-group fusion rule</u> of $\mathcal{C}$.

Example III.1.3.

1. $\operatorname{Rep}(S_3)$ is a braided near-group category with fusion rule $(\mathbb{Z}/2\mathbb{Z}, 1)$.
2. $\operatorname{Rep}(A_4)$ is a braided near-group category with fusion rule $(\mathbb{Z}/3\mathbb{Z}, 2)$.
3. Near-group categories with fusion rule $(G, 0)$ for some finite group G are known as Tambara-Yamagami categories. These categories are classified up to tensor equivalence in [TY]. When they come equipped with a braiding, they are classified up to braided tensor equivalence in [Si1].
4. The well-known Yang Lee (see [O1]) categories are precisely the near-group categories with fusion rule $(1, 1)$. Up to tensor equivalence, there are two such categories, each of these admitting two braidings.
5. Let $\mathcal{C}$ be the fusion category associated to the affine $\mathfrak{sl}_2$ on level 10 and let $A \in \mathcal{C}$ be the commutative $\mathcal{C}$-algebra of type E_6. The category $\operatorname{Rep}(A)$ of right A-modules contains a fusion subcategory (see [O2, Section 4.5]) which is a near-group category with fusion rule $(\mathbb{Z}/2\mathbb{Z}, 2)$.
6. The Izumi-Xu category $\mathcal{IX}$ (see [CMS, Appendix A.4]) is a near-group category with fusion rule $(\mathbb{Z}/3\mathbb{Z}, 3)$.

III.2. Near-Group Categories Are $\varphi-$Pseudounitary

The goal of this section is to prove that for any near-group category, $\mathcal{C}$, there is a field automorphism of $\varphi \in \operatorname{Gal}(\mathbb{A}/\mathbb{Q})$, such that $\mathcal{C}$ is φ-pseudounitary.

Now let $\mathcal{C}$ be a near-group category with non-invertible simple object X and fusion rule (G, k). Recall from Definition II.2.11, we may define the sphericalization $\tilde{\mathcal{C}}$ of $\mathcal{C}$. For $X_+, X_- \in \tilde{\mathcal{C}}$, let $d = \dim(X_+) = \mathrm{Tr}_{X_+}(i) = \mathrm{Tr}_X(\alpha)$, so that $\dim(X_-) = \mathrm{Tr}_{X_-}(i) = \mathrm{Tr}_X(-\alpha) = -d$. Similarly for $g \in G$, define $g_+ \in \tilde{\mathcal{C}}$ to be the simple object with $\dim(g_+) = 1$ and whose image under the forgetful functor $\mathcal{F} : \tilde{\mathcal{C}} \to \mathcal{C}$ is g.

III.2.1. Some Technical Lemmas

Lemma III.2.1. For $e \in G$ the identity, $e_- \otimes X_+ \simeq X_-$. Furthermore for $g, h \in G$, we have $g_+ \otimes X_\pm \simeq X_\pm$ and $g_+ \otimes h_+ \simeq (g \otimes h)_+$.

Proof. Applying the forgetful functor $\mathcal{F}(e_- \otimes X_+) \simeq \mathcal{F}(e_-) \otimes \mathcal{F}(X_+) \simeq X$, and $\dim(e_- \otimes X_+) = -\dim(X_+)$, so $e_- \otimes X_+ \simeq X_-$.

Similarly we have $\mathcal{F}(g_+ \otimes X_+) \simeq \mathcal{F}(g_+) \otimes \mathcal{F}(X_+) \simeq g \otimes X \simeq X$, and $\dim(g_+ \otimes X_+) = \dim(g_+)\dim(X_+) = 1 \cdot d = d$. Therefore $g_+ \otimes X_+ \simeq X_+$. An analogous proof shows $g_+ \otimes X_- \simeq X_-$.

Finally $\mathcal{F}(g_+ \otimes h_+) \simeq g \otimes h$, and $\dim(g_+ \otimes h_+) = 1$, so $g_+ \otimes h_+ \simeq (g \otimes h)_+$.

□

Lemma III.2.2. Let $\mathcal{C}$ be a near-group category with fusion rule (G, k) and non-invertible object X, and let $\tilde{\mathcal{C}}$ be the sphericalization of $\mathcal{C}$. We have $(X_\pm)^* \simeq X_\pm$, and furthermore

$$X_+ \otimes X_+ \simeq X_- \otimes X_- \simeq \bigoplus_{g \in G} g_+ \oplus sX_+ \oplus tX_-,$$

where $s + t = k$.

Proof. Clearly the forgetful functor maps $(X_+)^* \mapsto X^*$, therefore $(X_+)^* \simeq X_+$ or X_-. Since $\dim(X_-) = -\dim(X_+)$, we conclude that $(X_+)^* \simeq X_+$.

Since $(X_+)^* \simeq X_+$, we have for each $g \in G$,

$$\mathrm{Hom}(g_+, X_+ \otimes X_+) \cong \mathrm{Hom}(1, (g^{-1})_+ \otimes X_+ \otimes X_+) \cong \mathrm{Hom}(1, X_+ \otimes X_+) \neq 0.$$

Therefore g_+ appears as a summand of $X_+ \otimes X_+$ for each $g \in G$. By applying the forgetful functor, we see that g_+ appears as a summand at most once. This gives us

$$X_+ \otimes X_+ \simeq \bigoplus_{g \in G} g_+ \oplus sX_+ \oplus tX_-,$$

with no restriction on s, t. Again applying the forgetful functor gives

$$X \otimes X \simeq \bigoplus_{g \in G} g \oplus (s+t)X,$$

and the lemma is proved after noting

$$X_- \otimes X_- \simeq (e_- \otimes X_+) \otimes (e_- \otimes X_+) \simeq X_+ \otimes X_+.$$

□

After renaming of X_+, we may assume $s - t \geq 0$.

Lemma III.2.3. For $X_+ \in \tilde{C}$, we have $d = \dim(X_+) = \frac{r \pm \sqrt{r^2+4n}}{2}$ and $\dim(\mathcal{C}) = \frac{r^2+4n \pm r\sqrt{r^2+4n}}{2}$ where $n = |G|$ and $r = s - t \geq 0$.

Proof. We have

$$d^2 = \dim(X_+ \otimes X_+) = \dim\left(\bigoplus_{g \in G} \oplus sX_+ \oplus tX_-\right) = |G| + (s-t)d = n + rd.$$

And

$$\dim(\mathcal{C}) = |G| + d^2 = 2n + rd = \frac{r^2 + 4n \pm r\sqrt{r^2+4n}}{2}.$$

□

Lemma III.2.4. Recall $r = s - t$.

(a) If $\tilde{\mathcal{C}}$ is pseudo-unitary, then $r = k$,

(b) If $\sqrt{r^2 + 4n} \in \mathbb{Z}$, then $r = k$,

(c) If $\sqrt{k^2 + 4n} \in \mathbb{Z}$, then $r = k$.

Proof. (a) If $\tilde{\mathcal{C}}$ is pseudo-unitary, then $\dim(\tilde{\mathcal{C}}) = \mathrm{FPdim}(\tilde{\mathcal{C}}) = 2\,\mathrm{FPdim}(\mathcal{C})$. Therefore $r^2 + 4n + r\sqrt{r^2 + 4n} = k^2 + 4n + k\sqrt{k^2 + 4n}$, and since $0 \leq r = s - t \leq k$, we have $r = k$.

(b) If $\sqrt{r^2 + 4n} \in \mathbb{Z}$, then d is a rational algebraic integer, therefore $d \in \mathbb{Z}$. By [HR, Lemma A.1] $\tilde{\mathcal{C}}$ is pseudo-unitary, and $r = k$ by (a).

(c) If $\sqrt{k^2 + 4n} \in \mathbb{Z}$, then $\mathrm{FPdim}(X) \in \mathbb{Z}$. By [ENO, Proposition 8.24] $\tilde{\mathcal{C}}$ is pseudo-unitary and $r = k$ by (a).

□

We will also use the following well-known lemma about algebraic integers.

Lemma III.2.5. Let $a, b, c, d \in \mathbb{Z}$ such that $\sqrt{b}, \sqrt{d} \notin \mathbb{Z}$. Then $\frac{a+\sqrt{b}}{c+\sqrt{d}}$ is an algebraic integer if and only if $\frac{a-\sqrt{b}}{c-\sqrt{d}}$ is an algebraic integer.

Proof. Since $b \in \mathbb{Z}$ is not a square, we may write $b = m \cdot p_1^{\beta_1} \cdots p_k^{\beta_k}$ for some square $m \in \mathbb{Z}$ and primes $p_1, \ldots, p_k$ and odd integers $\beta_1, \ldots, \beta_k$. Similarly, we may write $d = n \cdot q_1^{\delta_1} \cdots q_l^{\delta_l}$ for some square n, primes $q_1, \ldots, q_l$ and odd integers $\delta_1, \ldots, \delta_l$. We will consider two cases.
Case (i): Up to ordering $p_1 = q_1$. Let $\sigma \in Gal(\mathbb{Q}(\sqrt{b}, \sqrt{d})/\mathbb{Q})$ be the element which maps $\sqrt{p_1}$ to $-\sqrt{p_1}$ and fixes $\sqrt{p_i}$ for $i \neq 1$ and $\sqrt{q_j}$ for $j \neq 1$. Then $\sigma(a + \sqrt{b}) = a - \sqrt{b}$, $\sigma(c + \sqrt{d}) = c - \sqrt{d}$ and $\sigma\left(\frac{a+\sqrt{b}}{c+\sqrt{d}}\right) = \frac{a-\sqrt{b}}{c-\sqrt{d}}$.
Case (ii): $p_i \neq q_j$ for all $1 \leq i \leq k$, $1 \leq j \leq l$. Then let $\sigma \in Gal(\mathbb{Q}(\sqrt{b}, \sqrt{d})/BQ)$ be the element which maps $\sqrt{p_1}$ to $-\sqrt{p_1}$, maps $\sqrt{q_1}$ to $-\sqrt{q_1}$ and fixes $\sqrt{p_i}$ for $i \neq 1$ and $\sqrt{q_j}$ for $j \neq 1$. Then as above $\sigma\left(\frac{a+\sqrt{b}}{c+\sqrt{d}}\right) = \frac{a-\sqrt{b}}{c-\sqrt{d}}$.

□

Theorem III.2.6. Any near-group category is spherical, moreover it is φ-pseudounitary for a suitable choice of φ.

