# Intrinsic Heterogeneity in Expectation Formation*

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**Abstract**

We introduce the concept of a *Misspecification Equilibrium* to dynamic macroeconomics. Agents choose between a list of misspecified econometric models and base their selection on relative forecast performance. A Misspecification Equilibrium is an equilibrium stochastic process in which agents forecast optimally given their choices, with the forecasting model parameters and predictor proportions endogenously determined. For appropriate conditions on the exogenous driving process and the degree of feedback of expectations, the Misspecification Equilibrium will exhibit *Intrinsic Heterogeneity*. With Intrinsic Heterogeneity more than one misspecified model receives positive weight in the distribution of predictors across agents, even in the neoclassical limit in which only the most successful predictors are used.

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## 1 Introduction

Despite its dominance in dynamic macroeconomic models, the Rational Expectations Hypothesis has limitations. A frequently cited drawback to the rational expectations approach is that in effect it assumes that agents know the underlying economic structure. In response to this criticism one popular alternative is to model agents as econometricians (Evans and Honkapohja 2001). This adaptive learning approach

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typically assumes agents have a correctly specified model with unknown parameters. Agents then use a reasonable estimator to obtain their coefficient estimates. In many models these beliefs converge to rational expectations.

In practice, however, econometricians often misspecify their models. Economic forecasters who use VAR's purposely limit the number of variables and the number of lags because of degree of freedom problems. If agents are expected to behave like econometricians then they can also be expected to misspecify their models. (Evans and Honkapohja 2001, Chapter 13) consider models with agents underparameterizing the law of motion, and show the existence of a Restricted Perceptions Equilibrium (RPE) in which agents form their beliefs optimally given their misspecification.[1] The issue of underparameterization is also emphasized by (Evans and Ramey 2001), who examine the implications of optimally chosen expectations within the simple adaptive expectations class.

In this paper we examine expectation formation in an environment where agents must forecast using an underparameterized econometric model. More specifically we confront agents with a list of misspecified econometric models, but, given this restriction, assume that agents forecast optimally. Agents choose between these optimal underparameterized models based on their relative mean success.

We investigate this approach in a linear stochastic framework, developing the analysis in the context of the cobweb model. Because the economic model is self-referential, in the sense that expectation formation affects the law of motion for the endogenous variables, the optimal parameters of each misspecified econometric model depend on the proportions of agents using the different models. We define a new equilibrium concept, called a *Misspecification Equilibrium*, in which these proportions are consistent with optimal forecasting from each econometric model. We show that for some economic model parameters and exogenous driving variables, agents will be distributed heterogeneously between the various predictors, even as we approach the limiting case in which agents choose only between the best performing statistical models. We say that a Misspecification Equilibrium with such a property exhibits *Intrinsic Heterogeneity*.

Heterogeneity in expectations has been considered previously in papers by (Townsend 1983), who takes a fully rational learning approach, starting with given priors, and (Haltiwanger and Waldman 1985) who assume that a certain fraction of agents are not rational. In adaptive learning models (Honkapohja and Mitra 2001) allow agents to have different specific learning rules. The seminal least squares learning paper by (Bray and Savin 1986) also allows for heterogeneity in priors. However, these papers all assume an *ad hoc* degree of heterogeneity, and, with least squares or Bayesian learning, the heterogeneity disappears in the limit. (Evans, Honkapohja and Marimon

[1] Sargent (1999) developed the implications of policy makers estimating and forecasting using a misspecified model.

2001) allow for stochastic heterogeneity in learning rules, but again the heterogenous expectations is only transitory.

(Brock and Hommes 1997) were among the first to model heterogeneous expectations as an endogenous outcome.[2] (Brock and Hommes 1997) examine a cobweb model in which agents choose a predictor from a set of costly alternatives. Agents base this choice on the most recent realized profits of the alternatives in a cobweb model. If agents are boundedly rational in the sense that their 'intensity of choice' between predictors is finite (that is, they do not fully optimize), then there will be heterogeneity and the degree of heterogeneity will vary in a complex manner.

Brock and Hommes illustrate these results in a particular case of rational versus myopic beliefs. Because agents always react to recent changes in profits their predictor choice will oscillate along with the equilibrium price. Our model is closely related to Brock and Hommes. Like their model, we assume that the map from predictor benefits to predictor choice resembles a multinomial logit. The multinomial logit has proven to be an important approach to modeling economic choices,[3] and has been increasingly employed in recent work in dynamic macroeconomics. Extensions of the (Brock and Hommes 1997) predictor selection dynamic appear in (Brock and deFountnouvelle 2000), (Brock and Hommes 1998, 2000), (Brock, Hommes, and Wagener 2001), (Branch 2002a, 2002b) and (Hommes 2001). (Brock and Durlauf 2001) extend the framework so that agent specific choices depend on the expected choices of others.

There are three important departures in our model. First, agents do not choose between a costly accurate forecast and a costless unsophisticated forecast; rather, they are forced to choose between equally misspecified costless models. Second, in line with the econometric learning literature, each forecasting model is optimal, given the misspecification. Third, we assume that agents make their choices based on unconditional mean payoffs rather than on the most recent period's realized payoff. This is more appropriate in a stochastic environment since otherwise agents would frequently be misled by single period anomalies. We will show that even if agents optimally choose between these misspecified models heterogeneity can arise. Given that agents base decisions on mean profits it is not at all obvious that heterogeneity would be possible if the 'intensity of choice' is large. Indeed, we will show that instances of asymptotically homogeneous expectations also arise.

The main difference in our results is that, unlike previous work, we derive heterogeneity as a possible equilibrium outcome of a self-referential model in which agents are constrained to underparameterize. In particular we examine the case in which agents are fully rational except that they misspecify by omitting at least one relevant

[2](Sethi and Franke 1995) also find heterogeneity, as an outcome of evolution in a model of stochastic strategic complementarities, and (Evans and Ramey 1992) permit heterogeneous expectations due to heterogeneous calculation costs.

[3]See, for example, (Manski and McFadden 1981).

variable or lag. We focus on the cobweb model for two reasons. First, we want to stay close to (Brock and Hommes 1997) in order to highlight the key differences. Second, the cobweb model is the simplest self-referential model that effectively illustrates the intuition of Intrinsic Heterogeneity.

We obtain conditions under which there is an equilibrium with agents heterogeneously split between the misspecified models even as the 'intensity of choice' becomes arbitrarily large. The intuition for this possibility is as follows. Suppose the cobweb price is driven by a two-dimensional vector of demand shocks. If both components of the demand shock matter for predicting prices, and if the feedback through expectations is sufficiently large, then there will be an incentive to deviate from homogeneity. If all agents coordinate on the same model the negative feedback through expectations will make the consensus model less useful for forecasting. In these instances an agent could profit by forecasting with the alternative model. With Intrinsic Heterogeneity the equilibrium is such that beliefs and predictor proportions drive expected profits to be identical.

The plan for this paper is as follows. Section 2 introduces the set-up in a general cobweb model. We obtain an existence result for Misspecification Equilibria, and give conditions under which the model exhibits Intrinsic Heterogeneity. Section 3 extends and illustates these results for the special case of a process driven by a two dimensional VAR(1) shock with agents choosing between two underparameterized models. Section 4 shows that a Misspecification Equilibrium can be attained under real-time learning. Section 5 concludes and describes future work.

# 2 Model

In this section we consider a self-referential stochastic process that is driven by vector autoregressive exogenous shocks. We assume that agents' expectations are based on one of a set of misspecified models of the economy, each taking the form of an underparameterization of the process. In the terminology of (Brock and Hommes 1997) we are in effect treating forecasts based on a fully correctly specified model as prohibitively costly, and those based on the misspecified models are equally and much less costly. (For convenience we will normalize this cost to zero). Much previous work has assumed a particular structure of agents' misspecification. We allow the choice of the misspecified model to be endogenous.

We develop our model as a version of the Adaptively Rational Equilibrium Dynamics (A.R.E.D.) of Brock and Hommes (1997) in which we constrain agents to choose between underparameterized models. Agents consider the unconditional expected payoff of the various possible underparameterizations and select between them according to their relative payoffs. Using the selected model they form their expectations as the optimal linear projection given this choice. In our Misspecification

Equilibrium, the projection parameters and predictor proportions are jointly determined and generate the equilibrium stochastic process.

We think that our emphasis on underparameterization is reasonable. The adaptive learning literature has argued in favor of modeling agents as econometricians as a plausible deviation from the rational expectations assumption. But, econometricians misspecify their econometric models. Computational time and limits on degrees of freedom make it impossible for an econometrician to include all economically relevant variables and lags. Our model in effect imposes such restrictions on agents, but otherwise requires them to behave optimally. A striking finding of our framework is that this can lead to the use of heterogeneous forecasting models.

We develop the model in stages. We first show that, for given predictor proportions, there exists a Restricted Perceptions Equilibrium (RPE) in which agents' misspecified beliefs are verified by the actual equilibrium process. We next allow for predictor proportions to be endogenously determined, and show the existence of a Misspecification Equilibrium. Finally, we formally define Intrinsic Heterogeneity and state a condition under which this will arise.

## 2.1 Set-up

We consider a cobweb model of the form

$$p_t = -\phi p_t^e + \gamma' z_t + v_t \tag{1}$$

where $v_t$ is white noise. Although there are several well-known economic models that fit the form (1), we focus on the "cobweb" model in order to keep a close connection between our model and (Brock and Hommes 1997). $z_t$ is a vector of observable demand disturbances, which will be further specified below.

