The Empirical Trap of Sign Reversals with Equality Restrictions

by

Stephen E. Haynes

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Abstract: This note explores the insidious empirical trap posed by two common equality restrictions in regression analysis. The trap is that restricted coefficients can lie outside the interval of unrestricted coefficients and even reverse sign when negatively correlated regressors are added to one another or when positively correlated regressors are subtracted from one another.

Key Words: Equality restrictions; Sign reversals; Invalid restrictions

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Author: Stephen E. Haynes
Department of Economics
University of Oregon
Eugene, OR 97403
Tel. 541-346-4665
Email: shaynes@uoregon.edu

1. INTRODUCTION

Two classes of equality restrictions -- additive constraints, where explanatory variables are added to one another, and subtractive constraints, where explanatory variables are subtracted from one another -- are routinely imposed in regression analysis and usually untested. Examples include estimation of models with linear aggregation of variables, distributed lags, measurement error, structural change, asymmetry, and where regressors have equal but opposite effects. It is commonly expected that the restricted coefficient in these regressions remains in the interval between the unrestricted coefficients, hence represents a simple "average" of the unrestricted coefficients and certainly does not reverse sign. Unfortunately, this expectation is incorrect in some cases, as additive and subtractive constraints can result in the empirical trap of sign reversals.

Generalizing results in Haynes and Stone (1981), this note explores the conditions that determine when the common "averaging" expectation is warranted and when it is not. These conditions are developed in four cases -- additive constraints, with directly or inversely correlated regressors, and subtractive constraints, with directly or inversely correlated regressors. The primary conclusion is that the "averaging" expectation may not be warranted in two of the four cases -- when negatively correlated regressors are added to one another or when positively correlated regressors are subtracted from one another. Several examples of documented and potential sign reversals and invalid hypothesis tests are also summarized.

2. EQUALITY RESTRICTION: ADDITIVE

The analysis is similar to Theil's (1971, pp. 556-67) examination of linear aggregation, where the expected value of the restricted macroparameter is expressed as a weighted average of the unrestricted microparameters, and the weights sum to unity but are not necessarily all positive. Consider first the simple three-variable linear model

$$Y = \alpha + \beta_1 X_1 + \beta_2 X_2 + u \tag{1}$$

where Y, X_1 and X_2 are column vectors of T observations; α, β_1, and β_2 are parameters, with $\beta_1\beta_2>0$; and u is a column vector of T disturbances. Imposing an additive restriction on eq. (1) is equivalent to estimating the model

$$Y = \alpha' + \delta(X_1 + X_2) + u' \tag{2}$$

under the equality assumption that $\delta=\beta_1=\beta_2$, where α' and δ are parameters and u' is the disturbance.

With variables in deviation form, define d as the least squares estimate of δ. The expected value of the least squares estimate of δ is

$$E(d) = w_1\beta_1 + w_2\beta_2 \tag{3}$$

where $w_1 = [\text{Var}(X_1)+\text{Cov}(X_1,X_2)]/D$; $w_2 = [\text{Var}(X_2)+(\text{Cov}(X_1,X_2)]/D$; and $D = \text{Var}(X_1)+\text{Var}(X_2)+2\text{Cov}(X_1,X_2)$.[1] E(d) is thus a weighted average of β_1 and β_2, where the weights sum to unity but one may be negative. As is apparent from the definition of the

[1] Eq. (3) follows the analysis of least-squares estimation subject to linear restrictions in Johnston and DiNardo (1997, pp. 95-97). Using their eq. (3.43), the OLS estimate of the coefficient column vector β of eq. (1) subject to the restriction that $\beta_1=\beta_2$ is $d = b - (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}Rb$, where b is the unrestricted OLS estimate of the coefficient column vector β, X is the observation matrix of the explanatory variables, and R=[0,1,-1] . The second element in the column vector d is the restricted OLS estimate of β_1 (and β_2), i.e., d in the text; taking expectations of d yields eq. (3).

weights, the sign of $Cov(X_1,X_2)$ is crucial in determining if one weight is negative, which is a necessary condition for a sign reversal.