Proof. Let $D = \dim(\tilde{\mathcal{C}}) = r^2+4n+r\sqrt{r^2+4n}$, and $\Delta = \mathrm{FPdim}(\tilde{\mathcal{C}}) = k^2+4n+k\sqrt{k^2+4n}$. Then by [ENO, Proposition 8.22]

$$\frac{D}{\Delta} = \frac{r^2+4n+r\sqrt{r^2+4n}}{k^2+4n+k\sqrt{k^2+4n}}$$

is an algebraic integer. Our goal is to prove that $r = k$, thus proving the theorem. When either $\sqrt{r^2+4n}$ or $\sqrt{k^2+4n}$ are integers, we know $r = k$ by Lemma III.2.4. Therefore assume $\sqrt{r^2+4n}, \sqrt{k^2+4n} \notin \mathbb{Z}$. Then

$$\frac{r^2+4n-r\sqrt{r^2+4n}}{k^2+4n-k\sqrt{k^2+4n}}$$

is an algebraic integer by Lemma III.2.5, and thus

$$\left(\frac{r^2+4n+r\sqrt{r^2+4n}}{k^2+4n+k\sqrt{k^2+4n}}\right)\left(\frac{r^2+4n-r\sqrt{r^2+4n}}{k^2+4n-k\sqrt{k^2+4n}}\right) = \frac{4n(r^2+4n)}{4n(k^2+4n)} = \frac{r^2+4n}{k^2+4n}$$

is an algebraic integer. Therefore $r^2 = k^2$, and $r = k$, since $r \geq 0$.

The full tensor category generated by simple objects $\{g_+\}_{g\in G} \cup \{X_+\}$ is tensor equivalent (by the forgetful functor) to $\mathcal{C}$. Therefore $\mathcal{C}$ is tensor equivalent to a full tensor subcategory of a spherical category and therefore spherical itself. Moreover if $d > 0$, then $\mathcal{C}$ is pseudo-unitary, and if $d < 0$, then $\mathcal{C}$ is φ-pseudounitary.

$\square$

III.2.2. Near-Group Categories with Integer Frobenius-Perron Dimension.

Proposition III.2.7. If a near-group category $\mathcal{C}$ with near-group fusion rule (G, k) has integer Frobenius-Perron dimension, then either $k = 0$ or $k = |G| - 1$. In the latter case $\mathrm{FPdim}(\mathcal{C}) = |G|(|G|+1)$.

Proof. $\mathrm{FPdim}(X) = \frac{1}{2}(k + \sqrt{k^2+4n})$. Therefore if $\mathrm{FPdim}(\mathcal{C}) \in \mathbb{Z}$, then $\mathrm{FPdim}(X)^2 = \frac{1}{2}(k^2 + 2n + k\sqrt{k^2+4n})$ is an integer, and $\sqrt{k^2+4n} \in \mathbb{Z}$. Therefore $k^2 + 4n = (k+l)^2$ for some $l \in \mathbb{Z}_{>0}$. Expanding, we get $4n = 2kl + l^2$. Therefore l is even and $l = 2p$ for $p \in \mathbb{Z}_{>0}$. Finally, $k+1 \le kp + p^2 \le n \le k+1$ by [Si2, Theorem 1.1] when $k \neq 0$.

Therefore $k = 0$ or $k = n-1 = |G|-1$. In the latter case, $\mathrm{FPdim}(X) = k+1 = |G|$, and $\mathrm{FPdim}(\mathcal{C}) = n + (k+1)^2 = n + n^2 = |G|(|G|+1)$. □

III.3. Müger Center of a Braided Near-Group Category

The goal of this section is to classify symmetric near-group categories. This is done in Proposition III.3.5. We will use mostly definitions and results from [DGNO].

III.3.1. The Müger Center of a Near-Group Category Contains All Invertible Objects.

Recall the following definitions for braided fusion categories.

Definition III.3.1. [DGNO, Section 2.2; Section 3.3] Let $\mathcal{C}$ be a fusion category:

(a) Define $\mathcal{C}_{\mathrm{ad}}$ to be the fusion subcategory generated by $Y \otimes Y^*$ for $Y \in \mathcal{O}(\mathcal{C})$.

(b) For $\mathcal{K}$ a fusion subcategory of $\mathcal{C}$, we define the *commutator of* $\mathcal{K}$ to be the fusion subcategory $\mathcal{K}^{\mathrm{co}} \subseteq \mathcal{C}$, generated by all simple objects $Y \in \mathcal{O}(\mathcal{C})$, where $Y \otimes Y^* \in \mathcal{O}(\mathcal{K})$.

Let $\mathcal{C}$ be a near-group category with near-group fusion rule (G,k). Recall that for this proposition we assume $k \neq 0$.

Lemma III.3.2. $\mathcal{C}_{\mathrm{ad}} = \mathcal{C}$.

Proof. This is clear as $X \simeq X^*$ and $X \otimes X \simeq G \oplus kX$, thus contains all simple objects of $\mathcal{C}$ as summands. □

Lemma III.3.3. [DGNO, Proposition 3.25] Let $\mathcal{K}$ be a fusion subcategory of a braided fusion category $\mathcal{C}$. Then $(\mathcal{K}_{ad})' = (\mathcal{K}')^{co}$.

Letting $\mathcal{K} = \mathcal{C}$ in Lemma III.3.3, we get

Proposition III.3.4. Let $\mathcal{C}$ be a braided near-group category with fusion rule (G, k). If $k > 0$, then the Müger center $\mathcal{C}' \subseteq \mathcal{C}$ is either $\mathcal{C}$ or Vec_G.

Proof. $\mathcal{C}' = (\mathcal{C}_{ad})' = (\mathcal{C}')^{co} \supset G$. □

In particular, $\mathcal{C}$ can only be modular if G is trivial.

III.3.2. Symmetric Tensor Categories

Let A be a group. Deligne [D] defines $\mathrm{Rep}(A, z)$ to be the category of finite dimensional super representations (V, ρ) of A, where $\rho(z)$ is the automorphism of parity of V. In [DGNO] this is presented as the fusion category $\mathrm{Rep}(G)$ with $z \in Z(G)$ satisfying $z^2 = 1$ and braiding σ' given by

$$\sigma'_{UV}(u \otimes v) = (-1)^{ij} v \otimes u \text{ if } u \in U, v \in V, zu = (-1)^i u, zv = (-1)^j v.$$

In [D, Corollaire 0.8] it is shown that any symmetric fusion category is equivalent to $\mathrm{Rep}(A, z)$ for some choice of finite group A, and central element $z \in A$ with $z^2 = 1$. If $z \neq 1$ we call such a category super-Tannakian. If $z = 1$, then $\mathrm{Rep}(A, z) = \mathrm{Rep}(A)$ and it is called Tannakian. Note that $\mathrm{Rep}(A/\langle z\rangle)$ is the subcategory of modules M where z acts trivially on M. This is a maximal Tannakian subcategory of $\mathrm{Rep}(A, z)$.

Proposition III.3.5. Let $\mathcal{C}$ be a symmetric near-group category with near-group fusion rule (G, k) and $k \neq 0$. Then $\mathcal{C}$ is braided tensor equivalent to $\mathrm{Rep}(\mathbb{F}_{p^l} \rtimes \mathbb{F}_{p^l}^{\times})$, for some $p^l \neq 2$.

Proof. Let $\mathcal{C}$ be a symmetric near-group category with near-group fusion rule (G, k). by [D, Corollaire 0.8] $\mathcal{C}$ is equivalent (as a tensor category) to $\mathrm{Rep}(H)$ for some finite group H. Since $\mathcal{C}$ is a near-group category, H has exactly one irreducible representation of dimension greater than one. The following lemma classifies such groups.

Lemma III.3.6. [Se] A group G has exactly one irreducible $\mathcal{C}$-representation of degree greater than one if and only if (i) $|G| = 2^k$, k is odd, $[G, G] = Z(G)$, and $|[G, G]| = 2$, or

(ii) G is isomorphic to the group of all transformations $x \mapsto ax + b$, $a \neq 0$, on a field of order $p^n \neq 2$.

By [D, Corollaire 0.8] and Lemma III.3.6, $\mathcal{C}$ is tensor equivalent to $\mathrm{Rep}(H)$ where $|H| = 2^l$, or H is isomorphic to the group of all transformations $x \mapsto ax + b$, $a \neq 0$, on a field of order $p^l \neq 2$.

If $|G| = 2^l$, then by Lemma III.2.7, $\mathrm{Rep}(G)$ is Tambara-Yamagami if it is near-group. Therefore we may assume that H is the latter group described above. Such a group H is isomorphic to $\mathbb{F}_{p^l} \rtimes \mathbb{F}_{p^l}^{\times}$ since there is a split short exact sequence

$$1 \to \mathbb{F}_{p^l} \to H \to \mathbb{F}_{p^l}^{\times} \to 1.$$

Therefore $\mathcal{C}$ is tensor equivalent to $\mathrm{Rep}(\mathbb{F}_{p^l} \rtimes \mathbb{F}_{p^l}^{\times})$. Since $Z(H) = 1$, there does not exist a braiding on $\mathrm{Rep}(H)$ making it a super-Tannakian category. Therefore $\mathcal{C}$ is braided tensor equivalent to $\mathrm{Rep}(\mathbb{F}_{p^l} \rtimes \mathbb{F}_{p^l}^{\times})$. □

Example III.3.7. It was seen in Example III.1.3 that $\mathrm{Rep}(S_3)$ and $\mathrm{Rep}(A_4)$ are near-group categories. It is not hard to see that $S_3 \cong \mathbb{F}_3 \rtimes \mathbb{F}_3^{\times}$ and $A_4 \cong \mathbb{F}_4 \rtimes \mathbb{F}_4^{\times}$.

III.3.3. Tannakian Centers of Braided Near-Group Categories

Recall (see [DGNO, Example 2.42]) sVec is defined to be the category $\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z}, z)$, where z is the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$. The following lemma is due to [Mu2, Lemma 5.4] and [DGNO, Lemma 3.28]. This lemma will be used to show that particular categories do not exist.

Lemma III.3.8. Let $\mathcal{C}$ be a braided fusion category and $\delta \in \mathcal{C}'$ an invertible object such that the fusion subcategory of $\mathcal{C}$ generated by δ is braided equivalent to $s\,\mathrm{Vec}$. Then for all $V \in \mathcal{O}(\mathcal{C})$, $\delta \otimes V$ cannot be mapped to V by some tensor automorphism.