We normally expect $\phi > 0$ in the cobweb model, which corresponds to upward sloping supply curves and downward sloping demand curves. Bray and Savin (1986) showed that $\phi > -1$ was the condition for the model to be stable under least squares learning. In this paper we focus on the negative feedback case of $\phi > 0$ and leave $\phi < 0$ for future work.[4]

In the cobweb model firms have a one-period production lag. We assume that firms have quadratic costs given by $FQ_t^* + \frac{1}{2}G(Q_t^*)^2$, where $Q_t^*$ is planned output and $F \geq 0$, $G > 0$. In addition we allow for exogenous productivity shocks realized after production decisions are made so that total quantity is $Q_t = Q_t^* + \kappa_t$. Here $\kappa_t$ is iid

[4] Equation (1) with $-1 < \phi < 0$ takes the same form as a Lucas-type monetary model. In future work we will pursue the possibility of heterogeneity in that model.

with zero mean. Firms aim to maximize expected profits.[5] Thus,

$$\begin{aligned}\max_{Q_t^*} \; E_{t-1}\pi_t &= E_{t-1}\left[p_t\left(Q_t^*+\kappa_t\right)-FQ_t^*-\frac{1}{2}G(Q_t^*)^2\right] \\ &= Q_t^*E_{t-1}p_t+E_{t-1}(p_t\kappa_t)-FQ_t^*-\frac{1}{2}G(Q_t^*)^2\end{aligned}$$

Solving this problem leads to the supply relation[6]

$$Q_t^* = G^{-1}p_t^e \tag{2}$$

where $p_t^e = E_{t-1}p_t$. Then actual supply follows $Q_t = G^{-1}p_t^e + \kappa_t$.

Demand is given by

$$Q_t = C - Dp_t + h'\zeta_t \tag{3}$$

where $\zeta_t$ is an $m \times 1$ vector of demand shocks that follows a zero-mean stationary VAR($n$) process and $D > 0$. The $\zeta_t$ process is assumed independent of $\kappa_t$. Setting demand equal to actual supply we have the following stochastic equilibrium price process

$$p_t = -(DG)^{-1}p_t^e + D^{-1}h'\zeta_t - D^{-1}\kappa_t, \tag{4}$$

where, for convenience, we have expressed $p_t$ and $p_t^e$ in deviation from the mean form.

It is convenient to rewrite the model in terms of an exogenous VAR(1) process. Defining

$$z_t' = (\zeta_t', \zeta_{t-1}', \cdots, \zeta_{t-n+1}')$$

we can write $z_t$ in its standard VAR(1) form

$$z_t = Az_{t-1} + \varepsilon_t$$

for appropriately defined $A$ and appropriately defined $\varepsilon_t$, which is exogenous white noise. Here $z_t$ is $mn \times 1$ and $A$ is $mn \times mn$. We denote the covariance matrix of $z_t$ as $\Omega = Ezz'$, and $\Omega$ is assumed to be positive definite. Setting

$$\phi = (DG)^{-1}, \gamma' = (D^{-1}h', 0, \ldots, 0) \text{ and } v_t = -D^{-1}\kappa_t$$

we can rewrite (4) in the form (1).

[5]It would be possible to extend the model to incorporate risk by assuming agents respond to variances of profits as well as expected profits. We make the expected profits assumption to keep the model as simple as possible.

[6]We have set, without loss of generality, $F = 0$. We are also assuming that agents treat $E_{t-1}(p_t\kappa_t)$ as a constant independent of the choice of $Q_t^*$. That this is a reasonable assumption can be verified by (4) below.

## 2.2 Model Misspecification

To close the model we need to specify the determination of $p_t^e$. We assume that there are $K$ econometric models available to form expectations and that model $j = 1, \ldots, K$ uses $k_j < mn$ explanatory variables. The market expectation is given by the weighted sum of the individual expectations

$$p_t^e = \sum_{j=1}^{K} n_j p_{j,t}^e \tag{5}$$

where $p_{j,t}^e = b^{j\prime} x_{t-1}^j$, $x_t^j = u^j z_t$. The $k_j \times m$ matrix $u^j$ is a selector matrix that picks out those elements of $z_t$ used in predictor $j$ and $b^j$ is $k_j \times 1$. Thus, $k_j$ is the number of elements in $z_t$ that predictor $j$ uses. We can rewrite (5) as

$$p_t^e = \sum_{j=1}^{K} n_j b^{j\prime} u^j z_{t-1}$$

This set-up forces agents to underparameterize the variables included in their information set and/or the number of lags of those variables. We believe this is a reasonable approximation of actual expectation formation. Cognitive and computing time constraints (as well as degrees of freedom) restrict the number of variables even the most diligent econometricians use in their models. Our form of misspecification makes agents be (at least somewhat) parsimonious in their expectation formation.

We next specify the determination of the parameters $b^j$. In a fully specified econometric model, and under rational expectations, all variables $z_t$ would be included and the coefficients used to form $p_t^e$ would be given by the least squares projection of $p_t$ on $z_t$. Here each predictor is constrained to use a subset $x_t^j$ of relevant variables, and thus each predictor differs from rational expectations. However, we will insist that the beliefs $b^j$ are formed optimally in the sense that $b^j$ is the least squares projection of $p_t$ on $u^j z_{t-1}$. That is, $b^j$ must satisfy

$$E u^j z_{t-1} \left( p_t - b^{j\prime} u^j z_{t-1} \right) = 0$$

Even though agents will never be "fully" accurate, they will be as accurate as possible given the variables in their information set.

## 2.3 Misspecification Equilibrium

Given the belief process (5) the actual law of motion (ALM) for this economy is

$$p_t = \left[ \gamma' A - \phi \left( \sum_{j=1}^{K} n_j b^{j\prime} u^j \right) \right] z_{t-1} + \gamma' \varepsilon_t + v_t$$

or

$$p_t = \xi' z_{t-1} + \gamma' \varepsilon_t + v_t, \tag{6}$$

where

$$\xi' = \gamma' A - \phi \left( \sum_{j=1}^{K} n_j b^{j\prime} u^j \right). \tag{7}$$

Here $n = [n_1, \cdots, n_K]'$ and $b = [b_1, \cdots, b_K]$. Given these equations and the parameter orthogonality condition we obtain

$$b^j = \left( u^j \Omega u^{j\prime} \right)^{-1} u^j \Omega \xi. \tag{8}$$

We now introduce the concept of Restricted Perceptions Equilibrium (RPE).[7] An RPE is an equilibrium process for $p_t$ such that the parameters $b^j$ are optimal given the misspecification. Note that, like a rational expectations equilibrium, an RPE is self-referential in that the optimal beliefs depend on the vector of parameters $\xi$ which depend in turn on the vector of beliefs $b$. Thus, an RPE can be defined as a process (6) such that $\xi$ is a solution to (7) and (8) for fixed $n$.

Substituting (8) into (7) yields

$$\xi' = \gamma' A - \phi \sum_{j=1}^{K} n_j \xi' \Omega u^{j\prime} \left( u^j \Omega u^{j\prime} \right)^{-1} u^j$$

or

$$\xi = \left[ I + \phi \sum_{j=1}^{K} n_j u^{j\prime} \left( u^j \Omega u^{j\prime} \right)^{-1} u^j \Omega \right]^{-1} A' \gamma \tag{9}$$

For a given $n$ an RPE exists (and is unique), provided the inverse in (9) exists.

In the Misspecification Equilibrium, which we define below, $n$ is determined endogenously. Equation (9) gives a well-defined mapping $\xi = \xi(n)$ provided the indicated inverse exists for all $n$ in the unit simplex. We therefore assume that the following condition holds:

***Condition*** $\triangle$*:* $\triangle \neq 0$ *for all* $n$ *in the unit simplex* $S = \{n \in \mathbb{R}^K : n_i \geq 0 \text{ and } \sum_{i=1}^{K} n_i = 1\}$*, where*

$$\triangle = \det \left( I + \phi \sum_{j=1}^{K} n_j u^{j\prime} \left( u^j \Omega u^{j\prime} \right)^{-1} u^j \Omega \right).$$

Condition $\triangle$ is a necessary and sufficient condition for the existence of a unique RPE for all $n \in S$.

We have the following result:

[7] See (Evans and Honkapohja 2001) for a definition and examples. The concept introduced here extends the concept of RPE to incorporate multiple misspecified models.

**Proposition 1** *For $\phi \geq 0$ sufficiently small, Condition $\triangle$ is satisfied and hence for all $n$ there exists a unique RPE.*

All proofs are contained in the Appendix. In the next Section we demonstrate that Condition $\triangle$ holds for all $\phi \geq 0$ in the case of a bivariate process.

We now embed the RPE into an equilibrium concept in which $n$ is endogenously determined by the mean profits of each predictor. We will call this a *Misspecification Equilibrium*. Note that the profits of each predictor depend on the parameters $\xi$ which in turn depend on $n$.