2.1 Directly Correlated and Uncorrelated Regressors

Assume initially that $Cov(X_1,X_2)\geq 0$. From eq. (3), $E(d)\in[\beta_1,\beta_2]$ if $Cov(X_1,X_2)\geq 0$, and $E(d)\in[\beta_1,\beta_2]$ iff $w_1w_2\geq 0$. Thus, imposition of an additive constraint with directly correlated (or uncorrelated) regressors leads to the common expectation that the expected value of the restricted estimate lies in the interval between the population parameters, i.e., is a simple "average" of these parameters, and certainly cannot reverse sign from these parameters. In this sense, the bias resulting from an invalid additive restriction is likely to be minor in that the restricted coefficient is bounded by the unrestricted population parameters.

Additive restrictions with directly correlated or uncorrelated regressors are common in empirical practice. Consider Theil's (1971, pp. 556-561) aggregation analysis. Microrelations are aggregated to macrorelations, and a macroparameter is a weighted average of microparameters, with the microvariables typically directed correlated.[2]

Another example involves measurement error, where the measured regressor is the sum of the true regressor and measurement error, with the latter two traditionally assumed uncorrelated. The expected value of the restricted coefficient (the coefficient on the measured regressor) lies between the two unrestricted population coefficients (the

[2] Also consider the moving-average restriction used in the estimation of distributed lags. Present and lagged values of an explanatory variable are added under the restriction that they have a common coefficient, and most time series are positively autocorrelated.

coefficient on the true regressor and the zero coefficient on the measurement error term), hence does not reverse sign but is biased towards zero.[3]

2.2 Inversely Correlated Regressors

Assume alternatively that $Cov(X_1,X_2)<0$. From eq. (3), $E(d)\notin[\beta_1,\beta_2]$ only if $Cov(X_1,X_2)<0$; $E(d)\notin[\beta_1,\beta_2]$ iff $w_1w_2<0$; and $E(d)\beta_1<0$ iff $w_1w_2<0$ and, for $w_i<0$, $|w_i\beta_i| > |w_j\beta_j|$ for $i\neq j$. Thus, with an additive constraint and inversely correlated regressors, the expected value of the restricted coefficient may not lie in the interval between the unrestricted parameters, i.e., is not a simple "average" of the unrestricted parameters, and may even reverse sign relative to these parameters. For this case, the bias is potentially serious because of the possibility of a sign reversal.

Unfortunately, additive equality restrictions with inversely correlated regressors are empirically relevant. Consider the example of structural change, i.e., estimating a regression equation where the parameters are assumed constant for a sample but in fact they are not constant. First, partition the data into two subsamples, i.e., define D_1 as a TxT diagonal selection matrix with zeros and ones on the diagonal, I as a TxT identity matrix, X as a Tx1 column vector on the regressor. Then define $D_2=I-D_1$, $X_1=D_1X$, and $X_2=D_2X$, which imply that $X_1+X_2=X$. Importantly, $Cov(X_1,X_2)<0$ if the elements of X are the same sign for all T observations, which is typically the case.

Given these interacted definitions for X_1 and X_2, unrestricted eq. (1) permits the response of Y to changes in X to differ by subsample if there is structural change, i.e., if $\beta_1\neq\beta_2$. However, eq. (2) restricts the response of Y to changes in X to be identical by

[3] To see this, rewrite eq. (1) as $Y=\alpha+\beta_1X_1+0\eta+u$, where X_1 is the true regressor and η is the measurement error. Estimation eq. (2) becomes $Y=\alpha'+\delta(X_1+\eta)+u'$, i.e., an equation with a regressor measured with error.

subsample, hence leads to bias if there is structural change. Furthermore, since the restriction in eq. (2) is additive and X_1 and X_2 are inversely correlated, the expected value of the pooled coefficient estimate may not lie in the interval between the subsample parameters, and may even reverse sign. Finally, unlike the other examples in this note, the restriction of a constant structure by subsample is not actively imposed. It is passively imposed by failing to account for structural change, i.e., by excluding the appropriately interacted regressors, and as a consequence the bias from structural change in eq. (2) may be especially insidious.[4]

3. EQUALITY RESTRICTION: SUBTRACTIVE

The analysis of additive equality restrictions can be converted to the analysis of subtractive equality restrictions by multiplying one of the regressors by minus one. For clarity, the analysis is presented explicitly, where X_2 is the regressor that has been multiplied by minus one.