Proof. For $V \in \mathcal{O}(\mathcal{C})$, let μ_V be defined to be the compostion

$$V \xrightarrow{\mathrm{id}_V \otimes \mathrm{coeval}_{V^*}} V \otimes V^* \otimes V^{**} \xrightarrow{\sigma_{V,V^*} \otimes \mathrm{id}_{V^{**}}} V^* \otimes V \otimes V^{**} \xrightarrow{\mathrm{eval}_V \otimes_{V^{**}}} V^{**}$$

where σ is the braiding on $\mathcal{C}$. It is well known (see [BK, Lemma 2.2.2]) that for $V, W \in \mathcal{O}(\mathcal{C})$, μ_V and μ_W satisfy

$$\mu_V \otimes \mu_W = \mu_{V \otimes W} \sigma_{W,V} \sigma_{V,W}.$$

Therefore since, $\delta \in \mathcal{C}'$, we have

$$\mu_{\delta \otimes V} = \mu_\delta \otimes \mu_V,$$

for all $V \in \mathcal{O}(\mathcal{C})$. Since δ generates $s\,\mathrm{Vec}$, we know that $\sigma'(\delta, \delta) = -\mathrm{id}_1$ and $\mu_\delta = -\mathrm{id}_\delta$. Recall [ENO] for a simple object $U \in \mathcal{O}(\mathcal{C})$, we define $d_+(U)$ to be the composition

$$1 \xrightarrow{\mathrm{coeval}_V} V \otimes V^* \xrightarrow{\mu \otimes \mathrm{id}_{V^*}} X^{**} \otimes V^* \xrightarrow{\mathrm{eval}_{V^*}} 1.$$

Then

$$\begin{aligned}
d_+(\delta \otimes V) &= \mathrm{eval}_{(\delta \otimes V)^*} \circ (\mu_{\delta \otimes V} \otimes \mathrm{id}_1) \circ \mathrm{coeval}_{\delta \otimes V} \\
&= \mathrm{eval}_{\delta^*} \circ (-\mathrm{id}_\delta \otimes \mathrm{id}_1) \circ \mathrm{coeval}_\delta \cdot \mathrm{eval}_{V^*} \circ (\mu_V \otimes \mathrm{id}_1)\, \mathrm{coeval}_V \\
&= -d_+(V).
\end{aligned}$$

Since $d_+(V) \neq 0$, we have $d_+(V) \neq d_+(\delta \otimes V)$, and V cannot be mapped to $\delta \otimes V$ by some automorphism.

$\square$

Let $\mathcal{C}$ be a braided near-group category with near-group fusion rule (G, k). Assume that $\mathcal{C}$ is not symmetric, so $\mathcal{C}' = \mathrm{Vec}_G$. Therefore $\mathcal{C}' = \mathrm{Rep}(A, z)$ for some choice of finite group A and $z \in A$. For the remainder of this section, we will assume that $z \neq 1$ and derive a contradiction.

Recall $H := \mathrm{Rep}(A/\langle z\rangle) \subseteq \mathcal{C}'$ is a maximal Tannakian subcategory of $\mathrm{Rep}(A, z)$. By Proposition II.2.23, there exists a braided fusion category $\mathcal{D}$ and an action of H on $\mathcal{D}$ so that $\mathcal{D}^H = \mathcal{C}$ and $\mathrm{Vec}^H = \mathrm{Rep}(H)$. Since $\mathcal{C}'$ is a braided tensor category over $\mathrm{Rep}(H)$, there also exists a category $\mathcal{D}_1 \subset \mathcal{D}$ such that $\mathcal{D}_1^H = \mathcal{C}'$. By [DGNO, Proposition 4.26], we have $\mathrm{FPdim}(\mathcal{D}_1) = \mathrm{FPdim}(\mathcal{C}')/|H| = 2$. Let $\mathcal{O}(\mathcal{D}_1) = \{1, Z\}$, and $\mathcal{O}_1, \ldots, \mathcal{O}_m$ denote the orbits of the simple objects of $\mathcal{D}$ under the action of H, where $\mathcal{O}_1 = \{1\}$ and $\mathcal{O}_2 = \{Z\}$. Since $\mathcal{D}_1^H = \mathcal{C}'$ and there is only one simple object of $\mathcal{C}$ not contained in $\mathcal{C}'$, there are only three orbits. The following proposition shows that no such category can exist.

Proposition III.3.9. There are no super-Tannakian categories $\mathcal{T}$ with the following structure:
(i) $\mathcal{O}(\mathcal{T}) = \{1, \delta, T_1, \ldots, T_q\}$ where the fusion subcategory of $\mathcal{T}$ generated by δ is braided equivalent to $s\,\mathrm{Vec}$,
(ii) An action of a group A on $\mathcal{T}$ transitively permuting $\{T_1, \ldots, T_q\}$.

Proof. Since δ is invertible, $\delta \otimes T_1 \simeq T_s$ for some $1 \leq s \leq q$. By Lemma III.3.8 there is no automorphism mapping $\delta \otimes T_1$ to T_s. This contradicts the assumption that A acts transitively on $\{T_1, \ldots, T_q\}$. $\square$

A result of Proposition III.3.5 is that a symmetric near-group category is Tannakian. Combined with what we just proved, we have the following proposition.

Proposition III.3.10. If $\mathcal{C}$ is a braided, near-group category and $k \neq 0$, then the Müger center $\mathcal{C}' = \mathrm{Rep}(H)$ for some abelian group H.

III.4. Classification of Non-Symmetric Braided Near-Group Categories

The goal of this section is to show there are 7 non-symmetric, braided, near-group categories (up to braided tensor equivalence) which are not Tambara-Yamagami. Again, we only care about the case when $k \neq 0$, as J. Siehler already did the classification when

$k = 0$ [Si1].

In the previous section, we proved $\mathcal{C}' = \mathrm{Rep}(H)$. By Theorem II.2.23, there exists a braided fusion category $\mathcal{D}$ and an action of H on $\mathcal{D}$ so that $\mathcal{D}^H = \mathcal{C}$ and $\mathrm{Vec}^H = G$. Let $\mathcal{O}_1, \ldots, \mathcal{O}_m$ denote the orbits of the simple objects of $\mathcal{D}$ under the action of H, where $\mathcal{O}_1 = \{1\}$. Since $\mathrm{Vec}^H = G$, we have $m = 2$, otherwise $G \cup \{X\} \subsetneq \mathcal{O}(\mathcal{C})$. For the remainder of this section, let $\mathcal{O}_2 = \{D_1, \ldots, D_s\}$.

Lemma III.4.1. If $s > 1$, then $\dim(D_j) = 1$ for $1 \leq j \leq s$.

Proof. If $s \geq 2$, then there exists $1 \leq i \leq s$, such that $D_i^* \not\cong D_1$. Therefore $D_1 \otimes D_i = \bigoplus_{j=1}^{s} a_j D_j$, and $d = \dim(D_j)$ satisfies the identity $d^2 = d \sum_{j=1}^{s} a_j$. This gives $d \in \mathbb{Z}$, so by the proof of [ENO, Proposition 8.22], d divides $\mathrm{FPdim}(D)$. Therefore, since $\mathrm{FPdim}(D) = 1 + sd^2$, we have $d = 1$. □

Lemma III.4.2. If $\mathcal{O}_2 = \{D_1, \ldots, D_s\}$, then s is either 1 or n.

Proof. By Lemma III.4.1, if $s \geq 2$, then $\mathrm{FPdim}(\mathcal{D}) = 1 + s$. Therefore $\mathrm{FPdim}(\mathcal{C}) = n(1+s) \in \mathbb{Z}$ and $1 + s = \mathrm{FPdim}(\mathcal{D}) = \frac{1}{n}\mathrm{FPdim}(\mathcal{C}) = n + 1$, since $\mathrm{FPdim}(\mathcal{C}) = n(n+1)$ by Proposition III.2.7. □

Proposition III.4.3. If $\mathcal{O}(\mathcal{D}) = \{1, D_1\}$, then either $\mathcal{C} = \mathcal{D}^H$ is a Yang-Lee category or $\mathcal{C}$ is Tambara-Yamagami. Moreover there are (up to braided equivalence) four braided near-group categories $\mathcal{C}$ which are not Tambara-Yamagami and $\mathcal{C}_H$ is of rank two.

Proof. Let $D = D_1$. Assume $D \otimes D = 1$. Let X be the non-invertible object of $\mathcal{C}$. Therefore X is an equivariant object under the action of H on $\mathcal{C}$. Therefore $X = mD$ for some integer m. Therefore $X \otimes X = m^2 1$ in $\mathcal{C}$ and must therefore lie in $\mathrm{Rep}(H)$ in $\mathcal{C}$. In this case $\mathcal{C}$ is Tambara-Yamagami.

Now assume $D \otimes D = 1 \oplus D$. Therefore H acts on $\mathcal{D}$ trivially. This gives $\mathcal{C} = \mathcal{D} \boxtimes \mathrm{Rep}(H)$, which is only a near-group category when H is trivial and $\mathcal{C} = \mathcal{D}$.

The last part of the proposition is simply a note that there are are four Yang-Lee categories up to braided equivalence [O1]. $\square$

Since we just classified the case when $\mathcal{D}$ is of rank two, we will assume for the remainder of this section that $s > 1$, and therefore by Lemma III.4.1, $\mathcal{D}$ is a pointed braided category which is non-degenerate by [DGNO, Corollary 4.30] since $\mathcal{D}^H = \mathcal{C}$, where $\mathcal{C}' = \mathrm{Rep}(H)$. It is shown (see [DGNO] or [JS]) that a non-degenerate pointed braided category is classified by an abelian group A and a non-degenerate quadratic form $q : A \to \mathbb{C}^\times$ on A. Note that A is the group of isomorphism classes of simple objects. They denote such a category by $\mathcal{C}(A, q)$. Recall that the data (A, q) for a finite abelian group A and a non-degenerate quadratic form $q : A \to \mathbb{C}^\times$ is called a metric group.

Proposition III.4.4. If $\mathcal{D}$ is of rank at least three, then A is either $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$. Moreover if:

(i) $A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, then $q(a) = -1$ for every non-trivial element of A,

(ii) $A = \mathbb{Z}/3\mathbb{Z}$, then $q(a) = q(b)$ are primitive third roots of unity for both non-trivial elements a, b of A.

Proof. $\mathcal{O}(\mathcal{D}) = \{1, D_1, \ldots, D_p\}$ where H acts transitively on $\{D_1, \ldots, D_s\}$ by braided-tensor functors. Let $A = \{e, d_1, \ldots, d_s\}$, then $o(d_i) = o(d_j)$ for $1 \leq i, j \leq s$. Therefore A is an elementary abelian group. Let $p = o(d_1)$. The action of H on $\mathcal{D}$ gives rise to an action of H on the metric group (A, q) by morphisms $\{\varphi_h\}_{h \in H}$ of metric groups. Since H acts transitively on $\mathcal{O}_2$, we have for any $1 \leq i, j \leq s$ we have $h \in H$ so that $\varphi_h(d_i) = d_j$. Since φ_h is a morphism of metric groups, we have $q(d_j) = q(\varphi_h(d_i)) = q(d_i)$. Therefore it makes sense to define $\omega = q(d_i)$. By [DGNO, Remark 2.37 (i)], we have $1 = q(e) = q(d_1^p) = \omega^{p^2}$, therefore ω is a root of unity.