In order to discuss the mapping for predictor proportions we need the profits for predictor $j$, which are given by

$$\begin{aligned}\pi_t^j &= p_t \left(\phi D p_{i,t}^e - D v_t\right) - \frac{1}{2}\phi D \left(p_{i,t}^e\right)^2 \\ &= \left[\xi(n)' z_{t-1} + \gamma' \varepsilon_t + v_t\right]\left[\phi D b^{j\prime} u^j z_{t-1} - D v_t\right] - \frac{1}{2}\phi D \left(b^{j\prime} u^j z_{t-1}\right)^2,\end{aligned}$$

where, again, we have expressed profits in deviation from mean form. Taking unconditional expectations of profits yields

$$E\pi_t^j = \phi D b^{j\prime} u^j \Omega \left(\xi(n) - \frac{1}{2} u^{j\prime} b^j\right) - D E v_t^2.$$

Evaluating expected profits in an RPE (i.e. plugging in (8)) leads to

$$E\pi^j = \phi D \xi(n)' \Omega u^{j\prime} (u^j \Omega u^{j\prime})^{-1} u^j \Omega \left(\xi(n) - \frac{1}{2} u^{j\prime} (u^j \Omega u^{j\prime})^{-1} u^j \Omega \xi(n)\right) - D E v_t^2. \quad (10)$$

Note that $E\pi^j$ is well-defined and finite for all $n$, provided Condition $\Delta$ holds so that $\xi(n)$ is well-defined. It will be convenient to denote the function given by (10) as

$$\tilde{F}_j(n) : S \to \mathbb{R}, \text{ for } j = 1, \ldots, K,$$

and to define $\tilde{F}(n) : S \to \mathbb{R}^{\mathbb{K}}$ by $\tilde{F}(n) = (\tilde{F}_1(n), \ldots, \tilde{F}_K(n))'$. Note that $\tilde{F}_j(n)$ and $\tilde{F}(n)$ are continuous on $S$ provided Condition $\Delta$ holds.

We now follow (Brock and Hommes 1997) in assuming that the predictor proportions follow a multinomial logit (MNL) law of motion. Brock and Hommes consider the cobweb model without noise where agents choose between rational and naive expectations. Agents adapt their choices based on the most recent relative predictor success.[8] This clearly would not be appropriate in the stochastic framework employed

[8] (Branch 2002) shows that many of the qualitative properties in the model without noise carry over to a model with small demand disturbances.

here and we instead assume that agents base their decision on unconditional expected relative payoffs.

The MNL approach leads to the following mapping, for each predictor $i$,

$$n_i = \frac{exp\{\alpha E\pi^i\}}{\sum_{j=1}^{K} exp\{\alpha E\pi^j\}}, \tag{11}$$

where $\alpha > 0$. Note that $n_i > 0$ for $\alpha$ and the $E\pi^j$ finite and that $\sum_j n_j = 1$. Again, it will be convenient to denote the map defined by (11) as

$$\tilde{H}_\alpha(E\pi^1, \ldots, E\pi^K) : \mathbb{R}^K \to S,$$

and clearly $\tilde{H}_\alpha$ is continuous. The parameter $\alpha$ is called the 'intensity of choice,' and parameterizes one dimension of agents' bounded rationality. As $\alpha \to +\infty$ we obtain the 'neoclassical' case of full optimization. We will be interested in the conditions in which heterogeneity can arise in the neoclassical case. We remark that our choice of payoff function $E\pi^j$ allows us to consider the fixed point of a map rather than the solution to a difference equation as in Brock and Hommes.

We now define the mapping

$$\tilde{T}_\alpha : S \to S \text{ where } \tilde{T}_\alpha = \tilde{H}_\alpha \circ \tilde{F}.$$

Under Condition $\Delta$ this map is well-defined and continuous. $\tilde{T}_\alpha$ maps a vector of predictor choices, $n$, through the belief parameter mapping $\xi$ into a vector of expected profits and then to a new predictor choice $n$. We are now in a position to present our central equilibrium concept:

***Definition*** *A Misspecification Equilibrium (ME) is a fixed point, $n^*$, of $\tilde{T}_\alpha$.*

Applying the Brouwer Fixed Point Theorem we immediately have:

**Theorem 2** *Assume Condition $\Delta$. There exists a Misspecification Equilibrium.*

In general we cannot rule out multiple equilibria. Let

$$N_\alpha = \{n^* | \tilde{T}_\alpha(n^*) = n^*\}.$$

For $\alpha$ finite and $E\pi^j$ finite, it is apparent that all components are positive for every fixed point $n^*$. Thus, heterogeneity for finite $\alpha$ is simply a by-product of the MNL assumption, which ensures that all predictors are used even if they differ in terms of their performance. However, it is of interest to know if heterogeneity continues to arise if agents are highly sensitive to relative performance, so that they only use predictors that are not dominated in performance. This leads to the following concept:

***Definition*** *A model is said to exhibit Intrinsic Heterogeneity if (i) an ME exists for all* $\alpha > 0$ *and (ii) there exists* $\bar{n} < 1$ *such that* $n_j^* \leq \bar{n}$, $j = 1, \ldots, K$, *for all* $\alpha$ *and all ME* $n^* \in N_\alpha$.

It can be shown that a model with intrinsic heterogeneity arises whenever the following additional condition is satisfied.

***Condition P:*** Let $e_i$ denote the $K \times 1$ coordinate vector with 1 in position $i$ and 0 elsewhere. Condition P is said to be satisfied if for each $i = 1, \ldots, K$ there exists $j \neq i$ such that $\tilde{F}_j(e_i) - \tilde{F}_i(e_i) > 0$.

**Theorem 3** *Assume Condition* $\Delta$ *and also Condition P. Then the model exhibits intrinsic heterogeneity.*

The next section will present a simple example to illustrate our concepts. In particular we present cases in which Condition P holds and the model exhibits Intrinsic Heterogeneity.

# 3 Example: Bivariate Case

To illustrate the properties of a Misspecification Equilibrium we will simplify the model by considering a special case in which detailed results can be obtained. In this section we assume that $z_t$ is a two-dimensional stationary VAR(1) $z_t = Az_{t-1} + \varepsilon_t$, where $A$ is $2 \times 2$, with eigenvalues inside the unit circle, and $E\varepsilon_t\varepsilon_t' = \Sigma_\varepsilon$ is positive definite. Each misspecified model will omit one explanatory variable and thus $K = 2$ and $k_j = 1$ for $j = 1, 2$. This is the simplest possible illustration of our framework, and we will see that it can generate cases with Intrinsic Heterogeneity.

With bivariate demand shocks the predictors are now

$$\begin{aligned} p_{1,t}^e &= b^1 u^1 z_{t-1} = b^1 z_{1,t-1} \\ p_{2,t}^e &= b^2 u^2 z_{t-1} = b^2 z_{2,t-1} \end{aligned}$$

Plugging these predictors into the law of motion for price and collecting terms leads to

$$p_t = \xi_1 z_{1,t-1} + \xi_2 z_{2,t-1} + \eta_t \tag{12}$$

$$\begin{bmatrix} 1 + n_1\phi & \phi n_1 \rho \\ \phi n_2 \tilde{\rho} & 1 + n_2\phi \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = A'\gamma, \tag{13}$$

where

$$\rho = \frac{Ez_{1t}z_{2t}}{Ez_{1t}^2}, \; \tilde{\rho} = \frac{Ez_{1t}z_{2t}}{Ez_{2t}^2},$$

and $\eta_t = \gamma'\varepsilon_t + v_t$. We remark that $Ez_tz_t'$ is entirely governed by $A$ and $\Sigma_\varepsilon$.

From the general results of the preceding section we know that a Misspecification Equilibrium exists for $\phi \geq 0$ sufficiently small. For the bivariate case existence can be shown for all $\phi \geq 0$. Furthermore, we will show that this equilibrium is unique.

## 3.1 Misspecification Equilibrium

If condition $\triangle$ is satisfied then this guarantees a unique $\xi_1, \xi_2$ for each $n' = (n_1, n_2)$, and a unique RPE. Since $n_2 = 1 - n_1$, in this section we define the key functions in terms of $n_1$ rather than $n$. Thus, in particular, if Condition $\Delta$ holds then (13) defines a continuous map $\xi = \xi(n_1)$.

**Proposition 4** *In the bivariate model, Condition $\triangle$ is satisfied for all $\phi \geq 0$. Hence there exists a unique RPE for every $n_1 \in [0, 1]$.*

From Theorem 2 it follows that there exists a ME. By developing the details we can obtain additional results. The profit functions are given by

$$
\begin{aligned}
E\pi^1 &= \frac{1}{2}\phi D\left(\xi_1^2(n_1) - \xi_2^2(n_1)\rho^2\right) Ez_{1t}^2 + \phi D\left(\xi_1(n_1) + \xi_2(n_1)\rho\right)\xi_2(n_1) Ez_1 z_2 - D\sigma_v^2 \\
E\pi^2 &= \frac{1}{2}\phi D\left(\xi_2^2(n_1) - \xi_1^2(n_1)\tilde{\rho}^2\right) Ez_{2t}^2 + \phi D\left(\xi_2(n_1) + \xi_1(n_1)\tilde{\rho}\right)\xi_1(n_1) Ez_1 z_2 - D\sigma_v^2,
\end{aligned}
$$

and we define

$$F(n_1) = E\pi^1 - E\pi^2.$$

In order to prove existence of a unique ME, we need to show that the profit difference function $F(\xi(n_1))$ is monotonic.

**Lemma 5** *In the bivariate model, the function $F(n_1)$ is monotonically decreasing for all $\phi \geq 0$.*

We remark that it is possible to instead have a positive slope for the profit difference function $F(n_1)$ when $\phi < 0$. In this case it will be possible to have multiple equilibria. Examples with $\phi < 0$ are the focus of future research.

The predictor proportion mapping (11) can be written

$$n_1 = \frac{1}{2}\tanh\left[\frac{\alpha}{2}\left(E\pi^1 - E\pi^2\right)\right] + \frac{1}{2} \equiv H_\alpha(E\pi^1 - E\pi^2),$$

where $H_\alpha : \mathbb{R} \rightarrow [0, 1]$ is a strictly increasing function. Note that we use $F$ and $H_\alpha$ in place of $\tilde{F}$ and $\tilde{H}_\alpha$ to emphasize that in contrast to the previous section the domain of $F$ and the range of $H_\alpha$ is now $[0,1]$ instead of the unit simplex $S$. This will simplify some of the arguments below.