Consider again eq. (1), repeated for convenience, but now assume $\beta_1\beta_2<0$.

$$Y = \alpha + \beta_1 X_1 + \beta_2 X_2 + u \tag{1}$$

Imposing a subtractive restriction on eq. (1) is equivalent to estimating the model

$$Y = a' + \zeta(X_1 - X_2) + u' \tag{4}$$

under the equality assumption that $\zeta=\beta_1=-\beta_2$. With the variables in deviation form and g the least squares estimate of ζ, the expected value of the least squares estimate of ζ is

$$E(g) = v_1\beta_1 - v_2\beta_2 \tag{5}$$

[4] An analogous problem to structural change is asymmetry, where the response of the dependent variable to a unit increase in a regressor is assumed identical, but of opposite sign, to a unit decrease in the same regressor. Haynes (1983) investigates with demographic data the danger of the restriction of symmetry, and shows that, contrary to common expectations, coefficient estimates based on symmetry do not lie in the interval between the unrestricted estimates, hence lead to spurious conclusions about the determinants of fertility.

where $v_1 = [Var(X_1)-Cov(X_1,X_2)]/F$; $v_2 = [Var(X_2)-Cov(X_1,X_2)]/F$; and $F = Var(X_1)+Var(X_2)-2Cov(X_1,X_2)$. As with additive restrictions, E(g) is a weighted average of β_1 and β_2, where the weights sum to unity but one may be negative, and the sign of $Cov(X_1,X_2)$ is crucial in determining if one weight is negative, a necessary condition for a sign reversal.

3.1 Directly Correlated Regressors

Again, initially assume $Cov(X_1,X_2)>0$. From eq. (5), $E(g)\notin[\beta_1,-\beta_2]$ only if $Cov(X_1,X_2)>0$; $E(g)\notin [\beta_1,-\beta_2]$ iff $v_1v_2<0$; and $E(\zeta)\beta_1<0$ iff $v_1v_2<0$ and, for $v_i<0$, $|v_i\beta_i|>|v_j\beta_j|$ for $i\neq j$. Thus, with a subtractive constraint and directly correlated regressors, the expected value of the restricted coefficient may not lie in the interval between the unrestricted parameters and may even reverse sign. This case is the analog to the previous case -- an additive restriction with inversely correlated regressors -- since in both cases the bias in estimating the restricted coefficient is potentially serious because the expected value of the restricted coefficient may differ in sign from the unrestricted parameters.

Unfortunately, subtractive restrictions with directly correlated regressors are also empirically relevant. Numerous models in international economics, including models of exchange rates/balance of payments, demand and supply equations for trade, international financial and direct investment, purchasing power parity, and covered interest rate arbitrage, impose subtractive constraints between corresponding domestic and foreign explanatory variables under the assumption that the variables have equal but opposite effects on the dependent variable. Furthermore, corresponding domestic and foreign variables are usually directly correlated, both in levels and first differences.

For one example, Haynes and Stone (1981) show that evidence by Frankel (1979) claimed to support the monetary model of exchange rates is spurious because of sign reversals induced by invalid subtractive restrictions between four pairs of domestic and foreign variables (money, income, interest rates, and inflation).[5] For a second example, Blonigen, Davies, and Head (2003) argue that evidence in Carr, Markusen, and Maskus (2001) regarding the appropriate model of foreign direct investment is spurious because of a sign reversal induced by an invalid subtractive restriction between domestic and foreign measures of skilled labor abundance.