[DGNO, Corollary 6.3] states that for (A, q) a metric group, we have

$$\Big|\sum_{a\in A} q(a)\Big|^2 = |A|.$$

Therefore if $|A| = m$, we have

$$m = |1 + (m-1)\omega|^2 \geq (|(m-1)\omega| - 1)^2 = (m-2)^2 = m^2 - 4m + 4.$$

This gives $(m-1)(m-4) \leq 0$, so $m = 2, 3$ or 4 and we assume $m \geq 3$. Since A is an elementary abelian group, we know that $m = 4$ implies $A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Assume $A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then $|1 + 3\omega|^2 = 4$, giving $\omega = -1$.

Finally, for $A = \mathbb{Z}/3\mathbb{Z}$, we have $\omega = q(d_1) = q(d_1^2) = \omega^4$, so ω is a third root of unity. This gives $|1 + 2\omega|^2 = 3$, and ω is a primitive third root of unity. □

Proposition III.4.5. Let $\mathcal{C}$ be a non-symmetric braided near group category with fusion rule (G, k) where $k \neq 0$. There are two such categories (up to braided tensor equivalence) when $G = \mathbb{Z}/2\mathbb{Z}$ and one such category (up to braided tensor equivalence) when $G = \mathbb{Z}/3\mathbb{Z}$.

Proof. Assume $G = \mathbb{Z}/2\mathbb{Z}$. We have shown above that $\mathcal{C} = \mathcal{C}(\mathbb{Z}/3\mathbb{Z}, q)^H$, where $\mathrm{Vec}_{\mathbb{Z}/2\mathbb{Z}} = \mathrm{Rep}(H)$ (therefore $H = \mathbb{Z}/2\mathbb{Z}$) and $q : \mathbb{Z}/3\mathbb{Z} \to \mathbb{C}^\times$ is defined by $q(a) = q(b)$ is a primitive third root of unity for non-trivial elements $a, b \in \mathbb{Z}/3\mathbb{Z}$. For each of the two choices of q, we have one non-trivial action of H on $\mathcal{C}(\mathbb{Z}/3\mathbb{Z}, q)$. Therefore there are two non-symmetric near-group categories with fusion rule $(\mathbb{Z}/2\mathbb{Z}, 1)$.

Assume $G = \mathbb{Z}/3\mathbb{Z}$. Then we showed that $\mathcal{C} = \mathcal{C}(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, q)^H$, where $\mathrm{Vec}_{\mathbb{Z}/3\mathbb{Z}} = \mathrm{Rep}(H)$ (therefore $H = \mathbb{Z}/3\mathbb{Z}$) and $q : \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \mathbb{C}^\times$ is defined by $q(a) = q(b) = -1$ for the generators of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Again, we only have one non-trivial action of $\mathbb{Z}/3\mathbb{Z}$ on

$\mathcal{C}(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, q)$. Therefore we get one non-symmetric near-group category with fusion rule $(\mathbb{Z}/3\mathbb{Z}, 2)$. $\square$

We have proved the following theorem.

Theorem III.4.6. Let $\mathcal{C}$ be a non-symmetric, braided, near-group category with fusion rule (G, k) where $k \neq 0$, then G is either the trivial group, $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$. Furthermore if G is trivial, then there are are four associated braided near group categories (up to braided tensor equivalence). All of these categories have fusion rule $(1, 1)$. If $G = \mathbb{Z}/2\mathbb{Z}$, then there are another two associated near-group categories, both with near-group fusion rule $(\mathbb{Z}/2\mathbb{Z}, 1)$. And finally, if $G = \mathbb{Z}/3\mathbb{Z}$, then there is one associated category with near-group fusion rule $(\mathbb{Z}/3\mathbb{Z}, 2)$.

CHAPTER IV

GENERALIZED NEAR-GROUP CATEGORIES

In Chapter III, we studied fusion categories with one non-invertible simple object. In this chapter we will study generalized near-group categories, which are fusion categories whose invertible objects act transitively on the non-invertible simple objects by tensor product. Similar to near-group categories, we will show in Theorem IV.3.6 that such categories have a $\varphi-$pseudounitary structure. We then focus our attention on braided categories. In Theorem IV.4.7, we show that every symmetric generalized near-group category is tensor equivalent to the category of representations of a finite metabelian group. Furthermore in Theorem IV.4.9, we give some strong structure to the metabelian groups which arise in symmetric generalized near-group categories. Finally we classify all modular generalized near-group categories in Theorem IV.5.2.

IV.1. Definition and Examples

Definition IV.1.1. A generalized near-group category is a fusion category such that the inverible objects transitively permute the non-invertible objects under the action of left tensor multiplication. A generalized near-group category is called braided if it comes equipped with a braiding.

Let $\mathcal{C}$ and $\mathcal{D}$ be fusion categories with simple objects $\mathcal{O}(\mathcal{C}) = \{X_1, \ldots, X_n\}$ and $\mathcal{O}(\mathcal{D}) = \{Y_1, \ldots, Y_n\}$. Let $\mathcal{C} \boxtimes \mathcal{D}$ be the fusion category with simple objects $X_i \boxtimes Y_j$ and morphisms $\mathrm{Hom}(X_i \boxtimes Y_j, X_s \boxtimes Y_t) := \mathrm{Hom}(X_i, X_s) \otimes \mathrm{Hom}(Y_j, Y_t)$.

Example IV.1.2. The following are examples of generalized near-group categories.

1. Let $\mathcal{C} = \mathrm{Rep}(S_3 \times \mathbb{Z}/2\mathbb{Z})$. Let $\rho : S_3 \to \mathbb{C}$ and $\sigma : \mathbb{Z}/2\mathbb{Z} \to \mathbb{C}^\times$ be irreducible characters for S_3 and $\mathbb{Z}/2\mathbb{Z}$ respectively. We claim that $\chi : S_3 \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{C}$ defined by $\chi(a,b) = \rho(a)\sigma(b)$ is an irreducible character of $S_3 \times \mathbb{Z}/2\mathbb{Z}$. Indeed

$$(\chi,\chi)_{\mathcal{C}} = \frac{1}{12}\sum_{\substack{a\in S_3\\ b\in\mathbb{Z}/2\mathbb{Z}}} \rho(a)\sigma(b)\overline{\rho(a)\sigma(b)} = \left(\frac{1}{6}\sum_{a\in S_3}\rho(a)\overline{\rho(a)}\right)\left(\frac{1}{2}\sum_{b\in\mathbb{Z}/2\mathbb{Z}}\sigma(b)\overline{\sigma(b)}\right)$$
$$= (\rho,\rho)_{\mathrm{Rep}(S_3)}\cdot(\sigma,\sigma)_{\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})} = 1.$$

   From this, it is clear that $\mathcal{C}$ has two non-invertible simple objects and they are transitively permuted by the representation of $S_3 \times \mathbb{Z}/2\mathbb{Z}$ associated to the non-trivial representation of $\mathbb{Z}/2\mathbb{Z}$. As with all categories of representations of finite groups, $\mathcal{C}$ comes equipped with a braiding.

2. More generally, let $A = \mathbb{F}_q \rtimes \mathbb{F}_q^\times$, where $q \neq 2$ and B be a cyclic group of finite order. Then an argument similar to above shows that $\mathrm{Rep}(A \times B)$ is a generalized near-group category. Note that we use the near-group structure of $\mathrm{Rep}(A)$.

3. Even more generally, we may let $\mathcal{C}$ be a near-group category and $\mathcal{D}$ be a pointed category. Then the category $\mathcal{C}\boxtimes\mathcal{D}$ is a generalized near-group category. If X is the non-invertible simple object of $\mathcal{C}$, then any non-invertible simple object of $\mathcal{C}\boxtimes\mathcal{D}$ is of the form $X\boxtimes Y$ for some invertible object $Y \in \mathcal{O}(\mathcal{D})$. It is clear that the invertible objects act transitively on the non-invertible simples since $X\boxtimes Y \simeq (X\boxtimes 1)\otimes(1\boxtimes Y)$.

A more interesting family of examples can be constructed in the following way.

Example IV.1.3. Let $G = \mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/2n\mathbb{Z} = \langle a, b | a^3 = b^{2n} = 1, ba = a^2 b\rangle$, where n is a positive integer. We will show that $\mathcal{C} = \mathrm{Rep}(G)$ is a generalized near-group category.

Let $a^ib^j, a^sb^t \in G$. Then

$$[a^ib^j, a^sb^t] = a^ib^ja^sb^tb^{-j}a^{-i}b^{-t}a^{-s} \in \{1, a, a^2\},$$

since commuting a power of a passed a power of b does not change the power of b. Therefore $G' = [G, G] \leq \{1, a, a^2\}$. Equality holds, since $aba^{-1}b^{-1} = aabb^{-1} = a^2$ and $a^{-1}bab^{-1} = a^2a^2bb^{-1} = a$. Therefore $G' = \{1, a, a^2\}$. This given the number of one-dimensional representation of $\mathcal{C}$ is $[G : G'] = \frac{6n}{3} = 2n$.

It is easy to see that the conjugacy classes of G are:

$$\underbrace{\{1\}, \{b^2\}, \{b^4\}, \ldots, \{b^{2n-2}\}}_{n}, \underbrace{\{a, a^2\}, \{ab^2, a^2b^2\}, \{ab^4, a^2b^4\}, \ldots \{ab^{2n-2}, a^2b^{2n-2}\}}_{n}$$
$$\underbrace{\{b, ab, a^2b\}, \{b^3, ab^3, a^2b^3\}, \ldots, \{b^{2n-1}, ab^{2n-1}, a^2b^{2n-1}\}}_{n}$$

Therefore G has $3n$ conjugacy classes, $3n$ irreducible representation and n non-invertible irreducible representations. Let $d_1, \ldots, d_n$ be the dimensions of the non-invertible simple representations. Then

$$6n = 2n + 4n \leq 2n + \sum_{i=1}^{n} d_i^2 = |G| = 6n.$$

Therefore all non-invertible irreducible representations of G are of dimension two. Let $\hat{\rho} : G' \to \mathbb{C}^\times$ be the irreducible character of G' given by $\hat{\rho}(a) = \omega$, where ω is a primitive third root of unity. Let $\rho := \hat{\rho} \upharpoonright^G$, be the induced representation of $\hat{\rho}$ to G. By [Ro, Theorem 8.142] we have

$$\rho(g) = \frac{1}{3} \sum_{x \in G} \dot{\hat{\rho}}(x^{-1}gx) = \begin{cases} 2n & \text{if } g = e, \\ n(\omega + \omega^2) = -n & \text{if } g \in \{a, a^2\}, \\ 0 & \text{if } g \notin G'. \end{cases}$$

Let $\chi_1, \dots \chi_{2n}$ be the invertible representations of G and $\rho_1, \dots, \rho_n$ be the irreducible, non-invertible representations of G. Then

$$(\rho, \chi_i) = \frac{1}{6n} \sum_{g \in G} \rho(g) \overline{\chi_i(g)} = \frac{1}{6n}(2n \cdot 1 - 2n \cdot \chi_i(a)) = \frac{1 - \chi_i(a)}{3} \in \mathbb{Z}_{\geq 0}.$$

Since $\chi_i(a)$ is a root of unity, we have $\chi_i(a) = 1$. We also have

$$(\rho, \rho_i) = \frac{1}{6n} \sum_{g \in G} \rho(g) \overline{\rho_i(g)} = \frac{1}{6n}(2n \cdot 2 - 2n \cdot \rho_i(a)) = \frac{2 - \rho_i(a)}{3} \in \mathbb{Z}_{\geq 0}.$$

Therefore $\rho_i(a) = 2 - 3d$, where d is a non-negative integer. Since $\rho_i(a)$ is a sum of two roots of unity, we have $|\rho_i(a)| = |2 - 3d| \leq 2$. Therefore $d = 1$. and $\rho_i(a) = -1$.