Because Condition $\triangle$ is satisfied for all $\phi \geq 0$, there exists a well defined mapping $T_\alpha = H_\alpha \circ F$. $T_\alpha : [0,1] \to [0,1]$, which is continuous. From Lemma 5 it follows that $T_\alpha$ is a continuous, decreasing function for each $\alpha$. It immediately follows that there is a unique fixed point, i.e., we have:

**Proposition 6** *Suppose $z_t$ is a bivariate VAR(1). If $\phi \geq 0$ the model has a unique Misspecification Equilibrium.*

Theorem 6 demonstrates that there is a unique equilibrium in the belief parameters and the proportion of agents using the two misspecified models. It does not tell us how agents are distributed between the predictors. Our main interest is in showing that it is possible for there to be intrinsic heterogeneity. Unlike (Brock and Hommes 1997) who obtain heterogeneity as an automatic implication of assuming that $\alpha$ is finite, we want to show that there exists cases of heterogeneity even in the limit as $\alpha \to \infty$. We now take up this issue.

## 3.2 Intrinsic Heterogeneity

The previous section established uniqueness of the misspecification equilibrium. We now discuss more specific properties of this equilibrium.

From the equations for expected profit, it can be shown that[9]

$$\begin{aligned} F(1) &\gtrless 0 \text{ iff } \xi_1^2(1) \gtrless \xi_2^2(1)Q, \text{ and} \\ F(0) &\gtrless 0 \text{ iff } \xi_1^2(0) \gtrless \xi_2^2(0)Q \end{aligned}$$

where $Q = \frac{Ez_2^2}{Ez_1^2} > 0$. Furthermore, from (13) we have

$$\begin{aligned} \frac{\xi_1^2(0)}{\xi_2^2(0)} &= \frac{(1+\phi)^2 (\gamma_1 a_{11} + \gamma_2 a_{21})^2}{(\gamma_1 a_{12} + \gamma_2 a_{22} - \phi\tilde{\rho}(\gamma_1 a_{11} + \gamma_2 a_{21}))^2} \equiv B_0 \\ \frac{\xi_1^2(1)}{\xi_2^2(1)} &= \frac{(\gamma_1 a_{11} + \gamma_2 a_{21} - \phi\rho(\gamma_1 a_{12} + \gamma_2 a_{22}))^2}{(1+\phi)^2 (\gamma_1 a_{12} + \gamma_2 a_{22})^2} \equiv B_1 \end{aligned}$$

These expressions assume that the denominators of the expressions are non-zero. Recall that $Q, \rho$, and $\tilde{\rho}$ are determined by $A$ and $\Sigma_\epsilon$. The above results and Lemma 5 imply:

**Lemma 7** *There are three possible cases depending on $\phi, \gamma, A$ and $\Sigma_\epsilon$.*

[9]The Appendix contains additional details of these derivations.

1. *Condition P: $F(0) > 0$ and $F(1) < 0$. Condition P is satisfied when $B_1 < Q < B_0$.*

2. *Condition P0: $F(0) < 0$ and $F(1) < 0$. Condition P0 is satisfied when $Q > B_0$.*

3. *Condition P1: $F(0) > 0$ and $F(1) > 0$. Condition P1 is satisfied when $Q < B_1$.*

Below we give numerical examples of when each condition may arise.

Under Condition P0, $F(1) < 0$ implies that model 2 is always more profitable. Under Condition P1, model 1 is always more profitable. In these cases we anticipate homogeneous expectations as the 'intensity of choice' $\alpha \to \infty$. However, if Condition P obtains there is an incentive to deviate from the consensus selection. We have the following result.

**Proposition 8** *Consider again the model with $z_t$ a bivariate VAR(1) and $\phi \geq 0$. The unique Misspecification Equilibrium $n_1^*$ has one of the following properties:*

1. *Condition P implies that as $\alpha \to \infty$, $n_1^* \to \hat{n}_1 \in (0,1)$ where $F(\hat{n}_1) = 0$. That is, the model has Intrinsic Heterogeneity.*

2. *Condition P0 implies that as $\alpha \to \infty$, $n_1^* \to 0$.*

3. *Condition P1 implies that as $\alpha \to \infty$, $n_1^* \to 1$.*

Proposition 8 establishes the possibility of Intrinsic Heterogeneity. We discuss the intuition further below. This result is novel because, for high $\alpha$, rationality of agents is bounded only through their model parameterizations. Agents fully optimize given their (misspecified) model of the economy. In Brock and Hommes' A.R.E.D. heterogeneity arises because of calculation costs and, most importantly, because with finite $\alpha$ a proportion of agents do not optimize in the sense that they do not fully respond to profit differences. Only in a (nonstochastic) steady-state will agents be evenly distributed across predictors.[10] In our model, agents optimize given their misspecification, all predictors are equally "sophisticated" and costless, and Intrinsic Heterogeneity can arise as part of a stochastic equilibrium. Most interestingly, it is the self-referential feature of the model, combined with underparameterization, that generates this heterogeneity.

[10] This is because in (Brock and Hommes' 1997) set-up all predictors return the same forecast in a steady-state. Hence if a predictor is costless, then it will return the same steady-state net benefit as all other costless predictors. In our model, the nature of the equilibrium forces each predictor to return the same mean profit as $\alpha \to \infty$.

### 3.3 Connection to the Rational Expectations Equilibrium

Our equilibrium differs from the Restricted Perceptions Equilibrium in (Evans and Honkapohja 2001). There agents also underparameterize, but the law of motion is imposed and all agents are homogeneous in their misspecification. These expectations differ from rational expectations by ignoring relevant information. Since all agents ignore the same information in their perceived law of motion it is clear that in equilibrium the parameters of the model will differ from a Rational Expectations Equilibrium (REE). In a Misspecification Equilibrium with Intrinsic Heterogeneity, each agent uses an underparameterized model, but aggregate expectations are conditioned on all available information. In principle, it is conceivable that a ME could reproduce the REE. In this subsection we use the bivariate example to show that this is not the case: the price process in a ME will differ from the process in an REE.

Recall that

$$p_t = -\phi p_t^e + \gamma' A z_{t-1} + \eta_t \tag{14}$$

where $\gamma$ is $(2 \times 1)$, $A$ is $(2 \times 2)$ with elements $a_{ij}$ for $j = 1, 2$, and $\eta_t = \gamma' \varepsilon_t + v_t$. Under rational expectations

$$p_t^e = E_{t-1} p_t \tag{15}$$

An REE is a stochastic process $p_t$ that satisfies (14) and (15). The cobweb model has a unique REE given by

$$p_t = \hat{\xi}_1 z_{1,t-1} + \hat{\xi}_2 z_{2,t-1} + \eta_t$$

where

$$\begin{aligned} \hat{\xi}_1 &= (1+\phi)^{-1} (\gamma_1 a_{11} + \gamma_2 a_{21}) \\ \hat{\xi}_2 &= (1+\phi)^{-1} (\gamma_1 a_{12} + \gamma_2 a_{22}) \end{aligned}$$

The parameters in a Misspecification Equilibrium are given by

$$\begin{bmatrix} 1 + n_1^* \phi & \phi n_1^* \rho \\ \phi(1 - n_1^*)\tilde{\rho} & 1 + (1 - n_1^*)\phi \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = A' \gamma, \tag{16}$$

where $n_1^* \in N_\alpha$. We saw that a non-trivial solution to (16) exists for all $\phi \geq 0$ and is given by

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \frac{1}{\triangle} \begin{bmatrix} (1 + (1 - n_1^*)\phi)(\gamma_1 a_{11} + \gamma_2 a_{21}) - \phi n_1^* \rho (\gamma_1 a_{12} + \gamma_2 a_{22}) \\ (1 + n_1^* \phi)(\gamma_1 a_{12} + \gamma_2 a_{22}) - \phi (1 - n_1^*) \tilde{\rho} (\gamma_1 a_{11} + \gamma_2 a_{21}) \end{bmatrix}$$

where $\triangle = (1 + n_1^* \phi)(1 + (1 - n_1^*)\phi) - \phi^2 n_1^* \rho \tilde{\rho}$.

Clearly the REE parameters $(\hat{\xi}_1, \hat{\xi}_2)'$ differ from the ME parameters $(\xi_1, \xi_2)'$. For example, consider the case when the random variables $z_{1,t}, z_{2,t}$ are uncorrelated. Then

$$\begin{aligned} \xi_1 &= (1 + n_1^* \phi)^{-1} (\gamma_1 a_{11} + \gamma_2 a_{21}) \\ \xi_2 &= (1 + (1 - n_1^*)\phi)^{-1} (\gamma_1 a_{12} + \gamma_2 a_{22}) . \end{aligned}$$

### 3.4 Further Discussion

The intuition behind Condition P and the existence of Intrinsic Heterogeneity is subtle. In a cobweb model the exogenous shocks $z$ have both a direct and an indirect effect on price. The direct effect is simply the $\gamma' z_t$ term. The indirect effect depends on the way in which agents incorporate $z$ into their expectations. It is the interplay between the direct and indirect effects that makes intrinsic heterogeneity possible. In this subsection we illustrate the intuition through a simple example.