Another example of subtractive restrictions and directly correlated regressors involves the forward premium paradox, a major enigma in International Finance. The paradox is that the forward premium (the log difference between the forward exchange rate and the spot exchange rate) predicts future movements in the spot exchange rate, but with the wrong sign. However, the regressor involves an untested subtractive restriction between the forward and spot exchange rates, and these two variables are highly directly correlated, both in levels and first differences. In Oh and Pippenger (1994), the impact of this restriction is examined and a partial resolution of the paradox is suggested.

3.2 Inversely Correlated and Uncorrelated Regressors

Assume alternatively that $Cov(X_1,X_2) \leq 0$. From eq. (5), $E(g) \in [\beta_1,-\beta_2]$ if $Cov(X_1,X_2) \leq 0$; and $E(g) \in [\beta_1,-\beta_2]$ iff $v_1v_2 \geq 0$. As with an additive restriction and directly correlated (or uncorrelated) regressors, a subtractive restriction with inversely correlated (or uncorrelated) regressors cannot lead to serious bias in that the expected value of the

[5] Also, Haynes and Stone (1982) demonstrate that increases in U.S. (foreign) income increase U.S. imports (exports), yet increases in the difference between U.S. and foreign income decrease U.S. net imports, i.e., imports minus exports.

restricted estimate lies in the interval between the population parameters, hence cannot reverse sign. Although examples of this case do not seem common in empirical work, it is presented for completeness.

4. EXTENSIONS

One extension of the above analysis involves the simultaneous imposition of several equality restrictions, both additive and subtractive. Consider the debate about the Ricardian equivalence proposition that an increase in the government deficit induces an equal offsetting increase in private saving because consumers internalize their implied future tax liabilities. The key test concerns regressing consumption against disposable personal income, defined as income minus taxes plus transfers plus government interest payments minus retained earnings. Modigliani and Sterling (1986) show that when the two additive and two subtractive restrictions in the definition are jointly imposed, i.e., when the effect on consumption of a change in income, taxes, transfers, government interest payments, and retained earnings are assumed identical in absolute value, the evidence does not support Ricardian equivalence. However, Kormendi and Meguire (1990, 1995) demonstrate that when these four restrictions are relaxed the evidence becomes consistent with Ricardian equivalence, implying that analysis of the determinants of consumption requires that the components of disposable personal income have distinguishable influences.

The analysis in this note ignores the effect of additional regressors which themselves do not involve restrictions. In the case where these additional regressors are independent of X_1 and X_2, the above analysis remains valid. In the general case where additional regressors are correlated with X_1 and X_2, the impact of equality restrictions becomes more complicated, depending also on the coefficients on the additional regressors and the variance-covariance matrix of all the regressors. However, in this case it remains true that, ceteris paribus, an additive (subtractive) restriction is more likely to cause a sign reversal when the two combined variables are inversely (directly) correlated.[6]

5. CONCLUSION

This note explores the danger of imposing invalid equality restrictions when the objective of regression estimation is to determine the sign of the unrestricted coefficients. The bias of invalid equality restrictions is not likely to be serious in two cases -- additive (subtractive) restrictions with directly (inversely) correlated regressors -- as in these cases the restricted coefficient is a simple "average" of the unrestricted coefficients and cannot reverse sign. However, for the other two cases -- additive (subtractive) restrictions with inversely (directly) correlated regressors -- the bias may well cause the expected value of the constrained coefficient to lie outside the interval of the unconstrained coefficients, and even reverse sign. For these latter two cases, hypothesis tests may frequently lead to either rejection of the true model or acceptance of a false one. The conclusion of the note is that, because of the empirical trap of sign reversals, equality restrictions should be tested whenever possible, especially when imposed for statistical convenience, e.g., to reduce multicollinearity.

[6] The analysis has obvious implications for out-of-sampling forecasting. Eqs. (3) and (5) show that the expected value of the restricted coefficient depends not only on the population parameters, but also on the variance-covariance matrix of the regressors. As a consequence, forecasts based on inappropriately restricted coefficients may perform poorly because they reflect a different variance-covariance matrix of the regressors over the forecast period.

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