Now consider the four dimensional representation $\rho_1 \otimes \rho_1^*$. We can either decompose $\rho_1 \otimes \rho_1^* \simeq 1 \oplus \chi_i \oplus \chi_j \oplus \chi_k$ or $\rho_1 \otimes \rho_1^* \simeq 1 + \chi_s \oplus \rho_t$. We must have the latter, since $(\rho_1 \otimes \rho_1^*)(a) = (-1)^2 = 1$ and $(1 \oplus \chi_i \oplus \chi_j \oplus \chi_k)(a) = 1 + 1 + 1 + 1 = 4$.
Now let $H = \{\chi_1, \dots, \chi_{2n}\}$ act on $\{\rho_1, \dots, \rho_n\}$ by tensoring on the left. Let $H_1 = \{\chi_i \in H; \chi_i \otimes \rho_1 \simeq \rho_1\}$ be the stabalizer of ρ_1. If $\chi_i \in H_1$, then

$$\mathrm{Hom}(\chi_i, \rho_1 \otimes \rho_1^*) \simeq \mathrm{Hom}(\chi_i, \chi_i \otimes \rho_1 \otimes \rho_1^*) = \mathrm{Hom}(1, \rho_1 \otimes \rho_1^*) \neq 0.$$

Therefore $|H_1| \leq 2$, and the orbit of ρ_1 has order n since, $n \geq |\mathcal{O}(\rho_1)| = [H, H_1] \geq n$. In particular, the invertible objects act transitively on the irreducible, non-invertible objects by tensoring on the left. Thus $\mathrm{Rep}(G)$ is a generalized near-group category.

IV.2. Fusion Rule

In this section, we discuss how objects in a generalized near-group category are tensored together. Our goal is to show that the tensor product is more complicated to that in near-group categories, but not much.

Lemma IV.2.1. Let $\mathcal{C}$ be a generalized near-group category. The group of invertible objects G acts transitively on the set of non-invertible objects by right multiplication.

Proof. Let X be any simple non-invertible oject of $\mathcal{C}$. Let $G_X^L = \{g \in G : g \otimes X \simeq X\}$ and $G_{X^*}^R = \{g \in G : X^* \otimes g \simeq X^*\}$. Then $G_X^L = G_{X^*}^R$, since for every $g \in G$,

$$\mathrm{Hom}(g \otimes X, X) \simeq \mathrm{Hom}(g, X \otimes X^*) \simeq \mathrm{Hom}(X^* \otimes g, X^*).$$

Let $\mathcal{O}_X^L$ be the orbit of X under the action of left multiplication by G, and $\mathcal{O}_{X^*}^R$ be the orbit of X^* under the action of right multiplication by G. Then

$$|\mathcal{O}_{X^*}^R| = |G| \cdot |G_{X^*}^R| = |G| \cdot |G_X^L| = |\mathcal{O}_X^L|.$$

By definition, $\mathcal{O}_X^L$ is all non-invertible simple objects of $\mathcal{C}$, therefore $\mathcal{O}_{X^*}^R$ is as well, and G acts transitively on the non-invertible objects of $\mathcal{C}$ by right multiplication. $\square$

Proposition IV.2.2. Let $\mathcal{C}$ be a generalized near-group category and let X_i be a non-invertible simple object in $\mathcal{C}$. Then

$$X_i \otimes X_i^* = \bigoplus_{h \in H} h \oplus k_1 X_1 \oplus \cdots \oplus k_m X_m,$$

where $H = \{g \in G; g \otimes X_1 = X_1\}$ and $k_1, \ldots, k_m$ are non-negative integers.

Proof. Since $\mathcal{C}$ is semisimple, we know that there exists non-negative integers l_g for $g \in G$ and $k_1, \ldots, k_m$ so that

$$X_1 \otimes X_1^* = \bigoplus_{g \in G} l_g \cdot g \oplus k_1 X_1 \oplus \cdots \oplus k_m X_m. \tag{IV.1}$$

By Lemma IV.2.1, for every $1 \leq i \leq m$, there exists an invertible object $g_i \in G$, such that

$X_1 \otimes g_i = X_i$. Therefore

$$X_i \otimes X_i^* = (X_1 \otimes g_i) \otimes (g_i^* \otimes X_1^*) = X_1 \otimes X_1 = \bigoplus_{g \in G} l_g \cdot g \oplus k_1 X_1 \oplus \cdots \oplus k_m X_m.$$

Now let $g \in G$ be an invertible object of $\mathcal{C}$. Then

$$l_g = \dim \operatorname{Hom}(X_1 \otimes X_1^*, g) = \dim \operatorname{Hom}(X_1, g \otimes X_1) \leq 1,$$

where equality holds if and only if $g \in H$. The proposition is proved. $\square$

Proposition IV.2.3. Let $\mathcal{C}$ be a generalized near-group category. Then $\mathcal{C}_{ad}$ is either a pointed category or a generalized near-group category.

Proof. Let $\mathcal{C}$ be a generalized near group category. Assume $\mathcal{C}_{\mathrm{ad}}$ is not pointed, that is it contains a non-invertible simple object. Let $X_i, X_j \in \mathcal{O}(\mathcal{C}_{\mathrm{ad}})$ be simple objects in $\mathcal{C}_{\mathrm{ad}}$. Since $\mathcal{C}$ is a generalized near-group category, there exists $g \in G$ such that $X_i \simeq g \otimes X_j$. Therefore

$$\operatorname{Hom}(g, X_i \otimes X_j^*) \cong \operatorname{Hom}(g, g \otimes X_j \otimes X_j^*) \cong \operatorname{Hom}(1, X_j \otimes X_j^*) \neq 0,$$

and $g \in \mathcal{C}_{\mathrm{ad}}$. Therefore the non-invertible objects of $\mathcal{C}_{\mathrm{ad}}$ are transitively permuted by the invertible objects and $\mathcal{C}_{\mathrm{ad}}$ is a generalized near-group category. $\square$

Definition IV.2.4. We say that a generalized near-group category $\mathcal{C}$ is <u>simple</u> if $\mathcal{C}_{\mathrm{ad}} = \mathcal{C}$.

The next result follows from Proposition IV.2.3.

Corollary IV.2.5. Every generalized near-group category is a graded extension of either a pointed category or a simple generalized near-group category.

IV.3. Generalized Near-Group Categories Are $\varphi-$Pseudounitary

This section is a generalization of Section III.2. The goal is to show that any genearlized near-group category admits a $\varphi-$pseudounitary structure for some automorphism $\varphi \in \mathrm{Gal}(\mathbb{A}/\mathbb{Q})$. The treatment is very similar to Section III.2.

Lemma IV.3.1. Let $\mathcal{C}$ be a generalized near group category with non-invertible simples $X_1, \ldots, X_m$. Let G be the group of invertible objects of $\mathcal{C}$ and $H = \{g \in G; g \otimes X_1 \simeq X_1\}$. Let $d = \mathrm{FPdim}(X_1)$, $n = |G|$ and $\tilde{n} = |H|$. Let $k_1, \ldots, k_m$ be defined by $X_1 \otimes X_1^* \simeq \bigoplus_{h \in H} h \oplus k_1 X_1 \oplus \cdots \oplus k_m X_m$, and $k = k_1 + \cdots + k_m$. Then

$$\mathrm{FPdim}(\mathcal{C}) = \frac{2n + 2\tilde{n}m + mk^2 + mk\sqrt{k^2 + 4\tilde{n}}}{2}.$$

Proof. Since $d = \mathrm{FPdim}(X_1)$, it is the largest root of $x^2 = kx + \tilde{n}$. Therefore

$$d = \frac{k + \sqrt{k^2 + 4\tilde{n}}}{2} \qquad \text{and} \qquad d^2 = \frac{2\tilde{n} + k^2 + k\sqrt{k^2 + 4\tilde{n}}}{2}$$

and

$$\mathrm{FPdim}(\mathcal{C}) = n + md^2 = \frac{2n + 2\tilde{n}m + mk^2 + mk\sqrt{k^2 + 4\tilde{n}}}{2}.$$

$\square$

Let $\mathcal{C}$ be a generalized near-group category and $\tilde{\mathcal{C}}$ be its sphericalization. For each non-invertible simple object $X_i \in \mathrm{ob}(\mathcal{C})$, define $X_{i,\pm} = (X_i)_\pm$ and $\delta_i = \dim(X_{i,+})$. Therefore $\dim(X_{i,-}) = -\delta_i$. For each $g \in G$, let g_+ be the simple object of $\tilde{\mathcal{C}}$ such that $\dim(g_+) = 1$ and g_+ is mapped to g by the forgetful functor $\mathcal{F} : \tilde{\mathcal{C}} \to \mathcal{C}$.

Lemma IV.3.2. For $e \in G$ the identity, $e_- \otimes X_{i,+} \simeq X_{i,-}$. Furthermore for $g, h \in G$, we have $g_+ \otimes X_{i,\pm} \simeq (g \otimes X_{i,\pm})$ and $g_+ \otimes h_+ \simeq (g \otimes h)_+$.

Proof. Let $\mathcal{F} : \tilde{\mathcal{C}} \to \mathcal{C}$ be the forgetful functor. Then $\mathcal{F}(e_- \otimes X_{i,+}) \simeq e \otimes X_i \simeq X_i$, and $\dim(e_- \otimes X_{i,+}) = -\delta$. Therefore $e_- \otimes X_{i,+} \simeq X_{i,-}$.

Let $g \in G$. Then $\mathcal{F}(g_+ \otimes X_{i,\pm}) \simeq g \otimes X_i$, and $\dim(g_+ \otimes X_{i,\pm}) = \pm\delta$. Therefore $g_+ \otimes X_{i,\pm} = (g \otimes X_i)_\pm$.