Suppose that the components $z_{1,t}, z_{2,t}$ are uncorrelated. Then the RPE is given by

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} (1+n_1\phi)^{-1} & 0 \\ 0 & (1+(1-n_1)\phi)^{-1} \end{bmatrix} \begin{bmatrix} \gamma_1 a_{11} + \gamma_2 a_{21} \\ \gamma_1 a_{12} + \gamma_2 a_{22} \end{bmatrix}$$

Recall that

$$p_t = \xi_1 z_{1,t-1} + \xi_2 z_{2,t-1} + \eta_t$$

Now set $\phi = 0$. This is the case where there is no feedback from expectations to price. In this special case

$$\begin{aligned} \xi_1 &= (\gamma_1 a_{11} + \gamma_2 a_{21}) \\ \xi_2 &= (\gamma_1 a_{12} + \gamma_2 a_{22}) \end{aligned}$$

The parameters $\xi_1, \xi_2$ are completely determined by the direct effect $\gamma' A$. For $\phi > 0$, the RPE parameters are

$$\begin{aligned} \xi_1 &= (1+n_1\phi)^{-1} (\gamma_1 a_{11} + \gamma_2 a_{21}) \\ \xi_2 &= (1+(1-n_1)\phi)^{-1} (\gamma_1 a_{12} + \gamma_2 a_{22}), \end{aligned}$$

and now depend both on the direct effect $\gamma' A$ and the indirect effect of expectations through $n_1$ and $\phi$. Note in particular that as $n_1 \to 1$ we have $|\xi_1(n_1)| \downarrow$ and $|\xi_2(n_1)| \uparrow$. For a fixed $\phi$ the indirect effect depends on $n_1$. As agents mass onto a particular predictor it diminishes the effect of that variable. This is because of the self-referential feature of the cobweb model that leads to an indirect effect on prices opposite to the direct effect of that variable. This makes $z_{1,t}$ a less useful predictor than before, and thus the $z_{2,t}$ component becomes more profitable. The opposite happens as $n_1 \to 0$ and consequently there is a unique $n_1$ in which both predictors fare equally well in terms of mean profits. This proportion is the limit point of Intrinsic Heterogeneity.

Condition P places conditions on the indirect and direct effects and on the relative importance of the two exogenous variables. In our simple example of uncorrelated shocks Condition P is equivalent to

$$\frac{(\gamma_1 a_{11} + \gamma_2 a_{21})^2}{(1+\phi)^2 (\gamma_1 a_{12} + \gamma_2 a_{22})^2} < Q < \frac{(1+\phi)^2 (\gamma_1 a_{11} + \gamma_2 a_{21})^2}{(\gamma_1 a_{12} + \gamma_2 a_{22})^2}$$

where $Q = \frac{Ez_2^2}{Ez_1^2}$. When there is no feedback ($\phi = 0$) there does not exist a matrix $A$ and $\Sigma_\varepsilon$ which satisfies Condition P. Intrinsic Heterogeneity does not exist in this instance. Because there is no indirect effect from expectations, and expectations have no bearing on price, agents will choose the model that forecasts price best. As $\phi$ increases, the range of admissible $Q$ increases. Given $A, \gamma$ and $Q$, condition $P$ will hold for $\phi > 0$ sufficiently large.

## 3.5 Numerical Examples

We illustrate our results numerically. Figure 1 gives the T-maps for various values of $\alpha$. The upper part of the figure shows the T-maps corresponding to (starting from $n_1 = 0$ and moving clockwise) $\alpha = 2, \alpha = 20, \alpha = 50, \alpha = 100, \alpha = 200, \alpha = 2000$. We set

$$A = \begin{bmatrix} .3 & .10 \\ .10 & .7 \end{bmatrix},$$

$\gamma' = [.7, .5]$,

$$\Sigma_\varepsilon = \begin{bmatrix} .7 & .2 \\ .2 & .6 \end{bmatrix},$$

and $\phi = 2$. The bottom portion of the figure is the profit difference function $F(n_1)$.

INSERT FIGURE 1 HERE

The matrix $A$ and parameter $\phi$ have been chosen so that Condition P holds, i.e. $F(1) < 0$. The proof of Proposition 8 shows that as $\alpha \to \infty$

$$H_\alpha(x) \to \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x < 0 \\ 1/2 \text{ if } x = 0 \end{cases}$$

and clearly this will govern the behavior of $T_\alpha = H_\alpha \circ F$. Figure 1 illustrates how as $\alpha$ increases the inverse S-shape becomes more pronounced. The dashed line is the 45-degree line and all fixed points of the T-map will intersect this line. As $\alpha$ increases the fixed point declines from above .5 to about .22, which is the point at which $F(\hat{n}_1) = 0$. The Misspecification Equilibrium continues to exhibit heterogeneity even as $\alpha \to \infty$.

Figures 2 illustrates how heterogeneity may disappear as $\alpha \to \infty$. We now set

$$A = \begin{bmatrix} .93 & .10 \\ .10 & .2 \end{bmatrix},$$

so that Condition P does not hold and instead condition P1 is satisfied. For low values of $\alpha$ some agents continue to use $z_2$ even though it returns a lower expected payoff. However, as $\alpha \to \infty$ all agents behave optimally and the proportion using $z_2$ goes to zero.

INSERT FIGURE 2 HERE

Figure 3 shows the role $\phi$ plays in the degree of Intrinsic Heterogeneity. This graph depicts the T-map for various increasing values of $\phi$. Notice that as $\phi$ increases the fixed point of the T-map moves further to the left. In this example, $z_1$ has a stronger influence on the price than $z_2$. When $z_2$ has a stronger effect, the fixed point will move to the right.

INSERT FIGURE 3 HERE

Note that in a Misspecification Equilibrium, in a model with Intrinsic Heterogeneity, all predictors have the same average return as the intensity of choice $\alpha$ becomes large. When $\alpha$ is finite there can be differences in the relative performance of predictors, but as $\alpha \to +\infty$ the mean returns across predictor must converge given our assumption of costless (or equally costly) predictors. Heterogeneity arises in the costless case of (Brock and Hommes 1997) only in the steady-state in which different predictors make identical forecasts. Our results arise in a stochastic equilibrium in which different predictors produce different forecasts, but achieve identical mean performance, as $\alpha \to +\infty$.

# 4 Stability under Real-Time Learning

In this section we address whether the equilibrium is attainable under real-time learning of the type emphasized in (Evans and Honkapohja 2001). In a Misspecification Equilibrium agents misspecify, but their forecasts are the optimal linear projections given their underparameterization. Furthermore, agents choose which component of the exogenous process to omit based on unconditional mean profits. We now substitute optimal linear projections with real-time estimates formed via recursive least squares (RLS).[11] We also assume that agents choose their model each period based on a real-time estimate of mean profits.

Prices now depend on time-varying parameters

$$p_t = \xi_1(b^1_{t-1}, n_{1,t-1})z_{1,t-1} + \xi_2(b^2_{t-1}, n_{1,t-1})z_{2,t-1} + \eta_t$$

in which $b^1_{t-1}, b^2_{t-1}$ are updated by RLS

$$\begin{aligned} b^1_t &= b^1_{t-1} + t^{-1}R^{-1}_{1,t}z_{1,t-1}\left(p_t - b^1_{t-1}z_{1,t-1}\right) \\ b^2_t &= b^2_{t-1} + t^{-1}R^{-1}_{2,t}z_{2,t-1}\left(p_t - b^2_{t-1}z_{2,t-1}\right) \end{aligned}$$

[11] For an overview of stability under RLS in dynamic macroeconomics see (Evans and Honkapohja 2001).

where

$$R_{1,t} = R_{1,t-1} + t^{-1}(z^2_{1,t-1} - R_{1,t-1})$$
$$R_{2,t} = R_{2,t-1} + t^{-1}(z^2_{2,t-1} - R_{2,t-1})$$

The $R_{j,t}, j = 1, 2$ are recursive estimates of the variances of the explanatory variables $z_j$.

Given these beliefs agents estimate the mean profits associated with each model

$$\hat{E}\pi_{1,t} = \hat{E}\pi_{1,t-1} + t^{-1}\left(\pi_{1,t} - \hat{E}\pi_{1,t-1}\right)$$
$$\hat{E}\pi_{2,t} = \hat{E}\pi_{2,t-1} + t^{-1}\left(\pi_{2,t} - \hat{E}\pi_{2,t-1}\right)$$

The mean profits map into predictor proportions according to the law of motion

$$n_{j,t} = \frac{\exp\left[\alpha \hat{E}\pi_{j,t}\right]}{\sum_{j=1}^{2} \exp\left[\alpha \hat{E}\pi_{k,t}\right]}$$

The dynamic version of the model exhibits real-time learning in the sense that agents adaptively update previous estimates of their belief parameters and the mean profits from those beliefs. Agents now choose their model in each time period based on these recursive estimates. We are interested in whether the sequence of estimates $b^1_t, b^2_t$ and predictor proportions $n_{1,t}$ converge to the Misspecification Equilibrium.[12] Our aim is to use numerical illustrations to show that the equilibrium can be stable under real-time learning. It is beyond the scope of this paper to establish analytical convergence results for this learning rule.

We continue with a particular parameterization that generated Intrinsic Heterogeneity in the previous section. We set

$$A = \begin{bmatrix} .3 & .1 \\ .1 & .7 \end{bmatrix}, \ \Sigma_\varepsilon = \begin{bmatrix} .7 & .2 \\ .2 & .6 \end{bmatrix},$$

$\gamma' = [.7, .5]$, and $\phi = 2$, and simulate the model for 100,000 time periods. We set the initial value of the VAR equal to a realization of its white noise shock, i.e., $z_0 = \varepsilon_0$. The initial value for $n_{1,0}$ is 0.82, a value that was chosen to lie away from the end points and the ME. Initial estimated mean profits are equal to the realized profits under the initial conditions. The initial belief parameters were set to $b^1_0 = 1, b^2_0 = 2$. The initial estimated variances $R_{1,0}, R_{2,0}$ are the identity matrices. We choose $\alpha = 100$.