Finally, let $g, h \in G$. then $\mathcal{F}(g_+ \otimes h_+) \simeq g \otimes h$, and $\dim(g_+ \otimes h_+) = 1$. Therefore $g_+ \otimes h_+ \simeq (g \otimes h)_+$, and the lemma is proved. □

Proposition IV.3.3. Let $\mathcal{C}$ be a generalized near-group category with non-invertible simple objects $X_1, \ldots, X_m$. Let $G, H, n, \tilde{n}, k_1, \ldots, k_m$ and k be defined as in Lemma IV.3.1. Let $X_{i,\pm}$ and δ_i be defined as above. Then $X_{i,+}^* := (X_{i,+})^* \simeq (X_i^*)_+$ and

$$X_{i,+} \otimes X_{i,+}^* \simeq \bigoplus_{h \in H} h_+ \oplus \bigoplus_{j=1}^{m} s_j X_{j,+} \oplus t_j X_{j,-},$$

where $s_j + t_j = k_j$.

Proof. Let X_i be a non-invertible object of $\mathcal{C}$. Then $X_{i,+}$ and $(X_{i,+})^*$ are both non-invertible simple objects of $\tilde{\mathcal{C}}$. Applying the forgetful functor $\mathcal{F} : \tilde{\mathcal{C}} \to \mathcal{C}$, we get

$$0 \neq \mathrm{Hom}(e_+, X_{i,+} \otimes (X_{i,+})^*) \simeq \mathrm{Hom}(e, X_i \otimes \mathcal{F}((X_{i,+})^*)).$$

Therefore $\mathcal{F}((X_{i,+})^*) \simeq X_i^*$. Since $\dim((X_i^*)_-) = -\dim((X_i^*)_+)$, we conclude that

$$(X_{i,+})^* \simeq (X_i^*)_+.$$

Now for each $g \in G$, we have

$$\begin{aligned}\dim \mathrm{Hom}(g_+, X_{i,+} \otimes X_{i,+}^*) &= \dim \mathrm{Hom}(g_+ \otimes X_{i,+}, X_{i,+}) \\ &= \dim \mathrm{Hom}((g \otimes X_i)_+, X_{i,+}) = \begin{cases} 1 & \text{if } g \in H, \\ 0 & \text{if } g \notin H. \end{cases}\end{aligned}$$

Therefore

$$X_{i,+} \otimes X_{i,+}^* \simeq (g_i \otimes X_1)_+ \otimes (g_i \otimes X_1)_+^* \simeq X_{1,+} \otimes X_{1,+}^* \simeq \bigoplus_{h \in H} h_+ \oplus \bigoplus_{j=1}^{m} s_j X_{j,+} \oplus t_j X_{j,-},$$

where $g_i \otimes X_1 \simeq X_i$ and $s_1, \ldots, s_m, t_1, \ldots, t_m$ are non-negative integers. Applying the forgetful functor, we get

$$X_i \otimes X_i = \bigoplus_{h \in H} h \oplus \bigoplus_{j=1}^{m} (s_j + t_j) X_j.$$

Therefore $s_j + t_j = k_j$ and the proposition is proved. $\square$

We may now rename $X_{i,+}$ so that $r_i := s_i - t_i \geq 0$. For the remainder of the section, assume that this has been done.

Lemma IV.3.4. Let $\mathcal{C}$ be a geralized near-group category with non-invertible simple objects $X_1, \ldots, X_m$. Let $G, H, n, \tilde{n}, k_1, \ldots, k_m, k, X_{i,\pm}, s_i, t_i$ and $r_i = s_i - t_i$ be defined as above. Then

$$\delta := \dim(X_{1,+}) = \frac{r \pm \sqrt{r^2 + 4\tilde{n}}}{2} \qquad \text{and} \qquad \dim(\tilde{\mathcal{C}}) = 2n + 2m\tilde{n} + mr^2 \pm mr\sqrt{r^2 + 4\tilde{n}},$$

where $r = r_1 + \cdots + r_m$.

Proof. By Proposition IV.3.3

$$\begin{aligned}
X_{i,+} \otimes X_{i,+}^* &\simeq \bigoplus_{h \in H} h_+ \oplus \bigoplus_{j=1}^{m} s_j X_{j,+} \oplus t_j X_{j,-} \\
\dim(X_{i,+})^2 &= \sum_{h \in H} \dim(h_+) + \sum_{j=1}^{m} s_j \dim(X_{j,+}) + t_j \dim(X_{j,-}) \\
\dim(X_{i,+})^2 &= |H| + \sum_{j=1}^{m} (s_j - t_j) \dim(X_{j,+})
\end{aligned}$$

Therefore $\delta_i = \dim(X_{i,+})$ is a root of $x^2 = \tilde{n} + rx$, and we have $\delta_i = \frac{r \pm \sqrt{r^2+4\tilde{n}}}{2}$ and $\delta_i^2 = \frac{2\tilde{n}+r^2 \pm r\sqrt{r^2+4\tilde{n}}}{2}$.

Let $\delta = \delta_i$ (for any i since this is constant). We have

$$\dim(\tilde{C}) = 2n + 2m\delta^2 = 2n + 2m\tilde{n} + mr^2 \pm mr\sqrt{r^2+4\tilde{n}}.$$

The lemma is proved. $\square$

Lemma IV.3.5. Recall $r = r_1 + \cdots + r_m$ and $k = k_1 + \cdots + k_m$.

1. If $\tilde{\mathcal{C}}$ is pseudo-unitary, then $r = k$,
2. If $\sqrt{r^2+4\tilde{n}} \in \mathbb{Z}$, then $r = k$,
3. If $\sqrt{k^2+4\tilde{n}} \in \mathbb{Z}$, then $r = k$.

Proof. (a) If $\tilde{\mathcal{C}}$ is pseudo-unitary, then $\dim(\tilde{\mathcal{C}}) = \mathrm{FPdim}(\tilde{\mathcal{C}}) = 2\,\mathrm{FPdim}(\mathcal{C})$. Therefore

$$2n + 2m\tilde{n} + mr^2 \pm mr\sqrt{r^2+4\tilde{n}} = 2n + 2\tilde{n}m + mk^2 + mk\sqrt{k^2+4\tilde{n}}$$
$$r^2 \pm r\sqrt{r^2+4\tilde{n}} = k^2 + k\sqrt{k^2+4\tilde{n}}$$

Since $|r| \leq k$, we have $r = k$.

(b) $\delta = \dim(X_{i,+})$ is a root of $x^2 = \tilde{n} + rx$ and thus an algebraic integer. If $\sqrt{r^2+4\tilde{n}} \in \mathbb{Z}$, then δ is a rational number and thus an integer. By [HR, Lemma A.1] $\tilde{C}$ is pseudo-unitary, and $r = k$ by (a).

(c) If $\sqrt{k^2+\tilde{n}} \in \mathbb{Z}$, then $\mathrm{FPdim}(X_i) \in \mathbb{Z}$. Therefore $\mathrm{FPdim}(\tilde{C}) = 2\,\mathrm{FPdim}(\mathcal{C}) \in \mathbb{Z}$. By [ENO, Proposition 8.24] $\tilde{C}$ is pseudo-unitary. Thus $r = k$ by (a). $\square$

Theorem IV.3.6. Every generalized near-group category is φ-pseudounitary.

Proof. Let $\mathcal{C}$ be a generalized near group category. Let $D = \dim(\tilde{\mathcal{C}}) = 2n + 2m\tilde{n} + mr^2 \pm mr\sqrt{r^2 + 4\tilde{n}}$ and $\mathrm{FPdim}(\tilde{\mathcal{C}}) = 2n + 2\tilde{n}m + mk^2 + mk\sqrt{k^2 + 4\tilde{n}}$, where $m, n, \tilde{n}, r$ and k are defined as above. Let $a = 2n + 2m\tilde{n} + mr^2$, $b = (mr)^2(r^2 + 4\tilde{n})$, $c = 2n + 2m\tilde{n} + mk^2$ and $d = (mk)^2(k^2 + 4\tilde{n})$. Then $D = a \pm \sqrt{b}$ and $\Delta = c + \sqrt{d}$. If D or Δ are integers, then by Lemma IV.3.5 $r = k$. Assume both D and Δ are not integers. By [ENO, Proposition 8.22]

$$\frac{D}{\Delta} = \frac{a \pm \sqrt{b}}{c + \sqrt{d}}$$

is an algebraic integer. By Lemma III.2.5 $\frac{a \mp \sqrt{b}}{c - \sqrt{d}}$ is an algebraic integer. Therefore

$$\begin{aligned}\left(\frac{a \pm \sqrt{b}}{c + \sqrt{d}}\right)\left(\frac{a \mp \sqrt{b}}{c - \sqrt{d}}\right) = \frac{a^2 - b}{c^2 - d} &= \frac{(2n + 2m\tilde{n} + mr^2)^2 - (mr)^2(r^2 + 4\tilde{n})}{(2n + 2m\tilde{n} + mk^2)^2 - (mk)^2(k^2 + 4\tilde{n})} \\ &= \frac{(n + m\tilde{n})^2 + nmr^2}{(n + m\tilde{n})^2 + nmk^2}\end{aligned}$$

Since this is a rational number, it is an integer. Moreover $r^2 \le k^2$, so

$$\frac{(n + m\tilde{n})^2 + nmr^2}{(n + m\tilde{n})^2 + nmk^2} = 1$$

and $r^2 = k^2$. Since $r > 0$, we have $r = k$. Therefore $r_i = k_i$, because $|r_i| \le k_i$.
The full tensor subcategory generated by simple objects $\{g_+\}_{g \in G} \cup \{X_{1,+}, \ldots, X_{m,+}\}$ is tensor equivalent to $\mathcal{C}$. Therefore $\mathcal{C}$ is tensor equivalent to a full tensor subcategory of a spherical category, and thus spherical. Moreover if $\delta = \dim(X_{i,+}) > 0$, then $\mathcal{C}$ is pseudo-unitary. If $\delta < 0$, then $\mathcal{C}$ is ϕ-pseudounitary. □

IV.4. Symmetric Generalized Near-Group Categories

One way that the study of generalized near-group categories differs from that of near-group categories is that symmetric generalized near-group categories need not be Tanakian. The following example illustrates that.

Example IV.4.1. Let $G = S_3 \times \mathbb{Z}/2\mathbb{Z}$. It was seen in Example IV.1.2 (1) that $\operatorname{Rep}(G)$ is a generalized near-group category. Let $x = (1, \tau) \in G$ be the non-trivial element of $Z(G)$. x has order two, therefore we may define a twisted braiding on $\operatorname{Rep}(G)$. $\operatorname{Rep}(G, x)$ (see Section III.3.2) is tensor equivalent to $\operatorname{Rep}(G)$, therefore it is genearlized near-group. Since x in non-trivial, we have $\operatorname{Rep}(G, x)$ is symmetric, but not Tanakian.

Let A be a group and $B \trianglelefteq A$ be a normal subgroup. Let χ be the character afforded by a representation ρ of B. For $a \in A$, define ρ_a to be the representation of B given by

$$\begin{aligned} \rho_b : A' &\longrightarrow \mathrm{GL}(V) \\ a &\mapsto \rho(a^{-1}ba) \end{aligned}$$

Then $\chi_a(b) := \operatorname{Tr}(\rho_a(b)) = \operatorname{Tr}(\rho(a^{-1}ba)) = \chi(a^{-1}ba)$.

Lemma IV.4.2. Let A be a finite group, $B \trianglelefteq A$ a normal subgroup and $a \in A$. If ρ is an irreducible represenation of B, then ρ_a is an irreducible representation of B.

Proof. Assume $\rho_a = \sigma \oplus \tau$ for some representations σ, τ of B. Then

$$\rho(b) = \rho_a(aba^{-1}) = \sigma(aba^{-1}) \oplus \tau(aba^{-1}) = \sigma_{a^{-1}}(b) \oplus \tau_{a^{-1}}(b) = (\sigma_{a^{-1}} \oplus \tau_{a^{-1}})(b).$$

Since $\sigma_{a^{-1}}, \tau_{a^{-1}}$ are representations of B, ρ is not irreducible over B. $\square$

Let σ be a represenation of A and $\sigma\downarrow_B$ be the restriction of σ to B. That is the representation given by the composition

$$\sigma\downarrow_B : B \hookrightarrow A \xrightarrow{\sigma} \mathrm{GL}(W).$$

Lemma IV.4.3. Let A be a finite group and $B \trianglelefteq A$ be a normal subgroup of A. Let

$\rho_1, \rho_2 \in \mathrm{Rep}(B)$ and $\sigma_1, \sigma_2 \in \mathrm{Rep}(A)$.

1. $(\rho_1 \otimes_B \rho_2) \upharpoonleft^A \simeq \left(\rho_1 \upharpoonleft^A\right) \otimes_A \left(\rho_2 \upharpoonleft^A\right)$.

2. $(\sigma_1 \otimes_A \sigma_2) \downharpoonleft_B \simeq (\sigma_1 \downharpoonleft_B) \otimes_B (\sigma_2 \downharpoonleft_B)$.

Proof. 1. Let V_1 and V_2 be the $\mathbb{C}B$ modules corresponding to ρ_1 and ρ_2, respectively. Then

$$\begin{aligned}(V_1 \otimes_{\mathbb{C}B} V_2) \upharpoonleft^A = \mathbb{C}A \otimes_{\mathbb{C}B} (V_1 \otimes_{\mathbb{C}B} V_2) &\simeq (\mathbb{C}A \otimes_{\mathbb{C}A} \mathbb{C}A) \otimes_{\mathbb{C}B} (V_1 \otimes_{\mathbb{C}B} \mathbb{C}B) \\ &\simeq (\mathbb{C}A \otimes_{\mathbb{C}B} V_1) \otimes_{\mathbb{C}A} (\mathbb{C}A \otimes_{\mathbb{C}B} V_2) \\ &\simeq V_1 \upharpoonleft^A \otimes_{\mathbb{C}A} V_2 \upharpoonleft^A .\end{aligned}$$

This proves part (1).

2. Let $b \in B$. Then

$$\begin{aligned}(\sigma_1 \otimes_A \sigma_2) \downharpoonleft_B (b) = (\sigma_1 \otimes_A \sigma_2)(b) = \sigma_1(b) \otimes \sigma_2(b) &= (\sigma_1 \downharpoonleft_B (b)) \otimes (\sigma_2 \downharpoonleft_B (b)) \\ &= (\sigma_1 \downharpoonleft_B) \otimes_B (\sigma_2 \downharpoonleft_B)(b).\end{aligned}$$

This proves part (2).

□

Theorem IV.4.4. [Ro, Theorem 8.142] Let A be a finite group and $B \trianglelefteq A$ a normal subgroup. If χ is the character afforded by a representation $\rho : A \to \mathrm{GL}(V)$, the the induced character $\chi \upharpoonleft^A$ is given by

$$\chi \upharpoonleft^A (a) = \frac{1}{|B|} \sum_{c \in A} \chi(c^{-1}ac).$$

Lemma IV.4.5. Let A be a finite group, $A' = [A, A]$ and ρ be a linear representation of A. Then $\rho \downharpoonleft_{A'}$ is the trivial representation of A'.

Proof. Let $\rho : A \to \mathbb{C}^\times$ be a linear representation and $b = xyx^{-1}y^{-1} \in A'$. Then

$$\rho(b) = \rho(xyx^{-1}y^{-1}) = \rho(x)\rho(y)\rho(x)^{-1}\rho(y)^{-1} = 1,$$

since $\mathbb{C}^\times$ is abelian. $\square$

Proposition IV.4.6. Let $\mathcal{C} = \mathrm{Rep}(A, z)$ be a symmetric generalized near-group category. Then all non-trivial representations of the commutator $A' = [A, A]$ are of the same dimension.

Proof. Let $\rho : A' \to \mathrm{GL}(V)$ be a non-trivial representation of A' of dimension δ and $\chi : A' \to \mathbb{C}^\times$ be its character. By [Ro, Theorem 8.142]

$$\chi\upharpoonright^A(a) = \frac{1}{|A'|}\sum_{c\in A}\dot{\chi}(c^{-1}ac)$$

for all $a \in A$, where $\chi\upharpoonright^A$ is the induced character on A of ρ and

$$\dot{\chi}(a) = \begin{cases} 0 & \text{if } a \notin A' \\ \chi(a) & \text{if } a \in A' \end{cases}.$$

Since $A' = [A, A] \trianglelefteq A$, $c^{-1}bc \in A'$ for all $b \in A'$ and $c \in A$. Therefore

$$\left(\chi\upharpoonright^A\right)\downharpoonright_{A'}(b) = \frac{1}{|A'|}\sum_{c\in A}\chi(c^{-1}bc)$$

for all $b \in A'$, where $\sigma\downharpoonright_{A'}$ is the restriction of $\sigma : A \to \mathrm{GL}(W)$ to A'.

For $c \in A$ and $b \in A'$, define $\chi_b(b) = \chi(c^{-1}bc)$. By Theorem IV.4.4

$$\left(\rho\upharpoonright^A\right)\downharpoonright_{A'} = \bigoplus_{b\in A}\rho_b.$$

Since, $\rho_e = \rho$, we have $\mathrm{Hom}_{A'}\left(\left(\rho\upharpoonright^A\right)\downharpoonright_{A'}, \rho\right) \neq 0$. By Lemma IV.4.5 there is a non-linear irreducibile represenation σ of A such that $\left(\rho \upharpoonleft^A, \sigma\right)_A \neq 0$. Since Rep($A$) is a generalized near group category, for every non-linear irreducible representation τ of A, there exists a linear represenation γ of A, such that $\tau = \gamma \otimes \sigma$. Therefore (by Frobenius Reciprocity)

$$\begin{aligned}\mathrm{Hom}_A\left(\rho \upharpoonleft^A, \tau\right) &\simeq \mathrm{Hom}_A\left(\rho \upharpoonleft^A, \gamma \otimes \sigma\right) \simeq \mathrm{Hom}_{A'}\left(\rho, (\gamma \otimes \sigma) \downharpoonleft_{A'}\right) \\ &\simeq \mathrm{Hom}_{A'}\left(\rho, \sigma \downharpoonleft_{A'}\right) \simeq \mathrm{Hom}_A\left(\rho \upharpoonleft^A, \sigma\right) \neq 0.\end{aligned}$$

Therefore every non-linear irreducible represenation of A appears as a direct summand of $\rho \upharpoonleft^A$. Furtheremore this means that the restriction of every non-linear irreducible representation, τ, of A to A' decomposes as

$$\tau \downharpoonleft_{A'} = \bigoplus_{a \in C} \rho_a$$

for some subset $C \subset A$. Since $\dim(\rho_a) = \rho_a(1) = \rho(a^{-1}1a) = \rho(1) = \dim(\rho) = \delta$, the restriction of every non-linear irreducible representation of A to A' decomposes into simples of dimension δ.

Let ρ' be any non-trivial represenation of A'. Then

$$\left(\rho' \upharpoonleft^A\right) \downharpoonleft_{A'} = \bigoplus_{b \in A} \rho'_b.$$

Therefore $\rho' = \rho'_e$ appears as a summand of the restriction of some non-linear irreducible representation of A. Therefore $\dim(\rho') = \delta$. This proves the proposition.

□

Theorem IV.4.7. Every symmetric generalized near-group category is tensor equivalent to the category of represenations of a metabelian group.

Proof. Let $\mathcal{C}$ be a symmetric generalized near-group category. By [D, Corollaire 0.8], $\mathcal{C} \simeq \mathrm{Rep}(A, z)$ as tensor categories for some finite group A. By Proposition IV.4.6 all non-trivial irreducible representation of $A' = [A, A]$ are of the same dimension, call this δ. Therefore $|A'| = 1 + k\delta^2$. Since δ divides $|A'|$, we must have $\delta = 1$. Therefore all irrecucible represenations of A' are linear and A' is abelian. $\square$

The proof of the following lemma follows very closely the proof of [EGO, Corollary 7.4]

Lemma IV.4.8. Let $\mathcal{C} \simeq \operatorname{Rep}(A, z)$ be a symmetric generalized near-group category. Let $x \in A'$ be a non-trivial element and define $B := (A/A')/\operatorname{Stab}(x)$. Then A' is isomorphic to the additive group of a field $\mathbb{F}_q$, and B is isomorphic to the multiplicative group $\mathbb{F}_q^\times$.

Proof. We have seen that all non-trivial irreducible representations of A' have the same dimension. Therefore A' is an elementry abelian group of order $q = p^n$, for some prime p. Since the abelian group B acts irreducibly on A', Schur's Lemma lets us identify A' with a one dimension vector space over $\mathbb{F}_q$. Furthermore since B acts simply on A', we may identify B with $\mathrm{GL}_1(\mathbb{F}_1) = \mathbb{F}_q^\times$. $\square$

Theorem IV.4.9. Let $\mathcal{C} = \operatorname{Rep}(A, z)$ be a symmetric generalized near-group category. Then either

1. $A' = \mathbb{Z}/2\mathbb{Z}$, or
2. $A = K \rtimes \mathbb{F}_q^\times$, for some q.