Figure 4 illustrates the results of a representative simulation. The top panel plots the simulated proportion $n_{1,t}$ against time. The middle and bottom panels plot

[12] Since we conduct the analysis numerically, we are being deliberately vague in what sense these sequences converge.

the simulated law of motion parameters $\xi_{1,t}, \xi_{2,t}$. In each plot the solid horizontal line represents the respective variables' values in the Misspecification Equilibrium with Intrinsic Heterogeneity. As can be seen, there appears to be convergence to the ME. Initially there is considerable volatility in the proportion of agents who choose predictor 1. This volatility gradually dampens until the proportion approaches its equilibrium value. The dampening is much quicker in belief parameters as they approach their equilibrium values in a short period of time. Similar convergence results were obtained for other parameter settings but the qualitative results were affected by $\alpha$. For larger values of $\alpha$ it takes longer for the predictor proportions to settle down near the equilibrium values. However, the system appears to be stable for all $\alpha > 0$.

INSERT FIGURE 4 HERE

The intuition behind the stability is as follows. In our parameterization there is a unique ME with Intrinsic Heterogeneity. Heterogeneity arises because Condition P guarantees that under, say, $z_1$ homogeneity agents will have an incentive to mass on $z_2$, and vice-versa. For large $\alpha$ agents mass on the predictor that returns the highest mean profit. In our simulations the proportions of agents are initially well away from the ME. This implies that one predictor has a higher profit than the other. In the next period agents mass onto that predictor. Because Condition P holds, in the next period agents mass onto the other predictor. As the rapid switching occurs agents update parameter estimates, which converge quickly, and accumulate data on relative mean forecast performance. As they learn about mean relative forecast performance, the volatility in predictor selection dampens and there is convergence towards the Misspecification Equilibrium.

In the light of (Brock and Hommes 1997) our results may seem surprising. However, in (Brock and Hommes 1997) the model is deterministic, the predictor choice is between a costly stabilizing predictor and a costless destabilizing predictor, and predictor fitness is the most recent period's realized profits. The stability results in our model are the result of agents looking at the mean relative performance of the predictors using the whole history of profits. This feature seems more appropriate within the stochastic model we examine, and is a key feature generating our numerical stability results.

# 5 Conclusion

This paper demonstrates how to obtain heterogeneous expectations as an equilibrium outcome in a model with optimizing agents. Our set-up is the standard cobweb model in which rational expectations was originally developed. We obtain our results with a discrete choice model for predictors, when agents are constrained to choose from a set of misspecified models. As in (Brock and Hommes 1997) the proportion of agents using the different predictors depends on their relative performance according to an intensity of choice parameter. As the intensity of choice increases agents will select only the most successful predictors. In (Brock and Hommes 1997) heterogeneity of expectations is a reflection of finite intensities of choice and disappears in the neoclassical limit. One of the main contributions of this paper is to show that heterogeneity can remain for high intensities of choice as a result of the availability of multiple misspecified models.

Because of limits in cognition, knowledge of the economy, degrees of freedom, etc., we assume that agents must underparameterize by neglecting a variable or lag from their forecasting model. The importance of misspecification is widely recognized in applied econometrics and one that we believe should be reflected in realistic models of bounded rationality. Although we constrain agents to choose from a list of misspecified models, at the same time we require that the parameters of each chosen model are formed optimally in the sense that forecast errors are orthogonal to the explanatory variables of that model.

Our major theoretical contribution is to obtain existence results for a Misspecification Equilibrium within this framework and to obtain a suitable condition under which heterogeneous expectations persists for high intensities of choice. When this condition is satisfied we say the model exhibits Intrinsic Heterogeneity.

Our central finding that misspecification can lead to heterogeneous expectations is not at all obvious. If the intensity of choice is large, a key requirement for this possibility is that the model be self-referential, i.e., that there be feedback from expectations to actual outcomes. Heterogeneous expectations are not a necessary outcome when the intensity of choice is large, but do arise under a suitable joint condition on the model and the exogenous driving processes. We illustrate the results in a simple bivariate model. In particular, we show that, *ceteris paribus*, Intrinsic Heterogeneity arises when the parameter governing the self-referential extent of the model is sufficiently large. This surprising feature of self-referential models has not been noted in previous work.

In this paper we have focused on the cobweb model. In future work, we will examine the framework in a Lucas-type monetary model. The Lucas-type model shares a similar reduced-form as the cobweb model, but expectations have a positive feedback on price. Since the self-referential feature of these models is central, a model with positive feedback can be expected to yield distinct results.

# A Appendix

**Proof of Proposition 1.** Consider the matrix

$$[\triangle] = \left( I + \phi \sum_{j=1}^{K} n^j \Omega' u^{j'} \left( u^j \Omega u^{j'} \right)^{-1} u^j \right).$$

The absolute value of the indicated sum has a maximum value when considered as a function of $n \in S$. Hence for $|\phi|$ sufficiently small $[\triangle]$ is strictly diagonally dominant (see Horn and Johnson (1985), p. 302) for all $n \in S$. Strictly diagonally dominant matrices have non-zero determinants and hence are invertible. ■

**Proof of Theorem 3.** Suppose to the contrary that the model does not exhibit intrinsic heterogeneity. From Theorem 2 we know that a ME exists for every $\alpha$. Since the model does not have intrinsic heterogeneity, then for all $\bar{n} < 1$ there are infinitely many $\alpha$ such that $n_k^* > \bar{n}$ for some component $k = 1, \ldots, K$ where $n^* \in N_\alpha$. Hence there exists a sequence indexed by $\hat{s}$ such that $\alpha(\hat{s}) \to \infty$ with fixed points $n^*(\hat{s})$ satisfying $n^*_{k(\hat{s})}(\hat{s}) \to 1$. It follows that for some $i \in \{1, \ldots, K\}$ there exists a subsequence indexed by $s$ such that $\alpha(s) \to \infty$ and $n_i^*(s) \to 1$. The expected profit functions $\tilde{F}_j(n)$ are continuous and hence for this sequence

$$E\pi^k(s) - E\pi^i(s) \to \tilde{F}_k(e_i) - \tilde{F}_i(e_i),$$

for all $k = 1, \ldots, K$, where $e_i$ is the unit coordinate vector with component $i$ equal to one. However, condition P implies that there exists $j \neq i$ such that $\tilde{F}_j(e_i) - \tilde{F}_i(e_i) > 0$. It follows from (11) that

$$n_i^*(s) = \frac{1}{1 + \sum_{k \neq i} exp\{\alpha(s)(E\pi^k(s) - E\pi^i(s)\}}.$$

Thus $n_i^*(s) \to 0$ as $s \to \infty$. This contradicts $n_i^*(s) \to 1$ and hence the model must exhibit intrinsic heterogeneity. ■

**Proof of Proposition 4.** We want to show

$$\triangle = (1 + n_1\phi)\left((1+\phi) - \phi n_1\right) - \phi^2 \rho\tilde{\rho} \left(n_1 - n_1^2\right) > 0$$

or equivalently

$$\triangle = \phi^2 \left(\rho\tilde{\rho} - 1\right) n_1^2 + \phi^2 \left(1 - \rho\tilde{\rho}\right) n_1 + (1+\phi)$$

The equation $\triangle$ is a quadratic concave in $\phi$. Evaluated at the end points ($n_1 = 0$ and $n_1 = 1$) $\triangle > 0$. The quadratic is maximized at $n_1 = 1/2$ and returns a value of $\triangle(1/2) = (1/2)\phi^2 + (1+\phi) > 0$. Since $\triangle$ is concave and is positive at both its extrema, we conclude that Condition $\triangle$ is satisfied. ■

**Proof of Lemma 5.** We can rewrite (13) as

$$S(n_1)\xi = A'\gamma,$$

where $\xi' = (\xi_1, \xi_2)$ and $S(n_1)$ is the indicated $2 \times 2$ matrix. Differentiating we obtain $(dS)\xi + S(d\xi) = 0$ and

$$\frac{d\xi}{dn_1} = -S^{-1}\frac{dS}{dn_1}\xi.$$

It is easily seen that

$$\frac{dS}{dn_1} = \phi \begin{pmatrix} 1 & \rho \\ -\tilde{\rho} & -1 \end{pmatrix}.$$

Somewhat abusing notation, it is now convenient to rewrite $F(n_1)$ as $F(\xi(n_1))$. To establish the result we compute $dF/dn_1 = (dF/d\xi)'(d\xi/dn_1)$. It can be verified that

$$(\frac{dF}{d\xi})' = \phi D(1-r^2)Ez_1^2\xi' \begin{pmatrix} 1 & 0 \\ 0 & -Q \end{pmatrix},$$

where $Q = Ez_2^2/Ez_1^2$.

Thus

$$\begin{aligned} dF/dn_1 &= -\phi^2 D(1-r^2)Ez_1^2\xi' K(n_1)\xi, \text{ where} \\ K &= \begin{pmatrix} 1 & 0 \\ 0 & -Q \end{pmatrix} S^{-1} \begin{pmatrix} 1 & \rho \\ -\tilde{\rho} & -1 \end{pmatrix}. \end{aligned}$$

Here $r^2 = \rho\tilde{\rho}$ with $0 \le r^2 < 1$. Computation of $K$ yields

$$K(n_1) = \begin{pmatrix} \frac{1+\phi(1-n_1(1-r^2))}{1+\phi+\phi^2(1-r^2)n_1(1-n_1)} & \frac{\sqrt{Q}r(1+\phi)}{1+\phi+\phi^2(1-r^2)n_1(1-n_1)} \\ \frac{\sqrt{Q}r(1+\phi)}{1+\phi+\phi^2(1-r^2)n_1(1-n_1)} & \frac{Q(1+\phi r^2+\phi(1-r^2)n_1)}{1+\phi+\phi^2(1-r^2)n_1(1-n_1)} \end{pmatrix}.$$

$K(n_1)$ is symmetric with $K_{11}(n_1) > 0$ and

$$\det(K(n_1)) = \frac{Q(1-r^2)}{1+\phi+\phi^2(1-r^2)n_1(1-n_1)} > 0.$$

Thus $K(n_1)$ is positive definite and $\xi' K(n_1)\xi \ge 0$ for all $\xi$. The result follows since $dF/dn_1 \le 0$ for all $0 \le n_1 \le 1$. ∎

**Further Details For Section 3.2.** Using the profit functions derived above we can find

$$\begin{aligned} \frac{F(1)}{Ez_1^2} &= -\phi D\{(\xi_1^2(1)\tilde{\rho} - \xi_2^2(1)\rho)\rho + (1/2)(\xi_2^2(1) - \\ &\quad \tilde{\rho}^2\xi_1^2(1))Q - (1/2)(\xi_1^2(1) - \rho^2\xi_2^2(1))\} \\ \frac{F(0)}{Ez_2^2} &= \phi D\{\tilde{\rho}[\xi_2^2(0)\rho - \xi_1^2(0)\tilde{\rho}] + (1/2)[(\xi_1^2(0) - \\ &\quad \xi_2^2(0)\rho^2)Q^{-1} - (\xi_2^2(0) - \xi_1^2(0)\tilde{\rho}^2)]\} \end{aligned}$$

Thus, for example,

$$F(1) < 0 \text{ if } \left[\xi_1^2(1) - \xi_2^2(1)\right]\left(Q\tilde{\rho}^2 - 1\right) > 0.$$

Using $Q\tilde{\rho}^2 = r^2 < 1$ it follows that

$$F(1) < 0 \text{ if } \left[\xi_1^2(1) - \xi_2^2(1)\right] < 0.$$

■

**Proof of Proposition 8.** Take part (1), which states that Condition P implies Intrinsic Heterogeneity. We will establish that $(i)$ for each $\alpha$, $\exists n_1^*(\alpha) \in N_\alpha$ uniquely, $(ii)$ $\exists \{\alpha(s)\}_s$ *s.t.* $\alpha(s) \to \infty \Rightarrow n_1^*(\alpha(s)) \to \hat{n}_1$ where $\hat{n}_1 \in N_\infty \equiv \{n_1 \in [0,1] :$ for $\alpha \to \infty$ $n_1 = T_\alpha(n_1)\}$ and $(iii)$ $F(\hat{n}_1) = 0$.

Claim $(i)$ that there exists a unique fixed point $n_1^*(\alpha)$ for each $\alpha$ comes directly from Theorem 6.

Claim $(ii)$ is that there is a sequence $\alpha(s)$ indexed by $s$ defined so that as $\alpha(s) \to \infty$ the corresponding sequence of fixed points from claim $(i)$ $n_1^*(\alpha(s)) \to \hat{n}_1$. That there exists a sequence $\alpha(s) \to \infty$ and a similarly corresponding sequence $n_1^*(\alpha)$ follows from claim $(i)$ and since $\alpha \in \mathbb{R}_+$ there are infinitely many such sequences. Theorem 6 used Brouwer's theorem and Lemma 5 to establish that there exists a unique fixed point for each $\alpha$. Hence there exists a limit to the sequence of fixed points indexed by $s$ and define it to be $n_1^*(\alpha(s)) \to \hat{n}_1$. By construction, $\hat{n}_1 \in N_\infty$.

Claim $(iii)$ is that $F(\hat{n}_1) = 0$. Assume $\hat{n}_1 \in N_\infty$, Condition P, and $F(\hat{n}_1) \neq 0$. It follows that $F(\hat{n}_1) > 0$ or $F(\hat{n}_1) < 0$. Recall, $n_1(\alpha) = H_\alpha(F(n_1))$. By definition, as $\alpha \to \infty$

$$H_\alpha(x) \to \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x < 0 \\ 1/2 \text{ if } x = 0 \end{cases}$$

So we have $n_1^*(\alpha) \to \hat{n}_1 \in \{0, 1\}$. But, assuming Condition P implies $F(1) < 0$ and $F(0) > 0$. Hence, $\hat{n}_1$ is not an ME which contradicts our initial assumption. It must be the case that, with Condition P, $F(\hat{n}_1) = 0$.

Note now that Lemma 5 establishes $\hat{n}_1$ is the unique point where $F(\hat{n}_1) = 0$. Thus, we conclude that Condition P implies $n_1^*(\alpha) \to \hat{n}_1$ where $F(\hat{n}_1) = 0$.

A similar argument establishes parts (2) and (3) of the proposition. Note that Condition P1 implies $F(1) > 0$ and $F(0) > 0$ and Condition P0 has $F(1) < 0$ and $F(0) < 0$. The monotonicity of $F$ means that $\forall n_1, \alpha$ $F(n_1(\alpha)) \neq 0$ and the result follows immediately from above. ■

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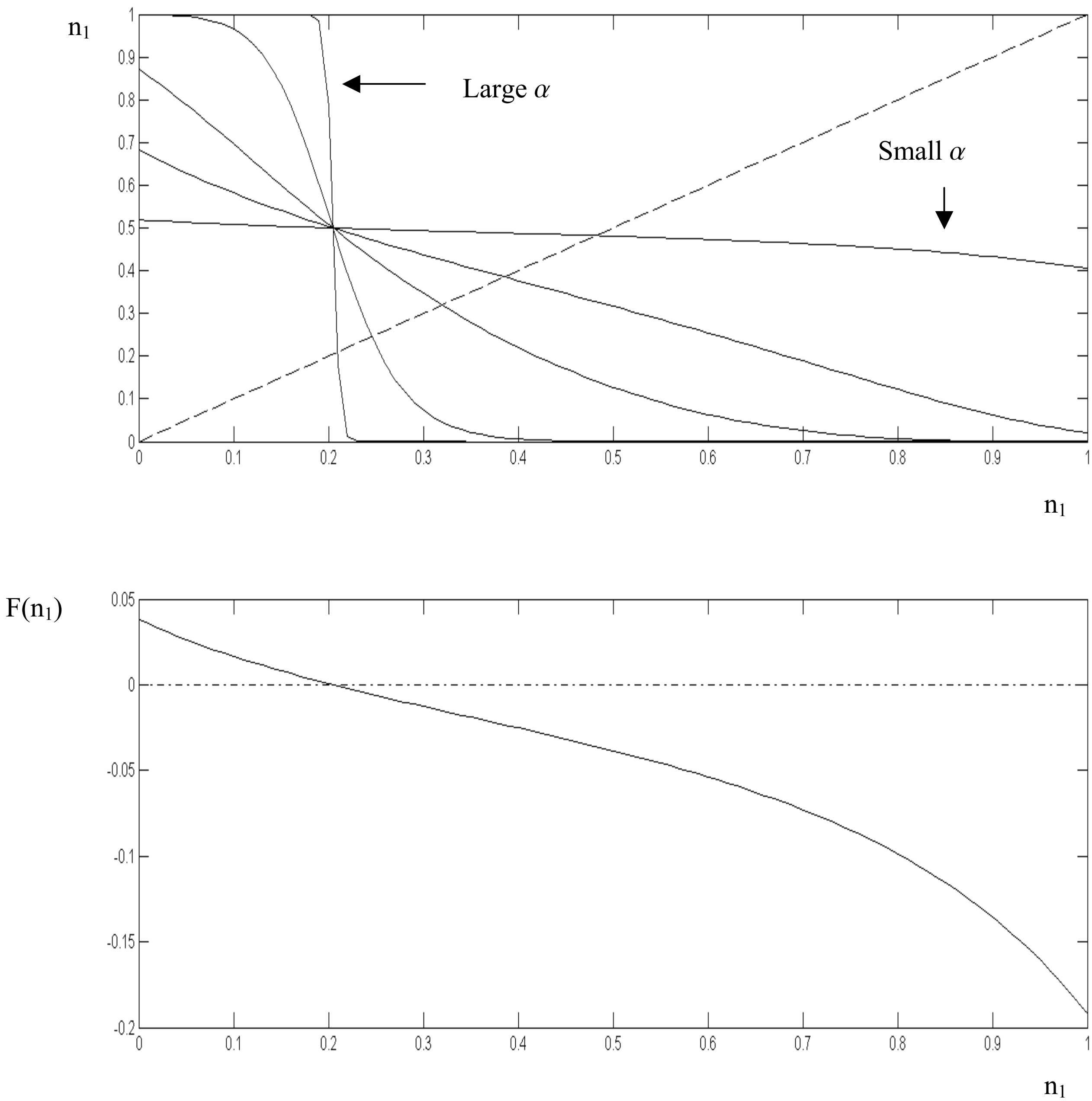


Figure 1: T-map for various values of α and $\phi$ =2.