Proof. By Lemma IV.4.8, A' is isomorphic to the additive group of a field $\mathbb{F}_q$. If $q = 2$, then $A' = \mathbb{Z}/2\mathbb{Z}$, and the theorem is proved. Assume $q \neq 2$. As in the proof of Lemma IV.4.8, we have the short exact sequence

$$1 \to K \to A/A' \to \mathbb{F}_q^\times \to 1.$$

The extensions of $\mathbb{F}_q^\times$ by K and then by $\mathbb{F}_q$ is given by the cohomology $H^j(\mathbb{F}_q^\times, H^i(K, \mathbb{F}_q))$. By the Hochschild-Serre Spectral Sequence, $H^j(\mathbb{F}_q^\times, H^i(K, \mathbb{F}_q)) = 0$ if $j > 0$. Since $q \neq 2$,

$\mathbb{F}_q^\times$ is not trivial and acts non-trivially by scalar multiplication on the $\mathbb{F}_q$-vectorspace $H^2(K, \mathbb{F}_q)$. Therefore $H^2(K, \mathbb{F}_q)$ as no $\mathbb{F}_q^\times$ invariants, $H^0(\mathbb{F}_q^\times, H^2(K, \mathbb{F}_q)) = 0$, and the only extension we can have is $A = K \rtimes \mathbb{F}_q^\times$.

□

IV.5. Modular Generalized Near-Group Categories

The goal of this section is to describe all generalized near-group categories that have a modular structure.

Let $\mathcal{C}$ be a modular generalized near-group category. By [DGNO, Lemma 3.31] the group of invertible objects of $\mathcal{C}$ is isomorphic as groups to the group $\mathrm{Aut}_\otimes(\mathrm{id}_\mathcal{C})$ of tensor automorphisms of the identity functor. By Proposition II.2.18, there exists an abelian (since $\mathcal{C}$ is braided) group $U_\mathcal{C}$ such that $\mathrm{Hom}(U_\mathcal{C}, \mathbb{C}^\times) \cong \mathrm{Aut}_\otimes(\mathrm{id}_\mathcal{C})$ and a faithful grading $o : \mathcal{O}(\mathcal{C}) \to U_\mathcal{C}$ where $o^{-1}(e) = \mathcal{C}_{\mathrm{ad}}$.

Lemma IV.5.1. Let $\mathcal{C}$ be a modular generalized near-group category with $\mathcal{C}_{\mathrm{ad}} \simeq \mathrm{sVec}$. Then $o(X) \in U_\mathcal{C}$ has order two for some non-invertible simple object $X \in \mathcal{O}(\mathcal{C})$.

Proof. Let (A, q) be the pre-metric group associated to the pointed, braided fusion category $\mathcal{C}_{\mathrm{pt}}$. Let $\mathcal{O}(\mathcal{C}_{\mathrm{ad}}) = \{1, z\}$. Since $\mathcal{C}' = \mathrm{sVec}$, we have $q(z) = -1$. By [DGNO, Corollary 3.27] $(\mathcal{C}_{\mathrm{ad}})' = \mathcal{C}_{\mathrm{pt}}$. Therefore $1 = b(z, a) = \frac{q(za)}{q(z)q(a)} = -\frac{q(za)}{q(a)}$ for all $a \in A$ and $za \neq a^{-1}$ for all $a \in A$. Therefore z is not a square.

Since the group of invertible objects of $\mathcal{C}$ has even order, there exists an element of $U_\mathcal{C}$ of order two. Let $V \in \mathcal{O}(\mathcal{C})$ be a simple object such that $o(V)$ has order two. If V is non-invertible, then the lemma is proved. Assume V is invertible. Since $o(V)$ has order two, $V \otimes V \in \{1, z\}$. Since z is not a square, we have $V \simeq V^*$. Similarly $z \otimes V \simeq (z \otimes V)^*$. Therefore $U_\mathcal{C}$ has two more elements of order two. We may repeat this process until we get a non-invertible simple object $X \in \mathcal{O}(\mathcal{C})$ such that $o(X)$ has order two. □

Theorem IV.5.2. Let $\mathcal{C}$ be a modular generalized near-group category. Then $\mathcal{C}_{\text{ad}}$ contains two simple objects. Moreover we have one of the following cases

1. If $\mathcal{C}_{\text{ad}}$ is pointed, then $\mathcal{C} \cong \mathcal{I} \boxtimes \mathcal{D}$, where D is a pointed category and $\mathcal{I}$ is an Ising category, that is a non-pointed fusion category with $\text{FPdim}(\mathcal{I}) = 4$ (see e.g. [DGNO, Appendix B]).

2. If $\mathcal{C}_{\text{ad}}$ is not pointed, then $\mathcal{C}_{\text{ad}}$ is Yang-Lee and $\mathcal{C} \cong \mathcal{C}_{\text{ad}} \boxtimes \mathcal{C}_{\text{pt}}$.

Proof. Let $o : \mathcal{O}(\mathcal{C}) \to U_{\mathcal{C}}$ be the grading described above.

1. Assume $\mathcal{C}_{\text{ad}}$ is pointed. Let m be the number of invertible objects in $\mathcal{C}$, k be the number of non-inverible simple objects and n be the number of invertible objects which fix the non-invertible simple objects. Then for ever non-invertible simple object X_i, we have $X_i \otimes X_i^* = \bigoplus_{h \in H} h$, where H is the group of invertible objects which fix X_1. Therefore $\text{FPdim}(X_i) = \sqrt{|H|} = \sqrt{n}$, and $\text{FPdim}(\mathcal{C}) = |\mathcal{O}(\mathcal{C}_{\text{pt}})| + k|H| = m + kn$. Since $\mathcal{C}$ is faithfully graded by $U_{\mathcal{C}}$ with trivial component $\mathcal{C}_{\text{ad}}$, [DGNO, Corollary 4.28] gives $\text{FPdim}(\mathcal{C}) = |U_{\mathcal{C}}| \cdot \text{FPdim}(\mathcal{C}_{\text{ad}}) = mn$. We also have that $\mathcal{O}(\mathcal{C}_{\text{pt}})/H$ acts simply transitively on the set of non-invertible simple objects of $\mathcal{C}$. Therefore $k = [\mathcal{O}(\mathcal{C}_{\text{pt}}) : H] = \frac{m}{n}$. When we combine these equations we get

$$mn = m + kn = m + \frac{m}{n}n = 2m.$$

Therefore $|H| = 2$ and $\mathcal{C}_{\text{ad}}$ has two simple objects.

Let X be a non-invertible simple object of $\mathcal{C}$. Then $\text{FPdim}(X) = \sqrt{2}$, and $\mathcal{C}_{\text{ad}}$ acts trivially on X by tensor product. Therefore $\mathcal{C}_{\text{ad}} = \text{sVec}$. Otherwise $\text{Rep}(\mathbb{Z}/2\mathbb{Z})$ is a braided fusion subcategory of $\mathcal{C}$ and we may deequivariantize by $\mathbb{Z}/2\mathbb{Z}$. Since $\text{Rep}(BZ/2\mathbb{Z}$ acts trivially on X, it must decompose into a sum of two simple objects of $\mathcal{C}_{\mathbb{Z}/2\mathbb{Z}}$. This is a contradiction since $\text{FPdim}(X) = \sqrt{2}$. Therefore $\mathcal{C}_{\text{ad}} = \text{sVec}$ as stated.

By Lemma IV.5.1, there exists a non-invertible simple object X such that $o(X) \in U_{\mathcal{C}}$ has order two. Therefore $X \simeq X^*$. Since $X \otimes X^* \in \mathcal{C}_{\text{ad}}$ and $\mathcal{C}_{\text{ad}}$ acts trivially on X, we have a fusion subcategory, $\mathcal{I}$, of $\mathcal{C}$ generated by $\mathcal{C}_{\text{ad}}$ and X. $\text{FPdim}(\mathcal{I}) = 1+1+2 = 4$, and $\mathcal{I}$ is not pointed, so $\mathcal{I}$ is an Ising category. By [DGNO, Corollary B.12], $\mathcal{I}$ is a modular category, therefore we may use [Mu3, Theorem 4.2] to get $\mathcal{C} \cong \mathcal{I} \boxtimes \mathcal{I}'$. Since $\mathcal{C}$ is modular, [DGNO, Corollary 3.26] gives $(\mathcal{I}')_{\text{ad}} = (\mathcal{I}^{\text{co}})' = (\mathcal{C})' = \text{Vec}$. Therefore $\mathcal{I}'$ is pointed.

2. Assume $\mathcal{C}_{\text{ad}}$ is not pointed. Let $x \in U_{\mathcal{C}}$. Let X be a non-invertible simple object of $\mathcal{C}_{\text{ad}}$. We want to show that $\mathcal{C}_x$ contains an invertible object. Since the graiding is faithful, there exists some simple object in $V \in \mathcal{C}_x$. If V is invertible, then we have shown that $\mathcal{C}_x$ contains an invertible object. Therefore assume that V is not invertible. Then since $\mathcal{C}$ is a generalized near-group category, there exists and invertible objects D such that $D \otimes X \simeq V$. Therefore $x = o(V) = o(D)o(X) = o(D) \cdot e = o(D)$, and $D \in \mathcal{C}_x$. Therefore $\mathcal{C}_x$ contains an invertible object for every $x \in U_{\mathcal{C}}$.

   Since $|U_{\mathcal{C}}| = |\operatorname{Aut}_{\otimes}(\text{id}_{\mathcal{C}})|$ is the number of invertible objects of $\mathcal{C}$, each $\mathcal{C}_x$ contains exactly one invertible object. Let $x \in U_{\mathcal{C}}$, and let D_x be the invertible object in $\mathcal{C}_x$. Then $D_x \otimes X$ is a non-invertible simple object with grading $o(D_x \otimes X) = o(D_x)o(X) = x \cdot e = x$. Therefore every $\mathcal{C}_x$ contains a simple non-invertible object. Since the invertible objects transitively permute the non-invertible objects, there cannot be more non-invertible objects than invertibles. Therefore every $\mathcal{C}_x$ contains exatly one invertible object and one simple non-invertible object. In particular, $\mathcal{O}(\mathcal{C}_{\text{ad}}) = \mathcal{O}(\mathcal{C}_e) = \{1, X\}$, and by assumption $\text{Hom}(X, X \otimes X) \neq 0$. By [O1] $\mathcal{C}_{\text{ad}}$ is

Yang-Lee.

By Lemma III.3.3, $(\mathcal{C}_{\mathrm{ad}})' = (\mathcal{C}')^{\mathrm{co}} = \mathrm{Vec}^{\mathrm{co}} = \mathcal{C}_{\mathrm{pt}}$. Therefore by [Mu3, Theorem 4.2], $\mathcal{C} \cong \mathcal{C}_{\mathrm{ad}} \boxtimes \mathcal{C}_{\mathrm{pt}}$.

$\square$

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