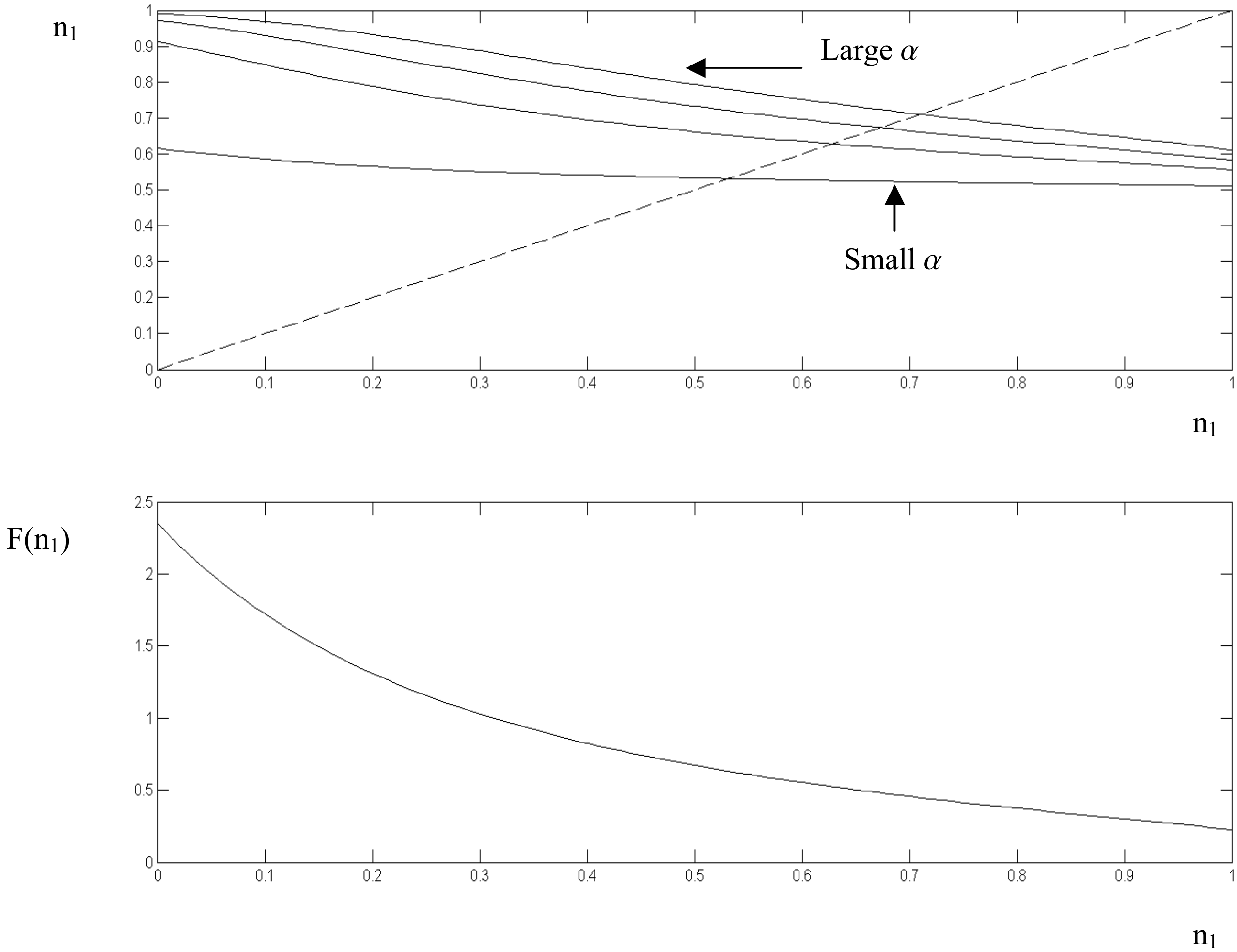


Figure 2. T-map for various values of α and $\phi$=2 for the case of no Intrinsic Heterogeneity and predictor 1 homogeneity.

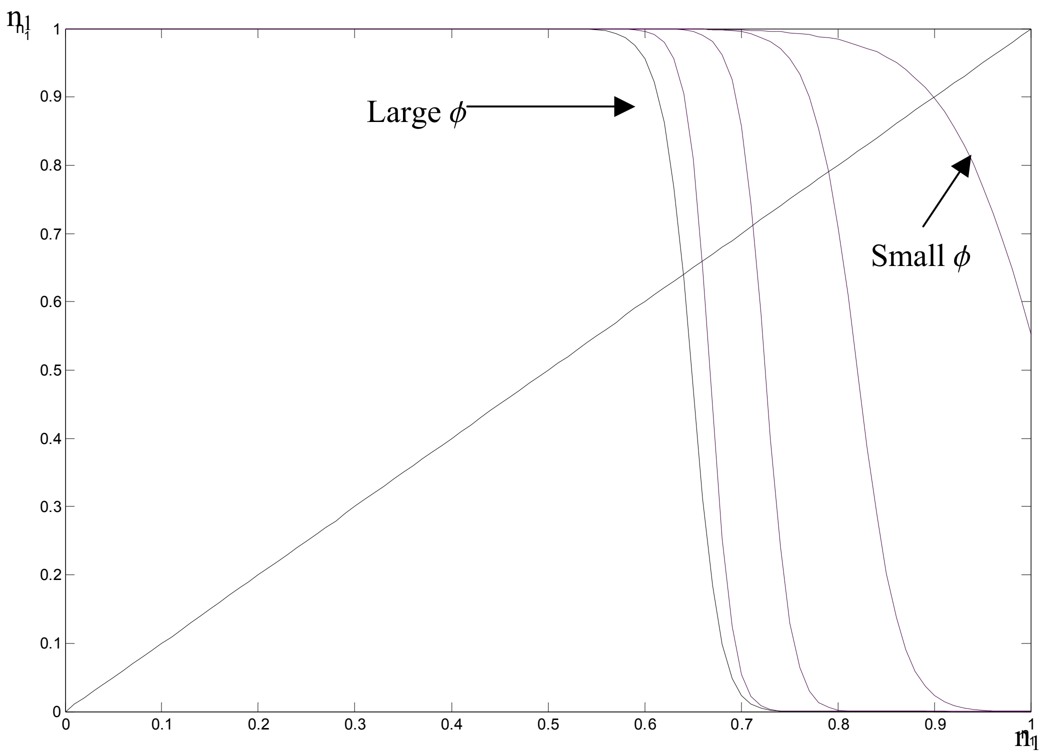


Figure 3. T-map for α=2000 and $\phi$=.5,1,2,5,10,20 for the case of Intrinsic Heterogeneity. Note that as $\phi$ increases the fixed point of the T-map.

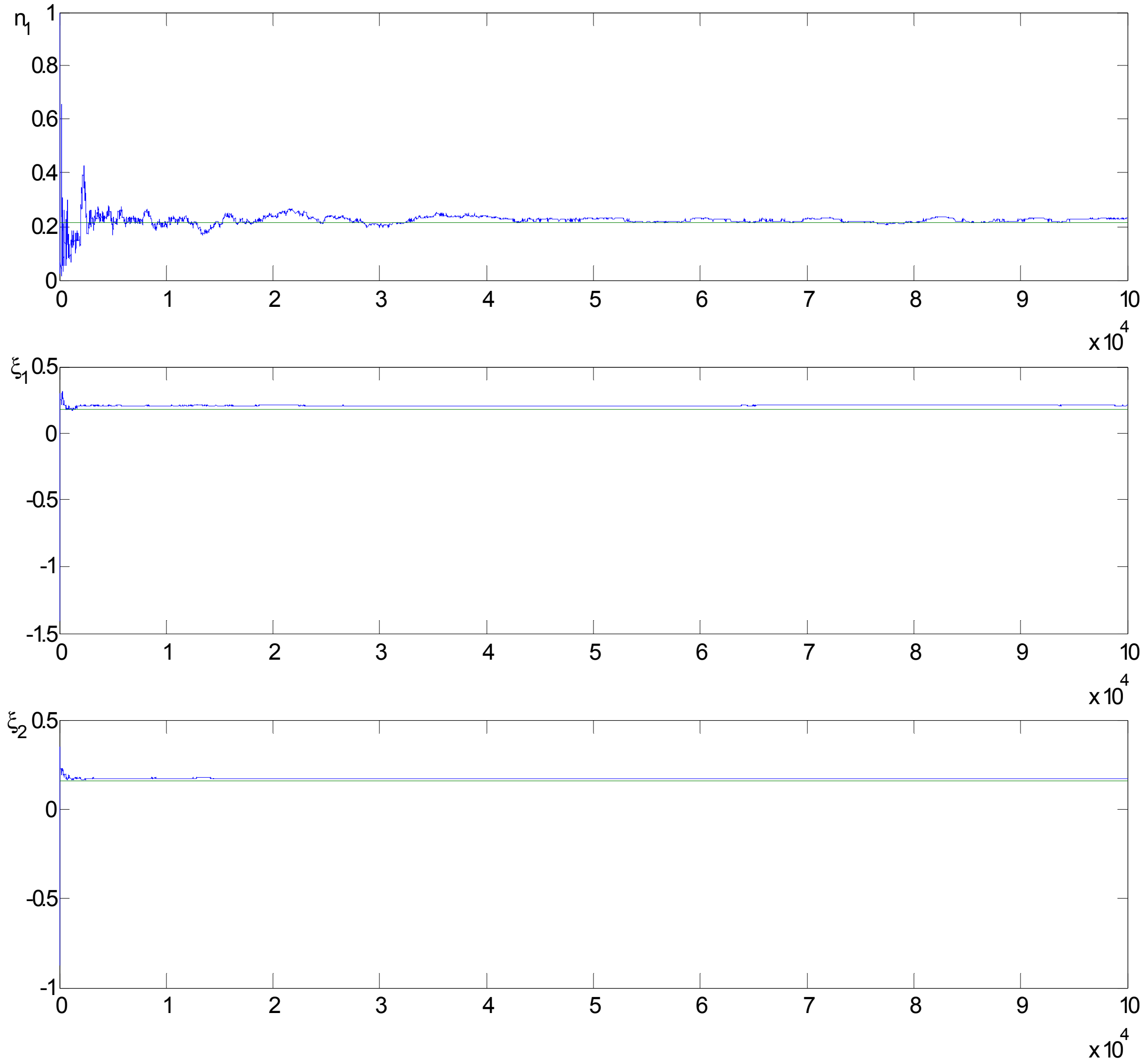


Figure 4. Real-time learning simulations.